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Light scalar fields in a dark universe: models of inflation, dark energy and dark matter

Campos escalares ligeros en un universo oscuro: modelos de inflación, energía oscura y materia oscura.

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Contents

Chapter 1

Introduction

1.1 Motivation

When I was a little child, I remember that sometimes I used to stare at the blue sky asking myself: How far could one fly up in the sky? How high could the airplanes fly? Beyond the clouds? And sometimes, when I could not sleep in the night, I was looking at the window asking myself: What happened with the blue sky? Why is it dark now? What are those glims that adult people call stars? How far are they? How far is the farthest star? Does the space has an end? And if it does, what is it beyond its end? Or is it infinite? What is infinite? But then I was remembering that I had to finish my homework for the next day and to solve problems like how much is it 3 apples plus 2 plums, and all those questions were vanishing in the black night.

Years passed by, and the questions were still travelling in the outer space, and I was more preoccupied with other things. Then, in 1995 I was invited to participate to a kind of summer camp where I assisted to nice talks on various issues related to science and astronomy. That summer I decided to follow the way of science, and of physics in particular. After years dedicated to the study of physics, I can now say that I have formed an idea of how the physical world works. Those questions I had in my childhood turned back recently, wrapped in a box together with their answers, carried on the shoulders by a strange giant that I had never seen before. And then my question was: Who is this giant?

In the last five years of doing the research work contained in this doctoral Thesis, I had the opportunity to learn a lot about what people think about the giant I mentioned above, and I realized that although its existence is clear to everybody, still nobody contrived to be its friend, to discover its origins, its real name and identity. What people have done is to assign different names to its different parts of the body and to estimate its weight and hight. I am grateful to have the chance to do the same as most of these people, I also tried to be the friend of the giant and to ask it why the box on its shoulders only contains the answers to the childish questions but does not show us anything about itself.

Of course, this strange giant I am talking about represents the mysterious universe, the hidden part of nature, which is still to be discovered by scientists or humanity in general. This was my main motivation of doing the research contained in this Thesis, and I hope that someday we will be able to shed some light on the dark side of our general knowledge and to be able to answer to almost (if not) all the questions that feed our curiosity.

1.2 Brief history of the universe

The most successful scientific theory today about the origin and evolution of the universe is the *Big Bang theory*, which is one of the most ambitious intellectual constructions of the humanity. It is based on two consolidated branches of theoretical physics, namely, the *theory of General Relativity* (GR) [1] and the *Standard Model* of Particle Physics (SM), and is able to make robust predictions, such as the expansion of the universe, the existence and properties of the cosmic microwave background radiation (CMB) and the relative primordial abundance of light elements.

These predictions have been tested by very precise experiments during the last decades, imposing in this way the Big Bang theory as the *standard cosmological model*. The first observational basis for the expansion of the universe came in 1929, when Edwin P. Hubble [2] observed that galaxies were receding from us with velocities proportional to the distance to us. The CMB radiation was first detected by Penzias and Wilson in 1964 [3].

According to the standard Big Bang theory, our universe emerged from a tremendous explosion, in which both space and time were created. The early universe was extremely hot, dense and rapidly expanding, while today the universe is cold − as suggested by the measured CMB temperature $T_{\rm CMB} \simeq 2.73$ K − almost empty and it is still expanding. Thus, one can say that the history of the universe is the history of its expansion, which involves various important qualitative changes of its characteristics.

The physics of the early universe is described by theories of high energy physics, such as the Grand Unified Theories (GUTs). As the universe cools, due to the expansion, phase transitions may occur, and the theory describing the physics in each phase is different. Because the range of energies probed in particle accelerators has a relatively low upper limit, the theories of the highest energies have a more speculative nature. This is the case of, e.g., Quantum Gravity, which would be the theory describing the universe in its initial states. In this PhD Thesis, some of the qualitative predictions of Quantum Gravity will be applied and their possible cosmological effects will be studied. By comparing the theoretical predictions with observations, we can constrain these effects and obtain valuable information about how the theory that correctly describes Quantum Gravity should look like.

Whatever is the correct theory of the highest energies, we can trace the evolution of the universe, starting from the Planck era, corresponding to a temperature $T \sim 10^{19}$ GeV and a time $t \sim 10^{-43}$ s. At a slightly lower temperature ($T \sim 10^{16}$ GeV), the universe would be described by GUTs, still in the field of speculations. Starting from that moment, it is believed that the universe suffers an accelerated stage of expansion, known as *inflation*, during which the universe expands exponentially and cools, such that at the end of inflation, the universe is practically frozen and empty. Inflation is one of the objects of study of this Thesis. After inflation, the universe is heated again (reheated), so that one can say that the thermal history of the universe actually starts after inflation ends, because all that happened before would have been wiped out.

After a reheating era in which a large amount of particles are produced, the universe consists of a hot plasma of relativistic particles in thermal equilibrium, in which particles are continuously created and destroyed. The universe cools and several processes occur. For example, if there is a Peccei-Quinn symmetry solving the strong CP problem, this symmetry should be broken at a temperature $T \sim 10^{12}$ GeV ($t \sim 10^{-30}$ s). It is also believed that at those scales an asymmetry between matter and antimatter is generated, which is indispensable for the existence of the matter contained in the universe. The generation of this asymmetry is called *baryogenesis*.

At a lower energy scale ($T \sim 10^{2}$ GeV, $t \sim 10^{-10}$ s), the electro-weak symmetry is broken, and starting from that moment, the spectrum of particles in thermal equilibrium is that of the known particles produced in terrestrial accelerators. In this symmetry breaking, an asymmetry between matter and antimatter might also be generated, through a process known as *electro-weak baryogenesis*.

Another important transition occurs at the scale of Quantum Chromodynamics (QCD), $T \sim 10^{2}$ MeV, when all the quarks − which since then were free − are confined to form hadrons. This process is known as *hadronization*.

When the universe has a temperature of about $T \sim 1$ MeV ($t \sim 1$ s), it contains neutrons, protons, electrons, positrons, neutrinos and photons. At that moment, fusion nuclear reactions may start and the lightest nuclei are formed (H, He, Li, ...). The processes of light nuclei formation occur out of equilibrium and they are known as *Big Bang* (or *Primordial*) *Nucleosynthesis* (BBN). Its predictions constitute a solid pillar for the Big Bang theory and they can be corroborated by observations. This is the earliest period in the history of the universe of which one has observational evidence, and it allows us to constrain a large number of theories and relevant parameters.

During nucleosynthesis, electrons and positrons annihilate producing photons, but a small fraction of electrons survive in the plasma. This is an important moment, since right before neutrinos had decoupled from the thermal bath, and are not affected by the e^+e^- pair annihilations, which introduce a temperature difference between the photons and neutrinos baths.

At temperature $T \sim 3$ eV ($t \sim 10^{11}$ s) the energy density in non-relativistic matter becomes equal to that in relativistic particles, after which the universe becomes matter-dominated. This is known as the epoch of *matter-radiation equality*, and is the epoch when structure formation becomes possible.

The next important moment in the history of the universe is when the nuclei, formed during Primordial Nucleosynthesis, combine with the existing electrons to form atoms. This occurs at a temperature $T \sim 0.3$ eV ($t \sim 10^{13}$ s) and is known as *recombination*. It is the moment when the universe becomes transparent to light, because there are no more electrons to scatter the photons, which can now travel freely. Thus, the photons that reach us from that epoch constitute the oldest "picture" of the universe, which can be seen in the CMB radiation. After that, the recent history of the universe begins, and the processes that occur are the formation of stars, galaxies, planets and so on.

At present, observations of the universe indicate the existence of two unknown forms of energy, which dominate the energy content of the universe: *dark matter* and *dark energy*. The nature of these components is still a mystery and it constitutes one of the most important problems of modern cosmology. In this Thesis, I will discuss this issue, and also the possibility of relating these recently dominating unknown components to the early inflationary stage will be investigated.

1.3 Objectives and contents of the Thesis

The main objective of this PhD Thesis is to investigate the physical theories that go beyond the standard model of particle physics, in the direction indicated by observations of the universe. In particular, there is strong evidence of the existence of dark matter and dark energy in the universe along with evidence of an early period of accelerated expansion, known as inflation. In this Thesis, on the one hand I try to investigate some of the above mentioned issues separately, and on the other hand I also try to relate the early inflationary epoch with the recent dominating dark components. Thus, another objective of my investigation is to try to unify different models explaining different problems in a single, consistent model, able to explain some of the most exciting problems of modern cosmology discussed here.

Next, I will shortly describe the content of the chapters of this Thesis, emphasizing those containing my original work.

- In **chapters 2, 3, 4, 5, 6** and **7**, I give an introduction to the theoretical framework in which my original work is developed. They are supposed to synthesize and summarize the existing work in the literature and to help making the original work more comprehensive to the reader.

- In **Chapter 8**, I construct a model in which the inflaton and the quintessence fields are components of the same complex scalar field. The model also introduces a new global symmetry and a new auxiliary field, which contributes during the inflationary period. The potential of the quintessence field arises from effects of Quantum Gravity, which is supposed to explicitly break all global symmetries. I investigate the required magnitude of this explicit breaking in order to satisfy all the observational constraints on dark energy candidates.

- In **Chapter 9**, based on similar ideas as those of the model of Chapter 8, I investigate the possibility of unifying inflation with dark matter. I also make a detailed numerical study to determine the parameters of the model in terms of the $U(1)$ symmetry breaking scale v.

- In **Chapter 10**, I present a complete particle physics model that is able to explain inflation, dark matter and dark energy in a unified description, and it also provides a mechanism for leptogenesis. The model introduces a new gauge group $SU(2)_Z$ that grows strong at a scale $\Lambda \sim 10^{-3}$ eV. In the work of Chapter 10, I emphasize the inflationary aspects of the whole model.

- In **Chapter 11**, the formalism described in Chapter 7 is used for investigating the possibility to obtain an effective equation of state $p < -\rho/3$ that can mimic dark energy, from coherent oscillations of a scalar field in a potential. The formalism is based on the use of the action-angle variables of analytical mechanics and I argue that it has some advantages, like the identification of adiabatic invariants.

- Finally, in **Chapter 12**, I present the conclusions of all the work exposed in this Thesis.

- In **Appendix A**, I calculate the angular power spectrum of curvature perturbations in general models of inflation, making the assumption that the universe has a non-zero (although small) curvature. I obtain that the angular power spectrum has significant deviations on large scales, as compared to the case of a perfectly flat universe. This subject is of importance from the point of view of constraining the cosmological parameters of the universe, such as its geometry.

- In **Appendix B**, I give some technical details related to the numerical analysis made in Chapter 9.

Note: In this Thesis, I use natural units with $c = \hbar = k_{\mathrm{B}} = 1$ and the signature of the Minkowski metric $x^2 = x_\mu x^\mu = g_{\mu\nu} x^\mu x^\nu = x_0^2 - \vec{x}^2$.

Chapter 2

Standard Big-Bang Cosmology

Cosmology has made a lot of progress during the last years and has at present a substantial observational and experimental basis that confirm that many aspects of the standard cosmological picture are a good approximation to reality. Still, the empirical basis has not reached the level of precision and accuracy of the Standard Model of particle physics (SM) and we cannot talk about a well-established theory of cosmology, in which one can measure the parameters with high precision. With all this, one can say that cosmology is living nowadays a golden epoch and the observational data, which become more and more precise, keeps cosmologists optimistic about establishing a true Standard Cosmological Theory in the near future.

In any case, all observations lead us to a rich and resolutive picture of the present state of the universe, which gives us important clues in constructing a cosmological model that is able to be extrapolated towards the past and also to make predictions. Based on fairly precise cosmological observations, there is strong evidence that the universe is *isotropic* to a very good approximation. This isotropy, when combined with the general copernican belief that we are not supposed to live at the ”center” of universe, leads to the conclusion that the universe must also be very *homogeneous* on cosmological scales. This is one of the basic principles of cosmology that motivates the study of the idealized case of a perfectly homogeneous and isotropic universe, which provide a great simplification of the analytical treatment. Observations also suggest that our universe is not static, rather it expands with velocities which grow with the distance, as formulated by the Hubble law [2].

2.1 Friedmann-Robertson-Walker metric

The dynamics of the universe can be properly described by the Einstein equations of the theory of General Relativity, which in general are complicated nonlinear equations. As discussed above, if we consider the idealized case of a perfect homogeneous and isotropic universe, things become much simpler. This is the case of the so-called standard cosmology, based upon the maximally symmetric Friedmann-Robertson-Walker (FRW) [4] line element:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{2.1}$$

where $\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the line element of some homogeneous isotropic 3-manifold and $a(t)$ is the scale factor at cosmic time t. The set x^1, x^2, x^3 are called *comoving coordinates* and an observer that has 4-velocity $u^\alpha = (1,0,0,0)$ in these coordinates is called *comoving observer.*

2.1.1 Spatial metric

The three-dimensional spatial metric γ_{ij} must describe a homogeneous and isotropic 3-manifold. The most simple homogeneous and isotropic 3-manifold is the three-dimensional Euclidean space. Its line element can be written both in terms of Cartesian and spherical polar coordinates as:

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}(x^1)^2 + \mathrm{d}(x^2)^2 + \mathrm{d}(x^3)^2 = \mathrm{d}\chi^2 + \chi^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{2.2}$$

The spherical polar comoving coordinates, (χ, ϕ, θ), are related to the Cartesian coordinates x^i by the following transformations:

$$\begin{cases} x^1 &= \chi\sin\theta\cos\phi \\ x^2 &= \chi\sin\theta\sin\phi \\ x^3 &= \chi\cos\theta. \end{cases} \tag{2.3}$$

We can generalize the above example of a sphere by adding a new dimension to obtain a hypersphere of radius $K^{-1/2}$, which can be viewed as a curved three-dimensional surface embedded in a four-dimensional Euclidean space:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \frac{1}{K} \tag{2.4}$$

where K is here a positive constant called *spatial curvature.*

By writing similar transformations between Cartesian and spherical polar coordinates, the spatial metric is given by:

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}\chi^2 + \frac{\sin^2(K^{1/2}\chi)}{K}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{2.5}$$

We can analytically continue the hypersphere to negative curvature, $K < 0$, to obtain a third homogeneous and isotropic 3-manifold, called hyperbolic space. Its line element is given by the analytic continuation of equation (2.5):

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}\chi^2 + \frac{\sinh^2[(-K)^{1/2}\chi]}{-K}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{2.6}$$

This metric is qualitatively different from the metric (2.5), in that while the metric (2.5) describes a finite 3-manifold, the metric (2.6) describes an infinite 3-manifold. Thus, the structure of the three spatial dimensions are used to classify the FRW universes: one has a spatially *flat* universe if $K = 0$, a *closed* universe if $K > 0$ or an *open* universe if $K < 0$.

It is possible to unify the three types of universes in a single form of the metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)[\mathrm{d}\chi^2 + f_K^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)] \tag{2.7}$$

by defining the generalized sine-function:

$$f_K(\chi) = \begin{cases} K^{-1/2}\sin(K^{1/2}\chi), & K > 0, \\ \chi, & K = 0, \\ (-K)^{-1/2}\sinh[(-K)^{1/2}\chi], & K < 0 \end{cases} \tag{2.8}$$

If we finally make the substitution $r = f_K(\chi)$, we obtained the well-known form of the FRW metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]. \tag{2.9}$$

By a proper redefinition of the scale factor, the curvature K can take integer values 1, 0, -1, for closed, flat and open universes, respectively. However, it is often convenient to define the scale factor such that its present value is equal to unity, $a_0 = 1$, and in this case one should keep the explicit curvature K in formulae.

2.1.2 Hubble expansion and conformalities

The scale factor $a(t)$ is a time-dependent dimensionless parameter describing the cosmological expansion. Thus, the scale factor $a(t)$ converts comoving coordinates into physical quantities and the expansion is therefore just a change of scale during the evolution of the universe. So, physical radial distances at some time t will be given by:

$$d_{\rm phys} = a(t)\int_0^r \frac{{\rm d}r}{\sqrt{1-Kr^2}}. \tag{2.10}$$

If one considers two neighboring comoving observers, the physical separation between them will grow when $a(t)$ is increasing and will become smaller if $a(t)$ is decreasing. The rate at which the comoving observers approach or recede from each other is given by:

$$\frac{\rm d}{{\rm d}t}\Delta l = \frac{\dot a(t)}{a(t)}\Delta l = H(t)\Delta l \tag{2.11}$$

where $H(t) \equiv \dot a/a$ is called the *Hubble parameter* (or *Hubble rate*) and a dot means derivative with respect to time, ${\rm d}/{\rm d}t$. The value of $H(t)$ today is called the Hubble constant, H_0. It receives this name in honor of Edwin P. Hubble, which in 1929 first observed [2] that the velocity at which galaxies were receding from us was proportional to the distance to us, i.e., $v = H_0 l$. This relation is analogous to equation (2.11). Since Hubble's discovery, astronomers have made great effort to determine the value of the Hubble constant. It has become common to parameterize its value by defining a dimensionless parameter h

$$h = \frac{H_0}{100{\rm km/s/Mpc}} \simeq \frac{H_0}{3.24\times 10^{-18}{\rm s}^{-1}} \tag{2.12}$$

and is h which contains all the observational uncertainties about H_0. Using the 3-year observations data from the Wilkinson Microwave Anisotropy Probe (WMAP)[5], the best fit for the value of h is $h = 0.732^{+0.031}_{-0.032}$.

A useful quantity is the *conformal time* τ, which is defined through the following relation:

$${\rm d}\tau = \frac{{\rm d}t}{a} = \frac{{\rm d}a}{\dot a a} = \frac{{\rm d}a}{a^2 H}. \tag{2.13}$$

If we introduce the conformal time variable into the FRW metric (2.9), we obtain

$${\rm d}s^2 = a^2(t)\left[{\rm d}\tau^2 - \frac{{\rm d}r^2}{1-Kr^2} + r^2({\rm d}\theta^2 + \sin^2\theta{\rm d}\phi^2)\right] \tag{2.14}$$

which makes it clear why τ is called conformal: the corresponding FRW line element is conformal to the Minkowski line element describing a static four-dimensional hypersurface. When working with the conformal time, it is useful to define the *conformal Hubble rate*, $\mathcal{H}$:

$$\mathcal{H} = \frac{{\rm d}a/{\rm d}\tau}{a} = \dot a = aH. \tag{2.15}$$

This helps us write the following general transformations for any function $f(t)$:

$$\dot{f}(t) = \frac{f'(\tau)}{a(\tau)}, \tag{2.16}$$

$$\ddot{f}(t) = \frac{f''(\tau)}{a^2(\tau)} - \mathcal{H}\frac{f'(\tau)}{a^2(\tau)} \tag{2.17}$$

where a prime indicates differentiation with respect to τ.

Another concept of interest is the *particle horizon*, which is related to the causal contact between different parts of the universe. Photons travel on the null paths characterized by $\mathrm{d}\chi = \mathrm{d}t/a(t)$ and since the bang until time t they could have travelled the physical distance

$$R_H = a(t)\int_0^t \frac{\mathrm{d}t'}{a(t')} = a(\tau)\int_{\tau_0}^{\tau} \mathrm{d}\tau \tag{2.18}$$

where τ_0 indicates the conformal time corresponding to $t = 0$. Equation (2.18) gives the distance to the particle horizon and is the maximum distance between two points that can be in causal contact. Note that in the standard cosmology $a(t) \propto t^n$ for which one gets that $R_H(t) \sim H^{-1}$, where H^{-1} is called the *Hubble radius*, and for this reason horizon and Hubble radius can be used interchangeably.

2.2 Kinematics of the FRW metric

Let us study how a particle propagates in a universe described by the metric (2.7). For simplicity, we consider a particle of mass M propagating radially with respect to an observer O, in which case the angular part of the metric is irrelevant, and the metric becomes:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\chi^2. \tag{2.19}$$

In this case, the metric components do not depend on χ and, as a consequence, the corresponding covariant component of the particle's momentum P_χ is constant. But P_χ is *not* the physical momentum as measured by the comoving observer, which would rather measure the physical momentum $P_{\text{phys}} = |P_\chi|/a$. We see that the physical momentum varies in inverse proportion to the scale factor of the universe, $P_{\text{phys}} \propto a^{-1}$. The energy of the particle also varies with $a(t)$ through the relation $\mathcal{E} = \sqrt{P_{\text{phys}}^2 + M^2}$, which, for a photon or any other massless particle, becomes $\mathcal{E} \propto a^{-1}$.

The light emitted by a distant object can be viewed quantum mechanically as a freely-propagating photon, or classically as a propagating plane wave. In the quantum mechanical description, the wavelength of light is inversely proportional to the photon momentum,

$\lambda = h/p$. We have seen that the momentum of a photon changes in proportion to a^{-1}, so that its wavelength will be proportional to a:

$$\lambda \propto a(t) \tag{2.20}$$

and as the universe expands, the wavelength of a freely-propagating photon increases. This means that there is a *redshift* of the wavelength of a photon due to the fact that the universe was smaller when the photon was emitted.

The same result can be obtained by considering the propagation of light from a distant galaxy as a classical wave phenomenon. Suppose a wave is emitted from a source at radial coordinate $\chi = \chi_1$ at time t_1, when the scale factor was $a_1 \equiv a(t_1)$, and arrives at the observer located at $\chi = 0$ at time t_0. The coordinate distance and time will be related by:

$$\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^{\chi_1} \mathrm{d}\chi = \chi_1. \tag{2.21}$$

The wavecrest emitted at a time $t_1 + \delta t_1$ will arrive at the observer at a time $t_0 + \delta t_0$. One can write a similar relation to (2.21) and obtain that:

$$\chi_1 = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} \tag{2.22}$$

which, after a simple rearrangement of the limits of integration gives:

$$\int_{t_1}^{t_1+\delta t_1} \frac{\mathrm{d}t}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)}. \tag{2.23}$$

For a sufficiently small difference δt between two consecutive wavecrests, the scale factor $a(t)$ can be considered constant over the integration time of (2.23), and one has:

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)}. \tag{2.24}$$

Since δt_1 (δt_0) is the time between successive crests of the emitted (detected) light, δt_1 (δt_0) is the wavelenght of the emitted (detected) light, so that one can write:

$$\frac{\lambda_1}{\lambda_0} = \frac{a_1}{a_0} \tag{2.25}$$

where a_0 is the scale factor now.

It is traditional to define the *redshift* of an object, z, in terms of the detected wavelength to the emitted wavelength:

$$1 + z \equiv \frac{\lambda_0}{\lambda_1} = \frac{a_0}{a_1}. \tag{2.26}$$

Since today astronomers observe distant galaxies to have red shifted spectra ($z > 0$), we can conclude that the universe is expanding.

2.3 Distances in cosmology

Another important concept related to observational tools in an expanding background is associated to the definition of a distance. In particular, we wish to define the distance from us to an object observed in the sky. For definiteness, we consider ourselves as a comoving observer O, at the origin and at time t_0, such that $a_0 = 1$. We want to measure the distance to another comoving observer O', which can be a galaxy. Then, a photon emitted by O', when the scale factor was $a = 1/(1+z)$, reach us at present, when $a_0 = 1$, and this fixes the radial coordinate χ of the galaxy. From equation (2.19) we have that $\mathrm{d}\chi/\mathrm{d}t = -a^{-1}$, where the minus sign comes from the fact that the photon is moving toward lower values of χ. We get:

$$\chi = \int_a^1 \frac{a^{-1}}{\mathrm{d}a/\mathrm{d}t}\mathrm{d}a = \int_a^1 \frac{\mathrm{d}a}{\dot{a}a} = \tau_0 - \tau \tag{2.27}$$

where we also used the conformal time definition, equation (2.13), and τ_0 is the conformal time now, at $t = t_0$. The physical distance between O and O' is obtained by integrating the line element:

$$\int_O^{O'} \mathrm{d}l = \int_0^{\chi} a(t)\mathrm{d}\chi = a(t)\chi \tag{2.28}$$

which we see is time-dependent. It is often convenient to use the *comoving distance*, which in the above example becomes χ and so it does not depend on time.

There are other useful measures of distance. Let us assume that there is a galaxy of proper diameter D at comoving coordinate $r = r_1$, which emitted light at $t = t_1$, detected by the observer O at $t = t_0$ and $r = 0$. From the metric (2.9) it follows that the observed angular diameter of the source, δ, is related to D by

$$\delta = \frac{D}{a(t_1)r_1}. \tag{2.29}$$

The *angular diameter distance*, d_A, is defined as

$$d_\mathrm{A} \equiv \frac{D}{\delta} = a(t_1)r_1 = \frac{r_1}{1+z}. \tag{2.30}$$

A related notion is the *comoving angular diameter distance*, in which the comoving diameter $D_\mathrm{c} = D/a(t_1)$ of the galaxy is used:

$$d_\mathrm{Ac} \equiv \frac{D_\mathrm{c}}{\delta} = r_1. \tag{2.31}$$

An alternative way of defining a distance is through the luminosity of a stellar object. The *luminosity distance* d_L plays a very important role in astronomy, for example in the

Supernova observations [6, 7]. Suppose we know the absolute luminosity L_s of the source, which we assume to emit radiation isotropically. The energy flux F measured by the observer O is $F = L_\mathrm{s}/(4\pi d_\mathrm{L}^2)$, and by using this relation one can define the luminosity distance as

$$d_\mathrm{L}^2 = \frac{L_\mathrm{s}}{4\pi F}. \tag{2.32}$$

In order to calculate the luminosity distance, we must find the relationship between the observed flux F and the luminosity of the source. In a static Euclidean geometry, the observed flux today ($a_0 = 1$) would be simply $F = L_\mathrm{s}/(4\pi r_1^2)$, where r_1 is the radial coordinate of the source. However, due to the expansion, there are two effects that have to be taken into account: first, due to the cosmological redshift, the energy of the emitted photons is reduced by a factor of $(1+z)$ and, second, there is a cosmological time dilation between the emitted photons, which arrive with less frequently than were emitted, also contributing a factor $(1 + z)$ in reducing the observed flux. Combining the two effects mentioned above, we have:

$$F = \frac{L_\mathrm{s}}{(4\pi r_1^2)}(1+z)^{-2}. \tag{2.33}$$

From equations (2.32) and (2.33) we obtain:

$$d_\mathrm{L} = (1+z)r_1 \tag{2.34}$$

which is also called the *luminosity distance-redshift relation*. It is used as a cosmological test if radiation sources of known luminosity, generally called *standard candles*, can be identified. At present, the most reliable standard candles are the Type Ia supernovae (SN Ia).

2.4 Dynamics of the universe

Since now, we have been concerned with the kinematics of a universe described by the FRW metric. Here, we would like to understand the time dependence of the scale factor, in which the dynamics of the expanding universe is implicitly contained.

2.4.1 The Friedmann Equation

The equations of motion for the FRW space-time are known as the *Friedmann equations*. To derive them, one must solve for the evolution of the scale factor using the Einstein equations [4]:

$$R^{\mu\nu} - \frac{1}{2}\mathcal{R}g^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{2.35}$$

where $R^{\mu\nu}$ is the Ricci tensor, $\mathcal{R}$ is the scalar curvature, $g^{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor for all the fields present, and G is the Newtonian gravitational constant. In order to proceed, we should make some assumptions about the stress energy tensor $T^{\mu\nu}$. There is no need for a detailed knowledge of the properties of the fundamental fields that contribute to $T^{\mu\nu}$, the only thing we must require is the consistency with the homogeneity and isotropy of the universe, i.e., it must be diagonal, with all the spatial components equal to one another. The simplest form of the stress-energy tensor is that of a perfect fluid characterized by a time-dependent energy density $\rho(t)$ and pressure $p(t)$. The precise form of the stress-energy is then:

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p). \tag{2.36}$$

By using this form of the stress-energy tensor in the Einstein equation (2.35) and the metric (2.9), we obtain the so-called Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{2.37}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.38}$$

The first equation (2.37) relates the dynamics of the scale factor $a(t)$ with the energy content and the curvature of the universe, while the second equation (2.38) gives the acceleration of the expansion. By historical reasons, one defines the *deceleration parameter* q_0 today ($t = t_0$) as:

$$q_0 \equiv -a(t_0)\frac{\ddot{a}(t_0)}{\dot{a}^2(t_0)} = -\frac{\ddot{a}_0}{a_0 H_0^2} \tag{2.39}$$

It is frequently useful to consider the equation of continuity $T^{\mu\nu}{}_{;\nu} = 0$ when solving the Friedmann equations. It implies a relation between ρ and p,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.40}$$

2.4.2 The critical density

The Friedmann equation (2.37) gives the expansion rate of the universe, which is characterized by the Hubble parameter $H = \dot{a}/a$. It is useful to define a time dependent *critical density*

$$\rho_{\mathrm{c}} \equiv \frac{3H^2}{8\pi G} \tag{2.41}$$

which corresponds to a universe with exactly flat spatial sections. The value of the critical density today is $\rho_{\mathrm{c},0} = 1.88h^2 10^{-29}\mathrm{g/cm}^3$. We have seen before that the dynamics of the

FRW metric depends on the energy content of the universe. Hence, it is also useful to define the *density parameter*

$$\Omega_{\rm total} \equiv \frac{\rho}{\rho_{\rm c}} \tag{2.42}$$

where in Ω all contributions are included: matter, radiation and cosmological constant. By replacing the definitions (2.41) and (2.42) in the Friedmann equation (2.37), the last can be recast as:

$$\frac{K}{H^2a^2} = \frac{\rho}{\rho_{\rm c}} - 1 = \Omega - 1. \tag{2.43}$$

This form of the Friedmann equation allows us to relate the total energy density of the universe with its local geometry,

$$\begin{aligned} \Omega > 1 \quad &\Leftrightarrow \quad K = 1 \quad \text{(CLOSED)} \\ \Omega = 1 \quad &\Leftrightarrow \quad K = 0 \quad \text{(FLAT)} \\ \Omega < 1 \quad &\Leftrightarrow \quad K = -1 \ \text{(OPEN)}. \end{aligned} \tag{2.44}$$

As we said, the energy density ρ includes all types of constituents: radiation, matter, vacuum, etc., so it can be written as the sum $\sum \rho_i$. This allows us to define a density parameter for each of the constituents

$$\Omega_i = \frac{\rho_i}{\rho_{\rm c}}. \tag{2.45}$$

From the FRW metric (2.9), it is clear that the effects of spatial curvature become relevant for $r \sim |K|^{-1/2}$, so one normally defines a physical *radius of curvature* of the universe

$$R_{\rm curv} \equiv a(t)|K|^{-1/2} = \frac{H^{-1}}{|\Omega - 1|^{1/2}}. \tag{2.46}$$

By taking the ratio of the Friedmann equations (2.37) and (2.38) and using the definition of the density parameter Ω_0 at present, the decelaration parameter (2.39) becomes:

$$q_0 = \frac{1}{2}\Omega_0 \left(1 + 3\frac{p}{\rho}\right) \equiv \frac{1}{2}\Omega_0(1 + 3w) \tag{2.47}$$

where $w = p/\rho$ is the *equation of state parameter*. It is a very important parameter because, depending on its value, the universe can be decelerating ($w > -1/3$) or accelerating ($w < -1/3$). In the standard situation, $p \geq 0$ and $\rho > 0$, which means that the universe is decelerating. Einstein introduced a positive cosmological constant Λ in his equations in order to obtain a static universe, motivated by the idea to avoid the initial singularity (Big Bang) suggested by an expanding decelerated universe. The presence of a cosmological constant into the Einstein equations implies that the universe will finally start to accelerate, supposing that the pressure and the energy density of other components are diluted by the expansion and then Λ starts to dominate. The Friedmann equations with cosmological constant would have to include the energy density ρ_Λ and pressure p_Λ defined as $\rho_\Lambda = -p_\Lambda \equiv \Lambda/(8\pi G)$. Deciding if Λ should be taken into account or not and explaining its small value suggested by observations is one of the biggest problem of physics [8]. In this Thesis I do not pretend to solve the cosmological constant problem, but just to propose alternative explanations of the present acceleration of the universe. It is also possible to obtain an accelerating universe $\ddot{a} > 0$ if it contains some unknown exotic component with a non-standard equation of state parameter that satisfies $w < -1/3$ and now dominates over all the other components. This is the so-called *dark energy* of the universe, which will be discussed in more detail in Chapter 5.

2.4.3 Single-component universe

Let us investigate the behavior of a flat universe ($K = 0$) dominated by a single component with equation of state parameter w. In general, w can be a function of time, but the case of constant w is simpler and common to the known forms of matter. For instance, we have $w = 0$ for non-relativistic matter, $w = 1/3$ for radiation and $w = -1$ for a cosmological constant (vacuum energy).

The Friedmann equation (2.37) in terms of w becomes

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \propto \rho(a) \propto a^{-3(1+w)} \tag{2.48}$$

which for $w \neq -1$ gives

$$a(t) \propto t^{2/[3(1+w)]}. \tag{2.49}$$

For a universe dominated by non-relativistic matter, $w = 0$, and we obtain that

$$\rho \propto a^{-3};\ a \propto t^{2/3} \propto \tau^2 \tag{2.50}$$

while for a radiation dominated universe, with $w = 1/3$, we have:

$$\rho \propto a^{-4};\ a \propto t^{1/2} \propto \tau. \tag{2.51}$$

This analysis does not apply to the case of a cosmological constant, with $w = -1$. In this case, the scale factor grows exponentially

$$a(t) = a_0 \exp\left(\sqrt{\frac{\Lambda}{3}} t\right) \tag{2.52}$$

where a_0 is the scale factor at $t = 0$. The Hubble rate is then a constant, $H_\Lambda = \sqrt{\Lambda/3}$ and the conformal time $\tau \propto -\exp(-\sqrt{\Lambda/3}\, t)$. The space corresponding to this case is called *de Sitter spacetime.*

2.4.4 Universe with vacuum energy and curvature

So far, we have considered various forms of energy in an exactly flat universe. In this Thesis, we are also interested in the situation in which the universe is only approximately flat and we consider a small curvature K. In Appendix A we work in the context of a non-flat vacuum energy dominated universe with curvature K. The Friedmann equation (2.37) in this context becomes

$$H^2 = H_\Lambda^2 - \frac{K}{a^2} \tag{2.53}$$

where H_Λ is the Hubble rate corresponding to a vacuum energy dominated flat universe. Equation (2.53) can be solved for $a(t)$ to obtain:

$$a(t) = \begin{cases} \sqrt{\frac{K}{H_\Lambda}} \cosh(H_\Lambda t), & K > 0 \\ \sqrt{\frac{-K}{H_\Lambda}} \sinh(H_\Lambda t), & K < 0 \end{cases} \tag{2.54}$$

or in terms of the conformal time τ:

$$a(\tau) = \begin{cases} -\frac{\sqrt{K/H_\Lambda}}{\sin(\sqrt{K}\tau)}, & K > 0, \\ -\frac{\sqrt{-K/H_\Lambda}}{\sinh(\sqrt{-K}\tau)}, & K < 0. \end{cases} \tag{2.55}$$

2.5 The early radiation-dominated universe

There are sufficient reasons to believe that the universe has evolved from an early hot and dense state to the present cold and almost empty universe. The CMB radiation we observe today is a relic picture of an earlier stage of the universe, when it was $(1+z) \sim 1100$ times smaller, and it possesses a thermal spectrum to a very good approximation. This makes us believe that the early universe consisted of a thermal bath of particles in equilibrium. The fundamental object for describing a hot plasma in thermal equilibrium is the *phase space distribution function*, $f(\vec{\mathbf{p}}, t)$. In a universe described by the FRW metric, f does not depend either on the direction of the momentum $\vec{\mathbf{p}}$, or on the position. The number density, n, the energy density, ρ, and the pressure, p, corresponding to a gas of particles with g internal degrees of freedom are given by [9]:

$$n = \frac{g}{(2\pi)^3} \int f(\vec{\mathbf{p}}) \mathrm{d}^3 p \tag{2.56}$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{\mathbf{p}}) f(\vec{\mathbf{p}}) \mathrm{d}^3 p \tag{2.57}$$

$$p = \frac{g}{(2\pi)^3} \int \frac{|\vec{\mathbf{p}}|^2}{3E} f(\vec{\mathbf{p}}) \mathrm{d}^3 p \tag{2.58}$$

where $E^2 = |\vec{\mathbf{p}}|^2 + m^2$. For a species in *kinetic equilibrium*, the phase space distribution function is given either by the familiar Fermi-Dirac (FD), or Bose-Einstein (BE) distributions

$$f(|\vec{\mathbf{p}}|) = \frac{1}{\mathrm{e}^{(E-\mu)/T} \pm 1} \tag{2.59}$$

where T is the temperature of the plasma, μ is the chemical potential of the species, and $+1$ corresponds to FD species and -1 to BE species.

By replacing $f(|\vec{\mathbf{p}}|)$ defined in (2.59) into equations (2.56)−(2.58), one can obtain the expressions of n, ρ and p. In the relativistic non-degenerate limit $T \gg m$ and $T \gg \mu$, we get simple expressions

$$\rho = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{(BE)} \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 & \text{(FD)} \end{cases} \tag{2.60}$$

$$n = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & \text{(BE)} \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3 & \text{(FD)} \end{cases} \tag{2.61}$$

$$p = \rho/3 \tag{2.62}$$

where $\zeta(3) = 1.202...$ is the Riemann zeta function of 3.

Another interesting limit corresponds to non-relativistic particles with $m \gg T$, for which we obtain:

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \mathrm{e}^{-(m-\mu)/T} \tag{2.63}$$

$$\rho = mn \tag{2.64}$$

$$p = nT \ll T. \tag{2.65}$$

This limit corresponds to the Maxwell-Boltzmann statistics.

Comparing the two limits considered above, we notice that the energy density of a non-relativistic species is exponentially suppressed as compared to the relativistic species. This is why, in a universe dominated by radiation, the total density can be very well approximated by the contribution of only the relativistic species.

It is convenient to express the total energy density and pressure of the relativistic species in terms of the photon temperature T:

$$\rho_{\mathrm{R}} = \frac{\pi^2}{30} g_* T^4 \tag{2.66}$$

$$p_{\mathrm{R}} = \rho_{\mathrm{R}}/3 = \frac{\pi^2}{90} g_* T^4 \tag{2.67}$$

where g_* counts the total number of effectively massless degrees of freedom and is defined as:

$$g_* = \sum_{i=\mathrm{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=\mathrm{fermions}} g_i \left(\frac{T_i}{T}\right)^4. \tag{2.68}$$

The value of g_* depends on the temperature T. For example, at high $T \sim 300\,\mathrm{GeV}$, all the species of the standard model will contribute and $g_* = 106.75$. At low temperature $T \ll \mathrm{MeV}$, the only relativistic species are 3 neutrino families and the photon, and $g_* = 3.36$.

With all this, we can express the Hubble rate H, the time t and the scale factor a in terms of the photon bath temperature, in a radiation dominated universe:

$$H = \sqrt{\frac{8\pi G}{3}\rho_{\mathrm{R}}} \simeq 1.66 g_*^{1/2} \frac{T^2}{M_{\mathrm{P}}} \tag{2.69}$$

$$t = \frac{1}{2H} \simeq 0.301 g_*^{-1/2} \frac{M_{\mathrm{P}}}{T^2} \sim \left(\frac{T}{\mathrm{MeV}}\right)^{-2} \mathrm{sec} \tag{2.70}$$

where $M_{\mathrm{P}} = G^{-1/2} \simeq 1.22 \times 10^{19}\mathrm{GeV}$ is the Planck mass. Finally, recalling that $w = 1/3$ in a radiation dominated universe in which $a(t) \propto t^{1/2}$, see equation (2.51), we find that

$$a \propto T^{-1}. \tag{2.71}$$

The *entropy in a comoving volume* provides a very useful fiducial quantity during the expansion of the universe. In the conditions of local thermal equilibrium of the particles in the thermal bath, the entropy per comoving volume remains constant

$$S = \frac{a^3(\rho + p)}{T} = \mathrm{const.} \tag{2.72}$$

This is a consequence of the second law of thermodynamics, applied to the expanding universe.

It is useful to define the *entropy density* s

$$s \equiv \frac{S}{V} = \frac{\rho + p}{T}. \tag{2.73}$$

The entropy density is dominated by the contribution of relativistic particles and can be written as:

$$s = \frac{2\pi^2}{45} g_{*s} T^3, \tag{2.74}$$

where g_{*s} is defined as:

$$g_{*s} = \sum_{i=\mathrm{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\mathrm{fermions}} g_i \left(\frac{T_i}{T}\right)^3. \tag{2.75}$$

We notice that s has the same temperature-dependence as the number density of relativistic particles. This allows us to relate the entropy density s with the photon density n_γ:

$$s = \frac{\pi^4}{45\zeta(3)} g_{*s} n_\gamma = 1.80 g_{*s} n_\gamma. \tag{2.76}$$

Since g_{*s} is a function of temperature, s and n_γ cannot always be used interchangeably.

Conservation of the entropy per comoving volume, $S \propto g_{*s} a^3 T^3$, implies that, first, the temperature and the scale factor are related by:

$$T \propto g_{*s}^{-1/3} a^{-1} \tag{2.77}$$

which for $g_{*s} = \text{const}$ leads to the familiar $T \propto a^{-1}$ relation.

Second, it provides a way of quantifying the net baryon number per comoving volume:

$$B = \frac{n_B}{s} \simeq (4 - 7) \times 10^{-11}. \tag{2.78}$$

The baryon number of the universe tells us two things: (1) the entropy per particle in the universe is extremely high, $\sim 10^{10}$, compared to about 10^{-2} in the sun, and a few, in the core of a newly formed neutron star; (2) the asymmetry between matter and antimatter is very small, about 10^{-10}. This asymmetry should be explained by baryogenesis, which is based on the idea that B, C and CP symmetries are violated in out-of-equilibrium interactions in the early universe.

Leptogenesis is the process of generation of a net lepton number through an out-of-equilibrium, lepton-number-violating, CP-asymmetric process. This asymmetry is then converted to a baryon asymmetry by the *sphaleron* process, which occurs in standard electroweak theory at temperatures above about 1 TeV. The model described in Chapter 10 provides a mechanism for leptogenesis by the decay of a new messenger scalar field $\tilde{\varphi}_1^{(Z)}$, associated to a new gauge symmetry $SU(2)_Z$, into an $SU(2)_Z$ fermion $\psi_i^{(Z)}$ and a SM lepton. This net lepton number then generates a net baryon number through the above mentioned mechanism.

2.6 The problems of the Big Bang cosmology

Up to here, we have seen the standard picture of how the universe evolved from very early epochs towards the present time. It is given by the standard Big Bang theory, which is based on the theory of General Relativity and on the Standard Model of elementary particles. Nevertheless, this standard cosmological model has some well-known problems: the *flatness* or the *oldness* problem, the *entropy* problem, the *horizon* or *large-scale smoothness* problem, and the *small-scale inhomogeneity* problem. They do not indicate any logical inconsistencies of the standard cosmology, rather they seem to require very special initial conditions for the evolution to a universe with the characteristics of our universe today. Let us explain these shortcomings in some more detail.

2.6.1 The flatness problem

Assuming that Einstein equations are valid up to times as early as the Planck era, when the temperature of the universe is $T_{\rm P} \sim M_{\rm P} \sim 10^{19}$GeV, let us calculate the curvature of the universe at that epoch. From equation (2.43) we see that if the universe is perfectly flat then $\Omega = 1$ at all times. Of course, there is *a priori* no reason for our universe to be perfectly flat. On the other hand, if there is even a small curvature term, the time dependence of $(\Omega - 1)$ should be taken into account. During a radiation-dominated (RD) period, one has that $H^2 \propto \rho_{\rm R} \propto a^{-4}$, and during matter domination (MD), $H^2 \propto \rho_{\rm M} \propto a^{-3}$, so that:

$$\Omega - 1 \propto \begin{cases} \frac{1}{a^2 a^{-4}} = a^2, & \text{RD} \\ \frac{1}{a^2 a^{-3}} = a, & \text{MD.} \end{cases} \tag{2.79}$$

In both cases $(\Omega - 1)$ decreases going backwards in time. Observations of the present universe indicate that $(\Omega - 1)$ is of order unity or less, which means that at the Planck epoch it had to be:

$$|\Omega - 1|_{T=T_{\rm P}} \approx |\Omega - 1|_{T=T_0} \frac{a_{\rm P}^2}{a_{\rm eq}^2} \frac{a_{\rm eq}}{a_0} \approx \frac{T_{\rm eq}^2}{T_{\rm P}^2} \frac{T_0}{T_{\rm eq}} \approx O(10^{-60}) \tag{2.80}$$

where we used the values $T_{\rm eq} \sim 10^{-9}$ GeV and $T_0 \sim 10^{-13}$ GeV, and the subscripts "eq" and "0" stand for the epoch of matter-radiation equality and today, respectively. Equation (2.80) indicates that at the Planck epoch, the universe should be unnaturally flat, without a special reason for that. Even if we go back simply to the epoch of nucleosynthesis, $T_{\rm N} \sim 1\,{\rm MeV}$, we have:

$$|\Omega - 1|_{T=T_{\rm N}} \approx |\Omega - 1|_{T=T_0} \frac{a_{\rm N}^2}{a_{\rm eq}^2} \frac{a_{\rm eq}}{a_0} \approx \frac{T_{\rm eq}^2}{T_{\rm N}^2} \frac{T_0}{T_{\rm eq}} \approx O(10^{-16}) \tag{2.81}$$

which does not alleviate much the initial condition for the curvature of the universe. For this reason, the flatness problem is also dubbed the "fine-tuning problem".

2.6.2 The entropy problem

The hypothesis of adiabatic expansion of the universe is connected with the flatness problem. To see this, recall the Friedmann equation (2.69) during a radiation-dominated period:

$$H^2 \sim \frac{\rho_{\rm R}}{M_{\rm P}^2} \sim \frac{T^4}{M_{\rm P}^2} \tag{2.82}$$

which applied to equation (2.43) gives:

$$\Omega - 1 \sim \frac{K M_{\rm P}^2}{a^2 T^4} \sim \frac{K M_{\rm P}^2}{S^{2/3} T^2}, \tag{2.83}$$

where the relation $S \propto a^3 T^3$ has been used. Under the hypothesis of adiabaticity, S is constant over the evolution of the universe and therefore

$$|\Omega - 1|_{t=t_{\rm P}} \sim \frac{M_{\rm P}^2}{T_{\rm P}^2}\frac{1}{S_{\rm U}^{2/3}} = \frac{1}{S_{\rm U}^{2/3}} \approx O(10^{-60}). \tag{2.84}$$

where $S_{\rm U} \sim O(10^{90})$ is the value of the universe entropy today. This means that $(\Omega - 1)$ was so close to zero at early epochs because the total entropy of our universe is incredibly large. Thus, the flatness problem is connected to the fact that the entropy in a comoving volume is conserved. It is possible, therefore, that the problem could be solved if the cosmic expansion was non-adiabatic for some finite time interval during the early history of the universe.

2.6.3 The horizon problem

According to the standard cosmology, photons decoupled from the rest of the components at a temperature of the order of 0.3 eV, when the universe was already dominated by non-relativistic matter. This corresponds to the so-called *last-scattering surface* (LS), which corresponds to a redshift of about 1100 and the universe was about 300 000 years old. These photons free-stream and reach us basically untouched, with a thermal spectrum consistent with that of a black body at temperature 2.73 K.

At the time of last-scattering, the length scale corresponding to our present Hubble radius $R_{\rm H}(t_0)$ was:

$$\lambda_{\rm H}(t_{\rm LS}) = R_{\rm H}(t_0)\frac{a_{\rm LS}}{a_0} = R_{\rm H}(t_0)\frac{T_0}{T_{\rm LS}}. \tag{2.85}$$

On the other hand, the Hubble rate has decreased as $H^2 \propto \rho_{\rm M} \propto T^3$, so that at last-scattering, the Hubble length was:

$$H_{\rm LS}^{-1} = R_{\rm H}(t_0)\left(\frac{T_0}{T_{\rm LS}}\right)^{3/2}. \tag{2.86}$$

By comparing the volumes corresponding to the scales calculated in (2.85) and (2.86) we obtain that

$$\frac{\lambda_{\rm H}^3(T_{\rm LS})}{H_{\rm LS}^{-3}} = \left(\frac{T_0}{T_{\rm LS}}\right)^{-3/2} \approx 10^6 \tag{2.87}$$

which means that there were $\sim 10^6$ causally disconnected regions within the volume that now corresponds to our horizon. It is very hard to believe that regions that have the same temperature today had never been in thermal contact before.

2.6.4 The low-scale inhomogeneity problem

We have seen that, at large scales, the universe seems to be very homogeneous and, without a reasonable explanation, this fact seems to be a rather astonishing coincidence, that 10^6 disconnected regions have the same temperature. Fluctuations in the temperature are related to the density inhomogeneity, which means that the universe should be very homogeneous at large scales. If so, one wonders what was the seed for the creation of the structure we observe today: stars ($\delta\rho/\rho \sim 10^{30}$), galaxies ($\delta\rho/\rho \sim 10^5$), cluster of galaxies ($\delta\rho/\rho \sim 10 - 10^3$), superclusters ($\delta\rho/\rho \sim 1$) and so on. The standard cosmology provides a general framework for understanding this picture. Once the universe becomes matter dominated, primeval density inhomogeneities ($\delta\rho/\rho \sim 10^{-5}$) are amplified by gravity and grow into the structure we see today [10]. However, the standard Big Bang theory does not provide a theoretical explanation of the origin of the primeval fluctuations, which are only considered as an input.

Chapter 3

Inflationary cosmology

The horizon and the flatness problems of the standard big bang cosmology are so serious that the theory seems to require some basic modifications of the hypothesis made so far. The most elegant solution is to suppose that the universe has gone through a non-adiabatic period and also through a period of accelerated expansion, during which physical scales λ evolved much faster than the horizon scale H^{-1}. This period of positive acceleration, $\ddot{a} > 0$, of the primeval universe is called *inflation*.

The inflationary hypothesis is attractive because it holds out the possibility of calculating cosmological quantities, given the Lagrangian describing the fundamental interactions. In the context of the Standard Model, it is not possible to incorporate inflation, but this should not be regarded as a serious problem because the Standard Model itself requires modifications at higher energy scales, for reasons that have nothing to do with cosmology.

In this chapter, I describe the concept of inflation and how it is related to the present state of the observable universe. I also give a survey of the main inflationary models proposed until now.

3.1 Solving the shortcomings of the standard Big Bang

3.1.1 Inflation and the horizon problem

As commented above, during inflation the universe is accelerating, i.e. $\ddot{a} > 0$. From the second Friedmann equation (2.38) we see that this is equivalent to $p < -\rho/3$, which is not satisfied either by radiation, or by matter. This means that we have to introduce another kind of substance able to satisfy the previous condition. In order to satisfy the requirement for having inflation, one usually assumes the extreme condition $p = -\rho$, which simplifies the analysis. Such a period of expansion of the universe is called *de Sitter* stage. In this case, from equation (2.37) we see that the energy density ρ is constant. Then, neglecting the curvature term ($K = 0$) in equation (2.40), we learn that H is also constant during the de Sitter phase. As a consequence, the universe is expanding exponentially,

$$a(t) = a_{\rm i} {\rm e}^{H(t-t_{\rm i})} \tag{3.1}$$

where a subscript "i" means the value at the initial time when inflation starts. Because the scale factor grows exponentially in time, the physical scales, which are proportional to a, will also grow exponentially, while the horizon scale H^{-1} remains constant. What this means is that if inflation lasts long enough, all the physical scales we observe today, which are supposed to have been outside the horizon in the past, can re-enter the horizon during the inflationary stage. This can explain the homogeneity of CMB.

Let us see how much inflation is needed to solve the horizon problem. It is useful to define *the number of e-foldings* of inflation, N, as a measure of the growth of the universe:

$$N = \ln \frac{a_{\rm f}}{a_{\rm i}} \tag{3.2}$$

where the subscripts "i" and "f" denote initial and final values, respectively. In a de Sitter phase of expansion, the scale factor $a(t)$ is given in (3.1) and the number of e-foldings becomes:

$$N(t) = H(t_{\rm f} - t_{\rm i}) \tag{3.3}$$

In order to solve the horizon problem, it is necessary that the largest scale we observe today, the present horizon H_0^{-1}, was reduced during inflation to a value $\lambda_{H_0}(t_{\rm i})$, smaller or equal to the value of horizon length $H_{\rm I}^{-1}$ during inflation:

$$\lambda_{H_0}(t_{\rm i}) = H_0^{-1} \left(\frac{a_{\rm f}}{a_0}\right) \left(\frac{a_{\rm i}}{a_{\rm f}}\right) = H_0^{-1} \left(\frac{T_0}{T_{\rm f}}\right) {\rm e}^{-N} \leq H_{\rm I}^{-1} \tag{3.4}$$

where $T_{\rm f}$ is the temperature at the end of inflation. The above relation gives a lower limit for N, which is usually around 70.

3.1.2 Inflation and the flatness problem

As we saw in 2.6.1, the flatness problem of the standard Big Bang cosmology consists in the fact that, according to equation (2.80), the curvature of the early universe should be fine-tuned to $|\Omega - 1| \sim 10^{-60}$ in order to reproduce a value $|\Omega_0 - 1|$ of order unity today. Inflation solves this problem in an elegant manner. Suppose that before inflation, $|\Omega - 1|_{t=t_{\rm i}}$ is of order unity; then, after inflation we have:

$$|\Omega - 1|_{t=t_{\rm f}} = \frac{|K|}{a_{\rm f}^2 H_{\rm f}^2} = \frac{|K|}{a_{\rm i}^2 H_{\rm i}^2} \frac{a_{\rm i}^2 H_{\rm i}^2}{a_{\rm f}^2 H_{\rm f}^2} \simeq |\Omega - 1|_{t=t_{\rm i}} \left(\frac{a_{\rm i}}{a_{\rm f}}\right)^2 \simeq {\rm e}^{-2N} \tag{3.5}$$

where we supposed that the Hubble rate H is approximately constant during inflation. Thus, if N is sufficiently large, say $N \approx 70$, it results that $|\Omega - 1|_{t=t_{\rm f}}$ can be of the required order of magnitude. In this way, the fine-tuning problem is solved or, at least, much ameliorated, by explaining a tiny number 10^{-60} with a number N of order 70.

If N is larger, then $|\Omega - 1|_{t=t_{\rm f}}$ is smaller, which means that a generic prediction of inflation is that $\Omega_0 = 1$ today, with a great precision. Nevertheless, we should specify that inflation does not change the global geometric properties of the space-time, meaning that if the universe was open/closed before inflation, it will always remain open/closed after it.

In Appendix A of this PhD Thesis we will investigate the modifications that a remanent curvature term may introduce to the otherwise standard perfectly flat universe scenario.

3.1.3 Inflation and the entropy problem

The entropy and the flatness problems have a common origin, which resides in the fact that the entropy in a comoving volume is conserved, and the universe seems to contain a large amount of entropy, $S_{\rm U} \sim 10^{90}$ [9]. If the cosmic expansion was non-adiabatic during a finite period in the early history of the universe, the entropy could have changed by an amount:

$$S_{\rm f} = Z^3 S_{\rm i} \tag{3.6}$$

from an epoch before inflation to some other epoch after inflation, where Z is a numerical factor. Assuming that the total entropy before inflation was of order unity, and that after the end of inflation the universe expands adiabatically, we have:

$$\begin{aligned} S_{\rm U} &= S_{\rm f} \sim (a_{\rm f} T_{\rm f})^3 \sim (a_{\rm i} T_{\rm i})^3 \left(\frac{a_{\rm f} T_{\rm f}}{a_{\rm i} T_{\rm i}}\right)^3 \sim S_{\rm i} \left(\frac{a_{\rm f}}{a_{\rm i}}\right)^3 \left(\frac{T_{\rm f}}{T_{\rm i}}\right)^3 \\ &\sim {\rm e}^{3N} \left(\frac{T_{\rm f}}{T_{\rm i}}\right)^3 \sim 10^{90} \end{aligned} \tag{3.7}$$

which, up to the logarithmic factor $\ln(T_{\rm i}/T_{\rm f})$, gives $N \sim 70$. I should specify that the large amount of entropy is not produced during inflation, but during the non-adiabatic phase transition after inflation (reheating), discussed in Section 3.3.

3.1.4 Other consequences of inflation

Initially, inflation was proposed as a solution to the flatness and horizon problems of the standard Big Bang cosmology, as well as that of the magnetic monopoles. Soon after, physicists realized that the inflationary scenario had other remarkable consequences.

For example, it can dilute any previous unwanted relics, like topological defects, which may form during the early universe phase transitions, see Section 6.2. But the most important feature of inflation is the fact that it provides a viable mechanism of generation of cosmological fluctuations [11]-[14], which are the seeds for the structure formation in our universe. These seeds are produced during inflation and are attributed to quantum fluctuations of the inflaton field. More details about this mechanism will be given in Section 3.5.

3.2 Basic picture of inflation

3.2.1 Historical review

The possibility of having an accelerated expansion of the universe has been contemplated by many authors, before any model of inflation was given. A comprehensive review of these pre-inflationary scenarios can be found in reference [15]. The first model of inflation as a physical model was proposed in 1981 by Alan Guth [16] as an elegant solution to the shortcomings of the standard Big Bang cosmological model. In his model, he supposes that some scalar field ψ is trapped in a local minimum at the origin of the potential. Inflation is produced by the energy of the false vacuum and ends when ψ tunnels through the barrier in the potential and evolves towards the true vacuum. This first inflationary model is called "old inflation", and later, it was noted that it was not a viable model of inflation because bubbles of the new phase could never coalesce.

A first viable model came in 1982. Linde [17], and Albrecht and Steinhardt [18], considered a model in which the inflaton was slowly rolling down a flat potential. In this "new inflation" model, the potential has a maximum at the origin. Inflation takes place near the maximum and ends when the inflaton starts to oscillate around the minimum. In new inflation, both the form of the potential and its possible GUT origin were given.

After a period of complicated model-building, Linde proposed [19] in 1983 a new scenario, in which the inflaton is rolling towards the origin and has field values bigger than the Planck mass $M_{\rm P}$. The potential was chosen to be of a simple form, say ϕ^2 or ϕ^4. The values of ϕ were supposed to be a chaotically varying function of position and for this reason, this model is usually called "chaotic inflation". A few years later, the chaotic model became the favored one, although it seemed difficult to find any connection with particle physics.

Around 1990, inflationary model-building resurrected, when La and Steinhardt [20] proposed an improved model of "old" inflation. Their purpose was to provide a mechanism for making the bubbles coalesce at the end of inflation. This mechanism was obtained by adding a slowly-rolling inflaton field ϕ, in the context of an extension of Einstein's General Relativity, the Brans-Dicke theory. For this reason, this model received the name of "extended inflation". It was ruled out immediately, in 1992, by the COBE detection [21] of CMB anisotropy, which indicated no sign of bubbles formed at the end of inflation.

In fact, extended inflation can be re-formulated as an Einstein gravity theory. This is why, in 1991, Linde [22] and Adams and Freese [23] proposed a crucial change in the idea behind extended inflation, but working with Einstein gravity. Their idea was to couple the trapped field ψ to the slowly-rolling inflaton field ϕ, making tunnelling completely impossible until the end of inflation. At the end of inflation, tunnelling takes place, the bubbles can coalesce very quickly and leave no imprint on the CMB.

Soon after, still in 1991, Linde [24] removed the idea of bubble formation at the end of inflation, by proposing a second-order phase transition at the end of inflation, instead of a first-order one as in the original model. This final paradigm is known as "hybrid inflation". We will discuss the hybrid inflation mechanism later on, in Section 3.4.

Nowadays, there are plenty of inflationary scenarios, related to SUSY theory, brane world, string theory, etc [25]-[33]. A survey of most of the present inflationary models is given in Section 3.4.

3.2.2 The inflaton field

In the previous section, we have seen that the shortcomings of the standard Big Bang cosmology can be solved by inflation, i.e., a period of accelerated expansion of the early universe. The most simple model of inflation consists of a single scalar field ϕ, called *inflaton*, which dominates the energy of the universe during inflation. We have also seen that, in order to produce acceleration ($\ddot{a} > 0$), its pressure should be negative and satisfy $p < -\rho/3$.

The action of the inflaton field is:

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_\phi = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \tag{3.8}$$

where $\sqrt{-g} = a^3$ for the FRW metric and $V(\phi)$ is the inflaton potential. The equation of motion of ϕ can be obtained from the Euler-Lagrange equations and is given by:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) = 0 \tag{3.9}$$

where a prime denotes $\mathrm{d}/\mathrm{d}\phi$ and an overdot means $\mathrm{d}/\mathrm{d}t$. The term $3H\dot{\phi}$ acts like a friction term and is a consequence of the expansion of the universe.

The energy density and pressure of the inflaton field can be calculated from the energy-momentum tensor $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_\phi$,

$$\rho_\phi = T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{(\nabla\phi)^2}{2a^2}, \tag{3.10}$$

$$p_\phi = T_{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{(\nabla\phi)^2}{6a^2} \tag{3.11}$$

(no sum over i). We can split the inflaton field as

$$\phi(x) = \phi_0(t) + \delta\phi(\mathbf{x}, t), \tag{3.12}$$

where $\phi_0(t)$ is the "classical" (homogeneous) field and $\delta\phi(\mathbf{x}, t)$ represents the quantum fluctuations around $\phi_0(t)$.

Let us study first the evolution of the classical field ϕ_0, which is much larger than the fluctuations. The fate of the fluctuations will be investigated later, in Section 3.5.

The energy density and the pressure of the classical field become:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.13}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.14}$$

in which I have dropped the subscript "0" for simplicity. If the potential energy dominates over the kinetic term, $V(\phi) \gg \dot{\phi}^2$, we obtain:

$$p_\phi \simeq -\rho_\phi \tag{3.15}$$

which is exactly what is needed to produce inflation.

3.2.3 The slow-roll parameters

We have seen that inflation requires that the energy density of the inflaton field be dominated by its potential, which means that the inflaton field should be slowly rolling down its potential. This is equivalent to say that V should be sufficiently flat and that the term $\ddot{\phi}$ is also small, $\ddot{\phi} \ll V'$.

The Friedmann equation (2.37) becomes:

$$H^2 \simeq \frac{8\pi G}{3}V(\phi) \tag{3.16}$$

when the potential energy of the inflaton field dominates the energy density of the universe. In this case, the equation of motion (3.9) is:

$$3H\dot{\phi} \simeq -V'(\phi). \tag{3.17}$$

In order to quantify the slow-roll conditions $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll V'$, it is useful to define the *slow-roll parameters*, ϵ and η, given by:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3\dot{\phi}^2}{2V} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{3.18}$$

$$\eta = \frac{V''}{3H^2} = \frac{3\ddot{\phi}}{V'} + \frac{3\dot{\phi}^2}{2V} = \frac{1}{8\pi G}\frac{V''}{V} \tag{3.19}$$

and a combination of these:

$$\delta = \eta - \epsilon = \frac{3\ddot{\phi}}{V'} = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{3.20}$$

From the above definitions and the second Friedmann equation (2.38), we have:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2, \tag{3.21}$$

which indicates that inflation can only occur if $\epsilon < 1$. Thus, during inflation, the slow-roll parameters should be much less than unity: $\epsilon, |\eta| \ll 1$. Inflation ends when one of these parameters becomes greater than 1: $\max(\epsilon, |\eta|) > 1$.

3.2.4 The epoch of horizon exit

Without inflation, the Hubble radius H^{-1} is a growing function of the scale factor, $H^{-1} \propto a^2$ during radiation-domination and $H^{-1} \propto a^{3/2}$ during matter-domination, while a physical scale $\lambda_{\rm phys}$ also grows due to the expansion, but only as $\lambda_{\rm phys} \propto a$. What this means is that in the standard cosmology, the horizon always grows faster than the physical scales, so that scales that were larger than the horizon in the past, may ultimately be smaller than the horizon. One says that a scale *enters* the horizon at the moment when it has the same size as the horizon.

The situation is different if there is a period of inflation, see Figure 3.1. The Hubble radius H^{-1} is approximately constant during inflation, while the physical scales grow exponentially as $\lambda_{\rm phys} \propto a(t) \propto \exp(Ht)$. This means that during inflation, scales that were smaller than the Hubble radius in the past become larger than this and one says that they *exit* the horizon. As suggested by the homogeneity problem, our present observable universe should have been inside the Hubble volume during inflation, and at some moment it exits the horizon. After the end of inflation, the Hubble radius starts growing faster than the observable scales, which will finally *re-enter* the horizon. The largest cosmological scales are re-entering the horizon at present. Regardless of the fact that the scales exit

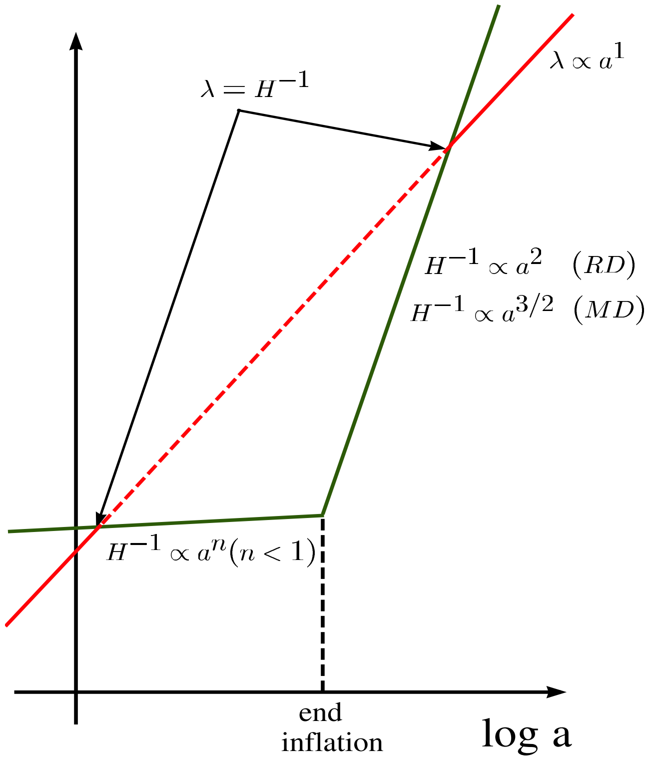

Figure 3.1: The evolution of the horizon and of a generic physical scale λ during and after inflation (from reference [34]).

or enter the horizon, one usually refers to these epochs as *horizon crossing*. At horizon crossing, one has:

$$k = a_* H_* \tag{3.22}$$

where k is the wave number corresponding to some comoving scale λ, and a star indicates the corresponding value at horizon crossing.

An important parameter is the number of e-foldings of inflation N_* that occur between a given scale k exits the horizon and the end of inflation. In order to give an expression for N_*, we should specify the main eras the universe goes through. After the scale k exits the horizon during inflation, the first important moment is the end of inflation, denoted by a subscript "end". After the end of inflation, the energy density stored in the inflaton field is converted into particles in a process called "reheating". We assume that during reheating, the universe is matter dominated. The end of this era will be denoted by a subscript "reh". After reheating, the produced particles thermalize and a new era starts, dominated by radiation. It lasts until the energy density in non-relativistic matter becomes equal to that in relativistic particles. The matter-radiation equality will receive the subscript "eq". Finally, after equality the universe becomes dominated by matter until the present epoch, which is denoted by the subscript "0". Here we have neglected the short period of recent dark energy domination, for simplicity.

By recalling that the radiation energy density $\rho_{\rm R}$ is proportional to a^{-4}, that the energy density of the matter $\rho_{\rm M}$ is proportional to a^{-3}, and that the total energy density is proportional to H^2, we can write:

$$\begin{aligned} \frac{k}{a_0 H_0} &= \frac{a_* H_*}{a_0 H_0} = \left(\frac{a_*}{a_{\rm end}}\right)\left(\frac{a_{\rm end}}{a_{\rm reh}}\right)\left(\frac{a_{\rm reh}}{a_{\rm eq}}\right)\left(\frac{a_{\rm eq}}{a_0}\right)\frac{H_*}{H_0} \\ &= {\rm e}^{-N_*}\left(\frac{\rho_{\rm reh}}{\rho_{\rm end}}\right)^{1/3}\left(\frac{\rho_{\rm eq}}{\rho_{\rm reh}}\right)^{1/4}\left(\frac{\rho_0}{\rho_{\rm eq}}\right)^{1/3}\left(\frac{\rho_*}{\rho_0}\right)^{1/2}. \end{aligned} \tag{3.23}$$

From the above equation, one can obtain [35, 36] an expression for N_*:

$$N_* = 61 - \ln h - \ln\frac{k}{a_0 H_0} + \ln\frac{V_*^{1/4}}{10^{16}{\rm GeV}} + \ln\frac{V_*^{1/4}}{V_{\rm end}^{1/4}} - \frac{1}{3}\ln\frac{V_{\rm end}^{1/4}}{\rho_{\rm reh}^{1/4}} \tag{3.24}$$

where $h \sim 0.7$ was defined in (2.12) and parameterizes the Hubble constant H_0. Thus, given a form for the potential $V(\phi)$ and an estimation of the temperature after reheating, one can calculate the number of e-foldings of inflation occurring after the scale k exits the horizon. The above relation also allows one to find the value of the inflaton field ϕ_* when a given scale k crosses the horizon. From the definition of N, equation (3.2), combined with equations (3.16) and (3.17), one has:

$$N_* = \frac{8\pi}{M_{\rm P}^2}\int_{\phi_{\rm end}}^{\phi_*}\frac{V(\phi)}{V'(\phi)}{\rm d}\phi \tag{3.25}$$

which combined with (3.24) gives $\phi_*(k)$. In general, $V_*^{1/4} \simeq V_{\rm end}^{1/4} \leq 10^{16}\,{\rm GeV}$ and $\rho_{\rm reh} < V_{\rm end}$, so that there is an upper limit for the number of e-foldings N_0 corresponding to the largest observable scales, $k = a_0 H_0$, which is $N_0 \sim 60$. For low-scale inflation, and a very low reheating temperature, this number is considerably reduced, being around 30 for the lowest energy scale $V^{1/4} \sim 1000\,{\rm GeV}$.

3.3 Reheating after inflation

After inflation was introduced to solve the problems of the standard cosmology discussed in Section 2.6, another problem arose, related to explaining the high temperatures required in the standard hot Big Bang picture. Due to the enormous expansion caused by inflation, the universe is left at effectively zero temperature. Thus, a successful theory of inflation must also give a *reheating* mechanism, by which the universe is *reheated* to a sufficiently high temperature, $T_{\rm rh} > 1000$ GeV, where $T_{\rm rh}$ is called the *reheating temperature*.

3.3.1 Standard reheating

The first theory of reheating [37]-[41] was based on the concept of single-body decays, and the inflaton field was considered as a collection of scalar particles, each particle having its finite probability of decaying. By coupling the inflaton ϕ to other scalar χ or fermion ψ fields, the inflaton can decay to these particles, which later thermalize.

Let us suppose that there are contributions to the Lagrangian of the form $\nu\sigma\phi\chi^2$ and $h\phi\bar{\psi}\psi$ resulting from inflaton couplings to scalars χ and fermions ψ, with dimensionless couplings ν and h, and σ has dimensions of mass. If the inflaton mass m_ϕ is much larger than those of χ and ψ, the corresponding decay rates are [37],[42]-[44]:

$$\Gamma_{\phi\to\chi\chi} = \frac{\nu^2\sigma^2}{8\pi m_\phi}, \tag{3.26}$$

$$\Gamma_{\phi\to\psi\psi} = \frac{h^2 m_\phi}{8\pi}. \tag{3.27}$$

Thermal equilibrium cannot be reached if $\Gamma < H$ because of the rapid expansion. An upper limit for the reheating temperature $T_{\rm rh}$ can be obtained by equating $\Gamma_{\rm tot} = \Gamma_{\phi\to\chi\chi} + \Gamma_{\phi\to\psi\psi}$ and the Hubble rate $H = (8\pi\rho/3M_{\rm P}^2)^{1/2}$, where $\rho = g_*\pi^2 T^4/30$ is the energy density of relativistic matter, see equation (2.66):

$$T_{\rm rh} \simeq 0.2 \left(\frac{100}{g_*}\right)^{1/4} \sqrt{\Gamma_{\rm tot} M_{\rm P}}. \tag{3.28}$$

There is an upper bound on the inflaton mass coming from observations of the CMB anisotropies, $m_\phi \sim 10^{-6} M_{\rm P}$, which puts an upper limit to the reheating temperature $T_{\rm rh} < 10^{16}$ GeV, which is below the GUT scale. This means that the GUT symmetries are not restored and the monopole problem is not affected. Inflation can solve [45] the gravitino problem [46] of supergravity models, but one can show that in such models, in general, the reheating temperature has an upper bound, $T_{\rm rh} < 10^9$ GeV [40],[47]-[51].

3.3.2 The theory of preheating

After inflation, the inflaton field starts oscillating around the minimum of its effective potential and produces particles. In the "old" reheating theory, the oscillating inflaton field was regarded as a collection of particles that decayed to other particles. The usual approach to reheating is through perturbation theory, but there are effects beyond the perturbation theory related to the stage of *parametric resonance*, also called *preheating*.

Preheating [52] is the stage of parametric resonance of non-perturbative nature and occurs far away from thermal equilibrium. The energy transfer from the inflaton field to other bosonic fields and particles during preheating is extremely efficient. In this Thesis, reheating/preheating after inflation is not investigated, but I recommend the reader interested in details to consult the work of reference [53]. Here, I briefly sketch the main ideas behind the theory of preheating.

Let us consider a simple model with a massive inflaton field ϕ coupled to another scalar field χ with the interaction term $g\phi^2\chi^2$, with g a dimensionless coupling. In the stage of parametric resonance, χ particles are produced by the decay of the inflaton field. If the occupation number of χ particles, $n_k > 1$, then the probability of decay becomes greatly enhanced due to effects related to Bose statistics (stimulated production), which may lead to explosive particle production. The main difference with respect to the perturbation theory is the fact that in the case of preheating, the amount of produced particles depends on the number of particles produced earlier. Thus, the elementary theory of reheating and preheating due to parametric resonance are two *different* ways of describing the decay of a scalar field.

There are different regimes in which parametric resonance may occur, depending on the amplitude of inflaton oscillations. For large amplitude, one has a *broad resonance* and it can be shown that in this regime reheating becomes extremely efficient. For small oscillation amplitude, one has a *narrow resonance*, which can be interpreted as a resonance with decay of two ϕ particles with mass m to two χ particles with momenta $k \sim m$. For each oscillation of the field $\phi(t)$ the growing modes of the field χ oscillate one time and the number of produced particles grows exponentially.

Stochastic resonance is the process of particle creation similar to parametric resonance in that, on average, the number of produced particles grows exponentially, but at some moments their number may decrease. This is due to the expansion of the universe that acts like a friction term and modifies the amplitude of the oscillations of the inflaton field, which make explosive particle production a rather stochastic process. Stochastic resonance only occurs during the first part of the process, when the amplitude of the oscillations are very large and the resonance is very broad. Gradually the amplitude of the field ϕ decreases and, finally, the expansion of the universe can be neglected and we can use the standard methods of investigation in Minkowski space.

Parametric resonance ends when the back reaction of the created particles becomes important and cannot be neglected. There are two important effects of the created particles: (i) back reactions of χ particles may increase the effective mass of the inflaton, m, which can make the resonance narrow and eventually shut it down; (ii) production of ϕ particles, which occurs due to interaction of χ particles with the oscillating field $\phi(t)$. This process is usually called *rescattering*. During rescattering, the effective mass of the field χ may change and it is possible that χ particles become so heavy that they can no longer be produced.

In the final stage of inflaton oscillations, parametric resonance terminates and reheating can be described by the elementary perturbative theory of reheating.

3.4 A survey of inflationary models

There is a large number of inflationary models at present. Some of them are based on possible extensions of the Standard Model of elementary particles, or are simply characterized by the form of inflaton potential. The first models of inflation only contained one scalar field whose potential energy started to dominate the energy density of the universe at a given moment, thus causing inflation. For most cases, the scale factor grows exponentially during inflation, $a(t) \propto \exp(Ht)$, but there are models where this is not true, i.e., in power-low inflation [54]-[56] it grows as $a(t) \propto t^p$, where $p \gg 1$. There are models which are based on modifications of the Einstein's General Relativity, as in the original extended inflation model [20, 57], but after a conformal modification of the metric it can be reduced to power-law inflation in Einstein's General Relativity [58]. Other interesting scenarios are warm inflation [59, 60], in which there is particle production during inflation and one does not require a reheating period after inflation, inflation in theories with more than four space-time dimensions [61]-[66], etc. Apart from one-field inflation models, there are models in which the inflaton field interacts with another auxiliary field [67], or one can have inflation resulting from the existence of a large number of fields [68].

Even restricting ourselves only to simple single-field models, we are still left with plenty of models [69], which might have some common features or differ substantially. Thus, it might be convenient to give general classification schemes [34, 70]. Models can be classified by various criteria: by the way inflation starts, by the various regimes that are possible during inflation or by the way it ends. With these criteria in mind, we can divide models in three general types: *large-field* models, in which the inflaton field can have values of the order of the Planck mass or larger, *small-field* models, in which the values of the inflaton field are smaller than M_{P} and generally the inflaton starts very close to the origin of the potential, and *hybrid* models, which include models with more than one relevant field during inflation.

Let us briefly describe the three types mentioned above and insist on the models that are relevant for the work developed in this Thesis.

3.4.1 Large-field models

In models of large-field type, the inflaton field is displaced from the minimum of the potential by an amount of order the Planck mass $M_{\rm P}$ or even larger. The most representative models of this type are the *chaotic* scenarios [19], in which one assumes that the universe emerged from a quantum gravitational state characterized by an energy density comparable with the Planck density, $M_{\rm P}^4$. The inflaton can take any value before inflation, and if in some region it is larger than $M_{\rm P}$, the friction term $3H\dot{\phi}$ is large and in that region inflation may occur. The generic types of potential in chaotic models are the polynomial form $V(\phi) = \Lambda^4(\phi/v)^n$ and the exponential form $V(\phi) = \Lambda^4 \exp(\phi/v)$. Such models are characterized by $V''(\phi) > 0$ and $-\epsilon < \delta < \epsilon$. In general, large-field models give a very large total number of e-foldings of inflation, so that our observable universe is only a very small part of the entire universe that suffered inflation.

3.4.2 Small-field models

If the inflaton field is smaller than the Planck mass, the corresponding model is of small-field type. Usually, the inflaton field starts at an unstable region of the potential, near the origin, and rolls down the potential towards a stable minimum. Representative examples of this type of models are *new* inflation [17, 18], in which the inflaton evolves after a spontaneous symmetry breaking of the potential, and *natural* inflation [71], where the inflaton is a pseudo Nambu-Goldstone boson. The generic potential in small-field models has the form $V(\phi) = \Lambda^4[1-(\phi/v)^n]$ and they are characterized by $V''(\phi) < 0$ and $\delta < -\epsilon$.

In Chapter 10 of this Thesis, I have used a particular small-field type of potential, the Coleman-Weinberg potential [72]. This is why, in what follows I will give more details about this kind of potential.

Coleman-Weinberg potential. The first model of new inflation type was based upon an $SU(5)$ GUT and had a potential of the Coleman-Weinberg (CW) type for the scalar Higgs field responsible for the spontaneous breaking of $SU(5)$ to $SU(3) \times SU(2) \times U(1)$ [17, 18]. The zero-temperature one-loop CW potential can be written in terms of the magnitude of the Higgs field in the $SU(3) \times SU(2) \times U(1)$ direction as:

$$V(\phi) = \frac{Av^4}{2} + A\phi^4 \left(\ln \frac{\phi^2}{v^2} - \frac{1}{2} \right) \tag{3.29}$$

where $A = 25\alpha_{\rm GUT}^2/16 \simeq 10^{-3}$, $\alpha_{\rm GUT} \simeq 1/45$ and v is the vacuum expectation value (vev) of ϕ. Note that the potential contains no mass term, so it is expected to be very flat near the origin. For $\phi \ll v$ it becomes:

$$V(\phi) \simeq \frac{1}{2}Av^4 - \frac{1}{4}\lambda\phi^4 \tag{3.30}$$

where $\lambda \simeq 4A\ln(\phi^2/v^2)$. Inflation in this model occurs for $\phi-$values very close to zero, so that $V(\phi) \simeq Av^4/2$ and from the first Friedmann equation (2.37) we have that the Hubble rate is given by:

$$H^2 \simeq \frac{4\pi}{3}\frac{Av^4}{M_{\rm P}^2}. \tag{3.31}$$

Temperature effects have been neglected here, but a more detailed treatment on finite temperature effects and spontaneous symmetry breaking will be given later, in Chapter 6.

Inflation ends when the slow-roll parameters become of order unity. The number of e-foldings of inflation as a function of the inflaton field is given by equation (8.18):

$$N(\phi) = \frac{8\pi}{M_{\rm P}^2}\int_{\phi}^{\phi_{\rm end}} \frac{V(\phi)}{-V'(\phi)}{\rm d}\phi = \frac{\pi}{2}\frac{v^4}{M_{\rm P}^2\phi^2\left|\ln\frac{\phi^2}{v^2}\right|}. \tag{3.32}$$

In the above equation I used the approximation $\phi \ll \phi_{\rm end} \simeq 3H^2/\lambda$, where $\phi_{\rm end}$ is the value of the inflaton field at the end of inflation.

In this Thesis, I use a CW-type of potential for inflation in a unified model of dark matter and dark energy, in Chapter 10. In that model, the coefficient A is estimated by taking into account one-loop contributions of new gauge "messenger" fields $\varphi_i^{(Z)}$ and fermions $\psi_i^{(Z)}$, charged under a new gauge symmetry $SU(2)_Z$, and also from inflaton self interactions.

3.4.3 Hybrid models

The models of hybrid type are based on the idea that, during inflation, the energy density of the universe is not dominated by the inflaton, but by the vacuum energy of a second field. Inflation ends because of the instability of this field. The generic inflaton potential in hybrid inflation has the form $V(\phi) = \Lambda^4[1 + (\phi/v)^n]$, and $V''(\phi) > 0$ and $0 < \epsilon < \delta$.

The hybrid inflation scenario [24, 67, 73] normally appears in supersymmetry and supergravity models. In this Thesis, I use a hybrid type of potential inspired from supergravity [74, 75], but here I prefer not to enter into the details of this type of theories. The interested reader may also consult reference [25].

The original idea of hybrid inflation belongs to A. Linde [24]. In the simplest realization of the model, he coupled the inflaton field ϕ to an auxiliary field χ, with a potential of the form:

$$V(\phi,\chi)=\frac{1}{4}\lambda(M^2-\chi^2)^2+\frac{1}{2}m^2\phi^2+\frac{1}{2}\lambda'\phi^2\chi^2 \tag{3.33}$$

where m is the mass of the inflaton, M is a mass scale, and λ and λ' are coupling constants. Inflation occurs due to the large vacuum energy $\Lambda^4=\lambda M^4/4$, and the inflaton slowly rolls down the potential towards the origin, while the field χ is fixed at a stable false minimum. Inflation ends when the inflaton reaches the critical value $\phi_c=M\sqrt{\lambda/\lambda'}$, where the field χ is destabilized and a phase transition with symmetry breaking occurs when χ "falls" into the true minimum of the potential.

Inverted hybrid inflation. Another interesting possibility is to reverse the sign of some of the terms appearing in the hybrid inflation potential. A simple example of such a potential is of the form [75]:

$$V(\phi,\chi)=V_0-\frac{1}{2}m_\phi^2\phi^2+\frac{1}{2}m_\chi^2\chi^2-\lambda\phi^2\chi^2+\ldots \tag{3.34}$$

The dots represent terms in the potential that are irrelevant during inflation, but they are needed to ensure that the potential is bounded from below. In this case, the inflaton is rolling away from the origin, while the field χ is fixed at the stable false minimum $\chi=0$. As in the standard hybrid case, inflation ends when the auxiliary field χ becomes unstable, for $\phi=\phi_c=m_\chi/\sqrt{\lambda}$. At that point, one requires that the fields evolve very quickly and oscillate around the absolute minimum of the potential. This rapid way in which inflation ends has inspired physicists to refer to it as the "waterfall mechanism".

In Figure 3.2 I illustrate an example of a potential used in inverted hybrid inflation models. Initially, the inflaton field ϕ is located at the origin point A, which is stable in the $\chi-$direction, but unstable in the $\phi-$direction. The inflaton ϕ may start to roll-down along the line characterized by $\chi=0$, until it reaches point B. There, the curvature of the potential in the $\chi-$direction changes sign and the field χ becomes unstable. The point B corresponds to the critical value of the inflaton field, $\phi_c=m_\chi/\sqrt{\lambda}$. If the curvature in the $\chi-$direction changes rapidly from positive values for $\phi<\phi_c$ to negative values for $\phi>\phi_c$, the evolution of the field χ may be much more rapid than that of ϕ. Then, the field χ may "fall" towards the absolute minimum located at the point C and reach it in a period of time comparable to the Hubble time, after which it starts rapid oscillations around the minimum. Thus, after the inflaton reaches the critical point B, no significant number of e-foldings of inflation is produced and inflation has a sudden end.

In this Thesis, I use a potential of inverted hybrid type, in order to describe a unified model of inflation and dark energy, in Chapter 8, and a unified model of inflation and

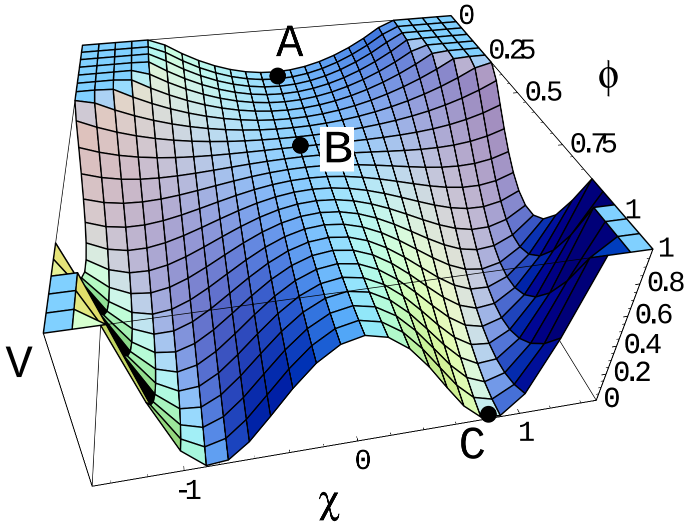

Figure 3.2: Schematic illustration of the potential of inverted hybrid inflation models. The inflaton field ϕ slowly-rolls between points A and B, after which the auxiliary field χ becomes unstable and it quickly "falls" towards the absolute minimum C.

dark matter, in Chapter 9. The potential I use in both models is inspired from SUSY, but I use a slightly modified form in order to bound it from below and to allow for a spontaneous breaking of the potential symmetry.

3.5 Quantum fluctuations during inflation

3.5.1 Fluctuations in pure de Sitter expansion

Let us consider a generic massless scalar field $\phi(\mathbf{x}, t)$ during a de Sitter phase of inflation. We assume that the field can be written as the sum of a homogeneous part and a small quantum fluctuation, $\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$. We will start first by studying the scalar perturbations produced during inflation, and discuss tensor perturbations later, in 3.5.7.

The fluctuation $\delta\phi(\mathbf{x}, t)$ can be expanded in Fourier modes as:

$$\delta\phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \mathrm{e}^{i\mathbf{k}\mathbf{x}} \delta\phi_{\mathrm{k}}(t). \tag{3.35}$$

The equation of motion for $\delta\phi_{\mathrm{k}}(t)$ can be obtained from the equation of motion (3.9) for

ϕ, and is given by:

$$\delta\ddot{\phi}_{\rm k} + 3H\delta\dot{\phi}_{\rm k} + \frac{k^2}{a^2}\delta\phi_{\rm k} = 0. \tag{3.36}$$

Note that for small scales $k \gg aH$ the friction term $3H\delta\dot{\phi}_{\rm k}$ can be neglected and the solution is that of an harmonic oscillator. For superhorizon scales $k \ll aH$, the last term in equation (3.36) is negligible and in this case the solution $\delta\phi_{\rm k}$ is approximately constant.

It is convenient to study the evolution of the fluctuations by performing the following redefinition:

$$\delta\sigma_{\rm k} \equiv a\delta\phi_{\rm k} \tag{3.37}$$

and work with conformal time $d\tau = dt/a$ defined in equation (2.13). With these changes, equation (3.36) becomes:

$$\delta\sigma_{\rm k}'' + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_{\rm k} = 0, \tag{3.38}$$

where a prime denotes derivative with respect to conformal time.

For a pure de Sitter expansion, $a(t) \propto e^{Ht}$ and $a(\tau) = -1/(H\tau)$, from which it results that $a''/a = 2/\tau^2$, and the solution to equation (3.38) is of the form:

$$\delta\sigma_{\rm k}(\tau) = C(k)e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right). \tag{3.39}$$

The constant $C(k)$ can be obtained from the canonical commutation relations satisfied by $\delta\sigma_{\rm k}$:

$$\delta\sigma_{\rm k}^*\delta\sigma_{\rm k}' - \delta\sigma_{\rm k}\delta\sigma_{\rm k}^{*'} = -i \tag{3.40}$$

and one obtains $C(k) = 1/\sqrt{2k}$. With this, the solution $\delta\sigma_{\rm k}$ becomes:

$$\delta\sigma_{\rm k}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{3.41}$$

Note that in the low-scale limit $k \gg aH$, which is equivalent to $-k\tau \gg 1$, the solution $\delta\sigma_{\rm k}(\tau)$ is just a plane wave $\delta\sigma_{\rm k}(\tau) = (1/\sqrt{2k})e^{-ik\tau}$.

Going back to $\delta\phi_{\rm k}$ variable, we see that in the large-scale limit $-k\tau \ll 1$ we have:

$$|\delta\phi_{\rm k}| = \frac{|\delta\sigma_{\rm k}|}{a} \simeq \frac{1}{a|\tau|}\frac{1}{\sqrt{2k^3}} \simeq \frac{H}{\sqrt{2k^3}} \tag{3.42}$$

which shows that, indeed, on superhorizon scales the fluctuations remain constant.

The main conclusion is that the fluctuations of a massless scalar field produced during pure de Sitter inflation behave as plane waves at scales smaller than the horizon, while after horizon exit they become constant and can be regarded as classical.

In the case of a massive field ϕ, with mass m_ϕ, one can show that the analogous of (3.42) is [76]:

$$|\delta\phi_{\rm k}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\eta_\phi}, \tag{3.43}$$

where $\eta_\phi = m_\phi^2/(3H^2)$ is defined in analogy with the slow roll parameters η and ϵ.

3.5.2 The power spectrum

Another useful quantity that characterizes the properties of the perturbations is the *power spectrum*. For a generic perturbation $\delta\phi_{\rm k}(t)$, the power spectrum $\mathcal{P}_{\delta\phi}(k)$ is defined by:

$$\langle 0|\delta\phi^*_{{\rm k}_1}\delta\phi_{{\rm k}_2}|0\rangle \equiv \delta^3(\mathbf{k}_1-\mathbf{k}_2)\frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi}(k), \tag{3.44}$$

where $|0\rangle$ is the vacuum quantum state of the system.

By using (3.44), the variance of the perturbations $\delta\phi(\mathbf{x},t)$ is:

$$\langle 0|\delta\phi^2(\mathbf{x},t)|0\rangle = \int\frac{{\rm d}^3k}{(2\pi)^3}|\delta\phi_{\rm k}|^2 = \int\frac{{\rm d}k}{k}\frac{k^3}{2\pi^2}|\delta\phi_{\rm k}|^2 = \int\frac{{\rm d}k}{k}\mathcal{P}_{\delta\phi}(k). \tag{3.45}$$

Thus, the power spectrum of the fluctuations of ϕ can be written as:

$$\mathcal{P}_{\delta\phi}(k) = \frac{k^3}{2\pi^2}|\delta\phi_{\rm k}|^2. \tag{3.46}$$

3.5.3 Fluctuations in a quasi de Sitter stage

During inflation, the Hubble rate is, in general, not exactly constant, but changes with time as $\dot{H} = -\epsilon H^2$. In this case, the scale factor is given by:

$$a(\tau) = -\frac{1}{H\tau}\frac{1}{1-\epsilon}. \tag{3.47}$$

Let us study the perturbations of a massive scalar field ϕ during a quasi de Sitter phase. For the fluctuations $\delta\phi_{\rm k}$ on superhorizon scales we obtain a similar result to equation (3.43) for pure de Sitter expansion,

$$|\delta\phi_{\rm k}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\eta_\phi-\epsilon} \tag{3.48}$$

with the difference that now the fluctuation also depends on the parameter ϵ.

By replacing equation (3.48) in (3.46) one obtains [76]:

$$\mathcal{P}_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\delta\phi}-1} \tag{3.49}$$

where one defines the *spectral index* $n_{\delta\phi}$ of the fluctuations as:

$$n_{\delta\phi} - 1 = \frac{\mathrm{d}\ln \mathcal{P}_{\delta\phi}}{\mathrm{d}\ln k} = 2\eta_\phi - 2\epsilon. \tag{3.50}$$

We conclude that the power spectrum of fluctuations produced during a quasi de Sitter phase of expansion is almost flat, $n_{\delta\phi} \sim 1$, because both η_ϕ and ϵ are small during inflation.

3.5.4 Consequences of inflaton fluctuations

In the previous subsections we have studied the perturbations of a generic scalar field ϕ, which was not necessary the inflaton, during an inflationary stage of expansion. Let us assume now that ϕ is the inflaton field, and see the consequences it has. Because the inflaton field dominates the energy density of the universe during inflation, one expects that its fluctuations produce certain changes in the ideal homogeneous and isotropic background. Indeed, the theory of structure formation is based on the assumption that the seeds for the inhomogeneities present in the universe are provided by inflaton field fluctuations, and it is nowadays the most popular and successful theory for explaining the observed structure of the universe [10].

The general philosophy is that the perturbations in the inflaton field induce perturbations in the stress energy-momentum tensor, which in turn induces perturbations of the metric. On the other hand, a perturbation of the metric induces a back-reaction on the evolution of the inflaton perturbations. This means that the perturbations of the inflaton field and those of the metric are tightly coupled to each other:

$$\delta\phi \Leftrightarrow \delta g_{\mu\nu}. \tag{3.51}$$

Thus, perturbations of the metric involve perturbations of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. In this way one can obtain the perturbed Einstein equation and also the perturbed Klein-Gordon equation for the inflaton field.

Because the theory of General Relativity is a gauge theory, one has to choose a gauge in order to compute the perturbations. This can be done in two ways: (i) the first is to define gauge-invariant quantities, which have physical meaning, but then the computation may be more complicated; (ii) the second is to choose a given gauge and perform the calculations in that gauge, which is technically simpler, but has the drawback of including possible gauge artifacts, which are not physical.

Let us give here a few examples of gauge-invariant quantities, related to different coordinate transformations on constant time hypersurfaces (slicing).

- the *comoving curvature perturbation*, defined as:

$$\mathcal{R} = \psi + \mathcal{H}\frac{\delta\phi}{\phi'} = \psi + H\frac{\delta\phi}{\dot{\phi}} \tag{3.52}$$

  where ψ is the gauge-dependent curvature perturbation on a generic slicing and $\delta\phi$ is the inflaton perturbation in that gauge. The curvature perturbation ψ is related to the intrinsic spatial curvature on hypersurphaces of constant conformal time τ, and for a flat universe gives ${}^{(3)}R = \frac{4}{a^2}\nabla^2\psi$. This means that $\mathcal{R}$ represents the gravitational potential on comoving hypersurfaces where $\delta\phi = 0$.

- the *curvature perturbation on slices of uniform density*, defined as:

$$\zeta = \psi + \mathcal{H}\frac{\delta\rho}{\rho'} = \psi + H\frac{\delta\rho}{\dot{\rho}} \tag{3.53}$$

  which is related to the gauge-dependent curvature perturbation ψ on a generic slicing and to the inflaton energy density perturbation $\delta\rho$ in that gauge. This means that ζ represents the gravitational potential on slices of uniform energy density where $\delta\rho = 0$. One can show that, on superhorizon scales, the curvature perturbation on slices of uniform density is equal to the comoving curvature perturbation, i.e., $\zeta \simeq \mathcal{R}$.

- the perturbation in *spatially flat gauge*, defined as:

$$Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\psi = \delta\phi + \frac{\dot{\phi}}{H}\psi = \frac{\dot{\phi}}{H}\mathcal{R} \tag{3.54}$$

  which is related to the inflaton perturbation $\delta\phi$ on a generic slicing and to the curvature perturbation ψ in that gauge. The meaning of Q is that it represents the inflaton potential on spatially flat slices where $\delta\psi = 0$.

3.5.5 Adiabatic and isocurvature perturbations

Before proceeding with the study of the fluctuations produced during inflation, let us discuss shortly the two possible types of primeval fluctuations.

- *adiabatic* or *curvature* perturbations, which are fluctuations in the total energy density $\delta\rho$. They can be characterized in a gauge-invariant manner as fluctuations in the local value of the curvature. For adiabatic perturbations, one can write:

$$\frac{\delta\rho}{\dot{\rho}} = \frac{\delta p}{\dot{p}} \tag{3.55}$$

  which implies that $p = p(\rho)$.

- *isocurvature* or *isothermal* perturbations, which leave the total energy density and the intrinsic curvature unperturbed, but there are relative fluctuations between the different components of the system. For isocurvature perturbations, one has:

$$\frac{\delta\rho}{\dot{\rho}} \neq \frac{\delta p}{\dot{p}} \tag{3.56}$$

which means that isocurvature perturbations may be interpreted as fluctuations in the form of the local equation of state. In order to have isocurvature perturbations, it is necessary to have more than one component. For a set of fluids with energy densities ρ_i, isocurvature perturbations are conventionally defined by gauge-invariant quantities

$$S_{ij} = 3H\left(\frac{\delta\rho_i}{\dot{\rho}_i} - \frac{\delta\rho_j}{\dot{\rho}_j}\right). \tag{3.57}$$

This means that if the inflaton field is the only field during inflation, the cosmological perturbations generated during inflation are of adiabatic type. On the other hand, if during inflation there is a massless axion-like field along with the inflaton field, isocurvature perturbations are expected to be generated. Isocurvature perturbations produced during inflation are highly constrained by recent observations of the CMB and large-scale structure (LSS) of the universe [77].

3.5.6 The power spectrum of comoving curvature perturbation

Our next task is to compute the curvature perturbation generated during inflation on superhorizon scales. We have seen that during inflation, quantum fluctuations of the inflaton field are generated and their wavelengths are stretched on large scales by the rapid expansion of the universe, while their amplitude becomes constant on superhorizon scales. This allows us to use either the comoving curvature perturbation $\mathcal{R}$ or the curvature on uniform energy density hypersurfaces ζ to describe curvature fluctuations on superhorizon scales.

Let us calculate the power spectrum of comoving curvature perturbation

$$\mathcal{R}_{\mathrm{k}} \simeq H\frac{\delta\phi_{\mathrm{k}}}{\dot{\phi}} \tag{3.58}$$

where $\delta\phi_{\mathrm{k}}$ is the fluctuation of the inflaton field on superhorizon scales.

The corresponding power spectrum is given by:

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}|\delta\phi_{\mathrm{k}}|^2 = \frac{2k^3}{\pi\epsilon M_{\mathrm{P}}^2}|\delta\phi_{\mathrm{k}}|^2. \tag{3.59}$$

The time evolution of $\delta\phi_{\mathrm{k}}$ can be evaluated by using the perturbed Klein-Gordon equation in the conformal Newtonian (or longitudinal) gauge, on superhorizon scales [76]:

$$\delta\ddot{\phi}_{\mathrm{k}} + 3H\delta\dot{\phi}_{\mathrm{k}} + (V'' + 6\epsilon H^2)\delta\phi_{\mathrm{k}} = 0 \tag{3.60}$$

which gives:

$$|\delta\phi_{\mathrm{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \tag{3.61}$$

where $\nu = (\frac{9}{4} + 9\epsilon - 3\eta)^{1/2}$ and ϵ and η are the slow-roll parameters defined in (3.18) and (3.19), respectively.

By replacing equation (3.61) in (3.59) we finally obtain [76]:

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{4\pi}{\epsilon M_{\mathrm{P}}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathrm{s}}-1} \equiv A_R^2\left(\frac{k}{aH}\right)^{n_{\mathrm{s}}-1} \tag{3.62}$$

where n_{s} is the *spectral index* of the comoving curvature perturbations defined as:

$$n_{\mathrm{s}} - 1 = \frac{\mathrm{d}\ln\mathcal{P}_{\mathcal{R}}}{\mathrm{d}\ln k} = 2\eta - 6\epsilon. \tag{3.63}$$

We conclude that the spectrum of curvature perturbations is almost scale-invariant with $n_{\mathrm{s}} \sim 1$.

3.5.7 Gravitational waves

Apart from scalar perturbations produced during inflation, there can also exist tensor perturbations, which describe the propagation of free gravitational waves [78]. A gravitational wave may be viewed as a ripple of spacetime in the FRW metric (2.9), which in the linear tensor perturbation theory may be written as:

$$g_{\mu\nu} = a^2(\tau)\left[\mathrm{d}\tau^2 - (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right] \tag{3.64}$$

where $|h_{ij}| \ll 1$.

The gauge-invariant tensor amplitude

$$v_{\mathrm{k}} = \frac{ah_{\mathrm{k}}}{2\sqrt{8\pi G}}, \tag{3.65}$$

where h_{k} is the amplitude of the gravitational waves, satisfies the following equation:

$$v_{\mathrm{k}}'' + \left(k^2 - \frac{a''}{a}\right)v_{\mathrm{k}} = 0. \tag{3.66}$$

In the slow-roll approximation and on superhorizon scales, the solution to equation (3.66) is given by:

$$|v_{\rm k}| = \frac{aH}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_{\rm T}} \tag{3.67}$$

where $\nu_{\rm T} \simeq \frac{3}{2} + \epsilon$.

This corresponds to a *tensor-power spectrum* of the form:

$$\mathcal{P}_T(k) \simeq \frac{64\pi}{M_{\rm P}^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{n_{\rm T}} \equiv A_T^2 \left(\frac{k}{aH}\right)^{n_{\rm T}} \tag{3.68}$$

where $n_{\rm T}$ is the *spectral index* of tensor perturbations, defined by:

$$n_{\rm T} = \frac{{\rm d}\ln \mathcal{P}_T}{{\rm d}\ln k} = 3 - 2\nu_{\rm T} = -2\epsilon. \tag{3.69}$$

This means that the tensor perturbations are almost scale-invariant, and the amplitude of the tensor modes only depends on the Hubble rate H during inflation.

An important observational quantity is the *tensor to scalar ratio*, which is defined as:

$$r \equiv \frac{A_T^2}{A_R^2} \simeq 16\epsilon. \tag{3.70}$$

Since $\epsilon \ll 1$ during inflation, the amplitude of tensor perturbations is very much suppressed relative to that of scalar perturbations.

From equations (3.69) and (3.70) we obtain the *consistency relation*:

$$r = -8n_{\rm T}. \tag{3.71}$$

3.6 Evolution of perturbations after inflation

So far in this chapter we have seen how perturbations are produced and how they evolve during inflation. A perhaps even more important question is the mechanism by which these perturbations evolve into the structure we observe today: stars, galaxies, clusters of galaxies, superclusters, voids, etc.

The theory of structure formation is a well-developed theory [10], in which one needs to know the initial conditions at the time structure formation began. The initial data should include knowledge about the composition of the universe, the amount of non-relativistic matter, and the spectrum and type of primeval density perturbations.

3.6.1 Angular power spectrum of CMB fluctuations

The observed CMB is a snapshot of the universe at the moment of the last-scattering, when the universe became neutral and CMB photons decoupled from the early hot plasma and could propagate towards us. The first precision data were taken by the COBE satellite [21], followed by MAXIMA [79], BOOMERANG [80], DASI [81] and WMAP [82]. They detected temperature fluctuations in the CMB at the level $\Delta T/T \sim 10^{-5}$. These fluctuations in the temperature are connected to fluctuations in the density at the epoch of recombination, which are of similar amplitude, $\Delta T/T \approx \Delta\rho/\rho$.

There are several physical processes responsible for the origin of the temperature fluctuations, which can contribute either to the large-angular scales, $\theta \gg 1^\circ$, or to the small-angular $\theta \ll 1^\circ$ anisotropy. The processes acting on small-angular scales are microphysical processes and will be shortly analyzed at the end of this section.

The temperature fluctuations on large-angular scales arise due to the *Sachs-Wolfe effect* [83]. They probe superhorizon scales at decoupling and provide the "virgin" spectrum of primeval fluctuations, because causality precludes microphysical processes from affecting the fluctuations on angular scales larger than about 1°. It consists in the fact that photons may gain or lose energy in the presence of gravitational potential wells produced by density fluctuations. The mathematical form of the Sachs-Wolfe effect is given by:

$$\frac{\delta T}{T} = \frac{1}{5}\mathcal{R}(\hat{\mathbf{x}}_{LS}) \tag{3.72}$$

where $\hat{\mathbf{x}}_{LS}$ is the coordinate of the observed photon on the last-scattering surface, and $\mathcal{R}$ is the comoving curvature perturbation defined in equation (3.52).

The temperature anisotropy is commonly expanded in spherical harmonics

$$\frac{\delta T}{T}(\mathbf{x_0}, \hat{\mathbf{n}}) = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\hat{\mathbf{n}}) \tag{3.73}$$

where $\mathbf{x_0}$ is our space-time position at present and $\hat{\mathbf{n}}$ is the direction of observation.

The *angular power spectrum* is defined by:

$$C_l = \langle |a_{lm}|^2\rangle = \frac{1}{2l+1}\sum_{m=-l}^{l} |a_{lm}|^2. \tag{3.74}$$

Due to homogeneity and isotropy, the C_l's do not depend on our spatial position x_0, nor on m.

The angular power spectrum is related to the power spectrum of curvature perturbations, $\mathcal{P}_{\mathcal{R}}$. One can show that [76]:

$$C_l = \frac{4\pi}{25}\int_0^\infty \frac{\mathrm{d}k}{k}\mathcal{P}_{\mathcal{R}}(k) j_l^2(k(\tau_0 - \tau_{\mathrm{LS}})) \tag{3.75}$$

where j_l is the spherical Bessel function of order l, τ_0 and $\tau_{\rm LS}$ are conformal times of the observer and of the last scattering surface, respectively, and $\mathcal{P}_{\mathcal{R}}(k)$ is the curvature power spectrum defined in equation (3.59). The above equation (3.75) is valid for $2 \leq l \ll (\tau_0 - \tau_{\rm LS})/\tau_{\rm LS} \sim 100$. The values of $\delta\phi_{\rm k}$ entering in $\mathcal{P}_{\mathcal{R}}(k)$ are obtained from the equation of motion for $\delta\phi_{\rm k}$, equation (3.36).

In Appendix A of this PhD Thesis, I calculate the angular power spectrum of fluctuations produced in a universe with a small curvature and compare it to the standard case of a perfectly flat universe. The curvature will affect the solution $\delta\phi_{\rm k}$ and, as a consequence, the resulting angular power spectrum will be modified on large scales, with respect to the flat case.

3.6.2 The linear growth of structure

The fluctuations in the inflaton field produced during inflation are stretched to scales larger than the horizon and after they exit the horizon their amplitude remains approximately constant. An interesting question is to study how these fluctuations evolve after they reenter the horizon and also to investigate the effects they may have on the universe.

As curvature perturbations enter the causal horizon, they create density fluctuations $\delta\rho_{\rm k}$ via gravitational attractions of the potential wells. It is useful to define the *density contrast* $\delta_{\rm k}$ by:

$$\delta_{\rm k} = \frac{\delta\rho_{\rm k}}{\bar{\rho}} \tag{3.76}$$

where $\bar{\rho}$ is the background average energy density.

Before investigating the fate of the density contrast after a perturbation enters the horizon, firstly we have to analyze how it behaves on superhorizon scales. It can be shown [10] that the superhorizon scales are unstable and the density contrast grows as:

$$\delta_{\rm k} \propto \begin{cases} a^2, & \text{RD} \\ a, & \text{MD} \end{cases} \tag{3.77}$$

After the perturbation enters the horizon, a Newtonian treatment of the evolution of perturbations suffices and it may be obtained from the *linear perturbation equation*:

$$\ddot{\delta}_{\rm k} + 2H\dot{\delta}_{\rm k} + \left(c_{\rm s}^2 \frac{k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta_{\rm k} = 0 \tag{3.78}$$

where $c_{\rm s}^2 = \dot{p}/\dot{\rho}$ defines the sound speed $c_{\rm s}$ of the perturbed component. If the mode enters the horizon during RD, baryons and photons are strongly coupled and the density contrast cannot grow due to the large pressure of the photon bath and it only oscillates.

If the mode enters the horizon during MD, the growth of the density contrast becomes possible, depending upon whether $c_s^2 k^2/a^2$ is larger, or smaller than $4\pi G\bar{\rho}$ in equation (3.78). This means that there is a scale characterized by $k_J = \sqrt{4\pi G\bar{\rho}}/c_s$, where k_J is the *Jeans wavenumber*, which separates the gravitationally stable and unstable modes. The short-wavelength modes $k \gg k_J$ are stable and correspond to oscillations, while for $k \ll k_J$ they are unstable and structure formation is possible. Thus, the epoch of matter-radiation equality sets an important scale for structure growth

$$k_{\mathrm{EQ}} = H^{-1}(a_{\mathrm{EQ}}) \simeq 0.08h\mathrm{Mpc}^{-1}. \tag{3.79}$$

We expect perturbations with $k \gg k_{\mathrm{EQ}}$ to be suppressed with respect to those having $k \ll k_{\mathrm{EQ}}$, by a factor $(a_{\mathrm{ENT}}/a_{\mathrm{EQ}})^2 = (k_{\mathrm{EQ}}/k)^2$, where a_{ENT} denotes the scale factor at the moment when the perturbations corresponding to scale k enter the horizon.

By defining a transfer function $T(k)$ by the relation $\mathcal{R}_{\mathrm{final}} = T(k)\mathcal{R}_{\mathrm{initial}}$, we obtain that:

$$T(k) = \begin{cases} 1 & k \ll k_{\mathrm{EQ}} \\ \left(\frac{k_{\mathrm{EQ}}}{k}\right)^2, & k \gg k_{\mathrm{EQ}} \end{cases} \tag{3.80}$$

which will suppress the matter power spectrum at scales $k \gg k_{\mathrm{EQ}}$. This prediction is confirmed by the observed shape of the CMB power spectrum.

After the universe becomes MD, the primordial power spectrum of density perturbations is reprocessed by gravitational instabilities and structure formation is possible. Solving equation (3.78) in the MD epoch, we obtain that the growing mode δ_{k}^{+} evolves as:

$$\delta_{\mathrm{k}}^{+} \propto t^{2/3} \propto a \propto (1+z)^{-1}. \tag{3.81}$$

It can be shown that in a baryonic dominated universe, the baryon density contrast today would be:

$$\delta_{\mathrm{B}}(t = t_0) < 0.1 \tag{3.82}$$

which is far too small to account for the large inhomogeneities observed in the universe. This means that in a pure baryonic universe, galaxies cannot form. The theory of structure formation requires the presence of non-baryonic non-relativistic weakly interacting matter. This type of matter is generically called *dark matter* and will be studied in the next chapter.

Chapter 4

Dark matter

4.1 Evidence for dark matter

There are many astrophysical and cosmological arguments in favor of the existence of *dark matter* in the universe, which is weakly interacting with normal matter and with photons. The adjective "dark" comes from the fact that it cannot absorb, nor emit any kind of electromagnetic radiation, including the visible spectrum.

The evidence of the existence of dark matter rely on observations of the dynamics of galaxies and clusters of galaxies and of some effects produced by their huge mass. One can infer the total mass of galaxies and clusters and then compare it to the observed luminous mass, and if they differ substantially, this means that some of the total mass in the universe is dark.

By measuring the radiation emitted by the baryonic matter in the visible, infrared and X-ray ranges, one can deduce the contribution of luminous matter to the universe density [84]:

$$0.002 \leq \Omega_{\mathrm{lum}} h \leq 0.006 \tag{4.1}$$

from which we can establish the conservative upper limit $\Omega_{\mathrm{lum}} < 0.01$.

In this section, I describe a few methods that are used to estimate the total amount of mass in a galaxy or cluster. First, I will present the limits on the baryonic density, and if it results to be larger than the luminous density, this means that some baryons should be dark. Second, I will present the constraints on the total matter density, Ω_{M}, which will be compared with the baryonic one, Ω_{B}. If $\Omega_{\mathrm{B}} \ll \Omega_{\mathrm{M}}$, it is a clear evidence of the existence of non-baryonic dark matter in the galactic halos.

4.1.1 Baryonic density

BBN limit. The overall baryonic content of the universe can be constrained by the BBN model, which predicts the abundances of the light elements such as deuterium (D), ^{3}He, ^{4}He and ^{7}Li. Within the standard Big Bang picture, their abundances only depend on one unknown cosmological parameter, the baryon number fraction relative to the present density of CMB photons, $\eta \equiv n_{\rm B}/n_\gamma$, which is usually parameterized as $\eta_{10} \equiv \eta/10^{-10}$. In terms of η_{10}, the baryon density is given by:

$$\Omega_{\rm B} h^2 = 3.65 \times 10^{-3} \eta_{10} \tag{4.2}$$

and η_{10} is inferred from observations of the primordial light elements abundances, and has values in the range [85]:

$$3.4 \leq \eta_{10} \leq 6.9. \tag{4.3}$$

The allowed range for η_{10} implies a range for $\Omega_{\rm B} h^2$:

$$0.012 \leq \Omega_{\rm B} h^2 \leq 0.024. \tag{4.4}$$

CMB limit. Another way to determine the matter content of the universe is by observations of the CMB, which contains a lot of information that can be used to constrain several key parameters. In the context of the "concordance" ΛCDM model, one can determine, among other parameters, the total matter density, $\Omega_{\rm M} h^2$, the baryon density $\Omega_{\rm B} h^2$ and the total density Ω, which is related to the curvature of the universe.

The most recent data come from 3-year WMAP observations and give for the baryon density the following value [5]:

$$\Omega_{\rm B} h^2 = 0.02229 \pm 0.00073 \tag{4.5}$$

with $h = 0.732^{+0.031}_{-0.032}$.

In conclusion, the data indicates that the baryon density $\Omega_{\rm B}$ is larger than the luminous matter density $\Omega_{\rm lum}$, which implies the existence of a baryonic dark matter component.

4.1.2 Matter density

Galactic rotational curves. Spiral galaxies are bound systems, gravitationally stable, whose matter content comprises stars and interstellar gas. The most part of the observed matter is concentrated in a relatively thin *disk*, where stars and gas spin around the galactic center, in quasi circular orbits. Our own galaxy is an example of a spiral galaxy.

Let us suppose that in a galaxy of mass M, concentrated in its center of masses, the rotational velocity at a distance R from the center is v. The stability condition requires that the centripetal acceleration is equal to the acceleration produced by gravitational forces,

$$\frac{v^2}{R} = \frac{GM}{R^2} \tag{4.6}$$

from which we obtain the velocity v as a function of the radius R:

$$v = \sqrt{\frac{GM}{R}}. \tag{4.7}$$

Thus, observing the rotation of galaxies we expect a behavior $v \propto 1/\sqrt{R}$. The rotational velocity v is measured [86] by observing 21 cm emission lines in HI regions (neutral hydrogen) beyond the point where most of the light in the galaxies ceases.

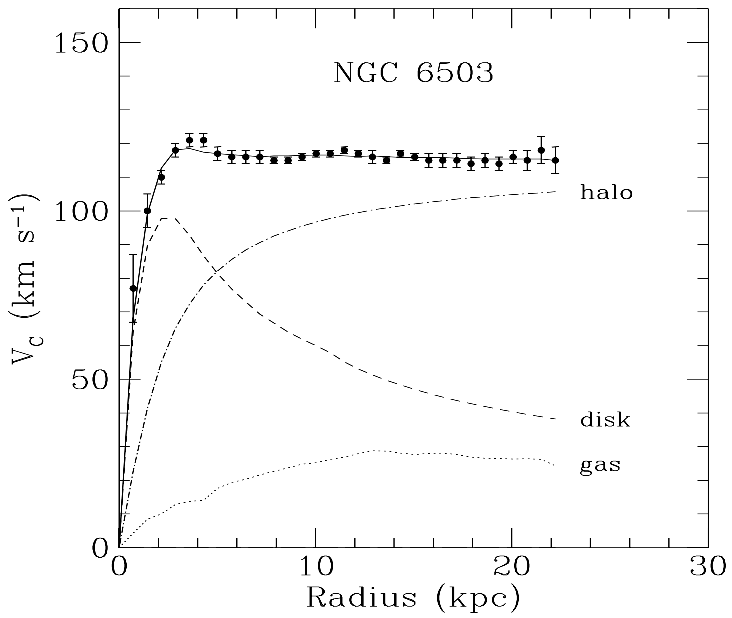

Figure 4.1: Rotation curve of the spiral galaxy NGC 6503 as established from radio observations of hydrogen gas in the disk. The dashed line shows the rotation curve expected for the visible component, the dot-dashed line is for the dark matter halo alone and the dotted line is for the gas (from reference [87]).

The measurements do not confirm expectations, instead they indicate that after a radius of about 5 kpc, the velocity becomes almost constant, see Figure 4.1. Supposing that the bulk of the mass is associated with light, the estimated velocity using equation (4.7) predicts a value which is three times smaller than measured, for the points situated at the extremity of the galaxy, at about 50 kpc from the center. This fact indicates that the gravitational field calculated only with luminous matter is a factor of 10 less than required to explicate observations.

One possible explanation to this problem is to suppose that gravity is modified at large scales. Another explanation is to consider galactic magnetic fields, in regions extending up to tens of kiloparsecs, where the interstellar gas density is low and the dynamics of the gas might be modified by these fields [88]. Nevertheless, the last argument would not affect the velocity of the stars.

The explanations mentioned above do not seem to be very appealing. Instead, it seems more attractive to suppose the existence of a large amount of dark matter in the visible halo of galaxies and even outside, which creates the gravitational field responsible for the observed rotational curves of galaxies.

In order to obtain a constant rotational velocity, as required by observations, the radial mass distribution $M(R)$ should be proportional to R,

$$v = \sqrt{\frac{GM(R)}{R}} \propto \sqrt{\frac{GR}{R}} = \text{const} \tag{4.8}$$

in which case the radial distribution of the density is:

$$\rho(R) \propto R^{-2}. \tag{4.9}$$

There is also evidence for dark matter in elliptical galaxies.

Large-scale motion of galaxies. The largest bound systems in the universe are the clusters of galaxies, whose typical radius are between $1-5$ Mpc and have masses between $2-9\times10^{14}M_{\odot}$, being $M_{\odot}$ the mass of our sun.

For any self-gravitating system like galaxies and clusters of galaxies, one can apply the virial theorem, which says that the mean kinetic energy $\langle E_{\rm kin}\rangle$ is equal to minus half the mean potential energy $\langle E_{\rm grav}\rangle$, due to the gravitational attraction between the objects that form the system:

$$\langle E_{\rm kin}\rangle = -\frac{1}{2}\langle E_{\rm grav}\rangle. \tag{4.10}$$

In this way, one can estimate [89] the dynamical mass of a cluster of galaxies characterized by a mean radius R and squared velocity $\langle v^2\rangle$:

$$M \simeq \frac{R\langle v^2\rangle}{0.4G}. \tag{4.11}$$

By measuring the velocity $\langle v^2\rangle$ observing the Doppler shift of the spectral lines of the constituent galaxies and estimating the size of the cluster, one can infer the approximate total mass of the cluster, M. Applying this technique to a set of cluster and assuming that the resulting mass density is representative for the entire universe, one obtains that the matter density of the universe is given by [90]:

$$\Omega_{\rm M} = 0.24 \pm 0.05 \pm 0.09 \tag{4.12}$$

where the first error is statistical and the second one is systematical.

Gravitational lensing. From the study of the dynamics of stellar objects, one deduces the existence of a large amount of dark matter in galaxies and clusters. Because of this huge concentration of mass, other interesting effects may appear. One of them is a consequence of the theory of General Relativity and consists in the deviation of light when it propagates in the gravitational field of very massive objects. This effect is similar to that of an optical lens, with the lens replaced by a galaxy or a cluster of galaxies, and is known as *gravitational lensing*. In Figure 4.2 is represented, schematically, the gravitational lensing effect. When the source, the deflecting mass M_{D} and the observer are situated on the same line, i.e., $r = 0$, the observer sees the image of a ring, called the *Einstein ring*, which has a radius r_{E} given by [91]:

$$r_{\mathrm{E}}^2 = 4GM_{\mathrm{D}}d \tag{4.13}$$

where $d = d_1 d_2/(d_1 + d_2)$, and d_1 and d_2 are defined in Figure 4.2.

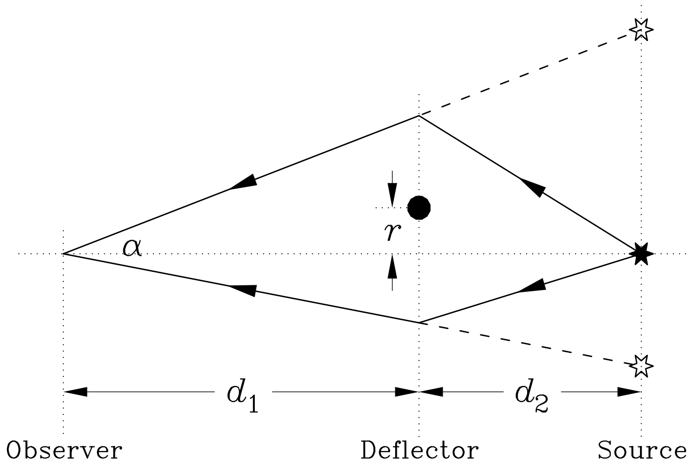

Figure 4.2: Geometry of the light deflection by a point-like mass which gives two images of a source viewed by an observer (from reference [92]).

In practice, the gravitational lensing effect has been observed starting from middle 80's, by using high-resolution telescopes. The observed images appear as arcs of a circle, which form around clusters of galaxies. The spectral analysis shows that the cluster and the image are very far away from each other, which can be interpreted as follows: the arc is the image of a distant galaxy, situated on the same line of sight as the cluster that produces a magnified and distorted image of the galaxy by the gravitational lensing effect. Thus, the arc is somehow a part of the corresponding Einstein ring.

Starting from a systematic analysis of the cluster's mass distribution, one can infer the gravitational field responsible for the distortion. The analysis suggests a total mass of the cluster much larger than the visible matter, which comes to confirm the need for dark matter in clusters. The required amount of dark matter is in concordance with the results coming from studying the dynamics of galaxies and clusters. The numerical estimations from gravitational lensing usually give slightly higher values as compared to other methods, $\Omega_{\rm M} = 0.2 - 0.3$ for scales less than $6h^{-1}$ Mpc, and $\Omega_{\rm M} = 0.4$ for superclusters with sizes of order of 20 Mpc.

If the resolution of the telescope used for observations is not good enough to measure the angular distance α in Figure 4.2, the resulting images forming the ring cannot be distinguished from each other and they appear as superposed. As a result, the observer perceives an enhancement of the source brightness. This effect is known as *gravitational microlensing*, and it has been used to detect dark objects with masses $M_{\rm D} \sim M_{\odot}$, at the Milky Way scale.

X-ray galaxy clusters. The galaxy clusters are a powerful X-ray source, explained by the large fraction of baryons in the form of hot gas. One calculates the baryon fraction of clusters, $f_{\rm B} h^{3/2} = 0.03 - 0.08$, which means that for $h = 0.72$ one obtains

$$\Omega_{\rm B}/\Omega_{\rm M} \simeq 0.13. \tag{4.14}$$

CMB. As mentioned in the previous subsection, recent observations of the CMB allow for accurate determination of several cosmological parameters. The value inferred from the recent WMAP observations for the total mass density is given by:

$$\Omega_{\rm M} h^2 = 0.1277^{+0.0080}_{-0.0079}. \tag{4.15}$$

In conclusion, the total matter density $\Omega_{\rm M}$ inferred by the methods presented in this subsection is clearly larger than the baryonic density $\Omega_{\rm B}$, which is a clear evidence for the presence of a non-baryonic dark matter component, with a relative density of about $\Omega_{\rm DM} \simeq 0.2$.

4.1.3 Structure formation with dark matter

In the previous chapter we have seen that if the universe only contained baryonic matter, structure formation would occur too slowly and galaxies would not have enough time to form. I will show here that dark matter can solve this problem, and that galaxies can form in the presence of a significant dark matter component in the universe.

Let us assume now that, during structure formation, the universe is dominated by dark matter, which only interacts with baryons through a gravitational coupling. Then, we can write down two coupled wave equations for the fluctuations in the two components:

$$\begin{cases} \ddot{\delta}_{\rm B} + 2H\dot{\delta}_{\rm B} - 4\pi G\bar{\rho}_{\rm B}\delta_{\rm B} & = & 4\pi G\bar{\rho}_{\rm DM}\delta_{\rm DM} \\ \ddot{\delta}_{\rm DM} + 2H\dot{\delta}_{\rm DM} - 4\pi G\bar{\rho}_{\rm DM}\delta_{\rm DM} & = & 4\pi G\bar{\rho}_{\rm B}\delta_{\rm B} \end{cases} \tag{4.16}$$

In the approximation $\Omega_{\rm B} \ll \Omega_{\rm DM} \simeq 1$, the second equation (4.16) above reduces to equation (3.78) with $c_{\rm s} = 0$. Its growing mode solution in a matter dominated universe was given at the end of the previous chapter, in equation (3.81). Applying it to dark matter, we have:

$$\delta_{\rm DM}(a) = \alpha a \tag{4.17}$$

where α is a constant. Inserting this solution in the first equation (4.16) for the baryonic component and using the same approximation $\bar{\rho}_{\rm B} \ll \bar{\rho}_{\rm DM}$, one finally obtains:

$$\delta_{\rm B}(a) = \alpha(a - a_{\rm dec}) = \delta_{\rm DM}(a)\left(1 - \frac{a_{\rm dec}}{a}\right) \tag{4.18}$$

where $a_{\rm dec}$ is the scale factor at decoupling.

From the solution (4.18) one can see the qualitative behavior of the baryonic fluctuations: for $a = a_{\rm dec}$, one has $\delta_{\rm B} \to 0$, while for $a \gg a_{\rm dec}$ one obtains $\delta_{\rm B} \simeq \delta_{\rm DM}$. This means that soon after decoupling baryons can fall into the potential wells created by dark matter, such that baryonic perturbations "catch up" with dark matter perturbations.

We have seen that baryonic perturbations cannot grow until the decoupling epoch and they alone would not have enough time to produce the observed structure of the universe, while with a dark matter component, structure formation becomes possible. The precise values of the baryon and dark matter densities are obtained from the higher order multipoles of the CMB power spectrum and from astrophysical constraints.

4.2 Candidates for dark matter

4.2.1 Baryons

Accepting the dark matter hypothesis, the first choice for a candidate should be something we know to exist, namely, baryons. The matter in the galactic halos appears to contribute at the level of $\Omega \sim 0.05$, consistent with the BBN predictions for baryons. Thus, we know that some of the baryons should be dark, since only a part $\Omega_{\rm lum} \leq 0.01$ of the galaxy is luminous. Also, baryonic dark matter cannot be the whole story if $\Omega_{\rm M} > 0.1$, as seems to be the case. Thus, the identity of the dark matter in galactic halos remains an important question needing to be resolved.

We should specify that dark baryons cannot form normal stars, because they would then be luminous. Neither can they be in the form of a hot gas, since this emits light, nor in the form of cold gas, since this absorbs radiation, which is immediately emitted in infrared. Finally, neither can they appear as relics of stars that have finished their fuel, because these should originate from an older population of stars, which in practice are not observed in the galactic halos.

Sites for halo baryons that have been discussed include frozen, cold or hot Hydrogen, remnants of massive stars such as white dwarfs [93, 94], which do not have enough mass to become supernovae. Other more plausible candidates are planets of Jupiter type or brown dwarfs, which are stars with masses less than $0.08 M_\odot$. These objects are known as *Massive Compact Halo Objects (MACHOs)*. Their pressure is not high enough as to be able to support Hydrogen combustion, and they only radiate the gravitational energy that is lost during their slow contraction, but this energy is very difficult to detect. Nevertheless, if a MACHO passes exactly in front of a distant star, it acts like a gravitational lens and can be detected by the microlensing effect [95]. It is precisely by this method that quite a few MACHOs have already been detected [96, 97].

Another type of candidates are black holes, which are not luminous and, if they are big enough, they can be long-lived. Black holes resulting from the collapse of stars with masses slightly larger than $8M_\odot$ cannot be dark matter, because during their formation process a considerable amount of unobserved metals would have been produced [98]. If the mass of the collapsing star is larger than $200M_\odot$, it would produce background light that would be detectable today in infrared. Since no such radiation has been detected, we conclude that black holes in these mass ranges cannot be dark matter.

Finally, very massive black holes, with masses larger than $10^5 M_\odot$, are not affected by the previous constraints [99], but the possibility of having a dynamical evidence of their existence is not clear enough, and one can only put upper limits on the density of these very massive black holes. The final conclusion is that baryonic dark matter cannot account for all the required amount of dark matter of the universe [100, 101].

4.2.2 Neutrinos

In the previous section we have seen that baryons can only account for a small part of the total amount of dark matter of the universe. It is natural then to keep searching for new candidates to the dark matter, other than baryons.

Since neutrinos seem to have a tiny mass and are long-lived particles, they are candidates to be non-baryonic hot dark matter [102]. Nevertheless, they cannot be the dominant form of dark matter, because in that case galaxies would form very late, at $z < 1$, which is ruled out by observations of galaxies and quasars at redshifts $z \geq 6$.

If neutrinos decouple while they are still relativistic, i.e. $m_\nu \leq 1$ MeV, their energy density can be expressed at late times as $\rho_\nu = m_\nu Y_\nu n_\gamma$, where $Y_\nu = n_\nu / n_\gamma$ is the number density of neutrinos, relative to the photon density. In an adiabatically expanding universe, one has $Y_\nu = 3/11$. This result is obtained by taking into account that photons and neutrinos have different statistics, and also the e^+e^- annihilation, which occurs after neutrino decoupling, and heats the photon bath relative to the neutrinos. Their final relic density is given by:

$$\Omega_\nu h^2 \simeq \frac{\sum_i m_{\nu_i}}{94\text{eV}}. \tag{4.19}$$

The sum of neutrinos masses can be constrained by various experiments. The recent WMAP data [5] combined with other astronomical data put a stringent upper limit, $\sum_i m_{\nu_i} < 0.66$ eV (95% CL), while experiments at Super-Kamiokande [103] put limits on the mass differences between distinct neutrinos families, $\Delta m^2_{\nu_i} \sim 3 - 19 \times 10^{-5}$ eV^2. Then, from equation (4.19) we obtain an upper bound on the neutrino fraction density, $\Omega_\nu < 0.01$, which confirms that neutrinos cannot be the dominant dark matter component.

Massive neutrinos are allowed from the cosmological point of view only if their mass is larger than a few GeV, but then they are ruled out by experiments at LEP, which put a lower limit on the neutrino mass, $m_\nu > 45$ GeV. It can be shown [104, 105] that this lower limit on the heavy neutrino mass leads to an upper limit on its abundance, $\Omega_\nu h^2 < 0.001$. Dirac neutrinos constituting all of dark matter are excluded for masses in the range 10 GeV $-$ 4.7 TeV by laboratory constraints [106]-[108], and only for masses in the range $200 - 400$ TeV [109] would they be the dominant dark matter component, with $\Omega_\nu h^2 \sim O(1)$.

4.2.3 WIMPs

The weakly interacting massive particles (WIMPs) are non-baryonic relic stable particles with masses between 10 GeV and a few TeV, and low cross-sections. Their relic density can be calculated [104, 110] by using the laws of thermodynamics in the expanding universe, for WIMPs that were in equilibrium with the plasma of SM particles:

$$\Omega_\chi h^2 \approx \frac{3 \times 10^{-27}\text{cm}^3\text{s}^{-1}}{\langle \sigma v \rangle}. \tag{4.20}$$

Here σ is the cross-section corresponding to WIMP pair annihilation to SM particles, v is the relative velocity between two WIMPs in the center of mass system, and $\langle ... \rangle$ means thermal average.

The best motivated WIMP candidate is the lightest supersymmetric particle (LSP) of supersymmetric (SUSY) models [110, 111]. In the minimal supersymmetric standard model (MSSM), if R-parity is unbroken, there is at least one SUSY particle, which must be stable. To be a dark matter candidate, the long lived particle should be electrically neutral and colorless. The sneutrino [112, 113] is one possibility, but it has been excluded as a dark matter candidate by direct [106] and indirect searches [114]. Another candidate is the gravitino, which is probably the most difficult to exclude.

The most popular candidate is the lightest neutralino [110, 115], which can have the adequate relic density for a large region in the relevant parameter space.

4.2.4 Pseudo Goldstone bosons

As will be seen in Chapter 6, the Goldstone theorem tells us that the breaking of a global symmetry implies the existence of a massless particle, called the Goldstone boson. If for some reason the global symmetry is also slightly explicitly broken, the Goldstone boson acquires a small mass, which converts it into a *pseudo-Goldstone boson* (PGB).

Apart from the known global symmetries, there might exist many other global symmetries in nature, which are not manifest due to the fact that they are broken. If we discovered a general mechanism of explicit breaking of all global symmetries, we would end up by having a large number of PGBs, which, under certain conditions, could play the role of non-baryonic dark matter.

In the work of reference [116], we introduce a new global $U(1)$ symmetry and assume that it is explicitly broken by Quantum Gravity effects, apart from the usual spontaneous symmetry breaking. The resulting PGB is a dark matter candidate in the conditions that the explicit symmetry breaking is exponentially small.

In Chapter 9 of this Thesis, I extend the model investigated in [116] to incorporate inflation, in such a way that a single complex scalar field ψ can be responsible for both the inflationary period and the non-baryonic dark matter of the universe.

4.2.5 Axions

The *axion* [117, 118] is a PGB associated to the spontaneous breaking of the global $U(1)_{\mathrm{PQ}}$ Peccei-Quinn symmetry, which was postulated to solve the strong CP problem [119, 120]. The energy scale f_{a} of the spontaneous breaking of $U(1)_{\mathrm{PQ}}$ is a free parameter, but from astrophysical [121]-[126] and cosmological [127]-[129] considerations, it is constrained to be in the range $10^{10} < f_{\mathrm{a}} < 10^{12}$ GeV. Apart from the spontaneous breaking, the Peccei-Quinn symmetry is also explicitly broken by non-perturbative QCD effects, which occur at a much lower scale, $\Lambda_{QCD} \sim 200$ MeV. Because of the explicit beaking, the axion receives a small mass, which is expected to be $m_a \sim m_\pi f_\pi / f_a$, where m_π is the pion mass and f_π the pion decay constant.

Axions can be produced both thermally and non-thermally. Non-thermal production may occur through coherent oscillations of the axion field a in its effective potential $V(a) \sim m_a^2 a^2$. For this reason, although its mass m_a can be very small, the axions produced in this way are non-relativistic and they contribute to cold dark matter. The axion relic density is [130]:

$$\Omega_a h^2 = C_a \left(\frac{f_a}{10^{12}\text{GeV}} \right)^{1.175} \theta_i^2 \tag{4.21}$$

where C_a is a constant in the range between 0.5 and 10, and $\theta_i \sim O(1)$ is the initial oscillation angle. Axions can account for the most part of the dark matter for values of $f_a \sim 10^{11}$ GeV, which corresponds to an axion mass $m_a \sim 0.1$ meV. Another non-thermal axion production mechanism is through decays of axion cosmic strings [131], which appear in the process of spontaneous symmetry breaking of the Peccei-Quinn symmetry. The relic density Ω_a contributed by string decays depends on the scale f_a and can be larger than (4.21), for large f_a values. In order for the axion to be dark matter, the density of string-decay produced axions should be subdominant, which implies lower values for f_a and larger values for m_a.

Thermal axion production occurs in a certain range of values of the temperature, for which there is thermal contact between the axion and the cosmic plasma. It is important to remark the fact that if there is a period of thermalization after a period of non-thermal axion production, the calculated axion relic density at present is considerably reduced, and this fact might affect the role of the axion as dark matter [132].

Axions may also be emitted in stars and supernovae, via axion-electron coupling or nucleon-nucleon bremsstrahlung [126]. Axion emission from red giants imply $f_a \geq 10^{10}$ GeV [121] and the supernova limit requires $f_a \geq 2 \times 10^{11}$ GeV [122]-[124] for a naive quark model coupling of the axion to nucleons. On the other hand, the cosmological density limit $\Omega_a < 0.3$ in (4.21) requires $f_a \leq 10^{12}$ GeV [127]-[129], which only leaves a narrow window open for the axion as a viable dark matter candidate.

4.3 Final remarks on dark matter

Let us summarize the discussion on dark matter and highlight its main characteristics. In Figure 4.3 are shown the contributions of different types of matter existing in the universe:

luminous matter $\Omega_{\rm lum}$, baryons $\Omega_{\rm B}$, matter in the galactic halo $\Omega_{\rm halo}$ and matter detected from cluster dynamics $\Omega_{\rm M}$.

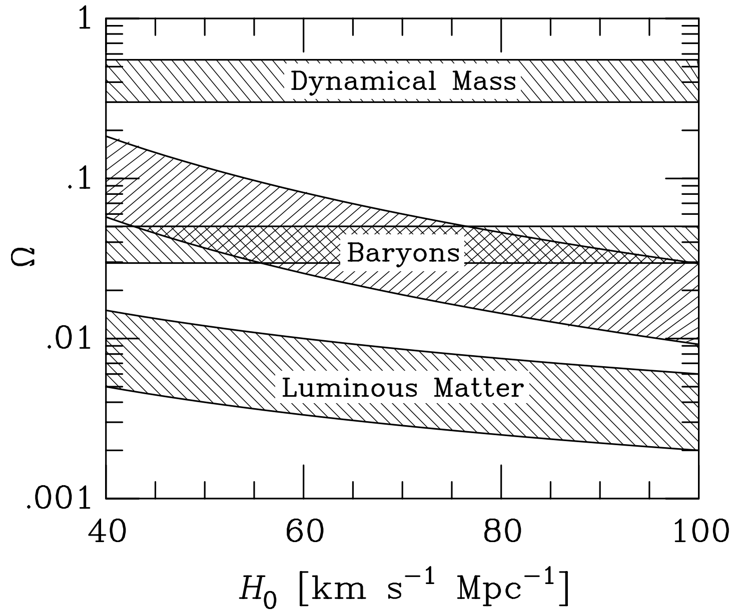

Figure 4.3: The observed cosmic matter components as functions of the Hubble constant. The graphic shows the luminous matter component, the galactic halo component, which is the horizontal band crossing the baryonic component from BBN, and the dynamical mass component from LSS analysis (from reference [133]).

As can be seen in Figure 4.3, there are three problems related to dark matter:

- the first problem is related to explaining 90% of the baryons. There are considerable discrepancies between $\Omega_{\rm B}$ deduced from BBN and CMB and direct observations of luminous stars, galaxies and interstellar gas, which implies that great part of the baryons should be dark. This is the *baryonic dark matter problem*;

- the second problem consists in explaining the nature of 90% of matter, in general. From measurements of $\Omega_{\rm M}$ and $\Omega_{\rm B}$, one can see that the nature of almost all the matter contained in the universe is unknown. This is the *non-baryonic dark matter problem* and is the standard dark matter problem;

- finally, according to recent WMAP [5] observations of the CMB, the universe is spatially flat, i.e., $\Omega_{\rm total} \simeq 1$. If we take into account all observations indicating that $\Omega_{\rm M} < 0.3$, we see that about 70% of the energy density of the universe does not consists of matter, and should be another form of unknown dark energy. This is the *dark energy problem*, which will be treated in the next chapter.

Chapter 5

Dark energy

5.1 Observational evidence for dark energy

We have seen in the previous chapter that baryonic matter only represents a small fraction of the total matter contained of the universe, suggesting the existence of non-baryonic cold dark matter. However, observations also indicate that the universe is approximately flat, i.e. $\Omega \sim 1$, while the fraction of the total matter is clearly less than one, $\Omega_{\rm M} \sim 0.3$. The difference between the total energy density and the matter energy density is attributed to an unknown form of energy, called dark energy. The presence of a dark energy component has the effect of accelerating the expansion of the universe, so that any observational evidence for accelerated expansion confirms the dark energy hypothesis.

5.1.1 Supernovae of type Ia

The first evidence for the accelerated expansion of the universe was provided by SN Ia observations [6, 7] and is related to the luminosity distance to these objects. The luminosity distance was defined in equation (2.32) of Section 2.3, in terms of the absolute luminosity of the source $L_{\rm s}$ and the observed flux F. A useful equation in cosmology is the luminosity distance-redshift relation, equation (2.34), which relates the luminosity distance $d_{\rm L}$ with the cosmological redshift z. This equation can be written in the form:

$$d_{\rm L} = a_0 f_{\rm K}(\chi_{\rm s})(1+z) \tag{5.1}$$

where $f_{\rm K}(\chi)$ is the generalized sine-function defined in equation (2.8), $\chi_{\rm s}$ is the radial coordinate of the source and a_0 is the present value of the scale factor.

In a flat FRW background, $f_{\rm K}(\chi) = \chi$, and the light travelling along the χ direction satisfies the geodesic equation $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\chi^2 = 0$. Then, one can obtain the following expression for d_L:

$$d_\mathrm{L} = a_0\chi_\mathrm{s}(1+z) = a_0(1+z)\int_{t_1}^{t_0}\frac{\mathrm{d}t}{a(t)} = (1+z)\int_0^z \frac{\mathrm{d}z'}{H(z')} \tag{5.2}$$

where we have also used the fact that $\dot{z} = -a_0\dot{a}/a^2 = -a_0 H/a$. If we assume that the universe contains all possible components, namely, non-relativistic and relativistic particles, cosmological constant (or vacuum energy with $w = -1$), and neglect the curvature term, from the Friedmann equation (2.37) we obtain:

$$H^2 = H_0^2 \sum_i \Omega_{i,0}(1+z)^{3(1+w_i)} \tag{5.3}$$

where each component is characterized by its density parameter, $\Omega_{i,0}$, at present time, and also by an equation of state parameter w_i. By replacing equation (5.3) in (5.2), we obtain a useful form of the luminosity distance-redshift relation, in a flat geometry:

$$d_\mathrm{L} = \frac{1+z}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\sum_i \Omega_{i,0}(1+z')^{3(1+w_i)}}}. \tag{5.4}$$

Thus, by measuring the luminosity distance of high redshift supernovae, one can infer the contribution of each component to the total energy density of the universe. The luminosity distance is obtained by measuring the apparent magnitude m of a source with an absolute magnitude M, via the relation [134, 135]

$$m - M = 5\log_{10}\left(\frac{d_\mathrm{L}}{\mathrm{Mpc}}\right) + 25 \tag{5.5}$$

where d_L has the general form (5.4).

The SN Ia occur when, by accreting matter from a companion, a white dwarf star exceed the Chandrasekhar mass limit and explode, emitting in this way a huge amount of energy. Their observed light curves (the luminosity as a function of time) have similar features, irrespective of their position in the universe, which suggests that the absolute magnitude M is independent of the redshift z. For this reason, the SN Ia can be treated as an *ideal standard candle*.

Figure 5.1 illustrates the observational values of the luminosity distance d_L versus the redshift z for a set of SN Ia data, together with three theoretical curves corresponding to three different combinations of the matter and cosmological constant densities, Ω_M and Ω_Λ, respectively. The best fit for a flat universe, based on different SN Ia observations [137], corresponds to the values $\Omega_\mathrm{M} \sim 0.3$ for matter, and $\Omega_\Lambda \sim 0.7$ for cosmological constant.

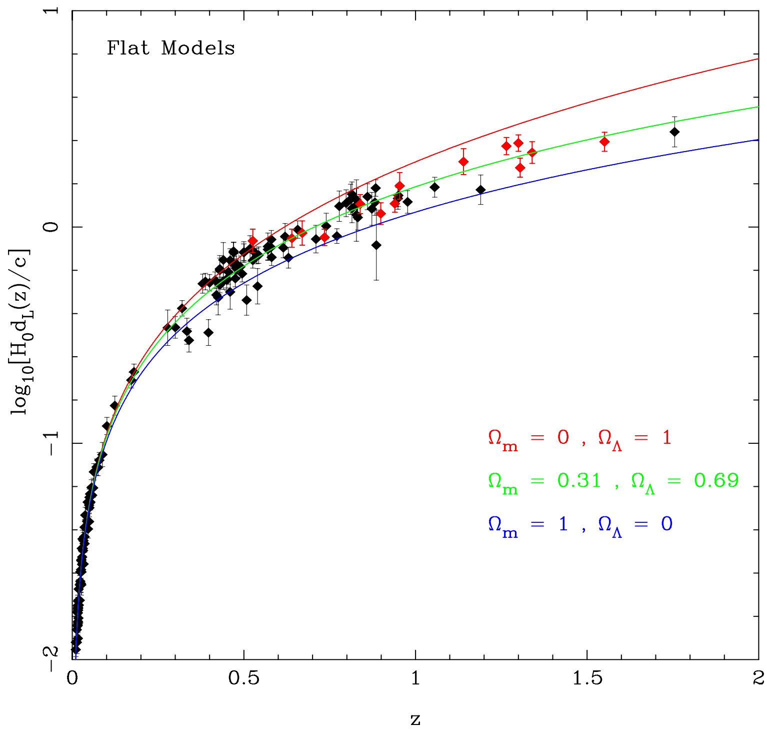

Figure 5.1: The luminosity distance $H_0 d_L$ versus the redshift z for a flat cosmological model. Three curves show the theoretical values of $H_0 d_L$ for (i) $\Omega_M = 0,\ \Omega_\Lambda = 1$, (ii) $\Omega_M = 0.31,\ \Omega_\Lambda = 0.69$ and (iii) $\Omega_M = 1,\ \Omega_\Lambda = 0$ (from reference [136]).

It is interesting to estimate the "coasting point", corresponding to the epoch of deceleration-acceleration transition. The corresponding redshift z_c is obtained by imposing the condition that the deceleration parameter q defined in equation (2.39) is zero at the coasting point. For the two component flat cosmology, $q(z)$ can be obtained by using equation (5.3), and the condition for acceleration becomes:

$$z < z_c \equiv \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{M,0}}\right)^{1/3} - 1 \tag{5.6}$$

which for $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$ gives $z_c = 0.67$.

Apart from SN Ia, there are other possible candles in the universe, such as the FRIIb radio galaxies [138, 139]. From the corresponding redshift-angular size data it is possible to constrain cosmological parameters in a dark energy scalar field model. In [140], a model-independent approach has been developed using a set of 20 radio galaxies out to a redshift $z \sim 1.8$. The derived constraints are consistent with − and generally weaker than − the SN Ia results.

Another suggested standard candle is the use of Gamma Ray Bursts (GRB) [141], which can test the expansion up to very high redshifts, $z > 5$, opening the possibility to probe the evidence for a dynamical equation of state for dark energy. Although the use of GRB is unlikely to be competitive with future supernovae missions like SNAP, they will be a very significant complement to the SN Ia data sets.

5.1.2 The age of the universe

The oldest stars have been observed in globular clusters in the Milky Way [142] and in the globular cluster M4 [143], and have ages around 11 or 12 Gyr. This means that the age of the universe (t_0) needs to be larger than these values: $t_0 > 11 - 12$ Gyr. If we calculate the age of the universe, which depends on its composition and geometry, the result should necessary satisfy the above age condition.

The age of the universe can be estimated by using the Friedmann equation (2.37), written in the form of equation (5.3):

$$H^2 = H_0^2 \left[\Omega_{\mathrm{M},0}(1+z)^3 + \Omega_{\Lambda,0} - K_0(1+z)^2\right] \tag{5.7}$$

in which we have neglected the radiation contribution $\Omega_{\mathrm{R},0}$ and included the curvature term, where $K_0 \equiv K/(a_0^2 H_0^2)$. With this, the age of the universe is given by:

$$t_0 = \int_0^{t_0} \mathrm{d}t = \int_0^{\infty} \frac{\mathrm{d}z}{H(1+z)} = \int_1^{\infty} \frac{\mathrm{d}x}{H_0 x \left(\Omega_{\mathrm{M},0} x^3 + \Omega_{\Lambda,0} - K_0 x^2\right)^{1/2}}, \tag{5.8}$$

where we introduced $x(z) \equiv 1 + z$ for simplicity.

Let us first evaluate the age of the universe in the absence of the cosmological constant ($\Omega_{\Lambda,0} = 0$). For a flat universe ($K_0 = 0$ and $\Omega_{\mathrm{M},0} = 1$), we obtain:

$$t_0 = \frac{2}{3H_0}. \tag{5.9}$$

If we take into account the constraints obtained from observations of the Hubble Space Telescope Key Project [144]

$$12.2 < H_0^{-1} < 15.3 \text{ Gyr} \tag{5.10}$$

it results that in a flat universe without cosmological constant, the age of the universe should be in the range $t_0 \sim 8 - 10$ Gyr, which is in conflict with the stellar age bound $t_0 > 11 - 12$ Gyr.

Instead of a flat universe, we may consider an open universe with $\Omega_{\mathrm{M},0} < 1$. In this case, the age of the universe is larger, which is understandable as it takes longer for gravitational interactions to slow down the expansion rate to its present value. However, in order to obtain an age consistent with observations, the matter density parameter $\Omega_{\mathrm{M},0}$ is constraint to be close to zero, meaning that the curvature term should dominate. The observations of the CMB [5] rule out this possibility and indicate that the universe is approximately flat, i.e., $|K_0| = |\Omega_{\mathrm{M},0} - 1| \ll 1$. We conclude that a universe without cosmological constant cannot live long enough as to be consistent with the oldest observed stellar objects.

An elegant solution to this problem is to take into account the cosmological constant, i.e., $\Omega_{\Lambda,0} \neq 0$. In this case, assuming a flat universe ($K_0 = 0$), the age of the universe is given by:

$$t_0 = \int_0^\infty \frac{\mathrm{d}x}{H_0 x \sqrt{\Omega_{\mathrm{M},0} x^3 + \Omega_{\Lambda,0}}} = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{1+\sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_{\mathrm{M},0}}}\right). \tag{5.11}$$

For $\Omega_{\mathrm{M},0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, one has $t_0 = 0.964 H_0^{-1} \sim 13$ Gyr, which satisfies the constraint coming from the oldest stellar population.

5.1.3 Constraints from CMB and LSS

There is strong evidence for dark energy in independent observations related to the CMB [5] and LSS [145, 146]. The CMB power spectrum encodes large amounts of information about the cosmological parameters. From the position of the first acoustic peak, at around $l \sim 200$, one can estimate the values of $\Omega_{\mathrm{B}} h^2$ and $\Omega_{\mathrm{M}} h^2$ [147], but there is a large degeneracy between the curvature K_0 and Ω_{M} due to the fact that h alone cannot be constrained [148, 149]. The baryon density Ω_{B} is most sensitive to the ratio of the amplitudes of the first two acoustic peaks, and one obtains an upper limit $\Omega_{\mathrm{B}} < 0.05$ regardless of the amount of dark matter [147]. The degeneracy between K_0 and Ω_{M} can be broken by using a prior for h. Independent measurements of the parameter h are possible from galaxy surveys [150] and SN Ia observations [6, 7], which suggest that $h \simeq 0.72$. With this, one can conclude that the universe is approximately flat, $|1-\Omega_0| \ll 1$, and that it should contain both dark matter and dark energy, with fractional densities of about $\Omega_{\mathrm{M}} \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$, respectively.

A nice resume of all the discussion above, whose goal was to constrain the cosmological parameters using SN Ia, CMB and LSS data, is illustrated in Figure 5.2, from which we can clearly conclude that a flat universe without a dark energy component is ruled out.

It should be mentioned that here we worked in the context of the ΛCDM model, which gives the best fit to observations. The model is based on the assumption that the equation of state of dark energy is constant, with $w = -1$, and that the dark matter

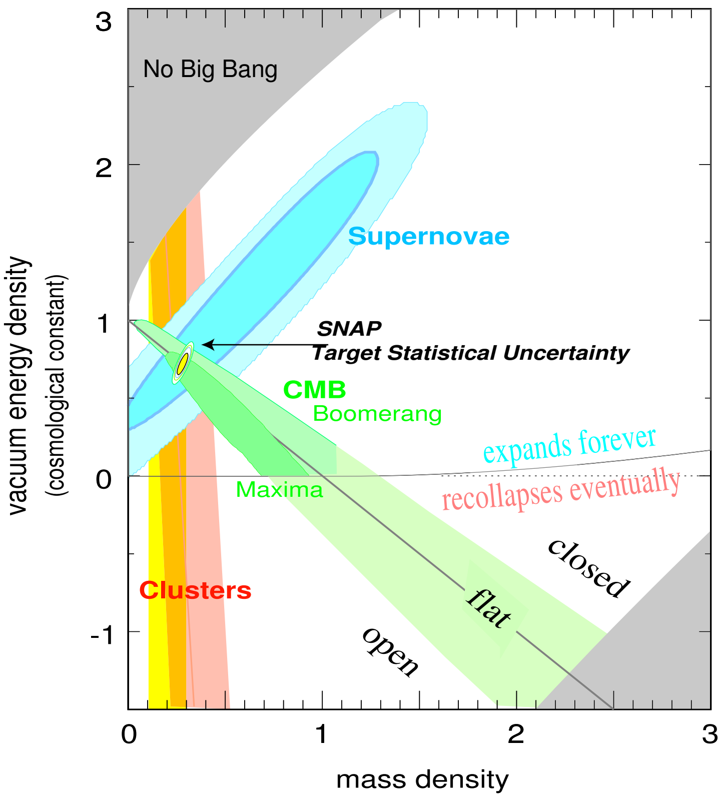

Figure 5.2: The $\Omega_M - \Omega_\Lambda$ confidence regions constrained from the observations of SN Ia, CMB and galaxy clusters. We also show the expected confidence region from a SNAP satellite for a flat universe with $\Omega_M = 0.28$ (from reference [151]).

component is cold. Other possibilities exist, in which the equation of state of dark energy can be dynamical, and dark energy can be explained by scalar fields. If we were able to distinguish between cosmological constant and dynamical dark energy from observations, we could say more about the origin of dark energy.

The observations of WMAP and SN Ia are consistent with a non-varying dark energy contributed by a cosmological constant, but this situation could be improved by future precise observations of SN Ia and of the Integrated Sachs-Wolfe (ISW) effect [83] on the CMB power spectrum [152].

5.2 Possible explanations of dark energy

We have seen [5]-[7], [85, 145, 146] that the combined observations of the SN Ia, CMB, LSS and BBN indicate that the universe is spatially flat and that it only contains a small part of baryonic matter, the resting part being in the form of two mysterious types of substance, a non-relativistic non-baryonic cold dark matter and a homogeneous fluid with negative pressure (or a pure cosmological constant), generically called dark energy. In the previous chapter we have seen some details about dark matter and a list of possible candidates that may contribute to it. Let us focus now on the possible origin and nature of dark energy and present a few common approaches for solving this problem [153].

5.2.1 Cosmological constant

A first simple and maybe natural solution to the dark energy problem would be the cosmological constant Λ, first introduced by Einstein in 1917 in order to obtain a static universe. Let us write the Einstein equation (2.35) with the explicit inclusion of the cosmological constant term:

$$R^{\mu\nu} - \frac{1}{2}\mathcal{R}g^{\mu\nu} = 8\pi G T^{\mu\nu} + \Lambda g^{\mu\nu}. \tag{5.12}$$

The Friedmann equations are:

$$H^2 = \frac{8\pi G}{3}\tilde{\rho} - \frac{K}{a^2} \tag{5.13}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\tilde{\rho} + 3\tilde{p}) \tag{5.14}$$

where in the modified energy density $\tilde{\rho}$ and pressure $\tilde{p}$ we distinguish between the cosmological constant contribution and the others:

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}, \quad \tilde{p} = p - \frac{\Lambda}{8\pi G}. \tag{5.15}$$

In the Einstein model of a static universe, the cosmological constant should be positive, and the universe is closed ($K > 0$) with a radius $a = \sqrt{K/\Lambda}$.

When Hubble discovered in 1929 the expansion of the universe [2], the static universe model was abandoned along with the cosmological constant. However, the issue of the cosmological constant returned in the 1990's, after the discovery of the acceleration of the universe, although there had been some previous discussion because of the age problem [154].

Although the cosmological constant is a simple and elegant solution to the dark energy problem, it suffers from severe problems if it originates from a vacuum energy density.

The expected value of the vacuum energy density evaluated by the sum of zero-point energies of quantum fields depends on the cut-off scale up to which quantum field theory is valid. If we set the cut-off at the Planck scale $M_{\rm P} = 1.22 \times 10^{19}$ GeV, the estimated vacuum energy density is $\rho_{\rm vac} \sim 10^{74}\,{\rm GeV}^4$. The problem arises when we compare this value to observations, which indicate that Λ should be of order the Hubble constant H_0, so that $\rho_\Lambda \sim H_0^2 M_{\rm P}^2/8\pi \sim 10^{-47}{\rm GeV}^4$, which is about 121 orders of magnitude smaller than the theoretical estimation. Even if we lower the cut-off scale to energy scales that are probed at accelerators, like the electroweak scale of about 100 GeV, the problem is far from being solved because we obtain $\rho_{\rm vac} \sim 10^6{\rm GeV}^4$, still much larger than ρ_Λ. This is the so-called *fine-tuning problem* of the cosmological constant [8].

An interesting solution to this problem is provided by SUSY, in which every bosonic degree of freedom contributing to the zero point energy is cancelled by its corresponding fermionic counter part, such that the net contribution to vacuum energy vanishes and Λ becomes zero. However, SUSY is not exact today and it should be broken around 10^3 GeV, in which case the fine-tuning problem of the cosmological constant still remains unsolved.

Other recently proposed scenarios are realized in string theory [155] or supergravity, by constructing de-Sitter vacua in which one has an effective cosmological constant, and one can arrange to have a value compatible with the observed Λ [156, 157]. In fact, the number of such de-Sitter vacua can be very large [158], up to order 10^{100}, and for this reason it receives the name of *string landscape*. In this PhD Thesis we do not consider such theories.

Whatever be the nature of the cosmological constant, because we know that today we have $\Omega_{\rm M} \sim 0.3$ and $\Omega_\Lambda \sim 0.7$, we are faced with another problem, called the *coincidence problem*, which consists in explaining why do we leave in a special period of transition from dark matter domination to dark energy domination, both having the same order of magnitude now.

An alternative explanation for the above problems is given by the *anthropic principle* [8, 159], according to which intelligent life in our universe can only appear if the fundamental constants of nature have specific values. This principle has generated much debate in the cosmology community, but the recent ideas about the existence of a vast landscape of de-Sitter vacua in string theory makes the anthropic principle an interesting approach.

We summarize the discussion on the cosmological constant by saying that although it is a simple and reasonable solution to the dark energy problem, it suffers from the fine-tuning and coincidence problems that need to be explained.

5.2.2 Scalar-field models

Another possibility, which has received great attention, is that of scalar fields as possible candidates to the dark energy of the universe. Scalar fields naturally arise in particle physics models and there is a wide variety of scalar-field dark energy models that have been proposed, such as: quintessence, K-essence, phantoms, tachyons, ghost condensates, just to mention a few of them.

Quintessence. Let us give here a brief description of quintessence and discuss it in more detail later, in Section 5.3. Quintessence means that there is a scalar field ϕ minimally coupled to gravity, which has a non-zero potential energy responsible for the present acceleration of the universe. It is similar to the inflaton case, with the difference that quintessence has started dominating recently. Thus, the action of the quintessence field and the expressions for the equation of motion, energy momentum tensor, energy density and pressure are the same as for the inflaton. Another difference with respect to the inflaton is that in the Friedmann equations we must keep the contributions of other constituents of the universe, i.e., we cannot neglect the contribution of dark matter.

In order to have accelerated expansion, the quintessence potential is required to be flat enough, so that one can also define here slow-roll parameters, such as $\epsilon = -\dot{H}/H^2$, where H depends now on both dark matter and dark energy.

The equation of state parameter for the quintessence field ϕ is defined as:

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{5.16}$$

where ρ_ϕ and p_ϕ are the energy density and pressure of the field ϕ, respectively. We note that w_ϕ is not constant, as in the cosmological constant case ($w_\Lambda = -1$), but can depend on the scale factor a. In the limit $\dot{\phi}^2 \ll V(\phi)$, we obtain $w_\phi \simeq -1$, which can mimic a cosmological constant.

In the original quintessence models [160]-[162], the potential of ϕ is of a power-law type

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}, \tag{5.17}$$

where α is a positive number and M is a constant mass scale. By matching the ϕ energy density with the present critical density, one can obtain an expression for the mass scale M

$$M = (\rho_{\phi,0} M_{\rm P}^\alpha)^{\frac{1}{4+\alpha}} \tag{5.18}$$

where $\rho_{\phi,0} \sim V(\phi_0)$ is the energy density of ϕ at present, and ϕ_0 is required to be of the order of the Planck mass ($\phi_0 \sim M_{\rm P}$). In this case, the severe fine-tuning problem of the cosmological constant can be alleviated if we choose, for example, $\alpha = 2$, which implies $M = 1$ GeV [163], compatible with some particle physics scale.

An interesting form of potential is the exponential potential

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi}{M_{\rm P}}\right), \tag{5.19}$$

where V_0 is a constant and $p > 1$. This potential causes a power-law expansion $a(t) \propto t^p$ and possesses cosmological scaling solutions [164, 165], in which the field energy density ρ_ϕ is proportional to the background fluid energy density, $\rho_{\rm M}$. More details about scaling solutions will be given later, in Section 5.3 .

K-essence. The K-essence models are based on modifications of the kinetic term in the Lagrangian density of a scalar field, in order to produce acceleration. The first model of this type, called K-inflation [166], was proposed for inflation, and was later applied to dark energy [167].

In K-essence, the most general scalar-field action is a function of the scalar field ϕ and $X \equiv -(1/2)(\nabla\phi)^2$, and is given by:

$$S = \int {\rm d}^4x\sqrt{-g}\, p(\phi, X) \tag{5.20}$$

where the Lagrangian density corresponds to a pressure density. Usually, in K-essence models, the Lagrangian density is of the form [167]-[169]:

$$p(\phi, X) = f(\phi)g(X) \tag{5.21}$$

which has its motivations in string theory.

A typical example of a K-essence Lagrangian density is [167]:

$$p(\phi, X) = f(\phi)(-X + X^2), \tag{5.22}$$

in which case the equation of state of the field ϕ is:

$$w_\phi = \frac{p}{\rho} = \frac{1 - X}{1 - 3X} \tag{5.23}$$

and gives acceleration for $1/2 < X < 2/3$. For a constant X, we notice that w_ϕ is also constant, and one can deduce the form of $f(\phi)$ from the continuity equation (2.40):

$$f(\phi) \propto (\phi - \phi_0)^{-\alpha}, \quad \alpha = \frac{2(1 + w_\phi)}{1 + w_{\rm M}} \tag{5.24}$$

where $w_{\rm M}$ is the equation of state parameter of the background fluid.

Tachyon field. Rolling tachyon condensates, in a class of string theories, may have interesting cosmological consequences, especially as a dark energy candidates, since one can show that the equation of state parameter of a rolling tachyon smoothly interpolates between -1 and 0 [170].

An example of an effective tachyonic Lagrangian is the following [171]:

$$\mathcal{L} = -V(\phi)\sqrt{1 - \partial_a\phi\partial^a\phi} \tag{5.25}$$

where $V(\phi)$ is the tachyon potential.

In a flat FRW background, the energy density ρ and the pressure density p are given by:

$$\rho = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} \ ; \quad p = -V(\phi)\sqrt{1-\dot{\phi}^2} \tag{5.26}$$

for which the equation of state parameter is:

$$w_\phi = \frac{p}{\rho} = \dot{\phi}^2 - 1. \tag{5.27}$$

Irrespective of the steepness of the potential, the equation of state parameter varies between 0 and -1, and accelerated expansion occurs for $\dot{\phi}^2 < 2/3$. The tachyon energy density is obtained from the continuity equation (2.40) and behaves as $\rho \propto a^{-m}$, with $0 < m < 3$.

As in the case of quintessence, for a special form of the tachyonic potential, one can have power-law expansion, $a \propto t^p$. In this case, the required form of the potential is [171]:

$$V(\phi) = \frac{p}{4\pi G}\left(1 - \frac{2}{3p}\right)^{1/2}\phi^{-2}. \tag{5.28}$$

Phantom (ghost) field. Phantom scalar field models for dark energy have been suggested by recent observational data that might indicate that the equation of state parameter could cross the $w = -1$ barrier and be less than -1 [172]. The proposed models are realized in the context of braneworlds or Brans-Dicke scalar-tensor gravity [173, 174], but the simplest example of a phantom dark energy is provided by a scalar field with a negative kinetic energy [175].

The action of a phantom field minimally coupled to gravity is given by:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \tag{5.29}$$

where the kinetic term has opposite sign as compared to the action of an ordinary scalar field.

The equation of state parameter in this case is:

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)} \tag{5.30}$$

and for $\dot{\phi}^2/2 < V(\phi)$, we obtain $w_\phi < -1$. Because the $w = -1$ barrier can be crossed by phantom fields, it received the name of the *phantom divide*.

In the case of a phantom scalar field dominating the energy density of the universe, there is a Big Rip future singularity and thus the universe has a finite lifetime. This singularity may be avoided if, e.g., the potential $V(\phi)$ has a maximum. In this case, the phantom field will execute damped oscillations about the maximum of the potential and, after a certain period of time, it settles at the top of the potential to mimic the de-Sitter like behavior.

From the viewpoint of quantum mechanics, phantom fields are generally plagued by severe ultra-violet quantum instabilities [176], which poses an interesting challenge for theoreticians.

Chaplygin gas. The Chaplygin gas [177] is a fluid with a non-canonical equation of state

$$p = -\frac{A}{\rho}, \tag{5.31}$$

where A is a positive constant. From the continuity equation (2.40), we can deduce the energy density of the fluid:

$$\rho = \sqrt{A + \frac{B}{a^6}} \tag{5.32}$$

where B is a constant. The nice feature of this type of fluid is that it provides an interesting possibility for the unification of dark energy and dark matter, because we notice that in the early time limit $a \ll (B/A)^{1/6}$ the Chaplygin gas energy density behaves as pressureless dust, $\rho \sim \sqrt{B}a^{-3}$, while in the late time limit $a \gg (B/A)^{1/6}$ it acts as a cosmological constant, $\rho \sim -p \sim \sqrt{A}$.

One can derive a corresponding potential for the Chaplygin gas by treating it as an ordinary scalar field ϕ. One obtains:

$$V(\phi) = \frac{\sqrt{A}}{2}\left(\cosh\sqrt{3}\kappa\phi + \frac{1}{\cosh\sqrt{3}\kappa\phi}\right) \tag{5.33}$$

where $\kappa \equiv \sqrt{8\pi G}$. Hence, a minimally coupled field with this potential is equivalent to the Chaplygin gas model.

However, this model has serious problems [178] when confronted with observations of the CMB anisotropy. The situation can be alleviated in the *generalized Chaplygin gas model* [179], in which $p = -A/\rho^\alpha$, with $0 < \alpha < 1$. In order to satisfy observational constraints, the allowed range of values for the parameter α is relatively small, i.e., $0 \leq \alpha < 0.2$ [178].

5.2.3 Modified gravity and other alternatives

So far, we have presented a few examples of models proposed to explain dark energy, based on new exotic contributions to the energy momentum tensor in the Einstein equation. However, there is still another possibility for having late time acceleration, i.e., to modify the geometry of spacetime itself and introduce higher curvature corrections to the Einstein-Hilbert action.

There are many proposed scenarios in the literature. In $f(R)$ gravities [180, 181], one supposes that the action is of the form:

$$S = \int \mathrm{d}^4x\sqrt{-g}f(R), \tag{5.34}$$

where $f(R)$ is an arbitrary function of the Ricci scalar R. In order to explain dark energy, the modifications introduced should only affect gravity on cosmological scales, while they should still be compatible with the newtonian limit. The original model (which adds a term $\delta f(R) \propto 1/R$) is not compatible with solar system experiments [182] and possesses instabilities [183], but there are forms of $f(R)$ that can pass all tests and be viable candidates for the dark energy.

The Gauss-Bonnet gravity [184] is inspired from string/M-theories and consists of an unusual coupling of a scalar field ϕ to the Gauss-Bonnet invariant, $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, which becomes important in the current low-curvature universe. An interesting feature of this theory is that it can mimic a phantom equation of state ($w < -1$), without necessity of dealing with the problematic phantom field, and it may prevent the Big Rip singularity.

Other interesting examples belong to the class of modified Gauss-Bonnet gravity [185], and consists of introducing general functions $f(G)$ or $f(G, R)$ in the Einstein Hilbert action.

There are also theories with large extra dimensions, inspired by string theory, in which our four dimensional spacetime "lives" on a *brane* embedded in a higher dimensional *bulk* spacetime, and gravity is the only interaction that can propagate into the anti de Sitter bulk. The Randall-Sundrum [33] and the Dvali-Gabadadze-Porrati [186] models are two different examples of this type of theories.

A totally different possibility explored recently is based on the idea that the observed acceleration is due to the effect of the back reaction of either super or sub-horizon cosmological perturbations [187, 188], and there is no need to modify gravity, nor to introduce any cosmological constant or some exotic negative-pressure fluid. This is an appealing idea, but it has raised a considerable amount of criticism that seems to demonstrate that this possibility is not plausible [189, 190].

5.3 The details of quintessence

In the previous section, we have seen a few ways of how to address the dark energy problem, i.e., by considering (*i*) a non-zero cosmological constant, (*ii*) scalar fields and (*iii*) modifications of gravity on large scales.

In this Thesis, we are interested in the study of scalar fields as candidates to the dark energy of the universe. In any viable model of dark energy, one requires that the energy density of the scalar field remains subdominant during all the history of the universe, emerging only at late times as the dominant component. However, the fact that the dark matter and dark energy densities are comparable today suggests that, in the past, they were many orders of magnitude different. This raises a fine-tuning problem for the initial dark energy density.

An interesting possibility is to consider a dark energy scalar field in the presence of a background fluid, because the system can have fixed points or it can enter a scaling regime, which may solve the initial conditions problem.

5.3.1 Fixed points and scaling regime

Let us study an important class of dynamical systems, the *autonomous systems* [164]. We consider the coupled system of two first-order differential equations, for two variables $x(t)$ and $y(t)$

$$\dot{x} = f(x,y,t), \quad \dot{y} = g(x,y,t) \tag{5.35}$$

which is said to be *autonomous* if the functions f and g do not contain explicit time-dependent terms, i.e., $\dot{x} = f(x,y)$ and $\dot{y} = g(x,y)$. We can then define a *fixed* (or *critical*) *point* $(x_{\rm c}, y_{\rm c})$ of the autonomous system as a point that satisfies the condition:

$$(f,g)\,\big|_{(x_{\rm c},y_{\rm c})} = 0\ . \tag{5.36}$$

A critical point $(x_{\rm c}, y_{\rm c})$ is an *attractor* if it satisfies the condition:

$$(x(t), y(t)) \longrightarrow (x_{\rm c}, y_{\rm c}) \;\; \text{for} \;\; t \to \infty. \tag{5.37}$$

One is interested in the study of the stability of fixed points, in order to ascertain if the system approaches one of the critical points. This can be done by considering small perturbations δx and δy around the critical point (x_c, y_c). From equations (5.35), one can find a first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}N}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \tag{5.38}$$

where $N = \ln a$ is the number of e-foldings. The general solution for the evolution of linear perturbations can be written as:

$$\delta x = C_1\,\mathrm{e}^{\mu_1 N} + C_2\,\mathrm{e}^{\mu_2 N} \tag{5.39}$$

$$\delta y = C_3\,\mathrm{e}^{\mu_1 N} + C_4\,\mathrm{e}^{\mu_2 N} \tag{5.40}$$

where C_1, C_2, C_3, C_4 are integration constants and μ_1, μ_2 are the eigenvalues of the matrix $\mathcal{M}$ appearing in equation (5.38). They satisfy the following equation:

$$\mu^2 - T\mu + D = 0 \tag{5.41}$$

where T is the trace of $\mathcal{M}$ and D is its determinant. The discriminant $\Delta \equiv T^2 - 4D$ appears in the solutions to the eigenvalues μ_1 and μ_2:

$$\mu_{1,2} = \frac{T \pm \sqrt{\Delta}}{2}. \tag{5.42}$$

The stability around the fixed points depends on the nature of the eigenvalues, and one generally uses the following classification [164, 191]:

(i) Stable node, if $\mu_1 < 0$ and $\mu_2 < 0$;

(ii) Unstable node, if $\mu_1 > 0$ and $\mu_2 > 0$;

(iii) Saddle point, if $\mu_1 < 0$ and $\mu_2 > 0$, or $\mu_1 > 0$ and $\mu_2 < 0$;

(iv) Stable spiral, if $\Delta < 0$ and $T < 0$.

In the cases (i) and (iv) the fixed point is an attractor.

Let us suppose that we have a system composed of a scalar field and a background barotropic fluid[1], with equation of state $p_\mathrm{M} = w_\mathrm{M}\rho_\mathrm{M}$, in the expanding universe. If the cosmological solutions are such that the energy density ρ_ϕ of the scalar field mimics that of the fluid ρ_M, they are called *scaling solutions* [164], and are of particular importance in the cosmology of dark energy.

[1] A barotropic fluid is a fluid whose pressure and density are related by an equation of state that does not depend on temperature, i.e., $\rho = \rho(p)$ or $p = p(\rho)$.

Scaling solutions are characterized by the relation:

$$\rho_\phi/\rho_{\rm M} = C, \tag{5.43}$$

where C is a constant. As long as the scaling solution is an attractor, the field will sooner or later enter the *scaling regime*, and in the radiation or matter dominating eras the field energy density should be subdominant. Ultimately, the system needs to exit from the scaling regime (5.43) in order to produce an accelerated expansion.

It is also possible to have a direct coupling Q between the scalar field ϕ and the barotropic fluid [192]. One can show that the presence of an interaction between ϕ and the background fluid might lead to accelerated expansion, although without interactions this is not possible.

We should also mention that exponential potentials $V(\phi) \sim \exp(\lambda\phi)$ correspond to scaling solutions, thus being of interest for cosmology [165].

Let us study the existence of fixed points and scaling regime for a system that consists of a scalar field ϕ in the presence of a barotropic background fluid, without interactions between them. From the Lagrangian of the scalar field ϕ

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{5.44}$$

one obtains the following equations:

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_{\rm M}\right] \tag{5.45}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\dot{\phi}^2 + (1+w_{\rm M})\rho_{\rm M}\right] \tag{5.46}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{{\rm d}V}{{\rm d}\phi} = 0. \tag{5.47}$$

In order to study the existence of fixed points, it is useful to introduce the following dimensionless new variables:

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H},\ y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H},\ \lambda \equiv -\frac{V'}{\kappa V},\ \Gamma \equiv \frac{VV''}{V'^2} \tag{5.48}$$

where a prime means differentiation with respect to ϕ.

In terms of these new variables, the equation of state parameter w_ϕ and the fraction of the energy density Ω_ϕ of the field ϕ become [164, 193]:

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{x^2 - y^2}{x^2 + y^2} \tag{5.49}$$

$$\Omega_\phi \equiv \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2 \tag{5.50}$$

and the total effective equation of state is defined by:

$$w_{\text{eff}} \equiv \frac{p_\phi + p_{\text{M}}}{\rho_\phi + \rho_{\text{M}}} = w_{\text{M}} + (1 - w_{\text{M}})x^2 - (1 + w_{\text{M}})y^2 \tag{5.51}$$

where $w_{\text{M}} = p_{\text{M}}/\rho_{\text{M}}$ is the equation of state parameter of the barotropic fluid. The fixed points can be obtained by imposing the following conditions:

$$\frac{\mathrm{d}x}{\mathrm{d}N} = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}N} = 0. \tag{5.52}$$

For instance, let us consider an exponential potential [164, 193]

$$V(\phi) = V_0 \mathrm{e}^{-\lambda\kappa\phi} \tag{5.53}$$

which corresponds to a constant λ defined in (5.48). The exponential potential (5.53) possesses two attractor solutions, depending on the values of λ and $\gamma_{\text{M}} \equiv 1 + w_{\text{M}}$:

(*i*) $\lambda^2 > 3\gamma_{\text{M}}$, for which the scalar field mimics the evolution of the barotropic fluid and one has $\gamma_\phi \equiv 1 + w_\phi = \gamma_{\text{M}}$ and $\Omega_\phi = 3\gamma_{\text{M}}/\lambda^2$;

(*ii*) $\lambda^2 < 3\gamma_{\text{M}}$, in which case the late time attractor is the scalar field dominated solution, with $\Omega_\phi = 1$ and $\gamma_\phi = \lambda^2/3$.

5.3.2 Constraints from nucleosynthesis

Any model of dark energy is constrained by the requirement that dark energy should not affect primordial nucleosynthesis, which puts an upper bound on the number of extra degrees of freedom, such as a light scalar field. The presence of a scalar field during the nucleosynthesis epoch, at a temperature around 1 MeV, has the effect of changing the expansion rate of the universe thus affecting the abundance of light elements.

Taking a conservative bound on the number of additional relativistic degrees of freedom, $\Delta N_{\text{eff}} \simeq 1.5$ [194], we obtain the following bound on the energy density of the scalar field [195, 196]:

$$\Omega_\phi(T \sim 1\text{MeV}) < 0.2. \tag{5.54}$$

Let us consider the two attractor solutions found in the previous subsection for the exponential potential. The first scaling solution (*i*) corresponds to the case in which the field energy density mimics that of the background, which at the epoch of nucleosynthesis consists of radiation ($\gamma = 4/3$). The constraint (5.54) gives:

$$\Omega_\phi = \frac{4}{\lambda^2} < 0.2 \tag{5.55}$$

from which we obtain $\lambda^2 > 20$. In this case, the equation of state is equal to that of the background and accelerated expansion is not possible. Late-time acceleration may be obtained if the system exits from the scaling solution (i) and changes to the second scaling solution (ii) near to the present.

5.3.3 Exit from a scaling regime

In the previous subsection, we have seen that in order to have a late-time accelerated expansion, the system composed of the scalar field ϕ and a background fluid needs a transition from one scaling solution to the other. There are some models in which this is possible, and we will consider here a few examples.

A first example is the double exponential potential [165, 197]:

$$V(\phi) = V_0(\mathrm{e}^{-\lambda\kappa\phi} + \mathrm{e}^{-\mu\kappa\phi}), \tag{5.56}$$

where λ and μ are some positive constants, with $\lambda > \mu$. By requiring that λ satisfies the condition (5.55), we ensure that the energy density of the scalar field mimics the background energy density and is subdominant during the radiation dominated era. The system exits the scaling regime and approaches the scalar-field dominated solution (ii), with $\Omega_\phi = 1$, when $\mu^2 < 3$, and gives accelerated expansion at late times if $\mu^2 < 2$. The advantage of this model is that, for a wide range of initial conditions, the solutions first enter the scaling regime, which is followed by a late-time accelerated expansion, once the potential becomes shallow.

A second example is the model suggested by Sahni and Wang [198], which use the following potential:

$$V(\phi) = V_0\left[\cosh(\lambda\kappa\phi) - 1\right]^n \tag{5.57}$$

which for $|\lambda\phi| \gg 1$ and $\phi < 0$ has the exponential form $V(\phi) \propto \mathrm{e}^{-n\lambda\kappa\phi}$, while for $|\lambda\phi| \ll 1$ it becomes $V(\phi) \propto \phi^{2n}$. This means that, initially, the energy density of the field mimics that of the background fluid, and its density parameter is $\Omega_\phi = 3\gamma/n^2\lambda^2$. When the field approaches the minimum of the potential, at $\phi = 0$, it exits from the scaling regime and starts oscillating about the minimum. The average equation of state during oscillations is:

$$\langle w_\phi \rangle = \frac{n-1}{n+1}. \tag{5.58}$$

Accelerated expansion is obtained if n is sufficiently small, $n < 1/2$, and it can be shown that the present-day values $\Omega_\phi \simeq 0.7$ and $\Omega_\mathrm{M} \simeq 0.3$ can be obtained for a wide range of initial conditions. An interesting feature of the model is that for $n = 1$ the field behaves as non-relativistic matter giving rise to a new form of cold dark matter, which the authors called *frustrated* dark matter.

Our last example is the model of Albrecht and Skordis [199], which can be derived from string theory. The potential they use is a combination of exponential and power-law terms:

$$V(\phi) = V_0 e^{-\lambda\kappa\phi} \left[A + (\kappa\phi - B)^2\right] . \tag{5.59}$$

The potential has a local minimum that can have today's value of the critical energy density. At early times the exponential term dominates and the energy density of ϕ tracks the background radiation or matter. At late times, the field can get trapped in the local minimum leading to $w_\phi \simeq -1$.

5.3.4 Dark energy from PGB

There are plenty of proposed dark energy models that at the phenomenological level are able to give a correct description and interpretation of observational data. A more delicate issue is that of relating models with particle physics in a natural way. Interesting approaches to addressing the origin of dark energy in particle physics are related to string/M-theory [200].

In this Thesis, however, I will not discuss this kind of approaches. I will focus on the possibility that the dark energy field is associated to a pseudo Goldstone boson (PGB) arising in models with spontaneously symmetry breaking (SSB) of global symmetries. In models of this type, the required flatness of the quintessence potential is justified and protected by the underlying symmetry.

A first PGB model is the axion dark energy model, introduced by Frieman *et al.* [201]. The axion potential is:

$$V(\phi) = \Lambda\left[1 + \cos(\phi/f)\right] . \tag{5.60}$$

When the field ϕ is near the maximum of the potential and its mass is much smaller than the expansion rate, i.e., $|m^2| \ll H^2$, the field is frozen and acts like quintessence, until its mass becomes comparable to the Hubble rate and ϕ starts to roll down the potential. This model requires that the scale f should be of the order of the Planck mass M_{P}.

A more recent axion dark energy model has been developed by Kim and Nilles [202], as well as by Hall and collaborators [203, 204]. In reference [202], the authors propose a dark energy candidate, called "quintaxion", which is a linear combination of two axions. One of them solves the strong CP-problem and is a cold dark matter candidate, while the other, the quintaxion, has a decay constant as expected from string theory ($f_{\mathrm{q}} \sim 10^{18}$ GeV) and is responsible for the observed vacuum energy. The flatness of the potential is due to the small masses of the quarks of some hidden sector of the model, which are protected by the existence of a global symmetry associated to the PGB.

Another way to protect the small mass scale of the quintessence field is through a seesaw mechanism [203, 204], which allows for two natural scales to play a vital role in determining all the other fundamental scales.

In reference [205], Caroll investigated the possibility that the couplings of the quintessence field to matter are suppressed by the presence of an approximate global symmetry, like in the case of long range forces. Although it is well constrained, a coupling between the field ϕ to the pseudoscalar $F_{\mu\nu}\tilde{F}^{\mu\nu}$ of electromagnetism could lead to the detection of a cosmological scalar field.

In Chapter 10, I propose a model in which the axion-like field resulting from the SSB of a global $U(1)$ symmetry mimics quintessence. The field is trapped at a local non-zero minimum of the $U(1)-$symmetric potential.

5.3.5 Quintessential inflation

The first model of quintessential inflation, proposed by Peebles and Vilenkin [206], introduces a scalar field ϕ with a potential such that it is able to explain both inflation and dark energy in a unified way, without the need of introducing a new weakly interacting field. During inflation, the potential has the form:

$$V(\phi) = \lambda(\phi^4 + M^4), \quad \phi < 0 \tag{5.61}$$

as in chaotic inflation, while at late-times, its form changes to:

$$V(\phi) = \frac{\lambda M^4}{1 + (\phi/M)^\alpha}, \quad \phi \geq 0. \tag{5.62}$$

This model has the advantage of explaining inflation and dark energy by using the same field ϕ, but has the drawback of requiring fine-tuning of the initial conditions, because there are no tracker solutions.

The authors of [207] proposed a model of quintessential inflation by introducing a renormalizable complex scalar field potential, invariant under a global $U(1)_{\rm PQ}$ symmetry, which is explicitly broken by small instanton effects at a lower energy scale. The resulting "axion" is then responsible for dark energy, while the real part of the scalar field produces inflation. This model combines the original idea of Natural Inflation [71, 208] with the idea of using a PGB for the quintessence field [201, 209].

The so-called "spintessence" models [210, 211] also use a complex scalar field, allowing for a unified description of both dark matter and dark energy. The complex scalar field ϕ is spinning in a $U(1)-$symmetric potential and has internal angular momentum. Depending on the nature of the spin and on the form of the potential, the field may act like an evolving dark energy component or like a self-interacting, fuzzy cold dark matter [211].

Other possibilities, like that of having PGB in higher dimensional theories [212] or quintessential inflation [213] in braneworld scenarios [33], have been investigated in the literature.

In this Thesis, I use the idea of quintessential inflation in Chapter 10 to obtain an ambitious model for the unification of inflation, dark matter and dark energy. Also, in Chapter 8, I propose that a scalar field ψ is responsible for both inflation and dark energy, in a model which possesses a global $U(1)$ symmetry, which is explicitly broken by Quantum Gravity effects.

5.4 Dark energy from scalar field oscillations

Let us consider now the evolution of the energy associated with coherent oscillations of a scalar field ϕ in a potential $V(\phi)$, in a homogeneous and isotropic universe [214]. The equation of motion for ϕ is given by the continuity equation:

$$\dot{\rho} = -3H(\rho + p) = -3H\dot{\phi}^2. \tag{5.63}$$

We ignore particle creation and couplings of the scalar field ϕ to other fields. Thus, the energy density of the scalar field ϕ decreases due to the expansion of the universe, which has the effect of decreasing the kinetic term $\dot{\phi}^2/2$.

We assume that ϕ oscillates about a minimum of $V(\phi)$, at $\phi = 0$, with a frequency $\omega \sim \dot{\phi}/\phi$ much larger than the expansion rate H. Then, the energy density ρ will be a slowly varying function of time, while $(\rho + p) = \dot{\phi}^2$ varies rapidly, compared to a time scale characterized by H^{-1}.

We can write $\dot{\phi}^2$ as the sum of a slowly varying part and a periodic one,

$$\dot{\phi}^2 = \rho + p = \gamma\rho + \gamma_{\rm p}\rho \tag{5.64}$$

where γ is the average of $\dot{\phi}^2/\rho$ over an oscillation, and $\gamma_{\rm p}$ is the periodic part. In the limit $\omega \gg H$, one obtains [214]:

$$\rho \propto a^{-3\gamma} \tag{5.65}$$

and if the energy density of ϕ dominates the total energy density of the universe, one also has:

$$a(t) \propto t^{2/3\gamma}. \tag{5.66}$$

One can obtain γ by averaging $\dot{\phi}^2/\rho$ over one cycle, and supposing that $\rho = V(\phi_{\max}) = V_{\max}$ is constant in one cycle:

$$\gamma = 2\frac{\int_0^{\phi_{\max}}(1 - V/V_{\max})^{1/2}\mathrm{d}\phi}{\int_0^{\phi_{\max}}(1 - V/V_{\max})^{-1/2}\mathrm{d}\phi} \tag{5.67}$$

where $\phi_{\max}$ is the amplitude of the oscillations and we have also assumed that $V(\phi) = V(-\phi)$. Because the average of the periodic part γ_{p} over one cycle is negligible, it results that $(\gamma - 1)\rho$ is the pressure averaged over one cycle.

The evolution of the energy density depends on the form of the potential $V(\phi)$. For a general simple polynomial,

$$V(\phi) = \lambda\phi^n \tag{5.68}$$

one obtains $\gamma = 2n/(n + 2)$ and [214]

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-6n/(n+2)} \tag{5.69}$$

where ρ_0 and a_0 are the values of the energy density and scale factor at a time t_0. For $n = 2$, the energy density of the scalar field behaves like non-relativistic matter with $\gamma = 1$, $\langle p\rangle = 0$ and $\rho \propto a^{-3}$, whereas for $n = 4$ it behaves like relativistic matter with $\gamma = 4/3$, $\langle p\rangle = \rho/3$ and $\rho \propto a^{-4}$.

In case the potential $V(\phi)$ is sufficiently flat, it can dominate over the kinetic term and the field ϕ may have an equation of state such that it may drive the acceleration of the universe, i.e., $w = \gamma - 1 < -1/3$. However, in general γ can have a slight variation in time, and then it may not always be possible to find an analytical solution for ρ.

If, in addition, $V(\phi)$ depends explicitly on time, the equation for ρ becomes:

$$\dot{\rho} = -3H\dot{\phi}^2 + \frac{\partial V}{\partial t}. \tag{5.70}$$

An interesting particular case is that of potentials of the form $V(\phi, t) = f(t)v(\phi)$, for which equation (5.70) can be integrated to give:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3\gamma} \left(\frac{f}{f_0}\right)^{1-\gamma/2}. \tag{5.71}$$

This result can be applied, for instance, to the invisible axion, which has a time-dependent potential that factorizes, $V(\phi, t) = \frac{1}{2}m_{\mathrm{a}}^2(t)\phi^2$, and for which one obtains $\gamma = 1$ and $\rho a^3/m_{\mathrm{a}} \propto$ constant.

In Chapter 7 of this Thesis, we will see a different way to obtain the above results, by using the action-angle formalism of analytical mechanics, and we will find that our formalism has some advantages. In Chapter 11, I apply this alternative formalism to obtain the evolution of the energy density associated to oscillations of a scalar field in a potential, and argue that it may be contributing to the dark energy of the universe.

Chapter 6

Symmetry breaking and phase transitions

6.1 Spontaneous breaking of global symmetries

One of the crucial aspects of quantum field theories are the symmetries of the Lagrangian that describes the system. Starting from these symmetries, one can construct the theory and describe the interactions between the particles that form the system. When these symmetries are *exact*, they give rise to conservation laws, such as, for instance, the electric charge conservation law. Also, when we gauge certain symmetries we get the Standard Model of particle physics.

Nevertheless, apart from the well-known symmetries of the SM, nature might have further symmetries, which are not observed today and whose corresponding conservation laws might be hidden. This is due to the fact that the vacuum state describing the system may not respect the symmetries of the Lagrangian. The vacuum of the system is the state of minimum energy, i.e., the state that minimizes the potential. There is no reason for this state to satisfy the same symmetries of the potential and, in this case, the symmetries shall not be perceptible. A very simple example is that of a parity symmetric potential, whose minima are displaced from the symmetry axis. In this case, the potential is invariant under the change of the field sign, while the configuration of the field located in one of the minima is not.

When the fundamental state (vacuum) of the system does not respect some of the symmetries of the Lagrangian, one says that the symmetry is *spontaneously broken*. The mechanism of spontaneous symmetry breaking (SSB) is a very important concept and often takes place in nature.

During the evolution of the universe, there are symmetries that were exact in the past and that later became spontaneously broken. This is because the shape of the potential may vary with the temperature. As we will see in this chapter, the fact that a plasma with a certain temperature is formed affects the shape of the potential. So, by changing the temperature, the shape of the potential changes, which means that the vacuum may also change. In this way, the symmetries respected by the vacuum at a higher temperature may not be respected at a lower temperature when they become spontaneously broken.

Today, the universe is very cold, $T \sim 3$K, but within the Big Bang theory there is strong evidence that the early universe consisted of a very high energy plasma. Then, as it was expanding, its temperature decreased proportionally to the scale factor, $T \propto a^{-1}$, and many of the symmetries that were exact at high temperature became broken. Symmetry breaking is related to phase transitions, which may have different effects, such as the production of new particles or topological defects. These effects can be the clue of hidden symmetries and they may help us to understand phenomena like inflation, baryogenesis, quark-hadron phase transition, etc.

For all these reasons, the study of SSB together with its effects is fundamental for understanding the evolution of the universe.

6.1.1 The Goldstone theorem

The Goldstone theorem [215, 216] states that when a global continuous symmetry is spontaneously broken, there appears a massless particle in the spectrum, a so-called *Goldstone boson*. There are many proofs of this theorem in the literature [217], so that we will not give here a rigorous demonstration of it. Instead, in order to understand this mechanism, we will give the well-known example of the linear sigma model.

The Lagrangian of the linear sigma model contains a set of N real scalar fields $\phi^i(x)$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left[(\phi^i)^2\right]^2 \tag{6.1}$$

where there is a sum over i in every factor $(\phi^i)^2$. The Lagrangian (6.1) is invariant under the following symmetry:

$$\phi^i \longrightarrow R_N^{ij}\phi^j \tag{6.2}$$

for any orthogonal $N \times N$ matrix R_N. The transformation group (6.2) is the group of rotations in N dimensions, i.e. the orthogonal $N-$dimensional group, $O(N)$.

The classical field configuration that represents the fundamental state, ϕ^i_0, is that which minimizes the potential

$$V(\phi^i) = -\frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}\left[(\phi^i)^2\right]^2 \tag{6.3}$$

and occurs for any value ϕ_0^i satisfying:

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}. \tag{6.4}$$

This condition only fixes the length of the vector ϕ_0^i, while its direction is arbitrary. Without losing generality, one can suppose that the vector ϕ_0^i points in the $N-$direction,

$$\phi_0^i = (0, 0, \ldots, 0, v) \tag{6.5}$$

where $v \equiv \mu/\sqrt{\lambda}$. For $N = 2$, the potential (6.3) has the typical "Mexican hat" shape, where the minimum can be at any point on a circle of radio v. In Figure 6.1 is illustrated the potential in this particular case, which helps us have a better understanding of the process of spontaneous breaking.

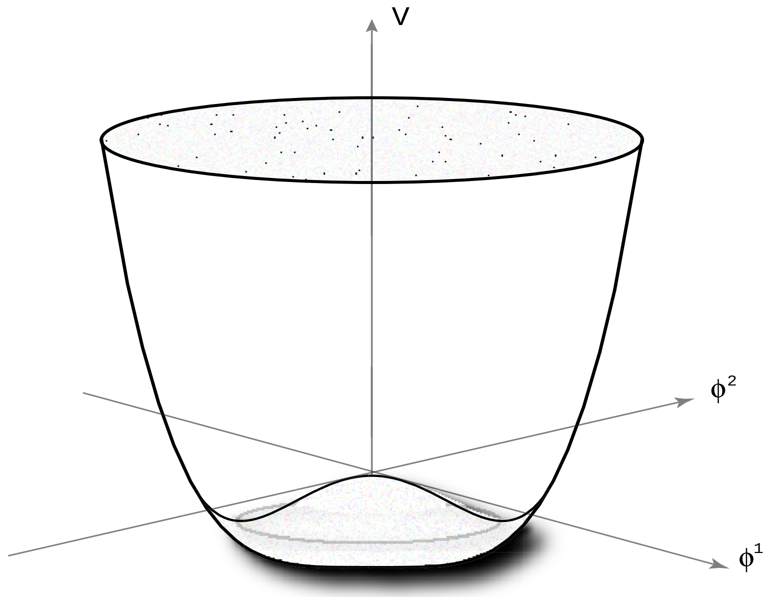

Figure 6.1: The potential (6.3) of the linear sigma model, for the case $N = 2$.

It is convenient to redefine the field $\phi^i(x)$, by introducing new fields $\pi^k(x)$ and $\sigma(x)$ as follows:

$$\phi^i(x) = \big(\pi^k(x), v + \sigma(x)\big) \quad k = 1, \ldots, N-1. \tag{6.6}$$

Introducing the change of variables (6.6) in (6.1), one finds the form of the Lagrangian after the SSB:

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\,\mu\sigma^3 \\ & -\sqrt{\lambda}\,\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}\left[(\pi^k)^2\right]^2 . \end{aligned} \tag{6.7}$$

We can see that the new Lagrangian (6.7) contains a massive field σ and a set of $N-1$ massless fields π^k. The original $O(N)$ symmetry becomes hidden, but the symmetry subgroup $O(N-1)$ is still explicit, and transforms the fields π^k as:

$$\pi^k \longrightarrow R^{kj}_{N-1}\pi^j. \tag{6.8}$$

The massive field σ describes oscillations of the field ϕ^i in radial direction, in which the second derivative of the potential is different from zero. The massless fields π^k describe angular oscillations of the field ϕ^i, along the potential valley. This valley is an $(N-1)-$dimensional surface, where all $N-1$ directions are flat and equivalent, which is a manifestation of the $O(N-1)$ symmetry.

In the example given here, the number of initial continuous symmetries was $N(N-1)/2$, corresponding to the different orthogonal axes, around which one can make an $O(N)-$rotation, in N dimensions. After the spontaneous breaking one is left with the subgroup $O(N-1)$, which contains $(N-1)(N-2)/2$ continuous symmetries. Thus, the difference $N-1$ will be the number of broken symmetries, which is exactly the number of massless particles that appear in the theory. This agrees with the predictions of the Goldstone theorem.

There are many examples of spontaneously broken symmetries in nature, which give rise to Goldstone bosons. The process of SSB is fundamental in many models that go beyond the SM. One example is the family symmetry, which is related to the number and properties of the SM families. The breaking of this symmetry is responsible for the *familions* [218]. Another example is the symmetry associated to the leptonic number, whose breaking would produce *majorons* [219, 220].

6.1.2 Explicit breaking and the pseudo Goldstone boson

The Lagrangian (6.1) of the linear sigma model is invariant under $O(N)$ transformations. If, for any reason, we add a term to the Lagrangian that explicitly breaks the $O(N)$ symmetry, the treatment made in the previous subsection has to be modified. For a sufficiently small explicit breaking, the framework used for the SSB is still valid. In order to have a better understanding of the explicit breaking effects, we will continue working with the previous example, the linear sigma model.

Let us assume that we add a small explicit symmetry breaking term to the Lagrangian (6.1). In the presence of this term, the direction in which the minimum ϕ_0^i will point is not arbitrary anymore. After the spontaneous breaking, some of the $N-1$ directions corresponding to oscillations of the massless π^k fields $-$ which in the case without explicit breaking were exactly flat − acquire a non-zero second derivative of the potential. This means that these directions acquire a small inclination, which is equivalent to the fields π^k acquiring a small mass. In the simple example when $N = 2$, illustrated in Figure 6.1, the effect of the explicit breaking is to slightly incline the "Mexican hat", which breaks the degeneracy between the infinite minima located on the circle of radio v, and we are left with only one absolute minimum.

Thus, in the presence of a small *explicit* symmetry breaking, the resulting massless Goldstone bosons acquire a small mass and become *pseudo Goldstone bosons* (PGBs). There are many examples of PGBs in nature, such as the pion or the hypothetical axion.

6.1.3 Quantum gravity effects

In this subsection, we will argue that quantum gravity effects may provide a mechanism of explicit breaking of global symmetries, and that these effects are expected to be very small. There are reasons to expect that non-perturbative quantum gravity effects break global symmetries: global charges can be absorbed by black holes which may evaporate, "virtual black holes" may form and evaporate in the presence of a global charge, wormholes may take a global charge away from our universe to some other one, etc.

There has been a lot of interest in studying global symmetries at high energies [221]-[224]. In [222], the loss of quantum coherence in a model of gravity coupled to axions is investigated. The loss of coherence opens the possibility that currents associated with global symmetries are not exactly conserved, while those associated with local symmetries are still exactly conserved. Coleman [223] argued that incoherence is not observed in a many-universe system in an equilibrium state, and he pointed out that if wormholes exist they can explain the vanishing of the cosmological constant. The authors of [224] pointed that even if incoherence is not observed in the presence of wormholes, other interesting consequences may emerge, such as the appearance of operators that violate global symmetries, of arbitrary dimensions, induced by baby universe interactions. In this context, the authors of [225] argue that if global symmetries are broken by virtual black holes or topology changing effects, they have to be exponentially suppressed.

In [116], we investigated the cosmological implications of quantum gravity effects, in a model with a global $U(1)$ symmetry. The model introduces a new complex scalar field $\Psi = \phi \exp(\theta/f)$ charged under the global $U(1)$ symmetry, which is spontaneous broken at a scale f.

The potential of the field Ψ contains a $U(1)-$symmetric term

$$V_{\rm sym}(\Psi) = \frac{1}{4}\lambda[|\Psi|^2 - f^2]^2 \tag{6.9}$$

where λ is a coupling constant, and an explicit symmetry breaking term describing quantum gravity effects:

$$V_{\text{non-sym}}(\Psi) = -g\frac{1}{M_P^{n-3}}|\Psi|^n \left(\Psi e^{-i\delta} + \Psi^\star e^{i\delta}\right) \tag{6.10}$$

where g is an effective coupling that characterizes the strength of the quantum gravity effects, δ is an arbitrary phase and n is an integer number, $n > 3$. The resulting PGB has a mass given by $m_\theta = \sqrt{2g}\,(f/M_{\text{P}})^{(n-1)/2}M_{\text{P}}$, and for a sufficiently small value of the explicit breaking, $g \sim 10^{-30}$, it is a dark matter candidate. In this Thesis, based on this model, in Chapter 8 I propose a model in which the real part of Ψ is responsible for inflation, while the imaginary part is a quintessence field. In Chapter 9, I propose a similar model in which the real part of Ψ is the inflaton and the resulting PGB is a dark matter candidate.

6.2 Finite temperature effects

The usual methods applied in quantum field theory are adequate for describing processes in vacuum, like those given in particle accelerators. Nevertheless, in the early universe, there are totally different conditions, because the universe consists of an extremely hot and dense plasma. In these conditions, one must find other methods, at half distance between thermodynamics and quantum field theories, which allow one make realistic computations in the conditions in which the environment is characterized by a thermal bath. These methods are developed in the finite temperature field theory. Much work has been done in this field and there are excellent references in the literature, where one can find all the details about it [226]-[228]. Here, we only pretend to highlight the basics of this theory and the main concepts that will be used later in the work of chapters 8, 10 and 9. We are only interested in the aspects related to scalar fields.

6.2.1 The effective potential

All the information about the effects of finite temperature can be included in the *effective potential*, V_{eff}^β. This may be written as the sum of the tree-level (classical) potential, V_0, and a term describing quantum and temperature effects, V^β:

$$V_{\text{eff}}^\beta(\phi_{\text{c}}) = V_0(\phi_{\text{c}}) + V^\beta(\phi_{\text{c}}) \tag{6.11}$$

where $\phi_{\text{c}} \equiv \bar{\phi}(x)$ is the constant field value in a translationally invariant theory.

There are two different formalisms for calculating the term V^β, which give the same results, at least at first order. One of them is the *imaginary time formalism*, usually applied in equilibrium situations, and the other is the *real time formalism*, which can be used in investigating out-of-equilibrium systems.

Let us focus on the imaginary time formalism and consider the finite temperature effective potential at one-loop. It is given by [228]:

$$V_1^\beta(\phi_\mathrm{c}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\omega}{2} + \frac{1}{\beta}\ln\left(1-\mathrm{e}^{-\beta\omega}\right)\right] \tag{6.12}$$

where $\beta \equiv 1/T$ and ω is defined as:

$$\omega^2 = |\vec{p}|^2 + m^2(\phi_\mathrm{c}). \tag{6.13}$$

In the above relation, the *shifted mass* $m^2(\phi_\mathrm{c})$ is given by the curvature of the potential V_0:

$$m^2(\phi_\mathrm{c}) \equiv \frac{\mathrm{d}^2V_0(\phi_\mathrm{c})}{\mathrm{d}\phi_\mathrm{c}^2}. \tag{6.14}$$

The first part of the integral (6.12) takes into account quantum corrections in the vacuum, thus given the effective potential at zero temperature, $V_1|_{T=0}$ [72]. The second part of (6.12) includes temperature effects and can be written as:

$$\frac{1}{\beta}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left(1-\mathrm{e}^{-\beta\omega}\right) = \frac{1}{2\pi^2\beta^4}J_\mathrm{B}[m^2(\phi_\mathrm{c})\beta^2] \tag{6.15}$$

where J_B is the thermal bosonic function defined as:

$$J_\mathrm{B}(m^2\beta^2) = \int_0^\infty \mathrm{d}x\, x^2 \ln\left[1-\mathrm{e}^{-\sqrt{x^2+\beta^2m^2}}\right]. \tag{6.16}$$

In this way, the one-loop effective potential is composed by the following parts:

$$V_\mathrm{eff}^\beta(\phi_\mathrm{c}) = V_0(\phi_\mathrm{c}) + V_1(\phi_\mathrm{c})|_{T=0} + \frac{1}{2\pi^2\beta^4}J_\mathrm{B}[m^2(\phi_\mathrm{c})\beta^2]. \tag{6.17}$$

At high temperatures, one can make some approximations and obtain a useful expression for the last term V_1^β in the effective potential (6.17) [228]:

$$\frac{1}{2\pi^2\beta^4}J_\mathrm{B}[m^2\beta^2] \simeq \frac{1}{24}\frac{m^2}{\beta^2} - \frac{1}{12\pi}\frac{m^3}{\beta} - \frac{1}{64\pi^2}m^4\ln\frac{m^2\beta^2}{a_b} + \mathcal{O}(m^6\beta^2) \tag{6.18}$$

where a_b is a constant ($\ln a_b = 5.4076$).

6.2.2 Phase transitions

One of the most relevant consequences of the finite temperature effects is the influence they have on phase transitions. The main point here is that at high temperature, the expectation value of the field, $\langle\phi\rangle$, which minimizes the potential, does not correspond to the minimum of the zero-temperature potential, $V_0(\phi)$, but to the minimum of the effective potential $V_{\rm eff}^{\beta}(\phi)$, given in (6.17).

If the minimum of the potential $V_0(\phi)$ occurs at a symmetry breaking value $\langle\phi\rangle = v \neq 0$, as for instance in the case of the potential (6.3) of the linear sigma model, at sufficiently high temperature the term V_1^{β} can be important and it can change the shape of the potential in such a way that the absolute minimum occurs at $\langle\phi(T)\rangle = 0$. This means that at high temperature, the $O(N)$ symmetry of the potential (6.3) is respected by the fundamental state. Generally, this phenomenon is known as *symmetry restoration* at high temperature and was discovered by Kirzhnits in the context of the electroweak theory [229]. Thus, when the temperature becomes less than a certain critical value, $T_{\rm c}$, a *phase transition* may occur from $\langle\phi(T)\rangle = 0$ to $\langle\phi\rangle = v$.

Symmetry restoration and phase transitions are very important in the context of cosmology. In the standard Big Bang cosmology, the universe is initially at very high temperatures, and one expects that the symmetries are not broken, due to temperature effects. This means that the symmetric phase $\langle\phi(T)\rangle = 0$ can be the stable absolute minimum. Then, when the temperature reaches the critical value $T_{\rm c}$, the minimum at $\phi = 0$ becomes metastable and the phase transition may proceed.

There are two types of phase transitions: first order and second order, and we will shortly describe them here.

Second-order phase transitions. This kind of phase transitions occurs, for example, in models of new inflation type [17, 18]. For a better understanding of this kind of phase transitions, we will give an example of a potential that leads to a second order phase transition:

$$V(\phi, T) = g(T^2 - T_0^2)\phi^2 + \frac{\lambda}{4}\phi^4 \tag{6.19}$$

where g and λ are some constants and T_0 is some temperature.

At zero temperature ($T = 0$), the potential has negative mass-squared term, $m^2 = -2gT_0^2$, and the absolute minimum of the potential corresponds to $\phi(0) = \pm\sqrt{2g/\lambda}\,T_0$. At finite temperature T, the curvature $V''(\phi, T) = \partial^2 V/\partial\phi^2$ of the potential depends on T,

$$V''(\phi, T) = 3\lambda\phi^2 + 2g(T^2 - T_0^2) \tag{6.20}$$

and the position of the minimum will also be $T-$dependent. At $T > T_0$, the curvature of

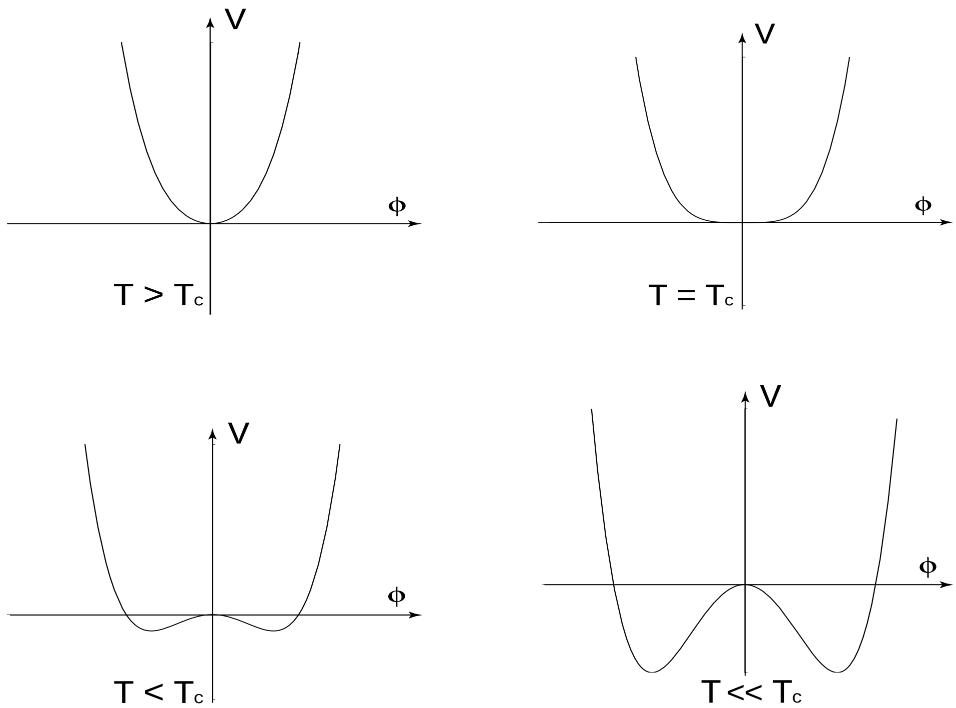

Figure 6.2: The shape of the potential (6.19) at different temperatures. The corresponding phase transition is of second order.

the potential is positive at any point, $V''(\phi,T) > 0$, and the minimum of the potential is located at the origin $\phi = 0$. At $T = T_0$, the potential becomes $V(\phi,T_0) = \lambda\phi^4/4$ and the origin $\phi = 0$ is still a minimum. When $T < T_0$, the curvature of the potential at $\phi = 0$ becomes negative, $V''(0,T) < 0$, and the origin becomes a maximum. Simultaneously, two minima appear at a $\phi \neq 0$:

$$\phi(T) = \pm\sqrt{\frac{2g(T_0^2 - T^2)}{\lambda}}. \tag{6.21}$$

This is an example of a second order phase transition, in which there is no barrier between the symmetric and the broken phases, and the symmetric phase $\phi = 0$ changes from a minimum to a maximum, when the temperature reaches the critical value $T_c = T_0$ (see Figure 6.2).

In chapters 8 and 9, I present two models which unify inflation with dark energy and dark matter, respectively, in which the effective potential suffers a second order phase transition before the beginning of inflation.

First-order phase transitions. In case there is a barrier between the symmetric and broken phases, the phase transition is said to be of first order, and in this case at the origin there is still a local minimum. A typical potential, which gives rise to a first-order phase transition is similar to (6.19), with an additional cubic term [230]:

$$V(\phi,T)=g(T^2-T_0^2)\phi^2-hT\phi^3+\frac{\lambda}{4}\phi^4 \tag{6.22}$$

where, as before, g, h, λ and T_0 are constants. The behavior of the potential (6.22) is somehow different than (6.19). At high temperature, the potential only has one minimum at the origin $\phi=0$. At some lower temperature, T_1, a local minimum at $\phi(T)\neq 0$ appears as an inflection point. At still lower temperatures, $T<T_1$, the new minimum becomes deeper and a barrier appears, between this minimum and the symmetric one at $\phi=0$. At the critical temperature T_c, the two minima become degenerate, and between them there is a local maximum at $\phi_\mathrm{M}=hT_\mathrm{c}/\lambda$. At $T<T_\mathrm{c}$ the minimum at $\phi=0$ becomes metastable and the other minimum at $\phi(T)\neq 0$ becomes the global one. In this example, the barrier disappears at a temperature $T=T_0$, and there are models in which T_0 can be equal to zero. Thus, a first-order phase transition may occur through tunneling from the false to the true minimum, in the temperature range $T_\mathrm{c}>T>T_0$ (see Figure 6.3).

The model presented in Chapter 10 contains a new complex scalar field ϕ, whose real and imaginary components are responsible for inflation and dark energy, respectively. In that model, the effective potential of the field ϕ suffers a first order phase transition before slow-roll inflation is produced.

6.2.3 Thermal tunneling

Thermal tunneling at finite temperature is the transition from the false to the true vacuum state, which implies the formation of bubbles of the true vacuum phase in the sea of the symmetric phase.

Let us consider the previous example of the potential given in (6.22). The tunneling probability per unit time per unit volume is given by [231]:

$$\frac{\Gamma}{\mathcal{V}}\sim A(T)\mathrm{e}^{-S_3/T} \tag{6.23}$$

where the prefactor $A(T)$ is roughly of order T^4, and S_3 is the three-dimensional Euclidean action, defined as:

$$S_3=\int \mathrm{d}^3x\left[\frac{1}{2}(\nabla\phi)^2+V(\phi,T)\right]. \tag{6.24}$$

Let us consider a bubble of the true vacuum, with spherical symmetry, and radius R.

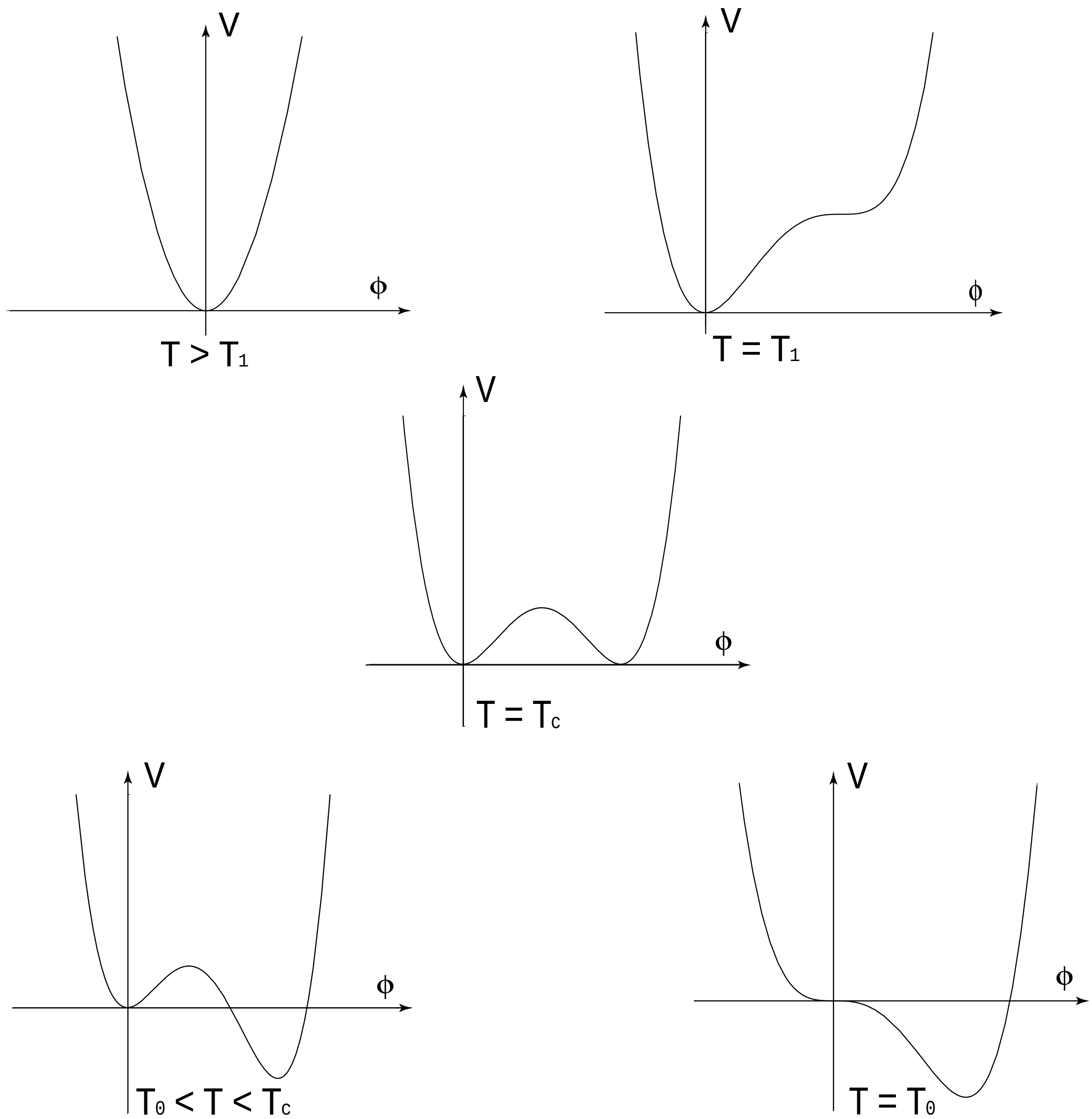

Figure 6.3: The shape of the potential (6.22) at different temperatures. At high temperatures, the potential only has one minimum at $\phi = 0$, while at temperatures in the range $T_0 < T < T_c$ it has two minima separated by a barrier. This is a characteristic of first-order phase transitions.

Then, the Euclidean action becomes [230]:

$$S_3 = 4\pi \int_0^R r^2 \mathrm{d}r \left[\frac{1}{2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}r} \right)^2 + V(\phi(r), T) \right] \tag{6.25}$$

where $r^2 = \vec{x}^2$. There are two contributions to the action (6.25): a surface term F_{S} coming from the first term in (6.25), and a volume term F_{V}, coming from the second term. They scale like:

$$S_3 \sim 2\pi R^2 \left(\frac{\delta\phi}{\delta R} \right)^2 \delta R + 4\pi R^3 \langle V \rangle \tag{6.26}$$

where δR is the thickness of the bubble wall, $\delta\phi$ is the value of the field at the minimum and $\langle V \rangle$ is the average of the potential inside the bubble.

When the hight of the barrier is large compared to the depth of the potential at the minimum, the solution of the minimal action corresponds to a very small bubble wall $\delta R/R \ll 1$, which means that we have a *thin wall* bubble.

When the temperature drops towards T_0, the hight of the barrier is small and the minimal action corresponds to a configuration where $\delta R/R \sim \mathcal{O}(1)$. In this case, we have a *thick wall* bubble.

In order for the bubble nucleation to be possible, the rate of bubble production should be larger than the expansion rate of the universe, otherwise the state will be trapped in the supercooled false vacuum.

In Chapter 10, I will present a model in which the dark energy field is trapped in a false vacuum, and has a very small probability of tunneling to the true vacuum.

6.2.4 Topological defects

In the process of SSB, there are other possible effects, like the production of *topological defects* [232], which are high concentrations of energy resulting from a nontrivial vacuum topology.

For instance, in the case of the potential illustrated in Figure 6.2, with a Z_2 symmetry $\phi \to -\phi$, the vacuum expectation value of the field ϕ can be $\langle\phi\rangle = +v$, as well as $\langle\phi\rangle = -v$. Thus, if there is no reason that the field can only take one of the two values, there are regions in which $\langle\phi\rangle$ takes one value and regions where $\langle\phi\rangle$ takes the other value, and in between these distinct regions the field must change continuously. This means that there must exist regions where $\phi = 0$, i.e., of false vacuum, in which a large amount of energy is concentrated. These regions, called *domain walls*, are two-dimensional, have a certain thickness, and appear every time a discrete symmetry is broken.

Other examples of topological defects are the cosmic strings, the magnetic monopoles and the textures. Cosmic strings may appear, for instance, in the case of a complex field $\psi = \phi e^{i\theta}$, where the vacuum expectation value of the modulus ϕ is fixed, $\langle\phi\rangle = v$, but the phase is arbitrary. Then, in the SSB process, there will appear one-dimensional regions in space where $\phi = 0$, which concentrate a large amount of energy. The cosmic strings must be infinite in extension or closed.

The magnetic monopoles are the zero-dimension analogous of cosmic strings and domain walls, and appear in the breaking of spherical symmetries. Cosmic textures are more complicated objects of higher dimensions.

Although topological defects may not probably be produced in terrestrial accelerators, their existence is predicted by the theories of phase transitions in the early universe [233]. Their production mechanism was first investigated by Kibble [234] and predicts that they are topologically stable. This fact may have important consequences, because their detection would be a probe of phase transitions in the early universe. Also, if there are too many domain walls or monopoles produced in the early universe, they may distort the CMB or may perturb the formation of structure, or their density may overclose the universe. Any cosmological model should then take into account all these problems related to the overproduction of topological defects.

An elegant solution to the topological defects problem is provided by the inflation hypothesis, described in Chapter 3, whose effect is to dilute them away.

Chapter 7

Action-angle formalism

7.1 Introduction

In this chapter, I will present a different method that can be used to obtain the same results of Section 5.4, referring to the law of variation of the energy density of a scalar field ϕ with respect to the scale factor a, and of a with respect to the time t, in the case that the coherent field oscillations dominate the energy density of the universe. This alternative method is based on an analytical mechanics formulation, by using the theory of canonical systems and action-angle variables, due to the similarities that exist between the field oscillations and the systems that are characterized by periodic motions [235].

Of special importance in many branches of physics are the systems in which motion is periodic. We are very often interested not so much in the details of the orbit as in the frequencies of the motion. A very elegant and powerful method of handling such systems is provided by a variation of the Hamilton-Jacobi procedure. In this technique the integration constants α_i appearing directly in the solution of the Hamilton-Jacobi equation are not themselves chosen to be the new momenta. Instead we use suitably defined constants J_i, which form a set of n independent functions of the α_i's, and which are known as the *action variables*. Here we will apply the general theory to our particular case, in which we only have one degree of freedom, $\alpha_1 = \rho$, the total energy density.

As discussed in Section 5.4, the scalar field ϕ has a general potential energy function, $V(\phi)$, which has an important role in the evolution of ϕ in the expanding universe. The equation that describes the evolution of ϕ is the equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{7.1}$$

Let us recall the definitions of the energy density of ϕ, $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) = T + V$ and of the pressure $p = \frac{1}{2}\dot{\phi}^2 - V(\phi) = T - V$, where T denotes the kinetic energy term and V

is the potential one. One can distinguish between two cases: the first one is to consider a static universe, with $H = 0$, and the second one is that of an expanding universe, with an expansion rate $H \neq 0$.

7.2 The static universe case $H = 0$

Let us consider first the simple case of scalar field oscillations in a static universe with $H = 0$. This is the analogue of the one-dimensional oscillator with no friction term. In this case the energy of the system is conserved, i.e., it does not depend on time t, and it is equal to the Hamilton function $\mathcal{H}(\phi, \Pi)$:

$$\mathcal{H}(\phi, \Pi) = \frac{1}{2}\Pi^2 + V(\phi) = \rho \tag{7.2}$$

with the momentum variable $\Pi = \dot{\phi}$. Solving for the momentum Π, we have that:

$$\Pi = \Pi(\phi, \rho), \tag{7.3}$$

which can be looked on as the equation of the orbit traced by the system point in the two-dimensional phase space, when the Hamiltonian has the constant value ρ. The meaning of the term "periodic motion" is determined by the characteristics of the phase space orbit. Two types of periodic motions may be distinguished:

1. In the first type the orbit is closed and the system point retraces its steps periodically. Both ϕ and Π are then periodic functions of time with the same frequency. Periodic motions of this nature will be found when the initial position lies between two zeros of the kinetic energy. It is often designated by the astronomical name *libration*, although to a physicist it is more likely to call to mind the common *oscillatory* systems, such as the one-dimensional harmonic oscillator.

2. In the second type of periodic motions the orbit in phase space is such that Π is some periodic function of ϕ. The most familiar example is that of a rigid body constrained to rotate about a given axis, with ϕ as the rotation angle. Increasing ϕ by 2π then produces no essential change in the state of the system. The position coordinate in this type of periodicity is invariably an angle of rotation, and the motion will be referred to simply as *rotation*.

Our case enters in the first type of periodic motions; the system will have a periodic *oscillatory* motion, and the oscillations will have constant amplitude because the total energy is conserved.

It is convenient to make use of the *action variable* to further describe the system:

$$J = \oint \Pi \, \mathrm{d}\phi = -\oint \phi \, \mathrm{d}\Pi \tag{7.4}$$

where the integration is made over a complete period of oscillation. From (7.3) and (7.4) one obtains that J is a function of ρ alone, or inverting, one can write the energy in terms of the new variable:

$$\rho = \rho(J). \tag{7.5}$$

The action variable J is chosen as the new momentum in the integration of the Hamilton-Jacobi equation. Because the Hamiltonian $\mathcal{H}(\phi, \Pi)$ does not depend on time, one can write the *time-independent Hamilton-Jacobi equation* as:

$$\mathcal{H}(\phi, \frac{\partial W}{\partial \phi}) = \rho \tag{7.6}$$

where $W(\phi, J)$ is the Hamilton's *characteristic function*. $W(\phi, J)$ is a generating function and defines a canonical transformation from old to new canonical coordinates, $(\phi, \Pi) \rightarrow (\alpha, J)$, where α is the variable canonical conjugate to J and is called *angle variable*. Thus, we can write the canonical transformations in terms of the new variables:

$$J = -\frac{\partial W}{\partial \alpha}, \quad \alpha = \frac{\partial W}{\partial J}. \tag{7.7}$$

Since W does not explicitly depend on time, the new hamiltonian $\bar{\mathcal{H}}$ coincides with the old one, and we have that $\bar{\mathcal{H}}(\alpha, J) = \mathcal{H}(\phi, \Pi) = \rho$. The energy ρ is only a function of J, which amounts to say that α is cyclic and J is constant,

$$\dot{J} = -\frac{\partial \bar{\mathcal{H}}}{\partial \alpha} = 0. \tag{7.8}$$

The other Hamilton equation is:

$$\dot{\alpha} = \frac{\partial \bar{\mathcal{H}}(J)}{\partial J} \equiv \nu, \tag{7.9}$$

where ν is a constant function only of J. We can integrate (7.9) to obtain:

$$\alpha(t) = \nu t + \alpha(0). \tag{7.10}$$

Let us see the physical interpretation of the new defined variable ν. Consider the change in α as ϕ goes through a complete cycle of oscillation, as given by:

$$\Delta\alpha = \oint \frac{\partial \alpha}{\partial \phi} \, \mathrm{d}\phi. \tag{7.11}$$

By using the second canonical equation (7.7) for α, the above equation becomes:

$$\Delta\alpha = \oint \frac{\partial^2 W}{\partial\phi\partial J}\,\mathrm{d}\phi. \tag{7.12}$$

Because J is a constant, the derivative with respect to J can be taken outside the integral sign:

$$\Delta\alpha = \frac{\mathrm{d}}{\mathrm{d}J}\oint \frac{\partial W}{\partial\phi}\,\mathrm{d}\phi = \frac{\mathrm{d}}{\mathrm{d}J}\oint \Pi\,\mathrm{d}\phi = 1, \tag{7.13}$$

where use has been made of the definition of J, equation (7.4), and of the equation $\Pi = \partial W/\partial\phi$.

Equation (7.13) states that α changes by 1 as ϕ goes through a complete period. But from equation (7.10) it follows that if τ is the period of a complete cycle of ϕ, then

$$\Delta\alpha = 1 = \nu\tau. \tag{7.14}$$

Hence, the constant ν can be identified with the reciprocal of the period,

$$\nu = \frac{1}{\tau} \tag{7.15}$$

and is therefore the usual frequency associated with the periodic motion of ϕ.

The use of action-angle variables thus, provides a powerful technique for obtaining the frequency of periodic motion without the need to find a complete solution to the motion of the system. If it is known *a priori* that the motion of a system with one degree of freedom is periodic, according to the definitions given above, the frequency can be found once $\bar{\mathcal{H}}$ is determined as a function of J. The derivative of $\bar{\mathcal{H}}$ with respect to J, by equation (7.9), then directly gives the frequency ν of the motion. We should also remark that J has dimensions of an angular momentum, and of course the coordinate conjugate to an angular momentum is an angle.

7.3 The expanding universe case $H \neq 0$

The more realistic case for describing the scalar field evolution corresponds to an expanding universe. In this case, equation (7.1) describes a one-dimensional oscillator with time-dependent friction that depends on the velocity $\dot{\phi}$. Nevertheless, in the present discussion we will assume that H is small and it also has a small relative change with time:

$$H \ll \omega\,; \quad \dot{H}/H \ll \omega. \tag{7.16}$$

This ensures that the motion is almost periodic, and it makes sense averaging over one cycle. Thus, we can still use the definition in equation (7.4) for J,

$$J = \oint \Pi \, \mathrm{d}\phi. \tag{7.17}$$

With $H \neq 0$, the total energy is not conserved but decreases with time, because the term that contains H acts like a dissipative force. Nevertheless, due to the smallness of H, one may assume that over a cycle the loss in the total energy is small so that ρ can be considered constant. We still define ρ as in the case $H = 0$,

$$\rho = T + V = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{7.18}$$

Thus, the equation of motion (7.1) can be written in the following form:

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H\frac{\dot{\phi}^2}{\rho} \tag{7.19}$$

which gives the relative change of ρ with time.

We can average $\dot{\phi}^2$ over one cycle,

$$\langle \dot{\phi}^2 \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t \, \dot{\phi}^2 \tag{7.20}$$

where τ is the period of a complete oscillation. In doing this, we see that the right hand side of equation (7.19) strongly depends on the shape of the potential.

We will now show that the relative change of J does not depend on the potential V. In order to do this, we have to make use of the virial theorem, which states that the time average of the total kinetic energy is equal to minus half the time average of the total potential energy. One can write this theorem in the form:

$$\langle \dot{\phi}^2 \rangle = \langle \phi \frac{\partial V}{\partial \phi} \rangle. \tag{7.21}$$

We can now proceed to calculate the time average

$$\begin{aligned} \langle \phi \frac{\partial V}{\partial \phi} \rangle &= \frac{1}{\tau}\int_0^\tau \mathrm{d}t \, \phi \frac{\partial V}{\partial \phi} \\ &= -\frac{1}{\tau}\int_0^\tau \mathrm{d}t \, \phi \, \dot{\Pi} \qquad (7.22) \\ &= -\frac{1}{\tau}\oint \phi \, \mathrm{d}\Pi = \frac{1}{\tau}J \qquad (7.23) \end{aligned}$$

where we have used the canonical equation for $\dot{\Pi}$ and the definition (7.4).

Thus, replacing $1/\tau = \nu$ and using the definition (7.9) for ν, we obtain that the virial theorem reads:

$$\langle \dot{\phi}^2 \rangle = \frac{\mathrm{d}\rho}{\mathrm{d}J} J \tag{7.24}$$

which can be introduced in equation (7.19) to obtain the relative change in J

$$\frac{1}{J}\frac{\mathrm{d}J}{\mathrm{d}t} = -3H \tag{7.25}$$

which demonstrates our previous statement that the relative change of J is V-independent. Solving this equation for J, and recalling that $H = \dot{a}/a$, we obtain that in the expanding universe

$$J \sim a^{-3}. \tag{7.26}$$

We would like to find a quantity that is conserved by the expansion. The solution comes immediately, since we know that $s \sim n_\gamma \sim a^{-3}$, where s is the entropy density and n_γ is the photon density of the universe. We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{J}{s}\right) = 0 \tag{7.27}$$

and so J/s is conserved by the expansion.

The formalism described above can be used to derive the effective equation of state parameter w of a scalar field ϕ,

$$w = \frac{p}{\rho} = \frac{\langle T \rangle - \langle V \rangle}{\rho} = \frac{2\langle T \rangle}{\rho} - 1 \tag{7.28}$$

or, using equation (7.24), we get:

$$w = \frac{J}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}J} - 1. \tag{7.29}$$

This equation allows us to calculate w without entering into details of motion. Once we have a general potential V for a scalar field, we calculate the energy density ρ of this field, define J as in equation (7.4), which is only a function of ρ, and then obtain w from equation (7.29).

We can also use the parameter γ, defined in equation (5.64) of Section 5.4, as being the average over one cycle of $\dot{\phi}^2/\rho$. In terms of the new variable J, this can be expressed as:

$$\gamma = \frac{J}{\rho}\frac{1}{\mathrm{d}J/\mathrm{d}\rho} = w + 1. \tag{7.30}$$

If w results to be constant for an arbitrary potential V, then equation (7.25) can be integrated to give:

$$\frac{1}{J}\frac{\mathrm{d}J}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{\gamma\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H \Longrightarrow \rho = \rho_0 (a/a_0)^{-3\gamma} \tag{7.31}$$

which is the same result obtained in (5.65). Again, considering that ρ dominates the total energy density of the universe, in the limit of zero spatial curvature of the universe, from the Friedmann equation (2.37) we obtain the same result as in (5.66), $a \propto t^{2/3\gamma}$, with γ given by (7.30).

7.4 Discussions

I have presented in this chapter an alternative language that can be applied to derive the same results obtained in Section 5.4, for a scalar field oscillating in a potential. I would like now to highlight one of the advantages of this new language, which has to do with a possible extension of the problem under consideration.

Let us suppose that for the physical system, which our methods are applied to, the Hamilton function depends explicitly on a parameter λ. This parameter can be either internal − and characterizes the properties of the system − or external, and in this case it characterizes the external field in which the system is found. We also suppose that for constant λ, the problem of motion of the system is solved by using the action-angle variables. In the following discussion we focus on the simple case $H = 0$. The question is: what would happen if λ was not constant, but it had a slow variation with time? In this case, the system is not conservative anymore and the total energy is not conserved. What about the action variable J? Would it still be conserved?

To answer this question, we have to enter into details and introduce another concept of analytical mechanics. If λ varies slowly with time,

$$\dot{\lambda}\tau \ll \lambda, \tag{7.32}$$

where τ is the time interval in which λ varies by $\Delta\lambda$, then the action variable J remains, practically, constant. A quantity that remains constant for a slow change of the parameter λ is called *adiabatic invariant*, and it is said that λ changes adiabatically.

The concept of adiabatic invariant is useful in many areas of physics and has been applied in the Bohr-Sommerfeld-Wilson quantization rules and in plasma physics, in the study of thermonuclear processes, charged particles accelerators, etc.

Let us demonstrate that J remains constant when λ changes adiabatically. Considering the parameter λ, equation (7.2) becomes:

$$\mathcal{H}(\phi, \Pi, \lambda) = \rho \tag{7.33}$$

and from here we can write:

$$\Pi = \Pi(\phi, \rho, \lambda). \tag{7.34}$$

Introducing equation (7.34) into the definition of J, equation (7.4), it results that J will be a function of time through the time-dependence of λ. In order to see whether J varies with time, one has to calculate its derivative with respect to time

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \oint \left(\frac{\partial \Pi}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial \Pi}{\partial \lambda} \dot{\lambda} \right) \mathrm{d}\phi \tag{7.35}$$

and then evaluate the average of $\mathrm{d}J/\mathrm{d}t$ over a complete period of motion, τ.

By taking the derivative of (7.33) with respect to time and using the canonical equations for Π and ϕ, one gets:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial \lambda} \dot{\lambda} \tag{7.36}$$

which, by averaging the first and the last sides of the equality, gives:

$$\langle \frac{\mathrm{d}\rho}{\mathrm{d}t} \rangle = \dot{\lambda} \langle \frac{\partial \mathcal{H}}{\partial \lambda} \rangle = \frac{1}{\tau} \dot{\lambda} \int_0^\tau \frac{\partial \mathcal{H}}{\partial \lambda} \mathrm{d}t. \tag{7.37}$$

In the above equation, we supposed that in the period τ, $\dot{\lambda}$ is practically constant. One can now change from the integration with respect to time, to integration with respect to ϕ, using the canonical equation $\dot{\phi} = \partial \mathcal{H}/\partial \Pi$, and write:

$$\langle \frac{\mathrm{d}\rho}{\mathrm{d}t} \rangle = \dot{\lambda} \frac{\oint \frac{\partial \mathcal{H}}{\partial \lambda} \left(\frac{\partial \mathcal{H}}{\partial \Pi} \right)^{-1} \mathrm{d}\phi}{\oint \left(\frac{\partial \mathcal{H}}{\partial \Pi} \right)^{-1} \mathrm{d}\phi}. \tag{7.38}$$

From equations (7.33) and (7.34) one can get:

$$\frac{\partial \mathcal{H}}{\partial \lambda} \left(\frac{\partial \mathcal{H}}{\partial \Pi} \right)^{-1} = -\frac{\partial \Pi}{\partial \lambda} \tag{7.39}$$

which can be put into (7.38) to obtain:

$$\langle \frac{\mathrm{d}\rho}{\mathrm{d}t} \rangle = -\dot{\lambda} \frac{\oint \frac{\partial \Pi}{\partial \lambda} \mathrm{d}\phi}{\oint \frac{\partial \Pi}{\partial \rho} \mathrm{d}\phi} \tag{7.40}$$

where we used the relation $(\partial \mathcal{H}/\partial \Pi)^{-1} = \partial \Pi/\partial \rho$.

If we take into account that in a complete cycle ϕ, $\dot{\lambda}$ and $\langle \dot{\rho} \rangle$ are constant, the last equation can be written down as follows:

$$\oint \left(\frac{\partial \Pi}{\partial \rho} \langle \dot{\rho} \rangle + \frac{\partial \Pi}{\partial \lambda} \dot{\lambda} \right) \mathrm{d}\phi = 0 \tag{7.41}$$

which, compared to equation (7.35), gives:

$$\langle \frac{\mathrm{d}J}{\mathrm{d}t} \rangle = 0. \tag{7.42}$$

This completes the demonstration that, in the approximations considered here ($H = 0$, $\dot{\lambda} =$ constant), the action variable J remains constant when λ changes adiabatically and thus, J is an adiabatic invariant.

In Chapter 11 of this PhD Thesis I apply the formalism described in this chapter to a scalar field ϕ oscillating in a potential, and I show that one can find potentials such that the resulting oscillations may lead to a sufficiently negative equation of state parameter as to describe dark energy.

Chapter 8

Unified Model for Inflation and Dark Energy with Planck-Scale Pseudo-Goldstone Bosons

In collaboration with *E. Massó*.
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8.1 Introduction

In spite of the fact that the standard model of elementary particles based on the gauge group $SU(3) \times SU(2) \times U(1)$ is able to accommodate all existing empirical data, few people believe that it is the ultimate theory. The reason is that the standard model leaves unanswered many deep questions. In any case, evidence (or disproval) of this belief can only be given by experiment. If we are able to discover a theory that indeed goes beyond the standard model, it will probably contain new symmetries. The global symmetries valid at high energies are expected to be only approximate, since Planck-scale physics breaks them explicitly [223, 224, 236, 237]. Even if the effect is probably extremely small, it may lead to very interesting effects. As has been discussed in [116], when a global symmetry is spontaneously broken and we have such a small explicit breaking the corresponding pseudo-Golstone boson (PGB) can have a role in cosmology. The focus in [116] was to show that the PGB could be a dark matter constituent candidate.

In the present paper we will rather be concerned with the periods of acceleration in the universe, namely with inflation in the very early universe and with dark energy dominance in the late stages in the evolution of the universe. Recent observational evidence for these two periods come mainly from the use of Supernovae as standard candles [6, 7], cosmic microwave background anisotropies [80, 82], [238]-[243], galaxy counts [244]-[247] and others [248]. Of course, the physics behind the two periods that are so distant in time may be completely unrelated. However, an appealing possibility is that they have a common origin. An idea for this kind of unification has been forwarded by Frieman and Rosenfeld [207]. Their framework is an axion field model where we have a global $U(1)_{\rm PQ}$ symmetry spontaneously broken at a high scale and explicitly broken by instanton effects at the low energy QCD scale. The real part of the field is able to inflate in the early universe while the axion boson could be the responsible for the dark energy period. The authors of that work [207] compare their model of quintessential inflation with other models of inflation and/or dark energy. We would like to show here that our model of Planck-scale broken symmetry offers an explicit scenario of a quintessential inflation.

In our model, we have a complex field Ψ that is charged under a certain global $U(1)$ symmetry, and in the potential we have the following $U(1)$-symmetric term

$$V_1(\Psi) = \frac{1}{4}\lambda[|\Psi|^2 - v^2]^2 \tag{8.1}$$

where λ is a coupling and v is the energy scale of the spontaneous symmetry breaking (SSB).

We do not need to know the details of how Planck-scale physics breaks our $U(1)$. It is enough to introduce the most simple effective $U(1)$-breaking term

$$V_{non-sym}(\Psi) = -g\frac{1}{M_P^{n-3}}|\Psi|^n\left(\Psi e^{-i\delta} + \Psi^\star e^{i\delta}\right) \tag{8.2}$$

($n > 3$). Here, $M_P^2 \equiv G_N^{-1}$, and the coupling g is expected to be very small [225]. The term vanishes when $M_P \to \infty$, as it should be. Previous work on explicit breaking of global symmetries can also be found in [249, 250], and related to Planck-scale breaking, in [251]-[253]. Cosmological consequences of some classes of PGBs are discussed in [201],[254]-[256].

Let us write the field as

$$\Psi = \phi\, e^{i\theta/v} \tag{8.3}$$

We envisage a model where ϕ, the real part of Ψ, is the inflaton and the PGB θ, the imaginary part of Ψ, is a quintessence field. In the proces of SSB at temperatures $T \sim v$ in the early universe, the scalar field ϕ develops in time, starting from $\phi = 0$ and going to values different from zero. A suitable model we will employ is the inverted hybrid inflation [74, 75], where one has a new real field χ that assists ϕ to inflate.

The new scalar field is massive and neutral under $U(1)$. We shall follow ref.[75] and couple χ to Ψ with a $-\Psi^*\Psi\chi^2$ term. More specifically we introduce the following contribution to the potential

$$V_2(\Psi,\chi) = \frac{1}{2}m_\chi^2\chi^2 + \left(\Lambda^2 - \frac{\alpha^2|\Psi|^2\chi^2}{4\Lambda^2}\right)^2 \tag{8.4}$$

Here α is a coupling and Λ and m_χ are mass scales. The interaction between the two fields will give the needed behavior of the real part of Ψ to give inflation. Also, we should mention that such models of inflation are realized in supersymmetry, using a globally supersymmetric scalar potential [75].

To summarize, our model has a complex field Ψ and a real field χ with a total potential

$$V(\Psi,\chi) = V_{sym}(\Psi,\chi) + V_{non-sym}(\Psi) + C \tag{8.5}$$

where C is a constant that sets the minimum of the effective potential at zero. The non-symmetric part is given by (8.2), whereas the symmetric part is the sum of (8.1) and (8.4),

$$V_{sym}(\Psi,\chi) = V_1(\Psi) + V_2(\Psi,\chi) \tag{8.6}$$

8.2 The model

In this section, we will explain in detail the model introduced in Section 8.1. Our basic idea is that the radial part ϕ of the complex scalar field Ψ is responsible for inflation, whereas the angular part θ plays the role of the present dominating dark energy of the universe. From now on, we will replace Ψ by its expression given in (8.3). The symmetric part of the potential is given by

$$V_{sym}(\phi,\chi) = \frac{1}{4}\lambda[\phi^2 - v^2]^2 + \frac{1}{2}m_\chi^2\chi^2 + \left(\Lambda^2 - \frac{\alpha^2\phi^2\chi^2}{4\Lambda^2}\right)^2 \tag{8.7}$$

while the symmetry-breaking term is

$$V_{non-sym}(\phi,\theta) = -2\,g\frac{\phi^{n+1}}{M_P^{n-3}}\cos\frac{\theta}{v} \tag{8.8}$$

where the following change of variables $\theta/v \longrightarrow \theta/v + \delta$ has been made.

8.2.1 Inflation

Let us study, firstly, the conditions to be imposed on our model to describe the inflationary stage of expansion of the primordial Universe. In order to do this, we will only work with the symmetric part of the effective potential,

$$V_{sym}(\phi,\chi) = \Lambda^4 + \frac{1}{2}\left(m_\chi^2 - \alpha^2\phi^2\right)\chi^2 + \frac{\alpha^4\phi^4\chi^4}{16\Lambda^4} + \frac{1}{4}\lambda(\phi^2 - v^2)^2, \tag{8.9}$$

which dominates over the non-symmetric one at early times. Here, ϕ is the inflaton field and χ is the field that plays the role of an auxiliary field, which is needed to have a sudden end of the inflationary regime, through a "waterfall" mechanism. This is important because in this way we can arrange for the right number of e-folds of inflation and for the right value of the spectral index of density perturbations produced during inflation, when the cosmological scales left the horizon. We also note that the $\phi^4\chi^4$ term in Eq.(8.9) does not play an important role during inflation, but only after it, and it sets the position of the global minimum of $V_{sym}(\phi,\chi)$, which will roughly be at $\phi \sim v$ and $\chi \sim M$, with $M = \frac{2\Lambda^2}{\alpha v}$.

The effective mass squared of the field χ is:

$$M_\chi^2 = m_\chi^2 - \alpha^2\phi^2 \tag{8.10}$$

so that for $\phi < \phi_c = \frac{m_\chi}{\alpha}$, the only minimum of $V_{sym}(\phi,\chi)$ is at $\chi = 0$. The curvature of the effective potential in the χ direction is positive, while in the ϕ direction is negative. This will lead to rollover of ϕ, while χ will stay at its minimum $\chi = 0$ until the curvature in χ direction changes sign. That happens when $\phi > \phi_c$ and χ becomes unstable and starts to roll down its potential. The mechanism and the conditions to be imposed on our model are similar to the original hybrid inflation model by Linde [67], so we will follow the same line of discussion. The main difference between the original model and our case consists in the fact that here the inflaton rollover is due to its negative squared mass $m_\phi^2 = -\lambda v^2$ and it starts moving from the origin $\phi = 0$ towards the minimum $\langle\phi\rangle \sim v \leq M_P$, so that there is no need to go to values for ϕ larger then the Planck scale. The fact that, initially, the inflaton field is placed at the origin is justified because in the very hot primordial plasma the symmetry of the effective potential is restored and the minimum of the potential is at $\phi = 0$. So we expect that, after the SSB, the radial field ϕ is set at the origin of the effective potential. However, due to quantum fluctuations, the field may be displaced from $\phi = 0$, such that it is unstable and may roll down the potential.

As is characteristic for hybrid models of inflation [69, 73], the dominant term in (8.9) is Λ^4. This is equivalent to writing:

$$\frac{1}{4}\lambda v^4 < \Lambda^4. \tag{8.11}$$

Another requirement is that the absolute mass squared of the inflaton be much less than the χ-mass squared,

$$|m_\phi^2| = \lambda v^2 \ll m_\chi^2, \tag{8.12}$$

which fixes the initial conditions for the fields: χ is initially constrained at the stable minimum $\chi = 0$, and ϕ may slowly roll from its initial position $\phi \simeq 0$.

Taking into account condition (8.11), the Hubble parameter at the time of the phase transition is given by:

$$H^2 = \frac{8\pi}{3M_P^2} V_{sym}(\phi_c, 0) \simeq \frac{8\pi}{3M_P^2}\Lambda^4. \tag{8.13}$$

We want ϕ to give sufficient inflation, that is, the potential $V_{sym}(\phi, 0)$ must fulfill the slow-roll conditions in ϕ direction, given by the two slow-roll parameters:

$$\epsilon \equiv \frac{M_P^2}{16\pi}\left(\frac{V'_{sym}}{V_{sym}}\right)^2 \ll 1, \tag{8.14}$$

$$|\eta| \equiv \left|\frac{M_P^2}{8\pi}\frac{V''_{sym}}{V_{sym}}\right| \ll 1 \tag{8.15}$$

where a prime means derivative with respect to ϕ. The first slow-roll condition Eq.(8.14) gives

$$\Lambda^4 \gg \frac{\lambda}{4\sqrt{\pi}} M_P v^3 \tag{8.16}$$

and the second slow-roll condition, Eq.(8.15), gives

$$\Lambda^4 \gg \frac{\lambda}{8\pi} M_P^2 v^2. \tag{8.17}$$

So, under these conditions, the universe undergoes a stage of inflation at values of $\phi < \phi_c$. In order to calculate the number of e-folds produced during inflation, we use the following equation [69]

$$N(\phi) = \int_t^{t_{end}} H(t)dt = \frac{8\pi}{M_P^2} \int_{\phi_{end}}^{\phi} \frac{V_{sym}}{V'_{sym}} d\phi \tag{8.18}$$

where $\phi_{end} \equiv \phi(t_{end}) = \phi_c$ marks the end of slow-roll inflation, and prime means derivative with respect to ϕ.

Let us study the behavior of the fields ϕ and χ just after the moment when the field $\phi = \phi_c$ for a period $\Delta t = H^{-1} = \sqrt{\frac{3}{8\pi}} \frac{M_P}{\Lambda^2}$. The equation of motion of the inflaton field, in the slow-roll approximation, is

$$3H\dot{\phi} + \frac{\partial V_{sym}(\phi, 0)}{\partial \phi} = 0. \tag{8.19}$$

In the time interval $\Delta t = H^{-1}$, the field ϕ increases from ϕ_c to $\phi_c + \Delta\phi$. If we suppose that ϕ_c takes an intermediate value between 0 and v, we can calculate $\Delta\phi$ using (8.19)

$$3H \frac{\Delta\phi}{\Delta t} \simeq \frac{3}{8}\lambda v^3 \tag{8.20}$$

where, for definiteness, we set $\phi_c \simeq v/2$. We finally get

$$\Delta\phi \simeq \frac{3}{64\pi} \frac{\lambda v^3 M_P^3}{\Lambda^4}. \tag{8.21}$$

The variation of M_χ^2 in this time interval is given by

$$\Delta M_\chi^2 \simeq -\frac{3}{64\pi} \frac{\lambda \alpha^2 v^4 M_P^2}{\Lambda^4}. \tag{8.22}$$

The field χ will roll down towards its minimum χ_{min} much faster than ϕ, if $|\Delta M_\chi^2| \gg H^2$. Taking into account Eqs.(8.13) and (8.22), this condition is equivalent to

$$\Lambda^4 \ll \frac{1}{16} \sqrt{\frac{\lambda}{2}} \alpha v^2 M_P^2. \tag{8.23}$$

In this time interval, χ rolls down to its minimum, oscillates around it with decreasing amplitude due to the expansion of the Universe, and finally stops at the minimum.

Once the auxiliary field χ arrives and settles down at the minimum, the inflaton field ϕ can roll down towards the absolute minimum of the potential, much faster than in the case when $\phi < \phi_c$, because the potential has a non-vanishing first derivative at that point

$$\frac{\partial V_{sym}}{\partial \phi} = \lambda\phi(\phi^2 - v^2) - \alpha^2 \chi_{min}^2 \phi \tag{8.24}$$

which we want to be large in order to assure that no significant number of e-folds is produced during this part of the field evolution. The requirement of fast-rolling of ϕ is translated into the following condition

$$v \leq M_P \tag{8.25}$$

(this was obtained considering the equation of motion of an harmonic oscillator with small friction term $3H\dot{\phi}$, and imposing the condition that the frequency $\omega^2 \geq H^2$).

8.2.2 Dark Energy

Let us now focus on the angular part θ of the complex scalar field Ψ, which we neglected when discussed about inflation. We want the PGB θ to be the field responsible for the present acceleration of the universe. For this to happen, we have to impose two conditions on our model: (i) the field θ must be stuck at an arbitrary initial value after the SSB of V, which we suppose is of order v, and will only start to fall towards its minimum in the future; (ii) the energy density of the θ field, ρ_0, must be comparable with the present critical density ρ_{c_0}, if we want θ to explain all of the dark energy content of our Universe. Conditions (i) and (ii) may be written as

$$m_\theta \leq 3H_0 \tag{8.26}$$

$$\rho_\theta \sim \rho_{c_0}. \tag{8.27}$$

where H_0 is the Hubble constant. Taking into account the expression for the mass of θ derived in [116], $m_\theta = \sqrt{2g}\left(\frac{v}{M_P}\right)^{\frac{n-1}{2}} M_P$, condition (8.26) becomes

$$g\left(\frac{v}{M_P}\right)^{n-1} \leq \frac{9H_0^2}{2M_P^2}. \tag{8.28}$$

The energy density of the θ field is given by the value of the non-symmetric part of the effective potential, $V_{non-sym}(\phi, \theta)$, with the assumption that the present values of both fields are of order v

$$\rho_\theta \simeq V_{non-sym}(v, v) = g\left(\frac{v}{M_P}\right)^{n-1} M_P^2 v^2. \tag{8.29}$$

Introducing (8.29) into (8.27) and remembering that the present critical energy density $\rho_{c_0} = \frac{3H_0^2 M_P^2}{8\pi}$, we have that

$$g\left(\frac{v}{M_P}\right)^{n-1} \simeq \frac{3H_0^2}{8\pi v^2}. \tag{8.30}$$

Combining (8.28) and (8.30) we get

$$\frac{3H_0^2}{8\pi v^2} \leq \frac{9H_0^2}{2M_P^2} \tag{8.31}$$

which finally gives

$$v \geq \frac{1}{6}M_P. \tag{8.32}$$

This is the restriction to be imposed on v in order for θ to be the field describing dark energy. Notice that it is independent of n. It is also interesting to obtain the restriction on the coupling g, which can be done if we introduce (8.32) into (8.30) giving

$$g \leq \frac{3 \times 6^{n+1}}{8\pi} \frac{H_0^2}{M_P^2}. \tag{8.33}$$

Replacing the value for $H_0 \sim 10^{-42}$ GeV and taking the smallest value $n = 4$, we obtain the limit

$$g \leq 10^{-119}. \tag{8.34}$$

8.3 Discussions and Conclusions

In the previous section, we derived the conditions to be imposed on the parameters of our model in order to give the right description for inflation and dark energy. Let us give here a numerical example and show the field evolution. In all the figures, we use the following values for the parameters: $v = 0.5 \times 10^{19}$ GeV, $\lambda = 10^{-16}$, $\Lambda = 9 \times 10^{14}$ GeV, $m_\chi = 2.5 \times 10^{12}$ and $\alpha = 10^{-6}$. The tiny value for λ is needed in order to generate the correct amplitude of density perturbations, $\delta\rho/\rho \sim 2 \times 10^{-5}$ [9, 257]. In Fig.8.1 we display the graphical representation of the symmetric part $V_{sym}(\phi, \chi)$ of the effective potential and in Fig.8.2 we show the numerical solution to the system of the two equations of motion of the fields $\tilde{\phi} = \phi/v$ and $\tilde{\chi} = \chi/M$. We have solved it for the interesting region starting from $\phi = \phi_c$ and $\chi = 0$ (because during inflation we know that $\chi = 0$ and ϕ slowly rolls down the potential). We notice that when χ approaches its minimum, the slow-rolling of ϕ ceases, and it rapidly evolves towards the minimum of the effective potential and oscillates around it.

In Fig.8.3, we plot the number of e-folds $N(\phi)$ defined in Eq.(8.18), as a function of the inflaton field. The maximum value for ϕ that we chose is ϕ_c, while the minimum value is $\simeq 0$. The interesting region is the one that gives "observable inflation", that is for values of ϕ that give $N(\phi) \leq 60$. This is because $N(\phi) \sim 60$ corresponds to the time when cosmological scales leave the horizon during inflation. All what happened before is outside our horizon and is totally irrelevant at the present time.

A way to confront the predictions of our model with observational data is through the spectral index n_s of density perturbations produced during inflation [248, 258]. The spectral index is defined in terms of the slow-roll parameters ϵ and η by the relation

$$n_s = 1 + 2\eta - 6\epsilon \tag{8.35}$$

and experimental data indicate a value of $n_s = 0.96 \pm 0.02$ [80, 82], [238]-[243]. We display in Fig.8.4 the dependence of the spectral index on the inflaton field ϕ.

One of our numerical conclusions is that g has to have an extremely small value, as we see in (8.34). It says that the effect of Planck-scale physics in breaking global symmetries should be exponentially suppressed. Let us mention at this point that there are arguments for such a strong suppression. Indeed, interest in the quantification of the effect came from the fact that the consequence of the explicit breaking of the Peccei-Quinn symmetry [119, 120] is that the Peccei-Quinn mechanism is no longer a solution to the strong CP-problem [259]-[261].

In ref. [225] it was shown that in string-inspired models there could be non-perturbative symmetry breaking effects of order $\exp[-\pi(M_P/M_{\text{string}})^2]$. For $M_{\text{string}} < 10^{18}$ GeV, we get (8.34). Although we considered perturbative effects, that analysis shows that perhaps the values (8.34) leading to θ being quintessence are realistic.

Finally we would like to comment on the possibility that instead of having one field Ψ we have N fields Ψ_1, Ψ_2, ... Ψ_N. We are motivated by the recent work [68] where N inflatons are introduced. The interesting case is when N is large, as suggested in some scenarios discussed in [68]. When having N fields, our relations should of course be modified. In the simple case that the parameters of the N fields are identical, to convert the formulae in the text to the new case, we should make the following changes: $v \to N^{1/2}v$, $\Lambda \to N^{1/4}\Lambda$, $g \to N^{-3/2}g$ and $\lambda \to N^{-1}\lambda$. This would allow to change the values of the parameters, for example with large values for N we can have smaller values for v, and so on.

To summarize, our purpose has been to give a step forward starting from the idea of Frieman and Rosenfeld [207] that fields in a potential may supply a unified explanation of inflation and dark energy. Our model contains two scalar fields, one complex and one real, and a potential that contains a non-symmetric part due to Planck-scale physics. We have determined the conditions under which our fields can act as inflaton and as quintessence. One of the conditions is that the explicit breaking has to be exponentially suppressed, as suggested by quantitative studies of the breaking of global symmetries by gravitational effects [225].

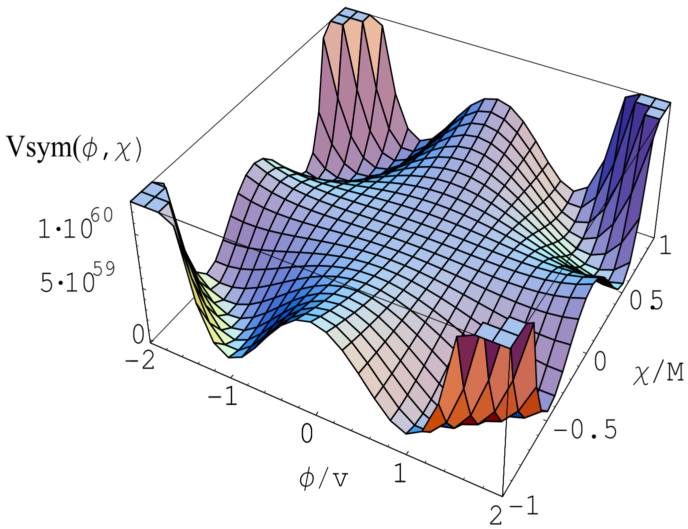


Figure 8.1: The symmetric part of the effective potential, V_{sym}, as function of the two normalized fields $(\phi/v, \chi/M)$

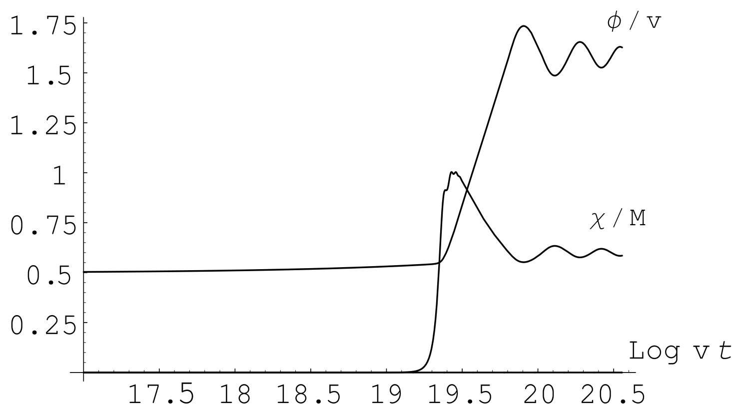


Figure 8.2: Numerical solution for the system of the coupled equations of motion of the two fields, $\tilde{\phi} = \phi/v$ and $\tilde{\chi} = \chi/M$. Notice the logarithmic time-scale on the abscise

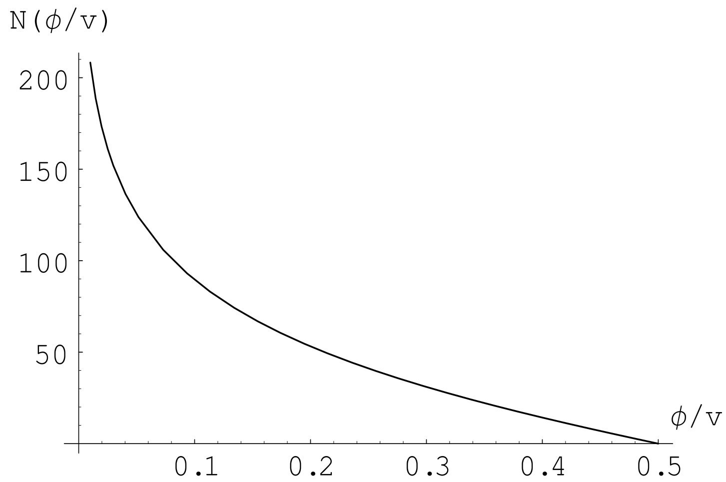
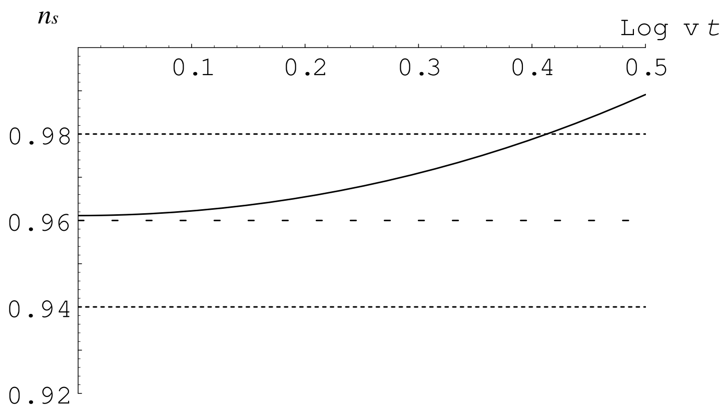
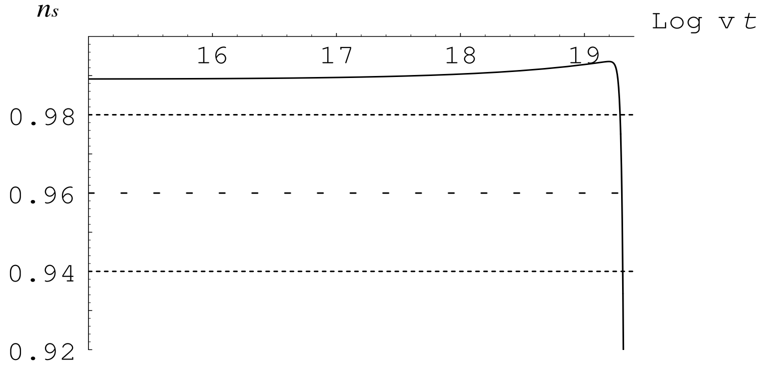


Figure 8.3: The number $N(\tilde{\phi})$ of e-folds of inflation as a function of $\tilde{\phi} = \phi/v$



(a)



(b)

Figure 8.4: (a) The spectral index n_s during inflation, as a function of the inflaton field, and (b) just after the time when $\phi = \phi_c$, as a function of the logarithm of time. In the above figures, the dashed line is the expected value for n_s within experimental errors, delimited by the pointed lines.

Chapter 9

Unified model for inflation and dark matter

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9.1 Introduction

Cosmology has made in the last few years enormous progress, especially in the accuracy of observational data, which is taken using technologies more and more precise. While in the past not too remote it seemed almost inconceivable, nowadays we may even talk about "precise cosmology". This is why, in analogy with the Standard Model (SM) of particle physics, many physicists are already talking about a "standard model" of cosmology. As suggested by recent observations of type Ia supernovas (SNIa) [137, 262], the matter power spectrum of large scale structure (LSS) [145] and anisotropy of the cosmic microwave background radiation (CMB) [5], the universe is presently dominated by too types of mysterious fluids: dark energy (DE), which has negative pressure whose consequence is to accelerate the expansion of the universe, and dark matter (DM), which is non-relativistic non-baryonic matter, very weakly coupled to normal matter and that only has gravitational effects on it.

The most simple explanation for DE is a cosmological constant Λ, but it raises another problem because its expected value is many orders of magnitude larger than the value suggested by observations. Another possible explanation is the existence of a slowly rolling scalar field, called quintessence, which is displaced from the minimum of its potential and started to dominate the energy density of the universe recently.

The same observations indicate that the universe is isotropic and homogeneous at large scales and spatially flat, for which in the old cosmological picture there is no reasonable explanation. The most successful and simple solution to the flatness and homogeneity problems is given by inflation [16], which in its simplest version is defined as a short period of accelerated expansion of the early universe caused by a single dominating scalar field, the inflaton. In addition, inflation gives the most popular mechanism of generation of cosmological fluctuations, which were the seed for the structure formation in our universe.

Although the SM based on the gauge group $SU(3) \times SU(2) \times U(1)$ is a solid theoretical construction able to accommodate all existing empirical data, it leaves many deep questions unanswered when trying to explain the origin and nature of the new ingredients introduced by modern cosmology, such as, for example, the inflaton, the DE and the DM. Thus, there are reasons to believe that the SM is not the ultimate theory and one has to look for extensions of it. If we are able to discover a theory that indeed goes beyond the SM, it will probably contain new symmetries, either local, or global.

A lot of effort has been done in studying global symmetries at high energies [223, 224, 236, 237], especially in trying to clarify the issue of quantum coherence loss in the presence of wormholes. It was argued that the loss of coherence opens the possibility that currents associated with global symmetries are not exactly conserved. Even if incoherence is not observed in the presence of wormholes, it was argued that other interesting consequences may emerge, such as the appearance of operators that violate global symmetries, of arbitrary dimensions, induced by baby universe interactions. There are other reasons to expect that quantum gravity effects break global symmetries: global charges can be absorbed by black holes which may evaporate, "virtual black holes" may form and evaporate in the presence of a global charge, etc.

In this context, the authors of [225] argue that if global symmetries are broken by virtual black holes or topology changing effects, they have to be exponentially suppressed. In particular, in order to save the axion theory, the suppression factor should have an extremely small value $g < 10^{-82}$. This suppression can be obtained in string theory, if the stringy mass scale is somewhat lower than the Planck-scale, $M_{\rm str} \sim 2 \times 10^{18}$GeV. Thus we expect to have an exponential suppression of the explicit breaking of global symmetries.

Even with such an extremely small explicit breaking, one can see that very interesting consequences may appear. In [116] (from now on Paper 1), it was shown that, when a global symmetry is spontaneously broken in the presence of a small explicit breaking, the resulting pseudo-Golstone boson (PGB) can be a DM particle. In [263] (from now on Paper 2), a similar study was made, but the purpose was to show that the resulting PGB could be a quintessence field explaining the present acceleration of the universe. In addition, based on the idea forwarded by Frieman and Rosenfeld [207], the model of Paper 2 also incorporated inflation. In this way, the two periods of accelerated expansion may have a common origin.

There is previous work related to explicit breaking of global symmetries [249, 250] and to Planck-scale breaking [251]-[253]. Cosmological consequences of some classes of PGBs are discussed in [201],[254]-[256].

Here, we extend the model in Paper 1 to also include inflation. The way we do it is similar to the work in Paper 2, the difference being that here we want the resulting PGB to be a DM particle, in contrast with Paper 2 where it was a quintessence field. Our result is that the parameter of the explicit breaking should be exponentially suppressed, $g < 10^{-30}$, as in Paper 1, but the level of suppressions is not that high as in the case of Paper 2, where a much smaller $g \sim 10^{-119}$ was needed in order for the PGB to explain DE. Inflation may occur here at scales as low as $V^{1/4} \sim 10^{10}$GeV.

The paper is structured as follows: in section 9.2 we make a short presentation of the model and then focus on the main features of it: inflation and dark matter. In section 9.3 we present our numerical results and, finally, in section 9.4 we make a discussion and give the conclusions. Technical details are given in the Appendix B.

9.2 The model

The model is, basically, the same as in Paper 2, so we just recall it here shortly. It contains a new complex scalar field, Ψ, charged under a certain global $U(1)$ symmetry, interacting with a massive real scalar field, χ, neutral under $U(1)$. It also contains a $U(1)-$symmetric potential

$$V_{\rm sym}(\Psi,\chi) = \frac{1}{4}\lambda(|\Psi|^2 - v^2)^2 + \frac{1}{2}m_\chi^2\chi^2 + \left(\Lambda^2 - \frac{\kappa^2|\Psi|^2\chi^2}{4\Lambda^2}\right)^2 \tag{9.1}$$

where λ and κ are coupling constants, m_χ and Λ are some energy scales and v is the $U(1)$ spontaneous symmetry breaking (SSB) scale.

The interaction term in (9.1) is of inverted hybrid type [74, 75] and can be realized in supersymmetry using a globally supersymmetric scalar potential [75]. However, in the present paper we are not preoccupied about the underlying theory in which this model can be realized, instead we only study the phenomenology of the potential (9.1).

Next, we allow terms in the potential that *explicitly* break $U(1)$. These terms are supposed to come from physics at the Planck-scale, and without knowledge of the exact theory at that scale, we introduce the most simple effective $U(1)-$breaking term [259]-[261]

$$V_{\rm non-sym}(\Psi) = -g\frac{1}{M_{\rm P}^{n-3}}|\Psi|^n\left(\Psi{\rm e}^{-{\rm i}\delta} + \Psi^\star{\rm e}^{{\rm i}\delta}\right) \tag{9.2}$$

where g is an effective coupling, $M_{\rm P} \equiv G_{\rm N}^{-1/2}$ is the Planck-mass and n is an integer ($n > 3$).

Summarizing, our effective potential is

$$V(\Psi,\chi) = V_{\rm sym}(\Psi,\chi) + V_{\rm non-sym}(\Psi) - C \tag{9.3}$$

where C is a constant that sets the minimum of the effective potential to zero.

By writing the field Ψ as

$$\Psi = \phi\,{\rm e}^{{\rm i}\tilde{\theta}} \tag{9.4}$$

we envisage a model in which the radial field ϕ is the inflaton, while the angular field $\tilde{\theta}$ is associated with a DM particle.

Thus, in this paper we consider the possibility of having a unified model of inflation and DM, improving in this way the model presented in Paper 1. We also want to present a more detailed numerical analysis of the part regarding inflation, which could also apply to the inflationary model of Paper 2.

9.2.1 Inflation

We first revisit the conditions that should be accomplished by our model in order to correctly describe the inflationary period of expansion of the universe. Inflation is supposed to have occurred in the early universe, when the energies it contained were huge. Thus, the appropriate term to deal with when describing inflation is the symmetric term $V_{\rm sym}$, while the non-symmetric one can be safely neglected, being many orders of magnitude smaller than $V_{\rm sym}$. Taking into account (9.4), we may write

$$V_{\rm sym}(\phi,\chi) = \Lambda^4 + \frac{1}{2}M_\chi^2(\phi)\chi^2 + \frac{\kappa^4\phi^4\chi^4}{16\Lambda^4} + \frac{1}{4}\lambda(\phi^2 - v^2)^2 - C \tag{9.5}$$

where $M_\chi^2(\phi) \equiv m_\chi^2 - \kappa^2\phi^2$, and we have also included the constant C. As commented above, ϕ is the inflaton field and χ is an auxiliary field that assists ϕ to inflate.

We assume that initially the fields ϕ and χ are in the vicinity of the origin of the potential, $\phi = \chi = 0$. At that point, the first derivatives of the potential are zero in both $\phi-$ and $\chi-$directions, but the second derivatives have opposite signs:

$$\left.\frac{\partial^2 V_{\rm sym}(\phi,\chi)}{\partial\phi^2}\right|_{\phi,\chi=0} = -\lambda v^2 < 0 \tag{9.6}$$

$$\left.\frac{\partial^2 V_{\rm sym}(\phi,\chi)}{\partial\chi^2}\right|_{\phi,\chi=0} = m_\chi^2 > 0. \tag{9.7}$$

This means that χ remains trapped at the false minimum in $\chi-$direction of the potential, $\chi = 0$, while ϕ becomes unstable and can roll down in the direction given by $\chi = 0$. If the potential in $\phi-$direction is sufficiently flat, ϕ can have a slow-roll and produce inflation. This regime lasts until the curvature in $\chi-$direction changes sign and inflation has a sudden end through the instability of χ, which triggers a waterfall regime and both fields rapidly evolve towards the absolute minimum of the potential. The critical point where inflation ends is given by the condition

$$M_\chi^2(\phi) = m_\chi^2 - \kappa^2\phi^2 = 0 \tag{9.8}$$

so that during inflation $\phi < \phi_{\rm c} = \frac{m_\chi}{\kappa} \lesssim v$.

The constraints related to the inflationary aspects of the model are the same of Paper 2. Let us just summarize them here.

- vacuum energy of field χ should dominate: $\frac{1}{4}\lambda v^4 \ll \Lambda^4$
- small $\phi-$mass as compared to $\chi-$mass: $|m_\phi^2| = \lambda v^2 \ll m_\chi^2 \lesssim \kappa^2 v^2$
- slow-roll conditions: $\epsilon \equiv \frac{1}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1$, $|\eta| \equiv \left|\frac{1}{8\pi}\frac{V''}{V}\right| \ll 1$

- rapid variation of $M_\chi^2(\phi)$ at the critical point: $|\Delta M_\chi^2(\phi_c)| > H^2$
- fast roll of ϕ after χ gets to its minimum: large $(\partial V_{\rm sym}/\partial\phi)|_{\chi_{\rm min}}$

These conditions have to be satisfied in order for the hybrid inflationary mechanism to work. There are other constraints related to fairly precise observational data:

- sufficient number of e-folds of inflation $N(\phi) = (8\pi)/M_{\rm P}^2 \int_{\phi_{\rm end}}^{\phi} (V_{\rm sym}/V'_{\rm sym}) {\rm d}\phi$ in order to solve the flatness and the horizon problems. The required number depends on the inflationary scale and on the reheating temperature, and is usually comprised between 35 for low-scale inflation and 60 for GUT-scale inflation
- the amplitude of the primordial curvature power-spectrum produced by quantum fluctuations of the inflaton field should fit the observational data [5], ${\mathcal{P}_{\mathcal{R}}}^{1/2} \simeq 4.86 \times 10^{-5}$
- the spectral index $n_{\rm s}$ should have the right value suggested by observations of the CMB [5], $n_{\rm s} = 0.951^{+0.015}_{-0.019}$ (provided tensor-to-scalar ratio $r \ll 1$).

Combining all the above constraints we obtain the following final relations that should be satisfied by the parameters of our model: $\lambda \ll \kappa^2$ and $v < M_{\rm P}$. We also obtain the dependence of some of the model parameters on the SSB scale v (for more details, see Paper 2 and the Appendix B). These will be used in section 9.3 for a numerical study. The range of values of the scale v will be fixed by the requirement that θ is a DM candidate.

9.2.2 Dark matter

As stated above, our idea is that θ, the PGB that appears after the SSB of $U(1)$ in the presence of a small explicit breaking, can play the role of a DM particle. Thus, after the end of inflation, θ finds itself in a potential given by the term $V_{\rm non-sym}$

$$V_{\rm non-sym}(\phi, \theta) = -2\, g \frac{\phi^{n+1}}{M_P^{n-3}} \cos\tilde{\theta} \tag{9.9}$$

where (9.4) has been used in (9.2) and the change of variables $\tilde{\theta} \longrightarrow \tilde{\theta} + \delta$ has been made. In Paper 1 it was shown that for exponentially small g the evolutions of the two components of Ψ are completely separated, so that we expect $\theta-$oscillations to start long after ϕ has settled down at its vacuum expectation value (vev),

$$\langle\phi\rangle \simeq M_P^{1/3} v^{2/3}. \tag{9.10}$$

A detailed study of the cosmology of the $\theta-$particle was made in Paper 1, for the lowest possible value $n = 4$. We do not want to enter into details here, but just to make use of the results of that work to obtain the values of the parameters of our model. The only difference here is the fact that the vev of the radial field ϕ is different from v, so that the constraints obtained in Paper 1 on v will apply here on $\langle\phi\rangle$. This will also affect the angular field $\tilde{\theta}$, which is here normalized as $\tilde{\theta} \equiv \theta/\langle\phi\rangle$.

Due to the small explicit breaking of the $U(1)$ symmetry, θ acquires a mass which depends on both g and $\langle\phi\rangle$

$$m_\theta^2 = 2g\left(\frac{\langle\phi\rangle}{M_P}\right)^3 M_P^2 \tag{9.11}$$

and this is why we should find constraints on both $\langle\phi\rangle$ and g in order for θ to be a suitable DM candidate. The constraints that should be imposed come from various astrophysical and cosmological considerations:

- θ should be a stable particle, with lifetime $\tau_\theta > t_0$, where t_0 is the universe's lifetime
- its present density should be comparable to the present DM density $\Omega_\theta \sim \Omega_{DM} \sim 0.25$
- it should not allow for too much energy loss and rapid cooling of stars [126]
- although stable, θ may be decaying at present, and its decay products should not distort the diffuse photon background

In Paper 1, all these constraints have been studied in detail. Because θ is massive, it can decay into two photons or two fermions, depending on its mass. The lifetime of θ depends on the effective coupling to the two photons/fermions and on its mass, which in turn depends on the two parameters $\langle\phi\rangle$ and g. It was shown that for the interesting value of $\langle\phi\rangle$ and g for which θ can be DM, the resulting $\theta-$mass has to be $m_\theta < 20$ eV, so the only decaying channel is into two photons.

There are different mechanism by which θ particles can be produced, as explained in Paper 1: (a) thermal production in the hot plasma, and (b) non-thermal production by $\theta-$field oscillations and by the decay of cosmic strings produced in the SSB. All these may contribute to the present energy density of θ particles, which was computed in Paper 1. By requiring it to be comparable to the present DM energy density of the universe, we obtain a curve in the space of parameters $\langle\phi\rangle$ and g, illustrated in Fig.9.1 as the line labelled "DM".

In Paper 1 it was argued that there are some similarities between our θ particle and the QCD axion [117, 118]. This is why when investigating its production in stars, we can apply similar constraints. The strongest one comes from the fact that θ may be produced in stars and constitute a novel energy loss channel, and these considerations put a limit on $\langle\phi\rangle$, but not on g

$$\langle\phi\rangle > 3.3 \times 10^9 \,\text{GeV}. \tag{9.12}$$

Another aspect about θ is that, although stable, a small fraction of its population may be decaying today and the resulting photons may produce distortions of the diffuse photon background. Thus, the calculated photon flux coming from $\theta-$decay is constraint to be less than the observed flux (see Paper 1 for details).

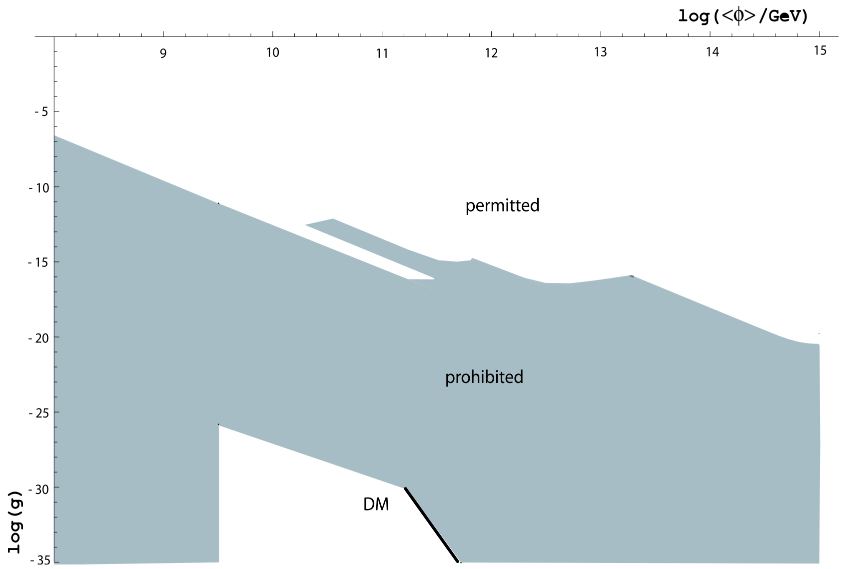

Figure 9.1: Permitted and prohibited regions in the $(\langle\phi\rangle, g)-$plane, taken from Paper 1. The interesting points are those which are near the line labelled "DM".

By combining astrophysical and cosmological constraints, we can obtain the interesting values for $\langle\phi\rangle$ and g for which θ is stable and its density is comparable to the DM density. These values can be easily read from Fig.9.1, being situated along the line labelled "DM". Note that there is an upper limit on g, which corresponds to a lower limit on $\langle\phi\rangle$, i.e.

$$g < 10^{-30}, \;\; \langle\phi\rangle > 10^{11} \,\text{GeV}. \tag{9.13}$$

These also put a lower limit on the value of $U(1)$ breaking scale,

$$v \sim \left(\frac{\langle\phi\rangle^3}{M_{\rm P}}\right)^{1/2} > 10^7 \,\text{GeV}. \tag{9.14}$$

Finally, according to the study made in Paper 1, for value $\langle\phi\rangle < 7.2 \times 10^{12}$ GeV, θ particles can be produced both thermally and non-thermally, but in the region characterized by $\langle\phi\rangle > 10^{11}$ GeV, the dominant energy density corresponds to non-thermally produced θ particles. Moreover, for $\langle\phi\rangle > 7.2\times10^{12}$ GeV, only non-thermal production is possible.

9.3 Numerical results

The constraints enumerated in subsection 9.2.1 will determine the values of some of the model parameters. Our scope is to make a general analysis of how these parameters depend on v. For λ we obtain an exact formula (see Appendix B)

$$\lambda = 4.4\times10^{-12}\frac{\phi_0^2(v^2-\phi_0^2)^2}{(v^2-3\phi_0^2)^3} \tag{9.15}$$

where ϕ_0 is the value of the inflaton field when the scale $k = 0.002$ Mpc^{-1} crossed the horizon during inflation and its $v-$dependence can be obtained numerically, see the Appendix B.

The exact formulae for the other parameters are too complicate to be shown here. Nevertheless, things get simplified in the limit $\lambda v^4 \ll \Lambda^4 (\equiv v \ll M_{\rm P})$, for which we obtain

$$\Lambda \simeq 1.6\times10^{-3}\left[\frac{\phi_0^2(v^2-\phi_0^2)^2}{(v^2-3\phi_0^2)^2}\right]^{1/4} M_{\rm P}^{1/2} \sim \lambda^{1/4}v^{1/2}M_{\rm P}^{1/2} \tag{9.16}$$

and

$$C \simeq \frac{3}{4}\Lambda^4\left(\lambda v^4/\Lambda^4\right)^{1/3} \ll \Lambda^4. \tag{9.17}$$

We notice that C can be neglected, as compared to Λ^4, in this limit.

The coupling κ is only constrained by the condition $\lambda \ll \kappa^2$, so that it can have any arbitrary value satisfying this inequality. In our numerical study we took the value $\kappa = 10^{-2}$. The mass of χ, namely m_χ, can have any arbitrary value satisfying $m_\chi < \kappa v$, but for the sake of simplicity we set it to $m_\chi = \kappa v/2$, without loss of generality.

In Fig.9.2 we display some graphics with the $v-$dependence of relevant parameters of our model. In Fig.9.2(a) we plot the numerical results for $\phi_0(v)$, which are then used to produce the other graphics. From Fig.9.2(b), one can see that λ does not vary too much with v and its values are around 10^{-13} for a large range of v. We also notice that the other parameters grow as different powers of v. For example in Fig.9.2(c), Λ, which sets the inflationary scale, varies as $v^{1/2}$ from $\sim 10^{10}$GeV, for $v = 10^7$GeV, to $\sim 10^{16}$GeV, for $v \sim M_{\rm P}$, i.e., from a relatively low-scale to a GUT-scale inflation. In Fig.9.2(d) are represented in the same graphic the values of Λ^4, C and λv^4 to confirm that, in the limit $v \ll M_{\rm P}$, one can use the approximation $\lambda v^4 \ll C \ll \Lambda^4$, while for $v \lesssim M_{\rm P}$ the three terms become of the same order and the above approximation is not valid anymore. In

Fig.9.2(e) and (f) we give additional results, such as the number of "observable" inflation N and the tensor-to-scalar ratio $r \equiv 16\epsilon$.

In particular, for the lowest possible value $v = 10^7$GeV, we get $N \simeq 47$ e-folds of inflation, and a very tiny value for the tensor-to-scalar ratio, $r \sim 10^{-27}$, making the detection of gravitational waves a practically impossible task. We specify that the spectral index $n_{\rm s} \simeq 0.95$ and the amplitude of curvature perturbations, ${\mathcal{P}_{\mathcal{R}}}^{1/2} \simeq 4.86 \times 10^{-5}$ for all v.

We also notice that for $v \sim M_{\rm P}$, we recover the inflationary scenario proposed in Paper 2, where θ was a quintessence field. The numerical analysis presented here can also be applied to that model, and one obtains $\langle\phi\rangle \sim v$, $N \sim 56$ e-folds of inflation and the more interesting result $r \sim 10^{-3} - 10^{-4}$, which makes gravitational waves detection more plausible in the future.

9.4 Discussions and Conclusions

In this work, we have presented a model that is able to describe inflation and dark matter in a unified scenario, by introducing a new complex scalar field $\Psi = \phi \exp(i\tilde{\theta})$ interacting with a real scalar, χ, and a potential invariant under certain global $U(1)$ symmetry. We allowed for a small explicit breaking term in the effective potential that is due to Planck-scale physics and investigated the possibility that ϕ is the inflaton and θ a dark matter particle. The corresponding constraints have been enumerated in subsections 9.2.1 and 9.2.2.

In this way, we improve the model of Paper 1, where θ was a DM particle, but the model did not include inflation. The results of Paper 1 are used here in subsection 9.2.2. For the part regarding inflation in our model, in subsection 9.2.1 we make a similar analysis as in Paper 2, which also improves Paper 1 by incorporating inflation, but the difference is that there θ was a quintessence field. The numerical analysis we present in section 9.3 extrapolates between the two scales considered in the models of Paper 1 and Paper 2.

In the present numerical study, we used the value $\kappa = 10^{-2}$, and we chose $m_\chi = \kappa v/2$ for simplicity. We observe that a tiny value is needed for $\lambda \sim 10^{-13}$, in order to generate the correct values of the amplitude of curvature perturbations and of the spectral index. We have no possible theoretical explanation for justifying this small $\lambda-$value, but this is a common problem of most of the inflationary models. Although we make a general numerical analysis to see how the parameters depend on the SSB scale v, we are finally interested in the value for which the angular field θ is a DM candidate, $v \ll M_{\rm P}$.

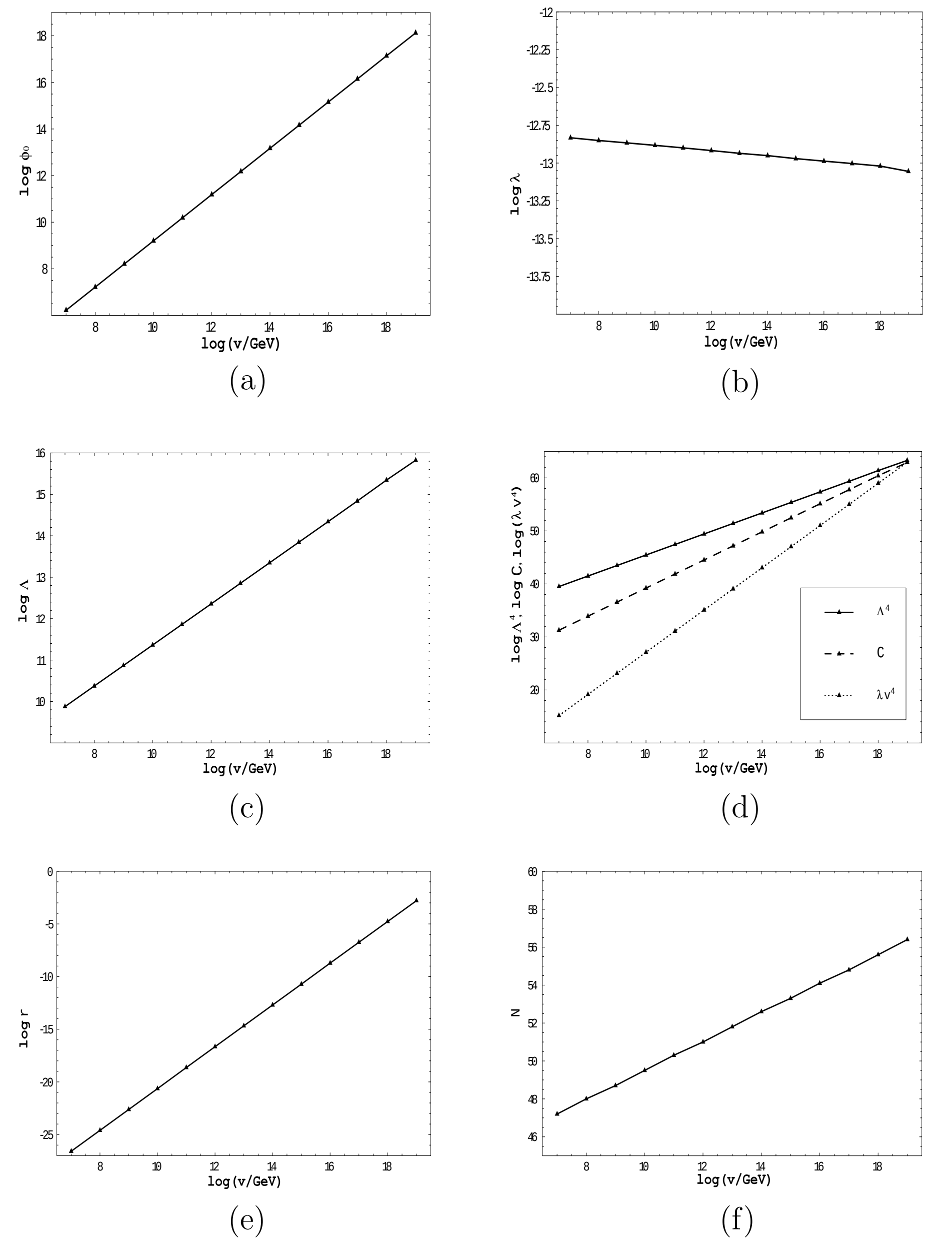

Figure 9.2: $v-$dependence of various parameters: (a)$-$the inflaton field value corresponding to the moment when the scale $k_0 = 0.002$ Mpc^{-1} left the horizon, $\phi_0(v)$; (b)$-$the inflaton self-coupling constant, $\lambda(v)$; (c)$-$the scale of inflation, $\Lambda(v)$; (d)$-$comparison between $\Lambda^4(v)$, $C(v)$ and λv^4, which tend to be of the same order for $v \sim M_{\rm P}$; (e)$-$the tensor-to-scalar ratio, $r(v)$; (f)$-$the number of e-folds of inflation that occur between the largest observable scale left the horizon and the end of inflation, $N(v)$.

Notice that for $v \ll M_{\rm P}$, the vev of the inflaton field ϕ is different from v and is approximately given by $\langle\phi\rangle \simeq v^{2/3}M_{\rm P}^{1/3} \neq v$, while for $v \sim M_{\rm P}$ they tend to be of the same order, $\langle\phi\rangle \sim O(v)$.

We included explicit $U(1)-$breaking terms in the potential and studied the possibility that the resulting PGB, θ, could be a DM particle. We found in Eq.(9.13) that the effective g coupling related to the explicit breaking should be exponentially suppressed, $g < 10^{-30}$. This confirms our expectations commented in the Introduction, that the effect of Planck-scale physics in breaking global symmetries should be exponentially suppressed [225]. With the extreme values $g = 10^{-30}$ and $v = 10^7$GeV, the mass of θ is fully determined, $m_\theta \sim 15$ eV.

It would be interesting to investigate reheating in our model to determine the exact reheating temperature $T_{\rm rh}$, and also to provide a specific mechanism for producing SM particles, but this goes beyond the scope of our paper.

As a final comment, we would like to add that such a strong suppression of g may be avoided if, for some reason, $n = 7$ and all smaller values prohibited. In this case, one obtains g of $O(1)$, but then one should find an argument why n cannot be smaller than 7.

Chapter 10

Low-scale inflation in a model of dark energy and dark matter

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10.1 Introduction

The recent three-year Wilkinson Microwave Anisotropy Probe (WMAP3) results [5] have put quite a severe constraint on inflationary models. In particular, new results on the value of the spectral index $n_s = 0.95 \pm 0.02$ are sufficiently "precise" as to rule out many models with an exact Harrison-Zel'dovich-Peebles scale-invariant spectrum with $n_s = 1$ and for which the tensor-to-scalar ratio $r \ll 1$. Any model purported to describe the early inflationary era will have to take into account this constraint. However, by itself, it is not sufficient to narrow down the various candidate models of inflation. In particular, "low-scale" inflationary models are by no means ruled out by the new data. By "low-scale" we refer to models in which the scale that characterizes inflation is several orders of magnitude smaller than a typical Grand Unified Theory (GUT) scale $\sim 10^{15} - 10^{16}$ GeV. It is in this context that we wish to present a model of low-scale inflation which could also describe the dark energy and dark matter [264, 265].

The model of dark energy and dark matter described in [265] involves a new gauge group $SU(2)_Z$ which grows strong at a scale $\Lambda_Z \sim 3 \times 10^{-3}$ eV starting with the value of the gauge coupling at $\sim 10^{16}$ GeV which is not too different from the Standard Model (SM) couplings at a similar scale. (This is nicely seen when we embed $SU(2)_Z$ and the SM in the unified gauge group E_6 [266].) The model of [265] contains, in addition to the usual SM content, particles which are SM singlets but $SU(2)_Z$ triplets, $\psi^{(Z)}_{(L,R),i}$ with $i = 1, 2$, particles which carry quantum numbers of both gauge groups, $\tilde{\varphi}^{(Z)}_{1,2}$, which are the so-called messenger fields with the decay of $\tilde{\varphi}^{(Z)}_1$ being the source of SM leptogenesis [267], and a singlet complex scalar field,

$$\phi_Z = (\sigma_Z + v_Z) \exp(i a_Z / v_Z) \,, \tag{10.1}$$

whose angular part a_Z is the axion-like scalar. We have defined the radial part of ϕ_Z as the sum of a field σ_Z and a vacuum-expectation-value (v.e.v.) v_Z. The $SU(2)_Z$ instanton-induced potential for a_Z (with two degenerate vacua) along with a soft-breaking term whose dynamical origin is discussed in [268], is one that is proposed in [265] as a model for dark energy. In that scenario, the present universe is assumed to be trapped in a false vacuum of the a_Z potential with an energy density $\sim \Lambda_Z^4$. The exit time to the true vacuum was estimated in [265] and was found to be enormous, meaning that our universe will eventually enter a late inflationary stage.

What might be interesting is the possibility that the real part of ϕ_Z, namely σ_Z, could play the role of the inflaton while the imaginary part, a_Z, plays the role of the "acceleron" as we have mentioned above. This unified description is attractive for the simple reason that *one complex* field describes both phenomena: Early and Late inflation. (This scenario has been exploited earlier [206, 263], [269]-[271] in the context of GUT scale inflation.) Although the structure of the potential describing the accelerating universe is determined, in the model of [265], by instanton dynamics of the $SU(2)_Z$ gauge interactions [265], the potential for σ_Z, which would describe the early inflationary universe, is arbitrary as with scalar potentials in general. In this case, the only constraint comes from the requirement that this potential should be of the type that gives the desired spectral index and the right amount of inflation corresponding to the characteristic scale of the model.

We now briefly describe the model of [265]. The key ingredient of that model is the postulate of a new, unbroken gauge group $SU(2)_Z$ which grows strong at a scale $\Lambda_Z \sim 3 \times 10^{-3}$ eV. The model also contains a global symmetry $U(1)_A^{(Z)}$ which is spontaneously broken by the v.e.v. of ϕ_Z, namely $\langle \phi_Z \rangle = v_Z$, and is also *explicitly* broken at a scale $\Lambda_Z \ll v_Z$ by the $SU(2)_Z$ gauge anomaly. Because of this, the pseudo-Nambu-Goldstone boson (PNGB) a_Z acquires a tiny mass as discussed in [265]. Its $SU(2)_Z$ instanton-induced potential used in the false vacuum scenario for the dark energy is given by

$$V_{tot}(a_Z, T) = \Lambda_Z^4 \left[1 - \kappa(T) \cos \frac{a_Z}{v_Z} \right] + \kappa(T) \Lambda_Z^4 \, \frac{a_Z}{2\pi \, v_Z} \,, \tag{10.2}$$

where $\kappa(T) = 1$ at $T < \Lambda_Z$. ($SU(2)_Z$ instanton effects become important when $\alpha_Z = g_Z^2/4\,\pi \sim 1$ at $\Lambda_Z \sim 3 \times 10^{-3}\, eV$.) The universe is assumed to be presently trapped in the false vacuum at $a_Z = 2\pi v_Z$ with an energy density $\sim (3 \times 10^{-3}\, eV)^4$. As such, this model mimicks the ΛCDM scenario with $w(a_Z) = \frac{\frac{1}{2}\dot{a}_Z^2 - V(a_Z)}{\frac{1}{2}\dot{a}_Z^2 + V(a_Z)} \approx -1$, at present and for a long time from now on, but not in the distant past [265].

What could be the form of the potential for the *real part*, namely σ_Z, of ϕ_Z? As with any scalar field, the form of the potential is rather arbitrary, with the general constraints being gauge invariance and renormalizability. In this paper, we would like to propose a form of potential for σ_Z which is particularly suited to the discussion of the "low-scale" inflationary scenario: a Coleman-Weinberg (CW) type of potential [72]. (The CW-type of potential has been recently used [272] to describe a GUT-scale inflation using the WMAP3 data.) There are three types of contributions to the potential. The sources of these three types are the following terms in the lagrangian:

a) $\phi_Z - \psi^{(Z)}_{(L,R),i}$ coupling

$$\sum_i K_i \, \bar{\psi}^{(Z)}_{L,i} \, \psi^{(Z)}_{R,i} \, \phi_Z + h.c. \tag{10.3}$$

Let us recall from [265] that (10.3) is invariant under the following global $U(1)_A^{(Z)}$ symmetry transformations: $\psi^{(Z)}_{L,i} \rightarrow e^{-i\alpha} \, \psi^{(Z)}_{L,i}$, $\psi^{(Z)}_{R,i} \rightarrow e^{i\alpha} \, \psi^{(Z)}_{R,i}$, and $\phi_Z \rightarrow e^{-2i\alpha} \, \phi_Z$.

b) $\phi_Z - \tilde{\varphi}_1^Z$ mixing (we ignore the $\phi_Z - \tilde{\varphi}_2^Z$ coupling since it is assumed to have a mass of the order of a typical GUT scale)

$$\tilde{\lambda}_{1Z}(\phi_Z^\dagger\,\phi_Z)(\tilde{\varphi}_1^{Z,\dagger}\,\tilde{\varphi}_1^Z) \tag{10.4}$$

c) σ_Z self-interaction

$$\frac{\lambda}{4!}\sigma_Z^4 \tag{10.5}$$

Both terms, (10.4) and (10.5), arise from the general potential for all fields.

Let us look into constraints on these couplings coming from issues discussed in [265]: dark matter and leptogenesis.

Since the coupling (10.3) will, in principle, contribute to the CW potential for σ_Z, it is crucial to have an estimate on the magnitude of the Yukawa couplings K_i. In [265], an argument was made as to why it might be possible that $\psi_i^{(Z)}$ could be Cold Dark Matter (CDM) provided

$$m_{\psi_i^{(Z)}} = |K_i| v_Z \leq O(200\,GeV)\,, \tag{10.6}$$

or

$$|K_i| \leq O(200\,GeV/v_Z) \tag{10.7}$$

Roughly speaking, in order for $\Omega_{CDM} \sim O(1)$, the annihilation cross sections for $\psi_i^{(Z)}$ are required to be of the order of weak cross sections. In this case, they are approximately $\sigma_{\text{annihilation}} \sim \alpha_Z^2(m_{\psi_i^{(Z)}})/m^2_{\psi_i^{(Z)}}$ ($\alpha_Z^2(m_{\psi_i^{(Z)}})$ is the coupling evaluated at $E = m_{\psi_i^{(Z)}}$) and have the desired magnitude when $m_{\psi_i^{(Z)}} \sim O(200\,GeV)$ with $\alpha_Z^2(m_{\psi_i^{(Z)}}) \sim \alpha^2_{SU(2)_L}(m_{\psi_i^{(Z)}})$, a characteristic feature of the model of [265].

A second requirement comes from a new mechanism for leptogenesis as briefly mentioned in [265] and described in detail in [267]. This new scenario of leptogenesis involves the decay of a messenger scalar field, $\tilde{\varphi}_1^{(Z)}$, into $\psi_i^{(Z)}$ and a SM lepton. In order to give the correct estimate for the net lepton number, a bound on the mass of $\tilde{\varphi}_1^{(Z)}$ was derived. In [267], it was found that

$$m_{\tilde{\varphi}_1^{(Z)}} \leq 1\,TeV\,. \tag{10.8}$$

This came about when one calculates the interference between the tree-level and one-loop contributions to the decays

$$\tilde{\varphi}_1^{(Z)} \to \bar{\psi}_{1,2}^{(Z)} + l \tag{10.9}$$

$$\tilde{\varphi}_1^{(Z),*} \to \psi_{1,2}^{(Z)} + \bar{l} \tag{10.10}$$

where l represents a SM lepton. By requiring that the asymmetry coming from this scenario to be $\epsilon_l^{\tilde{\varphi}_1} \sim -10^{-7}$ in order to obtain the right amount of baryon number asymmetry through the electroweak sphaleron process, [267] came up with the constraint (10.8) which could be interesting for searches of non-SM scalars at the Large Hadron Collider (LHC). On the other hand, as discussed in [265], the mixing between $\tilde{\varphi}_1^{(Z)}$ and ϕ_Z results in an additional term in the mass squared formula for $\tilde{\varphi}_1^{(Z)}$, namely $2\tilde{\lambda}_{1Z}v_Z^2$. Taking into account the leptogenesis bound (10.8), one can write

$$\tilde{\lambda}_{1Z} \leq (1\,TeV/v_Z)^2\,. \tag{10.11}$$

All these constraints will be used to estimate the contributions to the effective potential. The $\psi_i^{(Z)}$ fermion loop contribution to the σ_Z CW potential is given by $-|K_i|^4/16\pi^2$, and therefore will be bound by (10.7). The $\tilde{\varphi}_1^{(Z)}$ loop contribution to the potential is given by $\tilde{\lambda}_{1Z}^2/16\pi^2$ and is constrained by (10.11). The third contribution c) coming from the σ_Z loop is given by $\lambda^2/16\pi^2$. There are no constraints on it coming from dark matter or leptogenesis arguments, as we have in the other cases a) and b).

Below, we will constrain both v_Z and the coefficient of the CW potential (which includes contributions from various loops) using the latest WMAP3 data. Next, we use these results to further constrain $|K_i|$ and $\tilde{\lambda}_{1Z}$. We will finally comment on the implications of these constraints.

10.2 Inflation with a Coleman-Weinberg potential

Let us now see under which conditions we can obtain a viable scenario for inflation with our model. As previously mentioned, the scalar field ϕ_Z receives various contributions to its potential, which will have the generic CW form [72]

$$V_0(\phi_Z^\dagger\phi_Z) = A\left(\phi_Z^\dagger\phi_Z\right)^2\left(\log\frac{\phi_Z^\dagger\phi_Z}{v_Z^2} - \frac{1}{2}\right) + \frac{Av_Z^4}{2}. \tag{10.12}$$

After making the replacement (10.1) in (10.12), we obtain the potential for the real part σ_Z of ϕ_Z, which we want to be the inflaton field

$$V_0(\sigma_Z) = A(\sigma_Z + v_Z)^4\left[\log\frac{(\sigma_Z + v_Z)^2}{v_Z^2} - \frac{1}{2}\right] + \frac{Av_Z^4}{2}. \tag{10.13}$$

This expression corresponds to the zero temperature limit. If we take into consideration finite-temperature effects, we should add a new term depending on temperature T that will give the following effective potential to σ_Z

$$V_{\text{eff}}(\sigma_Z) = V_0(\sigma_Z) + \beta T^2(\sigma_Z + v_Z)^2 \tag{10.14}$$

where β is a numerical constant. At high temperature, the field σ_Z is trapped at the $U(1)_A^{(Z)}-$symmetric minimum $\sigma_Z = -v_Z$. As the universe cools, for a sufficiently low temperature a new minimum appears at the $U(1)_A^{(Z)}-$symmetry breaking value $\sigma_Z = 0$ ($\langle\phi_Z\rangle = v_Z$). The critical temperature is the temperature at which the two minima become degenerate and is equal to $T_{\rm cr} = v_Z\sqrt{A/\beta}\, e^{-1/4}$. The universe cools further with the field σ_Z being trapped at the false vacuum and inflation starts when the false vacuum energy of σ_Z becomes dominant. Nevertheless, when the universe reaches the Hawking temperature

$$T_H = \frac{H}{2\pi} \simeq \frac{1}{2\pi}\sqrt{\frac{8\pi}{3M_{\rm P}^2}V_0(-v_Z)} = \frac{\sqrt{A}\, v_Z^2}{\sqrt{3\pi}\, M_{\rm P}} \tag{10.15}$$

a first-order phase transition occurs and σ_Z may start its slow-rolling towards the true minimum of the potential. In (10.15), $M_{\rm P} \simeq 1.22 \times 10^{19}\,{\rm GeV}$ is the Planck-mass, H is the Hubble parameter at that epoch and we supposed that the energy density of σ_Z is the dominant one. Observable inflation occurs just after the false vacuum is destabilized and the inflaton slowly rolls down the potential. The evolution of σ_Z can be described classically.

Next, let us find the values for A and v_Z that are needed in order to obtain a viable model for inflation, compatible with observational data. The main constraints come from the combined observations of the Cosmic Microwave Background (CMB) and the Large Scale Structure (LSS) of the universe, which indicate the range of values for the spectral index $n_{\rm s}$, the tensor-to-scalar ratio r and, perhaps, evidence for a running in the spectral index. We also consider the constraint on the amplitude of the curvature perturbations, ${\mathcal{P}_\mathcal{R}}^{1/2}$, with the assumption that they were produced by quantum fluctuations of the inflaton field when the present large scales of the universe left the horizon during inflation. Finally, the number of e-folds of inflation produced between that epoch and the end of the inflationary stage should be large enough in order to solve the horizon and the flatness problems.

In our scenario, we have a theoretically motivated mechanism for generating a lepton asymmetry which then translates into a baryon asymmetry compatible with observations [265, 267]. Later on in this paper we will treat this aspect in more detail. For now, it is sufficient to say that after inflation, the σ_Z field starts to oscillate and to reheat the universe, mainly by decaying into two $\psi_i^{(Z)}$ fermions of masses given by (10.6), so that we want the inflaton to have sufficient mass as to decay into the two fermions. This is another condition to be considered for obtaining the adequate values for the parameters of our model.

Let us list the main constraints to be imposed on our model:

- the spectral index
$$n_{\rm s} \simeq 1 - 6\epsilon + 2\eta \tag{10.16}$$
should be in the range $n_{\rm s} = 0.95 \pm 0.02$, where $\epsilon = \frac{M_{\rm P}^2}{16\pi}\left(\frac{V'}{V}\right)^2$ and $\eta = \frac{M_{\rm P}^2}{8\pi}\frac{V''}{V}$ are the slow-roll parameters and a prime means σ_Z-derivative;

- the right number of e-folds of inflation between large scale horizon crossing and the end of inflation
$$N_0 = \int_{\sigma_{Z,end}}^{\sigma_{Z,0}} \frac{V}{V'} d\sigma_Z \tag{10.17}$$
where $\sigma_{Z,0}$ is the value of the inflaton field at horizon crossing and $\sigma_{Z,end}$ its value at the end of inflation;

- the amplitude of the curvature perturbations generated by the inflaton, evaluated at $\sigma_{Z,0}$
$$\mathcal{P_R}^{1/2} = \sqrt{\frac{128\pi}{3}} \frac{V^{3/2}}{M_{\rm P}^3 |V'|}|_{\sigma_{Z,0}} \tag{10.18}$$
should have the WMAP3 value $\mathcal{P_R}^{1/2} \simeq 4.7 \times 10^{-5}$;

- the inflaton mass $m_{\sigma_Z} = \sqrt{8A}\, v_Z$ should be at least 400 GeV or so in order for $\psi_i^{(Z)}$'s to be produced by the inflaton decay.

In our analysis, the parameters are functions of the inflaton field σ_Z and are evaluated when the present horizon scales left the inflationary horizon.

We have performed a complete numerical study. Imposing the requirements we have mentioned, we are interested in the lowest possible scale for inflation in our model. The scale is lowest for $v_Z \simeq 3 \times 10^9$ GeV. With this value, we obtain $A \simeq 3 \times 10^{-15}$ and $m_{\sigma_Z} \simeq 450$ GeV which, as commented above, is sufficient to produce two $\psi_i^{(Z)}$ fermions of masses $\mathcal{O}(200)$ GeV. We also obtain $n_{\rm s} = 0.923$ for the spectral index, not far away from the observed range, and $N \simeq 38$ e-folds of inflation between the present large scales horizon crossing until the end of inflation. The inflation scale is $V_0^{1/4} \equiv V(\sigma_{Z,0})^{1/4} \simeq 6 \times 10^5$ GeV. Low-scale inflation is interesting because it might be proved more easily in particle physics experiments.

Our model also satisfies the constraints for values of the parameters leading to higher values than $V_0^{1/4} \simeq 6 \times 10^5$ GeV. In Fig. 10.1 we show the dependence of the spectral index *vs* the energy scale of inflation. We see that the values of $n_{\rm s}$ are within 95% confidence level even at scales as low as 6×10^5 GeV, and increases with increasing inflationary scale. The graphic displayed in Fig. 10.1 was obtained with the assumption of instant reheating and a standard thermal history of the universe [35, 69]. In Section 10.3 we present a more detailed analysis on the reheating mechanism. Here we just mention that the reheating temperature in our model is smaller than $V_0^{1/4}$, in which case the values of $n_{\rm s}$ displayed in Fig.10.1 shift to smaller values.

From now on we will stick to the lowest possible example $v_Z = 3 \times 10^9$ GeV ($V_0^{1/4} \sim 6 \times 10^5$ GeV) and examine the consequences. We should stress that the values of A does not vary drastically when we raise $V_0^{1/4}$ and we can safely consider it constant, with the value $A \simeq 3 \times 10^{-15}$. We then study some of the consequences that arise when adopting this value for A.

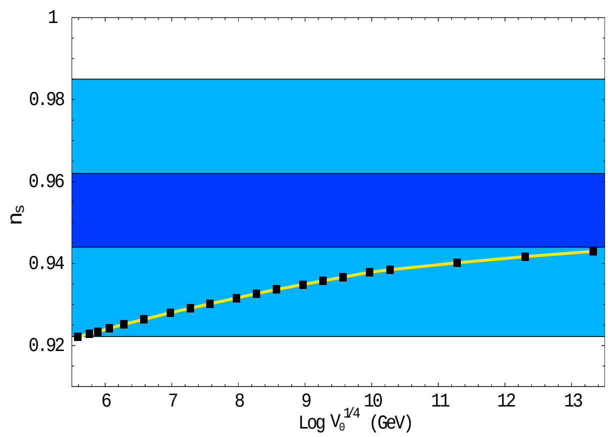

Figure 10.1: The spectral index $n_{\rm s}$ as a function of the logarithm of the scale of inflation, $\log V_0^{1/4}$, compared with WMAP3 [5] range for $n_{\rm s}$ (at 68% and 95% confidence levels)

As stated before, one can have one-loop contributions to the parameter A coming from loops containing a) fermions $\psi_i^{(Z)}$, b) the messenger field $\tilde{\varphi}_1^{(Z)}$, and c) the inflaton. The fermion loop contribution, of order $-|K_i|^4/16\pi^2$, can be estimated for the values of the parameters chosen in the previous numerical example. From (10.7) we get for $v_Z = 3 \times 10^9$, GeV

$$|K_i| \leq 6.7 \times 10^{-8} \tag{10.19}$$

which will then translate into the following contribution to the A parameter in the CW potential

$$A_\psi \approx -|K_i|^4/16\pi^2 \sim 10^{-31} \tag{10.20}$$

obviously being too small to be considered as contributing to it. Thus, fermion loops are completely negligible.

Next, we want to estimate what the contribution of the messenger field $\tilde{\varphi}_1^{(Z)}$ is. The leptogenesis bound (10.11) for $v_Z = 3 \times 10^9\,\text{GeV}$ becomes

$$\tilde{\lambda}_{1Z} \leq 10^{-13} \tag{10.21}$$

which gives the following contribution to the potential

$$A_{\tilde{\varphi}} \approx \tilde{\lambda}_{1Z}^2/16\pi^2 \sim 8 \times 10^{-29} \tag{10.22}$$

also being too small compared to the value $A = 3 \times 10^{-15}$. This means that the main contribution should come from σ_Z self-coupling λ. The necessary value of the λ coupling can be estimated by comparing its contribution $A_\sigma \approx \lambda^2/16\pi^2$ with A

$$A \simeq A_\sigma \approx \lambda^2/16\pi^2 \simeq 3 \times 10^{-15} \tag{10.23}$$

from which we obtain the constraint on λ

$$\lambda \simeq 6.9 \times 10^{-7}. \tag{10.24}$$

To end the discussion regarding inflation in our model, we would like to add that in our numerical study we obtained a small value for the running of the power spectrum, $\alpha \equiv \frac{\mathrm{d}n_\mathrm{s}}{\mathrm{d}\ln k} \simeq -0.002$. Other parameter that might be of interest is the tensor-to-scalar ratio r, which is defined usually as

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_R} \tag{10.25}$$

where $\mathcal{P}_T$ and $\mathcal{P}_R$ are the power spectra for tensor and scalar perturbations, respectively. In the slow-roll regime of inflation, r can be expressed in terms of the slow-roll parameters and, at first order, $r = 16\epsilon$, where ϵ has to be evaluated at horizon crossing. With the values used in our previous numerical example, we obtain a very small tensor-to-scalar ratio $r \sim 10^{-43}$, making the quest for gravitational wave detection from the inflationary epoch hopeless.

It is amusing to note that the value of the σ_Z self-coupling $\lambda \sim O(10^{-7})$ that is consistent with the data is of the same order as the constraint on the Yukawa coupling $|K_i|$ coming from the CDM scenario of [265].

10.3 Reheating

One of the most important questions of any inflationary scenario is the following: How do SM particles get generated at the end of inflation? In a generic inflationary model, it is not easy to answer this question since a generic inflaton is usually not coupled, either directly or indirectly, to SM particles. Although our inflaton is a SM-singlet field, we will show that its decay and the subsequent thermalization of the decay products can generate SM particles. In what follows, we will assume that the inflaton decays perturbatively as with the "old" reheating scenario and study its consequences. The interesting question of whether or not it can decay through the parametric resonance mechanism [52, 273] of "preheating" scenarios is beyond the scope of this paper and will be dealt with separately elsewhere.

At the end of inflation, the inflaton will rapidly roll down its potential to the true minimum. The reheating (or, equivalently, the damping of the inflaton oscillation) occurs via the decay

$$\sigma_Z \to \psi^{(Z)} + \bar{\psi}^{(Z)}. \tag{10.26}$$

The width of the decay (10.26) is given by

$$\Gamma(\sigma_Z \to 2\psi^{(Z)}) \simeq 9\left(\frac{m_\psi}{v_Z}\right)^2 \frac{m_\sigma}{8\pi}\beta \tag{10.27}$$

where $\beta = (1 - 4(m_\psi/m_\sigma)^2)^{3/2}$ and we remember that $m_\sigma = \sqrt{8A}\, v_Z$. To estimate the reheating temperature caused by the process (10.26) after the end of inflation, we write

$$\Gamma(\sigma_Z \to 2\psi^{(Z)}) \sim H_{\rm rh} \tag{10.28}$$

where $H_{\rm rh} \sim 1.66 T_{\rm rh}^2/M_{\rm P}$ is the Hubble parameter at the reheating temperature $T_{\rm rh}$. By combining Eqs. (10.27) and (10.28) we obtain the dependence of the reheating temperature $T_{\rm rh}$ on v_Z

$$T_{\rm rh} \simeq 1.3 \times 10^8 \left(\frac{v_Z}{\rm GeV}\right)^{-1/2}. \tag{10.29}$$

We see that the reheating temperature is a decreasing function of v_Z. This will set an upper bound on v_Z, because $T_{\rm rh}$ should be larger than twice the mass of $\psi^{(Z)}$ in order for the reheating mechanism to work, i.e.

$$T_{\rm rh} > 2m_\psi \sim 400\,{\rm GeV} \tag{10.30}$$

which combined with (10.29) gives

$$v_Z < 10^{11}\,{\rm GeV}. \tag{10.31}$$

This upper limit restrict us to a low-scale inflation range, $6 \times 10^5\,{\rm GeV} \leq {\rm V}_0^{1/4} \leq 2 \times 10^7\,{\rm GeV}$, and then great part of Fig. 10.1 will be excluded, unless some other reheating mechanism is invoked. The spectral index values as a function of the logarithm of the scale of inflation, in the allowed range, is shown in Fig. 10.2. Notice that the values of $n_{\rm s}$ are a bit smaller now than in the case of instant reheating, but still marginally compatible with the WMAP3 value for $n_{\rm s}$.

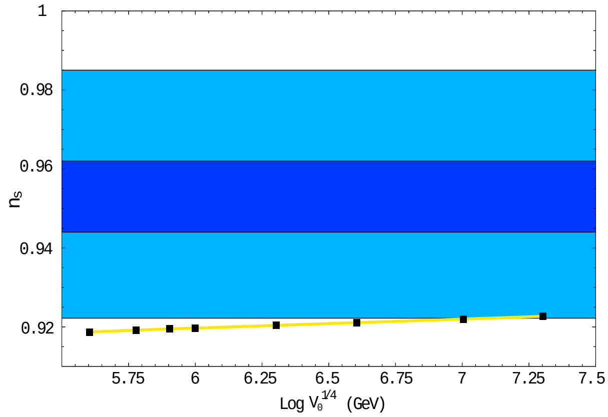

Figure 10.2: The spectral index n_s as a function of the logarithm of the inflationary scale $V_0^{1/4}$, compared with WMAP3 [5] range for n_s (at 68% and 95% confidence levels), in the range allowed after imposing constraints coming from the reheating mechanism

Let us focus now on the mechanism by which SM particles are produced. For the sake of clarity in the following discussion, we will denote the QCD gluons by $\tilde{g}$ and the $SU(2)_Z$ "gluons" by $\mathbf{G}$. The chain of reactions which finally leads to the SM particles can be seen as follows:

$$\psi^{(Z)} + \bar{\psi}^{(Z)} \to \mathbf{G}\,\mathbf{G} \to \tilde{\varphi}_1^{(Z)}\,\tilde{\varphi}_1^{(Z)} \to W\,W,\; Z\,Z \to q\,\bar{q},\; l\,\bar{l}\,, \tag{10.32}$$

and

$$q\,\bar{q} \to \tilde{g}\tilde{g}\,. \tag{10.33}$$

We end up with a thermal bath of SM and $SU(2)_Z$ particles. This thermalization is possible because of the simple fact that $\tilde{\varphi}_1^{(Z)}$ carries *both SM and $SU(2)_Z$ quantum numbers*. Another important point concerns the various reactions rate in (10.32,10.33). The corresponding amplitudes are proportional to $O(g^2)$, where g stands for either the $SU(2)_Z$ coupling or a typical SM coupling at an energy above the electroweak scale. From [265] and [266], it can be seen that the various gauge couplings are of the same order of magnitude for a large range of energy, from a typical GUT scale down to the electroweak scale. One can safely conclude that the various reaction rates are comparable in magnitudes and the thermalization process shown above is truly effective. In principle, the messenger field also couples to $\psi^{(Z)}$ and a SM lepton, as shown in [265], but this is irrelevant in the thermalization process because the corresponding Yukawa couplings are too small.

It is remarkable to notice also that, because of the quantum numbers of the messenger field, the decay of $\tilde{\varphi}_1^{(Z)}$ into a SM lepton and $\psi^{(Z)}$ can generate a net SM lepton number which is subsequently transmogrified into a net baryon number through the electroweak sphaleron process as shown in [267]. In other words, the crucial presence of the messenger field $\tilde{\varphi}_1^{(Z)}$ facilitates both the generation of SM particles through thermalization and the subsequent leptogenesis through its decay.

10.4 Conclusions

In this paper, we show that the model presented in [265], which explains dark matter and dark energy, also provides a mechanism for inflation in the early universe. We find that it is conceivable to have a low-scale inflation.

The complete model contains a new gauge group $SU(2)_Z$ which grows strong at a scale $\Lambda \sim 3 \times 10^{-3}$ eV, with the gauge coupling at GUT scale comparable to the SM couplings at the same scale. In addition to the SM particles, the model contains new particles: $\psi^{(Z)}_{(L,R),i} (i = 1, 2)$ which are $SU(2)_Z$ triplets and SM singlets; $\tilde{\varphi}^{(Z)}_{1,2}$ which are the so-called messenger fields and carry charges of both $SU(2)_Z$ and SM groups; and ϕ_Z, which is a singlet complex scalar field. The model also contains a new global symmetry $U(1)^{(Z)}_A$, which is spontaneously broken by the v.e.v. of the scalar field, $\langle\phi_Z\rangle = v_Z$, and also explicitly broken at the scale $\Lambda_Z \ll v_Z$ by the $SU(2)_Z$ gauge anomaly. The real part of the complex scalar field, namely σ_Z, is identified with the inflaton field. We considered a CW-type of potential for σ_Z and obtained the constraints on the parameters of the model in order to have a right description of inflation. The angular part of the complex scalar field, namely a_Z, acquires a small mass due to the explicit breaking of $SU(2)_Z$ and is trapped in a false vacuum, being responsible for the dark energy of the universe. The new particles $\psi^{(Z)}_{(L,R),i}$, with masses of order 200 GeV, explain the dark matter. They are produced at reheating, by the decay of the inflaton. For values of the $SU(2)_Z$ breaking scale $v_Z \sim 3 \times 10^9$ GeV, we obtain a low-scale model of inflation, namely a scale of $\sim 6 \times 10^5$ GeV. Notice that, in order to have a realistic reheating mechanism, this "low-scale" is also bounded from above by $\sim 2 \times 10^7$ GeV as we have discussed in the last section. Because of this fact, our model is a bona-fide "low-scale" inflationary scenario. It is an exciting possibility because the model might be indirectly probed at future LHC experiments.

Chapter 11

Scalar Field Oscillations Contributing to Dark Energy

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11.1 Introduction

The standard model of cosmology assumes the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{11.1}$$

corresponding to an homogeneous and isotropic universe. Here k is the curvature signature and R is the expansion factor, whose time change is given by the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_T - \frac{k}{R^2}. \tag{11.2}$$

We have written the equation in such a way that the cosmological constant is included in the total energy density ρ_T.

There are different contributions to ρ_T. Matter and radiation are among them. They can be introduced as a fluid with pressure proportional to the energy density, $p = w\rho$; $w = 0$ corresponds to matter and $w = 1/3$ to radiation. Recent results coming from high-redshift supernovae [6, 7], cosmic background radiation [82], [240]-[243] and galaxy survey [244]-[247] suggest contributions with $w \approx -1$. This is the so-called dark energy that leads to the present acceleration of the universe. This acceleration may be produced by a cosmological constant, since it has an equation of state with $w = -1$. However, there might be other causes, like quintessence [160]-[162], [201, 274], cardassian expansion [275], etc. Of course, there might be several components building up dark energy.

In this paper, we would like to discuss some aspects of the contribution to the energy density of coherent scalar field oscillations in a potential. The physics of such oscillations was analyzed in a pioneer work by Turner [214]. The purpose of our paper is the following. First, in Sect.11.2, we find the analytical form of the field oscillations in a couple of interesting cases. In Sect.11.3, we treat the general case and derive relevant physical results using the action and angle variables of analytical mechanics [235],[276]-[278]. We find this language very useful, particularly when an adiabatic change of the potential is present. Another objective of our paper is to show that scalar field oscillations may contribute to the dark energy of the universe. We do this in Sect.11.4, where we present some instances of potentials leading to an accelerated universe.

11.2 Analytical solutions for scalar field oscillations

A classical scalar field in a potential has the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi). \tag{11.3}$$

In the metric (11.1), and considering spatially homogeneous configurations, the equation of motion for the field is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{11.4}$$

Here dot means time-derivative and prime means $d/d\phi$.

In this section we concentrate on two particular potentials, $V \sim \phi^2$ and $V \sim \phi^4$, where one can find analytical solutions. While the solution for the harmonic potential is in the literature [127]-[129], the solution we find for ϕ^4 is new, as far as we know.

For the quadratic potential

$$V = \frac{1}{2}m^2\phi^2 \tag{11.5}$$

we have harmonic oscillations, $\phi(t) = A\sin(mt + \varphi)$, in the case $H = 0$.

When $H \neq 0$, but in the case $H \ll m$ and $\dot{H}/H \ll m$, we expect a time dependent amplitude and, perhaps, a different frequency. We make the following ansatz

$$\widetilde{\phi}(t) = A(t)\sin(\lambda mt + \varphi). \tag{11.6}$$

We find $\lambda = 1$ and the following equation

$$\dot{A} = -\frac{3}{2}HA. \tag{11.7}$$

The energy density ρ corresponding to $\widetilde{\phi}$ evolves as

$$\rho \propto A^2 \propto R^{-3} \tag{11.8}$$

where in the last proportionality we have used the solution of (11.7). This corresponds to the behavior of non-relativistic matter in the expanding universe.

The second potential to be solved is

$$V = \frac{1}{2}m^2\phi^4. \tag{11.9}$$

We shall work in complete analogy with the previous example. We first find the solution to (11.4) for $H = 0$, which is

$$\phi(t) = B\,\mathrm{sn}(Bmt + \varphi) \tag{11.10}$$

where sn is the Jacobi elliptic function $\mathrm{sn}(Bmt + \varphi, -1)$ [279].

For $H \neq 0$ we choose an ansatz of the form

$$\widetilde{\phi}(t) = B(t)\mathrm{sn}(B(t)\lambda mt + \varphi). \tag{11.11}$$

The calculus here is more complicated than in the previous example, when introducing (11.11) in (11.4). We obtain a differential equation for the evolution of the amplitude B:

$$\frac{d^2(Bt)^3}{dt^2} + \left(3H - \frac{2}{t}\right)\frac{d(Bt)^3}{dt} = 0. \tag{11.12}$$

This equation can be easily solved for the usual form $H = ct^{-1}$, where the value of the constant c depends on which kind of energy dominates the universe ($c = 1/2$ for a radiation-dominated universe and $c = 2/3$ for a matter-dominated one). From (11.12) we obtain

$$\frac{d(Bt)^3}{dt} \propto t^{2-3c} \tag{11.13}$$

that gives

$$B \propto t^{-c}. \tag{11.14}$$

Since we are assuming $H = ct^{-1}$, which gives $R \propto t^c$, we conclude that, for any value of c,

$$B \propto R^{-1}. \tag{11.15}$$

Also, we get an equation for λ, which now depends on the relative variation of B

$$\lambda = \left(1 + \frac{\dot{B}}{B}t\right)^{-1} = \frac{1}{1-c} \tag{11.16}$$

where in the last equality we took into account (11.14).

The energy density in this case is given by

$$\rho = \frac{1}{2}(\dot{\widetilde{\phi}})^2 + \frac{1}{2}m^2(\widetilde{\phi})^4 = \frac{1}{2}m^2B^4 \tag{11.17}$$

where we have used properties of the Jacobi elliptic functions [279]. Thus, taking into account (11.15), we have

$$\rho \propto B^4 \propto R^{-4}. \tag{11.18}$$

This type of R-dependence corresponds to a fluid of relativistic particles, or equivalently, to radiation. The form of the oscillations in a ϕ^4 potential with small friction H was also found in [280], in the context of preheating after inflation. The method used in [280] was to make a conformal transformation of the space-time metric and the fields, as well as a number of re-scalings. Even if we get the same final results, we present our method because it is somewhat simpler.

The two examples discussed here are particular cases in which it is possible to analytically solve the equation of motion that describes the oscillations of the field ϕ in the expanding universe. For other potentials that can be considered, it might be impossible to find analytical solutions to (11.4), so the method applied before might not succeed in obtaining the dependence of the energy density ρ on the scale factor R. For this purpose, one needs to find another method, in which the evolution of $\rho(R)$ can be obtained without necessity of solving the equation of motion (11.4). This is what we develop in the next section.

11.3 Action-angle formalism

We are concerned with the oscillations of the ϕ-field about some minimum of the potential, but we shall *not* restrict to the case that the oscillation amplitude is small. Thus, we are faced with a system with a periodic motion, whose details might be complicated. Often, we are not specially interested in these details, but rather in the frequencies of the oscillations, the contribution to the energy density, etc. A method of analytical mechanics tailored for such a situation is provided by the use of action-angle variables.

As before, it is convenient to start with $H = 0$ in (11.4); we have then a conservative system with hamiltonian density $\mathcal{H}$

$$\mathcal{H}(\phi, \Pi) = \frac{1}{2}\Pi^2 + V(\phi) \equiv \rho \tag{11.19}$$

with the momentum $\Pi = \dot{\phi}$ and energy density ρ. When having a periodic motion, one introduces [235],[276]-[278] the action variable

$$J \equiv \oint \Pi \, d\phi \tag{11.20}$$

where the integration is over a complete period of oscillation. Using $\Pi = \sqrt{2(\rho - V)}$ from (11.19), J can be written as

$$J = 2\int_{\phi_{\min}}^{\phi_{\max}} \sqrt{2(\rho - V)} d\phi \tag{11.21}$$

where ϕ_{min} and ϕ_{max} are the return points, $V(\phi_{min}) = V(\phi_{max}) = \rho$. J is chosen as the new (conserved) momentum in the integration of the Hamilton-Jacobi equation. The generating function given by the abbreviated action

$$W = \int \Pi d\phi \tag{11.22}$$

allows to canonically transform (ϕ, Π) into (α, J), with the angle variable defined by

$$\alpha = \frac{\partial W}{\partial J}. \tag{11.23}$$

Since W does not explicitly depend on time, the new hamiltonian $\bar{\mathcal{H}}$ coincides with the old one $\mathcal{H} = \rho$. The energy ρ is only a function of J, which amounts to say that α is cyclic and J is constant

$$\dot{J} = -\frac{\partial \bar{\mathcal{H}}}{\partial \alpha} = 0. \tag{11.24}$$

The other Hamilton equation is

$$\dot{\alpha} = \frac{\partial \bar{\mathcal{H}}(J)}{\partial J} \equiv \nu, \tag{11.25}$$

with $\nu = \nu(J)$ constant. We can integrate (11.25) to obtain

$$\alpha(t) = \nu t + \alpha(0) \tag{11.26}$$

and in a complete period, $\alpha(\tau) = \alpha(0)$. In this way, we identify ν in (11.25) with the frequency of the motion:

$$\nu = \frac{1}{\tau} = \frac{d\rho}{dJ}. \tag{11.27}$$

Until here, we have reminded ourselves of the standard approach that uses action-angle variables to find the frequency [235],[276]-[278].

The realistic case of the expanding universe, with $H \neq 0$, can be treated with the same method, provided H and $\dot{H}/H$ are small compared to the frequency of the ϕ oscillations, that is to say,

$$H \ll \nu\,; \quad \dot{H}/H \ll \nu. \tag{11.28}$$

Notice that for the usual form $H = ct^{-1}$, $\dot{H}/H \sim H$ so that the two conditions in (11.28) are actually the same. The conditions (11.28) ensure that the motion is almost periodic, and that it makes sense averaging over one cycle. The H-term in (11.4) represents a time-dependent friction, so we expect energy to decrease; however, we are assuming this friction small enough so that we can consider the energy constant in one-cycle period.

We still define J as in (11.20) and (11.21), and also the energy density ρ as in (11.19). The decrease rate of ρ can be obtained rewriting the equation of motion (11.4),

$$\frac{d\rho}{dt} = -3H\dot{\phi}^2. \tag{11.29}$$

When averaging the r.h.s. over one cycle, we can take H out of the average, $\langle H\dot{\phi}^2\rangle \simeq H\langle\dot{\phi}^2\rangle$. The average of $\dot{\phi}^2$ is related to J:

$$\begin{aligned} \langle\dot{\phi}^2\rangle &= \tfrac{1}{\tau}\oint \Pi\dot{\phi}\,dt \\ &= \tfrac{1}{\tau}\oint \Pi d\phi = \tfrac{1}{\tau}J \end{aligned} \tag{11.30}$$

(this is essentially the virial theorem). Thus,

$$\langle\dot{\phi}^2\rangle = \frac{d\rho}{dJ}J. \tag{11.31}$$

With the help of (11.31), (11.29) reads

$$\frac{1}{J}\frac{dJ}{dt} = -3H. \tag{11.32}$$

What is remarkable about this equation is that the change in the action variable J is *independent* of the form and details of V. The equation can be readily integrated to show that J dilutes as a volume in an expanding universe

$$J \propto R^{-3}. \tag{11.33}$$

Eqs.(11.32) and (11.33) are one of our main results. To find more, we notice that the pressure can also be averaged in one oscillation period,

$$p = \langle\frac{1}{2}\dot{\phi}^2\rangle - \langle V\rangle = \langle\dot{\phi}^2\rangle - \rho = w\rho. \tag{11.34}$$

To calculate w, we first use (11.31) and get

$$w = \frac{J}{\rho}\frac{1}{dJ/d\rho} - 1 \tag{11.35}$$

and, introducing (11.21), we finally obtain

$$w = \frac{2}{\rho}\frac{\int_{\phi_{\min}}^{\phi_{\max}}(\rho - V)^{1/2}d\phi}{\int_{\phi_{\min}}^{\phi_{\max}}(\rho - V)^{-1/2}d\phi} - 1. \tag{11.36}$$

Eq.(11.36) coincides with the corresponding expression in Ref.[214], where another formalism was used.

There is another useful relation that can be easily obtained using our formalism. Whenever w is constant, we can integrate (11.35) to obtain

$$\rho \propto J^{w+1}. \tag{11.37}$$

Then, making use of (11.33), we get the well-known relation

$$\rho \propto R^{-3(w+1)}. \tag{11.38}$$

As an example, let us apply the method described above for the following power-law potential

$$V = a\phi^n \tag{11.39}$$

for which the action variable can be exactly calculated

$$J = \frac{4\sqrt{2\pi}\,\Gamma\left(\frac{1}{n}\right)}{(n+2)\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}\sqrt{\rho}\left(\frac{\rho}{a}\right)^{1/n}. \tag{11.40}$$

Using (11.27), we get the frequency of the motion

$$\nu = \frac{n\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}{2\sqrt{2\pi}\,\Gamma\left(\frac{1}{n}\right)}\sqrt{\rho}\left(\frac{\rho}{a}\right)^{-1/n} \tag{11.41}$$

and using (11.35), we get the parameter w

$$w = \frac{n-2}{n+2}. \tag{11.42}$$

Let us show that these results coincide with what we obtained in Sect.11.2. For $n = 2$, we get from (11.42) that $w = 0$, which corresponds to the equation of state for non-relativistic matter, $p = 0$. Equivalently, from eq.(11.38), we obtain that

$$\rho \propto R^{-3} \tag{11.43}$$

which is the same as (11.8). Still for the harmonic potential, $n = 2$, taking $a = \frac{1}{2}m^2$, we can calculate J by making use of (11.40)

$$J = 2\pi\frac{\rho}{m} \tag{11.44}$$

which is nothing else than the number density of particles of mass m, except the factor of 2π. We now see that, in this case, (11.33) is equivalent to assert that the number density decreases as R^{-3} in an expanding universe.

For $n = 4$, (11.42) gives $w = 1/3$, and (11.38) gives $\rho \propto R^{-4}$ which coincides with (11.18) and corresponds to the equation of state for relativistic matter.

As expected, we have recovered the same results of Sect.11.2 without the need to solve the equation of motion, eq(11.4). For this reason, the action-angle formalism will allow us to study more general cases of potentials for which there is no analytical solution to (11.4). This is what we will do in the next section.

As another application of the action-angle variables, we discuss the issue of adiabatic invariants. When a parameter a of the potential changes with time slowly compared with the natural frequency of the system,

$$\frac{\dot{a}}{a} \ll \nu, \tag{11.45}$$

we say it changes adiabatically. When there is such an explicit time change in the potential, W in (11.22) depends on time, and $\bar{\mathcal{H}}$ and ρ do not coincide any longer

$$\bar{\mathcal{H}} = \rho + \frac{\partial W}{\partial t} = \rho + W_{,a}\dot{a} \tag{11.46}$$

where the derivative $W_{,a} \equiv \partial W/\partial a$ is taken at constant J. Instead of (11.32), we now have

$$\begin{aligned} \dot{J} &= -3HJ - \langle \partial \bar{\mathcal{H}}/\partial \alpha \rangle \\ &= -3HJ - \langle (\partial W_{,a}/\partial \alpha)\dot{a} \rangle . \end{aligned} \tag{11.47}$$

When averaging this identity, due to (11.45), we can consider $\dot{a}$ as constant during a cycle, i.e., $\langle (\partial W_{,a}/\partial \alpha)\dot{a} \rangle \simeq \langle (\partial W_{,a}/\partial \alpha) \rangle \dot{a}$. The average $\langle \partial W_{,a}/\partial \alpha \rangle$ vanishes, since $W_{,a}$ is a periodic function of α. It follows that

$$\dot{J} = -3HJ, \tag{11.48}$$

namely, eqs.(11.32) and (11.33) are still valid, although we stress that V is changing with time. For $H = 0$, the system is conservative and we have $\dot{J} = 0$, i.e., J is identified with an adiabatic invariant [235],[276]-[278].

An instance where we can apply these results is the invisible axion model. In this model we have a time-dependent axion mass, $m = m_a(t)$ in (11.5). Even if $\rho(t)$ and $m_a(t)$ are complicated functions of time, we conclude after our study that, provided the change is adiabatic, the combination

$$\frac{J}{2\pi} = \frac{\rho(t)}{m_a(t)} \propto R^{-3} \tag{11.49}$$

behaves as the axion number density. This behavior is what is expected on general grounds. This result was derived in [127]-[129] when calculating the relic axion density. What we have done in the present paper is a kind of generalization of (11.49), that may be applied when having other potentials and/or other types of adiabatic change.

11.4 Dark energy from field oscillations

In the standard model of cosmology, besides eq.(11.2), there is another equation that one can get from Einstein equations. It can be put in the form

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p). \tag{11.50}$$

For the simple equation of state $p = w\rho$, we see that $w < -1/3$ leads to an accelerated universe. We will show in this section that such values of w can be obtained from an oscillating scalar field whose energy density dominates. An application based on this idea in the context of inflation was developed by Damour and Mukhanov in [281] (see also [282]).

In order to check whether a given potential may lead to acceleration, we will calculate w using the expression (11.35), that involves an average over one cycle [1]. The value of w

[1] In [281], a nice geometric interpretation of the condition $w < -1/3$ is given.

depends on the form of the potential and on the return values ϕ_{max} and ϕ_{min}. In turn, these change with time due to the friction produced by the expansion of the universe, so that, in general, w depends on time.

A remarkable exception to this dependence on the return values is given by the power-law potential (11.39). As shown in (11.42), w only depends on the power n. For integer n, we do not have oscillations neither for $n = 0$, nor for $n = 1$. The potential $V = a\phi^2$ leads to oscillations that correspond to non-relativistic matter with $w = 0$. The next potential leading to oscillations is $V = a\phi^4$, that we saw it implies $w = 1/3$. Higher values of the power n lead to higher values for w. No integer value of n, leading to oscillations, can correspond to a fluid with $w < -1/3$ [2].

In order to have smaller w values, one clearly needs a field ϕ that, even oscillating, has a slow-roll for enough time. To illustrate that this is possible, we shall investigate a few potentials. In all of them, we shall put the center of the oscillations at $\phi = 0$. In addition, we will work with symmetric potentials, so that $-\phi_{min} = \phi_{max} \equiv \phi_0$.

Let us start with the "Mexican hat" potential shown in Fig.11.1,

$$V_1(\phi) = \lambda_1(\phi^2 - v_1^2)^2 \tag{11.51}$$

with λ_1 a coupling and v_1 an energy scale. We consider $\phi_0 > \sqrt{2}v_1$, to have oscillations around $\phi = 0$. The parameter w, calculated numerically, is a function of ϕ_0/v_1 and it is displayed in Fig.11.2. We see that when ϕ_0 is close enough to $\sqrt{2}v_1$, we have values $w < -1/3$. However, an unsatisfactory feature is that one has to tune the return value ϕ_0.

A potential that does not have this fine-tuning problem is given by

$$V_2(\phi) = \frac{{v'_2}^4\phi^2}{\phi^2 + v_2^2} \tag{11.52}$$

(see Fig.11.3). Here, v_2 and v'_2 are energy scales. The values of w are shown in Fig.11.4, where we see that $w < -1/3$ for $\phi_0 > 1.2\,v_2$. To get the observed acceleration of the universe, we should have $v'_2 \simeq 2 \times 10^{-3}$ eV, as expected. Notice that in the past, field oscillations give $w \simeq -1$ always. As happens in the case of a cosmological constant, the contribution we are discussing was subdominant in the past, as soon as non-relativistic matter enters in the stage and dominates.

One could find other potentials giving acceleration. One such example was discussed in [281]. There, the objective was to get first slow-roll inflation and, after, oscillations

[2] The case of a non-integer $n < 2$ is considered in [283]. Power law potentials with scaling solutions are studied in [284].

provoking further inflation. In contrast to [281], we work in the period of dark energy domination and we would like to know the future of the universe. We see in Fig.11.4 that the oscillations will end up giving $w = 0$, i.e., being harmonic and consequently matter dominated.

One may think that for all potentials, when the field is close enough to the minimum, one ends with the usual harmonic potential (11.5) that well approximates oscillations around the minimum, so that one obtains $w = 0$. We would like to point out that there is at least one exception. Consider the potential

$$V_3(\phi) = {v'_3}^4 e^{-v_3^2/\phi^2} \tag{11.53}$$

with v_3 and v'_3 some energy scales (see Fig.11.5). An inspection to the potential shows that for ϕ_0 large enough, we should have $w < -1/3$. We have calculated w and our results are shown in Fig.11.6. Indeed, we see that $w < -1/3$ for $\phi_0 > 1.8\, v_3$. For smaller values of ϕ_0, higher values of w are obtained. We notice that in the limit $\phi_0/v_3 \to 0$ the value $w = 0$ is not obtained, but rather $w = 1$. This corresponds to a fluid with $p = \rho$. The reason for this very particular behavior is the well-known fact that $V_3(\phi)$ in (11.53) can not be developed in a Taylor-Maclaurin series around $\phi = 0$. Again we should check that V_3 is consistent with the observational constraints. With $v'_3 \simeq 2 \times 10^{-3}$ eV the oscillations fit the acceleration. Also, for early times $w \simeq -1$ so that the oscillations do not contradict the observed evolution of the universe.

Let us finally address the requirement that dark energy should have very homogeneous density, with spatial irregularities rearranging at an effective relativistic speed of sound c_s, given by

$$c_s^2 = \frac{dp}{d\rho} \tag{11.54}$$

In the main spirit of our paper, we should point out that (11.54) can be calculated following the technique of action-angle variables. This allows to find c_s without reference to the details of the solution to the equation of motion of the field. The way we do it is through the equation

$$c_s^2 = \frac{d(w\rho)}{d\rho} = \frac{\rho dw/d\phi_0}{d\rho/d\phi_0} + w = \frac{V_0 dw/d\phi_0}{dV_0/d\phi_0} + w \tag{11.55}$$

where $V_0 \equiv V(\phi_0) = \rho$ and w was found in (11.35) or (11.36).

As a check, we have calculated the integrals (11.55) for the potential (11.5) and obtained $c_s^2 \simeq 0$, as expected. The implications of a scalar field with the potential (11.5) when it dominates the energy density of the universe have been fully investigated in ref.[285]. Here we do not pretend to do such a complete study for the potentials of Sect.11.4 since our emphasis is in the techniques developed in Sect.11.3. Let us point out, however, that in fact we find that for the potentials (11.51), (11.52) and (11.53), c_s^2 is near zero or even negative, which indicates that collapse at small scales would happen or even would lead to dangerous instabilities.

It is not difficult, however, to find potentials giving $w \leq -1/3$ and $c_s^2 \simeq 1$. One example is

$$V_4(\phi) = v_4'^4 \frac{\phi^2}{v_4^2 + \phi^2} + \frac{1}{4}\lambda\phi^4 \tag{11.56}$$

We have verified that w from (11.36) and c_s^2 from (11.55) for the potential V_4 in (11.56) have the expected values for a dark energy contribution, for a suitable range for the potential free parameters.

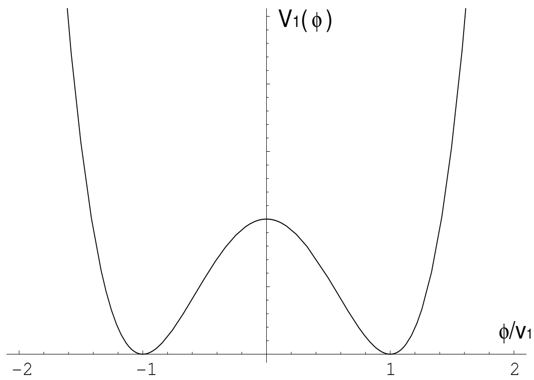

Figure 11.1: "Mexican hat" potential V_1 defined in (11.51).

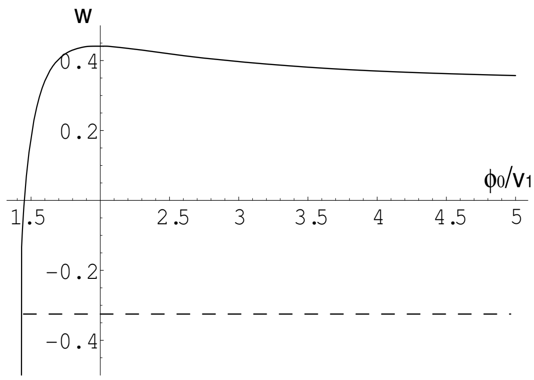

Figure 11.2: w as a function of the amplitude ϕ_0 of the oscillations in the potential V_1. The dashed line indicates $w < -1/3$.

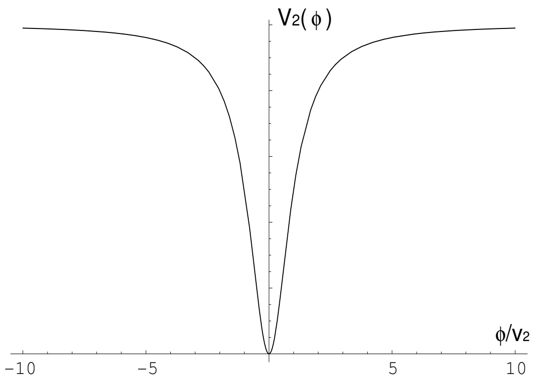

Figure 11.3: The shape of the potential V_2 defined in (11.52).

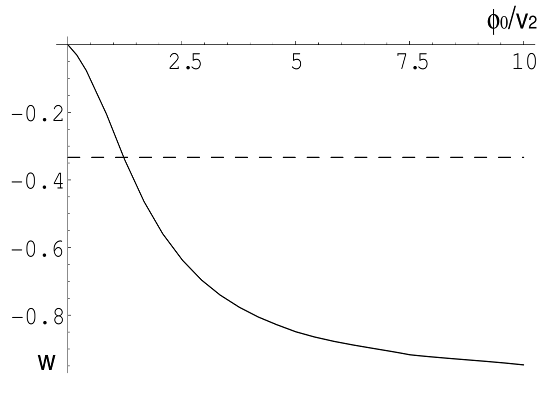

Figure 11.4: w as a function of the amplitude ϕ_0 of the oscillations in the potential V_2. The dashed line indicates $w < -1/3$.

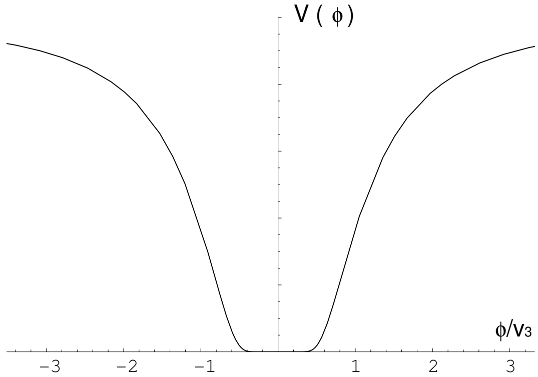

Figure 11.5: The shape of the potential V_3 defined in (11.53).

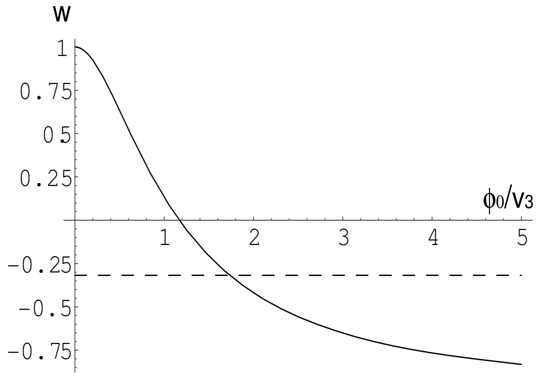

Figure 11.6: w as a function of the amplitude ϕ_0 of the oscillations in the potential V_3. The dashed line indicates $w < -1/3$.

Chapter 12

Conclusions

In this PhD Thesis, I presented a few original models proposed to solve some of the shortcomings of the standard cosmology, as possible extensions of the Big Bang model. In the introductory chapters I described some of the main features of the standard Big Bang cosmological model, which is the best cosmological theory today. It is able to explain a large number of observational data and also to describe the evolution of the universe from its very early states. As the observations become more and more precise, the level of our knowledge of the observable universe becomes more and more elevated. Based on these observations, we can corroborate the theoretical predictions and, also, we are able to indicate the points where the theory needs to be modified or extended.

Thus, today we are able to answer to many of the questions that humanity has always made. We know that our universe originated some 13.7 Gyr ago from a tremendous explosion known as the "Big Bang", and we can describe the universe back in time up to its very early states, when it was a small fraction of a second old.

Nevertheless, our theoretical knowledge of the universe is limited, and we cannot extrapolate the theory back to time zero, nor can we explain the recent observations that indicate that the nature of 95% of the energy content of the universe is unknown to us. The substance we are made of and that is so familiar to us, i.e. baryons, has always been a subdominant component during the evolution of the universe, and today it only represents a small fraction of the total content of the universe.

The objectives of this PhD Thesis are to find some possible extensions of the standard cosmological theory, in two opposite directions: backward in time towards higher energies, and forward in time towards smaller energies. In the primeval universe, about 10^{-30} s after the bang, it is believed that the universe suffered a short period of accelerated expansion known as inflation. In the recent history of the universe, there is evidence of dark matter and dark energy, which cannot be explained by the standard theory. Although the energy scales characteristic to inflation, dark matter and dark energy are very different, one of the objectives of this doctoral Thesis is to give a unitary description of the physics at these different scales. A large part of this doctoral Thesis was dedicated to construct a few models for the unification of inflation, dark energy and/or dark matter.

Symmetries play an important role in any physical theory describing different processes that occur in the universe. It is expected that the early universe should possess additional symmetries, which today are not observed because they are broken. If the additional symmetries are global, one also expects that Quantum Gravity effects break them explicitly. In chapters 8 and 9 of this Thesis I investigated this possibility, by considering a new simple global $U(1)$ symmetry. The Quantum Gravity effects on this new global symmetry are parameterized by the presence in the Lagrangian of effective operators of dimensions higher than 4, suppressed by powers of Planck mass, which are of the form:

$$-g\frac{1}{M_P^{n-3}}|\psi|^n\left(\psi e^{-i\delta}+\psi^\star e^{i\delta}\right) \tag{12.1}$$

where g is an effective coupling. The models of chapters 8 and 9 introduce a new complex scalar field ψ, invariant under the global $U(1)$ symmetry, and a new scalar field χ neutral under $U(1)$. In both models the real part of the field ψ is responsible for inflation, by interacting with the field χ, and the real part θ becomes a PGB due to the explicit $U(1)-$breaking terms. In Chapter 8, imaginary component θ plays the role of a quintessence field contributing to the dark energy of the universe. I obtained that for this to be the case, the effective coupling g, parameterizing the explicit $U(1)-$breaking, should be exponentially suppressed, at the level $g \lesssim 10^{-119}$.

In Chapter 9 instead, I studied the possibility that the particle associated to the field θ could be a dark matter particle. In this case, I conclude that the explicit breaking effects should also be exponentially suppressed, but at a less stringent level $g \lesssim 10^{-30}$. The work of Chapter 9 also contains a detailed numerical analysis for determining the parameters of the model, which can also be applied to the work developed in Chapter 8.

In this way, the models of chapters 8 and 9 relate a primeval process like inflation with the recent dominating dark energy and dark matter, respectively, as being components of the same field ψ. This mechanism for producing inflation has the features of an inverted hybrid model of inflation. After inflation, the angular field θ settles at an arbitrary value of order the $U(1)$ breaking scale, v, and has a non-zero potential energy due to Quantum Gravity effects.

Apart from global symmetries, in the early universe one may have new gauge symmetries and fields associated to these symmetries. In the work of Chapter 10, a new gauge group $SU(2)_Z$ is postulated, which is supposed to become strong at a scale $\Lambda \sim 10^{-3}$ eV, with the gauge coupling at GUT scale comparable to the SM couplings at the same scale. The model introduces new particles, like fermions $\psi^{(Z)}_{(L,R),i} (i = 1, 2)$, which are $SU(2)_Z$ triplets and SM singlets; messenger fields $\tilde{\phi}^{(Z)}_{1,2}$, which carry charges of both $SU(2)_Z$ and SM groups, and a singlet complex scalar field ϕ_Z. This scalar is invariant under a new global symmetry $U(1)^{(Z)}_A$, which is spontaneously broken by the vev of the scalar field, $\langle \phi_Z \rangle = v_Z$, and explicitly broken at the scale $\Lambda_Z \ll v_Z$ by the $SU(2)_Z$ gauge anomaly. In this model, inflation is produced by the real part of the field ϕ_Z, while the angular field a_Z gets trapped in a false vacuum of its potential and is responsible for the dark energy of the universe. The new fermions $\psi^{(Z)}_{(L,R),i}$ are produced during reheating after inflation, and for masses of order of 200 GeV they may explain dark matter. The model also provides a mechanism for producing a lepton number asymmetry, which leads to a baryon asymmetry through instanton effects.

Usually, a scalar field in a sufficiently flat potential may be slowly rolling down the potential and, consequently, may cause the acceleration of the universe. This is the case of inflation or dark energy, which occur when the inflaton or the quintessence field dominates the energy density of the universe. However, in Chapter 11, I investigated the possibility of having an accelerated expansion of the universe produced by a scalar field that is oscillating in a potential, with a frequency much larger than the Hubble expansion rate. These oscillations are described in the framework of analytical mechanics, by using the action-angle variables, and has the advantage of inferring the frequency of the oscillations without the need to solve the equation of motion. The effective equation of state parameter can be obtained as a function of the potential, by using this formalism. I gave a few instances of potentials for which the resulting equation of state associated to field oscillations satisfied the condition for having an accelerating universe, thus being regarded as a possible explanation of the dark energy of the universe. Another interesting feature of the formalism is the identification of adiabatic invariants, which are quantities that are preserved by the expansion in the condition in the condition that the system contains an adiabatically changing parameter.

Appendix A

Was the universe open or closed before inflation?

In collaboration with *E. Massó* and *S. Mohanty*
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A.1 Introduction

The curvature of the universe goes down exponentially after the start of inflation [16]. If there is a residual curvature still present by the time the scales which are entering our horizon at present were leaving the inflationary horizon there will be deviation from the scale invariant perturbations due to non-zero curvature. The corrections to the power spectrum at horizon scales are multiplicative powers of $(1 - K/\beta^2)$, where the curvature $K = (\Omega_0 - 1)(a_0 H_0)^2$ and $\beta = \sqrt{k^2 + K}$ is the comoving canonical wavenumber. We calculate the primordial power spectrum for the case closed and open universe at the time of inflation. We choose the Bunch-Davies boundary condition to normalize the wave-functions. We find that for the case of a closed universe the power spectrum of curvature perturbations is

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1+\frac{K}{\beta^2}\right)^2}, \qquad \frac{\beta}{\sqrt{K}} = 3, 4, 5\cdots \ (\text{for } K > 0). \tag{A.1}$$

and for the case of inflation in an open universe

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1-\frac{|K|}{\beta^2}\right)^2\left(1+\frac{|K|}{\beta^2}\right)}, \qquad \frac{\beta}{\sqrt{|K|}} > 1, \ (\text{for } K < 0). \tag{A.2}$$

where β are the eigenvalues of the radial-part of the Laplacian. In the case of closed universe β takes discrete values in units of $\sqrt{K} = R_c^{-1}$ (R_c being the curvature radius), the modes corresponding to $\beta/\sqrt{K} = 1, 2$ can be eliminated by gauge transformations [286] so there is a large-wavelength cutoff at $\beta_c^{-1} = R_c/3$. This large wavelength cut-off in a closed universe has been used to explain the observed low CMB anisotropy at low multi-poles [287] and [288]. In the case of open universe only modes with $\beta > \sqrt{|K|}$ cross the inflationary horizon.

Our result for the power spectrum in the closed and open universe cases differs from the phenomenological power spectrum [289],

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\ \frac{1}{1+\frac{K}{\beta^2}} \tag{A.3}$$

used in the calculation of CMB anisotropies in both the closed and open cases. Our results agrees qualitatively with (A.3) in that the power at small β is suppressed in the closed universe inflation (A.1) and enhanced in the open universe (A.2).

According to inflation [16], the curvature of the present universe $\Omega_0 - 1$ is related to the curvature at any time during inflation $\Omega_i - 1$ as

$$\frac{\Omega_0 - 1}{\Omega_i - 1} = \left(\frac{a_i H_i}{a_0 H_0}\right)^2 \tag{A.4}$$

If a_i is the scale factor at the time during inflation when scales of the size of our present horizon were exiting the inflationary horizon then $a_0 H_0 = a_i H_i$ and $\Omega_0 = \Omega_i$. If the curvature at the beginning of inflation $(\Omega_{start} - 1) = O(1)$ then in order to have a deviation of say one-percent from unity in the present curvature, the number of e-foldings prior to the a_i must be small. Putting an upper bound on the present curvature $(\Omega_0 - 1)$ from observations also puts a lower bound on the number of extra e-foldings necessary in inflation in addition to the minimum number needed to solve the horizon problem [290].

The geometry of the universe can be determined from the CMB anisotropy from the angular size of the acoustic peak. However the constraints on the density of the universe Ω depends upon priors like the value H_0 and Ω_λ. For example the combination of WMAP, LSS and HST supernovae observations gives a constraint on the density of the universe as $(\Omega - 1) = 0.06^{+0.02}_{-0.02}$ [291] which means that the curvature at one-σ could be as large as $K/(a_0 H_0)^2 = 0.08$. In the case of the closed universe the power spectrum (A.1) $P_{\mathcal{R}} \propto (1 + K/\beta^2)^{-2}$ at the scale of the horizon $\beta = a_0 H_0$ is suppressed by about 16% compared to the power for the flat universe. The effect of curvature shows in the temperature anisotropy at low-multipoles in addition to the location of the angular peaks. In a following paper we will use our formulae for the power spectrum in closed (A.1) and open universe inflation (A.2) to determine the best fit value of curvature K from the WMAP data.

A.2 Scalar power spectrum

We expand the inflaton field $\phi(\mathbf{x}, t) \equiv \phi(t) + \delta\phi(\mathbf{x}, t)$, where the perturbations $\delta\phi$ around the constant background $\phi(t)$ obey the minimally coupled KG equation

$$\ddot{\delta\phi} + 3\frac{\dot{a}}{a}\dot{\delta\phi} - \frac{1}{a^2}\nabla^2\delta\phi = 0. \tag{A.5}$$

With the separation of variables

$$\delta\phi(\mathbf{x}, t) = \sum_k \delta\phi_k(t) Q(\mathbf{x}, k) \tag{A.6}$$

the KG equation can be split as

$$\ddot{\delta\phi}_k + 3\frac{\dot{a}}{a}\dot{\delta\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0 \tag{A.7}$$

$$\nabla^2 Q(\mathbf{x}, k) = -k^2 Q(\mathbf{x}, k) \tag{A.8}$$

where ∇^2 is the Laplacian operator for the spatial part. Making the transformation $d\eta = dt/a$ and $\sigma(\eta, k) = a(\eta)\delta\phi_k(\eta)$ we get the KG equation for $\sigma(\eta, k)$

$$\sigma'' + (k^2 - \frac{a''}{a})\sigma = 0 \tag{A.9}$$

where primes denote derivatives w.r.t conformal time η.

The Friedman equations in conformal time are,

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho\, a^2 - K \tag{A.10}$$

$$\left(\frac{a'}{a}\right)' = -\frac{4\pi G}{3}(\rho + 3p)a^2. \tag{A.11}$$

Consider the universe with cosmological constant and curvature, then $\rho = \rho_\lambda$ and $p = -\rho_\lambda$ and we get using the Friedman equations,

$$\frac{a''}{a} = \frac{16\pi G}{3}\rho_\lambda\, a^2 - K \equiv 2a^2 H_\lambda^2 - K \tag{A.12}$$

where $H_\lambda = (\frac{8\pi G}{3}\rho_\lambda)^{1/2}$ is the Hubble parameter during pure inflation. Substituting (A.12) in the KG equation (A.9) we obtain,

$$\sigma'' + (k^2 - 2a^2 H_\lambda^2 + K)\sigma = 0. \tag{A.13}$$

The curvature affects the wave equation of $\sigma(\eta)$ in the explicit dependence K and also in the changed dynamics of $\eta-$dependence of the scale factor a which is important in the early stages of inflation.

The scalar field perturbation can be written as

$$\delta\phi(\mathbf{x}, \eta) = \sum_k \frac{\sigma(\eta, k)}{a(\eta)}\, Q(\mathbf{x}, k) \tag{A.14}$$

where $\sigma(\eta)$ is the solution of equation (A.13) and the spatial harmonics $Q(\mathbf{x}, k)$ are solutions of equation (A.8) [286]. One can separate the radial and angular modes of $Q_\beta^{lm}(r, \theta, \phi)$ as

$$Q_\beta^{lm}(r, \theta, \phi) = \Phi_\beta^l(r)\, Y_l^m(\theta, \phi) \tag{A.15}$$

where $\beta = (k^2 + K)^{1/2}$ are the eigenvalues of the radial-part of the Laplacian with eigenfunctions given by the hyperspherical Bessel functions $\Phi_\beta^l(r)$ which are listed in [286]. In the limit $K \to 0$, the radial eigenfunctions $\Phi_\beta^l(r) \to j_l(k\, r)$. The main properties that are needed for the calculation of the power spectrum are orthogonality

$$\int \gamma r^2 dr d\Omega Q_\beta^{lm}(r, \theta, \phi) Q_{\beta'}^{*\,l'm'}(r, \theta, \phi) = \frac{1}{\beta^2}\delta_{ll'}\delta_{mm'}\delta_{\beta\beta'} \tag{A.16}$$

where $\gamma = (1 + \frac{Kr^2}{4})^{-3}$ is the determinant of the spatial metric, and completeness

$$\sum_{l,m} \int \beta^2 d\beta Q_\beta^{lm}(r,\theta,\phi) Q_\beta^{*lm}(r',\theta',\phi') = \gamma^{-1} \frac{1}{r^2} \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \tag{A.17}$$

In case of closed universe the integral over β is replaced by sum over the integers $\beta/\sqrt{K} = 3, 4, 5....$ For open and flat universes β is a real non-negative variable.

The gauge invariant perturbations are a combination of metric and inflaton perturbations. The curvature perturbations are gauge invariant and at super-horizon scales are related to the inflaton perturbations as

$$\mathcal{R}(\mathbf{x},\eta) = \frac{H}{\dot{\phi}} \delta\phi(\mathbf{x},\eta). \tag{A.18}$$

Curvature perturbations generated during inflations are frozen outside the horizon till they re-enter in the radiation or matter era. CMB anisotropies at large angles are caused by curvature perturbations in the surface of last scattering which enter in the matter era. The Sachs-Wolfe effect at large angles, relates the temperature perturbation in the direction $\hat{\mathbf{n}}$ observed by the observer located at the point $(\mathbf{x_0}, \eta_0)$ to the curvature perturbation at the point $(\mathbf{x}_{LS}, \eta_{LS})$ in the LSS,

$$\frac{\delta T(\mathbf{x_0}, \hat{\mathbf{n}}, \eta_0)}{T} = \frac{1}{5} \mathcal{R}(\mathbf{x}_{LS}, \eta_{LS}) \tag{A.19}$$

where $\mathbf{x}_{LS} = \hat{\mathbf{n}}(\eta_{LS} - \eta_0)$. Using the completeness of $Q_\beta^{lm}(r,\theta,\phi)$ we can expand $\mathcal{R}$ as a sum-over the eigenmodes,

$$\mathcal{R}(\mathbf{x}_{LS}, \eta_{LS}) = \sum_{lm} \int \beta^2 d\beta \left[\frac{H}{\dot{\phi}} \, \delta\phi_\beta(\eta) \right]_{\eta=\eta_*} Q_\beta^{lm}(\mathbf{x}_{LS}). \tag{A.20}$$

Here we have used the fact that $\mathcal{R}$ does not change after exiting the horizon during inflation (at a conformal time which we shall denote by η_*) till it re-enters the horizon close to the LS era. Using the Sachs-Wolfe relation (A.19) and the mode expansion of the curvature perturbation (A.20) and using the orthogonality (A.16) of Q_β^{lm}, we obtain

$$\left\langle \frac{\delta T(\hat{\mathbf{n}}_1)}{T} \frac{\delta T(\hat{\mathbf{n}}_2)}{T} \right\rangle = \sum_l \frac{2l+1}{4\pi} P_l(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) \int \beta^2 d\beta \frac{1}{25} |\mathcal{R}(\beta,\eta_*)|^2 \, |\Phi_\beta^l(\eta_0 - \eta_{LS})|^2 \tag{A.21}$$

The angular spectrum C_l of temperature anisotropy defined by

$$\left\langle \frac{\delta T(\hat{\mathbf{n}}_1)}{T} \frac{\delta T(\hat{\mathbf{n}}_2)}{T} \right\rangle = \sum_l \frac{2l+1}{4\pi} P_l(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) \, C_l \tag{A.22}$$

can be written in terms of the power spectrum of curvature perturbations by comparing (A.22) with (A.21),

$$C_l = 4\pi \int \frac{d\beta}{\beta}\frac{1}{25}|P_{\mathcal{R}}(\beta)|^2\,|\Phi^l_\beta(\eta_0-\eta_{LS})|^2 \tag{A.23}$$

where the power spectrum of curvature perturbations is defined as

$$P_{\mathcal{R}}(\beta) = \frac{\beta^3}{2\pi^2}\left[\left(\frac{H}{\dot\phi}\right)^2|\delta\phi_\beta(\eta)|^2\right]_{\eta=\eta_*}. \tag{A.24}$$

We shall now derive the power spectrum for the open and closed inflation universes.

A.3 Closed universe inflation

For a closed universe, $K>0$, from the Friedman equation (A.10) we get

$$\dot a = H_\lambda a\sqrt{1-\frac{K}{H_\lambda^2 a^2}} \tag{A.25}$$

which can be integrated to give

$$a(t) = \frac{\sqrt{K}}{H_\lambda}\cosh H_\lambda t. \tag{A.26}$$

We consider as initial conditions the moment when the inflaton energy density starts to dominate over the curvature, i.e. for $t=0$ we have $a(0)=\frac{\sqrt{K}}{H_\lambda}$. The conformal time is then given by

$$\eta(a) = \frac{-1}{\sqrt{K}}\arcsin\frac{\sqrt{K}}{aH_\lambda}. \tag{A.27}$$

The conformal time spans the interval $\eta=(-\frac{\pi}{2\sqrt{K}},0)$ as the scale factor a varies between $(\frac{\sqrt{K}}{H_\lambda},\infty)$, so for $K\neq 0$ our initial conditions are different from the standard inflation case. The dependence of the scale factor on the conformal time is obtained from (A.27)

$$a(\eta) = -\frac{\sqrt{K}}{H_\lambda}\frac{1}{\sin\sqrt{K}\eta}. \tag{A.28}$$

The conformal time KG equation (A.13) for the closed-inflationary universe is of the form

$$\sigma''(\eta) + \left[k^2 - K\left(2\mathrm{cosec}^2\sqrt{K}\eta - 1\right)\right]\sigma(\eta) = 0. \tag{A.29}$$

This equation can be solved exactly and the solutions are

$$\begin{aligned}\sigma(\eta) &= c_1\left(-\sqrt{K}\cot\sqrt{K}\eta + i\sqrt{k^2+K}\right)e^{i\sqrt{k^2+K}\eta} \\ &+ c_2\left(-\sqrt{K}\cot\sqrt{K}\eta - i\sqrt{k^2+K}\right)e^{-i\sqrt{k^2+K}\eta}. \end{aligned} \tag{A.30}$$

The normalization constants c_1 and c_2 are determined by imposing the Bunch-Davies vacuum condition i.e the assumption that in the infinite past limit $\eta \to -\pi/(2\sqrt{K})$, σ is a plane wave which obeys the canonical commutation relation

$$\sigma^*\sigma' - \sigma^{*\prime}\sigma = i \tag{A.31}$$

from which we obtain

$$|c_1| = \frac{1}{\sqrt{2}(k^2+K)^{3/4}}, \quad c_2 = 0. \tag{A.32}$$

Replacing these constants into (A.30) we find that in the limit $\eta \to -\pi/(2\sqrt{K})$,

$$\sigma(\eta \to -\pi/(2\sqrt{K})) \equiv \sigma_{BD} = \frac{1}{\sqrt{2\beta}}e^{i\beta\eta} \tag{A.33}$$

where

$$\beta = (k^2+K)^{1/2}. \tag{A.34}$$

We will assume that the vacuum state of the universe $|0\rangle$ is the state in which there are no σ_{BD} particles. The creation and annihilation operators are for the the BD vacuum can be written as

$$\sigma_{BD}(\mathbf{x},\eta) = \sum_{lm}\int \beta^2 d\beta\left(a_{\beta lm}\, Q_\beta^{lm}(\mathbf{x})\,\frac{e^{i\beta\eta}}{\sqrt{2\beta}} \;+\; a^\dagger_{\beta lm}\, Q^{*lm}_\beta(\mathbf{x})\,\frac{e^{-i\beta\eta}}{\sqrt{2\beta}}\right) \tag{A.35}$$

Using the commutation relation (A.31) of σ_{BD} and the orthogonality of Q_β^{lm} (A.16) we see that the creation and annihilation operators obey the canonical commutation relations

$$[a_{\beta lm}, a^\dagger_{\beta' l' m'}] = \frac{1}{\beta^2}\delta(\beta-\beta')\delta_{ll'}\delta_{mm'}. \tag{A.36}$$

From the foregoing discussion it is clear that β is the radial canonical momentum. The quantum fluctuations become classical when $\beta = aH$. We shall evaluate the power spectrum at horizon crossing, as the modes do not change after exiting the inflation horizon till they re-enter the horizon in the radiation or matter era.

Substituting the constants c_1 and c_2 in the general solution (A.30) and going back to the $\delta\phi$, we find that

$$\langle 0|\delta\phi_\beta(\eta)^2|0\rangle = \frac{1}{a(\eta)^2}\left[\frac{\beta^2 + K\cot^2\sqrt{K}\eta}{2\beta^3}\right]. \tag{A.37}$$

We want to evaluate the spectrum of perturbations at horizon crossing. The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_* \left(H_\lambda^2 - \frac{K}{a_*^2} \right)^{1/2} \tag{A.38}$$

from which we obtain the values of the scale factor

$$a_* = \frac{(\beta^2 + K)^{1/2}}{H_\lambda} \tag{A.39}$$

and of the conformal time

$$\eta_* = -\frac{1}{\sqrt{K}} \arctan \frac{\sqrt{K}}{\beta} \tag{A.40}$$

at horizon crossing. The corresponding value of the Hubble parameter is

$$H(a_*) = H_\lambda \frac{\beta}{(\beta^2 + K)^{1/2}}. \tag{A.41}$$

The power spectrum $\mathcal{P}(\beta)$ in this case is given by

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{\left(1 + \frac{K}{\beta^2}\right)^2}. \tag{A.42}$$

A.4 Open universe inflation

Now we consider the case of an open universe with $K < 0$. From the Friedman equation (A.10) we have

$$\dot{a} = H_\lambda a \sqrt{1 + \frac{|K|}{H_\lambda^2 a^2}} \tag{A.43}$$

where we work with the absolute value of the curvature, taking into account that $|K| = -K$ in this case. The above expression can be integrated to give

$$a(t) = \frac{\sqrt{|K|}}{H_\lambda} \sinh H_\lambda t \tag{A.44}$$

with initial condition $a(t = 0) = 0$. The conformal time is

$$\eta(a) = \frac{-1}{\sqrt{|K|}} \text{arcsinh} \frac{\sqrt{|K|}}{H_\lambda a}. \tag{A.45}$$

The conformal time spans the interval $\eta = (-\infty, 0)$ as the scale factor varies in the interval $a = (0, \infty)$. We can solve for $a(\eta)$ and obtain

$$a(\eta) = -\frac{\sqrt{|K|}}{H_\lambda} \frac{1}{\sinh \sqrt{|K|} \eta}. \tag{A.46}$$

The conformal time KG equation for the open-inflationary universe is of the form

$$\sigma''(\eta) + \left[k^2 - |K|\left(2\,\mathrm{cosech}^2\sqrt{|K|}\eta + 1\right)\right]\sigma(\eta) = 0. \tag{A.47}$$

This equation has exact solutions

$$\begin{aligned}\sigma(\eta) &= c_1\left(-\sqrt{|K|}\coth\sqrt{|K|}\eta + i\sqrt{k^2-|K|}\right)e^{i\sqrt{k^2-|K|}\,\eta} \\ &+ c_2\left(\sqrt{|K|}\coth\sqrt{|K|}\eta + i\sqrt{k^2-|K|}\right)e^{-i\sqrt{k^2-|K|}\,\eta}.\end{aligned} \tag{A.48}$$

The normalization constants c_1 and c_2 are chosen so that in the infinite past $\eta \to -\infty$ limit one gets plane waves satisfying the following relation

$$\sigma^*\sigma' - \sigma^{*\prime}\sigma = i \tag{A.49}$$

and obtain,

$$|c_1| = \frac{1}{\sqrt{2}k(k^2-|K|)^{1/4}}, \qquad c_2 = 0. \tag{A.50}$$

We then obtain for the magnitude of $\delta\phi_\beta(\eta) = \sigma(\eta)/a(\eta)$ the expression,

$$|\delta\phi_\beta(\eta)|^2 = \frac{1}{a(\eta)^2}\left[\frac{\beta^2 + |K|\coth^2\sqrt{|K|}\eta}{2(\beta^2+|K|)\,\beta}\right]. \tag{A.51}$$

where for the open universe,

$$\beta = (k^2 - |K|)^{1/2}. \tag{A.52}$$

The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_*\left(H_\lambda^2 + \frac{|K|}{a_*^2}\right)^{1/2} \tag{A.53}$$

and we obtain for the scale factor at Hubble crossing

$$a_* = \frac{(\beta^2 - |K|)^{1/2}}{H_\lambda} \tag{A.54}$$

and the corresponding conformal time is given by

$$\eta_* = -\frac{1}{\sqrt{|K|}}\mathrm{arctanh}\frac{\sqrt{|K|}}{\beta}. \tag{A.55}$$

The Hubble parameter at horizon crossing is

$$H(a_*) = H_\lambda\frac{\beta}{\sqrt{\beta^2-|K|}}. \tag{A.56}$$

We notice that in an open-universe stage of inflation, only the modes satisfying the condition $\beta^2 > |K|$ will cross the Hubble radius.

With this, we obtain the following expression for the curvature power spectrum at Hubble crossing

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1-\frac{|K|}{\beta^2}\right)^2\left(1+\frac{|K|}{\beta^2}\right)}. \tag{A.57}$$

A.5 Conclusions

There are earlier calculations of power spectrum in a non-flat inflationary universe [292]-[298]. Our results for the primordial power spectra for both the closed and open pre-inflation universe cases differ in some details from these earlier papers because of difference in the way we have implemented the initial conditions. Our results of the primordial power spectrum have been derived assuming that the vacuum state in the infinite past was the Bunch-Davies vacuum and we have evaluated the primordial power spectrum at horizon crossing of the perturbation modes.

The power spectrum obtained by Ratra and Peebles [299] and Lyth and Stewart [292] for the open universe case, obtained by assuming conformal boundary condition for the initial state at $\eta \to -\infty$ is

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1+\frac{|K|}{\beta^2}\right)}. \tag{A.58}$$

This is sometime written in the form

$$\beta^{-3}P_{\mathcal{R}}(\beta) \propto \frac{1}{\beta(\beta^2+K)} \equiv \frac{1}{\beta(\beta^2+1)}. \tag{A.59}$$

Bucher, Goldhaber and Turok [297] consider an open universe with a tunneling solution and assume that the initial states annihilate the Bunch-Davies vacuum and obtain a power spectrum ,

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1+\frac{|K|}{\beta^2}\right)}\coth\left[\frac{\pi\beta}{\sqrt{|K|}}\right]. \tag{A.60}$$

In our paper we also assume a Bunch-Davies vacuum but we consider the standard slow roll inflation model, where the expansion was dominated by the curvature term prior to inflation, and evaluate the power spectrum at the horizon exit $a_*\,H(a_*) = \beta$. In our solution for the power spectrum of the open universe case (A.2) we have a factor of $1/(1-|K|/\beta^2)$ instead of $\coth(\pi\beta/\sqrt{|K|})$ of (A.60). All three solutions for the power spectrum (A.2), (A.62) and (A.60) agree in the limit of small curvature $|K|/\beta^2 \to 0$.

The experimental bounds on the total density of the universe from a combination of WMAP, LSS and HST supernovae observations is $\Omega_0 = 1.06 \pm 0.02$ [291]. This implies that the curvature

$$K = (\Omega - 1)\,H_0^2 a_0^2 = (0.06 \pm 0.02)\,H_0^2 a_0^2 \tag{A.61}$$

If one uses the Ratra-Peebles form of the power spectrum for the closed universe

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1-\frac{K}{\beta^2}\right)} \tag{A.62}$$

we see that for perturbations of the horizon size $\beta \simeq H_0 a_0$, the power spectrum is suppressed by up to 8% (compared to the flat universe), on the other hand if one uses the power spectrum (A.1) for the closed universe derived in this paper the suppression of large scale power can be as large as 16%. We do not advocate determination of the form of power spectrum from the data, however the choice of power spectrum used as an input (in numerical programmes like CMBFAST) will affect the determination cosmological parameters like Ω, H_0, n_s etc from the CMB data.

At the beginning of inflation the curvature $\Omega - 1$ is expected to be of order one. By the time perturbations of our horizon size exit the inflation horizon, the curvature drops to $\Omega_0 - 1$ which is the present value. A non-zero observation of the curvature will tell us whether the universe prior to inflation was open or closed (even though it is almost flat now) and put constraints on the number of extra e-foldings that must have occurred beyond the minimum number needed to solve the horizon problem. Spatial curvature is a threshold effect which can give us information on the pre-inflation universe from observations of the CMB anisotropy at large angles, similar to the effect of a possible pre-inflation thermal era [300, 301].

The spatial curvature is measured in the CMB mainly from the angular size of the acoustic horizon. From the power spectrum of the curved and open inflation cases we see that if $K > 0$, power is suppressed at large angles and if $K < 0$ power is enhanced at large angles. A more accurate determination of the curvature of the universe from the observations of CMB anisotropy spectrum can be achieved by using (A.1) and (A.2) as the primordial power spectrum generated in inflation for the closed and open universe respectively.

Appendix B

Numerical details on the model of inflation and dark matter

Here we give the details of how to obtain the parameters related to inflation in our model. We base our analysis on 3 constraints, which are:

1. The number of e-folds of inflation between a given scale k crosses the horizon and the end of inflation is given by [35, 36, 69, 302]

$$N \simeq 62 - \ln \frac{k}{a_0 H_0} - \ln \left(\frac{10^{16}\text{GeV}}{V_{\text{sym}}(\phi_0, 0)^{1/4}} \right) + \frac{1}{3} \ln \left(\frac{T_{\text{rh}}}{V_{\text{sym}}(\phi_0, 0)^{1/4}} \right) \tag{B.1}$$

   where T_{rh} is the reheating temperature and $a_0/k = H_0^{-1} \simeq 4000$ Mpc is the biggest observable scale. A scale of interest is $k_0 = 0.002\ \text{Mpc}^{-1}$, for which we have reliable observational data [5]. The number of e-folds corresponding to k_0 can be expressed in terms of the inflaton field ϕ_0

$$\begin{aligned} N(\phi_0) &= \frac{8\pi}{M_{\text{P}}^2} \int_{\phi_{\text{end}}}^{\phi_0} \frac{V_{\text{sym}}}{V'_{\text{sym}}} \text{d}\phi \\ &= \frac{\pi}{M_{\text{P}}^2} \left[\frac{4(\Lambda^4 - C)}{\lambda v^2} \ln \frac{(v^2 - \phi_0^2)\phi_{\text{end}}^2}{(v^2 - \phi_{\text{end}}^2)\phi_0^2} + v^2 \ln \frac{\phi_{\text{end}}^2}{\phi_0^2} - (\phi_{\text{end}}^2 - \phi_0^2) \right]. \end{aligned} \tag{B.2}$$

   From now on we set $T_{\text{rh}} = 10^9\text{GeV}$ and $\phi_{\text{end}} = v/2 (\equiv m_\chi = \kappa v/2)$, for simplicity. By equating the two expressions (B.1) and (B.2) for $N(\phi_0)$, we finally obtain

$$\begin{aligned} 60 &- \ln \frac{10^{16}\text{GeV}}{\left[\Lambda^4 - C + \frac{1}{4}\lambda(v^2 - \phi_0^2)^2\right]^{1/4}} + \frac{1}{3} \ln \frac{10^9\text{GeV}}{\left[\Lambda^4 - C + \frac{1}{4}\lambda(v^2 - \phi_0^2)^2\right]^{1/4}} \\ &= \frac{\pi}{M_{\text{P}}^2} \left[\frac{4(\Lambda^4 - C)}{\lambda v^2} \ln \frac{v^2 - \phi_0^2}{3\phi_0^2} + v^2 \ln \frac{v^2}{4\phi_0^2} - \left(\frac{v^2}{4} - \phi_0^2 \right) \right]. \end{aligned} \tag{B.3}$$

2. The amplitude of the curvature perturbations ${\mathcal{P}_{\mathcal{R}}}^{1/2}$ has the observed value ${\mathcal{P}_{\mathcal{R}}}^{1/2} \simeq 4.86 \times 10^{-5}$ corresponding to the scale k_0 [5]. This means that

$$\begin{aligned} {\mathcal{P}_{\mathcal{R}}}^{1/2} &= \sqrt{\frac{128\pi}{3}} \left| \frac{V_{\rm sym}(\phi_0,0)^{3/2}}{M_{\rm P}^3 V'_{\rm sym}(\phi_0,0)} \right| \\ &= \sqrt{\frac{128\pi}{3}} \frac{\left[\Lambda^4 - C + \frac{1}{4}\lambda(v^2-\phi_0^2)^2\right]^{3/2}}{\lambda M_{\rm P}^3 \phi_0 (v^2-\phi_0^2)} \simeq 4.86 \times 10^{-5}. \end{aligned} \tag{B.4}$$

3. The value of the spectral index $n_{\rm s} \simeq 1 + 2\eta$ should have a value close to $n_{\rm s} = 0.95$ for the same scale $k_0 = 0.002$ Mpc^{-1}, where $\eta = (M_{\rm P}^2/8\pi)(V''_{\rm sym}/V_{\rm sym})$ is a slow-roll parameter. This becomes

$$2\eta = \frac{-\lambda M_{\rm P}^2 (v^2 - 3\phi_0^2)}{4\pi\left[\Lambda^4 - C + \frac{1}{4}\lambda(v^2-\phi_0^2)^2\right]} \simeq -0.05. \tag{B.5}$$

One can see that by combining equations (B.4) and (B.5) one obtains an expression for λ in terms of ϕ_0 and v

$$\lambda = 4.4 \times 10^{-12} \frac{\phi_0^2 (v^2-\phi_0^2)^2}{(v^2-3\phi_0^2)^3}. \tag{B.6}$$

Next, by replacing (B.6) into (B.5) one obtains

$$\Lambda^4 - C = 4.4 \times 10^{-12} \frac{\phi_0^2 (v^2-\phi_0^2)^2}{(v^2-3\phi_0^2)^2} \left[\frac{5}{\pi} M_{\rm P}^2 - \frac{(v^2-\phi_0^2)^2}{4(v^2-3\phi_0^2)}\right]. \tag{B.7}$$

Finally, by introducing (B.6) and (B.7) into (B.3), one obtains an equation which relates ϕ_0 and v. We solved it numerically for a few $v-$values in the interval $(10^7 - 10^{19})$GeV and we obtained the corresponding values for ϕ_0, which are shown in Fig.9.1(a). Once we have $\phi_0(v)$, we can turn back to (B.6) and (B.7) and find the values of λ and $\Lambda^4 - C$, respectively.

Still, we would like to find Λ and C, separately. This can be done by requiring that the absolute minimum of $V_{\rm sym}(\phi,\chi)$ is equal to zero. The position of the absolute minimum is given by the following conditions

$$\frac{\partial V_{\rm sym}(\phi,\chi)}{\partial \phi} = -\lambda v^2 \phi + \lambda \phi^3 - \kappa^2 \phi \chi^2 + \frac{\kappa^4 \phi^3 \chi^4}{4\Lambda^4} = 0, \quad \frac{\partial^2 V_{\rm sym}(\phi,\chi)}{\partial \phi^2} > 0 \tag{B.8}$$

$$\frac{\partial V_{\rm sym}(\phi,\chi)}{\partial \chi} = m_\chi^2 \chi - \kappa^2 \phi^2 \chi + \frac{\kappa^4 \phi^4 \chi^3}{4\Lambda^4} = 0, \quad \frac{\partial^2 V_{\rm sym}(\phi,\chi)}{\partial \chi^2} > 0. \tag{B.9}$$

Solving the above equation system, one can obtain $\phi_{\rm min}$ and $\chi_{\rm min}$. We do not show here the analytical solutions because they are very complicated. From the condition

$V_{\rm sym}(\phi_{\rm m}, \chi_{\rm m}) = 0$, one can obtain a relation between C and Λ. With this, going back to equation (B.7), one obtains the dependence $\Lambda(v)$ and subsequently $C(v)$. The results we obtained are shown in Fig.9.1 (c) and (d), respectively.

Things become much simpler in the limit $v \ll M_{\rm P}$. As shown in Fig.9.1 (d), when $v \ll M_{\rm P}$, the following relations are satisfied

$$\lambda v^4 \ll C \ll \Lambda^4. \tag{B.10}$$

In this case, from (B.5) we obtain

$$\Lambda^4 \simeq \frac{5}{\pi}\lambda M_{\rm P}^2(v^2 - 3\phi_0^2) \sim \lambda v^2 M_{\rm P}^2 \tag{B.11}$$

and the solutions of (B.8) and (B.9) become very simple

$$\phi_{\rm m} \simeq \frac{v^{1/3}\Lambda^{2/3}}{\lambda^{1/6}} \sim v^{2/3} M_{\rm P}^{1/3}\,, \quad \chi_{\rm m} \simeq \frac{2\lambda^{1/6}\Lambda^{4/3}}{\kappa v^{1/3}}. \tag{B.12}$$

With the above expressions, the approximate solution for C is also very simple

$$C \simeq \frac{3}{4}\Lambda^4 \left(\frac{\lambda v^4}{\Lambda^4}\right)^{1/3} \sim \Lambda^4 \left(\frac{v}{M_{\rm P}}\right)^{2/3} \sim \lambda v^4 \left(\frac{M_{\rm P}}{v}\right)^{4/3} \tag{B.13}$$

where we made use of (B.11). This also helps us understand why the following inequalities $\lambda v^4 \ll C \ll \Lambda^4$ are satisfied for $v \ll M_{\rm P}$.

In the same limit ($v \ll M_{\rm P}$), we get simple expressions for the $v-$dependence of the number of e-folds of observable inflation, $N(v) \propto \ln v$, and of the tensor-to-scalar ratio, $r(v) \propto v^2/M_{\rm P}^2$.

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