Localization: On Division Rings and Tilting Modules

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Memòria presentada per a
aspirar al grau de doctor
en Matemàtiques

Certifico que la present Memòria ha estat
realitzada per en Javier Sánchez Serdà,
sota la direcció de la
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Bellaterra, Juny 2008

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"And here we are again
the door is closed behind us
and the long road lies ahead
where do we go from here?"

Fates Warning, *Leave the past behind*

Introduction

Let R be a ring. Suppose that R embeds in a division ring E (i.e. E is a not necessarily commutative field). By the *division ring of fractions* of R inside E we mean the embedding $R \hookrightarrow E(R)$ where $E(R)$ denotes the intersection of all subdivision rings of E that contain R. When R is a commutative ring, the answer to whether R is embeddable in a division ring is well known:

(a) *Existence*: R has a division ring of fractions $Q(R)$ if and only if R is a *domain*, i.e. R is a nonzero ring such that $xy = 0$ implies that $x = 0$ or $y = 0$.
(b) *Uniqueness*: If $\lambda\colon R \hookrightarrow Q(R)$ is a division ring of fractions of R, given any embedding $\psi\colon R \hookrightarrow E$ in a division ring E, there exists a morphism of rings $\bar{\psi}\colon Q(R) \to E$ such that $\bar{\psi}\lambda = \psi$. In particular $E(R) \cong Q(R)$.
(c) *Form of the elements*: The division ring of fractions $Q(R)$ is constructed in an analogous way as the rationals from the integers. The elements of $Q(R)$ are of the form $s^{-1}r$ for some $r \in R$ and $s \in R \setminus \{0\}$. Moreover, we have a rule to decide when two fractions $s_1^{-1}r_1,\ s_2^{-1}r_2$ represent the same element of E (iff $s_1r_2 = s_2r_1$).

This situation extends to the non-commutative setting provided that R is a left (right) Ore domain [**Ore31**], but in general the picture is not like that.

Certainly, being a domain is a necessary condition for the ring R to be embeddable in a division ring, but it is not sufficient as A.I. Mal'cev showed [**Mal37**]. Necessary and sufficient conditions can be found in [**Coh95**], but they are difficult to verify for a given domain R and, in addition, the proof of the existence of the division ring is not constructive.

There are also many (non-Ore) domains R with more than one division ring of fractions [**Fis71**] (or see also Chapter 7). For example, for any field k, the free k-algebra $k\langle X\rangle$ on a set X of cardinality at least two.

Furthermore, if $R \hookrightarrow D$ is a division ring of fractions of R, the elements of D can be built up from the elements of R in stages, using addition, subtraction, multiplication, and division by nonzero elements. It can happen that the subset $\{s^{-1}r \in D \mid r \in R,\ s \in R \setminus \{0\}\}$ is not a division ring, and it may not be possible to simplify expressions like $uv^{-1}w + xy^{-1}z$ or $(w_2 - x_2(w_1 - x_1y_1^{-1}z_1)^{-1}z_2)^{-1}$. This leads to the concept of inversion height. The inversion height of a rational expression is the maximum number of nested inversions that occur in it. For example, the inversion height of $uv^{-1}w+xy^{-1}z$ is one and of $(w_2-x_2(w_1-x_1y_1^{-1}z_1)^{-1}z_2)^{-1}$ is three. The *inversion height of an element* of D is the least inversion height of the rational expressions of elements of R that represent f. Observe that the inversion height of an element is not so easy to compute. It depends of course on the nature of the ring R, but there are also rational identities. To illustrate this point, consider elements $x, y \in R$. At first sight the element $f = (x^{-1} + (y^{-1} - x)^{-1})^{-1} \in D$ seems to be of inversion height 3. However, by Hua's identity, $f = x - xyx$, and therefore is of inversion height zero. The *inversion height of a division ring of fractions* $R \hookrightarrow D$ is the supremum of the inversion height of the elements of D (it may be infinite). Notice that if R is a left (right) Ore domain, $R \hookrightarrow D$ is of inversion height 1.

Therefore, description of division rings of fractions turns out to be a difficult task.

About two thirds of this work is concerned with the problem of the uniqueness of division rings of fractions and the form in which the elements of a division ring of fractions can be expressed. We proceed to outline our main results in this direction.

Let G be a group and k a division ring. One can construct a crossed product group ring kG, for example the usual group ring $k[G]$. We deal with the problem of embedding kG in division rings when G is a *locally indicable group*, that is, a group such that every nontrivial finitely generated subgroup H of G has a normal subgroup N with H/N infinite cyclic. These groups form a large class: locally free groups, torsion-free abelian groups, orderable groups, torsion-free one-relator groups and extensions of groups in these classes are locally indicable groups. Observe that if $1 \neq H \leq G$ is finitely generated and the class of t generates H/N, then the powers of t are kN-linearly independent.

An embedding $kG \hookrightarrow D$, with D a division ring, is *Hughes-free* if the powers of t are $D(kN)$-linearly independent for all $1 \neq H \leq G$ finitely generated, $N \lhd H$ such that H/N is infinite cyclic and t such that the coset of t generates H/N, that is, we can extend the kN-linear independence of t to the division ring $D(kN)$ generated by kN inside D. An important example of Hughes-free embedding goes back to A.I. Mal'cev [**Mal48**] and B.H. Neumann [**Neu49a**]. They independently showed: for each ordered group $(G, <)$, division ring k and crossed product group ring kG, the embedding $kG \hookrightarrow k((G,<))$ is a Hughes-free embedding, where $k((G,<))$ denotes the *Mal'cev-Neumann* series ring, i.e.

$$k((G,<)) = \{\gamma = \sum_{g\in G} a_g g \mid \operatorname{supp}\gamma \text{ is well-ordered}\}.$$

In particular, this holds for (locally) free groups.

A locally indicable group G is *Hughes-free embeddable* if kG has a Hughes-free division ring of fractions for every division ring k and every crossed product group ring kG. Orderable groups are examples of Hughes-free embeddable groups.

We give new proofs of two results by I. Hughes, [**Hug70**] and [**Hug72**]. More precisely, we show:

Hughes' Theorem I. *Let k be a division ring and G a locally indicable group. If $kG \hookrightarrow D_1$ and $kG \hookrightarrow D_2$ are two Hughes-free division ring of fractions, then there exists a (unique) isomorphism $\varphi\colon D_1 \to D_2$*

$$\begin{tikzcd} & D_1 \arrow[dd, dashed, "\varphi"] \\ kG \arrow[ur, hook] \arrow[dr, hook] & \\ & D_2 \end{tikzcd}$$

making the diagram commutative. □

Hughes' Theorem II. *Suppose that G is a locally indicable group with a normal subgroup L such that G/L is locally indicable. If both L and G/L are Hughes-free embeddable, then G is Hughes-free embeddable.* □

The proof of Hughes' Theorem I and the machinery developed to show both results is a joint work with W. Dicks and D. Herbera [**DHS04**]. Hughes' Theorem I has played an important role in the study of division rings of fractions of the free algebra and of crossed product group rings of free groups. It has been used by J. Lewin in [**Lew74**] to describe the universal division ring of fractions of the free algebra and of the crossed product group ring of a free group as the division ring of fractions of kG inside the Mal'cev-Neumann series rings. Also, P. Linnell [**Lin00**], [**Lin93**] made use of it to prove that the division ring of fractions that arises from group von Neumann algebras of free groups is the same as the universal division ring of fractions of the group ring of a free group. Because of these important results and the fact that, in words of P. Linnell [**Lin06**, p. 52], “the proof given by Hughes in [**Hug70**] is extremely condensed, and ... it is difficult to follow”, a new proof was needed.

We also give an easier proof of the aforementioned result by J. Lewin following closely the work by C. Reutenauer [**Reu99**].

Hughes’ Theorem II gives a large supply of Hughes-free embeddings and partial positive answers to Mal’cev problem of whether the group ring of a right orderable group is embeddable in a division ring [**MMC83**, Question 1.6]. Moreover, it allows us to define a division ring coproduct, and it is also useful for computing the inversion height of some embeddings of rings in division rings.

In this direction, we close a conjecture by B.H. Neumann [**Neu49a**, p. 215]. More precisely, we prove

THEOREM. *Let G be a free group on a set X of at least two elements and k a field. Let $<$ be a total order on G such that $(G,<)$ is an ordered group. Then the embedding $kG \hookrightarrow k((G,<))$ is of infinite inversion height.* □

This result was showed by C. Reutenauer [**Reu96**] for X an infinite set. Indeed, he proved that the inversion height of the entries of the inverse of an $n \times n$ generic matrix (i.e. a matrix of the form (x_{ij}) where the x_{ij}’s are distinct noncommuting variables in X) equals n with respect to $k\langle X\rangle \hookrightarrow k((G,<))$. In the finite case we are able to reduce the problem to the situation proved by C. Reutenauer.

J.L. Fisher [**Fis71**], using results by A.V. Jategaonkar [**Jat69**], gave embeddings of the free k-algebra $k\langle x,y\rangle$ in division rings of inversion height one and two. We continue these investigations, and following the pattern of the embeddings of Fisher of inversion height 2, we provide examples of inversion height one and two of the free k-algebra $k\langle X\rangle$ for any set X of at least two elements. Moreover, we use these examples to give embeddings of the free group k-algebra of inversion height one and two. Our results on finite inversion height are part of the joint work with D. Herbera [**HS07**].

The last chapter of this dissertation is a joint work with L. Angeleri Hügel [**AHS08**]. It is oriented towards an application of localization to module theory. We focus on the construction of tilting modules. Tilting modules were introduced as an abstraction of the ideas contained in [**BGP73**] and [**GP72**] to generalize the theory of Morita equivalence. They were used to carry information between module categories of finitely generated algebras, especially hereditary algebras. Later M. Auslander and I. Reiten [**AR91**] found an important relationship between tilting modules and homologically finite categories which allowed to link tilting theory to Homological Conjectures, quasi-hereditary algebras or Cohen-Macaulay rings. Tilting theory has also been useful in the general theory of modules since a lot of results for classic tilting modules are valid in Mod-R (not necessarily finitely generated modules). In [**AHC01**] the definition of tilting modules was extended to the category Mod-R over a ring R, and was shown that there exists a relationship between tilting modules and homologically finite categories analogous to that discovered by Auslander-Reiten. This connection has been used to get new results on the Homological Conjectures. For example in [**AHHT06**] a conjecture on the finitistic dimension is shown to be valid for Gorenstein rings not necessarily commutative.

Another interesting application of tilting theory was discovered in [**AHHT05**] where it is considered the localization $\mathfrak{S}^{-1}R$ of a (not necessarily commutative) ring with respect to a left Ore set $\mathfrak{S}$ consisting of non-zero-divisors. The authors focused on the case that $\mathfrak{S}^{-1}R$ has projective dimension at most one and extended some classical results on localization of commutative domains due to Kaplansky, Hamsher and Matlis to arbitrary commutative rings. For proving their results it was essential to use tilting modules. In fact, they constructed a tilting module $\mathfrak{S}^{-1}R\oplus\mathfrak{S}^{-1}R/R$ which generates the class of $\mathfrak{S}$-divisible modules provided R is commutative or $\mathfrak{S}^{-1}R/R$ is countably generated. This also shed new light on some important examples of tilting modules over 1-Gorenstein rings or over valuation domains.

We push forward the idea of constructing tilting modules using localization techniques. We show that every injective ring epimorphism $\lambda\colon R\to S$ with the property that $\operatorname{Tor}_1^R(S,S)=0$ and $\operatorname{pd} S_R\leq 1$ gives rise to a tilting right R-module $S\oplus S/R$.

If $\mathcal{U}$ is a class of finitely presented right R-modules of projective dimension one such that $\operatorname{Hom}_R(\mathcal{U},R)=0$, we can consider the universal localization $\lambda\colon R\to R_{\mathcal{U}}$ of R at $\mathcal{U}$ in the sense of [**Sch85**]. It is known that $\lambda\colon R\to R_{\mathcal{U}}$ is a ring epimorphism with $\operatorname{Tor}_1^R(R_{\mathcal{U}},R_{\mathcal{U}})=0$. Suppose that R embeds in $R_{\mathcal{U}}$, and $\operatorname{pd}(R_{\mathcal{U}})_R\leq 1$. Then $T_{\mathcal{U}}=R_{\mathcal{U}}\oplus R_{\mathcal{U}}/R$ is a tilting right R-module. If we further assume that $R_{\mathcal{U}}/R$ is a direct limit of $\mathcal{U}$-filtered right R-modules, then the tilting class $\operatorname{Gen} T_{\mathcal{U}}$ coincides with the class $\mathcal{U}^{\perp}$ of all modules M satisfying $\operatorname{Ext}_R^1(\mathcal{U},M)=0$. This allows us to extend the aforementioned result on [**AHHT05**] and prove that if $\mathfrak{S}$ is a left Ore set of non-zero-divisors of R such that $\operatorname{pd}(\mathfrak{S}^{-1}R_R)\leq 1$, then $T_{\mathfrak{S}}=\mathfrak{S}^{-1}R\oplus\mathfrak{S}^{-1}R/R$ is a tilting right R-module whose tilting class $\operatorname{Gen} T_{\mathfrak{S}}$ coincides with the class of $\mathfrak{S}$-divisible right R-modules.

This way of constructing tilting modules also fits in the context of finite dimensional tame hereditary algebras. We obtain a tilting right R-module $T_{\mathcal{U}}=R_{\mathcal{U}}\oplus R_{\mathcal{U}}/R$ with tilting class $\mathcal{U}^{\perp}$ for every set $\mathcal{U}$ of simple regular modules.

Two more interesting results are shown when R is a hereditary noetherian prime ring. We recover a classification result from [**BET05**] and show that the tilting modules

$$T_{\mathfrak{P}}=R_{\mathcal{U}_{\mathfrak{P}}}\oplus R_{\mathcal{U}_{\mathfrak{P}}}/R$$

arising from universal localization at $\mathfrak{U}_{\mathfrak{P}}=\{R/\mathfrak{m}\mid \mathfrak{m}\in\mathfrak{P}\}$, where $\mathfrak{P}$ runs through all subsets of max-spec(R), form a representative set up to equivalence of the class of all tilting R-modules. And more generally, we prove that if R is a classical maximal order, then

$$\mathbb{T}=\{T_{\mathcal{W}}=R_{\mathcal{W}}\oplus R_{\mathcal{W}}/R\mid \mathcal{W}\subseteq\mathcal{U}_r\}$$

is a representative set up to equivalence of the class of all tilting right R-modules where $\mathcal{U}_r$ denotes a representative set of all simple right R-modules.

Now we proceed to explain how this work is structured, and to discuss in more detail some of the main results presented in this dissertation.

An important part of Chapter 1 consists of elementary material that can be found in many undergraduate text books and thus most of the proofs are omitted. We include it in order to fix notation and to be as self-contained as possible. However some algebraic background such as basics on the theory of groups, rings, modules and homological algebra is needed. For example, commutative localization, free product of groups (with or without amalgamation), exact sequences induced by the functors Ext and Tor,... are concepts we use along these pages and that we assume are known. The reader is referred to, for example, [**Rot73**], [**Rot70**], [**Lam01**], [**Lam99**], [**Jac85**], [**Jac89**] for unexplained terminology. It is recommended to skip this chapter and go back to clarify terms when needed.

The first three sections present basic definitions, examples and results on monoid, groups, rings and modules.

In the fourth section we present ordinal and cardinal arithmetic because it will be used in Chapter 7.

Section 5 consists of results on homological algebra. We concentrate on the behavior of the functor Ext with respect to direct limits. This material will be used in Chapter 8.

In Section 6 we introduce the concepts of graph, rooted trees and graph of groups. This last concept will be used to show some closure properties of locally indicable groups in Chapter 2. Rooted trees will be very useful in Chapter 5 to define the complexity of elements of a certain semiring.

Section 7 is devoted to present the concept of semiring which is relevant to the proof of Hughes' Theorems I and II. We also provide important examples of semirings.

In Section 8 we state well known results and definitions on completions and valuations of rings that are needed in Chapter 7 to give embeddings of the free group algebra in division rings.

The last section of this chapter concerns itself with the proof of the so-called Magnus-Fox embedding. This already known result asserts that if H is the free group on a set X and R is a ring, then the group ring $R[H]$ embeds in the formal power series ring $R\langle\langle X\rangle\rangle$ via the morphism of R-rings given by $x \mapsto 1 + x$ for all $x \in X$. It will be used in Chapter 7.

In Chapter 2 we collect together well known material on locally indicable groups. This class of groups was introduced by G. Higman in his PhD Thesis and his paper [**Hig40**] on group rings. Although there are some monographs that cover parts of this chapter and other interesting related subjects, for example [**Gla99**] and [**BMR77**], we have not found a survey or text book where all the results in this section are proved. So we felt it was necessary to make this compilation for the sake of completion as these groups play an important role in our work. We concentrate more on the properties which will have a counterpart when dealing with Hughes-free division rings of fractions of crossed product group rings kG of a locally indicable group G over a division ring k.

We begin by showing some closure properties of the class $\mathfrak{X}$ of locally indicable groups. Among others, $\mathfrak{X}$ is closed under free products and extensions, or more generally, it contains the groups with a subnormal series with locally indicable factors (see Definition 2.5). We will do something similar with Hughes-free division rings of fractions in Chapter 6. We go on discussing some properties of an important subclass of $\mathfrak{X}$, the class of (two sided) orderable groups. Torsion-free abelian groups and (locally) free groups are examples of orderable groups. It turns out that $\mathfrak{X}$ is a class properly between the classes of orderable groups and right orderable groups.

We also show the characterization of locally indicable groups as those groups G with a chain of subgroups Σ of G such that

(i) $\{1\}, G \in \Sigma$.
(ii) Σ contains all unions and intersections of its members.
(iii) For each pair (L, H) of subgroups in Σ such that there is no element in Σ that lies properly between L and H, then $L \lhd H$ and H/L is torsion-free abelian.

We provide the recent proof of this fact by A. Navas-Flores [**NF07**] which greatly simplifies the original one by S. Brodskii [**Bro84**].

We end this chapter enumerating some other important examples of locally indicable groups, e.g. torsion-free one-relator groups [**How82**] and right orderable amenable groups [**Mor06**].

In Chapter 3 we present the results on localization that will be needed later. Some of the results in this chapter are very important in the development of this work, but most of the proofs and details are omitted. They can be found in the books [**Lam01**], [**Coh95**] or [**Sch85**]. We begin with Ore localization and some classical results, followed by some results on the key concept of *division ring of fractions* of a domain and by some useful examples.

We then go on with matrix localization. We define the relevant concepts of *fir*, *semifir*, *specializations*, *universal division ring of fractions* and *coproduct of division rings*. They all appear throughout these pages.

Universal localization is discussed in the next section. We give the definition of *rank function* and state the results that we will need in Chapter 8.

Chapter 4 is probably the best starting point for a reading of this work. We define some of the most important objects that will be considered along these pages. The first four sections consist of well known material. The first two are made of standard material on crossed product group rings (monoid semirings). They are generalizations of the usual group ring. An interesting feature of crossed product group rings, unlike group rings, is the following: Let kG be a crossed product group ring of a group G over a ring k, and let N be a normal subgroup of G. Then kG can be seen as a crossed product group ring $(kN)(G/N)$ of the group G/N over the ring kN. We also show that a crossed product group ring kG of a right orderable group over a domain k is a domain, in particular, this holds for locally indicable groups G.

In the next section we present the Mal'cev-Neumann series ring $k((G,<))$ associated with a crossed product group ring kG of an ordered group $(G,<)$ over a division ring k. It consists of the series $\sum\limits_{x\in G} a_x\bar{x}$ with well-ordered support with the expected operations of sum and product that extend the ones of kG. Following [**DL82**], we show the celebrated result by A.I. Mal'cev [**Mal48**] and B.H. Neumann [**Neu49a**] which proves that $k((G,<))$ is a division ring. A very useful property is that the computation of an expression for the inverse of a nonzero element in $k((G,<))$ is "easy". There is an algorithm to invert series. More precisely, if $f = \sum\limits_{x\in G} a_x\bar{x} \in k((G,<))$ and $x_0 = \min\{x \in \operatorname{supp} f\}$, then

$$f^{-1} = \sum_{m\geq 0} (a_{x_0}\bar{x}_0)^{-1}(g(a_{x_0}\bar{x}_0)^{-1})^m,$$

where $g = a_{x_0}\bar{x}_0 - f$. This is a point, which will be corroborated in succeeding chapters, where becomes more relevant the fact that working with series makes life easier.

The results above imply that if k is a division ring, G a (locally) free group on a set X and $<$ a total order on G such that $(G,<)$ is an ordered group, then any crossed product group ring kG is embeddable in its associated Mal'cev-Neumann series ring $k((G,<))$. We go on to show that kG is a semifir if G is a locally free group, and that kG and its polynomial ring $k\langle X\rangle$ are firs. Thus, in any case, kG and $k\langle X\rangle$ have a universal division ring of fractions.

It was proved by J. Lewin in [**Lew74**] that the universal division ring of fractions of the group ring $k[G]$ and $k\langle X\rangle$, called the *free division ring of fractions*, is the division ring of fractions of $k[G]$ inside $k((G,<))$. Later, in [**LL78**], it was noted that the proof given in [**Lew74**] worked for any crossed product group ring kG. The proof of [**Lew74**] relied on the highly nontrivial result by I. Hughes [**Hug70**], i.e. Hughes' Theorem I. More recently, C. Reutenauer gave an easier proof of the fact that the universal division ring of fractions of the free algebra $k\langle X\rangle$ of a finite set over a field k is the division ring of fractions of the group ring $k[G]$ inside the Mal'cev-Neumann series ring $k((G,<))$ where G is the free group on X and $<$ is a total order on G such that $(G,<)$ is an ordered group. In Section 5 we closely follow this proof to show the result in all its generality. That is, let k be a division ring, G a free group on a set X and kG any crossed product group ring. Let $<$ be a total order on G such that $(G,<)$ is an ordered group. Then the universal division ring of fractions of kG and its polynomial algebra $k\langle X\rangle$ is the division ring of fractions of kG inside $k((G,<))$. Moreover, we observe that this result holds true for G a locally free group.

The objects and techniques used to demonstrate Hughes' Theorems I and II are explained in Chapters 5 and 6. We proceed to give an outline of the proof of Hughes' Theorem I which we think useful to explain why the different concepts are introduced. Notice that the statement we prove as Hughes' Theorem I in Chapter 6 is more general than the one we have stated at the beginning of the introduction, which in turn is the same as the ones in [**Hug70**] and [**DHS04**]. But they could be the same because we have not found an example that verifies one and not the other. It is not the moment to go into details, and since, anyway, the proofs are the same, we think it is more instructive to outline the proof of the theorem at the beginning of the introduction.

Let k be a division ring, let G be a group, and let kG be a crossed product group ring. Suppose that kG has a division ring of fractions D. For a subgroup N of G, we denote by $D(kN)$ the division ring of fractions of kN inside D. Let $k^\times G$ denote the group of trivial units of kG.

As we said before, a division ring of fractions of a domain R is constructed in stages, using addition, subtraction, multiplication, and division by nonzero elements. Then, given two division rings of fractions of R, there is at most one morphism of rings which is the identity on R because the image of the elements of R prescribes the image of the elements of the division ring of fractions. Moreover, this morphism, if it exists, must be an isomorphism. Therefore, if we want to prove that there exists one ring isomorphism between two division rings of fractions of R, we need to prove that the only possible one is well defined. This is the objective of the machinery developed in Chapter 5.

In case $R = kG$, the division ring of fractions is determined by the group of trivial units $k^\times G$. Notice that when we construct a division ring of fractions from $k^\times G$, subtraction is not needed, it is multiplication by the trivial unit -1. This leads to the concept of semiring and rational U-semiring where U is a group, see Section 7 of Chapter 1. It is easily seen that if D' is a division ring of fractions of kG, then $D' \cup \{\infty\}$ has a natural structure of rational $k^\times G$-semiring. The symbol ∞ is needed because in a rational semiring every element has a "formal inverse" (∞ stands for the inverse of zero and of itself).

In the division ring of fractions of a domain, we can distinguish different levels which coincide with the elements of inversion height $\leq n$. Level zero is the domain, level one the subring generated by the domain and its inverses, level two is the subring generated by level one and its inverses, and so on. For example, in an Ore domain, the construction ends at the first level. In Section 4 we define an object that imitates this construction of the division ring of fractions in a formal way, the *rational U-semiring of formal rational expressions*, denoted $\mathrm{Rat}(U)$. As in a polynomial ring we have all possible products, here we get all possible inverses, and formal-inverting an element is going up a level. These levels are built using the *free multiplicative U-monoid* that is defined in Section 3.

$\mathrm{Rat}(U)$ is an initial object in the category of rational U-semirings, that is, given a rational U-semiring T, there exists a unique morphism of rational U-semirings $\mathrm{Rat}(U) \longrightarrow T$. In addition, if V is a subgroup of U, $\mathrm{Rat}(V)$ is naturally embedded in $\mathrm{Rat}(U)$.

Thus, for two division rings of fractions of kG, D_1, D_2 say, we get

$$\Phi_i\colon\ \mathrm{Rat}(k^\times G) \longrightarrow D_i \cup \{\infty_i\}, \quad i = 1, 2,$$

onto morphisms of rational $k^\times G$-semirings. It turns out that the only possible isomorphism between D_1 and D_2 is the restriction $\beta\colon D_1 \to D_2$ of the morphism of rational $k^\times G$-semirings

$$\beta\colon D_1 \cup \{\infty_1\} \longrightarrow D_2 \cup \{\infty_2\}$$

defined by $\beta(\Phi_1(f)) = \Phi_2(f)$ for all $f \in \mathrm{Rat}(k^\times G)$. To show that β is well defined reduces to prove that $\Phi_1(f) = 0_{D_1}, \infty_1$ if and only if $\Phi_2(f) = 0_{D_2}, \infty_2$. In Theorem 6.2 we prove this fact in case G is locally indicable and D_1, D_2 are Hughes-free division rings of fractions.

A lot of results on polynomials or on free groups are proved by induction on the degree or the length of the word, respectively. The natural numbers are used to measure the "complexity" of the polynomials or of the words. In Section 2 we make the set of (isomorphism classes of) finite rooted trees into a well ordered set and a rational U-semiring for any group U. Moreover, the order defined is compatible with the operations. Then we measure the "complexity" of the elements of the rational U-semiring of formal rational expressions $\mathrm{Rat}(U)$ by assigning a rooted tree to each element of $\mathrm{Rat}(U)$. This correspondence is compatible with the operations in $\mathrm{Rat}(U)$. In particular, we get a complexity function for the elements of $\mathrm{Rat}(k^\times G)$. The proof of Hughes' Theorem I is by induction on the complexity of $f \in \mathrm{Rat}(k^\times G)$.

Vaguely, the idea of the order defined is that, for example, if $g \in G$, the element $u_1 = (1+g+g^2)^{-1} + (g^3+g^4)^{-1}$ is more complex than $u_2 = (1+g+g^4)^{-1}$, but less complex than $u_3 = ((1+g)^{-1}+g)^{-1}$. Notice that u_1 and u_2 are elements of level one and u_3 of level two. The element u_1 is made of two pieces, one of them as "complex" as u_2, and u_3 consists of one piece which is more "complex" than any of the ones in which u_1 or u_2 are divided.

In the original proof of Hughes' Theorem [**Hug70**], the order defined to measure the complexity of the elements, although equivalent, is more difficult to understand and to work with.

Let G be a locally indicable group, and let H be a finitely generated subgroup of G. Hence there exists $N \lhd H$ such that H/N is infinite cyclic. Let $t \in H$ be such that its coset generates H/N. Then kH can be seen as the skew polynomial ring $kN[t, t^{-1}; \alpha]$ where α is left conjugation by t. The condition D_i, $i = 1, 2$, being Hughes-free implies that the Ore domain $D_i(kN)[t, t^{-1}; \alpha]$ is embedded in D_i for $i = 1, 2$. Hence, the division ring of fractions of kH inside D_i can be seen as the division ring of fractions of kH inside the skew Laurent series ring $D_i(kH)((t; \alpha_i))$. Notice that in $D_i(kH)((t; \alpha_i))$ we have a "formula" for inverting series.

In Section 6 we take advantage of these ideas to factor Φ_i locally. Given a finitely generated subgroup H of G, the restriction of Φ_i to $k^\times H$ can be seen as

$$\Phi_i \colon \mathrm{Rat}(k^\times H) \longrightarrow D_i(kN)((t; \alpha_i)) \cup \{\infty_i\}.$$

Then we construct a rational $k^\times H$-semiring, $\mathrm{Rat}(k^\times N)((t; \alpha)) \cup \{\infty\}$ which depends on Φ_i. It can be seen as a formal model for $D_i(kN)((t; \alpha_i))$. The elements of $\mathrm{Rat}(k^\times N)((t; \alpha))$ are series whose coefficients are zero or elements of $\mathrm{Rat}(k^\times N)$. To define the formal inverses of the elements of $\mathrm{Rat}(k^\times N)((t; \alpha)) \cup \{\infty\}$ we imitate the "formula" for inverting series in a skew Laurent series ring.

Then we obtain a commutative diagram of morphisms of rational $k^\times H$-semirings for $i = 1, 2$

$$\begin{tikzcd}
\mathrm{Rat}(k^\times H) \arrow[rr, "\Phi_i"] \arrow[dr, "\Psi_i"'] & & D_i(kN)((t; \alpha_i)) \cup \{\infty_i\} \\
& \mathrm{Rat}(k^\times N)((t; \alpha)) \cup \{\infty\} \arrow[ur, "\Omega_i"'] &
\end{tikzcd}$$

To construct a division ring of fractions of a crossed product group ring kG it is enough to consider elements that contain $1 \in G$ in its support. For example: if $g_1, g_2 \in G \setminus \{1\}$, then

$$\begin{aligned}\Big((g_1+g_2)^{-1}+(g_1^2+g_1g_2)^{-1}\Big)^{-1} &= \Big(g_1^{-1}(1+g_2g_1^{-1})^{-1}+(g_1^2+g_1g_2)^{-1}\Big)^{-1}\\ &= \Big((1+g_2g_1^{-1})^{-1}+(g_1+g_1g_2g_1^{-1})^{-1}\Big)^{-1}g_1\\ &= p\cdot g_1,\end{aligned}$$

where p is an element "with 1 in its support".

In Section 5 we present a formal analogue of elements in the division ring of fractions with "1 in its support": the subset P of *primitive elements* of $\mathrm{Rat}(U)$. It is proved that, for every $f \in \mathrm{Rat}(U)$, there exist $p \in P$ and $u \in U$ such that $f = pu$. And what is more important, for every $p \in P$, a finitely generated subgroup of U, called $\mathrm{source}(p)$, is defined. This subgroup $\mathrm{source}(p)$ is such that $p \in \mathrm{Rat}(\mathrm{source}(p))$. This construction gives a canonical way to associate a finitely generated subgroup of U to a primitive element in $\mathrm{Rat}(U)$. We illustrate this with the most trivial situation. Consider the element $p = 1 + a_1g_1 + a_2g_2 + a_3g_3 \in kG$ with $a_i \in k \setminus \{0\}$ and $g_i \in G$. Then

$$\begin{aligned}p &= \Big((a_1g_1)^{-1}+1+a_2g_2(a_1g_1)^{-1}+a_3g_3(a_1g_1)^{-1}\Big)a_1g_1\\ &= \Big((a_2g_2)^{-1}+a_1g_1(a_2g_2)^{-1}+1+a_3g_3(a_2g_2)^{-1}\Big)a_2g_2,\end{aligned}$$

$$\begin{aligned}\mathrm{source}(p) &= \langle a_1g_1, a_2g_2, a_3g_3\rangle\\ &= \langle (a_1g_1)^{-1}, a_2g_2(a_1g_1)^{-1}, a_3g_3(a_1g_1)^{-1}\rangle\\ &= \langle (a_2g_2)^{-1}, a_1g_1(a_2g_2)^{-1}, a_3g_3(a_2g_2)^{-1}\rangle \le k^\times G.\end{aligned}$$

We can suppose that f is a primitive element in the proof of Hughes' Theorem I since Φ_i are morphisms of rational $k^\times G$-semirings. Then there exists a finitely generated subgroup H of G, obtained in a natural way from $\mathrm{source}(f)$, such that $f \in \mathrm{Rat}(k^\times H)$ and the coefficients of $\Psi_i(f)$ are of lesser complexity than f. Therefore the image of f is determined by elements of lesser complexity than the complexity of f. Then the result follows by induction on the complexity.

An important consequence of Hughes' Theorem I, already noted in [**Hug70**], is that if k is a division ring, G an orderable group and kG a crossed product group ring, then, for any total orders $<$ and $<'$ such that $(G,<)$ and $(G,<')$ are ordered groups, the division rings of fractions of kG inside $k((G,<))$ and $k((G,<'))$ are isomorphic.

The proof of Hughes' Theorem II in [**Hug72**] depends heavily on [**Hug70**], and thus it is also difficult to follow. As it has interesting consequences we give a proof of it based on the objects and techniques of [**DHS04**].

Let G be a locally indicable group and $L \lhd G$. Suppose that we are in the hypothesis of Hughes' Theorem II. Recall that the crossed product group ring kG can be expressed as the crossed product group ring $(kL)(G/L)$. By hypothesis, kL has a Hughes-free division ring of fractions D. From Hughes' Theorem I, it follows that the crossed product group ring structure of $(kL)(G/L)$ can be (uniquely) extended to $D(G/L)$. Again by hypothesis, $D(G/L)$ has a Hughes-free division ring of fractions E. The aim is to show that

$$kG \hookrightarrow D(G/L) \hookrightarrow E$$

is a Hughes-free embedding. So let H be a nontrivial finitely generated subgroup of G and $N \lhd H$ with H/N infinite cyclic generated by the class of t. We have to show that the powers of t are $E(kN)$-linearly independent, i.e. if $d_{-n}, \dots, d_n \in E(kN)$, then

$$\sum_{i=-n}^{n} d_i t^i = 0 \text{ implies that } d_{-n} = \cdots = d_n = 0. \tag{1}$$

There is a morphism of rational $k^\times H$-semirings $\Phi\colon \operatorname{Rat}(k^\times H) \to E(kH) \cup \{\infty\}$, which extends to a morphism of additive monoids $\Phi'\colon \operatorname{Rat}(k^\times H) \cup \{0\} \to E(kH) \cup \{\infty\}$. To prove (1) it is enough to show that

$$\sum_{i=-n}^{n} \Phi(f_i) t^i = \Phi\Big(\sum_{i=-n}^{n} f_i t^i\Big) = 0 \text{ implies that } \Phi(f_i) = 0,$$

for $f_{-n}, \dots, f_n \in \operatorname{Rat}(k^\times N) \cup \{0\}$. The result is then proved by induction on the elements of the form $f = \sum\limits_{i=-n}^{n} f_i t^i$ with $f_i \in \operatorname{Rat}(k^\times N) \cup \{0\}$.

The problem can be reduced to the situation of f a primitive element and $H = \operatorname{source}(f)$. The proof is then divided in four cases depending on how the group LH/LN looks like. If either $H \subseteq L$, or $\frac{LH}{LN}$ is infinite cyclic, the result follows directly from the hypothesis. Difficulties appear when $H \nsubseteq L$ and $\frac{LH}{LN}$ is finite. Since G/L is locally indicable, there exists a subgroup B of H such that $\frac{LH}{B} \cong \frac{LH/L}{B/L}$ is infinite cyclic. Then the proof splits into two parts, depending on whether $LH \neq BN$ or $LH = BN$. Technical arguments, somehow similar to a change of basis (one diferent for each case), allow us to prove the result by induction on the complexity of f.

As a consequence of Hughes' Theorem II, we obtain that many of the closure properties that hold for locally indicable groups also hold for Hughes-free embeddable groups. From these we obtain a lot of new locally indicable groups (hence right orderable groups) which are (Hughes-free) embeddable in division rings. For example every poly-ordered group is Hughes-free embeddable. We also get that the free product of Hughes-free embeddable groups is a Hughes-free embeddable group. This allows us to define the *Hughes-free coproduct of division rings* as follows. Let $\{G_i\}_{i\in I}$ be a set of Hughes-free embeddable groups. Let k be a division ring and kG_i a crossed product group ring with Hughes-free division ring of fractions D_i for each $i \in I$. The ring coproduct $\mathop{*}\limits_k kG_i$ can be seen as a crossed product group ring $k(\mathop{*}\limits_k G_i)$. Then we define the Hughes-free coproduct of $\{D_i\}_{i\in I}$ as the Hughes-free division ring of fractions of $k(\mathop{*}\limits_k G_i)$. We prove that in some cases it agrees with the coproduct of division rings $\mathop{\circ}\limits_k D_i$ defined by P.M. Cohn (see Definition 3.44). As Hughes' Theorem II gives some insight into how the division ring of fractions looks like, we think that it may help to reach a better understanding of $\mathop{\circ}\limits_k D_i$.

Let k be a division ring, G a free group, $<$ a total order on G such that $(G,<)$ is an ordered group and kG a crossed product group ring. J. Lewin [**Lew74**] proved that the embedding of kG inside its universal division ring of fractions is Hughes-free. Then, from Hughes' Theorem I, he deduced that it is isomorphic to the division ring of fractions of kG inside $k((G,<))$. We generalize this proof to a greater class of locally indicable groups, and show that if a crossed product group ring kG, with G in this class, has a universal division ring of fractions, then it has to be its Hughes-free division ring of fractions.

In Chapter 7 we deal with inversion height. We begin by presenting the basic definitions and easy properties of this invariant. Then we deal with embeddings of finite inversion height from [**HS07**], the so called JF-embeddings. They are embeddings of the free k-ring $k\langle X\rangle$ for X a set of at least two elements and k a division ring. They are obtained from results by

A.V. Jategaonkar [**Jat69**] and generalize embeddings of the free k-algebra $k\langle x, y\rangle$ given by J.L Fisher [**Fis71**] of inversion height 2. We prove that the inversion height of JF-embeddings is at most two, and provide examples of inversion height one and two for any set X of at least two elements.

Let us discuss our results with an example. Consider a field K with a morphism $\alpha\colon K \to K$ which is not onto, k a subfield of K such that $\alpha(a) = a$ for all $a \in k$, a nonempty set $\{t_i\}_{i\in I}$ of elements in K which are $\alpha(K)$-linearly independent, and the skew polynomial ring $K[x;\alpha]$. Then the ring generated by k and $X = \{x_i = t_i x\}_{i\in I}$ is the free k-algebra on X. Observe that $K[x;\alpha]$ is a left Ore domain. Let $Q = Q^l_{\mathrm{cl}}(K[x;\alpha])$. Then $k\langle X\rangle \hookrightarrow K[x;\alpha] \hookrightarrow Q$ gives an embedding of the free k-algebra $k\langle X\rangle$ in the division ring Q [**Jat69**].

The JF-embeddings of inversion height one for X a finite set of $n \geq 2$ elements are obtained from $K = k(t)$, $\alpha\colon K \to K$ defined by $t \mapsto t^n$ and $X = \{x, tx, \dots, t^{n-1}x\}$. For the case of X an infinite set of cardinality λ, we need the results on Chapter 1 on ordinal and cardinal arithmetic. We replace the semiring of exponents $\mathbb{N}$ of the polynomial ring $k[t]$ by the ordinal number λ^ω. Then, with the help of the *natural sum* $\oplus$ and *natural product* $\otimes$ of ordinals, we endow $M_\lambda = \{\gamma \mid \gamma < \lambda^\omega\}$ with a structure of semiring. We express the monoid ring kM_λ in multiplicative notation $k[t^\gamma \mid \gamma < \lambda^\omega]$. It has a field of fractions K. Then $\alpha\colon K \to K$, defined by $t^\gamma \mapsto t^{\lambda\otimes\gamma}$, and $X = \{t^\gamma x\}_{\gamma<\lambda}$ does the work.

JF-embeddings of inversion height two are obtained very much as the ones by J.L. Fisher. For each set I of cardinality at least two, fix $i_0 \in I$, and consider $J = I \setminus \{i_0\}$. Let $K = k(t_{in} \mid i \in J, n \geq 0)$, and define $\alpha\colon K \to K$ by $t_{in} \mapsto t_{in+1}$. Then the set $\{t_{i1}x\} \cup \{x\}$ generates a free k-algebra inside Q.

It is interesting to note that the embedding of the free algebra given by a JF-embedding does not extend to an embedding of the free group. We study this question in Section 5 and compute how the group G generated by the set X inside $Q^\times$ looks like in our examples of JF-embeddings. On the other hand, using the techniques and results in [**Lic84**], we show that an extension of the division ring Q contains the power series ring $k\langle\langle X\rangle\rangle$. Then the Magnus-Fox embedding of Chapter 1 implies that the free group algebra $k[H]$ of the free group H on a set Y of cardinality $|X|$ is contained in Q, providing examples of embeddings of the free group algebra of inversion height one and two.

In Section 6 we pay attention to embeddings of the the free algebra of infinite inversion height. The key result in this section is the following:

Theorem A. Let R be a domain with a division ring of fractions D. Let $(L,<)$ be an ordered group. Consider a crossed product group ring RL that can be extended to DL. Let $E = D((L,<))$ be the associated Mal'cev-Neumann series ring. Thus E is a division ring and $RL \hookrightarrow E$. If $f \in D$ is of inversion height n (with respect to $R \hookrightarrow D$), then $f \in E$ is of inversion height n (with respect to $RL \hookrightarrow E$). □

Let H be a free group on a set X of cardinality at least two, and $<$ a total order on H such that $(H,<)$ is an ordered group. Let k be a field and consider the group ring $k[H]$. Set $E = k((H,<))$. As we said before, B.H. Neumann [**Neu49a**] conjectured that $k[H] \hookrightarrow E$ is of infinite inversion height, and it was proved by C. Reutenauer [**Reu96**] that this holds for X an infinite set. In fact he proved that $k\langle X\rangle \hookrightarrow E$ is of infinite inversion height, which is equivalent to $k[H] \hookrightarrow E$ of infinite inversion height. We proceed to sketch how we reduce the problem to the situation of X a finite set of cardinality at least two.

Let $x \in X$. By a well-known result of groups, H can be expressed as an extension of the free group N on the infinite set $\{x^n y x^{-n} \mid y \in X \setminus \{x\},\ n \in \mathbb{Z}\}$ by the infinite cyclic group generated by x. Then $k[H]$ can be seen as a skew polynomial ring $k[N][x, x^{-1};\alpha]$ where α is defined by left conjugation by x. Moreover, we show that the polynomial ring

$E(k[N])[x,x^{-1};\alpha]$, where $E(k[N])$ denotes the division ring of fractions of $k[N]$ inside E, is contained in E in the natural way. Then, as a corollary of Theorem A, we infer that every element $f \in E(k[N])$ has the same inversion height with respect to $k[H] \hookrightarrow E$ as with respect to $k[N] \hookrightarrow E$. Thus, as N is the free group on an infinite set, Reutenauer's result implies that the inversion height of $k[H] \hookrightarrow E$ is infinite. In our proof we also exhibit elements of any given inversion height with respect to $k\langle X\rangle \hookrightarrow E$ and with respect to $k[H] \hookrightarrow E$.

We end this section by showing that infinite inversion height does not characterize the embedding $k\langle X\rangle \hookrightarrow E$. Indeed, with the help of Hughes-Theorem II we prove: Let k be a field. For each finite set X of at least two elements, there exist infinite non-isomorphic division rings of fractions D of $k\langle X\rangle$ such that $k\langle X\rangle \hookrightarrow D$ is of infinite inversion height.

The main object of Chapter 8 is the study of tilting modules obtained from universal localization. Let R be a ring. A right R-module T is *tilting* if the following three conditions are satisfied

(T1) $\operatorname{pd} T \leq 1$.
(T2) $\operatorname{Ext}^1_R(T, T^{(I)}) = 0$ for any set I.
(T3) There exists an exact sequence $0 \to R \to T_1 \to T_2 \to 0$ with $T_1, T_2 \in \operatorname{Add} T$.

Two tilting modules T and T' are *equivalent* if $T^\perp = T'^\perp$. Section 1 contains well known results on tilting modules. For example, we show the characterization from [**CT95**] of tilting right R-modules as the right R-modules T such that the class $\operatorname{Gen} T$ coincides with the class $T^\perp = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(T,M) = 0\}$. Recall that $\operatorname{Gen} T$ denotes the class of right R-modules which are homomorphic images of arbitrary direct sums of copies of T.

In Section 2 we mainly deal with homological properties of ring epimorphisms and the most important example of ring epimorphism in our discussion, universal localization.

The main results of Chapter 8 begin in Section 3. They are from [**AHS08**], and some of them are generalizations of the ones in [**AHHT05**]. We begin by showing that if R is a ring, and $\lambda\colon R \to S$ is an injective ring epimorphism with $\operatorname{Tor}^R_1(S,S) = 0$, then, among others, the following are equivalent

(i) $\operatorname{pd}(S_R) \leq 1$.
(ii) $T = S \oplus S/R$ is a tilting right R-module.

Tilting modules constructed in this way generalize the classical tilting $\mathbb{Z}$-module $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$. It also allows us to give many examples of tilting right R-modules. For example the ones studied in [**AHHT05**], where $S = \mathfrak{S}^{-1}R$, the left Ore localization of R at a left Ore set $\mathfrak{S}$ consisting of non-zero-divisors. Or, more generally, if $\mathcal{U}$ is a class of finitely presented modules of projective dimension one such that $\operatorname{Hom}_R(\mathcal{U}, R) = 0$ (i.e. $\mathcal{U}$ consists of *bound modules*) such that R-embeds in the universal localization $R_\mathcal{U}$ and $\operatorname{pd}((R_\mathcal{U})_R) \leq 1$, then $T_\mathcal{U} = R_\mathcal{U} \oplus R_\mathcal{U}/R$ is a tilting right R-module.

In Section 4 we concentrate on tilting right R-modules $T_\mathcal{U}$ obtained as before from universal localization. We give necessary and sufficient conditions for $T_\mathcal{U} = \mathcal{U}^\perp$. Most of the results are consequences of the following sufficient condition: $R_\mathcal{U}/R = \varinjlim N_i$ where N_i is a $\mathcal{U}$*-filtered* right R-module for each i, that is, each N_i has an ascending chain of submodules $(N_{i\nu} \mid \nu \leq \kappa_i)$ indexed at a cardinal κ_i such that

(a) N_{i0}=0 and $N_i = \bigcup\limits_{\nu<\kappa} N_{i\nu}$.
(b) $N_{i\nu} = \bigcup\limits_{\beta<\nu} N_{i\beta}$ for all limit ordinals $\nu < \kappa$.
(c) $N_{i(\nu+1)}/N_{i\nu}$ is isomorphic to some module in $\mathcal{U}$.

Although it is rather technical, it has interesting applications. It permits us to generalize already known results in the Ore situation. More precisely, we prove that if $\mathfrak{S}$ is a left Ore set and $\mathrm{pd}(\mathfrak{S}^{-1}R_R) \leq 1$, then $T_{\mathfrak{S}} = \mathfrak{S}^{-1}R \oplus \mathfrak{S}^{-1}R/R$ is a tilting right R-module with tilting class the $\mathfrak{S}$-*divisible modules*, i.e. modules M such that $\mathrm{Ext}^1_R(R/sR, M) = 0$ for all $s \in \mathfrak{S}$.

From that condition, best results are obtained when R is a hereditary ring with a faithful rank function $\rho\colon K_0(R) \to R$ such that the localization R_ρ of R at ρ is a semisimple artinian ring (see Section 4 in Chapter 3) as the next two sections show.

In Section 5 we are concerned with hereditary noetherian prime rings. They are equipped with a rank function: the normalized uniform dimension. In some situations all tilting right R-modules can be described up to equivalence in terms of universal localization. This is the case when R is a hereditary noetherian prime ring R such that there are no faithful simple right R-modules, and that $\mathrm{Ext}^1_R(U, V) = 0$ for all pairwise non-isomorphic simple right R-modules U and V. In particular, this holds for classical maximal orders and Dedekind domains. In both cases

$$\mathbb{T} = \{T_{\mathcal{W}} = R_{\mathcal{W}} \oplus R_{\mathcal{W}}/R \mid \mathcal{W} \subseteq \mathcal{U}_r\}$$

is a representative set up to equivalence of the class of all tilting right R-modules where $\mathcal{U}_r$ denotes a representative set of all simple right R-modules. For Dedekind domains it was obtained in a different form in [**BET05**]. We also show that, in general, not all these tilting modules $T_{\mathcal{W}}$ can be obtained from Ore or matrix localization.

In Section 6, via universal localization, we can construct tilting right R-modules in the context of finite dimensional tame hereditary algebras R where there is defined a rank function, the *normalized defect*. We show that for every set of simple regular modules $\mathcal{U}$ the module $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is tilting and that $T_{\mathcal{U}}^{\perp} = \mathcal{U}^{\perp}$. We also show that, in general, $T_{\mathcal{V}}^{\perp}$ does not coincide with $\mathcal{V}^{\perp}$, for $\mathcal{V}$ a class of bound modules. We give an example of this for a set $\mathcal{V}$ of regular modules.

Acknowledgments

I would like to take this opportunity to thank the people who has contributed, in some way or another, to the realization of this work.

Voldria començar agraint a la meva directora, la Dolors Herbera, tot el seu ajut en la realització d'aquest treball. Res d'això no hauria estat possible sense la seva guia ni els seus coneixements e idees. Però no només per la part cientfica ha estat un plaer ser alumne seu, també pel tracte cordial, el seu support i ànims quan els he necessitat, la seva paciència perquè no sempre he estat l'alumne ideal, el seu entusiasme per les Matemàtiques,... Gràcies!!

Many of these pages are based on joint works with Warren Dicks and Dolors Herbera [**DHS04**], Dolors Herbera [**HS07**], Lidia Angeleri Hügel [**AHS08**], thank you for your contribution.

Gràcies al Departament de Matemàtiques de la Universitat Autònoma de Barcelona perquè als que venim de fora ens feu sentir com a casa i per tots els mitjans que heu posat al meu abast per l'elaboració d'aquest treball. En especial al Grup de Teoria d'Anells per tots els seminaris i cursos que han organitzat, aquests han estat importants per a la meva formació matemàtica. Gràcies també al personal de secretaria, els de ara i els que ja no hi són, la vostra tasca és essencial pel bon funcionament del departament.

Vorrei ringraziare il Dipartimento di Informatica e Comunicazione dell'Università degli Studi dell'Insubria di Varese per la sua ospitalità, é stato un piacere lavorare nel dipartimento.

Grazie Lidia e Maria, abbiamo lavorato e imparato tanto nei seminari! Il vostro entusiasmo nel lavoro e la vita sono stati veramente importanti per me quando avevo più bisogno.

I would like to thank Robert Lee Wilson and Vladimir Retakh for the interest shown.

Gràcies als meus companys de despatx al llarg d'aquests anys: Jalila, Manel, Jorge, Vladimir, Enrique, Ramón, Lidia, Joaquim, Judit, Noemí, Sara, Noèlia, Josep, Lluís i Laura, heu fet que em senti molt bé al nostre lloc de treball.

Gracias a mis amigos en Málaga, Simón, Víctor, Dani, Raúl, Óscar, Curro, Sergio, Domingo, José María, Josefran,... porque siempre estáis ahí a pesar de que no mantengo mucho el contacto. Para mí es muy importante "recargar las pilas" con vosotros, además, por aquí se os echa de menos! Y a los "malagueños" en Barcelona Aude, Javier (Gómez Bello) y 13 Vientos.

Gracias a mis compañeros de piso a lo largo de todos estos años: mi hermano Alberto, Ramón, Diego, Micky y Judit, porque, al fin y al cabo, habéis sido vosotros los que me habéis tenido que aguantar durante este proceso. Lo he pasado muy bien y me he sentido muy a gusto compartiendo experiencias con vosotros.

Gracias a Carlos, Ander i Mireia, Alfonso, Susana, Mari, Francisca, Efrén, Francesc,... del grup de Joves de la Parròquia de Sant Martí de Cerdanyola, me gustaban mucho las reuniones que manteníamos.

Grazie Maria e Marco, Lidia e famiglia , Nir (il contrato! ma, ma,... che cosa fai?) perchè senza di voi la vita a Varese non avrebbe stata così piacevole.

Gracias a los futbolistas de los viernes.

Gràcies a tots els amics i companys que he fet i conegut en el dia a dia a la universitat. Hem compartit dinars, cafés, alguna festa,... coses que fan que la vida valgui la pena: Àlex, Punk is not dead; Berta, Judit, María y Noemí por tener siempre una sonrisa a punto; Birgit y su libreta; Dani, per tenir Catalunya i molt més al cor; Eduard, mucha suerte por Dinamarca!; Enrique, por su pasión por la música; Geir Arne, has dejado huella en todos nosotros; Gerard i Albert, perqué gairebé acabeu amb mi a la neu; Isa, mi profesora favorita de Estadística; Lluís Q. per ser el meu professor de Català i tantes altres coses; Lola, por tener un coche sobre el que uno puede expresarse libremente; Miquel, per resoldre'm mil i un problemes, per cert, vés entrenant per la propera cursa; Nacho, por tener un par de aplausos siempre a punto; Natàlia, por ser una caja de (agradables) sorpresas; Noèlia, amb tu mai falta dinar; Pere, *per nada* en particular; Ramón, hay razones *como pa meté los pies*, por todos los grandes momentos que hemos pasado juntos, por las conversaciones y risas justo antes de irme a dormir, por hacer del piso de Cerdanyola un hogar, portuario,... en resumen, por ser un verdadero amigo; Rosa, la próxima vez ganaré yo; Sara, mi "hermana de sangre", por cuidar mi cultura literaria; Yago, Nexpresso, what else? De esto sí que me acuerdo; Francesco, David, Joan, Jesús, Lluís B, Anna, Margarida,... Carlos, Daniel, Jalila, Iñaki, María del Mar, Jordi, Jorge, Jaume, Francesc,...

Gracias a Micky y Laura, vosotros sois los que habéis sufrido más las consecuencias de este largo último arreón para acabar, siempre con alegría.

Gràcies a la meva família: Nuri, Josep Maria, Fèlix i Nico, perquè sense vosaltres la meva adaptació a Barcelona hauria estat molt més difícil; Pere Joan i Assumpció per vetllar per la meva cultura, m'ho passo molt bé quan veniu a Barcelona; Bernat i Esteve, sempre és un plaer poder trobar-nos; Laura, Joan, Mireia (Enhorabona!), Laia, Blanca i Joan pels diumenges de canelons, pollastre i cargols en família.

Gracias a mi familia: Pepe, Loli, Pepe, Pilar, Loli y Juana por hacerme sentir como en casa cada vez que os visito, muchas menos de las que me gustaría; Aurora y Juan, porque ahora que vais más a menudo a Málaga, mis padres se están "despendolando"; Jesús (enhorabuena!) y Alejo, por todos los buenos momentos que hemos pasado juntos; Jesús, porque si cuando voy a Málaga y no te metes conmigo, pues no es lo mismo.

Gracias a mis padres, Rafael y Marta, a mis hermanos Marta, Alberto y Rafa por estar siempre ahí, por vuestro cariño y apoyo,... porque sin vosotros nada de esto tendría sentido. Va por ustedes!

This research has been partially supported by:

- DGESIC (Spain) through Project PB1998-0873
- The DGI and the European Regional Development Fund, jointly, through Project BFM2002-01390
- Comissionat per Universitats i Recerca of the Generalitat de Catalunya
- MEC-DGESIC (Spain) through Project MTM2005-00934
- Part of this research was carried out during two visits at Università dell'Insubria, Varese, in 2005 and 2006, supported by a grant of the Facoltà di Scienze dell'Università dell' Insubria, Varese, and by Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya

Partial list of notation

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\emptyset$	empty set
$\subseteq$	inclusion
$\subsetneq$	strict inclusion
$A \setminus B$	set theoretic difference
$A \cong B$	A is isomorphic to B
A^t	transpose of the matrix A
$\mathbb{M}_{m\times n}(R)$	set of $m \times n$ matrices with entries from R
$\mathbb{M}_n(R)$	set of $n \times n$ matrices with entries from R
$\mathrm{GL}_n(R)$	group of invertible $n \times n$ matrices over R
Mod- R, R -Mod	category of right, left modules over the ring R
$M_R,\ {}_RN$	right R-module M, left R-module N
${}_RM_S$	R-S-bimodule M
$\mathrm{tr}_{\mathcal{B}}(M)$	trace submodule of M, i.e. $\sum\{f(B) \mid f \in \mathrm{Hom}_R(B, M),\ B \in \mathcal{B}\}$
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$\mathrm{Hom}_R(M, N)$	group of morphisms of R-modules from M to N
$\mathrm{End}_R(M)$	ring of endomorphisms of the R-module M
$\mathrm{Aut}(R)$	group of automorphisms of the ring R
$\bigoplus\limits_{i\in I} M_i$	direct sum of the modules (groups) $\{M_i\}_{i\in I}$
$M^{(I)}$	direct sum of I copies of M
$\prod\limits_{i\in I} M_i$	direct product of $\{M_i\}_{i\in I}$
M^I	direct product of I copies of M
M^*	R-dual of an R-module M
$R^\times$	group of units of the ring R
$\mathrm{Spec}(R)$	set of prime ideals of the ring R
max-spec(R)	set of maximal ideals of the commutative ring R
$\varinjlim$	direct limit
$\varprojlim$	inverse limit
$H \lhd G$	H is a normal subgroup of the group G
$\mathop{*}\limits_{i\in I} G_i$	free product of the family of groups $\{G_i\}_{i\in I}$
$*_H^{i\in I} G_i$	free product of the family of groups $\{G_i\}_{i\in I}$ amalgamating H
length(M)	composition length of the module M

N^X	additive monoid of all functions from X to the additive monoid N	3
$\operatorname{supp}(f)$	support of f	3
$\ker f$	kernel of the morphism f	4, 5, 9
$\operatorname{im} f$	image of the morphism f	4, 5, 9
$N \rtimes C$	semidirect product of the group N by the group C	4
$A \wr B$	standard wreath product of A and B	4
$R^\times$	group of units of the ring R	5
$R\langle X\rangle$	free R-ring on the set X, polynomial ring of a crossed product group ring (semiring)	6, 63
$k\langle\langle X\rangle\rangle$	formal power series ring	6
$R[[x;\alpha]]$	skew power series ring (semiring)	7, 68
$R[x]$	polynomial ring (semiring)	7, 63
$R[x;\alpha]$	skew polynomial ring (semiring)	7, 63
$R[x,x^{-1}]$	Laurent polynomial ring (semiring)	7, 63
$R[x,x^{-1};\alpha]$	skew Laurent polynomial ring (semiring)	7, 63
$R[G]$	group (monoid) ring (semiring)	8, 63
$R((x;\alpha))$	skew Laurent series ring (semiring)	7, 68
$\operatorname{coker} f$	cokernel of the morphism f	9
M^*	dual of the module M	9
α^*	dual of the morphism α	9
$\operatorname{pd}(M)$	projective dimension of the module M	10
$\operatorname{id}(M)$	injective dimension of the module M	10
$\lvert A\rvert$	cardinality of the set A	15
$\mathcal{C}(G,X)$	Cayley graph of the group G with respect to the subset X	18
$(G(-),\Delta)$	Graph of groups	19
$\pi(G(-),\Delta,\Delta_0)$	fundamental group of the graph of groups $(G(-),\Delta)$	19
$R\cup\{0\}$	the semiring R with an absorbing zero	20
$R\setminus\{0\}$	the semiring R without a zero	20
$R\cup\{\infty\}$	the semiring R together with infinity	20
$\omega(R[G])$	augmentation ideal of the group ring $R[G]$	25
$S^G(x_1,\dots,x_n)$	normal subsemigroup of the group G generated by $\{x_1,\dots,x_n\}$	33
$S(x_1,\dots,x_n)$	subsemigroup generated by $\{x_1,\dots,x_n\}$	33
$S^C(x_1,\dots,x_n)$	Conrad subsemigroup generated by $\{x_1,\dots,x_n\}$	33
$\mathfrak{S}^{-1}R$	left Ore localization of the ring R at the left denominator set $\mathfrak{S}$	43
$\mathcal{T}_{\mathfrak{S}}(M)$	$\mathfrak{S}$-torsion submodule of M	46
$E(R)$, $D(R)$	division ring of fractions of the domain R inside the division ring E, D	47
$Q_n(R,E)$	subring of E consisting of the elements of inversion height at most n with respect to the embedding $R\hookrightarrow E$	47, 141
$Q^l_{\mathrm{cl}}(R)$	left Ore division ring of fractions of the left Ore domain R	48
R_Σ	localization of R at Σ	49, 55
$R_{\mathcal{P}}$	localization of R at the prime matrix ideal $\mathcal{P}$	49
$\underset{k}{*} R_i$	ring coproduct of the family of k-rings $\{R_i\}_{i\in I}$	54
$\underset{k}{\circ} D_i$	division ring coproduct of the division rings $\{D_i\}_{i\in I}$	54

$\mathcal{P}_R$, ${}_R\mathcal{P}$	category of all finitely generated projective right (left) R-modules	54
$K_0(R)$	Grothendieck group of finitely generated projective R-modules	57
$[P]$	isomorphism class of the module P	57
R_ρ	localization of R at the rank function ρ	58
$R((G,<))$	Mal'cev-Neumann series ring (semiring)	67
ρ_H	canonical morphism of groups from $k^\times H$ to H	85
$\mathcal{A}$, $\mathcal{A}_H$, $\theta(\mathcal{A})$	atlas	85
$\mathcal{T}$	set (semiring) of finite rooted trees	88
$\mathrm{fam}(X)$	family of the finite rooted tree X	88
$\mathrm{width}(X)$	width of the finite rooted tree X	89
$\mathrm{h}(X)$	height of the finite rooted tree X	89
$\exp(X)$	expanded of the finite rooted tree X	89
$\log(X)$	largest element of $\mathrm{fam}(X)$	91
$X \times_U Y$	tensor product of the right U-set X and the left U-set Y	94
$\bigcup\limits_{\gamma\in\Gamma} X_\gamma$	coproduct of the bisets $\{X_\gamma\}_{\gamma\in\Gamma}$	95
$U\natural X$	free multiplicative monoid on the U-biset X over the group U	98
$\mathrm{Rat}(U)$	universal rational U-semiring	99
$\mathrm{Tree}(f)$	complexity of the element $f \in \mathrm{Rat}(U)$	102
$\mathrm{source}(p)$	source subgroup of the element $p \in \mathrm{Rat}(U)$	104
$\mathrm{h}_E(R)$	inversion height of the domain R inside the division ring E	141
$\gamma \oplus \delta$	natural sum of the ordinal numbers γ and δ	149
$\gamma \otimes \delta$	natural product of the ordinal numbers γ and δ	149
A^{ij}	matrix obtained from the matrix A by deleting the i-th row and the j-th column	165
r_p^j	row vector obtained from the p-th row of a matrix deleting the j-th entry	165
s_q^i	column vector obtained from the q-th column of a matrix deleting the i-th entry	165
$\lvert A\rvert_{ij}$	(i,j)-th quasideterminant of the matrix A	166
$\mathcal{L}^\perp$	modules M such that $\mathrm{Ext}^1_R(L,M)=0$ for all L in the class $\mathcal{L}$	179
$\mathrm{Add}\,M$	modules isomorphic to direct summands of direct sums of copies of M	179
$\mathrm{Prod}\,M$	modules isomorphic to direct summands of product of copies of M	179
$\mathrm{Gen}\,M$	modules generated by M	179
$\mathrm{Pres}\,M$	M-presented modules	179
$\mathcal{X}_\mathcal{S}$	perpendicular category to $\mathcal{S}$	188
$\mathbf{T}M$	torsion part of the right R-module M over the semihereditary ring R	196
$\mathbf{P}M$	projective part of the right R-module M over the semihereditary ring R	196
$u(P)$	normalized uniform dimension of P	204
$\mathcal{U}_r$, $\mathcal{V}_r$	set of representatives of all isomorphism classes of finitely presented simple,torsion right modules	206
$\mathcal{D}_r$	$\{R/sR \mid s \text{ a non-zero-divisor}\}$	206
$\mathrm{Cl}(\mathfrak{p})$	clique of the prime ideal $\mathfrak{p}$	210
mod-R, R-mod	subcategory of finitely generated right, left R-modules	212

Contents

Part 1

Background Material

CHAPTER 1

Basic Terminology and Examples

1. Monoids and Groups

DEFINITIONS 1.1. (a) A nonempty set M with an associative binary operation

$$\begin{aligned} M \times M &\to M \\ (x, y) &\mapsto xy \end{aligned}$$

is called a *monoid* if it has an *identity element*, that is, there exists $1_M \in M$ such that $1_M x = x 1_M = x$ for all $x \in M$.

(b) A monoid M is said to be *abelian* or *commutative* if $xy = yx$ for all $x, y \in M$. Usually, when a monoid is commutative, the binary operation is denoted by $+$ and the identity element by 0. By an *additive monoid* we mean a commutative monoid with a binary operation $+$.

(c) Let M and M' be monoids. A *morphism of monoids* is a map $f\colon M \to M'$ such that
   (i) $f(x_1 x_2) = f(x_1) f(x_2)$ for every $x_1, x_2 \in M$, and
   (ii) $f(1_M) = 1_{M'}$.

An *isomorphism of monoids* is a bijective morphism of monoids. Two monoids M and M' are *isomorphic*, denoted $M \cong M'$, if there is an isomorphism $f\colon M \to M'$.

(d) Let X be a set and M be a monoid containing X. We say that M is the *free monoid on* X if for every monoid M', every map $f\colon X \to M'$ has a unique extension to a morphism of monoids of M to M'.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & M \arrow[d, dashed] \\ & M' \end{tikzcd}$$

It is known that for every set X there exists the free monoid on X. Observe that if M_1 and M_2 are free monoids on X, then M_1 is isomorphic to M_2. □

A monoid that will be used throughout is the following.

EXAMPLE 1.2. Let N be an additive monoid and let X be a set. We write N^X for the set of all functions from X to N. The map which sends every $x \in X$ to 0 is also denoted by 0. We will denote the elements f of N^X as formal sums $\sum_{x \in X} a_x x$, where a_x is the image of $x \in X$ by f. The set N^X can be made an additive monoid with the sum

$$f + g = \sum_{x \in X} (a_x + b_x) x,$$

for all $f = \sum_{x \in X} a_x x$ and $g = \sum_{x \in X} b_x x$ in N^X.

For an element $f = \sum_{x \in X} a_x x \in N^X$, we define the *support* of f, denoted by $\operatorname{supp}(f)$, as

$$\operatorname{supp}(f) = \{x \in X \mid a_x \neq 0\}.$$

Notice that $f = 0$ if and only if $\operatorname{supp}(f) = \emptyset$. □

Now we turn our attention to some basic definitions on groups.

Definitions 1.3. (a) A *group* G is a monoid G satisfying the further requirement that for each $x \in G$ there is an *inverse element*, that is, there is an element $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = 1$. The group G is *abelian* if it is an abelian monoid.

(b) A nonempty subset H of a group G is a *subgroup* of G in case H is a group under the binary operation of G.

(c) A subgroup N of a group G is a *normal subgroup* of G, denoted $N \lhd G$, in case $gNg^{-1} \subseteq N$ for every $g \in G$. The cosets of N in G form a group, denoted G/N, called the *quotient group*.

(d) Given two groups G and G', a *morphism of groups* is a map $f\colon G \to G'$ such that $f(xy) = f(x)f(y)$ for all $x, y \in G$. Notice that as a consequence $f(1_G) = 1_{G'}$. An *isomorphism of groups* is a bijective morphism of monoids. Two groups G and G' are *isomorphic*, denoted $G \cong G'$, if there is an isomorphism $f\colon G \to G'$.

(e) For a morphism of groups $f\colon G \to G'$, we define the subgroups *kernel* and *image* of f as $\ker f = \{x \in G \mid f(x) = 1_{G'}\}$ and $\operatorname{im} f = \{y \in G' \mid y = f(x) \text{ for some } x \in G\}$ respectively. Note that $\ker f$ is a normal subgroup of G.

(f) Let X be a set and G be a group containing X. We say that G is the *free group on X* if for every group G', every map $f\colon X \to G'$ has a unique extension to a morphism of groups of G to G'.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & G \arrow[d, dashed] \\ & G' \end{tikzcd}$$

Observe that if G_1 and G_2 are free groups on X, then G_1 is isomorphic to G_2.

(g) A group G is *locally free* if every finitely generated subgroup of G is free (on some set).

(h) It is known that for every set X there exists the free monoid on X and that every group G is a quotient of a free group. A group G is defined by *generators* $X = \{x_i \mid i \in I\}$ and *relations* $\Delta = \{r_j \mid j \in J\}$ in case F is the free group on X, R is the normal subgroup of F generated by $\{r_j \mid j \in J\}$, and $G \cong F/R$. We say that $\langle X \mid \Delta \rangle$ is a *presentation* of G. □

Important constructions of groups are the next ones.

Example 1.4. (a) A group H is a *semidirect product* of N by C in case H contains subgroups N and C such that:

(i) $N \lhd H$
(ii) $NC = H$
(iii) $N \cap C = \{1\}$.

We will write $H = N \rtimes C$. Observe that each element g of G can be uniquely expressed as a product $g = nc$ with $n \in N$ and $c \in C$, and these expressions are multiplied in the following way $(n_1c_1)(n_2c_2) = (n_1c_1n_2c_1^{-1})(c_1c_2)$. So if we know the groups N and C, to determine H, we only need to know the action of C in N by left conjugation of its elements.

(b) If A and B denote groups, then the standard wreath product of A and B, denoted $A \wr B$, is constructed as follows. Let $F = A^B$ be the direct product of copies of A indexed by the set B. Explicitly, F is the set of all functions from B into A, made into a group by componentwise multiplication. For $f \in F$ and $b \in B$, define $f^b \in F$ by $f^b(y) = f(yb^{-1})$ for all $y \in B$. Then, for each $b \in B$, the mapping $f \mapsto f^b$ is an automorphism of F, and the group of all such automorphisms is isomorphic to B; the *standard wreath product of*

A *and* B, $A \wr B$, is the extension of F by this group of automorphisms. Each element of $A \wr B$ can be written uniquely as fb, with $f \in F$ and $b \in B$ and these elements multiply according to the rule $f_1b_1f_2b_2 = f_1f_2^{b_1^{-1}}b_1b_2$. So $A \wr B$ is a semi-direct product of F and B: $F \lhd A \wr B$, $FB = A \wr B$, and $F \cap B = \{1\}$.

(c) By $A \wr B$ we also denote the *restricted standard wreath product* of A and B. It consists of the elements fb of the standard wreath product such that f has finite support, that is, $F = A^{(B)} = \bigoplus_B A$ in the foregoing construction. □

2. Rings

Definitions 1.5. (a) A *ring* is an additive abelian group $(R, +)$ with a second associative binary operation, multiplication, the two operations being related by the *distributive laws*:

$$x(y + z) = xy + xz \quad \text{and} \quad (y + z)x = yx + zx.$$

for all $x, y, z \in R$. We also require the existence of an *identity element* $1 \in R$ such that $x1 = 1x = x$ for all $x \in R$. That is, all rings considered here will have an identity element. A ring is said to be a *commutative ring* if its multiplicative monoid is commutative.

(b) If R is a ring, then $x \in R$ is called a *unit* if it has a multiplicative inverse, i.e. if there exists $x^{-1} \in R$ such that $xx^{-1} = x^{-1}x = 1$. The set of all units in R is denoted by $R^\times$ and is called the *group of units*. Notice it is a group.

If $x, y \in R \setminus \{0\}$ and $xy = 0$, we say that x is a *left zero divisor* and y is a *right zero divisor*.

(c) A nonzero ring is called a *domain* if it has no zero divisors. A nonzero ring in which every nonzero element has a multiplicative inverse is called a *division ring*. A commutative division ring is a *field*. Notice that every division ring is a domain.

(d) Let R be a ring and I a subset of R. We say that I is a *right ideal* of R if it is an additive subgroup of R such that for all $y \in I$ and $r \in R$, $yr \in I$. And we say that I is a *left ideal* if it is an additive subgroup of R such that for all $y \in I$ and $r \in R$, $ry \in I$. If I is a left and a right ideal, we say that I is an *ideal*.

(e) Given an ideal I of a ring R, the *quotient ring* of R modulo I is the ring constructed from the additive group R/I and with the multiplication defined by $\bar{r} \cdot \bar{s} = \overline{rs}$ for all $r, s \in R$.

(f) Given two rings, R and R', a *morphism of rings* is an additive morphism of groups $f\colon R \to R'$ such that

$$f(xy) = f(x)f(y) \text{ for every } x, y \in R, \qquad \text{and} \qquad f(1_R) = 1_{R'}.$$

When $R = R'$ we call f a *ring endomorphism*. A bijective morphism of rings is called an *isomorphism*. The rings R and R' are *isomorphic*, denoted $R \cong R'$, if there is an isomorphism of rings $f\colon R \to R'$. A morphism of rings is a *monomorphism* if it is an injective map.

(g) For a morphism of rings $f\colon R \to R'$, we define the *kernel* and *image* of f as the ideal of R $\ker f = \{r \in R \mid f(x) = 0\}$ and the subring of R' $\operatorname{im} f = \{r' \in R' \mid r' = f(r) \text{ for some } r \in R\}$ respectively.

(h) Let R be a ring. The *Jacobson radical* of R, denoted $J(R)$, is the intersection of all maximal right ideals of R (i.e. right ideals different from R and not contained in any other right ideal different from R and themselves). It can be proved that J is an ideal of R. Among other important properties of $J(R)$ are the following:

(a) An element r of R is invertible (in R) if and only if $\bar{r} \in R/J(R)$ is invertible (in $R/J(R)$).

(b) If $\mathbb{M}_n(R)$ denotes the ring of $n \times n$ matrices over R, then $J(\mathbb{M}_n(R)) = \mathbb{M}_n(J(R))$.

(i) A ring R that has a unique maximal right ideal or, equivalently, that the set of non-units of R form an ideal, is said to be a *local ring*.
(j) A ring is *left artinian* if it satisfies the descending chain condition on left ideals. Similarly is defined a right artinian ring. A ring is *artinian* if it is both left and right artinian.
(k) A ring is *left noetherian* if it satisfies the ascending chain condition on left ideals. Similarly is defined a right noetherian ring. A ring is *noetherian* if it is both left and right noetherian.
(l) Let k be a division ring and consider the set $\mathbb{M}_{m\times n}(k)$ of $m \times n$ matrices over k. For a matrix $A \in \mathbb{M}_{m\times n}(k)$, the *rank* of A is the number of left k-linear independent rows of A, which coincides with the number of right k-linear independent columns of A. $\square$

In the next examples we follow very close the exposition in [**Lam01**, Section 1].

EXAMPLES 1.6. We present examples of rings that will be used throughout this dissertation.

(a) Let k be any ring, and $X = \{x_i\}_{i\in I}$ be a system of independent, non-commuting indeterminates over k. Then we can form the *free k-ring* generated by X, which we denote by $k\langle X\rangle$. The elements of $k\langle X\rangle$ are polynomials in the non-commuting variables in X with coefficients from k. The freeness of $k\langle X\rangle$ refers to the following universal property: if $\varphi_0\colon k \to k'$ is any morphism of rings and $\{a_i \mid i \in I\}$ is any subset of k' such that each a_i commutes with each element of $\varphi_0(k)$, then there exists a unique morphism of rings $\varphi\colon k\langle X\rangle \to k'$ such that $\varphi_{|k} = \varphi_0$, and $\varphi(x_i) = a_i$ for each $i \in I$. When k is a field, we usually call $k\langle X\rangle$ the *free k-algebra* on X.
(b) Let k be a ring and $X = \{x_i\}_{i\in I}$ be independent variables over k. In this example, the variables may be taken to be either pairwise commuting or otherwise, but we shall assume that they all commute with the elements of k. With this convention, we can form the *ring of formal power series* $k\langle\langle X\rangle\rangle$, if the variables are non-commuting, and $k[[x_i \mid i \in I]]$ if the variables are pairwise commuting. In either case the elements have the form

$$F = \sum_{n=0}^{\infty} f_n = f_0 + f_1 + f_2 + \cdots , \tag{2}$$

where f_n is a homogeneous polynomial in X over k with degree n, and we sum and multiply these power series formally, i.e. given $G = \sum\limits_{n=0}^{\infty} g_n$, then

$$F + G = \sum_{n=0}^{\infty}(f_n + g_n), \qquad FG = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} f_i g_{n-i}\right).$$

It is not difficult to calculate the units of $k\langle\langle X\rangle\rangle$ and $k[[x_i \mid i \in I]]$; indeed, F as in (2) is a unit if and only if the constant term f_0 is a unit in k. It suffices to do the if part, so let us assume that $f_0 \in k^\times$. To find a power series $G = g_0 + g_1 + g_2 + \cdots$ such that $FG = 1$, we have to solve the equations:

$$1 = f_0g_0, \quad 0 = f_0g_1 + f_1g_0, \quad 0 = f_0g_2 + f_1g_1 + f_2g_0, \quad 0 = \sum_{i=0}^{n} f_i g_{n-i}, \text{ for } n \geq 1.$$

Since $f_0 \in k^\times$, we can solve for $g_0, g_1, g_2, \ldots$ inductively. This shows that F is right invertible, and by symmetry we see that F is also left invertible in $k\langle\langle X\rangle\rangle$ or $k[[x_i \mid i \in I]]$. Observe that $k\langle X\rangle$ is contained in $k\langle\langle X\rangle\rangle$ as the series with only a finite number of nonzero f_n.

(c) Let k be a ring and $\alpha\colon k \to k$ be a ring endomorphism. We construct the *skew power series ring* $k[[x;\alpha]]$. The elements of $k[[x;\alpha]]$ are formal power series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

with $a_n \in k$ for each n. The sum and multiplication are defined by

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n, \quad \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i \alpha^i(b_{n-i})\right) x^n.$$

Now the units of $k[[x;\alpha]]$ are not difficult to determine. Indeed

$$k[[x;\alpha]]^\times = \left\{\sum_{n=0}^{\infty} a_n x^n \mid a_0 \in k^\times\right\}.$$

The contention $k[[x;\alpha]]^\times \subseteq \left\{\sum\limits_{n=0}^{\infty} a_n x^n \mid a_0 \in k^\times\right\}$ is clear by the way multiplication of series is defined.

Now let $F = \sum\limits_{n=0}^{\infty} a_n x^n$ with $a_0 \in k^\times$. In order to find $G = \sum\limits_{n=0}^{\infty} b_n x^n$ such that $FG = 1$, we have to solve the equations:

$$\begin{cases} 1 = a_0 b_0 \\ 0 = a_0 b_n + \sum\limits_{j=1}^{n} a_j \alpha^j(b_{n-j}) \quad n = 1, 2, \dots \end{cases}$$

And to find $H = \sum\limits_{n=0}^{\infty} c_n x^n$ such that $HF = 1$ we have to solve the equations:

$$\begin{cases} 1 = c_0 a_0 \\ 0 = c_n \alpha^n(a_0) + \sum\limits_{j=0}^{n-1} c_j \alpha^j(a_{n-j}) \quad n = 1, 2, \dots \end{cases}$$

Since a_0 (and then $\alpha^n(a_0)$) is a unit in k, we can solve for $b_0, b_1, \dots$ and $c_0, c_1, \dots$ inductively. This shows that F is right and left invertible in $k[[x;\alpha]]$. Hence F is invertible in $k[[x;\alpha]]$.

An important subring of $k[[x;\alpha]]$ is the *skew polynomial ring* $k[x;\alpha]$ consisting of the series $\sum\limits_{n=0}^{\infty} a_n x^n$ of $k[[x;\alpha]]$ such that only a finite number of a_n are different from zero.

Observe that when α is the identity map we obtain the usual *polynomial ring* $k[x]$ and *power series ring* $k[[x]]$.

(d) Let k be a ring and $\alpha\colon k \to k$ a ring automorphism. Then we can form the *skew Laurent series ring* $k((x;\alpha))$. The elements of $k((x;\alpha))$ are the set of formal power series $\sum\limits_{n\in\mathbb{Z}} a_n x^n$ where $a_n \in k$ for each $n \in \mathbb{Z}$, and among the coefficients a_n with $n < 0$ only finitely many can be nonzero. The sum and multiplication are defined by

$$\sum_{n\in\mathbb{Z}} a_n x^n + \sum_{n\in\mathbb{Z}} b_n x^n = \sum_{n\in\mathbb{Z}} (a_n + b_n) x^n, \quad \left(\sum_{n\in\mathbb{Z}} a_n x^n\right)\left(\sum_{n\in\mathbb{Z}} b_n x^n\right) = \sum_{n\in\mathbb{Z}}\left(\sum_{i\in\mathbb{Z}} a_i \alpha^i(b_{n-i})\right) x^n.$$

Observe that the rings $k[[x;\alpha]]$ and $k[x;\alpha]$ are subrings of $k((x;\alpha))$. Another important subring of $k((x;\alpha))$ is the *skew Laurent polynomial ring* $k[x, x^{-1};\alpha]$ consisting of the power series with only a finite number of nonzero coefficients. When α is the identity we obtain the *Laurent series ring* $k((x))$ and *Laurent polynomial ring* $k[x, x^{-1}]$.

One particularly good feature of $k((x;\alpha))$ is that if k is a division ring, then so is $k((x;\alpha))$. To see this note that each nonzero series can be written as $\left(\sum_{n=0}^{\infty} a_n x^n\right) x^r$ where $r \in \mathbb{Z}$ and $a_0 \neq 0$. If $\sum_{n=0}^{\infty} b_n x^n$ is the inverse of $\sum_{n=0}^{\infty} a_n x^n$, which can be computed as in the foregoing example, then $x^{-r}\left(\sum_{n=0}^{\infty} b_n x^n\right)$ is the inverse of $\left(\sum_{n=0}^{\infty} a_n x^n\right) x^r$. Therefore, when α is a ring automorphism, the rings $k[x;\alpha]$, $k[x,x^{-1};\alpha]$ and $k[[x;\alpha]]$ can be embedded in a division ring. If α is only an injective morphism of rings (so that these rings are domains), then $k[x;\alpha]$ and $k[[x;\alpha]]$ are also embeddable in a division ring, see Proposition 3.10.

(e) Let k be any ring, and G be a group or a monoid. Then we can form the *group/monoid ring* $k[G] = \bigoplus_{g\in G} kg$. The elements of $k[G]$ are finite formal sums of the shape $\sum_{g\in G} a_g g$, with $a_g \in k$ for each $g \in G$. Then the elements are summed and multiplied as follows

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g, \qquad \left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}\left(\sum_{\{(p,q)\mid pq=g\}} a_p b_q\right) g.$$

Observe that if G is the free monoid on a set X, then $k[G]$ is the free k-ring $k\langle X\rangle$ defined in (a).

Very important for us will be the generalization of this kind of rings called crossed product group/monoid rings, see Chapter 4. □

3. Modules

We give some of the definitions and results that will be used in this dissertation. The reader is referred to [**Rot70**], [**Lam01**] and [**Lam99**] for details and the missing concepts such as the definitions of tensor product $A \otimes_R B$ and the functors $\mathrm{Tor}_n^R(\_,\_)$ and $\mathrm{Ext}_R^1(\_,\_)$.

Definitions 1.7. Let R be a ring.

(a) A *right R-module* is an additive group M such that there is a map $M \times R \to M$, $(m,r) \mapsto mr$, satisfying:
  (i) $(m+m')r = mr + m'r$;
  (ii) $m(r+r') = mr + mr'$;
  (iii) $m(rr') = (mr)r'$;
  (iv) $m1_R = m$,
for all $r, r' \in R$ and $m, m' \in M$. We will sometimes write M_R to indicate that M is a right R-module. Similarly are defined a left R-module and ${}_RM$.

(b) A *submodule* of a right R-module M is an additive subgroup M' of M with $m'r \in M'$ for all $m' \in M'$ and $r \in R$. The *quotient module* M/M' is the quotient group made into a right R-module by $\bar{m}r = \overline{mr}$ for all $m \in M$ and $r \in R$. A right R-module M is said to be a *simple module* if it has no other submodules other than $\{0\}$ and M.

(c) If M and N are right R-modules, a *morphism of right R-modules* is a morphism of abelian groups $f\colon M \to N$ such that $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$. We say that the morphism of right R-modules f is an *isomorphism* (respectively *monomorphism*, *epimorphism*) if f is bijective (respectively injective, onto).

(d) Let M and N be two right R-modules. The set of all morphisms of right R-modules between M and N is denoted by $\mathrm{Hom}_R(M,N)$. It is an abelian group where the sum of $f, g \in \mathrm{Hom}_R(M,N)$ is the morphism defined by $(f+g)(m) = f(m) + g(m)$ for all $m \in M$.

(e) Let $f\colon M \to N$ be a morphism of right R-modules. We define the *kernel* of f as $\ker f = \{m \in M \mid f(m) = 0\}$; the *image* of f as $\operatorname{im} f = \{n \in N \mid f(m) = n \text{ for some } m \in M\}$; the *cokernel* of f as $\operatorname{coker} f = N/\operatorname{im} f$.

(f) A sequence of morphisms of right R-modules

$$\cdots \to M_{n+1} \stackrel{f_{n+1}}{\to} M_n \stackrel{f_n}{\to} M_{n-1} \to \cdots$$

is *exact* if $\operatorname{im} f_{n+1} = \ker f_n$ for all n. A *short exact sequence* is an exact sequence of morphisms of right R-modules of the form $0 \to A \to B \to C \to 0$.

(g) If M is a right R-module such that there exists an exact sequence of right R-modules $0 \to X \to Y \to M \to 0$ where Y is a finitely generated free module, and X is finitely generated, then we say that M is *finitely presented.*

(h) An important example of right R-module is R_R. Its submodules coincide with the right ideals of R. A right R-module F is called *free* if it is isomorphic to a direct sum of copies of R_R. Equivalently, F_R is free iff it has a *basis*, i.e. a set $\{e_i : i \in I\} \subseteq F$ such that any element of F is a unique finite right R-linear combination of the e_i's. Also, a right R-module F_R with a subset $B = \{e_i \mid i \in I\}$ is free with B as basis iff the following universal property holds: for any family of elements $\{m_i : i \in I\}$ in any right R-module M, there is a unique morphism of right R-modules $f\colon F \to M$ with $f(e_i) = m_i$ for all $i \in I$.

(i) A right R-module E is *injective* if for any exact sequence of right R-modules $0 \to A \to B \to C \to 0$ the induced sequence of abelian groups

$$0 \to \operatorname{Hom}_R(C, E) \to \operatorname{Hom}_R(B, E) \to \operatorname{Hom}_R(A, E) \to 0$$

is exact.

(j) A right R-module E is *projective* if for any exact sequence of right R-modules $0 \to A \to B \to C \to 0$ the induced sequence of abelian groups

$$0 \to \operatorname{Hom}_R(P, A) \to \operatorname{Hom}_R(P, B) \to \operatorname{Hom}_R(P, C) \to 0$$

is exact. Free modules are examples of projective right R-modules. In fact, projective right R-modules can be characterized as the right R-modules which are isomorphic to direct summands of free right R-modules.

(k) A right R-module P is *flat* if for any short exact sequence of left R-modules $0 \to A \to B \to C \to 0$ the induced sequence of abelian groups

$$0 \to P \otimes_R A \to P \otimes_R B \to P \otimes_R C \to 0$$

is exact. Examples of flat modules are projective modules.

(l) Let P be a right R-module, we define P^* as the left R-module $\operatorname{Hom}_R(P, R)$, where, for each $r \in R$, (rf) is defined by $p \mapsto rf(p)$ for all $p \in P$. Let $\alpha\colon P \to Q$ be a morphism of right R-modules, then $\alpha^*\colon Q^* \to P^*$ is the morphism of left R-modules defined by $\gamma \mapsto \gamma\alpha$. Similarly are defined P^* and α^* in case P is a left R-module and α a morphism of left R-modules. □

Useful properties of finitely generated projective right R-modules are the following two results.

LEMMA 1.8. *Let R be a ring, let P, Q be projective right R-modules, and let $\alpha\colon P \to Q$ be a morphism of right R-modules. Then the following hold:*

(i) *P^* is a projective left R-module.*

(ii) *If P and Q are finitely generated, then the following diagram is commutative*

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\cong"'] & Q \arrow[d, "\cong"] \\ P^{**} \arrow[r, "\alpha^{**}"] & Q^{**} \end{tikzcd}$$

where the vertical arrows are isomorphisms of right R-modules defined by $p \mapsto \hat{p}$ and $\hat{p}\colon P^* \to R$, $\gamma \mapsto \gamma(p)$. □

Lemma 1.9. *Let R be a ring. Let P and Q be finitely generated projective right R-modules, and let M be a right R-module. Then there exists an isomorphism $M \otimes_R P^* \cong \mathrm{Hom}_R(P, M)$ such that for any morphism of right R-modules $\alpha\colon P \to Q$, the following diagram is commutative*

$$\begin{tikzcd} M \otimes_R Q^* \arrow[r, "1_M \otimes_R \alpha^*"] \arrow[d, "\cong"'] & M \otimes P^* \arrow[d, "\cong"] \\ \mathrm{Hom}_R(Q, M) \arrow[r] & \mathrm{Hom}_R(P, M) \end{tikzcd}$$

where the map in the last row is defined by $f \mapsto f\alpha$. □

Definition 1.10. Let R be a ring and M a right R-module.

(a) A right R-module E that contains M is an *injective hull* of M if E is injective and every nonzero submodule of E intersects M nontrivially. It is known that any right R-module has an injective hull and that injective hulls are unique up to isomorphism, that is, there is an isomorphism between any two injective hulls of M which is the identity on M.

(b) An *injective coresolution* of M is an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots$$

where each E_n is an injective right R-module. Note that every right R-module has an injective coresolution.

(c) A *projective resolution* of M is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

where each P_n is a projective right R-module. Note that every right R-module has a projective resolution.

(d) We say that $\mathrm{pd}(M) \le n$ if there is a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

(e) We say that $\mathrm{id}(M) \le n$ if there is an injective coresolution

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0. \quad \square$$

The following well-known result will be used without any reference.

Lemma 1.11. *Let R be a ring and M a right R-module.*

(i) *The following are equivalent:*
  (a) $\mathrm{pd}\, M \le n$.
  (b) $\mathrm{Ext}^r_R(M, N) = 0$ *for all right R-modules N and all $r \ge n+1$.*

(ii) *The following are equivalent:*
  (a) $\mathrm{id}(M) \le n$.
  (b) $\mathrm{Ext}^r_R(N, M) = 0$ *for all right R-modules N and all $r \ge n+1$.* □

Definitions 1.12. (a) A ring R is *right (left) hereditary* if all right (left) ideals of R are projective as right (left) R-modules. A ring R is *hereditary* if it is right and left hereditary.
(b) A ring R is *right (left) semihereditary* if all finitely generated right (left) ideals are projective as right (left) R-modules. A ring R is *semihereditary* if it is right and left semihereditary. □

Lemma 1.13. *Let R be a ring.*

(i) *The following are equivalent:*
  (a) *R is right hereditary.*
  (b) *Any submodule of a projective right R-module is a projective right R-module.*
  (c) $\mathrm{pd}(M) \leq 1$ *for all right R-modules M.*

(ii) *The following are equivalent:*
  (a) *R is right semihereditary.*
  (b) *Any finitely generated submodule of a projective right R-module is a projective right R-module.* □

Definition 1.14. Let R and S be two rings. We say that M is an *R-S-bimodule*, denoted ${}_RM_S$, if M is a left R-module and a right S-module, and there is an associative law: $r(ms) = (rm)s$ for all $r \in R,\ s \in S$ and $m \in M$. □

The following result will be referred as the *Hom-Tensor adjunction*

Lemma 1.15. *Let R and S be two rings. Let A be a right R-module, B an R-S-bimodule and C a right S-module, then there is an isomorphism of abelian groups*

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(A \otimes_R B, C). \quad \square$$

4. Ordinal and cardinal numbers

In this section we collect standard material on ordinal and cardinal numbers. Most of it is taken from [**Sie58**].

4.1. Ordinal numbers. These concepts will be used throughout this section.

Definitions 1.16. (a) Given a set A and a binary relation $<$ defined in A, we say that $(A, <)$ is an *ordered set* (or $<$ is an *order* in A) if the following conditions hold:
  (i) For all $a, a' \in A$, $a < a'$ implies that $a' < a$ does not hold.
  (ii) The relation $<$ is *transitive*, i.e. if $a, b, c \in A$ with $a < b$ and $b < c$, then $a < c$.
(b) The ordered set $(A, <)$ is a *totally ordered set* (or $<$ is a *total order* in A) if for each pair $a, b \in A$ with $a \neq b$, then either $a < b$ or $b < a$.
(c) An ordered set $(A, <)$ is said to be *well-ordered* if each non-empty subset of A has a least element, i.e. for each $\emptyset \neq B \subseteq A$ there exists $a \in B$ such that $a < b$ for all $b \in B \setminus \{a\}$. Notice that a well-ordered set is a totally ordered set.
(d) Two ordered sets $(A, <)$ and $(B, <')$ are *similar* if there exists a bijective map $f\colon A \to B$ such that $f(a_1) < f(a_2)$ for all $a_1, a_2 \in A$ with $a_1 < a_2$.
  Observe that similarity is an equivalence relation. Thus ordered sets are divided into disjoint classes. These classes are called *order types*. Hence every ordered set $(A, <)$ determines a certain order type, namely the type determined by the class of all ordered sets similar to $(A, <)$. The order type of $(A, <)$ is denoted by $\overline{(A, <)}$.
(e) Order types of well-ordered sets are called *ordinal numbers*. Number 0 (order type of the empty set) is included among ordinal numbers. Notice that if $(A, <)$ is a well-ordered set, then any ordered set $(B, <)$ similar to $(A, <)$ is also a well-ordered set.

The finite ordinals, that is, those which are the order type of a finite set, are denoted by $0, 1, 2, \dots$ according to the number of elements of the set. The ordinal number corresponding to the order type of $(\mathbb{N}, <)$ is denoted by ω.

(f) Let $(A, <)$ be a totally ordered set. The subset A' of A is called a *segment* if for each pair of elements $a \in A'$, $b \in A$ such that $b < a$, then $b \in A'$. □

4.2. Some properties and ordinal arithmetic. The next result is well-known:

PROPOSITION 1.17. *Let $(A, <)$ and $(A', <')$ be two well-ordered sets. Then one and only one of the following cases hold:*

(i) *$(A, <)$ and $(A', <')$ are similar.*
(ii) *$(A, <)$ is similar to a certain segment of $(A', <')$ different from A' and $(A', <')$ is not similar to any subset of $(A, <)$.*
(iii) *$(A', <')$ is similar to a certain segment of $(A, <)$ different from A and $(A, <)$ is not similar to any subset of $(A', <')$.* □

DEFINITION 1.18. (a) Let $(A, <)$ and $(A', <')$ be two well-ordered sets, $\alpha = \overline{(A, <)}$ and $\beta = \overline{(A', <')}$. We say that $\alpha < \beta$ if case (ii) in Proposition 1.17 holds. On the other hand if case (iii) in Proposition 1.17 holds, we say that $\beta < \alpha$. Therefore given two ordinals α, β one and only one of the following possibilities hold:

$$\text{(i) } \alpha = \beta \qquad \text{(ii) } \alpha < \beta \qquad \text{(iii) } \beta < \alpha.$$

Ordinal number 0 is regarded as the smallest ordinal number.

(b) Given a well-ordered set $(A, <)$ and an element $a \in A$, we denote by A_a the segment of A

$$A_a = \{x \in A \mid x < a\}.$$

Notice that A_a is different from A and that it may be empty if a is the first element of $(A, <)$. Conversely, let A' be a segment of A different from A. Since $A \setminus A'$ is not empty, it has a least element a. Then clearly $A' = A_a$. Therefore the set of all segments of the well-ordered set A that are different from A is the set of all sets A_a for $a \in A$. □

Let $\alpha > 0$ be an ordinal number and let $(A, <)$ be a nonempty well-ordered set of type α. For each $a \in A$ denote by ξ_a the order type of A_a. For all $a \in A$ we have that $\xi_a < \alpha$ and if $a, b \in A$ with $a < b$, then $\xi_a < \xi_b$. In this way to each element of A corresponds a certain ordinal number $\xi < \alpha$, a greater number always corresponding to a later element. On the other hand, every ordinal number $\xi < \alpha$ corresponds to a certain element of the set A; indeed, if $\xi < \alpha$ and $(A', <')$ is a well-ordered set of type ξ, then A' is similar to a segment of A different from A, and if $a \in A$ is the first element not in this segment we get that $(A', <')$ is similar to A_a, hence $\xi = \xi_a$. Therefore

PROPOSITION 1.19. *Let α be an ordinal number. A well-ordered set of type $\alpha \neq 0$ is similar to the set of ordinal numbers $A_\alpha = \{\xi \mid \xi < \alpha\}$, ordered according to their magnitude.* □

DEFINITIONS 1.20. Let α and β be two ordinal numbers. Thus there exist well-ordered sets $(A, <_A)$ and $(B, <_B)$ of order types α and β respectively.

(a) Let $(S, <_{A+B})$ be the well-ordered set formed as follows. As a set S is the disjoint union of A and B. The binary relation $<_{A+B}$ is defined by the conditions: if $s_1, s_2 \in S$, then $s_1 <_{A+B} s_2$ holds if and only if $s_1, s_2 \in A$ and $s_1 <_A s_2$, or $s_1, s_2 \in B$ and $s_1 <_B s_2$, or $s_1 \in A$ and $s_2 \in B$. In other words, we order the disjoint union S of A and B in such a manner that for each two elements of S both belonging to A or both belonging to B we retain the ordering in which they appeared in those sets, and two elements of the set S belonging one to A and the other to B we regard the one belonging to A as preceding the other. It is easy to prove that $(S, <_{A+B})$ is a well-ordered set. Let σ be the order type of $(S, <_{A+B})$. Then we define the *sum* of α and β as

$$\alpha + \beta = \sigma.$$

The concept of sum of ordinal numbers can be immediately generalized by induction to the sum of an arbitrary finite number of ordinal numbers, and it is easy to prove that this sum has the property of associativity, that is

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

for any ordinal numbers α, β, γ. From the definition of the sum of ordinal numbers it follows that

$$\alpha + 0 = 0 + \alpha = \alpha$$

for any ordinal number α.

On the other hand, the sum of ordinal numbers is not commutative. For example

$$1 + \omega = \omega \neq \omega + 1.$$

(b) Let $(P, <_{A \cdot B})$ be the well-ordered set constructed as follows. As a set $P = A \times B$. The order $<_{A \cdot B}$ in $A \times B$ is defined as: for $(a_1, b_1), (a_2, b_2) \in A \times B$, $(a_1, b_1) <_{A \cdot B} (a_2, b_2)$ if and only if $b_1 <_B b_2$, or $b_1 = b_2$ and $a_1 < a_2$. It is easy to prove that $(P, <_{A \cdot B})$ is a well-ordered set. The order type ρ of P is the *product* of α and β, i.e.

$$\alpha \cdot \beta = \rho.$$

From the definition it follows that

$$1 \cdot \alpha = \alpha \cdot 1 = \alpha$$

for any ordinal number α. The concept of the product of ordinal numbers is generalized by induction to the product of an arbitrary finite number of ordinals and it is not difficult to prove that this product is associative, that is

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

for any ordinal numbers α, β, γ.

On the other hand, the product of ordinal numbers is not commutative. For example

$$2 \cdot \omega = \omega \neq w \cdot 2.$$

As regards the law of distributivity of the multiplication of ordinals with respect to their addition, it holds only in one of its forms, namely when the second factor is a sum. It is not too difficult to prove that for all ordinal numbers α, β, γ we have

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \tag{3}$$

but

$$(1 + 1) \cdot \omega = 2 \cdot \omega = \omega \neq \omega + \omega = \omega \cdot 2 = \omega \cdot (1 + 1).$$

By induction formula (3) can be generalized to an arbitrary finite number of ordinals and obtain the formula

$$\alpha \cdot (\beta_1 + \cdots + \beta_n) = \alpha \cdot \beta_1 + \cdots + \alpha \cdot \beta_n.$$

for every finite sequence of types $\alpha, \beta_1, \ldots, \beta_n$. In particular for $\beta_1 = \cdots = \beta_n = 1$, we obtain that

$$\alpha \cdot n = \alpha + \overset{(n}{\cdots} + \alpha$$

for each ordinal number α and natural number n.

(c) Let μ and γ be two ordinal numbers such that $\mu, \gamma > 0$. Let us denote by $Z(\mu, \gamma)$ the set of all sequences of type γ whose terms are ordinal numbers δ with $0 \leq \delta < \mu$, the number of δ which are different from zero being finite. That is, $a = \{a_\xi\}_{\xi<\gamma} \in Z(\mu, \gamma)$ if and only if for each ξ a_ξ is an ordinal number with $0 \leq a_\xi < \mu$ and the number of ξ such that $a_\xi \neq 0$ is finite.

Suppose that $a = \{a_\xi\}_{\xi<\gamma}$, $b = \{b_\xi\}_{\xi<\gamma}$ are two different elements in $Z(\mu, \gamma)$. Hence the number of those ordinal numbers ξ such that $a_\xi \neq b_\xi$ is finite and there exists at least one because a and b are different. Let ν be the greatest of such ξ. We say that $a \prec b$ if $a_\nu \prec b_\nu$ and $b \prec a$ if $b_\nu < a_\nu$. It can be proved that $(Z(\mu, \gamma), \prec)$ is a well-ordered set.

Let ε be the order type of $(Z(\mu, \gamma), \prec)$. We define the exponentiation of μ to the γ as ε, i.e.

$$\mu^\gamma = \varepsilon.$$

Clearly $\mu^1 = \mu$ and $1^\mu = 1$ for each ordinal number $\mu > 0$. We also define $\mu^0 = 1$ for any ordinal number $\nu > 0$. □

The following remarks will prove useful for us in Chapter 7.

REMARKS 1.21. Let γ, δ and μ be ordinal numbers.

(a) If $\gamma > \delta$, then

$$\mu + \gamma > \mu + \delta.$$

In particular for $\delta = 0$ we get

$$\mu + \gamma > \mu.$$

(b) If $\mu > 0$ and $\gamma, \delta \geq 0$, then

$$\mu^{\gamma+\delta} = \mu^\gamma \cdot \mu^\delta.$$

(c) If $\mu > 1$, and $0 < \gamma < \delta$, then

$$\mu^\gamma < \mu^\delta. \quad \square$$

4.3. Cardinal numbers. Recall the next two important statements which are known to be equivalent to the Axiom of Choice [**Sie58**].

ZERMELO'S THEOREM 1.22. *For any set A there exists a binary relation $<$ defined on it such that $(A, <)$ is a well-ordered set.* □

DEFINITION 1.23. An ordered set $(A, <)$ is *inductive* if for every non-vacuous subset C of A such that $(C, <)$ is totally ordered, then C has a least upper bound in A, that is, there exists $u \in A$ such that for each $a \in A$ either $u = a$ or $a < u$, and if $v \in A$ is also such that for each $a \in A$ either $v = a$ or $a < v$, then $v = u$ or $u < v$. □

ZORN'S LEMMA 1.24. *Let $(A, <)$ be an ordered set that is inductive. Then A has maximal elements, that is, there exists $m \in A$ such that no $a \in A$ satisfies $m < a$.* □

DEFINITIONS 1.25. (a) We say that the sets A and B have the *same cardinality*, and indicate this by $|A| = |B|$, if there exists a bijection among A and B. To have the same cardinality is an equivalence relation. Thus ordinal numbers are divided into disjoint classes consisting of the ordinal numbers α such that the sets $A_\alpha = \{\xi \mid 0 \leq \xi < \alpha\}$ have the same cardinality. We will denote the cardinality $|A_\alpha|$ of A_α by $|\alpha|$ for any ordinal number α.

(b) A *cardinal number* is an ordinal number α such that if β is a different ordinal number with $|\beta| = |\alpha|$, then $\alpha < \beta$, that is, cardinal numbers are the "least" ordinal numbers in the equivalence classes.

(c) Given a set A, by Zermelo's Theorem, there exists a binary relation $<$ on A such that $(A, <)$ is well-ordered. Let γ be the order type of $(A, <)$. Then γ is an ordinal number. Let α be the least ordinal number with the same cardinality as γ (and as A). The ordinal number α is called the *cardinality* of A. In this way we associate to each set a cardinal number α. Observe that any other relation $<'$ such that $(A, <')$ is well-ordered gives an ordinal number γ' but the same ordinal α because there exists a bijection between A, A_γ and $A_{\gamma'}$. We denote the cardinality of A by $|A|$ since it only depends on A itself.

Hence one and only one of the relations

$$|A| < |B|, \quad |A| = |B|, \quad |A| > |B|$$

holds for any given sets A and B. Thus given cardinal numbers α and β we say that $\alpha < \beta$ (as cardinal numbers) if $|\alpha| < |\beta|$, and $\beta < \alpha$ if $|\beta| < |\alpha|$. □

4.4. Cardinal arithmetic. We proceed to give the definition of cardinal number and the main properties of cardinal arithmetic.

Definition 1.26. Let α, β and γ be cardinal numbers. Let A and B be sets of cardinality α and β respectively.

(a) We define the *sum of cardinal numbers* α and β, denoted $\alpha + \beta$, as the cardinality of the disjoint union $A \cup B$. It is not difficult to see that the sum of cardinal numbers has the following properties

$$\alpha + \beta = \beta + \alpha \qquad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma),$$

that is the sum of cardinal numbers is commutative and associative.

(b) We define the *product of cardinal numbers* α and β, denoted $\alpha \cdot \beta$, as the cardinality of the set $A \times B$. It can be seen that the product of cardinal numbers has the following properties

$$\alpha \cdot \beta = \beta \cdot \alpha \qquad \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma,$$

that is, the product of cardinal numbers is commutative and associative. Moreover the product of cardinal numbers satisfies the distributive law with respect to the sum, i.e.

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

(c) We define the cardinal number α^β as the cardinality of the set of maps $A^B = \{f : B \to A\}$. Cardinal exponentiation satisfies the following properties

$$\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma \qquad (\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma \qquad (\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}. \quad \square$$

Cardinal arithmetic has the following important properties

Remark 1.27. Let α be an infinite cardinal number and β a smaller or equal cardinal number to α. Then

$$\alpha + \beta = \alpha \qquad \alpha \cdot \beta = \alpha. \quad \square$$

5. Homological tools

In this section we state well known results on homological algebra that will be used in Chapter 8.

Definitions 1.28. Let R be a ring.

(a) An ordered set $(I, <)$ is *upper directed*, provided that for all $i, j \in I$ there exists $l \in I$ such that $i \leq l$ and $j \leq l$.

(b) Given an upper directed set $(I, <)$, a system $\mathcal{M} = (M_i, f_{ji} \mid i \leq j \in I)$ is a *direct system of modules*, provided that M_i is a right R-module for each $i \in I$ and f_{ji} is a morphism of right R-modules such that $f_{ji} \colon M_i \to M_j$ for $i \leq j \in I$, with f_{ii} the identity on M_i and $f_{li} = f_{lj} f_{ji}$ whenever $i \leq j \leq l \in I$.

(c) Let $\mathcal{M}$ be as in (b). Viewing $\mathcal{M}$ as a diagram in the category of all right R-modules, we can form its colimit $(M, f_i \mid i \in I)$. In particular, M is a right R-module and $f_i \in \operatorname{Hom}_R(M_i, M)$ satisfies $f_i = f_j f_{ji}$ for all $i \leq j \in I$. This colimit (or sometimes just the module M itself) is called the *direct limit* of the direct system $\mathcal{M}$; it is denoted by $\varinjlim_{i \in I} M_i$. There is a unique morphism of right R-modules $\pi \colon \bigoplus_{i \in I} M_i \to \varinjlim_{i \in I} M_i$ such that $f_i = \pi_{|M_i}$. The morphism π is onto.

(d) An ascending chain $(N_\nu | \nu < \kappa)$ of submodules of a right R-module N indexed by a cardinal κ is called *continuous* if $N_\nu = \bigcup_{\beta < \nu} N_\beta$ for all limit ordinals $\nu < \kappa$. The continuous chain is called a *filtration* of N if $N_0 = 0$ and $N = \bigcup_{\nu < \kappa} N_\nu$.

(e) Given a class $\mathcal{U}$ of right R-modules, we say that a right R-module N is *$\mathcal{U}$-filtered* if it admits a filtration $(N_\nu | \nu < \kappa)$ such that $N_{\nu+1}/N_\nu$ is isomorphic to some module in $\mathcal{U}$ for every $\nu < \kappa$. $\square$

Now we state some well-known results that explain the behavior of the functor Ext with respect to direct limits.

The proof of the first three results can be found for example in [**GT06**, Lemmas 3.1.2, 3.1.4, 3.1.6].

Eklof-Lemma 1.29. *Let R be a ring. Let M be a right R-module, and let $\mathcal{U}$ be a class of right R-modules such that* $\operatorname{Ext}^1_R(U, M) = 0$ *for all* $U \in \mathcal{U}$. *If N is a $\mathcal{U}$-filtered right R-module, then* $\operatorname{Ext}^1_R(N, M) = 0$. $\square$

Auslander-Lemma 1.30. *Let R be a ring. Let $n \in \mathbb{N}$. Let $\mathcal{U}$ be a class of right R-modules such that* $\operatorname{pd} U \leq n$ *for any* $U \in \mathcal{U}$. *Let M be a $\mathcal{U}$-filtered right R-module. Then* $\operatorname{pd} M \leq n$. $\square$

Lemma 1.31. *Let $1 \leq n \in \mathbb{N}$. Let R be a ring, and suppose that M is a right R-module that has a projective presentation*

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

with P_i a finitely generated projective right R-module for each $i \leq n+1$. Let $(N_i, f_{ji} \mid i \leq j \in I)$ be a direct system of modules. Then, for all $i \leq n$,

$$\operatorname{Ext}^i_R(M, \varinjlim_{i \in I} N_i) \cong \varinjlim_{i \in I} \operatorname{Ext}^i_R(M, N_i). \quad \square$$

Definitions 1.32. Let R be a ring.

(a) An exact sequence $0 \to A \to B \to C \to 0$ of right R-modules is said to be *pure* if

$$0 \to A \otimes_R X \to B \otimes_R X \to C \otimes_R X \to 0$$

is an exact sequence for any left R-module X. In this case, we say that A is a *pure submodule* of B.

(b) Modules that are injective with respect to pure sequences are called *pure-injective modules*. In other words, a module M is pure-injective if the sequence

$$0 \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(A, M) \to 0$$

is an exact sequence for each pure sequence $0 \to A \to B \to C \to 0$. $\square$

The following is a characterization of pure sequences. For a proof of it see for example [**Lam99**, Theorem 4.89].

LEMMA 1.33. *Let R be a ring. Let $0 \to A \to B \to C \to 0$ be an exact sequence of right R-modules. The following are equivalent:*

(i) *The sequence $0 \to A \to B \to C \to 0$ is pure.*

(ii) *For each finitely presented right R-module M the sequence*

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is an exact sequence. □

Now we give an important property proved by M. Auslander of the behavior of pure injective modules with respect to $\mathrm{Ext}^n_R(\_, \_)$. It can be found in [**GT06**, Lemma 3.3.4]

LEMMA 1.34. *Let R be a ring and M be a pure injective module. Let $(N_{ji}, f_{ji} \mid i \le j \in I)$ be a direct system of right R-modules. Then for each $n \in \mathbb{N}$,*

$$\mathrm{Ext}^n_R(\varinjlim_{i\in I} N_i, M) \cong \varprojlim_{i\in I} \mathrm{Ext}^n_R(N_i, M). \quad \square$$

6. Graphs

We make a quick summary of basic concepts and results which is mostly taken from [**DD89**].

Rooted trees will play an important role in proving Hughes' Theorems 6.3, 6.10 and will be intensively used in Section 2.

6.1. U-sets.

DEFINITIONS 1.35. Let U, V be groups. Let X, X_1, X_2 be sets.

We say that X is a *left U-set* if a map $U \times X \longrightarrow X$, $(u, x) \longmapsto ux$, is given such that $1x = x$ for all $x \in X$, and $u(u'x) = (uu')x$ for all $u, u' \in U$, $x \in X$.

We say that X is a *right U-set* if a map $X \times U \longrightarrow X$, $(x, u) \longmapsto xu$, is given such that $x1 = x$ for all $x \in X$, and $(xu)u' = x(uu')$ for all $u, u' \in U$, $x \in X$.

A map $\alpha\colon X_1 \longrightarrow X_2$ between left (right) U-sets is said to be a *morphism of left* (*right*) *U-sets* if $\alpha(ux) = u(\alpha(x))$ $(\alpha(xu) = (\alpha(x))u)$ for all $u \in U$, $x \in X$.

We say that X is a *V-U-biset* if it is a left V-set and a right U-set, and $v(xu) = (vx)u$ for all $v \in V, u \in U$, $x \in X$.

If $V = U$, we say that X is a *U-biset.* If $x \in X$, $U = V$ and $u \in U$, by x^u we denote $u^{-1}xu$.

A map $\alpha\colon X_1 \longrightarrow X_2$ between two V-U-bisets is said to be a *morphism of V-U-bisets* if α is a morphism of left V-sets and of right U-sets.

6.2. Graphs.

DEFINITIONS 1.36. (a) By a *graph* $(\Gamma, V, E, \iota, \tau)$ we mean a nonempty set Γ with a specified nonempty subset V, its complement $E = \Gamma \setminus V$, and two maps $\iota, \tau\colon E \longrightarrow V$. In this event we say simply that Γ is a graph. For any subset Δ of Γ we write $V\Delta = V \cap \Delta$, $E\Delta = E \cap \Delta$. If Δ is nonempty, and for each $e \in E\Delta$ both ιe and τe belong to $V\Delta$, then Δ is said to be a *subgraph* of Γ.

We call $V = V\Gamma$ and $E = E\Gamma$ the *vertex set* and *edge set* of Γ, and the elements *vertices* and *edges* of Γ, respectively. The functions $\iota, \tau\colon E \longrightarrow V$ are the *incidence functions* of Γ.

If e is any edge then ιe and τe are the vertices *incident* to e, and are called the *initial* and *terminal* vertices of e, respectively. A vertex which is the terminal vertex of some edge is sometimes called a *head*.

For graphs Γ, Δ, a *morphism of graphs* $\alpha\colon \Gamma \longrightarrow \Delta$ is a map such that $\alpha(E\Gamma) \subseteq E\Delta$, $\alpha(V\Gamma) \subseteq V\Delta$ and for each $e \in E\Gamma$, $\alpha(\iota e) = \iota(\alpha e)$, $\alpha(\tau e) = \tau(\alpha e)$. A *graph isomorphism* is a bijective morphism of graphs.

(b) For a set S we write $S^{\pm 1}$ for $S \times \{1, -1\}$, and denote an element (s, ε) by s^ε.

More incidence functions, again denoted ι, τ, are defined on $E\Gamma^{\pm 1}$ by setting $\iota e^1 = \iota e$, $\tau e^1 = \tau e$, and $\iota e^{-1} = \tau e$, $\tau e^{-1} = \iota e$. We think of e^1, e^{-1} as traveling along e the right way and the wrong way, respectively.

A *path* p in Γ is a finite sequence

$$v_0, e_1^{\varepsilon_1}, v_1, e_2^{\varepsilon_2}, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n,$$

where $n \geq 0$, $v_i \in V\Gamma$ for each $i \in \{0, \ldots, n\}$, and $e_i^{\varepsilon_i} \in E\Gamma^{\pm 1}$, $\iota e_i^{\varepsilon_i} = v_{i-1}$, $\tau e_i^{\varepsilon_i} = v_i$ for each $i \in \{1, \ldots, n\}$. We say that p is a path of *length* n from v_0 to v_n, and

$$v_0, \ldots, v_n, e_1, ..., e_n, e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$$

are said to *occur* in p.

If for each $i \in \{0, \ldots, n-1\}$, $e_{i+1}^{\varepsilon_{i+1}} \neq e_i^{-\varepsilon_i}$ then p is said to be *reduced*. Notice that if $e_{i+1}^{\varepsilon_{i+1}} = e_i^{-\varepsilon_i}$ for some $i \in \{0, \ldots, n-1\}$ then

$$v_0, e_1^{\varepsilon_1}, v_1, \ldots, e_{i-1}^{\varepsilon_{i-1}}, v_{i-1}, e_{i+2}^{\varepsilon_{i+2}}, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n$$

is a path of length $n-2$ from v_0 to v_n.

We say that Γ is a *tree* if for any vertices v, w of Γ there is a unique reduced path from v to w.

A path p is said to be a *closed* path at a vertex v if $v_0 = v_n = v$, and is said to be a *simple closed* path if it is nonempty and there are no other repetitions of vertices and no repetitions of edges. A graph with no simple closed paths is called a *forest*.

Two elements of Γ are said to be *connected* in Γ if there exists a path in Γ in which they both occur. Being connected is an equivalence relation.

(c) Let G be a group. The *Cayley graph* of G with respect to a subset X of G, denoted $\mathcal{C}(G, X)$, is the graph with vertex set G, edge set $X \times G$, and incidence functions $\iota(x, g) = g$, $\tau(x, g) = xg$ for all $(x, g) \in X \times G$. □

When Γ is a finite graph, it is customary to describe or define a graph by means of a diagram in which each vertex is represented by a point and each edge e by an arrow starting at ιe and finishing at τe.

The following useful results are not too difficult to prove, making use of Zorn's lemma for (v).

Proposition 1.37. (i) *A graph is a tree if and only if it is a connected forest.*

(ii) *A subgraph of a tree is a union of trees.*

(iii) *Suppose that Γ is a graph that has a finite number of vertices and that it is a union of trees. Then the total number of edges is smaller or equal to the number of vertices minus one.*

(iv) *Let G be a free group on a set X. Then the Cayley graph $\mathcal{C}(G, X)$ is a tree.*

(v) *If Γ is a connected graph then Γ has a maximal subtree. Any maximal subtree of Γ has vertex set all of $V\Gamma$.* □

Definition 1.38. A *rooted tree* (Γ, v_0) is a tree Γ with a distinguished vertex $v_0 \in V\Gamma$, called the *root*, such that for any other vertex $v \in V\Gamma$ no inverse edges occur in the reduced path p from v_0 to v, i.e. $p = v_0, e_1^1, v_1, \ldots, v_{n-1}, e_n^1, v_n$.

For rooted trees $(\Gamma, v_0), (\Delta, w_0)$, we say that a morphism of graphs $\alpha\colon \Gamma \to \Delta$ is a *morphism of rooted trees* if $\alpha(v_0) = w_0$. An *isomorphism of rooted trees* is a bijective morphism of rooted trees. □

The customary way to draw a rooted tree (Γ, v_0) is to place the root at the top. Then the terminal vertices of the edges with initial vertex v_0 are placed one level below the root, and so on. In fact, since all arrows are directed downwards we may redraw (Γ, v_0) without directed arrows, just drawing the edges e as a (non-directed) segment joining ιe and τe.

6.3. Graphs of groups. Everything in this subsection will only be used to prove Proposition 2.8 and Corollary 2.9.

Definitions 1.39. (a) By a *graph of groups* $(G(-), \Delta)$ we mean a connected graph $(\Delta, V, E, \bar{\iota}, \bar{\tau})$ together with a function $G(-)$ which assigns to each $v \in V$ a group $G(v)$, and to each edge $e \in E$ a distinguished subgroup $G(e)$ of $G(\bar{\iota}e)$ and an injective morphism of groups $t_e\colon G(e) \to G(\bar{\tau}e),\ g \mapsto g^{t_e}$. We call the $G(v),\ v \in V$, the *vertex groups*, the $G(e),\ e \in E$, the *edge groups*, and the t_e the *edge functions*.

(b) Let $(G(-), \Delta)$ be a graph of groups as in (a). Choose a maximal subtree Δ_0 of Δ, so $V\Delta_0 = V\Delta$ by Proposition 1.37(v). We define the associated *fundamental group* $\pi(G(-), \Delta, \Delta_0)$ to be the group presented with

- generating set: $\{t_e \mid e \in E\} \vee \bigvee_{v \in V} G(v)$.
- relations:
  - · the relations for $G(v)$, for each $v \in V$,
  - · $t_e^{-1} g t_e = g^{t_e}$ for all $e \in E,\ g \in G(e) \subseteq G(\bar{\iota}e)$, so $g^{t_e} \in G(\bar{\tau}e)$,
  - · $t_e = 1$, for all $e \in E\Delta_0$. □

Remark 1.40. Let $(G(-), \Delta)$ be a graph of groups. Let $G = \pi(G(-), \Delta, \Delta_0)$ be its fundamental group. Then the $G(v)$ embed in G. For a proof of this fact see for example [**DD89**, Corollary 7.5]. □

Following the notation of Definitions 1.39 we state the following theorem whose proof can be found for example in [**DD89**, Theorem 7.7].

Theorem 1.41. *Let $G = \pi(G(-), \Delta, \Delta_0)$. If K is a subgroup of G which intersects each G-conjugate of each edge group $G(e)$ trivially, i.e.: $K \cap g^{-1}G(e)g = \{1\}$ for all $g \in G,\ e \in E$, then $K = F * \mathop{*}_{i \in I} K_i$ for some free subgroup F, and subgroups K_i of K of the form $K \cap g^{-1}G(v)g$ as g ranges over a certain set of elements of G and v ranges over $V\Delta$.* □

7. Semirings

Throughout chapters 5 and 6 we will use the concept of semiring which is an algebraic structure that suits our purposes.

Definitions 1.42. (a) By a *semiring* R we mean a set R endowed with two binary operations $+$ and $\cdot$ which give R the structure of an additive semigroup, and a multiplicative monoid, respectively, and such that the multiplication is left and right distributive over the addition. Note that a semiring has an identity element which will be denoted by 1.

By a *morphism of semirings* $\Phi\colon R_1 \longrightarrow R_2$ we mean a map between two semirings such that, for all $r, r' \in R_1$, $\Phi(r+r') = \Phi(r)+\Phi(r'),\ \Phi(rr') = \Phi(r)\Phi(r')$, and $\Phi(1_{R_1}) = \Phi(1_{R_2})$.

(b) Let R be a semiring. If $(R,+)$ has a neutral element it is called the *zero* and we denote it by 0, i.e. $0+r=r$ for all $r\in R$. Moreover, it is called an *absorbing zero* if $0r=r0=0$ for all $r\in R$.

Now we introduce the semiring $R\cup\{0\}$. If R has an absorbing zero, then $R\cup\{0\}=R$. Otherwise, $R\cup\{0\}$ is R together with a new element called 0, such that R is a subsemiring of $R\cup\{0\}$ with $0+r=r$ and $0r=r0=0$ for all $r\in R\cup\{0\}$.

(c) Let R be a semiring with a zero element 0. We say R is *zero-sum free* if $r_1+r_2=0$ implies $r_1=r_2=0$. And R is called *zero-divisor free* if $r_1r_2=0$ implies $r_1=0$ or $r_2=0$.

Let R be a semiring which is zero-sum and zero-divisor free. Observe that the set $R\setminus\{0\}$ is a subsemiring of R.

(d) If R is a semiring, then we define $R\cup\{\infty\}$ to be the semiring which is the set consisting of R together with a new element, ∞, such that R is a subsemiring of $R\cup\{\infty\}$, and $\infty+r=\infty\cdot r=r\cdot\infty=\infty$ for all $r\in R\cup\{\infty\}$.

(e) Let U be a group. By a *U-semiring* R we mean a semiring R given with a morphism of monoids $\phi\colon U\to R$. Thus the identity elements are respected, and R has a U-biset structure.

By a morphism of U-semirings we mean a map between two U-semirings such that it is a morphism of semirings and a map of U-bisets, that is, $\phi(urv)=u\phi(r)v$ for all $r\in R$ and $u,v\in U$.

(f) By a *rational semiring* R we simply mean a semiring R endowed with a map, called the *$*$-map*, denoted $R\to R$, $r\mapsto r^*$.

(g) By a *rational U-semiring* R we mean a rational semiring R which is a U-semiring such that the $*$-map on R is an *anti-map of U-bisets*, that is,

$$(urv)^*=v^{-1}r^*u^{-1} \text{ for all } r\in R, u,v\in U.$$

By a *morphism of rational U-semirings* $\Phi\colon R_1\to R_2$ we mean a map between two rational U-semirings which is a morphism of U-semirings such that $\Phi(r_1^*)=\Phi(r_1)^*$ for all $r_1\in R_1$. $\square$

EXAMPLES 1.43. (a) Rings are semirings and morphisms of rings are morphisms of semirings.

(b) Let U be a group. Consider the group semiring $\mathbb{N}[U]$. It is the submonoid of $\mathbb{N}^U$ consisting of the maps with finite support where the product is given by

$$\left(\sum_{u\in U}n_uu\right)\left(\sum_{u\in U}m_uu\right)=\sum_{u\in U}\left(\sum_{vw=u}n_vm_w\right)u.$$

It has a natural structure of U-semiring. It is zero-sum and zero-divisor free. Thus $\mathbb{N}[U]\setminus\{0\}$ is also a U-semiring. Moreover it satisfies that for every U-semiring R with morphism of monoids $\phi\colon U\to R$, there exists a unique morphism of U-semirings $\Phi\colon\mathbb{N}[U]\setminus\{0\}\to R$ extending ϕ. It is defined by $\Phi(\sum\limits_{u\in U}n_uu)=\sum\limits_{u\in U}n_u\phi(u)$.

(c) Crossed product monoid semirings, see Section 1 of Chapter 4.

(d) Let R be a ring. The zero element of R is an absorbing zero. Hence $R\cup\{0\}=R$. On the other hand consider now the semiring $R\cup\{\infty\}$. Let U be the group of units of R. The inclusion map $U\hookrightarrow R\cup\{\infty\}$ endows $R\cup\{\infty\}$ with a U-semiring structure. The map $R\cup\{\infty\}\to R\cup\{\infty\}$ defined by

$$r\mapsto r^*=\begin{cases} r^{-1} \text{ if } r\in U\\ \infty \quad \text{if } r\in\Big(R\cup\{\infty\}\Big)\setminus U\end{cases}$$

makes $R\cup\{\infty\}$ a rational U-semiring.

(e) Let T be a ring and $\alpha\colon T \to T$ an automorphism of rings. Consider the skew Laurent series ring $T((x;\alpha))$ (see Definitions 4.17 or Examples 1.6). Let

$$U = \{f = \sum_{n\in\mathbb{Z}} a_n x^n \mid a_N \in T^\times \text{ where } N = \min \operatorname{supp}(f)\}.$$

It is known that every element in U is invertible. Indeed, if we define $g = a_N x^N - f$, then

$$\begin{aligned} f^{-1} &= (a_N x^N - g)^{-1} \\ &= (a_N x^N)^{-1}(1 - g(a_N x^N)^{-1})^{-1} \\ &= (a_N x^N)^{-1} \sum_{m\geq 0} (g(a_N x^N)^{-1})^m \end{aligned}$$

It is not easy to see that U is a group. Thus the map $T((x;\alpha)) \cup \{\infty\} \to T((x;\alpha)) \cup \{\infty\}$ defined by

$$r \mapsto r^* = \begin{cases} r^{-1} & \text{if } r \in U \\ \infty & \text{if } r \in \Big(T((x;\alpha)) \cup \{\infty\}\Big) \setminus U \end{cases}$$

makes $T((x;\alpha)) \cup \{\infty\}$ a rational U-semiring.

Notice that if k is a division ring and α is an automorphism of k, then $k((x;\alpha))$ is a division ring (see Corollary 4.20 or Examples 1.6). The structures of rational semirings of $k((x;\alpha)) \cup \{\infty\}$ given for $R = k((x;\alpha))$ as in (b) and for $T = k$ as in (c) coincide. $\square$

Remark 1.44. If R is a rational U-semiring and V is a subgroup of U, then R has a natural structure of rational V-semiring. $\square$

8. Completions and valuations

Definitions 1.45. Let R be a ring. Let $\{I_n\}_{n\geq 1}$ be a family of (two sided) ideals of R such that $I_{n+1} \subset I_n$ for all $n \geq 1$. Consider the natural map $f_{n+1}\colon R/I_{n+1} \to R/I_n$ for each $n \geq 1$. We define the *completion of R with respect to* $\{I_n\}_{n\geq 1}$ as the ring

$$\widehat{R} = \varprojlim\, (R/I_n, f_n).$$

Let I be an ideal of R. If $I_n = I^n$ for all $n \geq 1$, then we say $\widehat{R}$ is the *completion of R with respect to I*. $\square$

Remarks 1.46. Let R be a ring. Let $\{I_n\}_{n\geq 1}$, $\{J_n\}_{n\geq 1}$ be two families of ideals of R such that $I_{n+1} \subset I_n$ and $J_{n+1} \subset J_n$ for all $n \geq 1$. Denote by $\widehat{R}_I$ and $\widehat{R}_J$ the completions of R with respect to $\{I_n\}_{n\geq 1}$ and $\{J_n\}_{n\geq 1}$ respectively. Then

(a) $\widehat{R}_I$ can be seen as the subring of $\prod_{n\geq 0} R/I_{n+1}$ consisting of

$$\Big\{(\bar a_n)_{n\geq 0} \in \prod_{n\geq 0} R/I_{n+1} \;\mid\; a_{n+1} - a_n \in I_{n+1},\ n \geq 0\Big\}.$$

(b) There is a natural morphism of rings $R \to \widehat{R}_I$ defined by $a \mapsto (\bar a, \bar a, \bar a, \ldots)$. Moreover, it is injective if and only if $\bigcap_{n\geq 1} I_n = 0$.

(c) $\widehat{R}_I$ is also the completion of R with respect to $\{I_n\}_{n\geq n_0}$ for each $n_0 \geq 1$.

(d) Suppose that for each $n \geq 1$ there exists $n' \geq 1$ such that $I_n \subseteq J_{n'}$ and that for each $m \geq 1$ there exists $m' \geq 1$ such that $J_m \subseteq I_{m'}$. Then $\widehat{R}_I \cong \widehat{R}_J$. This can be deduced from the universal properties of the inverse limit. $\square$

Examples 1.47. The main examples for us are the following

(a) Let K be a ring and $\alpha\colon K \to K$ an injective ring endomorphism. Consider the skew polynomial ring $R = K[x;\alpha]$. Let $I = \langle x\rangle$. Then the completion $\widehat{R}$ of R with respect to I is $K[[x;\alpha]]$ via the isomorphism $\varphi\colon K[[x;\alpha]] \to \widehat{R}$ defined by $\varphi(f) = (\overline{a_n})_{n\geq 0}$ where $a_n = r_0 + \cdots + r_n x^n$ for each $f = \sum\limits_{n\geq 0} r_n x^n \in K[[x;\alpha]]$.

(b) Let k be a ring and X a set. Consider the free k-ring on X, $R = k\langle X\rangle$. Let $I = \langle X\rangle$. Then the completion of R with respect to I is $k\langle\langle X\rangle\rangle$ via the isomorphism $\varphi\colon k\langle\langle X\rangle\rangle \to \widehat{R}$ defined by $\varphi(f) = (\overline{a_n})_{n\geq 0}$ where $a_n = f_0 + \cdots + f_n$ for each $f = \sum\limits_{n\geq 0} f_n$ where f_n is a homogeneous polynomial of degree n. $\square$

Definitions 1.48. Consider the monoid $\mathbb{N}\cup\{\infty\}$ with the ordering defined by the usual ordering on $\mathbb{N}$ and $\infty > n$ for all $n \in \mathbb{N}$. The sum on $\mathbb{N}$ is defined as usual and $n + \infty = \infty + n = \infty$ for all $n \in \mathbb{N}\cup\{\infty\}$.

Let R be a ring. By a *valuation* on R we understand an onto map $v\colon R \to \mathbb{N}\cup\{\infty\}$ such that

(V.1) $v(x) = \infty$ if and only if $x = 0$.
(V.2) $v(xy) = v(x) + v(y)$ for all $x, y \in S$.
(V.3) $v(x + y) \geq \min\{v(x), v(y)\}$.

A ring R endowed with a valuation v is called a *valuation ring*. The *completion of R with respect to v* is the ring $\widehat{R} = \varprojlim R/I_n$ where $I_n = \{r \in R \mid v(r) \geq n\}$ for each $n \geq 1$. $\square$

Remarks 1.49. Let R be a valuation ring with valuation v.

(a) R is a domain by (V.2) and (V.1).
(b) $v(1) = v(-1) = 0$, and therefore
  (b.1) $v(r) = 0$ if $r \in R$ is invertible.
  (b.2) $v(-r) = v(r)$ for all $r \in R$.
(c) For each $n \in \mathbb{N}$, $I_n = \{r \in R \mid v(r) \geq n\}$ is an ideal of R. $\square$

Examples 1.50. We will deal with the following examples of valuation rings.

(a) Let K be a ring and $\alpha\colon K \to K$ an injective ring endomorphism. Consider the skew polynomial ring $R = K[x;\alpha]$. Each $f \in R$ is of the form $f = \sum\limits_{n\geq 0} a_n x^n$ where only a finite number of $f_n \neq 0$. Define $v\colon R \to \mathbb{N}\cup\{\infty\}$ by $v(f) = \min\{n \mid a_n \neq 0\}$. Then v is a valuation. Observe that for each $n \geq 1$, $I_n = \langle x\rangle^n$.

(b) Let k be a ring and X a set. Consider the free k-ring on X, $R = k\langle X\rangle$. Each $f \in R$ is of the form $f = \sum\limits_{n\geq 0} f_n$ where f_n is a homogeneous polynomial of degree n and only a finite number of $f_n \neq 0$. Define $v\colon R \to \mathbb{N}\cup\{\infty\}$ by $v(f) = \min\{f_n \mid f_n \neq 0\}$. Then v is a valuation. Observe that for each $n \geq 1$, $I_n = \langle X\rangle^n$. $\square$

Lemma 1.51. *Let R be a ring with a valuation $v\colon R \to \mathbb{N}\cup\{\infty\}$. Consider the completion $\widehat{R}$ of R with respect to v. Then the following statements hold*

(i) $R \hookrightarrow \widehat{R}$.
(ii) *$\widehat{R}$ is a valuation ring with a valuation that extends v.*
(iii) *If we denote by v the extension of v to $\widehat{R}$, then $\widehat{\widehat{R}}$, the completion of $\widehat{R}$ with respect to v is isomorphic to $\widehat{R}$.*

Proof. (i) Clearly $\bigcap\limits_{n\geq 1} I_n = 0$. Thus Remarks 1.46(b) implies that $R \hookrightarrow \widehat{R}$.

(ii) Let $(\overline{a_n})_{n\geq 0} \in \widehat{R} \setminus \{0\}$. Define $v((\overline{a_n})_{n\geq 0}) = \min\{n \mid \overline{a_n} \neq 0\}$. If $(\overline{a_n})_{n\geq 0} = 0$, then $v((\overline{a_n})_{n\geq 0}) = \infty$. Thus (V.1) holds.

Notice that $\overline{a_n} = 0$ if and only if $v(a_n) > n$.

Let $(\overline{a_n})_{n\geq 0}, (\overline{b_n})_{n\geq 0} \in \widehat{R}$. Let $n_1 = v((\overline{a_n})_{n\geq 0})$ and $n_2 = v((\overline{b_n})_{n\geq 0})$.

Then $(\overline{a_n})_{n\geq 0} + (\overline{b_n})_{n\geq 0} = (\overline{a_n + b_n})_{n\geq 0}$. Now $a_n, b_n \in I_n$ for all $n < n_1$ or $n < n_2$ respectively. Thus

$$\begin{aligned} v((\overline{a_n + b_n})_{n\geq 0}) &= \min\{n \mid \overline{a_n + b_n} \neq 0\} \\ &= \min\{n \mid a_n + b_n \notin I_n\} \\ &\geq \min\{n_1, n_2\} \\ &= \min\{v((\overline{a_n})_{n\geq 0}), v((\overline{b_n})_{n\geq 0})\}. \end{aligned}$$

So (V.3) is satisfied.

To verify (V.2), first observe that if either $p < n_1$ and $q \leq n_2$ or $p \leq n_1$ and $q < n_2$ then $v(a_p b_q) = v(a_p) + v(b_q) > p + q$. Thus $v((\overline{a_n})_{n\geq 0} \cdot (\overline{b_n})_{n\geq 0}) = v((\overline{a_n b_n})_{n\geq 0}) \geq n_1 + n_2$.

On the other hand, $a_{n_1+n_2} = (a_{n_1+n_2} - a_{n_1+n_2-1}) + \cdots + (a_{n_1+1} - a_{n_1}) + a_{n_1}$, and $b_{n_1+n_2} = (b_{n_1+n_2} - b_{n_1+n_2-1}) + \cdots + (b_{n_2+1} - b_{n_2}) + b_{n_2}$. Then $a_{n_1+n_2} \cdot b_{n_1+n_2} = a_{n_1} b_{n_2} + \sum_j c_j$ where $v(c_j) > n_1 + n_2$, i.e. $c_j \in I_{n_1+n_2+1}$. Thus $v(a_{n_1+n_2} \cdot b_{n_1+n_2}) = n_1 + n_2$. Hence $v((\overline{a_n})_{n\geq 0} \cdot (\overline{b_n})_{n\geq 0}) = n_1 + n_2$.

(iii) Set $I_n = \{r \in R \mid v(r) \geq n\}$ and $\widehat{I_n} = \{z \in \widehat{R} \mid v(z) \geq n\}$ for each $n \geq 1$. Observe that $R \hookrightarrow \widehat{R}$ induces $R/I_n \to \widehat{R}/\widehat{I_n}$. So we get a morphism of rings $\widehat{R} \to \widehat{\widehat{R}}$ by the universal property of the completion.

On the other hand, let $\overline{(\overline{a_n})_{n\geq 0}} \in \widehat{R}/\widehat{I_m}$. We define a morphism $\widehat{R}/\widehat{I_m} \to R/I_m$ by $\overline{(\overline{a_n})_{n\geq 0}} \mapsto \overline{a_{m-1}}$. Note that the map is well-defined because if $(\overline{a_n})_{n\geq 0} - (\overline{b_n})_{n\geq 0} \in \widehat{I_m}$, then $a_{m-1} - b_{m-1} \in I_m$. Moreover the following diagram is commutative

$$\begin{tikzcd} \widehat{R}/\widehat{I_{m-1}} \arrow[r] & R/I_{m-1} \\ \widehat{R}/\widehat{I_m} \arrow[r] \arrow[u] & R/I_m \arrow[u] \end{tikzcd}$$

because $a_{m-1} - a_{m-2} \in I_{m-1}$ for each $(\overline{a_n})_{n\geq 0} \in \widehat{R}$. So it induces a morphism of rings $\widehat{\widehat{R}} \to \widehat{R}$. Now both compositions have to be the identity, as desired. □

Examples 1.52. (a) Let R be as in Examples 1.50(a). Then the valuation defined on the completion $\widehat{R} = K[[x;\alpha]]$ is $o(\sum_{n\geq 0} a_n x^n) = \min\{n \mid a_n \neq 0\}$.

(b) Let R be as in Examples 1.50(b). Then the valuation defined on the completion $\widehat{R} = k\langle\langle X\rangle\rangle$ is $o(\sum_{n\geq 0} f_n) = \min\{n \mid f_n \neq 0\}$. □

Definition 1.53. The valuations in Examples 1.52 are called *order*. And $o(\sum_{n\geq 0} f_n)$, $o(\sum_{n\geq 0} a_n x^n)$ are the *order of the series* $\sum_{n\geq 0} f_n$, $\sum_{n\geq 0} a_n x^n$ respectively. □

9. Magnus-Fox embedding

Let H be a free group on the set $\{h_i\}_{i\in I}$. Let $X = \{x_i\}_{i\in I}$. Consider the series ring $\mathbb{Z}\langle\langle X\rangle\rangle$. It was proved by W. Magnus [**Mag35**, Satz I] that the map $\varphi\colon H \to \mathbb{Z}\langle\langle X\rangle\rangle^{\times}$ defined by $h_i \mapsto 1 + x_i$ is an injective morphism of groups. Slight additions to the proof of this fact given in [**MKS76**, Theorem 5.5.6] work to show that $\varphi\colon H \to R\langle\langle X\rangle\rangle^\times$ defined by $h_i \mapsto 1+x_i$ is an injective morphism of groups for any ring R.

PROPOSITION 1.54. *Let $R \neq 0$ be a ring. Let H be the free group on the set $\{h_i\}_{i\in I}$. Let $X = \{x_i\}_{i\in I}$. Consider the series ring $R\langle\langle X\rangle\rangle$. Then the morphism of groups, $\varphi\colon H \to R\langle\langle X\rangle\rangle^\times$ defined by $h_i \mapsto 1 + x_i$ and hence $h_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \cdots$, is an injective morphism of groups.*

PROOF. Let R_0 be the image of the morphism of rings $\mathbb{Z} \to R$. Thus R_0 is either isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for some natural n. Observe that the image of φ is contained in $R_0\langle\langle X\rangle\rangle$. Therefore to prove the result it is enough to show that the morphism of groups $\varphi_R\colon H \to R\langle\langle X\rangle\rangle^\times$, $h_i \mapsto 1 + x_i$, is injective where R is either $\mathbb{Z}$ of $\mathbb{Z}_n$. Note that for each natural $n > 1$ there exists a prime number p (a divisor of n) such that the projection $\mathbb{Z}_n \overset{\pi_p}{\to} \mathbb{Z}_p$ is an onto morphism of rings. It can be extended in a natural way to $\mathbb{Z}_n\langle\langle X\rangle\rangle \overset{\pi_p}{\to} \mathbb{Z}_p\langle\langle X\rangle\rangle$. Notice that $\pi_p\varphi_{\mathbb{Z}_n} = \varphi_{\mathbb{Z}_p}$. Thus if we prove the result for $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ we are done.

So suppose that $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$.

Let W be a nonempty reduced word,

$$W = h_{i_1}^{e_1} \cdots h_{i_r}^{e_r},$$

where e_j are nonzero integers, $i_j \in I$ for $j = 1, \ldots, r$, and $i_j \neq i_{j+1}$. We must show that $\varphi(W) \neq 1$.

Let e be a positive integer, then $(1+x_i)^e$ is a polynomial on x_i different from 1 since it has a monomial on x_i^e. If e is a negative integer, then $(1 + x_i)^e$ is a series on x_i different from 1, since it is the inverse of the polynomial $(1 + x_i)^{-e}$. In either case $\varphi(h_i)$ is a series of the form $1 + a_1x_i + a_2x_i^2 + \cdots$ If e is a nonzero integer, let m_i be the nonzero coefficient of smallest degree $l_i \geq 1$ of $(1 + x_i)^e - 1$. Hence

$$(1 + x_i)^e = 1 + m_ix_i^{l_i} + x_i^{l_i+1}h_i(x_i),$$

where $h_i(x_i)$ is a series on x_i alone. So

$$\varphi(W) = (1 + m_{i_1}x_{i_1}^{l_{i_1}} + x_{i_1}^{l_{i_1}+1}h_{i_1}(x_{i_1})) \cdots (1 + m_{i_r}x_{i_r}^{l_{i_r}} + x_{i_r}^{l_{i_r}+1}h_{i_r}(x_{i_r})),$$

a series that contains the unique monomial

$$m_{i_1} \cdots m_{i_r}x_{i_1}^{l_{i_1}} \cdots x_{i_r}^{l_{i_r}},$$

because $m_{i_1} \cdots m_{i_r} \neq 0$ in R since $R(= \mathbb{Z} \text{ or } \mathbb{Z}_p)$ is a domain. Therefore $\varphi(W) \neq 1$ as desired. □

REMARKS 1.55. The group ring $R[H]$ is a free left R-module with basis H and $R\langle\langle X\rangle\rangle$ is a left R-module, thus we can extend φ in Proposition 1.54 to a morphism of left R-modules which turns out to be a morphism of R-rings. □

ERRATA.
In what follows, to prove Proposition 1.60 it is used Propositon 1.57 from the paper by R.H. Fox [**Fox53**], but this result was proved by R.H. Fox using Proposition 1.60.
Proposition 1.60 still holds true. In cite [**AD07**, Section 2.12] it is noted that the proof of Proposition 1.60 of R.H. Fox for the ring $R = \mathbb{Z}$ works for any ring R.

Definition 1.56. Let R be a ring and G a group. Consider the group ring $R[G]$. The morphism of R-rings $\varepsilon\colon R[G] \to R$ which sends every $g \in G$ to 1 is called the *augmentation map* of $R[G]$. The kernel of ε is known as the *augmentation ideal* of $R[G]$. The two-sided ideal $\ker \varepsilon$ is usually denoted by $\omega(R[G])$. We will denote the n-th power of $\omega(R[G])$ as $\omega^n(R[G])$.

Every element $\sum\limits_{g\in G} r_g g \in R[G]$ can be expressed as $\sum\limits_{g\in G} r_g(g-1) - \sum\limits_{g\in G} r_g$. Thus

$$R \oplus \omega(R[G]) \tag{4}$$

as a left (or right) R-module and $\omega(R[G])$ is generated by the elements of the form $g-1$ with $g \in G$. □

The ideal $\omega(R[G])$ is an important object in the study of group rings. A well-known result is the following [**Fox53**, Corollary 4.4]

Proposition 1.57. *Let R be a ring and H a free group. Then* $\bigcap\limits_{n\geq 1} \omega^n(R[H]) = 0$. □

Now we need the next definitions.

Definitions 1.58. (a) An *rng* is an additive abelian group $(S,+)$ with a second associative binary operation, multiplication, the two operations being related by the distributive laws. That is, S is a ring not necessarily with an identity element.

(b) Let R be a ring. By an *R-rng* we mean an R-bimodule S which has structure of rng and such that the abelian group of S as an R-bimodule and as a rng coincide and the following relations connecting the R-action and ring multiplication hold

$$r(s_1s_2) = (rs_1)s_2, \quad (s_1s_2)r = s_1(s_2r), \quad (s_1r)s_2 = s_1(rs_2)$$

for $s_1, s_2 \in S,\ r \in R$.

(c) Given R-rngs S_1, S_2, by a *morphism of R-rngs* we mean an R-bimodule homomorphism $f\colon S_1 \to S_2$ such that $f(st) = f(s)f(t)$ for all $s,t \in S_1$. □

Lemma 1.59. *Let R be a ring. Let H be the free group on the set $\{h_i\}_{i\in I}$. Consider the group ring $R[H]$. Let $X = \{x_i\}_{i\in I}$. For each natural $n \geq 1$ the following statements hold:*

(i) *The ideal $\omega^n(R[H])$ is generated by the set $\mathcal{B}_n = \{(h_{i_1}-1)\cdots(h_{i_n}-1) \mid (i_1,\dots,i_n) \in I^n\}$ as a left ideal.*

(ii) *The R-rng $\omega(R[H])/\omega^{n+1}(R[H])$ is isomorphic to the R-rng $\langle X\rangle/\langle X\rangle^{n+1}$.*

Proof. (i) For each $i \in I$, let $\epsilon_i = \pm 1$. Let $w \in H$. From the equalities

$$(h_i^{\epsilon_i} w - 1) = (h_i^{\epsilon_i} - 1) + h_i^{\epsilon_i}(w-1) \tag{5}$$

$$(h_i^{-1} - 1) + (h_i - 1) = -(h_i^{-1} - 1)(h_i - 1) \in \omega^2(R[H]) \tag{6}$$

we infer by induction on the length of w that the set $\mathcal{B}_1 = \{h_i - 1 \mid i \in I\}$ generates $\omega(R[H])$ as a left ideal.

Notice that in a product of the form $p_2(h_{i_2}-1)p_1(h_{i_1}-1)$ where $p_1, p_2 \in R[H]$, the element $p_2(h_{i_2}-1)p_1 \in \omega(R[H])$. Therefore, again by induction, it can be seen that $\mathcal{B}_n$ generates $\omega^n(R[H])$ as a left ideal.

(ii) By (i), every element in $\omega^n(R[H])$ can be expressed as a left $R[H]$-linear combination of the elements in $\mathcal{B}_n$. From (4) we get that the classes of the elements in $\mathcal{B}_n$ generate $\omega^n(R[H])/\omega^{n+1}(R[H])$ as a left R-module.

Extend the morphism φ of groups given in Proposition 1.54 to a morphism of R-rings. Since $\varphi(h_i - 1) = x_i$, every element in $\omega^n(R[H]) \setminus \{0\}$ is sent by φ to a series of order at least m, and if $(h_{i_1}-1)\cdots(h_{i_n}-1) \in \mathcal{B}_n$, its image is $x_{i_1}\cdots x_{i_n}$. Therefore the classes of the elements in $\mathcal{B}_n$ are R-linearly independent in $\omega^n(R[H])/\omega^{n+1}(R[H])$.

The foregoing shows that the classes of the elements in $\bigcup\limits_{j=1}^{n} \mathcal{B}_j$ form a left R-basis of $\omega(R[H])/\omega^{n+1}(R[H])$.

Now observe that the morphism of free left R-modules

$$\langle x_1, \ldots, x_m \rangle / \langle x_1, \ldots, x_m \rangle^{n+1} \longrightarrow \omega(R[H])/\omega^{n+1}(R[H])$$

defined by $\bar{x}_i \mapsto \overline{(h_i - 1)}$ is an isomorphism of R-rngs. $\square$

Now we come to the main result in this section.

Proposition 1.60. *Let R be a ring. Let H be the free group on $\{h_i\}_{i\in I}$. Let $X = \{x_i\}_{i\in I}$. Consider the morphism of R-rings $\varphi\colon R[H] \to R\langle\langle X \rangle\rangle$ defined by $\varphi(h_i) = 1 + x_i$. Then φ is injective.*

Proof. For each $n \geq 1$, consider the morphism $\psi_n\colon R\langle\langle X\rangle\rangle \to R\langle X\rangle/\langle X\rangle^n$ given by $\psi_n(\sum\limits_{m\geq 0} f_m) = \overline{f_0 + \cdots + f_n}$. Notice that $\ker \psi_n = \{f \in R\langle\langle X\rangle\rangle \mid o(f) \geq n\}$.

We claim that $\ker \psi_n\varphi = \omega^n(R[H])$ for all $n \geq 1$.

First notice that $\omega^n(R[H]) \subseteq \ker \psi_n\varphi$ for each $n \geq 1$, because as it is done in the proof of Lemma 1.59(ii), the image by φ of an element in $\omega^n(R[H])$ is a series of order at least n.

Let $h \in \ker \psi_n\varphi$. Then $h = \lambda_1 g_1 + \cdots + \lambda_t g_t$ for some $\lambda_1, \cdots, \lambda_t \in R$, $g_1, \ldots, g_t \in H$. Since for every $g \in H$, $\varphi(g)$ is a series with independent term 1, we get that $\lambda_1 + \cdots + \lambda_t = 0$. That is, $h \in \omega(R[H])$. By the commutativity of the following diagram

$$\begin{tikzcd}
\omega(R[H]) \arrow[rr, "\psi_n\varphi|_{\omega(R[H])}"] \arrow[dr] & & R\langle X\rangle/\langle X\rangle^n \\
& \frac{\omega(R[H])}{\omega^n(R[H])} \cong \frac{\langle X\rangle}{\langle X\rangle^n} \arrow[ur, hook] &
\end{tikzcd}$$

we get that $h \in \omega^n(R[H])$. This finishes the proof of our claim.

If $h \in \ker\varphi$, then $h \in \ker \psi_n\varphi = \omega^n(R[H])$ for all $n \geq 1$. Therefore $h \in \bigcap\limits_{n\geq 1} \omega^n(R[H])$, and $h = 0$ by Proposition 1.57. $\square$

The result in Proposition 1.60 is well-known. This embedding is known as the *Magnus-Fox embedding.* Our proof is from [**Lic84**, Proposition 3]. There the result is stated for R an Ore domain, but as we have just seen the proof also works for any ring R. Another proof is given in [**AD07**, Theorem 2.11]. In paragraph 2.12 of this paper a historical remark on the Magnus-Fox embedding is given. There it is explained that this result was already noted in [**She06**] and that the proof of Proposition 1.60 for $R = \mathbb{Z}$ was given in the paper by R.H. Fox [**Fox53**].

"Something is starting to breathe
Something is coming alive
What which should never be
Spawned by the demon seed
Don't let the fetus survive"

Dead Soul Tribe, *Feed Part I: Stone by stone*

CHAPTER 2

Locally indicable groups

In this chapter we deal with locally indicable groups. They were introduced by G. Higman in his PhD thesis and his paper [**Hig40**] on group rings. These groups are very important in this dissertation because a lot of our embeddability results are about embedding crossed product group rings kG of a locally indicable group G over a division ring k. We give important properties and examples of locally indicable groups and discuss relations with some other classes of groups.

1. Definition and closure properties

Before defining locally indicable groups we give the following useful lemma which will be used throughout without any further reference. We also provide some trivial but important examples of locally indicable groups.

LEMMA 2.1. *Let G be a nontrivial group. The following statements are equivalent.*

(i) $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$
(ii) *There exists an onto morphism of groups $G \to \mathbb{Z}$*
(iii) *There exists a normal subgroup H of G such that G/H is infinite cyclic*
(iv) *$G = H \rtimes C$, the semidirect product of H by C where C is an infinite cyclic group.*

PROOF. Clearly (i)-(iii) are equivalent and (iv) implies (iii). Suppose (iii) holds. Let t be an element such that Ht generates G/H. Let C be the subgroup of G generated by t. Then C is infinite cyclic since G/H is a homomorphic image of C; $H \cap C = \{1\}$ since G/H would be finite if $t^n \in H$ for some $n \in \mathbb{N} \setminus \{0\}$; $HC = G$ because if $g \in G$, then $Hg = Ht^m$, for some $m \in \mathbb{Z}$, which implies $g = ht^m$ for some $h \in H$. Therefore $G = H \rtimes C$. □

DEFINITION 2.2. Let G be a group. We say that G is *indicable* if either G is trivial or G satisfies the equivalent statements of Lemma 2.1. And G is said to be *locally indicable* if every finitely generated subgroup of G is indicable. □

REMARKS 2.3. (a) An indicable group need not be a locally indicable group. For example, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is indicable but not locally indicable since there does not exist a nontrivial morphism of groups $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$.

(b) A subgroup of a locally indicable group is again a locally indicable group.

(c) Locally indicable groups are torsion-free groups. If a is not the identity element of a locally indicable group, there exists a morphism of groups from $\langle a \rangle$ onto $\mathbb{Z}$. Thus a has infinite order. □

EXAMPLE 2.4. The following classes of groups consist of locally indicable groups.

(a) Locally free groups and, in particular, free groups.
(b) Torsion-free abelian groups.

PROOF. (a) Let H be a nontrivial finitely generated subgroup of a (locally) free group F. Then H is a free group on a subset $X \neq \emptyset$ of H. Choose an element $x \in X$. Then, by the universal property of free groups, there is a (unique) morphism of groups $\varphi\colon H \longrightarrow \mathbb{Z}$ such that $x \longmapsto 1$ and $x' \longmapsto 0$ for every $x' \in X \setminus \{x\}$.

(b) Let G be a torsion-free abelian group and H a nontrivial finitely generated subgroup. By the classification of finitely generated abelian groups, $H \cong \mathbb{Z}^k$ for some $k \in \mathbb{N} \setminus \{0\}$. Then the composition of this isomorphism with the projection over any of the components of $\mathbb{Z}^k$ defines a morphism of groups from H onto $\mathbb{Z}$. □

The following definition is very important because, as we will state later, every locally indicable group is of this form with torsion free abelian factors.

Definitions 2.5. Let G be a group. Let $\mathfrak{X}$ be a class of groups closed under isomorphisms.

(a) Let Σ be a chain of subgroups of G. We say that the pair (L, H) is a *jump in* Σ if $L, H \in \Sigma$, $L < H$ and no subgroup in Σ lies properly between L and H.

(b) A *subnormal system of G with factors in $\mathfrak{X}$* is a chain of subgroups Σ of G such that
   (i) $\{1\}, G \in \Sigma$,
   (ii) Σ contains all unions and intersections of its members,
   (iii) for each jump (L, H) in Σ, $L \lhd H$ and $H/L \in \mathfrak{X}$.

(c) If Σ is a chain of subgroups of G satisfying (i) and (ii) above, then, for $x \in G \setminus \{1\}$, we define the *jump associated with* x in Σ as the jump (L_x, H_x), where L_x is the union of the groups of Σ that do not contain x, and H_x is the intersection of all the subgroups of Σ containing x.

Some subnormal systems have their own names:

(d) A *subnormal series of G with factors in $\mathfrak{X}$* is an ascending chain of subgroups of G, $(G_\gamma)_{\gamma\leq\tau}$, indexed at some ordinal τ such that $G_0 = \{1\}$, $G_\tau = G$, G_γ is a normal subgroup of $G_{\gamma+1}$, $G_{\gamma+1}/G_\gamma \in \mathfrak{X}$, and if ρ is a limit ordinal smaller or equal to τ then $G_\rho = \bigcup_{\gamma<\rho} G_\gamma$.

(e) We say that G is *poly-$\mathfrak{X}$* if G has a finite subnormal series

$$\langle 1\rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with each quotient G_{i+1}/G_i belonging to the family $\mathfrak{X}$. □

Now we proceed to give some of the closure properties of the class of locally indicable groups. The following result is a concrete case of [**BH72**, Theorem 3]. A proof of it for poly-{locally indicable} groups as well as proofs of Proposition 2.6 and Corollary 2.7(i) for the finite case were given in [**Hig40**, Appendix].

Proposition 2.6. *Let G be a group. Let $\mathfrak{X}$ be the class of locally indicable groups. Suppose that G has a subnormal system with factors in $\mathfrak{X}$. Then G is locally indicable. In particular if G has a subnormal series with factors in $\mathfrak{X}$ or if G is a poly-$\mathfrak{X}$ group, then G is locally indicable.*

Proof. Let Σ be a subnormal system of G with factors in $\mathfrak{X}$. Suppose that B is a nontrivial finitely generated subgroup of G generated by $g_1, \ldots, g_n \in G$. For each g_i, let (L_i, H_i) be the jump associated with g_i. By (iii) of Definition 2.5(b) we get that $L_i \lhd H_i$ and H_i/L_i is locally indicable. Since Σ is a chain, there is $i_0 \in \{1, \ldots, n\}$ such that $H_i \subseteq H_{i_0}$ for all $i \in \{1, \ldots, n\}$. Thus B is contained in H_{i_0}. Now the projection $\bar{B}$ of B in H_{i_0}/L_{i_0} is a nontrivial finitely generated subgroup. Therefore there exists an onto morphism of groups $\bar{B} \to \mathbb{Z}$. So the composition $B \to \bar{B} \to \mathbb{Z}$ is an onto morphism of groups. □

Corollary 2.7. *The following statements hold*

(i) *The cartesian product of locally indicable groups is a locally indicable group.*
(ii) *The direct sum of locally indicable groups is a locally indicable group.*

(iii) *Let G, H be locally indicable groups. Then $G \wr H$, the (restricted) standard wreath product of G by H, is a locally indicable group.*

(iv) *The subdirect product of locally indicable groups is locally indicable, i.e. let $\{H_i\}_{i\in I}$ be a family of normal subgroups of G with $\bigcap_{i\in I} H_i = \{1\}$ such that each quotient G/H_i is a locally indicable group, then G is a locally indicable group.*

PROOF. (i) Suppose $(G_i)_{i\in I}$ is a family of locally indicable groups. Consider $\prod_{i\in I} G_i$. Well-order the index set identifying I with the set of ordinal numbers γ smaller than a certain ordinal number τ. For each $\gamma \leq \tau$, let $H_\gamma = \{(x_i) \in \prod_{i\in I} G_i \mid x_i = 1,\ i \geq \gamma\}$. The set $\Sigma = \{H_\gamma\}_{\gamma\leq\tau}$ is a chain of subgroups of $\prod_{i\in I} G_i$. The groups $H_0 = \{1\}$ and $H_\tau = \prod_{i\in I} G_i$ are in Σ. For any subset of Σ, since I is well ordered, the intersection and the union of its members belong to Σ. For each $\gamma \leq \tau$, since H_γ is a normal subgroup of $\prod_{i\in I} G_i$, $H_\gamma \lhd H_{\gamma+1}$ and $H_{\gamma+1}/H_\gamma \cong G_\gamma$ is locally indicable. Therefore Σ is a subnormal series of G with locally indicable factors. Now apply Proposition 2.6 to get $\prod_{i\in I} G_i$ is locally indicable.

(ii) It is a subgroup of the cartesian product. It will be useful to have another way of showing this: note that the same proof of (i) works for $H_\gamma = \{(x_i) \in \bigoplus_{i\in I} G_i \mid x_i = 1,\ i \geq \gamma\}$.

(iii) $G \wr H$ is the extension of $\prod_{h\in H} G_h$ (or $\bigoplus_{h\in H} G_h$) by H, where $G_h = G$ for all $h \in H$.

(iv) G can be seen as a subgroup of the locally indicable group $\prod_{i\in I} G/H_i$ via the diagonal map. □

In [**Ber90**, Section 9], Corollary 2.9 is given for right orderable groups. We state it for locally indicable groups and realize that the same proof works to show it and Proposition 2.8. We follow the notation in Section 6.3 of Chapter 1.

PROPOSITION 2.8. *Let $(G(-), \Delta)$ be a graph of groups and $G = \pi(G(-), \Delta, \Delta_0)$ its fundamental group. The following conditions are equivalent:*

(i) *The $G(v)$ can be embedded in a common locally indicable group L by morphisms $f_v\colon G(v) \to L$ that can be extended to $f\colon G \to L$.*

(ii) *G is locally indicable.*

PROOF. (ii) ⇒ (i) Recall that $G(v)$ embeds in G for each vertex v by Remark 1.40. Then take $L = G$.

(i) ⇒ (ii) Let $K = \ker f$. Since f_v is an injective morphism for each vertex v, we infer that $K \cap gG(e)g^{-1} \subseteq K \cap gG(v)g^{-1} = \{1\}$ for all $g \in G$ and vertex v. Then, by Theorem 1.41, $K = F * \mathop{*}_{q\in Q} K_q$ for some free subgroup F, and subgroups K_q of K of the form $K \cap gG(v)g^{-1}$ as g ranges over a certain set of elements of G and v ranges over $V\Delta$. Hence K is a free group. Consider the subnormal series of G

$$\{1\} \lhd K \lhd G.$$

Notice that K is a locally indicable group by Example 2.4(a). Also G/K is locally indicable, since it is isomorphic to the image of f, a subgroup of L. Therefore G is a locally indicable group by Proposition 2.6. □

COROLLARY 2.9. *Let H be a group. Let $\{G_i\}_{i\in I}$ be a set of groups such that H is a subgroup of G_i for each $i \in I$. The following conditions are equivalent*

(i) *The G_i can be embedded in a common locally indicable group L by morphisms that agree on H.*

(ii) *The free product amalgamating H, $*_H^{i\in I} G_i$, is locally indicable.*

Therefore

(a) *If G_i is locally indicable for each $i \in I$ then the free product $\underset{i\in I}{*} G_i$ is a locally indicable group. Moreover, $\underset{i\in I}{*} G_i$ is the extension of a free group K by the group $\prod\limits_{i\in I} G_i$.*

(b) *For every subgroup H of a locally indicable group G, every free product $*_H^{i\in I} G$ amalgamating H is locally indicable.*

Proof. (ii) $\Rightarrow$ (i) Just take $L = *_H^{i\in I} G_i$.

(i) $\Rightarrow$ (ii) First we construct a connected graph $(\Delta, V, E, \bar{\iota}, \bar{\tau})$. Set $V = I$. Fix $i_0 \in I$. Let $E = \{e_i\}_{i\in I\setminus\{i_0\}}$. For each $e_j \in E$, $\bar{\iota}(e_j) = i_0$, $\bar{\tau}(e_j) = j$. Thus Δ is a tree, and clearly coincides with its maximal subtree Δ_0. For each $i \in V$, let $G(i) = G_i$ and for each $e_j \in E$, $G(e_j) = H$. Hence $(G(-), \Delta)$ is a graph of groups. The fundamental group of $(G(-), \Delta)$ is $G = *_H^{i\in I} G(i) = *_H^{i\in I} G_i$. By hypothesis and the universal property of $*_H^{i\in I} G_i$, there exists a morphism of groups $f\colon *_H^{i\in I} G_i \to L$ which extends the embeddings of G_i in L for each $i \in I$. Now apply Proposition 2.8.

(a) Apply the foregoing proof with $H = \{1\}$ and $L = \prod\limits_{i\in I} G_i$, which is locally indicable by Corollary 2.7(i).

(b) The condition (i) is satisfied with $L = G$. $\square$

It was proved by A. Karrass and D. Solitar in [**KS70**, Theorem 9] that the free product of two locally indicable groups amalgamating an infinite cyclic group is always a locally indicable group. On the other hand, they also showed that the free product of two locally indicable groups amalgamating a subgroup is not always a locally indicable group [**KS70**, p. 250].

We illustrate the results in Proposition 2.8 and Corollary 2.9 with some examples.

Examples 2.10. The following groups are locally indicable

(a) $G = \langle a, b \mid a^2 = b^3 \rangle$.

(b) $\Gamma = \langle X, T \mid TXTX^{-1} = XTX^{-1}T \rangle = \langle X, T \mid TXTX^{-1}T^{-1}XT^{-1}X^{-1} = 1 \rangle$.

Proof. (a) $G = A \underset{C}{*} B$ where $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$ are infinite cyclic groups and $C \hookrightarrow A$, $c \mapsto a^2$, $C \hookrightarrow B$, $c \mapsto b^3$. Let $D = \langle d \rangle$ be another infinite cyclic group and consider the embeddings $f_1\colon A \to D$, $a \mapsto d^3$ and $f_2\colon B \to D$, $b \mapsto d^2$. Since D is locally indicable, we infer from Corollary 2.9 that G is locally indicable.

(b) As it is done in [**LS77**, Chapter IV Section 5], since the relator of Γ has exponent sum zero on X, we can express Γ as an HNN-extension of a free commutative group with stable letter X as follows:

$$\Gamma' = \langle X', T_0, T_1 \mid T_0T_1T_0^{-1}T_1^{-1} = 1,\ X'T_0X'^{-1} = T_1 \rangle.$$

The isomorphism $\Gamma \to \Gamma'$ is given by $X \mapsto X'$, $T \mapsto T_0$. And the isomorphism $\Gamma' \to \Gamma$ by $T_0 \mapsto T$, $T_1 \mapsto XT_0X^{-1}$, $X' \mapsto X$.

We can also view Γ as a semidirect product of N and the infinite cyclic group $C = \langle X \rangle$, where

$$N = \langle T_i,\ i \in \mathbb{Z} \mid T_iT_{i+1} = T_{i+1}T_i \rangle,$$

and X acts on N as $T_i \mapsto T_{i+1}$. The isomorphism $N \rtimes C \to \Gamma$ is given by $X \mapsto X$, $T_i \mapsto X^iTX^{-i}$, $i \in \mathbb{Z}$. And the isomorphism $\Gamma \to N \rtimes C$ by $X \mapsto X$, $T \mapsto T_0$.

Observe that N is the fundamental group of the graph of groups

$$\cdots \longrightarrow \underset{\bullet}{G(i)} \xrightarrow{G(e_i)} \underset{\bullet}{G(i+1)} \xrightarrow{G(e_{i+1})} \cdots$$

where $G(i)$ is the free abelian group in $\{T_i, T_{i+1}\}$ and $G(e_i) = \langle T_{i+1}\rangle$ for each $i \in \mathbb{Z}$. And we have the morphism of groups

$$N \longrightarrow \langle T_{\bar{0}}, T_{\bar{1}}\rangle \quad \text{defined by } T_i \mapsto \begin{cases} T_{\bar{0}} \text{ if } i \text{ is even} \\ T_{\bar{1}} \text{ if } i \text{ is odd} \end{cases}$$

where $\langle T_{\bar{0}}, T_{\bar{1}}\rangle$ is the free abelian group on the set $\{T_{\bar{0}}, T_{\bar{1}}\}$. Moreover it can be deduced from the proof of Proposition 2.8 that N is the extension of a non-cyclic free group K by the free abelian group $\langle T_{\bar{0}}, T_{\bar{1}}\rangle$. Then N is locally indicable by Proposition 2.8. And Γ is locally indicable because it is the extension of two locally indicable groups. □

Of course these examples can be obtained from Theorem 2.37, but it will be useful for us to express these groups in this way to illustrate some results on Hughes-free embeddings in Chapter 6. We will generalize Example 2.10(b) in Corollary 7.60.

Proposition 2.11. *The directed union of locally indicable groups is a locally indicable group.*

Proof. Suppose $G = \varinjlim G_\gamma$. For each γ, denote by $f_\gamma\colon G_\gamma \longrightarrow G$ the morphism of groups given by the definition of directed union. Notice that f_γ is injective for each γ. Let $g_1, \dots, g_n \in G\setminus\{1\}$. Define $H = \langle g_1, \dots, g_n\rangle$. Then there exists γ such that H is contained in the image of f_γ. Hence H is isomorphic to a nontrivial finitely generated subgroup of a locally indicable group G_γ. □

2. Orderable and right orderable groups

2.1. Definition and some properties.

Definition 2.12. (a) Let M be a monoid. We say that M is a *right orderable monoid* if the elements of M can be totally ordered in a manner compatible with right multiplication by the elements of M. To be more precise, if there exists a total order $<$ on M such that,

(i) for all $x, y, z \in M,\ x < y$ implies $xz < yz$.

M is a *left orderable monoid* if there exists a total order $<$ on M such that,

(ii) for all $x, y, z \in M,\ x < y$ implies $zx < zy$.

If there exists a total order $<$ on M such that (i) and (ii) hold, we say that M is an *orderable monoid*.

In these cases we will say that $(M, <)$ is a *(right, left) ordered monoid*.

(b) A group G is said to be a *(left, right) orderable group* if G is a (left, right) orderable monoid. Analogously $(G, <)$ is a *(left, right) ordered group*.

Let $(G, <)$ be an ordered group. Of particular importance in (left, right) ordered groups is the so-called *positive cone*, namely

$$P = P(G, <) = \{x \in G \mid 1 < x\}. \quad \square$$

Remark 2.13. Let $(M, <)$ be an ordered monoid (group). Let $x, y, a, b \in M$. If $x < y$ and $a < b$, then $ax < by$ and $xa < yb$.

Proof. Since $x < y$, then $ax < ay$ and $xa < ya$. In the same way, $a < b$ implies $ya < yb$ and $ay < by$. Hence $ax < ay < by$ and $xa < ya < yb$. □

The next lemma, in essence, yields an alternate definition of a (right) ordered group. The proof is not difficult and can be found in [**Pas77**, Lemmas 13.1.3 and 13.1.4]. In many cases, to endow a group with a structure of (right) ordered group, we will prove the existence of a set verifying the conditions of Lemma 2.14 (i) (or (ii)).

Lemma 2.14. *Let G be a group and $<$ a total order on G.*

(i) *If $(G,<)$ is an ordered group with positive cone P, then P has the following properties:*
  (a) *P is a subsemigroup of G, that is, P is multiplicatively closed.*
  (b) *$G = P \cup \{1\} \cup P^{-1}$ is a disjoint union.*
  (c) *P is a normal subset of G, that is, $x^{-1}Px = P$ for all $x \in G$.*

  Conversely, suppose that G has a subset P satisfying conditions (a), (b) *and* (c). *If we define $x < y$ to mean that $yx^{-1} \in P$, then $(G,<)$ is an ordered group with positive cone P.*

(ii) *If $(G,<)$ is a right ordered group with positive cone P, then P has the following properties:*
  (a) *P is a subsemigroup of G.*
  (b) *$G = P \cup \{1\} \cup P^{-1}$ is a disjoint union.*

  Conversely, suppose that G has a subset satisfying conditions (a) *and* (b). *If we define $x < y$ to mean that $yx^{-1} \in P$, then $(G,<)$ is a right ordered group with positive cone P.* □

Remarks 2.15. (a) Orderable groups are right orderable groups.

(b) Observe that conditions of Lemma 2.14(ii) on P are right-left symmetric. Thus a right orderable group must also be left orderable, but, of course, not necessarily under the same ordering. Indeed, if $(G,<)$ is a right ordered group, then $\prec$ defined by $x \prec y$ if and only if $y^{-1} < x^{-1}$ makes $(G,\prec)$ a left ordered group with the same positive cone as $(G,<)$.

(c) Right orderable groups are torsion free. Note that if $1 < g$, then $1 < g^n$ for all $n \geq 1$ and order $<$ such that $(G,<)$ is an ordered group. Analogously if $g < 1$. However not all torsion-free groups are right orderable groups. For example it is proved in [**Pas77**, Lemma 13.3.3] that the group $G = \langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2}\rangle$ is torsion free but not right orderable.

(d) If G is a (right) orderable group, then so is every subgroup of G. On the other hand, these properties are not inherited by quotient groups. As an example consider the (right) ordered group $\mathbb{Z}$. Given any nonzero $n \in \mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ is not (right) orderable since it is not torsion free.

(e) Let G be a group and $H \lhd G$. If H and G/H are right orderable, then so is G. Moreover, if $(H,<_H)$ and $(G/H,<_{G/H})$ are right ordered groups with positive cones P_H and $P_{G/H}$ respectively, then $P_G = \{x \in G \mid x \in P_H \text{ or } \bar{x} \in P_{G/H}\}$ is a positive cone for G.

(f) Let G be a group and $H \lhd G$. If H and G/H are orderable, then G need not be orderable. For example, let $G = \langle x, y \mid y^{-1}xy = x^{-1}\rangle$ and $H = \langle x\rangle$. Then $G/H \cong \langle y\rangle$. Both H and G/H are infinite cyclic and thus orderable. But G is not orderable because if x belongs to some positive cone P_G of G then $x^{-1} = y^{-1}xy \in P_G$.

(g) Let G be a group and $H \lhd G$. Suppose that $(H,<_H)$ and $(G/H,<_{G/H})$ are ordered groups with positive cones P_H and $P_{G/H}$ respectively such that P_H is a normal subset of G, then G can be ordered (as in (e)) via the positive cone $P_G = \{x \in G \mid x \in P_H \text{ or } \bar{x} \in P_{G/H}\}$. □

2.2. Examples of orderable groups. The most important classes of orderable groups for us are given in Proposition 2.16 and Corollary 2.24. As it is shown in Section 3, orderable groups are locally indicable. So, at first sight, it seems that these results do not add anything new to our discussions on locally indicable groups; but in the following chapters we will realize how important is that these groups are orderable, not only locally indicable.

Now we proceed to give our first important class of examples, due to [**Lev13**]. The proof is taken from [**Lam01**, Theorem 6.31]. Notice that Proposition 2.16 can be seen as a consequence of Corollary 2.20(i) since a product $\mathbb{Z}\times \overset{n)}{\cdots} \times\mathbb{Z}$ is an ordered group with the lexicographical order, see the proof of Proposition 2.21(i).

Proposition 2.16. *Let G be a torsion-free abelian group. Then G is an orderable group.*

Proof. We consider G as an additive group. Then G is a $\mathbb{Z}$-module. Since $S = \mathbb{Z} \setminus \{0\}$ is a multiplicative set, we can localize at S. We obtain $GS^{-1} \cong G \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that G embeds into $G_1 = G \otimes_{\mathbb{Z}} \mathbb{Q}$ because G is torsion free. We are going to construct a positive cone P for G_1, then G will be an ordered group with the inherited order. G_1 is a $\mathbb{Q}$-vector space. Choose a $\mathbb{Q}$-basis $\{g_i\}_{i \in I}$ for G_1, and fix a total order " $<$ " on the indexing set I. Using the additive notation for G_1, we can define P to be the set of elements

$$g_{i_1} a_1 + \cdots + g_{i_n} a_n,$$

with $a_1, \ldots, a_n \in \mathbb{Q}$, $i_1, \ldots, i_n \in I$, where $i_1 < i_2 < \ldots < i_n$ and $a_1 > 0$ in $\mathbb{Q}$. Then it is not very difficult to see that the sum of two elements in P is in P, that G_1 equals the disjoint union $P \cup \{0\} \cup (-P)$ and that P is a normal subset in G_1 because the group is abelian. $\square$

Now we introduce some useful results and definitions in order to prove that (Conrad) (right) orderability is a local property. Lemma 2.19 is due in various forms (among others) to [**Lor49**], [**Łos54**] and [**Ohn52**]. The proof we provide is taken from [**NF07**], while the proof of Corollary 2.20 is taken from [**Pas77**, Corollary 13.2.2] where a different proof of Lemma 2.19 is also given.

Definitions 2.17. Let G be a group. Suppose that $x_1, x_2, \ldots, x_n \in G$.

(a) We denote by $S^G(x_1, x_2, \ldots, x_n)$ the normal subsemigroup of G which they generate. Thus $S^G(x_1, \ldots, x_n)$ consists of all finite products of the form $x_{i_1}^{g_1} x_{i_2}^{g_2} \cdots x_{i_j}^{g_j}$ with $g_i \in G$ and $j \geq 1$.

(b) We define $S(x_1, x_2, \ldots, x_n)$ to be the semigroup of G generated by these elements. That is, $S(x_1, x_2, \ldots, x_n)$ consists of all finite products of the form $x_{i_1} x_{i_2} \cdots x_{i_j}$ with $j \geq 1$.

(c) We denote $S^C(x_1, \ldots, x_n)$ the *Conrad subsemigroup* of G generated by these elements. That is, $S^C(x_1, \ldots, x_n)$ is the smallest semigroup among all semigroups W which contain $x_1, \ldots, x_n$ and $x^2 y x^{-1}$ for all $x, y \in W$. Thus,

$$S^C(x_1, \ldots, x_n) = \bigcup_{m \geq 0} S^C_m(x_1, \ldots, x_n),$$

where $S^C_0(x_1, \ldots, x_n) = S(x_1, \ldots, x_n)$, and, for all $m \geq 0$, $S^C_{m+1}(x_1, \ldots, x_n)$ is the semigroup generated by the sets $S^C_m(x_1, \ldots, x_n)$ and $\{z^2 w z^{-1} \mid z, w \in S^C_m(x_1, \ldots, x_n)\}$. $\square$

The proof of the next result is topological. Recall the following facts:

Remarks 2.18. (a) Given a family $\{X_i\}_{i \in I}$ of topological spaces, the cartesian product $\prod_{i \in I} X_i$ is a topological space with the *product topology*, i.e. the least one such that the projections $p_j \colon \prod_{i \in I} X_i \to X_j$ are continuous for all $j \in I$.

(b) *Tychonov Theorem*: Given a family $\{X_i\}_{i \in I}$ of topological spaces, the cartesian product $\prod_{i \in I} X_i$ is compact if and only if X_i is compact for all $i \in I$.

(c) A topological space X is compact if and only if it satisfies the *Finite Intersection Property*, i.e. for any family $\{F_i\}_{i \in I}$ of closed subsets of X such that any finite intersection $F_{i_1} \cap \cdots \cap F_{i_r} \neq \emptyset$, then $\bigcap_{i \in I} F_i \neq \emptyset$. $\square$

Lemma 2.19. *Let G be a group. The following statements hold true.*

(i) *G is an orderable group if and only if for all nonidentity elements $x_1, \ldots, x_n \in G$ there exist suitable signs $\varepsilon_i = \pm 1$ such that $1 \notin S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$.*

(ii) *G is a right orderable group if and only if for all nonidentity elements $x_1, \ldots, x_n \in G$ there exist suitable signs $\varepsilon_i = \pm 1$ such that $1 \notin S(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$.*

Proof. We consider only part (i), because (ii) is proved in the same way.

Suppose that G is orderable. Given $x_1, \ldots, x_n \in G \setminus \{1\}$, choose signs ε_i such that $x_i^{\varepsilon_i} > 1$ for all i. Then $x > 1$ for each $x \in S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$. Therefore $1 \notin S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ as desired.

To prove the converse, endow $\{+,-\}^{G\setminus\{1\}}$ with the product topology. Hence $\{+,-\}^{G\setminus\{1\}}$ is a compact topological space. For each finite family $x_1, \ldots, x_n \in G \setminus \{1\}$, and for each family of *suitable signs* $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ (i.e. $1 \notin S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$). Consider the closed subset $\mathcal{X}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n)$ of $\{+,-\}^{G\setminus\{1\}}$ formed by all functions sgn which satisfy the following property: $\operatorname{sgn}(x) = +$ and $\operatorname{sgn}(x^{-1}) = -$ for every $x \in S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$. Notice that $\mathcal{X}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n)$ is not empty because $1 \notin S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$. For fixed $x_1, \ldots, x_n$, let $\mathcal{X}(x_1, \ldots, x_n)$ be the union of all the sets $\mathcal{X}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n)$ for suitable family of signs $\{\varepsilon_1, \ldots, \varepsilon_n\}$. Notice that it is a closed subset because it is a finite union (at most 2^n) of closed subsets. Note also that for any finite family of subsets $\{\mathcal{X}_i = \mathcal{X}(x_{i1}, \ldots, x_{in_i})\}_{1\le i\le m}$ the intersection $\bigcap_{i=1}^{m} \mathcal{X}_i$ is not empty because $\mathcal{X}(x_{11}, \ldots, x_{1n_1}, \ldots, x_{m1}, \ldots, x_{mn_m}) \subseteq \bigcap_{i=1}^{m} \mathcal{X}_i$. By the Finite Intersection Property, the intersection $\mathcal{X}$ of all the sets of the form $\mathcal{X}(x_1, \ldots, x_n)$ is not empty. Moreover, if $f \in \mathcal{X}$, then $P = \{x \in G \mid f(x) = +\}$ is a positive cone, and therefore G is orderable. $\square$

Corollary 2.20. *Let G be a group. The following statements hold true.*

(i) *If all finitely generated subgroups of G are orderable groups then G is an orderable group.*

(ii) *If all finitely generated subgroups of G are right orderable groups then G is a right orderable group.*

Proof. We prove (i), while (ii) can be shown in the same way. Suppose that G is not an orderable group. Then by Lemma 2.19(i) there exist nonidentity elements $x_1, \ldots, x_n \in G$ such that $1 \in S^G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ for all 2^n choices of the signs $\varepsilon_i = \pm 1$. By writing the identity element as an explicit product in each of the 2^n cases, we see that there exists a finitely generated subgroup H of G with $x_1, \ldots, x_n \in H$ and $1 \in S^H(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ for all choices of signs. But H is assumed to be an orderable group. Hence we have a contradiction, and G is therefore an orderable group. $\square$

About the closure properties of the class of orderable groups, although we have already seen that the class of orderable groups is not closed under extensions, we can state the following results. The proof of the following Proposition is mainly taken from [**BMR77**, Theorem 2.1.1].

Proposition 2.21. *The following statements hold*

(i) *The cartesian product of (right) orderable groups is a (right) orderable group.*

(ii) *The directed union of (right) orderable groups is a (right) orderable group.*

(iii) *The restricted standard wreath product of two (right) orderable groups is a (right) orderable group.*

(iv) *The subdirect product of (right) orderable groups is a (right) orderable group.*

Proof. (i) Let I be a set, and for each $i \in I$, let G_i be a (right) orderable group. Fix an order $<_i$ of G_i such that $(G_i, <_i)$ is a (right) ordered group for each $i \in I$. Well-order the index set I. Set

$$P = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} G_i \mid \text{ if } i_0 = \min\{i \mid x_i \neq 1\}, \text{ then } 1 <_{i_0} x_{i_0} \right\}.$$

Then P is a positive cone.

(ii) Suppose that $G = \varinjlim_{i\in I} G_i$, where G_i is a (right) orderable group for each $i \in I$. Given $g_1, \dots, g_n \in G$, there exists $i \in I$ such that $g_1, \dots, g_n \in f_i(G_i)$. Since f_i is injective, the subgroup generated by $g_1, \dots, g_n$ is (right) orderable. Now apply Corollary 2.20.

(iii) Suppose that $(G, <_G)$ and $(H, <_H)$ are (right) ordered groups. Every element of $G \wr H$ can be expressed uniquely as a product gh where $h \in H$ and $g \in \bigoplus_{h\in H} G_h$, $G_h = G$ for all $h \in H$. We say that $g \in \bigoplus_{h\in H} G_h$ is positive if the first component different from 1_G (the identity element of G) is positive. Then

$$P = \{gh \in G \wr H \mid 1 <_H h,\ \text{or}\ h = 1_H \text{ and } g \text{ is positive } \}$$

is a positive cone.

(iv) Let G be a group and $\{H_i\}_{i\in I}$ be a set of normal subgroups of G such that $\bigcap_{i\in I} H_i = \{1\}$ and G/H_i is orderable for all $i \in I$. Then G is (right) orderable because G embeds in $\prod_{i\in I} G/H_i$ and this group is (right) orderable by (i). □

Before giving the next result we need this definition

Definition 2.22. Let R be a ring. We say that R is an *ordered ring* if there exists a subset Q of R satisfying the following properties

(i) $Q + Q \subseteq Q$ (ii) $Q \cdot Q \subseteq Q$ (iii) $R = Q \cup \{0\} \cup (-Q)$ is a disjoint union.

Given $x, y \in R$ we can define a total order on R by $x < y$ if and only if $y - x \in Q$. Notice that the following properties are verified for $x, y, z \in Q$

(a) if $x < y$, then $x + z < y + z$, (b) if $0 < x, y$, then $0 < xy$.

Conversely, if there exists a total order $<$ on R satisfying (a) and (b), then the *positive cone* $Q = \{x \in R \mid 0 < x\}$ satisfies (i), (ii), (iii) and $<$ is the total order induced by Q. □

The following Proposition was first proved in [**Vin49**], but the proof given here is from [**Ber90**, Sections 3,4] where more general results are proved.

Proposition 2.23. *The free product of (right) orderable groups is a (right) orderable group.*

Proof. Let F, G be two (right) orderable groups. Set $H = F \times G$. By Proposition 2.21, H is (right) orderable. Fix a total order $\prec$ on H such that $(H, \prec)$ is an ordered group. Let R be the group ring $\mathbb{Z}[H]$. We endow R with a structure of ordered ring with positive cone

$$Q = \Big\{ r = \sum_{h\in H} a_h h \in R \mid \ \text{if } h_0 = \min \operatorname{supp} r,\ \text{then } 0 < a_{h_0} \Big\}.$$

Notice that F and G are embedded in Q.

Consider the polynomial ring $R[t]$ and the matrix ring $\mathbb{M}_2(R[t]) \cong \mathbb{M}_2(R)[t]$. We now embed F and G inside the group of units of $\mathbb{M}_2(R[t])$. The map $F \to \mathbb{M}_2(R[t])$, $f \mapsto \left(\begin{smallmatrix} f & t(f-1) \\ 0 & 1 \end{smallmatrix}\right)$ is the composition of the maps $F \to \mathbb{M}_2(R[t])$ defined by $f \mapsto \left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\mathbb{M}_2(R[t]) \to \mathbb{M}_2(R[t])$ given by right conjugation by $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Similarly the embedding $G \to \mathbb{M}_2(R[t])$, $g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ t(g-1) & g \end{smallmatrix}\right)$, is the composition of the maps $G \to \mathbb{M}_2(R[t])$ defined by $g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$, and $\mathbb{M}_2(R[t]) \to \mathbb{M}_2(R[t])$ given by right conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$.

These two maps induce a morphism of groups from $F * G$ to the group of units of $\mathbb{M}_2(R[t])$. We claim that it is injective. The nonidentity elements of $F * G$ can be expressed uniquely as an alternating product of nonidentity elements of F and G. The claim follows if we prove that the images of such elements are never the identity matrix. Suppose that A is the image of some element $w \in F * G$. Apply A to the column vector $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \in \mathbb{M}_2(R[t])$. We show that if the leftmost factor in the expression of w as an alternating product of elements of F and G is in F then the upper entry of $A\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ has strictly greater degree (on t) than the lower entry; and if the leftmost factor of w is in G then the lower entry of $A\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ has strictly greater degree (on t) than the upper entry. Thus the image A of w is never the identity matrix. We prove the claim by induction on the length of the element w. If w has length 1, then $w = f \in F$ or $w = g \in G$, and the result is clear from the definition of the map. If w has length greater than 1 and the leftmost factor of w is in F, then $w = fgw'$, where $f \in F$ and $g \in G$. Call B the image of gw'. We apply the induction hypothesis to gw' to obtain that the lower entry of the image $B\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ has strictly greater degree (on t) than the upper entry. Then the upper entry of $A\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}f & t(f-1)\\0 & 1\end{smallmatrix}\right) B\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ has degree on t strictly greater than the lower entry. In the same way the claim can be proved if the leftmost factor of w is in G.

Notice that the image of $F * G$ in $\mathbb{M}_2(R[t]) \cong \mathbb{M}_2(R)[t]$ is contained in the monoid U of matrices whose constant term is a diagonal matrix with positive entries. Now we introduce an order $\lessdot$ in U compatible with the product of matrices so that $(F * G, \lessdot)$ is an ordered group.

Choose an order among the four entries in a 2×2 matrix. We say that $B \in \mathbb{M}_2(R)$ is positive (in $\mathbb{M}_2(R)$) if and only if the first nonzero entry of B is positive (in R). Given $A, B \in U \subseteq \mathbb{M}_2(R)[t]$, let $n \geq 0$ be the least integer such that t^n has nonzero coefficient in $A - B$. We say that $B \lessdot A$ if and only if such coefficient is positive (in $\mathbb{M}_2(R)$). This gives a total ordering in U and it is compatible with the product since the product, in either order, of a positive element of $\mathbb{M}_2(R)$ and a diagonal matrix of $\mathbb{M}_2(R)$ with positive diagonal entries is still positive (in $\mathbb{M}_2(R)$). Hence $F * G$ is (right) orderable.

It can be showed by induction that $G_1 * \cdots * G_n$ is orderable for any $n \geq 1$ and orderable groups $G_1, \ldots, G_n$. Since (right) orderability is a local property by Corollary 2.20, it follows that $\mathop{*}\limits_{i \in I} G_i$ is (right) orderable for any set I and (right) orderable groups G_i, $i \in I$. ☐

Corollary 2.24. *Locally free groups, and in particular free groups, are orderable groups.*

Proof. A free group on a set X is isomorphic to the free product $\mathop{*}\limits_{x \in X} C_x$, where C_x is an infinite cyclic group for each $x \in X$. Since infinite cyclic groups are orderable, the result for free groups follows from Proposition 2.23. That locally free groups are orderable is a consequence of the foregoing observation and Corollary 2.20(i). ☐

The fact that a free group is orderable is due to Birkhoff [**Bir42**], Iwasawa [**Iwa48**], Neumann [**Neu49b**]. Another way of showing that was given in Bergman [**Ber90**, Section 1] using the Magnus-Fox embedding 1.60, or in more detail [**Reu99**, Section 2.3].

3. Relations between locally indicable groups and (right) orderable groups

The first important result which we want to prove is that orderable groups are locally indicable [**Lev43**]. So we collect some properties of the convex subgroups of a right ordered group which are taken from [**Fuc63**, Chapter IV] and [**Con59**].

Definitions 2.25. (a) Let $(G, <)$ be a right ordered group. A subgroup H of G is said to be *convex* if for all $a, b \in H$ and $g \in G$, the inequality $a \leq g \leq b$ implies that $g \in H$.

(b) A (right) orderable group G is said *Archimedean* if there exists a total order $<$ on G such that $(G, <)$ is a (right) ordered group and for every $a, b \in P(G, <)$ there exists $n \in \mathbb{N} \setminus \{0\}$ such that $b < a^n$. We also say $(G, <)$ is an *Archimedean (right) ordered group.* ☐

Remarks 2.26. Let $(G,<)$ be a (right) ordered group. Let Σ be the set of convex subgroups of $(G,<)$. Then

(a) The partially ordered set (by inclusion) Σ is in fact a chain.
(b) The arbitrary union and intersection of convex subgroups is a convex subgroup.
(c) Let $N \lhd G$. The group G/N can be (right) ordered with an ordering compatible with $<$ if and only if N is a convex subgroup. By an order compatible with $<$ we mean that for every $a,b \in G$ such that $aN \neq bN$, $aN < bN$ if and only if $a < b$.
(d) If N is a normal convex subgroup of $(G,<)$, then there is a bijective correspondence between the convex subgroups of $(G,<)$ that contain N and the convex subgroups of $(G/N,<)$.

Moreover, if $(G,<)$ is an ordered group,

(e) If (L,H) is a jump in Σ, then $L \lhd H$.
(f) If $(G,<)$ has no other convex subgroups other than $\{1\}$ and G, then G is Archimedean.

Proof. (a) Let C,D be two convex subgroups of a (right) ordered group $(G,<)$. Suppose that $d \in D \setminus C$. Then for every $c \in C$, $d^{-1} < c < d$ or $d < c < d^{-1}$ (otherwise $d \in C$). So $c \in D$.

(b) Follows by the definition of convex subgroups.

(c) Suppose that N is convex, we show that the (right) order on G/N is well defined. Let $a,b \in G$ be such that $a < b$. Let a', b' be other representatives of the cosets aN and bN respectively. Then $a' = n_1 a$, $b' = n_2 b$, for certain $n_1, n_2 \in N$. Suppose that $b' < a'$. Hence $n_2 b a^{-1} < n_1$. On the other hand $ab^{-1} < n_2$ because otherwise $n_2 < ab^{-1} < 1$ and $aN = bN$, a contradiction. Hence $ab^{-1}n_2^{-1} < 1$ and $1 < n_2 b a^{-1}$. Therefore $1 < n_2 b a^{-1} < n_1$. Since N is convex, $n_2 b a^{-1} \in N$, and $ba^{-1} \in N$. From this, $aN = bN$, a contradiction. The other properties needed to see that G/N is a (right) ordered group are easily verified.

Conversely, suppose that G/N is (right) ordered. Given $n_1, n_2 \in N$ and $g \in G$ such that $n_1 < g < n_2$, then $g \in N$ because the order is well defined.

(d) Follows from (c).

(e) For each $h \in H$, the subgroup $hLh^{-1} \subseteq H$ is convex because $(G,<)$ is (two-sided) ordered. Thus $hLh^{-1} = L$ for all $h \in H$ since (L,H) is a jump.

(f) If it is not Archimedean, there exist $h, g \in G$ with $1 < h < g$ such that $h^n < g$ for every $n \in \mathbb{N} \setminus \{0\}$. Since $(G,<)$ is a (two-sided) ordered group, the smallest convex subgroup containing h is of the form $\{x \in G \mid h^{-i} \leq x \leq h^i\}$. Thus G has a nontrivial convex subgroup different from G, a contradiction. ☐

Lemma 2.27. *An Archimedean right orderable group is a torsion-free abelian group.*

Proof. Let G be an Archimedean right orderable group. We already know that right orderable groups are torsion free, so we only need to show that G is abelian. Let $<$ be a total order on G such that $(G,<)$ is a right ordered Archimedean group. Let $P = P(G,<)$.

Step one: Any Archimedean right ordered group is an ordered group.

By Lemma 2.14, it is enough to prove that $xPx^{-1} = P$ for all $x \in G$. Let $y \in P$ and $x \in G$. Suppose that $x > 1$. Let n be the least natural such that $1 < x < y^n$. If $xyx^{-1} < 1$, then $xy < x < y^n$. Hence $x < y^{n-1}$, a contradiction. Thus $xPx^{-1} \subseteq P$ for all $x \in P$. Hence $xP^{-1}x^{-1} \subseteq P^{-1}$ for all $x \in P$. Since G is the disjoint union $P \cup \{1\} \cup P^{-1}$ we get that $xPx^{-1} = P$ for all $x \in P$. This also implies that $P = x^{-1}Px$. Therefore $P = xPx^{-1}$ for all $x \in G$.

Step two: Any Archimedean ordered group is torsion free abelian.

Assume the existence of $g \in P$ such that $1 \leq x < g$ implies that $x = 1$. Because of the Archimedean property, for every $a \in G$ there exists an integer n such that $g^n \leq a < g^{n+1}$, and then $1 \leq ag^{-n} < g$. Hence $ag^{-n} = 1$, and $a = g^n$. Consequently, $G = \langle g \rangle$ is a commutative group.

Next assume that no such g exists. For every $x \in P$, there is $y \in G$ such that $1 < y < x$. Here either $y^2 \leq x$ or $x \leq y^2$. This second option implies that $y^{-1}x \leq y$. Multiplying by x on the left and by y^{-1} on the right we get $(xy^{-1})^2 \leq x$. Thus for each $x > 1$ there exists $z \in G$ with $1 < z < x$ and $z^2 \leq x$.

Notice that it suffices to show that the elements of P commute. Let a, b be positive elements of G with $ab \neq ba$. Suppose that $ba < ab$. Then for $x = aba^{-1}b^{-1}$ choose $z \in G$ with $1 < z < x$ and $1 < z^2 \leq x$. By the Archimedean property, there exist natural integers m, n satisfying

$$z^m \leq a < z^{m+1}, \quad z^n \leq b < z^{n+1}. \tag{7}$$

Since $a^{-1}b^{-1} \leq z^{-n-m}$, (7) implies that $x < z^2$, contrary to $z^2 \leq x$. Thus G is commutative. □

In fact, an Archimedean right orderable group is isomorphic to a subgroup of the additive group of the real numbers. The proof of the foregoing lemma is part of Cartan's proof [**Car39**] of this stronger result.

THEOREM 2.28. *Orderable groups are locally indicable groups.*

PROOF. Let G be an orderable group. Suppose that $(G, <)$ is an ordered group. Let Σ be the set of convex subgroups of $(G, <)$. We claim that Σ is a subnormal system with torsion-free abelian factors. Then Example 2.4(b) and Proposition 2.6 imply that G is locally indicable. The claim follows because $\{1\}, G \in \Sigma$, and from Remarks 2.15 (a), (b), (e), (f) and Lemma 2.27. □

The converse of Theorem 2.28 is not true. There are locally indicable groups which are not orderable as Remark 2.15(f) shows. On the other hand, locally indicable groups are right orderable [**BH72**]. The proof that we give of this fact is taken from [**Pas77**, Exercise 13.9].

PROPOSITION 2.29. *Let G be a group. Suppose that every nonidentity finitely generated subgroup of G can be mapped homomorphically onto a nonidentity right orderable group. Then G is a right orderable group. In particular, a locally indicable group is a right orderable group.*

PROOF. If G is not right orderable, then, by Lemma 2.19(ii), there exists a minimal integer n and nonidentity elements $x_1, \dots, x_n \in G$ with $1 \in S(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$ for all choices of sign. Set $H = \langle x_1, \dots, x_n \rangle$ and let $\overline{H} = H/N$ be a nonidentity homomorphic image of H which is right orderable. Assume that $x_1, \dots, x_t \in N$ and $x_{t+1}, \dots, x_n \notin N$. Notice that $t \geq 1$ because otherwise, as H/N is a right orderable group, there would exist signs such that $1 \notin S(\bar{x}_{t+1}^{\delta_{t+1}}, \dots, \bar{x}_n^{\delta_n})$ by Lemma 2.19(ii) which contradicts the choice of $x_1, \dots, x_n$. Also $t < n$ because $\overline{H}$ is not trivial. So there exist signs $\delta_1, \dots, \delta_n$ with $1 \notin S(x_1^{\delta_1}, \dots, x_t^{\delta_t})$ and $1 \notin S(\bar{x}_{t+1}^{\delta_{t+1}}, \dots, \bar{x}_n^{\delta_n})$. By the minimality of n, if $1 \in S(x_1^{\delta_1}, \dots, x_n^{\delta_n})$, then $\bar{1}$, the image in $\overline{H}$ of $1 \in H$, is in $S(\bar{x}_{t+1}^{\delta_{t+1}}, \dots, \bar{x}_n^{\delta_n})$. Thus $1 \notin \mathrm{S}(x_1^{\delta_1}, \dots, x_n^{\delta_n})$, a contradiction. □

But then, is the converse of Proposition 2.29 true? That is, do the classes of locally indicable groups and right orderable groups coincide? The answer is no. It was proved independently by G. M. Bergman [**Ber91**] and V. M. Tararin [**Tar93**] that there exist right orderable groups which are not locally indicable. In [**Ber91**, Section 6] it is proved that the group G with presentation $\langle x, y, z \mid x^2 = y^3 = z^7 = xyz \rangle$ is right orderable and perfect i.e. $[G, G] = G$. Thus it has no infinite cyclic quotient group.

So now arises the question of which right orderable groups are locally indicable. This can be answered in two ways, restricting the class of right orderings or the class of groups. We give an answer to the former and some partial answers to the latter in the next section.

4. Characterization of locally indicable groups and some recent advances

DEFINITION 2.30. Let $(G,<)$ be a right ordered group. Let Σ be the chain of convex subgroups of $(G,<)$. We say that $<$ is a *Conrad right order* if $L \lhd H$ and H/L is Archimedean for every jump (L,H) in Σ. Notice that H/L is a (right orderable) torsion-free abelian group by Lemma 2.27.

We say that a group G is *Conrad right orderable* if there exists a total order $<$ on G such that $(G,<)$ is a Conrad right ordered group. These right ordered groups were introduced by P. Conrad in [**Con59**]. Many equivalent conditions to Conrad orderability can be found in [**BMR77**, Section 7.4]. □

Before stating the main result of this section we need some preliminary results. The proof of Proposition 2.31 is taken from [**Gla99**, Lemma 6.6.2], while Proposition 2.32 from [**NF07**]. Condition (ii) in Proposition 2.31 was first introduced in [**Jim07**].

PROPOSITION 2.31. *Let $(G,<)$ be a right ordered group. The following are equivalent.*

(i) *$(G,<)$ is Conrad right ordered.*
(ii) *For all $x,y\in P(G,<)$, $y^2x>y$.*
(iii) *For all $x,y\in P(G,<)$, $y^nx>y$ for some $n\in\mathbb{N}$.*
(iv) *For all $x,y\in G$, $1<x<y$ implies $xy^nx^{-1}>y$ for some $n\in\mathbb{N}$.*

PROOF. (i) ⇒ (ii) Let Σ be the chain of convex subgroups of G. Let $x,y\in P(G,<)$. If $x\leq y$, consider the jump (L_y,H_y) in Σ. Then H_y/L_y is a torsion-free abelian group. By Remarks 2.26(c), $(H_y/L_y,<)$ is an ordered abelian group. Then $1<yL_y$ and $1\leq xL_y$. Moreover $y^2xL_y\geq y^2L_y>yL_y$. Hence $y^2x>y$ by Remarks 2.26(c).

(ii) ⇒ (iii) Clear.

(iii) ⇒ (iv) First notice that if $a,b\in P(G,<)$, there exists $m\in\mathbb{N}$ such that $(ab)^m>ba$. Otherwise $(ab)^m\leq ba$ for all $m\in\mathbb{N}$. Since $1<a$, then $(ba)^mb<a(ba)^mb=(ab)^{m+1}\leq ba$. Thus $(ba)^mb<ba$ for all $m\in\mathbb{N}$, contradicting (iii). By what we have just proved, there exists $n\in\mathbb{N}$ such that $xy^nx^{-1}=(xyx^{-1})^n>x^{-1}xy=y$, as desired.

(iv) ⇒ (i) Let Σ be the class of convex subgroups of G.

Step one: Let $w\in P(G,<)$. If $a,b\in G$ and $n\in\mathbb{N}$ with $a<w^n$ and $b<w^n$, then $ab<w^m$ for some $m\in\mathbb{N}$.

If there is $m\in\mathbb{N}$ such that $w^nb\leq w^m$, then $ab<w^nb\leq w^m$ as desired. So we suppose that no such m exists. Thus $w^m<w^nb$ for all $m\in\mathbb{N}$. Therefore $w^mb^{-1}<w^n$ for all $m\in\mathbb{N}$. But $b<w^n$ implies $bw^m<w^{n+m}$. Hence $bw^mb^{-1}<w^{n+m}b^{-1}<w^n$ for all $m\in\mathbb{N}$. This contradicts our hypothesis (iv).

Step two: Let (L,H) be a jump in Σ. If $x,y\in P(H,<)\setminus L$, then $y^m>x$ for some $m\in\mathbb{N}$.

Suppose that there does not exist such m. Consider the set

$$C=\{c\in G\mid \exists n\in\mathbb{N}\text{ such that }1<c<y^n\text{ or }1<c^{-1}<y^n\}.$$

Notice that if $c\in C$, then $c^{-1}\in C$. Moreover, if $c_1,c_2\in C$, then $c_1c_2\in C$ by Step one. Hence C is a subgroup of G. Furthermore C is convex. Let $c,d\in C$ and $z\in G$ such that $c\leq z\leq d$. Then $1<zc^{-1}<dc^{-1}$. For $dc^{-1}\in C$, $zc^{-1}\in G$. Moreover $zc^{-1}\cdot c=z\in G$. Therefore, since $y\in C$, C is a convex subgroup of G which strictly contains L, but, since $x\notin C$, C is strictly contained in H, a contradiction with the fact that (L,H) is a jump.

Step three: If (L,H) is a jump in Σ, then $L\lhd H$.

If $z \in L$ and $x \in P(H,<) \setminus L$, notice that $xz^{-1} \in P(H,<) \setminus L$. By Step two, let $n \in \mathbb{N}$ be the least one such that $x < (xz^{-1})^n$. If $x(xz^{-1})x^{-1} \leq 1$, then $x(xz^{-1}) \leq x < (xz^{-1})^n$ and $x < (xz^{-1})^{n-1}$ contradicting the minimality of n. Therefore $x(xz^{-1})x^{-1} > 1$. Hence $x(zx^{-1})x^{-1} = (x(xz^{-1})x^{-1})^{-1} < 1$, that is, $xzx^{-1} < x$. Since z was arbitrary in L, we obtain that $xz^m x^{-1} < x$ for any $m \in \mathbb{Z}$. If $xz^m x^{-1} \notin L$, and $xzx^{-1} > 1$, then $x < xz^m x^{-1} = (xzx^{-1})^m$ for some $m \in \mathbb{N}$ by Step two, contradicting what we have just showed. In the same way, if $xz^m x^{-1} \notin L$ and $xzx^{-1} < 1$, we get that $x < xz^{-m}x^{-1}$ for some $m \in \mathbb{N}$, a contradiction. Therefore $xLx^{-1} \subseteq L$. To show that $L \lhd H$ it is enough to prove that $x^{-1}Lx \subseteq L$. To prove this, let $y \in P(H,<) \setminus L$. By Step two, there exists $m \in \mathbb{N}$ such that $x < y^m$. By hypothesis there exists $r \in \mathbb{N}$ such that $y^m < xy^{mr}x^{-1}$. Hence $xyx^{-1} \in P(H,<) \setminus L$. Therefore $x(P(H,<) \setminus L)x^{-1} \subseteq P(H,<) \setminus L$ and, moreover, $x(H \setminus L)x^{-1} \subseteq H \setminus L$. This implies that $x^{-1}Lx \subseteq L$ as desired.

Thus $L \lhd H$ and H/L is Archimedean by Step 2. □

Proposition 2.32. *A group G is Conrad right orderable if and only if for all nonidentity elements $x_1, \ldots, x_n \in G$ there exist suitable signs $\varepsilon_i = \pm 1$ such that $1 \notin S^C(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$.*

Proof. If $(G,<)$ is Conrad right ordered and $x_1, \ldots, x_n \in G \setminus \{1\}$, choose signs $\varepsilon_1, \ldots, \varepsilon_n$ such that $x_i^{\varepsilon_i} > 1$. Then, by Proposition 2.31(ii), every element $x \in S^C(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ verifies $x > 1$, and thus $1 \notin S^C(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$.

Conversely, suppose that G satisfies the condition stated on Conrad semigroups. Consider the compact topological space $\{+,-\}^{G \setminus \{1\}}$ with the product topology. For each finite family $x_1, \ldots, x_n \in G \setminus \{1\}$ and each family of suitable signs $\varepsilon_1, \ldots, \varepsilon_n$, consider the closed subset $\mathcal{CX}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n)$ of $\{+,-\}^{G \setminus \{1\}}$ that consists of all functions sgn which satisfy the following property: $\operatorname{sgn}(x) = +$ and $\operatorname{sgn}(x^{-1}) = -$ for all $x \in S^C(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$. Notice that $\mathcal{CX}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n)$ is not empty because $1 \notin S^C(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$. For fixed $x_1, \ldots, x_n \in G \setminus \{1\}$, let $\mathcal{CX}(x_1, \ldots, x_n)$ be the union of all the sets $\mathcal{CX}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n)$ for suitable signs $\varepsilon_1, \ldots, \varepsilon_n$. Notice that $\mathcal{CX}(x_1, \ldots, x_n)$ is a closed subset. Moreover, for any finite family of subsets $\{\mathcal{CX}_i = \mathcal{CX}(x_{i1}, \ldots, x_{in_i})\}_{1 \leq i \leq m}$, the intersection $\bigcap_{i=1}^{m} \mathcal{CX}_i$ is not empty because $\mathcal{CX}(x_{11}, \ldots, x_{1n_1}, \ldots, x_{m1}, \ldots, x_{mn_m}) \subseteq \bigcap_{i=1}^{m} \mathcal{CX}_i$. Now, by the Finite Intersection Property, the intersection $\mathcal{X}$ of all the sets of the form $\mathcal{CX}(x_1, \ldots, x_n)$ is not empty. Then, if $f \in \mathcal{X}$, $P = \{x \in G \mid f(x) = +\}$ is a positive cone that defines a Conrad right order. □

What follows is the main result of this section. The equivalence of conditions (i) and (iii) was conjectured in [**BH72**], but the whole result was first proved in [**Bro84**]. Another proof can be found in [**RR02**]. Both proofs use nontrivial results on group varieties. The more direct proof provided here is from [**NF07**, Proposition 3.11].

Theorem 2.33. *Let G be a group. The following statements are equivalent.*

(i) *G is a locally indicable group.*
(ii) *G is a Conrad right orderable group.*
(iii) *G has a subnormal series Σ with torsion-free abelian factors.*

Proof. (i) $\Rightarrow$ (ii) We prove that G satisfies the condition of Proposition 2.32. So let $x_1, \ldots, x_n \in G \setminus \{1\}$. Let $I_1 = \{1, \ldots, n\}$. Since G is locally indicable, there exists a nontrivial morphism $\phi_1 \colon \langle x_1 \ldots, x_n \rangle \to \mathbb{Z}$. Let $I_2 = \{i \in I_1 \mid \phi_1(x_i) = 0\}$. If $I_2 \neq \emptyset$, by local indicability, there exists a nontrivial morphism of groups $\phi_2 \colon \langle x_i \mid i \in I_2 \rangle \to \mathbb{Z}$. Continuing in this way, if $I_{m+1} = \{i \in I_m \mid \phi_m(x_i) = 0\} \neq \emptyset$, there exists a nontrivial morphism of groups $\phi_{m+1} \colon \langle x_i \mid i \in I_{m+1} \rangle \to \mathbb{Z}$. Notice that for some $m \in \{1, \ldots, n\}$, $I_{m+1} = \emptyset$. For each $i \in \{1, \ldots, n\}$, let $j(i)$ be the unique index such that $\phi_{j(i)}$ is defined on x_i and $\phi_{j(i)}(x_i) \neq 0$.

Let $\varepsilon_i \in \{-1,1\}$ be such that $\phi_{j(i)}(x_i^{\varepsilon_i}) > 0$. We claim that $1 \notin S^C(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$. Notice that it is enough to prove that for each $x \in S^C(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$ there exists j such that $\phi_j(x) > 0$. So let $x \in S^C(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$. First notice that for each m such that ϕ_m exists, if $x \in S^C(x_i^{\varepsilon_i} \mid i \in I_m)$, then $\phi_m(x) \geq 0$ because $S^C(x_i^{\varepsilon_i} \mid i \in I_m) = \bigcup_{r=0}^{\infty} S_r^C(x_i^{\varepsilon_i} \mid i \in I_m)$. Moreover, for every $y, z \in \langle x_i \mid i \in I_m \rangle$, $\phi_m(y^2 z y^{-1}) = \phi_m(y) + \phi_m(z)$. Then, it is not difficult to realize that $\phi_m(x) = 0$ implies that $I_{m+1} \neq \emptyset$ and $x \in S^C(x_i^{\varepsilon_i} \mid i \in I_{m+1})$. Since $I_{m+1} = \emptyset$ for some m, we get that $\phi_j(x) \neq 0$ for some j.

(ii) $\Rightarrow$ (iii) Suppose that G is Conrad right orderable. Let $<$ be a total order on G such that $(G,<)$ is Conrad right ordered. Then Σ, the chain of convex subgroups of $(G,<)$, is a subnormal system of G with torsion-free abelian factors by Remarks 2.26(a), (b), the definition of Conrad right order and Lemma 2.27.

(iii) $\Rightarrow$ (i) Example 2.4(b) and Proposition 2.6 imply that G is locally indicable. □

Now we proceed to state some recent advances on which classes of groups are such that every right orderable group is locally indicable. For that we need the following definition

Definition 2.34. A group G is *amenable* if there is a *measure* -a function that assigns to each subset of G a number from 0 to 1- such that

(a) The measure is a *probability measure*: the measure of the whole group G is 1.
(b) The measure is *finitely additive*: given finitely many disjoint subsets of G, the measure of the union of the sets is the sum of the measures.
(c) the measure is *left-invariant*: given a subset A and an element g of G, the measure of A equals the measure of gA. □

It is known that subgroups of amenable groups are amenable and extensions of amenable groups by amenable groups are amenable. Some examples of amenable groups are polycyclic, solvable-by-finite, supramenable and elementary amenable groups. On the other hand. the direct product of an infinite family of amenable groups need not be amenable and, if a group contains a non-abelian free subgroup, then it is not amenable [**Wag93**].

That a right orderable amenable group is locally indicable has been proved by D. W. Morris in [**Mor06**]. This theorem generalizes results in [**Rhe81**], [**CK93**], [**Kro93**], where it was respectively proved that right orderable groups in the classes of polycyclic, supramenable or solvable-by-finite groups are locally indicable. It also implies the result in [**Lin99**] that every right orderable group in the smallest class of groups closed under extensions and directed unions that contains supramenable and elementary amenable groups is locally indicable.

On the other hand it is known that right orderable groups in the class $\mathfrak{X}$ of groups with no nonabelian free subsemigroups are locally indicable [**LMR95**] (for example nilpotent groups are contained in $\mathfrak{X}$) and also that a right orderable group G with a normal series $(G_\alpha)_{\alpha\leq\tau}$ (i.e. a subnormal series where $G_\alpha \lhd G$ for all α) whose factors are locally nilpotent groups is locally indicable [**Tar91**]. Along these lines, a further result was given in [**LMR00**]: a right orderable group with a normal series with factors in $\mathfrak{X}$ is locally indicable.

Furthermore, an even stronger result than [**Mor06**] and [**LMR00**] has been conjectured in [**Lin01**]: every right orderable group which contains no non-abelian free subgroup is locally indicable.

5. Torsion-free one-relator groups

An important class of examples of locally indicable groups is that of torsion-free one-relator groups. In this section it is briefly explained what a torsion-free one-relator group is and stated two important results without proofs.

DEFINITION 2.35. A *one-relator group* is a group which has a presentation of the form $\langle X \mid w\rangle$, where w is a word in the free group on X.

Given F the free group on X, a *freely reduced word* in F on X is a word of the form $x = x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ where $x_i \in X$, $e_i = \pm 1$ and $x_i \neq x_{i+1}$ if $e_i = -e_{i+1}$.

A freely reduced word $w = x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ is *cyclically reduced* if $x_1 \neq x_n$ or if $x_1 = x_n$ then $e_1 \neq -e_n$. Every element of a free group is conjugate to an element given by a cyclically reduced word. Hence every one-relator group has a presentation of the form $\langle X \mid w\rangle$, where w is a cyclically reduced word in the free group on X. □

The following well-known result was proved in [**KMS60**].

PROPOSITION 2.36. *Let $G = \langle X \mid w\rangle$ where w is cyclically reduced in the free group on X. The group G is torsion free if w is not a proper power in the free group on X. If $w = u^n$, $n > 1$, where u itself is not a proper power, then u has order n in G and all elements of G of finite order are conjugates of powers of u.* □

The following theorem was proved independently by S.D. Brodskii [**Bro84**] and J. Howie [**How82**].

THEOREM 2.37. *Torsion-free one-relator groups are locally indicable groups.* □

REMARKS 2.38. (a) Let F be the free group with basis $\{x, y, z\}$, and let $f \in \mathrm{End}(F)$ be defined by $x \longmapsto x^2yx^{-1}y^{-1}$, $y \longmapsto y^2zy^{-1}z^{-1}$, $z \longmapsto z^2xz^{-1}x^{-1}$. Then for each $n \geq 0$ the group

$$G_n = \langle x, y, z \mid f^n(x), f^n(y)\rangle$$

is locally indicable and it is not a one-relator group. A proof of this can be found in [**How85**].

(b) By Theorem 2.37 and Proposition 2.29, every torsion-free one-relator group is right orderable. □

DEFINITION 2.39. Let $\{G_i\}_{i\in I}$ be a set of groups. A word $w = a_1a_2\cdots a_n$ in the free product $\mathop{*}\limits_{i\in I} G_i$ is *reduced* if two contiguous letters belong to different groups. The *length* of the reduced word w is n. We say that w is *cyclically reduced* if w is reduced and, whenever the length of w is at least two, a_1 and a_n belong to different factors. □

The following generalization of Theorem 2.37 can be found in [**How82**, Theorem 4.2]

THEOREM 2.40. *Let $\{G_i\}_{i\in I}$ be a set of groups, and G be the quotient of $\mathop{*}\limits_{i\in I} G_i$ by the normal closure of a cyclically reduced word w of length at least 2. The following are equivalent:*

(i) *G is locally indicable*
(ii) *G is torsion free*
(iii) *w is not a proper power in $\mathop{*}\limits_{i\in I} G_i$.* □

"... Feeding on you
Feeding on me
Feeding on everyone
Feeding on everything"

Dead Soul Tribe, *Feed Part I: Stone by stone*

CHAPTER 3

Localization

1. Ore localization

Here we collect some well known results on Ore localization. Details and proofs will be omitted. Most of them can be found for example in the book of T.Y. Lam [**Lam99**, Sections 10-11], where a nice introduction about the problem of embedding domains in division rings is given.

Definitions 3.1. Let R be a ring.

(a) By a *multiplicative set* of R, we shall mean a subset $\mathfrak{S} \subset R$ that satisfies the following three properties:
   (i) $\mathfrak{S}$ is closed under multiplication,
   (ii) $0 \notin \mathfrak{S}$, and
   (iii) $1 \in \mathfrak{S}$.

(b) A morphism of rings $\psi\colon R \longrightarrow R'$ is said to be $\mathfrak{S}$-*inverting* if $\psi(\mathfrak{S}) \subset R'^{\times}$.

(c) Given a multiplicative set $\mathfrak{S}$ of R, a ring R' is said to be a *left Ore ring of fractions* of R (with respect to $\mathfrak{S}$) if there is a given morphism of rings $\varphi\colon R \longrightarrow R'$ such that:
   (i) φ is $\mathfrak{S}$-inverting.
   (ii) Every element of R' has the form $\varphi(s)^{-1}\varphi(a)$ for some $a \in R$ and $s \in \mathfrak{S}$.
   (iii) $\ker\varphi = \{r \in R : sr = 0 \text{ for some } s \in \mathfrak{S}\}$.

   In this event, R' is denoted by $\mathfrak{S}^{-1}R$. $\square$

The construction of a left Ore ring of fractions is done very much in the same way as the one of the ring of fractions of a commutative ring.

Theorem 3.2. *The ring R has a left Ore ring of fractions $\mathfrak{S}^{-1}R$ if and only if the following properties hold:*

(i) *For any $a \in R$ and $s \in \mathfrak{S}$, $\mathfrak{S}a \cap Rs \neq \emptyset$.*
(ii) *For $a \in R$, if $as' = 0$ for some $s' \in \mathfrak{S}$, then $sa = 0$ for some $s \in \mathfrak{S}$.*

Moreover, the morphism of rings $\varphi\colon R \longrightarrow \mathfrak{S}^{-1}R$ has the following universal property: for any $\mathfrak{S}$-inverting morphism of rings $\psi\colon R \longrightarrow T$, there exists a unique morphism of rings $f\colon \mathfrak{S}^{-1}R \longrightarrow T$ such that $\psi = f\varphi$.

Sketch of the proof. It is not very difficult to prove that if R has a left Ore ring of fractions $\mathfrak{S}^{-1}R$, then (i) and (ii) hold.

Conversely, suppose that $\mathfrak{S}$ and R satisfy (i) and (ii). We construct a left Ore ring of fractions $\mathfrak{S}^{-1}R$.

We start the construction by working with $\mathfrak{S} \times R$. We define an equivalence relation as follows

$$(s,a) \sim (s',a') \text{ iff there exist } b,b' \in R \text{ such that } bs = b's' \in \mathfrak{S} \text{ and } ba = b'a' \in R.$$

We write $s^{-1}a$ for the equivalence class of (s,a).

Observe that

$$\text{two fractions } s_1^{-1}a_1, s_2^{-1}a_2 \text{ can be brought to a common denominator} \tag{8}$$

From $\mathfrak{S}s_1 \cap Rs_2 \neq \emptyset$, we get elements $r \in R$, $s \in \mathfrak{S}$ such that $ss_1 = rs_2 \in \mathfrak{S}$, so now $s_1^{-1}a_1 = (ss_1)^{-1}sa_1$ and $s_2^{-1}a_2 = (rs_2)^{-1}(ra_2)$.

We can define $s_1^{-1}a_1 + s_2^{-1}a_2 = t^{-1}(sa_1 + ra_2)$ where $t = ss_1 = rs_2$.

The zero element of the commutative group $(\mathfrak{S}^{-1}R, +)$ is $1^{-1}0$.

We define $\varphi\colon R \longrightarrow \mathfrak{S}^{-1}R$ by $\varphi(r) = 1^{-1}r$.

In order to multiply $s_1^{-1}a_1$ with $s_2^{-1}a_2$, we use $Rs_2 \cap \mathfrak{S}a_1 \neq \emptyset$ to find $r \in R$ and $s \in \mathfrak{S}$ such that $rs_2 = sa_1$. Then we define $s_1^{-1}a_1s_2^{-1}a_2 = s_1^{-1}s^{-1}ra_2 = (ss_1)^{-1}ra_2$.

The identity element is $1^{-1}1$.

It can be proved that φ is a morphism of rings with

$$\ker\varphi = \{a \in R \mid (1, a) \sim (1, 0)\} = \{a \in R \mid sa = 0 \text{ for some } s \in \mathfrak{S}\}.$$

For every $s \in \mathfrak{S}$, $\varphi(s)$ is invertible with inverse $s^{-1}1$.

Hence every element of $\mathfrak{S}^{-1}R$ is written as $\varphi(s)^{-1}\varphi(a) = s^{-1}a$. □

Definitions 3.3. Let $\mathfrak{S}$ be a multiplicative set of a ring R.

(a) If $\mathfrak{S}$ satisfies conditions (i) and (ii) of Theorem 3.2, then $\mathfrak{S}$ is called a *left denominator set*.

(b) If $\mathfrak{S}$ satisfies only condition (i) of Theorem 3.2, then $\mathfrak{S}$ is said to be a *left Ore set*. □

An important property of Ore localization is that a finite number of fractions can be brought to a common denominator. More precisely

Remark 3.4. Let R be a ring and $\mathfrak{S}$ a left denominator set. By an iteration of what has been done in (8), given a finite number of elements $s_1^{-1}a_1, \ldots, s_n^{-1}a_n \in \mathfrak{S}^{-1}R$, there exist $s \in \mathfrak{S}$ and $b_1, \ldots, b_n \in R$ such that $s^{-1}b_1 = s_1^{-1}a_1, \ldots, s^{-1}b_n = s_n^{-1}a_n$. □

Most of the situations that we will be interested in, to verify that a certain multiplicative set $\mathfrak{S}$ is a left denominator set, it is enough to show that $\mathfrak{S}$ is a left Ore set. We proceed to state this result which will be used without any further reference. But before that, we need to give a definition.

Definition 3.5. Let R be a ring. A left ideal I of R is called an *annihilator left ideal* if it is the left annihilator of a non-empty set X of R, i.e. $I = \{r \in R \mid rx = 0 \text{ for all } x \in X\}$. □

Lemma 3.6. *Let R be a ring. Let $\mathfrak{S}$ be a multiplicative set of R. If either*

(i) *$\mathfrak{S}$ consists of non-zero-divisors, or*

(ii) *$\mathfrak{S}$ consists of central elements, or*

(iii) *R satisfies the ascending chain condition on annihilator left ideals,*

then $\mathfrak{S}$ is a left denominator set provided $\mathfrak{S}$ is a left Ore set. □

Definition 3.7. Let R be a ring.

(a) Suppose that R is a subring of a ring A. We say that R is a *left order* in A if
   (i) every non-zero-divisor of R is invertible in A, and
   (ii) every element of A has the form $s^{-1}r$, where $a \in R$, and s is a non-zero-divisor of R.

   In this event the set $\mathfrak{S}$ consisting on all non-zero-divisors of R is a left Ore set and $A = \mathfrak{S}^{-1}R$.

(b) The ring R is said to be *left Goldie* if it satisfies:
   (i) the ascending chain condition on annihilator left ideals, and
   (ii) R does not contain an infinite direct sum of non-zero left ideals. □

The following result is known as *Goldie's Theorems*.

THEOREM 3.8. *Let R be a ring.*

(i) *R is a left order in a semisimple ring A if and only if R is a semiprime left Goldie ring.*
(ii) *R is a left order in a simple artinian ring if and only if R is a prime left Goldie ring.* □

Of course there exist right versions of these results and definitions: right Ore ring of fractions, right Ore set, right order, right Goldie ring,...

DEFINITIONS 3.9. (a) A ring R is an *order* in a ring A provided R is a left and a right order in A.
(b) A subset $\mathfrak{S}$ of a ring R is an *Ore set* if it is both a left and a right Ore set.
(c) If a subset $\mathfrak{S}$ of a ring R is a left and a right denominator set, then the universal properties of the left and the right Ore ring of fractions imply that $R\mathfrak{S}^{-1} \cong \mathfrak{S}^{-1}R$ as R-rings. In this case $\mathfrak{S}^{-1}R$ is called the *Ore ring of fractions*. □

Now we present an example that we will need later in Chapter 7. The proof can be found in [**Coh95**, Theorem 2.3.1].

PROPOSITION 3.10. *Let k be a division ring, and let $\alpha\colon k \to k$ be a morphism of rings. Consider the skew polynomial ring $k[x;\alpha]$ and the skew power series ring $k[[x;\alpha]]$. Let $\mathfrak{S} = \{1, x, \dots, x^n, \dots\}$. Then $\mathfrak{S}$ is a left Ore set whose inversion yields the division ring $\mathfrak{S}^{-1}k[[x;\alpha]]$ which contains $k[x;\alpha]$ and $k[[x;\alpha]]$, consisting of all power series of the form*

$$x^{-r}\sum_{n=0}^{\infty} a_n x^n,\ r \geq 0,\ a_n \in k \text{ for each } n \geq 0.$$

Moreover, if α is an automorphism, then every element of $\mathfrak{S}^{-1}k[[x;\alpha]]$ can be expressed as a series $\sum\limits_{n\geq s} b_n x^n$, with $s \in \mathbb{Z}$ and $b_n \in k$ for each n. Therefore $\mathfrak{S}^{-1}k[[x;\alpha]] = k((x;\alpha))$. □

REMARK 3.11. Let k be a division ring, and let $\alpha\colon k \to k$ be a morphism of rings. Consider the division ring $\mathfrak{S}^{-1}k[[x;\alpha]]$. In general, the expression of a series $x^{-r}\sum\limits_{n=0}^{\infty} a_n x^n$ is not unique. Indeed, let $A = x^{-r}\sum\limits_{n=0}^{\infty} a_n x^n$ and $B = x^{-s}\sum\limits_{n=0}^{\infty} b_n x^n$ be series in $\mathfrak{S}^{-1}k[[x;\alpha]]$ with $r \geq s$, then

$$A = B \text{ iff } a_n = 0 \text{ for } 0 \leq n < r-s \text{ and } a_{n+r-s} = \alpha^{r-s}(b_n) \text{ for } n \geq r-s. \tag{9}$$

On the other hand, if α is an isomorphism, then every series in $\mathfrak{S}^{-1}k[[x;\alpha]]$ can be expressed in the form $\sum\limits_{n\geq s} a_n x^n$, and this expression is unique. □

In general, if a ring R is not commutative, given a right R-module M and a subset $\mathfrak{S}$ of R, we cannot talk about the $\mathfrak{S}$-torsion submodule of M. On the other hand, this can be done if $\mathfrak{S}$ happens to be a right Ore subset.

LEMMA 3.12. *Let R be a ring and $\mathfrak{S}$ a multiplicative subset of R. Then*

(i) *$\mathfrak{S}$ is a right Ore set if and only if the set $\mathcal{T}_{\mathfrak{S}}(M) = \{m \in M \mid ms = 0 \text{ for some } s \in \mathfrak{S}\}$ is a submodule for each $M \in$ Mod-R.*
(ii) *$\mathfrak{S}$ is a left Ore set if and only if the set $\mathcal{T}_{\mathfrak{S}}(M) = \{m \in M \mid sm = 0 \text{ for some } s \in \mathfrak{S}\}$ is a submodule for each $M \in R$-Mod.*

Moreover, if (i) *holds, then $\mathcal{T}_{\mathfrak{S}}(M)$ is the trace submodule $\sum\limits_{s\in\mathfrak{S}}\{f(R/sR) \mid f \in \mathrm{Hom}_R(R/sR, M)\}$; and if* (ii) *holds, then $\mathcal{T}_{\mathfrak{S}}(M)$ is the trace submodule $\sum\limits_{s\in\mathfrak{S}}\{f(R/Rs) \mid f \in \mathrm{Hom}_R(R/Rs, M)\}$.*

Proof. We prove (i), then (ii) follows by symmetric arguments. Suppose that $\mathfrak{S}$ is a right Ore set and M a right R-module. Let $m \in \mathcal{T}_{\mathfrak{S}}(M)$. Then there exists $s \in \mathfrak{S}$ such that $ms = 0$. For each $r \in R$, there exist $s' \in \mathfrak{S}$ and $r' \in R$ such that $rs' = sr'$ because $\mathfrak{S}$ is a right Ore set. Hence $(mr)s' = msr' = 0$, i.e. $mr \in \mathcal{T}_{\mathfrak{S}}(M)$. Suppose now that $m_1, m_2 \in \mathcal{T}_{\mathfrak{S}}(M)$. Let $s_1, s_2 \in \mathfrak{S}$ such that $m_1 s_1 = m_2 s_2 = 0$. By the foregoing, $m_2 s_1 \in \mathcal{T}_{\mathfrak{S}}(M)$. Thus there exists $w_2 \in \mathfrak{S}$ such that $m_2 s_1 w_2 = 0$. Notice that $s_1 w_2 \in \mathfrak{S}$. Then

$$(m_1 + m_2)s_1 w_2 = (m_1 s_1)w_2 + (m_2 s_1)w_2 = 0,$$

that is, $m_1 + m_2 \in \mathcal{T}_{\mathfrak{S}}(M)$.

Conversely, suppose that $\mathcal{T}_{\mathfrak{S}}(M)$ is a submodule for each $M \in \text{Mod-}R$. For any $s \in \mathfrak{S}$ and $r \in R$, consider the right R-module $M = R/sR$. Clearly $\bar{1} \in \mathcal{T}_{\mathfrak{S}}(M)$. Hence, $\bar{r} = \bar{1}r \in \mathcal{T}_{\mathfrak{S}}(M)$ because $\mathcal{T}_{\mathfrak{S}}(M)$ is a submodule. Thus there exist $s' \in \mathfrak{S}$ and $r' \in R$ such that $rs' = sr'$.

Now suppose that (i) holds and that M is a right R-module. Then $m \in \mathcal{T}_{\mathfrak{S}}(M)$ iff there exists $s \in \mathfrak{S}$ such that $ms = 0$ iff there exists a morphism of right R-modules $f\colon R/sR \to M$ with $f(\bar{1}) = m$. Now observe that for each $s \in \mathfrak{S}$ and morphism of right R-modules $f\colon R/sR \to M$, then $f(\bar{1}) \in \mathcal{T}_{\mathfrak{S}}(M)$. Thus $f(R/sR) = f(\bar{1})R \subseteq \mathcal{T}_{\mathfrak{S}}(M)$, and then

$$\sum \{f(R/Rs) \mid s \in \mathfrak{S}\} \subseteq \mathcal{T}_{\mathfrak{S}}(M),$$

because $\mathcal{T}_{\mathfrak{S}}(M)$ is a submodule of M. □

Definition 3.13. Let R be a ring, $\mathfrak{S}$ a right (left) Ore subset of R and M a right (left) R-module. The submodule $\mathcal{T}_{\mathfrak{S}}(M)$ is called the *$\mathfrak{S}$-torsion submodule* of M. We say that M is *$\mathfrak{S}$-torsion-free* if $\mathcal{T}_{\mathfrak{S}}(M) = 0$, and that M is *$\mathfrak{S}$-torsion* if $\mathcal{T}_{\mathfrak{S}}(M) = M$.

Suppose that $\mathfrak{S}$ is a right (left) Ore subset that consists of all non-zero-divisors of R. If $\mathcal{T}_{\mathfrak{S}}(M) = 0$, we usually say that M is *torsion-free* instead of $\mathfrak{S}$-torsion-free. If $\mathcal{T}_{\mathfrak{S}}(M) = M$, we usually say that M is *torsion* instead of $\mathfrak{S}$-torsion. □

Some other important properties of Ore localization for us are contained in the following result.

Proposition 3.14. *Let R be a ring and $\mathfrak{S}$ a left denominator set. Then the following statements hold:*

(i) *$\mathfrak{S}^{-1}R$ is a flat right R-module. That is, for every exact sequence of left R-modules $0 \to B \to C \to D \to 0$, then $0 \to \mathfrak{S}^{-1}R \otimes_R B \to \mathfrak{S}^{-1}R \otimes_R C \to \mathfrak{S}^{-1}R \otimes_R D \to 0$ is an exact sequence of left $\mathfrak{S}^{-1}R$-modules.*

(ii) *Let M be a left R-module. Then the kernel of the natural map $\psi\colon M \to \mathfrak{S}^{-1}R \otimes_R M$, $m \mapsto 1 \otimes m$, equals $\mathcal{T}_{\mathfrak{S}}(M)$.*

Sketch of the proof. (i) can be proved by verifying the conditions of a flatness test: the map $\mathfrak{S}^{-1}R \otimes_R I \to \mathfrak{S}^{-1}RI$, defined by $s^{-1}r \otimes y \mapsto s^{-1}ry$, is an isomorphism of abelian groups for each left ideal I of the ring R, see [**Lam99**, Lemma 4.12].

(ii) It can be proved that there exists a "left Ore module of fractions" $\mathfrak{S}^{-1}M$ in the same way as the construction of $\mathfrak{S}^{-1}R$. The elements are of the form $s^{-1}m$ for each $s \in \mathfrak{S}$ and $m \in M$. The kernel of the natural map $\varphi\colon M \to \mathfrak{S}^{-1}M$ is $\mathcal{T}_{\mathfrak{S}}(M)$. It has the following universal property: for each morphism of left R-modules $\psi\colon M \to N$ such that the action of s on N is bijective, there exists a unique morphism of left R-modules $f\colon \mathfrak{S}^{-1}M \to N$ such that $f\varphi = \psi$. Therefore there exists a unique morphism of left R-modules $f\colon \mathfrak{S}^{-1}M \to \mathfrak{S}^{-1}R \otimes_R M$ such that $f\varphi = \psi$. By the universal property of the tensor product, there exists a morphism $g\colon \mathfrak{S}^{-1}R \otimes_R M \to \mathfrak{S}^{-1}M$ Then both compositions of f and g give the identity. Then ψ has kernel $\mathcal{T}_{\mathfrak{S}}(M)$. □

2. Division rings of fractions

In this section we present concepts and examples that will be used throughout these pages.

DEFINITIONS 3.15. Let R be a ring.

(a) By an *R-ring* we understand a ring L with a given morphism of rings $p\colon R \longrightarrow L$. Given another R-ring L' with morphism of rings $p'\colon R \longrightarrow L'$, by a *morphism of R-rings* we understand a morphism of rings $q\colon L \longrightarrow L'$ such that $qp = p'$. When q is an isomorphism (automorphism) we say q is an *R-isomorphism* (*R-automorphism*). If R is a commutative ring and the image of the morphism of rings $p\colon R \longrightarrow L$ is contained in the center of L, we usually say that L is an *R-algebra*.

Suppose that R is embedded in a division ring E, $R \hookrightarrow E$.

(b) We define the *division ring of fractions of R inside E*, denoted $E(R)$, as the R-ring which is the intersection of all division subrings of E that contain (the image of) R. Of course $E(R)$ is a division ring. Notice that different embeddings of R inside E give different division rings of fractions of R inside E.

(c) We say that a division ring D is a *division ring of fractions of R* if there exists an embedding of R inside D such that $D = D(R)$.

(d) Let $R \hookrightarrow D_1$ and $R \hookrightarrow D_2$ be two division rings of fractions of R. We say that they are *isomorphic division rings of fractions* (of R), if D_1 and D_2 are isomorphic as R-rings. $\square$

The following remark is very important. The situation described in it will be most helpful for us through Chapters 5-7.

REMARK 3.16. The division ring of fractions D of R inside E can be constructed from R in the following way. Let S be a set of generators (as a ring) of R. Let X_0 be the empty set and $X_1 = (R \setminus \{0\})^{-1}$. Now suppose that we have defined X_r for all natural numbers $r \leq n$. Then we define $Q_r(R,E)$ to be the subring generated by $S \cup X_r$, and set

$$X_{n+1} = (Q_n(R,E) \setminus Q_{n-1}(R,E))^{-1} \cup X_n.$$

Let $X = \bigcup_{n\geq 0} X_n$. We have defined an ascending chain of subsets $(X_n)_{n\in\mathbb{N}}$ and of subrings $(Q_n(R,E))_{n\in\mathbb{N}}$ of E. Hence $\bigcup_{n\geq 0} Q_n(R,E)$ is a subring of E. Moreover, it contains R and every nonzero element of it is invertible. Thus $D \subseteq \bigcup_{n\geq 0} Q_n(R,E)$. Notice that for each $n \geq 0$, $Q_n(R,E)$ is contained in the subring of E generated by S and X, and that $X_n \subseteq D$ for each n. Thus $D = \bigcup_{n\geq 0} Q_n(R,E)$ and D equals the subring of E generated by S and X.

Notice that we could also have defined $Q_0(R,E) = R$, and for each $n \geq 0$

$$Q_{n+1}(R,E) = \text{subring of } E \text{ generated by } \{s^{-1}, r \mid r, s \in Q_n(R,E),\ s \neq 0\}$$

as it is done in [**Fis71**]. Being this last definition easier, we are inspired by the first one to produce Section 4 in Chapter 5.

In summary the elements of D can be built up from elements of R (or of S) in stages using addition, subtraction, multiplication and division by nonzero elements, and for each $n \geq 0$, $Q_n(R,E)$ is the set of elements of D which can be obtained using at most n nested inversions. $\square$

In general, as we have explained in the Introduction, unlike in the commutative case, a domain may not have a division ring of fractions or may have more than one (see Chapter 7). However the Ore situation is very similar to the commutative one.

Definition 3.17. Let R be a ring. When R is a domain and $\mathfrak{S} \setminus \{0\}$, then the left Ore condition can be re-expressed in the equivalent form: $Rr_1 \cap Rr_2 \neq (0)$ for any $r_1, r_2 \in R \setminus \{0\}$. If $\mathfrak{S}$ satisfies the left Ore condition, we say that R is a *left Ore domain*. The left Ore ring of fractions $\mathfrak{S}^{-1}R$ is usually denoted by $Q^l_{\mathrm{cl}}(R)$. Observe that $Q^l_{\mathrm{cl}}(R)$ is a division ring because the inverse of $s^{-1}r$ is $r^{-1}s$ for each $s, r \in R \setminus \{0\}$. So we usually call $Q^l_{\mathrm{cl}}(R)$ the *left Ore division ring of fractions* of R. □

Now the universal property of $Q^l_{\mathrm{cl}}(R)$ implies that a (left/right) Ore domain has a unique division ring of fractions.

Theorem 3.18. *Let R be a (left) Ore domain. Then $\varphi \colon R \hookrightarrow Q^l_{cl}(R)$ is a division ring of fractions of R. Furthermore, for every embedding $\psi \colon R \hookrightarrow E$ of R in a division ring E, there exists a unique monomorphism $f \colon Q^l_{cl}(R) \to E$ such that $f\varphi = \psi$. Therefore we can suppose that $Q^l_{cl}(R)$ is embedded in every division ring which contains R.* □

Observe that if R is a left Ore domain and $D = Q^l_{\mathrm{cl}}(R)$, then, under the notation of Remark 3.16, $D = Q_1(R, D) = \bigcup_{n \geq 0} Q_n(R, D)$.

Some (left) Ore division rings of fractions we will deal with are the following ones. For a proof see for example [**Lam99**] and [**GW89**].

Proposition 3.19. *The following statements hold:*

(i) *If R is a left (right) noetherian domain, then R is a left (right) Ore domain.*

(ii) *Let k be a division ring and $\alpha \colon k \to k$ be a ring endomorphism. Then $k[x; \alpha]$ is a principal left ideal domain. Thus $k[x; \alpha]$ is a left Ore domain with left Ore division ring of fractions denoted by $k(x; \alpha)$. Moreover, if α is an automorphism, then $k[x; \alpha]$ and $k[x, x^{-1}; \alpha]$ are principal right and left ideal domains and have as Ore division ring of fractions $k(x; \alpha)$.*

(iii) *Let R be a domain and $\alpha \colon R \to R$ be a ring endomorphism. If R is a left Ore domain, then so is $R[x; \alpha]$. Indeed, if D is the left Ore division ring of fractions of R, then $D(x; \alpha)$ is the left Ore division ring of fractions of $R[x; \alpha]$. Moreover, if α is an automorphism, then $R[x, x^{-1}; \alpha]$ is a left Ore domain.* □

3. Matrix Localization

In this section we present the concepts and results on matrix localization that we will need in the forthcoming chapters. Most of the proofs are omitted. They can be found in [**Coh95**] and [**Coh85**], where the concepts in this section are discussed in full detail.

3.1. Epic R-division rings. We now study the morphisms of rings from a given ring to division rings.

Definitions 3.20. Let R be a ring, and let $p \colon R \to F$ be an R-ring.

(a) If F is a division ring, we say that F is an *R-division ring*.

(b) If $p \colon R \to F$ is a division ring of fractions of $\operatorname{im} p$, we say that F is an *epic R-division ring*. □

If R is commutative, epic R-division rings $p \colon R \to L$ are determined up to R-isomorphism by the kernel of p. More precisely,

Remark 3.21. Let R be a commutative ring. The map

$$\begin{aligned} \{\text{Prime ideals of } R\} &\longrightarrow \{R\text{-isomorphism classes of epic } R\text{-division rings}\} \\ \mathfrak{p} &\longmapsto Q(R/\mathfrak{p}) \end{aligned}$$

is bijective, where $Q(R/\mathfrak{p})$ is the field of fractions of $R/\mathfrak{p}$ and has the natural R-ring structure given by $R \longrightarrow R/\mathfrak{p} \hookrightarrow Q(R/\mathfrak{p})$. □

Let R be a ring and F an epic R-division ring with $p\colon R \to F$. In the noncommutative case, the kernel of the morphism p is not sufficient to describe R-division rings. For example, in Chapter 7 we present non-isomorphic division rings of fractions of the free algebra (therefore $\ker p = 0$).

If R is commutative, the epic R-division ring F can be also constructed from the localization $R_{\mathfrak{p}}$, where $\mathfrak{p} = \ker p$, $p\colon R \to F$. Since the image of the elements of $R \setminus \mathfrak{p}$ by p are invertible, we get an R-morphism of rings $R_{\mathfrak{p}} \to F$ whose kernel is $\mathfrak{p}R_{\mathfrak{p}}$. Hence, $F \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ as R-division rings. It is this second way of constructing epic R-division rings the one that generalizes to the noncommutative setting, but, considering elements which map to zero is not enough, the matrices which become singular must be taken into account. What follows tries to explain this last assertion in more detail.

We begin with the construction of the localization of a ring at a certain set of matrices.

Definitions 3.22. Let R be a ring. Let Σ be a set of matrices over R.

(a) A morphism of rings $f\colon R \to R'$ is called *Σ-inverting* if, for each $A = (a_{ij}) \in \Sigma$, $fA = (f(a_{ij}))$ is an invertible matrix over R'. Notice that for morphisms to division rings only square matrices will play a role.
(b) A Σ-inverting morphism of rings $f\colon R \to R'$ is *universal Σ-inverting* if, for each Σ-inverting morphism $\psi\colon R \to S$, there exists a unique morphism of rings $\bar{\psi}\colon R' \to S$ such that $\bar{\psi} f = \psi$. □

Theorem 3.23. *Let R be a ring. Let Σ be a set of matrices over R. Then there exists a ring R_Σ and a universal Σ-inverting morphism of rings $\lambda\colon R \to R_\Sigma$.*

Proof. For every $m \times n$ matrix $A = (a_{ij}) \in \Sigma$, we choose mn symbols a'_{ji} which we adjoin to R, with defining relations the ones given by entries of the matrices equalities

$$AA' = I_m, \quad A'A = I_n,$$

where $A' = (a'_{ji})$, and I_m, I_n denote the identity matrices of orders m and n respectively. We denote the resulting ring by R_Σ. Notice that, by construction, the natural morphism of rings $\lambda\colon R \to R_\Sigma$ is Σ-inverting and the inverse of λA is A'. Given any Σ-inverting morphism of rings $f\colon R \to R'$, we define $f'\colon R_\Sigma \to R'$ by mapping λa (for $a \in R$) to $f(a)$ and a'_{ji} to the (j,i)-th entry of the inverse of fA. Any relation in R_Σ is a consequence of the relations in R and the relations expressing that A' is the inverse of λA. All these relations hold in R', so f' is well-defined and it is a morphism of rings. It is unique because its values on λR are prescribed, as well as on $(\lambda A)^{-1}$, by the uniqueness of inverses. □

Definition 3.24. Let R be a ring. Let Σ be a subset of matrices over R. Then the ring R_Σ constructed in Theorem 3.23 is called the *universal Σ-inverting ring* or the *localization of R at Σ*. □

From the localization of a ring R at certain sets of matrices it is possible to obtain all epic R-division rings.

Definition 3.25. Let R be a ring. Given an R-division ring F, with map $p\colon R \to F$, by the *prime matrix ideal* of F (or of p), we understand the collection of all square matrices over R, of all orders, which map to non-invertible matrices over F. If $\mathcal{P}$ is the set of all such matrices, then we can define a localization $R_{\mathcal{P}}$, analogous to $R_{\mathfrak{p}}$ in the commutative case. Let Σ be the complement of $\mathcal{P}$ in the set of all square matrices over R. Thus Σ consists of all square matrices over R which become invertible over F. Then the universal Σ-inverting ring R_Σ is usually written $R_{\mathcal{P}}$, just as we write $R_{\mathfrak{p}}$ in the commutative case. $\square$

The construction of an epic R-division ring can be described in terms of its prime matrix ideal.

THEOREM 3.26. *Let R be any ring and F an epic R-division ring with prime matrix ideal $\mathcal{P}$. Then the localization $R_{\mathcal{P}}$ is a local ring with residue class division ring F.* $\square$

So we have as many epic R-division rings as prime matrix ideals, analogously as in the commutative case we have as many epic R-division rings as prime ideals.

What matrices are the candidates to become invertible in a division ring? Full matrices.

DEFINITION 3.27. Let R be a ring.

(a) Let A be an $n \times n$ square matrix over R. Consider the different ways of writing A as a product
$$A = PQ, \text{where } P \text{ is } n \times r \text{ and } Q \text{ is } r \times n \tag{10}$$
for varying r. If in every representation of A we have $r \geq n$, A is said to be *full.* Observe that, if R is a division ring, then a matrix A is full if and only if A is invertible.

(b) Two square matrices A, A' over R of the same size are *associated* if $A' = PAQ$ with P, Q invertible over R. Notice that if A, A' are associated, then A is full if and only if A' is full.

(c) Let A be an $n \times n$ square matrix over R. We say that A is *hollow* if it has a zero submatrix of size $r \times s$ with $r + s > n$. $\square$

The set of full matrices is the biggest set of matrices we can try to invert in a division ring. For some rings R, it is possible to find an epic R-division ring such that all full matrices become invertible, see Section 3.2. In fact, this epic R-division ring will be a division ring of fractions of R, for note that any nonzero element of R is a 1×1 full matrix. On the other hand, in general, not all full matrices over a ring R become invertible over an epic R-division ring, see Example 3.34. Furthermore, this happens for many rings as the following useful remark shows. It is taken from [**Lew74**].

REMARK 3.28. Let R be a ring that has a non-free finitely generated projective module P. Then R has a full matrix which is not invertible in any division ring which R embeds.

PROOF. Let M be a free module of least rank n such that $M = P_1 \oplus P \cong R^n$. Let A be the $n \times n$ matrix which gives the projection over P. Then A is an idempotent, not zero and not the identity. Hence A is not invertible in any division ring in which A embeds. Now we prove that A is full. Suppose that A is not full, then

$$\begin{tikzcd} R^n \arrow[rr, "A"] \arrow[dr] & & R^n \\ & R^m \arrow[ur, "C"'] & \end{tikzcd}$$

with $m < n$. Let N be the image of the matrix C. Then $P \subseteq N$ and P is a direct summand of N since it is of R^n. So we have $R^m \xrightarrow{C} N \to P \to 0$. Hence P is a direct summand of R^m, contradicting the minimality of n. $\square$

Hollow matrices are an important set of non-full matrices.

LEMMA 3.29. *Let R be a ring and A an $n \times n$ hollow matrix over R. Then A is not full.*

Proof. Suppose that A has a zero submatrix of size $r \times s$ with $r+s>n$. Since any two associated matrices are simultaneously full or not, we can suppose

$$A = \begin{array}{cc} & \begin{array}{cc} n-s & s \end{array} \\ & \begin{pmatrix} P & 0 \\ Q & S \end{pmatrix} \begin{array}{l} r \\ n-r \end{array} \end{array}$$

Then

$$A = \begin{pmatrix} P & 0 \\ Q & S \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} I_{n-s} & 0 \\ Q & S \end{pmatrix}$$

where the matrices are $n \times (n-s+n-r)$ and $(n-s+n-r) \times n$. Therefore, if $r+s>n$, we get $n-s+n-r<n$. ☐

Let R be a ring. The epic R-division rings can be made into a category. To take morphisms of R-rings as morphisms would be too restrictive, as all maps would then be isomorphisms. To see that let $f\colon F_1 \to F_2$ be a morphism of R-rings between epic R-division rings. Since F_1 is a division ring and $\ker f$ is a proper ideal of F_1, f is injective. Then f is onto because $f(F_1)$ is a subdivision ring of F_2 which contains the image of R and F_2 is an epic R-division ring.

A workable notion of morphism in this category is that of specialization.

Definitions 3.30. Let R be a ring and F, L, T be epic R-division rings.

(a) A *local morphism* between F, L is a morphism of R-rings $f\colon F_0 \to L$, whose domain F_0 is an R-subring of F, which maps non-units to non-units. Since L is a division ring, this means that the non-units in F_0 form an ideal ($= \ker f$). Hence F_0 is a local ring. Moreover, since L is an epic R-division ring, then $F_0/\ker f \cong L$, because the image of f is a division ring isomorphic to $F_0/\ker f$ that contains R.
(b) Suppose that $f\colon F \to L$ and $g\colon L \to T$ are local morphisms with domains F_0 and L_0 respectively. Let $F_0' = f^{-1}(L_0)$. We define the composition gf as the local morphism obtained by the restriction $gf\colon F_0' \to T$.
(c) Two local morphisms between F and L are said to be *equivalent* if there is a subring of F on which both are defined, and on which they agree and again define a local morphism. It is easy to see that it is an equivalence relation.
(d) An equivalence class of local morphisms between epic R-division rings is called a *specialization*. In this way we obtain for each ring R, a category with epic R-division rings as objects and specializations as morphisms. ☐

Example 3.31. Let R be a commutative ring. Let F_1 and F_2 be epic R-division rings. By Lemma 3.21, there exist $\mathfrak{p}_1, \mathfrak{p}_2$ prime ideals of R such that $\mathfrak{p}_1 = \ker p_1$, $\mathfrak{p}_2 = \ker p_2$ and $F_1 \cong R_{\mathfrak{p}_1}/\mathfrak{p}_1 R_{\mathfrak{p}_1}$, $F_2 \cong R_{\mathfrak{p}_2}/\mathfrak{p}_2 R_{\mathfrak{p}_2}$. Then there exists a specialization $f\colon F_1 \to F_2$ if and only if $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. ☐

We have just seen that, in the commutative case, a specialization between epic R-division rings can be characterized in terms of the corresponding prime ideals, i.e. kernels. In general, a specialization between epic R-division rings can be characterized in terms of the corresponding prime matrix ideals:

Theorem 3.32. *Let R be a ring. Let F_1, F_2 be epic R-division rings with prime matrix ideals $\mathcal{P}_1, \mathcal{P}_2$ and corresponding localizations $R_{\mathcal{P}_1}$, $R_{\mathcal{P}_2}$. Then the following conditions are equivalent:*

(i) *There is a specialization $\beta\colon F_1 \to F_2$.*
(ii) $\mathcal{P}_1 \subseteq \mathcal{P}_2$.
(iii) *there is a morphism of R-rings $R_{\mathcal{P}_2} \to R_{\mathcal{P}_1}$.*

Further, if there are specializations $F_1 \to F_2$ and $F_2 \to F_1$, then F_1 and F_2 are R-isomorphic. □

DEFINITIONS 3.33. Let R be a ring.

(a) A *universal R-division ring* is an epic R-division ring F such that for every other epic R-division ring T there exists a specialization $F \to T$. That is, F is an initial object in the category of epic R-division rings. By Theorem 3.32, it means that R has a least prime matrix ideal.

(b) If F is a universal R-division ring such that $R \hookrightarrow F$, we say that F is the *universal division ring of fractions* of R. Notice that if R has a universal R-division ring F and a division ring of fractions T, then F is a universal R-division ring of fractions. This is because the specialization from F to T is a morphism of R-rings. □

EXAMPLE 3.34. (a) Let R be a commutative ring. Suppose that R has a least prime ideal $\mathfrak{p}$. Then $Q(R/\mathfrak{p})$, the field of fractions of $R/\mathfrak{p}$, is a universal R-division ring by Example 3.31. In particular, if R is a commutative domain, the field of fractions of R is the universal R-division ring of fractions.

Let $R = k[x,y,z]$. The field of fractions $F = k(x,y,z)$ is the universal R-division ring of fractions. It can be proved that the matrix $A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$ is full over R, but it is not invertible over F because $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

(b) More generally, let R be a left Ore domain. If R has a universal division ring of fractions, then it is $Q^l_{\mathrm{cl}}(R)$. Indeed, if there is a specialization from an epic R-division ring D to $Q^l_{\mathrm{cl}}(R)$, then R embeds in D since R embeds in $Q^l_{\mathrm{cl}}(R)$. Then the fact that D is an epic R-division ring and the universal property of Ore localization implies that D is R-isomorphic to $Q^l_{\mathrm{cl}}(R)$. Therefore the prime matrix ideal of $Q^l_{\mathrm{cl}}(R)$ is a minimal prime matrix ideal. □

The following result will be useful for us in Chapter 6. The statement is a slight generalization of [**LL78**, Lemma 1] while the proof remains the same.

LEMMA 3.35. *Let R be a ring. Let F and L be epic R-division rings, and ρ an R-specialization from F to L. Suppose that S is a subring of F contained in the domain of ρ. If $L(\rho(S))$ is an S-division ring of fractions with a minimal prime matrix ideal, then $F(S)$ is an S-division ring of fractions contained in the domain of ρ, and so ρ maps $F(S)$ isomorphically onto $L(\rho(S))$.*

PROOF. Let F_0 be the domain of ρ. Then F and L are F_0-division rings, via the inclusion $F_0 \hookrightarrow F$ and via $\rho\colon F_0 \to L$, and ρ is an F_0-specialization. Let Σ be the set of matrices over S that become invertible over $L(\rho(S))$. Observe that each matrix of Σ is invertible over F_0 because it is invertible over its residue class division ring, $F_0/\ker\rho \cong L$. Thus they are invertible over F. The matrices of Σ are also invertible over $F(S)$ because, when considered as endomorphisms of $F(S)$-vectorial spaces of finite dimension, they are injective. Therefore $F(S)$ is R-isomorphic to $L(\rho(S))$ by Theorem 3.32 and the minimality of the prime matrix ideal of $L(\rho(S))$. Note that there exists a morphism of S-rings $f\colon S_\Sigma \to F_0$ whose image is contained in $F(S)$ and that composed with ρ has image $L(\rho(S))$. Which implies that there is an onto R-morphism of rings from a subring of $F(S)$ to $L(\rho(S))$. Therefore, the image of f is exactly $F(S)$, i.e. $F(S)$ is contained in F_0. □

3.2. Firs, Semifirs and Sylvester domains. In this section we briefly talk about some classes of rings with a universal division ring of fractions. They are the classes of firs and semifirs, introduced by P.M. Cohn [**Coh64**], and Sylvester domains, introduced by W. Dicks and E. Sontag [**DS78**].

DEFINITIONS 3.36. Let R be a non-zero ring.

(a) R is a *semifir* if every finitely generated left ideal is free, of unique rank. This condition is known to be left-right symmetric.

(b) R is a *left fir* if every left ideal is free, of unique rank. *Right fir* is defined similarly. R is a *fir* if it is left and right fir. This condition is not left-right symmetric. Observe that clearly a left fir is a semifir. □

EXAMPLES 3.37. Some trivial, but important examples of firs are the following:

(a) A division ring is a fir.

(b) A principal left (right) ideal domain is a left (right) fir. In particular, if k is a division ring and α is an automorphism of k, the skew Laurent polynomial ring $k[x, x^{-1}; \alpha]$ is a fir. □

Other important examples, constructed from these ones, will be given in Example 3.43 and Theorem 4.22.

DEFINITIONS 3.38. Let R be a ring.

(a) A relation of r terms
$$a \cdot b = a_1 b_1 + \ldots + a_r b_r = 0, \tag{11}$$
$a_1, b_1, \ldots, a_r, b_r \in R$, is said *trivial* if for each $i = 1, \ldots, r$, either $a_i = 0$ of $b_i = 0$. If there is an invertible $r \times r$ matrix P over R such that $aP^{-1} \cdot Pb = 0$ is a trivial relation, then (11) is said *trivializable*.

(b) More generally, a matrix product
$$AB = 0, \tag{12}$$
where A is $m \times r$ and B is $r \times n$ is *trivializable* if there is an invertible $r \times r$ matrix P over R which trivializes (12), i.e. such that for each $i = 1, \ldots, r$ either the i-th column of AP^{-1} or the i-th row of PB is zero.

(c) Let A be an $m \times n$ matrix over R. Consider all factorizations
$$A = PQ, \text{ where } P \text{ is } m \times r \text{ and } Q \text{ is } r \times n.$$
When r has the least possible value, r is called the *inner rank* of A, written $\mathrm{r}\,A$. Observe that if A is an $s \times s$ full matrix, then $\mathrm{r}\,A = s$.

(d) If R is a non-zero ring, we say that R is a *Sylvester domain* if for any matrices A and B over R such that the number of columns of A equals the number of rows of B, equal to n, say, the following condition holds:
$$AB = 0 \implies \mathrm{r}\,A + \mathrm{r}\,B \leq n.$$

(e) Two square matrices A, A' are *stably associated* if $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A' & 0 \\ 0 & I' \end{smallmatrix}\right)$ are associated for some identity matrices I, I' of appropiate size. □

The following result relates all the foregoing concepts.

PROPOSITION 3.39. *Let R be a non-zero ring. R is a semifir if and only if for every r, every relation as* (11) *of r terms is trivializable. Moreover, the following statements hold:*

(i) *If R is a semifir, then every matrix product $AB = 0$ as* (12) *is trivializable for every r.*

(ii) *If R is a semifir, then R is a Sylvester domain.* □

Now we state one of the main results of this section.

THEOREM 3.40. *Let R be a ring. The following conditions are equivalent:*

(i) *R is a Sylvester domain.*

(ii) *If Φ denotes the set of all full matrices over R, then R_Φ, the localization of R at Φ, is a division ring.*

In this event R_Φ is the universal division ring of fractions of R.

In particular, for any fir or any semifir R, the localization R_Φ at the set Φ of all full matrices is the universal division ring of fractions for R, and all full matrices become invertible via $R \to R_\Phi$. □

DEFINITION 3.41. Let k be a division ring. Let $(R_i)_{i\in I}$ be a family of k-rings. By $\underset{k}{*} R_i$ we mean the *ring coproduct* of the family $(R_i)_{i\in I}$ over k. It is a k-ring with a morphism of k-rings $u_i\colon R_i \to \underset{k}{*} R_i$ for every i such that if R' is a k-ring with morphism of k-rings $v_i\colon R_i \to R'$ for every i, then there exists a unique morphism of k-rings $f\colon \underset{k}{*} R_i \to R'$ such that $fu_i = v_i$ for all $i \in I$. □

The following result is our key to construct the firs and semifirs we are interested in. The part corresponding to Sylvester domains can be found in [**DS78**].

THEOREM 3.42. *Let k be a division ring. Let $(R_i)_{i\in I}$ be a family of k-rings. The following statements hold:*

(i) *If $\{1\} \cup S_i$ is a left k-basis of R_i, then the monomials on $\underset{i\in I}{\cup} S_i$, such that no two successive letters of which are in the same factor, form together with 1, a left k-basis for the coproduct $\underset{k}{*} R_i$.*

(ii) *Suppose that R_i is a fir (semifir, Sylvester domain) for each $i \in I$. Then $\underset{k}{*} R_i$ is a fir (semifir, Sylvester domain).* □

EXAMPLE 3.43. Let k be a division ring, and let $(D_i)_{i\in I}$ be a family of k-division rings. By Examples 3.37, each D_i is a fir. Hence $\underset{k}{*} D_i$ is a fir by Theorem 3.42. Therefore $\underset{k}{*} D_i$ has a universal division ring of fractions by Theorem 3.40. □

DEFINITION 3.44. Let k be a division ring. Let $(D_i)_{i\in I}$ be a family of k-division rings. The universal division ring of fractions of $\underset{k}{*} D_i$, denoted by $\underset{k}{\circ} D_i$, will be called *the division ring coproduct* of $\{D_i\}_{i\in I}$. □

Observe that for each division ring of fractions D in which all D_i embed, there exists a specialization from $\underset{k}{\circ} D_i$ to the epic $\underset{k}{*} D_i$-division ring inside D.

4. Universal Localization

In this section we present the results on universal localization that will be needed in Chapter 8. Most of the proofs and details are omitted. The reader is referred to [**Sch85**] from where we have taken the results. We begin by fixing some notation.

NOTATION 3.45. Let R be a ring. Let P and Q be projective right R-modules. Let Σ be a class of morphisms between finitely generated projective right R-modules.

(a) By $\mathcal{P}_R$ (respectively ${}_R\mathcal{P}$) we denote the category of all finitely generated projective right (left) R-modules.
(b) By P^* we denote the projective left R-module $\mathrm{Hom}_R(P, R)$.
(c) If $\alpha \in \mathrm{Hom}_R(P, Q)$, we denote by α^* the morphism of left R-modules $\alpha^*\colon Q^* \to P^*$ defined by $\gamma \mapsto \gamma\alpha$.
(d) By Σ^* we will denote the class $\{\alpha^* \mid \alpha \in \Sigma\}$. □

Definition 3.46. Let R be a ring. Let Σ be a class of morphisms between finitely generated projective right R-modules. The *universal localization of R at Σ* is a ring R_Σ with a morphism of rings $\lambda\colon R \to R_\Sigma$ such that:

(a) λ is *Σ-inverting,* i.e. if $\alpha\colon P \to Q,\ \alpha \in \Sigma$, then $\alpha \otimes_R 1_{R_\Sigma}\colon P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism of right R_Σ-modules.
(b) λ is *universal Σ-inverting*, i.e. if S is a ring such that there exists a Σ-inverting morphism $\psi\colon R \to S$, then there exists a unique morphism of rings $\bar\psi\colon R_\Sigma \to S$ such that $\bar\psi\lambda = \psi$.

In the same way can be defined the universal localization at a class of morphisms between finitely generated projective left R-modules. □

Remarks 3.47. Let R be a ring, and let Σ be a class of morphisms between finitely generated projective right R-modules.

(a) The universal localization of R at Σ, $\lambda\colon R \to R_\Sigma$ is unique up to isomorphism of R-rings because λ is universal Σ-inverting, i.e. if $\lambda_i\colon R \to S_i,\ i = 1, 2$, are universal localizations of R at Σ, there exists a unique isomorphism of rings $\varphi\colon S_1 \to S_2$ such that $\varphi\lambda_1 = \lambda_2$.
(b) R_{Σ^*} is isomorphic to R_Σ as R-rings. Notice that applying Lemma 1.9 and the Hom-tensor adjunction we get the following natural isomorphisms for every $Q \in \mathcal{P}_R$

$$R_\Sigma \otimes_R Q^* \cong \operatorname{Hom}_R(Q, R_\Sigma) \cong \operatorname{Hom}_R(Q, \operatorname{Hom}_{R_\Sigma}(R_\Sigma, R_\Sigma)) \cong \\ \cong \operatorname{Hom}_{R_\Sigma}(Q \otimes_R R_\Sigma, R_\Sigma) \cong (Q \otimes_R R_\Sigma)^*.$$

Hence $\alpha \otimes 1_{R_\Sigma}$ is invertible if and only if $1_{R_\Sigma} \otimes \alpha^*$ is invertible if and only if $(\alpha \otimes 1_{R_\Sigma})^*$ is invertible for each $\alpha \in \Sigma$. □

Examples 3.48. Let R be a ring.

(a) If $\mathfrak{S} \subset R$ is a left denominator set, then the left Ore localization $\mathfrak{S}^{-1}R$ is the universal localization of R at all the morphisms of right R-modules $\alpha_s\colon R \to R$, defined by $r \mapsto sr$, for each $s \in \mathfrak{S}$. Equivalently, $\mathfrak{S}^{-1}R$ is the universal localization of R at the morphisms of left R-modules α_s^* for each $s \in \mathfrak{S}$ given by right multiplication by s.

First notice that $1_{\mathfrak{S}^{-1}R} \otimes \alpha_s\colon R \otimes_R \mathfrak{S}^{-1}R \cong \mathfrak{S}^{-1}R \longrightarrow R \otimes_R \mathfrak{S}^{-1}R \cong \mathfrak{S}^{-1}R$, $b^{-1}a \mapsto sb^{-1}a$ is invertible since s is invertible in $\mathfrak{S}^{-1}R$.

Let $\psi\colon R \to S$ be such that the morphism of right S-modules $\alpha_s \otimes 1_S\colon R \otimes_R S \cong S \to S$, $x \to sx$ is invertible for each $s \in \mathfrak{S}$. Let t be the image of 1 by the inverse of $\alpha_s \otimes 1_S$. Then $st = ts = 1$. That is, the image of s in S is invertible. Now there exists a unique morphism of rings $\bar\psi\colon \mathfrak{S}^{-1}R \to S$ such that $\psi = \bar\psi\lambda$ by the universal property of Ore localizations.

Since α_s^* is given by right multiplication by s for each $s \in \mathfrak{S}$, an analogous argument as before shows that $\mathfrak{S}^{-1}R$ is the universal localization of R at the morphisms of left R-module α_s^* for each $s \in \mathfrak{S}$.

In the same way, if $\mathfrak{S}$ is a right denominator set, then $R\mathfrak{S}^{-1}$ is the universal localization of R at the maps α_s, for each $s \in \mathfrak{S}$ and $R\mathfrak{S}^{-1}$ is the universal localization at the maps α_s^* for each $s \in \mathfrak{S}$.
(b) If Σ is a set of matrices over R, then Σ can be seen as a set of morphisms between finitely generated free right (left) R-modules. Then the localization of R at these morphisms coincides with the localization of R at the set of matrices Σ by Theorem 3.23. □

Definitions 3.49. Let R be a ring and $\alpha\colon P \to Q$, $\alpha'\colon P' \to Q'$ be two morphisms of right R-modules.

(a) The morphisms α and α' are said to be *associated* if there is a commutative square of morphisms of right R-modules

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & Q \arrow[d] \\ P' \arrow[r, "\alpha'"] & Q' \end{tikzcd}$$

where the vertical maps are isomorphisms. Observe that this defines an equivalence relation among the morphisms of right R-modules.

(b) Suppose that $P, Q \in \mathcal{P}_R$. Let $E = E^2 \in \mathbb{M}_n(R)$ and $F = F^2 \in \mathbb{M}_m(R)$ be such that $P \cong ER^n$ and $Q \cong FR^m$. Since $\operatorname{Hom}_R(ER^n, FR^m) \cong FM_{m\times n}(R)E$, there exists a unique matrix $A \in \mathbb{M}_{m\times n}(R)$ such that the following diagram of morphisms of right R-modules is commutative

$$\begin{tikzcd} P \arrow[r, "\alpha"] & Q \arrow[d, "\cong"] \\ ER^n \arrow[u, "\cong"] \arrow[r, dashed, "A"] & FR^m \end{tikzcd}$$

and $FAE = A$. We say that (E, A, F) *represents* α or that (E, A, F) is a *representative* of α □

THEOREM 3.50. *Let R be a ring, and let Σ be a class of morphisms between finitely generated projective right R-modules. Then the universal localization of R at Σ exists.*

PROOF. We claim that we may suppose that Σ is a set. For each $\alpha \in \Sigma$, let $(E_\alpha, A_\alpha, F_\alpha)$ be a representative of α. Observe that the different representatives form a set since they are elements of $\mathbb{M}_n(R) \times \mathbb{M}_{m\times n}(R) \times \mathbb{M}_m(R)$, $m, n \in \mathbb{N}$. Moreover, let $R \to S$ be a Σ-inverting morphism of rings. Then, the commutativity of the diagram

$$\begin{tikzcd} P \otimes_R S \arrow[r, "\alpha\otimes 1_S"] & Q \otimes_R S \arrow[d, "\cong"] \\ E_\alpha R^n \otimes_R S \arrow[u, "\cong"] & F_\alpha R^m \otimes_R S \arrow[d, "\cong"] \\ E_\alpha S^n \arrow[u, "\cong"] \arrow[r, dashed, "A_\alpha"] & F_\alpha S^m \end{tikzcd} \tag{13}$$

for each $\alpha \in \Sigma$, implies that $\alpha \otimes 1_S$ is invertible if and only if there exists a (unique) matrix $B_\alpha \in \mathbb{M}_{m\times n}(S)$ such that $B_\alpha A_\alpha = E$, $A_\alpha B_\alpha = F$ and $EB_\alpha F = B_\alpha$. This proves our claim.

Suppose that Σ is a set. For each $\alpha \in \Sigma$, let $(E_\alpha, A_\alpha, F_\alpha)$ be a representative of α. For every $m_\alpha \times n_\alpha$ matrix $A_\alpha = (a^\alpha_{ij})$, we choose mn symbols b^α_{ji} which we adjoin to R, with defining relations the ones given by the entries of the matrix equalities

$$B_\alpha A_\alpha = E_\alpha, \quad A_\alpha B_\alpha = F_\alpha, \quad E_\alpha B_\alpha F_\alpha = B_\alpha \tag{14}$$

where $B_\alpha = (b^\alpha_{ji})$. We denote the resulting ring by R_Σ. Notice that, by construction, the natural morphism of rings $\lambda\colon R \to R_\Sigma$ is Σ-inverting and $(F_\alpha, B_\alpha, E_\alpha)$ represents the inverse of α for each $\alpha \in \Sigma$. Let $\psi\colon R \to R'$ be a Σ-inverting morphism of rings. For each $\alpha \in \Sigma$, by (13), there exists a unique $m \times n$ matrix C_α over R' such that $C_\alpha A_\alpha = E_\alpha$, $A_\alpha C_\alpha = F_\alpha$, $E_\alpha C_\alpha F_\alpha = C_\alpha$. We define $\bar\psi\colon R_\Sigma \to R'$ by mapping λa (for $a \in R$) to $\psi(a)$ and b^α_{ji} to the (j, i)-th entry of C_α. Any relation in R_Σ is a consequence of the relations in R and the relations in (14). All these relations hold in R', so $\bar\psi$ is well-defined and it is a morphism of rings. It is unique because the values on R are prescribed, as well as the ones on b^α_{ji} by the uniqueness of C_α. $\square$

It follows from the proof of Theorem 3.50:

Remark 3.51. Let R be a ring and Σ a class of morphisms between finitely generated projective right R-modules. Let $\lambda\colon R \to R_\Sigma$ be the universal localization of R at Σ. Let $\psi\colon R \to S$ be a morphism of rings. Let $(E_\alpha, A_\alpha, F_\alpha)$ be a representative of α for each $\alpha \in \Sigma$. Then there exists a (unique) morphism of rings $\bar\eta\colon R_\Sigma \to S$ such that $\bar\eta\lambda = \eta$ if and only if there exists a (unique) matrix B_α over S such that $B_\alpha A_\alpha = E_\alpha,\ A_\alpha B_\alpha = F_\alpha,\ E_\alpha B_\alpha F_\alpha = B_\alpha$ for each $\alpha \in \Sigma$. $\square$

Definition 3.52. Let R be a ring. If there are two right R-modules Z, T such that $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1_Z\end{smallmatrix}\right)$ is associated with $\left(\begin{smallmatrix}\alpha' & 0\\ 0 & 1_T\end{smallmatrix}\right)$, then α and α' are said to be *stably associated*, that is, if there exists a commutative square where the vertical maps are isomorphisms

$$\begin{array}{ccc} P \oplus Z & \xrightarrow{\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1_Z\end{smallmatrix}\right)} & Q \oplus Z \\ \downarrow & & \downarrow \\ P' \oplus T & \xrightarrow{\left(\begin{smallmatrix}\alpha' & 0\\ 0 & 1_T\end{smallmatrix}\right)} & Q' \oplus T. \end{array}$$

Observe that stably association is an equivalence relation. Note also that if Σ_1 and Σ_2 are classes of morphisms between finitely generated projective right R-modules, then $R_{\Sigma_1} \cong R_{\Sigma_2}$ provided every element of Σ_i is stably associated with an element of Σ_j for $i, j \in \{1, 2\}$. $\square$

The proof of the following result can be found in [**Coh85**, Theorem 0.6.2]. Notice that (iii)⇒(i) is the assertion of Schanuel's lemma.

Lemma 3.53. *Let R be a ring, and let $\alpha\colon P \to Q$ and $\alpha'\colon P' \to Q'$ be two morphisms of right R-modules. Suppose that Q, Q' are projective right R-modules and α, α' are injective morphisms. Then the following conditions are equivalent:*

(i) *There is an isomorphism $\mu\colon P \oplus Q' \to Q \oplus P'$ of the form*
$$\mu = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \text{ with inverse } \mu^{-1} = \begin{pmatrix}\delta' & \beta'\\ \gamma' & \alpha'\end{pmatrix}.$$
(ii) *α is stably associated with α'.*
(iii) coker $\alpha \cong$ coker α'. $\square$

The ring R_Σ and the kernel of $\lambda\colon R \to R_\Sigma$ are fairly well understood when R is equipped with a rank function.

Definitions 3.54. Let R be a ring.

(a) We denote by $K_0(R)$ the *Grothendieck group of finitely generated projective right R-modules*, modulo direct sums, that is, the abelian group generated by the isomorphism classes $[P]$ of $P \in \mathcal{P}_R$, modulo the relations $[P] + [Q] - [P \oplus Q]$ for all $P, Q \in \mathcal{P}_R$.
(b) A *(projective) rank function* on a ring R is a morphism of groups $\rho\colon K_0(R) \to \mathbb{R}$ such that
   (i) $\rho([P]) \geq 0$, for all $P \in \mathcal{P}_R$, and
   (ii) $\rho([R]) = 1$.

   If $\rho([P]) > 0$ for every nonzero $P \in \mathcal{P}_R$, we say that ρ is a *faithful rank function.* For the sake of simplicity we will write $\rho(P)$ instead of $\rho([P])$ for every $P \in \mathcal{P}_R$.

   Suppose that R has a rank function $\rho\colon K_0(R) \to \mathbb{R}$.

(c) Let $\alpha\colon P \to Q$ be a morphism between finitely generated projective right R-modules. Consider the finitely generated projective right R-modules P' such that there exist morphisms β, γ making the following diagram commutative

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & Q \\ & P' \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{15}$$

We define the *inner rank* of α as $\rho(\alpha) = \inf\{\rho(P') \mid P' \text{ satisfies (15)}\}$.

(d) We say that a morphism between finitely generated projective right R-modules $\alpha\colon P \to Q$ is *full* in case $\rho(\alpha) = \rho(P) = \rho(Q)$. We will denote the localization of R at all full maps by R_ρ, and will call it the *universal localization of R at ρ.* If ρ is faithful and α is full, we define α to be an *atomic full morphism* if, in any nontrivial factorization as in (15), $\rho(P') > \rho(P) = \rho(Q) = \rho(\alpha)$.

(e) Suppose that R is a semihereditary ring with a faithful rank function ρ. Notice that every full map is injective under these assumptions since $P = \ker\gamma \oplus \operatorname{im}\gamma$ for every morphism $\gamma\colon P \to Q$ between finitely generated projective right R-modules. Let M be a right R-module. We say that M is *ρ-torsion* if M is the cokernel of a full morphism. We say that M is *ρ-simple* if M is the cokernel of an atomic full morphism. For a characterization of ρ-torsion and ρ-simple modules in the hereditary case see [**CB91**, Definition 1.3]. ☐

Let R be a ring such that all finitely generated projective right R-modules are free of unique rank (for example Sylvester domains [**DS78**]). Then it has a unique rank function ρ. Note that morphisms among finitely generated projective right R-modules are given by matrices. Thus ρ-full matrices are exactly full matrices. If R is a Sylvester domain, then the localization of R at ρ is a division ring by Theorem 3.40.

The following results are proved in more general situations in [**Sch85**]. Here we state them in the form needed in Chapter 8, for R a hereditary ring.

The first result describes the ρ-torsion modules.

Theorem 3.55. *Let R be a hereditary ring with a faithful projective rank function. The full subcategory of* Mod-R *consisting of all ρ-torsion modules is closed under images, kernels, cokernels and extensions, so it is an exact abelian subcategory of* Mod-R. *The simple objects are the ρ-simples and every object has finite length in this subcategory. Moreover, every full morphism factorizes uniquely into full atomic morphisms , i.e. if $\alpha = \alpha_n \cdots \alpha_1$ and $\alpha = \beta_m \cdots \beta_1$ where α_i, β_i are full atomic morphisms, then $m = n$ and there is a permutation $\sigma \in S_n$ such that α_i is stably associated with $\beta_{\sigma(i)}$.* ☐

This second one gives us a sufficient condition for $\lambda\colon R \to R_\Sigma$ to be injective, and the next theorem computes the kernel in some situations.

Theorem 3.56. *Let R be a hereditary ring with a faithful rank function ρ. Let Σ be a collection of full morphisms. Then the universal localization $\lambda\colon R \to R_\Sigma$ is injective.* ☐

Theorem 3.57. *Let R be a hereditary ring with a rank function ρ. If ρ takes values in $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{Z}$, then the kernel of the universal localization $\lambda\colon R \to R_\rho$ is precisely the trace ideal of the finitely generated projectives of rank zero, i.e. $\lambda(r) = 0$ if and only if there exist a finitely generated projective right R-module P with $\rho(P) = 0$ and a morphism of right R-modules $f\colon P \to R$ such that $r \in \operatorname{im} f$.* ☐

Sometimes we know how the ring R_ρ looks like.

Theorem 3.58. *Let R be a hereditary ring with a projective rank function ρ taking values in $\mathbb{Z}$. Then R_ρ is a division ring.* ☐

The following results explain how are the intermediate localizations of R that embed in R_ρ.

THEOREM 3.59. *Let R be a hereditary ring with a faithful rank function ρ taking values in $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{Z}$. Let Ξ be a class of full morphisms. Then R_Ξ embeds in R_ρ if and only if there exists a set Σ of atomic full morphisms such that $R_\Xi = R_\Sigma$. Moreover, given two sets Σ_1, Σ_2 of atomic full morphisms, then R_{Σ_1} is the same subring of R_ρ as R_{Σ_2} if and only if every element of Σ_1 is stably associated with an element of Σ_2 and every element of Σ_2 is stably associated with an element of Σ_1.* $\square$

THEOREM 3.60. *Let R be a hereditary ring with a rank function ρ, and let $\lambda\colon R \to R_\rho$ be the universal localization of R at ρ. Suppose that R_ρ is a simple artinian ring. Let S be a subring of R_ρ such that $\operatorname{im}\lambda \subseteq S$ and $\lambda\colon R \to S$ is a ring epimorphism. Then S is the universal localization of R at those full morphisms between finitely generated projective right R-modules that become invertible over S.* $\square$

Let R be a commutative ring. Then the localization of R at a set Σ of square matrices is the same as the (Ore) localization at the multiplicative set

$$\mathfrak{S} = \{(\det H_1)\cdots(\det H_n) \mid n \in \mathbb{N},\ H_i \in \Sigma\}.$$

On the other hand, there exist a commutative ring R and a set Σ of morphisms between finitely generated projective R-modules such that the universal localization R_Σ of R at Σ is not an Ore localization, i.e. $\mathfrak{S}^{-1}R \ncong R_\Sigma$ as R-rings for all multiplicative sets $\mathfrak{S}$ of R. For a proof of these facts see Remark 8.74.

"Tortured tongues feast their frenzy
They hiss out all that is nothing
The night time of the hearing flower
Has put aside the laugh dancing flame
No longer warming the wings
Of their fluttering dust angel mistress
The petals have closed for this long night
Their brittle limbs are thinning
Their meek and weeping gesture fares their well
To the falling paper blossoms
One by one, down into the everflow
One by one, drown in the overflow
Gliding through the emptiness
Flying through the emptiness"

Psychotic Waltz, *Into the Everflow*

CHAPTER 4

Crossed product group rings and Mal'cev-Neumann series rings

1. Crossed product monoid semirings

In this section we introduce the concept of crossed product monoid (group) semiring (ring) and some of their basic properties. It is crucial in our work. Being our concepts more general, we follow the exposition in [**Pas89**, Chapter 1].

Definitions 4.1. Let R be a semiring, and let G be a monoid. The set $(R * G) \cup \{0\}$ is the subset of $(R \cup \{0\})^G$ consisting of the maps with finite support expressed as $\sum\limits_{x \in G} r_x \bar{x}$. The element $\sum\limits_{x \in G} r_x \bar{x}$ with $r_x = 0$ for all $x \in G$ is (also) denoted by 0.

We define a *crossed product monoid semiring* $R * G$ (or simply RG) of G over R as a semiring which contains R constructed in the following way. As a set $R * G = (R * G) \cup \{0\}$ if R has an absorbing zero; otherwise $R * G = ((R * G) \cup \{0\}) \setminus \{0\}$. Addition is as expected

$$\sum_{x \in G} r_x \bar{x} + \sum_{x \in G} s_x \bar{x} = \sum_{x \in G} (r_x + s_x) \bar{x},$$

and multiplication is determined by the two rules below:

Twisting. For $x, y \in G$ we have

$$\bar{x}\bar{y} = \tau(x, y)\overline{xy}$$

where $\tau \colon G \times G \longrightarrow R^{\times}$.

Action. For $x \in G$ and $r \in R$ we have

$$\bar{x} r = r^{\sigma(x)} \bar{x}$$

where $\sigma \colon G \to \operatorname{Mon}(R)$ and $\operatorname{Mon}(R)$ denotes the monoid of injective semiring endomorphisms of R.

More concretely

$$\left(\sum_{x \in G} r_x \bar{x}\right) \left(\sum_{x \in G} s_x \bar{x}\right) = \sum_{x \in G} \left(\sum_{\{(y,z) \mid yz = x\}} r_y s_z^{\sigma(y)} \tau(y, z)\right) \bar{x}.$$

Notice that neither σ nor τ need to preserve any kind of structure.

If R is a ring or G is a group, we say that RG is a *crossed product group (monoid) ring (semiring)*.

If R is a ring, G a group and $\sigma \colon R \to \operatorname{Aut}(R)$, then $R*G$ is a crossed product group ring as defined in [**Pas89**, Page 2]. Notice that σ is always like this if G is a group by Remarks 4.3(b).

If H is a submonoid of G, then $RH = \{\eta \in RG \mid \operatorname{supp} \eta \subseteq H\}$ is the naturally embedded sub-crossed product monoid semiring. $\square$

Note that, by definition, a crossed product monoid semiring is merely a semiring which happens to have a particular structure relative to R and G. The following is useful.

LEMMA 4.2. *Let $R{*}G$ be a crossed product monoid semiring of the monoid G over the semiring R.*

(i) *The associativity of $R*G$ is equivalent to the assertions that for all $x, y, z \in G$*
  (a) $\tau(x,y)\tau(xy,z) = \tau(y,z)^{\sigma(x)}\tau(x,yz)$
  (b) $\sigma(xy)\mu(x,y) = \sigma(y)\sigma(x)$ *(composition from left to right) where $\mu(x,y)$ denotes the automorphism of R defined by $r \mapsto r^{\mu(x,y)} = \tau(x,y)r\tau(x,y)^{-1}$.*

(ii) *The existence of an identity element is equivalent to the fact that $\sigma(1)\colon R \to R$ is a semiring automorphism.*

PROOF. (i) The associativity of $R*G$ is equivalent to the equality

$$[(r\bar{x})(s\bar{y})](t\bar{z}) = (r\bar{x})[(s\bar{y})(t\bar{z})]$$

for all $r, s, t \in R$ and $x, y, z \in G$. The left-hand expression equals

$$rs^{\sigma(x)}\tau(x,y)t^{\sigma(xy)}\tau(xy,z)\overline{xyz} = rs^{\sigma(x)}t^{\sigma(xy)\mu(x,y)}\tau(x,y)\tau(xy,z)\overline{xyz},$$

while the right-hand expression becomes

$$rs^{\sigma(x)}t^{\sigma(y)\sigma(x)}\tau(y,z)^{\sigma(x)}\tau(x,yz)\overline{xyz}.$$

Certainly, if (a) and (b) hold, then the product in $R*G$ is associative. Suppose that $R*G$ is associative. Then, by setting $r = s = t = 1$, (i) follows. To prove (ii), let $r = s = 1$, and t be any element in R.

(ii) Suppose that $\sigma(1)$ is an automorphism. Let $u = \left(\tau(1,1)^{-1}\right)^{\sigma(1)^{-1}}$, and let $e = u\bar{1}$. Then e is an idempotent. $R*G$=$e(R*G)$= $(R*G)e$ because given any $r\bar{x} \in R*G$ then $e(u^{-1}r\tau(1,x)^{-1})^{\sigma(1)^{-1}}\bar{x} = r\bar{x}$ (the other case is analogous). Now, for $f \in R*G$, there exist $g, h \in R*G$ such that $f = eg$ and $f = he$. Then $ef = e^2g = eg = f$ and $fe = he^2 = he = f$. This implies that e is the identity element of $R*G$.

Conversely, let $\eta = \sum_{x\in G} r_x\bar{x}$ be the identity element of $R*G$. Since $\eta\bar{y} = \bar{y}$ for all $y \in G$, we get that $r_x = 0$ for all $x \in G \setminus \{1\}$. Hence $\eta = v\bar{1}$ for some $v \in R$. Moreover, for all $r \in R$, $r\bar{1} = v\bar{1}r\bar{1} = vr^{\sigma(1)}\tau(1,1)\bar{1}$. For $\tau(1,1)$ is invertible, if $r = 1$, then $v = \tau(1,1)^{-1}$. Hence $r^{\sigma(1)} = \tau(1,1)r\tau(1,1)^{-1}$. Therefore $\sigma(1)$ is a semiring automorphism. □

REMARKS 4.3. (a) Crossed product monoid semirings do not have a natural basis. If $d\colon G \to R^\times$ assigns to each element $x \in G$ a unit d_x, then $\widetilde{G} = \{\tilde{x} = d_x\bar{x} \mid x \in G\}$ is another R-basis for $R*G$ which still exhibits the basic crossed product monoid semiring structure. Here we have $\tilde{\tau}\colon G \times G \to R^\times$ and $\tilde{\sigma} : G \to \mathrm{Aut}(R)$ defined by $\tilde{\tau}(x,y) = d_x {d_y}^{\sigma(x)}\tau(x,y)d_{xy}^{-1}$ and $\tilde{\sigma}(x)\colon R \to R$ by $r \mapsto r^{\tilde{\sigma}(x)} = d_x r^{\sigma(x)} d_x^{-1}$.

(b) As we have seen in the proof of Lemma 4.2(ii), the identity element of $R*G$ is of the form $1 = u\bar{1}$ for some $u \in R^\times$. Thus, because of (a), we will assume that $\bar{1} = 1_{R*G}$. The embedding of R into $R*G$ is then given by $r \mapsto r\bar{1}$. Moreover, $\tau(1, \_) = \tau(\_, 1) = 1$ and $\sigma(1)$ is the identity on R. Hence, in case G is a group, looking at Lemma 4.2(i)(b) for $y = x^{-1}$, we obtain that for each $x \in G$, the morphism $\sigma(x) \in \mathrm{Aut}(R)$.

(c) Let $G^\times$ denote the group of units in G. The group $G^\times$ is, in general, not contained in $R*G$. Rather, for each $x \in G^\times$, the element $\bar{x}$ is a unit of the semiring with inverse $\bar{x}^{-1} = (\tau(x,x^{-1})^{-1})^{\sigma(x)^{-1}}\overline{x^{-1}} = \tau(x^{-1},x)^{-1}\overline{x^{-1}}$, and

$$R^\times G^\times = \{u\bar{x} \mid x \in G^\times, u \in R^\times\},$$

the *group of trivial units of* $R*G$, satisfies $R^\times G^\times/R^\times \cong G^\times$. Notice that $R^\times G^\times$ acts on both $R*G$ and R by conjugation and that for each $x \in G^\times$, $r \in R$ we have $\bar{x}r\bar{x}^{-1} = r^{\sigma(x)}$.

We will mostly be concerned with the case of a crossed product group ring kG of a group G over a division ring k. In this case the group of trivial units is

$$k^{\times}G = \{u\bar{x} \mid x \in G, u \in k^{\times}\},$$

and $k^{\times}G/k^{\times} \cong G$. □

CONVENTIONS 4.4. The notation $R * G$ for a crossed product monoid semiring is ambiguous since it does not convey the full σ, τ-structure. Nevertheless it is simpler and hence preferable to something like (R, G, σ, τ). Moreover all crossed products may (and in many cases will) have the same notation RG. □

Certain special cases of crossed product group rings have their own names. Some of them have already been defined in Examples 1.6.

DEFINITIONS 4.5. Let $R * G$ be a crossed product monoid semiring.

(a) If the twisting is trivial, that is, $\tau(x, y) = 1$ for all $x, y \in G$, then $R*G = RG$ is a *skew group (monoid) ring (semiring)*. In this case G is naturally embedded in $R * G$.

(b) If the twisting and the action are trivial, then $R*G = R[G](= RG)$ is a *group (monoid) ring (semiring)*.

(c) Suppose that G equals $\mathbb{N}$ but in multiplicative notation, i.e. $G = \{x^n \mid n \in \mathbb{N}\}$. Let $\alpha\colon R \to R$ be an injective morphism of semirings. If the twisting is trivial and $\sigma\colon G \to \mathrm{Mon}(R)$ is the only morphism of monoids such that $x \to \alpha$, we obtain the *skew polynomial semiring (ring)* $R[x; \alpha]$. Hence given $a = \sum\limits_{n\in\mathbb{N}} a_n x^n$, $b = \sum\limits_{n\in\mathbb{N}} b_n x^n \in R[x; \alpha]$,

$$a + b = \sum_{n\in\mathbb{N}} (a_n + b_n)x^n, \qquad ab = \sum_{n\in\mathbb{N}} \sum_{0\leq l\leq n} a_l \alpha^l(b_{n-l})x^n.$$

When α is the identity on R we get the *polynomial semiring (ring)* $R[x]$.

(d) Suppose that G equals $\mathbb{Z}$ but in multiplicative notation, i.e. $G = \{x^n \mid n \in \mathbb{Z}\}$. Let $\alpha \in \mathrm{Aut}(R)$. If the twisting is trivial and $\sigma\colon G \to \mathrm{Aut}(R)$ is the only morphism of groups such that $x \mapsto \alpha$, we obtain the *skew Laurent polynomial semiring (ring)* $R[x, x^{-1}; \alpha]$. Hence given $a = \sum\limits_{n\in\mathbb{Z}} a_n x^n$, $b = \sum\limits_{n\in\mathbb{Z}} b_n x^n \in R[x; \alpha]$,

$$a + b = \sum_{n\in\mathbb{Z}} (a_n + b_n)x^n, \qquad ab = \sum_{n\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_l \alpha^l(b_{n-l})x^n.$$

When α is the identity on R we get the *Laurent polynomial semiring (ring)* $R[x, x^{-1}]$.

(e) Let R be a semiring (ring). Let G be a free group on a set X. Form a crossed product group ring RG. By the *polynomial semiring (ring)* $R\langle X\rangle$ *of* RG, we mean the subsemiring (subring) of RG generated by R and the set $\{\bar{x} \mid x \in X\}$. Notice that $R\langle X\rangle$ is a crossed product monoid semiring (ring) of the free monoid generated by X over R. □

What follows is an easy but important remark.

REMARK 4.6. When $G = \mathbb{N}$ or $G = \mathbb{Z}$, RG can always be seen as a skew polynomial semiring (ring) $R[x; \alpha]$ or as a skew Laurent polynomial semiring (ring) $R[x, x^{-1}; \alpha]$ respectively. Indeed, making a change of basis as in Remarks 4.3(a), with $\widetilde{x^n} = \bar{x}^n$ for all $n \in \mathbb{N}$ ($n \in \mathbb{Z}$), then $\widetilde{x}^n = \widetilde{x^n}$, $\tau(\widetilde{x^n}, \widetilde{x^m}) = 1$ and $\widetilde{x^n}r = r^{\sigma(x)^n}\widetilde{x^n}$. □

2. Some useful results on crossed product group rings

Our first result is an important property of crossed product group rings. The proof will also be useful for us in Chapter 6.

Lemma 4.7. *Let R be a ring, and let G be a group with a normal subgroup N. Let RG be a crossed product group ring. Then*

$$RG = (RN)(G/N)$$

where the latter is some crossed product group ring of the group G/N over the ring RN.

Proof. For each $1 \neq \beta \in G/N$, let $x_\beta \in G$ be a fixed representative of the class β. For the class $1 \in G/N$ choose $x_1 = 1 \in G$. Then $G = \dot{\bigcup} N x_\beta$ and $\widehat{G} = \{\bar{x}_\beta : \beta \in G/N\} \subseteq \bar{G}$ is a copy of G/N. This shows that $RG = \bigoplus_\beta RN\bar{x}_\beta$. Therefore $\widehat{G}$ is an RN-basis for RG. Moreover, we have the maps:

$$\begin{aligned} \hat{\sigma}\colon G/N &\longrightarrow \mathrm{Aut}(RN) \\ \beta &\longmapsto \hat{\sigma}(\beta)\colon RN \longrightarrow RN \\ &\qquad\qquad\quad y \longmapsto \bar{x}_\beta y \bar{x}_\beta^{-1} \end{aligned}$$

$$\begin{aligned} \hat{\tau}\colon G/N \times G/N &\longrightarrow (RN)^\times \\ (\beta,\gamma) &\longmapsto \hat{\tau}(\beta,\gamma) = \tau(x_\beta, x_\gamma)\tau(n_{\beta\gamma}, x_{\beta\gamma})^{-1}\bar{n}_{\beta\gamma} \end{aligned}$$

where $n_{\beta\gamma}$ is the unique element in N such that $n_{\beta\gamma}x_{\beta\gamma} = x_\beta x_\gamma$.
Then $\bar{x}_\beta(\sum_{n\in N} r_n\bar{n}) = (\sum_{n\in N} r_n\bar{n})^{\hat{\sigma}(\beta)}\bar{x}_\beta$ and $\bar{x}_\beta\bar{x}_\gamma = \hat{\tau}(\beta,\gamma)\bar{x}_{\beta\gamma}$ as desired.

Notice that the assumption ($x_1 = 1$) imply that RN embeds in RG via $r\bar{n} \mapsto r\bar{n}\cdot 1$. □

When R and G have *good* properties, sometimes it is possible to infer from them a certain behavior of the crossed product group ring RG as for example the following proposition shows. It was proved by G. Higman [**Hig40**, Section 4] in the case of locally indicable groups.

Proposition 4.8. *Let G be a right orderable group. Let R be a domain. Consider a crossed product group ring RG of G over R. Then RG is a domain and has no units other than the trivial units. In particular, this applies when G is a locally indicable group.*

Proof. We show that if $\eta \in RG \setminus \{0\}$ is not a trivial unit, then η is neither a zero divisor nor invertible.

Let $\mu \in RG \setminus \{0\}$. Call $A = \operatorname{supp}\eta$ and $B = \operatorname{supp}\mu$. Since A is finite there exist $a_{\min}, a_{\max} \in A$ such that $a_{\max}$ is the largest element in A and $a_{\min}$ the smallest element in A. Choose $b', b'' \in B$ such that $a_{\max}b'$ and $a_{\min}b''$ are the largest and the smallest element in the finite sets $a_{\max}B$ and $a_{\min}B$, respectively. If $a \in A$ and $b \in B$, then $a_{\min} \leq a \leq a_{\max}$ yields $a_{\min}b \leq ab \leq a_{\max}b$, and hence

$$a_{\min}b'' \leq a_{\min}b \leq ab \leq a_{\max}b \leq a_{\max}b'$$

Moreover, $a_{\min}b'' = ab$ implies that $a_{\min} = a$ and then that $b'' = b$. Similarly, $ab = a_{\max}b'$ gives $a = a_{\max}$ and $b = b'$. Therefore, the two products $a_{\max}b'$ and $a_{\min}b''$ are uniquely represented. Moreover, if $|A| \geq 2$ or $|B| \geq 2$, then $a_{\min}b'' \neq a_{\max}b'$. Thus $|\operatorname{supp}(\eta\nu)| \geq 2$ and $\eta\nu \neq 0$ and $\eta\nu \neq 1$. If $|A| = |B| = 1$. Then $\eta = r\bar{x}$, $\nu = s\bar{y}$ with $x, y \in G$ and r neither zero nor invertible. The products $\eta\mu = rs^{\sigma(x)}\tau(x,y)\overline{xy}$ and $\mu\eta = sr^{\sigma(y)}\tau(y,x)\overline{yx}$ show that $\eta\mu \neq 0$ and η is not invertible because r is not.

For the second part recall that a locally indicable group is a right orderable group by Proposition 2.29. □

Concerning the Ore condition on crossed product group rings we can state the following results. The first one tells us that the Ore condition is in fact a local property.

Proposition 4.9. *Let R be a ring and G a group. Consider a crossed product group ring RG. Then the following are equivalent:*

(i) *RG is a (left, right) Ore domain.*
(ii) *RH is a (left, right) Ore domain for each subgroup H of G.*
(iii) *RH is a (left, right) Ore domain for each finitely generated subgroup H of G.*

PROOF. Clearly (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) because given $a, b \in RG$, there exists a finitely generated subgroup H of G such that $a, b \in RH$.

(i) $\Rightarrow$ (ii) We prove the result of the right Ore condition. The left one is shown similarly. Let $H \leq G$. Define the equivalence relation on G by $x \sim y$ if and only if $xy^{-1} \in H$. Then $G = \bigcup_\alpha Hx_\alpha$, where the union is disjoint and $\{x_\alpha\}$ is a complete set of representatives of the (left) cosets of $\sim$. Moreover RG is a free RH-module with basis $\{x_\alpha\}$. Suppose that RH is not right Ore. Thus, given $a, b \in RH$, $aRH \cap bRH = 0$. Hence

$$aRG \cap bRG = a(\bigoplus_\alpha RHx_\alpha) \cap b(\bigoplus_\alpha RHx_\alpha) = 0.$$ □

PROPOSITION 4.10. *Let k be a division ring. Let G be a group. Consider a crossed product group ring kG. Suppose that G has a normal subgroup N such that G/N is torsion-free abelian and kN is a left (right) Ore domain. Then kG is a left (right) Ore domain.*

PROOF. Notice that for each $a, b \in kG$ there exists a subgroup U such that U contains N, U/N is finitely generated and $a, b \in kU$.

The ring $kG = kN(G/N)$ is a domain by Proposition 4.8 for $R = kN$.

If we prove that kU is left Ore for each such U, then it follows that kG is a left Ore domain.

Observe that $U/N \cong \bigoplus_{i=1}^{n} \mathbb{Z}$. We show that kU is left Ore by induction on n. If $n = 0$ the result is clear by hypothesis. Suppose that $x_1, \ldots, x_n$ are such that $\{Nx_1, \ldots, Nx_i\}$ generate $\mathbb{Z} \oplus \overset{(i}{\cdots} \oplus \mathbb{Z}$ for $1 \leq i \leq n$. By induction hypothesis, if $U_{n-1} = \langle N, x_1, \ldots, x_{n-1}\rangle$ then kU_{n-1} is left Ore. Since $U_{n-1} \lhd U$, left conjugation by $\bar{x}_n$ gives an automorphism α of kU_{n-1}. Moreover the powers of $\bar{x}_n$ are left kU_{n-1}-linearly independent. Thus kU is isomorphic to $kU_{n-1}[t, t^{-1}; \alpha]$. Then Proposition 3.19 implies that kU is a left Ore domain. □

COROLLARY 4.11. *Let k be a division ring. Let G be a group. Suppose that $(G_\gamma)_{\gamma \leq \tau}$ is a subnormal series with torsion-free abelian factors. Consider a crossed product group ring kG. Then kG is an Ore domain.*

PROOF. We prove by transfinite induction that the result holds true for each G_γ. If $\gamma = 0$ the result is clear. Suppose that $\gamma = \beta + 1$ for some ordinal $\beta < \tau$. For kG_β satisfies the induction hypothesis, the result follows for kG_γ by Proposition 4.10. If γ is a limit ordinal and H a finitely generated subgroup of G_γ, then $H \subseteq G_\beta$ for some $\beta < \gamma$. By induction hypothesis and Proposition 4.9, kH is an Ore domain. Thus Proposition 4.9 implies that kG_β is an Ore domain. □

The following well known result will be useful to construct the example in Proposition 7.22.

LEMMA 4.12. *Let k be a division ring. Let M be an ordered commutative monoid. Consider the monoid k-ring kM. Then kM is a left (and right) Ore domain.*

PROOF. A similar argument as the one of Proposition 4.8 shows that kM is a domain. Let X be a finite subset of M. Let $k[X]$ denote the monoid algebra on the free commutative monoid generated by X. Observe that $k[X]$ is a noetherian domain. Let M_X denote the submonoid of M generated by X. The monoid subring kM_X of the domain kM is a homomorphic image of the noetherian ring $k[X]$, so it is a noetherian domain. Thus kM_X is a (two-sided) Ore domain by Proposition 3.19. Therefore we can conclude that kM is an Ore domain. □

The next one is a well known obstruction to the Ore condition. The proof we provide is by P.A. Linnell in [**Lin06**, Proposition 2.2].

PROPOSITION 4.13. *Let G be a group which has a subgroup isomorphic to the free group F on two generators, let k be a division ring, and let kG be a crossed product group ring. Then kG does not satisfy the right (left) Ore condition.*

PROOF. By Corollary 2.24 and Proposition 4.8, kF is a domain. And by Proposition 4.9, it is enough to show that kF is not right Ore. Suppose that F is free on a, b. We prove that $(\bar{a}-1)kF \cap (\bar{b}-1)kF = 0$. Write $A = \langle a \rangle$ and $B = \langle b \rangle$. Suppose that $\eta \in (\bar{a}-1)kF \cap (\bar{b}-1)kF$. Then we may write

$$\eta = \sum_i (u_i - 1)\overline{x_i} d_i = \sum_i (v_i - 1)\overline{y_i} e_i \tag{16}$$

where $u_i = \bar{a}^{q(i)}$ for some $q(i) \in \mathbb{Z}$, $v_i = \bar{b}^{r(i)}$ for some $r(i) \in \mathbb{Z}$, $d_i, e_i \in k$ and $x_i, y_i \in F$. The general element g of F can be written in a unique way as $g_1 \cdots g_l$, where g_i are alternately in A and B, and $g_i \neq 1$ for all i; we shall define the length $\lambda(g)$ of g to be l. Of course $\lambda(1) = 0$. Let L be the maximum of all $\lambda(x_i), \lambda(y_i)$, let s denote the number of x_i with $\lambda(x_i) = L$, and let t be the number of y_i with $\lambda(y_i) = L$. We shall use induction on L and then on $s + t$, to show that $\eta = 0$. If $L = 0$, then $x_i, y_i = 1$ for all i and the result is obvious. If $L > 0$, then without loss of generality, we may assume that $s > 0$. Suppose that $\lambda(x_i) = L$ and x_i starts with an element from A, so $x_i = a^p h$ where $0 \neq p \in \mathbb{Z}$, and $\lambda(h) = L - 1$. Then

$$(u_i - 1)\overline{x_i} d_i = (\bar{a}^{q(i)} - 1)\bar{a}^p \bar{h} d d_i = (\bar{a}^{q(i)+p} - 1)\bar{h} d d_i - (\bar{a}^p - 1)\bar{h} d d_i$$

for some $d \in k$. This means that we have found an expression for η with smaller $s + t$, so all the x_i with $\lambda(x_i) = L$ start with an element from B. Therefore if $\vartheta = \sum_i u_i \overline{x_i} d_i$ where the sum is over all i such that $\lambda(x_i) = L$, then each x_i starts with an element of B and hence $\lambda(a^{q(i)} x_i) = L+1$. We now see from (16) that $\vartheta = 0$. Since $s > 0$ by assumption, the expression for ϑ above is nontrivial and therefore there exists $i \neq j$ such that $a^{q(i)} x_i = a^{q(j)} x_j$. This forces $q(i) = q(j)$ and $x_i = x_j$. Thus $u_i = u_j$ and we may replace $(u_i - 1)\overline{x_i} d_i + (u_j - 1)\overline{x_j} d_j$ with $(u_i - 1)\overline{x_i}(d_i + d_j)$, thereby reducing s by 1 and the proof that $(\bar{a} - 1)kF \cap (\bar{b} - 1)kF = 0$ is complete. ☐

The following result explains how semirings come into play when trying to elucidate whether two division rings of fractions of a crossed product group ring are isomorphic.

LEMMA 4.14. *Let D_1 and D_2 be two division rings of fractions of a crossed product group ring kG. Then $D_1 \cup \{\infty_1\}$ and $D_2 \cup \{\infty_2\}$ are rational $k^\times G$-semirings as in* Examples 1.43(d). If there exists a morphism of rational $k^\times G$-semirings $\beta \colon D_1 \cup \{\infty_1\} \longrightarrow D_2 \cup \{\infty_2\}$, then $\beta(D_1) \subseteq D_2$ and $\beta_{|D_1} \colon D_1 \longrightarrow D_2$ is a ring isomorphism which is the identity on kG.

Conversely, if $\beta \colon D_1 \longrightarrow D_2$ is a ring isomorphism which is the identity on kG, then its extension, $\beta \colon D_1 \cup \{\infty_1\} \longrightarrow D_2 \cup \{\infty_2\}$, sending ∞_1 to ∞_2, is a morphism of rational $k^\times G$-semirings.

PROOF. Let $\beta \colon D_1 \cup \{\infty_1\} \longrightarrow D_2 \cup \{\infty_2\}$ be a morphism of rational $k^\times G$-semirings. Let $x \in D_1 \setminus \{0_{D_1}\}$. Because β is a morphism of rational $k^\times G$-semirings and $1 \in k^\times G$,

$$1_{D_2} = \beta(1_{D_1}) = \beta(xx^{-1}) = \beta(xx^*) = \beta(x)\beta(x^*) = \beta(x)\beta(x)^*.$$

Therefore $\beta(x) \in D_2 \setminus \{0_{D_2}\}$, and $\beta(D_1) \subseteq D_2$. We can consider $\beta_{|D_1} \colon D_1 \longrightarrow D_2$, which is an (injective) morphism of rings because it is a morphism of additive semigroups and a morphism of multiplicative monoids. Since β is a morphism of $k^\times G$ semirings, $\beta(x) = x$ for all $x \in k^\times G$.

Then $\beta(x) = x$ for all $x \in kG$ because β is a morphism of additive semigroups. Hence, $\beta_{|D_1}$ is an isomorphism because $\beta(D_1)$ is a division ring containing kG.

Conversely, suppose that $\beta\colon D_1 \longrightarrow D_2$ is a kG-isomorphism, then it is straightforward to prove the remaining part. $\square$

3. Mal'cev-Neumann series rings

As we have just seen in Proposition 4.8, given a division ring k and a right orderable group G, any crossed product group ring kG is a domain. So now arises the still open question of whether kG is embeddable in a division ring. This is known as *Mal'cev's problem* [**MMC83**, Question 1.6]. We will give examples of locally indicable groups G in which the answer is yes in Chapter 6.

Some important results related to Mal'cev's problem have been proved by J. Lewin and T. Lewin in [**LL78**] and by N. Dubrovin in [**Dub94**]. In the first one it is proved that if k is a division ring and G is a torsion-free one-relator group, then the crossed product group ring kG can be embedded in a division ring. Another proof of this result was given by W. Dicks in [**Dic83**].

In [**Dub94**, Theorem 10.1] it is shown that if G is the universal covering group of $\mathrm{SL}(2,\mathbb{R})$, then the group ring $k[G]$ is embeddable in a division ring. It was proved by G.M. Bergman [**Ber91**] that this G is right orderable, but it is not locally indicable.

Mal'cev's problem arose as a consequence of the result that independently proved A.I. Mal'cev [**Mal48**] and B.H. Neumann [**Neu49a**]: when G is an orderable group, kG embeds in a division ring, the so-called Mal'cev-Neumann series ring. This section is mainly devoted to prove this result. It is a key result in this dissertation that will be used throughout.

Definitions 4.15. Let R be a semiring, and let $(G,<)$ be an ordered monoid. Form a crossed product monoid semiring RG. We want to construct a new semiring in which RG embeds.

(a) Let $R((G,<)) \cup \{0\}$ denote the subset of $(R\cup\{0\})^G$ consisting of all the maps with well-ordered support, expressed as $\sum\limits_{x\in G} r_x\bar{x}$. We proceed to define the *Mal'cev-Neumann series semiring* $R((G,<))$. As a set, $R((G,<)) = R((G,<))\cup\{0\}$ whenever R has an absorbing zero. Otherwise $R((G,<)) = (R((G,<))\cup\{0\})\setminus\{0\}$. Let $f = \sum\limits_{x\in G} a_x\bar{x}$ and $g = \sum\limits_{x\in G} b_x\bar{x}$ be elements in $R((G,<))$. Then addition is defined by
$$f+g = \sum_{x\in G}(a_x+b_x)\bar{x},$$
and multiplication
$$fg = \sum_{x\in G}\Big(\sum_{M_x} a_y b_z^{\sigma(y)}\tau(y,z)\Big)\bar{x}$$
where $M_x = \{(y,z)\in \operatorname{supp}(f)\times\operatorname{supp}(g) \mid yz = x\}$.

The sum and product are well defined. The sum since $\operatorname{supp}(f+g)\subseteq\operatorname{supp}(f)\cup\operatorname{supp}(g)$ is a well-ordered subset of G. The multiplication because M_x is a finite set and $\operatorname{supp}(fg)$ is contained in $\{yz \mid y\in\operatorname{supp}(f),\ z\in\operatorname{supp}(g)\}$, a well-ordered subset of G.

If R is a ring, we say that $R((G,<))$ is a *Mal'cev-Neumann series ring.*

(b) Let $G\cup\{\infty\}$ be the totally ordered set consisting of G together with a new element ∞ such that $x<\infty$ for all $x\in G$. We define a map $\omega\colon R((G,<))\to G\cup\{\infty\}$ by $f\mapsto\omega(f)=\inf\{x\in\operatorname{supp}(f)\}$. Here we are suposing that the infimum of the empty set is ∞, thus $\omega(0)=\infty$ if $0\in R((G,<))$. This map induces a *partial ordering on* $R((G,<))$ by
$$f<g \quad\text{if}\quad \omega(f)<\omega(g)$$

for all $f, g \in R((G,<))$. Notice that w is multiplicative, i.e. $w(fg) = w(f)w(g)$, provided we define $g\infty = \infty g = \infty$ for all $g \in G \cup \{\infty\}$. □

REMARK 4.16. (a) RG is embedded in $R((G,<))$ as RG can be identified with the subsemiring of elements of $R((G,<))$ with finite support.
(b) On the semiring $R((G,<))$ there is defined the R-action $R \times R((G,<)) \to R((G,<))$, $(\lambda, f) \mapsto \lambda f$, where $\lambda f = \sum_{x\in G} (\lambda a_x)\bar{x}$ if $f = \sum_{x\in G} a_x \bar{x}$. Notice that $\mathrm{supp}(\lambda f) \subseteq \mathrm{supp}(f)$. □

As for crossed product monoid semirings, certain special cases of Mal'cev-Neumann series semirings have their own names. They have already been defined in Examples 1.6.

EXAMPLES 4.17. Let R be a semiring (ring). Let $\alpha \in \mathrm{Mon}(R)$. The associated Mal'cev-Neumann series semiring (ring) with
(a) $R[x;\alpha]$ is denoted by $R[[x;\alpha]]$ and is called *skew series semiring (ring)*
(b) $R[x]$ is denoted by $R[[x]]$ and is called *series semiring (ring).*

If α is an automorphism. The associated Mal'cev-Neumann series semiring (ring) to
(a) $R[x,x^{-1};\alpha]$ is denoted by $R((x;\alpha))$ and is called *skew Laurent series semiring (ring).*
(b) $R[x,x^{-1}]$ is denoted by $R((x))$ and is called *Laurent series semiring (ring).* □

DEFINITION 4.18. Let R be a semiring. Let $(G,<)$ be an ordered monoid. Let RG be a crossed product monoid semiring. Consider the associated Mal'cev-Neumann series semiring $R((G,<))$.

Let I be a set. Suppose that there exists a map $I \longrightarrow R((G,<))$, $i \longmapsto f_i = \sum_{x\in G} a_{ix}\bar{x}$, such that the following two conditions hold:

(1) $\bigcup_{i\in I} \mathrm{supp}(f_i)$ is well ordered,
(2) for each $x \in G$ the set $\{i \in I \mid x \in \mathrm{supp}(f_i)\}$ is finite.

Then we say that $\sum_{i\in I} f_i$ is *defined in* $R((G,<))$.

In this situation $\sum_{i\in I} f_i$ will be used to denote $\sum_{x\in G}\Big(\sum_{\{i\mid x\in\mathrm{supp}(f_i)\}} a_{ix}\Big)\bar{x}$. Note that $\sum_{i\in I} f_i$ is then an element of $R((G,<))$. □

The proof of the next theorem is taken from the paper by W. Dicks and J. Lewin [**DL82**, Proposition 2.1]. They state the result for monoid rings, but as we see here the proof works weakening these assumptions.

THEOREM 4.19. *Let R be a semiring. Let $(G,<)$ be an ordered monoid. Let RG be a crossed product monoid semiring. Consider the associated Mal'cev-Neumann series semiring $R((G,<))$. Let I, J be sets and $\nu(I)$ the set of all finite sequences in I. Let $\sum_{i\in I} f_i$, $\sum_{i\in I} g_i$, $\sum_{j\in J} h_j$ be defined in $R((G,<))$. Then the following hold*

(i) *For any $\lambda \in R$, $\sum_{i\in I} \lambda f_i$ is defined in $R((G,<))$ and equals $\lambda \sum_{i\in I} f_i$.*
(ii) *$\sum_{i\in I}(f_i + g_i)$ is defined in $R((G,<))$ and equals $\sum_{i\in I} f_i + \sum_{i\in I} g_i$.*
(iii) *$\sum_{(i,j)\in I\times J}(f_i h_j)$ is defined in $R((G,<))$ and equals $(\sum_{i\in I} f_i)(\sum_{j\in J} h_j)$.*
(iv) *If each $f_i > 1$, then*

$$\sum_{(i_1,\ldots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n}$$

is defined in $R((G,<))$.

(v) *If each* $f_i > 1$ *and* R *is a ring, then* $\sum_{(i_1,\ldots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n}$ *is a two-sided inverse of* $1-\sum_{i\in I} f_i$.

PROOF. (i) Since $\bigcup_{i\in I}\operatorname{supp}(\lambda f_i) \subseteq \bigcup_{i\in I}\operatorname{supp}(f_i)$, the first set is well ordered. Also for each $x \in G$

$$\{i \in I \mid x \in \operatorname{supp}(\lambda f_i)\} \subseteq \{i \in I \mid x \in \operatorname{supp}(f_i)\},$$

hence the first set is finite. This implies that $\sum_{i\in I}\lambda f_i$ is defined in $R((G,<))$. Clearly it follows that $\sum_{i\in I}\lambda f_i = \lambda\sum_{i\in I} f_i$.

(ii) Since $\bigcup_{i\in I}\operatorname{supp}(f_i+g_i) \subseteq (\bigcup_{i\in I}\operatorname{supp}(f_i))\cup(\bigcup_{i\in I}\operatorname{supp}(g_i))$, which is a well-ordered set, then $\bigcup_{i\in I}\operatorname{supp}(f_i+g_i)$ is well ordered.

For every $x \in G$, $\{i \in I \mid x \in \operatorname{supp}(f_i+g_i)\}$ is finite because it is contained in the finite set $\{i \in I \mid x \in \operatorname{supp}(f_i)\} \cup \{i \in I \mid x \in \operatorname{supp}(g_i)\}$. Thus $\sum_{i\in I}(f_i+g_i)$ is defined in $R((G,<))$ and $\sum_{i\in I}(f_i+g_i) = \sum_{i\in I} f_i + \sum_{i\in I} g_i$.

(iii) The set $\bigcup_{(i,j)\in I\times J}\operatorname{supp}(f_i h_j)$ is contained in $T = \{yz \mid y \in \bigcup_{i\in I}\operatorname{supp}(f_i),\ z \in \bigcup_{j\in J}\operatorname{supp}(h_j)\}$. We prove that T is well ordered and consequently $\bigcup_{(i,j)\in I\times J}\operatorname{supp}(f_i h_j)$ is too. Suppose that T is not well ordered. Then there exists an infinite decreasing sequence in T,

$$y_1z_1 > y_2z_2 > \ldots > y_nz_n > \ldots,$$

where $y_n \in \bigcup_{i\in I}\operatorname{supp}(f_i)$, $z_n \in \bigcup_{j\in J}\operatorname{supp}(h_j)$ for all n. There exists a subsequence $\{y_{n_k}\}_{k\geq 1}$ of $\{y_n\}_{n\geq 1}$ such that

$$y_{n_1} \leq y_{n_2} \leq \cdots \leq y_{n_k} \leq \cdots.$$

Hence

$$z_{n_1} > z_{n_2} > \cdots > z_{n_k} > \cdots,$$

that is, $\bigcup_{j\in J}\operatorname{supp}(h_j)$ is not well ordered, a contradiction.

To prove that for each $x \in G$ the set $\{(i,j) \in I \times J \mid x \in \operatorname{supp}(f_i h_j)\}$ is finite, suppose that there exists $x \in G$ such that this set is infinite. Since the sets $\{i \in I \mid y \in \operatorname{supp}(f_i)\}$ and $\{j \in J \mid z \in \operatorname{supp}(h_j)\}$ are finite for each $y, z \in G$, then the set

$$L = \{(y,z) \in \bigcup_{i\in I}\operatorname{supp}(f_i) \times \bigcup_{j\in J}\operatorname{supp}(h_j) \mid x = yz\}$$

is infinite. Hence either $L_1 = \{y \in \bigcup_{i\in I}\operatorname{supp}(f_i) \mid \exists z \text{ such that } (y,z) \in L\}$ is infinite or $L_2 = \{z \in \bigcup_{j\in J}\operatorname{supp}(h_j) \mid \exists y \text{ such that } (y,z) \in L\}$ is infinite. Suppose that L_1 is infinite. Then, for $L_1 \subseteq \bigcup_{i\in I}\operatorname{supp}(f_i)$ is well ordered, there exists a strictly increasing sequence $\{y_{1n}\}_{n>0}$ in $\bigcup_{i\in I}\operatorname{supp}(f_i)$. As before there exists $\{z_{2n}\}_{n>0} \subseteq \bigcup_{j\in J}\operatorname{supp}(h_j)$ with

$$z_{21} > z_{22} > \ldots > z_{2n} > \ldots,$$

contradicting that $\bigcup_{j\in J}\operatorname{supp}(h_j)$ is well ordered.

Now it is not very difficult to prove that $\sum_{(i,j)\in I\times J}(f_i h_j) = (\sum_{i\in I} f_i)(\sum_{j\in J} h_j)$.

(iv) Suppose that $\sum_{(i_1,\ldots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n}$ is not defined in $R((G,<))$. Then either

(a) $\bigcup_{(i_1,\ldots,i_n)\in\nu(I)}\operatorname{supp}(f_{i_1}\cdots f_{i_n})$ is not well ordered or

(b) the set $S_x = \{(i_1,\ldots,i_n) \in \nu(I) \mid x \in \operatorname{supp}(f_{i_1}\cdots f_{i_n})\}$ is infinite for some $x \in G$.

(a) implies that there exists a sequence $\{x_n\}_{n\geq 1} \subseteq \bigcup\limits_{(i_1,\dots,i_n)\in\nu(I)} \operatorname{supp}(f_{i_1}\cdots f_{i_n})$, such that

$$x_1 > x_2 > \cdots > x_n > \cdots$$

Hence there exists a sequence of elements in $\nu(\bigcup\limits_{i\in I}\operatorname{supp}(f_i))$,

$$(x_{11}, x_{12}, \dots, x_{1k_1}), (x_{21}, x_{22}, \dots, x_{2k_2}), \dots, (x_{n1}, x_{n2}, \dots, x_{nk_n}), \dots,$$

such that

$$x_1 = x_{11}\cdots x_{1k_1} > x_2 = x_{21}\cdots x_{2k_2} > \cdots > x_n = x_{n1}\cdots x_{nk_n} > \cdots$$

Clearly $(x_{l1}, \dots, x_{lk_l}) \neq (x_{s1}, \dots, x_{sk_s})$ if $l \neq s$.

(b) implies that there exists a sequence without repetitions in $\nu(\bigcup\limits_{i\in I}\operatorname{supp}(f_i))$,

$$(x_{11}, x_{12}, \dots, x_{1k_1}), (x_{21}, x_{22}, \dots, x_{2k_2}), \dots, (x_{n1}, x_{n2}, \dots, x_{nk_n}), \dots$$

such that

$$x = x_{11}\cdots x_{1k_1} = x_{21}\cdots x_{2k_2} = \cdots = x_{n1}\cdots x_{nk_n} = \cdots .$$

We now prove that certainly the sequence can be chosen without repetitions. Given the element $(x_{l1}, \dots, x_{lk_l}) \in G^{k_l}$, we define

$$S_{xl} = \{(i_{l1}, \dots, i_{lk_l}) \in \nu(I) \mid x_{l1} \in \operatorname{supp}(f_{i_{l1}}), \dots, x_{lk_l} \in \operatorname{supp}(f_{i_{lk_l}})\}.$$

Notice that S_{xl} is finite because for every x_{lt} the set $\{i \in I \mid x_{lt} \in \operatorname{supp}(f_i) \neq 0\}$ is finite.

Let $(i_{11}, \dots, i_{1k_1}) \in S_x$, then there exists $(x_{11}, \dots, x_{1k_1})$ such that $x = x_{11}\cdots x_{1k_1}$ and $x_{11} \in \operatorname{supp}(f_{i_{11}}), \dots, x_{1k_1} \in \operatorname{supp}(f_{i_{1k_1}})$.

Suppose that we are given $(x_{l1}, \dots, x_{lk_l})$, $l \geq 1$, then $S_x \setminus (S_{x1} \cup \cdots \cup S_{xl})$ is infinite. Choose

$$(i_{l+11}, \dots, i_{l+1k_{l+1}}) \in S_x \setminus (S_{x1} \cup \cdots \cup S_{xl}).$$

Therefore there exists $(x_{l+11}, \dots, x_{l+1k_{l+1}}) \in G^{k_{l+1}}$ such that $x = x_{l+11}\cdots x_{l+1k_{l+1}}$ with $x_{l+11} \in \operatorname{supp}(f_{i_{l+11}}), \dots, x_{l+1k_{l+1}} \in \operatorname{supp}(f_{i_{l+1k_{l+1}}})$. Because of the way it has been chosen, $(x_{l+11}, \dots, x_{l+1k_{l+1}}) \neq (x_{t1}, \dots, x_{tk_t})$ for every $1 \leq t < l+1$.

Then, in any case (a) or (b), there exists a sequence without repetitions in $\nu(\bigcup\limits_{i\in I}\operatorname{supp}(f_i))$,

$$(x_{11}, \dots, x_{1k_1}), \dots, (x_{n1}, \dots, x_{nk_n}), \dots \tag{17}$$

such that

$$x_{11}\cdots x_{1k_1} \geq \cdots \geq x_{n1}\cdots x_{nk_n} \geq \cdots \tag{18}$$

Call $\mathcal{S}$ the set of sequences in $\nu(\bigcup\limits_{i\in I}\operatorname{supp}(f_i))$ like (17) and satisfying (18).

In $\nu(\bigcup\limits_{i\in I}\operatorname{supp}(f_i))$ there is defined a (partial) order by length. That is, given the elements $(x_1, \dots, x_s), (x'_1, \dots, x'_l) \in \nu(\bigcup\limits_{i\in I}\operatorname{supp}(f_i))$,

$$(x_1, \dots, x_s) > (x'_1, \dots, x'_l) \text{ if and only if } s > l.$$

From this we can define a (partial) order $\succ$ on $\mathcal{S}$, the lexicographic order. We say that

$$\big((x_{n1}, \dots, x_{nk_n})\big)_{n\geq 1} \succ \big((x'_{n1}, \dots, x'_{nk'_n})\big)_{n\geq 1}$$

if and only if there exists $n_0 \geq 0$ such that

$$\begin{gathered}(x_{11}, \dots, x_{1k_1}) = (x'_{11}, \dots, x'_{1k'_1}), \dots, (x_{n_01}, \dots, x_{n_0k_{n_0}}) = (x'_{n_01}, \dots, x'_{n_0k'_{n_0}}) \\ \text{and} \quad (x_{n_0+11}, \dots, x_{n_0+1k_{n_0+1}}) > (x'_{n_0+11}, \dots, x'_{n_0+1k'_{n_0+1}}).\end{gathered}$$

We want to find a minimal sequence in $(\mathcal{S}, \succ)$. To do that we define the "reverse order" in $\mathcal{S}$ by,

$$((x'_{n1}, \ldots, x'_{nk'_n}))_{n\geq 1} \ {}^*\!\!\succ ((x_{n1}, \ldots, x_{nk_n}))_{n\geq 1}$$

if and only if $((x_{n1}, \ldots, x_{nk_n}))_{n\geq 1} \succ ((x'_{n1}, \ldots, x'_{nk'_n}))_{n\geq 1}$.

Suppose that $\left(((x^\gamma_{n1}, \ldots, x^\gamma_{nk^\gamma_n}))_{n\geq 1}\right)_{\gamma\in\Gamma}$ is a chain in $(\mathcal{S}, {}^*\!\succ)$. We construct an upper bound of this chain in the following way.

Consider $((x^\gamma_{11}, \ldots, x^\gamma_{1k^\gamma_1}))_{\gamma\in\Gamma}$ and the subset $\{k^\gamma_1\}_{\gamma\in\Gamma}$ of nonzero naturals. Because $\mathbb{N}$ is a well ordered set, $\{k^\gamma_1\}_{\gamma\in\Gamma}$ has a least element $k^{\gamma_1}_1$. The definition of ${}^*\!\succ$ implies that, for all $\gamma \geq \gamma_1$,

$$k^\gamma_1 = k^{\gamma_1}_1 \text{ and } (x^\gamma_{11}, \ldots, x^\gamma_{1k^\gamma_1}) = (x^{\gamma_1}_{11}, \ldots, x^{\gamma_1}_{1k^{\gamma_1}_1}).$$

Suppose that $r \geq 1$ and that we have

$$(x^{\gamma_1}_{11}, \ldots, x^{\gamma_1}_{1k^{\gamma_1}_1}), \ldots, (x^{\gamma_r}_{r1}, \ldots, x^{\gamma_r}_{rk^{\gamma_r}_r}) \tag{19}$$

such that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_r$, and, for $i = 1, \ldots, r$,

$$\text{if } \gamma \geq \gamma_i, \text{ then } k^\gamma_i = k^{\gamma_i}_i \text{ and } (x^\gamma_{i1}, \ldots, x^\gamma_{ik^\gamma_i}) = (x^{\gamma_i}_{i1}, \ldots, x^{\gamma_i}_{ik^{\gamma_i}_i}).$$

In particular, by the definition of ${}^*\!\succ$, there are no repetitions in (19) and

$$x^{\gamma_1}_{11} \cdots x^{\gamma_1}_{1k^{\gamma_1}_1} \geq \cdots \geq x^{\gamma_r}_{r1} \cdots x^{\gamma_r}_{rk^{\gamma_r}_r}.$$

Consider $\{k^\gamma_{r+1}\}_{\gamma\geq\gamma_r}$. Let $\gamma_{r+1} \in \Gamma$ be such that $k^{\gamma_{r+1}}_{r+1}$ is the least element of $\{k^\gamma_{r+1}\}_{\gamma\geq\gamma_r}$. Then, for all $\gamma \geq \gamma_{r+1}$,

$$k^\gamma_{r+1} = k^{\gamma_{r+1}}_{r+1} \text{ and } (x^\gamma_{r+11}, \ldots, x^\gamma_{r+1k^\gamma_{r+1}}) = (x^{\gamma_{r+1}}_{r+11}, \ldots, x^{\gamma_{r+1}}_{r+1k^{\gamma_{r+1}}_{r+1}}).$$

Moreover, $(x^{\gamma_{r+1}}_{r+11}, \ldots, x^{\gamma_{r+1}}_{r+1k^{\gamma_{r+1}}_{r+1}}) \neq (x^{\gamma_i}_{i1}, \ldots, x^{\gamma_i}_{ik^{\gamma_i}_i})$ for $i = 1, \ldots, r$.

In this way we obtain $((x^{\gamma_n}_{n1}, \ldots, x^{\gamma_n}_{nk^{\gamma_n}_n}))_{n\geq 1} \in \mathcal{S}$, such that, for all $\gamma \in \Gamma$,

$$((x^{\gamma_n}_{n1}, \ldots, x^{\gamma_n}_{nk^{\gamma_n}_n}))_{n\geq 1} \ {}^*\!\!\succeq ((x^\gamma_{n1}, \ldots, x^\gamma_{nk^\gamma_n}))_{n\geq 1}.$$

That is, the sequence $((x^{\gamma_n}_{n1}, \ldots, x^{\gamma_n}_{nk^{\gamma_n}_n}))_{n\geq 1}$ is an upper bound in $(\mathcal{S}, {}^*\!\succ)$ of the chain $\left(((x^\gamma_{n1}, \ldots, x^\gamma_{nk^\gamma_n}))_{n\geq 1}\right)_{\gamma\in\Gamma}$.

Therefore, by Zorn's Lemma, $(\mathcal{S}, {}^*\!\succ)$ has a maximal element. This means that $(\mathcal{S}, \succ)$ has a minimal element.

We summarize what we have shown until now. If $\sum\limits_{(i_1,\ldots,i_n)\in\nu(I)} f_{i_1} \cdots f_{i_n}$ is not defined in $R((G,<))$, the partially ordered set $(\mathcal{S}, \succ)$ is not empty and has a minimal element. Let $((x^0_{n1}, \ldots, x^0_{nk_n}))_{n\geq 1}$ be a minimal element in $(\mathcal{S}, \succ)$.

The sequence of first components $\{x^0_{n1}\}_{n\geq 1}$ is a subset of the well ordered set $\bigcup\limits_{i\in I} \operatorname{supp}(f_i)$. Hence there exists a subsequence $\{x^0_{n_l1}\}_{l\geq 1}$ such that

$$x^0_{n_11} \leq x^0_{n_21} \leq \cdots \leq x^0_{n_l1} \leq \cdots.$$

We prove that $((x^0_{n_l2}, \ldots, x^0_{n_lk_{n_l}}))_{l\geq 1}$ is a sequence in $\mathcal{S}$.

First notice that $k_{n_l} \geq 2$ for all l, otherwise $((x^0_{n1}, \ldots, x^0_{nk_n}))_{n\geq 1} \notin \mathcal{S}$ because there would be repetitions since $x_{n_ls} > 1$.

Suppose that $l < s$. Consider $(x^0_{n_l1}, \ldots, x^0_{n_lk_{n_l}})$ and $(x^0_{n_s1}, \ldots, x^0_{n_sk_{n_s}})$.

If $x^0_{n_l 1} = x^0_{n_s 1}$, then $(x^0_{n_l 2},\dots,x^0_{n_l k_{n_l}})$ and $(x^0_{n_s 2},\dots,x^0_{n_s k_{n_s}})$ are not empty and not the same, since $(x^0_{n_l 1},\dots,x^0_{n_l k_{n_l}}) \neq (x^0_{n_s 1},\dots,x^0_{n_s k_{n_s}})$.

Suppose that $x^0_{n_l 1} < x^0_{n_s 1}$. We already know that $(x^0_{n_l 2},\dots,x^0_{n_l k_{n_l}})$ and $(x^0_{n_s 2},\dots,x^0_{n_s k_{n_s}})$ are not empty. Since $x^0_{n_l 1}\cdots x^0_{n_l k_{n_l}} \geq x^0_{n_s 1}\cdots x^0_{n_s k_{n_s}}$, then $x^0_{n_l 2}\cdots x^0_{n_l k_{n_l}} > x^0_{n_s 2}\cdots x^0_{n_s k_{n_s}}$. Thus $(x^0_{n_l 2},\dots,x^0_{n_l k_{n_l}}) \neq (x^0_{n_s 2},\dots,x^0_{n_s k_{n_s}})$.

Also,

$$x^0_{n_1 2}\cdots x^0_{n_1 k_{n_1}} \geq \cdots \geq x^0_{n_l 2}\cdots x^0_{n_l k_{n_l}} \geq \cdots,$$

since $x^0_{n_1 1} \leq x^0_{n_2 1} \leq \cdots \leq x^0_{n_l 1} \leq \cdots$.

Consider now the sequence

$$\begin{gathered}(x^0_{11},\dots,x^0_{1k_1}), \quad \dots, \quad (x^0_{n_1-11},\dots,x^0_{n_1-1k_{n_1-1}}),\\ (x^0_{n_1 2},\dots,x^0_{n_1 k_{n_1}}), \quad (x^0_{n_2 2},\dots,x^0_{n_2 k_{n_2}}), \quad \dots,\\ (x^0_{n_l 2},\dots,x^0_{n_l k_{n_l}}), \quad \dots\end{gathered} \tag{20}$$

As before, since by hypothesis $x^0_{n_l 1} > 1$, for all $n_l \geq 1$, then

$$x^0_{n1}\cdots x^0_{nk_n} \geq x^0_{n_l 1}x^0_{n_l 2}\cdots x^0_{n_l k_{n_l}} > x^0_{n_l 2}\cdots x^0_{n_l k_{n_l}}$$

for $1 \leq n \leq n_1 - 1$ and $l \geq 1$. Then, since $\big((x^0_{n_l 2},\dots,x^0_{n_l k_{n_l}})\big)_{l\geq 1} \in \mathcal{S}$, the foregoing implies that (20) is a sequence without repetitions in $\mathcal{S}$. But $\big((x^0_{n1},\dots,x^0_{nk_n})\big)_{n\geq 1} \succ$ (20) since they are equal until $n_1 - 1$; a contradiction with the minimality of $\big((x^0_{n1},\dots,x^0_{nk_n})\big)_{n\geq 1}$.

Therefore $\sum\limits_{(i_1,\dots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n}$ is defined in $R((G))$.

(v) Suppose now that R is a ring. Then it makes sense to consider $1 - \sum\limits_{i\in I} f_i$. Notice that $I \times \nu(I) = \nu(I) \setminus \{\{\emptyset\}\}$. Then,

$$\begin{aligned}&(1-\sum_{i\in I} f_i)(\sum_{(i_1,\dots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n}) =\\ &= \sum_{(i_1,\dots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n} - (\sum_{i\in I} f_i)(\sum_{(i_1,\dots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n}) =\\ &= \sum_{(i_1,\dots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n} - \sum_{(i,i_1,\dots,i_n)\in I\times\nu(I)} f_i f_{i_1}\cdots f_{i_n} =\\ &= 1 + \sum_{(i,i_1,\dots,i_n)\in I\times\nu(I)} f_i f_{i_1}\cdots f_{i_n} - \sum_{(i,i_1,\dots,i_n)\in I\times\nu(I)} f_i f_{i_1}\cdots f_{i_n} = 1.\end{aligned}$$

In the same way $(\sum\limits_{(i_1,\dots,i_n)\in\nu(I)} f_{i_1}\cdots f_{i_n})(1-\sum\limits_{i\in I} f_i) = 1$. □

As a consequence we get the result by A.I. Mal'cev [**Mal48**] and B.H. Neumann [**Neu49a**]. It is very important how the inverse of a nonzero series is constructed. Observe that it is an algorithmic process.

Corollary 4.20. *Let R be a ring. Let $(G,<)$ be an ordered group. Let RG be a crossed product group ring. Consider the associated Mal'cev-Neumann series ring $R((G,<))$. Let $f = \sum\limits_{x\in G} a_x\bar{x} \in R((G,<))$, and $\omega(f) = x_0$. If $a_{x_0} \in R^\times$, then f is invertible and*

$$f^{-1} = \sum_{m\geq 0} (a_{x_0}\bar{x}_0)^{-1}(g(a_{x_0}\bar{x}_0)^{-1})^m,$$

where $g = a_{x_0}\bar{x}_0 - f$. In particular, $R((G,<))$ is a division ring provided that R is a division ring.

Proof. Since a_{x_0} is invertible, $a_{x_0}\bar{x}_0$ is too. Notice that $g(a_{x_0}\bar{x}_0)^{-1} = 1 - f(a_{x_0}\bar{x}_0)^{-1} > 1$. Then Theorem 4.19(iv) implies that $f(a_{x_0}\bar{x}_0)^{-1} = 1 - g(a_{x_0}\bar{x}_0)^{-1}$ is invertible and

$$(1 - g(a_{x_0}\bar{x}_0)^{-1})^{-1} = \sum_{m\geq 0} (g(a_{x_0}\bar{x}_0)^{-1})^m.$$

Since $f(a_{x_0}\bar{x}_0)^{-1}$ and $a_{x_0}\bar{x}_0$ are invertible, then f is invertible and

$$f^{-1} = \sum_{m\geq 0} (a_{x_0}\bar{x}_0)^{-1}(g(a_{x_0}\bar{x}_0)^{-1})^m. \qquad \square$$

The following is [**DL82**, Corollary 2.1] for crossed product monoid rings. The proof given there also works for the more general setting.

Corollary 4.21. *Let k be a division ring. Let $(G,<)$ be an ordered monoid. Let kG be a crossed product monoid ring. Consider the associated Mal'cev-Neumann series ring $k((G,<))$. Let H be a subset of $k((G,<))$ that is a subgroup of $k((G,<))^\times$. Suppose that H is totally ordered under the partial order defined on $k((G,<))$ and that $\lambda H\lambda^{-1} \subseteq H$ for all $\lambda \in k^\times$. Let*

$$D = \Big\{ d = \sum_{h\in H} \lambda_h h \mid \lambda_h \in k,\ d \text{ is defined in } k((G,<)) \Big\}.$$

Then D is a division ring and $\omega(D^\times) = \omega(H)$. Moreover, D embeds in the subring $k((\omega(H),<))$ of $k((G,<))$.

Proof. By Theorem 4.19(i)-(iii), we get that D is a subring of $k((G,<))$. Consider any $d = \sum_{h\in H} \lambda_h h \in D \setminus \{0\}$. Since $\bigcup_{h\in H, \lambda_h\neq 0} \operatorname{supp}(h)$ is well ordered, the set $\{\omega(h) \mid h \in H,\ \lambda_h \neq 0\}$ is well ordered. Therefore $H_d = \{h \in H \mid \lambda_h \neq 0\}$ is well ordered. Let h_d be the least element of H_d. Then $(\lambda_{h_d}\bar{h}_d)^{-1}d = 1 - \sum_{h\in H} \lambda'_h h$, for some $\lambda'_h \in k$, is defined in $k((G,<))$. Notice that $(\lambda_{h_d}\bar{h}_d)^{-1} \in D$ and $1 < (\lambda_{h_d}\bar{h}_d)^{-1}d$. By Theorem 4.19(iv) it has a two-sided inverse which belongs to D, and hence so has d. Thus D is a division ring. Moreover $\omega(d) = \omega(h_d) \in \omega(H)$, and since for each $x \in \omega(H)$ there exists $h' \in H$ with $\omega(h') = x$, we obtain $\omega(D^\times) = \omega(H)$.

To see that D embeds in $k((\omega(H),<))$ consider the map $\Phi\colon D \to k((\omega(H),<))$ defined by $\sum_{h\in H} \lambda_h h \mapsto \sum_{h\in H} \lambda_h \omega(h)$. Then Φ is well defined because $\{\omega(h) \mid h \in H,\ \lambda_h \neq 0\}$ is well ordered. Moreover Φ is a morphism of rings by Theorem 4.19, and, since H is totally ordered, Φ is injective. $\square$

Let k be a division ring. Let G be an orderable group. Let $<_1, <_2$ be two different total orders on G such that $(G,<_1)$ and $(G,<_2)$ are different ordered groups. For $i = 1,2$, let D_i be the division ring of fractions of kG inside $k((G,<_i))$. At first sight, it is not clear whether D_1 is isomorphic to D_2. Certainly they are different as sets. Let $P_i = \{g \in G : 1 <_i g\}$, $i = 1,2$. As we have seen in Lemma 2.14, P_i determines $<_i$ and $G = P_i \cup \{1\} \cup P_i^{-1}$. Therefore there exists $g_1 \in P_1 \setminus P_2$. Then the series $\sum_{n\geq 0} \bar{g}_1^n$ is the inverse of $1 - \bar{g}_1$ in $k((G,<_1))$, but it does not exist in $k((G,<_2))$ since $g_1^{n+1} <_2 g_1^n$. The inverse of $1 - \bar{g}_1$ in $k((G,<_2))$ is $-\sum_{n\geq 0} \bar{g}_1^{-(n+1)}$. Moreover, when $<_2$ is the order given by $h <_2 g$ if and only if $g <_1 h$, we get that the only series which are in $k((G,<_1)) \cap k((G,<_2))$ are the ones with finite support, that is kG. This follows because a series with infinite support A in $k((G,<_i))$ has a strictly ascending sequence $\{a_n\}$ with respect $<_i$ inside A, but $\{a_n\}$ is strictly decreasing with respect $<_{i'}$.

We will see in Corollary 6.5 that, as a consequence of Hughes' Theorem I 6.3, D_1 and D_2 are kG-isomorphic. For the concrete case of the free group, it can also be seen as a result of Lewin's construction of the free division ring as Corollary 4.41 shows.

4. The free division ring of fractions

In this section we briefly present P.M. Cohn's construction of the free division ring of fractions. This concept is one of the main objects of study in this dissertation. The results in this section follow from the ones in Section 3.2 of Chapter 3.

In the next theorem, the proof of (i) is taken from [**LL78**, Section 2], and (ii) is [**Coh85**, Exercise 1.1.3].

THEOREM 4.22. *Let k be a division ring. Let G be a group. Form a crossed product group ring kG. The following hold,*

(i) *If G is a free group on the set X, then kG and the polynomial ring $k\langle X\rangle$ are firs.*
(ii) *If G is a locally-free group, then kG is a semifir.*

In any of these events, kG (respectively $k\langle X\rangle$) has a universal division ring of fractions. Moreover, if Φ is the set of full matrices over kG (respectively over $k\langle X\rangle$), then the localization of kG ($k\langle X\rangle$) at Φ is the universal division ring of fractions of kG ($k\langle X\rangle$).

PROOF. (i) Suppose that G is the free group on the set X. Notice that for every $x\in X$ the subring R_x generated by k and $\{\bar{x},\bar{x}^{-1}\}$ (generated by k and $\bar{x}$) is isomorphic to the skew Laurent polynomial ring $k[x,x^{-1};\alpha]$ (skew polynomial ring $k[x;\alpha]$), where $\alpha(a)=\bar{x}a\bar{x}^{-1}$, for all $a\in k$ is an isomorphism, see Remark 4.6. Hence R_x a fir by Examples 3.37(b).

Consider the family $(R_x)_{x\in X}$ and the coproduct $\mathop{*}\limits_{k} R_x$. By Theorem 3.42(ii), $\mathop{*}\limits_{k} R_x$ is a fir.

By Definition 3.41, since for every $x\in X$ we have the inclusion map $u_x\colon R_x\hookrightarrow kG$ ($u_x\colon R_x\hookrightarrow k\langle X\rangle$), there exists an onto morphism of k-rings $f\colon \mathop{*}\limits_{k} R_x\to kG$ ($f\colon \mathop{*}\limits_{k} R_x\to k\langle X\rangle$). For each x, a k-basis of R_x is $S_x=\{\bar{x}^n\mid n\in\mathbb{Z}\}$ ($S_x=\{\bar{x}^n\mid n\in\mathbb{N}\}$). Then f is injective because the image of the k-basis of $\mathop{*}\limits_{k} R_x$ given by Theorem 3.42(i) is a k-basis of kG ($k\langle X\rangle$).

(ii) By definition of locally free group, for every finitely generated subgroup H of G, H is a free group. Then, by (i), kH is a fir.

On the other hand, $kG=\varinjlim\limits_{H\leq_{\text{f.g}}G} kH$. So if we prove that the direct limit of a directed set of semifirs is a semifir we are done.

Let $(\Gamma,<)$ be an upper directed set. Let $(R_\gamma,\varphi_{\delta\gamma}\mid\gamma\leq\delta\in\Gamma)$ be a direct system of semifirs. Thus, if $\gamma,\delta\in\Gamma$, $\gamma\leq\delta$, $\varphi_{\delta\gamma}\colon R_\gamma\to R_\delta$ is a morphism of rings. Suppose that R_γ is a semifir for every $\gamma\in\Gamma$. Let $R=\varinjlim R_\gamma$. Then, for every $\gamma\leq\delta\in\Gamma$, there exist morphisms of rings $\varphi_\gamma\colon R_\gamma\to R$, $\varphi_\delta\colon R_\delta\to R$, such that $\varphi_\gamma=\varphi_\delta\varphi_{\delta\gamma}$.

Suppose that $a\cdot b=a_1b_1+\cdots+a_rb_r=0$ in R. It is known that by the construction of the direct limit, there exists $\gamma\in\Gamma$ such that $a_1,b_1,\dots,a_r,b_r$ are the image of $a_1',b_1',\dots,a_r',b_r'\in R_\gamma$ by φ_γ, and $a'\cdot b'=a_1'b_1'+\cdots+a_r'b_r'=0$. Since R_γ is a semifir, by Proposition 3.39, there exists an invertible $r\times r$ matrix P with entries in R_γ such that $a'P^{-1}\cdot Pb'=0$ is a trivial relation. Then $\varphi_\gamma P$ verifies that $a(\varphi_\gamma P)^{-1}\cdot\varphi_\gamma Pb=0$ is a trivial relation. Then the result follows by Proposition 3.39.

The last part follows from Theorem 3.40. □

Let R be a ring, and let $G\neq 1$ be a group. In [**Won78**] it is proved that the group ring $R[G]$ is a fir if and only if R is a division ring and G is a free group. In [**DM79**] it is proved that the group ring $R[G]$ is a semifir if and only if R is a division ring and G is locally free. The if part of this result is proved analogously for crossed product group rings (see the proof given in Theorem 4.22). But the only if part does not generalize to crossed product group rings. Indeed, let $G \neq 1$ be a free group, and let N be a nontrivial normal subgroup of G. By Lemma 4.7, $kG = kN(G/N)$ and kG is a fir (therefore a semifir), but kN is not a division ring, and G/N is not a locally free group in many cases. As an example, let G be an infinite cyclic group generated by g. Let $N = \langle g^n \rangle$, for some $n \geq 1$. Then $kG = kN * C_n$ where C_n is the cyclic group of n elements.

Definition 4.23. Let k be a division ring, and let G be a free group on a set X. Consider a crossed product group ring kG. The universal division ring of fractions of kG will be called *the free division ring* associated with kG. □

Usually, when k is a field, the term *free field* refers to the universal division ring of fractions of the free algebra $k\langle X \rangle$ associated with the free group ring $k[G]$.

The following consequences of Proposition 3.39 will be very useful for us in Section 5 to give another description of the universal division ring of fractions of a crossed product group ring kG with k a division ring and G a free group.

Corollary 4.24. *Let R be a semifir. A matrix A is full if and only if $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ is full for any identity matrix I. Hence a square matrix A is full if and only if any stably associated matrix A' is full.*

Proof. Let A be an $n \times n$ matrix over R. Notice that if A is not full, then $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ is not full. Conversely, suppose that the $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not full. Then there exist matrices B_1, B_2, C_1, C_2 over R of sizes $n \times n$, $1 \times n$, $n \times n$, $n \times 1$, respectively such that

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

In particular $B_1 C_2 = 0$. By Proposition 3.39, there exists an $n \times n$ invertible matrix P that trivializes $B_1 C_2 = 0$. Since $B_2 C_2 = B_2 P^{-1} P C_2 = 1$, we obtain that $PC_2 \neq 0$. Thus a column, say the i-th one, of $B_1 P^{-1}$ is zero. If B_1' is the matrix obtained erasing the i-th column of $B_1 P^{-1}$ and C_1' is the matrix obtained erasing the i-th row of PC_1, we get that $A = B_1' C_1'$. Hence A is not full. Now the result follows by induction on the size of I. □

Definition 4.25. Corollary 4.24 allows us to use *Higman's trick*: Let R be a ring and A an $n \times n$ matrix over R such that the (i,j)-th entry of A is of the form $f + ab$ with $f, a, b \in R$. Then by enlarging the matrix and applying row and column elementary transformations over R we obtain successively:

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & \vdots & & 0 \\ \cdots & f+ab & \cdots & 0 \\ & \vdots & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} & \vdots & & 0 \\ \cdots & f+ab & \cdots & a \\ & \vdots & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} & \vdots & & 0 \\ \cdots & f & \cdots & a \\ & \vdots & & 0 \\ \hline 0 & -b & 0 & 1 \end{array}\right)$$

Thus, applying Higman's trick repeatedly, the matrix A is stably associated over R to a matrix B whose entries are "less complex" than the entries of A. □

Higman's trick will be very useful in the following situation. The proof follows from the definition.

Lemma 4.26. *Let k be a division ring, and let G be the free group on a set X. Consider a crossed product group ring kG and its polynomial ring $k\langle X \rangle$. The following hold:*

(i) *If A is a matrix with entries over kG, then A is stably associated over kG to a matrix B such that the support of each entry of B is contained in $X \cup \{1\} \cup X^{-1}$, where X^{-1} denotes the subset of G consisting of the inverses of X.*

(ii) *If A is a matrix with entries over $k\langle X\rangle$, then A is stably associated over $k\langle X\rangle$ to a matrix B such that the support of each entry of B is contained in $X \cup \{1\}$.* $\square$

5. The free division ring of fractions inside the Mal'cev-Neumann series ring: Reutenauer's approach

We have seen in Theorem 4.22 that if G is a free group on a set X (or more generally, a locally free group), and k a division ring, then any crossed product group ring kG and any polynomial ring $k\langle X\rangle$ are semifirs. Therefore they have a universal division ring of fractions. This division ring of fractions is constructed in terms of generators and relations, and thus it is not easy to work with it in many situations. So a more explicit description of the universal division ring of fractions is needed.

J. Lewin and T. Lewin showed in [**Lew74**, Theorem 2] and [**LL78**, Section 2] that the universal division ring of fractions of a crossed product group ring kG, where G is a free group and k a division ring, is the division ring of fractions of kG inside the Mal'cev-Neumann series ring $k((G,<))$ for any total order $<$ such that $(G,<)$ is an ordered group. When kG is a group ring J. Lewin also showed that the universal division ring of fractions of $k\langle X\rangle$ is the division ring of fractions of $k\langle X\rangle$ inside $k((G,<))$. More recently, C. Reutenauer [**Reu99**] gave an easier proof of this fact with X finite. We follow Reutenauer's proof and extend it to the general case stated by J. Lewin and T. Lewin. We also observe that Lewin's theorem can be extended to locally free groups.

5.1. Technical definitions and results. What follows was proved in [**Reu99**, Section 3.2] for a finitely generated free group G, a field k and the group ring $k[G]$. But as we see, they can be stated for any ordered group G, any division ring k and any crossed product group ring kG.

Definitions 4.27. Let $(G,<)$ be an ordered group and X a subset of G with $1 \notin X$. Let k be a division ring. Consider a crossed product group ring kG and the corresponding Mal'cev-Neumann series ring $k((G,<))$.

(a) Let $g_1,\dots,g_n$, $n \geq 1$, be different elements in G. We associate to them a graph $\Gamma(X) = (\Gamma(X), V, E, \iota, \tau)$ as follows. The set of vertices V consists of $g_1,\dots,g_n$ together with the elements $g \in G$ such that there exist $i \neq j$ and $x, y \in X \cup \{1\}$ such that $g = xg_i = yg_j$. This second kind of vertices are said to be *special*. Notice that some g_i may be a special vertex. For $x \in X \cup \{1\}$, there is an edge, labeled (x, g_i), with $\iota(x, g_i) = g_i$ and $\tau(x, g_i) = xg_i$ (i.e. $xg_i \overset{(x,g_i)}{\longleftarrow} g_i$) if and only if xg_i is a special vertex. Note that the heads of $\Gamma(X)$ are exactly the special vertices.

(b) Let $S_1,\dots,S_n \in k((G,<)) \setminus \{0\}$, $n \geq 1$. For each $i = 1,\dots,n$, let $g_i = \min \operatorname{supp} S_i$. Suppose that all g_i are different. Let M be a $p \times n$ matrix with entries in the k-subspace of kG spanned by all $\bar{x}$ with $x \in X \cup \{1\}$. Denote by m_{ji} the (j,i)-th entry of M. Consider the graph associated with $g_1,\dots,g_n$ given in (a). We associate to M a subgraph $\Gamma(X, M)$ of $\Gamma(X)$. $\Gamma(X,M)$ has the same vertices as $\Gamma(X)$ and an edge $xg_i \overset{(x,g_i)}{\longleftarrow} g_i$ in $\Gamma(X)$ is an edge of $\Gamma(X,M)$ if x is in the support of some entry in the i-th column of M. $\square$

Remark 4.28. Note that if we remove the loops $g_i \overset{(1,g_i)}{\longleftarrow} g_i$ in $\Gamma(X)$, then $\Gamma(X)$ becomes a subgraph of the Cayley graph $\mathcal{C}(G,X)$. $\square$

Lemma 4.29. *Under the notation and assumptions of* Definitions 4.27(b), *suppose that* $M\begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} = 0$, *that is,*

$$m_{j1}S_1 + m_{j2}S_2 + \cdots + m_{jn}S_n = 0 \tag{21}$$

for $j = 1, \ldots, p$. *Then the following hold true:*

(i) *If the* j*-th row of* M *is not zero then*

$$h_j = \min_{t=1,\ldots,n}\{x_{jt}g_t\}, \tag{22}$$

where $x_{jt} = \min \operatorname{supp} m_{jt}$, *is a head of* $\Gamma(X, M)$.

(ii) *If* $M \neq 0$, *the smallest head of* $\Gamma(X, M)$ *is* $h = \min_{j=1,\ldots,p}\{h_j\}$, *and* h *can only be obtained in* (21) *with the products* $x_{jt}g_t$.

(iii) *If* $\Gamma(X, M)$ *has no edges, then* $M = 0$.

Proof. (i) Note that if $m_{jt} \neq 0$ for some $t \in \{1, \ldots, n\}$, then $x_{jt}g_t \in \operatorname{supp} m_{jt}S_t$ because, since $(G, <)$ is an ordered group,

$$x_{jt}g_t < ba \text{ for each } b \in \operatorname{supp} m_{jt},\ a \in \operatorname{supp} S_t \text{ with } b \neq x_{jt} \text{ or } a \neq g_t. \tag{23}$$

In particular $h_j \in \operatorname{supp} m_{jt_0}S_{t_0}$ for some $t_0 \in \{1, \ldots, n\}$. Then, by (21), (23) and the definition of h_j, there exists $t_1 \neq t_0$ such that $x_{jt_1}g_{t_1} = x_{jt_0}g_{t_0} = h_j$. Hence h_j is a head of $\Gamma(X)$ and of $\Gamma(X, M)$.

(ii) Since $M \neq 0$, h is a head by (i). The result follows from (23) and the definition of h.

(iii) If $M \neq 0$, by (i), there exists a head, and therefore an edge, in $\Gamma(X, M)$. □

Remark 4.30. If $P \in \mathrm{GL}_p(k)$, then the set of heads of $\Gamma(X, PM)$ is contained in the set of heads of $\Gamma(X, M)$. Similarly for the set of heads of $\Gamma(X, M')$ where M' is any submatrix of M. □

Lemma 4.31. *Under the notation and assumptions of* Definitions 4.27, *suppose that* $M \neq 0$ *and that* $M\begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} = 0$. *Let* h *be the smallest head of* $\Gamma(X, M)$. *Let* $e(h)$ *be the number of edges in* $\Gamma(X)$ *with head* h. *Assume that* $p \geq e(h) - 1$. *Then there exists* $P \in \mathrm{GL}_p(k)$ *such that* $PM = \begin{pmatrix} M_1 \\ M' \end{pmatrix}$, *where* M' *is of size* $(p - e(h) + 1) \times n$. *Moreover,* $M'\begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} = 0$, *and* h *is not a head of* $\Gamma(X, M')$.

Proof. We may assume that the edges in $\Gamma(X)$ with head h are $h \overset{(x_1,g_1)}{\longleftarrow} g_1, \ldots, h \overset{(x_f,g_f)}{\longleftarrow} g_f$, with $f = e(h)$. Denote by $m_{ji}(x)$ the coefficient of $\bar{x}$ in m_{ji}. We claim that the coefficient of $\bar{h}$ in (21) is

$$m_{j1}(x_1)\alpha_1^{\sigma(x_1)}\tau(x_1, g_1) + m_{j2}(x_2)\alpha_2^{\sigma(x_2)}\tau(x_2, g_2) + \cdots + m_{jf}(x_f)\alpha_f^{\sigma(x_f)}\tau(x_f, g_f),$$

where α_i is the coefficient of $\bar{g}_i$ in S_i. Certainly this is a summand of the coefficient of $\bar{h}$ in (21). So suppose that there is some $l \in \{1, \ldots, n\}$, $g \in G$ and some $x \in X \cup \{1\}$ such that $g \in \operatorname{supp} S_l$, $x \in \operatorname{supp} m_{jl}$, and $xg = h$. By Lemma 4.29(ii), $g = g_l$ and $x = x_{jl} = x_l$. Therefore $l \in \{1, \ldots, f\}$, and the claim is proved.

From (21) we obtain that for any $j = 1, \ldots, p$,

$$m_{j1}(x_1)\alpha_1^{\sigma(x_1)}\tau(x_1, g_1) + m_{j2}(x_2)\alpha_2^{\sigma(x_2)}\tau(x_2, g_2) + \cdots + m_{jf}(x_f)\alpha_f^{\sigma(x_f)}\tau(x_f, g_f) = 0.$$

This means that the columns of the $p \times f$ matrix

$$A = (m_{ji}(x_i))_{\substack{1 \le j \le p \\ 1 \le i \le f}}$$

are right k-linearly dependent. Thus its rank is at most $f-1$. Since $p \ge f-1$, there exists $P \in \mathrm{GL}_p(k)$ such that $PA = \begin{pmatrix} A_1 \\ A' \end{pmatrix}$, where $A' = 0$ is of size $(p-f+1) \times f = p' \times f$. Then $PM = \begin{pmatrix} M_1 \\ M' \end{pmatrix}$ where M' is of size $(p-f+1) \times n$, and such that x_i does not appear in the i-th column of M', for $i = 1, \dots, f$. Hence the edge $h \overset{(x_i, g_i)}{\longleftarrow} g_i$ does not exist in $\Gamma(X, M')$. □

This is [**Reu99**, Lemma 1] which turns out to be very important.

Lemma 4.32. *Let G be a free group on a finite set X. For $n \ge 1$, let $g_1, \dots, g_n$ be distinct elements of G. Associate to them the graph $\Gamma(X)$ defined in* Definitions 4.27(a). *Let e be the number of edges in $\Gamma(X)$, and let s be the number of special vertices. Then $e \le n + s - 1$.*

Proof. Let l be the number of loops, and s' be the number of special vertices without loop around them. Then $s = s' + l$ and $n = n' + l$ where n' is the number of g_i without loop. So the total number of vertices in $\Gamma(X)$ is $n + s' = n + s - l$. By Remark 4.28, if we remove the loops in $\Gamma(X)$, then we obtain a subgraph of the Cayley graph $\mathcal{C}(G, X)$, hence a union of trees by Proposition 1.37. Therefore if we replace each loop around a vertex v by a new edge $v \to \bar{v}$, where $\bar{v}$ is a new vertex, we obtain a union of trees. This graph has e edges, and $n + s - l + l = n + s$ vertices. Recall that in a union of trees the number of edges is smaller or equal than the number of vertices minus one by Proposition 1.37. Thus $e \le n + s - 1$ as desired. □

Before stating Theorem 4.35, the extension of [**Reu99**, Theorem] to crossed product group rings, we need to give some definitions and one easy technical result.

Definitions 4.33. Let G be a free group on a set X. Let k be a division ring and kG a crossed product group ring. A matrix M over kG is a

(a) *linear matrix* if each entry of M has the support contained in $X \cup \{1\} \cup X^{-1}$.
(b) *polynomial linear matrix* if each entry of M has the support contained in $X \cup \{1\}$.

Suppose now that G is a locally free group. A matrix M over kG is a

(c) *polynomial linear matrix* if there exists a finite subset Y of G such that Y is the basis of a free subgroup H of G and the support of each entry of M is contained in $Y \cup \{1\}$. □

Lemma 4.34. *Let k be a division ring. Let G be a free group on a set X. Consider a crossed product group ring kG. Then a linear matrix M over kG is stably associated with a polynomial linear matrix M'. Furthermore, M is full over kG if and only if M' is full over kG.*

Proof. We proceed by induction on the number m of elements of the form $\overline{x^{-1}}$ that appear in M (counting multiplicities). If $m = 0$, then M is polynomial linear and the result follows. Suppose that $m > 0$. Thus M is not a polynomial linear matrix. If in the i-th row of M there is an element of the form $\overline{x^{-1}}$, multiply M by the invertible matrix that correspond to multiply the i-th row by $\bar{x}$. Now use Higman's trick 4.25 to obtain again a linear matrix. The matrix so obtained is stably associated with M and has less elements of the form $\overline{x^{-1}}$ among its entries. Then apply induction hypothesis to obtain the result.

For the second part, recall that a matrix over a semifir is full if and only if any stably associated matrix is full by Corollary 4.24. □

Theorem 4.35. *Let k be a division ring. Let G be a free group on a finite set X. Let $<$ be a total order on G such that $(G,<)$ is an ordered group. Consider a crossed product group ring kG and the associated Mal'cev-Neumann series ring $k((G,<))$. Then a polynomial linear matrix which is not invertible over $k((G,<))$ is not full over $k\langle X\rangle$ and hence not full over kG.*

Proof. Let A be a $p\times p$ polynomial linear matrix. Suppose that A is not invertible over $k((G,<))$. Thus A, as an endomorphism of the right $k((G,<))$-vector space $k((G,<))^n$, is not injective. Hence there exist $S_1,\dots,S_p$ Ma'lcev-Neumann series, not all 0, such that $A\begin{pmatrix} S_1\\ \vdots\\ S_p\end{pmatrix}=0$.

Multiplying $\begin{pmatrix} S_1\\ \vdots\\ S_p\end{pmatrix}$ on the left by a matrix $Q\in \mathrm{GL}_p(k)$ (and A on the right by Q^{-1}), we may assume that $\begin{pmatrix} S_1\\ \vdots\\ S_p\end{pmatrix}=\begin{pmatrix} S_1\\ \vdots\\ S_n\\ 0\\ \vdots\\ 0\end{pmatrix}$, and that the leading elements $g_1,\dots,g_n$ of $S_1,\dots,S_n$ are distinct.

Now $A=(MN)$, where M is of size $p\times n$. We then have $M\begin{pmatrix} S_1\\ \vdots\\ S_n\end{pmatrix}=0$.

The result will follow applying Lemma 4.31 repeatedly. First notice that by Lemma 4.32, we have

$$e=\sum_{\substack{h\in V\Gamma(X)\\ \text{special}}} e(h)\leq n+s-1\leq p+s-1,$$

where $e(h)$ is the number of edges in $\Gamma(X)$ with head h and s the number of special vertices. Thus $p\geq 1+\sum_{\substack{h\in V\Gamma(X)\\ \text{special}}}(e(h)-1)$. Now, if $\Gamma(X,M)$ has an edge, let h_1 be its smallest head. By Lemma 4.31 we obtain a matrix $P_1\in\mathrm{GL}_p(k)$ such that $P_1M=\left(\begin{smallmatrix} M_1\\ M'\end{smallmatrix}\right)$ where M' is of size $(p-e(h_1)+1)\times n$, and h_1 is not a head of $\Gamma(X,M')$. If M' has an edge, let h_2 be its smallest head. Again by Lemma 4.31, we obtain a matrix $P_2\in\mathrm{GL}_p(k)$ such that $P_2P_1M=\left(\begin{smallmatrix} M_2\\ M''\end{smallmatrix}\right)$ where M'' is of size $\left(p-\sum_{i=1}^{2}(e(h_i)-1)\right)\times n$ such that neither h_1 nor h_2 are heads of $\Gamma(X,M'')$. Continuing in this way, in each step we reduce the number of heads by Remark 4.30. So after $r\leq s$ steps, we obtain matrices $P_1,\dots,P_r\in\mathrm{GL}_p(k)$ such that $P_r\cdots P_1M=\left(\begin{smallmatrix} M_r\\ M^{(r)}\end{smallmatrix}\right)$ where $M^{(r)}$ is of size $\left(p-\sum_{i=1}^{r}(e(h_i)-1)\right)\times n$ and such that $\Gamma(X,M^{(r)})$ has no edges. Hence $M^{(r)}=0$ by Lemma 4.29(iii). Notice that $\left(p-\sum_{i=1}^{r}(e(h_i)-1)\right)\geq p-e+s$.

This shows that there exists a matrix $P\in\mathrm{GL}_p(k)$ such that PM has a rectangle of $0'$s of size at least $(p-e+s)\times n$. By Lemma 4.32, $n+p-e+s\geq p+1$, showing that M is associated with a hollow matrix.

So we have proved that there exist invertible matrices $P,Q\in\mathrm{GL}_p(k)$ such that PAQ is a hollow matrix, and therefore A is not full by Lemma 3.29. ☐

5.2. Consequences. We proceed to give the main results of this section. They are corollaries of the ones in the foregoing subsection.

THEOREM 4.36. *Let k be a division ring. Let G be a free group on a set X. Let $<$ be a total order on G such that $(G,<)$ becomes an ordered group. Consider a crossed product group ring kG and the associated Mal'cev-Neumann series ring $k((G,<))$. Let K be the division ring of fractions of kG inside $k((G,<))$ (which is the same as the one of $k\langle X\rangle$). Then every full matrix over kG (or over $k\langle X\rangle$) is invertible over K. Therefore K is the universal division ring of fractions of both kG and $k\langle X\rangle$.*

PROOF. Let A be a full matrix over kG (or over $k\langle X\rangle$). It is enough to show that A is invertible over some subdivision ring K' of K. Let H be a free subgroup of G generated by a finite subset Y of X such that the entries of A are in kH (or $k\langle Y\rangle$). Notice that A is full over kH (or $k\langle Y\rangle$), that $k((H,<)) \leq k((G,<))$ and that if K' is the division ring of fractions of kH inside $k((H,<))$, then $K' \leq K$. Now A is stably associated over kH (or $k\langle Y\rangle$) to a square polynomial linear matrix A' by Lemmas 4.26 and 4.34. Since kH and $k\langle Y\rangle$ are semifirs, A' is full over kH and $k\langle Y\rangle$ by Corollary 4.24. Then, by Theorem 4.35, A' is invertible over $k((H,<))$. Notice that A', as an endomorphism of the right K'-vector space K'^n, is injective, and thus invertible. Therefore the coeficients of A'^{-1} are in K'. Hence A' is invertible over K'. But A' is stably associated with A also over K'. Since stably associated matrices over a division ring are simultaneously invertible or not, A is invertible over K'. □

COROLLARY 4.37. *Let k be a division ring. Let L be a locally free group. Let $<$ be a total order on L such that $(L,<)$ is an ordered group. Consider a crossed product group ring kL and the associated Mal'cev-Neumann series ring $k((L,<))$. Let K' be the division ring of fractions of kL inside $k((L,<))$. Then every full matrix over kL is invertible over K'. Therefore K' is the universal division ring of fractions of kL.*

PROOF. Let A be a full matrix over kL. Then there exists a finitely generated free subgroup G of L such that the entries of A are in kG. Notice that $k((G,<)) \leq k((L,<))$. By Theorem 4.36, A is invertible over K, the division ring of fractions of kG inside $k((L,<))$. Therefore it is invertible over K'. □

COROLLARY 4.38. *Let k be a division ring. Let L be a locally free group. Consider a crossed product group ring kL. Let $U(kL)$ be the universal division ring of fractions of kL. Let H be a subgroup of L. Then the division ring of fractions of kH inside $U(kL)$ is the universal division ring of fractions of kH.*

PROOF. Let $<$ be a total order on L such that $(L,<)$ is an ordered group. Then $(H,<)$ is also an ordered group. By Corollary 4.37, $U(kL)$ is the division ring of fractions of kL inside $k((L,<))$. Notice that $k((H,<)) \leq k((L,<))$ and that the division ring of fractions of kH inside $k((H,<))$ and $k((L,<))$ is the same. The result now follows because the division ring of fractions of kH inside $k((H,<))$ is the universal field of fractions of kH by Corollary 4.37. □

COROLLARY 4.39. *Let k be a division ring. Let G be a locally free group on a set X. Consider a crossed product group ring kG. A square polynomial linear matrix A over kG of order p is not full if and only if for some invertible matrices P,Q over k, the matrix PAQ is hollow.*

PROOF. If the matrix PAQ is hollow, then PAQ is not full by Lemma 3.29. Now clearly A is not full.

Suppose that A is not full. Let $<$ be a total order on G such that $(G,<)$ is an ordered group. Consider the Mal'cev-Neumann series ring $k((G,<))$ associated with kG. Let Y be a finite subset of G such that Y is the basis of a free subgroup H of G and the support of each entry of A is contained in $Y \cup \{1\}$. Now $k((H,<)) \subseteq k((G,<))$. Observe that A is not full over kH, otherwise it would be invertible over $k((H,<))$ by Theorem 4.36, and therefore over

$k((G,<))$, a contradiction. Hence A is not full over kH, A is a polynomial linear matrix (over $k\langle Y\rangle$), so we can apply the proof of Theorem 4.35 to obtain the desired result. □

COROLLARY 4.40. *Let k be a subdivision ring of k'. Let L be a locally free group. Consider a crossed product group ring $k'L$ such that*

(i) *The image of k by $\sigma(x)$ is contained in k for each $x \in L$*
(ii) *$\tau(x,y) \in k$ for all $x, y \in L$*

then each full matrix over kL (or $k\langle X\rangle$ when L is a free group on a set X) is also full over $k'L$ (or $k'\langle X\rangle$).

PROOF. By conditions (i) and (ii), we can consider kL as a subring of $k'L$. Suppose that $(L,<)$ is an ordered group. Then, by Corollary 4.37, a full matrix A over kL becomes invertible over $k((L,<))$ which is a subdivision ring of $k'((G,<))$. Thus A is also invertible over $k'((G,<))$. Thus it has to be full over $k'G$. □

COROLLARY 4.41. *Let k be a division ring. Let L be a locally free group. Let $<_1$ and $<_2$ be two different total orders on L such that $(L,<_1)$ and $(L,<_2)$ are ordered groups. Let D_i, $i = 1, 2$, be the division ring of fractions of kL inside $k((L,<_i))$. Then D_1 is kL-isomorphic to D_2.*

PROOF. D_1 and D_2 are the universal division ring of fractions of kL. □

As we said before, Theorem 4.36 and Corollary 4.38 were first proved in [**Lew74**, Theorems 1 and 2] and [**LL78**, Section 2]. There, Corollary 4.38 was used to prove Theorem 4.36. We will do something similar in Section 2.1 of Chapter 6. Our proof of Theorem 4.36 is patterned on the one given in [**Reu99**, Corollary 1].

Statements of Corollaries 4.39 and 4.40 are the natural generalization of [**Reu99**, Corollaries 2 and 3] to crossed product group rings. They were first proved by P.M. Cohn for tensor rings, see for example [**Coh95**, Corollary 6.3.6 and Theorem 6.4.6].

Corollary 4.37 is probably well known, but we have not found it in the literature.

"Do you hear the hounds they call
Scan the dark eyes aglow
Through the bitter rain and cold
They hunt you down
Hunt you down"

Savatage, *Hounds*

Part 2

Hughes' Theorems

CHAPTER 5

Towards Hughes' Theorems

Our aim is to define the objects needed to prove the results in Chapter 6, and to show their main properties. Almost everything in this chapter appears in the joint work with W. Dicks and D. Herbera [**DHS04**]. Most of the objects deal with the concepts on semirings defined in Section 7 of Chapter 1.

For the sake of simplicity we fix the notation at the beginning of some sections and it remains valid until the end of that section unless otherwise stated.

1. Hughes-free division ring of fractions

In Theorem 6.3 it is proved that two division rings of fractions satisfying a certain property are isomorphic. It is useful to define first such a property because what follows tries to imitate that situation in a formal way.

NOTATION 5.1. Throughout this section let k be a division ring, G a locally indicable group and kG a crossed product group ring. □

Let H be a nontrivial finitely generated subgroup of G. Since G is locally indicable, there exists $N \lhd H$ such that H/N is infinite cyclic. So H can be expressed as an internal semidirect product $N \rtimes C$ with C infinite cyclic. By Remarks 4.3(c), $k^\times H/k^\times \cong H$. Hence we have a morphism of groups

$$\begin{array}{ccccc} k^\times H & \xrightarrow{\rho_H} & k^\times H/k^\times & \cong & H \\ a\bar{x} & \longmapsto & [a\bar{x}] & \mapsto & x \end{array} \tag{24}$$

Let $t \in k^\times H$ be such that $\rho_H(t)$ generates C. Clearly $t \in k^\times C$. Observe that left conjugation by the trivial unit t induces an automorphism α of kN, i.e. $\alpha\colon kN \to kN,\ z \mapsto tzt^{-1}$. Moreover the powers of t are left (and right) kN-linearly independent because every element of H can be uniquely expressed as a product of an element of N and a power of $\rho_H(t)$. Thus $kH = \bigoplus_{n\in\mathbb{Z}} kNt^n$. Therefore kH can be seen as the skew Laurent polynomial ring $kN[t, t^{-1}; \alpha]$.

Suppose that kG has a division ring of fractions D. We say that D is a Hughes-free division ring of fractions if the kN-linear independence of the powers of t extends to $D(kN)$. More precisely:

DEFINITIONS 5.2. Suppose that kG has a division ring of fractions D.

(a) An *atlas* $\mathcal{A} = \{N_H\}_H$ of G, where H ranges over all nontrivial finitely generated subgroups of G, is a set consisting of subgroups N_H of H, one for each H, such that $N_H \lhd H$ and H/N_H is infinite cyclic. If L is a fixed subgroup of G, we denote by $\mathcal{A}_L$ the atlas of L $\{N_H \in \mathcal{A} \mid H \leq L\}$. If θ is a monomorphism of G, by $\theta(\mathcal{A})$ we denote the atlas of $\theta(G)$ $\{N'_{\theta(H)} = \theta(N_H)\}_{\theta(H)}$. Notice that an atlas $\mathcal{A}$ always exists provided G is a locally indicable group. If G is trivial, then $\mathcal{A}$ is the empty atlas.

(b) Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. We say that D is an *$\mathcal{A}$-Hughes-free division ring of fractions* (of kG) if for each nontrivial finitely generated subgroup H of G, any expression of H as an internal semidirect product $N_H \rtimes C$ with C infinite cyclic and any $t \in k^\times C$

such that $\rho_H(t)$ generates C, then the powers of t are $D(kN_H)$-linearly independent, i.e. if $d_0, \dots, d_n \in D(kN_H)$, then

$$d_0 + d_1 t + \cdots + d_n t^n = 0 \quad \text{implies that} \quad d_0 = \cdots = d_n = 0.$$

(c) If D is an $\mathcal{A}$-Hughes-free division ring of fractions for each atlas $\mathcal{A}$ of G, then we say that D is a *Hughes-free division ring of fractions.*

(d) If there exists an embedding of kG in a division ring E, $kG \hookrightarrow E$, such that $E(kG)$ is an ($\mathcal{A}$-)Hughes-free division ring of fractions, we say that $kG \hookrightarrow E$ is an ($\mathcal{A}$-)*Hughes-free embedding.* We also say that kG is ($\mathcal{A}$-)*Hughes-free embeddable.* $\square$

Some other ways of expressing $\mathcal{A}$-Hughes-freeness are given in the next result.

LEMMA 5.3. *Suppose that kG has a division ring of fractions D. Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. The following are equivalent:*

(i) *D is an $\mathcal{A}$-Hughes-free division ring of fractions.*

(ii) *For each nontrivial finitely generated subgroup H of G there exist an expression of H as an internal semidirect product $N_H \rtimes C$ with C infinite cyclic and $t \in k^\times C$ such that $\rho_H(t)$ generates C and the powers of t are $D(kN_H)$-linearly independent.*

(iii) *For each nontrivial finitely generated subgroup H of G there exists an element $x \in H$ such that $N_H x$ generates H/N_H and the powers of $\bar{x}$ are $D(kN_H)$-linearly independent.*

PROOF. (i) $\Rightarrow$ (ii) It is clear because (i) is more general than (ii).

(ii) $\Rightarrow$ (iii) Let H be a nontrivial finitely generated subgroup of G. Let t be given by (ii). Set $x = \rho_H(t)$. Then $H/N_H \cong C$ is infinite cyclic and $N_H x$ generates H/N_H. The powers of $\bar{x}$ are $D(kN_H)$-linearly independent because $t = a\bar{x}$ for some $a \in k$.

(iii) $\Rightarrow$ (i) Let H be a nontrivial finitely generated subgroup of G. Let x be given by (iii). Suppose that $H = N_H \rtimes C$ with C infinite cyclic. Let $t \in k^\times C$ be such that $\rho_H(t)$ generates C. Then $t = a\bar{x}^{\pm 1}$ for some $a \in k^\times N_H$. Suppose that $d_0 + d_1 t + \cdots + d_n t^n = 0$. For each i, $d_i t^i = d_i a_i \overline{x^{\pm 1}}^i$ for some $a_i \in k^\times N_H$. Then $d_i a_i = 0$ for $i = 1, \dots, n$. Since a_i is invertible for each i, $d_0 = d_1 = \cdots = d_n = 0$. $\square$

Following the notation of Definitions 5.2 we get the following important remarks.

REMARKS 5.4. Suppose that kG has an $\mathcal{A}$-Hughes-free division ring of fractions D. For each nontrivial finitely generated subgroup H of G and $t \in k^\times C$ such that $\rho_H(t)$ generates C we have the following:

(a) Left conjugation by t induces an automorphism α of $D(kN_H)$. Moreover, the subring generated by $D(kN_H)$ and t (respectively $\{t, t^{-1}\}$) is isomomorphic to the skew polynomial ring $D(kN_H)[t;\alpha]$ (skew Laurent polynomial ring $D(kN_H)[t, t^{-1};\alpha]$).

(b) The division ring of fractions of kH inside D, $D(kH)$, is isomorphic to

$$Q^l_{\mathrm{cl}}(D(kN_H)[t, t^{-1};\alpha]) = D(kN_H)(t;\alpha).$$

Therefore $D(kH)$ embeds in the skew Laurent series ring $D(kN_H)((t;\alpha))$ in a natural way.

(c) The powers of t are also right $D(kN_H)$-linearly independent. Hence $\mathcal{A}$-Hughes-freeness is a left and right symmetric concept.

PROOF. (a) Recall that $D(kN_H) = \bigcup\limits_{n=0}^{\infty} Q_n(kN_H, D)$that , and $Q_{n+1}(kN_H, D)$ is the subring generated by the elements of $Q_n(kN_H, D)$ and its inverses, see Remark 3.16. We have already noted at the beginning of this section that left conjugation by t induces an automorphism of $kN_H = Q_0(kN_H, D)$. Suppose that left conjugation by t induces an automorphism of $Q_n(kN_H, D)$. This implies that $tQ_{n+1}(kN_H, D)t^{-1} = Q_{n+1}(kN_H, D)$ because both

tzt^{-1} and $t^{-1}zt$ belong to $Q_{n+1}(kN_H, D)$ for each generator z of $Q_{n+1}(kN_H, D)$. Therefore $tD(kN_H)t^{-1} = D(kN_H)$.

The second part follows because the powers of t are $D(kN_H)$-linearly independent since D is an $\mathcal{A}$-Hughes-free division ring of fractions.

(b) $D(kH)$ contains $D(kN_H)$ and $\{t, t^{-1}\}$. By (a), it contains the left Ore domain $D(kN_H)[t, t^{-1}; \alpha]$. By the universal property of Ore localization, $D(kN_H)(t; \alpha)$ embeds in $D(kH)$. Moreover, the image of $D(kN_H)(t; \alpha)$ is a division ring containing $kN_H[t, t^{-1}; \alpha] = kH$. Therefore $D(kH) \cong D(kN_H)(t; \alpha)$. Now, $D(kN_H)((t; \alpha))$ is a division ring that contains $D(kN_H)[t, t^{-1}; \alpha]$. Again the universal property of Ore localization implies that $D(kN_H)(t; \alpha)$, and therefore $D(kH)$, is contained in $D(kN_H)((t; \alpha))$.

(c) Let $d_0, \dots, d_n \in D(kN_H)$. Suppose that $d_0 + td_1 + \cdots + t^n d_n = 0$. Thus

$$d_0 + \alpha(d_1)t + \cdots + \alpha^n(d_n)t^n = 0.$$

By (a), $\alpha(d_i) \in D(kN_H)$ for all i. Hence $d_0 = \alpha(d_1) = \cdots = \alpha^n(d_n) = 0$. Since α is an isomorphism, $d_0 = d_1 = \cdots = d_n$. ☐

Definitions 5.5. Let G be a group.

(a) Suppose that G is locally indicable. We say that the group G is *Hughes-free embeddable* if kG is Hughes-free embeddable for each division ring k and each crossed product group ring kG.
(b) We say that G is *embeddable* if kG can be embedded in a division ring for each division ring k and each crossed product group ring kG. ☐

We proceed to give some important examples of Hughes-free embeddings and Hughes-free embeddable groups. Example 5.6(d) was already given by I. Hughes in [**Hug70**].

Examples 5.6. Let G be a group.

(a) Suppose that G is locally indicable. Let k be a division ring. Consider a crossed product group ring kG. Suppose that kG is a left (right) Ore domain. Then the embedding $kG \hookrightarrow D$ is Hughes-free provided D is the left (right) Ore division ring of fractions of kG.
(b) Suppose that G has a subnormal series $(G_\gamma)_{\gamma \leq \tau}$ with torsion-free abelian factors. Then G is Hughes-free embeddable.
(c) Suppose that G is a right orderable amenable group. Then G is Hughes-free embeddable.
(d) If G is an orderable group, then G is Hughes-free embeddable. Indeed, if $(G, <)$ is an ordered group, then $kG \hookrightarrow k((G, <))$ is a Hughes-free embedding for any division ring k and any crossed product group ring kG.
(e) If G is a locally free group, then G is Hughes-free embeddable.

Proof. (a) We prove the result for the left Ore case. The proof of the right Ore case is similar.

Let H be a nontrivial finitely generated subgroup of G. Let $N \lhd H$ be such that H/N is infinite cyclic. Let $x \in H$ such that Nx generates H/N. Suppose that

$$d_0 + d_1\bar{x} + \cdots + d_n\bar{x}^n = 0$$

where $d_0, d_1, \dots, d_n \in D(kN)$. Notice that kN is a left Ore domain by Proposition 4.9, and that $D(kN)$ is the left Ore division ring of fractions of kN. By Remark 3.4, there exist $u, v_0, \dots, v_n \in kN$ such that $d_i = u^{-1}v_i$. Multiplying by u on the left we get

$$v_0 + v_1\bar{x} + \cdots + v_n\bar{x}^n = 0.$$

Since the powers of $\bar{x}$ are kN-linearly independent, $v_i = 0$ for all i. Hence $d_0 = \cdots = d_n = 0$.

(b) The result follows from (a) and Corollary 4.11.

(c) By the comments at the end of Section 4 of Chapter 2, G is locally indicable.

It is known that kG has an Ore division ring of fractions whenever kG is a domain by a result of D. Tamari [**Tam57**], see also [**DLM$^+$03**, Theorem 6.3] or [**Lüc02**, Example 8.16]. Certainly this is the case by Proposition 4.8. Now apply (a).

(d) From Theorem 2.28, G is a locally indicable group. Let k be a division ring and kG a crossed product group ring. Suppose that $(G,<)$ is an ordered group. Set

$$E = k((G,<)) = \{\gamma = \sum_{z\in G} d_z\bar{z} \mid \operatorname{supp}(\gamma) \text{ is well ordered}\}$$

the Mal'cev-Neumann series ring associated to kG, see Chapter 4. Let H be a nontrivial finitely generated subgroup of G. Let $N \lhd H$, and $x \in H$ such that $H/N = \langle Nx\rangle$ is infinite cyclic.

Notice that if $d \in E(kN)$, then $d \in \{\gamma = \sum\limits_{h\in N} d_h\bar{h} \mid \operatorname{supp}(\gamma) \text{ is well ordered}\}$. Suppose that there exist $d_i = \sum\limits_{h\in N} d_{hi}\bar{h} \in E(kN)$ for $i \in \{1,\dots,n\}$ such that $0 = \sum\limits_{i=0}^{n} d_i(\bar{x})^i$. Then

$$0 = \sum_{i=0}^{n} d_i\bar{x}^i = \sum_{i=0}^{n}\Big(\sum_{h\in N} d_{hi}\bar{h}\Big)\bar{x}^i.$$

Since $h_1x^{i_1} = h_2x^{i_2}$ if and only if $h_1 = h_2$ and $i_1 = i_2$, for $h_1, h_2 \in N$, it follows that $d_{hi} = 0$ for all $h \in N$ and $i \in \{1,\dots,n\}$. Hence $d_0 = d_1 = \cdots = d_n = 0$.

(e) The group G is orderable by Corollary 2.24. Thus (c) implies that $kG \hookrightarrow k((G,<))$ is a Hughes-free embedding for any division ring k, any crossed product group ring kG and any total order $<$ on G such that $(G,<)$ is an ordered group. □

As a consequence of Theorem 6.10 we will obtain more Hughes-free embeddings. On the other hand, we remark that there exist embeddings of crossed product group rings kH, with H a locally indicable group, inside division rings Q such that Q is not $\mathcal{A}$-Hughes-free for any atlas $\mathcal{A}$ of H. The first example of this situation was given by J. Lewin in [**Lew74**, Section V]. We will see more examples in Chapter 7, see Corollary 7.61.

2. A measure of complexity

In this section, we collect together standard material on finite rooted trees, and construct the rational semiring $\mathcal{T}$ which will be used to measure the complexity of elements of another rational semiring, $\operatorname{Rat}(U)$, that will be defined in Section 4. $\mathcal{T}$ will turn out to be a well-ordered set. We also give the main properties of the behavior of this order with respect to the operations defined on $\mathcal{T}$.

Definitions 5.7. Let $\mathcal{T}$ denote the set of all (isomorphism classes of) finite rooted trees. We give $\mathcal{T}$ the structure of semiring as follows. Let $X, Y \in \mathcal{T}$.

(a) The *sum* $X + Y$ is obtained from the disjoint union $X \cup Y$ by identifying the root of X with the root of Y and taking the resulting vertex to be the new root. It is not difficult to realize that the sum is associative. Then $(\mathcal{T}, +)$ is an additive monoid, and the zero element $0_{\mathcal{T}}$ is the tree with exactly one vertex. Notice that $X + Y = 0_{\mathcal{T}}$ if and only if $X = Y = 0_{\mathcal{T}}$.

(b) We define the *family of* X, denoted $\operatorname{fam}(X)$, as the set of components of the graph obtained by deleting the root of X and all incident edges. We view $\operatorname{fam}(X)$ as a finite family of finite rooted trees, with multiplicities, where the root of each component is the vertex that was incident to the deleted edge. Notice that $\operatorname{fam}(0_{\mathcal{T}})$ is empty. We remark that fam$(X+Y)$ can be thought of as fam$(X)\cup$fam(Y) provided multiplicities are taken into account.

(c) The *width* of X, denoted width(X), is the number of elements in fam(X). For example, in Example 5.9 the width of Y_0 is 3. In a tree of width one, the root is incident to a unique edge, called the *stem*. Note that width is additive, i.e.

$$\mathrm{width}(X+Y)=\mathrm{width}(X)+\mathrm{width}(Y).$$

(d) We recursively define the *height* of X as follows. We say that $0_{\mathcal{T}}$ has height 0, and that, if $X\neq 0_{\mathcal{T}}$, then the height of X is one more than the maximum of the heights of the elements of fam(X). We denote the height of $X\in\mathcal{T}$ by h(X).

(e) We define *expanded* X, denoted $\exp(X)$, as the tree obtained from X by adding a stem, that is, we add a new vertex, and a new edge which joins the new vertex to the root of X, and then the new vertex is taken as the root of $\exp(X)$. So the height increases by 1 and fam$(\exp(X))=\{X\}$.

We have $X=\sum\limits_{X'\in\mathrm{fam}(X)}\exp(X')$, a (possibly empty) sum of trees with stems.

(f) We define the *product*

$$X\cdot Y=\sum_{X'\in\mathrm{fam}(X)}\sum_{Y'\in\mathrm{fam}(Y)}\exp(X'+Y').$$

Thus, the product of two trees with stems identifies the stems, and the multiplication is then extended distributively. The multiplication is commutative since the sum is. The identity element $1_{\mathcal{T}}=\exp(0_{\mathcal{T}})$ is the tree with exactly one edge. Notice that $X\cdot Y=1_{\mathcal{T}}$ if and only if $X=Y=1_{\mathcal{T}}$.

We remark that fam$(X\cdot Y)$ can be thought of as fam$(X)+$fam(Y), where the elements of fam$(X)+$fam(Y) are the rooted trees of the form $X'+Y'$ for $X'\in$ fam(X), $Y'\in$ fam(Y) and multiplicities are taken into account. Notice that width is multiplicative, i.e.

$$\mathrm{width}(X\cdot Y)=\mathrm{width}(X)\cdot\mathrm{width}(Y).\qquad\square$$

Now it is not difficult to prove that $\mathcal{T}$ is a semiring.

Lemma 5.8. *The product defined on $\mathcal{T}$ is distributive with respect to the addition and it is associative. Therefore $\mathcal{T}$ is a semiring with absorbing zero $0_{\mathcal{T}}$.*

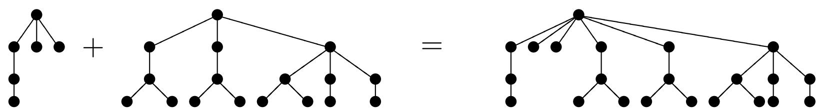
Proof. Let $X,Y,Z\in\mathcal{T}$.

$$\begin{aligned}X\cdot(Y+Z) &= \sum_{X'\in\mathrm{fam}(X)}\sum_{W\in\mathrm{fam}(Y+Z)}\exp(X'+W)\\ &= \sum_{X'\in\mathrm{fam}(X)}\sum_{Y'\in\mathrm{fam}(Y)}\exp(X'+Y')+\sum_{X'\in\mathrm{fam}(X)}\sum_{Z'\in\mathrm{fam}(Z)}\exp(X'+Z')\\ &= X\cdot Y+X\cdot Z.\end{aligned}$$

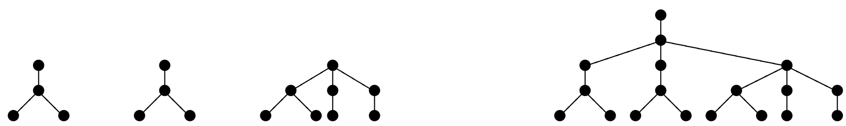
And because of the commutativity, $(Y+Z)\cdot X = Y\cdot X + Z\cdot X$.

$$\begin{aligned}
X\cdot(Y\cdot Z) &= X\cdot\left(\sum_{Y'\in\mathrm{fam}(Y)}\sum_{Z'\in\mathrm{fam}(Z)}\exp(Y'+Z')\right)\\
&= \sum_{Y'\in\mathrm{fam}(Y)}\sum_{Z'\in\mathrm{fam}(Z)} X\cdot\exp(Y'+Z')\\
&= \sum_{X'\in\mathrm{fam}(X)}\sum_{Y'\in\mathrm{fam}(Y)}\sum_{Z'\in\mathrm{fam}(Z)}\exp(X'+Y'+Z')\\
&= \sum_{X'\in\mathrm{fam}(X)}\sum_{Y'\in\mathrm{fam}(Y)}\exp(X'+Y')\cdot Z\\
&= \left(\sum_{X'\in\mathrm{fam}(X)}\sum_{Y'\in\mathrm{fam}(Y)}\exp(X'+Y')\right)\cdot Z = (X\cdot Y)\cdot Z. \qquad \square
\end{aligned}$$

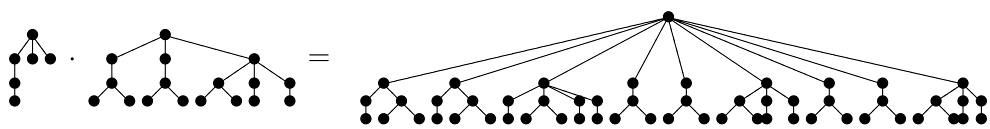
Examples 5.9.



X_0 Y_0 $X_0 + Y_0$



$\mathrm{fam}(Y_0)$ $\exp(Y_0)$



X_0 Y_0 $X_0\cdot Y_0$

Definitions 5.10. (a) We make $\mathcal{T}$ into a rational semiring with $*$-map defined by $X^* = \exp^2(X)$ for $X \in \mathcal{T}$. That is, we add a double-length stem to X.

(b) Let U be a group. We endow $\mathcal{T}$ with a structure of rational U-semiring via the trivial multiplicative map $U \to \mathcal{T}$ which sends every element of U to $1_{\mathcal{T}}$. Here the U-biset structure is trivial. $\square$

We next want to well-order $\mathcal{T}$.

Definitions 5.11. For $m, n \in \mathbb{N}$, we let $\mathcal{T}_{n,m}$ denote the subset of $\mathcal{T}$ consisting of all the elements of height at most $n-1$, together with all the elements of height exactly n and width at most m.

We now define a total order $\geq$ on $\mathcal{T}$.

We let $0_{\mathcal{T}}$ be the least element of $\mathcal{T}$.

In particular, we have ordered $\mathcal{T}_{1,0} = \{0_\mathcal{T}\}$.

Suppose that $n \geq 1$, and that we have ordered $\mathcal{T}_{n,0}$.

Suppose that $m \geq 1$, and that we have ordered $\mathcal{T}_{n,m-1}$.

Consider any nonzero $X, Y \in \mathcal{T}_{n,m}$.

Notice that $\mathrm{fam}(X)$ is a finite family of elements of $\mathcal{T}_{n,0}$. Thus $\mathrm{fam}(X)$ has a largest element. We define $\log X$ to be the largest element of $\mathrm{fam}(X)$.

Note that $\exp(\log X)$ is a summand of X, and we denote the unique complement by $X - \exp\log X$. Thus, $X - \exp\log X$ is formed by deleting from X a suitable edge incident to the root, and then deleting the component ($= \log X$) which does not contain the root.

Since $\log X$ and $\log Y$ lie in $\mathcal{T}_{n,0}$, they can be compared. Also, $X - \exp\log X$ and $Y - \exp\log Y$ lie in $\mathcal{T}_{n,m-1}$, and therefore they can be compared. We define $X > Y$ to mean

$$(\log X > \log Y) \text{ or } (\log X = \log Y \text{ and } X - \exp\log X > Y - \exp\log Y).$$

By induction on m, $\mathcal{T}_{n,m}$ is ordered for all $m \in \mathbb{N}$, that is, $\mathcal{T}_{n+1,0}$ is ordered.

By induction on n, $\mathcal{T}_{n,0}$ is ordered for all $n \in \mathbb{N}$, that is, $\mathcal{T}$ is ordered. □

Remarks 5.12. Let $X, Y \in \mathcal{T} \setminus \{0_\mathcal{T}\}$.

(a) $X = \exp\log X + (X - \exp\log X)$ and $Y \cdot X = Y \cdot \exp\log X + Y \cdot (X - \exp\log X)$.

(b) If $\log X = \log Y$, then

(b1) $X \leq Y$ if and only if $X - \exp\log X \leq Y - \exp\log Y$.

(b2) $X = Y$ if and only if $X - \exp\log X = Y - \exp\log Y$.

(c) $X - \exp\log X < X$.

Proof. (a) Follows directly from the definition and the distributive law.

(b1) Follows directly from the definition.

(b2) The only if part is clear. For the if part apply (a).

(c) Notice that $\mathrm{fam}(X - \exp\log X) \subset \mathrm{fam}(X)$.

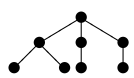
The proof is by induction on the multiplicity of $\log X$ in $\mathrm{fam}(X)$, denoted by m_X. If $m_X = 1$, $\log X \notin \mathrm{fam}(X - \exp\log X)$. Then $\log X > \log(X - \exp\log X)$. This implies $X - \exp\log X < X$.

Suppose that $m_X > 1$ and that $Y > Y - \exp\log Y$ for all $Y \in \mathcal{T}$ such that $m_Y < m_X$. Since $m_X > 1$, $\log(X) = \log(X - \exp\log X)$ and $m_{X-\exp\log X} = m_X - 1 < m_X$. Hence $\log X = \log(X - \exp\log X)$ and, by the induction hypothesis,

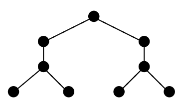
$$(X - \exp\log X) - \exp\log(X - \exp\log X) < X - \exp\log X.$$

That is, $X > X - \exp\log X$. □

Example 5.13. With the same notation as in Examples 5.9



$\log(Y_0)$



$Y - \exp\log(Y_0)$ □

Remark 5.14. The ordering on $\mathcal{T}$ refines the partial ordering given by the height. That is, if $\mathrm{h}(X) < \mathrm{h}(Y)$, then $X < Y$ for $X, Y \in \mathcal{T}$. Moreover, $\mathrm{h}(\log Y) = \mathrm{h}(Y) - 1$.

Proof. Let $X, Y \in \mathcal{T}$ with $\mathrm{h}(X) < \mathrm{h}(Y)$ (hence $Y \neq 0_\mathcal{T}$). We prove the result by induction on $\mathrm{h}(Y) \geq 1$.

Suppose that $\mathrm{h}(Y) = 1$. If $\mathrm{h}(X) < \mathrm{h}(Y)$, then $\mathrm{h}(X) = 0$. Therefore $0_\mathcal{T} = X < Y$.

Suppose that $1 < \mathrm{h}(Y)$. Notice that $\mathrm{h}(Y') < \mathrm{h}(Y)$ for all $Y' \in \mathrm{fam}(Y)$, and there exists $Y' \in \mathrm{fam}(Y)$ with $\mathrm{h}(Y') = \mathrm{h}(Y) - 1$. Then $\mathrm{h}(\log Y) = \mathrm{h}(Y) - 1$ because $\log Y \in \mathrm{fam}(Y)$ and the induction hypothesis. If $X = 0_{\mathcal{T}}$, obviously $X < Y$. Suppose that $X \neq 0_{\mathcal{T}}$. Hence $\mathrm{h}(\log X) \leq \mathrm{h}(X) - 1 < \mathrm{h}(Y) - 1 = \mathrm{h}(\log Y) < \mathrm{h}(Y)$. The induction hypothesis implies that $\log X < \log Y$. So $X < Y$. $\square$

LEMMA 5.15. *$\mathcal{T}$ is well ordered.*

PROOF. Suppose not, so that there exists an infinite, strictly descending sequence (T_i) in $\mathcal{T}$; thus

$$T_0 > T_1 > T_2 > \cdots . \tag{25}$$

We shall obtain a contradiction.

For each i, T_i is then nonzero, and we let n_i denote the height of T_i, and m_i denote the multiplicity of $\log T_i$ in $\mathrm{fam}(T_i)$. Hence $n_i \geq 1$, $m_i \geq 1$.

Since $\mathbb{N}$ is well-ordered, we may assume that (25) has been chosen to minimize n_0.

It follows that the set of elements of $\mathcal{T}$ of height strictly less than n_0 is well-ordered. Thus, we can ignore n_0, and assume, by Remarks 5.14, that (25) has been chosen to minimize $\log T_0$, and, with $\log T_0$ fixed, to minimize its multiplicity m_0.

By the definition of the ordering,

$$\log T_0 \geq \log T_1 \geq \log T_2 \geq \cdots . \tag{26}$$

If some term of (26) is less than $\log T_0$, we can omit finitely many terms from (25), and obtain a contradiction to the minimality of $\log T_0$. Thus

$$\log T_0 = \log T_1 = \log T_2 = \cdots .$$

By the definition of the ordering,

$$T_0 - \exp\log T_0 > T_1 - \exp\log T_1 > T_2 - \exp\log T_2 > \cdots . \tag{27}$$

If $m_0 = 1$, then (27) contradicts the minimality of $\log T_0$. If $m_0 \geq 2$, then (27) contradicts the minimality of m_0. $\square$

In $\mathcal{T}$, $\log(0_{\mathcal{T}})$ was not defined, and it is convenient to have an interpretation for this expression.

CONVENTIONS 5.16. Let $-\infty_{\mathcal{T}}$ denote the empty tree, and say it has height -1 and width 0.

We extend the structure of semiring of $\mathcal{T}$ to $\mathcal{T} \cup \{-\infty_{\mathcal{T}}\}$ as in Definitions 1.42(d). That is, $(-\infty_{\mathcal{T}}) + X = (-\infty_{\mathcal{T}}) \cdot X = X \cdot (-\infty_{\mathcal{T}}) = -\infty_{\mathcal{T}}$ for all $X \in \mathcal{T} \cup \{-\infty_{\mathcal{T}}\}$.

We extend the order on $\mathcal{T}$ to an order on $\mathcal{T} \cup \{-\infty_{\mathcal{T}}\}$ so that $-\infty_{\mathcal{T}}$ is the new smallest element.

Define $\log(0_{\mathcal{T}}) = \log(-\infty_{\mathcal{T}}) = -\infty_{\mathcal{T}}$ and $\exp(-\infty_{\mathcal{T}}) = 0_{\mathcal{T}}$. $\square$

LEMMA 5.17. *If $X, Y, X', Y' \in \mathcal{T}$, then the following hold in $\mathcal{T}$:*

(i) $(\mathcal{T}, +)$ *is an ordered monoid.*
(ii) $(\mathcal{T} \setminus \{0_{\mathcal{T}}\}, \cdot)$ *is an ordered monoid.*
(iii) $\log(X + Y) = \max\{\log X, \log Y\}$.
(iv) $\log(X \cdot Y) = \log X + \log Y$.
(v) $\log^2(X + Y) = \max\{\log^2 X, \log^2 Y\}$.
(vi) $\log^2(X \cdot Y) \leq \max\{\log^2 X, \log^2 Y\}$, *and equality holds if X and Y are nonzero.*

PROOF. (iii) holds true because of the remarks about the family of the sum made on Definitions 5.7(b).

(i) Let $X, Y, Y' \in \mathcal{T}$ with $Y < Y'$. We prove that $X + Y < X + Y'$ by induction on the complexity of X. If $X = 0_{\mathcal{T}}$ the result is clear.

Suppose that $X > 0_{\mathcal{T}}$. If $\exp\log X \neq X$, then $X - \exp\log X < X$ and $\exp\log X < X$ by Remarks 5.12(c). Thus applying twice the induction hypothesis

$$X + Y = (X - \exp\log X) + \exp\log X + Y < (X - \exp\log X) + \exp\log X + Y' = X + Y'.$$

Suppose that $X = \exp\log X$. Since $\log Y \leq \log Y'$, we get that $\log(X + Y) \leq \log(X + Y')$ by (iii). Moreover

$$\max\{\log X, \log Y\} = \log(X + Y) \leq \log(X + Y') = \max\{\log X, \log Y'\}.$$

It is clear that $X + Y < X + Y'$ if either $\log(X + Y) = \log Y < \log Y' = \log(X + Y')$, or $\log(X + Y) = \log X < \log Y' = \log(X + Y')$.

If $\log X = \log(X + Y) = \log(X + Y') = \log X$, then

$$\begin{aligned} (X + Y) - \exp\log(X + Y) &= (X - \exp\log X) + Y \\ &< (X - \exp\log X) + Y' \\ &= (X + Y') - \exp\log(X + Y'), \end{aligned}$$

and the result follows.

If $\log Y = \log(X + Y) = \log(X + Y') = \log Y' \neq \log X$, we proceed by induction on Y. Notice that $Y \geq 1_{\mathcal{T}}$. If $Y = 1_{\mathcal{T}}$, then $X = 0_{\mathcal{T}}$ and the result follows. If $Y > 0_{\mathcal{T}}$, then by the induction hypothesis

$$\begin{aligned} (X + Y) - \exp\log(X + Y) &= X + (Y - \exp\log Y) \\ &< X + (Y' - \exp\log Y') \\ &= X + Y' - \exp\log(X + Y'), \end{aligned}$$

as desired.

(iv) If $X = 0_{\mathcal{T}}$ or $Y = 0_{\mathcal{T}}$, the result is clear. If X and Y are nonzero, it follows from the remarks about the family of the product made on Definitions 5.7(f) and by (i).

(ii) Suppose that $X, Y, Y' \in \mathcal{T} \setminus \{0_{\mathcal{T}}\}$ with $Y < Y'$. There are two possibilities either $\log Y < \log Y'$, or $\log Y = \log Y'$ and $Y - \exp\log Y < Y' - \exp\log Y'$. In the first case, by (i) and (iv) we get that

$$\log(X \cdot Y) = \log X + \log Y < \log X + \log Y' = \log(X \cdot Y').$$

For the second possibility we proceed by induction on Y. Suppose that $Y = 1_{\mathcal{T}}$. Notice that $X \cdot (Y' - \exp\log Y') > 0_{\mathcal{T}}$. Then, by (i), we obtain

$$\begin{aligned} X \cdot Y &= X \cdot \exp\log Y + X \cdot (Y - \exp\log Y) \\ &= X + 0_{\mathcal{T}} \\ &< X + X \cdot (Y' - \exp\log Y') \\ &= X \cdot \exp\log Y' + X \cdot (Y' - \exp\log Y') \\ &= X \cdot Y'. \end{aligned}$$

If $Y > 1_{\mathcal{T}}$, again by (i) together with the induction hypothesis

$$\begin{aligned} X \cdot Y &= X \cdot \exp\log Y + X \cdot (Y - \exp\log Y) \\ &< X \cdot \exp\log Y' + X \cdot (Y' - \exp\log Y') \\ &= X \cdot Y'. \end{aligned}$$

To prove (v), apply log to (ii).

(vi) $\log^2(X \cdot Y) = \log(\log X + \log Y) = \max\{\log^2 X, \log^2 Y\}$, by (iv) and (iii), provided $\log X \neq -\infty$ and $\log Y \neq -\infty$, that is, if X and Y are nonzero. $\square$

What follows is just Lemma 5.17(i)-(ii) in the way we will use it.

REMARKS 5.18. Let $X, Y, X', Y' \in \mathcal{T}$.

(a) If $X \leq X'$ and $Y \leq Y'$, then $X + Y \leq X' + Y'$, and equality holds if and only if $X = X'$ and $Y = Y'$. Hence, $X \leq X + Y$, and equality holds if and only if $Y = 0_{\mathcal{T}}$.

(b) If X, Y, X', Y' are nonzero, $X \leq X'$ and $Y \leq Y'$, then $X \cdot Y \leq X' \cdot Y'$, and equality holds if and only if $X = X'$, $Y = Y'$. Hence, $X \leq X \cdot Y$, and equality holds if and only if $Y = 1_{\mathcal{T}}$.

PROOF. (a) and (b) follow from Lemma 5.17(i)-(ii) and Remark 2.13. $\square$

3. The free multiplicative U-monoid on a U-biset X

We present how to construct a U-semiring from a U-biset in the same way the tensor algebra is built from a bimodule. In fact we follow very close the exposition in [**Jac89**].

NOTATION 5.19. Throughout this section U, V and W will be multiplicative groups. $\square$

DEFINITIONS 5.20. Let X_1 be a V-U-biset and X_2 be a U-W-biset.

(a) There is a natural V-W-biset structure on $X_1 \times X_2$ given by $v(x_1, x_2) = (vx_1, x_2)$, $(x_1, x_2)w = (x_1, x_2w)$ for $v \in V$, $w \in W$ and $(x_1, x_2) \in X_1 \times X_2$.

(b) Let Y be a V-W-biset. A morphism of V-W-bisets $f\colon X_1 \times X_2 \longrightarrow Y$ is said to be a *balanced morphism* if $f(x_1u, x_2) = f(x_1, ux_2)$ for all $x_1 \in X_1$, $x_2 \in X_2$, $u \in U$.

(c) We define $X_1 \times_U X_2$ to be $(X_1 \times X_2)/\sim$ where $(x_1, x_2) \sim (x_1', x_2')$ if and only if there exists $u \in U$ such that $x_1u = x_1'$ and $u^{-1}x_2 = x_2'$. It is not difficult to prove that $\sim$ is an equivalence relation. The equivalence class of $(x_1, x_2) \in X_1 \times X_2$ will be denoted by $x_1 \cdot x_2$.

(d) There is a V-W-biset structure on $X_1 \times_U X_2$ defined by $v(x_1 \cdot x_2) = (vx_1) \cdot x_2$, and $(x_1 \cdot x_2)w = x_1 \cdot (x_2w)$ for all $v \in V$, $w \in W$, $x_1 \cdot x_2 \in X_1 \times_U X_2$. $\square$

Observe that the natural map $X_1 \times X_2 \stackrel{\cdot}{\longrightarrow} X_1 \times_U X_2$, defined by $(x_1, x_2) \longmapsto x_1 \cdot x_2$, is an onto balanced morphism of V-W-bisets.

LEMMA 5.21. *Let $f\colon X_1 \times X_2 \longrightarrow Y$ be a balanced morphism of V-W-bisets. Then there exists a unique morphism of V-W-bisets, $\tilde{f}\colon X_1 \times_U X_2 \longrightarrow Y$ with $\tilde{f}(x_1 \cdot x_2) = f(x_1, x_2)$ for all $(x_1, x_2) \in X_1 \times X_2$. That is, $\tilde{f}$ makes the following diagram commutative*

$$\begin{tikzcd} X_1 \times X_2 \arrow[rr, "f"] \arrow[dr, "\cdot"'] & & Y \\ & X_1 \times_U X_2 \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

PROOF. Clearly $\tilde{f}$, if it exists, is unique because $\cdot$ is onto. So we only need to show it is a well defined morphism of V-W-bisets. Suppose that $x_1 \cdot x_2 = x_1' \cdot x_2'$. Then there exists $u \in U$ such that $x_1u = x_1'$ and $u^{-1}x_2 = x_2'$, and

$$\tilde{f}(x_1' \cdot x_2') = f(x_1', x_2') = f(x_1u, u^{-1}x_2) = f(x_1, u(u^{-1}x_2)) = f(x_1, x_2) = \tilde{f}(x_1 \cdot x_2).$$

Therefore $\tilde{f}$ is well defined. Moreover, $\tilde{f}$ is a morphism of V-W-bisets because

$$\tilde{f}(v(x_1 \cdot x_2)) = \tilde{f}((vx_1) \cdot x_2) = f(vx_1, x_2) = vf(x_1, x_2) = v\tilde{f}(x_1 \cdot x_2)$$

for any $v \in V$, $x_1 \cdot x_2 \in X_1 \times_U X_2$. Similarly, $\tilde{f}((x_1 \cdot x_2)w) = \tilde{f}(x_1 \cdot x_2)w$ for any $w \in W$, $x_1 \cdot x_2 \in X_1 \times_U X_2$. $\square$

DEFINITIONS 5.22. Let X_1 be a V-U-biset and X_2 a U-W-biset.

(a) Suppose that $f_1\colon X_1 \longrightarrow Y_1$ is a morphism of V-U-bisets and $f_2\colon X_2 \longrightarrow Y_2$ is a morphism of U-W-bisets. The composition

$$X_1 \times X_2 \overset{(f_1,f_2)}{\longrightarrow} Y_1 \times Y_2 \longrightarrow Y_1 \times_U Y_2,$$

is a balanced morphism of V-W-bisets. We define $f_1 \cdot f_2\colon X_1 \times_U X_2 \longrightarrow Y_1 \times_U Y_2$ as the unique morphism of V-W-bisets given by Lemma 5.21. Thus $f_1 \cdot f_2$ is defined by

$$f_1 \cdot f_2(x_1 \cdot x_2) = f_1(x_1) \cdot f_2(x_2) \text{ for all } x_1 \cdot x_2 \in X_1 \times_U X_2.$$

Note that if f_1 is a V-U-biset isomorphism and f_2 is a U-W-biset isomorphism, then $f_1 \cdot f_2$ is a U-W-biset isomorphism.

(b) As the map $X_1 \times U \longrightarrow X_1$, defined by $(x_1, u) \longmapsto x_1u$, is balanced, there is a map $X_1 \times_U U \longrightarrow X_1$, defined by $x_1 \cdot u \longmapsto x_1u$, that is an isomorphism of V-U-bisets with inverse $X_1 \longrightarrow X_1 \times_U U$, $x_1 \longmapsto x_1 \cdot 1$. By symmetry, the map $U \times_U X_2 \longrightarrow X_2$, defined by $u \cdot x_2 \longmapsto ux_2$, is an isomorphism of U-W-bisets.

(c) Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of V-U-bisets. Consider the disjoint union $\bigcup\limits_{\gamma\in\Gamma} X_\gamma$. It is a V-U-biset with the actions given by $v(x,\gamma) = (vx,\gamma)$, $(x,\gamma)u = (xu,\gamma)$ for all $v \in V$, $u \in U$, $(x,\gamma) \in \bigcup\limits_{\gamma\in\Gamma} X_\gamma$. The *coproduct of* $(X_\gamma)_{\gamma\in\Gamma}$ is the disjoint union together with the inclusion maps $i_\gamma\colon X_\gamma \longrightarrow \bigcup\limits_{\gamma\in\Gamma} X_\gamma$, $i_\gamma(x) = (x,\gamma)$. The coproduct verifies that if Y is a V-U-biset, and for every $\gamma \in \Gamma$ we have a morphism of V-U-bisets, $f_\gamma\colon X_\gamma \longrightarrow Y$, then there exists a unique morphism of V-U-bisets $f\colon \bigcup\limits_{\gamma\in\Gamma} X_\gamma \longrightarrow Y$, such that $fi_\gamma = f_\gamma$. $\square$

LEMMA 5.23. *Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of V-U-bisets. Let Y be a U-W-biset. Then the map*

$$\eta_Y\colon \Big(\bigcup_{\gamma\in\Gamma} X_\gamma\Big) \times_U Y \longrightarrow \bigcup_{\gamma\in\Gamma} (X_\gamma \times_U Y)$$

$$(x,\gamma)\cdot y \longmapsto (x\cdot y,\gamma)$$

is an isomorphism of V-W-bisets. If $(Y_\delta)_{\delta\in\Delta}$ is a family of U-W-bisets, then the map

$$\varphi\colon \Big(\bigcup_{\gamma\in\Gamma} X_\gamma\Big) \times_U \Big(\bigcup_{\delta\in\Delta} Y_\delta\Big) \longrightarrow \bigcup_{(\gamma,\delta)\in\Gamma\times\Delta} (X_\gamma \times_U Y_\delta)$$

$$((x,\gamma)\cdot(y,\delta)) \longmapsto (x\cdot y,(\gamma,\delta))$$

is an isomorphism of V-W-bisets.

PROOF. The map $\Big(\bigcup\limits_{\gamma\in\Gamma} X_\gamma\Big) \times Y \longrightarrow \bigcup\limits_{\gamma\in\Gamma} (X_\gamma \times_U Y)$, $((x,\gamma),y) \longmapsto (x\cdot y,\gamma)$ is a balanced morphism of V-W-bisets. Then, by Lemma 5.21, there exists a morphism of V-W-bisets $\eta_Y\colon \Big(\bigcup\limits_{\gamma\in\Gamma} X_\gamma\Big) \times_U Y \longrightarrow \bigcup\limits_{\gamma\in\Gamma} (X_\gamma \times_U Y)$ defined by $\eta_Y\big((x,\gamma)\cdot y\big) = (x\cdot y,\gamma)$.

On the other hand, for each $\gamma \in \Gamma$, consider $i_\gamma\colon X_\gamma \longrightarrow \bigcup\limits_{\gamma\in\Gamma} X_\gamma$, the inclusion map, which is a morphism of V-U-bisets. The identity map on Y, 1_Y, is a morphism of U-W-bisets. Thus we have the morphism of V-W-bisets $i_\gamma \cdot 1_Y\colon X_\gamma \times_U Y \longrightarrow \Big(\bigcup\limits_{\gamma\in\Gamma} X_\gamma\Big) \times_U Y$, for each $\gamma \in \Gamma$, such that $x\cdot y \longmapsto (x,\gamma)\cdot y$. By the universal property of the coproduct, there exists a unique morphism of V-W-bisets $\tau_Y\colon \bigcup_{\gamma\in\Gamma}(X_\gamma \times_U Y) \longrightarrow \Big(\bigcup_{\gamma\in\Gamma} X_\gamma\Big) \times_U Y$ such that $\tau_Y(x\cdot y,\gamma) = ((x,\gamma)\cdot y)$. Then $\eta_Y\tau_Y = 1_{\bigcup_{\gamma\in\Gamma}(X_\gamma\times_U Y)}$ and $\tau_Y\eta_Y = 1_{(\bigcup_{\gamma\in\Gamma} X_\gamma)\times_U Y}$.

In the same way it can be proved that

$$X_\gamma \times_U \Big(\bigcup_{\delta\in\Delta} Y_\delta\Big) \longrightarrow \bigcup_{\delta\in\Delta}\Big(X_\gamma \times_U Y_\delta\Big)$$

$$x\cdot(y,\delta) \longmapsto (x\cdot y,\delta)$$

is an isomorphism of V-W-bisets. Hence

$$\Big(\bigcup_{\gamma\in\Gamma} X_\gamma\Big) \times_U \Big(\bigcup_{\delta\in\Delta} Y_\delta\Big) \longrightarrow \bigcup_{\gamma\in\Gamma}\Big(X_\gamma \times_U \Big(\bigcup_{\delta\in\Delta} Y_\delta\Big)\Big) \longrightarrow \bigcup_{\gamma\in\Gamma}\Big(\bigcup_{\delta\in\Delta}(X_\gamma\times_U Y_\delta)\Big)$$

$$((x,\gamma)\cdot(y,\delta)) \longmapsto (x\cdot(y,\delta),\gamma) \longmapsto ((x\cdot y,\delta),\gamma)$$

$$\bigcup_{\gamma\in\Gamma}\Big(\bigcup_{\delta\in\Delta}(X_\gamma\times_U Y_\delta)\Big) \longrightarrow \bigcup_{(\gamma,\delta)\in\Gamma\times\Delta}(X_\gamma\times_U Y_\delta)$$

$$((x\cdot y,\delta),\gamma) \longmapsto ((x\cdot y),(\gamma,\delta))$$

is an isomorphism of V-W-bisets. □

Definitions 5.24. (a) Let M be a monoid. We say that M is a *U-monoid* if there exists a morphism of monoids $U \longrightarrow M$. Then M has a U-biset structure. Moreover, since $M\times M \longrightarrow M$, $(m,n)\longmapsto mn$, is a balanced morphism of U-bisets, it factors through $M\times M \longrightarrow M\times_U M$, $(m,n)\longmapsto m\cdot n$.

(b) Let M and N be U-monoids. A *morphism of U-monoids* is a morphism of monoids $M \longrightarrow N$ such that the following diagram is commutative

$$\begin{tikzcd} & M \arrow[d] \\ U \arrow[ur] \arrow[dr] & \\ & N \end{tikzcd}$$

(c) Consider the monoid semiring $\mathbb{N}[M]$. It has a natural structure of U-semiring. It is zero-sum and zero-divisor free. Thus $\mathbb{N}[M]\setminus\{0\}$ is also a U-semiring. Moreover it has the following universal property: for every U-semiring R with a morphism of U-monoids $\phi\colon M\to R$ there exists a unique morphism of U-semirings $\Phi\colon \mathbb{N}[M]\setminus\{0\}\to R$ that coincides with ϕ on M. It is given by $\Phi(\sum_{m\in M} n_m m) = \sum_{m\in M} n_m\phi(m)$. □

Lemma 5.25. *The map $(X_1\times_U X_2)\times_U X_3 \longrightarrow X_1\times_U(X_2\times_U X_3)$, $(x_1\cdot x_2)\cdot x_3 \longmapsto x_1\cdot(x_2\cdot x_3)$ is a U-biset isomorphism.*

Proof. Observe that the maps $(X_1\times_U X_2)\times X_3 \to X_1\times_U(X_2\times_U X_3)$, given by $(x_1\cdot x_2, x_3)\mapsto x_1\cdot(x_2\cdot x_3)$, and $X_1\times(X_2\times_U X_3) \to (X_1\times_U X_2)\times_U X_3$, defined by $(x_1, x_2\cdot x_3)\mapsto(x_1\cdot x_2)\cdot x_3$, are balanced morphisms of U-bisets. Then, by Lemma 5.21, we get unique morphisms of U-bisets $(X_1\times_U X_2)\times_U X_3\to X_1\times_U(X_2\times_U X_3)$, $(x_1\cdot x_2)\cdot x_3\mapsto x_1\cdot(x_2\cdot x_3)$, and $X_1\times_U(X_2\times_U X_3)\to(X_1\times_U X_2)\times_U X_3$, $x_1\cdot(x_2\cdot x_3)\mapsto(x_1\cdot x_2)\cdot x_3$. Notice that these morphisms of U-bisets are mutually inverse. □

DEFINITIONS 5.26. (a) Let $X_1,\dots,X_n$ be U-bisets. We define $X_1\times_U X_2\times_U\cdots\times_U X_n$ inductively by $X_1\times_U X_2\times_U\cdots\times_U X_i=(X_1\times_U\cdots\times_U X_{i-1})\times_U X_i$. Also, if $x_j\in X_j$, $j=1,\dots,n$, we define $x_1\cdot x_2\cdots x_n$ inductively by $x_1\cdots x_i=(x_1\cdots x_{i-1})\cdot x_i$.

We claim that if $1\le m<n$ we have a unique isomorphism

$$\pi_{m,n}\colon (X_1\times_U\cdots\times_U X_m)\times_U(X_{m+1}\times_U\cdots\times_U X_n)\longrightarrow X_1\times_U\cdots\times_U X_n$$

such that $\pi_{m,n}\big((x_1\cdots x_m)\cdot(x_{m+1}\cdots x_n)\big)=x_1\cdots x_m\cdot x_{m+1}\cdots x_n$. We prove it by induction on $n-m$. If $n-m=1$ the isomorphism is the identity by definition. Suppose that $1<n-m$ and the claim true for nonzero naturals smaller than $n-m$, then

$$\begin{array}{cl}
 & (X_1\times_U\cdots\times_U X_m)\times_U(X_{m+1}\times_U\cdots\times_U X_n)\\
= & (X_1\times_U\cdots\times_U X_m)\times_U\big((X_{m+1}\times_U\cdots X_{n-1})\times_U X_n\big)\\
\underset{\text{by Lemma 5.25}}{\cong} & \big((X_1\times_U\cdots\times_U X_m)\times_U(X_{m+1}\times_U\cdots\times_U X_{n-1})\big)\times_U X_n\\
\underset{\text{by induc. hypoth.}}{\cong} & (X_1\times_U\cdots\times_U X_m\times_U X_{m+1}\times_U\cdots\times_U X_{n-1})\times_U X_n\\
= & X_1\times_U\cdots\times_U X_n.
\end{array}$$

Via the isomorphism defined by

$$\begin{array}{cl}
 & (x_1\cdots x_m)\cdot(x_{m+1}\cdots x_n)=(x_1\cdots x_m)\cdot\big((x_{m+1}\cdots x_{n-1})\cdot x_n\big)\\
\longmapsto & \big((x_1\cdots x_m)\cdot(x_{m+1}\cdots x_{n-1})\big)\cdot x_n\\
\overset{\pi_{m,n-1}\cdot 1_{X_n}}{\longmapsto} & (x_1\cdots x_{n-1})\cdot x_n=x_1\cdots x_n.
\end{array}$$

(b) Let X be a U-biset. Put $X^{\times_U^0}=U$, and $X^{\times_U^i}=X\times_U\overset{i)}{\cdots}\times_U X$, $i\ge 1$. Define $\pi_{0,n}\colon U\times_U X^{\times_U^n}\longrightarrow X^{\times_U^n}$ and $\pi_{n,n}\colon X^{\times_U^n}\times_U U\longrightarrow X^{\times_U^n}$, $n\ge 0$, to be the canonical isomorphisms of U-bisets given by $\pi_{0,n}(u\cdot x)=ux$ and $\pi_{n,n}(x\cdot u)=xu$, for all $x\in X^{\times_U^n}$ and $u\in U$.

Form the U-biset,

$$U\natural X=\bigcup_{n\in\mathbb{N}}X^{\times_U^n}=U\cup X\cup \mathrm{X}^{\times_U^2}\cup\dots$$

We proceed to endow $U\natural X$ with a structure of U-monoid.

Using that $\bigcup_{i,j}(X^{\times_U^i}\times_U X^{\times_U^j})$ is the coproduct of the U-bisets $X^{\times_U^i}\times_U X^{\times_U^j}$ we obtain a morphism of U-bisets $\pi'\colon\bigcup_{i,j}(X^{\times_U^i}\times_U X^{\times_U^j})\longrightarrow U\natural X$, which coincides with $\pi_{m,n}$ on $X^{\times_U^m}\times_U X^{\times_U^{n-m}}$. Composing π' with the U-biset isomorphism

$$(U\natural X)\times_U(U\natural X)=\left(\bigcup_{n\in\mathbb{N}}X^{\times_U^n}\right)\times_U\left(\bigcup_{n\in\mathbb{N}}X^{\times_U^n}\right)\longrightarrow\bigcup_{i,j}(X^{\times_U^i}\times_U X^{\times_U^j})$$

given by Lemma 5.23, we obtain a morphism of U-bisets $\pi\colon(U\natural X)\times_U(U\natural X)\longrightarrow U\natural X$. We claim that

$$(U\natural X)\times(U\natural X)\longrightarrow(U\natural X)\times_U(U\natural X)\xrightarrow{\pi}U\natural X$$

endows $U\natural X$ with a structure of U-monoid. Clearly the inclusion map $U\longrightarrow U\natural X$ is a morphism of monoids. The associativity is clear if one of the factors is in U, since, by definition, $u\cdot x=ux$ and $x\cdot u=xu$ for all $x\in U\natural X$, $u\in U$. In particular $1\in U$ is the identity element of $U\natural X$. Now consider $x = x_1 \cdots x_m$, $y = y_1 \cdots y_n$, $z = z_1 \cdots z_p$, $m, n, p > 0$. The definition of π gives

$$\begin{aligned}(xy)z &= \big((x_1 \cdots x_m)(y_1 \cdots y_n)\big)(z_1 \cdots z_p)\\ &= (x_1 \cdots x_m \cdot y_1 \cdots y_n)(z_1 \cdots z_p)\\ &= x_1 \cdots x_m \cdot y_1 \cdots y_n \cdot z_1 \cdots z_p\\ &= (x_1 \cdots x_m)(y_1 \cdots y_n \cdot z_1 \cdots z_p)\\ &= (x_1 \cdots x_m)\big((y_1 \cdots y_m)(z_1 \cdots z_p)\big)\\ &= x(yz).\end{aligned}$$

We call $U\natural X$ the *free multiplicative U-monoid on X over U.* □

LEMMA 5.27. *Let M be a multiplicative U-monoid, and let $f\colon X \longrightarrow M$ be a U-biset map. Then there exists a unique morphism of U-monoids $\bar{f}\colon U\natural X \longrightarrow M$ that coincides with f on X. Moreover, if M is a U-semiring, there exists a unique morphism of U-semirings $\bar{f}\colon \mathbb{N}[U\natural X] \setminus \{0\} \longrightarrow M$ that coincides with f on X.*

PROOF. We define, inductively, a morphism of U-bisets $f^{(n)}\colon X^{\times_U^n} \longrightarrow M$, $n = 1, 2, \ldots$, by $f^{(1)} = f$, and $f^{(i)}$ the morphism of U-bisets from $X^{\times_U^i} = X^{\times_U^{i-1}} \times_U X$ obtained by composing $f^{(i-1)} \cdot f$ with the morphism of U-bisets $M \times_U M \longrightarrow M$, $m \cdot n \longmapsto mn$. Let $x_i \in X$, $i = 1, 2, \ldots, n$, then

$$f^{(n)}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n). \tag{28}$$

We define $f^{(0)}\colon U \longrightarrow M$ as the morphism of monoids which gives M the structure of U-monoid. We let $\bar{f}\colon U\natural X \longrightarrow M$ coincide with $f^{(n)}$ on $X^{\times_U^n}$, $n = 0, 1, 2, \ldots$. It is immediate from (28) that $\bar{f}$ is a morphism of U-monoids. The uniqueness is also clear by the definition of $U\natural X$.

If M is a U-semiring, the morphism of U-monoids $\bar{f}\colon U\natural X \to M$ induces a unique morphism of U-semirings $\bar{f}\colon \mathbb{N}[U\natural X] \setminus \{0\} \to M$ that coincides with f on X by the universal property of $\mathbb{N}[U\natural X] \setminus \{0\}$. □

DEFINITION 5.28. Suppose that U is a subgroup of some group W. If Y is a W-biset, then a subset X of Y is said to be an *admissible U-sub-biset of the W-biset Y* if X is closed under left and right multiplication by the elements of U, and, moreover, for all $w \in W \setminus U$ both $X \cap wX$ and $X \cap Xw$ are empty. □

LEMMA 5.29. *Suppose that U is a subgroup of a group W. Let X_1, X_2 be U-bisets, and let Y_1, Y_2 be W-bisets. For each $i = 1, 2$, suppose that there exists $f_i\colon X_i \longrightarrow Y_i$, an injective morphism of U-bisets such that $f_i(X_i)$ is an admissible U-sub-biset of Y_i. Then $f_1 \cdot f_2\colon X_1 \times_U X_2 \longrightarrow Y_1 \times_W Y_2$ is an injective morphism of U-bisets, and $f_1 \cdot f_2(X_1 \times_U X_2)$ is an admissible U-sub-biset of $Y_1 \times_W Y_2$.*

PROOF. Suppose that $x_1 \cdot x_2, z_1 \cdot z_2 \in X_1 \times_U X_2$ are such that $f_1 \cdot f_2(x_1 \cdot x_2) = f_1 \cdot f_2(z_1 \cdot z_2)$, that is, $f_1(x_1) \cdot f_2(x_2) = f_1(z_1) \cdot f_2(z_2)$. It means there exists $w \in W$ such that $f_1(x_1)w = f_1(z_1)$ and $w^{-1} f_2(x_2) = f_2(z_2)$. Hence $w \in U$ because $f_1(X_1)$ is an admissible U-sub-biset of Y_1. Then the injectivity of the morphisms of U-bisets f_i, $i = 1, 2$, implies that $x_1 \cdot x_2 = z_1 \cdot z_2$.

Let $x_i \in X_i$, $i = 1, 2$. Let $w \in W$. Suppose that $w(f_1(x_1) \cdot f_2(x_2)) \in f_1 \cdot f_2(X_1 \times_U X_2)$. That is, there exist $z_1 \in X_1$, $z_2 \in X_2$ such that $(wf_1(x_1)) \cdot f_2(x_2) = f_1(z_1) \cdot f_2(z_2)$. Then there exists $v \in W$ such that

$$\begin{aligned}&\text{(a) } (wf_1(x_1))v = f_1(z_1),\\ &\text{(b) } v^{-1} f_2(x_2) = f_2(z_2).\end{aligned}$$

Because $f_2(X_2)$ is an admissible U-sub-biset of Y_2, (b) implies that $v \in U$. Condition (a) means that $w(f_1(x_1 v)) = f_1(z_1)$. This together with the fact that $f_1(X_1)$ is an admissible U-sub-biset of Y_1 yields $w \in U$. Thus we have proved that $w(f_1 \cdot f_2(X_1 \times_U X_2)) \cap f_1 \cdot f_2(X_1 \times_U X_2) = \emptyset$, for all $w \in W \setminus U$. Analogously $f_1 \cdot f_2(X_1 \times_U X_2)w \cap f_1 \cdot f_2(X_1 \times_U X_2) = \emptyset$ for all $w \in W \setminus U$. So $f_1 \cdot f_2(X_1 \times_U X_2)$ is an admissible U-sub-biset of $Y_1 \times_W Y_2$ as desired. $\square$

Corollary 5.30. *Let U be a subgroup of a group W. Let X be a U-biset. Let Y be a W-biset. Suppose that there exists an injective morphism of U-bisets, $\imath\colon X \longrightarrow Y$, such that $\imath(X)$ is an admissible U-sub-biset of Y. Then $\imath$ can be extended to a unique injective morphism of U-monoids (respectively U-semirings), $\bar{\imath}\colon U\natural X \longrightarrow W\natural Y$ ($\bar{\imath}\colon \mathbb{N}[U\natural X]\setminus\{0\} \longrightarrow \mathbb{N}[W\natural Y]\setminus\{0\}$). Moreover $\bar{\imath}(U\natural X)$ ($\bar{\imath}(\mathbb{N}[U\natural X] \setminus \{0\})$) is an admissible U-sub-biset of $W\natural Y$ ($\mathbb{N}[W\natural Y] \setminus \{0\}$).*

Proof. Notice that $\imath$ can be seen as a morphism of U-bisets from X to $W\natural Y$. Then, by Lemma 5.27, there exists a unique morphism of U-monoids $\bar{\imath}$ (respectively, U-semirings) which extends $\imath$. If $x_1, \ldots, x_n \in X$, then $\bar{\imath}$ sends $u \in U \subseteq W$ to u, and $x_1 \cdots x_n \in U\natural X$ to $\imath(x_1)\cdots\imath(x_n) = \bar{\imath}(x_1\cdots x_{n-1}) \cdot \bar{\imath}(x_n) \in W\natural Y$. Observe that $\bar{\imath}(X^{\times_U^n}) \subseteq Y^{\times_W^n}$. Thus, if we prove $\bar{\imath}\,|_{X^{\times_U^n}}$ is injective and $\bar{\imath}(X^{\times_U^n})$ is an admissible U-sub-biset of $Y^{\times_W^n}$, the result will follow. We do it by induction on n.

If $n = 0$, then $\bar{\imath}$ is the inclusion map of U into W. If $n = 1$, then $\bar{\imath}\,|_X = \imath$. In either case, $\bar{\imath}\,|_{X^{\times_U^n}}$ is injective, and its image is an admissible U-sub-biset. If $n \geq 1$, the result follows from Lemma 5.29 because $\bar{\imath}\,|_{X^{\times_U^{n+1}}} = \bar{\imath}\,|_{X^{\times_U^n}} \cdot \bar{\imath}\,|_X$. $\square$

4. The Rational U-Semiring of Formal Rational Expressions $\mathrm{Rat}(U)$

We now construct a rational U-semiring, $\mathrm{Rat}(U)$, which is a formal analog to the construction of the division ring of fractions of a ring R given in Remark 3.16. This object has a universal property which allows us to give important examples of morphisms of rational U-semirings. One of these defines the complexity of the elements in $\mathrm{Rat}(U)$.

Notation 5.31. Throughout this section:

(a) U is a multiplicative group.
(b) If B is a U-biset, $B^\dagger$ denotes a disjoint copy of B, with bijective map $B \to B^\dagger$, $b \mapsto b^*$, and we endow $B^\dagger$ with a U-biset structure by defining $ub^*v = (v^{-1}bu^{-1})^*$, for all $u, v \in U$, $b \in B$. $\square$

Definitions 5.32. Consider the U-semiring, and thus a U-biset, $\mathbb{N}[U] \setminus \{0\}$. We set

$$X_1 = (\mathbb{N}[U] \setminus \{0\})^\dagger, \text{ and } X_0 = \emptyset.$$

Clearly, X_0 is a U-sub-biset of X_1.

Now suppose that $n \geq 1$, and that we are given a U-sub-biset X_{n-1} of a U-biset X_n.

Then $\mathbb{N}[U\natural X_n]$ is a U-semiring, and, since $\mathbb{N}[U\natural X_{n-1}]$ is a U-sub-biset of $\mathbb{N}[U\natural X_n]$, then $\mathbb{N}[U\natural X_n] \setminus \mathbb{N}[U\natural X_{n-1}]$ is a U-sub-biset. We define

$$X_{n+1} = (\mathbb{N}[U\natural X_n] \setminus \mathbb{N}[U\natural X_{n-1}])^\dagger \cup X_n.$$

Thus we have recursively defined an ascending chain (X_n) of U-bisets.

We denote the union of this chain by X, a U-biset, and define the *universal rational U-semiring* as $\mathrm{Rat}(U) = \mathbb{N}[U\natural X] \setminus \{0\}$, a rational U-semiring with a $*$-map which carries $\mathbb{N}[U] \setminus \{0\}$ to X_1, and $\mathbb{N}[U\natural X_n] \setminus \mathbb{N}[U\natural X_{n-1}]$ to $X_{n+1} \setminus X_n$, for each $n \geq 1$.

The U-semiring with absorbing zero $\mathrm{Rat}(U) \cup \{0\}$ is (isomorphic to) the U-semiring $\mathbb{N}[U\natural X]$. $\square$

Remark 5.33. The $*$-map carries bijectively $\mathbb{N}[U\natural X_n]\setminus\{0\}$ to X_{n+1}, for each $n \geq 0$. Therefore the $*$-map carries bijectively $\mathrm{Rat}(U)$ to X.

PROOF. For $n = 0$ we get the result by the definition of X_1. Suppose that $n > 0$ and that the $*$-map carries $\mathbb{N}[U\natural X_{n-1}] \setminus \{0\}$ to X_n. The result follows because

$$\mathbb{N}[U\natural X_n] \setminus \{0\} = (\mathbb{N}[U\natural X_n] \setminus \mathbb{N}[U\natural X_{n-1}]) \cup (\mathbb{N}[U\natural X_{n-1}] \setminus \{0\}),$$

and the $*$-map carries bijectively $\mathbb{N}[U\natural X_n] \setminus \mathbb{N}[U\natural X_{n-1}]$ to $X_{n+1} \setminus X_n$, and $\mathbb{N}[U\natural X_{n-1}] \setminus \{0\}$ to X_n by induction hypothesis.

The second part follows taking unions. □

We now present the universal property of $\mathrm{Rat}(U)$ which we shall apply in different situations.

LEMMA 5.34. *If U is a multiplicative group, and R a rational U-semiring, then there exists a unique morphism $\Phi\colon \mathrm{Rat}(U) \to R$ of rational U-semirings.*

Moreover, if R has a zero element, 0_R, then Φ extends to a morphism of additive monoids $\Phi'\colon \mathrm{Rat}(U) \cup \{0\} \to R$, and this is a morphism of U-semirings if 0_R is an absorbing zero.

PROOF. We use the notation of Definitions 5.32.

Let $\phi_0\colon X_0\ (= \emptyset) \to R$ be the inclusion map which is a map of U-bisets.

Suppose that $n \geq 0$, and that $\phi_n\colon X_n \to R$ is a map of U-bisets.

Then, by Lemma 5.27, ϕ_n induces a morphism of U-semirings

$$\phi_n\colon \mathbb{N}[U\natural X_n] \setminus \{0\} \to R.$$

Now we define $\phi_{n+1}\colon X_{n+1} \to R$, by

$$\phi_{n+1}(f^*) = (\phi_n(f))^* \text{ for all } f \in \mathbb{N}[U\natural X_n] \setminus \{0\}.$$

This is a U-biset map. Note that ϕ_{n+1} is well defined by Remark 5.33.

Thus we have recursively defined a sequence (ϕ_n) of morphisms of U-bisets.

We prove, by induction, that ϕ_{n+1} agrees with ϕ_n on X_n, for all $n \geq 0$.

If $n = 1$, it is clear because $X_0 = \emptyset$, and ϕ_0 is the inclusion map. Suppose that $n \geq 1$, and ϕ_n agrees with ϕ_{n-1}. Let $g \in X_n$, $g = f^*$ for some $f \in \mathbb{N}[U\natural X_{n-1}] \setminus \{0\}$. Then

$$\phi_{n+1}(g) = \phi_{n+1}(f^*) = (\phi_n(f))^* = (\phi_{n-1}(f))^* = \phi_n(f^*) = \phi_n(g).$$

Taking unions, or limits, we get a morphism of U-bisets, $\Phi\colon X \to R$, and this induces a morphism of rational U-semirings $\Phi\colon \mathbb{N}[U\natural X] \setminus \{0\} \to R$, as desired.

Observe that the U-monoid structure of R determines the image of the elements of X_n because of the way $\mathrm{Rat}(U)$ is constructed. This proves that Φ is unique.

The second part follows easily defining $\Phi'(0) = 0_R$. □

The following examples turn out to be very important.

EXAMPLE 5.35. If U is a subgroup of some group W, then, by Lemma 5.34 with $R = \mathrm{Rat}(W)$, we get a morphism of rational U-semirings

$$\Phi\colon \mathrm{Rat}(U) \longrightarrow \mathrm{Rat}(W).$$

Furthermore, Φ is injective, and $\mathrm{Rat}(U)$ is (identified with) an admissible U-sub-biset of the W-biset $\mathrm{Rat}(W)$.

PROOF. We denote the U-bisets needed to construct $\mathrm{Rat}(U)$ by $X_0 = \emptyset$, $X_1 = (\mathbb{N}[U] \setminus \{0\})^\dagger$ and $X_{n+1} = (\mathbb{N}[U\natural X_n] \setminus \mathbb{N}[U\natural X_{n-1}])^\dagger \cup X_n$. We denote by $Y_0 = \emptyset$, $Y_1 = (\mathbb{N}[W] \setminus \{0\})^\dagger$ and $Y_{n+1} = (\mathbb{N}[W\natural Y_n] \setminus \mathbb{N}[W\natural Y_{n-1}])^\dagger \cup Y_n$ the W-bisets needed to construct $\mathrm{Rat}(W)$. We are going to prove by induction on n that $\phi_n(= \Phi\mid_{X_n})$ is an injective map of U-bisets such that $\phi_n(X_n) \subseteq Y_n$ is an admissible U-sub-biset of Y_n for every $n \in \mathbb{N}$. Assume that this is proved. Then $\Phi(X = \cup X_n) \subseteq Y = \cup Y_n$, $\Phi\mid_X$ is an injective map of

U-bisets, and $\Phi(X)$ is an admissible U-sub-biset of Y. Also, by Corollary 5.30, it follows that $\Phi\colon \operatorname{Rat}(U) \longrightarrow \operatorname{Rat}(W)$ is an injective morphism of rational U-semirings and that $\Phi(\operatorname{Rat}(U))$ is an admissible U-sub-biset of $\operatorname{Rat}(W)$.

For $n = 0$ it is clear since $X_0 = Y_0 = \emptyset$, and ϕ_0 is the inclusion map.

Suppose that $n \geq 0$, and $\phi_n\colon X_n \longrightarrow \operatorname{Rat}(W)$ is an injective U-biset map such that $\phi_n(X_n) \subseteq Y_n$ is an admissible U-sub-biset of Y_n. Then ϕ_n induces an injective morphism of U-semirings $\phi_n\colon \mathbb{N}[U\natural X_n]\setminus\{0\} \longrightarrow \mathbb{N}[W\natural Y_n]\setminus\{0\}$ such that $\phi_n(\mathbb{N}[U\natural X_n]\setminus\{0\})$ is an admissible U-sub-biset of $\mathbb{N}[W\natural Y_n]\setminus\{0\}$ by Corollary 5.30. Recall that $\phi_{n+1}\colon X_{n+1} \longrightarrow \operatorname{Rat}(W)$ is defined by $\phi_{n+1}(f^*) = (\phi_n(f))^*$ for all $f \in \mathbb{N}[U\natural X_n] \setminus \{0\}$. Then the fact that the $*$-map bijectively carries $\mathbb{N}[W\natural Y_n] \setminus \{0\}$ to Y_{n+1} implies that $\phi_{n+1}(X_{n+1}) \subseteq Y_{n+1}$, and ϕ_{n+1} is injective.

If $w \in W$, then $w\phi_{n+1}(f^*) = ((\phi_n f)w^{-1})^*$, and $\phi_{n+1}(f^*)w = (w^{-1}\phi_n(f))^*$ for all $f \in \mathbb{N}[U\natural X_n] \setminus \{0\}$. Since $\phi_n(\mathbb{N}[U\natural X_n] \setminus \{0\})$ is an admissible U-sub-biset of $\mathbb{N}[W\natural Y_n] \setminus \{0\}$ and the $*$-map carries bijectively $\mathbb{N}[W\natural Y_n] \setminus \{0\}$ to Y_{n+1}, it follows that $\phi_{n+1}(X_{n+1})$ is an admissible U-sub-biset of Y_{n+1}. ☐

Example 5.36. Let D be a division ring, and let $D \cup \{\infty\}$ have the structure of rational U-semiring as described in Examples 1.43(d) for each subgroup U of $D^\times$.

Suppose that $U \leq W \leq D^\times$ is a subgroup of W. Then there exist $\Phi_U\colon \operatorname{Rat}(U) \to D \cup \{\infty\}$, a morphism of rational U-semirings, $\Phi_W\colon \operatorname{Rat}(W) \to D \cup \{\infty\}$, a morphism of rational W-semirings, and a commutative diagram of morphisms of rational U-semirings

$$\begin{tikzcd} \operatorname{Rat}(W) \arrow[rr, "\Phi_W"] & & D \cup \{\infty\} \\ & \operatorname{Rat}(U) \arrow[ul, "\Psi_{U,W}"] \arrow[ur, "\Phi_U"'] & \end{tikzcd}$$

Moreover, $\Psi_{U,W}$ is injective, so we think of Φ_U as the restriction of Φ_W to $\operatorname{Rat}(U)$.

Proof. The structure of rational U-semiring of $D \cup \{\infty\}$ gives a unique morphism of rational U-semirings $\Phi_U\colon \operatorname{Rat}(U) \to D \cup \{\infty\}$ by Lemma 5.34. In the same way we obtain $\Phi_W\colon \operatorname{Rat}(W) \to D\cup\{\infty\}$. The morphism $\Psi_{U,W}$ exists and is injective by Example 5.35. The commutativity of the diagram is given by the uniqueness of Φ_U. ☐

Example 5.37. Let D be a division ring. Let U be a subgroup of $D^\times$ such that $-1 \in U$. Let D' be the smallest subdivision ring of D that contains U. Let $D \cup \{\infty\}$ have the structure of rational U-semirings as in Examples 1.43(d). As before, there exists a morphism of rational U-semirings $\Phi\colon \operatorname{Rat}(U) \to D \cup \{\infty\}$. Then the image of Φ is $D' \cup \{\infty\}$.

Proof. Let $R = \Phi(\mathbb{N}[U] \setminus \{0\})$. Since $-1 \in U$, then R is a ring, the one generated by U. Recall from Remark 3.16 that $D' = \bigcup_{n=0}^{\infty} Q_n(R, D)$. Now we prove that, for each $n \geq 1$, $\Phi(\mathbb{N}[U\natural X_n] \setminus \{0\}) = Q_n(R, D) \cup \{\infty\}$.

Suppose that $n = 1$. Since Φ is a morphism of rational U-semirings, the way D' is constructed implies that $\Phi(\mathbb{N}[U\natural X_1] \setminus \{0\}) \subseteq Q_1(R, D) \cup \{\infty\}$. Moreover, if $d \in R \setminus \{0\}$, and $\Phi(f) = d$ for $f \in \mathbb{N}[U] \setminus \{0\}$, then $\Phi(f^*) = d^{-1}$ with $f^* \in X_1$. Hence, because $\mathbb{N}[U\natural X_1] \setminus \{0\}$ is a U-semiring and $-1 \in U$, there exists $f \in \mathbb{N}[U\natural X_1] \setminus \{0\}$ such that $\Phi(f) = d$ for every $d \in Q_1(R, D)$. Observe that $1 + (-1) \in \mathbb{N}[U] \setminus \{0\}$ and $\Phi((1 + (-1))^*) = \infty$.

Suppose that $n \geq 1$ and $\Phi(\mathbb{N}[U\natural X_n] \setminus \{0\}) = Q_n(R, D) \cup \{\infty\}$. Then $\Phi(X_{n+1})$ and $\Phi(\mathbb{N}[U\natural X_{n+1}] \setminus \{0\})$ are contained in $Q_{n+1}(R, D) \cup \{\infty\}$ because Φ is a morphism of rational U-semirings. If $d \in Q_n(R, D) \setminus \{0\}$, then there exists $f \in \mathbb{N}[U\natural X_n] \setminus \{0\}$ such that $\Phi(f) = d$. Then $\Phi(f^*) = d^{-1}$. Notice that $f^* \in X_{n+1}$. Hence, because $\mathbb{N}[U\natural X_{n+1}] \setminus \{0\}$ is a U-semiring and $(-1) \in U$, there exists $f \in \mathbb{N}[U\natural X_{n+1}] \setminus \{0\}$ such that $\Phi(f) = d$ for every $d \in Q_{n+1}(R, D)$. ☐

Example 5.38. Let k be a division ring. Let G be a group. Let kG be a crossed product group ring. Suppose that kG has a division ring of fractions D. Let N be a subgroup of G. Then $U = k^{\times}N$ is a subgroup of $D^{\times}$. Endow $D \cup \{\infty\}$ with the structure of rational U-semiring as in Examples 1.43(d). By Lemma 5.34, there exists $\Phi\colon \operatorname{Rat}(U) \longrightarrow D \cup \{\infty\}$, a morphism of rational U-semirings. Then the image of Φ is $D(kN) \cup \{\infty\}$. Moreover, $\Phi(\mathbb{N}[U] \setminus \{0\}) = kN$ and

$$\Phi(\mathbb{N}[U \natural X_n] \setminus \{0\}) = Q_n(kN, D) \cup \{\infty\} \quad \text{for } n \geq 1.$$

Proof. Follows from Example 5.37 and its proof. □

Example 5.39. By Definition 5.10(b), $\mathcal{T}$ is a rational U-semiring. By Lemma 5.34 with $R = \mathcal{T}$, we get a morphism of U-semirings,

$$\operatorname{Tree}\colon \operatorname{Rat}(U) \cup \{0\} \to \mathcal{T}.$$

For $f \in \operatorname{Rat}(U) \cup \{0\}$, $\operatorname{Tree}(f)$ is called the *complexity* of f. We can prove

Lemma 5.40. *If $f, g, f', g' \in \operatorname{Rat}(U) \cup \{0\}$, then the following hold.*

(i) $\operatorname{Tree}(f) = 0_{\mathcal{T}}$ *if and only if* $f = 0$.
(ii) $\operatorname{Tree}(f) = 1_{\mathcal{T}}$ *if and only if* $f \in U$.
(iii) $\operatorname{Tree}(f + g) = \operatorname{Tree}(f) + \operatorname{Tree}(g)$*; hence, if* $\operatorname{Tree}(f) \leq \operatorname{Tree}(f')$ *and* $\operatorname{Tree}(g) \leq \operatorname{Tree}(g')$, *then* $\operatorname{Tree}(f + g) \leq \operatorname{Tree}(f' + g')$, *and equality holds if and only if* $\operatorname{Tree}(f) = \operatorname{Tree}(f')$ *and* $\operatorname{Tree}(g) = \operatorname{Tree}(g')$. *Then it follows* $\operatorname{Tree}(f) \leq \operatorname{Tree}(f + g)$, *and equality holds if and only if* $g = 0$.
(iv) $\operatorname{Tree}(fg) = \operatorname{Tree}(f) \cdot \operatorname{Tree}(g)$*; therefore, if f, f', g and g' are nonzero,* $\operatorname{Tree}(f) \leq \operatorname{Tree}(f')$ *and* $\operatorname{Tree}(g) \leq \operatorname{Tree}(g')$, *then* $\operatorname{Tree}(f \cdot g) \leq \operatorname{Tree}(f' \cdot g')$, *and equality holds if and only if* $\operatorname{Tree}(f) = \operatorname{Tree}(f')$ *and* $\operatorname{Tree}(g) = \operatorname{Tree}(g')$. *It follows that if f and g are nonzero, then* $\operatorname{Tree}(f) \leq \operatorname{Tree}(fg)$, *and equality holds if and only if $g \in U$.*
(v) $\log(\operatorname{Tree}(f + g)) = \max\{\log(\operatorname{Tree}(f)), \log(\operatorname{Tree}(g))\}$.
(vi) $\log(\operatorname{Tree}(fg)) = \log(\operatorname{Tree}(f)) + \log(\operatorname{Tree}(g))$.
(vii) $\log^2(\operatorname{Tree}(f + g)) = \max\{\log^2(\operatorname{Tree}(f)), \log^2(\operatorname{Tree}(g))\}$.
(viii) $\log^2(\operatorname{Tree}(f \cdot g)) \leq \max\{\log^2(\operatorname{Tree}(f)), \log^2(\operatorname{Tree}(g))\}$, *and equality holds if f and g are nonzero.*
(ix) *If $f \neq 0$, then* $\log^2(\operatorname{Tree}(f^*)) = \operatorname{Tree}(f)$.

Proof. We prove (i) and (ii) at the same time. $\operatorname{Rat}(U)$ is the disjoint union of the following subsets:

$$U, \quad X, \quad (U \natural X) \setminus (U \cup X), \quad \mathbb{N}[U \natural \mathrm{X}] \setminus (U \natural X \cup \{0\}).$$

If $f \in U$, then $\operatorname{Tree}(f) = 1_{\mathcal{T}} \neq 0_{\mathcal{T}}$.

If $f \in X$, then $f = r^*$ with $r \in \mathbb{N}[U \natural X] \setminus \{0\}$. Since $\operatorname{Tree}|_{\operatorname{Rat}(U)}$ is a morphism of rational U-semirings, $\operatorname{Tree}(f) = \exp^2(\operatorname{Tree}(r))$. Hence $\mathrm{h}(\operatorname{Tree}(f)) \geq 2$. So $\operatorname{Tree}(f) > 1_{\mathcal{T}} > 0_{\mathcal{T}}$ by Remark 5.14.

If $f \in (U \natural X) \setminus (U \cup X)$, $f \in X^{\times_U^n}$, $n > 1$. Thus $f = f_1 \cdots f_n$ for $f_i \in X$. Then

$$\operatorname{Tree}(f) = \operatorname{Tree}(f_1 \cdots f_n) = \operatorname{Tree}(f_1) \cdots \operatorname{Tree}(f_n) \geq \operatorname{Tree}(f_1) \cdot \operatorname{Tree}(f_2) > \operatorname{Tree}(f_1) > 1_{\mathcal{T}} > 0_{\mathcal{T}},$$

by Remarks 5.18(b).

If $f \in \mathbb{N}[U \natural X] \setminus (U \natural X \cup \{0\})$, then $f = \sum_{i=1}^{n} f_i$, $n > 1$, $f_i \in U \natural X$. We already know that $\operatorname{Tree}(f_i) \geq 1_{\mathcal{T}} > 0_{\mathcal{T}}$, $i = 1, \ldots, n$. Hence

$$\operatorname{Tree}(f) = \sum_{i=1}^{n} \operatorname{Tree}(f_i) \geq \operatorname{Tree}(f_1) + \operatorname{Tree}(f_2) > \operatorname{Tree}(f_1) \geq 1_{\mathcal{T}} > 0_{\mathcal{T}}$$

by Remarks 5.18(a).

(iii)-(vii) follow from (i), (ii) and Lemma 5.17(i)-(v) using that Tree is a morphism of U-semirings.

(ix) follows from the fact that $\log(\exp(X)) = X$ and $\mathrm{Tree}(f^*) = (\mathrm{Tree}(f))^*$.

(viii) follows from (i) and Lemma 5.17(vi). □

A somehow similar thing is done in [**DGH03**]. They define the complexity of the elements of the division ring instead of using an auxiliary object ($\mathrm{Rat}(U)$) as we do. They also measure the complexity of the elements assigning to them an ordinal instead of a rooted tree.

5. Source subgroups

We use the notation of Definitions 5.32 throughout this section.

We want to prove that, for each $f \in \mathrm{Rat}(U)$, there exists a (unique) smallest subgroup V of U such that $f \in \mathrm{Rat}(V) \cdot U$; we will then show that V is finitely generated.

Definitions 5.41. We define a subset Q of X, and a subset P of $\mathrm{Rat}(U) = \mathbb{N}[U\natural X] \setminus \{0\}$.

Let $Q_0 = X_0$ $(= \emptyset)$. Let $n \geq 0$. Suppose that we have defined Q_n a subset of X_n. Denote by $\langle Q_n \rangle$ the multiplicative submonoid of $U\natural X_n$ generated by Q_n. We define:

$$P_n = \langle Q_n \rangle + \mathbb{N}[U\natural X_n], \qquad Q_{n+1} = P_n^*. \tag{29}$$

Notice that $P_n \subseteq \mathbb{N}[U\natural X_n] \setminus \{0\}$. Then, by Remark 5.33,

$$Q_{n+1} = P_n^* \subseteq (\mathbb{N}[U\natural X_n] \setminus \{0\})^* = X_{n+1}.$$

Thus $(Q_n)_{n\in\mathbb{N}}$ is an ascending sequence of subsets of X, and $(P_n)_{n\in\mathbb{N}}$ is an ascending sequence of subsets of $\mathbb{N}[U\natural X] \setminus \{0\}$.

We prove by induction that $(Q_n)_{n\in\mathbb{N}}$ form an ascending chain. Obviously, $Q_0 \subseteq Q_1$. Suppose that $n \geq 1$ and $Q_{n-1} \subseteq Q_n$. Then $\langle Q_{n-1} \rangle \subseteq \langle Q_n \rangle$, and (29) implies that $P_{n-1} \subseteq P_n$. Thus, $Q_n = P_{n-1}^* \subseteq P_n^* = Q_{n+1}$. By (29), $(P_n)_{n\in\mathbb{N}}$ is an ascending chain of subsets of $\mathbb{N}[U\natural X] \setminus \{0\}$. We define $Q = \bigcup_{n\geq 0} Q_n \subseteq X$ and $P = \bigcup_{n\geq 0} P_n \subseteq \mathbb{N}[U\natural X] \setminus \{0\}$. We call P the *set of primitive elements* of $\mathrm{Rat}(U)$. □

Lemma 5.42. *The following hold:*

(i) $P = \langle Q \rangle + \mathbb{N}[U\natural X]$, $Q = P^*$.

(ii) Q, $\langle Q \rangle$ *and* P *are closed under* U*-conjugation. In fact* Q_n, $\langle Q_n \rangle$ *and* P_n *are closed under* U*-conjugation for every* $n \in \mathbb{N}$.

(iii) $QU = UQ = X$. *In fact* $Q_nU = UQ_n = X_n$ *for every* $n \in \mathbb{N}$.

(iv) $\langle Q \rangle U = U\langle Q \rangle = U\natural X$. *In fact* $\langle Q_n \rangle U = U\langle Q_n \rangle = U\natural X_n$ *for every* $n \in \mathbb{N}$.

(v) $PU = UP = \mathbb{N}[U\natural X] \setminus \{0\} = \mathrm{Rat}(U)$. *In fact,* $P_nU = UP_n = \mathbb{N}[U\natural X_n] \setminus \{0\}$ *for every* $n \in \mathbb{N}$.

Proof. (i) $Q = \bigcup_{n\geq 0} Q_n = \bigcup_{n\geq 0} P_n^* = (\bigcup_{n\geq 0} P_n)^* = P^*$.

$P = \bigcup_{n\geq 0} P_n = \bigcup_{n\geq 0} (\langle Q_n \rangle + \mathbb{N}[U\natural X_n]) = \langle Q \rangle + \mathbb{N}[U\natural X]$.

We prove (ii)-(v) at the same time by induction on n.

Clearly, $Q_0 = \emptyset$ is closed under U-conjugation and $UQ_0 = Q_0U = X_0 = \emptyset$. Suppose that $n \geq 0$, Q_n is closed under U-conjugation, and $UQ_n = Q_nU = X_n$. Then $\langle Q_n \rangle$ is closed under U-conjugation. Hence $P_n = \langle Q_n \rangle + \mathbb{N}[U\natural X_n]$ and $P_n^* = Q_{n+1}$ are also closed under U-conjugation. Moreover, $U\langle Q_n \rangle = \langle Q_n \rangle U$. This last object is a submonoid of $U\natural X_n$ that contains U and $UQ_n = Q_nU (= X_n$ by induction hypothesis). Therefore $U\natural X_n = U\langle Q_n \rangle = \langle Q_n \rangle U$.

Since for every $u \in U$, $u\mathbb{N}[U\natural X_n] = \mathbb{N}[U\natural X_n]u = \mathbb{N}[U\natural X_n]$, the foregoing implies

$$UP_n = P_nU = \mathbb{N}[U\natural X_n] \setminus \{0\}.$$

Now,

$$UQ_{n+1} = UP_n^* = (P_nU)^* = (\mathbb{N}[U\natural X_n] \setminus \{0\})^* = X_{n+1}.$$

Analogously $Q_{n+1}U = X_{n+1}$. The remaining parts of (ii)-(v) follow by taking unions. $\square$

Definition 5.43. Let $p \in P$ and $u \in U$. Recall $p^u \in P$ by Lemma 5.42(ii). We are going to define recursively a subgroup of U, the *source subgroup of* p, denoted by $\mathrm{source}_{\mathrm{U}}(p)$. It will satisfy the following two properties:

(a) $\mathrm{source}_{\mathrm{U}}(p)^u = \mathrm{source}_{\mathrm{U}}(p^u)$.
(b) If $pu \in P$, then $u \in \mathrm{source}_{\mathrm{U}}(pu) = \mathrm{source}_{\mathrm{U}}(p)$.

(In fact we will need that $\mathrm{source}_{\mathrm{U}}(p)$ satisfies these properties to make this recursive definition).

We partition P into the following four subsets,

$$\{1\}, \qquad Q, \qquad \langle Q\rangle \setminus (Q \cup \{1\}), \qquad \langle Q\rangle + (\mathbb{N}[U\natural X] \setminus \{0\}).$$

We define $\mathrm{source}_{\mathrm{U}}(1)$ to be the trivial subgroup of U. It is clear that (a) and (b) are satisfied.

Suppose that we have (well) defined $\mathrm{source}_{\mathrm{U}}(q)$, and it satisfies (a) and (b) for all $q \in P$ with $\mathrm{Tree}(q) < \mathrm{Tree}(p)$. We call this "the transfinite induction hypothesis". We now define $\mathrm{source}_{\mathrm{U}}(p)$ depending on which of the four subsets of P it belongs to.

Case 1. $p \in \langle Q\rangle + (\mathbb{N}[U\natural X] \setminus \{0\})$. Then,

$$p = \sum_{i=1}^{n} f_i, \tag{30}$$

where $n \geq 2$, $f_i \in U\natural X = \langle Q\rangle U$ for each i, and $f_{i_0} \in \langle Q\rangle \subseteq P$ for some i_0. By Lemma 5.42(iv), there exist $u_i \in U$, $p_i \in \langle Q\rangle \subseteq P$ such that $f_i = p_iu_i$, $i = 1, \ldots, n$.

Because $n \geq 2$, by Lemma 5.40(iii), $\mathrm{Tree}(p_i) = \mathrm{Tree}(p_iu_i) < \mathrm{Tree}(p)$ for $i = 1, \ldots, n$. Thus $\mathrm{source}_{\mathrm{U}}(p_i)$ satisfies the transfinite induction hypothesis for each i. We define $\mathrm{source}_{\mathrm{U}}(p)$ to be the subgroup of U generated by

$$\bigcup_{i=1}^{n} (\mathrm{source}_{\mathrm{U}}(p_i) \cup \{u_i\}).$$

Consider the following argument. Suppose that $pu = \sum\limits_{i=1}^{n} f_iu \in P$. Then

$$pu \in \langle Q\rangle + (\mathbb{N}[U\natural X] \setminus \{0\}).$$

Hence $pu = \sum\limits_{j=1}^{m} f'_j$, $m \geq 2$, $f'_j \in U\natural X$ for each j, and $f_{j_0} \in \langle Q\rangle$ for some j_0. Then $n = m$, and we can suppose that $f'_i = f_iu$ for each $i \in \{1, \ldots, n\}$. By Lemma 5.42(iv), there exist $u'_i \in U$ and $p'_i \in \langle Q\rangle$ such that $f'_i = p'_iu'_i$, $i = 1, \ldots, n$. Hence $p'_iu'_i = p_iu_iu$ and

$$p'_i = p_iu_iu{u'_i}^{-1} \text{ for each } i.$$

Notice that $\mathrm{Tree}(p'_i) = \mathrm{Tree}(p'_iu'_i) < \mathrm{Tree}(p)$ for $i = 1, \ldots, n$. Then, by the transfinite induction hypothesis,

$$u_iu{u'_i}^{-1} \in \mathrm{source}_{\mathrm{U}}(p'_i) = \mathrm{source}_{\mathrm{U}}(p_i) \text{ for each } i. \tag{31}$$

If $u = 1$, by the foregoing argument, $u_i {u'_i}^{-1} \in \mathrm{source}_{\mathrm{U}}(p'_i) = \mathrm{source}_{\mathrm{U}}(p_i)$. Then the subgroup generated by $\bigcup_{i=1}^{n} (\mathrm{source}_{\mathrm{U}}(p_i) \cup \{u_i\})$ is the same as the one generated by $\bigcup_{i=1}^{n} (\mathrm{source}_{\mathrm{U}}(p'_i) \cup \{u'_i\})$. Therefore $\mathrm{source}_{\mathrm{U}}(p)$ is well defined.

We may assume that $u_{i_0} = 1$ and $u'_{j_0} = 1$ because $p \in \langle Q \rangle + (\mathbb{N}[U \natural X] \setminus \{0\})$ and $\mathrm{source}_{\mathrm{U}}(p)$ is well defined. Then $f_{i_0} = p_{i_0} \in \langle Q \rangle$, $f'_{j_0} = p'_{j_0} \in \langle Q \rangle$. Thus, (31) implies,

$$\begin{gathered} u {u'_{i_0}}^{-1} \in \mathrm{source}_{\mathrm{U}}(p'_{i_0}) = \mathrm{source}_{\mathrm{U}}(p_{i_0}) \\ u_{j_0} u \in \mathrm{source}_{\mathrm{U}}({p_{j_0}}') = \mathrm{source}_{\mathrm{U}}(p_{j_0}). \end{gathered}$$

The first one implies that $u \in \mathrm{source}_{\mathrm{U}}(pu)$. The second one that $u \in \mathrm{source}_{\mathrm{U}}(p)$. Thus, by (31), $u'_i \in \mathrm{source}_{\mathrm{U}}(p)$ and $u_i \in \mathrm{source}_{\mathrm{U}}(pu)$ for each i. Therefore $\mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_{\mathrm{U}}(pu)$.

If p is as in (30), $p^u = \sum_{i=1}^{n} f_i^u = \sum_{i=1}^{n} (p_i u_i)^u = \sum_{i=1}^{n} p_i^u u_i^u$. Notice that $p_i^u \in P$ and $f_{i_0}^u \in \langle Q \rangle$ by Lemma 5.42(ii). Also $\mathrm{Tree}(p_i^u) = \mathrm{Tree}(p_i) < \mathrm{Tree}(p) = \mathrm{Tree}(p^u)$ for $i = 1, \ldots, n$. Hence,

$$\mathrm{source}_{\mathrm{U}}(p_i^u) = \mathrm{source}_{\mathrm{U}}(p_i)^u \text{ for each } i.$$

So $\mathrm{source}_{\mathrm{U}}(p^u)$, the subgroup generated by $\bigcup_{i=1}^{n} (\mathrm{source}_{\mathrm{U}}(p_i^u) \cup \{u_i^u\})$, equals $\mathrm{source}_{\mathrm{U}}(p)^u$.

Case 2. Suppose that $p \in \langle Q \rangle \setminus (Q \cup \{1\})$. Then p can be expressed as

$$p = q \cdot r \tag{32}$$

with $q \in \langle Q \rangle \setminus \{1\}$, $r \in Q \subseteq X$ (in fact, if $p \in X^{\times_U^n}$, then $q \in X^{\times_U^{n-1}}$ and $r \in X$). By Lemma 5.40(iv), $\mathrm{Tree}(q) < \mathrm{Tree}(p)$ and $\mathrm{Tree}(r) < \mathrm{Tree}(p)$. Thus q and r satisfy the induction hypothesis. We define $\mathrm{source}_{\mathrm{U}}(p)$ to be the subgroup of U generated by

$$\mathrm{source}_{\mathrm{U}}(q) \cup \mathrm{source}_{\mathrm{U}}(r).$$

Consider the following argument. Suppose that $pu \in P$. Then $pu \in \langle Q \rangle \setminus (Q \cup \{1\})$. Thus $pu = q' \cdot r'$ with $q' \in \langle Q \rangle \setminus \{1\}$, $r' \in Q$. Because $pu = q \cdot ru = q' \cdot r'$, there exists $v \in U$ such that $q' = qv$, $r' = v^{-1} r u = r^v v^{-1} u$. Notice that $\mathrm{Tree}(q') = \mathrm{Tree}(q) < \mathrm{Tree}(p)$ and $\mathrm{Tree}(r') = \mathrm{Tree}(r) < \mathrm{Tree}(p)$. Then, by the transfinite induction hypothesis,

$$\begin{gathered} v \in \mathrm{source}_{\mathrm{U}}(q') = \mathrm{source}_{\mathrm{U}}(q) \\ v^{-1} u \in \mathrm{source}_{\mathrm{U}}(r') = \mathrm{source}_{\mathrm{U}}(r^v) = \mathrm{source}_{\mathrm{U}}(r)^v. \end{gathered}$$

If $u = 1$, then $p = q \cdot r = q' \cdot r'$, and the subgroup generated by $\mathrm{source}_{\mathrm{U}}(q) \cup \mathrm{source}(r)$ is the same as the one generated by $\mathrm{source}_{\mathrm{U}}(q') \cup \mathrm{source}_{\mathrm{U}}(r')$. Hence $\mathrm{source}_{\mathrm{U}}(p)$ does not depend on the expression of $p = q \cdot r$ as in (32), so it is well defined.

In the general case, because $v \in \mathrm{source}_{\mathrm{U}}(p) \cap \mathrm{source}_{\mathrm{U}}(pu)$,

$$u \in \mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_{\mathrm{U}}(pu).$$

Notice that $p^u = (q \cdot r)^u = q^u \cdot r^u$, and $q^u, r^u \in P$, by Lemma 5.42(ii). Then, because $\mathrm{Tree}(q^u) = \mathrm{Tree}(q)$ and $\mathrm{Tree}(r^u) = \mathrm{Tree}(r)$, the transfinite induction hypothesis implies that $\mathrm{source}_{\mathrm{U}}(p^u)$, the subgroup generated by $\mathrm{source}(q^u) \cup \mathrm{source}(r^u)$, equals $\mathrm{source}_{\mathrm{U}}(p)^u$.

Case 3. Suppose that $p \in Q = P^*$ by Lemma 5.42(i). Then there exists a unique $q \in P$ such that

$$p = q^*. \tag{33}$$

Because $\mathrm{h}(\mathrm{Tree}(q)) < \mathrm{h}(\mathrm{Tree}(p))$, then $\mathrm{Tree}(q) < \mathrm{Tree}(p)$ by Remark 5.14. Thus q satisfies the induction hypothesis. We define

$$\mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_{\mathrm{U}}(q).$$

Since q is unique, $\mathrm{source}_{\mathrm{U}}(p)$ is well defined.

Suppose that $pu \in P$, then $pu = q^*u = (u^{-1}q)^* = (q^u u^{-1})^*$. For $Q = P^*$ and Remark 5.33, $q^u u^{-1} \in P$. By definition $\mathrm{source}_{\mathrm{U}}(pu) = \mathrm{source}_{\mathrm{U}}(q^u u^{-1})$. By Lemma 5.42(ii), $q^u \in P$. Since

$$\mathrm{Tree}(q^u) = \mathrm{Tree}(q) < \mathrm{Tree}(p), \tag{34}$$

the transfinite induction hypothesis implies that

$$u^{-1} \in \mathrm{source}_{\mathrm{U}}(q)^u = \mathrm{source}_{\mathrm{U}}(q^u) = \mathrm{source}(q^u u^{-1}) = \mathrm{source}_{\mathrm{U}}(pu).$$

Hence $u \in \mathrm{source}_U(q)$ and $\mathrm{source}_U(q) = \mathrm{source}_U(q)^u$. Therefore

$$u \in \mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_{\mathrm{U}}(q) = \mathrm{source}_{\mathrm{U}}(pu).$$

Because $p^u = q^{*u} = q^{u*}$ and $q^u \in P$, (34) and the transfinite induction hypothesis imply that

$$\mathrm{source}_{\mathrm{U}}(p^u) = \mathrm{source}_{\mathrm{U}}(q^u) = \mathrm{source}_{\mathrm{U}}(q)^u = \mathrm{source}_{\mathrm{U}}(p)^u. \qquad \square$$

Lemma 5.44. *If p is a primitive element of $\mathrm{Rat}(U)$, then the following hold:*

(i) $\mathrm{source}_{\mathrm{U}}(p)$ *is finitely generated.*
(ii) $p \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p))$.
(iii) *If U is a subgroup of some group W, then p is primitive element of $\mathrm{Rat}(W)$ and* $\mathrm{source}_W(p) = \mathrm{source}_{\mathrm{U}}(p)$.

In particular we can write $\mathrm{source}_U(p)$ *as* $\mathrm{source}(p)$.

Proof. We call P_U and P_W the set of primitive elements of $\mathrm{Rat}(U)$ and $\mathrm{Rat}(W)$ respectively. The objects $(X, Q, \ldots)$ involved in the construction of P_U and $\mathrm{Rat}(U)$, P_W and $\mathrm{Rat}(W)$ will be denoted as in Definitions 5.41 and 5.32 with a subscript U, if they are used to construct P_U or $\mathrm{Rat}(U)$, and with a subscript W, if they are used to construct P_W or $\mathrm{Rat}(W)$. As in Definition 5.43 we partition P_U and P_W into four sets.

$$P_U^1 = \{1\}, \quad P_U^2 = Q_U, \quad P_U^3 = \langle Q_U\rangle \setminus (Q_U \cup \{1\}), \quad P_U^4 = \langle Q_U\rangle + (\mathbb{N}[U\natural X_U] \setminus \{0\})$$
$$P_W^1 = \{1\}, \quad P_W^2 = Q_W, \quad P_W^3 = \langle Q_W\rangle \setminus (Q_W \cup \{1\}), \quad P_W^4 = \langle Q_W\rangle + (\mathbb{N}[W\natural X_W] \setminus \{0\})$$

We prove (i)-(iii) at the same time by transfinite induction. In fact, for (iii) we are going to prove

(iii′) If $p \in P_U^j$, then $p \in P_W^j$, $j = 1, \ldots, 4$. And $\mathrm{source}_W(p) = \mathrm{source}_{\mathrm{U}}(p)$.

If $p = 1$, then $\mathrm{source}_{\mathrm{U}}(p) = \{1\}$ is finitely generated, $p \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p))$, $p \in P_W^1$ and $\mathrm{source}_W(p) = \{1\} = \mathrm{source}_{\mathrm{U}}(p)$.

Let $1 \neq p \in P_U$ and suppose that for elements of lesser complexity (i), (ii) and (iii') hold.

Case 1. If $p \in \langle Q_U\rangle + (\mathbb{N}[U\natural X_U] \setminus \{0\})$, then $p = \sum_{i=1}^{n} p_i u_i$ as in (30). By definition, $\mathrm{source}_{\mathrm{U}}(p)$ is the subgroup generated by $\bigcup_{i=1}^{n}(\mathrm{source}_{\mathrm{U}}(p_i) \cup \{u_i\})$. The transfinite induction hypothesis implies that, for each i, $\mathrm{source}_{\mathrm{U}}(p_i)$ is finitely generated, $p_i \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p_i))$, $p_i \in \langle Q_W\rangle$, and $\mathrm{source}_{\mathrm{U}}(p_i) = \mathrm{source}_W(p_i)$. Then $\mathrm{source}_{\mathrm{U}}(p)$ is finitely generated, $p \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p))$, $p \in \langle Q_W\rangle + (\mathbb{N}[W\natural X_W] \setminus \{0\})$ (notice that if $p_{i_0}u_{i_0} \in \langle Q_U\rangle$, then $p_{i_0}u_{i_0} \in \langle Q_W\rangle$) and $\mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_W(p)$.

Case 2. If $p \in \langle Q_U\rangle \setminus (Q_U \cup \{1\})$, then $p = q \cdot r$ as in (32). By definition, $\mathrm{source}_{\mathrm{U}}(p)$ is the subgroup generated by $\mathrm{source}_{\mathrm{U}}(q)$ and $\mathrm{source}_{\mathrm{U}}(r)$. Hence $\mathrm{source}_{\mathrm{U}}(p)$ is finitely generated because $\mathrm{source}_{\mathrm{U}}(q)$ and $\mathrm{source}_{\mathrm{U}}(r)$ are finitely generated by the transfinite induction hypothesis. For $q \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(q)) \subseteq \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p))$ and $r \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(r)) \subseteq \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p))$, then $p = q \cdot r \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p))$.

By the transfinite induction hypothesis, q and r are primitive in $\mathrm{Rat}(W)$ with $q \in \langle Q_W\rangle \setminus \{1\}$, $r \in Q_W$. Then $p = q \cdot r \in \langle Q_W\rangle \setminus (Q_W \cup \{1\})$.

By definition, $\mathrm{source}_W(p)$ is the subgroup of W generated by $\mathrm{source}_W(q)$ and $\mathrm{source}_W(r)$. By the transfinite induction hypothesis,

$$\mathrm{source}_{\mathrm{U}}(q) = \mathrm{source}_W(q) \text{ and } \mathrm{source}_{\mathrm{U}}(r) = \mathrm{source}_W(r).$$

Hence $\mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_W(p)$.

CASE 3. If $p \in Q_U$, then $p = r^*$ as in (33). Since $\mathrm{Tree}(r) < \mathrm{Tree}(p)$, then

$$\mathrm{source}_{\mathrm{U}}(r) = \mathrm{source}_{\mathrm{U}}(p)$$

is finitely generated and $r \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(r))$. Hence

$$p = r^* \in \mathrm{Rat}(\mathrm{source}_{\mathrm{U}}(p)).$$

By the transfinite induction hypothesis, $p = r^* \in Q_W$ because r is primitive in $\mathrm{Rat}(W)$. Also $\mathrm{source}_W(p) = \mathrm{source}_W(r) = \mathrm{source}_{\mathrm{U}}(r) = \mathrm{source}_{\mathrm{U}}(p)$. □

DEFINITION 5.45. Let $f \in \mathrm{Rat}(U)$. By Lemma 5.42(v), $f = pu$, with $p \in P$ and $u \in U$. We define

$$\mathrm{source}(f) = \mathrm{source}(p).$$

It is well defined for suppose that $f = p'u'$, with $p' \in P$ and $u' \in U$, then $p' = puu'^{-1}$. Thus, by Definition 5.43(b), $\mathrm{source}(p') = \mathrm{source}(p)$. □

REMARK 5.46. By Lemma 5.44(i), $\mathrm{source}(f)$ is finitely generated, and, by Definition 5.45, $f \in \mathrm{Rat}(\mathrm{source}(f)) \cdot U$. □

LEMMA 5.47. *If $f \in \mathrm{Rat}(U)$, then $\mathrm{source}(f)$ is the smallest subgroup of U among the subgroups V of U such that $f \in \mathrm{Rat}(V) \cdot U$. That is, if $V \leq U$ such that $f \in \mathrm{Rat}(V) \cdot U$, then $\mathrm{source}(f) \leq V$.*

PROOF. By Remark 5.46, $f \in \mathrm{Rat}(\mathrm{source}(f)) \cdot U$.

Let $V \leq U$ such that $f \in \mathrm{Rat}(V) \cdot U$. Then there exist $q \in \mathrm{Rat}(V)$ and $u \in U$ such that $f = qu$. By Lemma 5.42(v), $q = pv$ with p a primitive element of $\mathrm{Rat}(V)$ and $v \in V$. By Lemma 5.44(iii), p is also a primitive element of $\mathrm{Rat}(U)$ and $\mathrm{source}_U(p) = \mathrm{source}_V(p)$. By Definition 5.45,

$$\mathrm{source}(f) = \mathrm{source}_{\mathrm{U}}(p) = \mathrm{source}_V(p) \leq V. \qquad \square$$

6. Skew Laurent series constructions

NOTATION 5.48. Throughout this section R is a ring, α an automorphism of R and V is a subgroup of $R^\times$ such that $-1 \in V$ and $\alpha(V) = V$. □

Consider the skew Laurent series ring $R((t;\alpha))$. Recall from examples 1.43(e) that the set

$$R((t;\alpha))^{\curlywedge} = \Big\{ f = \sum_{n\in\mathbb{Z}} d_n t^n \mid d_N \in R^\times \text{ where } N = \min\mathrm{supp}(f) \Big\}$$

is a subgroup of $R((t;\alpha))^\times$ that contains $R^\times$. Moreover, if $f \in R((t;\alpha))^{\curlywedge}$, then

$$f^{-1} = \sum_{m\geq 0} (d_N t^N)^{-1} (g (d_N t^N)^{-1})^m,$$

where $g = d_N t^N - f$.

Let $V\langle t\rangle$ denote the subset of $R((t;\alpha))$ consisting of the polynomials whose support contains exactly one element and its nonzero coefficient is in V. Observe that $V\langle t\rangle$ is a subgroup of $R((t;\alpha))^{\times}$. In fact it is an internal semidirect product $V \rtimes \langle t\rangle$, with $tv = \alpha(v)t$. Therefore $R((t;\alpha)) \cup \{\infty\}$ is a $V\langle t\rangle$-semiring as in Examples 1.43(d) where the $*$-map is defined as follows:

$$\infty^* = \infty, \quad f^* = f^{-1} \text{ if } f \in R((t;\alpha))^{\times} \quad \text{and}$$
$$f^* = \infty \quad \text{for all } f \in R((t;\alpha)) \setminus R((t;\alpha))^{\times}.$$

By Lemma 5.34, there is a morphism of rational $V\langle t\rangle$-semirings,

$$\Phi\colon \operatorname{Rat}(V\langle t\rangle) \longrightarrow R((t;\alpha)) \cup \{\infty\},$$

and Φ extends to a morphism of additive monoids

$$\Phi'\colon \operatorname{Rat}(V\langle t\rangle) \cup \{0\} \longrightarrow R((t;\alpha)) \cup \{\infty\}.$$

This section is devoted to construct the rational $V\langle t\rangle$-semiring, $\operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\}$, factor Φ through it and give some important properties of this factorization.

Step 1. Construction of the $V\langle t\rangle$-semiring $\operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\}$.

Notice that α induces a group automorphism of V. Thus $\alpha\colon V \to \operatorname{Rat}(V)$ endows $\operatorname{Rat}(V)$ with a new structure of rational V-semiring. It gives rise to a morphism of rational V-semirings $\alpha\colon \operatorname{Rat}(V) \longrightarrow \operatorname{Rat}(V)$ by Lemma 5.34. In the same way α^{-1} induces a morphism of rational V-semirings $\beta\colon \operatorname{Rat}(V) \longrightarrow \operatorname{Rat}(V)$. Since $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1_V$, again Lemma 5.34 implies that $\beta\alpha = \alpha\beta = 1_{\operatorname{Rat}(V)}$. Hence $\alpha\colon \operatorname{Rat}(V) \longrightarrow \operatorname{Rat}(V)$, without considering the different structures of V-bisets, is a semiring automorphism. We extend it to

$$\alpha\colon \operatorname{Rat}(V) \cup \{0\} \to \operatorname{Rat}(V) \cup \{0\}.$$

Consider the skew Laurent series semiring $\operatorname{Rat}(V)((t;\alpha))$. The semiring $\operatorname{Rat}(V\langle t\rangle) \cup \{0\}$ contains copies of $\operatorname{Rat}(V)$ and $\langle t\rangle$, and we denote the product by $\operatorname{Rat}(V)\langle t\rangle$, a multiplicative submonoid of $\operatorname{Rat}(V\langle t\rangle)$. Moreover, we have the multiplicative monoid inclusions

$$V\langle t\rangle \subseteq \operatorname{Rat}(V)\langle t\rangle \subseteq \operatorname{Rat}(V)((t;\alpha)),$$

which implies that $\operatorname{Rat}(V)((t;\alpha))$ and $\operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\}$ are $V\langle t\rangle$-semirings. An element $f \in \operatorname{Rat}(V)((t;\alpha))$ is represented by

$$f = \sum_{n\in\mathbb{Z}} d_n t^n = \sum_{n\in\mathbb{Z}} f_n \in \operatorname{Rat}(V)((t;\alpha)), \tag{35}$$

where we understand $f_n = d_n t^n \in \operatorname{Rat}(V)\langle t\rangle \cup \{0\}$.

Notice that for all $x \in \operatorname{Rat}(V) \cup \{0\} \subseteq \operatorname{Rat}(V\langle t\rangle) \cup \{0\}$,

$$\Phi'(\alpha(x))t = \Phi'(\alpha(x)t) = \Phi'(tx) = t\Phi'(x).$$

Hence if $\Phi'(x) \neq \infty$, then $\Phi'(\alpha(x)) = \alpha(\Phi'(x))$. If $\Phi'(x) = \infty$, then $\Phi'(\alpha(x)) = \infty$.

Step 2. Definition of $\Omega\colon \operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\} \longrightarrow R((t;\alpha)) \cup \{\infty\}$, a morphism of $V\langle t\rangle$-semirings.

We define $\Omega(\infty) = \infty$.
Suppose that we are given $f \in \operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\}$ as in (35).
If $\Phi'(f_n) = \infty$ for some $n \in \mathbb{Z}$, then we define $\Omega(f) = \infty$.
If $\Phi'(f_n) \neq \infty$ for all $n \in \mathbb{Z}$, we define

$$\Omega(f) = \sum_{n\in\mathbb{Z}} \Phi'(f_n) = \sum_{n\in\mathbb{Z}} \Phi'(d_n)t^n \in R((t;\alpha)).$$

Now we prove that Ω is a morphism of $V\langle t\rangle$-semirings.

If $f=\infty$ or $g=\infty$, then $\Omega(f+g)=\Omega(f)+\Omega(g)=\infty$, $\Omega(f\cdot g)=\Omega(f)\Omega(g)=\infty$, and $\Omega(at^kfbt^l)=at^k\Omega(f)bt^l=\infty$ for all $at^k, bt^l\in V\langle t\rangle$.

Let $f=\sum\limits_{n\in\mathbb{Z}} f_n=\sum\limits_{n\in\mathbb{Z}} d_nt^n$, $g=\sum\limits_{n\in\mathbb{Z}} g_n=\sum\limits_{n\in\mathbb{Z}} e_nt^n$ as in (35). Let $at^k, bt^l\in V\langle t\rangle$.

$$\Omega(f+g)=\Omega\Big(\sum_{n\in\mathbb{Z}}(f_n+g_n)\Big),\quad \Omega(f\cdot g)=\Omega\Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}} d_m\alpha^m(e_{n-m})\Big)t^n\Big).$$

Suppose that there exists $n_0\in\mathbb{Z}$ such that $\Phi'(f_{n_0})=\infty$. Then

$$\Phi'(f_{n_0}+g_{n_0})=\Phi'(f_{n_0})+\Phi'(g_{n_0})=\infty.$$

Thus $\Omega(f+g)=\Omega(f)+\Omega(g)=\infty$.

Because there exists $n_1\in\mathbb{Z}$ such that $e_{n_1}\neq 0$, then,

$$\Phi'\Big(\sum_{m\in\mathbb{Z}} d_m\alpha^m(e_{n_0+n_1-m})t^{n_0+n_1}\Big)=\infty.$$

Therefore $\Omega(f\cdot g)=\Omega(f)\cdot\Omega(g)=\infty$. In the same way it is proved when there exists $n_0\in\mathbb{Z}$ such that $\Phi'(g_n)=\infty$. The foregoing also proves that $\Omega(at^kfbt^l)=at^k\Omega(f)bt^l=\infty$.

Suppose that $\Phi'(f_n)\neq\infty$ and $\Phi'(g_n)\neq\infty$ for all $n\in\mathbb{Z}$. Then,

$$\begin{aligned}\Omega(f+g) &= \Omega\Big(\sum_{n\in\mathbb{Z}}(f_n+g_n)\Big)=\sum_{n\in\mathbb{Z}}\Phi'(f_n+g_n)=\sum_{n\in\mathbb{Z}}(\Phi'(f_n)+\Phi'(g_n))\\ &= \sum_{n\in\mathbb{Z}}\Phi'(f_n)+\sum_{n\in\mathbb{Z}}\Phi'(g_n)=\Omega(f)+\Omega(g).\end{aligned}$$

Now,

$$\begin{aligned}\Phi'\Big(\sum_{m\in\mathbb{Z}} d_m\alpha^m(e_{n-m})\Big) &= \sum_{m\in\mathbb{Z}}\Phi'(d_m)\Phi'(\alpha^m(e_{n-m}))\\ &= \sum_{m\in\mathbb{Z}}\Phi'(d_m)\alpha^m(\Phi'(e_{n-m}))\neq\infty \text{ for all } n\in\mathbb{Z}.\end{aligned}$$

Hence,

$$\begin{aligned}\Omega(f\cdot g) &= \Omega\Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}} d_m\alpha^m(e_{n-m})\Big)t^n\Big)\\ &= \sum_{n\in\mathbb{Z}}\Phi'\Big(\sum_{m\in\mathbb{Z}} d_m\alpha^m(e_{n-m})\Big)t^n\\ &= \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\Phi'(d_m)\alpha^m(\Phi'(e_{n-m}))\Big)t^n\\ &= \Big(\sum_{n\in\mathbb{Z}}\Phi'(d_n)t^n\Big)\Big(\sum_{n\in\mathbb{Z}}\Phi'(e_n)t^n=\Omega(f)\Omega(g)\Big).\end{aligned}$$

It also proves that $\Omega(at^kfbt^l)=at^k\Omega(f)bt^l$.

Therefore Ω is a morphism of $V\langle t\rangle$-semirings.

Step 3. We make $\mathrm{Rat}(V)((t;\alpha))\cup\{\infty\}$ into a rational $V\langle t\rangle$-semiring so that Ω is a morphism of rational $V\langle t\rangle$-semirings.

We need to define a $*$-map such that $\Omega(f^*)=\infty$ for all f with $\Omega(f)^*=\infty$, and $\Omega(f^*)=\Omega(f)^{-1}$ for all $f\in\mathrm{Rat}(V)((t;\alpha))$ with $\Omega(f)\in R((t;\alpha))^{\curlywedge}$.

We define $\infty^*=\infty$.

Suppose that we are given f as in (35).

If $\Omega(f)^*=\infty$, then we define $f^*=\infty$.

If $\Omega(f)^* \neq \infty$, then $\Omega(f) \in R((t;\alpha))^{\curlywedge}$. Thus $\Phi'(f_n) \neq \infty$ for all $n \in \mathbb{Z}$, and $\Phi'(f_n) \neq 0$ for some $n \in \mathbb{Z}$. Let N be the least integer such that $\Phi'(f_N) \neq 0$. Then $f_N = d_N t^N$ for some $d_N \in \mathrm{Rat}(V)$ such that $\Phi'(d_N) \in R^{\times}$. Notice that $\mathrm{Rat}(V)$ contains d_N^* and $\alpha^{-N}(d_N^*)$. So if we set

$$f_N^* = t^{-N} d_N^* = \alpha^{-N}(d_N^*)t^{-N} \in \mathrm{Rat}(V)\langle t\rangle \quad \text{and} \quad g = \sum_{n \geq N+1} (-1) f_n, \tag{36}$$

then $\sum\limits_{m\geq 0} f_N^*(gf_N^*)^m$ is defined in $\mathrm{Rat}(V)((t;\alpha))$, by Theorem 4.19(iv) and (i). We define

$$f^* = \sum_{m\geq 0} f_N^*(gf_N^*)^m. \tag{37}$$

Notice that the least element of the support of f^* is $-N$.

Now we prove that $\mathrm{Rat}(V)((t;\alpha)) \cup \{\infty\}$ is a rational $V\langle t\rangle$-semiring with this $*$-map .

Let $at^k, bt^l \in V\langle t\rangle$. Let $f \in \mathrm{Rat}(V)((t;\alpha))$ as in (35) or $f = \infty$.

Suppose that $f^* = \infty$, that is, $\Omega(f)^* = \infty$. By Step 2, Ω is a morphism of $V\langle t\rangle$-semirings, thus $\Omega(at^k f bt^l) = at^k \Omega(f) bt^l$. Then

$$\Omega(at^k f bt^l)^* = (bt^l)^{-1}\Omega(f)^*(at^k)^{-1} = \infty.$$

Hence, $(at^k f bt^l)^* = \infty = (bt^l)^{-1} f^* (at^k)^{-1}$.

Suppose that $f^* \neq \infty$, then $f^* = \sum\limits_{m\geq 0} f_N^*(gf_N^*)^m$, where f, f_N^* and g are as in (35) and (36), respectively.

Now,

$$h = at^k f bt^l = at^k\Big(\sum_{n\in\mathbb{Z}} d_n t^n\Big) bt^l = \sum_{n\in\mathbb{Z}} a\alpha^k(d_n)\alpha^{n+k}(b)t^{n+k+l} = \sum_{i\in\mathbb{Z}} h_i,$$

where $e_{r+k+l} = a\alpha^k(d_r)\alpha^{r+k}(b)$ and $h_{r+k+l} = e_{r+k+l}t^{r+k+l} = at^k d_r t^r bt^l$. Then, for all $r \in \mathbb{Z}$, $\Phi'(h_{r+k+l}) = at^k \Phi'(f_r) bt^l$. Thus, $\Omega(f)^* \neq \infty$ and $N + k + l$ is the first integer such that $\Phi'(h_{N+k+l}) \neq 0$. Moreover $\Phi'(e_{N+k+l}) \in R^{\times}$. So we proceed as in (36) and (37) to calculate h^*.

If we set

$$h_{N+k+l}^* = t^{-N-k-l} e_{N+k+l}^* = \alpha^{-N-k-l}(e_{N+k+l}^*)t^{-N-k-l} \quad \text{and} \quad q = \sum_{r\geq N+1} (-1) h_{r+k+l},$$

then $h^* = \sum\limits_{m\geq 0} h_{N+k+l}^*(qh_{N+k+l}^*)^m$.

We express h_{N+k+l}^* in a more suitable way for our purpose

$$\begin{aligned}
h_{N+k+l}^* &= \alpha^{-N-k-l}(\alpha^{N+k}(b^{-1})\alpha^k(d_N^*)a^{-1})t^{-N-k-l} \\
&= \alpha^{-l}(b^{-1})\alpha^{-N-l}(d_N^*)\alpha^{-N-k-l}(a^{-1})t^{-N-k-l} \\
&= \alpha^{-l}(b^{-1})t^{-l}\alpha^{-N}(d_N^*)t^{-N}\alpha^{-k}(a^{-1})t^{-k} \\
&= \alpha^{-l}(b^{-1})t^{-l} f_N^* \alpha^{-k}(a^{-1})t^{-k} \\
&= (bt^l)^{-1} f_N^* (at^k)^{-1}.
\end{aligned}$$

Now observe that

$$(h_{r+k+l}h_{N+k+l}^*)^{at^k} = d_r t^r f_N^*.$$

So $(qh^*_{N+k+l})^{at^k} = gf^*_N$. Hence,

$$\begin{aligned} h^* &= \sum_{m\geq 0} h^*_{N+k+l}(qh^*_{N+k+l})^m \\ &= (bt^l)^{-1} \sum_{m\geq 0} f^*_N (at^k)^{-1}(qh^*_{N+k+l})^m \\ &= (bt^l)^{-1} \sum_{m\geq 0} f^*_N ((qh^*_{N+k+l})^{at^k})^m (at^k)^{-1} \\ &= (bt^l)^{-1} \Big(\sum_{m\geq 0} f^*_N (gf^*_N)^m \Big)(at^k)^{-1} \\ &= (bt^l)^{-1} f^* (at^k)^{-1}. \end{aligned}$$

We prove that Ω is a morphism of rational $V\langle t\rangle$-semirings. It only remains to prove that $\Omega(f^*) = \Omega(f)^*$ for all $f \in \mathrm{Rat}(V)((t;\alpha)) \cup \{\infty\}$.

Let f be as in (35) or $f = \infty$.

If $\Omega(f)^* = \infty$, then, since $f^* = \infty$, $\Omega(f^*) = \Omega(f)^*$.

Suppose that $f \in \mathrm{Rat}(V)((t;\alpha))$, $\Omega(f) \in R((t;\alpha))^{\curlywedge}$. Let f_N and g be as in (36). Notice that if $h = \sum_{n\geq N+1} -\Phi'(d_n)t^n$,

$$\begin{aligned} \Omega(f)^* &= \Omega(f)^{-1} = \sum_{m\geq 0} (\Phi'(d_N)t^N)^{-1}(h(\Phi'(d_N)t^N)^{-1})^m \\ &= \sum_{m\geq 0} \Omega(f^*_N)(\Omega(g)\Omega(f^*_N))^m. \end{aligned}$$

Then because Ω is a morphism of $V\langle t\rangle$-semirings, then $\Omega(g) = h$ and

$$\begin{aligned} \Omega(f^*) &= \Omega\Big(\sum_{m\geq 0} f^*_N(gf^*_N)^m \Big) \\ &= \Omega\Big(\sum_{m=0}^{r} f^*_N(gf^*_N)^m \Big) + \Omega\Big(\sum_{m\geq r+1} f^*_N(gf^*_N)^m \Big) \text{ for every } r \in \mathbb{N}, \end{aligned}$$

and $\Omega\Big(\sum_{m=0}^{r} f^*_N(gf^*_N)^m \Big) = \sum_{m=0}^{r} \Omega(f^*_N)(\Omega(g)\Omega(f^*_N))^m$.

This shows that the least element of the support of $\Omega(f^*) - \Omega(f)^*$ is larger than $-N + k$ for every $k \in \mathbb{N}$. Therefore $\Omega(f^*) = \Omega(f)^*$.

STEP 4. Φ factors through Ω.

Because $\mathrm{Rat}(V)((t;\alpha)) \cup \{\infty\}$ is a rational $V\langle t\rangle$-semiring, we get a unique morphism $\Psi\colon \mathrm{Rat}(V\langle t\rangle) \longrightarrow \mathrm{Rat}(V)((t;\alpha)) \cup \{\infty\}$ of rational $V\langle t\rangle$-semirings by Lemma 5.34.

Since Φ and $\Omega\Psi$ are morphisms of $V\langle t\rangle$-semirings which coincide on $V\langle t\rangle$, Lemma 5.34 implies that $\Phi = \Omega\Psi$. So we get the following commutative diagram:

$$\begin{tikzcd} \mathrm{Rat}(V\langle t\rangle) \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & R((t;\alpha)) \cup \{\infty\} \\ & \mathrm{Rat}(V)((t;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega"'] & \end{tikzcd}$$

We view Φ, Ψ and Ω as acting as the identity on $V\langle t\rangle$.

For each $f \in \mathrm{Rat}(V\langle t\rangle)$ such that $\Psi(f) \neq \infty$, we abuse notation and write $\Psi(f) = \sum\limits_{n\in\mathbb{Z}} f_n$, where we understand that $f_n \in \mathrm{Rat}(V)t^n \cup \{0\}$.

If $f \in \mathrm{Rat}(V\langle t\rangle)$, then we write $\Psi(f) = f$ if there exist $n \in \mathbb{Z}$ and $d_n \in \mathrm{Rat}(V)$ such that $f = d_n t^n \in \mathrm{Rat}(V)\langle t\rangle$ and $\Psi(f)$ is the Laurent series having one nonzero summand $f_n = d_n t^n$. $\square$

Theorem 5.49. *With notation as in* Step 4, *if* $f \in \mathrm{Rat}(V\langle t\rangle)$ *and*

$$\infty \neq \Psi(f) = \sum_{n\in\mathbb{Z}} f_n \in \mathrm{Rat}(V)((t;\alpha)),$$

then, for each $n \in \mathbb{Z}$, $\mathrm{Tree}(f_n) \leq \mathrm{Tree}(f)$, *and equality holds if and only if*

$$\Psi(f) = f_n = f.$$

Proof. To simplify notation, put $U = V\langle t\rangle$, then $\mathrm{Rat}(U) = \mathbb{N}[U\natural X] \setminus \{0\}$ as in notation of Definitions 5.32.

If $f \in \mathrm{Rat}(U)$ we argue by induction on the complexity of f, $\mathrm{Tree}(f)$.

If f is an element of least complexity, that is, $f \in U$, then $\Psi(f) = f$, as desired.

Suppose that $\mathrm{Tree}(f) > 1_{\mathcal{T}}$ and the result holds for elements of lesser complexity.

As in Lemma 5.40 we partition $\mathbb{N}[U\natural X] \setminus \{0\}$ into the four sets

$$U, \quad X, \quad (U\natural X) \setminus (X \cup U), \quad \mathbb{N}[U\natural X] \setminus (U\natural X \cup \{0\}).$$

We only have to prove the result in the case that f is in one of the last three subsets.

Case 1. Suppose that $f \in \mathbb{N}[U\natural X] \setminus (U\natural X \cup \{0\})$. There exist $g, h \in \mathbb{N}[U\natural X] \setminus \{0\}$ such that $f = g + h$, in fact we could assume $h \in U\natural X$. Since $\Psi(f) \neq \infty$, then $\infty \neq \Psi(g) = \sum\limits_{n\in\mathbb{Z}} g_n$ and $\infty \neq \Psi(h) = \sum\limits_{n\in\mathbb{Z}} h_n$. By Lemma 5.40(iii), $\mathrm{Tree}(g) < \mathrm{Tree}(f)$ and $\mathrm{Tree}(h) < \mathrm{Tree}(f)$. Therefore the theorem holds for g and h by the induction hypothesis, .

Since Ψ is a morphism of semirings, $f_n = g_n + h_n$ for every $n \in \mathbb{Z}$. So if $n \in \mathbb{Z}$, Lemma 5.40(iii) implies that

$$\mathrm{Tree}(f_n) = \mathrm{Tree}(g_n) + \mathrm{Tree}(h_n) \leq \mathrm{Tree}(g) + \mathrm{Tree}(h) = \mathrm{Tree}(f).$$

Suppose that $\mathrm{Tree}(f_n) = \mathrm{Tree}(f)$. By the induction hypothesis, $\mathrm{Tree}(g_n) \leq \mathrm{Tree}(g)$ and $\mathrm{Tree}(h_n) \leq \mathrm{Tree}(h)$. Thus $\mathrm{Tree}(g_n) = \mathrm{Tree}(g)$ and $\mathrm{Tree}(h_n) = \mathrm{Tree}(h)$ by Lemma 5.40(iii). Again the induction hypothesis implies that $\Psi(g) = g_n = g$ and $\Psi(h) = h_n = h$. Hence $\Psi(f) = \Psi(g) + \Psi(h) = g + h = g_n + h_n = f_n = f$.

Case 2. Suppose that $f \in (U\natural X) \setminus (X \cup U)$, then $f = g \cdot h$ for some $g, h \in U\natural X \setminus U$, in fact, we could suppose $h \in X$.

Let us make the following observation. For each $x \in X$, $x = q^*$ for some $q \in \mathbb{N}[U\natural X] \setminus \{0\}$, and $\mathrm{Tree}(x) = \mathrm{Tree}(q)^*$. Thus $\mathrm{width}(\mathrm{Tree}(x)) = 1$. Also $\mathrm{width}(\mathrm{Tree}(u)) = \mathrm{width}(1_{\mathcal{T}}) = 1$ for every $u \in U$. Then, since $\mathrm{width}(X \cdot Y) = \mathrm{width}(X) \cdot \mathrm{width}(Y)$ for all $X, Y \in \mathcal{T}$, $\mathrm{width}(\mathrm{Tree}(r)) = 1$ for all $r \in U\natural X$. Hence

$$Z \geq \mathrm{Tree}(y) \text{ if and only if } \log Z \geq \log(\mathrm{Tree}(y)) \text{ for } y \in U\natural X \text{ and } Z \in \mathcal{T}. \tag{38}$$

By Lemma 5.40(iv), $\mathrm{Tree}(g) < \mathrm{Tree}(f)$ and $\mathrm{Tree}(h) < \mathrm{Tree}(f)$. Hence the result holds for g and h. Let $n \in \mathbb{Z}$,

$$\begin{aligned}
\log(\mathrm{Tree}(f_n)) &= \log\Big(\mathrm{Tree}\Big(\sum_{m\in\mathbb{Z}} g_m h_{n-m}\Big)\Big)\\
&= \log\Big(\sum_{m\in\mathbb{Z}} \mathrm{Tree}(g_m)\,\mathrm{Tree}(h_{n-m})\Big)\\
&= \max_{m\in\mathbb{Z}}\{\log \mathrm{Tree}(g_m) + \log \mathrm{Tree}(h_{n-m})\}\\
&\leq \log(\mathrm{Tree}(g)) + \log(\mathrm{Tree}(h)) \qquad (39)\\
&= \log(\mathrm{Tree}(g)\,\mathrm{Tree}(h))\\
&= \log(\mathrm{Tree}(gh))\\
&= \log(\mathrm{Tree}(f)).
\end{aligned}$$

Notice that there is only a finite number of $m \in \mathbb{Z}$ such that $g_m h_{n-m} \neq 0$, the only possible nonzero products are those with $\min\operatorname{supp}\Psi(g) \leq m \leq n - \min\operatorname{supp}\Psi(h)$ (perhaps there does not exist such m). So we can apply Lemma 5.40(v)-(vi).

Suppose that there exists $n \in \mathbb{Z}$ such that $\log(\mathrm{Tree}(f_n)) = \log(\mathrm{Tree}(f))$. By (39) there exists $m_0 \in \mathbb{Z}$ such that

$$\log(\mathrm{Tree}(g_{m_0})) + \log(\mathrm{Tree}(h_{n-m_0})) = \log(\mathrm{Tree}(g)) + \log(\mathrm{Tree}(h)).$$

Then the induction hypothesis and Lemma 5.40(iii) imply that

$$\log(\mathrm{Tree}(g_{m_0})) = \log(\mathrm{Tree}(g)) \quad \text{and} \quad \log(\mathrm{Tree}(h_{n-m_0})) = \log(\mathrm{Tree}(h)).$$

By (38), $\mathrm{Tree}(g_{m_0}) \geq \mathrm{Tree}(g)$ and $\mathrm{Tree}(h_{n-m_0}) \geq \mathrm{Tree}(h)$. Thus, by the induction hypothesis, $\mathrm{Tree}(g_{m_0}) = \mathrm{Tree}(g)$, $\mathrm{Tree}(h_{n-m_0}) = \mathrm{Tree}(h)$ and $\Psi(g) = g_{m_0} = g$, $\Psi(h) = h_{n-m_0} = h$. Hence $\Psi(f) = \Psi(gh) = \Psi(g)\Psi(h) = g_{m_0}h_{n-m_0} = f_n = gh = f$.

In particular, if there exists $n \in \mathbb{Z}$ such that $\mathrm{Tree}(f_n) = \mathrm{Tree}(f)$, then $f_n = f$.

If there does not exist $n \in \mathbb{Z}$ such that $\mathrm{Tree}(f_n) = \mathrm{Tree}(f)$, by the foregoing, $\log(\mathrm{Tree}(f_n)) \neq \log(\mathrm{Tree}(f))$ for all $n \in \mathbb{Z}$. Now (39) implies that $\log(\mathrm{Tree}(f_n)) < \log(\mathrm{Tree}(f))$. Therefore $\mathrm{Tree}(f_n) < \mathrm{Tree}(f)$ for all $n \in \mathbb{Z}$.

Case 3. Suppose that $f \in X$.

First we prove the following. Suppose that $q = \sum\limits_{l\geq 1} q_l = \sum\limits_{l\geq 1} e_l t^l \in \mathrm{Rat}(V)((t;\alpha))$. If $m \geq 2$, then $q^m = \sum\limits_{l\geq m} (q^m)_l$ where

$$(q^m)_l = \sum_{k_{m-1}=1}^{l-(m-1)} \;\sum_{k_{m-2}=1}^{\substack{l-k_{m-1}\\-(m-2)}} \;\sum_{k_{m-3}=1}^{\substack{l-k_{m-1}\\-k_{m-2}\\-(m-3)}} \cdots \sum_{k_1=1}^{\substack{l-k_{m-1}\\-\cdots\\-k_2-1}} q_{k_{m-1}}\cdots q_{k_1} q_{l-\sum\limits_{i=1}^{m-1} k_i}$$

The proof is by induction on m.

If $m = 2$, $(q^2)_l = \sum\limits_{k_1=1}^{l-1} q_{k_1} q_{l-k_1}$.

Suppose that $m > 2$, and the result is valid for $m-1$. Then,

$$(q^m)_l = \sum_{k_{m-1}=1}^{l-(m-1)} q_{k_{m-1}} (q^{m-1})_{l-k_{m-1}} =$$

$$= \sum_{k_{m-1}=1}^{l-(m-1)} \sum_{k_{m-2}=1}^{\substack{l-k_{m-1}\\-(m-2)}} \sum_{k_{m-3}=1}^{\substack{l-k_{m-1}\\-k_{m-2}\\-(m-3)}} \cdots \sum_{k_1=1}^{\substack{l-k_{m-1}\\-\cdots\\-k_2-1}} q_{k_{m-1}} \cdots q_{k_1} q_{l-\sum_{i=1}^{m-1} k_i},$$

as desired.

Because $f \in X$, there exists $g \in \mathbb{N}[U \natural X] \setminus \{0\}$ such that $f = g^*$ and $\Psi(g) \neq \infty$. Suppose that $\Psi(g) = \sum_{n\in\mathbb{Z}} g_n$. Notice that $\log^2(\mathrm{Tree}(f)) = \log^2(\mathrm{Tree}(g^*)) = \mathrm{Tree}(g)$, by Lemma 5.40(ix). Since $\mathrm{h}(\mathrm{Tree}(g)) < \mathrm{h}(\mathrm{Tree}(f))$, Remark 5.14 implies that $\mathrm{Tree}(g) < \mathrm{Tree}(f)$. Thus the result holds for g. Also, $\log^2(\mathrm{Tree}(g_n)) < \mathrm{Tree}(g_n)$ for all $n \in \mathbb{Z}$. Hence,

$$\log^2(\mathrm{Tree}((-1)g_n)) = \log^2(\mathrm{Tree}(g_n)) < \mathrm{Tree}(g_n) \leq \mathrm{Tree}(g) \text{ for all } n \in \mathbb{Z}. \tag{40}$$

Let N be the least integer such that $\Phi'(g_N) \neq 0$, then

$$\Psi(f) = \Psi(g^*) = \sum_{r\in\mathbb{Z}} f_r = \sum_{m\geq 0} g_N^* (hg_N^*)^m \quad \text{where } h = \sum_{n\geq N+1} (-1)g_n,$$

and $\min\operatorname{supp}\Psi(f) = -N$.

If we set $q = hg_N^*$, by the observation at the beginning of this case, $(hg_N^*)^m = \sum_{l\geq m} (q^m)_l$, where

$$(q^1)_l = (-1)g_{l+N} g_N^*,$$

and, if $m \geq 2$,

$$(q^m)_l = \sum_{k_{m-1}} \sum_{k_{m-2}} \sum_{k_{m-3}} \cdots \sum_{k_1} ((-1)g_{k_{m-1}+N} g_N^*) \cdots ((-1)g_{k_1+N} g_N^*)((-1)g_{l+N-\sum_{i=1}^{m-1} k_i} g_N^*)$$

with

$$\begin{array}{rcl} 1 \leq & k_1 & \leq l - k_{m-1} - \cdots - k_2 - 1 \\ \vdots & \vdots & \vdots \\ 1 \leq & k_{m-3} & \leq l - k_{m-1} - k_{m-2} - (m-3) \\ 1 \leq & k_{m-2} & \leq l - k_{m-1} - (m-2) \\ 1 \leq & k_{m-1} & \leq l - (m-1). \end{array}$$

Therefore, $\sum_{m\geq 0} (hg_N^*)^m = 1 + \sum_{l\geq 1} (\sum_{m=1}^{l} (q^m)_l)$, and

$$\Psi(f) = \sum_{m\geq 0} g_N^* (hg_N^*)^m = g_N^* + \sum_{l\geq 1} \sum_{m=1}^{l} g_N^* (q^m)_l.$$

Hence,

$$\log^2(\mathrm{Tree}(f_{-N})) = \log^2(\mathrm{Tree}(g_N^*)) = \mathrm{Tree}(g_N) \leq \mathrm{Tree}(g) = \log^2(\mathrm{Tree}(f)). \tag{41}$$

Let $l \geq 1$, by Lemma 5.40(vii)-(viii),

$$
\begin{aligned}
\log^2(\mathrm{Tree}(f_{l-N})) &= \log^2\Big(\mathrm{Tree}\Big(\sum_{m=1}^{l} g_N^*(q^m)_l\Big)\Big) \\
&= \max\{\log^2(\mathrm{Tree}(g_N^*)(q^m)_l) \mid 1 \leq m \leq l\} \\
&\leq \max\{\log^2(\mathrm{Tree}(g_N^*)), \log^2(\mathrm{Tree}((-1)g_{k+N})) \mid 1 \leq k \leq l\} \\
&= \max\{\mathrm{Tree}(g_N), \log^2(\mathrm{Tree}(g_{k+N})) \mid 1 \leq k \leq l\} \qquad (42) \\
&\overset{(40)}{\leq} \mathrm{Tree}(g) = \log^2(\mathrm{Tree}(f)).
\end{aligned}
$$

There are two cases for $l \geq 0$:

(a) $\log^2(\mathrm{Tree}(f_{l-N})) < \log^2(\mathrm{Tree}(f))$

(b) $\log^2(\mathrm{Tree}(f_{l-N})) = \log^2(\mathrm{Tree}(f))$

In (a) $\log(\mathrm{Tree}(f_{l-N})) < \log(\mathrm{Tree}(f))$. So $\mathrm{Tree}(f_{l-N}) < \mathrm{Tree}(f)$.

In (b) by (40), (41) and (42), $\mathrm{Tree}(g_N) = \mathrm{Tree}(g)$. By the induction hypothesis, $\Psi(g) = g_N = g$. Then $\Psi(f) = g_N^* = f_{-N} = f$.

We can now prove Case 3.

Notice that $\mathrm{Tree}(f_n) < \mathrm{Tree}(f)$ if $n < -N$, because $f_n = 0$ for $n < -N$.

If there exists $n \in \mathbb{Z}$ such that $\mathrm{Tree}(f_n) = \mathrm{Tree}(f)$, then

$$\log^2(\mathrm{Tree}(f_n)) = \log^2(\mathrm{Tree}(f)).$$

So we are in case (b). Hence $f = f_{-N}$.

Suppose that, for all $n \in \mathbb{Z}$, $\mathrm{Tree}(f_n) \neq \mathrm{Tree}(f)$. Then we are in case (a) for all $n \geq -N$. Therefore $\mathrm{Tree}(f_n) < \mathrm{Tree}(f)$ for all $n \in \mathbb{Z}$. □

Lemma 5.50. *If $f \in \mathrm{Rat}(V\langle t\rangle)$ and*

$$\infty \neq \Psi(f) = \sum_{n\in\mathbb{Z}} f_n \in \mathrm{Rat}(V)((t;\alpha)),$$

then the following hold

(i) *If W is a subgroup of $V\langle t\rangle$ such that $f \in \mathrm{Rat}(W) \cup \{0\}$ and $-1, t \in W$, then $f_n \in \mathrm{Rat}(W)$ for each $n \in \mathbb{Z}$. In particular, if $f_n = e_n t^n$, then $e_n \in \mathrm{Rat}(W) \cup \{0\}$ for each n.*

(ii) *If $f \in \mathrm{Rat}(V\langle t^r\rangle)$ for some $r \in \mathbb{Z}$, then $\operatorname{supp}\psi(f) \subseteq r\mathbb{Z}$; i.e. $\psi(f) \in \mathrm{Rat}(V)((t^r;\alpha))$.*

Proof. First we introduce some notation that will be useful since both statements are proved in the same way by induction on the complexity of f.

$\mathrm{Rat}(W)$ is the disjoint union of the subsets

$$Z_1 = W, \quad Z_2 = X_W, \quad Z_3 = W\natural X_W \setminus (W \cup X_W), \quad Z_4 = \mathbb{N}[W\natural X_W] \setminus (W\natural X_W \cup \{0\})$$

If $U_r = V\langle t^r\rangle$, then $\mathrm{Rat}(U_r)$ is the disjoint union of the subsets

$$Z_1 = U_r, \quad Z_2 = X_r, \quad Z_3 = U_r\natural X_r \setminus (U_r \cup X_r), \quad Z_4 = \mathbb{N}[U_r\natural X_r] \setminus (U_r\natural X_r \cup \{0\})$$

Case 1. If $f \in Z_1$, $\psi(f) = f$. Thus (i) and (ii) are satisfied.

Case 2. If $f \in Z_2$, then $f = g^*$ for some $g \in Z_4 \cup Z_3 \cup Z_1$. Notice that $\psi(g) \neq \infty$ and $\mathrm{Tree}(g) < \mathrm{Tree}(f)$. Thus the implications hold for g, that is, if $\psi(g) = \sum_{n\in\mathbb{Z}} g_n$, then g_n satisfies the result for each $n \in \mathbb{Z}$. Let N be the least integer such that $\Phi'(g_N) \neq 0$. Then

$$\psi(f) = \sum_{n\in\mathbb{Z}} f_n = \sum_{m\geq 0} g_N^*(hg_N^*)^m$$

where $h = \sum_{n \geq N+1} (-1)g_n$. Note that the result holds for g_N^* because of the way g_N^* is constructed. Now f_n is built up from $\{g_N^*,\ (-1)g_m \mid m = N+1, \ldots, 2N+n\}$ using multiplication and addition. Hence f_n satisfies the result for all $n \in \mathbb{Z}$.

Case 3. If $f \in Z_3$, then $f = g \cdot h$ for some $g, h \in Z_3 \setminus Z_1$. Hence $\psi(g) \neq \infty \neq \psi(h)$, $\mathrm{Tree}(g) < \mathrm{Tree}(f)$ and $\mathrm{Tree}(h) < \mathrm{Tree}(f)$. If $\psi(g) = \sum_{n \in \mathbb{Z}} g_n$ and $\psi(h) = \sum_{n \in \mathbb{Z}} h_n$, then

$$\psi(f) = \psi(gh) = \psi(g)\psi(h).$$

Thus $f_n = \sum_{m \in \mathbb{Z}} g_m h_{n-m}$ for each $n \in \mathbb{Z}$, where only a finite number of $g_m,\ h_l$ are nonzero. Since the result holds for all such $g_m,\ h_l$, it is also verified by any finite product and sum. Therefore the result holds for for each $f_n,\ n \in \mathbb{Z}$.

Case 4. If $f \in Z_4$, then either $f = g+h$ with $g, h \in \mathrm{Rat}(W)$ in case (i), or $g, h \in \mathrm{Rat}(V\langle t^r \rangle)$ in case (ii). In any case, $\mathrm{Tree}(g) < \mathrm{Tree}(f)$, $\mathrm{Tree}(h) < \mathrm{Tree}(f)$ and $\psi(g) \neq \infty \neq \psi(h)$. Thus, if we suppose that $\psi(g) = \sum_{n \in \mathbb{Z}} g_n$ and $\psi(h) = \sum_{n \in \mathbb{Z}} h_n$, then $g_n,\ h_n$ satisfy the result for all $n \in \mathbb{Z}$. For $\psi(f) = \psi(g+h) = \psi(g) + \psi(h)$, we get that $f_n = g_n + h_n$ for each $n \in \mathbb{Z}$. Hence the result holds for f_n for each $n \in \mathbb{Z}$. □

"Hey Nikki you know everything
That there is to do
Here's a gun take it home
Wait by the phone
We'll send someone over
To bring you what you need
You're a one man death machine
Make this city bleed"

Queensrÿche, *Operation Mindcrime*

CHAPTER 6

Proofs and Consequences

The objective of this chapter is to prove Hughes' Theorems I and II and some of its consequences. They first appeared in the papers by I. Hughes [**Hug70**] and [**Hug72**] respectively. The proof of Hughes' Theorem I we provide is a slight variation on the one given in [**DHS04**].

1. Hughes' Theorem I

Before proving Hughes' Theorem I 6.3, let us give a quick summary of what we have proved until now. It will be useful in the proof of Theorem 6.2 and Hughes' Theorem II 6.10.

Let k be a division ring. Let G be a locally indicable group. Let kG be a crossed product group ring. Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. Suppose that D_1 and D_2 are $\mathcal{A}$-Hughes-free division rings of fractions of kG. Then $D_i \cup \{\infty_i\}$ is a rational $k^\times G$-semiring, and we have unique morphisms of rational $k^\times G$-semirings $\Phi_i\colon \operatorname{Rat}(k^\times G) \to D_i \cup \{\infty_i\}$ for $i = 1, 2$ by Lemma 5.34.

Let H be a fixed nontrivial finitely generated subgroup of G. Then H equals an internal semidirect product $N_H \rtimes C$ with C infinite cyclic and $N_H \in \mathcal{A}$. Consider the morphism of groups $\rho_H\colon k^\times H \to k^\times H/k^\times \cong H$ from (24) in page 85. Let $t \in k^\times C$ be such that $\rho_H(t)$ generates C. We have already seen in Remarks 5.4(a) that left conjugation by t induces an automorphism α_i of $D_i(kN_H)$ for $i = 1, 2$. Moreover $D_i(kH)$ is isomorphic to $D_i(kN_H)(t; \alpha_i)$ for $i = 1, 2$ by Remarks 5.4(b).

As in Section 6 of Chapter 5, call $V = k^\times N_H$, a subgroup of $D_i(kN_H)^\times$ such that $-1 \in V$ and $\alpha_i(V) = V$. Let $V\langle t\rangle$ denote the subset of $D_i(kN_H)((t; \alpha_i))$ consisting of the polynomials whose support contains exactly one element and its nonzero coefficient is in V. Notice that $V\langle t\rangle = k^\times H$ is a subgroup of $k^\times G$. Then the restriction of Φ_i to $\operatorname{Rat}(V\langle t\rangle)$ can be seen as the unique morphism of rational $V\langle t\rangle$-semirings $\Phi_i\colon \operatorname{Rat}(V\langle t\rangle) \to D_i(kN_H)((t; \alpha_i)) \cup \{\infty\}$ for $i = 1, 2$ by Example 5.36. Thus, as in Step 4 of Section 6 of Chapter 5, we get the following commutative diagram of morphisms of rational $V\langle t\rangle$-semirings

$$\begin{tikzcd}
\operatorname{Rat}(V\langle t\rangle) \arrow[rr, "\Phi_i"] \arrow[dr, "\Psi_i"'] & & D_i(kN_H)((t;\alpha_i)) \cup \{\infty_i\}, \\
& \operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega_i"'] &
\end{tikzcd}$$

and the morphism of additive monoids $\Phi_i'\colon \operatorname{Rat}(V\langle t\rangle) \cup \{0\} \to D_i(kN_H)((t; \alpha_i)) \cup \{\infty_i\}$ for $i = 1, 2$.

Observe that the structure of rational $V\langle t\rangle$-semirings of $\operatorname{Rat}(V)((t; \alpha)) \cup \{\infty\}$ could be different for each i because the $*$-map of $\operatorname{Rat}(V)((t; \alpha)) \cup \{\infty\}$ depends on Φ_i' for each i. If $w = \sum_{n\in\mathbb{Z}} q_n \in \operatorname{Rat}(V)((t; \alpha))$, Φ_i' is the key to construct w^*. Since $D_i(kN_H)$ is a division ring, $\Omega_i(w)^* = \infty_i$ if and only if either $\Phi_i'(q_n) = \infty_i$ for some $n \in \mathbb{Z}$ or $\Phi_i'(q_n) = 0$ for all $n \in \mathbb{Z}$. And $\Omega_i(w)^* \neq \infty_i$ if and only if $\Phi_i'(q_n) \neq \infty$ for all $n \in \mathbb{Z}$ and $\Phi_i'(q_n) \neq 0$ for some $n \in \mathbb{Z}$. Therefore

REMARK 6.1. If $\Phi_i'(q_n) = \infty_i$ for some $n \in \mathbb{Z}$, then $w^* = \infty$.
If $\Phi_i'(q_n) = 0_{D_i}$ for all $n \in \mathbb{Z}$, then $w^* = \infty$.
Otherwise, if N is the first integer such that $\Phi_i'(q_n) \neq 0_{D_i}$, and $z = \sum_{n \geq N+1} (-1)q_n$, then $w^* = \sum_{m \geq 0} q_N^*(zq_N^*)^m$ as in (36) and (37). $\square$

THEOREM 6.2. *Let k be a division ring. Let G be a locally indicable group. Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. Consider a crossed product group ring kG. Suppose that kG has $\mathcal{A}$-Hughes-free division rings of fractions D_1 and D_2. Consider the natural structure of rational $k^\times G$-semirings of $D_i \cup \{\infty_i\}$ for $i = 1, 2$. Let $\Phi_i\colon \operatorname{Rat}(k^\times G) \to D_i \cup \{\infty_i\}$ be the unique morphism of rational $k^\times G$-semirings for $i = 1, 2$. Let $f \in \operatorname{Rat}(k^\times G)$. Then*

(i) $\Phi_1(f) = \infty_1$ *if and only if* $\Phi_2(f) = \infty_2$.
(ii) $\Phi_1(f) = 0_{D_1}$ *if and only if* $\Phi_2(f) = 0_{D_2}$.

PROOF. We proceed by induction on the complexity $\operatorname{Tree}(f)$ of f.
If $f \in k^\times G$, i.e. $\operatorname{Tree}(f) = 1_{\mathcal{T}}$, then

$$\Phi_1(f) = f \neq 0_{D_1}, \infty_1 \quad \text{and} \quad \Phi_2(f) = f \neq 0_{D_2}, \infty_2.$$

So the result holds.

Suppose that $\operatorname{Tree}(f) > 1_{\mathcal{T}}$, and the result holds for elements of $\operatorname{Rat}(k^\times G)$ of lesser complexity than $\operatorname{Tree}(f)$.

By Lemma 5.42(v), $f = pu$, where $u \in k^\times G$ and p is a primitive element of $\operatorname{Rat}(k^\times G)$. Since $\Phi_i, i = 1, 2$, are morphisms of rational $k^\times G$-semirings and $\operatorname{Tree}(p) = \operatorname{Tree}(f)$, a proof of the result for p implies that the result holds for f.

Consider $\operatorname{source}(p)$. It is a finitely generated subgroup of $k^\times G$ such that $p \in \operatorname{Rat}(\operatorname{source}(p))$ by Lemma 5.44.

Consider also the morphism of groups, $\rho_G\colon k^\times G \to k^\times G/k^\times \cong G$. Observe that if $H \leq G$, then $\rho_{G|_{k^\times H}} = \rho_H$. Set $H = \rho_G(\operatorname{source}(p))$, a finitely generated subgroup of G.

If H is trivial, then $\operatorname{source}(p) \leq k^\times$. Therefore $\Phi_i\colon \operatorname{Rat}(\operatorname{source}(p)) \longrightarrow D_i$, $i = 1, 2$, both can be seen as the unique morphism of rational $\operatorname{source}(p)$-semirings given by Lemma 5.34, $\Phi\colon \operatorname{Rat}(\operatorname{source}(p)) \longrightarrow k \cup \{\infty\}$. Then the result holds since $\Phi_{1|\operatorname{Rat}(\operatorname{source}(p))} = \Phi_{2|\operatorname{Rat}(\operatorname{source}(p))}$.

Suppose that H is a nontrivial finitely generated subgroup of G. Then $H = N_H \rtimes C$ where C is infinite cyclic and $N_H \in \mathcal{A}$. Let $t \in k^\times C$ be such that $\rho(t)$ generates C.

If we call $V = k^\times N_H$, proceeding as at the beginning of this section, we get the following commutative diagram of rational $V\langle t\rangle$-semirings for $i = 1, 2$:

$$\begin{tikzcd}
\operatorname{Rat}(V\langle t\rangle) \arrow[rr, "\Phi_i"] \arrow[dr, "\Psi_i"'] & & D_i(kN_H)((t;\alpha_i)) \cup \{\infty_i\} \\
& \operatorname{Rat}(V)((t;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega_i"'] &
\end{tikzcd}$$

Claim 1: $\Psi_1(h) = \Psi_2(h)$, for all $h \in \operatorname{Rat}(V\langle t\rangle)$ with $\operatorname{Tree}(h) \leq \operatorname{Tree}(p)$. In particular, $\Psi_1(p) = \Psi_2(p)$.

Obviously, for the elements $h \in V\langle t\rangle$, $\Psi_1(h) = \Psi_2(h) = h$.

Let $h \in \operatorname{Rat}(V\langle t\rangle)$, $\operatorname{Tree}(h) \leq \operatorname{Tree}(p)$. Suppose that the claim is true for elements of lesser complexity than $\operatorname{Tree}(h)$.

If h is a sum or a product of elements of $\operatorname{Rat}(V\langle t\rangle)$, since Ψ_1 and Ψ_2 are morphisms of rational $V\langle t\rangle$-semirings, and each of this elements is of lesser complexity than h, then $\Psi_1(h) = \Psi_2(h)$.

If $h = q^*$ for some $q \in \mathrm{Rat}(V\langle t\rangle)$, then $\mathrm{Tree}(q) < \mathrm{Tree}(h) \leq \mathrm{Tree}(p)$ by Remark 5.14. Hence $\Psi_1(q) = \Psi_2(q) = \sum\limits_{n\in\mathbb{Z}} q_n$. By Theorem 5.49, $\mathrm{Tree}(q_n) \leq \mathrm{Tree}(q) < \mathrm{Tree}(p)$. Thus, by the induction hypothesis,

$$\Phi_1'(q_n) = 0_{D_1} \text{ if and only if } \Phi_2'(q_n) = 0_{D_2},$$
$$\Phi_1'(q_n) = \infty_1 \text{ if and only if } \Phi_2'(q_n) = \infty_2,$$

for each $n \in \mathbb{Z}$. This implies that $\Psi_1(q)^* = \Psi_2(q)^*$ by Remark 6.1, and therefore $\Psi_1(h) = \Psi_2(h)$. Hence Claim 1 is proved.

Claim 2: $p \notin \mathrm{Rat}(V)\langle t\rangle$.

Suppose that $p \in \mathrm{Rat}(V)\langle t\rangle$. In particular, $p \in \mathrm{Rat}(V)(k^\times G)$. Then $\mathrm{source}(p) \leq V$ by Lemma 5.47. Hence $\rho(\mathrm{source}(p)) = N_H \neq H$, a contradiction. So Claim 2 is proved.

We go back to the proof of our result. By Claim 1, $\Psi_1(p) = \Psi_2(p) = \sum\limits_{n\in\mathbb{Z}} p_n$. Because of Claim 2 and Theorem 5.49, $\mathrm{Tree}(p_n) < \mathrm{Tree}(p)$ for all $n \in \mathbb{Z}$.

Now, by the induction hypothesis and because $\Phi_i = \Omega_i\Psi_i$, $i = 1, 2$,

$$\begin{aligned}
\Phi_1(p) = 0_{D_1} \quad & \text{if and only if} \quad \Omega_1(\Psi_1(p)) = 0_{D_1},\\
& \text{if and only if} \quad \Phi_1'(p_n) = 0_{D_1} \text{ for all } n \in \mathbb{Z},\\
& \text{if and only if} \quad \Phi_2'(p_n) = 0_{D_2} \text{ for all } n \in \mathbb{Z},\\
& \text{if and only if} \quad \Omega_2(\Psi_2(p)) = 0_{D_2},\\
& \text{if and only if} \quad \Phi_2(p) = 0_{D_2}.
\end{aligned}$$

Also,

$$\begin{aligned}
\Phi_1(p) = \infty_1 \quad & \text{if and only if} \quad \Omega_1(\Psi_1(p)) = \infty_1,\\
& \text{if and only if} \quad \text{there exists } n_0 \in \mathbb{Z} \text{ such that } \Phi_1'(p_{n_0}) = \infty_1,\\
& \text{if and only if} \quad \text{there exists } n_0 \in \mathbb{Z} \text{ such that } \Phi_2'(p_{n_0}) = \infty_2,\\
& \text{if and only if} \quad \Omega_2(\Psi_2(p)) = \infty_2,\\
& \text{if and only if} \quad \Phi_2(p) = \infty_2. \quad \square
\end{aligned}$$

Finally we can give Hughes' Theorem I.

Hughes' Theorem I 6.3. *Let k be a division ring. Let G be a locally indicable group. Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. Let kG be a crossed product group ring. Suppose that D_1 and D_2 are ($\mathcal{A}$-)Hughes-free division rings of fractions of kG. Then there exists a unique ring isomorphism $\beta\colon D_1 \longrightarrow D_2$ such that β is the identity on kG.*

Proof. By Lemma 4.14, it is enough to define a morphism of rational $k^\times G$-semirings $\beta\colon D_1 \cup \{\infty_1\} \to D_2 \cup \{\infty_2\}$. Then $\beta_{|D_1}\colon D_1 \to D_2$ is the unique kG-ring isomorphism.

Because of Example 5.38, there exist morphisms of $k^\times G$-rational semirings,

$$\Phi_i\colon \mathrm{Rat}(k^\times G) \longrightarrow D_i \cup \{\infty_i\} \quad i = 1, 2,$$

which are onto.

We want β to be a morphism of rational $k^\times G$-semirings. In this event $\beta\Phi_1$ would be a morphism of $k^\times G$-semirings from $\mathrm{Rat}(k^\times G)$ into $D_2 \cup \{\infty_2\}$, and by Lemma 5.34 there exists only one. So it has to coincide with Φ_2. Thus we have to define

$$\beta(\Phi_1(f)) = \Phi_2(f) \text{ for all } f \in \mathrm{Rat}(k^\times G). \tag{43}$$

First we show that β is well-defined as in (43). We must show that for all $f, g \in \mathrm{Rat}(k^\times G)$ with $\Phi_1(f) = \Phi_1(g)$, then $\Phi_2(f) = \Phi_2(g)$.

Let $f, g \in \mathrm{Rat}(k^\times G)$, such that $\Phi_1(f) = \Phi_1(g)$.

If $\Phi_1(f) = \infty_1$, then, by Theorem 6.2, $\Phi_2(f) = \Phi_2(g) = \infty_2$.

If $\Phi_1(f) \neq \infty_1$,

$$\Phi_1(f + (-1)g) = \Phi_1(f) + (-1)\Phi_1(g) = \Phi_1(f) - \Phi_1(g) = 0_{D_1}.$$

By Theorem 6.2, $0_{D_2} = \Phi_2(f + (-1)g)$. Then

$$0_{D_2} = \Phi_2(f + (-1)g) = \Phi_2(f) + (-1)\Phi_2(g) = \Phi_2(f) - \Phi_2(g)$$

because Φ_2 is a morphism of $k^\times G$-semirings. Hence $\Phi_2(f)$ and $\Phi_2(g)$ are different from ∞_2, and

$$\Phi_2(f) = \Phi_2(g),$$

as desired.

As β is well defined as in (43), it only remains to prove that certainly β is a morphism of rational $k^\times G$-semirings. Let $f, g \in \mathrm{Rat}(k^\times G)$, $u, v \in k^\times G$,

$$\begin{aligned}\beta(\Phi_1(f) + \Phi_1(g)) &= \beta(\Phi_1(f+g)) = \Phi_2(f+g) = \Phi_2(f) + \Phi_2(g)\\ &= \beta(\Phi_1(f)) + \beta(\Phi_1(g)),\end{aligned}$$

$$\begin{aligned}\beta(\Phi_1(f)\Phi_1(g)) &= \beta(\Phi_1(fg)) = \Phi_2(fg) = \Phi_2(f)\Phi_2(g)\\ &= \beta(\Phi_1(f))\beta(\Phi_1(g)),\end{aligned}$$

$$\beta(\Phi_1(f)^*) = \beta(\Phi_1(f^*)) = \Phi_2(f^*) = \Phi_2(f)^* = \beta(\Phi_1(f))^*,$$

$$\beta(u\Phi_1(f)v) = \beta(\Phi_1(ufv)) = \Phi_2(ufv) = u\Phi_2(f)v = u\beta(\Phi_1(f))v,$$

$$\beta(1) = \beta(\Phi_1(1)) = \Phi_2(1) = 1. \quad \square$$

Remarks 6.4. (a) Observe that if D_1 is a Hughes-free division ring of fractions, and D_2 is an $\mathcal{A}$-Hughes-free division ring of fractions, then, since D_1 is $\mathcal{A}$-Hughes-free, D_1 and D_2 are kG-isomorphic, and D_2 is also a Hughes-free division ring of fractions. Therefore, if kG is Hughes-free embeddable and we are given a concrete embedding of kG inside a division ring D, it is enough to show that $kG \hookrightarrow D$ is an $\mathcal{A}$-Hughes-free embedding for some atlas $\mathcal{A}$ of G to ensure that $kG \hookrightarrow D$ is a Hughes-free embedding.

(b) Notice that if kG is Hughes-free embeddable, and we are given an embedding of kG inside a division ring D such that $kG \hookrightarrow D$ is not Hughes-free, then (a) implies that there exists a nontrivial finitely generated subgroup H of G such that for each expression of H as an internal semidirect product $H = N \rtimes C$ with C infinite cyclic and $t \in k^\times C$ such that $\rho_H(t)$ generates C, the powers of t are $D(kN)$-linearly dependent.

(c) The original proof of Theorem 6.3 in [**Hug70**] and the one in [**DHS04**] deal with concept of Hughes-freeness instead of $\mathcal{A}$-Hughes-freeness. We do not know any example of an $\mathcal{A}$-Hughes-free division ring of fractions which is not Hughes-free, so both concepts could be the same. On the other hand (b) does not seem to follow directly from the definition of Hughes-freeness. $\square$

The following Corollary was already noted by I. Hughes in [**Hug70**].

Corollary 6.5. *Let k be a division ring. Let G be an orderable group. Consider a crossed product group ring kG. Suppose that $<$ and $<'$ are total orders on G such that $(G, <)$ and $(G, <')$ are ordered groups. Consider the associated Mal'cev-Neumann series rings $k((G, <))$*

and $k((G,<'))$. Then the division rings of fractions D and D' of kG inside $k((G,<))$ and $k((G,<'))$ respectively are kG-isomorphic.

PROOF. By Examples 5.6(d), $kG \hookrightarrow D$ and $kG \hookrightarrow D'$ are Hughes-free embeddings. Then Hughes' Theorem I 6.3 implies that D and D' are kG-isomorphic division rings. $\square$

The following shows that ($\mathcal{A}$-)Hughes-freeness is a local property.

COROLLARY 6.6. *Let k be a division ring. Let G be a locally indicable group. Consider a crossed product group ring kG. The following hold:*

(i) *If kH is Hughes-free embeddable for each finitely generated subgroup H of G, then kG is Hughes-free embeddable.*
(ii) *Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. If kH is $\mathcal{A}_H$-Hughes-free embeddable for each finitely generated subgroup H of G, then kG is $\mathcal{A}$-Hughes-free embeddable.*

Indeed if D_H is an ($\mathcal{A}_H$-)Hughes-free division ring of fractions of kH, then $D = \varinjlim_{H \leq_{f.g.} G} D_H$ is an ($\mathcal{A}$-)Hughes-free division ring of fractions of kG.

PROOF. For each finitely generated subgroup H of G, consider the ($\mathcal{A}_H$-)Hughes-free embedding $kH \hookrightarrow D_H$. If $H \leq L$, then we infer from Hughes' Theorem I 6.3 that there exists the following commutative diagram of morphisms of rings

$$\begin{tikzcd} kH \arrow[r, hook] \arrow[d, hook] & D_H \arrow[d, hook, dashed] \\ kL \arrow[r, hook] & D_L \end{tikzcd}$$

Then, taking limits, we get that $\varinjlim_{H \leq_{\text{f.g.}} G} kH = kG \hookrightarrow D = \varinjlim_{H \leq_{\text{f.g.}} G} D_H$ is an ($\mathcal{A}$-)Hughes-free embedding because for each finitely generated subgroup H of G, $D(kH) = D_H$. $\square$

The next result is a slight variation on [**DHS04**, Corollary 7.3]. It probably was implicit in the proof of [**Hug70**, Corollary].

Let G be a locally indicable group, and let k be a division ring. Consider a crossed product group ring kG. Let $\Gamma\colon kG \longrightarrow kG$ be an injective morphism of rings (respectively an automorphism) such that $\Gamma(k) = k$. Recall from Proposition 4.8 that the only units in kG are the elements of $k^\times G$. So $\Gamma_{|k^\times G}\colon k^\times G \longrightarrow k^\times G$ is a group monomorphism (automorphism) and Γ is of the form $\bar{g} \longmapsto d_g\overline{\gamma(g)}$ with $d_g \in k^\times$ and $\gamma(g) \in G$. Since $\Gamma(k) = k$, then Γ induces the following monomorphism (automorphism) of G:

$$G \cong k^\times G/k^\times \xrightarrow{\ \Gamma\ } k^\times G/k^\times \cong G$$
$$g \longmapsto [\bar{g}] \longmapsto \left[\overline{\gamma(g)}\right] \longmapsto \gamma(g)$$

which we call γ. With this notation

LEMMA 6.7. *Let $\mathcal{A} = \{N_H\}_H$ be an atlas of G. If either*

(i) *kG has a Hughes-free division ring of fractions D, or*
(ii) *kG has an $\mathcal{A}$-Hughes-free division ring of fractions D such that $\Gamma(kG) \hookrightarrow D$ is $\gamma(\mathcal{A})$-Hughes-free,*

then Γ can be uniquely extended to a monomorphism of D. If moreover Γ is an automorphism of kG, then Γ can be extended to an automorphism of D.

Proof. By Hughes' Theorem I 6.3, it is enough to show that $\Gamma\colon kG \longrightarrow kG \subseteq D$ is an $(\mathcal{A}$-)Hughes-free embedding, since we already have that $kG \hookrightarrow D$ is $(\mathcal{A}$-)Hughes-free.

Let H be a non-trivial finitely generated subgroup of G. Suppose that $H = N_H \rtimes \langle t \rangle$. Then $\gamma(H)$ is a finitely generated subgroup of G and $\gamma(H) = \gamma(N_H) \rtimes \langle \gamma(t) \rangle$. Suppose that there are $b_0, \dots, b_n \in D(\gamma(N_H))$ with

$$0 = \sum_{i=0}^{n} b_i \Gamma(t)^i = \sum_{i=0}^{n} b_i d'_{ti} \overline{\gamma(t)}^i \qquad \text{for some } d'_{ti} \in k^\times.$$

Because of our hypothesis on D and the fact that $b_i d'_{ti} \in D(\gamma(N_H))$, then $b_i d'_{ti} = 0$ for $i = 0, \dots, n$, which implies that $b_i = 0$ for $i = 0, \dots, n$. $\square$

Some examples of Γ as in Lemma 6.7 are given by morphisms of k-rings and conjugation by a trivial unit. This allows us to prove the $\mathcal{A}$-Hughes-free generalization of [**Hug70**, Corollary].

Corollary 6.8. *Let k be a division ring. Let G be a locally indicable group and N a normal subgroup of G. Let $\mathcal{R} = \{x_\beta \mid \beta \in G/N\}$ be a complete set of representatives of the classes of G/N such that $1 \in \mathcal{R}$. Let $\mathcal{A} = \{L_H\}_H$ be an atlas of N. Suppose that kN has a division ring of fractions D such that $kN \hookrightarrow D$ is $(x_\beta \mathcal{A} x_\beta^{-1}$-)Hughes-free embeddable (for each $x_\beta \in \mathcal{R}$). Then:*

(i) *kG embeds in a crossed product group ring $D(G/N)$ in a natural way.*
(ii) *If moreover G/N is embeddable, then kG embeds in a division ring.*

Proof. (i) Recall from Lemma 4.7 that $kG = (kN)(G/N)$. We construct a crossed product group ring $D(G/N)$ which extends $(kN)(G/N)$. To this aim we need to define the twisting $\breve{\tau}$ and the action $\breve{\sigma}$, and see that the product is associative. We make reference to the proof of Lemma 4.7.

In order to get an extension of $(kN)(G/N)$, the twisting $\breve{\tau}$ has to be the same as the one given in the proof of Lemma 4.7.

By Remarks 4.3(c), for every $\beta \in G/N$, $\hat{\sigma}(\beta) \in \operatorname{Aut}(kN)$ is left conjugation by $\bar{x}_\beta$. Lemma 6.7 and our hypothesis on D imply that $\hat{\sigma}(\beta)$ can be extended to a unique automorphism $\breve{\sigma}(\beta)$ of D. In this way we define $\breve{\sigma}\colon G/N \to \operatorname{Aut}(D)$, $\beta \mapsto \breve{\sigma}(\beta)$.

We prove that the product in $D(G/N)$ is associative by showing that (a) and (b) in Lemma 4.2 are satisfied.

$D(G/N)$ verifies (a) because $\breve{\tau} = \hat{\tau}$, $\breve{\sigma}(\beta)_{|kN} = \hat{\sigma}(\beta)$ and $\hat{\tau}$ and $\hat{\sigma}$ already satisfy (a) of Lemma 4.2.

$D(G/N)$ satisfies (b) because $\breve{\sigma}(\alpha\beta)\mu(\alpha,\beta)$ and $\breve{\sigma}(\beta)\breve{\sigma}(\alpha)$ both are automorphism of D which agree on kN since $\hat{\sigma}(\alpha\beta)\mu(\alpha,\beta) = \hat{\sigma}(\beta)\hat{\sigma}(\alpha)$ and both are $(\mathcal{A})$-Hughes-free embeddings.

(ii) We have just embedded kG in the crossed product group ring $D(G/N)$. Then, because G/N is embeddable, we can embed $D(G/N)$ in a division ring. $\square$

2. Hughes' Theorem II

The following is a technical result on groups that will be useful in the course of the proof of Hughes' Theorem II 6.10.

LEMMA 6.9. *Let W be a group. Suppose that $X, Y \lhd W$ with X finite cyclic, $Y = \langle y \rangle$ infinite cyclic, W/X infinite cyclic and $W/Y = \langle zY \rangle$ cyclic of order p.*

$$\begin{tikzcd}
& W \arrow[ddl, dash, "\infty"'] \arrow[d, dash] \arrow[ddr, dash, "p"] & \\
& XY \arrow[dl, dash] \arrow[dr, dash] & \\
X \arrow[dr, dash] & & Y \arrow[dl, dash] \\
& 1 &
\end{tikzcd}$$

Suppose that $z^p = y^q$ with $q \in \mathbb{N}$ and $(p, q) = d$. Let $r, m \in \mathbb{Z}$ be such that $p = rd$ and $q = md$. Let $\alpha, \beta \in \mathbb{Z}$ with $r\alpha + m\beta = 1$. Set $u = z^\beta y^\alpha$ and $v = z^r y^{-m}$. Then

(i) $W = \langle y, z \mid [y, z],\ z^p = y^q \rangle$
(ii) $X = \langle v \rangle$ *and has order* d
(iii) $z = u^m v^\alpha$ *and* $y = u^r v^{-\beta}$
(iv) $W/X \cong \langle u \rangle$
(v) W/XY *has order* r *and* XY/Y *has order* d

PROOF. First observe that W is generated by $\{y, z\}$. Secondly, X consists of all the elements of finite order in W and therefore $X \cap Y = 1$. For a proof of the latter, notice that if $w \in W$ is of finite order, then $\bar{w} \in W/X$ has finite order. Thus $\bar{w} = 1$, i.e. $w \in X$.

(i) These remarks imply that $Y \cong XY/X \leq W/X$. Moreover, since W/X is commutative, we get that $\overline{y^z} = \overline{y} \in W/X$, and $y^{-1}y^z \in X$. Thus $yz = zy$.

The foregoing implies that there exists an onto morphism of groups

$$\varphi\colon \langle y, z \mid [y, z], z^p = y^q \rangle \longrightarrow W.$$

In W, any relation is of the form $z^\delta = y^\gamma$ for some $\delta, \gamma \in \mathbb{Z}$, because y and z commute. By the definition of p, $\delta = p\varepsilon$ for some $\varepsilon \in \mathbb{Z}$. Thus $z^\delta = (z^p)^\varepsilon = (y^q)^\varepsilon = y^\gamma$, and $\gamma = q\varepsilon$. Therefore φ is an isomorphism.

(ii) Consider the morphism of groups $\eta\colon W \to C$, where C is infinite cyclic, given by $y \mapsto c^r$ and $z \mapsto c^m$. Clearly $X \subseteq \ker \eta$. An element $n = z^\gamma y^\delta \in \ker \eta$ if and only if $m\gamma + \delta r = 0$, if and only if $n = (z^r y^{-m})^\varepsilon = v^\varepsilon$ for some $\varepsilon \in \mathbb{Z}$ because $(r, m) = 1$. Since v has finite order, $v^d = 1$, we get that $\ker \eta = X = \langle v \rangle$. In the isomorphism $X \cong XY/Y \leq W/Y$, v is sent to the element $\bar{v} = \overline{z^r y^{-m}} = \overline{z^r} \in XY/Y$ which has order d. Hence v has order d.

(iii) It is straightforward.

(iv) From (iii) we infer that u and v generate W. Hence $W/X \cong \langle u \rangle$ because $v \in X$ and $\eta(u) = c$.

(v) For W/Y has order p, $\frac{XY}{Y} \cong X$ and $|W/Y| = |\frac{W}{XY}||\frac{XY}{Y}| = |\frac{W}{XY}|d$, we obtain that $|W/Y| = r$. □

HUGHES' THEOREM II 6.10. *Suppose that G is a locally indicable group with a normal subgroup L such that G/L is locally indicable. If both L and G/L are Hughes-free embeddable, then G is Hughes-free embeddable.*

PROOF. Let k be a division ring and kG a crossed product group ring. Since L is Hughes-free embeddable, kL has a Hughes-free division ring of fractions. Let D be the Hughes-free division ring of fractions of kL. By Corollary 6.8(i), $kG = kL(G/L)$ embeds in $D(G/L)$ in a natural way. Moreover, since G/L is also Hughes-free embeddable, $D(G/L)$ has a Hughes-free division ring of fractions E. Therefore we have the embedding of kG inside the division ring E

$$kG \hookrightarrow D(G/L) \hookrightarrow E.$$

We will show that this embedding is Hughes-free.

Consider the morphisms of groups from (24) in page 85 that will be used throughout the proof

$$k^\times G \xrightarrow{\rho_G} k^\times G/k^\times \cong G, \quad k^\times G = k^\times L(G/L) \subseteq D^\times(G/L) \xrightarrow{\rho_{G/L}} \frac{D^\times(G/L)}{D^\times} \cong G/L,$$

defined by $\rho_G(a\bar{x}) = x$ and $\rho_{G/L}(d\bar{z}) = z$, and observe that if $H \leq G$, then $(\rho_G)_{|H} = \rho_H$.

Let H be a nontrivial finitely generated subgroup of G. Suppose that $H = N \rtimes C$ is an expression of H as an internal semidirect product with C infinite cyclic. Let $t \in k^\times C$ be such that $\rho_H(t)$ generates C. Suppose that $\sum\limits_{i=-n}^{n} d_i t^i = 0$, where $d_i \in E(kN)$ for all i. We have to show that $d_{-n} = \cdots = d_n = 0$.

Endow $E(kH) \cup \{\infty\}$ with the structure of rational $k^\times H$-semiring of Example 1.43(d). Let $\Phi\colon \operatorname{Rat}(k^\times H) \to E(kH) \cup \{\infty\}$ be the morphism of rational $k^\times H$-semirings and $\Phi'\colon \operatorname{Rat}(k^\times H) \cup \{0\} \to E(kH) \cup \{\infty\}$ be the morphism of additive monoids from Lemma 5.34. Let $f \in \operatorname{Rat}(k^\times H)$ defined by

$$f = \sum_{i=-n}^{n} f_i t^i \tag{44}$$

with $f_i \in \operatorname{Rat}(k^\times N) \cup \{0\}$ for each i. By Example 5.38, to prove the result it is enough to show that for each f as in (44) then

$$\Phi(f) = \sum_{i=-n}^{n} \Phi'(f_i) t^i = 0, \text{ implies that } \Phi'(f_i) = 0 \text{ for each } i. \tag{45}$$

We proceed by induction on the complexity $\operatorname{Tree}(f)$ of f. The result is clear if there exists only one $f_i \neq 0$. This implies that the result holds for $\operatorname{Tree}(f) = 1_{\mathcal{T}}$ and $\operatorname{Tree}(f) = 1_{\mathcal{T}} + 1_{\mathcal{T}}$. In the former case, $f = f_{i_0} t^{i_0} \in k^\times H$. In the latter case, $f = h_1 t^{n_1} + h_2 t^{n_2}$ with $h_i \in k^\times N$, and $\Phi(f) = 0$ if and only if $n_1 = n_2$ and $h_1 = -h_2$. Thus $f = (h_1 + h_2) t^{n_1}$.

Suppose that the result holds for elements of lesser complexity than $\operatorname{Tree}(f)$, and that there exists more than one $f_i \neq 0$.

By Lemma 5.42(v), $f = pu$ with p a primitive element of $\operatorname{Rat}(k^\times H)$ and $u \in k^\times H$. Thus $pu = f = \sum\limits_{i=-n}^{n} f_i t^i$ if and only if $p = \sum\limits_{i=-n}^{n} f_i t^i u^{-1}$.

Suppose that $u^{-1} = vt^m$ with $v \in k^\times N$, then $t^i v t^{-i} \in k^\times N$ for each i and

$$p = \sum_{i=-n}^{n} f_i t^i v t^m = \sum_{i=-n}^{n} f_i t^i v t^{-i} t^{i+m}.$$

Again by Lemma 5.42(v), $f_i = p_i w_i$ with p_i a primitive element of $\operatorname{Rat}(k^\times N)$ and $w_i \in k^\times N$ for each i. Hence $p = \sum\limits_{i=-n}^{n} p_i w_i t^i v t^{-i} t^{i+m}$. Put $v_i = w_i t^i v t^{-i}$ for each i. Then, relabeling the indexes if necessary, we can suppose that $p = \sum\limits_{i=-n}^{n} p_i v_i t^i$ with $v_i \in k^\times N$.

By Case 1 of Definition 5.43, source(p) is the finitely generated subgroup of $k^\times H$ generated by $\bigcup\limits_{\{i \mid p_i \neq 0\}} (\operatorname{source}(p_i) \cup \{v_i t^i\})$, and since there are at least two summands and hence one $t^i \neq 1$, $\frac{\rho_G(\operatorname{source}(p))}{\rho_G(\operatorname{source}(p)) \cap N}$ is a nontrivial cyclic subgroup of the infinite cyclic group H/N.

If we choose $z \in \operatorname{source}(p)$ such that the class of $\rho_G(z)$ generates $\frac{\rho_G(\operatorname{source}(p))}{\rho_G(\operatorname{source}(p)) \cap N}$, then $v_i t^i = y_i z^{l_i}$ where $y_i \in \operatorname{source}(p) \cap k^\times N$. Notice that $l_i \neq l_j$ provided $i \neq j$ because $\rho_G(t)$

generates H/N. Moreover, $p_iy_i \in \operatorname{Rat}(k^\times N')$ and $\rho_G(\operatorname{source}(p)) = N' \rtimes \langle \rho_G(z)\rangle$ where $N' = \rho_G(\operatorname{source}(p)) \cap N$.

Hence $p = \sum\limits_{i=-n}^{n} p_iy_iz^{l_i}$, and $\operatorname{Tree}(p) = \operatorname{Tree}(f)$ by Lemma 5.40(ii). Moreover, $\Phi(p) = 0$ if and only if $\Phi(f) = 0$, and $\Phi'(f_i) = 0$ if and only if $\Phi'(p_iy_i) = 0$.

By the foregoing, we may assume that $H = \operatorname{source}(p)$, $H = N \rtimes C$ with C infinite cyclic, $t \in k^\times C$ with $\rho_H(t)$ that generates C, and $p = \sum\limits_{i=-n}^{n} f_it^i$ where $f_i \in \operatorname{Rat}(k^\times N) \cup \{0\}$.

Note that $\frac{LH}{LN}$ is cyclic generated by the class of $\rho_G(t)$. If H is not contained in L, then, since G/L is locally indicable, $\frac{LH}{L}$ is indicable and $\frac{LH}{L} = \frac{B}{L} \rtimes \frac{K}{L}$ where $\frac{K}{L}$ is infinite cyclic for some subgroups B and K of G. The proof is divided in four cases, one for each of the following possibilities

(1) $H \leq L$,
(2) $H \nsubseteq L$ and $\frac{LH}{LN}$ is infinite cyclic,
(3) $H \nsubseteq L$, $\frac{LH}{LN}$ is finite cyclic and $LH \neq BN$,
(4) $H \nsubseteq L$, $\frac{LH}{LN}$ is finite cyclic and $LH = BN$.

Case 1. H is a subgroup of L. Then kH is contained in kL and D, the Hughes-free division ring of fractions of kL. Therefore, if $\Phi(f) = 0$, then $\Phi'(f_{-n}) = \cdots = \Phi'(f_n) = 0$.

Case 2. $H \nsubseteq L$ and $\frac{LH}{LN}$ is infinite cyclic. If we set $C_L = \langle \rho_G(t)L\rangle = \langle \rho_{G/L}(t)\rangle \leq G/L$, then $\frac{LH}{L} = \frac{LN}{L} \rtimes C_L$. Notice that C_L is infinite cyclic, $t \in D^\times C_L$ and that $\Phi'(f_i) \in E(D\frac{LN}{L})$, the division ring of fractions of $D\frac{LN}{L}$ inside E. Therefore $\Phi'(f_{-n}) = \cdots = \Phi'(f_n) = 0$ because $D\frac{G}{L} \hookrightarrow E$ is Hughes-free.

Case 3. $H \nsubseteq L$, $\frac{LH}{LN}$ is finite cyclic and $LH \neq BN$. Suppose that the group $\frac{LH}{LN}$ generated by the class of $\rho_G(t)$ is finite with p elements. Set $A = B \cap LN$, $W = \frac{LH}{A}$ and $Y = \frac{LN}{A}$. Thus $\frac{\frac{LH}{A}}{\frac{LN}{A}} = W/Y$ has p elements and is generated by the class of $\rho_G(t)$. Now

$$Y = \frac{LN}{A} \cong \frac{LNB}{B} = \frac{NB}{B} \leq \frac{LH}{B} \cong K/L.$$

Hence Y is a subgroup of an infinite cyclic group. Notice that Y is not trivial. Otherwise $A = LN$ and $LN \leq B$. Then $\frac{LH}{LN} \to \frac{LH}{B}$ would be an onto morphism of groups and $\frac{LH}{LN}$ an infinite cyclic group, a contradiction. Therefore Y is an infinite cyclic group generated by the class of some $h \in N$. Set $X = B/A \leq \frac{LH}{A} = W$. Since $X \cong \frac{BLN}{LN} \leq \frac{LH}{LN}$, X is finite cyclic. Then $XY = \frac{B}{A}\frac{LN}{A} = \frac{BN}{A}$. So we have the following diagram as in Lemma 6.9

$$\begin{tikzcd}
 & W = \frac{LH}{A} \arrow[d, dash, "r"] \arrow[ddl, dash, "\infty"'] \arrow[ddr, dash, "p"] & \\
 & XY = \frac{BN}{A} \arrow[dl, dash] \arrow[dr, dash, "d"] & \\
X = B/A \arrow[dr, dash, "d"'] & & Y = \frac{LN}{A} \arrow[dl, dash, "\infty"] \\
 & 1 &
\end{tikzcd}$$

Suppose that $\rho_G(t)^p \equiv h^q \mod A$. Put $(p,q) = d$ and $p = rd,\ q = md$. Let α, β be such that $\alpha r + \beta m = 1$. Set $u' = \rho_G(t)^\beta h^\alpha,\ v = \rho_G(t)^r h^{-\alpha}$. Then, by Lemma 6.9,

a) $\rho_G(t) \equiv {u'}^m v^\alpha \mod A$,
b) $h \equiv {u'}^r v^{-\beta} \mod A$,

c) X is generated by the class of v and $v^d \equiv 1 \mod A$,
d) B equals the subgroup generated by $\{v, A\}$,
e) $\frac{W}{X} \cong \frac{LH}{B}$ is infinite cyclic generated by the class of u',
f) $\frac{W}{XY} \cong \frac{LH}{BN}$ has order $r > 1$ (otherwise $LH = BN$).

Then $\frac{LH}{L} = \frac{B}{L} \rtimes \langle u'L\rangle$ by e). Let $u \in k^\times H \subseteq D^\times \frac{LH}{L}$ be such that $\rho_{\frac{LH}{L}}(u) = u'L$.

For each $i \in \{-n, \dots, n\}$ we have that $i = l_i r + j$ with $0 \leq j < r$. Then

$$p = \sum_{i=-n}^{n} f_i t^i = \sum_{j=0}^{r-1} \left(\sum_{\{i|i=l_i r+j\}} f_i t^{l_i r} \right) t^j = \sum_{j=0}^{r-1} g_j t^j,$$

where we set $g_j = \sum_{\{i|i=l_i r+j\}} f_i t^{l_i r}$.

If there exists j such that $p = g_j t^j$ (i.e. all i such that f_i is nonzero belong to the set $\{i \mid i = l_i r + j\}$), then $pt^{-j} = g_j$. Thus $\operatorname{source}(g_j) = \operatorname{source}(p)$. On the other hand, $g_j \in \operatorname{Rat}(k^\times N\langle t^r\rangle)$. Therefore, by Lemma 5.47, $\operatorname{source}(p) \subseteq k^\times N\langle t^r\rangle$ and

$$H = \rho_G(\operatorname{source}(p)) = N \rtimes \langle \rho_G(t^r)\rangle,$$

a contradiction. Thus $\operatorname{Tree}(g_j) < \operatorname{Tree}(p)$ for each j by Corollary 5.40(iii).

Notice that $\rho_G(t^{l_i r}) \in BN$ because $\frac{LH}{BN}$ has order r. Hence $g_j \in \operatorname{Rat}(k^\times BN)$. Moreover, since $XY = \frac{BN}{A}$ and Y is generated by the class of $h \in N$, we get that BN is generated by $B \cup \{h\}$. Since $h = u'^r v^{-\beta}$ and $v \in B$, then BN is generated by $B \cup \{u'^r\}$. Therefore $g_j \in \operatorname{Rat}(k^\times BN) = \operatorname{Rat}(k^\times B\langle u^r\rangle) \subseteq \operatorname{Rat}(D^\times \frac{B}{L}\langle u^r\rangle)$ for each $j \in \{0, dotsc, r-1\}$.

Now we proceed as at the beginning of Section 1. Consider $E(D\frac{LH}{L})$, the division ring of fractions of $D\frac{LH}{L}$ inside E. Consider the skew Laurent series ring $E(D\frac{B}{L})((u;\alpha))$ where α is given by left conjugation by u. We can suppose that $E(D\frac{LH}{L})$ is contained in $E(D\frac{B}{L})((u;\alpha))$. Then $D^\times \frac{LH}{L} = (D^\times \frac{B}{L})\langle u\rangle$ can be identified with the series with exactly one element in its support whose nonzero coefficient is in $D^\times \frac{B}{L}$. We can construct the following commutative diagram of morphisms of rational $D^\times \frac{LH}{L}$-semirings

$$\begin{tikzcd}
\operatorname{Rat}(D^\times \frac{B}{L}\langle u\rangle) \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & E(D\frac{B}{L})((u;\alpha)) \cup \{\infty\} \\
& \operatorname{Rat}(D^\times \frac{B}{L})((u;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega"'] &
\end{tikzcd}$$

Notice that $kH \hookrightarrow D\frac{LH}{L} \hookrightarrow E(D\frac{LH}{L})$ and $k^\times H \hookrightarrow (k^\times L)\frac{LH}{L} \hookrightarrow D^\times \frac{LH}{L}$. Therefore $E(kH) \hookrightarrow E(D\frac{LH}{L})$ and $\operatorname{Rat}(k^\times H) \hookrightarrow \operatorname{Rat}((D^\times \frac{B}{L})\langle u\rangle)$. So we can consider the commutative diagram of rational $k^\times H$-semirings

$$\begin{tikzcd}
& \operatorname{Rat}(k^\times H) \arrow[dl, hook'] \arrow[dr, "\Phi"] & \\
\operatorname{Rat}((D^\times \frac{B}{L})\langle u\rangle) \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & E(D\frac{B}{L})((u;\alpha)) \cup \{\infty\} \\
& \operatorname{Rat}(D^\times \frac{B}{L})((u;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega"'] &
\end{tikzcd}$$

By Lemma 5.50(ii), $\Psi(g_j) = h_j$ is a series in u^r. By a), we may suppose that $t^j = b_j u^{mj}$, where $b_j \in k^\times B$ for each j. Then $\Psi(f) = \sum_{j=0}^{r-1} h_j b_j u^{mj}$. Recall that $\Phi(f) = 0$ implies $\Omega(\Psi(f)) = 0$. Now since r and m are coprime and $0 \leq j \leq r-1$, then $\Omega(h_j) = 0$ for each j. Therefore $\Phi(g_j) = \Omega(\Psi(g_j)) = \Omega(h_j) = 0$ for each j. Now, since $\mathrm{Tree}(g_j) < \mathrm{Tree}(p)$, the induction hypothesis implies that $\Phi(f_i) = 0$ for each $i \in \{-n, \dots, n\}$.

Case 4. $H \not\subseteq L$, $\frac{LH}{LN}$ is finite cyclic and $LH = NB$.

We can suppose that $\rho_G(t) \in B$. Otherwise, if $\rho_G(t) = h'b$ with $h' \in N$, we choose $h \in k^\times N$ such that $\rho_G(ht) = b$ and we work with $t' = ht$ instead of t. Notice that $H = N \rtimes \langle \rho_G(t') \rangle$ and Lemma 5.3(b).

Recall that $\frac{LH}{L} = \frac{B}{L} \rtimes \frac{K}{L}$ for some subgroups B and K of G such that $\frac{K}{L}$ is infinite cyclic. Hence N is not contained in B. Let $u \in k^\times N \leq D^\times \frac{LH}{L} = D^\times \frac{NB}{L}$ be such that $\rho_{G/L}(u)$ generates K/L.

Consider $E(D\frac{LH}{L})$, the division ring of fractions of $D\frac{LH}{L}$ inside E. Consider the skew Laurent series ring $E(D\frac{B}{L})((u;\alpha))$, where α is defined by left conjugation by the element u. Then $D^\times \frac{LH}{L} = D^\times \frac{B}{L}\langle u \rangle$ can be identified with the series with exactly one element in its support and whose nonzero coefficient is in $D^\times \frac{B}{L}$. And as we made at the beginning of Section 1, we can construct the following diagram of morphisms of rational $D^\times \frac{LH}{L}$-semirings

$$\begin{tikzcd}
\mathrm{Rat}(D^\times \frac{B}{L}\langle u \rangle) \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & E(D\frac{B}{L})((u;\alpha)) \cup \{\infty\} \\
& \mathrm{Rat}(D^\times \frac{B}{L})((u;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega"'] &
\end{tikzcd}$$

Note that $kH \hookrightarrow D(\frac{LH}{L})$ and $k^\times H \hookrightarrow D^\times \frac{B}{L}\langle u \rangle$. Thus $E(kH) \hookrightarrow E(D\frac{LH}{L}) \hookrightarrow E(D\frac{B}{L})((u;\alpha))$ and $\mathrm{Rat}(k^\times H) \hookrightarrow \mathrm{Rat}(D^\times \frac{B}{L}\langle u \rangle)$. So we get the following commutative diagram of morphisms of rational $k^\times H$-semirings

$$\begin{tikzcd}
& \mathrm{Rat}(k^\times H) \arrow[dl, hook'] \arrow[dr, "\Phi"] & \\
\mathrm{Rat}((D^\times \frac{B}{L})\langle u \rangle) \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & E(D\frac{B}{L})((u;\alpha)) \cup \{\infty\} \\
& \mathrm{Rat}(D^\times \frac{B}{L})((u;\alpha)) \cup \{\infty\} \arrow[ur, "\Omega"'] &
\end{tikzcd}$$

If $p \in \mathrm{Rat}(D^\times \frac{B}{L})\langle u \rangle$, since $u \in k^\times N$, Lemma 5.47 implies that $\mathrm{source}(p) \leq D^\times \frac{B}{L}$. Observe that the only elements of $k^\times H$ inside $D^\times \frac{B}{L}$ are contained in $k^\times B$. Hence

$$\mathrm{source}(p) \leq (k^\times H) \cap D^\times \frac{B}{L} \leq k^\times L \frac{B}{L} = k^\times B.$$

Then $\mathrm{source}(p) = H \leq B$ and $\frac{LH}{L} \leq B/L$, a contradiction. Therefore, by Theorem 5.49, $\Psi(p) = \sum_{m \in \mathbb{Z}} g_m$ is a series with $\mathrm{Tree}(g_m) < \mathrm{Tree}(p)$, where $g_m = e_m u^m$.

Now we compute $\Psi(p)$ in another way. Suppose that $\Psi(f_i) = \sum_{m\in\mathbb{Z}} g_{im}$ if $f_i \neq 0$. Since $f_i \in \mathrm{Rat}(k^\times N)$ and $-1, u \in k^\times N$, Lemma 5.50(i) implies that $g_{im} \in \mathrm{Rat}(k^\times N)$ where $g_{im} = e_{im}u^m$.

Then

$$\begin{aligned}\Psi(p) &= \sum_{i=-n}^{n}\Big(\sum_{m\in\mathbb{Z}} g_{im}\Big)t^i\\ &= \sum_{i=-n}^{n}\Big(\sum_{m\in\mathbb{Z}} e_{im}u^m\Big)t^i\\ &= \sum_{i=-n}^{n}\Big(\sum_{m\in\mathbb{Z}} e_{im}u^m t^i u^{-m}\Big)u^m\\ &= \sum_{m\in\mathbb{Z}}\Big(\sum_{i=-n}^{n} e_{im}u^m t^i u^{-m}\Big)u^m.\end{aligned}$$

Notice that $\sum\limits_{i=-n}^{n} e_{im}u^m t^i u^{-m} \in \mathrm{Rat}(D^\times \frac{B}{L})$ for each m because $t \in k^\times B$. Hence

$$g_m = \sum_{i=-n}^{n} g_{im}t^i u^{-m} = \sum_{i=-n}^{n} g_{im}t^i u^{-m}t^{-i}t^i = \sum_{i=-n}^{n} g_{im}w_{im}t^i,$$

where $w_{im} = t^i u^{-m} t^{-i} \in k^\times N$ because $u \in k^\times N$. Then $g_{im}w_{im} \in \mathrm{Rat}(k^\times N)$.

Note that $\Phi(p) = 0$ if and only if $\Omega(\Psi(f)) = 0$, if and only if $\sum_{m\in\mathbb{Z}} \Phi'(g_m) = 0$, if and only if $\Phi'(g_m) = 0$ for each m.

Now, for each $m \in \mathbb{Z}$, $\Phi'(g_m) = 0$ implies that $\Phi'\Big(\sum\limits_{i=-n}^{n} g_{im}w_{im}t^i\Big) = 0$. Since $\mathrm{Tree}(g_m) < \mathrm{Tree}(p)$, then $\Phi'(g_{im}w_{im}) = 0$ for each i and m by the induction hypothesis. This implies that $\Phi'(g_{im}) = 0$ for each m and i because $w_{im} \in k^\times N$. Therefore

$$\Phi'(f_i) = \Omega\Psi(f_i) = \Omega\Big(\sum_{m\in\mathbb{Z}} g_{im}\Big) = \sum_{m\in\mathbb{Z}} \Phi'(g_{im}) = 0. \quad \square$$

Remark 6.11. Let G and L be as in Hughes' Theorem II 6.10. Let k be a division ring and kG a crossed product group ring. In the proof of Theorem 6.10 we see how the Hughes-free division ring of fractions E of kG is. If D is the Hughes-free division ring of fractions of kL, then E is the Hughes-free division ring of fractions of $D\frac{G}{L}$. $\square$

We now present some consequences of Hughes' Theorem II. They are closure properties for Hughes-free embeddability in the same vein as the results of Section 1 of Chapter 2 are closure properties for local indicability.

The following consequence of Hughes' Theorem II 6.10 was already noted in [**Hug72**] for poly-orderable groups.

Corollary 6.12. *Suppose that $(G_\gamma)_{\gamma\leq\tau}$ is a subnormal series of a group G with Hughes-free embeddable factors. Then G is Hughes-free embeddable. In particular, if $(G_\gamma)_{\gamma\leq\tau}$ is a subnormal series with orderable factors, then G is Hughes-free embeddable.*

Proof. We show that for every ordinal $\gamma \leq \tau$, G_γ is Hughes-free embeddable.

If $\gamma = 1$, G_γ is Hughes-free embeddable by hypothesis.

Let $\gamma > 1$. Suppose that the result holds for all ordinals $\rho < \gamma$. If $\gamma = \rho + 1$ for some ordinal ρ, then G_γ is the extension of the Hughes-free embeddable groups G_ρ and G_γ/G_ρ. By Hughes' Theorem II 6.10 G_γ is Hughes-free embeddable.

Suppose that γ is a limit ordinal. Let k be a division ring and kG_γ a crossed product group ring. Each finitely generated subgroup H of G_γ is contained in some G_ρ with $\rho < \gamma$. Thus kH is Hughes-free embeddable. Now we infer from Corollary 6.6(i) that kG_γ is Hughes-free embeddable. Since k and kH were arbitrary, we obtain that G_γ is Hughes-free embeddable.

Let k be a division ring and kG any crossed product group ring. Let D_{G_γ} be the Hughes-free division ring of fractions of kG_γ. Observe that if $\gamma < \delta$, then $D_{G_\gamma} \hookrightarrow D_{G_\delta}$ by Hughes' Theorem I 6.3. Then $D = \varinjlim_{\gamma \leq \tau} D_{G_\gamma}$ is a Hughes-free division ring of fractions for kG. □

Recall from Theorem 2.33 that a locally indicable group has a subnormal system with torsion-free abelian factors. The groups G with a subnormal series with torsion-free abelian factors are Hughes-free embeddable by Examples 5.6(b). Any crossed product group ring kG with k a division ring is then an Ore domain by Corollary 4.11.

Recall that a free group is Hughes-free embeddable because it is orderable, see Corollary 2.24 and Examples 5.6(d). A free group G on a set X has a subnormal system with torsion-free abelian factors, the lower central series

$$G \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots \supseteq \{1\} = \bigcap_{n \geq 0} G^{(n)},$$

where $G^{(1)} = [G, G]$ and $G^{(n+1)} = [G, G^{(n)}]$, see for example [**MKS76**, Section 5.7]. But, provided $|X| \geq 2$, G has no subnormal series with torsion-free abelian factors because kG is neither a right nor a left Ore domain by Proposition 4.13. Thus Corollary 6.12 gives a lot of new examples of Hughes-free embeddable groups.

Corollary 6.13. *Let $\{G_i\}_{i \in I}$, G and H be Hughes-free embeddable groups. Then*

(i) $\prod_{i \in I} G_i$ *is Hughes-free embeddable.*
(ii) $\bigoplus_{i \in I} G_i$ *is Hughes-free embeddable.*
(iii) $G \wr H$ *is Hughes-free embeddable.*
(iv) $\ast_{i \in I} G_i$ *is Hughes-free embeddable.*
(v) $\ast_H^{i \in I} G$ *is Hughes-free embeddable.*

Proof. All are consequences of Corollary 6.12.

(i) As in Corollary 2.7(i), $\prod_{i \in I} G_i$ has a subnormal series $(H_\gamma)_{\gamma \leq \tau}$ such that $H_{\gamma+1}/H_\gamma \cong G_\gamma$ where G_γ is isomorphic to some G_{i_0}.

(ii) Same proof as (i), see Corollary 2.7(ii).

(iii) $G \wr H$ is the extension of $\prod_{h \in H} G_h$ by H, where $G_h = G$.

(iv) By Corollary 2.9(a), $\ast_{i \in I} G_i$ is the extension of a free group K by the group $\prod_{i \in I} G_i$.

(v) By Corollary 2.9(b), $\ast_H^{i \in I} G$ is the extension of a free group K by the group G. □

Corollary 6.14. *Let $(G(-), \Delta)$ be a graph of groups and $G = \pi(G(-), \Delta, \Delta_0)$ its fundamental group. Suppose that the $G(v)$ can be embedded in a group L by morphisms $f_v \colon G(v) \to L$ that can be extended to $f \colon G \to L$. If the image of f is Hughes-free embeddable, then G is Hughes-free embeddable.*

Proof. It was shown in the proof of Proposition 2.8 that G is the extension of a free group by the image of f. $\square$

Observe that in the last and the next corollaries, we cannot say "such that L is Hughes-free embeddable" as in Proposition 2.8, because a subgroup H of a Hughes-free embeddable group L need not be Hughes-free embeddable since any crossed product group ring structure kH perhaps cannot be extended to kL.

Corollary 6.15. *Let $\{G_i\}_{i\in I}$ be a family of groups. Suppose that a group H is embedded in each G_i. Suppose that there exist a group L and embeddings $G_i \overset{f_i}{\hookrightarrow} L$ that agree on H. Suppose that the image of the extension of the f_i's, $f\colon *_H^{i\in I} G_i \to L$, is Hughes-free embeddable. Then $*_H^{i\in I} G_i$ is Hughes-free embeddable.*

Proof. It was shown in the proof of Corollary 2.9(i) that $*_H^{i\in I} G_i$ is the extension of a free group by the image of f. $\square$

Example 6.16. The group Γ with presentation

$$\Gamma = \langle X, T \mid TXTX^{-1} = XTX^{-1}T\rangle = \langle X, T \mid TXTX^{-1}T^{-1}XT^{-1}X^{-1} = 1\rangle$$

is Hughes-free embeddable.

Proof. As we saw in Examples 2.10, Γ is locally indicable. We proved that by writing Γ as an extension of a fundamental group N of a graph of groups by an infinite cyclic group. Moreover N is Hughes-free embeddable because we gave an onto morphism of groups $f\colon N \to F$, where F was a torsion-free abelian group (and hence Hughes-free embeddable by Examples 5.6), such that the restriction to every vertex group was injective. $\square$

Remark 6.17. Not all Hughes-free embeddings $kG \hookrightarrow D$ invert all full matrices. Indeed there exist Hughes-free embeddable groups G such that a crossed product group ring kG is not a Sylvester domain.

Proof. As we saw in Examples 2.10, $G = \langle a, b \mid a^2 = b^3\rangle$ is locally indicable and equals $G = A \underset{C}{*} B$ where $A = \langle a\rangle$, $B = \langle b\rangle$, $C = \langle c\rangle$ are infinite cyclic groups and $C \hookrightarrow A$, $c \mapsto a^2$, $C \hookrightarrow B$, $c \mapsto b^3$. Moreover, let $D = \langle d\rangle$ be another infinite cyclic group and consider the embeddings $f_1\colon A \to D, a \mapsto d^3$ and $f_2\colon B \to D, b \mapsto d^2$. Since any subgroup of C is Hughes-free embeddable, we infer from Corollary 6.15 that G is Hughes-free embeddable. The group ring $k[G]$ has a nonfree finitely generated projective module [**Dun72**]. Then any embedding of the group ring $k[G]$ in a division ring does not invert all full matrices by Remark 3.28. $\square$

2.1. Around Lewin's proof. Let G be a free group. Let k be a division ring and kG any crossed product group ring. As we said in Section 5.2, the universal division ring of fractions U of kG coincides with the division ring of fractions of kG inside $k((G,<))$ for any total order $<$ on G such that $(G,<)$ is an ordered group. It was proved by J. Lewin [**Lew74**], who showed first that $U(kH)$ was the universal division ring of fractions of kH for any subgroup H of G. Then he used this fact to prove that the embedding $kG \hookrightarrow U$ is Hughes-free, and then Hughes' Theorem I to show that U coincides with the division ring of fractions of kG inside $k((G,<))$. We will generalize this in Proposition 6.21.

Motivated by these results we give the following definitions.

Definitions 6.18. Let G be a locally indicable group.

(a) Let k be a division ring and kG a crossed product group ring. We say that kG is a *Lewin crossed product group ring* if it satisfies the following properties

(i) For each finitely generated subgroup H of G, kH has a universal division ring of fractions U_H.
(ii) If $H_1 \leq H_2$ are finitely generated subgroups of G, then U_{H_1} embeds in U_{H_2} in a natural way, that is, the following diagram of morphisms of rings is commutative

$$\begin{tikzcd} kH_1 \arrow[r, hook] \arrow[d, hook] & kH_2 \arrow[d, hook] \\ U_{H_1} \arrow[r, hook] & U_{H_2} \end{tikzcd}$$

(b) We say that G is a *Lewin group* provided that kG is a Lewin crossed product group ring for each division ring k and each crossed product group ring kG. $\square$

EXAMPLES 6.19. (a) Free groups and locally free groups are Lewin groups by Corollary 4.38.
(b) If G is a poly-{infinite cyclic} group and k a field, then the group ring $k[G]$ is a Lewin crossed product group ring. It was shown in [**Pas82**, Theorem 2.7] that $k[G]$ has as universal division ring of fractions its Ore division ring of fractions. Since each finitely generated subgroup H of G is again poly-{infinite cyclic} the result follows. Moreover, by the foregoing, if G has a subnormal series $\{G_n\}_{n=0}^{\infty}$ with infinite cyclic factors, then the group ring $k[G]$ is a Lewin crossed product group ring for any field k. More generally, if for every nontrivial finitely generated subgroup H of G the group ring $k[H]$ is poly-{infinite cyclic}, then the group ring $k[G]$ is a Lewin crossed product group ring. $\square$

The following result is due to W. Dicks.

PROPOSITION 6.20. *Let G be a locally indicable group. Let k be a division ring. Suppose that kG is a Lewin crossed product group ring. Then kL has a universal division ring of fractions U_L for every subgroup $L \leq G$. Moreover, U_L is the division ring of fractions of kL inside U_G.*

PROOF. Suppose that $L \leq G$. For each pair of finitely generated subgroups $H_1, H_2 \leq L$ such that $H_1 \leq H_2$, we have

$$\begin{tikzcd} kH_1 \arrow[r, hook] \arrow[d, hook] & kH_2 \arrow[d, hook] \\ U_{H_1} \arrow[r, hook] & U_{H_2} \end{tikzcd}$$

Then $U_L = \varinjlim_{H \leq_{\text{f.g.}} L} U_H$ is a division ring. Observe that U_H embeds in U_L for each finitely generated subgroup H of L. Also $kL \hookrightarrow U_L$ because $kL = \varinjlim_{H \leq_{\text{f.g.}} L} kH$, and U_L is a division ring of fractions of kL since for each $d \in U_L$ there exists a finitely generated subgroup H of L such that $d \in U_H$.

Let $f\colon kL \to D$ be a morphism of rings with D a division ring. Suppose that the image of a matrix A by f becomes invertible over D. Then there exists a finitely generated subgroup H of L such that the entries of A are in kH. Now if we consider $f_{|kH}\colon kH \to D$, since fA is invertible over D, then A is invertible over U_H because U_H is the universal division ring of fractions of kH. Thus A is invertible over U_L. Hence U_L is the universal division ring of fractions of kL.

If $L \leq G$, then $U_G = \varinjlim_{H \leq_{\text{f.g.}} G} U_H$. Since $U_L = \varinjlim_{H \leq_{\text{f.g.}} L} U_H$, we get that $U_L \hookrightarrow U_G$ by the universal property of the direct limit and the fact that U_L is a division ring. $\square$

As we have seen in Proposition 6.20 for $L = G$, if kG is a Lewin crossed product group ring, then we can glue together all the universal division rings of fractions U_H for each finitely generated subgroup H of G. In general, let G be a locally indicable group. For each finitely generated subgroup H of G, we consider the minimal prime matrix ideals that determine division rings of fractions U_H. If we try to glue together one U_H for each H as in the Lewin group case, then only one minimal prime matrix ideal can be used for that. Moreover, if such embedding exists, then it is Hughes-free. This is done generalizing Lewin's proof of [**Lew74**, Proposition 6]:

PROPOSITION 6.21. *Let k be a division ring. Let G be a locally indicable group. Let kG be a crossed product group ring with a division ring of fractions U. Suppose that for each finitely generated subgroup H of G, the prime matrix ideal associated with $U(kH)$ is a minimal prime matrix ideal. Then*

(i) *$kG \hookrightarrow U$ is Hughes-free and therefore any two such U are isomorphic and the minimal prime matrix ideals are the same for each finitely generated subgroup H of G.*
(ii) *If kG has a universal division ring of fractions V, then U and V are kG-isomorphic.*
(iii) *If kG is a Lewin crossed product group ring, then $kG \hookrightarrow U_G$ is Hughes-free where U_G is the universal division ring of fractions of kG.*

PROOF. (i) Let H be a nontrivial finitely generated subgroup of G. Express H as an internal semidirect product $H = N \rtimes C$ with C infinite cyclic. If $t \in k^\times C$ such that $\rho_H(t)$ generates C, then $kH = kN[t, t^{-1}; \alpha]$.

We claim that left conjugation by t induces an automorphism of $U(kN)$ which extends α. Indeed left conjugation by t and by t^{-1} induce automorphisms of $Q_0(kN, U) = kN$. Suppose that left conjugation by t and t^{-1} induce automorphisms of $Q_n(kN, U)$ for $n \geq 0$. Let $f \in Q_n(kN, U) \setminus \{0\}$. Then

$$tf^{-1}t^{-1} = (tft^{-1})^{-1},\ t^{-1}f^{-1}t = (t^{-1}ft)^{-1} \in Q_{n+1}(kN, U) \setminus \{0\}.$$

This implies that left conjugation by t and t^{-1} induce ring endomorphisms of $Q_{n+1}(kN, U)$. Notice that the composition of these morphisms is the identity. Hence left conjugation by t induces an automorphism of $Q_{n+1}(kN, U)$. Therefore it induces an automorphism of $U(kN) = \bigcup_{n \geq 0} Q_n(kN, U)$ which extends α as claimed. We call the extension of α again α.

Consider the skew polynomial ring $P = U(kN)[x, x^{-1}; \alpha]$. Notice that $kH \hookrightarrow P$ and there exists a morphism of rings $g\colon P \to U(kH) \subseteq U$ which is the identity on $U(kN)$ and sends $x \mapsto t$. Therefore g is the identity on kH.

P is a principal ideal domain. Hence it is a fir by Examples 3.37. The Ore division ring of fractions of P is its universal division ring of fractions D, and every full matrix over P is invertible over D. Notice that $U(kH)$ is an epic P-field. Therefore there exists a specialization $h\colon D \to U(kH)$.

$$\begin{tikzcd} kH \arrow[r] \arrow[dr] & P \arrow[r] \arrow[d, "g"] & D \arrow[dl, dashed, "h"] \\ & U(kH) & \end{tikzcd}$$

Notice that D is an epic kH-field. Then, since the prime matrix ideal of $U(kH)$ is minimal, we get that h is in fact an isomorphism by Theorem 3.32. Therefore the powers of t are $U(kN)$-linearly independent since the powers of x are $U(kN)$-linearly independent. Hence $kG \hookrightarrow U$ is Hughes-free.

(ii) Suppose that V is the universal division ring of fractions of kG. First observe that for each finitely generated subgroup H of G, any matrix that becomes invertible over $U(kH)$, then it also becomes invertible over V by Theorem 3.32. Now each matrix that becomes invertible over V also becomes invertible over U because it can be seen as a matrix over $U(kH)$ for some finitely generated subgroup H of G, and the minimality of the prime matrix ideal associated with $U(kH)$.

(iii) If kG is a Lewin crossed product group ring, then the prime matrix ideal of $U_G(kH)$ is the least one for each finitely generated subgroup H of G by definition of Lewin crossed product group ring. Now apply (i). ☐

This result generalizes [**Lew74**, Proposition 6] because there it is proved that the embedding of the (Lewin) crossed product group ring kH with H a free group inside its universal division ring of fractions is Hughes-free.

Observe that Proposition 6.21 also generalizes the Ore situation. That is, let k be a division ring and G a locally indicable group such that a crossed product group ring kG is left (right) Ore. Let U be the left (right) Ore division ring of fractions of kG. By Proposition 4.9, for each finitely generated subgroup H of G, kH is a left Ore domain. The prime matrix ideal associated with $U(kH) = Q^l_{\mathrm{cl}}(kH)$ is minimal. If kG has a universal division ring of fractions it has to be U by the universal property of Ore localization. Also $kG \hookrightarrow U$ is Hughes-free by Examples 5.6(a).

Remarks 6.22. (a) Let G be a locally indicable group and k a division ring. Suppose that a crossed product group ring kG is a Lewin crossed product group ring. If G is an orderable group and $<$ is a total order on G such that $(G,<)$ is an ordered group, then the universal division ring of fractions of kG given by Proposition 6.20 is the division ring of fractions of kG inside the Mal'cev-Neumann series ring $k((G,<))$ associated with kG. This follows from Proposition 6.21, Examples 5.6(d) and Hughes' Theorem I 6.3.

(b) Lewin groups are Hughes-free embeddable groups by Proposition 6.21. ☐

Proposition 6.23. *Suppose that a group G has a subnormal series $(G_\gamma)_{\gamma\leq\tau}$ whose factors are Lewin groups. Let k be a division ring and kG a crossed product group ring. Let D be the Hughes-free division ring of fractions of kG. If there exists another division ring of fractions E of kG such that there exists a specialization $\theta\colon E \to D$, then E and D are kG-isomorphic. In particular, if kG has a universal division ring of fractions, then it coincides with D.*

Proof. We prove that $E(kG_\gamma)$ is kG_γ-isomorphic to $D(kG_\gamma)$ via θ for $\gamma \leq \tau$ by induction on γ.

If $\gamma = 0$, then G_0 is a Lewin group. Since $D(kG_0)$ is the universal division ring of fractions of kG_0, by Propositions 6.20-6.21, the result follows by Lemma 3.35. Suppose that the result holds for all ordinals β smaller than γ. If $\gamma = \beta + 1$, then notice that by Remark 6.11, $D(kG_\gamma)$ is the Hughes-free division ring of fractions of $(D(kG_\beta))\frac{G_\gamma}{G_\beta}$. Hence $D(G_\gamma)$ is the universal division ring of fractions of kG_γ by Propositions 6.20-6.21. So if we consider the subring of E generated by $E(kG_\beta)$ and G_γ, again Lemma 3.35 implies the desired result. If γ is a limit ordinal, taking limits we get the isomorphism of kG_γ-rings

$$D(kG_\gamma) = \varinjlim_{\beta<\gamma} D(kG_\beta) \cong \varinjlim_{\beta<\gamma} E(kG_\beta) = E(kG_\gamma). \qquad \square$$

Again Proposition 6.23 generalizes the Ore situation, in which the factors are infinite cyclic groups, and the result by J. Lewin because he proved the result for the subnormal series $1 \lhd G$ with G a free group.

All these results can probably be strengthened. For example something that generalizes this

Remark 6.24. Let G be a locally indicable group, k a division ring and kG a crossed product group ring. If G has a subnormal series $1 \lhd H \lhd G$ such that kH is a left (right) Ore and

G/H is a Lewin group, then there exists a specialization from the Hughes-free division ring of fractions D of kG to any division ring of fractions E of kG.

Proof. Observe that D is the universal division ring of fractions of $D(kH)\frac{G}{H}$ by Remark 6.11, Propositions 6.20-6.21(iii). Notice that since $kG \hookrightarrow E$, then the universal property of Ore localization implies that there is an embedding $D(kH) \hookrightarrow E$ extending $kH \hookrightarrow E$. Hence there is an embedding $D(kH)\frac{G}{H} \hookrightarrow E$. Now there exists a specialization of $D(kH)\frac{G}{H}$-rings (which is of kG-rings) from D to E. $\square$

2.2. Hughes-free coproduct of division rings. Let k be a division ring. Suppose that $\{D_i\}_{i\in I}$ is a family of k-division rings. In general it is not known how $\underset{k}{\circ} D_i$, the division ring coproduct of $\{D_i\}_{i\in I}$, looks like because it is constructed via generators and relations. With the aim of shedding new light to this problem we want to define a coproduct of division rings provided that, for each $i \in I$, D_i is the Hughes-free division ring of fractions of some crossed product group ring kG_i of some Hughes-free embeddable group G_i over k. We prove that these two notions of coproduct coincide in some cases.

So let $\{G_i\}_{i\in I}$ be a family of Hughes-free embeddable groups. Let kG_i be a crossed product group ring and D_i the Hughes-free division ring of fractions of kG_i for each i.

Lemma 6.25. *$\underset{k}{*} kG_i$ is a crossed product group ring $k(\underset{i\in I}{*} G_i)$ of $\underset{i\in I}{*} G_i$ over k.*

Proof. For each $i \in I$, define $T_i = \{\bar{x} \in kG_i \mid x \in G_i \setminus \{1\}\}$. Then $\{1\} \cup T_i$ is a k-basis for kG_i. Set $T = \bigcup_{i\in I} T_i$. By Theorem 3.42(i), $\underset{k}{*} kG_i$ has a basis $\mathcal{B}$ consisting of all the monomials $x = \overline{x_{i_1}} \cdots \overline{x_{i_n}}$, where $n \geq 0$, $x_{i_j} \in T_{i_j}$ and $G_{i_{j-1}} \neq G_{i_j}$. Moreover, for each $x \in \mathcal{B}$, we obtain an automorphism of k defined by $r^{\sigma(x)} = \overline{x_{i_1}} \cdots \overline{x_{i_n}}\, r(\overline{x_{i_1}})^{-1} \cdots (\overline{x_{i_n}})^{-1}$. And for each $x, y \in \mathcal{B}$, with $y = \overline{y_{j_1}} \cdots \overline{y_{j_m}}$, an element $\tau(x,y) \in k^\times$. If x_{i_n} and y_{j_1} are in different factors then $xy \in \mathcal{B}$ and $\tau(x,y) = 1$. If x_{i_n} and y_{j_1} are in the same factor, then xy is a $k^\times$-multiple of $\overline{x_{i_1}} \cdots \overline{x_{i_{n-1}}} \cdot \overline{x_{i_n} y_{j_1}} \cdot \overline{y_{j_2}} \cdots \overline{y_{j_m}} \in \mathcal{B}$.

The foregoing proves the lemma, but in fact we can give the explicit form of the twisting and the action:

For each $i \in I$ there is defined a twisting and an action

$$\tau_i \colon G_i \times G_i \to k^\times, \qquad \sigma_i \colon G_i \to \mathrm{Aut}(k).$$

Any element of $\underset{i\in I}{*} G_i$ can be uniquely expressed as $x = x_{i_1} \cdots x_{i_n}$ where $n \geq 0$, $x_{i_j} \in G_{i_j}$ and $i_{j-1} \neq i_j$. Let $x = x_{i_1} \cdots x_{i_n}$ and $y = y_{j_1} \cdots y_{j_m} \in \underset{i\in I}{*} G_i$. We define the twisting

$$\tau \colon (\underset{i\in I}{*} G_i) \times (\underset{i\in I}{*} G_i) \to k \quad \text{as} \quad \tau(x,y) = \begin{cases} 1 & \text{if } i_n \neq y_{j_1} \\ \tau(x_{i_n}, j_1)^{\sigma_{i_{n-1}}(x_{i_{n-1}}) \cdots \sigma_{i_1}(x_{i_1})} & \text{if } i_n = j_1. \end{cases}$$

The action $\sigma \colon \underset{i\in I}{*} G_i \to \mathrm{Aut}(k)$ as $\sigma(x_{i_1} \cdots x_{i_n}) = \sigma_{i_1}(x_{i_n}) \cdots \sigma_{i_1}(x_{i_1})$ (recall that in this case the composition is from left to right, i.e. $\sigma_{i_n}(x_{i_n})$ acts first. $\square$

Definition 6.26. We define the *Hughes-free coproduct* of $\{D_i\}_{i\in I}$ as the Hughes-free division ring of fractions D of $k(\underset{i\in I}{*} G_i)$. $\square$

Proposition 6.27. *Let k be a division ring. Let $\{G_i\}_{i\in I}$ be a family of Hughes-free embeddable groups. Consider a crossed product group ring kG_i for each $i \in I$. Suppose that D_i is the Hughes-free division ring of fractions of kG_i for each $i \in I$. Let D be the Hughes-free coproduct of $\{D_i\}_{i\in I}$. The following hold:*

(i) *D is faithful, i.e. $D_i \hookrightarrow D$.*

(ii) *D is separating, i.e. $D_i \cap D_j = k$ for $i \neq j$, provided that either*
  (a) *G_i is an orderable group for each i, or*
  (b) *kG_i is an Ore domain for each $i \in I$.*

(iii) *Suppose that G_i has a subnormal series $\{G_{i\gamma}\}_{\gamma \leq \tau_i}$ with Lewin factors for each $i \in I$. Then D coincides with $\underset{k}{\circ} D_i$.*

(iv) *If G_i is a Lewin orderable group for each $i \in I$, and $(\underset{i\in I}{\ast} G_i, <)$ is an ordered group, then $\underset{k}{\circ} D_i$ is the division ring of fractions of $k(\underset{i\in I}{\ast} G_i)$ inside $k((\underset{i\in I}{\ast} G_i, <))$. In particular, if G_i is a locally free group for each $i \in I$, and $(\underset{i\in I}{\ast} G_i, <)$ is an ordered group, then $\underset{k}{\circ} D_i$ is the division ring of fractions of $k(\underset{i\in I}{\ast} G_i)$ inside $k((\underset{i\in I}{\ast} G_i, <))$.*

Proof. (i) Notice that the embedding $kG_i \hookrightarrow k(\underset{i\in I}{\ast} G_i) \hookrightarrow D$ is Hughes-free. Therefore $D(kG_i) = D_i$ as desired.

(ii) (a) $\underset{i\in I}{\ast} G_i$ is an orderable group by Proposition 2.23. Fix an order $<$ on $\underset{i\in I}{\ast} G_i$ such that $(\underset{i\in I}{\ast} G_i, <)$ is an ordered group. Then the restriction of $<$ to G_i makes $(G_i, <)$ an ordered group. Thus D_i is the division ring of fractions of kG_i inside $k((G_i, <))$ by Example 5.6(d), and D is the division ring of fractions of $k(\underset{i\in I}{\ast} G_i)$ inside $k((\underset{i\in I}{\ast} G_i, <))$. Moreover, for each $i \in I$, the image of D_i in D is the set of series γ such that $\operatorname{supp}\gamma \subseteq G_i$. Therefore $D_i \cap D_j = k$ for $i \neq j$.

(b) Suppose that there exists $x \in D_i \cap D_j \subseteq D$. Then x can be expressed as $a_i s_i^{-1}$ and as $s_j^{-1} a_j$ for some $a_i, s_i \in kG_i$ and $a_j, s_j \in kG_j$. So we get that $s_j a_i = a_j s_i$. Express a_i, s_i and a_j, s_j in the k-basis $\{\bar{x} \mid x \in G_i\}$ of kG_i and $\{\bar{y} \mid y \in G_j\}$ of kG_j respectively. Then, expressing the products in the corresponding k-basis of $\underset{k}{\ast} kG_i$, we realize that $a_i = \alpha$ and $s_i = \beta$ for some $\alpha, \beta \in k$.

(iii) Notice that, since D is faithful by (i), we have a morphism of rings $f\colon \underset{i\in I}{\ast} D_i \to D$ which extends the inclusions $D_i \hookrightarrow D$. Observe that $\underset{k}{\circ} D_i$ is a division ring of fractions of $k(\underset{i\in I}{\ast} G_i)$ because $\underset{k}{\ast} D_i \hookrightarrow \underset{k}{\circ} D_i$ by the definition of the coproduct $\underset{k}{\circ} D_i$. From that we obtain the following commutative diagram of morphisms of rings

$$\begin{tikzcd}
 & D & \\
k(\underset{i\in I}{\ast} G_i) = \underset{k}{\ast} kG_i \arrow[ur, hook] \arrow[dr, hook] & & \\
 & \underset{k}{\ast} D_i \arrow[uu, "f"'] \arrow[r, hook] & \underset{k}{\circ} D_i
\end{tikzcd}$$

Since $\underset{k}{\circ} D_i$ is the universal division ring of fractions of $\underset{k}{\ast} D_i$, then f induces a $\underset{k}{\ast} D_i$-specialization which is also a $k(\underset{i\in I}{\ast} G_i)$-specialization.

On the other hand, recall from Corollary 2.9(a) that $\underset{i\in I}{\ast} G_i$ is an extension of a free group by $\prod_{i\in I} G_i$. Thus G has a subnormal series with Lewin factors by the proof of Corollary 2.7(i), and the fact that G_i has a subnormal series with Lewin factors for each $i \in I$. Now apply Proposition 6.23 to get the $k(\underset{i\in I}{\ast} G_i)$-isomorphism.

(iv) Observe that $k(\underset{i\in I}{\ast} G_i) \hookrightarrow k((\underset{i\in I}{\ast} G_i, <))$ is Hughes-free. Hence D is the division ring of fractions of $k(\underset{i\in I}{\ast} G_i)$ inside $k((\underset{i\in I}{\ast} G_i, <))$ by Example 5.6(d) and Hughes' Theorem I 6.3. Now apply (iii).

For the second part notice that a locally free group is a Lewin group by Corollary 4.37. □

This result should be compared with the result by P.M. Cohn that states: Let A_1, A_2 be semifirs with a common division ring k. Then

$$U\Big(A_1 \underset{k}{*} A_2\Big) = U\Big(A_1 \underset{k}{*} U(A_2)\Big) = U\Big(U(A_1) \underset{k}{*} U(A_2)\Big)$$

where $U(R)$ denotes the universal division ring of fractions of the semifir R.

For example, when $A_1 = kG_1$ and $A_2 = kG_2$ are crossed product group rings where G_1 and G_2 are locally free groups, then Proposition 6.27 coincides with the result of P.M. Cohn. On the other hand, let $G = \langle a, b \mid a^2 = b^3 \rangle$. Then G is a Hughes-free embeddable group, G has a subnormal series with Lewin factors, and the group ring $k[G]$ is not a semifir by Remark 6.17 and its proof. Denote by $\mathcal{H}(\mathcal{R})$ the Hughes-free division ring of fractions of a ring R. If $A_1 = A_2 = k[G]$, the Hughes-free coproduct of $A_1 \underset{k}{*} A_2$ is defined and coincides with $\mathcal{H}(A_1) \underset{k}{\circ} \mathcal{H}(A_2) = U(\mathcal{H}(A_1) \underset{k}{\circ} \mathcal{H}(A_2))$ by Proposition 6.27(iii). Then we could write

$$\mathcal{H}(A_1 \underset{k}{*} A_2) = \mathcal{H}(\mathcal{H}(A_1) \underset{k}{*} \mathcal{H}(A_2)).$$

So in this way it generalizes the result by P.M. Cohn, since $U(A_i)$ is not defined.

3. Some other Hughes-freeness conditions

Here we present open problems which are inspired on Hughes' Theorem I 6.3.

3.1. The following problems were raised by P.A. Linnell in [**Lin06**, Section 4].

Motivated by Hughes' Theorem I 6.3, P.A. Linnell extends the definition of Hughes-freeness to a more general situation.

Definition 6.28. Given a morphism of rings $A \to B$, we denote by $\mathcal{R}_B(A)$ the rational closure of A in B. Let k be a division ring, let G be a group, let kG be a crossed product group ring, and let Q be a ring containing kG such that $\mathcal{R}_Q(kG) = Q$, and every element of Q is either a zero-divisor or invertible. In this situation we say that Q is *strongly Hughes-free* if whenever $N \lhd H \leqslant G$, $h_1, \dots, h_n \in H$ are in distinct cosets of N and $q_1, \dots, q_n \in \mathcal{R}_Q(kN)$, then $q_1 h_1 + \cdots q_n h_n = 0$ implies $q_i = 0$ for all i (i.e. the h_i are linearly independent over $\mathcal{R}_Q(kN)$). □

It would be interesting to extend Hughes' Theorem I 6.3 to more general groups. With this aim he states

Problem 6.29. Let k be a division ring, let G be a group, let kG be a crossed product group ring, and let Q be a ring containing kG such that $\mathcal{R}_Q(kG) = Q$, and every element of Q is either a zero-divisor or invertible. Suppose that P and Q are strongly Hughes-free rings for kG. Does there exists an isomorphism $P \to Q$ which is the identity on kG? [**Lin06**, Problem 4.7] □

When G is a locally indicable group and Q is a division ring of fractions for kG, then Q is strongly Hughes-free implies that Q is a Hughes-free division ring of fractions of kG. So now arises the following problem

Problem 6.30. Let G be a locally indicable group, let k be a division ring, let kG be a crossed product group ring, and let Q be a division ring of fractions for kG that is Hughes-free. Is Q strongly Hughes-free? [**Lin06**, Problem 4.8] □

The answer to Problem 6.30 is yes in the cases when G is an orderable group [**Lin06**] or kG is a left or right Ore domain. So it would seem likely that the answer is always yes.

3.2. Let k be a field. Let L be an arbitrary Lie algebra over k and $\mathfrak{e}(L)$ be its universal enveloping algebra. P.M. Cohn proved that $\mathfrak{e}(L)$ can be embedded in a division ring $\widetilde{D(L)}$ (for example see [**Coh95**, Theorem 2.6.6]). The division ring of fractions of $\mathfrak{e}(L)$ inside $\widetilde{D(L)}$ will be denoted by $D(L)$.

It is known that if H is a free Lie algebra on X over k, then its universal enveloping algebra $\mathfrak{e}(H)$ is $k\langle X\rangle$. Consider $\widetilde{D(H)}$. A.I. Lichtman proved in [**Lic00**, Theorem 1] that the division ring $D(H)$ is isomorphic to the universal division ring of fractions of $k\langle X\rangle$.

Furthermore, let H_1 be a subalgebra of H and consider $\widetilde{D(H_1)}$. Then $\Delta(H_1)$, the division ring of fractions of $\mathfrak{e}(H_1)$ inside $D(H)$, is isomorphic to $D(H_1)$. Moreover, every basis of $\mathfrak{e}(H)$ over $\mathfrak{e}(H_1)$ remains linearly independent over $\Delta(H_1)$ [**Lic00**, Corollary 3].

These results should be compared with J. Lewin's result 4.37. Then A.I. Lichtman [**Lic00**, Section 8] conjectures an analogue of Hughes' Theorem I 6.3. First he gives a notion similar to Hughes-free embeddings.

Definition 6.31. Let L be a Lie algebra over a field k. An embedding of $\mathfrak{e}(L)$ in a division ring E is *Lichtman-free* if for every subalgebra L_1 of L a basis of $\mathfrak{e}(L)$ over $\mathfrak{e}(L_1)$ remains linearly independent over the subdivision ring of E generated by $\mathfrak{e}(L_1)$. □

Then he conjectures the existence of an analogue of Hughes' Theorem I 6.3

Problem 6.32. If L is a Lie algebra such that every finitely generated subalgebra of it does not coincide with its commutator ideal, then every two Lichtman-free embeddings of $\mathfrak{e}(L)$ are isomorphic. □

A positive answer to this problem would show that if H is a free Lie algebra, then any Lichtman-free embedding of $\mathfrak{e}(H)$ would give the division ring $D(H)$. And this should be seen as the analogue of J. Lewin's result [**Lew74**, Theorem 1] (see also Section 2.1 in this chapter) who showed that any Hughes-free enbedding of the crossed product free group k-ring kH gives the universal division ring of fractions of kH.

"Slowly I return to the familiar,
Spiralling in constant flux
What have I created?"

Nevermore, *42147*

Part 3

Inversion height

CHAPTER 7

Inversion height

All the results in this chapter can be found in the joint work with D. Herbera [**HS07**] except Section 7 and otherwise stated.

1. Basic definitions and properties

We begin this section fixing some notions that will be used throughout.

DEFINITIONS 7.1. Suppose that R is a domain embedded in a division ring E. Recall from Definition 3.15 that we defined inductively:
$Q_0(R,E) = R$, and, for $n \geq 0$,
$Q_{n+1}(R,E) = \begin{array}{c}\text{subring of } E\\ \text{generated by}\end{array} \{r, s^{-1} \mid r, s \in Q_n(R,E),\ s \neq 0\}$. Then $D = \bigcup_{n=0}^{\infty} Q_n(R,E)$ is the division ring of fractions of R inside E.

(a) We define $\mathrm{h}_E(R)$, the *inversion height of* R (inside E), as ∞ if there is no $n \in \mathbb{N}$ such that $Q_n(R,E)$ is a division ring. Otherwise,
$$\mathrm{h}_E(R) = \min\{n \mid Q_n(R,E) \text{ is a division ring}\}.$$
(b) An element $x \in D$ is said to have *inversion height* 0 if $x \in R$, while x is said to have *inversion height* $n \geq 1$ if $x \in Q_n(R,E) \setminus Q_{n-1}(R,E)$. □

We present the most trivial examples one can think of when dealing with inversion height. In the following sections we provide some more.

EXAMPLES 7.2. (a) R is a division ring if and only if $\mathrm{h}_E(R) = 0$ for any division ring E containing R if and only if $\mathrm{h}_E(R) = 0$ for a division ring E containing R.
(b) If R is a left (right) Ore domain, but not a division ring, then the universal property of the Ore localization implies that $E(R) = Q^l_{\mathrm{cl}}(R)$ (respectively $E(R) = Q^r_{\mathrm{cl}}(R)$). Thus every element of $E(R)$ is of the form $s^{-1}r$ (rs^{-1}) for some $r, s \in R$. Therefore $\mathrm{h}_E(R) = 1$ for any division ring E containing R. □

The converse of Examples 7.2(b) is not true. If R has an embedding in a division ring of inversion height one, then it does not need to satisfy any kind of Ore condition. J.L. Fisher [**Fis71**] gave an example of an embedding of the free algebra $k\langle x,y\rangle$ inside a division ring of inversion height one. Also there are embeddings of inversion height different from zero or one. Again J.L. Fisher presented an example of an embedding of the free algebra $k\langle x,y\rangle$ inside a division ring of inversion height two in [**Fis71**]. One of our main aims is to show that embeddings of the free algebra of inversion height one and two exist for a free algebra on an arbitrary set with at least two elements, generalizing the ones given by J.L. Fisher. In fact our proofs follow his patterns. We do it in Section 3 and Section 4 respectively.

There are also embeddings of infinite inversion height, see Section 7.

Now we give some easy but important remarks that will be used throughout this chapter sometimes without any reference.

REMARKS 7.3. In the notation of Definitions 7.1, the following statements hold:

(a) If $\mathrm{h}_E(R) = n < \infty$, then $D = Q_m(R, E)$ for $m \geq n$ because $\mathrm{h}_E(R)$ is the first natural n such that $D = Q_n(R, E)$.
(b) $\mathrm{h}_E(R)$ depends on the embedding $R \hookrightarrow E$ considered. Hence we will talk about the *inversion height of the embedding* $R \hookrightarrow E$.
(c) The inversion height of an element $x \in D$ is the minimal number of successive inversions required to express the element x (from elements of R) as defined in [**GR97**]. The same concept is called *level complexity* in [**DGH03**].
(d) If S is another subring of E such that $R \subseteq S$, then $Q_n(R, E) \subseteq Q_n(S, E)$, and the division ring of fractions of R inside E is contained in the division ring of fractions of S inside E.
(e) If F is a division ring that contains E, then $Q_n(R, E) = Q_n(R, F)$ and $\mathrm{h}_E(R) = \mathrm{h}_F(R)$.
(f) Let $\iota_1 : R \hookrightarrow D_1$ and $\iota_2 : R \hookrightarrow D_2$ be two division rings of fractions of R. If D_1 is isomorphic to D_2 as division rings of fractions, i.e. there exists a ring isomorphism $\varphi : D_1 \to D_2$ such that $\iota_2\varphi = \iota_1$, then $\mathrm{h}_{D_1}(R) = \mathrm{h}_{D_2}(R)$. Moreover $\varphi(Q_n(R, D_1)) = Q_n(R, D_2)$ for all $n \in \mathbb{N}$. □

We illustrate the definitions and remarks with the following beautiful result that will be useful later because it is a source for the construction of other examples of a fixed inversion height from a given one. More concretely, we prove that given an embedding of the free algebra on a finite number of generators of inversion height m, with $1 \leq m \leq \infty$, there exists an embedding of the same inversion height m of the free algebra on an infinite countable number of generators. It is a slight generalization of [**HS07**, Proposition 2.3].

PROPOSITION 7.4. *Let k be a ring. Let $Z = \{z_1, z_2, \dots\}$ be an infinite countable set. Let E be a division ring. Fix $r \geq 1$, and consider the free k-ring $k\langle x_0, x_1, \dots, x_r\rangle$. Suppose that $k\langle x_0, x_1, \dots, x_r\rangle \hookrightarrow E$ is an embedding of inversion height $1 \leq m \leq \infty$. Then there exists an embedding $k\langle Z\rangle \hookrightarrow E$ of inversion height m, and the division ring of fractions of $k\langle x_0, x_1, \dots, x_r\rangle$ and $k\langle Z\rangle$ inside E coincide.*

PROOF. If $i \in \mathbb{N} \setminus \{0\}$, then i can be uniquely expressed as

$$i = rn + j \quad \text{with } n, j \in \mathbb{N} \text{ and } 1 \leq j \leq r. \tag{46}$$

Consider the embedding

$$\begin{aligned} k\langle Z\rangle &\hookrightarrow k\langle x_0, x_1, \dots, x_r\rangle \hookrightarrow E \\ z_i &\mapsto x_j x_0^n. \end{aligned}$$

where each i is expressed as in (46).

We identify $k\langle Z\rangle$ with its image in $k\langle x_0, x_1, \dots, x_r\rangle$.

We show that $Q_1(k\langle x_0, x_1, \dots, x_r\rangle, E) = Q_1(k\langle Z\rangle, E)$. Then the result will follow by the definition of inversion height.

By Remarks 7.3 (d), since $k\langle Z\rangle \subseteq k\langle x_0, x_1, \dots, x_r\rangle$, $Q_1(k\langle Z\rangle, E) \subseteq Q_1(k\langle x_0, x_1, \dots, x_r\rangle, E)$.

Note that $x_1 = z_1, \dots, x_r = z_r, x_0 = z_1^{-1} z_{r+1} \in Q_1(k\langle Z\rangle, E)$. Let $p \in k\langle x_0, x_1, \dots, x_r\rangle \setminus \{0\}$. Observe that

$$p = f_0 + x_0 f_1 + \cdots + x_0^n f_n,$$

where $f_0, f_1, \dots, f_n \in k\langle Z\rangle$. Hence

$$\begin{aligned} p^{-1} &= (f_0 + x_0 f_1 + \dots + x_0^n f_n)^{-1} \\ &= \left(x_1^{-1} x_1 \left(f_0 + x_0 f_1 + \dots + x_0^n f_n\right)\right)^{-1} \\ &= \left(x_1^{-1} \left(x_1 f_0 + x_1 x_0 f_1 + \dots + x_1 x_0^n f_n\right)\right)^{-1} \\ &= (x_1 f_0 + x_1 x_0 f_1 + \dots + x_1 x_0^n f_n)^{-1} x_1 \\ &= (z_1 f_0 + z_{r+1} f_1 + \dots + z_{rn+1} f_n)^{-1} z_1. \end{aligned}$$

Therefore $p^{-1} \in Q_1(k\langle Z\rangle, E)$. So the generators of $Q_1(k\langle x_0, x_1, \dots, x_r\rangle, E)$ belong to $Q_1(k\langle Z\rangle, E)$. Hence $Q_1(k\langle x_0, x_1, \dots, x_r\rangle, E) \subseteq Q_1(k\langle Z\rangle, E)$. □

2. JF-embeddings

We want to find embeddings of the free k-ring on a set X in division rings. So we need a way to recognize free k-rings inside rings. With this purpose we present the following two results. The first one was proved by A.V. Jategaonkar in [**Jat69**]. The second one is a generalization of the example given by A.V. Jategaonkar in his paper. We give the proofs for the sake of completion. They are taken from [**Lam99**, Section 9C].

JATEGAONKAR'S LEMMA 7.5. *Let $X = \{x_i\}_{i\in I}$ be a subset of a ring R with $|X| \geq 2$. Let k be a subring of R. Suppose that*

(i) *The elements of X are right linearly independent over R.*
(ii) *The elements of k commute with X, i.e. $ax_i = x_i a$ for all $i \in I$ and $a \in k$.*

Then the subring of R generated by k and X is $k\langle X\rangle$, the free k-ring on X.

PROOF. Let $Y = \{y_i\}_{i\in I}$ be a disjoint copy of X. Consider the free k-ring $k\langle Y\rangle$. Suppose that the subring generated by k and X is not the free k-ring on X. Choose a nonconstant polynomial $f \in k\langle Y\rangle$ of least degree n such that the evaluation $f(X) = 0$. Express f in the form $f = a + \sum\limits_{i\in I} y_i g_i(Y)$ where $a \in k$ and $g_i(Y) \neq 0$ for only a finite number of $i \in I$. Suppose that $g_{i_0} \neq 0$. Let $j_0 \in I \setminus \{i_0\}$. From $0 = f(X)x_{j_0}$, using (ii) we see that

$$0 = f(X)x_{j_0} = x_{i_0}(g_{i_0}(X)x_{j_0}) + x_{j_0}(a + g_{j_0}x_{j_0}) + \sum_{i\in I\setminus\{i_0,j_0\}} x_i g_i(X)x_{j_0}.$$

Thus $g_{i_0}(X)x_{j_0} = 0$ by (i). Now write g_{i_0} in the form $g_{i_0} = b + \sum\limits_{i\in I} y_i h_i(Y)$ where $b \in k$ and $h_i(Y) \neq 0$ for only a finite number of $i \in I$. Then we have

$$\deg g_{i_0} \leq n-1, \quad \deg h_i \leq n-2 \ \text{ for all } i \in I, \tag{47}$$

and as before

$$0 = g_{i_0}x_{j_0} = x_{i_0}(h_{i_0}(X)x_{j_0}) + x_{j_0}(b + h_{j_0}(X)x_{j_0}) + \sum_{i\in I\setminus\{i_0,j_0\}} x_i h_i(X)x_{j_0}.$$

Again by (i), $h_{i_0}(X)x_{j_0} = b + h_{j_0}(X)x_{j_0} = h_i(X)x_{j_0} = 0$. Hence $b + h_{j_0}(Y)y_{j_0}$ and $h_i(Y)y_{j_0}$ with $i \in I \setminus \{i_0\}$ are satisfied by X. Then, by the minimality of n and using (47), we get that $b = 0$ and $h_i(Y) = 0$ for all $i \in I$, contradicting $g_{i_0} \neq 0$. □

LEMMA 7.6. *Let $\alpha\colon K \to K$ be an injective endomorphism of the ring K, and let $R = K[x;\alpha]$. If $\{t_i\}_{i\in I} \subseteq K$ are right linearly independent over $\alpha(K)$, then $\{t_i x\}_{i\in I} \subseteq R$ are right linearly independent over R.*

PROOF. Suppose that $\sum_{i\in I}(t_ix)f_i = 0$, where $f_i \in R$ are almost all 0. Write $f_i = \sum_{j\in\mathbb{N}} a_{ij}x^j$ ($a_{ij} \in K$). Then

$$0 = \sum_{i\in I} t_ix \sum_{j\in\mathbb{N}} a_{ij}x^j = \sum_{j\in\mathbb{N}} \left(\sum_{i\in I} t_i\alpha(a_{ij})\right) x^{j+1}.$$

Therefore, for each j, we have $\sum_{i\in I} t_i\alpha(a_{ij}) = 0$, and so $\alpha(a_{ij}) = 0$ for all i, j. Since α is injective, it follows that $f_i = \sum_{j\in\mathbb{N}} a_{ij}x^j = 0$ for all $i \in I$. $\square$

As an example of the foregoing, and following the notation of Lemma 7.6, let K be a field, α a non-onto morphism of rings, k a subfield of K with $\alpha(a) = a$ for all $a \in k$ and $t \in K\setminus\alpha(K)$. Then 1 and t are right linearly independent over $\alpha(K)$. Thus the subring generated by k, x, tx is the free k-algebra on $\{x, tx\}$. Notice that $R = K[x; \alpha]$ is a left Ore domain by Proposition 3.19. Hence, if $Q = Q^l_{\text{cl}}(R)$, we obtain an embedding of the free k-algebra on two generators inside Q. Such examples were considered by J.L. Fisher in [**Fis71**] to produce the embedding of the free k-algebra on two generators inside a division ring of inversion height 2.

Motivated by these results we give the following definition that is very important for the rest of the chapter. It singles out a class of embeddings of the free k-ring into a division ring which gives an abstract setting to the situation considered by J.L. Fisher and allows to generalize it to an arbitrary number of indeterminates. Moreover, it fixes the notation that will be used throughout.

DEFINITION 7.7. Let K, k be division rings. Suppose that K has a fixed structure of k-ring. Let $\alpha\colon K \to K$ be a morphism of k-rings which is not onto. Consider the skew polynomial ring $K[x; \alpha]$.

Let I be a set with $|I| \geq 2$. Let $\{t_i\}_{i\in I} \subseteq K$. Suppose that

(a) The elements of $\{t_i\}_{i\in I}$ are right linearly independent over $\alpha(K)$
(b) For all $a \in k$ and $i \in I$, $at_i = t_ia$
(c) There exists $i_0 \in I$ such that t_{i_0} is in the center of K
(d) The subring T of K generated by $k \cup \{\alpha^n(t_it_{i_0}^{-1}) \mid i \in I,\ n \geq 0\}$ is left Ore.

Set $x_i = t_ix,\ i \in I$. By (a) and Lemma 7.6, the elements of $X = \{x_i\}_{i\in I} \subseteq K[x; \alpha]$ are right linearly independent over $K[x; \alpha]$. Since α is a morphism of k-rings $\alpha(a) = a$ for all $a \in k$. Then, by (b), the elements of k commute with the elements of X. Hence (i) and (ii) of Jategaonkar's Lemma 7.5 are satisfied. Therefore the subring generated by X and k is the free k-ring on X, $k\langle X\rangle$.

Notice that the skew polynomial ring $K[x; \alpha]$ is a left Ore domain by Proposition 3.19. Let $Q = Q^l_{\text{cl}}(K[x; \alpha])$ be its left Ore division ring of fractions.

So as in [**Jat69**] we get an embedding of $k\langle X\rangle$, the free k-ring on X, in a division ring

$$k\langle X\rangle \hookrightarrow Q.$$

In this setting we say that $k\langle X\rangle \hookrightarrow Q$ is a $(K, k, \alpha, I, \{t_i\}_{i\in I}, t_{i_0})$-*JF-embedding.* $\square$

As expected, JF stands for Jategaonkar and Fisher. Now we present some useful remarks that also fix notation for the rest of this chapter.

REMARKS 7.8. In the notation of Definition 7.7, the following statements hold

(a) If K is a commutative field, conditions (b),(c) and (d) are superfluous. Since the set $k \cup \{\alpha^n(t_it_{i_0}^{-1}) \mid i \in I,\ n \geq 0\}$ is contained in K, T is clearly a (commutative) left (and right) Ore domain.

(b) In order to obtain an embedding of a free k-ring inside a division ring, only conditions (a), (b) in Definition 7.7 are needed. Conditions (c), (d) are necessary to make computations easier and obtain a bound for the inversion height of the embedding. So sometimes we will talk about a $(K,k,\alpha,I,\{t_i\}_{i\in I})$-*J-embedding* to express that conditions (a), (b) in Definition 7.7 hold, but perhaps (c) and (d) do not. Clearly JF-embeddings are J-embeddings.

(c) It will be useful to consider Q as a subring of a Laurent power series division ring as follows. Consider the power series ring $K[[x;\alpha]]=\{\sum_{l\geq 0} a_lx^l \mid a_l\in K\}$. Let $\mathfrak{S}=\{1,x,x^2,\dots\}$. By Proposition 3.10, $\mathfrak{S}$ is a left Ore set and the localization of $K[[x;\alpha]]$ at $\mathfrak{S}$,

$$E=\mathfrak{S}^{-1}K[[x;\alpha]]=\{x^{-n}\sum_{l=0}^{\infty} a_lx^l \mid a_l\in K,\ n\geq 0\}, \tag{48}$$

is a division ring. Therefore we get

$$k\langle X\rangle \hookrightarrow K[x;\alpha]\hookrightarrow Q\hookrightarrow E.$$

So sometimes we will talk about the $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-JF-embedding $k\langle X\rangle\hookrightarrow E$. □

The next lemma yields in essence that all JF-embeddings can be supposed with $t_{i_0}=1$, which makes computations easier.

Lemma 7.9. *Suppose that $k\langle X\rangle\hookrightarrow Q$ is a $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-JF-embedding. Let Z be a disjoint copy of X. Then there exists a $(K,k,\alpha,I,\{s_i=t_it_{i_0}^{-1}\}_{i\in I},s_{i_0}=1)$-JF-embedding $k\langle Z\rangle\hookrightarrow Q$ and an isomorphism of k-rings, $\varphi\colon K[z;\alpha]\to K[x;\alpha]$, which induces the following commutative diagram*

$$\begin{array}{ccc} k\langle Z\rangle & \xrightarrow{\cong} & k\langle X\rangle \\ \downarrow & & \downarrow \\ K[z;\alpha] & \xrightarrow{\varphi} & K[x;\alpha] \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\cong} & Q \end{array}$$

In particular $\mathrm{h}_Q(k\langle Z\rangle)=\mathrm{h}_Q(k\langle X\rangle)$.

Proof. Consider the skew polynomial ring $K[z;\alpha]$. The elements of the set $\{s_i=t_it_{i_0}^{-1}\}_{i\in I}$ satisfy the conditions of Definition 7.7:

(a) the elements of $\{s_i\}_{i\in I}$ are right linearly independent over $\alpha(K)$ since t_{i_0} is in the center of K; (b) for all $a\in k,\ i\in I,\ as_i=s_ia$; (c) $s_{i_0}=1$ is in the center of K; (d) the subring generated by $k\cup\{\alpha^n(s_is_{i_0}^{-1})=\alpha^n(t_it_{i_0}^{-1})\mid i\in I,\ n\geq 0\}$ is T, and therefore it is left Ore.

Put $z_i=s_iz$ for all $i\in I$. Set $Z=\{z_i\mid i\in I\}$. Hence the subring generated by Z and k is the free k-ring $k\langle Z\rangle$, and $k\langle Z\rangle\hookrightarrow Q^l_{\mathrm{cl}}(K[x;\alpha])$ is a $(K,k,\alpha,I,\{s_i\}_{i\in I},s_{i_0}=1)$-JF-embedding. Notice that $Q=Q^l_{\mathrm{cl}}(K[z;\alpha])$. Because t_{i_0} is in the center of K, we can define the isomorphism $\varphi\colon K[z;\alpha]\to K[x;\alpha]$, where $\varphi(a)=a$ for all $a\in K$ and $\varphi(z)=t_{i_0}x$. Then φ induces the following commutative diagram of morphisms of rings

$$\begin{array}{ccc} k\langle Z\rangle & \xrightarrow{\cong} & k\langle X\rangle \\ \downarrow & & \downarrow \\ K[z;\alpha] & \xrightarrow{\varphi} & K[x;\alpha] \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\cong} & Q \end{array}$$

Observe that $\varphi(z_i)=x_i$ for all $i\in I$. Then, from the commutativity of the diagram, we infer that $\mathrm{h}_Q(k\langle X\rangle)=\mathrm{h}_Q(k\langle Z\rangle)$. □

THEOREM 7.10. *Let* $k\langle X\rangle \hookrightarrow Q$ *be a* $(K, k, \alpha, I, \{t_i\}_{i\in I}, t_{i_0})$*-JF-embedding. Then* $\mathrm{h}_Q(k\langle X\rangle)$ *is at most two.*

PROOF. First suppose that $t_{i_0} = 1$. Notice that $T \subseteq K$ and $\alpha(T) \subseteq T$. Let $i \in I$ and $n \geq 0$, then

$$\alpha^n(t_i) = x^n t_i x x^{-n-1} = x_{i_0}^n x_i x_{i_0}^{-n-1} \in Q_1(k\langle X\rangle, Q).$$

Therefore $k\langle X\rangle \subseteq T[x;\alpha] \subseteq Q_1(k\langle X\rangle, Q)$. Since T is a left Ore domain, by Proposition 3.19, $T[x;\alpha]$ is a left Ore domain and

$$k\langle X\rangle \subseteq Q^l_{\mathrm{cl}}(T[x;\alpha]) \subseteq Q_2(k\langle X\rangle, Q). \tag{49}$$

So $Q_2(k\langle X\rangle, Q)$ contains a division ring that contains $k\langle X\rangle$, but on the other hand $\bigcup_{n=0}^{\infty} Q_n(k\langle X\rangle, E)$ is the smallest division ring inside E that contains $k\langle X\rangle$. Therefore $Q_2(k\langle X\rangle, Q) = Q^l_{\mathrm{cl}}(T[x;\alpha])$ and $\mathrm{h}_Q(k\langle X\rangle) \leq 2$.

If $t_{i_0} \neq 1$, by Lemma 7.9, we get a $(K, k, \alpha, I, \{s_i\}_{i\in I}, s_{i_0} = 1)$-JF-embedding $k\langle Z\rangle \hookrightarrow Q$. By the preceding case $\mathrm{h}_Q(k\langle Z\rangle) \leq 2$. Again by Lemma 7.9, $\mathrm{h}_Q(k\langle X\rangle) = \mathrm{h}_Q(k\langle Z\rangle) \leq 2$. □

So the inversion height of a JF-embedding is at most two, but do there exist embeddings of inversion height one and two? The answer is yes. The next two sections are devoted to give examples of JF-embeddings of $k\langle X\rangle$ of inversion height one and two respectively.

It is worth mentioning the following.

REMARK 7.11. Observe that condition (d) in Definition 7.7 has only been used in (49). If we replace it with

(d') The subring T of K generated by $k \cup \{\alpha^n(t_i t_{i_0}^{-1}) \mid i \in I,\ n \geq 0\}$ has inversion height n,

then Theorem 7.10 could have been stated as

> *Let* $k\langle X\rangle \hookrightarrow Q$ *be a* $(K, k, \alpha, I, \{t_i\}_{i\in I}, t_{i_0})$*-JF-embedding. Then* $\mathrm{h}_Q(k\langle X\rangle)$ *is at most* $n+1$. □

3. JF-embeddings of inversion height one

As stated before, in this section we provide examples of JF-embeddings of the free k-ring $k\langle X\rangle$ of inversion height one. Although the strategy for the construction of these embeddings is the same when X is finite or not, the way of carry it through is not, so we have divided this section in two subsections depending on whether X is finite or infinite.

3.1. The finite case. Let k be a division ring. Consider $k[t]$, the polynomial ring with coefficients in k with its natural structure of k-ring. Fix $n \geq 2$. Let $\alpha_n \colon k[t] \to k[t]$ be the morphism of k-rings defined by $t \mapsto t^n$. Observe that α_n is injective.

Let K be the (left and right) Ore division ring of fractions of $k[t]$. By the universal property of the Ore localization, α_n can be extended to a morphism of k-rings $\alpha_n \colon K \to K$.

Consider $K[x;\alpha_n]$. Notice that $Q_n = Q^l_{\mathrm{cl}}(K[x;\alpha_n]) = Q^l_{\mathrm{cl}}(k[t][x;\alpha_n])$ by Proposition 3.19. Let $I = \{0, 1, \ldots, n-1\}$. Set $t_0 = 1, t_1 = t, \ldots, t_{n-1} = t^{n-1}$. Define

$$x_0 = x,\ x_1 = tx, \ldots,\ x_{n-1} = t^{n-1}x, \text{ and } X_n = \{x_0, \ldots, x_{n-1}\}.$$

It is not difficult to verify conditions (a)−(d) in Definition 7.7: (b) and (c) are clear, (d) holds because $T = k[t]$, and (a) is satisfied because $1, \ldots, t^{n-1}$ are right linearly independent over $\alpha_n(k[t]) = k[t^n]$.

Hence we obtain a $(K, k, \alpha_n, I, \{t_i\}_{i=0}^{n-1}, 1)$-JF-embedding $k\langle X_n\rangle \hookrightarrow Q_n$.

PROPOSITION 7.12. Q_n *is a division ring of fractions of* $k\langle X_n\rangle$*, and* $k\langle X_n\rangle \hookrightarrow Q_n$ *has inversion height* 1.

Proof. Let M be the (free) monoid generated by X_n.

Step 1: Let $r, s \in \mathbb{N}$. There exists $m \in \mathbb{N}$ such that $(t^r x^s)x^m \in M$.

We prove it by induction on r.

If $r = 0, \dots, n-1$, $(t^r x^s)x = t^r x x^s = x_r x_0^s \in M$.

Consider $t^r x^s$, with $r > n-1$. Suppose that for each $0 \leq b < r$ there exists $m_1 \in \mathbb{N}$ such that $(t^b x^{s_1})x^{m_1} \in M$. By the division algorithm there exists $1 \leq b < r$ such that $r = bn + l$ for some $l \in \{0, \dots, n-1\}$. Then

$$(t^r x^s)x = (t^r x)x^s = (t^{bn+l}x)x^s = (t^l t^{bn} x)x^s = t^l x t^b x^s = x_l t^b x^s.$$

Now, by induction hypothesis, there exists $m_1 \in \mathbb{N}$ such that $(t^b x^s)x^{m_1} \in M$. Hence

$$(t^r x^s)x^{m_1+1} = (t^r x^s)x x^{m_1} = x_l (t^b x^s)x^{m_1} \in M.$$

Step 2: Notice that if $(t^r x^s)x^m \in M$, then $(t^r x^s)x^p \in M$ for all $p \geq m$. Therefore, given a finite number of pairs $(r_1, s_1), \dots, (r_l, s_l) \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $(t^{r_i} x^{s_i})x^m \in M$ for $i = 1, \dots l$. Thus, given polynomials $p, q \in k[t][x;\alpha]$, there exists $m \in \mathbb{N}$ such that $px^m, qx^m \in k\langle X_n\rangle$.

Step 3: Let $h \in Q_n$, then there exist $p, q \in k[t][x;\alpha_n]$ such that $h = q^{-1}p$. By Step 2, there exists $m \in \mathbb{N}$ such that $qx^m, px^m \in k\langle X\rangle$. Thus,

$$q^{-1}p = (qx^m x^{-m})^{-1}(px^m)x^{-m} = x_0^m (qx^m)^{-1}(px^m)x_0^{-m} \in Q_1(k\langle X_n\rangle, Q_n). \qquad \square$$

The following result did not appear in [**HS07**].

Corollary 7.13. *Let k be a division ring. Let $Z = \{z_1, z_2, \dots\}$ be an infinite countable set. There exist infinite non-isomorphic division rings of fractions D of $k\langle Z\rangle$ such that $k\langle Z\rangle \hookrightarrow D$ is of inversion height one.*

Proof. In Proposition 7.12 we have proved that the embedding

$$k\langle x_0, \dots, x_{n-1}\rangle \hookrightarrow Q_n = Q^l_{\mathrm{cl}}(k[t][x;\alpha_n]),$$

where $\alpha_n \colon k[t] \to k[t]$ is a morphism of k-rings given by $t \mapsto t^n$, has inversion height one for each $n \geq 2$.

By Proposition 7.4, the embedding $\delta_n \colon k\langle Z\rangle \hookrightarrow Q_n$ defined by

$$\begin{aligned} k\langle Z\rangle &\hookrightarrow k\langle x_0, x_1, \dots, x_{n-1}\rangle \hookrightarrow Q_n, \\ z_i &\mapsto x_j x_0^s. \end{aligned}$$

where $i = (n-1)s + j$ with $1 \leq j \leq n-1$, has inversion height one.

Observe that Q_n is the division ring of fractions of $k\langle Z\rangle$ for each $n \geq 2$ by Propositions 7.4 and 7.12. Thus it remains to prove that there does not exist a morphism of rings $f \colon Q_m \to Q_n$ for $n \neq m$ making the following diagram commutative

$$\begin{tikzcd} & Q_m \arrow[d, "f"] \\ k\langle Z\rangle \arrow[ur, hook, "\delta_m"] \arrow[r, hook, "\delta_n"'] & Q_n \end{tikzcd}$$

Fix $n > m$, and suppose that such an f exists. For f is a morphism of $k\langle Z\rangle$-rings, then $f(\delta_m(z_i)) = f(\delta_n(z_i))$ for all $i \geq 1$.

On the one hand, $\delta_m(z_m) = x_1 x_0 = txx = tx^2$, $\delta_m(z_1) = tx$. Thus

$$\delta_m(z_1)^{-1}\delta_m(z_m) = x_0 = x, \qquad \delta_m(z_1)\delta_m(z_m)^{-1}\delta_m(z_1) = tx(tx^2)^{-1}tx = t.$$

On the other hand, $\delta_n(z_m) = x_m = t^m x$, $\delta_n(z_1) = tx$. Thus

$$\delta_n(z_1)^{-1}\delta_n(z_m) = x^{-1}t^{-1}t^m x = x^{-1}t^{m-1}x, \quad \delta_n(z_1)\delta_n(z_m)^{-1}\delta_n(z_1) = tx(t^m x)^{-1}tx = t^{-m+2}x.$$

Hence $f(x) = x^{-1}t^{m-1}x$ and $f(t) = t^{-m+2}x$. In Q_m, $xt = t^m x$. Therefore $f(xt) = f(t^m x)$ in Q_n. Now

$$\begin{aligned} f(xt) = f(x)f(t) &= x^{-1}t^{m-1}xt^{-m+2}x \\ &= x^{-1}t^{m-1}(t^{-m+2})^n x^2 \\ &= x^{-1}t^{m-1}t^{(-m+2)n}x^2, \end{aligned}$$

and

$$\begin{aligned} f(t^m x) = f(t)^m f(x) &= (t^{-m+2}x)^m x^{-1}t^{m-1}x \\ &= (t^{-m+2}x)^{m-1}t^{-m+2}xx^{-1}t^{m-1}x \\ &= (t^{-m+2}x)^{m-1}tx. \end{aligned}$$

Multiplying both expressions on the left by x we get

$$t^{m-1}t^{(-m+2)n}x^2 = x(t^{-m+2}x)^{m-1}tx.$$

Therefore we have an equality in $K[x;\alpha_n]$ where the polynomial on the left is of degree 2 in x while the polynomial on the right is of degree at least 3 in x since $m \geq 2$, a contradiction. □

Remark 7.14. For all $n \geq m \geq 2$, consider the JF-embedding

$$k\langle x_0, x_1, \ldots, x_{m-1}\rangle \hookrightarrow k\langle x_0, x_1, \ldots, x_{n-1}\rangle \hookrightarrow Q_n$$

defined by $x_i \mapsto x_i$ for $i = 0, \ldots, m-1$. Since $x_0 \mapsto x$ and $x_1 \mapsto tx$, then the division ring of fractions of $k\langle x_0, x_1, \ldots, x_{m-1}\rangle$ inside Q_n contains k, x and t. Thus Q_n is the division ring of fractions of $k\langle x_0, x_1, \ldots, x_{m-1}\rangle$. Moreover, if $n' \geq m$, Q_n is not isomorphic to $Q_{n'}$ as division rings of fractions of $k\langle x_0, x_1, \ldots, x_{m-1}\rangle$ whenever $n \neq n'$, see for example [**Lam99**, Theorem 9.2.7]. By Theorem 7.10, $\mathrm{h}_{Q_n}(k\langle x_0, x_1, \ldots, x_{m-1}\rangle) \leq 2$. On the other hand, it is not known whether $\mathrm{h}_{Q_n}(k\langle x_0, x_1, \ldots, x_{m-1}\rangle) = 1$. □

3.2. The infinite case. In the finite case, to construct the embedding of the free k-ring on X_n inside a division ring, we have used the ring $k[t]$ and the morphism $\alpha\colon k[t] \to k[t]$ defined by $t \mapsto t^n$. Then the powers of t smaller than n determined the free k-ring inside $Q^l_{\mathrm{cl}}(k[t][x;\alpha])$. Notice that the exponents of t form an ordered semiring, the natural numbers. We try following this pattern for the infinite case. If X is an infinite set of cardinality λ, then we have to construct an ordered semiring of exponents M_λ, of at least λ elements, then the powers of t smaller than t^λ will determine the free k-ring.

The semiring M_λ is built from an ordinal number, λ^ω. But recall from Section 1.4 that the usual sum and product of ordinal numbers are neither commutative nor cancellative, for example $1 + \omega = \omega \neq w + 1$ and $2 \cdot \omega = \omega \neq \omega \cdot 2$. These would produce things like $t^{\omega+1} = t \cdot t^\omega = t^\omega$ and $t^2 t^\omega \neq t^\omega t^2$, which are not desirable for a set of exponents, moreover t^ω would be a zero divisor. This difficulty is overcome with what are called natural sum and natural product of ordinal numbers. Then we proceed to the construction of the semiring M_λ and the proof of the main result.

We shall need some additional definitions and results on ordinal numbers which can be found for example in [**Sie58**].

We begin with the following key result to define the natural sum and product of ordinals. It is [**Sie58**, XIV.19 Theorem 2].

Proposition 7.15. *Every ordinal number $\lambda > 0$ may be represented uniquely in the form*

$$\lambda = \omega^{\lambda_1}a_1 + \omega^{\lambda_2}a_2 + \cdots + \omega^{\lambda_r}a_r \tag{50}$$

where r and $a_1, \ldots, a_r$ are nonzero natural numbers, while $\lambda_1 > \cdots > \lambda_r$ is a decreasing sequence of ordinal numbers. □

Definitions 7.16. (a) Let λ be an ordinal. Formula (50) is called the *normal form* of the ordinal number λ. Sometimes we abuse notation and allow some a_l to be zero.

(b) Let γ and β be nonzero ordinal numbers. Abusing notation, with suitable re-labeling, the normal forms for these ordinals can be written using the same strictly decreasing set of exponents $\gamma_1 > \gamma_2 > \cdots > \gamma_r$. Thus

$$\gamma = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_r} m_r \quad \text{and} \quad \beta = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_r} n_r,$$

where $n_i, m_i \in \mathbb{N}$. Then the *natural sum* $\oplus$ and *natural product* $\otimes$ of γ and β are defined by

$$\gamma \oplus \beta = \sum_{i=1}^{r} \omega^{\gamma_i}(m_i + n_i), \qquad \gamma \otimes \beta = \bigoplus_{i,j=1}^{r} \omega^{\gamma_i \oplus \gamma_j} m_i n_j.$$

In addition we define $0 \oplus \gamma = \gamma \oplus 0 = \gamma$, and $0 \otimes \gamma = \gamma \otimes 0 = 0$ for any ordinal number γ. □

Thus, in order to form the natural sum and product of the ordinal numbers γ and β, we sum and multiply their normal forms as if they were polynomials in the indeterminate ω.

The proof of the next remark is straightforward from the definitions.

Remarks 7.17. Let β, γ, δ be ordinal numbers. The operations $\oplus$ and $\otimes$ satisfy:

(a) $\gamma \oplus \beta = \beta \oplus \gamma$ and $(\delta \oplus \gamma) \oplus \beta = \delta \oplus (\gamma \oplus \beta)$, i.e. $\oplus$ is commutative and associative.
(b) If $\delta \oplus \beta = \gamma \oplus \beta$, then $\delta = \gamma$, i.e. $\oplus$ is cancellative.
(c) $\gamma \otimes \beta = \beta \otimes \gamma$ and $(\delta \otimes \gamma) \otimes \beta = \delta \otimes (\gamma \otimes \beta)$, i.e. $\otimes$ is commutative and associative.
(d) If $\delta \otimes \beta = \gamma \otimes \beta$, with $\beta \neq 0$, then $\delta = \gamma$.
(e) $\delta \otimes (\gamma \oplus \beta) = \delta \otimes \gamma \oplus \delta \otimes \beta$, i.e. the distributive law holds.
(f) $\gamma \oplus \beta = \gamma + \beta$ and $\gamma \otimes \beta = \gamma \cdot \beta$ if γ and β are finite ordinal numbers. □

Definition 7.18. Every ordinal number $\lambda > 0$ that is not a sum of two ordinal numbers smaller than λ is called a *prime component*. Thus if an ordinal number λ is a prime component, then there exists no decomposition $\lambda = \beta + \gamma$, where $\beta < \lambda$ and $\gamma < \lambda$. □

For example, among finite ordinal numbers, only 1 is a prime component. The ordinal number ω is a prime component.

Now we give some useful properties of prime components.

Remarks 7.19. (a) Let $\gamma > 0$ be an ordinal number. Then γ is a prime component if and only if for every ordinal number $\varepsilon < \gamma$ we have $\varepsilon + \gamma = \gamma$.

(b) Prime components are powers of the ordinal number ω (whose exponents are ordinal numbers) and conversely, powers of the ordinal number ω are prime components.

(c) Let $\lambda > 0$ be a prime component. Let $\gamma, \beta < \lambda$. Then $\gamma + \beta < \lambda$ and $\gamma \oplus \beta < \lambda$.

Proof. The proof of (a) and (b) can be found in [**Sie58**, XIV.6 Theorem 1] and [**Sie58**, XIV.19 Theorem 1] respectively.

(c) Suppose that $\gamma + \beta \geq \lambda$. By Remarks 1.21, $(\gamma + \beta) + \lambda > \gamma + \beta \geq \lambda$. By (a), $(\gamma + \beta) + \lambda = \gamma + (\beta + \lambda) = \gamma + \lambda = \lambda$. Hence $\lambda > \lambda$, a contradiction.

If the normal forms of γ, β are

$$\gamma = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_r} m_r \quad \text{and} \quad \beta = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_r} n_r,$$

then $\gamma \oplus \beta = \sum_{i=1}^{r} \omega^{\gamma_i}(m_i + n_i)$. Since $\gamma, \beta < \lambda$, then $\omega^{\gamma_i} < \lambda$. Therefore $\gamma \oplus \beta < \lambda$. □

The following result is probably well-known since it is only an easy application of the cardinal arithmetic, but has not been found in the literature.

Lemma 7.20. *Let λ be an infinite cardinal number. Then the following hold*

(i) *λ is a prime component.*
(ii) *$\lambda = \omega^{\omega^\delta}$ for some ordinal number δ.*
(iii) *The ordinal λ^ω is a prime component with normal form $\omega^{\omega^{\delta+1}}$.*

Proof. (i) If λ is not a prime component, then $\lambda = \beta + \gamma$ with $\beta < \lambda$ and $\gamma < \lambda$. But then $|\lambda| = |\beta| + |\gamma| = \max\{|\beta|, |\gamma|\}$ by Remark 1.27, which contradicts λ is a cardinal number.

(ii) By Remarks 7.19(b), $\lambda = \omega^\gamma$ for some ordinal γ. Again by Remarks 7.19(b), we have to show that γ is a prime component. Suppose that γ is not a prime component. Then there exist nonzero ordinal numbers ϵ, ν such that $\gamma = \epsilon + \nu$ with $\epsilon, \nu < \gamma$. Hence

$$|\lambda| = |\omega^\gamma| = |\omega^{\epsilon+\nu}| = |\omega^\epsilon \cdot \omega^\nu| = |\omega^\epsilon| \cdot |\omega^\nu| = \max\{|\omega^\epsilon|, |\omega^\nu|\}.$$

By Remarks 1.21, $\omega^\epsilon < \omega^\gamma$ and $\omega^\nu < \omega^\gamma$, thus we get a contradiction with the fact that λ is a cardinal number.

(iii) By Remarks 1.21, $\lambda^\omega = \left(\omega^{\omega^\delta}\right)^\omega = \omega^{\omega^\delta \omega} = \omega^{\omega^{\delta+1}}$. By the uniqueness of the normal form and Remarks 7.19(b), λ^ω is a prime component with normal form $\omega^{\omega^{\delta+1}}$. $\square$

Now we are ready to present our semiring of exponents.

Lemma 7.21. *Let λ be an infinite cardinal. Consider the set of ordinal numbers*

$$M_\lambda = \{\gamma \mid \gamma < \lambda^\omega\}.$$

Then the following hold:

(i) *M_λ is a commutative, cancellative and ordered monoid with respect to $\oplus$.*
(ii) *M_λ is a semigroup with respect to $\otimes$. Moreover, if $\gamma < \beta$, then $\nu \otimes \gamma < \nu \otimes \beta$ for all nonzero $\nu, \gamma, \beta \in M_\lambda$.*
(iii) *The map $M_\lambda \xrightarrow{\bar{\lambda}} M_\lambda$, defined by $\gamma \mapsto \lambda \otimes \gamma$, is an injective morphism of monoids.*

In particular M_λ is a commutative semiring.

Proof. Let $\gamma, \beta < \lambda^\omega$. We may suppose that

$$\gamma = \omega^{\gamma_1} m_1 + \cdots + \omega^{\gamma_r} m_r, \quad \beta = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_r} n_r$$

where $\gamma_1 > \cdots > \gamma_r$ and $n_i, m_i \in \mathbb{N}$. By Lemma 7.20(iii), $\lambda = \omega^{\omega^\delta}$ for some ordinal number δ.

(i) We already know that $\oplus$ is associative and cancellative. By definition, $0 \oplus \gamma = \gamma \oplus 0 = \gamma$ for all $\gamma \in M_\lambda$. Since λ^ω is a prime component by Lemma 7.20(iii), then Remarks 7.19(c) implies that $\gamma \oplus \beta < \lambda^\omega$. That is, $\gamma \oplus \beta \in M_\lambda$. Therefore M_λ is a commutative monoid.

To prove that M_λ is an ordered monoid, first notice that $\omega^\nu < \omega^\varepsilon$ for ordinal numbers $\nu < \varepsilon$ by Remarks 1.21. Now Remarks 7.19(c) implies that $\omega^{\nu_1} a_1 + \cdots + \omega^{\nu_s} a_s < \omega^\varepsilon$ for ordinal numbers $\varepsilon > \nu_1 > \cdots > \nu_s > 0$. Hence

$$\gamma > \beta \text{ iff } \exists\, i_0 \in \{1, \dots, r\} \text{ such that } n_1 = m_1, \dots, m_{i_0-1} = n_{i_0-1} \text{ and } m_{i_0} > n_{i_0}. \tag{51}$$

From this it is not difficult to show that if δ is an ordinal number, then $\gamma > \beta$ implies that $\gamma \oplus \delta > \beta \oplus \delta$, i.e. M_λ is an ordered monoid.

(ii) Consider $\gamma \otimes \beta = \bigoplus_{i,j=1}^{r} \omega^{\gamma_i \oplus \gamma_j}(m_i n_j)$. Notice that $\gamma_i < \omega^{\delta+1}$ for $i = 1, \dots, r$, because $\gamma, \beta < \lambda^\omega = \omega^{\omega^{\delta+1}}$. Hence $\gamma_i \oplus \gamma_j < \omega^{\delta+1}$ and $\omega^{\gamma_i \oplus \gamma_j} < \omega^{\omega^{\delta+1}}$ for all $i, j = 1, \dots, r$. By Remarks 7.19(c), $\bigoplus_{i,j=1}^{r} \omega^{\gamma_i \oplus \gamma_j}(m_i n_j) < \lambda^\omega$.

Let $0 \neq \delta = \omega^{\delta_1}v_1 + \cdots + \omega^{\delta_s}v_s$. Now $\delta\gamma = \bigoplus_{i,j} \omega^{\delta_i \oplus \gamma_j}(v_i m_j)$ and $\delta\beta = \bigoplus_{i,j} \omega^{\delta_i \oplus \gamma_j}(v_i n_j)$. If j_0 is such that $m_{j_0} > n_{j_0}$ and $m_j = n_j$ for $1 \leq j < j_0$, then $v_1 m_{j_0} > v_1 n_{j_0}$ and $v_i m_j = v_i n_j$ for all other i, j such that $\delta_i \oplus \gamma_j \geq \delta_1 \oplus \gamma_{j_0}$. Hence $\delta \otimes \gamma > \delta \otimes \beta$.

(iii) Observe that $\bar{\lambda}$ is well defined by (ii) and that $\bar{\lambda}$ is injective by Remarks 7.17(d). By definition $\lambda \otimes 0 = 0$. Moreover, since the distributive laws are satisfied, $\bar{\lambda}$ is a morphism of monoids. □

Let λ be an infinite cardinal. Let k be a division ring. Consider the monoid k-ring $R_\lambda = kM_\lambda$ expressed in multiplicative notation. Thus, as a set,

$$R_\lambda = \{a_1 t^{\gamma_1} + \cdots + a_r t^{\gamma_r} \mid r \in \mathbb{N},\ a_1, \dots, a_r \in k,\ \gamma_1, \dots, \gamma_r < \lambda^\omega\}.$$

Given $a, b \in R_\lambda$, we can suppose that $a = a_1 t^{\gamma_1} + \cdots + a_s t^{\gamma_s},\ b = b_1 t^{\gamma_1} + \cdots + b_s t^{\gamma_s}$. Then the sum and product are defined as

$$a + b = \sum_{i=1}^{s} (a_i + b_i) t^{\gamma_i}, \qquad ab = \sum_{i,j=1}^{s} a_i b_j t^{\gamma_i \oplus \gamma_j}.$$

By Lemma 7.21(iii), $\bar{\lambda}$ is an injective morphism of monoids. Hence $\bar{\lambda}$ induces the injective morphism of k-rings $\alpha\colon R_\lambda \to R_\lambda$ defined by $\alpha(a_1 t^{\gamma_1} + \cdots + a_s t^{\gamma_s}) = a_1 t^{\lambda \otimes \gamma_1} + \cdots + a_s t^{\lambda \otimes \gamma_s}$.

Note that R_λ is an Ore domain by Lemma 4.12. Let $K = Q_{\mathrm{cl}}(R_\lambda)$. Since α is injective, α can be extended to K. Let $\alpha\colon K \to K$ be its extension.

Consider now the skew polynomial ring $K[x; \alpha]$ and the set $\{t^\gamma\}_{\gamma < \lambda} \subseteq K$. Define $Q = Q^l_{\mathrm{cl}}(K[x;\alpha]) = Q^l_{\mathrm{cl}}(R_\lambda[x;\alpha])$. Then conditions (a)-(d) in Definition 7.7 are satisfied. Indeed, expanding the normal forms of the ordinal numbers, it is not difficult to prove that given ordinal numbers $\gamma_1, \gamma_2 < \lambda$ and $\varepsilon_1, \varepsilon_2 < \lambda^\omega$, then $\lambda \otimes \varepsilon_1 \oplus \gamma_1 = \lambda \otimes \varepsilon_2 \oplus \gamma_2$ if and only if $\gamma_1 = \gamma_2$ and $\varepsilon_1 = \varepsilon_2$. Hence the elements elements of $\{t^\gamma\}_{\gamma<\lambda}$ are right linearly independent over $\alpha(R_\lambda)$ and thus over $\alpha(K)$. Therefore (a) holds.

(b) for all $\gamma < \lambda$ and $a \in k$, $at^\gamma = t^\gamma a$,

(c) for $\gamma = 0$, $t^0 = 1$, is in the center of K,

(d) the subring T of K generated by $k \cup \{\alpha^n(t^\gamma) \mid \gamma < \lambda,\ n \geq 0\}$ is left Ore by Lemma 4.12.

Set $x_\gamma = t^\gamma x$, for all $\gamma < \lambda$. Let $X = \{x_\gamma\}_{\gamma<\lambda}$. By the foregoing, $k\langle X\rangle \hookrightarrow Q$ is a $(K, k, \alpha, \lambda, \{t^\gamma\}_{\gamma<\lambda}, 1)$-JF-embedding. Observe that $|X| = \lambda$.

Theorem 7.22. *Q is a division ring of fractions of $k\langle X\rangle$, and $k\langle X\rangle \hookrightarrow Q$ has inversion height 1.*

Proof. This proof follows the same structure as the proof of Proposition 7.12. Let M be the free monoid generated by X.

By Lemma 7.20, there exists an ordinal number $\delta > 0$ such that $\lambda = \omega^{\omega^\delta}$ and $\lambda^\omega = \omega^{\omega^{\delta+1}}$.

We claim Step one: Let γ be an ordinal number smaller than λ^ω and let $s \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $(t^\gamma x^s)x^m \in M$.

We can suppose that the normal form of γ is

$$\gamma = \omega^{\gamma_1} m_1 + \cdots + \omega^{\gamma_r} m_r + \omega^{\gamma_{r+1}} m_{r+1} + \cdots + \omega^{\gamma_{r+d}} m_{r+d},$$

where $\omega^{\delta+1} > \gamma_1 > \cdots > \gamma_r \geq \omega^\delta > \gamma_{r+1} > \cdots > \gamma_{r+d} \geq 0$ and $m_i, r, d \in \mathbb{N}$. If we define $\varepsilon = \omega^{\gamma_1} m_1 + \cdots + \omega^{\gamma_r} m_r$ and $\eta = \omega^{\gamma_{r+1}} m_{r+1} + \cdots + \omega^{\gamma_{r+d}} m_{r+d}$, then

$$\gamma = \eta \oplus \varepsilon. \tag{52}$$

Notice that $\eta < \omega^{\omega^\delta} = \lambda$.

For $i=1,\dots,r$, we can suppose that

$$\gamma_i=\omega^{\delta}l_{i1}+\omega^{\beta_2}l_{i2}+\cdots+\omega^{\beta_p}l_{ip} \tag{53}$$

for some ordinal numbers $\delta>\beta_2>\cdots>\beta_p\geq 0$ and $l_{i1},\dots,l_{ip}\in\mathbb{N}$.

Note that $l_{11}\geq\cdots\geq l_{r1}$ because $\gamma_1>\gamma_2>\cdots>\gamma_r$.

Call l_{11} the *leading natural exponent of* γ. To prove Step one we proceed by induction on the leading natural exponent l_{11} of γ.

If $l_{11}=0$, then $\varepsilon=0$, $\gamma<\omega^{\omega^\delta}=\lambda$, and $t^\gamma x=x_\gamma$. Hence $(t^\gamma x^s)x=(t^\gamma x)x^s=x_\gamma x^s\in M$.

Suppose that $l_{11}\geq 1$. For $i=1,\dots,r$,

$$\gamma_i=\omega^{\delta}\oplus\left(\omega^{\delta}(l_{i1}-1)+\omega^{\beta_2}l_{i2}+\cdots+\omega^{\beta_p}l_{ip}\right). \tag{54}$$

Define for $i=1,\dots,r$

$$\nu_i=\omega^{\delta}(l_{i1}-1)+\omega^{\beta_2}l_{i2}+\cdots+\omega^{\beta_p}l_{ip}. \tag{55}$$

Then $\gamma_i=\omega^\delta\oplus\nu_i$ for $i=1,\dots,r$. Therefore

$$\varepsilon=\omega^{\omega^\delta\oplus\nu_1}m_1+\cdots+\omega^{\omega^\delta\oplus\nu_r}m_r=\omega^{\omega^\delta}\otimes(\omega^{\nu_1}m_1+\cdots+\omega^{\nu_r}m_r).$$

Call

$$\nu=\omega^{\nu_1}m_1+\cdots+\omega^{\nu_r}m_r. \tag{56}$$

Then $\varepsilon=\omega^{\omega^\delta}\otimes\nu=\lambda\otimes\nu$. Notice that $\nu_1>\nu_2>\cdots>\nu_r$.

Hence

$$(t^\gamma x^s)x=(t^\gamma x)x^s=(t^{\eta\oplus\varepsilon}x)x^s=t^\eta(t^{\lambda\otimes\nu}x)x^s=t^\eta x t^\nu x^s=x_\eta t^\nu x^s.$$

Now looking at (54), (55), (56), we see that we can apply induction hypothesis to the leading natural exponent of ν, $l_{11}-1$. Hence there exists $m\in\mathbb{N}$ such that $(t^\nu x^s)x^m\in M$.

Thus $(t^\gamma x^s)x^{m+1}=(t^\gamma x^s)xx^m=x_\eta(t^\nu x^s x^m)\in M$.

Now the proof of the result follows with Step 2 and Step 3 as in Proposition 7.12. □

4. JF-embeddings of inversion height two

The following J-embedding appeared in [**Jat69**]. It also can be seen as a generalization of the example given by J.L. Fisher in [**Fis71**] of an embedding of the free algebra $k\langle x,y\rangle$ inside a division ring of inversion height 2. In fact his example is recovered when $|J|=1$ in the notation of the next discussion. The proofs that we provide are almost the same as the ones given by J.L. Fisher.

Let J be a set with $|J|\geq 1$. Let k be a commutative field. Set $K=k\left(t_{in}\mid i\in J,\ n\geq 1\right)$, the field of fractions of the commutative polynomial ring $k[t_{in}\mid i\in J,\ n\geq 1]$ on the variables $\{t_{in}\}_{\substack{n\geq 1\\ i\in J}}$. Let $\alpha\colon K\to K$ be the monomorphism of k-rings given by $\alpha(t_{in})=t_{in+1}$ for any $i\in J$, $n\geq 1$. Consider the skew polynomial ring $K[x;\alpha]$, its left Ore division ring of fractions Q, and E the Laurent power series ring containing Q as in Remarks 7.8(c). The elements of $\{1\}\cup\{t_{i1}\}_{i\in J}$ are right linearly independent over $\alpha(K)$ since they are right linearly independent over $\alpha\left(k[t_{in}\mid i\in J,\ n\geq 1]\right)$. Define $x_i=t_{i1}x$. From Remarks 7.8(a), we infer that the k-algebra generated by $X=\{x\}\cup\{x_i\}_{i\in J}$ is a free k-algebra, and we obtain a $(K,k,\alpha,\{1\}\cup J,\{1\}\cup\{t_i\}_{i\in J},1)$-JF-embedding $k\langle X\rangle\hookrightarrow Q\hookrightarrow E$.

We intend to show that $\mathrm{h}_E(k\langle X\rangle)=2$. For that we need the following lemma that tells us how are represented the elements of the free k-algebra $k\langle X\rangle$ inside E.

Lemma 7.23. *If $r\in k\langle X\rangle$, then $r=\sum\limits_{j=0}^{n}f_jx^j$, where $f_j=\sum\limits_{\varepsilon,\gamma}a_{\varepsilon\gamma}t_{i_11}^{\varepsilon_{i_1}}\cdots t_{i_jj}^{\varepsilon_{i_j}}$, $a_{\varepsilon\gamma}\in k$, $\varepsilon=(\varepsilon_{i_1},\dots,\varepsilon_{i_j})\in\{0,1\}^j$, $\gamma=(i_1,\dots,i_j)\in J^j$ and almost all $a_{\varepsilon\gamma}$ are zero.*

PROOF. We show by induction that if m is a monomial of degree j on X, then $m = t_{i_1 1}^{\varepsilon_{i_1}} \cdots t_{i_j j}^{\varepsilon_{i_j}} x^j$. If this is proved, the result follows by gathering monomials of the same degree.

If m has degree $-\infty$ or zero, the result is clear. If m has degree one, $m = x$ or $m = t_{i1}x$ for some $i \in J$. Hence the result holds true for degree one. Suppose that if m' is a monomial on X of degree $j \geq 1$, then $m' = t_{i_1 1}^{\varepsilon_{i_1}} \cdots t_{i_j j}^{\varepsilon_{i_j}} x^j$. Then, if m has degree $j+1$, $m = m'u$ where $u = t_{i_{j+1}1}^{\varepsilon_{i_{j+1}}} x$. Thus $m = t_{i_1 1}^{\varepsilon_{i_1}} \cdots t_{i_j j}^{\varepsilon_{i_j}} x^j t_{i_{j+1}1}^{\varepsilon_{i_{j+1}}} x = t_{i_1 1}^{\varepsilon_{i_1}} \cdots t_{i_j j}^{\varepsilon_{i_j}} t_{i_{j+1} j+1}^{\varepsilon_{i_{j+1}}} x^{j+1}$. □

PROPOSITION 7.24. $\mathrm{h}_Q(k\langle X\rangle) = \mathrm{h}_E(k\langle X\rangle) = 2$, *and* Q *is the division ring of fractions of* $k\langle X\rangle$ *inside* E.

PROOF. We already know that $\mathrm{h}_Q(k\langle X\rangle) = \mathrm{h}_E(k\langle X\rangle) \leq 2$ by Proposition 7.10 and Remarks 7.3(e).

The first assertion follows because $k \subseteq Q_1(k\langle X\rangle, Q)$ and $t_{in} = x^{n-1}t_{i1}xx^{-n} \in Q_1(k\langle X\rangle, Q)$. Hence $k[t_{in} \mid i \in J, n \geq 1][x;\alpha] \subseteq Q_1(k\langle X\rangle, Q)$, and thus $Q = Q_2(k\langle X\rangle, Q)$ since Q is the left Ore division ring of fractions of $k[t_{in} \mid i \in J, n \geq 1][x;\alpha]$.

To show that $\mathrm{h}_Q(k\langle X\rangle) = \mathrm{h}_E(k\langle X\rangle) = 2$ we need to prove that $Q_1(k\langle X\rangle, Q)$ is not a division ring. Define

$$\mathcal{S} = \left\{ \sum_{\varepsilon,\gamma} a_{\varepsilon\gamma} t_{i_1 1}^{\varepsilon_{i_1}} \cdots t_{i_j j}^{\varepsilon_{i_j}} \neq 0 \;\middle|\; \begin{array}{l} a_{\varepsilon\gamma}\in k \text{ almost all zero },\ j\in\mathbb{N}, \\ \varepsilon=(\varepsilon_{i_1},\dots,\varepsilon_{i_j})\in\{0,1\}^j,\ \gamma=(i_1,\dots,i_j)\in J^j \end{array} \right\}$$

and

$$\mathcal{M} = \{\text{finite products of elements of } \mathcal{S}\}.$$

Note that if $s \in \mathcal{S}$, the degree on t_{il} is at most one.

The set $\mathcal{M}$ is a multiplicative set of the commutative ring $k[t_{in} \mid i \in J,\ n \geq 1]$, so we can localize at $\mathcal{M}$. Call $V = k[t_{in} \mid i \in J,\ n \geq 1]_{\mathcal{M}}$. Since $\alpha(\mathcal{S}) \subseteq \mathcal{S}$, $\alpha(\mathcal{M}) \subseteq \mathcal{M}$. Hence $\alpha(V) \subseteq V$. Thus

$$U = \left\{ x^{-n} \sum_{l=0}^{\infty} v_l x^l \in E \mid v_l \in V \right\}$$

is a subring of E.

Let $r \in k\langle X\rangle \setminus \{0\} \subseteq E$. We proceed to find an expression of $r^{-1} \in E$. By Lemma 7.23, $r = \left(\sum_{j=0}^{n} f_j x^j\right) x^{j_0}$, $f_j \in \mathcal{S} \cup \{0\}$, $f_0 \neq 0$. Now

$$r^{-1} = x^{-j_0} \sum_{l=0}^{\infty} b_l x^l,$$

where $b_0 = f_0^{-1} \in V$, and if $l \geq 1$, $b_l = -f_0^{-1} \sum_{s=1}^{l} f_s \alpha^s(b_{l-s}) \in V$. Therefore $r^{-1} \in U$ and $Q_1(k\langle X\rangle, E) \subseteq U$.

On the other hand, $t_{i2} = x(t_{i1}x)x^{-2} \in Q_1(k\langle X\rangle, E)$. Now $Q_1(k\langle X\rangle, E)$ is not a division ring because $t_{i1} - t_{i2}^2 \in Q_1(k\langle X\rangle, E)$ for each $i \in J$, but its inverse, $(t_{i1} - t_{i2}^2)^{-1} \notin U$ for each $i \in J$. To prove this, suppose that $(t_{i1} - t_{i2}^2)^{-1} \in U$. Hence $(t_{i1} - t_{i2}^2)^{-1} = x^{-n} \sum_{l=0}^{\infty} v_l x^l$ for some series in U. Then $x^n(t_{i1} - t_{i2}^2)^{-1} = \sum_{l=0}^{\infty} v_l x^l$ and $\alpha^n((t_{i1} - t_{i2}^2)^{-1})x^n = \sum_{l=0}^{\infty} v_l x^l$. Thus $(t_{i(n+1)} - t_{i(n+2)}^2)^{-1} x^n = v_n x_n$. Therefore $(t_{i(n+1)} - t_{i(n+2)}^2)^{-1} \in V$. Let $f \in k[t_{in} \mid i \in J,\ n \geq 1]$ and $g \in \mathcal{M}$ be such that $(t_{i(n+1)} - t_{i(n+2)}^2)^{-1} = fg^{-1}$. Thus $g = f(t_{i(n+1)} - t_{i(n+2)}^2)$. For

$k[t_{in} \mid i \in J,\ n \geq 1]$ is a unique factorization domain, and $t_{i(n+1)} - t_{i(n+2)}^2$ is irreducible in $k[t_{in} \mid i \in J,\ n \geq 1]$, $t_{i(n+1)} - t_{i(n+2)}^2$ is a prime element in $k[t_{in} \mid i \in J,\ n \geq 1]$. Hence, if $g = g_1 \cdots g_m$, where $g_1, \dots, g_m \in \mathcal{S}$, then $(t_{i(n+1)} - t_{i(n+2)}^2) \mid g_l$ for some $l \in \{1, \dots, m\}$, but this is a contradiction because $\deg_{t_{i(n+2)}}(g_l) = 1$. $\square$

Now we present the analogous result to Corollary 7.13 which does not appear in [**HS07**].

Corollary 7.25. *Let k be a field. Let $Z = \{z_1, z_2, \dots\}$ be an infinite countable set. There exist infinite non-isomorphic division rings of fractions D of $k\langle Z\rangle$ such that $k\langle Z\rangle \hookrightarrow D$ is of inversion height two.*

Proof. First we fix the notation. Let $p \geq 2$. Set $K_p = k\,(t_{in} \mid i \in \{1, \dots, p-1\},\ n \geq 1)$, the field of fractions of the polynomial ring $k[t_{in} \mid i \in \{1, \dots, p-1\},\ n \geq 1]$ on the commuting variables $\{t_{in} \mid i \in \{1, \dots, p-1\},\ n \geq 1\}$. Let $\alpha_p \colon K_p \to K_p$ be the monomorphism of k-rings given by $\alpha_p(t_{in}) = t_{in+1}$ for $i \in \{1, \dots, p-1\}$ and $n \geq 1$. Consider $K_p[x; \alpha_p]$, and its left Ore division ring of fractions Q_p. Then we obtain the JF-embedding of k-algebras $k\langle x_0, x_1, \dots, x_{p-1}\rangle \hookrightarrow Q_p$, where $x_0 \mapsto x$ and $x_i \mapsto t_{i1}x$ for $1 \leq i \leq p-1$.

We have proved in Proposition 7.24 that it has inversion height two for each $p \geq 2$.

By Proposition 7.4, the embedding $\delta_p \colon k\langle Z\rangle \hookrightarrow Q_p$ defined by

$$\begin{array}{rcl} k\langle Z\rangle & \hookrightarrow & k\langle x_0, x_1, \dots, x_{p-1}\rangle \hookrightarrow Q_p, \\ z_j & \mapsto & x_i x_0^s \end{array}$$

where $j = (p-1)s + i$ with $1 \leq i \leq p-1$, has inversion height two.

Observe that Q_p is the division ring of fractions of $k\langle Z\rangle$ for each $p \geq 2$ by Propositions 7.4 and 7.24.

Thus it remains to prove that there does not exist a morphism of rings $f \colon Q_p \to Q_q$ for $p \neq q$ making the following diagram commutative

$$\begin{tikzcd} & Q_p \arrow[dd, "f"] \\ k\langle Z\rangle \arrow[ur, hook, "\delta_p"] \arrow[dr, hook, "\delta_q"'] & \\ & Q_q \end{tikzcd}$$

Fix $q > p$, and suppose that such an f exists. For f is a morphism of $k\langle Z\rangle$-rings, then $f(\delta_p(z_j)) = \delta_q(z_j)$ for all $j \geq 1$.

On the one hand, we have that $\delta_p(z_1) = t_{11}x$, $\delta_p(z_p) = t_{11}x^2$, $\delta_p(z_1)^{-1}\delta_p(z_p) = x$,

$$\delta_p(z_1)\delta_p(z_p)^{-1}\delta_p(z_1) = t_{11}xx^{-2}t_{11}^{-1}t_{11}x = t_{11}, \qquad \delta_p(z_{2p-1}) = \delta_p(z_{2(p-1)+1}) = x_1x_0^2 = t_{11}xx^2,$$

and

$$\delta_p(z_1)\delta_p(z_p)^{-1}\delta_p(z_1) = \delta_p(z_{2p-1})(\delta_p(z_1)^{-1}\delta_p(z_p))^{-3} = \delta_p(z_{2p-1})(\delta_p(z_p)^{-1}\delta_p(z_1))^3.$$

Thus $\delta_p(z_1) = \delta_p(z_{2p-1})(\delta_p(z_p)^{-1}\delta_p(z_1))^2$ in Q_p. Therefore

$$\delta_q(z_1) = \delta_q(z_{2p-1})(\delta_q(z_p)^{-1}\delta_q(z_1))^2 \tag{57}$$

holds in Q_q because f is a morphism of $k\langle Z\rangle$-rings.

On the other hand, $\delta_q(z_1) = t_{11}x$ and $\delta_q(z_p) = t_{p1}x$. If $2p-1 < q$, then $\delta_q(z_{2p-1}) = t_{(2p-1)1}x$. If $2p-1 \geq q$, write $2p-1 = l(q-1)+i$ with $1 \leq i \leq q-1$, and then $\delta_q(z_{2p-1}) = x_ix_0^l = t_ixx^l$.

We now show that (57) does not hold in Q_q.

Suppose that $2p-1 < q$, then (57) implies that

$$t_{11}x = t_{(2p-1)1}x((t_{p1}x)^{-1}t_{11}x)^2 = t_{(2p-1)1}x(x^{-1}t_{p1}^{-1}t_{11}xx^{-1}t_{p1}^{-1}t_{11}x) = t_{(2p-1)1}t_{p1}^{-1}t_{11}t_{p1}^{-1}t_{11}x,$$

a contradiction.

Suppose that $2p-1\geq q$, then (57) implies that

$$t_{11}x = t_ixx^l((t_{p1}x)^{-1}t_{11}x)^2 = t_ixx^l(x^{-1}t_{p1}^{-1}t_{11}xx^{-1}t_{p1}^{-1}t_{11}x) = t_ix^lt_{p1}^2t_{11}^2x.$$

Hence $t_{11}x$ equals $t_ix^lt_{p1}^2t_{11}^2x$, a polynomial of degree at least two on x in $K[x;\alpha_q]$, a contradiction. □

5. The group ring point of view

Let X be a set and F the free group on X. Let k be a division ring. In general, given an embedding $k\langle X\rangle \overset{\iota}{\hookrightarrow} Q$ where Q is a division ring, there does not exist an embedding $k[F]\hookrightarrow Q$ extending ι. In fact not even the group generated by X inside $Q^\times$ is isomorphic to F. For some examples see the JF-embeddings below. In this section we will briefly study the structure of the group G generated by X inside $Q^\times$, and some situations in which kG embeds in Q. Then we go on to see how G looks like in our examples of JF-embeddings in the foregoing sections.

In this section we consider $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0}=1)$-JF-embeddings $k\langle X\rangle\hookrightarrow Q$, and, from now on, *we assume that the elements in* $\{\alpha^n(t_i)\}_{\substack{i\in I\\ n\geq 0}}$ *are in the center of* K. The free group on $\{x_i\}_{i\in I}$ is not contained in Q because, if $i\neq i_0$, the commutativity of $\{\alpha^n(t_i)\}_{\substack{i\in I\\ n\geq 0}}$ implies that

$$x_i^2x^{-2} = (t_ixt_ix)x^{-2} = t_i\alpha(t_i) = \alpha(t_i)t_i = (xt_ix^{-1})t_ixx^{-1} = xx_ix^{-2}x_ix^{-1},$$

that is,

$$x_i^2x_{i_0}^{-1} = x_{i_0}x_ix_{i_0}^{-2}x_i. \tag{58}$$

Notice that (58) also holds if $t_{i_0}\neq 1$ by Lemma 7.9.

Let G be the subgroup of $Q\setminus\{0\}$ generated by $\{x_i\}_{i\in I}$. By definition $x_i=t_ix$, hence, in our situation, $x_{i_0}=x$; therefore

$$G=\langle\{x_i\}_{i\in I}\rangle = \langle x,\{t_i\}_{i\in I}\rangle \leq Q\setminus\{0\}.$$

Set $N=\langle\{x^nt_ix^{-n}\}_{\substack{n\in\mathbb{Z}\\ i\in I}}\rangle\leq G$. The map $Q\to Q$ given by left conjugation by x is clearly an automorphism of Q. Moreover, it coincides with α on K. We call this extension again α. So $\alpha\colon Q\to Q$, $\alpha(q)=xqx^{-1}$ for all $q\in Q$. Therefore $x^nt_ix^{-n}=\alpha^n(t_i)$ for every $n\in\mathbb{Z}$. Moreover, if $n\geq 0$, $\alpha^n(t_i)\in K$.

LEMMA 7.26. *The following statements hold:*

(i) *N is an abelian group, and it is the normal subgroup of G generated by $\{t_i\}_{i\in I}$.*
(ii) *The elements $x^nt_ix^{-n}$ are transcendental over k for each $n\in\mathbb{Z}$, $i\in I\setminus\{i_0\}$. In particular, they are torsion-free.*
(iii) *G/N is the infinite cyclic group generated by Nx.*
(iv) *$G=N\rtimes\langle x\rangle$.*

PROOF. (i) To show that N is abelian, it is enough to show that the generators commute. Now note that the commutativity of the product of elements in $\{\alpha^n(t_i)\}_{\substack{i\in I\\ n\geq 0}}$ implies the commutativity of the product of elements in $\{\alpha^n(t_i)\}_{\substack{i\in I\\ n\in\mathbb{Z}}}$. The rest of the statement is clear.

(ii) Let $p(z)\in k[z]\setminus\{0\}$ be such that $p(\alpha^n(t_i))=0$ for some $n\in\mathbb{Z}$, $i\in I$. Since $\alpha^m(\alpha^n(t_i))=x^m\alpha^n(t_i)x^{-m}$ is a root of $p(z)$ for each $m\in\mathbb{N}$, we obtain that there exist $m_1<m_2\in\mathbb{N}$ such that $\alpha^{m_1+n}(t_i)=\alpha^{m_2+n}(t_i)$. This implies that $\alpha^{m_2-m_1}(t_i)=t_i$, a contradiction because $t_i\notin\alpha(K)$ since $\{t_i\}_{i\in I}$ are right linearly independent over $\alpha(K)$ and $i\neq i_0$.

(iii) Since $xt_i = \alpha(t_i)x$ for any $i \in I$, G/N is generated by Nx. Suppose that there exists $n \geq 1$ such that $x^n \in N$. Let $i \in I \setminus \{i_0\}$. By (i), $\alpha(\alpha^{n-1}(t_i)) = \alpha^n(t_i) = x^n t_i x^{-n} = t_i$, a contradiction.

(iv) We know that $N \lhd G$ and that G/N is infinite cyclic generated by Nx. Hence $G = N\langle x\rangle$. We prove that $N \cap \langle x\rangle = \{1\}$. Let $r, n_1, \dots, n_l, m_1, \dots, m_l \in \mathbb{Z}$, $i_1, \dots, i_l \in I$, such that $x^r = x^{n_1} t_{i_1}^{m_1} x^{-n_1} \cdots x^{n_l} t_{i_l}^{m_l} x^{-n_l}$. Let $n = \min\{n_1, \dots, n_l\}$. Then

$$x^r = x^{-n}x^r x^n = x^{n_1-n}t^{m_1}x^{-n_1+n}\cdots x^{n_l-n}t^{m_l}x^{-n_l+n} = \alpha^{n_1-n}(t_{i_1}^{m_1})\cdots\alpha^{n_l-n}(t_{i_l}^{m_l}) \in K.$$

Therefore $r = 0$, and $x^r = 1$. □

Proposition 7.27. *Suppose that the evaluation homomorphism*

$$ev\colon \alpha(K)[z_i \mid i \in I \setminus \{i_0\}] \to Q,$$

where $z_i \mapsto t_i$, and $a \mapsto a$ for all $a \in \alpha(K)$ is injective. The following statements hold:

(i) *N is a torsion-free abelian group.*
(ii) *The group rings $k[N]$ and $k[G] = k[N][x, x^{-1}; \alpha]$ are contained in Q.*
(iii) *$k[G]$ is a two-sided Ore domain with the division ring of fractions of $k\langle X\rangle$ inside Q as Ore division ring of fractions. In particular $\mathrm{h}_Q(k[G]) = 1$.*

Proof. (i) We already know that N is an abelian group by Lemma 7.26(i). Suppose that there exist $s > 1$, and integers $n_1 < n_2 < \cdots < n_l$ such that

$$\left(\alpha^{n_1}(t_{i_{11}}^{\varepsilon_{i_{11}}}\cdots t_{i_{r_1 1}}^{\varepsilon_{i_{r_1 1}}})\cdots\alpha^{n_l}(t_{i_{1l}}^{\varepsilon_{i_{1l}}}\cdots t_{i_{r_l l}}^{\varepsilon_{i_{r_l l}}})\right)^s = 1,$$

where $\varepsilon_{i_{uv}} = \pm 1$. Then

$$\left(t_{i_{11}}^{\varepsilon_{i_{11}}}\cdots t_{i_{r_1 1}}^{\varepsilon_{i_{r_1 1}}}\right)^s = \left(\alpha^{n_2-n_1}(t_{i_{12}}^{-\varepsilon_{i_{12}}}\cdots t_{i_{r_2 2}}^{-\varepsilon_{i_{r_2 2}}})\right)^s\cdots\left(\alpha^{n_l-n_1}(t_{i_{1l}}^{-\varepsilon_{i_{1l}}}\cdots t_{i_{r_l l}}^{-\varepsilon_{i_{r_l l}}})\right)^s \in \alpha(K).$$

If $\varepsilon_{i_{11}} = \cdots = \varepsilon_{i_{r_1 1}} = 1$, there is a contradiction. If $\varepsilon_{i_{11}} = \cdots = \varepsilon_{i_{r_1 1}} = -1$, we invert the left and right hand side of the equality to get a contradiction. If some $\varepsilon_{i_{u1}}$ are 1 and some -1, we move the negative to the right hand side to obtain a contradiction.

(ii) Suppose that

$$d_1 n_1 + \cdots + d_m n_m = 0, \tag{59}$$

where $n_l \in N,\ n_l \neq n_s$, if $l \neq s$ and $d_l \in k$.

Only a finite number of t_i appear in the expression of n_l. Call them $t_{i_1}, \dots, t_{i_r}$. We may suppose that

$$n_l = \alpha^{r_{l1}}(t_{i_1}^{s_{li_1 1}}\cdots t_{i_r}^{s_{li_r 1}})\cdots\alpha^{r_{lu_l}}(t_{i_1}^{s_{li_1 u_l}}\cdots t_{i_r}^{s_{li_r u_l}}),\quad l = 1, \dots, m. \tag{60}$$

We prove that $d_1 = \cdots = d_m = 0$ by induction on m.

If $m = 1$, the result follows because Q is a domain.

Suppose that the result holds for $m - 1 \geq 1$. Conjugating by a suitable power of x, using that the elements in $\{\alpha^n(t_i)\}_{\substack{i\in I\\ n\geq 0}}$ are in the center of K and reordering the summands, we may suppose that $0 \leq r_{l1} < r_{l2} < \cdots < r_{lu_l}$, $0 \leq r_{11} \leq r_{21} \leq \cdots \leq r_{m1}$.

If $(r_{l1}, s_{li_1 1}, \dots, s_{li_r 1})$ is the same for all l, then we can factor out $\alpha^{r_{l1}}(t_{i_1}^{s_{li_1 1}}\cdots t_{i_r}^{s_{li_r 1}})$ from (59). Since $n_l \neq n_s$, if $l \neq s$, we go on this way until we find j_0 such that not all $(r_{lj_0}, s_{li_1 j_0}, \dots, s_{li_r j_r})$ are equal. So we can suppose that the $(r_{l1}, s_{li_1 1}, \dots, s_{li_r 1})$ are not equal for all l in (60).

If $r_{11} = r_{21} = \cdots = r_{m1}$, since α and $\alpha^{r_{11}}$ are injective, we could express (59) as

$$d_1 t_{i_1}^{s_{1i_1 1}}\cdots t_{i_r}^{s_{1i_r 1}}\alpha(a_1) + \cdots + d_m t_{i_1}^{s_{mi_1 1}}\cdots t_{i_r}^{s_{mi_r 1}}\alpha(a_m) = 0,$$

where $\alpha(a_l) \in N$. Multiplying by a certain product of t_i's so that everything is in the image of ev, putting together all the resulting $(s_{li_11}, \dots, s_{li_r1})$ which are equal, applying that ev is injective and the induction hypothesis we get the result.

Hence suppose that there exists l_0 such that $r_{11} = r_{21} = \cdots = r_{l_0-11} < r_{l_01} \leq \ldots \leq r_{m1}$.

We claim that $0 \neq d_1n_1+\cdots+d_{l_0-1}n_{l_0-1} \notin \alpha^{r_{l_01}}(K)$. Suppose that there exists $b_0 \in K \setminus \{0\}$ such that

$$\begin{aligned} d_1n_1 + \cdots + d_{l_0-1}n_{l_0-1} &= \alpha^{r_{l_01}}(b_0) \\ &= \alpha^{r_{11}}(t_{i_1}^{s_{1i_11}} \cdots t_{i_r}^{s_{1i_r1}})\alpha^{r_{11}+1}(b_1) + \cdots + \\ &\quad\ \alpha^{r_{11}}(t_{i_1}^{s_{l_0-1i_11}} \cdots t_{i_r}^{s_{l_0-1i_r1}})\alpha^{r_{11}+1}(b_{i_0-1}). \end{aligned}$$

Hence, $t_{i_1}^{s_{1i_11}} \cdots t_{i_r}^{s_{1i_r1}}\alpha(b_1)+\cdots+t_{i_1}^{s_{l_0-1i_11}} \cdots t_{i_r}^{s_{l_0-1i_r1}}\alpha(b_{i_0-1})-\alpha^{r_{l_01}-r_{11}}(b_0) = 0$, a contradiction with the injectivity of ev.

Now, since $d_{l_0}n_{l_0}+\cdots+d_mn_m \in \alpha^{r_{l_01}}(K)$, the claim implies that $d_1n_1+\cdots+d_{l_0-1}n_{l_0-1} = 0$ and $d_{l_0}n_{l_0} + \cdots + d_mn_m = 0$. By the induction hypothesis, it follows that $k[N] \subseteq Q$.

Observe that given any element q in the subring of Q generated by $k[N]$ and $\{x, x^{-1}\}$, conjugating by a suitable power r of x, we get that $x^rqx^{-r} \in K[x; \alpha]$. This shows that $k[G] = k[N][x, x^{-1}; \alpha]$ is contained in Q.

(iii) Since N is torsion free abelian by (i), then $k[N]$ is a two-sided Ore domain. Hence $k[G] \cong k[N][x, x^{-1}; \alpha]$ is a two-sided Ore domain by Proposition 3.19. The universal property of the Ore localization implies that the Ore division ring of fractions of $k[G]$ is contained in Q. By the construction of X and G, it is clear that the division ring of fractions of $k\langle X\rangle$ inside Q is the same as the division ring of fractions of $k[G]$. $\square$

Notice that the group ring $k[N]$ is not always contained in the division ring of fractions of $k\langle X\rangle$.

Example 7.28. Let k be a field. Let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$. Let $\alpha\colon K \to K$ be given by $t \mapsto t^2 + t$. Then $\alpha(K) = k(t^2 + t)$, the field of fractions of $k[t^2 + t] \subseteq k[t]$. Notice that $k[t^2 + t] \subseteq k[t]$ is an integral extension and that $k[t^2 + t] \cong k[t]$. Therefore $k[t^2 + t]$ is integrally closed in its field of fractions. We claim that $t^n \notin \alpha(K)$ for all $n \geq 1$. Suppose that $t^n \in \alpha(K) = k(t^2 + t)$ for some $n \geq 1$. Since t^n is integral over $k[t^2 + t]$ and $k[t^2 + t]$ is integrally closed, our assumption on t^n implies that $t^n \in k[t^2 + t]$. Hence there exist $a_0, \dots, a_m \in k$, with $a_m \neq 0$, such that

$$t^n = a_m(t^2 + t)^m + \cdots + a_1(t^2 + t) + a_0.$$

But this is not possible because the foregoing equality forces $t^{2m} = t^n$, and if $0 \leq j \leq m$ is the least one such that $a_j \neq 0$, then there exists a monomial on t^j. So the claim is proved.

Consider now the $(K, k, \alpha, I = \{1, 2\}, \{1, t\}, 1)$-JF-embedding. This gives the free k-algebra on two generators inside $K[x; \alpha]$ generated by x and tx. In this situation $N = \langle\{x^ntx^{-n}\}_{n\in\mathbb{Z}}\rangle = \langle\{\alpha^n(t)\}_{n\in\mathbb{Z}}\rangle$. Note that t is algebraic over $\alpha(K)$ for t is a root of the polynomial $z^2 + z - (t^2 + t)$ over the polynomial ring $\alpha(K)[z]$. Therefore the ring $k[N]$ is not contained in Q because $t^2, t, xtx^{-1} = \alpha(t) \in N$ and $t^2 + t - (xtx^{-1}) = 0$, so these elements are not k-linear independent inside Q. $\square$

5.1. The examples revisited. Observe that our examples of JF-embeddings in Sections 3 and 4 satisfy that the elements in the set $\{\alpha^n(t_i)\}_{\substack{i\in I\\ n\geq 0}}$ are in the center of K. So we can specialize the previous results to these examples.

(a) Consider the $(K, k, \alpha, \{1\} \cup J, \{1\} \cup \{t_{i1}\}_{i\in J}, 1)$-JF-embedding of Section 4. Consider the group G generated by $\{x_i\}_{i\in\{1\}\cup J}$. $G = \langle\{x_i\}_{i\in\{1\}\cup J}\rangle = \langle\{x\} \cup \{x_i\}_{i\in J}\rangle \leq Q \setminus \{0\}$. Note that

the evaluation map
$$ev\colon k(t_{in} \mid i \in J,\ n \geq 2)[z_i \mid i \in J] \to Q,$$
$z_i \mapsto t_{i1},\ a \mapsto a$ for all $a \in k$, is injective. Therefore N is torsion free abelian by Proposition 7.27(i). Furthermore, N is the free abelian group on $\{t_{in}\}_{\substack{n\geq 1\\ i\in J}} \cup \{x^n t_{i1} x^{-n}\}_{\substack{n<0\\ i\in J}}$ since any relation among the generators imply a relation among the t_{in}'s. If we relabel $x^n t_{i1} x^{-n},\ n < 0$, as t_{in+1}, we get that N is the free abelian group on $\{t_{in}\}_{\substack{n\in\mathbb{Z}\\ i\in J}}$. And $G = N \rtimes \langle x\rangle$, where x acts as $xt_{in} = t_{in+1}x$ for all $n \in \mathbb{Z},\ i \in J$. That is,
$$G = \langle x, t_{in}, i \in J, n \in \mathbb{Z} \mid t_{in+1} = xt_{in}x^{-1},\ t_{in}t_{jm} = t_{jm}t_{in}\rangle.$$

Again Proposition 7.27 implies that $k[N]$ and $k[G] = k[N][x, x^{-1}; \alpha]$ embed in Q, and that, together with Proposition 7.24, Q is the Ore division ring of fractions of $k[G]$. Hence $\mathrm{h}_Q(k[G]) = 1$.

(b) Consider the $(K = k(t), k, \alpha_n, \{0, 1, \dots, n-1\}, \{t_i\}_{i=0}^{n-1}, 1)$-JF-embedding of Section 3.1. Then $G = \langle x_0 = x, x_1, \dots, x_{n-1}\rangle = \langle x, t\rangle \subseteq Q \setminus \{0\}$, and $N = \langle x^m t x^{-m} \mid m \in \mathbb{Z}\rangle$. Consider $\mathbb{Z}[\frac{1}{n}]$ as a group with multiplicative notation, i.e. $\mathbb{Z}[\frac{1}{n}] = \{y^{n^l q} \mid q, l \in \mathbb{Z}\}$.

The proof of the following result is analogous to the one of Proposition 7.32.

Proposition 7.29. *The following statements hold:*

(i) *If $z \in N$, there exist $l, m \in \mathbb{Z}$ such that $z = x^m t^l x^{-m}$.*
(ii) *The map $\mathbb{Z}[\frac{1}{n}] \to N$, $y^{n^l q} \mapsto x^l t^q x^{-l}$ is an isomorphism of groups.*
(iii) *$G \cong \mathbb{Z}[\frac{1}{n}] \rtimes C$, where $C = \langle x\rangle$ is the infinite cyclic group and $xy^{n^l q} = y^{n^{l+1} q}x$.*
(iv) *$k[N]$ embeds in Q.*
(v) *$k[G] = k[N][x, x^{-1}; \alpha] \hookrightarrow Q$, and Q is the division ring of fractions of $k[G]$.* □

(c) Let λ be an infinite cardinal. Consider the monoid M_λ of Section 3.2.

The following is the general construction of the universal group of a monoid specialized to the monoid M_λ.

Definition 7.30. Define the equivalence relation $\sim$ of pairs $(\gamma, \delta), (\varepsilon, \eta) \in M_\lambda \times M_\lambda$
$$(\gamma, \delta) \sim (\varepsilon, \eta) \text{ iff } \exists\ \mu \in M_\lambda \text{ such that } \gamma \oplus \eta \oplus \mu = \varepsilon \oplus \delta \oplus \mu. \tag{61}$$

Since $(M_\lambda, \oplus)$ is cancellative, (61) is equivalent to
$$\gamma \oplus \eta = \varepsilon \oplus \delta. \tag{62}$$

We denote by $\gamma \ominus \delta$ the equivalence class of (γ, δ).

Let H_λ denote the set of equivalence classes, i.e. $H_\lambda = \{\gamma \ominus \delta \mid (\gamma, \delta) \in M_\lambda \times M_\lambda\}$. The set H_λ can be endowed with a group structure via the binary operation
$$(\gamma \ominus \delta) \oplus (\varepsilon \ominus \eta) = (\gamma \oplus \varepsilon) \ominus (\delta \oplus \eta),$$
for all $(\gamma, \delta), (\varepsilon, \eta) \in M_\lambda \times M_\lambda$. The zero element of H_λ is $0 \ominus 0 = \gamma \ominus \gamma$ for any $\gamma \in M_\lambda$. The opposite of $\gamma \ominus \delta$ is $\delta \ominus \gamma$. Moreover, $M_\lambda \hookrightarrow H_\lambda$ via $\gamma \mapsto \gamma \ominus 0$.

The group H_λ is called the *universal group of the monoid M_λ*. □

The group H_λ can be turned into an ordered ring (see Definition 2.22) as follows

Lemma 7.31. *The group H_λ can be endowed with an ordered ring structure via the binary operation*
$$(\gamma \ominus \delta) \otimes (\varepsilon \ominus \eta) = (\gamma \otimes \varepsilon \oplus \delta \otimes \eta) \ominus (\delta \otimes \varepsilon \oplus \gamma \otimes \eta)$$

for all $\gamma \ominus \delta, \varepsilon \ominus \eta \in H_\lambda$. In particular, H_λ is a (commutative) domain and $(H_\lambda, \oplus)$ is an ordered group with positive cone $P = \{\gamma \ominus \delta \mid \gamma > \delta\}$.

Proof. First we prove that the product is well defined. Observe that $\otimes$ is commutative. Suppose that $\gamma, \gamma', \delta, \delta', \varepsilon, \eta \in M_\lambda$ and $\gamma \ominus \delta = \gamma' \ominus \delta' \in H_\lambda$. Then

$$(\gamma \ominus \delta) \otimes (\varepsilon \ominus \eta) = (\gamma \otimes \varepsilon \oplus \delta \otimes \eta) \ominus (\delta \otimes \varepsilon \oplus \gamma \otimes \eta)$$

$$(\gamma' \ominus \delta') \otimes (\varepsilon \ominus \eta) = (\gamma' \otimes \varepsilon \oplus \delta' \otimes \eta) \ominus (\delta' \otimes \varepsilon \oplus \gamma' \otimes \eta).$$

Both expressions are the same if and only if

$$\gamma \otimes \varepsilon \oplus \delta \otimes \eta \oplus \delta' \otimes \varepsilon \oplus \gamma' \otimes \eta = \gamma' \otimes \varepsilon \oplus \delta' \otimes \eta \oplus \delta \otimes \varepsilon \oplus \gamma \otimes \eta,$$

if and only if

$$(\gamma \oplus \delta') \otimes \varepsilon \oplus (\delta \oplus \gamma') \otimes \eta = (\gamma' \oplus \delta) \otimes \varepsilon \oplus (\delta' \oplus \gamma) \otimes \eta,$$

where we have used that the natural sum and product of ordinals satisfy the distributive law. This last equality holds because the equality of classes imply that $\gamma \oplus \delta' = \gamma' \oplus \delta$. Now the product is well defined, indeed if $\varepsilon', \eta' \in M_\lambda$ and $\varepsilon' \ominus \eta' = \varepsilon \ominus \eta$, applying twice what we have just proved, we get

$$(\gamma \ominus \delta) \otimes (\varepsilon \ominus \eta) = (\gamma' \ominus \delta') \otimes (\varepsilon \ominus \eta) = (\gamma' \ominus \delta') \otimes (\varepsilon' \ominus \eta')$$

as desired.

The associative and the distributive laws follow straightforward from the definition and the fact that they hold in M_λ.

Clearly the identity element is $1 \ominus 0$ because $(1 \ominus 0) \otimes (\gamma \ominus \delta) = \gamma \otimes \delta$ for all $\gamma, \delta \in M_\lambda$. Therefore H_λ is a ring.

Let $P = \{\gamma \ominus \delta \mid \gamma > \delta\}$. To prove that H_λ is an ordered ring we have to show that the next three properties hold

$$P \oplus P \subseteq P, \qquad P \cup (\ominus P) = H_\lambda \setminus \{0 \ominus 0\}, \qquad P \otimes P \subseteq P.$$

Let $\gamma \ominus \delta, \varepsilon \ominus \eta \in P$. Consider $(\gamma \ominus \delta) \oplus (\varepsilon \ominus \eta) = (\gamma \oplus \varepsilon) \ominus (\delta \oplus \eta)$. Then $\gamma > \delta$ and $\varepsilon > \eta$ because they are elements from P. Since M_λ is an ordered monoid by Lemma 7.21, we obtain that $\gamma \oplus \varepsilon > \delta \oplus \eta$, i.e. the sum of elements in P is in P.

Observe that $\ominus P = \{\varepsilon \ominus \eta \mid \varepsilon < \eta\}$. Note that the zero element correspond exactly to the classes $\gamma \ominus \gamma$ with $\gamma \in M_\lambda$. Now P and P' have no intersection because if $\gamma \ominus \delta \in P$, then $\gamma' > \delta'$ for all representatives $\gamma' \ominus \delta'$ of the class $\gamma \ominus \delta$.

Now we show that $P \otimes P \subseteq P$. To do this we need to take a closer look at how the elements in M_λ look like. Let $\gamma, \delta, \varepsilon, \eta \in M_\lambda$ such that $\gamma \ominus \delta, \varepsilon \ominus \eta \in P$. We may express the normal form of these ordinal numbers with the same set of exponents. Thus suppose that

$$\gamma = \omega^{\gamma_1} m_1 + \cdots + \omega^{\gamma_r} m_r,\ \delta = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_r} n_r,\ \varepsilon = \omega^{\gamma_1} p_1 + \cdots + \omega^{\gamma_r} p_r,\ \eta = \omega^{\gamma_1} q_1 + \cdots + \omega^{\gamma_r} q_r$$

where $\gamma_1 > \cdots > \gamma_r$ are ordinal numbers and $m_i, n_i, p_i, q_i \in \mathbb{N}$ for all $i \in \{1, \dots, r\}$. Since $\gamma > \delta$ and $\varepsilon > \eta$, there exist $i_0, j_0 \in \{1, \dots, r\}$ such that

$$m_{i_0} > n_{i_0} \text{ and } m_i = n_i \text{ for all } i < i_0, \qquad p_{j_0} > q_{j_0} \text{ and } p_i = q_i \text{ for all } i < j_0. \tag{63}$$

Now $(\gamma \ominus \delta) \otimes (\varepsilon \ominus \eta) = (\gamma \otimes \varepsilon \oplus \delta \otimes \eta) \ominus (\delta \otimes \varepsilon \oplus \gamma \otimes \eta)$. If we prove that $\gamma \otimes \varepsilon \oplus \delta \otimes \eta > \delta \otimes \varepsilon \oplus \gamma \otimes \eta$ we are done. So we make the computations

$$\gamma \otimes \varepsilon \oplus \delta \otimes \eta = \bigoplus_{i,j=1}^{r} \omega^{\gamma_i \oplus \gamma_j} (m_i p_j + n_i q_j), \qquad \delta \otimes \varepsilon \oplus \gamma \otimes \eta = \bigoplus_{i,j=1}^{r} \omega^{\gamma_i \oplus \gamma_j} (n_i p_j + m_i q_j).$$

We claim that the coefficient of $\omega^{\gamma_{i_0}\oplus\gamma_{j_0}}$ in $\gamma\otimes\varepsilon\oplus\delta\otimes\eta$ is greater than the coefficient of $\omega^{\gamma_{i_0}\oplus\gamma_{j_0}}$ in $\delta\otimes\varepsilon\oplus\gamma\otimes\eta$ and that the coefficients of $\omega^{\gamma_i\otimes\gamma_j}$ in $\gamma\otimes\varepsilon\oplus\delta\otimes\eta$ and $\delta\otimes\varepsilon\oplus\gamma\otimes\eta$ are the same for all i,j such that $\omega^{\gamma_i\oplus\gamma_j}>\omega^{\gamma_{i_0}\oplus\gamma_{j_0}}$.

Observe that $m_ip_j+n_iq_j\geq n_ip_j+m_iq_j$ if and only if $(m_i-n_i)(p_j-q_j)\geq 0$. In this expression, for i_0,j_0 we get that $(m_{i_0}-n_{i_0})(p_{j_0}-q_{j_0})>0$ by (63). For any different pair i,j such that $\gamma_i\oplus\gamma_j\geq\gamma_{i_0}\oplus\gamma_{j_0}$, either $m_i=n_i$ or $p_j=q_j$ by (63), and thus $m_ip_j+n_iq_j=n_ip_j+m_iq_j$. This proves our claim and that H_λ is an ordered ring.

Every ordered ring, and in particular H_λ, is a domain. The proof is as follows. Let $\gamma\ominus\delta,\varepsilon\ominus\eta\in H_\lambda\setminus\{0\ominus 0\}$. Then there exist signs $\oplus$ or $\ominus$ such that $\gamma\ominus\delta$ or $\ominus(\gamma\ominus\delta)$, and $\varepsilon\ominus\eta$ or $\ominus(\varepsilon\ominus\eta)$ belong to P. Notice that $(\gamma\ominus\delta)\otimes(\varepsilon\ominus\eta)=0$ if and only if $(\ominus(\gamma\ominus\delta))\otimes(\ominus(\varepsilon\ominus\eta))=0$ if and only if $(\ominus(\gamma\ominus\delta))\otimes(\varepsilon\ominus\eta)=(\gamma\ominus\delta)\otimes(\ominus(\varepsilon\ominus\eta))=0$.

So we may suppose that $\gamma\ominus\delta$ and $\varepsilon\ominus\eta$ are in P, and thus their product is not zero because $P\otimes P\subseteq P$, as desired. $\square$

Consider the multiplicative subset of H_λ

$$S=\{1,\lambda,\lambda\otimes\lambda=\lambda^{\otimes^2},\dots,\lambda\otimes\overset{(n}{\cdots}\otimes\lambda=\lambda^{\otimes^n},\dots\}.$$

We can localize H_λ at S to obtain

$$S^{-1}H_\lambda=\left\{\frac{\gamma\ominus\delta}{\lambda^{\otimes^n}}\mid\gamma,\delta\in M_\lambda,\ n\in\mathbb{N}\right\}.$$

We will express the elements of $S^{-1}H_\lambda$ as $\lambda^{\otimes^n}(\gamma\ominus\delta)$ with $n\in\mathbb{Z}$. Notice that when $n>0$, $\lambda^{\otimes^n}(\gamma\ominus\delta)=(\lambda^{\otimes^n}\ominus 0)\otimes(\gamma\ominus\delta)$, and if $n<0$, $\lambda^{\otimes^n}(\gamma\ominus\delta)=\frac{\gamma\ominus\delta}{\lambda^{\otimes^{-n}}}$.

Consider the $(K=Q^l_{\mathrm{cl}}(R_\lambda),k,\alpha,\lambda,\{t^\gamma\}_{\gamma<\lambda},1)$-JF-embedding of Section 3.2. Then $G=\langle\{x_\gamma\}_{\gamma<\lambda}\rangle=\langle\{x,t^\gamma\}_{\gamma<\lambda}\rangle\leq Q\setminus\{0\}$ and $N=\langle\{x^mt^\gamma x^{-m}\mid m\in\mathbb{Z},\ \gamma<\lambda\}\rangle$.

Proposition 7.32. *The following statements hold:*

(i) *There is an injective morphism of groups $H_\lambda\to Q^\times$ defined by $\gamma\ominus\delta\mapsto(t^\delta)^{-1}t^\gamma$.*

(ii) *We will denote $(t^\delta)^{-1}t^\gamma$ by $t^{\gamma\ominus\delta}$. If $z\in N$, there exist $m\in\mathbb{Z}$, $\gamma\ominus\delta\in H_\lambda$ such that $z=x^mt^{\gamma\ominus\delta}x^{-m}$.*

(iii) *The map $\psi\colon S^{-1}H_\lambda\to N$, $\lambda^{\otimes^m}(\gamma\ominus\delta)\mapsto x^mt^{\gamma\ominus\delta}x^{-m}$ is an isomorphism of abelian groups.*

(iv) *Consider $S^{-1}H_\lambda$ as a group with multiplicative notation i.e.*

$$S^{-1}H_\lambda=\{y^{\lambda^{\otimes^n}(\gamma\ominus\delta)}\mid n\in\mathbb{Z},\ \gamma,\delta\in M_\lambda\}.$$

Then $G\cong S^{-1}H_\lambda\rtimes C$, where $C=\langle x\rangle$ is the infinite cyclic group and $xy^{\lambda^{\otimes^m}(\gamma\ominus\delta)}=y^{\lambda^{\otimes^{m+1}}(\gamma\ominus\delta)}x$.

(v) *$k[N]$ embeds in Q.*

(vi) *$k[G]=k[N][x,x^{-1};\alpha]\hookrightarrow Q$, and Q is the division ring of fractions of $k[G]$.*

Proof. (i) Straightforward.

(ii) Let $n_1<n_2\in\mathbb{Z},\ \gamma_1\ominus\delta_1,\ \gamma_2\ominus\delta_2\in H_\lambda$. Then

$$\begin{aligned}x^{n_1}t^{\gamma_1\ominus\delta_1}x^{-n_1}x^{n_2}t^{\gamma_2\ominus\delta_2}x^{-n_2}&=x^{n_1}t^{\gamma_1\ominus\delta_1}t^{\lambda^{\otimes^{n_2-n_1}}(\gamma_2\ominus\delta_2)}x^{-n_2+n_2-n_1}\\&=x^{n_1}t^{(\gamma_1\oplus\lambda^{\otimes^{n_2-n_1}}\otimes\gamma_2)\ominus(\delta_1\oplus\lambda^{\otimes^{n_2-n_1}}\otimes\delta_2)}x^{-n_1}.\end{aligned}$$

Observe that the elements of N are finite products of elements of the form $x^mt^{\gamma\ominus 0}x^{-m}$ and their inverses. Thus the foregoing observation proves the result since N is abelian by Lemma 7.26.

(iii) The map ψ is well defined. Let $n > m \in \mathbb{Z}$, $\gamma\ominus\delta$, $\gamma'\ominus\delta'$, with $\lambda^{\otimes^n}(\gamma\ominus\delta) = \lambda^{\otimes^m}(\gamma'\ominus\delta')$.

$$\begin{aligned}
\psi(\lambda^{\otimes^n}(\gamma\ominus\delta)) &= x^n t^{\gamma\ominus\delta} x^{-n} \\
&= x^m x^{n-m} t^{\gamma\ominus\delta} x^{m-n} x^{-m} \\
&= x^m t^{\lambda^{\otimes^{(n-m)}}\otimes(\gamma\ominus\delta)} x^{-m} \\
&= x^m t^{\gamma'\ominus\delta'} x^{-m} \\
&= \psi(\lambda^{\otimes^m}(\gamma'\ominus\delta')).
\end{aligned}$$

The map ψ is a morphism,

$$\begin{aligned}
\psi(\lambda^{\otimes^n}(\gamma\ominus\delta)\oplus\lambda^{\otimes^m}(\varepsilon\ominus\eta)) &= \psi(\lambda^{\otimes^m}(\lambda^{\otimes^{n-m}}(\gamma\ominus\delta)\oplus(\varepsilon\ominus\eta))) \\
&= x^m t^{\lambda^{\otimes^{n-m}}(\gamma\ominus\delta)\oplus(\varepsilon\ominus\eta)} x^{-m} \\
&= x^m x^{n-m} t^{\gamma\ominus\delta} x^{-n+m} x^{-m} x^m t^{\varepsilon\ominus\eta} x^{-m} \\
&= x^n t^{\gamma\ominus\delta} x^{-n} x^m t^{\varepsilon\ominus\eta} x^{-m} \\
&= \psi(\lambda^{\otimes^n}(\gamma\ominus\delta))\psi(\lambda^{\otimes^m}(\varepsilon\ominus\eta)).
\end{aligned}$$

The morphism ψ is injective because $1 = \psi(\lambda^{\otimes^n}(\gamma\ominus\delta)) = x^n t^{\gamma\ominus\delta} x^{-n}$. Hence $t^{\gamma\ominus\delta} = 1$. Therefore $\gamma = \delta$ and $\lambda^{\otimes^n}(\gamma\ominus\delta) = 0$.

The morphism ψ is onto by (ii).

(iv) By (iii) and Lemma 7.26(iv).

(v) Suppose that

$$a_1 x^{m_1} t^{\gamma_1\ominus\delta_1} x^{-m_1} + \cdots + a_r x^{m_r} t^{\gamma_r\ominus\delta_r} x^{-m_r} = 0$$

with $x^{m_i} t^{\gamma_i\ominus\delta_i} x^{-m_i} \neq x^{m_j} t^{\gamma_j\ominus\delta_j} x^{-m_j}$. Let $m = \min\{m_1,\dots,m_r\}$. Then

$$a_1 x^{-m+m_1} t^{\gamma_1\ominus\delta_1} x^{-m_1+m} + \cdots + x^{-m+m_r} t^{\gamma_r\ominus\delta_r} x^{-m_r+m} = 0.$$

Since $-m + m_i \geq 0$, then $\lambda^{\otimes^{m_i-m}} \otimes (\gamma_i\ominus\delta_i) \in M_\lambda$ for all i, and

$$a_1 t^{\lambda^{\otimes^{m_1-m}}\otimes(\gamma_1\ominus\delta_1)} + \cdots + a_r t^{\lambda^{\otimes^{m_r-m}}\otimes(\gamma_r\ominus\delta_r)} = 0.$$

Thus there are $i \neq j$ such that $\lambda^{\otimes^{m_i-m}}\otimes(\gamma_i\ominus\delta_i) = \lambda^{\otimes^{m_j-m}}\otimes(\gamma_j\ominus\delta_j)$, unless $a_1 = \cdots = a_r = 0$. This implies that $\lambda^{\otimes^{m_i}} \otimes (\gamma_i\ominus\delta_i) = \lambda^{\otimes^{m_j}} \otimes (\gamma_j\ominus\delta_j)$.

(vi) Note that given q that belongs to the subring of Q generated by $k[N]$ and $\{x, x^{-1}\}$, there exists $r \in \mathbb{N}$ such that $x^r q x^{-r} \in K[x;\alpha]$. This shows that the powers of x are $k[N]$-linearly independent. ☐

6. JFL-embeddings

In this section we use J-embeddings to obtain embeddings of the free group algebra of inversion height at most two. The techniques displayed to get these embeddings are from the paper by A.I. Lichtman [**Lic84**]. We also use the Magnus-Fox embedding of Section 9 in Chapter 1.

We begin with a result implicit in the proof of [**Lic84**, Proposition 4].

LEMMA 7.33. *Let R be a valuation ring with valuation $v\colon R \to \mathbb{N}\cup\{\infty\}$. Consider the completion $\widehat{R}$ with respect to v. Then $\widehat{R}$ is again a valuation ring with valuation also denoted by v. Let k be a subring of $\widehat{R}$ such that $v_{|_{k\setminus\{0\}}} = 0$. Let $X \subseteq \widehat{R}$. Suppose that k and X generate a free k-ring $k\langle X\rangle$ inside $\widehat{R}$. If there exists $m \geq 1$ with*

$$1 \leq v(x) \leq m \quad \textit{for all } x \in X, \tag{64}$$

then $k\langle\langle X\rangle\rangle \hookrightarrow \widehat{R}$.

PROOF. Certainly $\widehat{R}$ is a valuation ring by Lemma 1.51. For each $n \geq 1$, set $\widehat{I_n} = \{z \in \widehat{R} \mid v(z) \geq n\}$, a sequence of ideals of $\widehat{R}$, and $J_n = k\langle X\rangle \cap \widehat{I_n}$, a sequence of ideals of $k\langle X\rangle$. Observe that the completion of $\widehat{R}$ with respect to $\widehat{I_n}$, that is, with respect to v, is again $\widehat{R}$ by Lemma 1.51. Hence the completion of $k\langle X\rangle$ with respect to $\{J_n\}_{n\geq 1}$ is contained in $\widehat{R}$. By (64), $\langle X\rangle^n \subseteq J_n$. On the other hand, since $v_{|k\setminus\{0\}} = 0$, we have that $J_m \subseteq \langle X\rangle$. Hence $J_{nm} \subseteq \langle X\rangle^n$ for all $n \geq 1$. Therefore, $k\langle\langle X\rangle\rangle$, the completion of $k\langle X\rangle$ with respect to $\langle X\rangle^n$, and the completion of $k\langle X\rangle$ with respect to $\{J_n\}_{n\geq 1}$ are isomorphic. Therefore $k\langle\langle X\rangle\rangle \subseteq \widehat{R}$. $\square$

The next Corollary is [**Lic84**, Proposition 4].

COROLLARY 7.34. *Let R be a valuation ring with valuation $v\colon R \to \mathbb{N}\cup\{\infty\}$. Consider the completion $\widehat{R}$ with respect to v. Then $\widehat{R}$ is again a valuation ring with valuation also denoted by v. Let k be a subdivision ring of $\widehat{R}$. Let $X \subseteq \widehat{R}$. Suppose that k and X generate a free k-ring $k\langle X\rangle$ inside $\widehat{R}$. Suppose that*

$$1 \leq v(x) \quad \text{for all } x \in X. \tag{65}$$

Then k and $Y = \{1+x\}_{x\in X}$ generate a free group k-ring on Y inside $\widehat{R}$.

PROOF. Notice that $v_{|k\setminus\{0\}} = 0$ by Remarks 1.49(b). Let Z be a finite subset of X. By Lemma 7.33,

$$k\langle Z\rangle \hookrightarrow k\langle\langle Z\rangle\rangle \hookrightarrow \widehat{R}.$$

By Proposition 1.60, we get that the subring of $\widehat{R}$ generated by k and $\{1+z, (1+z)^{-1}\}_{z\in Z}$ is the free group k-ring on $\{1+z,(1+z)^{-1}\}_{z\in Z}$. Since this can be done for any finite subset Z of X, we obtain that the subring of $\widehat{R}$ generated by k and $\{1+x,(1+x)^{-1}\}_{x\in X}$ is the free group k-ring. $\square$

From these results we are ready to prove how to obtain an embedding of the free group k-ring inside a division ring from a J-embedding. We proceed as in [**Lic84**, Corollary 1].

LEMMA 7.35. *Suppose that we have a $(K,k,\alpha,I,\{t_i\}_{i\in I})$-J-embedding $k\langle X\rangle \hookrightarrow K[x;\alpha] \hookrightarrow Q$. Let H be the free group on $\{h_i\}_{i\in I}$. Then there exist embeddings of k-rings*

(i) *$k\langle\langle X\rangle\rangle \hookrightarrow K[[x;\alpha]]$, defined by $x_i \mapsto x_i = t_ix$.*
(ii) *$k[H] \hookrightarrow Q$, defined by $h_i \mapsto 1+x_i$.*

PROOF. Recall from Examples 1.50 that $K[x;\alpha]$ is a valuation ring with valuation given by $v\left(\sum\limits_{n\geq 0} a_nx^n\right) = \min\{n \mid a_n \neq 0\}$. Notice that the completion of $K[x;\alpha]$ with respect to $I_n = \langle x\rangle^n$ is $K[[x;\alpha]]$. So we have $k\langle X\rangle \hookrightarrow K[[x;\alpha]]$ and $v(x_i) = 1$. Then, by Lemma 7.33, $k\langle\langle X\rangle\rangle \hookrightarrow K[[x;\alpha]]$. By Corollary 7.34, we get the embedding of k-rings $k[H] \hookrightarrow K[[x;\alpha]] \hookrightarrow E$ defined by $h_i \mapsto 1+x_i$. Notice that $Q = Q_{\mathrm{cl}}(K[x;\alpha])$ is contained in E by the universal property of Ore localization. Moreover, $k[H]$ is contained in Q because $1+x_i \in Q$ for all $i \in I$. $\square$

DEFINITION 7.36. Suppose that $k\langle X\rangle \hookrightarrow Q$ is a $(K,k,\alpha,I,\{t_i\}_{i\in I})$-J-embedding (respectively a $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-JF-embedding). Consider the free group on $\{h_i\}_{i\in I}$. The embedding $k[H] \hookrightarrow Q$ given in Lemma 7.35 will be called a $(K,k,\alpha,I,\{t_i\}_{i\in I})$-*JL-embedding* (respectively $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-*JFL-embedding*). $\square$

Note that JL stands for Jategaonkar Lichtman and JFL for Jategaonkar Fisher and Lichtman.

REMARKS 7.37. Consider a $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-JF-embedding $k\langle X\rangle \hookrightarrow Q$ and the induced JFL-embedding $k[H] \hookrightarrow Q$. Then the following hold

(a) $k\langle X\rangle$ and $k[H]$ have the same division ring of fractions inside Q (and E).
(b) $\mathrm{h}_Q(k[H]) \leq \mathrm{h}_Q(k\langle X\rangle) \leq 2$.

Proof. (a) Let D be the division ring of fractions of $k\langle X\rangle$ inside Q. Clearly $k\langle X\rangle \subseteq k[H]$ because $x_i = (1+x_i)-1$. Since $1+x_i \in D$ for all $i \in I$, $(1+x_i)^{-1} \in D$ and $k[H] \subseteq D$. Therefore $k\langle X\rangle \subseteq k[H] \subseteq D$. This implies that the division ring of fractions of $k[H]$ inside Q is contained in D. Now the result follows by Remarks 7.3(d).

(b) Follows by (a), Remarks 7.3(d) and Proposition 7.10 □

Now it is not difficult to give JL-embeddings of inversion height one.

Proposition 7.38. *Let $k\langle X\rangle \hookrightarrow Q$ be a $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-JF-embedding of inversion height one. Consider the induced $(K,k,\alpha,I,\{t_i\}_{i\in I},t_{i_0})$-JFL-embedding $k[H] \hookrightarrow Q$. Then $\mathrm{h}_Q(k[H]) = 1$. In particular, for each JF-embedding of Section 3, we get a JFL-embedding of inversion height one.*

Proof. Follows from Remarks 7.37(b), and the fact that $k[H]$ is not a division ring. □

We now show that there are examples of embeddings of the free group algebra of inversion height 2.

Proposition 7.39. *Consider the $(K,k,\alpha,\{1\}\cup J,\{1\}\cup\{t_{i1}\}_{i\in J},1)$-JFL-embedding obtained from the $(K,k,\alpha,\{1\}\cup J,\{1\}\cup\{t_{i1}\}_{i\in J},1)$-JF-embedding of Section 4 in this chapter. Then $\mathrm{h}_E(k[H]) = 2$.*

Proof. Define $\mathcal{S}_0 = k$, and for $n \geq 1$,

$$\mathcal{S}_n = \left\{ \sum_{\varepsilon,\gamma} a_{\varepsilon\gamma} t_{i_1 1}^{\varepsilon_{i_1}1} \cdots t_{i_n n}^{\varepsilon_{i_n}n} \neq 0 \;\middle|\; \begin{array}{l} a_{\varepsilon\gamma}\in k \text{ almost all zero },\ \varepsilon=(\varepsilon_{i_1},\dots,\varepsilon_{i_n})\in\{0,1\}^n, \\ \gamma=(i_1,\dots,i_n)\in J^n \end{array} \right\}.$$

Let $W = \left\{ \sum\limits_{n=0}^{\infty} a_n x^n \mid a_n \in \mathcal{S}_n \cup \{0\} \right\}$. We claim that W is a subring of E. Clearly W is an additive subgroup of E. Let $b = \sum\limits_{n=0}^{\infty} b_n x^n$, $c = \sum\limits_{n=0}^{\infty} c_n x^n \in W$. For each $l \in \mathbb{N}$, we have $b_l = \sum\limits_{\varepsilon\gamma} b_{l\varepsilon\gamma} t_{i_1 1}^{\varepsilon_{i_1}1} \cdots t_{i_l l}^{\varepsilon_{i_l}l}$, and if $n \geq l$, $c_{n-l} = \sum\limits_{\varepsilon\gamma} c_{n-l\varepsilon\gamma} t_{i_{l+1}1}^{\varepsilon_{i_{l+1}}l+1} \cdots t_{i_n n-l}^{\varepsilon_{i_n}n}$, then $\alpha^l(c_{n-l}) = \sum\limits_{\varepsilon\gamma} c_{n-l\varepsilon\gamma} t_{i_{l+1}l+1}^{\varepsilon_{i_{l+1}}l+1} \cdots t_{i_n n}^{\varepsilon_{i_n}n}$. Then $bc = \sum\limits_{n=0}^{\infty} \left(\sum\limits_{l=0}^{n} b_l \alpha^l(c_{n-l}) \right) x^n \in W$, and the claim is proved.

Now we show that $k[H] \subseteq W$. First observe that $1+x_i = 1+t_{i1}x \in W$ for all $i \in J$, and $1+x \in W$.

$$\begin{aligned} (1+t_{i1}x)^{-1} &= \sum_{n=0}^{\infty} (-t_{i1}x)^n = \sum_{n=0}^{\infty} (-1)^n t_{i1} \cdots t_{in} x^n \in W. \\ (1+x)^{-1} &= \sum_{n=0}^{\infty} (-x)^n \in W. \end{aligned}$$

Therefore $k[H]$, the k-algebra generated by $\{1+x_i, (1+x_i)^{-1} \mid i \in J\} \cup \{1+x, (1+x)^{-1}\}$, is contained in W.

Let $p = \left(\sum\limits_{n=0}^{\infty} a_n x^n \right) x^l \in W \setminus \{0\}$ with $a_0 \neq 0$. Then

$$p^{-1} = x^{-l} \sum_{n=0}^{\infty} b_n x^n,$$

where $b_0 = a_0^{-1} \in V$, and, if $n \geq 1$, $b_n = -a_0^{-1} \sum_{j=1}^{n} a_j \alpha^j(b_{n-j}) \in V$. Therefore $p^{-1} \in U$. This shows that $Q_1(k[H], E) \subseteq U$. By Remarks 7.37, the ring $Q_2(k[H], E)$ is the division ring of fractions of $k[H]$ inside Q. As in the proof of Proposition 7.24, $(t_{i1} - t_{i2}^2)^{-1}$ belongs to the division ring of fractions of $k[H]$ (or $k\langle X\rangle$) inside Q, but $(t_{i1} - t_{i2}^2)^{-1} \notin U$. Therefore $(t_{i1} - t_{i2}^2)^{-1} \notin Q_1(k[H], E)$. □

We can also state a result for the group ring that looks like Corollaries 7.13 and 7.25. It does not appear in [**HS07**].

COROLLARY 7.40. *Let k be a field. Let $\{h_i\}_{i\geq 1}$ be an infinite countable set and H the free group on $\{h_i\}_{i\geq 1}$. Then there exist infinite non-isomorphic division rings of fractions D of $k[H]$ such that $k[H] \hookrightarrow D$ is of inversion height one and inversion height two.*

PROOF. Fix a natural $p \geq 2$. Let $Z = \{z_1, z_2, \dots\}$ be an infinite countable set. In Corollaries 7.13 and 7.25 we proved that the embedding $\delta_p \colon k\langle Z\rangle \hookrightarrow Q_p \hookrightarrow E$ of k-algebras defined by $z_i \mapsto x_j x_0^s$, where $i = (p-1)s + j$ with $1 \leq j \leq p-1$, is of inversion height one or two depending on whether we are talking about the embedding of Corollary 7.13 or Corollary 7.25 respectively. Moreover Q_p is not isomorphic to Q_q as division rings of fractions of $k\langle Z\rangle$ if $p \neq q$.

In the case of inversion height one observe that $v(z_i) = v(x_j x_0^s) = v(t^j x x^s) \geq 1$ for all $i \geq 1$. In the case of inversion height two note that $v(z_i) = v(x_j x_0^s) = v(t_{j1} x x^s) \geq 1$ for all $i \geq 1$, where v is the usual valuation defined on $K[[x;\alpha_p]] \subseteq E$ in both cases. Hence, by Corollary 7.34, we obtain that $k[H] \hookrightarrow K[[x;\alpha_p]] \subseteq E$, where $h_i \mapsto 1 + z_i$. Moreover $k[H] \hookrightarrow Q_p \subseteq E$ because $1 + z_i \in Q_p$ for each $i \geq 1$. Thus both $k[H]$ and $k\langle Z\rangle$ have Q_p as division ring of fractions. Notice that Q_p and Q_q are not isomorphic as division rings of fractions of $k[H]$ if $p \neq q$, because if they were, then they would be isomorphic as division rings of fractions of $k\langle Z\rangle$ since $h_i - 1 = z_i$.

Now in the case of Corollary 7.13, $k[H] \hookrightarrow Q_p$ is of inversion height one by Proposition 7.4 and Remarks 7.3(d) since $k\langle Z\rangle \subseteq k[H]$.

In the case of Corollary 7.25, observe that $h_j \mapsto 1 + t_{i+1} x^{s+1}$ and then

$$(1 + t_{i1} x^{s+1})^{-1} = \sum_{n=0}^{\infty} (-t_{i1} x^{s+1})^n = \sum_{n=0}^{\infty} (-1)^n t_{i1} t_{i(s+2)} \cdots t_{i(n-1)(s+1)+1} x^{n(s+1)} \in W.$$

Thus $k[H] \subseteq W$. Then the same proof of Proposition 7.39 shows that $k[H] \hookrightarrow Q_p$ is of inversion height two. □

7. Embeddings of infinite inversion height: A solution to a conjecture by B.H. Neumann

Let X be a set with $|X| \geq 2$, H the free group on X and k a field. Choose a total order on H such that $(H, <)$ is an ordered group. Consider the Mal'cev Neumann series ring $k((H, <))$ associated with the group ring $k[H]$. B.H. Neumann conjectured in [**Neu49a**, p. 215] that

(N) the inversion height of the embedding $k[H] \hookrightarrow k((H, <))$ is infinite.

Related to this conjecture, C. Reutenauer proved in [**Reu96**, Theorem 2.1] that this conjecture holds when X is infinite (see below for more details). As far as we know the conjecture has not been proved in the finite case, although it was expected to be true [**Reu96**, Section 5.2]. This section is devoted to confirm this conjecture when X is a finite set with at least two elements.

We begin with some definitions.

DEFINITIONS 7.41. Let k be a ring and X a set. Let A be an $n \times n$ matrix with entries over the free k-ring $k\langle X\rangle$.

(a) Let $i, j, p, q \in \{1, \ldots, n\}$. By A^{ij} we denote the matrix obtained from A by deleting the i-th row and the j-th column. By r_p^j we mean the row vector obtained from the p-th row of A deleting the j-th entry. And by s_q^i we denote the column vector obtained from the q-th column of A by deleting the i-th entry.

(b) The matrix $A = (x_{ij})$ is said to be a *generic matrix* (over $k\langle X\rangle$) if the x_{ij}'s are distinct variables in X. $\square$

Following the works by I. Gelfand, V. Retakh et al, for example [**GGRW05**, Section 1.2], we have the following

LEMMA 7.42. *Let k be a division ring, X a non-empty set and H the free group on X. Choose a total order on H such that $(H, <)$ is an ordered group. Let $E = k((H, <))$ be the Mal'cev-Neumann series ring associated with $k[H]$. Consider the embedding*

$$k\langle X\rangle \hookrightarrow k[H] \hookrightarrow E.$$

The following statements hold

(i) *Any generic matrix (over $k\langle X\rangle$) is invertible over E.*

(ii) *Let $B = (y_{pq})$ be the inverse of an $n \times n$ generic matrix $A = (x_{ij})$. Then*

$$y_{ji} = \Big(x_{ij} - r_i^j (A^{ij})^{-1} s_j^i\Big)^{-1},$$

and the inversion height of y_{ji} with respect to $k\langle X\rangle \hookrightarrow E$ is at most n for any $1 \le i, j \le n$.

PROOF. (i) Observe that a generic matrix is full. Indeed, let $A = (x_{ij})$ be an $n \times n$ generic matrix. Consider the morphism of k-rings $\varphi\colon k\langle X\rangle \to k$ defined by $x_{ii} \mapsto 1$ and $x \mapsto 0$ for all $x \in X \setminus \{x_{11}, \ldots, x_{nn}\}$. Then the image of A by φ is the identity $n \times n$ matrix over k which is clearly full. Hence the image of A is invertible in E by Theorem 4.36.

(ii) Let A be an $n \times n$ generic matrix. First we see how y_{nn} looks like. If $n = 1$, it is clear that $y_{11} = x_{11}^{-1}$. If $n \ge 2$, decompose A as $\left(\begin{array}{c|c} A^{nn} & s_n^n \\ \hline r_n^n & x_{nn} \end{array}\right)$ and consider the formulas

$$\begin{aligned}
Y_{11} &= \Big(A^{nn} - r_n^n x_{nn}^{-1} s_n^n\Big)^{-1} \\
Y_{12} &= -(A^{nn})^{-1} s_n^n \Big(x_{nn} - r_n^n (A^{nn})^{-1} s_n^n\Big)^{-1} \\
Y_{21} &= -x_{nn}^{-1} r_n^n \Big(A^{nn} - s_n^n x_{nn}^{-1} r_n^n\Big)^{-1} \\
Y_{22} &= \Big(x_{nn} - r_n^n (A^{nn})^{-1} s_n^n\Big)^{-1}.
\end{aligned}$$

Note that they all make sense in E. For example, A^{nn} is an $(n-1)\times(n-1)$ generic matrix and thus invertible by (i). The element $r_n^n (A^{nn})^{-1} s_n^n$ is a series whose support is contained in the subgroup generated by $X \setminus \{x_{nn}\}$. Therefore $x_{nn} - r_n^n (A^{nn})^{-1} s_n^n$ is non-zero and thus invertible. A similar reasoning shows that $A^{nn} - r_n^n x_{nn}^{-1} s_n^n$ is not zero.

It is not difficult to prove that the product $\left(\begin{array}{c|c} A^{nn} & s_n^n \\ \hline r_n^n & x_{nn} \end{array}\right)\left(\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right)$ is the identity matrix, and thus $\left(\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right)$ is the inverse of A.

Consider the (i, j)-th entry of A. Let P be the permutation (of rows) matrix such that moves the i-th row to the n-th row and the p-th row to the $(p-1)$-th row for $i < p \le n$. Let Q

be the permutation (of columns) matrix such that moves the j-th column to the n-th column and the q-th column to the $(q-1)$-th column for $j < q \leq n$. Therefore $PAQ = \left(\begin{array}{c|c} A^{ij} & s_j^i \\ \hline r_i^j & x_{ij} \end{array}\right)$. By the foregoing, the (n,n)-th entry of $Q^{-1}A^{-1}P^{-1}$ is $\left(x_{ij} - r_i^j(A^{ij})^{-1}s_j^i\right)^{-1}$. Therefore the (j,i)-th entry of $B = A^{-1} = Q(Q^{-1}A^{-1}P^{-1})P$ is $\left(x_{ij} - r_i^j(A^{ij})^{-1}s_j^i\right)^{-1}$.

We prove by induction on n that the inversion height of y_{ji} is at most n. For $n=1$, the result is clear, $B = x_{11}^{-1}$. Suppose that $n \geq 2$, and the result holds for $n-1$, i.e. the entries of the inverse of an $(n-1)\times(n-1)$ generic matrix are of inversion height at most $n-1$. We have just proved that $y_{ji} = \left(x_{ij} - r_i^j(A^{ij})^{-1}s_j^i\right)^{-1}$. Our induction hypothesis implies that the inversion height of $x_{ij} - r_i^j(A^{ij})^{-1}s_j^i$ is at most $n-1$. Thus the inversion height of y_{ji} is at most n. $\square$

The following notion was introduced by I. Gelfand and V. Retakh in [**GR91**] [**GR92**] [**GR93**]. Usually it is used in a more general context, see for example [**GGRW05**].

Definition 7.43. Let k be a division ring, X a set and H the free group on X. Consider the free k-ring $k\langle X\rangle$. Let $A = (x_{ij})$ be an $n\times n$ generic matrix over $k\langle X\rangle$. For $i,j \in \{1,\dots,n\}$ the *(i,j)-th quasideterminant* $|A|_{ij}$ of A is the element of $k((H,<))$ defined by the formula

$$|A|_{ij} = x_{ij} - r_i^j(A^{ij})^{-1}s_j^i.$$

Thus, $|A|_{ij}$ is the inverse of the (j,i)-th entry of the inverse of A by Lemma 7.42. $\square$

Observe that the definition of the (i,j)-th quasideterminant does not depend on the total order $<$ chosen such that $(H,<)$ is an ordered group. Indeed if $<'$ is such an order and $E' = k((H,<'))$, then there exists an isomorphism of $k[H]$-rings (in particular of $k\langle X\rangle$-rings) $\phi\colon E(k[H]) \to E'(k[H])$ by Corollary 6.5 or Corollary 4.41. Therefore the image by ϕ of $|A|_{ij}$ defined in E is $|A|_{ij}$ defined in E'.

Let X be a finite set, H the free group on X and k a field. Let $<$ be a total order on H such that $(H,<)$ is an ordered group. Consider the Mal'cev-Neumann series ring $E = k((H,<))$ associated with the group ring $k[H]$. It was conjectured by I. Gelfand and V. Retakh

(GR) Let A be an $n\times n$ generic matrix over $k\langle X\rangle$. The inversion height of $|A|_{ij}$ with respect to $k\langle X\rangle \hookrightarrow E$ is $n-1$ for each $i,j \in \{1,\dots,n\}$.

The matrix A^{ij} is an $(n-1)\times(n-1)$ generic matrix. Hence Lemma 7.42 implies that the inversion height of $|A|_{ij}$ is at most $n-1$. Furthermore C. Reutenauer proved in [**Reu96**, Theorem 2.1] that (GR) holds. More concretely, he showed that in the notation of (GR)

Theorem 7.44. *Each entry of the inverse of the $n\times n$ generic matrix A is of inversion height n with respect to $k\langle X\rangle \hookrightarrow E$.* $\square$

Note that Theorem 7.44 implies (GR). Otherwise it would contradict Theorem 7.44 since the inversion height of $|A|_{ij}$ is at most $n-1$ and the (j,i)-th entry of A^{-1} is $|A|_{ij}^{-1}$.

Now it is not difficult to prove, as we said at the beginning of this section, that C. Reutenauer's result implies (N) when X is an infinite set.

Remark 7.45. Let X be an infinite set, H the free group on X and k a field. Choose a total order on H such that $(H,<)$ is an ordered group. Consider the Mal'cev-Neumann series ring $E = k((H,<))$ associated with the group ring $k[H]$. Then the inversion height of $k\langle X\rangle \hookrightarrow E$ and $k[H] \hookrightarrow E$ is infinite. Indeed, if A_n is an $n\times n$ generic matrix, then the inversion height of the entries of A_n^{-1} is n with respect to $k\langle X\rangle \hookrightarrow E$ and $n-1$ with respect to $k[H] \hookrightarrow E$.

Proof. The result is a consequence of the following claim. Let f be an element in $E(k\langle X\rangle)$, the division ring of fractions of $k\langle X\rangle$ inside E, of inversion height $\leq m$ with respect to $k\langle X\rangle \hookrightarrow E$. Then there exists a finite subset Y of X such that $f \in E(k\langle Y\rangle)$, and the inversion height of f with respect to $k\langle Y\rangle \hookrightarrow E$ is $\leq m$. For $m = 0$ the claim is clear. So suppose that the claim is true for $m-1 \geq 0$. Since $f \in Q_m(k\langle X\rangle, E)$, f is a finite sum of elements of the form $f_1 \cdots f_l$ where either $f_i \in Q_{m-1}(k\langle X\rangle, E)$ or f_i is the inverse of some nonzero element in $Q_{m-1}(k\langle X\rangle, E)$. Hence, if $f = \sum\limits_{j=1}^{r} f_{1j}\cdots f_{l_j j}$, the induction hypothesis implies that there exist $Y_{1j}, \ldots, Y_{l_j j}$ such that $f_{ij} \in E(k\langle Y_{ij}\rangle)$ and the inversion height of f_{ij} with respect to $k\langle Y_{ij}\rangle \hookrightarrow E$ is $\leq m$. Let $Y = \bigcup\limits_{i,j} Y_{ij}$. Then $f \in E(k\langle Y\rangle)$, and the inversion height of f with respect to $k\langle Y\rangle \hookrightarrow E$ is $\leq m$ because $Q_m(k\langle Y_{ij}\rangle, E) \subseteq Q_m(k\langle Y\rangle, E)$. So the claim is proved.

Since X is an infinite set, there exist $n\times n$ generic matrices A_n for each natural $n \geq 1$. The entries of the inverse of A_n are of inversion height n with respect to $k\langle Y\rangle \hookrightarrow E$ for any finite subset Y of X that contains the entries of A_n by Theorem 7.44, and the fact that $E(k\langle Y\rangle)$ is the division ring of fractions of $k\langle Y\rangle$ inside $k((H_Y, <))$ where H_Y is the free (sub)group on Y. Hence our claim implies that the inversion height of the entries of the inverse of A_n is exactly n with respect to $k\langle X\rangle \hookrightarrow E$. Hence, there exist elements of inversion height n with respect to $k\langle X\rangle \hookrightarrow E$ for each n. Thus $k\langle X\rangle \hookrightarrow E$ is of infinite inversion height.

Now note that $k[H] \subseteq Q_1(k\langle X\rangle, E) \subseteq Q_1(k[H], E)$. Thus

$$Q_{n-1}(k[H], E) \subseteq Q_n(k\langle X\rangle, E) \subseteq Q_n(k[H], E)$$

for all $n \geq 1$. This implies that the inversion height of the entries of A_n^{-1} is either n or $n-1$ with respect to $k[H] \hookrightarrow E$. We prove by induction on n that it is at most $n-1$.

For $n = 1$, it is clear that if $x \in X$, then $x^{-1} \in H \subseteq k[H]$, and thus it is of inversion height $n-1 = 0$. Suppose that $n \geq 2$ and the inversion height of the entries on an $(n-1)\times(n-1)$ generic matrix is at most $n-2$ with respect to $k[H] \hookrightarrow E$. Consider an $n\times n$ generic matrix $A_n = (x_{ij})$. Then the (j,i)-th entry of A_n^{-1} is $\left(x_{ij} - r_i^j (A_n^{ij})^{-1} s_j^i\right)^{-1}$, which is of inversion height at most $n-1$ with respect to $k[H] \hookrightarrow E$ because of the induction hypothesis applied to $(A_n^{ij})^{-1}$. $\square$

After these well-known results we proceed to prepare the proof of (N) when X is a finite set of cardinality at least two. We begin with the following important result that will allow us to reduce our problem to the case of Theorem 7.44.

Theorem 7.46. *Let R be a domain with a division ring of fractions D. Let $(L, <)$ be an ordered group. Consider a crossed product group ring RL that can be extended to DL. Let $E = D((L, <))$ be the associated Mal'cev-Neumann series ring. Thus E is a division ring and $RL \hookrightarrow E$. Then*

$$Q_n(RL, E) \subseteq \mathcal{S}_n = \left\{ \sum_{x\in L} a_x \bar{x} \in E \mid a_x \in Q_n(R, D) \right\} \tag{66}$$

for each $n \in \mathbb{N}$. Moreover, if $f \in D$ is of inversion height n (with respect to $R \hookrightarrow D$), then $f \in E$ is of inversion height n (with respect to $RL \hookrightarrow E$). Therefore $\mathrm{h}_E(RL) \geq \mathrm{h}_D(R)$.

Proof. First we claim that for each $x \in L$, $n \in \mathbb{N}$ and $a \in Q_n(R, D)$, $\bar{x}a\bar{x}^{-1} \in Q_n(R, D)$. We proceed to prove this by induction on n. The claim is clear for $n = 0$ because

$$\bar{x}a\bar{x}^{-1} = a^{\sigma(x)} \in R = Q_0(R, D).$$

Suppose that the result holds for $n \geq 0$. Let $a \in Q_n(R,D) \setminus \{0\}$. Then
$$\bar{x}a^{-1}\bar{x}^{-1} = (\bar{x}a\bar{x}^{-1})^{-1} \in Q_{n+1}(R,D) \setminus \{0\}.$$
Now the claim follows from the fact that $Q_{n+1}(R,D)$ is the subring of D generated by the set $\{a, b^{-1} \mid a, b \in Q_n(R,D),\ b \neq 0\}$.

Clearly (66) holds for $n = 0$ because $Q_0(RL,E) = RL \subseteq \mathcal{S}_0$. So suppose that (66) holds for $n \geq 0$, and we prove (66) for $n+1$. Let $f \in Q_n(RL,E)$ with $f \neq 0$. Suppose that $f = \sum\limits_{x \in L} a_x\bar{x}$ with $a_x \in Q_n(R,D)$. Let $x_0 = \omega(f) = \min\{x \in L \mid x \in \operatorname{supp} f\} \in L$. Then, from Corollary 4.20,
$$f^{-1} = \sum_{m \geq 0} (a_{x_0}\bar{x}_0)^{-1}(g(a_{x_0}\bar{x}_0)^{-1})^m,$$
where $g = a_{x_0}\bar{x}_0 - f \in \mathcal{S}_n$. Note that $(a_{x_0}\bar{x}_0)^{-1} = \bar{x}_0^{-1}a_{x_0}^{-1} = \bar{x}_0^{-1}a_{x_0}^{-1}\bar{x}_0\bar{x}_0^{-1} \in \mathcal{S}_{n+1}$, and thus $g(a_{x_0}\bar{x}_0)^{-1} \in \mathcal{S}_{n+1}$. Now observe that $\mathcal{S}_{n+1}$ is a ring, indeed if $p = \sum\limits_{y \in L} b_y\bar{y},\ q = \sum\limits_{z \in L} c_z\bar{z} \in \mathcal{S}_{n+1}$, then $p + q \in \mathcal{S}_{n+1}$ and
$$pq = \sum_{x \in L}\left(\sum_{yz=x} b_y\bar{y}c_z\bar{y}^{-1}\tau(y,z)\right)\bar{x} \in \mathcal{S}_{n+1}.$$
Thus $(g(a_{x_0}\bar{x}_0)^{-1})^m \in \mathcal{S}_{n+1}$ for each $m \in \mathbb{N}$. Now recall that the series $\sum\limits_{m \geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m$ is defined in $D((L,<))$ by Theorem 4.19(iv). Thus, for each $x \in L$, the set
$$\{m \in \mathbb{N} \mid x \in \operatorname{supp}\left((g(a_{x_0}\bar{x}_0)^{-1})^m\right)\}$$
is finite. Hence, for each $x \in L$, the coeficient of $\bar{x}$ in $\sum\limits_{m \geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m$ is an element of $Q_{n+1}(R,D)$, i.e. $\sum\limits_{m \geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m \in \mathcal{S}_{n+1}$. Therefore
$$f^{-1} = (a_{x_0}\bar{x}_0)^{-1}\sum_{m \geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m \in \mathcal{S}_{n+1}.$$
As before, it is not difficult to prove that (66) holds for $n+1$ by the definition of $Q_{n+1}(RL,E)$, and the fact that $\mathcal{S}_{n+1}$ is a ring.

If R is a division ring, the remaining part is clear. So suppose that R is not a division ring. Fix $n \in \mathbb{N}$ such that there exists $f \in Q_{n+1}(R,D) \setminus Q_n(R,D)$. Since $R \subseteq RL$, note that $f \in Q_{n+1}(R,D) \subseteq Q_{n+1}(RL,E)$. On the other hand, observe that two series $\sum b_x\bar{x}, \sum c_x\bar{x} \in E$, where $a_x, b_x \in D$ for all $x \in L$, are equal if and only if $b_x = a_x$ for all $x \in L$. Therefore $f \notin \mathcal{S}_n$ and thus $f \notin Q_n(RL,E)$ as desired. □

Recall that if R is a ring and α an automorphism of R, then $R[x,x^{-1};\alpha]$ is a crossed product group ring with associated Mal'cev-Neumann series ring the skew Laurent series ring $R((x;\alpha))$. Thus from Theorem 7.46 we obtain

Corollary 7.47. *Let R be a ring and α an automorphism of R. Suppose that R has a division ring of fractions D and that α extends to an automorphism of D. So that $R[x,x^{-1};\alpha]$ embeds in the division ring $Q = Q^l_{cl}(D[x,x^{-1};\alpha])$. If $f \in D$ is of inversion height n (with respect to $R \hookrightarrow D$) then f is of inversion height n (with respect to $R[x,x^{-1};\alpha] \hookrightarrow Q$). Therefore* $\mathrm{h}_Q(R[x,x^{-1};\alpha]) \geq \mathrm{h}_D(R)$. □

In case that α is not necessarily a ring automorphism we can prove an analogous result as the previous corollary proceeding as in the proof of Theorem 7.46. More concretely

Corollary 7.48. *Let R be a domain, and $\alpha\colon R \to R$ be a ring monomorphism. Suppose that R has a division ring of fractions D, that α extends to D and that*

$$\alpha\Big(Q_{n+1}(R,D) \setminus Q_n(R,D)\Big) \subseteq Q_{n+1}(R,D) \setminus Q_n(R,D) \tag{67}$$

for all $n \in \mathbb{N}$. Observe that $R[x;\alpha]$ embeds in the division ring $Q = Q_{cl}(D[x;\alpha])$. If $f \in D$ is of inversion height n (with respect to $R \hookrightarrow D$), then f is of inversion height n (with respect to $R[x;\alpha] \hookrightarrow Q$). Therefore $\mathrm{h}_Q(R[x;\alpha]) \geq \mathrm{h}_D(R)$.

Proof. It is not difficult to realize that for each $a \in Q_n(R,D)$, $\alpha(a) \in Q_n(R,D)$.

By the universal property of the Ore localization and Proposition 3.10, we can see Q embedded in the division ring

$$E = \left\{ x^{-r} \sum_{m=0}^{\infty} a_m x^m \mid a_m \in D,\ r \geq 0 \right\}.$$

As before, we prove that

$$Q_n(R[x;\alpha],Q) \subseteq \mathcal{S}_n = \left\{ x^{-r} \sum_{m=0}^{\infty} a_m x^m \mid a_m \in Q_n(R,D),\ r \geq 0 \right\}.$$

For $n=0$, it is clear since $Q_0(R[x;\alpha],Q) = R[x;\alpha] \subseteq \mathcal{S}_0$. Let $n \geq 0$ and

$$f = x^{-r} \sum_{r=0}^{\infty} a_m x^m \in Q_n(R[x;\alpha],Q) \setminus \{0\}.$$

Let $m_0 = \min\{m \mid a_m \neq 0\}$. Then $f = x^{-r}\left(\sum_{l=0}^{\infty} a_{m_0+l}x^l\right)x^{m_0}$. Set $h = \sum_{l=0}^{\infty} a_{m_0+l}x^l$ and $g = a_{m_0}x^{m_0} - h$. Thus

$$h^{-1} = \sum_{m=0}^{\infty} (a_{m_0}x^{m_0})^{-1}(g(a_{m_0}x^{m_0})^{-1})^m \in \mathcal{S}_{n+1}.$$

Then $f^{-1} = x^{-m_0}h^{-1}x^r \in \mathcal{S}_{n+1}$. From this and the definition of $Q_{n+1}(R[x;\alpha],Q)$, it follows that $Q_{n+1}(R[x;\alpha],Q) \subseteq \mathcal{S}_{n+1}$.

If R is a division ring, the remaining part is clear. So suppose that R is not a division ring. Fix $n \in \mathbb{N}$ such that there exists $f \in Q_{n+1}(R,D) \setminus Q_n(R,D)$. If $f \in \mathcal{S}_n$, then there exists a series $x^{-r}\sum_{m=0}^{\infty} a_m x^m$, with $r \geq 0$ and $a_m \in Q_n(R,D)$ for all $m \in \mathbb{N}$, such that $f = x^{-r}\sum_{m=0}^{\infty} a_m x^m$. Hence $x^r f = \sum_{m=0}^{\infty} a_m x^m$ and $\alpha^r(f)x^r = \sum_{m=0}^{\infty} a_m x^m$. Therefore $\alpha^r(f) = a_r \in Q_n(R,D)$, a contradiction with (67). Thus, $f \notin \mathcal{S}_n$, as desired. $\square$

Note that if α is an automorphism then (67) in the foregoing corollary holds. Hence we obtain the following useful result.

Corollary 7.49. *Let R be a domain and $\alpha\colon R \to R$ be a ring isomorphism. Suppose that R has a division ring of fractions D and that α extends to D. Observe that $R[x;\alpha]$ embeds in the division ring $Q = Q_{cl}(D[x;\alpha])$. If $f \in D$ is of inversion height n (with respect to $R \hookrightarrow D$) then f is of inversion height n (with respect to $R[x;\alpha] \hookrightarrow Q$). Therefore $\mathrm{h}_Q(R[x;\alpha]) \geq \mathrm{h}_D(R)$.* $\square$

Proof. This result follows from Corollary 7.48 once we show that (67) holds. This can be proved by induction on n. Suppose that α induces an automorphism of $Q_n(R,D)$. Since $\alpha(f^{-1}) = \alpha(f)^{-1}$ and $\alpha^{-1}(f^{-1}) = (\alpha^{-1}(f))^{-1}$ for each $f \in Q_n(R,D) \setminus \{0\}$, α induces an automorphism of $Q_{n+1}(R,D)$. Since $\alpha(Q_n(R,D)) = Q_n(R,D)$, (67) holds. Now notice that α induces an automorphism of $Q_0(R,D) = R$. $\square$

Before giving Theorem 7.53, we need to introduce a standard result on free groups that can be found, for example, in [**Kur60**, Section 36].

DEFINITIONS 7.50. Let H be a free group on a set X and N a subgroup of H.

(a) A *transversal* of N in H is a representative from each right coset Nh of N in H. We will denote the representative of Nh by $\overline{Nh}$.

(b) A *Schreier transversal* of N in H is a transversal S such that whenever $h = x_1^{\epsilon_1} \cdots x_m^{\epsilon_m} \in S$ is a reduced word (where $x_i \in X$ and $\epsilon_i = \pm 1$), then every initial segment $x_1^{\epsilon_1} \cdots x_l^{\epsilon_l}$, $l \leq m$, also is in S. $\square$

THEOREM 7.51. *Let H be a free group on the set X, and N be a subgroup of H. If the set $S = \{\overline{Nh} \in Nh \mid h \in H\}$ is a Schreier transversal of N in H, then the set*

$$Y = \{(\overline{Nh})y(\overline{Nhy})^{-1} \neq 1 \mid y \in X,\ \overline{Nh} \in S\}$$

is a basis for N, that is, N is the free group on the set Y. $\square$

EXAMPLE 7.52. Let H be the free group on a set X with at least two elements. Let $C = \langle c \rangle$ be the infinite cyclic group. Let $x \in X$. Consider the morphism of groups $\varphi\colon H \to C$ given by $x \mapsto c$ and $y \mapsto 1$ for all $y \in X \setminus \{x\}$. Let $N = \ker \varphi$. Note that for each $h \in H$ there exists $n \in \mathbb{Z}$ such that $Nh = Nx^n$. Hence $S = \{x^n \mid n \in \mathbb{Z}\}$ is a Schreier transversal of N in H. The representative for $Nx^m y = Nx^m y x^{-m} x^m = Nx^m$ in S is x^m. Hence the set $\{x^n y x^{-n} \mid y \in X \setminus \{x\},\ n \in \mathbb{Z}\}$ is a basis for N by Theorem 7.51. $\square$

Now we are ready to prove the desired result on the conjecture (N).

THEOREM 7.53. *Let k be a field, X a finite set with at least two elements and H the free group on X. Choose a total order on H such that $(H, <)$ is an ordered group. Consider the Mal'cev-Neumann series ring $E = k((H, <))$ associated with the group ring $k[H]$. Then the inversion height of $\iota_X\colon k\langle X\rangle \hookrightarrow E$ and $\iota_H\colon k[H] \hookrightarrow E$ is infinite. Moreover, if x and y are different elements in X, then the entries of the inverse of the matrix*

$$A_n = \begin{pmatrix} xyx^{-1} & x^2yx^{-2} & \cdots & x^nyx^{-n} \\ x^{n+1}yx^{-n-1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x^{n^2-n+1}yx^{-n^2+n-1} & \cdots & \cdots & x^{n^2}yx^{-n^2} \end{pmatrix}$$

are of inversion height n with respect to ι_X and of inversion height $n-1$ with respect to ι_H.

PROOF. Let $x \in X$. By Example 7.52, H is the extension of the free group N on the infinite set $Z = \{x^n y x^{-n} \mid y \in X \setminus \{x\},\ n \in \mathbb{Z}\}$ by the infinite cyclic group L generated by x. Then $k[H]$ can be seen as a crossed product group ring $(k[N])L$ by Lemma 4.7. Taking a closer look at $k[H] = k[N]L$, for example as in the proof of Lemma 4.7, one sees that in fact $k[H] = k[N][x, x^{-1}; \alpha]$ where α is left conjugation by x, i.e. $\alpha(f) = xfx^{-1}$ for all $f \in k[N]$.

Now we prove that α can be extended to an automorphism of $E(k[N])$. It can be seen as a consequence of Hughes' Theorem I, but we show it in a more elementary way. First notice that left conjugation by x and x^{-1} induce automorphisms of $E(k[H])$ because $x \in E(k[H])$. We show by induction on n that left conjugation by x extends to an automorphism of $Q_n(k[N], E)$ for each $n \in \mathbb{N}$ using an argument that we have already used. For $n = 0$ it is clear because $Q_0(k[N], E) = k[N]$. Suppose that $n \geq 0$, and that left conjugation by x and x^{-1} induce automorphisms of $Q_n(k[N], E)$. Let $f \in Q_n(k[N], E) \setminus \{0\}$. Then $xf^{-1}x^{-1} = (xfx^{-1})^{-1}$ and $x^{-1}fx = (x^{-1}fx)^{-1}$ belong to $Q_{n+1}(k[N], E) \setminus \{0\}$. Hence left conjugation by x and x^{-1} induce endomorphisms of $Q_{n+1}(k[H], E)$, and their composition in any order is the identity. Thus left conjugation by x induces an automorphism of $Q_{n+1}(k[H], E)$. Hence left conjugation by x induces an automorphism of $E(k[N]) = \bigcup_{n\geq 0} Q_n(k[N], E)$ as desired. Call again α this extension.

Note that the powers of x are $E(k[N])$-linearly independent because the series of $E(k[N])$ have their support contained in N, and $n_1x^{r_1} = n_2x^{r_2}$ for $n_1, n_2 \in N$ and $r_1, r_2 \in \mathbb{Z}$ if and only if $n_1 = n_2$ and $r_1 = r_2$. Hence $E(k[N])[x, x^{-1}; \alpha]$ is contained in $E(k[H])$. Also α induces an automorphism of $k\langle Z\rangle$ and clearly $E(k\langle Z\rangle) = E(k[N])$. Hence $k\langle Z\rangle[x; \alpha]$ and $E(k\langle Z\rangle)[x; \alpha]$ are contained in $E(k[H])$. Both $E(k[N])[x, x^{-1}; \alpha]$ and $E(k\langle Z\rangle)[x; \alpha]$ have the same Ore division ring of fractions, both embed in the division ring $E(k[H])$, and both have $E(k[H])$ as Ore division ring of fractions because both contain $k\langle X\rangle$.

Note that the entries of A_n are in $k[N]$. Therefore the entries of its inverse are in $E(k[N])$. Now Corollary 7.47 implies that the inversion height of the entries of the inverse of A_n with respect to $k[N] \hookrightarrow E(k[N])$ and with respect to $k[H] \hookrightarrow E(k[H])$ are the same. By Remark 7.45, the inversion height of the entries of the inverse of A_n is $n - 1$ with respect to $k[N] \hookrightarrow E(k[N])$. Therefore the entries of the inverse of A_n have inversion height $n - 1$ with respect to ι_H. Hence ι_H is of infinite inversion height.

Notice that $k[H] \subseteq Q_1(k\langle X\rangle, E) \subseteq Q_1(k[H], E)$. Hence

$$Q_{n-1}(k[H], E) \subseteq Q_n(k\langle X\rangle, E) \subseteq Q_n(k[H], E)$$

for all $n \geq 1$. Thus the inversion height of the entries of the inverse of A_n with respect to ι_X is n or $n - 1$ and the inversion height of ι_X is infinite.

Finally we prove that the inversion height of the entries of the inverse of A_n is at least n with respect to ι_X.

Note that the entries of A_n are in $k\langle Z\rangle$. Therefore the entries of its inverse are in $E(k\langle Z\rangle)$. Now Corollary 7.49 implies that the inversion height of the entries of the inverse of A_n with respect to $k\langle Z\rangle \hookrightarrow E(k\langle Z\rangle)$ and with respect to $k\langle Z\rangle[x; \alpha] \hookrightarrow E(k\langle Z\rangle[x; \alpha])$ are the same. By Remark 7.45, the inversion height of the entries of the inverse of A_n is n with respect to $k\langle Z\rangle \hookrightarrow E(k\langle Z\rangle)$. Hence it is also n with respect to $k\langle Z\rangle[x; \alpha] \hookrightarrow E(k\langle Z\rangle[x; \alpha])$.

Since $k\langle X\rangle \subseteq k\langle Z\rangle[x; \alpha] \subseteq Q_1(k\langle X\rangle, E)$, then

$$Q_{n-1}(k\langle X\rangle, E) \subseteq Q_{n-1}(k\langle Z\rangle[x; \alpha], E) \subseteq Q_n(k\langle X\rangle, E) \subseteq Q_n(k\langle Z\rangle[x; \alpha], E)$$

for all $n \geq 1$. Thus the inversion height of the entries of the inverse of A_n is at least n with respect to ι_X, as desired. $\square$

Remark 7.54. Let X be a set of cardinality at least 2, H the free group on X and $<$ a total order on H such that $(H, <)$ is an ordered group. Consider a crossed product group ring kH, its polynomial ring $k\langle X\rangle$ and the associated Mal'cev-Neumann series ring $E = k((H, <))$. Consider the embeddings $k\langle X\rangle \hookrightarrow E$ and $kH \hookrightarrow E$. It is reasonable to think that $k\langle X\rangle \hookrightarrow E$ is of infinite inversion height. Moreover, if Theorem 7.44 is true for the polynomial algebra $k\langle X\rangle$ of the crossed product group ring kH, then the conjecture (N) would also hold for crossed product group rings since the proof of Theorem 7.53 is valid for any crossed product group ring kH. $\square$

Now we show that Theorem 7.46 is in fact a theorem for groups with a subnormal series with orderable factors.

Theorem 7.55. *Let k be a division ring. Let G be a group with a subnormal series $(G_\gamma)_{\gamma\leq\tau}$ with orderable factors. Consider a crossed product group ring kG and the embedding $kG \hookrightarrow D$ where D is the Hughes-free division ring of fractions of kG. Let $f \in kG_\gamma$ for some ordinal number $\gamma \leq \tau$. Then the inversion height of f with respect to $kG \hookrightarrow D$ is the inversion height* *of f with respect to $kG_\gamma \hookrightarrow D(kG_\gamma)$. In particular, if k is a field and either G_γ or any factor $G_{\gamma+1}/G_\gamma$ is a non-commutative free group, then* $\mathrm{h}_D(kG) = \infty$.

PROOF. First observe that $D(kG_\gamma)$ is the Hughes-free division ring of fractions of kG_γ for each ordinal number $\gamma \leq \tau$.

Fix an ordinal number $\gamma \leq \tau$ and $f \in D(kG_\gamma)$. We prove by induction that the inversion height of f with respect to $kG_\delta \hookrightarrow D(kG_\delta)$ is exactly the inversion height of f with respect to $kG_\gamma \hookrightarrow D(kG_\gamma)$ for all $\gamma \leq \delta \leq \tau$.

For $\delta = \gamma$ the result is clear. Suppose that the result holds for kG_δ with $\delta \geq \gamma$. By Lemma 4.7, $kG_{\delta+1} = kG_\delta \frac{G_{\delta+1}}{G_\delta}$. Since $kG_{\delta+1} \hookrightarrow D(kG_{\delta+1})$ is a Hughes-free embedding, and $D(kG_\delta)$ is the Hughes-free division ring of fractions of kG_δ, then $D(kG_{\delta+1})$ is the Hughes-free division ring of fractions of $D(kG_\delta)\frac{G_{\delta+1}}{G_\delta}$ by Remark 6.11. Fix an order $<$ on $G_{\delta+1}/G_\delta$ such that $(G_{\delta+1}/G_\delta, <)$ is an ordered group. Therefore $D(kG_{\delta+1})$ can be seen as the division ring of fractions of $D(kG_\delta)\frac{G_{\delta+1}}{G_\delta}$ inside $D(kG_\delta)((\frac{G_{\delta+1}}{G_\delta}, <))$ by Examples 5.6(d) and Hughes' Theorem I 6.3. Now Theorem 7.46 implies that the inversion height of f with respect to $kG_{\delta+1} \hookrightarrow D(kG_{\delta+1})$ equals the inversion height of f with respect to $kG_\delta \hookrightarrow D(kG_\delta)$. Suppose now that δ is a limit ordinal number $\leq \tau$, and that we have proved the result for all ordinals $\gamma \leq \eta < \delta$. Notice that the inversion height of f with respect to $kG_\delta \hookrightarrow D(kG_\delta)$ is smaller or equal than with respect to $kG_\eta \hookrightarrow D(kG_\eta)$ for any $\gamma \leq \eta$ by Remarks 7.3(d). As in the proof of Remark 7.45, it can be proved that there exists a finitely generated subgroup L of G_δ such that the inversion height of f with respect to $kG_\delta \hookrightarrow D(kG_\delta)$ equals the inversion height of f with respect $kL \hookrightarrow D(kL)$. Let $\gamma \leq \eta < \delta$ be such that L is contained in G_η. The inversion height of f with respect to $kG_\eta \hookrightarrow D(kG_\eta)$ equals the inversion height of f with respect to $kG_\gamma \hookrightarrow D(kG_\gamma)$ by induction hypothesis, and it is smaller or equal than the inversion height of f with respect to $kL \hookrightarrow D(kL)$. Thus we have equality. Therefore the inversion height of f with respect to $kG_\gamma \hookrightarrow D(kG_\gamma)$ equals the inversion height of f with respect to $kG_\delta \hookrightarrow D(kG_\delta)$.

For the second part observe that the foregoing and Theorem 7.53 imply the result when some G_γ is a non-commutative free group. If some factor $G_{\gamma+1}/G_\gamma$ is a non-commutative free group, observe that we have proved

$$kG_{\gamma+1} \hookrightarrow D(kG_\gamma)\frac{G_{\gamma+1}}{G_\gamma} \hookrightarrow D(kG_{\gamma+1}) \hookrightarrow D(kG_\gamma)\Big(\Big(\frac{G_{\gamma+1}}{G_\gamma}, <\Big)\Big).$$

By Theorem 7.53, $D(kG_\gamma)\frac{G_{\gamma+1}}{G_\gamma} \hookrightarrow D(kG_\gamma)((\frac{G_{\gamma+1}}{G_\gamma}, <))$ is of infinite inversion height. Then $kG_{\gamma+1} \hookrightarrow D(kG_{\gamma+1})$ is of infinite inversion height by Remarks 7.3(d). □

From the first part of Theorem 7.55 we also obtain another proof of Theorem 7.46.

COROLLARY 7.56. *Let k be a field, X a finite set with at least two elements and H the free group on X. Choose a total order on H such that $(H, <)$ is an ordered group. Consider the Mal'cev-Neumann series ring $E = k((H, <))$ associated with the group ring $k[H]$. Then the inversion height of $\iota_H \colon k[H] \hookrightarrow E$ is infinite.*

PROOF. Consider any non-trivial normal subgrup N of H such that H/N is a non-trivial orderable group. It is known that if H/N is of infinite order, then N is (free and) not finitely generated [**Mas91**, Exercise 8.3]. For example, we can take $N = [H, H]$, the commutator subgroup. Then H/N is the free abelian group on X. The embedding $k[H] \hookrightarrow E$ is Hughes-free by Examples 5.6. The group H has a subnormal series $1 \lhd N \lhd H$ with orderable factors. By Remark 7.45, $k[N] \hookrightarrow E(k[N])$ is of infinite inversion height. Therefore $k[H] \hookrightarrow E$ is of infinite inversion height by the first part of Theorem 7.55. □

Remark 7.57. The advantage of the normal subgroup N in the proof of Theorem 7.53 is that we can avoid Hughes' Theorem II, and it also allows us to show elements of exactly inversion height n with respect to $k\langle X\rangle \hookrightarrow E$. This last thing is not so immediate from the choice of other normal subgroups N, for example $N = [H, H]$. $\square$

Corollary 7.58. *Let k be a division ring, I a set of cardinality at least two and $\{G_i\}_{i\in I}$ a family of non-trivial orderable groups. Let $<$ be a total order on $\underset{i\in I}{*} G_i$ such that $(\underset{i\in I}{*} G_i, <)$ is an ordered group. Consider a crossed product group ring $k(\underset{i\in I}{*} G_i)$ and the embedding $k(\underset{i\in I}{*} G_i) \hookrightarrow E = k((\underset{i\in I}{*} G_i, <))$ in its associated Mal'cev-Neumann series ring. Then*

$$\mathrm{h}_E(k(\underset{i\in I}{*} G_i)) = \infty.$$

Proof. First notice that $\underset{i\in I}{*} G_i$ is orderable by Proposition 2.23. By Corollary 2.9(a), $\underset{i\in I}{*} G_i$ is the extension of a free group K by the group $\prod_{i\in I} G_i$. The same arguments of [**Hig40**, Appendix] prove that K is a free group on an infinite set, or also by [**Mas91**, Exercise 8.3]. Indeed, if we fix a total order $\prec$ on I, then K is the free group on the non-trivial elements of the set

$$\Big\{[g_{i_1}g_{i_2}\cdots g_{i_r}, g_{i_{r+a}}\cdots g_{i_s}] \mid g_{i_j} \in G_{i_j},\ i_1 \prec i_2 \prec \cdots \prec i_s \in I\Big\}.$$

Therefore $\mathrm{h}_E(k(\underset{i\in I}{*} G_i)) = \infty$ by Remark 7.45 and Theorem 7.46.

Another proof follows from Theorem 7.55. The embedding $k(\underset{i\in I}{*} G_i) \hookrightarrow E$ is Hughes-free by Examples 5.6(d). By Corollary 2.9(a), $\underset{i\in I}{*} G_i$ is the extension of a free group K and $\prod_{i\in I} G_i$. Moreover K is not commutative. In fact if $i_1 \neq i_2 \in I$, $g_1 \neq h_1 \in G_{i_1}$, $g_2 \neq h_2 \in G_{i_2}$, then $1 \neq g_1g_2g_1^{-1}g_2^{-1}$ and $1 \neq h_1h_2h_1^{-1}h_2^{-1}$ are elements of K and $g_1g_2g_1^{-1}g_2^{-1} \neq h_1h_2h_1^{-1}h_2^{-1}$. $\square$

Until now, the embedding of the free k-algebra $k\langle X\rangle$ inside its universal field of fractions is the only embedding of infinite inversion height we have seen. One might think that infinite inversion height could determine this embedding, but as the following two results show this is not true.

Corollary 7.59. *Let k be a field and $Z = \{z_1, z_2, \dots\}$ be an infinite countable set. Then $k\langle Z\rangle$, the free k-algebra on Z, has infinite non-isomorphic division rings of fractions D for which $k\langle Z\rangle \hookrightarrow D$ is of infinite inversion height.*

Proof. For each $r \geq 1$, let $X_r = \{x_0, x_1, \dots, x_r\}$, H_r the free group on X_r, $<_r$ a total order on H_r such that $(H_r, <_r)$ is an ordered group, and D_r the division ring of fractions of $k\langle X_r\rangle$ and $k[H_r]$ inside $k((H_r, <_r))$. In the proof of Proposition 7.4 we gave embeddings $k\langle Z\rangle \hookrightarrow k\langle X_r\rangle \hookrightarrow D_r$ such that D_r is the division ring of fractions of $k\langle Z\rangle$ and $\mathrm{h}_{D_r}(k\langle Z\rangle) = \mathrm{h}_{D_r}(k\langle X\rangle_r) = \infty$ for each $r \geq 1$ by Theorem 7.53.

Now we show that D_r is not isomorphic to $D_{r'}$ as division rings of fractions of $k\langle Z\rangle$ for different even naturals r, r'. Note that if D_r is isomorphic to $D_{r'}$, then the image a full matrix over D_r is full over $D_{r'}$.

For each even r consider the matrix $A_r = \left(\begin{smallmatrix} z_1 & z_{r+1} \\ z_2 & z_{r+2}\end{smallmatrix}\right)$ over $k\langle Z\rangle$. The image of A_r in D_r is the matrix

$$\begin{pmatrix} x_1 & x_1x_0 \\ x_2 & x_2x_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_0 \end{pmatrix},$$

and thus it is not full. But the image of A_r in $D_{r'}$ for even $r' > r$ is the full matrix $\left(\begin{smallmatrix} x_1 & x_{r+1} \\ x_2 & x_{r+2}\end{smallmatrix}\right)$. Hence $D_{r'}$ is not isomorphic to D_r as division rings of fractions of $k\langle Z\rangle$ for different even naturals r, r', and $k\langle Z\rangle \hookrightarrow D_r$ is of infinite inversion height. $\square$

COROLLARY 7.60. *Let k be a field. For each finite set X with $|X| \geq 2$, there exist infinite non-isomorphic division rings of fractions D of $k\langle X\rangle$ such that $\mathrm{h}_D(k\langle X\rangle) = \infty$.*

PROOF. Let $r \geq 1$ be a natural number. Consider the group

$$\Gamma_r = \left\langle X, T \;\middle|\; \begin{array}{c} TXTX^{-1}T^{-1}XT^{-1}X^{-1}=1 \\ TX^2TX^{-2}T^{-1}X^2T^{-1}X^{-2}=1 \\ \cdots \\ TX^rTX^{-r}T^{-1}X^rT^{-1}X^{-r}=1 \end{array} \right\rangle.$$

Proceeding as in Examples 2.10(b), Γ_r is isomorphic to the group

$$\Gamma'_r = \left\langle X, T_0, T_1, \dots T_r \;\middle|\; \begin{array}{cc} T_0T_1T_0^{-1}T_1^{-1}, & XT_0X^{-1}=T_1 \\ T_0T_2T_0^{-1}T_2=1, & XT_1X^{-1}=T_2 \\ \cdots & \cdots \\ T_0T_rT_0^{-1}T_r^{-1}=1, & XT_{r-1}X^{-1}=T_r \end{array} \right\rangle, \tag{68}$$

where the isomorphism $\Gamma_r \to \Gamma'_r$ is given by $X \mapsto X$, $T \mapsto T_0$, and the inverse $\Gamma'_r \to \Gamma_r$ by $X \mapsto X$, $T_i \mapsto X^iTX^{-i}$, $i = 0, \dots, r$. Another way of looking at Γ_r is as the semidirect product of the group

$$N_r = \left\langle T_i,\ i \in \mathbb{Z} \;\middle|\; \begin{array}{c} T_iT_{i+1}=T_{i+1}T_i \\ T_iT_{i+2}=T_{i+2}T_i \\ \cdots \\ T_iT_{i+r}=T_{i+r}T_i \end{array} \right\rangle$$

and the infinite cyclic group $C = \langle X\rangle$ where X acts on N_r as $T_i \mapsto T_{i+1}$, i.e. $XT_iX^{-1} = T_{i+1}$. The isomorphism $\Gamma_r \to N_r \rtimes C$ is given by $X \mapsto X$, $T \mapsto T_0$ with inverse $N_r \rtimes C \to \Gamma_r$ defined by $X \mapsto X$, $T_i \mapsto X^iTX^{-i}$, $i \in \mathbb{Z}$. Observe that N_r is the fundamental group of the graph of groups

$$\cdots \longrightarrow \underset{\bullet}{G(i)} \xrightarrow{G(e_i)} \underset{\bullet}{G(i+1)} \xrightarrow{G(e_{i+1})} \cdots$$

where $G(i)$ is the free abelian group in $\{T_i, T_{i+1}, \dots, T_{i+r}\}$ and $G(e_i)$ the free abelian group in $\{T_{i+1}, \dots, T_{i+r}\}$ for each $i \in \mathbb{Z}$. Also N_r can be seen as

$$\cdots G(i-1) *_{G(e_{i-1})} G(i) *_{G(e_i)} G(i+1) *_{G(e_{i+1})} \cdots \tag{69}$$

Consider the morphism of groups $N_r \to A_r = \langle T_{\bar 0}, T_{\bar 1}, \dots, T_{\bar r}\rangle$, where A_r is the free abelian group on $\{T_{\bar 0}, T_{\bar 1}, \dots, T_{\bar r}\}$, defined by $T_i \mapsto T_{\bar j}$ if $i = n(r+1)+j$ for some $j, n \in \mathbb{Z}$, $0 \leq j \leq r$. Then N is locally indicable by Proposition 2.8. Moreover, it can be deduced from the proof of that result that N_r is the extension of a free group L_r, which is the kernel of the morphism $N_r \to A_r$, by the free abelian group A_r. Note that L_r is not cyclic because the elements $T_0T_{r+1}T_0^{-1}T_{r+1}^{-1}$ and $T_{2r+2}T_{3r+3}T_{2r+2}^{-1}T_{3r+3}^{-1}$ belong to L_r, but they do not commute as can be deduced from (69). Observe that Γ_r has a subnormal series

$$1 \lhd L_r \lhd N_r \lhd \Gamma_r,$$

with L_r, $\Gamma_r/N_r = C$ and $N_r/L_r = A_r$ orderable groups. By Corollary 6.12, $k[\Gamma_r]$ is Hughes-free embeddable. Let E_r be the Hughes-free division ring of fractions of $k[\Gamma_r]$. By Theorem 7.55, $\mathrm{h}_{E_r}(k[\Gamma_r]) = \infty$.

Consider the JF-embedding of Section 3.1 defined by the following data. Let $K = k(t)$, the field of fractions of the polynomial ring $k[t]$. Let $\alpha_{r+2}\colon K \to K$ be the morphism of k-rings defined by $t \mapsto t^{r+2}$. Let $Q_{r+2} = Q^l_{\mathrm{cl}}(K[x;\alpha_{r+2}])$. Let $I = \{0, \dots, r+1\}$. Set $t_0 = 1$, $t_1 = t, \dots, t_{r+1} = t^{r+1}$ and $x_0 = x$, $x_1 = tx, \dots, x_{r+1} = t^{r+1}x$. Set $\mathcal{X}_{r+2} = \{x_0, \dots, x_{r+1}\}$. In this way we obtain a $(K, k, \alpha_{r+2}, I, \{t_i\}_{i=0}^{r+1}, 1)$-JF-embedding $k\langle \mathcal{X}_{r+2}\rangle \hookrightarrow Q_{r+2}$.

Observe that there exists a morphism of groups $\Gamma_r \to Q_{r+2}^{\times}$ defined by $X \mapsto x$, $T \mapsto t$, or better, in the notation of (68), $X \mapsto x$, $T_0 \mapsto t, \dots,$ $T_r \mapsto t_{r+1}$. Hence, if we set $X_0 = X$, $X_1 = T_0X$, $\dots, X_{r+1} = T_rX$, then $\{X_0, X_1, \dots, X_{r+1}\}$ generate a free monoid inside Γ'_r. Therefore the group ring $k[\Gamma_r]$ contains the free k-algebra $k\langle X_0, \dots, X_{r+1}\rangle$, and clearly also the free k-algebra $k\langle X_0, \dots, X_r\rangle$. Moreover, E_r is a division ring of fractions of $k\langle X_0, \dots, X_r\rangle$

because $T = T_0 = X_1 X_0^{-1}$, $X_0 = X$ belong to the division ring of fractions of $k\langle X_0, \dots, X_r\rangle$ inside E_r. Then, since $k\langle X_0, \dots, X_r\rangle \subseteq k[\Gamma_r]$, we get that $\mathrm{h}_{E_r}(k\langle X_0, \dots, X_r\rangle) \geq \mathrm{h}_{E_r}(k[\Gamma_r]) = \infty$. For each $r' > r$, consider the embedding $k\langle X_0, \dots, X_r\rangle \hookrightarrow k\langle X_0, \dots, X_{r'}\rangle \hookrightarrow k[\Gamma_{r'}]$. Again $E_{r'}$ is the division ring of fractions of $k\langle X_0, \dots, X_r\rangle$ because $T = X_1 X_0^{-1}$ and $X_0 = X$. Hence $\mathrm{h}_{E_{r'}}(k\langle X_0, \dots, X_r\rangle) = \infty$ for each $1 \leq r < r'$ because $k\langle X_0, \dots, X_r\rangle \subseteq k\langle X_0, \dots, X_{r'}\rangle$. Note that, for $r \leq r' < r''$, $E_{r'}$ is not isomorphic to $E_{r''}$ as division rings of fractions of $k\langle X_0, \dots, X_r\rangle$ because such isomorphism would imply that $k[\Gamma_{r'}]$ is isomorphic to $k[\Gamma_{r''}]$ as $k\langle X_0, \dots, X_r\rangle$-rings, which is impossible since $\Gamma_{r'}$ is not isomorphic to $\Gamma_{r''}$ via a morphism which sends $X \mapsto X$ and $T \mapsto T$. □

Note that Corollary 7.59 could also be deduced from Corollary 7.60 using Proposition 7.4.

In Section 6, we have given embeddings of the free group k-ring $k[H]$ inside division rings. One question arises, are they Hughes-free embeddings? At first sight, by equality (58), it seems that they are not because, roughly speaking, there is "too much commutativity". The following result confirms this first impression. So we can give many examples of non-Hughes-free embeddings of the free group algebra. The first example of a non-Hughes-free embedding was given in [**Lew74**, Section V], we give an infinite family of such examples, some more than in [**HS07**, Proposition 8.4].

Corollary 7.61. *Let k be a field. Let H be the free group on a set X of at least two elements. Suppose that the free group ring $k[H]$ has a division ring of fractions D such that $k[H] \hookrightarrow D$ is of finite inversion height. Then $k[H] \hookrightarrow D$ is not a Hughes-free embedding. In particular, all JFL-embeddings of $k[H]$ are not Hughes-free.*

Proof. Let $<$ be a total order on H such that $(H, <)$ is an ordered group. Consider the associated Mal'cev-Neumann series ring $E = k((H, <))$. We know that $k[H] \hookrightarrow E$ is a Hughes-free embedding by Examples 5.6(d). By Remark 7.45, $k[H] \hookrightarrow E$ is of infinite inversion height. Recall that inversion height is preserved by isomorphisms of division ring of fractions by Remark 7.3(f). Then, since $k[H] \hookrightarrow D$ is of finite inversion height, $k[H] \hookrightarrow D$ is not a Hughes-free division ring of fractions of $k[H]$ by Hughes' Theorem I 6.3. □

"A universe not without end
But with an infinite number of endings and beginnings
Fragments becoming galaxies
Created, destroyed and recreated"

Ark, *Absolute Zero*

Part 4

Tilting modules

CHAPTER 8

Tilting modules arising from ring epimorphisms

In this chapter we generalize the classical construction of the tilting $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$, see Example 8.4, and study the properties of the tilting modules constructed in this way. The main results of this chapter are from a joint work with L. Angeleri Hügel [**AHS08**]. We inform the reader that the homological tools of Section 5 in Chapter 1 will be used throughout.

1. Basics on tilting modules

In this section we present a few of the basics on tilting modules. We intend to give proofs of most of the elementary results that will be needed later as well as to give some flavor of the techniques displayed when dealing with tilting modules. We also state some deep results on the relations between tilting classes and definable classes that will be helpful in the following sections. For a much more detailed exposition of the subjects in this section see [**GT06**].

We begin by setting the notation.

Notation 8.1. Let R be a ring. Let M be a right R-module.

(a) By Mod-R we denote the category of right R-modules, and by R-Mod the category of left R-modules.
(b) Let $\mathcal{L}$ be a class of right R-modules, by $\mathcal{L}^\perp$ we mean
$$\mathcal{L}^\perp = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(L, M) = 0 \text{ for all } L \in \mathcal{L}\}.$$
If $\mathcal{L} = \{L\}$ we will write $L^\perp$ instead of $\{L\}^\perp$.
(c) We denote by $\operatorname{Add} M$ the class of all modules isomorphic to direct summands of arbitrary direct sums of copies of M, i.e. $N \in \operatorname{Add} M$ iff there exist a set I and right R-modules B and C such that $M^{(I)} = B \oplus C$ and $N \cong B$.
(d) Dually, we denote by $\operatorname{Prod} M$ the class of all modules isomorphic to direct summands of arbitrary direct products of copies of M, i.e. $N \in \operatorname{Prod} M$ iff there exist a set I and right R-modules B, C such that $M^I = B \oplus C$ and $N \cong B$.
(e) We denote by $\operatorname{Gen} M$ the class of right R-modules generated by M, that is, the right R-modules which are epimorphic images of arbitrary direct sums of copies of M.
(f) We denote by $\operatorname{Pres} M$ the class of all M-presented right R-modules, that is, the right R-modules N such that there exist sets I, J and an exact sequence $M^{(I)} \to M^{(J)} \to N \to 0$. Notice that $\operatorname{Pres} M \subseteq \operatorname{Gen} M$. □

We give now some remarks that will be used throughout.

Remarks 8.2. Let R be a ring.

(a) Let $\mathcal{L}$ be a class of right R-modules with $\operatorname{pd} L \leq 1$ for all $L \in \mathcal{L}$, then $\mathcal{L}^\perp$ is closed under images. Indeed, if $0 \to X \to Y \to Z \to 0$ is an exact sequence with $Y \in \mathcal{L}^\perp$, then, applying $\operatorname{Hom}_R(L, \_)$, we obtain the exact sequence
$$0 = \operatorname{Ext}^1_R(L, Y) \to \operatorname{Ext}^1_R(L, Z) \to \operatorname{Ext}^2_R(L, X) = 0$$
for each $L \in \mathcal{L}$.

(b) $\mathrm{Hom}(R, M) \cong M$ as right R-modules via the map $f \mapsto f(1)$. □

Now we proceed to give the definitions of some of the key concepts of this chapter.

DEFINITIONS 8.3. Let R be a ring.

(a) A right R-module T is said to be a *tilting module* if it satisfies
 (T1) $\mathrm{pd}\, T \leq 1$.
 (T2) $\mathrm{Ext}^1_R(T, T^{(I)}) = 0$ for each set I.
 (T3) There exists an exact sequence

$$0 \to R \to T_0 \to T_1 \to 0$$

 where $T_0, T_1 \in \mathrm{Add}\, T$.

(b) A class $\mathcal{C}$ of right R-modules is a *tilting class* if there exists a tilting right R-module T such that $\mathcal{C} = T^\perp$.

(c) Two tilting right R-modules T and T' are said to be *equivalent* if $T^\perp = T'^\perp$.

(d) Let M be a right R-module and $\mathcal{C}$ a class of right R-modules closed under isomorphic images. A morphism $f \in \mathrm{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-*preenvelope* of M provided the morphism of abelian groups $\mathrm{Hom}_R(f, C')\colon \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$, that is, for each morphism $f'\colon M \to C'$ there is a morphism $g\colon C \to C'$ such that the following diagram is commutative.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f'"'] & C \arrow[d, dashed, "g"] \\ & C' \end{tikzcd}$$

(e) A $\mathcal{C}$-preenvelope $f \in \mathrm{Hom}_R(M, C)$ is a $\mathcal{C}$-*envelope* of M provided that f is *left minimal*, that is, every $g \in \mathrm{End}_R(C)$ such that $f = gf$ is an automorphism.

(f) Let $\mathfrak{V}$ be a set of R consisting of non-zero-divisors of R. Let M be a right R-module. We say that M is $\mathfrak{V}$-*divisible* if for each $y \in M$ and $v \in \mathfrak{V}$ there exists $x \in M$ such that $y = xv$, i.e. "y is divisible by v". If R is a domain, we say that M is *divisible* in case M is $\mathfrak{V}$-divisible for $\mathfrak{V} = R \setminus \{0\}$. □

Now we present the example of tilting module that we want to generalize.

EXAMPLE 8.4. Consider the $\mathbb{Z}$-module $T = \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$. Then T is a tilting $\mathbb{Z}$-module.

PROOF. The ring $\mathbb{Z}$ is hereditary, i.e. $\mathrm{pd}\, M \leq 1$ for all $M \in \text{Mod-}\mathbb{Z}$. Thus $\mathrm{pd}(T) \leq 1$, that is, (T1) is satisfied.

It is known that if R is a principal ideal domain, then the classes of injective and divisible modules coincide, see for example [**Lam99**, Corollary 3.17']. In our case $R = \mathbb{Z}$ is certainly a principal ideal domain, and the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are clearly divisible. Also the direct sum of divisible modules is divisible. Hence $T^{(I)}$ is divisible, and thus injective, for each set I. Therefore $\mathrm{Ext}^1_R(T, T^{(I)}) = 0$ for any set I, and (T2) is satisfied.

The condition (T3) is verified because

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0 \tag{70}$$

is an exact sequence with $\mathbb{Z},\ \mathbb{Q}/\mathbb{Z} \in \mathrm{Add}\, T$.

Now we compute the tilting class $T^\perp$. Clearly the class of divisible (=injective) $\mathbb{Z}$-modules is contained in $T^\perp$. On the other hand, if $M \in T^\perp$, then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, M) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, M) = 0$

by the properties of $\operatorname{Ext}^1_{\mathbb{Z}}(\_,\_)$ with respect to direct sums in the first component. Thus, applying $\operatorname{Hom}(\_, M)$ to (70), we obtain the exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, M) \xrightarrow{\pi^*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, M) \xrightarrow{\iota^*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \to \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, M) = 0.$$

Let $r \in \mathbb{Z} \setminus \{0\}$ and $m \in M$. Consider the map $f_m \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$ defined by $f(1) = m$. By the exactness of the sequence above, there exists $g \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, M)$ such that $g\iota = f_m$. Then $m = f_m(1) = g\iota(1) = g(1) = g(\frac{r}{r}) = g(\frac{1}{r})r$. Therefore M is divisible, as desired.

Observe that the foregoing argument also shows that $\operatorname{Gen} T = \operatorname{Gen} \mathbb{Q}$ is the class $T^{\perp}$ of divisible $\mathbb{Z}$-modules. □

Remark 8.5. Observe that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a ring epimorphism, $\operatorname{pd}_{\mathbb{Z}}(\mathbb{Q}) \leq 1$ and $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) = 0$ because $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. Also the tilting class is described in terms of divisibility. □

The following example shows that preenvelopes generalize the concept of injective hulls.

Example 8.6. Let R be a ring and $\mathcal{C}$ the class of injective right R-modules. Then every right R-module has a $\mathcal{C}$-envelope.

Proof. Let M be a right R-module and $\iota\colon M \to E$ its injective hull. Then ι is a $\mathcal{C}$-envelope. First note that if E' is another injective right R-module, then

$$\operatorname{Hom}_R(\iota, E')\colon \operatorname{Hom}_R(E, E') \to \operatorname{Hom}_R(M, E')$$

is surjective. Thus ι is a preenvelope. Let $g \in \operatorname{End}_R(E)$ such that $g\iota = \iota$. Then $\ker g = 0$, otherwise $\ker g \cap \iota(M) \neq 0$ because $\iota(M)$ is essential in E, a contradiction with $g\iota = \iota$. Hence g is injective. Now $\operatorname{im} g$ is an injective module which is a submodule of E. Thus it is a direct summand of E. Then $g(E) = E$ because $\iota(M) \subseteq g(E)$ is essential in E. Therefore g is an isomorphism. □

Now we present an important result by P. Eklof and J. Trlifaj [**ET01**]. It can be found in [**GT06**, Theorem 3.2.1] from where we have taken the proof. But first we need to give the following definitions.

Definitions 8.7. (a) Let $\{\gamma_\nu\}_{\nu<\kappa}$ be a transfinite sequence of ordinal numbers of length κ. We say that it is an *increasing sequence* if $\gamma_\nu < \gamma_\mu$ when $\nu < \mu < \kappa$. In this case we define $\gamma = \lim\limits_{\nu\to\kappa} \gamma_\nu = \sup\{\gamma_\nu \mid \nu < \kappa\}$.

(b) An infinite cardinal number δ is said to be a *regular cardinal number* if there does not exist an increasing transfinite sequence of ordinal numbers $\{\gamma_\nu\}_{\nu<\kappa}$ of length $\kappa < \delta$ and $\delta = \lim\limits_{\nu\to\kappa} \gamma_\nu$. For example, the ordinal number ω is a regular cardinal. It is known that for each cardinal number γ there exists a regular cardinal δ such that $\gamma < \delta$. Indeed for each ordinal number γ, the cardinal number $\aleph_{\gamma+1}$ is a regular cardinal such that $\gamma < \aleph_{\gamma+1}$, see for example [**Jec03**, Corollary 5.3]. □

Theorem 8.8. *Let R be a ring and M a right R-module. Let $\mathcal{S}$ be a set of modules. Then there is a short exact sequence*

$$0 \to M \hookrightarrow B \to B/M \to 0$$

where $B \in \mathcal{S}^{\perp}$ and B/M is $\mathcal{S}$-filtered. In particular, $M \hookrightarrow B$ is an $\mathcal{S}^{\perp}$-preenvelope of M with $B/M \in {}^{\perp}(\mathcal{S}^{\perp})$.

If moreover $\operatorname{pd} X \leq n$ *for all* $X \in \mathcal{S}$*, then* $\operatorname{pd} B/M \leq n$.

Proof. For each $X \in \mathcal{S}$ fix a presentation

$$0 \to P_X \xrightarrow{\alpha_X} F_X \to X \to 0 \tag{71}$$

with F_X a free right R-module.

Given a right R-module Y, $\mathrm{Hom}_R(\_, Y)$ applied to (71) induces the exact sequence

$$0 \to \mathrm{Hom}_R(X,Y) \to \mathrm{Hom}_R(F_X,Y) \xrightarrow{\mathrm{Hom}_R(\alpha_X,Y)} \mathrm{Hom}_R(P_X,Y) \to \\ \to \mathrm{Ext}^1_R(X,Y) \to \mathrm{Ext}^1_R(F_X,Y) = 0$$

Therefore $Y \in \mathcal{S}^\perp$ if and only if

$$\mathrm{Hom}(F_X,Y) \xrightarrow{\mathrm{Hom}_R(\alpha_X,Y)} \mathrm{Hom}_R(P_X,Y) \quad \text{is onto for all } X \in \mathcal{S}. \tag{72}$$

Let δ be an infinite regular cardinal such that each $X \in \mathcal{S}$ is generated by a set of cardinality strictly smaller than δ.

To construct B we need to build a continuous chain of right R-modules $(B_\gamma \mid \gamma < \delta)$ such that

(i) $B_0 = M$.
(ii) $B_{\gamma+1}/B_\gamma$ is isomorphic to a direct sum of modules in $\mathcal{S}$ for each $\gamma < \delta$. More precisely, $B_{\gamma+1}/B_\gamma \cong \bigoplus\limits_{X\in\mathcal{S}} X^{(\mathrm{Hom}_R(P_X,B_\gamma))}$.
(iii) $B = \varinjlim B_\gamma$ where $\gamma < \delta$. Therefore B/M is $\mathcal{S}$-filtered.
(iv) B satisfies (72), i.e. $B \in \mathcal{S}^\perp$.
(v) If there exists $n \in \mathbb{N}$ such that $\operatorname{pd} X \leq n$ for all $X \in \mathcal{S}$, then $\operatorname{pd}(B/M) \leq n$.

Define $B_0 = M$. Suppose that we have constructed B_γ for $\gamma < \delta$, then $B_{\gamma+1}$ is built as follows. For each $X \in \mathcal{S}$, let $a_{X,\gamma}$ denote a disjoint copy of $\mathrm{Hom}_R(P_X, B_\gamma)$ and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus\limits_{X\in\mathcal{S}} P_X^{(a_{X,\gamma})} \arrow[r, "\oplus\alpha_X^{(a_{X,\gamma})}"] \arrow[d, "\varphi_\gamma"'] & \bigoplus\limits_{X\in\mathcal{S}} F_X^{(a_{X,\gamma})} \\ & B_\gamma & \end{tikzcd} \tag{73}$$

where φ_γ is given by $\varphi_\gamma(p_{(X,f)}) = f(p)$ for each $p_{(X,f)} \in \bigoplus\limits_{X\in\mathcal{S}} P_X^{(a_{X,\gamma})}$, i.e. $p_{(X,f)}$ has all its components zero but for the one corresponding to $f \in \mathrm{Hom}_R(P_X, B_\gamma)$ which is $p \in P_X$. Thus φ_γ restricted to the (X,f) component with $f \in \mathrm{Hom}_R(P_X, B_\gamma)$ equals f.

Let $B_{\gamma+1}$ be the pushout module obtained from the pushout diagram of (73)

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus\limits_{X\in\mathcal{S}} P_X^{(a_{X,\gamma})} \arrow[r, "\oplus\alpha_X^{(a_{X,\gamma})}"] \arrow[d, "\varphi_\gamma"'] & \bigoplus\limits_{X\in\mathcal{S}} F_X^{(a_{X,\gamma})} \arrow[d, "g_\gamma"] \\ & B_\gamma \arrow[r, "\iota_\gamma"] & B_{\gamma+1} \end{tikzcd} \tag{74}$$

Since $\oplus\alpha_X^{(a_{X,\gamma})}$ is injective, then $B_\gamma \overset{\iota_\gamma}{\hookrightarrow} B_{\gamma+1}$ is too. For $\mathrm{coker}(\oplus\alpha_X^{(a_{X,\gamma})}) \cong \bigoplus\limits_{X\in\mathcal{S}} X^{(a_{X,\gamma})}$, we get the isomorphism of right R-modules $B_{\gamma+1}/B_\gamma \cong \bigoplus\limits_{X\in\mathcal{S}} X^{(a_{X,\gamma})}$. Thus (ii) follows.

If $\beta < \delta$ is a limit ordinal, we set $B_\beta = \varinjlim\limits_{\gamma<\beta} B_\gamma$.

Define $B = \varinjlim B_\gamma$. By the foregoing, (iii) is satisfied.

Now we prove that B satisfies (72). To do that, we will show that for each $Z \in \mathcal{S}$ and $f \in \operatorname{Hom}_R(P_Z, B)$, there exists $h \in \operatorname{Hom}_R(F_Z, B)$ such that $h\alpha_Z = f$

$$\begin{tikzcd} P_Z \arrow[r, "\alpha_Z"] \arrow[d, "f"'] & F_Z \arrow[dl, dashed, "h"] \\ B & \end{tikzcd}$$

Fix $Z \in \mathcal{S}$ and $f \in \operatorname{Hom}_R(P_Z, B)$. Since P_Z is generated by a set of cardinality $< \delta$, there exists $\gamma < \delta$ such that $f(P_Z) \subseteq B_\gamma$. So we may think of f as a morphism $f\colon P_Z \to B_{\gamma+1}$. Let $\varepsilon_{(Z,f,\gamma)}$ be the natural identification of F_Z with the (Z, f) direct summand of $\bigoplus\limits_{X\in\mathcal{S}} F_X^{(a_{X,\gamma})}$, and $\epsilon_{(Z,f,\gamma)}$ the natural identification of P_Z with the (Z, f) direct summand of $\bigoplus\limits_{X\in\mathcal{S}} P_X^{(a_{X,\gamma})}$. Notice that $(\oplus\alpha_X^{(a_{X,\gamma})})\epsilon_{(Z,f,\gamma)} = \varepsilon_{(Z,f,\gamma)}\alpha_Z$. Consider a diagram as in (74). Then

$$f = \iota_\gamma\varphi_\gamma\epsilon_{(Z,f,\gamma)} = g_\gamma(\oplus\alpha_X^{(a_{X,\gamma})})\epsilon_{(Z,f,\gamma)} = g_\gamma\varepsilon_{(Z,f,\gamma)}\alpha_Z,$$

as desired. Thus (iv) is satisfied.

Suppose that there exists $n \in \mathbb{N}$ such that $\operatorname{pd} X \leq n$ for all $X \in \mathcal{S}$. To prove (v), let $\bar{B}_\gamma = B_\gamma/B_0 = B_\gamma/M$ for all $\gamma < \delta$. Thus $\bar{B}_0 = 0$. Notice that $\operatorname{pd}(\bar{B}_0) = 0 \leq n$ and $\bar{B}_{\gamma+1}/\bar{B}_\gamma \cong \frac{B_{\gamma+1}/B_0}{B_\gamma/B_0} \cong B_{\gamma+1}/B_\gamma$. Thus $\operatorname{pd}(\bar{B}_{\gamma+1}/\bar{B}_\gamma) \leq n$ for all $\gamma \geq 1$, because of (ii). Hence $\operatorname{pd}(B/M) \leq n$ because of Auslander-Lemma 1.30.

If $Z \in \mathcal{S}^\perp$, then $\operatorname{Ext}^1_R(B/M, Z) = 0$ by Eklof Lemma 1.29 since B/M is $\mathcal{S}$-filtered and $\mathcal{S} \subseteq {}^\perp(\mathcal{S}^\perp)$. Thus $B/M \in {}^\perp(\mathcal{S}^\perp)$. Now the exactness of

$$\operatorname{Hom}_R(B, Z) \to \operatorname{Hom}_R(M, Z) \to \operatorname{Ext}^1_R(B/M, Z) = 0,$$

for all $Z \in \mathcal{S}^\perp$ implies that $M \hookrightarrow B$ is an $\mathcal{S}^\perp$-preenvelope. □

Corollary 8.9. *Let R be a ring. If T is a right R-module, then every right R-module M has a $T^\perp$-preenvelope $M \hookrightarrow B$ such that $B/M \in {}^\perp(T^\perp)$. Moreover, if $\operatorname{Ext}^1_R(T, T^{(I)}) = 0$ for any set I, then B/M is a direct sum of copies of T.*

Proof. The first part follows directly from Theorem 8.8 for $\mathcal{S} = \{T\}$.

By Theorem 8.8, we know that B/M is $\{T\}$-filtered. Let $(N_\gamma \mid \gamma < \delta)$ be a $\{T\}$-filtration of B/M. The module $N_0 = 0$ is an (empty) direct sum of copies of T. Suppose now that N_γ is a direct sum of copies of T, then $N_{\gamma+1}$ is too because the exact sequence

$$0 \to N_\gamma \to N_{\gamma+1} \to N_{\gamma+1}/N_\gamma \to 0$$

splits since $N_{\gamma+1}/N_\gamma \cong T$ and $\operatorname{Ext}^1_R(N_{\gamma+1}/N_\gamma, N_\gamma) = 0$ by hypothesis. Thus $N_{\gamma+1}$ is a direct sum of copies of T. Suppose now that β is a limit ordinal number, and that the result holds true for all ordinal numbers smaller than β. Then $N_\beta = \bigcup\limits_{\gamma<\beta} N_\gamma$ is a direct sum of copies of T. Therefore $B/M = \bigcup\limits_{\gamma<\delta} N_\gamma$ is a direct sum of copies of T. □

Now we present an important characterization of tilting modules from [**CT95**, Proposition 1.3]. The proof is from [**GT06**, Lemma 6.1.12].

Proposition 8.10. *Let R be a ring and T a right R-module. The following hold true*

(i) *If $\operatorname{Gen} T = T^\perp$, then $\operatorname{Gen} T = \operatorname{Pres} T$.*
(ii) *T is a tilting module if and only if $\operatorname{Gen} T = T^\perp$.*

Proof. (i) By definition $\operatorname{Pres} T \subseteq \operatorname{Gen} T$. Let $M \in \operatorname{Gen} T$. Then we can give the following presentation of M

$$0 \to K \to T^{(\operatorname{Hom}_R(T,M))} \xrightarrow{\varphi} M \to 0 \tag{75}$$

where φ is the morphism of right R-modules such that φ restricted to the component $f \in \operatorname{Hom}_R(T, M)$ equals f, and $K = \ker \varphi$. Applying $\operatorname{Hom}_R(T, \_)$ to (75) we obtain

$$\begin{aligned} 0 \to \operatorname{Hom}_R(T, K) \to \operatorname{Hom}_R(T, T^{(\operatorname{Hom}_R(T,M))}) \xrightarrow{\varphi_*} \operatorname{Hom}_R(T, M) \to \\ \to \operatorname{Ext}^1_R(T, K) \to \operatorname{Ext}^1_R(T, T^{(\operatorname{Hom}_R(T,M))}) \to \cdots \end{aligned}$$

Since $T^{(\operatorname{Hom}_R(T,M))} \in \operatorname{Gen} T$, then $\operatorname{Ext}^1_R(T, T^{(\operatorname{Hom}_R(T,M))}) = 0$. Now observe that φ_* is onto. Indeed, for each $f \in \operatorname{Hom}_R(T, M)$, if ι_f denotes the natural inclusion of T inside the f component of $T^{(\operatorname{Hom}_R(T,M))}$, then $\varphi_*(\iota_f) = \varphi\iota_f = f$. Hence $\operatorname{Ext}^1_R(T, K) = 0$, that is, $K \in T^\perp = \operatorname{Gen} T$, as desired.

(ii) Suppose that T is a tilting module. By (T2), $T^{(I)} \in T^\perp$ for each set I. By (T1) and Remarks 8.2(a), $T^\perp$ is closed under images. Hence $\operatorname{Gen} T \subseteq T^\perp$.

On the other hand, let $M \in T^\perp$. Applying $\operatorname{Hom}_R(\_, M)$ to the exact sequence given by (T3)

$$0 \to R \xrightarrow{\iota} T_0 \to T_1 \to 0$$

with $T_0, T_1 \in \operatorname{Add} T$, we obtain

$$\cdots \to \operatorname{Hom}_R(T_0, M) \to \operatorname{Hom}_R(R, M) \to \operatorname{Ext}^1_R(T_1, M) \to \cdots$$

Notice that $T_1 \oplus T_1' = T^{(I)}$ for some set I and right R-module T_1'. Then $\operatorname{Ext}^1_R(T_1, M) = 0$ because $\operatorname{Ext}^1_R(T^{(I)}, M) = \bigoplus_I \operatorname{Ext}^1_R(T, M) = 0$. Hence $\operatorname{Hom}_R(T_0, M) \to \operatorname{Hom}_R(R, M)$ in onto. Thus, for each $f \in \operatorname{Hom}_R(R, M)$, there exists $g \in \operatorname{Hom}_R(T_0, M)$ such that $g\iota = f$. By Remarks 8.2(b), $\operatorname{Hom}_R(R, M) \cong M$ via $f \mapsto f(1)$. Hence there exists $g \in \operatorname{Hom}_R(T_0, M)$ such that $g(\iota(1)) = m$ for each $m \in M$, i.e. $M \in \operatorname{Gen} T_0 \subseteq \operatorname{Gen} T$.

Conversely, suppose now that $\operatorname{Gen} T = T^\perp$. Let $N \in \text{Mod-}R$ and consider the exact sequence $0 \to N \to E \to E/N \to 0$ with E an injective right R-module. Applying $\operatorname{Hom}_R(T, \_)$ to this sequence we get

$$\cdots \to \operatorname{Ext}^1_R(T, E/N) \to \operatorname{Ext}^2_R(T, N) \to \operatorname{Ext}^2_R(T, E) \to \cdots$$

Since E is injective, then $\operatorname{Ext}^2_R(T, E) = 0$ and $E \in T^\perp = \operatorname{Gen} T$. Thus $E/N \in \operatorname{Gen} T = T^\perp$ and $\operatorname{Ext}^1_R(T, E/N) = 0$. Therefore $\operatorname{Ext}^2_R(T, N) = 0$. As N was arbitrary, we have just proved that $\operatorname{pd} T \leq 1$, that is, (T1) is satisfied.

Obviously (T2) is satisfied because $T^{(I)} \in \operatorname{Gen} T = T^\perp$ for each set I.

By Corollary 8.9, with $M = R$, there exist right R-modules T_0 and T_1 such that $T_0 \in T^\perp$, $T_1 \in {}^\perp(T^\perp)$, T_1 is a direct sum of copies of T, and $0 \to R \to T_0 \to T_1 \to 0$ is an exact sequence. In particular, $T_1 \in \operatorname{Add} T$. Observe that $T_0 \in {}^\perp(T^\perp)$ because R and T_1 belong to ${}^\perp(T^\perp)$. By (i), $T_0 \in T^\perp = \operatorname{Gen} T = \operatorname{Pres} T$. Let $0 \to X \to T^{(J)} \to T_0 \to 0$ be a presentation of T_0 with $X \in \operatorname{Gen} T = T^\perp$ and J a set. It splits because $\operatorname{Ext}^1_R(T_0, X) = 0$. Therefore $T_0 \in \operatorname{Add} T$, and (T3) is satisfied. □

Corollary 8.11. *Let T and T' be two tilting right R-modules. Then T is equivalent to T' if and only if* $\operatorname{Add} T = \operatorname{Add} T'$.

Proof. If $\operatorname{Add} T = \operatorname{Add} T'$, then $T \in \operatorname{Add} T' \subseteq \operatorname{Gen} T'$ and $T' \in \operatorname{Add} T \subseteq \operatorname{Gen} T$. This and Proposition 8.10(ii) imply that $T^\perp = \operatorname{Gen} T = \operatorname{Gen} T' = T'^\perp$.

Conversely, suppose that $T^\perp = T'^\perp$. Let $X \in \operatorname{Add} T' \subseteq \operatorname{Gen} T' = \operatorname{Gen} T = \operatorname{Pres} T$ by Proposition 8.10. Then X has a presentation

$$0 \to K \to T^{(I)} \to X \to 0 \tag{76}$$

with $K \in \operatorname{Gen} T = \operatorname{Gen} T' = T'^\perp$, then (76) splits because $\operatorname{Ext}^1_R(X, K) = 0$. Hence $X \in \operatorname{Add} T$. By symmetry, $\operatorname{Add} T' = \operatorname{Add} T$. □

Definitions 8.12. Let R be a ring.

(a) A tilting right R-module T is said to be of *finite type* provided that there is a set $\mathcal{S}$ of finitely presented right R-modules of projective dimension at most one such that $T^\perp = \mathcal{S}^\perp$.
(b) A tilting class $T^\perp$ is of *finite type* if T is equivalent to a tilting module T' of finite type.
(c) A class $\mathcal{C}$ of right R-modules is called *definable* if
   (i) $\mathcal{C}$ is closed under products, i.e. for any set I, if $L_i \in \mathcal{C}$ for all $i \in I$, then $\prod_{i \in I} L_i \in \mathcal{C}$.
   (ii) $\mathcal{C}$ is closed under direct limits, i.e. for any direct system $\{L_i, f_{ji} \mid i \leq j \in I\}$, if $L_i \in \mathcal{C}$ for all $i \in I$, then $\varinjlim L_i \in \mathcal{C}$.
   (iii) $\mathcal{C}$ is closed under pure submodules, i.e. if L_1 is a pure submodule of $L_2 \in \mathcal{C}$, then $L_1 \in \mathcal{C}$. □

The following result is due to S. Bazzoni.

Proposition 8.13. *Let R be a ring, and let $\mathcal{S}$ be a set of finitely presented right R-modules of projective dimension at most one. Then there exists a tilting right R-module T with $T^\perp = \mathcal{S}^\perp$.*

Proof. For each $X \in \mathcal{S}$ fix a projective presentation

$$0 \to P_X \overset{\alpha_X}{\to} F_X \to X \to 0 \tag{77}$$

with F_X a finitely generated free right R-module and P_X a finitely generated projective right R-module.

Proceeding as in the proof of Theorem 8.8 with $M = R$, since all modules in $\mathcal{S}$ are finitely generated, then the regular cardinal δ can be supposed to be ω. Hence we obtain a module B with a continuous chain of submodules $\{B_n\}_{n \in \mathbb{N}}$ satisfying

(i) $B_0 = R$.
(ii) B_{n+1}/B_n isomorphic to a direct sum of modules in $\mathcal{S}$ for each $n \in \mathbb{N}$. More precisely, $B_{n+1}/B_n \cong \bigoplus_{X \in \mathcal{S}} X^{(\operatorname{Hom}_R(P_X, B_n))}$.
(iii) $B = \lim_{n \to \infty} B_n$. Therefore B/R is $\mathcal{S}$-filtered.
(iv) $B \in \mathcal{S}^\perp$.
(v) $B/R \in {}^\perp(\mathcal{S}^\perp)$.
(vi) $\operatorname{pd}(B/R) \leq 1$ because $\operatorname{pd} X \leq 1$ for all $X \in \mathcal{S}$.

Our tilting module will be $T = B \oplus B/R$. From

$$0 \to R \to B \to B/R \to 0 \tag{78}$$

we clearly have $B, B/R \in \operatorname{Add} T$. Hence (T3) is satisfied. Now we prove that T verifies (T1) and (T2).

(T1) Let K be any right R-module. Applying $\operatorname{Hom}(\_, K)$ to (78) we obtain

$$\cdots \to 0 = \operatorname{Ext}^1_R(R, K) \to \operatorname{Ext}^2_R(B/R, K) \to \operatorname{Ext}^2_R(B, K) \to \operatorname{Ext}^2_R(R, K) = 0.$$

So to check that $\operatorname{pd}(T) \leq 1$, it is enough to see that $\operatorname{pd}(B/R) \leq 1$. But it has already been proved in (vi).

(T2) Since $B \in \mathcal{S}^{\perp}$ by (iv), Remarks 8.2(a) shows that $B/R \in \mathcal{S}^{\perp}$. Therefore $T \in \mathcal{S}^{\perp}$ by the properties of $\operatorname{Ext}^1_R(\_,\_)$ with respect to direct products on the second component.

Let I be a set. Since $\operatorname{Ext}^1_R(T, T^{(I)}) \cong \operatorname{Ext}^1_R(B, T^{(I)}) \oplus \operatorname{Ext}^1_R(B/R, T^{(I)})$, applying $\operatorname{Hom}_R(\_, T^{(I)})$ to (78), we get that $\operatorname{Ext}^1_R(T, T^{(I)}) = 0$ if and only if $\operatorname{Ext}^1_R(B/R, T^{(I)}) = 0$.

Denote B_n/R by $\bar{B}_n$ for each $n \in \mathbb{N}$, and let $a_{X,n}$ denote a disjoint copy of $\operatorname{Hom}_R(P_X, B_n)$. By (ii), $\bar{B}_{n+1}/\bar{B}_n \cong \bigoplus_{X \in \mathcal{S}} X^{(a_{X,n})}$ if $n \in \mathbb{N}$. Therefore $\operatorname{Ext}^1_R(\bar{B}_0, T^{(I)}) = 0$ and

$$\operatorname{Ext}^1_R(\bar{B}_{n+1}/\bar{B}_n, T^{(I)}) \cong \operatorname{Ext}^1_R(\bigoplus_{X \in \mathcal{S}} X^{(a_{X,n})}, T^{(I)}) \cong \prod_{X \in \mathcal{S}} \Big(\prod_{a_{X,n}} \Big(\bigoplus_I \operatorname{Ext}^1_R(X, T) \Big) \Big) = 0,$$

because $T \in \mathcal{S}^{\perp}$, and the fact that $\operatorname{Ext}^1_R(X, T^{(I)}) \cong \bigoplus_I \operatorname{Ext}^1_R(X, T)$ by Lemma 1.31 because X is finitely presented of projective dimension at most one. Thus $\operatorname{Ext}^1_R(B/R, T^{(I)}) = 0$ by Eklof-Lemma 1.29.

It only remains to check that $T^{\perp} = \mathcal{S}^{\perp}$. Again, applying $\operatorname{Hom}_R(\_, T^{(I)})$ to (78), we get that $Z \in T^{\perp}$ if and only if $Z \in (B/R)^{\perp}$. If $Z \in \mathcal{S}^{\perp}$, then $Z \in (B/R)^{\perp}$ because of (iii) and Eklof-Lemma 1.29. Hence $\mathcal{S}^{\perp} \subseteq T^{\perp}$.

We have already seen that $T \in \mathcal{S}^{\perp}$. Now $\mathcal{S}^{\perp}$ is closed under direct sums by Lemma 1.31 because $\mathcal{S}$ consists of finitely presented modules of projective dimension at most one. Note that $\mathcal{S}^{\perp}$ is closed under images by Remarks 8.2(a). Thus $\operatorname{Gen} T = T^{\perp} \subseteq \mathcal{S}^{\perp}$. □

Proposition 8.13 is very useful because it tells us that many classes are tilting classes. On the other hand, the tilting module constructed is somehow unmanageable, and not very useful when trying to give a classification of tilting modules over some ring R.

REMARK 8.14. Let R be a ring. Let $\mathcal{C}$ be a class of right R-modules such that for each $X \in \mathcal{C}$ there exists a projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0$$

with P_0, P_1, P_2 finitely generated projective right R-modules. Then $\mathcal{C}^{\perp}$ is a definable class. In particular, if $\mathcal{C}$ is a set of finitely presented modules of projective dimension at most one, then $\mathcal{C}^{\perp}$ is definable.

PROOF. The class $\mathcal{C}^{\perp}$ is closed under products because $\prod_{i \in I} \operatorname{Ext}^1_R(M, B_i) \cong \operatorname{Ext}^1_R(M, \prod_{i \in I} B_i)$ holds in general for any set I, and $M, B_i \in \text{Mod-}R$, $i \in I$.

The class $\mathcal{C}^{\perp}$ is closed under direct limits because $\operatorname{Ext}^1_R(X, \varinjlim N_i) \cong \varinjlim \operatorname{Ext}^1_R(X, N_i)$ for any direct system $\{(N_i, f_{ij}) \mid i \leq j \in I\}$ of right R-modules and $X \in \mathcal{C}$ by Lemma 1.31.

Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of right R-modules such that $B \in \mathcal{C}^{\perp}$. If $X \in \mathcal{C}$, applying $\operatorname{Hom}_R(X, \_)$, we get

$$\begin{aligned} 0 \to \operatorname{Hom}_R(X, A) \to \operatorname{Hom}_R(X, B) \to \operatorname{Hom}_R(X, C) \to \\ \to \operatorname{Ext}^1_R(X, A) \to \operatorname{Ext}^1_R(X, B) = 0. \end{aligned}$$

On the other hand, since X is finitely presented, $\operatorname{Hom}_R(X, B) \to \operatorname{Hom}_R(X, C)$ is onto by Lemma 1.33. Therefore $\operatorname{Ext}^1_R(X, A) = 0$. Thus $\mathcal{C}^{\perp}$ is closed under pure submodules. □

Let R be a ring. Let T be a tilting right R-module. The class $T^{\perp}$ is closed under direct products by the properties of $\operatorname{Ext}^1_R(\_,\_)$. Observe that $T^{\perp}$ is closed under direct sums because $T^{\perp} = \operatorname{Gen} T$ by Proposition 8.10. Hence $T^{\perp}$ is closed under direct limits because for any direct system $\{L_i, f_{ji} \mid i \leq j \in I\}$ there exists an onto morphism $\bigoplus_{i \in I} L_i \to \varinjlim L_i$ and $T^{\perp}$ is closed under images by Remarks 8.2(a). Furthermore we have the following result from

[**BH08**, Theorem 2.6] (or see also [**GT06**, Chapter 5]) whose proof is far from the scope of this dissertation. It also gives the converse of Proposition 8.13.

THEOREM 8.15. *Let R be a ring. Then any tilting class is definable and of finite type.* □

As noted in Section 2 of [**BH08**], a combination of Ziegler's result [**Zie84**, Theorem 6.9] and the Keisler-Shelah Theorem (cf. [**Kei61**] and [**She71**]) implies that two definable classes are the same if and only if they contain the same indecomposable pure injective modules. Hence we obtain [**BH08**, Corollary 2.7]:

COROLLARY 8.16. *Let R be a ring. Let $\mathcal{C}$ and $\mathcal{C}'$ be two tilting classes of right R-modules. Then $\mathcal{C} = \mathcal{C}'$ if and only if they contain the same indecomposable pure-injective modules.* □

Notice that Corollary 8.16 makes easier to work with tilting classes because of Lemma 1.34.

2. Ring epimorphisms

DEFINITION 8.17. Let R, S be two rings and $\lambda\colon R \to S$ a morphism of rings. λ is a *ring epimorphism* if, for every pair of morphism of rings $\delta_i\colon S \to S'$, $i = 1, 2$, the condition $\delta_1\lambda = \delta_2\lambda$, implies that $\delta_1 = \delta_2$. □

EXAMPLES 8.18. Let R be a ring.

(a) Let Σ be a class of morphisms between finitely generated projective right R-modules, then $\lambda\colon R \to R_\Sigma$, the universal localization of R at Σ, is a ring epimorphism.
(b) Let $\mathfrak{S}$ be a left denominator set of R. Then $R \to \mathfrak{S}^{-1}R$ is a ring epimorphism.
(c) Let $\lambda\colon R \to D$ be a morphism of rings such that D is a division ring of fractions of $\operatorname{im}\lambda$, then λ is a ring epimorphism.

PROOF. (a) Suppose that $\delta\colon R_\Sigma \to S$ is a morphism of rings. Then

$$\alpha \otimes_R 1_S = \alpha \otimes_R 1_{R_\Sigma} \otimes_{R_\Sigma} 1_S,$$

for every $\alpha \in \Sigma$. Now $\alpha \otimes_R 1_{R_\Sigma}$ is invertible. Hence $\alpha \otimes_R 1_S$ is invertible, and δ is the only possible morphism of rings extending λ by the universal property of R_Σ.

(b) Follows from (a) because the left Ore localization $\mathfrak{S}^{-1}R$ is the universal localization of R at the morphisms of right R-modules $R \to R$, $r \mapsto sr$, for all $s \in \mathfrak{S}$, by Examples 3.48(a).

(c) Let $\delta_1, \delta_2\colon D \to S$ be morphisms of rings. Recall that $Q_{n+1}(\operatorname{im}\lambda, D)$ is the subring of D generated by the subring $Q_n(\operatorname{im}\lambda, D)$ and the inverses of its nonzero elements. Hence if δ_1 agrees with δ_2 in $d \in Q_n(\operatorname{im}\lambda, D) \setminus \{0\}$, then $\delta_1(d^{-1}) = \delta_1(d)^{-1} = \delta_2(d)^{-1} = \delta_2(d^{-1})$. Hence if δ_1 coincides with δ_2 on $\operatorname{im}\lambda = Q_0(\operatorname{im}\lambda, D)$, then they agree on $D = \bigcup_{n\geq 0} Q_n(\operatorname{im}\lambda, D)$. □

Observe that if $\lambda\colon R \to S$ is a morphism of rings, then every right (left) S-module is a right (left) R-module "by restriction of scalars", and every morphism of right (left) S-modules is a morphism of right (left) R-modules. When λ is a ring epimorphism we can say more about this. For example [**Ste75**, Chapter XI, Proposition 1.2]:

LEMMA 8.19. *Let $\lambda\colon R \to S$ be a morphism of rings. The following are equivalent:*

(i) *λ is a ring epimorphism.*
(ii) *Mod-S is a full subcategory of Mod-R.*
(iii) *$S \otimes_R S \to S$, given by $s_1 \otimes s_2 \mapsto s_1 s_2$, is an isomorphism of S-bimodules.* □

REMARKS 8.20. Let $\lambda\colon R \to S$ be a ring epimorphism.

(a) Given two right S-modules M, N and a morphism of right R-modules $f\colon M \to N$, then f is a morphism of right S-modules by Lemma 8.19(ii).

(b) The map $S \to S \otimes_R S$ defined by $s \mapsto s \otimes 1$ is the inverse of the isomorphism given in Lemma 8.19(iii).
(c) Let M be a right S-module, and N a left S-module. Then

$$M \otimes_R N \cong M \otimes_S S \otimes_R S \otimes_S N \cong M \otimes_S S \otimes_S N \cong M \otimes_S N$$

by Lemma 8.19(iii). □

We will deal with ring epimorphisms $\lambda\colon R \to S$ such that $\mathrm{Tor}_1^R(S,S) = 0$. This condition has been characterized by A.I. Schofield [**Sch85**, Theorem 4.8] as follows:

THEOREM 8.21. *Let $\lambda\colon R \to S$ be a ring epimorphism. The following statements are equivalent*
(i) $\mathrm{Tor}_1^R(S,S) = 0$.
(ii) $\mathrm{Tor}_1^R(M,N) = \mathrm{Tor}_1^S(M,N)$ *for all* $M \in \text{Mod-}S$ *and* $N \in S\text{-Mod}$.
(iii) $\mathrm{Ext}_R^1(M,N) = \mathrm{Ext}_S^1(M,N)$ *for all* $M, N \in \text{Mod-}S$.
(iv) $\mathrm{Ext}_R^1(M,N) = \mathrm{Ext}_S^1(M,N)$ *for all* $M, N \in S\text{-Mod}$. □

We want to characterize the S-modules among the R-modules, the following notion from [**GL91**] will be useful for our discussion.

DEFINITION 8.22. If $\mathcal{S}$ is a class of right R-modules, the *(right) perpendicular category to* $\mathcal{S}$ is defined to be the full subcategory $\mathcal{X}_{\mathcal{S}}$ of Mod-R consisting of all modules A satisfying the following two conditions:
(a) $\mathrm{Hom}_R(S,A) = 0$ for all $S \in \mathcal{S}$.
(b) $\mathrm{Ext}_R^1(S,A) = 0$ for all $S \in \mathcal{S}$.
If $\mathcal{S} = \{S\}$ we will write $\mathcal{X}_S$ instead of $\mathcal{X}_{\{S\}}$. □

The following result is proved in [**GL91**, Proposition 4.12] but in a less general context.

THEOREM 8.23. *Let $\lambda\colon R \to S$ be an injective ring epimorphism with* $\mathrm{Tor}_1^R(S,S) = 0$. *Then the following are equivalent for* $M \in \text{Mod-}R$.
(i) $M \in \text{Mod-}S$.
(ii) $\mathrm{Ext}_R^1(S/R,M) = \mathrm{Hom}_R(S/R,M) = 0$, *i.e.* $M \in \mathcal{X}_{S/R}$.

PROOF. Applying $\mathrm{Hom}_R(\_,M)$ to the exact sequence $0 \to R \to S \to S/R \to 0$, we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(S/R,M) \to \mathrm{Hom}_R(S,M) \xrightarrow{\gamma} \mathrm{Hom}_R(R,M) \to \\ \to \mathrm{Ext}_R^1(S/R,M) \to \mathrm{Ext}_R^1(S,M) \end{aligned}$$

(i) $\Rightarrow$ (ii): If $M \in \text{Mod-}S$, then $\mathrm{Ext}_R^1(S,M) = \mathrm{Ext}_S^1(S,M) = 0$ by Theorem 8.21. Moreover, the composition of maps $M \cong \mathrm{Hom}_S(S,M) = \mathrm{Hom}_R(S,M) \xrightarrow{\gamma} \mathrm{Hom}_R(R,M) \cong M$ is the identity on M, and γ is an isomorphism. Hence $\mathrm{Ext}_R^1(S/R,M) = \mathrm{Hom}_R(S/R,M) = 0$.
(ii) $\Rightarrow$ (i): Assume that $\mathrm{Ext}_R^1(S/R,M) = \mathrm{Hom}_R(S/R,M) = 0$. Then γ is an isomorphism, and $\mathrm{Hom}_R(S,M) \xrightarrow{\gamma} \mathrm{Hom}_R(R,M) \cong M$, $f \mapsto f_{|R} \mapsto f(1)$, endows M with a structure of right S-module. □

REMARK 8.24. As a consequence of the last proof, we see that for a right R-module M, the only possible structure as right S-module is the one given by $\mathrm{Hom}_R(S,M)$. □

As we have seen in Examples 8.18, if R is a ring, then $\lambda\colon R \to R_\Sigma$, the universal localization of R at a class of morphisms between finitely generated projective right R-modules, provides examples of ring epimorphisms. Now we want to show that $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$ [**BD78**, Section 5]. For that we proceed as in [**Sch85**, Theorem 4.7]. We begin with a lemma that will also be useful later.

Lemma 8.25. *Let R be a ring, and let Σ be a class of morphisms between finitely generated projective right R-modules. A left R-module X is a left R_Σ-module if and only if $\alpha\otimes 1_X$ is an isomorphism for each $\alpha\in\Sigma$.*

Proof. Suppose that X is an R_Σ-module. Let $\alpha\colon P\to Q$ be a morphism from Σ. Then $\alpha\otimes 1_X$ is the composition of the following isomorphisms

$$P\otimes_R X\cong P\otimes_R R_\Sigma\otimes_{R_\Sigma} X\overset{\alpha\otimes 1_{R_\Sigma}\otimes 1_X}{\longrightarrow} Q\otimes_R R_\Sigma\otimes_{R_\Sigma} X\cong Q\otimes_R X.$$

Thus $\alpha\otimes 1_X$ is an isomorphism.

Conversely, suppose that $\alpha\otimes 1_X$ is an isomorphism for each $\alpha\in\Sigma$. Consider the canonical morphism of rings given by the structure of left R-module of X, $\eta\colon R\to \mathrm{End}_{\mathbb{Z}}(X)$. Let $\alpha\colon P\to Q$ be any morphism from Σ. Let E,F and A be matrices over R that represent P,Q and α respectively. Then, since $\alpha\otimes 1_X$ is an isomorphism, then $A\in\mathrm{End}_{\mathbb{Z}}(X)$ induces the isomorphism of abelian groups

$$EX^n\cong ER^n\otimes_R X\cong P\otimes_R X\overset{\alpha\otimes 1_X}{\longrightarrow} Q\otimes_R X\cong FR^m\otimes_R X\cong FX^m.$$

Hence there exists a unique matrix $B\in\mathbb{M}_{n\times m}(\mathrm{End}_{\mathbb{Z}}(X))$ such that $BA=E,\ AB=F$ and $EBF=B$. By Remark 3.51, there exists a unique morphism of rings $\bar\eta\colon R_\Sigma\to\mathrm{End}_{\mathbb{Z}}(X)$ such that $\bar\eta\lambda=\eta$. Then $\bar\eta$ endows X with a structure of left R_Σ-modules. ☐

Lemma 8.26. *Let R be a ring and Σ a class of morphisms between finitely generated left R-modules. Let $\lambda\colon R\to R_\Sigma$ be the universal localization of R at Σ. Let L,N be two right R_Σ-modules. Then $\mathrm{Ext}^1_R(M,N)=\mathrm{Ext}^1_{R_\Sigma}(M,N)$.*

Proof. Consider a right R-module X which is an extension (as R-module) of M by N. Thus there exists an exact sequence of right R-modules $0\to N\to X\to M\to 0$. Let $\alpha\colon P\to Q$ be any morphism from Σ. Applying $P\otimes_R {\_}$ and $Q\otimes_R {\_}$, we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P\otimes_R N \arrow[r] \arrow[d, "\alpha\otimes 1_N"] & P\otimes_R X \arrow[r] \arrow[d, "\alpha\otimes 1_X"] & P\otimes_R M \arrow[r] \arrow[d, "\alpha\otimes 1_M"] & 0\\
0 \arrow[r] & Q\otimes_R N \arrow[r] & Q\otimes_R X \arrow[r] & Q\otimes_R M \arrow[r] & 0
\end{tikzcd}$$

By Lemma 8.25, $\alpha\otimes 1_N$ and $\alpha\otimes 1_M$ are isomorphisms. Thus $\alpha\otimes 1_X$ is an isomorphism for all $\alpha\in\Sigma$. Again by Lemma 8.25, X is a left R_Σ-module. ☐

Now, by Lemma 8.26 and Theorem 8.21, we can state our desired result.

Theorem 8.27. *Let R be a ring and Σ a class of morphisms between finitely generated projective right R-modules. Let $\lambda\colon R\to R_\Sigma$ be the universal localization of R at Σ. Then* $\mathrm{Tor}_1^R(R_\Sigma,R_\Sigma)=0$. ☐

From this result, as it is done in [**Sch85**, Theorem 4.9], we obtain [**BD78**, Theorem 5.3]:

Theorem 8.28. *Let R be a ring and Σ a class of morphisms between finitely generated projective right R-modules. Let $\lambda\colon R\to R_\Sigma$ be the universal localization of R at Σ. If R is right hereditary, then R_Σ is right hereditary.*

Proof. In general, a ring S is right hereditary if and only if $\mathrm{Ext}^1_S(M,{\_})$ is a right exact functor for all $M\in\text{Mod-}S$. By Theorems 8.21 and 8.27, $\mathrm{Ext}^1_{R_\Sigma}(M,N)=\mathrm{Ext}^1_R(M,N)$ for all $M,N\in\text{Mod-}R_\Sigma$. Now R_Σ is right hereditary because $\mathrm{Ext}^1_R(M,{\_})$ is right exact for all $M\in\text{Mod-}R$. ☐

3. Tilting modules arising from ring epimorphisms

We will now study tilting modules, like $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$ (see Example 8.4), constructed from injective ring epimorphisms. We start out with a generalization of some results from [**AHHT05**, Section 6], which in turn generalized part of [**Mat73**, Chapter 1]. In [**AHHT05**] it is considered the ring epimorphism $R \to S = \mathfrak{S}^{-1}R$ where $\mathfrak{S}$ is a left Ore set consisting of non-zero-divisors of the (not necessarily commutative) ring R, while in [**Mat73**] S is the *full quotient ring* (i.e. $S = \mathfrak{S}^{-1}R$ and $\mathfrak{S}$ the set consisting of all non-zero-divisors of R) of the commutative ring R.

The following two technical lemmas generalize [**AHHT05**, Lemma 6.2] and [**Mat73**, Lemma 1.8], respectively. They are proved in the same way as the original ones.

LEMMA 8.29. *Let $\lambda\colon R \to S$ be a morphism of rings, and let M be a right R-module. The image of the morphism $\operatorname{Hom}_R(S, M) \to M$, $f \mapsto f(1)$ coincides with the trace*

$$\operatorname{tr}_S(M) = \sum\{f(S) \mid f \in \operatorname{Hom}_R(S, M)\}$$

of S in M.

PROOF. Notice that $\operatorname{Hom}_R(S, M)$ is a right S-module. Now $m \in \operatorname{tr}_S(M)$ if and only if there exist $s_1, \dots, s_n \in S$, and $f_1, \dots, f_n \in \operatorname{Hom}_R(S, M)$ such that

$$m = f_1(s_1) + \cdots + f_n(s_n) = (f_1 s_1 + \cdots + f_n s_n)(1).$$

Hence $m = g(1)$ for $g = f_1 s_1 + \cdots + f_n s_n \in \operatorname{Hom}_R(S, N)$. $\square$

LEMMA 8.30. *Let $\lambda\colon R \to S$ be a morphism of rings. Then the following statements are equivalent*

(i) $\operatorname{tr}_S(M/\operatorname{tr}_S(M)) = 0$ *for all* $M \in \text{Mod-}R$, *that is,* $\operatorname{Hom}_R(S, M/\operatorname{tr}_S(M)) = 0$ *for all* $M \in \text{Mod-}R$.

(ii) $\operatorname{Gen} S_R$ *is closed under extensions.*

PROOF. (i) $\Rightarrow$ (ii): Let $0 \to A \to B \to B/A \to 0$ be an exact sequence of right R-modules with $A, B/A \in \operatorname{Gen} S_R$. Since A is contained in $\operatorname{tr}_S(B)$, we get the surjective morphism of right R-modules $B/A \to B/\operatorname{tr}_S(B)$. Hence $B/\operatorname{tr}_S(B) \in \operatorname{Gen} S_R$, but by hypothesis $\operatorname{Hom}_R(S, B/\operatorname{tr}_S(B)) = 0$. Therefore $B/\operatorname{tr}_S(B) = 0$ and $B = \operatorname{tr}_S(B) \in \operatorname{Gen} S_R$.

(ii) $\Rightarrow$ (i): Suppose that $\operatorname{tr}_S(M/\operatorname{tr}_S(M)) \neq 0$ for a right R-module M. Then there exists a submodule X of M such that X contains $\operatorname{tr}_S(M)$, $X/\operatorname{tr}_S(M) \neq 0$ and $X/\operatorname{tr}_S(M) \in \operatorname{Gen} S_R$. Consider the exact sequence $0 \to \operatorname{tr}_S(M) \to X \to X/\operatorname{tr}_S(M) \to 0$. By hypothesis, $X \in \operatorname{Gen} S_R$, which implies that $X = \operatorname{tr}_S(M)$, a contradiction. $\square$

In the following result, (ii) is the generalization of [**AHHT05**, Lemma 6.1] and it is proved in the same way. And (i) could be stated in a more general way, but we prove it in our context.

LEMMA 8.31. *Let $\lambda\colon R \to S$ be an injective ring epimorphism with $\operatorname{Tor}_1^R(S, S) = 0$. Denote by $\mathcal{X}_S$ the perpendicular category to S_R. Then*

(i) *For every injective right R-module N,*

$$\operatorname{Ext}^1_R(S, \operatorname{Hom}_R(S/R, N)) \cong \operatorname{Hom}_R(\operatorname{Tor}_1^R(S, S/R), N) = 0.$$

(ii) $\operatorname{Hom}_R(S/R, M) \in \mathcal{X}_S$ *for any right R-module M.*

PROOF. From the exact sequence of R-bimodules

$$0 \to R \to S \to S/R \to 0, \tag{79}$$

since N_R is injective, we get the exact sequence of right R-modules

$$0 \to \mathrm{Hom}_R(S/R, N) \to \mathrm{Hom}_R(S, N) \to \mathrm{Hom}_R(R, N) \to 0.$$

Applying $\mathrm{Hom}_R(S, \_)$ we obtain

$$\begin{aligned}0 \to \mathrm{Hom}_R(S, \mathrm{Hom}_R(S/R, N)) \to \mathrm{Hom}_R(S, \mathrm{Hom}_R(S, N)) \to \mathrm{Hom}_R(S, \mathrm{Hom}_R(R, N)) \to \\ \to \mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S/R, N)) \to \mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S, N)) = 0.\end{aligned} \tag{80}$$

Notice that $\mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S, N)) = \mathrm{Ext}^1_S(S, \mathrm{Hom}_R(S, N)) = 0$ because $\mathrm{Hom}_R(S, N)$ is a right S-module and Theorem 8.21.

Applying $S \otimes_R \_$ to (79), we get the exact sequence of R-bimodules

$$0 = \mathrm{Tor}^R_1(S, S) \to \mathrm{Tor}^R_1(S, S/R) \to S \otimes_R R \to S \otimes_R S \to S \otimes_R S/R \to 0. \tag{81}$$

Now, from the natural isomorphisms $S \otimes_R S \cong S \cong S \otimes_R R$ in Lemma 8.19(iii), we see that

$$\mathrm{Tor}^R_1(S, S/R) = S \otimes_R S/R = 0. \tag{82}$$

In particular, $\mathrm{Hom}_R(\mathrm{Tor}^R_1(S, S/R), N) = 0$.

By the injectivity of N,

$$\begin{aligned}0 \to \mathrm{Hom}_R(S \otimes_R S/R, N) \to \mathrm{Hom}_R(S \otimes_R S, N) \to \mathrm{Hom}_R(S \otimes_R R, N) \to \\ \to \mathrm{Hom}_R(\mathrm{Tor}^R_1(S, S/R), N) \to 0\end{aligned} \tag{83}$$

Notice that by the Hom-tensor adjunction the first three elements in (80) are naturally isomorphic to the first three elements in (83). Hence

$$\mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S/R, N)) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(S, S/R), N) = 0,$$

and (i) is proved.

To prove (ii), denote by $E(M)$ the injective hull of M. Applying the functor $\mathrm{Hom}_R(S/R, \_)$ to the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ we obtain

$$0 \to \mathrm{Hom}_R(S/R, M) \to \mathrm{Hom}_R(S/R, E(M)) \xrightarrow{\alpha} \mathrm{Hom}_R(S/R, E(M)/M). \tag{84}$$

For any right R-module M the Hom-tensor adjunction gives that

$$\mathrm{Hom}_R(S, \mathrm{Hom}_R(S/R, M)) \cong \mathrm{Hom}_R(S \otimes_R S/R, M).$$

Then, by (82), we get that

$$\mathrm{Hom}_R(S, \mathrm{Hom}_R(S/R, M)) = 0, \text{ for any right } R\text{-module } M. \tag{85}$$

Hence $\mathrm{Hom}_R(S/R, N) \in \mathcal{X}_S$ provided N is injective.

Applying $\mathrm{Hom}_R(S, \_)$ to (84) we obtain the exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_R(S, \mathrm{Hom}_R(S/R, M)) \to \mathrm{Hom}_R(S, \mathrm{Hom}_R(S/R, E(M))) \to \mathrm{Hom}_R(S, \mathrm{im}\,\alpha) \to \\ \to \mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S/R, M)) \to \mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S/R, E(M))).\end{aligned}$$

By (i), $\mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S/R, E(M))) = 0$. And (85) implies that $\mathrm{Hom}_R(S, \mathrm{im}\,\alpha) = 0$ because $\mathrm{im}\,\alpha \leq \mathrm{Hom}_R(S/R, E(M)/M)$.

Therefore $\mathrm{Hom}_R(S, \mathrm{Hom}_R(S/R, M)) = \mathrm{Ext}^1_R(S, \mathrm{Hom}_R(S/R, M)) = 0$, as desired. □

Now it is time to give the generalization of [**AHHT05**, Proposition 1.3].

THEOREM 8.32. *Let R be a ring. Let $\lambda\colon R \to S$ be an injective ring epimorphism with $\mathrm{Tor}^R_1(S, S) = 0$. Denote by $\mathcal{X}_S$ the perpendicular category to S_R. The following conditions are equivalent.*

(i) $\mathrm{pd}(S_R) \leq 1$.

(ii) *$\mathcal{X}_S$ is closed under cokernels of monomorphisms.*

(iii) $\operatorname{Ext}^1_R(S/R, M)$ *belongs to* $\mathcal{X}_S$ *for any right* R*-module* M.
(iv) $(S/R)^\perp = \operatorname{Gen} S_R$.
(v) $T = S \oplus S/R$ *is a tilting right* R*-module.*
(vi) $\operatorname{pd}((S/R)_R) \leq 1$.

Moreover, under (i)-(vi), $\operatorname{Hom}_R(S, M/\operatorname{tr}_S(M)) = 0$ for any right R-module M.

Proof. (i) $\Rightarrow$ (ii): Let $0 \to L \to M \to N \to 0$ be an exact sequence with $L, M \in \mathcal{X}_S$. Applying $\operatorname{Hom}_R(S, \_)$ we obtain the exact sequence

$$\to \operatorname{Hom}_R(S, M) \to \operatorname{Hom}_R(S, N) \to \operatorname{Ext}^1_R(S, L) \to \operatorname{Ext}^1_R(S, M) \to \operatorname{Ext}^1_R(S, N) \to \operatorname{Ext}^2_R(S, L)$$

Now, $\operatorname{Hom}_R(S, M) = \operatorname{Ext}^1_R(S, L) = \operatorname{Ext}^1_R(S, M) = 0$ because $L, M \in \mathcal{X}_S$, and $\operatorname{Ext}^2_R(S, N) = 0$ because of (i). Therefore $\operatorname{Hom}_R(S, N) = \operatorname{Ext}^1_R(S, N) = 0$.
(ii) $\Rightarrow$ (iii): Let M be any right R-module. Denote by $E(M)$ the injective hull of M. Applying the functor $\operatorname{Hom}_R(S/R, \_)$ to the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ we get

$$\begin{aligned} 0 \to \operatorname{Hom}_R(S/R, M) \to \operatorname{Hom}_R(S/R, E(M)) \xrightarrow{\beta} \operatorname{Hom}_R(S/R, E(M)/M) \to \\ \to \operatorname{Ext}^1_R(S/R, M) \to \operatorname{Ext}^1_R(S/R, E(M)) = 0. \end{aligned}$$

By (ii) and Lemma 8.31(ii) applied to $0 \to \operatorname{Hom}_R(S/R, M) \to \operatorname{Hom}_R(S/R, E(M)) \to \operatorname{im}\beta \to 0$, we obtain that $\operatorname{im}\beta \in \mathcal{X}_S$. Repeating the argument with

$$0 \to \operatorname{im}\beta \to \operatorname{Hom}_R(S/R, E(M)/M) \to \operatorname{Ext}^1_R(S/R, M) \to 0,$$

we see that $\operatorname{Ext}^1_R(S/R, M) \in \mathcal{X}_S$.
(iii)$\Rightarrow$ (iv): Let M be a right R-module. Applying $\operatorname{Hom}_R(\_, M_R)$ to the exact sequence $0 \to R \to S \to S/R \to 0$ we obtain

$$\begin{aligned} 0 \to \operatorname{Hom}_R(S/R, M) \to \operatorname{Hom}_R(S, M) \xrightarrow{\alpha} \operatorname{Hom}_R(R, M) \to \\ \to \operatorname{Ext}^1_R(S/R, M) \to \operatorname{Ext}^1_R(S, M) \to \operatorname{Ext}^1_R(R, M) = 0. \end{aligned}$$

The natural isomorphism $\operatorname{Hom}_R(R, M) \to M$, defined by $f \mapsto f(1)$, gives a map $\alpha\colon \operatorname{Hom}_R(S, M) \to M$ whose image is the trace of S in M by Lemma 8.29. Hence $M \in \operatorname{Gen} S_R$ if and only if α is surjective. If $M \in (S/R)^\perp$, then clearly α is surjective and $M \in \operatorname{Gen} S_R$. Conversely, suppose that α is surjective. Then $\operatorname{Ext}^1_R(S/R, M) \cong \operatorname{Ext}^1_R(S, M)$, so $\operatorname{Ext}^1_R(S, M)$ belongs to $\mathcal{X}_S$ by (iii). But $\operatorname{Ext}^1_R(S, M)$ is a right S-module, and the only right S-module which belongs to $\mathcal{X}_S$ is the zero module. Hence $\operatorname{Ext}^1_R(S, M) = \operatorname{Ext}^1_R(S/R, M) = 0$, and $M \in (S/R)^\perp$.
(iv) $\Rightarrow$ (v): Observe that $T^\perp = (S/R)^\perp$. Then, by (iv), $\operatorname{Gen} T_R = \operatorname{Gen} S_R = (S/R)^\perp = T^\perp$, and so T is a tilting right R-module by Proposition 8.10(ii).
(v) $\Rightarrow$ (vi): If T_R is a tilting right R-module, then $\operatorname{pd} T_R \leq 1$, which clearly implies that $\operatorname{pd}((S/R)_R) \leq 1$ by the properties of $\operatorname{Ext}^i_R(\_, \_)$ with respect to direct sums in the first component.
(vi) $\Rightarrow$ (i) follows from the exact sequence $0 \to R \to S \to S/R \to 0$ and the long exact sequence induced by $\operatorname{Hom}_R(\_, M)$ for any right R-module M.

To prove the last assertion of the Theorem, notice that $\operatorname{Gen} S_R$ is closed under extensions by (iv). Now apply Lemma 8.30. $\square$

Remarks 8.33. Suppose that $\lambda\colon R \to S$ is a morphism of rings as in Theorem 8.32.

(1) When R is a commutative ring, and S is the full ring of quotients of R, the objects of the perpendicular category $\mathcal{X}_S$ are precisely the R-modules that Matlis called *cotorsion* in [**Mat73**].
(2) When $S = \mathfrak{S}^{-1}R$, where $\mathfrak{S}$ is a left Ore set consisting of non-zero-divisors of R, the objects in $\mathcal{X}_S$ are called $\mathfrak{S}$-*cotorsion* in [**AHHT05**].

(3) In many cases, for example if R is a hereditary ring, $S \oplus S/R$ is a two-sided tilting R-module. □

Examples 8.34. Let R be a ring.

(a) Denote by $Q^r_{\max}(R)_R$ the *maximal right ring of quotients* of R, see e.g. [**Ste75**, p. 200]. Assume that $\mathrm{pd}(Q^r_{\max}(R)_R) \le 1$ and that one of the following conditions is satisfied:
  (i) R is a right non-singular ring such that every finitely generated non-singular right R-module can be embedded in a free module, or
  (ii) $Q^r_{\max}(R)$ is right Kasch (for example, this holds true whenever $Q^r_{\max}(R)$ is semisimple).

  Then $Q^r_{\max}(R) \oplus Q^r_{\max}(R)/R$ is a tilting right R-module. This follows combining Theorem 8.32 with [**Ste75**, Chapter XII, Theorem 7.1] in case (a), or with [**Ste75**, Chapter XI, Proposition 5.3] in case (b).

(b) By [**Ste75**, Chapter XI, Theorem 4.1], there exist a ring $Q^r_{\mathrm{tot}}(R)$ and a ring epimorphism $\varphi\colon R \to Q^r_{\mathrm{tot}}(R)$ such that
  (i) φ is an injective ring epimorphism and $Q^r_{\mathrm{tot}}(R)$ is flat as a left R-module.
  (ii) For every injective epimorphism of rings $\gamma\colon R \to T$ such that ${}_RT$ is flat, there is a unique morphism of rings $\delta\colon T \to Q^r_{\mathrm{tot}}(R)$ such that $\delta\gamma = \varphi$.

  If $\mathrm{pd}(Q^r_{\mathrm{tot}}(R)_R) \le 1$, then we infer from Theorem 8.32 that $Q^r_{\mathrm{tot}}(R) \oplus Q^r_{\mathrm{tot}}(R)/R$ is a tilting right R-module.

(c) Let $\mathfrak{S}$ be a left Ore set of R consisting of non-zero-divisors. Then $R \hookrightarrow \mathfrak{S}^{-1}R$ is a ring epimorphism by Examples 8.18(b), and $\mathrm{Tor}^R_1(\mathfrak{S}^{-1}R, \mathfrak{S}^{-1}R) = 0$ because $\mathfrak{S}^{-1}R$ is a flat right R-module. Therefore, if $\mathrm{pd}(\mathfrak{S}^{-1}R_R) \le 1$, then $\mathfrak{S}^{-1}R \oplus \mathfrak{S}^{-1}R/R$ is a tilting right R-module by Theorem 8.32.

(d) Let Σ be a class of morphisms between finitely generated projective right R-modules such that the universal localization $\lambda\colon R \to R_\Sigma$ is an embedding and $\mathrm{pd}(R_\Sigma)_R \le 1$. Then $R_\Sigma \oplus R_\Sigma/R$ is a tilting right R-module by Examples 8.18(a) and Theorems 8.27 and 8.32.

(e) Until now all the examples we have provided in (a), (b) and (c) are such that S_R is a flat right R-module. This is not always the case. For example (see also [**Nee07**, Example 0.2]), let X be a nonempty set. Let G be the free group on X. Let k be a field. Consider the free algebra $R = k\langle X\rangle$, and the free group algebra kG with the natural embedding $k\langle X\rangle \hookrightarrow kG$ which sends $x \mapsto x$ for every $x \in X$. Then $T_X = kG \oplus kG/k\langle X\rangle$ is a tilting right (and left) R-module. In fact, if $\Sigma = \{\alpha_x \mid x \in X\}$ where $\alpha_x\colon k\langle X\rangle \to k\langle X\rangle$ is defined by $p \mapsto xp$, then kG can be regarded as the universal localization of R at Σ. Since $k\langle X\rangle$ is hereditary, $\mathrm{pd}(kG_{k\langle X\rangle}) \le 1$, so T_X is a tilting right (left) R-module by (d). Finally, observe that kG is not a flat right (left) $k\langle X\rangle$-module if $|X| \ge 2$. Indeed, let $x \ne y \in X$. Consider the unique embedding of left (right) $k\langle X\rangle$-modules such that

$$\begin{array}{ccc} k\langle X\rangle \oplus k\langle X\rangle & \xrightarrow{\alpha} & k\langle X\rangle \\ (1,0) & \longmapsto & x \\ (0,1) & \longmapsto & y \end{array}$$

  Consider $1_{kG} \otimes \alpha\colon kG \oplus kG \longrightarrow kG$. Then $(x^{-1}, 0)$ and $(0, y^{-1})$ have the same image 1. Thus $1_{kG} \otimes \alpha$ ($\alpha \otimes 1_{kG}$) is not injective. □

Remark 8.35. We said at the beginning of this chapter that we wanted to construct tilting modules in the same way as we did with the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ in Example 8.4. So given a ring R with a division ring of fractions D, one could think of forming a tilting right R-module $D \oplus D/R$. If R happens to be a hereditary ring, certainly $R \hookrightarrow D$ is a ring epimorphism by Examples 8.18(c) and $\mathrm{pd}\, D_R \le 1$. On the other hand, $\mathrm{Tor}^R_1(D, D) \ne 0$ in general. Now we show this.

Let Σ be the set of matrices over R that become invertible over D. Then R_Σ is a local ring and, if J is its Jacobson radical, $R_\Sigma/J \cong D$ by Theorem 3.26. So we have the exact sequence of right R-modules (and of R_Σ-modules by Remarks 8.20(a))

$$0 \to J \xrightarrow{\iota} R_\Sigma \xrightarrow{\pi} D \to 0.$$

Then the functor $\_ \otimes_R D$ induces the exact sequence

$$\mathrm{Tor}_1^R(R_\Sigma, D) \to \mathrm{Tor}_1^R(D, D) \to J \otimes_R D \to R_\Sigma \otimes_R D \to D \otimes_R D \to 0.$$

Observe that $R \to R_\Sigma$ is a ring epimorphism with $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$ by Examples 8.18(a) and Theorem 8.27. Thus $\mathrm{Tor}_1^R(R_\Sigma, D) = \mathrm{Tor}_1^{R_\Sigma}(R_\Sigma, D) = 0$ by Theorem 8.21. Applying repeatedly Remarks 8.20(c) we obtain

$$D \otimes_R D \cong D \otimes_{R_\Sigma} D \cong R_\Sigma/J \otimes_{R_\Sigma} R_\Sigma/J \cong R_\Sigma/J$$

$$R_\Sigma \otimes_R D \cong R_\Sigma \otimes_{R_\Sigma} D \cong D \cong R_\Sigma/J.$$

$$J \otimes_R D \cong J \otimes_{R_\Sigma} D \cong J \otimes_{R_\Sigma} R_\Sigma/J \cong J/J^2.$$

Hence $\pi \otimes 1_D$ is an isomorphism, and $\mathrm{Tor}_1^R(D, D) = 0$ if and only if $J/J^2 = 0$.

It is known that, if S is a ring with Jacobson radical I and P a nonzero projective right S-module, then $PI \subsetneq P$ (see for example [**Lam01**, Theorem 24.7]). Now, since R_Σ is hereditary by Theorem 8.28, then J is a projective right (and left) R_Σ-module. Thus $J^2 \neq J$, provided $J \neq 0$.

And $J = 0$ if and only if Σ is the complement of a minimal prime matrix ideal, because of the following result [**Coh95**, Theorem 4.6.14]: Let R be a weakly semihereditary ring. Then there are natural bijections between the set of minimal prime matrix ideals over R and the universal localizations (at matrices) that are division rings.

Hence, for $R = k\langle X\rangle$, R has a universal division ring of fractions, thus a unique minimal prime matrix ideal. Hence $\mathrm{Tor}_1^R(D, D) \neq 0$ for all division rings of fractions D of R which are not its universal division ring of fractions. □

Tilting modules that arise from injective ring epimorphisms as in Theorem 8.32 can be characterized as we do in [**AHS08**]:

Theorem 8.36. *Let R be a ring and T be a tilting right R-module. The following statements are equivalent.*

(i) *There is an injective ring epimorphism $\lambda : R \to S$ such that $\mathrm{Tor}_1^R(S, S) = 0$ and $S \oplus S/R$ is a tilting module equivalent to T.*
(ii) *There is an exact sequence $0 \to R \xrightarrow{a} T_0 \to T_1 \to 0$ such that $T_0, T_1 \in \mathrm{Add}\, T$ and $\mathrm{Hom}_R(T_1, T_0) = 0$.*

Moreover, under these conditions, $a : R \to T_0$ is a $T^\perp$-envelope of R, and $\lambda : R \to S$ is a ring epimorphism with $\mathrm{Tor}_i^R(S, S) = 0$ for all $i \geq 1$. □

This result allows us to show that there are tilting modules that do not arise from ring epimorphisms as in Theorem 8.32.

Examples 8.37. (a) Let R be a hereditary (indecomposable) artin algebra of infinite representation type. Denote by $\mathbf{p}$ the preprojective component of R (see Section 6). There is a countably infinitely generated tilting right R-module generating $\mathbf{p}^\perp$, called the *Lukas tilting module*, and denoted by L, cf. [**Luk91, KT05**]. It has the property that there are non-zero morphisms between any two non-zero modules from $\mathrm{Add}\, L$, see [**Luk91**, Theorem 6.1(b)] and [**Luk93**, Lemma 3.3(a)]. So, there cannot be an exact sequence $0 \to R \xrightarrow{a} L_0 \to L_1 \to 0$ such that $L_0, L_1 \in \mathrm{Add}\, L$ and $\mathrm{Hom}_R(L_1, L_0) = 0$, and therefore L does not arise from a ring epimorphism as above by Theorem 8.36.

(b) Let R be a Prüfer domain which is not a Matlis domain, that is, the quotient field Q of R has projective dimension > 1 over R. Then R has no divisible envelope, see [**GT06**, Corollary 6.3.18]. So, the Fuchs tilting module δ, which is a tilting module generating the class of all divisible modules [**GT06**, Example 5.1.2], is another example of a tilting module that does not arise from a ring epimorphism as above by Theorem 8.36. □

4. Tilting modules arising from universal localization

Our main example of epimorphism of rings is given by universal localization. As we have seen in Examples 8.34(d), if $\lambda\colon R \to R_\Sigma$ is injective and $\operatorname{pd}(R_\Sigma)_R \leq 1$, then $R_\Sigma \oplus R_\Sigma/R$ is a tilting right R-module. In this section we investigate in more detail this example. But first we begin with some definitions.

Lemma 3.53 is useful to define universal localizations R_Σ in terms of the cokernels of the morphisms between finitely generated projective modules of the class Σ when the cokernels are like follows.

Definitions 8.38. Let R be a ring.

(a) Let U be a right (left) R-module. We say that U is a *bound right (left) R-module* if U is finitely presented, $\operatorname{pd} U = 1$ and $\operatorname{Hom}_R(U, R) = 0$. In other words, U is a bound right (left) R-module if and only if U is the cokernel of some morphism $\alpha\colon P \to Q$ with $P, Q \in \mathcal{P}_R$ (${}_R\mathcal{P}$) such that α and α^* are injective.

(b) If U is a bound right R-module with projective presentation $0 \to P \overset{\alpha}{\to} Q \to U \to 0$ with $P, Q \in \mathcal{P}_R$, then we have an exact sequence $0 \to Q^* \overset{\alpha^*}{\to} P^* \to \operatorname{coker}\alpha^* \to 0$, and $\operatorname{coker}\alpha^*$ is the *Auslander-Bridger transpose* of U denoted by $\operatorname{Tr} U = \operatorname{coker}\alpha^*$, see for example [**ARS95**]. In our situation $\operatorname{Tr} U = \operatorname{Ext}^1_R(U, R)$. Observe that by the duality between finitely generated projective right R-modules and finitely generated projective left R-modules, U is a bound right R-module if and only if $\operatorname{Tr} U$ is a bound left R-module. For a class $\mathcal{U}$ of bound right R-modules we denote $\operatorname{Tr}\mathcal{U} = \{\operatorname{Tr} U \mid U \in \mathcal{U}\}$.

(c) Let $\mathcal{U}$ be a class of bound right R-modules. For each $U \in \mathcal{U}$, consider a morphism α_U between finitely generated projective right R-modules such that

$$0 \to P \overset{\alpha_U}{\to} Q \to U \to 0 \tag{86}$$

is exact. We will denote by $R_\mathcal{U}$ the universal localization of R at $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$. In fact, $R_\mathcal{U}$ does not depend on the chosen class Σ by Lemma 3.53, and we will also call it the *universal localization of R at $\mathcal{U}$*. Observe that $R_\mathcal{U} = R_{\operatorname{Tr}\mathcal{U}}$ by Remarks 3.47. By abuse of notation, we will write $\alpha_U \in \mathcal{U}$ for any morphism α_U between finitely generated projective right R-modules as in (86) with $U \in \mathcal{U}$.

(d) Let $\mathcal{U}$ be a class of bound right R-modules. A right R-module N is said to be *$\mathcal{U}$-torsion-free* if $\operatorname{Hom}_R(U, N) = 0$ for all $U \in \mathcal{U}$, and N is said to be *$\mathcal{U}$-divisible* if $\operatorname{Ext}^1_R(U, N) = 0$ for all $U \in \mathcal{U}$. □

Remarks 8.39. Let R be a ring.

(a) Observe that bound right R-modules are isomorphic to cokernels of morphisms between finitely generated projective right R-modules. We may even suppose that the projectives are free (but the morphisms need not be injective). Hence, given a class $\mathcal{V}$ of bound right R-modules, there exists a set of bound right R-modules $\mathcal{U}$ such that $\mathcal{U}^\perp = \mathcal{V}^\perp$. That is, we may suppose that $\mathcal{V}$ is a set when dealing with the class $\mathcal{V}^\perp$.

(b) Suppose that R is a semihereditary ring with a faithful rank function ρ, see Section 4 in Chapter 3. Then the ρ-torsion and the ρ-simple right R-modules are examples of bound right R-modules. □

The following results intend to explain why the classes $\mathcal{U}$ of bound right R-modules are the best candidates to construct tilting right R-modules from universal localization of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. We begin with a lemma whose proof is taken from [**Nee07**, Proposition 2.2].

LEMMA 8.40. *Let R be a ring and Σ a class of morphisms between finitely generated projective right R-modules such that the universal localization of R at Σ, $\lambda\colon R \to R_\Sigma$, is injective. Then every morphism $\alpha\colon P \to Q$ in Σ is injective.*

PROOF. First note that for each projective right R-module P, the map $\iota_P\colon P \to P \otimes_R R_\Sigma$, defined by $p \mapsto p \otimes 1$, is injective because $\lambda\colon R \to R_\Sigma$ is injective and P_R flat.

Secondly, the morphism $\alpha \otimes 1_{R_\Sigma}\colon P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism for each $\alpha\colon P \to Q$ in Σ.

Now the commutativity of the diagram

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\iota_P"] & Q \arrow[d, "\iota_Q"] \\ P \otimes_R R_\Sigma \arrow[r, "\alpha\otimes 1_{R_\Sigma}"] & Q \otimes_R R_\Sigma \end{tikzcd}$$

shows that α is injective for each $\alpha \in \Sigma$ by the foregoing observations. □

The converse of Lemma 8.40 is not true even in not so bad situations, see Theorem 3.57.

Now we give a definition of torsion submodule useful for finitely presented modules over a semihereditary ring.

DEFINITION 8.41. Let R be a ring and M a right R-module. We define

$$\mathbf{T}M = \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}.$$

Observe that $\mathbf{T}M$ is a submodule of M. Then we can define $\mathbf{P}M = M/\mathbf{T}M$. □

REMARKS 8.42. Let R be a ring and M a right R-module.

(a) $\mathbf{T}M$ is the kernel of the canonical map $\iota(M)\colon M \to (M^*)^*$ which sends $x \in M$ to the morphism $H_x\colon M^* \to R$ defined by $f \mapsto f(x)$.

(b) $\mathbf{TP}M = 0$, and given the natural projection $\pi\colon M \to \mathbf{P}M$, then $\mathrm{Hom}_R(\_, R)$ induces an isomorphism $\pi^*\colon (\mathbf{P}M)^* \to M^*$ of left R-modules. To prove these assertions observe that $\pi^*\colon (\mathbf{P}M)^* \to M^*$ is injective by the properties of $\mathrm{Hom}_R(\_, R)$. Now, if $f \in M^*$, then $\mathbf{T}M \subseteq \ker f$. Thus f factorizes through $M/\mathbf{T}M$, which proves that π^* is onto. If $x \in M$ such that $\bar{x} \in \mathbf{TP}M$, then $x \in \mathbf{P}M$ because π^* is surjective. Hence $\bar{x} = 0$. □

The proof of the next result is from [**Lüc97**, Theorem 1.2.3].

LEMMA 8.43. *Let R be a semihereditary ring and M a finitely presented right R-module. Then M is isomorphic to the direct sum of the projective right R-module $\mathbf{P}M$ and of the bound right R-module $\mathbf{T}M$.*

PROOF. Consider a presentation $R^m \xrightarrow{\varphi} R^n \to M \to 0$ of M. From it we obtain the exact sequence $0 \to M^* \to (R^n)^* \xrightarrow{\varphi^*} (R^m)^*$. Since R is semihereditary, and $(R^m)^*$ is a projective left R-module, then the finitely generated submodule $\operatorname{im}\varphi^*$ of $(R^m)^*$ is a projective left R-module. Then the exact sequence $0 \to M^* \to (R^n)^* \xrightarrow{\varphi^*} \operatorname{im}\varphi^* \to 0$ splits, and thus M^* is a direct summand of $(R^n)^*$. Hence M^* is a finitely generated projective left R-module.

By Remarks 8.42(b), $(\mathbf{P}M)^* \cong M^*$. The right R-module $(M^*)^* \cong ((\mathbf{P}M)^*)^*$ is a finitely generated projective right R-module. By Remarks 8.42(a), the kernel of the canonical map

$\iota(\mathbf{P}M)\colon \mathbf{P}M \to ((\mathbf{P}M)^*)^*$ is $\mathbf{TP}M$, which is zero by Remarks 8.42(b). Thus $\mathbf{P}M$ is isomorphic to a finitely generated submodule of $((\mathbf{P}M)^*)^*$. Therefore $\mathbf{P}M$ is a finitely generated projective right R-module because R is semihereditary. Hence the exact sequence $0 \to \mathbf{T}M \to M \xrightarrow{\pi} \mathbf{P}M \to 0$ splits. Thus $M \cong \mathbf{T}M \oplus \mathbf{P}M$ and $\mathbf{T}M$ is finitely presented. Moreover, applying $\mathrm{Hom}_R(\_, R)$ to it, we get the exact sequence

$$0 \to (\mathbf{P}M)^* \overset{\cong}{\to} M^* \to (\mathbf{T}M)^* \to \mathrm{Ext}^1_R(\mathbf{P}M, R) = 0.$$

Therefore $(\mathbf{T}M)^* = 0$, and $\mathbf{T}M$ is a bound right R-module. □

REMARKS 8.44. Let R be a ring.

(a) Let Σ be a class of morphisms between finitely generated projective right R-modules. If we want the universal localization $\lambda\colon R \to R_\Sigma$ to be injective, Lemma 8.40 implies that $\alpha\colon P \to Q$ has to be injective for each $\alpha \in \Sigma$. Moreover, since $R_\Sigma = R_{\Sigma^*}$, the morphism between finitely generated projective left R-modules $\alpha^*\colon Q^* \to P^*$ must be injective for each $\alpha \in \Sigma$. Therefore $\operatorname{coker}\alpha$ is a bound right R-module for each $\alpha \in \Sigma$.

(b) By Theorem 8.15, for each tilting right R-module T, there exists a set $\mathcal{V}$ of finitely presented right R-modules of projective dimension at most one such that $T^\perp = \mathcal{V}^\perp$. If R is a semihereditary ring, then we can suppose that $\mathcal{V}$ is a set of bound right R-modules by Lemma 8.43. Then the tilting class is expressed in terms of $\mathcal{V}$-divisibility. □

Let us disgress to show that our definitions of $\mathcal{U}$-torsion-freeness and $\mathcal{U}$-divisibility generalize the classical notion of torsion-freeness and divisibility over right (left) Ore sets of non-zero-divisors.

LEMMA 8.45. *Let R be a ring, $\mathfrak{V}$ a right Ore subset of R consisting of non-zero-divisors, and M a right R-module. Let $\mathcal{V} = \{R/vR \mid v \in \mathfrak{V}\}$. Then*

(i) *R/vR is a bound right R-module with $\mathrm{Tr}(R/vR) = R/Rv$ for each $v \in \mathfrak{V}$.*
(ii) *M is $\mathfrak{V}$-torsion-free if and only if M is $\mathcal{V}$-torsion-free*
(iii) *M is $\mathfrak{V}$-divisible if and only if M is $\mathcal{V}$-divisible*
(iv) *$R\mathfrak{V}^{-1} = R_\mathcal{V} = R_{\mathrm{Tr}\,\mathcal{V}}$.*
(v) *M is a right $R\mathfrak{V}^{-1}$-module if and only if M is $\mathfrak{V}$-torsion-free and $\mathfrak{V}$-divisible.*
(vi) *Let N be a left R-module. Then N is a left $R\mathfrak{V}^{-1}$-module if and only if N is $\mathfrak{V}$-torsion-free and $\mathfrak{V}$-divisible.*

Analogous results can be stated if $\mathfrak{V}$ is a left Ore subset of R consisting of non-zero-divisors and M is a left R-module.

PROOF. (i) Let $v \in \mathcal{V}$. Then R/vR has a presentation $0 \to R \xrightarrow{l_v} R \to R/vR \to 0$ where $l_v\colon R \to R$ is defined by $x \mapsto vx$ and l_v^* is the morphism $r_v\colon R \cong R^* \to R \cong R^*$ defined by $x \mapsto xv$. Clearly l_v^* is injective because $\mathfrak{V}$ consists of non-zero-divisors and $\operatorname{coker} l_v^* = R/Rv$.

(ii) It follows by the characterization of $\mathcal{T}_\mathfrak{V}(M)$ given in Lemma 3.12.

(iii) Observe that M is $\mathfrak{V}$-divisible if and only if the morphism of abelian groups $M \to M$, defined by $m \mapsto mv$, is surjective for each $v \in \mathfrak{V}$. Consider the presentation

$$0 \to R \xrightarrow{l_v} R \to R/vR \to 0$$

where $l_v(x) = vx$. Applying $\mathrm{Hom}_R(\_, M)$ to it we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(R/vR, M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Ext}^1_R(R/vR, M) \to 0. \qquad (87)$$

Note that $M \cong \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(R, M) \cong M$ is defined by $m \mapsto mv$, and thus it is surjective if and only if $\mathrm{Ext}^1_R(R/vR, M) = 0$.

(iv) Consider the canonical morphisms of rings $\iota\colon R \to R\mathfrak{V}^{-1}$ and $\lambda\colon R \to R_{\mathcal{V}}$. For each $v \in \mathfrak{V}$, let $l_v\colon R \to R$ be defined as in (i). Clearly $l_v \otimes 1_{R\mathfrak{V}^{-1}}$ is an isomorphism. Thus there exists a unique morphism of rings $\bar{\iota}\colon R_{\mathcal{V}} \to R\mathfrak{V}^{-1}$ such that $\iota = \bar{\iota}\lambda$. Now observe that, since $l_v \otimes 1_{R_{\mathcal{V}}}$ is invertible, $L_v\colon R_{\mathcal{V}} \to R_{\mathcal{V}}$ defined by $x \mapsto vx$ is an isomorphism of right $R_{\mathcal{V}}$-modules. Therefore v is invertible. The universal property of the Ore localization implies that there exists a unique morphism of rings $\bar{\lambda}\colon R\mathfrak{V}^{-1} \to R_{\mathcal{V}}$ such that $\lambda = \bar{\lambda}\iota$. It follows from the universal properties of $R\mathfrak{V}^{-1}$ and $R_{\mathcal{V}}$ that both compositions of $\bar{\iota}$ and $\bar{\lambda}$ are the identity.

(v) M is an $R\mathfrak{V}^{-1}$-module iff there exists a morphism of rings $R\mathfrak{V}^{-1} \to \mathrm{End}_{\mathbb{Z}}(M)$ extending the morphism of rings $R \to \mathrm{End}_{\mathbb{Z}}(M)$ given by the structure of right R-module of M. Such an extension exists iff the image of v in $\mathrm{End}_{\mathbb{Z}}(M)$ is invertible for each $v \in \mathfrak{V}$ iff the map $M \to M$, $m \mapsto mv$, is bijective for each $v \in \mathfrak{V}$. This last condition is equivalent to say that M is $\mathfrak{V}$-torsion-free and $\mathfrak{V}$-divisible by (87).

(vi) It is proved as (v). $\square$

Now suppose that R is a semihereditary ring and M a finitely presented right R-module. As we have seen in Lemma 8.43, $M \cong \mathbf{P}M \oplus \mathbf{T}M$. Let $\mathcal{W}$ be the class of all bounded right R-modules. Then, as $\mathbf{T}M$ is a bound right R-module and $\mathbf{P}M$ is a projective right R-module, M is $\mathcal{W}$-torsion-free if and only if $\mathbf{T}M = 0$. Moreover, $\mathbf{T}M$ coincides with the trace submodule of the class $\mathcal{W}$. If moreover the subset $\mathfrak{V}$ of R consisting of all non-zero-divisors is a right Ore subset, then clearly $\mathcal{T}_{\mathfrak{V}}(M) \subseteq \mathbf{T}M$. Both coincide for all finitely presented right R-modules M if and only if every finitely presented torsion-free right R-module can be embedded in a free module (or equivalently is projective). For example, this happens when R is a two-sided order in a semisimple ring Q (i.e. $\mathfrak{V}$ is a two sided Ore subset and $R\mathfrak{V}^{-1}$ is a semisimple ring), see [**Jat86**, Theorem 2.2.15].

By definition, the perpendicular category $\mathcal{X}_{\mathcal{U}}$ of a class of bound modules $\mathcal{U}$ consists of the $\mathcal{U}$-torsion-free and $\mathcal{U}$-divisible modules. It can also be interpreted as the category of modules over the universal localization of R at $\mathcal{U}$, as noted by W. Crawley-Boevey [**CB91**, Property 2.5] in a slightly less general situation. Observe that the next result also generalizes Lemma 8.45(v) and (vi).

PROPOSITION 8.46. *Let R be a ring. Let $\mathcal{U}$ be a class of bound right R-modules. The following statements are equivalent for $M \in \mathrm{Mod}\text{-}R$.*

(i) $M \in \mathrm{Mod}\text{-}R_{\mathcal{U}}$.
(ii) $1_M \otimes_R \alpha_U^*$ *is invertible for all morphisms* $\alpha_U \in \mathcal{U}$.
(iii) $\mathrm{Tor}_R^1(M, \mathrm{Tr}\,U) = M \otimes_R \mathrm{Tr}\,U = 0$ *for every right R-module* $U \in \mathcal{U}$.
(iv) $\mathrm{Hom}_R(U, M) = \mathrm{Ext}_R^1(U, M) = 0$ *for every right R-module* $U \in \mathcal{U}$.

PROOF. The implication (i) $\Leftrightarrow$ (ii) follows from the fact that $R_{\mathcal{U}} = R_{\mathrm{Tr}\,\mathcal{U}}$ and the left version of Lemma 8.25.
(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): Take $\alpha_U \in \mathcal{U}$. Consider the exact sequences

$$0 \to P \overset{\alpha_U}{\to} Q \to U \to 0$$

$$0 \to Q^* \overset{\alpha_U^*}{\to} P^* \to \mathrm{Tr}\,U \to 0.$$

Applying $M \otimes_R \_$ to the second one and $\mathrm{Hom}_R(\_, M)$ to the first one, we get the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^R(M, \mathrm{Tr}\,U) \arrow[r] & M \otimes_R Q^* \arrow[r, "1_M \otimes_R \alpha_U^*"] \arrow[d] & M \otimes_R P^* \arrow[r] \arrow[d] & M \otimes_R \mathrm{Tr}\,U \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_R(U, M) \arrow[r] & \mathrm{Hom}_R(Q, M) \arrow[r] & \mathrm{Hom}_R(P, M) \arrow[r] & \mathrm{Ext}_R^1(U, M) \arrow[r] & 0 \end{tikzcd} \tag{88}$$

where the vertical arrows are isomorphisms by Lemma 1.9. Hence $1_M \otimes_R \alpha_U^*$ is an isomorphism iff $\operatorname{Tor}_R^1(M, \operatorname{Tr} U) = M \otimes_R \operatorname{Tr} U = 0$ iff $\operatorname{Hom}_R(U, M) = \operatorname{Ext}_R^1(U, M) = 0$. □

REMARK 8.47. Let R be a ring. Let $\mathcal{U}$ be a class of bound right R-modules. Then the class $\mathcal{U}^\perp$ of $\mathcal{U}$-divisible modules is a tilting class. Moreover

$$\mathcal{U}^\perp = \{M \in \text{Mod-} R \mid M \otimes_R \operatorname{Tr} U = 0 \text{ for all } U \in \mathcal{U}\}.$$

PROOF. For the first statement, notice that we can suppose that $\mathcal{U}$ is a set by Remark 8.39(a). Then apply Proposition 8.13. For the second part notice that the commutative diagram of (88) holds for any $U \in \mathcal{U}$ and any right R-module M. Then $\operatorname{Ext}_R^1(U, M) = 0$ if and only if $M \otimes_R \operatorname{Tr} U = 0$. □

A candidate for the tilting class $\mathcal{U}^\perp$ given by a class $\mathcal{U}$ of bounded right R-modules is $R_\mathcal{U} \oplus R_\mathcal{U}/R$. When $\lambda : R \to R_\mathcal{U}$ is injective and $\operatorname{pd}(R_\mathcal{U})_R \leq 1$, we have a tilting right R-module $R_\mathcal{U} \oplus R_\mathcal{U}/R$ by Example 8.34(d). In general, however, its tilting class $\operatorname{Gen} R_\mathcal{U}$ does not coincide with the tilting class $\mathcal{U}^\perp$, as we will see in Example 8.79. The next result describes the case when $\operatorname{Gen} R_\mathcal{U} = \mathcal{U}^\perp$.

THEOREM 8.48. *Let R be a ring. Let $\mathcal{U}$ be a class of bound right R-modules. Let further $\lambda : R \to S$ be an injective ring epimorphism with $\operatorname{Tor}_1^R(S, S) = 0$ and $\operatorname{pd} S_R \leq 1$. The following statements are equivalent:*

(i) $\operatorname{Gen} S_R = \mathcal{U}^\perp$.
(ii) *The map $\lambda \colon R \to S$ is a $\mathcal{U}^\perp$-(pre)envelope.*
(iii) *$S_R \in \mathcal{U}^\perp$, and every (pure-injective) module $M \in \mathcal{U}^\perp$ belongs to $(S/R)^\perp$.*

In particular, conditions (i)-(iii) *hold true if $S_R \in \mathcal{U}^\perp$ and S/R is a direct limit of $\mathcal{U}$-filtered right R-modules.*

PROOF. We already know by Theorem 8.32 that $T = S \oplus S/R$ is a tilting right R-module with $\operatorname{Gen} T_R = \operatorname{Gen} S_R = (S/R)^\perp$.
(i) ⇒ (ii): If $M \in \mathcal{U}^\perp = \operatorname{Gen} S_R$, then $\operatorname{Ext}_R^1(S/R, M) = 0$ because clearly $S/R \in \operatorname{Gen} S_R$. Therefore, applying $\operatorname{Hom}_R(\_, M)$ to the exact sequence $0 \to R \xrightarrow{\lambda} S \to S/R \to 0$, we get that $\operatorname{Hom}_R(\lambda, M)$ is surjective. So, $\lambda : R \to S$ is a $\mathcal{U}^\perp$-preenvelope. Suppose now that $g \in \operatorname{End}_R(S)$ satisfies $\lambda = g\lambda$. Since Mod- S is a full subcategory of Mod- R by Lemma 8.19(ii), $g \in \operatorname{End}_S(S)$. Now, since $g(1) = 1$, we get that g is the identity and therefore an isomorphism. So λ is even a $\mathcal{U}^\perp$-envelope.
(ii) ⇒ (i): By the definition of a preenvelope, we have that S_R belongs to $\mathcal{U}^\perp$. Since $\mathcal{U}$ consists of finitely presented modules, Lemma 1.31 implies that direct sums of copies of S_R are in $\mathcal{U}^\perp$. Now it follows that $\operatorname{Gen} S_R \subseteq \mathcal{U}^\perp$ because $\mathcal{U}^\perp$ is closed under images by Remark 8.2.

For the reverse inclusion, note that S_R is a generator of $\mathcal{U}^\perp$ by [**AHTT01**, Lemma 1.1]. Indeed, if $M \in \mathcal{U}^\perp$, then, for each $m \in M$, the morphism of right R-modules $R \to M$ defined by $1 \mapsto m$ factors through λ. Therefore there exists an onto morphism of right R-modules $\bigoplus_{m \in M} S \to M$.
(i) ⇒ (iii) follows from $(S/R)^\perp = \operatorname{Gen} S_R$ and the remark at the beginning of the proof.
(iii) ⇒ (i): We deduce as above that $\operatorname{Gen} S_R \subseteq \mathcal{U}^\perp$. To prove equality, first observe that $\mathcal{U}^\perp$ is a tilting class by Remark 8.39(a) and Proposition 8.13. So both classes are tilting classes. By Corollary 8.16, $\operatorname{Gen} S_R$ and $\mathcal{U}^\perp$ coincide if and only if they contain the same pure-injective right R-modules. The latter holds true by (iii).

We now prove the last statement. Suppose that $S_R \in \mathcal{U}^\perp$ and $S/R = \varinjlim N_i$ where all N_i are $\mathcal{U}$-filtered right R-modules. By condition (iii), it is enough to show that every pure-injective module $M \in \mathcal{U}^{\perp}$ belongs to $(S/R)^{\perp}$. Now, for such module M, Lemma 1.34 implies that

$$\operatorname{Ext}^1_R(S/R, M) = \operatorname{Ext}^1_R(\varinjlim N_i, M) \cong \varprojlim \operatorname{Ext}^1_R(N_i, M).$$

Since N_i is $\mathcal{U}$-filtered for any $i \in I$, $\operatorname{Ext}^1_R(N_i, M) = 0$ by Eklof-Lemma 1.29. Therefore $\operatorname{Ext}^1_R(S/R, M) = 0$. ☐

Corollary 8.49. *Let R be a ring. Let $\mathcal{U}$ be a class of bound right R-modules. Suppose that R embeds in $R_{\mathcal{U}}$, and $\operatorname{pd}(R_{\mathcal{U}})_R \leq 1$. Assume further that $R_{\mathcal{U}}/R$ is a direct limit of $\mathcal{U}$-filtered right R-modules. Then $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting right R-module with $\operatorname{Gen} T_{\mathcal{U}} = \operatorname{Gen}(R_{\mathcal{U}})_R = \mathcal{U}^{\perp}$.*

Proof. Notice that $R_{\mathcal{U}} \in \mathcal{U}^{\perp}$ because $R_{\mathcal{U}}$ is a right $R_{\mathcal{U}}$-module, see Proposition 8.46. So the statement follows immediately from Examples 8.18(a), Theorems 8.27 and 8.48. ☐

Recall from Section 4 of Chapter 3 that if R is a hereditary ring with a faithful rank function ρ, then the universal localization $\lambda\colon R \to R_{\rho}$ of R at ρ has a fairly well understood behavior. It produces a source of examples of tilting modules from universal localization satisfying the hypothesis of Corollary 8.49 thanks to the following result. It was stated in [**Sch86**], and there it was noted that it follows from the proof of [**Sch85**, Theorem 12.6].

Theorem 8.50. *Let R be a hereditary ring with a faithful rank function ρ such that R_{ρ} is a simple artinian ring. Let $\mathcal{U}$ be a class of ρ-simple modules. The following statements hold true:*

(i) *As a right R-module, $R_{\mathcal{U}}/R$ is a directed union of finitely presented modules N_i such that each N_i is a finite extension of modules from $\mathcal{U}$.*
(ii) *As a left R-module, $R_{\mathcal{U}}/R$ is a directed union of finitely presented modules M_j such that each M_j is a finite extension of modules of the form $\operatorname{Tr} U$ with $U \in \mathcal{U}$.* ☐

Now we give our result on tilting modules.

Corollary 8.51. *Let R be a hereditary ring with a faithful rank function ρ. The following statements hold true.*

(i) *If $\mathcal{V}$ is a class of ρ-torsion right R-modules, then $T_{\mathcal{V}} = R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R$ is a tilting right R-module.*
(ii) *Suppose that R_{ρ} is simple artinian. If $\mathcal{U}$ consists of ρ-simple modules, then*

$$T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

is a tilting right R-module with tilting class $T_{\mathcal{U}}^{\perp} = \mathcal{U}^{\perp}$, and it is also a tilting left R-module with tilting class ${}_R T_{\mathcal{U}}{}^{\perp} = (\operatorname{Tr} \mathcal{U})^{\perp}$.

Proof. Since ρ is faithful, then $R \to R_{\mathcal{V}}$ is an embedding for any class of ρ-torsion modules by Theorem 3.56.

(i) It follows from Examples 8.34(d).

(ii) By (i), $T_{\mathcal{U}}$ is a tilting right R-module and a tilting left R-module. Now Theorem 8.50 implies that $R_{\mathcal{U}}/R$ is a directed union of finitely presented right (left) R-modules N_i such that each N_i is a finite extension of modules from $\mathcal{U}$ ($\operatorname{Tr}\mathcal{U}$). Therefore Corollary 8.49 implies the result. ☐

Many rings satisfy the conditions in Corollary 8.51(ii). By Theorem 3.58, every hereditary ring R with a faithful rank function ρ taking values on the integers is such that R_{ρ} is a division ring. Firs are one example of this kind. If R is a fir, projectives are free of unique rank, thus the only possible rank function sends $[R^n] \mapsto n$, so it is faithful, and certainly takes values on the integers. Recall that hereditary local rings, crossed product group rings kG of a free group G over a division ring k and free algebras are examples of firs.

Suppose now that R is a hereditary ring embeddable in a division ring D. Then we can define a correspondence on the class of finitely generated projectives as $\rho P = \dim_D(P \otimes_R D)$. Clearly $P \cong P'$ implies that $\rho P = \rho P'$. Also $\rho R = 1$, and, since $P \cong P_1 \oplus P_2$ implies $P_1 \otimes_R D \cong P_1 \otimes_R D \oplus P_2 \otimes_R D$, we see that $\rho(P_1 \oplus P_2) = \rho(P_1) + \rho(P_2)$. Thus ρ is a rank function with values on the integers. The sequence $0 \to R \to D$ is exact, and tensoring with a projective right R-module P, we obtain the exact sequence $0 \to P \to P \otimes_R D$ which shows that $P \otimes_R D \neq 0$, and therefore ρ is faithful. Again, by Theorem 3.58, R_ρ is a division ring (maybe different from D).

Some other examples of rings satisfying the conditions of Corollary 8.51(ii) are tame hereditary algebras (see Section 6) and hereditary noetherian prime rings, in particular maximal classical orders and Dedekind prime rings (see Section 5).

The following lemma will be useful in giving some other applications of Corollary 8.49.

LEMMA 8.52. *Let $\lambda\colon R \to S$ be a ring epimorphism. Let $\mathcal{V}$ be a class of bound right R-modules such that $S_R \in \mathcal{V}^\perp$. If $\operatorname{Gen} S_R$ coincides with the class of $\mathcal{V}$-divisible modules, then $\operatorname{pd} S_R \leq 1$. In particular, if $\lambda\colon R \to R_\mathcal{V}$ is the universal localization of R at $\mathcal{V}$, and $\operatorname{Gen}(R_\mathcal{V})_R$ coincides with the class of $\mathcal{V}$-divisible modules, then $\operatorname{pd}(R_\mathcal{V})_R \leq 1$.*

PROOF. Remark 8.39(a) implies that we can suppose that $\mathcal{V}$ is a set.

By Theorem 8.8, with $M = R$ and $\mathcal{S} = \mathcal{V}$, we get a $\mathcal{V}^\perp$-preenvelope $R \hookrightarrow B$ of R with $\operatorname{pd}(B/R)_R \leq 1$. Thus, for each $N \in \mathcal{V}^\perp$ and $n \in N$, there exists a morphism $\varphi_n\colon B \to N$ such that $\varphi_n(1) = n$ which is an extension of the morphism of right R-modules $R \to N$ defined by $1 \mapsto n$. In particular, there exists a morphism of right R-modules $\varphi\colon B \to S$ such that $\varphi(1) = 1$.

By hypothesis, there exists a set I and an onto morphism of right R-modules $\eta\colon S^{(I)} \to B$. Therefore there exists a morphism of right R-modules $\varphi\eta\colon S^{(I)} \to S$ with $1 \in \operatorname{im}\varphi\eta$. Now $\varphi\eta$ is a morphism of right S-modules by Remarks 8.20(a). Thus $\varphi\eta$ is onto and it splits. So there exists $\psi\colon S \to S^{(I)}$ such that $\varphi\eta\psi = 1_S$. Hence φ is an epimorphism and it splits. Then S_R is a direct summand of B_R, a module of projective dimension at most one. Indeed, B_R is an extension of two modules, R and B/R, of projective dimension ≤ 1. This implies that $\operatorname{pd} S_R \leq 1$.

For the last part recall that $\lambda\colon R \to R_\mathcal{V}$ is a ring epimorphism by Examples 8.18(a), and that $R_\mathcal{V} \in \mathcal{V}^\perp$ by Proposition 8.46(iv) because clearly $R_\mathcal{V}$ is an $R_\mathcal{V}$-module. □

The proof of the last result is an extension of the one given in the last paragraph of [**AHHT05**, Proposition 6.4]. There S is the localization $\mathfrak{V}^{-1}R$ of R at a left Ore set of non-zero-divisors $\mathfrak{V}$ and the role of B is played by the so called Fuchs tilting module relative to $\mathfrak{V}$. With Lemma 8.52 in mind we can state the following generalization of [**AHHT05**, Proposition 6.4]. They proved it assuming that either R is commutative or $\mathfrak{V}^{-1}R$ is countably generated as a right R-module.

COROLLARY 8.53. *Let R be a ring. Let $\mathfrak{U}$ be a left Ore set of non-zero-divisors of R. Then $\operatorname{pd}(\mathfrak{U}^{-1}R_R) \leq 1$ if and only if $\operatorname{Gen}(\mathfrak{U}^{-1}R_R)$ coincides with the class of $\mathfrak{U}$-divisible right R-modules. In this case $T_\mathfrak{U} = \mathfrak{U}^{-1}R \oplus \mathfrak{U}^{-1}R/R$ is a tilting right R-module whose tilting class coincides with the class of $\mathfrak{U}$-divisible right R-modules.*

PROOF. Suppose that $\operatorname{pd}(\mathfrak{U}^{-1}R_R) \leq 1$. Since $\mathfrak{U}$ consists of non-zero-divisors, R embeds in $\mathfrak{U}^{-1}R$. Setting $\mathcal{U} = \{R/uR \mid u \in \mathfrak{U}\}$, we know by the left version of Lemma 8.45(i) and (iv) that $\mathcal{U}$ is a set of bound right R-modules and $R_{\mathcal{U}} \cong \mathfrak{U}^{-1}R$. On the other hand, given $u, v \in \mathfrak{U}$, there exist $z \in \mathfrak{U}$, $w \in R$, such that $wu = zv$. Then

$$u^{-1}R + v^{-1}R \subseteq (zv)^{-1}R = (wu)^{-1}R.$$

Hence every finitely generated right submodule of $\mathfrak{U}^{-1}R$ is contained in $u^{-1}R$ for some $u \in \mathfrak{U}$. Therefore $\mathfrak{U}^{-1}R/R = \varinjlim_{u\in\mathfrak{U}} u^{-1}R/R$. Moreover, notice that for every $u \in \mathfrak{U}$, $u^{-1}R/R \cong R/uR$, thus, $\mathfrak{U}^{-1}R/R$ is a direct limit of the $\mathcal{U}$-filtered modules $u^{-1}R/R$. Then we obtain that $T_{\mathfrak{U}} = \mathfrak{U}^{-1}R \oplus \mathfrak{U}^{-1}R/R$ is a tilting right R-module and $\mathrm{Gen}(T_{\mathfrak{U}})_R = \mathrm{Gen}(\mathfrak{U}^{-1}R)_R = \mathcal{U}^{\perp}$ by Corollary 8.49.

The proof of the other implication is Lemma 8.52 with $\mathcal{V} = \mathcal{U}$ and $S = \mathfrak{U}^{-1}R$. □

Observe that Corollary 8.53 cannot be generalized to universal localization because of Example 8.79.

REMARK 8.54. If $\mathfrak{U}$ is a two-sided Ore set of non-zero-divisors, then $\mathrm{pd}({}_R\mathfrak{U}^{-1}R) \leq 1$ if and only if $\mathrm{Gen}({}_R\mathfrak{U}^{-1}R)$ coincides with the class of $\mathfrak{U}$-divisible left R-modules. In fact, in this case $\mathfrak{U}^{-1}R = R\mathfrak{U}^{-1} = R_{\mathcal{U}}$, and we can apply the left version of Corollary 8.49 on ${}_RR\mathfrak{U}^{-1}$.

However, if $\mathfrak{U}$ is just a left Ore set of non-zero-divisors of R, and $\mathrm{pd}({}_R\mathfrak{U}^{-1}R) \leq 1$, then $T_{\mathfrak{U}} = \mathfrak{U}^{-1}R \oplus \mathfrak{U}^{-1}R/R$ is a tilting left R-module by Theorem 8.32, but we cannot compute $T_{\mathfrak{U}}^{\perp}$ as we do not know whether $\mathfrak{U}^{-1}R/R$ can be written as a direct limit of $\{R/Ru \mid u \in \mathfrak{U}\}$-filtered left R-modules.

Stronger results will be obtained in Theorem 8.68 under the assumption that R is a hereditary noetherian prime ring. □

COROLLARY 8.55. *Let R be a commutative valuation domain with field of fractions Q. Suppose that $\mathrm{pd}(R_{\mathfrak{p}})_R \leq 1$ for each prime ideal $\mathfrak{p}$ of R (equivalently, suppose that $R_{\mathfrak{p}}$ is countably generated as an R-module for every prime ideal $\mathfrak{p}$ of R). Then the set*

$$\mathbb{T} = \{T_{\mathfrak{p}} = R_{\mathfrak{p}} \oplus R_{\mathfrak{p}}/R \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$$

is a representative set up to equivalence of the class of all tilting right R-modules.

PROOF. For each prime ideal $\mathfrak{p}$ of R, let $\mathfrak{U}_{\mathfrak{p}} = R \setminus \mathfrak{p}$. By Corollary 8.53 we know that $T_{\mathfrak{p}} = R_{\mathfrak{p}} \oplus R_{\mathfrak{p}}/R$ is a tilting R-module and $T_{\mathfrak{p}}^{\perp}$ equals the class of $\mathfrak{U}_{\mathfrak{p}}$-divisible R-modules.

It is known that the set of Fuchs tilting modules $\{\delta_{\mathfrak{U}_{\mathfrak{p}}} \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$ is a representative set up to equivalence of the class of all tilting R-modules, and $\delta_{\mathfrak{U}_{\mathfrak{p}}}^{\perp}$ is the class of $\mathfrak{U}_{\mathfrak{p}}$-divisible R-modules [**GT06**, Theorem 6.2.21].

The assumption that $\mathrm{pd}(R_{\mathfrak{p}})_R \leq 1$ for each prime ideal $\mathfrak{p}$ of R is satisfied if and only if $R_{\mathfrak{p}}$ is countably generated as an R-module for every prime ideal $\mathfrak{p}$ of R. In fact, if R is a commutative local ring and $\mathfrak{U}$ is a multiplicative subset of non-zero-divisors, then $\mathrm{pd}\, R\mathfrak{U}_R^{-1} \leq 1$ if and only if $R\mathfrak{U}_R^{-1}$ is a countably generated R-module [**AHHT05**, Page 531]. In the particular case when R is a valuation domain see [**FS85**, Theorem IV.3.1]. □

Now we show that Theorem 8.48 can help to compute tilting classes.

EXAMPLE 8.56. Let X be a nonempty set. Let G be the free group on X. Let k be a field. Consider the free algebra $R = k\langle X\rangle$ and the free group algebra kG with the natural embedding $k\langle X\rangle \hookrightarrow kG$ which sends $x \mapsto x$ for every $x \in X$. We saw in Examples 8.34(d) that $T_X = kG \oplus kG/k\langle X\rangle$ is a tilting right R-module. Let $\mathcal{X} = \{k\langle X\rangle/xk\langle X\rangle \mid x \in X\}$, a set of bound right R-modules. Notice that $kG = R_{\mathcal{X}}$. We proceed to show that $T_X^{\perp} = \mathcal{X}^{\perp}$ by verifying condition (ii) in Theorem 8.48.

Let $M \in \mathcal{X}^{\perp}$. We have to show that for every $k\langle X\rangle \overset{\widetilde{f}}{\to} M$, there exists $f\colon kG \to M$ extending $\widetilde{f}$. We will define f on the elements of G, and then extend it by linearity.

Every element $g \in G$ can be uniquely expressed as a word of the form

$$g = x_1^{e_1} \cdots x_r^{e_r} \text{ where } x_i \in X,\ e_i = \pm 1 \text{ and } x_i \neq x_{i+1} \text{ if } e_i = -e_{i+1}. \tag{89}$$

We proceed by induction on the length of g. If $r = 0$, that is, $g = 1$, then we define $f(1)$ as $\widetilde{f}(1)$. Let $r+1 > 0$, and suppose that we have defined $f(g)$ for all $g \in G$ of length $\leq r$. Let g be a word of length $r+1$, suppose that $g = x^{e_1} \cdots x_{r+1}^{e_{r+1}} = hx_{r+1}^{e_{r+1}}$ as in (89).

For every $x \in X$, applying $\operatorname{Hom}_{k\langle X\rangle}(\_, M)$ to the exact sequence

$$0 \to k\langle X\rangle \overset{\alpha_x}{\to} k\langle X\rangle \to k\langle X\rangle/xk\langle X\rangle \to 0$$

we get

$$\begin{aligned} 0 \to \operatorname{Hom}_{k\langle X\rangle}(k\langle X\rangle/xk\langle X\rangle, M) \to \operatorname{Hom}_{k\langle X\rangle}(k\langle X\rangle, M) \to \operatorname{Hom}_{k\langle X\rangle}(k\langle X\rangle, M) \to \\ \to \operatorname{Ext}^1_{k\langle X\rangle}(k\langle X\rangle/xk\langle X\rangle, M) = 0. \end{aligned}$$

This implies that for every $m \in M$ and $x \in X$ there exists an $n \in M$ such that $m = nx$. So fix $n \in M$ such that $nx_{r+1} = f(h)$. Then

$$f(g) = \begin{cases} f(h)x_{r+1}^{e_{r+1}} & \text{if } e_{r+1} = 1 \\ n & \text{if } e_{r+1} = -1. \end{cases}$$

Hence $k\langle X\rangle \hookrightarrow kG$ is an $\mathcal{X}^{\perp}$-preenvelope, and $T_X^{\perp} = \operatorname{Gen} kG = \mathcal{X}^{\perp}$.

Analogously, it can be proved that T_X is also a tilting left R-module with tilting class ${}_RT_X^{\perp} = \{k\langle X\rangle/k\langle X\rangle x \mid x \in X\}^{\perp}$. □

Now we intend to give a remark relating tilting modules and its dual concept, cotilting modules. For that we first give the definition of cotilting modules and state without proof a particular situation of [**AHHT06**, Theorem 2.2].

Definition 8.57. Let R be a ring.

(a) A left R-module C is a *cotilting module* provided it satisfies
   - (C1) C is of injective dimension at most one.
   - (C2) $\operatorname{Ext}^1_R(C^I, C) = 0$ for each $i \geq 1$ and all sets I.
   - (C3) There exists an exact sequence $0 \to C_1 \to C_2 \to W \to 0$ where W is an injective cogenerator of R-Mod and $C_1, C_2 \in \operatorname{Prod} C$.

(b) A class of left R-modules $\mathcal{F}$ is a *cotilting class* if there exists a cotilting left R-module C such that $\mathcal{F} = {}^{\perp}C$, i.e. $\mathcal{F} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(M, C) = 0\}$.

(c) A cotilting class $\mathcal{F} = {}^{\perp}C$ is of *cofinite type* provided that there exists a set $\mathcal{S}$ of finitely copresented modules of injective dimension at most one such that ${}^{\perp}C = {}^{\perp}\mathcal{S}$. □

Theorem 8.58. *Let R be a ring. There exists a bijective correspondence between the tilting classes in* Mod-R *and cotilting classes of cofinite type in* R-Mod. *If $\mathcal{S}^{\perp}$ is a tilting class, with $\mathcal{S}$ a set of finitely presented right R-modules of projective dimension at most one, then the corresponding cotilting class in* R-Mod *is* $\mathcal{S}^{\top} = \{{}_RX \mid \operatorname{Tor}_1^R(Y, X) = 0$ *for all* $Y \in \mathcal{S}\}$. □

The following result should be compared with [**AHHT05**, Remark 5.8], there this result is stated for the left Ore situation, i.e.: there exists a left Ore set $\mathfrak{V}$ consisting of nonzero divisors of R such that $\mathcal{U} = \{R/vR \mid v \in \mathfrak{V}\}$.

Proposition 8.59. *Let R be a ring. Let $\mathcal{U}$ be a class of bound right R-modules, and consider the class* $\operatorname{Tr}\mathcal{U} = \{\operatorname{Tr} U \mid U \in \mathcal{U}\}$. *Then the class $\mathcal{U}^{\perp}$ of $\mathcal{U}$-divisible modules is a tilting class. Moreover, the class of* $\operatorname{Tr}\mathcal{U}$*-torsion-free modules is a cotilting class of left R-modules. More* *precisely, it is the cotilting class of cofinite type that corresponds to* $\mathcal{U}^{\perp}$ *under the bijective correspondence from* Theorem 8.58.

PROOF. The first statement follows from Remark 8.47. For the second statement, recall that the correspondence in Theorem 8.58 sends the tilting class $\mathcal{U}^{\perp}$ to the cotilting class $\mathcal{U}^{\top} = \{{}_RX \mid \operatorname{Tor}_1^R(U,X) = 0 \text{ for all } U \in \mathcal{U}\}$.

If $U \in \mathcal{U}$, and $0 \to P \xrightarrow{\alpha} Q \to U \to 0$ is a projective presentation of U with P and Q finitely presented, we obtain the exact sequences $0 \to Q^* \to P^* \to \operatorname{Tr} U \to 0$, and

$$0 \longrightarrow \operatorname{Tor}_1^R(U,X) \longrightarrow P \otimes X \longrightarrow Q \otimes X \longrightarrow U \otimes X \longrightarrow 0$$

$$0 \to \operatorname{Hom}_R(\operatorname{Tr} U, X) \to \operatorname{Hom}_R(P^*, X) \to \operatorname{Hom}_R(Q^*, X) \to \operatorname{Ext}_R^1(\operatorname{Tr} U, X) \to 0.$$

Since $P \otimes_R X \cong \operatorname{Hom}_R(P^*, X)$ and $Q \otimes_R X \cong \operatorname{Hom}_R(Q^*, X)$ are naturally isomorphic by Lemma 1.9, we get $\operatorname{Tor}_1^R(U,X) \cong \operatorname{Hom}_R(\operatorname{Tr} U, X)$. Therefore $X \in \mathcal{U}^{\top}$ if and only if $\operatorname{Hom}_R(\operatorname{Tr} U, X) = 0$ for all $U \in \mathcal{U}$, that is, X is $\operatorname{Tr}\mathcal{U}$-torsion free. □

5. Noetherian prime rings

Throughout this section R will be a right order in a semisimple ring A. Recall that this means that the subset $\mathfrak{V}$ consisting of all non-zero-divisors is a right Ore set and that the right Ore localization of R at $\mathfrak{V}$ is A (see Definition 3.7). We remind the reader that ${}_RA$ is a flat left R-module and that a right R-module M is a torsion right R-module if and only if $M \otimes_R A = 0$ by Proposition 3.14.

We begin this section giving an easy result that will be useful. Notice that the proof works for any ring R which embeds in a semisimple ring A.

LEMMA 8.60. *Let R be a right order in a semisimple ring A. Let n be the length of A as a right A-module. Then the correspondence $u\colon K_0(R) \to \frac{1}{n}\mathbb{Z}$ defined by*

$$u(P) = \frac{\operatorname{length}(P \otimes_R A_A)}{n}$$

is a faithful rank function.

PROOF. Let P, P', Q be finitely generated projective right R-modules.

Clearly, if $P \cong P'$, then $P \otimes_R A \cong P' \otimes_R A$ and $u(P) = u(P')$. Also, the fact that $(P \oplus Q) \otimes_R A \cong P \otimes_R A \oplus Q \otimes_R A$ implies that

$$\begin{aligned}\operatorname{length}((P \oplus Q) \otimes_R A_A) &= \operatorname{length}((P \otimes_R A_A) \oplus (Q \otimes_R A_A)) \\ &= \operatorname{length}(P \otimes_R A_A) + \operatorname{length}(Q \otimes_R A_A).\end{aligned}$$

Hence $u(P \oplus Q) = u(P) + u(Q)$. Now $A_A \cong R \otimes_R A$ gives that $u(R) = 1$. Therefore u is a rank function.

Observe that $0 \to R \to A$ is exact, and $0 \to P \to P \otimes_R A$ is again exact because P is projective. Thus, if $P \neq 0$, then $u(P) \neq 0$. □

DEFINITION 8.61. Let R be a right order in a semisimple ring A. The faithful rank function u of Lemma 5 is called the *normalized uniform dimension of R.* □

Recall the following result which can be found, for example, in [**Jat86**, Corollary 2.2.12].

LEMMA 8.62. *Let R be a right order in a simple artinian ring. If there exists a simple torsion-free right R-module, then R itself is a simple artinian ring.* □

What follows is the trivial generalization of the foregoing Lemma to the semisimple situation.

LEMMA 8.63. *Let R be a right order in a semisimple ring A. If there exists a simple torsion-free right R-module, then there exists a primitive central idempotent e of A such that eRe is a simple artinian ring.*

PROOF. Suppose that M is a simple torsion-free right R-module. It is known that there exists a right ideal I of R such that M and I have isomorphic essential submodules, see for example [**Jat86**, Proposition 2.2.11]. Hence, since M is simple, M embeds in R. So we can suppose that M is a right ideal of R. There exists a primitive central idempotent e of A such that $Me \neq 0$. Then $Me \cong M$ as right R-modules, and Me is a simple torsion-free right eRe-module. Notice that eRe is a right order in the simple artinian ring eAe. Thus eRe is simple artinian by Lemma 8.62. □

The following is stated for hereditary noetherian prime rings in [**CB91**, Section 3].

PROPOSITION 8.64. *Let R be a semihereditary right order in a semisimple ring A. Let u be the normalized uniform dimension of R. Suppose that there is no primitive central idempotent e of A such that eRe is simple artinian. Then*

(i) *The class of finitely presented torsion right R-modules coincides with the class of u-torsion right R-modules.*
(ii) *The class of finitely presented simple right R-modules coincides with the class of u-simple right R-modules.*
(iii) *A equals R_u, the universal localization of R at u.*

PROOF. (i) Given a finitely presented torsion right R-module V_R (hence $\operatorname{pd} V_R = 1$) with finite projective presentation $0 \to P \xrightarrow{\alpha} Q \to V \to 0$, applying $\_ \otimes_R A$, we get

$$0 \to P \otimes_R A \xrightarrow{\alpha\otimes 1_A} Q \otimes_R A \to V \otimes_R A = 0.$$

Hence $u(P) = u(Q) = u(\alpha)$, and V is a u-torsion module. Conversely, if V is a u-torsion module, then V is a finitely presented right R-module with $\operatorname{pd} V_R = 1$. Let $0 \to P \xrightarrow{\alpha} Q \to V \to 0$ be a presentation of V with P and Q finitely generated projective right R-modules. Notice that $\operatorname{length}(P \otimes_R A_A) = \operatorname{length}(Q \otimes_R A_A)$. Hence $\alpha \otimes 1_A$ is an isomorphism and $V \otimes_R A = 0$. Thus V is a torsion right R-module.

(ii) Let U be a finitely presented simple right R-module with finite projective presentation $0 \to P \xrightarrow{\alpha} Q \to U \to 0$. Since U is simple, and there is no primitive central idempotent e of A such that eRe is simple artinian, Lemma 8.63 implies that U is a torsion right R-module, and therefore u-torsion with $\operatorname{pd} U_R = 1$ by (i). Now suppose that

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & Q \\ & P' \arrow[ur, "\gamma"'] & \end{tikzcd}$$

with $u(P') = u(P) = u(Q) = u(\alpha)$. Hence $\operatorname{length}(Q \otimes_R A_A) = \operatorname{length}(P' \otimes_R A_A)$. Since $\alpha \otimes 1_A$ is surjective, we get that $\gamma \otimes 1_A$ is an isomorphism. Hence we have the commutative diagram

$$\begin{tikzcd} P' \otimes_R A \arrow[r, "\cong"] & Q \otimes_R A \\ P' \arrow[u] \arrow[r, "\gamma"'] & Q \arrow[u] \end{tikzcd}$$

where the vertical arrows are injective. Hence γ is injective. Clearly β is injective. Now, since $U \cong Q/P$ is simple, we get that β or γ is an isomorphism. This shows that U is u-simple.

On the other hand, if U is a u-torsion module which is not a simple module, then it contains a finitely generated submodule $0 \neq V \subsetneqq U$. Suppose that $0 \to P \xrightarrow{\alpha} Q \to U \to 0$ is a projective presentation of U with P and Q finitely generated. Then there exists a finitely generated submodule $0 \neq P' \subsetneqq Q$ such that $P'/P \cong V$. Since R is semihereditary, P' is a projective right R-module. Now α factors through P' in the following way

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, hook] & & Q \\ & P' \arrow[ur, hook] & \end{tikzcd}$$

and U cannot be a u-simple module since $u(P) = u(Q) = u(P')$.

(iii) If $0 \to P \overset{\alpha}{\to} Q \to W \to 0$ is an exact sequence with W torsion and P, Q finitely generated projective right R-modules, then $\alpha \otimes 1_A \colon P \otimes_R A \to Q \otimes_R A$ is an isomorphism. Therefore condition (a) in Definition 3.46 is satisfied.

Let $b \colon R \to B$ be a morphism of rings such that $\alpha \otimes 1_B$ is invertible for every full morphism $P \overset{\alpha}{\to} Q$. By (i), α is full if and only if $\operatorname{coker} \alpha$ is torsion. If s is a non-zero-divisor of R, then R/sR is torsion. Therefore the map $\alpha_s \colon R \to R$, defined by $r \mapsto sr$, is a full morphism and $\alpha_s \otimes 1_B$ is invertible. Hence $b(s)$ is invertible in B. By the universal property of Ore localization, there exists a unique morphism of rings $\psi \colon A \to B$ such that $\psi_{|R} = b$. Thus condition (b) in Definition 3.46 is satisfied. □

The following result is [**ER70**, Theorem 1.3] in the semisimple situation. We prove it using the theory of rank functions.

Theorem 8.65. *Let R be a hereditary noetherian semiprime ring which is a right order in the semisimple ring A. Suppose that there is no primitive central idempotent e of A such that eRe is simple artinian. Let $J \subseteq I$ be right ideals of R. Then I/J is an artinian right R-module if and only if J is an essential submodule of I.*

Proof. We use the following known fact, see for example [**Jat86**, Proposition 2.2.2]: If R is a right order in a semisimple ring, then a submodule N of a torsion-free right R-module M is essential in M if and only if M/N is torsion.

Suppose that I/J is artinian. Then it has finite length, that is, it is a finite extension of simple right R-modules, and hence torsion right R-modules by Lemma 8.63. Thus I/J is torsion.

On the other hand, suppose that J is an essential submodule of I. By the remark at the beginning of the proof, I/J is a finitely presented torsion right R-module. By Theorem 3.55, it follows that I/J has finite length. □

Now we concentrate on the noetherian prime situation where best results are obtained.

Notation 8.66. Let R be a hereditary noetherian prime ring. From now on let $\mathcal{U}_r$ be a set of representatives of all isomorphism classes of finitely presented simple right R-modules. Let $\mathcal{V}_r$ be a set of representatives of all isomorphism classes of finitely presented torsion right R-modules. Let finally $\mathcal{D}_r = \{R/sR \mid s \text{ a non-zero-divisor of } R\}$. In the same way we define $\mathcal{U}_l$, $\mathcal{V}_l$, $\mathcal{D}_l$. □

First we give [**CB91**, Remark 3.3].

Proposition 8.67. *Let R be a hereditary noetherian prime ring which is an order in the simple artinian ring A. If S is a subring of A, then there exists a unique subset $\mathcal{U}_S$ of $\mathcal{U}_r$ (or $\mathcal{U}_l$) such that $S = R_{\mathcal{U}_S}$, the universal localization of R at $\mathcal{U}_S$.*

Proof. We show that the inclusion $R \to S$ is a ring epimorphism. Observe that A is a flat left and right R-module. Indeed, since S is an R-submodule of A and R is hereditary (in particular its weak dimension is at most one), then S is also a flat right and left R-module.

Consider the following commutative diagram whose columns are injective morphisms and its rows are exact sequences, where the first two rows are obtained from $0 \to S \to A \to A/S \to 0$ tensoring by $\_ \otimes_R S$ and $A \otimes_R \_$ respectively

$$\begin{tikzcd}
0 \arrow[r] & S \otimes_R S \arrow[r] \arrow[d] & A \otimes_R S \arrow[r] \arrow[d] & (A/S) \otimes_R S \arrow[r] & 0 \\
0 \arrow[r] & A \otimes_R S \arrow[r] & A \otimes_R A \arrow[r] \arrow[d] & A \otimes_R (A/S) \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r] & A \arrow[r] & A/S \arrow[r] & 0
\end{tikzcd}$$

They imply that $S \otimes_R S$ embeds in $A \otimes_R A$ and in A by Lemma 8.19. Moreover the image of $S \otimes_R S$ in A is S. Thus $R \to S$ is a ring epimorphism by Lemma 8.19.

Then Theorem 3.60 shows that S is the universal localization of R at some set of u-torsion right R-modules. Now Theorem 3.59 implies the result because it asserts that the universal localizations of R at sets of full morphisms embedding in R_u are in bijective correspondence with collections of stable association classes of atomic full morphisms . $\square$

Now we come to the main result of this section.

THEOREM 8.68. *Let R be a hereditary noetherian prime ring which is not simple artinian. Let A_R be the simple artinian quotient ring of R. Then*

(i) *$T = A \oplus A/R$ is a tilting right R-module with $T^{\perp} = \mathcal{U}_r^{\perp} = \mathcal{V}_r^{\perp} = \mathcal{D}_r^{\perp}$.*
(ii) *$T = A \oplus A/R$ is a tilting left R-module with $T^{\perp} = \mathcal{U}_l^{\perp} = \mathcal{V}_l^{\perp} = \mathcal{D}_l^{\perp}$.*
(iii) *For any overring $R < S < A$ there exists a unique subset $\mathcal{U}_S$ of $\mathcal{U}_r$ (respectively, of $\mathcal{U}_l$) such that $S \oplus S/R$ is a tilting right (left) R-module with tilting class $\mathcal{U}_S^{\perp}$.*
(iv) *For any right Ore subset $\mathfrak{S}$ of R consisting of non-zero-divisors, let*

$$\begin{aligned} \mathcal{U}_{\mathfrak{S}} &= \{U \in \mathcal{U}_r \mid \textit{for each } v \in U \textit{ there exists } s \in \mathfrak{S} \textit{ with } vs = 0\} \\ &= \{U \in \mathcal{U}_r \mid U \otimes_R R\mathfrak{S}^{-1} = 0\}. \end{aligned}$$

Then $R\mathfrak{S}^{-1}$ is the universal localization of R at $\mathcal{U}_{\mathfrak{S}}$. Moreover, $T_{\mathfrak{S}} = R\mathfrak{S}^{-1} \oplus R\mathfrak{S}^{-1}/R$ is a tilting right R-module with tilting class $T_{\mathfrak{S}}^{\perp} = \mathcal{U}_{\mathfrak{S}}^{\perp}$, and $T_{\mathfrak{S}}$ is a tilting left R-module with tilting class $T_{\mathfrak{S}}^{\perp} = \{R/Rs \mid s \in \mathfrak{S}\}^{\perp} = \{\operatorname{Tr} U \mid U \in \mathcal{U}_{\mathfrak{S}}\}^{\perp}$.
(v) *For any (two-sided) Ore subset $\mathfrak{S}$ of R consisting of non-zero-divisors, let $\mathcal{U}_{\mathfrak{S}}$ be as in* (iv). *Then $T_{\mathfrak{S}} = R\mathfrak{S}^{-1} \oplus R\mathfrak{S}^{-1}/R$ is a tilting right R-module with tilting class $T_{\mathfrak{S}}^{\perp} = \{R/sR \mid s \in \mathfrak{S}\}^{\perp} = \mathcal{U}_{\mathfrak{S}}^{\perp}$.*

PROOF. (i) By Proposition 8.64 and Theorem 8.50, A/R is a directed union of modules N_i where each N_i is a finite extension of simple right R-modules. This can also be proved in a more classical way as we proceed to see. In the proof of Corollary 8.53 we showed that A is the directed union of the right R-modules $v^{-1}R/R \cong R/vR$ where v is a non-zero-divisor of R. Now the right R-module R/vR is a torsion right R-module. Then R/vR is of finite length and so a finite extension of simple right R-modules by Theorem 8.65.

So we can apply Corollary 8.49 to obtain that T is a tilting right R-module with tilting class $T^{\perp} = \mathcal{U}_r^{\perp}$. Moreover, $\mathcal{U}_r^{\perp} = \mathcal{V}_r^{\perp}$ since every element in $\mathcal{V}_r$ is a finite extension of elements in $\mathcal{U}_r$ by Theorem 3.55. Again in a more classical way, if $M \in \mathcal{V}_r$, M is finitely generated and torsion. Let $m_1, \dots, m_n$ be a set of generators for M. For each m_i there exists a non-zero-divisor v_i

such that $R/v_iR \to m_iR$ is an onto morphism of right R-modules. Thus $\bigoplus_{i=1}^{n} R/v_iR \to M$ is surjective. By Theorem 8.65, $\bigoplus_{i=1}^{n} R/v_iR$ is of finite length, thus M is of finite length.

On the other hand, by Corollary 8.53, the tilting class of $A \oplus A/R = T$ is $\mathcal{D}_r^{\perp}$ because A is the left Ore localization of R at the subset of R consisting on all non-zero-divisors of R.

(ii) is proven with symmetric arguments.

(iii) By Proposition 8.67, S is the universal localization of R at a unique subset $\mathcal{U}_S$ of $\mathcal{U}_r$ ($\mathcal{U}_l$). Now, because of Proposition 8.64(ii) and (iii), we can apply Corollary 8.51(ii).

(iv) $T_{\mathfrak{S}}$ is a tilting left R-module with $T_{\mathfrak{S}}^{\perp} = \{R/Rs \mid s \in \mathfrak{S}\}^{\perp}$ by (the right version of) Corollary 8.53. Suppose that we have proved that $R\mathfrak{S}^{-1}$ is the universal localization of R at $\mathcal{U}_{\mathfrak{S}}$. By Proposition 8.64, we can apply Corollary 8.51(ii) to obtain the desired results.

We now prove that $R\mathfrak{S}^{-1}$ is the universal localization of R at $\mathcal{U}_{\mathfrak{S}}$. The argument is very similar to the one of [**CB91**, Lemma 3.4].

First of all, notice that ${}_RR\mathfrak{S}^{-1}$ is flat, and for every $U \in \mathcal{U}_{\mathfrak{S}}$, $U \otimes_R R\mathfrak{S}^{-1} = 0$. Hence, if $0 \to P \xrightarrow{\alpha} Q \to U \to 0$ is a projective presentation of U with P and Q finitely generated, then $\alpha \otimes 1_{R\mathfrak{S}^{-1}}$ is invertible. Hence condition (a) in Definition 3.46 is satisfied.

Let B be a ring with a morphism of rings $b\colon R \to B$ such that for every $U \in \mathcal{U}_{\mathfrak{S}}$ and any finite projective presentation $0 \to P \xrightarrow{\alpha} Q \to U \to 0$, $\alpha \otimes_R 1_B$ becomes invertible. Let $s \in \mathfrak{S}$. Consider R/sR. Since $\mathfrak{S}$ is right Ore, by Theorem 8.65, R/sR has finite length, and therefore it has a finite filtration of simple right R-modules. Recall that $\mathfrak{S}$ consists of non-zero-divisors. Hence $R/sR \cong s^{-1}R/R$. Since $\mathfrak{S}$ is a right Ore set, for every $s^{-1}r \in s^{-1}R$, there exist $t \in \mathfrak{S}$, $x \in R$ such that $s^{-1}r = xt^{-1}$. Therefore, for every $z \in R/sR$ there exists $t \in \mathfrak{S}$ with $zt = 0$. This implies that all the composition factors of R/sR are in $\mathcal{U}_{\mathfrak{S}}$.

For each $s \in \mathfrak{S}$, define the morphism $\delta_s\colon R \to R$, given by $r \to sr$. By the foregoing, $\delta_s \otimes 1_B$ is invertible for every $s \in \mathfrak{S}$. Notice that $\delta_s \otimes 1_B$ can be regarded as the morphism $B \to B$ defined by $x \mapsto b(s)x$. Thus $b(s)$ is invertible for all $s \in \mathfrak{S}$. By the universal property of Ore localization there exists a morphism of rings $\gamma\colon R\mathfrak{S}^{-1} \to B$ making the following diagram commutative

$$\begin{tikzcd}[column sep=small] R \arrow[rr] \arrow[dr, "b"'] & & R\mathfrak{S}^{-1} \arrow[dl, "\gamma"] \\ & B & \end{tikzcd}$$

Therefore condition (ii) in Definition 3.46 is satisfied.

For (v) apply the left and the right versions of (iv). □

Remark 8.69. Suppose that we are under the notation of Theorem 8.68(iii) and (iv). In [**Goo74**] it is shown that there is a bijection between the collection of overrings $R \leq S \leq A$ and the collections $\mathcal{Y}$ of isomorphism classes of simple right R-modules. In this correspondence, S is the overring such that $W \otimes_R S = 0$ for all $W \in \mathcal{Y}$ and no other simple right R-module. On the other hand, we have proved in Proposition 8.67 that all overrings $R \leq S \leq A$ are universal localizations of R at some set $\mathcal{U}_S$ of isomorphism classes of simple right R-modules. Moreover, it is proved in [**Sch86**, Theorem 10] (in a more general context) that $W \otimes_R R_{\mathcal{U}_S} = 0$ for all $W \in \mathcal{U}_S$ and no other simple right R-module. Therefore, $\mathcal{U}_S = \{W \in \mathcal{U}_r \mid W \otimes_R S = 0\}$ in Theorem 8.68(iii). From this we also obtain another proof of the fact stated in Theorem 8.68(iv), that $R\mathfrak{S}^{-1} = R_{\mathcal{U}_{\mathfrak{S}}}$. □

Before stating the next results on tilting modules we recall the following definitions.

Definitions 8.70. (a) A right R-module M is *faithful* if the ideal

$$\operatorname{ann}(M) = \{r \in R \mid mr = 0 \text{ for all } m \in M\} = 0.$$

The right R-module M is *unfaithful* if it is not a faithful module.

(b) Let Z be a commutative noetherian domain with quotient field K, and let Q be a central simple K-algebra. A *Z-order* in Q is a Z-subalgebra R of Q, finitely generated as Z-module and such that R contains a K-basis of Q. A *hereditary order* R is a hereditary ring R which is a Z-order in some central simple K-algebra Q, where Z is some Dedekind domain with quotient field $K \neq Z$. The hereditary order R is a *maximal order* if it is not properly contained in any other Z-order in Q. □

For the proof of the following result we will use some unexplained results about noetherian prime rings. Most of them, with the terminology we use, can be found in the survey [**Lev00**]. Full proofs of these results can be found in the references given there.

THEOREM 8.71. *Let R be a hereditary noetherian prime ring which is not simple artinian. Let $\mathcal{U}_r$ be a set of representatives of all isomorphism classes of all simple right R-modules. Suppose that there are no simple faithful right R-modules, and that $\mathrm{Ext}^1_R(U_1, U_2) = 0$ for any two non-isomorphic simple right R-modules U_1, U_2. Then*

$$\mathbb{T} = \{T_{\mathcal{W}} = R_{\mathcal{W}} \oplus R_{\mathcal{W}}/R \mid \mathcal{W} \subseteq \mathcal{U}_r\}$$

is a representative set up to equivalence of the class of all tilting right R-modules.

In particular, the statement holds true when R is a maximal order or a hereditary local noetherian prime ring which is not a simple artinian ring (for example a not necessarily commutative discrete valuation domain).

PROOF. For the first part we follow the terminology of [**Lev00**]. If M is a finitely generated right R-module, then $M \cong \mathbf{P}M \oplus \mathbf{T}M$ by Lemma 8.43. It is known, see for example [**Jat86**, Theorem 2.2.15], that any finitely generated torsion-free right R-module is embeddable in a finitely generated free right R-module. Therefore, provided that $\mathfrak{U}$ is the subset of R consisting of all non-zero-divisors of R, $\mathbf{T}M = \mathcal{T}_{\mathfrak{U}}M$. Moreover, $\mathcal{T}_{\mathfrak{U}}M$ is of finite length, see for example the proof of Theorem 8.68(iii). Furthermore, $\mathcal{T}_{\mathfrak{U}}M$ has a decomposition $\mathcal{T}_{\mathfrak{U}}M = V_1 \oplus V_2$ where V_1 is a (finite) direct sum of uniserial modules whose composition factors are all unfaithful, and V_2 is a direct sum of modules whose composition factors belong to so-called faithful towers [**Kuz72**, Theorem 2.19] or [**KL95**, Theorem 4.6]. Since we are assuming that there are no faithful simple right R-modules, $V_2 = 0$.

So $M^{\perp} = (W_1 \oplus \cdots \oplus W_n)^{\perp}$ where W_i are indecomposable finitely generated uniserial modules. Since $\mathrm{Ext}^1_R(U, U') = 0$ for any two non-isomorphic simple right R-modules U, U', we obtain that all composition factors of W_i are isomorphic to the same simple right R-module U_i. Hence $W_i^{\perp} = U_i^{\perp}$. Therefore $M^{\perp} = U_1^{\perp} \cap \cdots \cap U_n^{\perp}$. So for every set of finitely generated right R-modules $\mathcal{V}$, there exists a subset $\mathcal{W}$ of $\mathcal{U}_r$ such that $\mathcal{V}^{\perp} = \mathcal{W}^{\perp}$.

It was proved in [**GW79**] that if U is an unfaithful simple right R-module there exists a unique $V \in \mathcal{U}_r$ such that $\mathrm{Ext}^1_R(V, U) \neq 0$. So in our situation, $\mathrm{Ext}^1_R(U, U) \neq 0$ and $\mathrm{Ext}^1_R(U', U) = 0$ for nonisomorphic simple right R-modules U, U'. Hence given a subset $\mathcal{W}$ of $\mathcal{U}_r$ the class $\mathcal{W}^{\perp}$ is uniquely determined by the elements of $\mathcal{W}$ because $(U_r \setminus \mathcal{W}) \subset \mathcal{W}^{\perp}$ and for each $W \in \mathcal{W}$, $W \notin \mathcal{W}^{\perp}$.

From Remarks 8.44(b), Proposition 8.64 and Corollary 8.51(ii) we infer that $\mathbb{T}$ is a representative set up to equivalence of the class of all tilting right R-modules.

Let R be a maximal order. Observe first that every right ideal which contains a non-zero-divisor is an essential submodule of R because the set of non-zero-divisors is a right Ore set. By Lemma 8.62, for every maximal right ideal $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is a torsion right R-module. Hence there exists a non-zero-divisor $s \in R$ such that $\bar{1}s = 0$, i.e. $s \in \mathfrak{m}$. Thus $\mathfrak{m}$ is essential in R_R. Since R is finitely generated as a module over the commutative ring Z, then R is *right bounded*, i.e. every essential right ideal of R contains a two sided ideal which is essential as a right ideal [**GW89**, Proposition 8.1]. Hence $\mathfrak{m}$ contains a nonzero two-sided ideal. Therefore $\operatorname{ann}(R/\mathfrak{m}) \neq 0$, that is, $R/\mathfrak{m}$ is unfaithful.

If R is a hereditary order, there exists a bijection between $\mathcal{U}_r$ and the set of the nonzero prime ideals of R, $\operatorname{Spec}(R)$. The correspondence sends a simple right R-module M in $\mathcal{U}_r$ to the (unique, because $R/\mathfrak{p}$ is simple artinian) simple $R/\mathfrak{p}$-module.

Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals of R with corresponding simple right R-modules $M_{\mathfrak{p}}$ and $M_{\mathfrak{q}}$. We say that $\mathfrak{q} \rightsquigarrow \mathfrak{p}$ if $\operatorname{Ext}^1_R(M_{\mathfrak{q}}, M_{\mathfrak{p}}) \neq 0$. The connected components of the graph constructed from the set of prime ideals of R and the relation $\rightsquigarrow$ are called *cliques*. The clique that contains $\mathfrak{p}$ is denoted by $\operatorname{Cl}(\mathfrak{p})$. It is known that in a hereditary order $\operatorname{Cl}(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{q} \cap Z = \mathfrak{p} \cap Z\}$, see [**GW89**, Theorem 11.20].

In a maximal order R there is a bijection between $\operatorname{Spec}(R)$ and $\operatorname{Spec}(Z)$ given by $\mathfrak{p} \mapsto \mathfrak{p} \cap Z$, see [**Rei75**, Theorem 22.4].

Therefore, if R is a maximal order, and $U, V \in \mathcal{U}_r$, $\operatorname{Ext}^1_R(U, V)$ if and only if $U = V$.

If R is a hereditary local noetherian prime ring which is not a simple artinian ring, there is only one simple right R-module up to isomorphism, and it is unfaithful since it is isomorphic to the quotient of R by its maximal ideal. $\square$

We now recover the classification of tilting modules over Dedekind domains obtained in [**BET05**, Theorem 5.3].

Corollary 8.72. *Let R be a Dedekind domain.*

(i) *Let $\mathfrak{M}$ be a subset of* $\operatorname{max-spec}(R)$. *Consider the multiplicative subset $\mathfrak{S} = R \setminus \bigcup_{\mathfrak{p} \in \mathfrak{M}} \mathfrak{p}$ of R and the set of simple R-modules $\mathcal{U}_{\mathfrak{S}} = \{R/\mathfrak{m} \mid \mathfrak{m} \nsubseteq \bigcup_{\mathfrak{p} \in \mathfrak{M}} \mathfrak{p}\}$ (if $\mathfrak{M} = \emptyset$, then $\mathfrak{S} = R \setminus \{0\}$). Then $R\mathfrak{S}^{-1}$ is the universal localization of R at $\mathcal{U}_{\mathfrak{S}}$, and $T_{\mathfrak{S}} = R\mathfrak{S}^{-1} \oplus R\mathfrak{S}^{-1}/R$ is a tilting R-module with $T_{\mathfrak{S}}^{\perp} = \mathcal{U}_{\mathfrak{S}}^{\perp} = \{R/sR \mid s \in \mathfrak{S}\}^{\perp}$.*

(ii) *Let $\mathfrak{P}$ be a subset of* $\operatorname{max-spec}(R)$. *Consider the set of simple R-modules $\mathcal{U}_{\mathfrak{P}} = \{R/\mathfrak{m} \mid \mathfrak{m} \in \mathfrak{P}\}$. Then $T_{\mathfrak{P}} = R_{\mathcal{U}_{\mathfrak{P}}} \oplus R_{\mathcal{U}_{\mathfrak{P}}}/R$ is a tilting right R-module with $T_{\mathfrak{P}}^{\perp} = \mathcal{U}_{\mathfrak{P}}^{\perp}$. Therefore the set*

$$\mathbb{T} = \{T_{\mathfrak{P}} \mid \mathfrak{P} \subseteq \operatorname{max-spec}(R)\}$$

is a representative set up to equivalence of the class of all tilting R-modules.

Proof. (i) Recall that a Dedekind domain is a commutative noetherian prime ring which is not simple artinian. Now notice that

$$\mathcal{U}_{\mathfrak{S}} = \{R/\mathfrak{m} \mid \mathfrak{m} \nsubseteq \bigcup_{\mathfrak{p} \in \mathfrak{M}} \mathfrak{p}\} = \{R/\mathfrak{m} \mid \text{ for every } v \in R/\mathfrak{m} \text{ there is } s \in \mathfrak{S} \text{ with } vs = 0\}.$$

Then apply Theorem 8.68(v).

(ii) By Proposition 8.64 and Corollary 8.51(ii), we obtain that $T_{\mathfrak{P}}$ is a tilting module with $T_{\mathfrak{P}}^{\perp} = \mathcal{U}_{\mathfrak{P}}^{\perp}$.

For the second statement, we prove that R satisfies the conditions of Theorem 8.71.

All simple R-modules are unfaithful because they are isomorphic to the quotient of R by a maximal ideal.

Let now M be an extension of the non-isomorphic simple R-modules $R/\mathfrak{p}$ and $R/\mathfrak{q}$ with $\mathfrak{p}$ and $\mathfrak{q}$ nonzero prime ideals of R. Obviously $\mathfrak{pq} \subseteq \operatorname{ann}(M)$. On the other hand, the ideal $\operatorname{ann}(M)$ annihilates $R/\mathfrak{p}$ and $R/\mathfrak{q}$. Thus $\operatorname{ann}(M) \subseteq \mathfrak{p} \cap \mathfrak{q} = \mathfrak{pq}$. Therefore $\operatorname{ann}(M) = \mathfrak{pq}$. The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are comaximal. Thus $1 = p + q$ for some $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$. Hence, for each $m \in M$, $m = mp + mq$. Moreover, if $m \in M\mathfrak{p} \cap M\mathfrak{q}$, then

$$m = m(p+q) = mp + mq \in M\mathfrak{qp} + M\mathfrak{pq} = 0.$$

Therefore $M = M\mathfrak{p} \oplus M\mathfrak{q}$. Note that $M\mathfrak{p}$ and $M\mathfrak{q}$ are nonzero, otherwise $\mathfrak{p}\mathfrak{q} \subsetneq \operatorname{ann}(M)$. Now, since M is the extension of $M\mathfrak{p}$, $M\mathfrak{q}$ and, on the other hand, of the simple R-modules $R/\mathfrak{p}$, $R/\mathfrak{q}$, then $R/\mathfrak{p} \cong M\mathfrak{q}$ and $R/\mathfrak{q} \cong M\mathfrak{p}$. □

Let us make the following remark that will be useful in the proof of Remark 8.74.

Remark 8.73. Let R be a commutative ring and $\lambda\colon R \to S$ a ring epimorphism. Then S is a commutative ring.

Proof. For each $t \in S$, consider the morphisms of rings $\delta_1\colon S \to \mathbb{M}_2(S)$ and $\delta_{2t}\colon S \to \mathbb{M}_2(S)$ defined by $\delta_1(s) = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$ and $\delta_{2t}(s) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -t & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} s & 0 \\ ts-st & s \end{smallmatrix}\right)$ for all $s \in S$. If $t = \lambda(r)$ for some $r \in R$, observe that $\delta_1\lambda = \delta_{2t}\lambda$ by the commutativity of R, and $\delta_1 = \delta_{2t}$ because λ is a ring epimorphism. Hence, $\lambda(r)$ commutes with every element of S. As r was an arbitrary element of R, we obtain that $\lambda(r)s = s\lambda(r)$ for any $r \in R$ and $s \in S$. Now $\delta_1\lambda = \delta_{2t}\lambda$ for any $t \in S$. Therefore, since λ is a ring epimorphism, $\delta_1 = \delta_{2t}$, and $ts - st = 0$ for any $s, t \in S$, that is, S is a commutative ring. □

Remark 8.74. There exist Dedekind domains R for which the set of all tilting right R-modules of R cannot be expressed neither in terms of Ore localization nor of matrix localization. In particular there exists a universal localization which is neither an Ore nor a matrix localization.

Proof. We first note that given a multiplicative set $\mathfrak{S}$ of R there exists a subset $\mathfrak{Q} \subseteq \operatorname{Spec}(R)$ such that if $\mathfrak{S}' = R \setminus \bigcup_{\mathfrak{q}\in\mathfrak{Q}} \mathfrak{q}$, then $R\mathfrak{S}^{-1} = R\mathfrak{S}'^{-1}$. Indeed, for each $v \in R$ such that $\mathfrak{S} \cap vR = \emptyset$, consider a maximal ideal $\mathfrak{q}_v$ among the ideals I with $v \in I$ and $\mathfrak{S} \cap I = \emptyset$. Then $\mathfrak{q}_v$ is known to be a prime ideal. Now the set $\mathfrak{S}' = R \setminus \bigcup_v \mathfrak{q}_v$ is such that $\mathfrak{S} \subseteq \mathfrak{S}'$. On the other hand, if $z \in \mathfrak{S}'$, there exists $r \in R$ such that $zr \in \mathfrak{S}$. Hence z is invertible in $R\mathfrak{S}^{-1}$. Therefore $R\mathfrak{S}^{-1} = R\mathfrak{S}'^{-1}$.

It is known that there exist Dedekind domains with subsets $\mathfrak{M}$ of $\operatorname{Spec}(R)$ such that there exist primes $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \subseteq \bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$, and $\mathfrak{p} \notin \mathfrak{M}$, see for example [**Sal04**, Example 4.14]. Consider such example R with field of fractions D. Let $\mathcal{U} = \{R/\mathfrak{m} \mid \mathfrak{m} \notin \mathfrak{M}\}$.

Suppose that there exists a multiplicative subset $\mathfrak{S}$ of R such that $R\mathfrak{S}^{-1} = R_{\mathcal{U}}$. By the remark at the beginning of the proof, we may suppose that $\mathfrak{S}$ is the complement of the union of the prime ideals in $\mathfrak{Q} \subseteq \operatorname{Spec}(R)$, i.e. $\mathfrak{S} = R \setminus \bigcup_{\mathfrak{q}\in\mathfrak{Q}} \mathfrak{q}$. Then $R\mathfrak{S}^{-1}$ is the universal localization of R at $\mathcal{V} = \{R/\mathfrak{n} \mid \mathfrak{n} \nsubseteq \bigcup_{\mathfrak{q}\in\mathfrak{Q}} \mathfrak{q}\}$ by Corollary 8.72(i). Since $R\mathfrak{S}^{-1} = R_{\mathcal{V}} = R_{\mathcal{U}}$, then $\mathcal{U} = \mathcal{V}$ because of Proposition 8.67. On the other hand, $\mathfrak{p} \in \mathcal{U}$, but $\mathfrak{p} \notin \mathcal{V}$, a contradiction. Therefore there does not exists such an Ore set $\mathfrak{S}$.

Suppose now that there exists a set of matrices Σ such that its universal localization $\lambda\colon R \to R_\Sigma$ produces the tilting R-module $R_\Sigma \oplus R_\Sigma/R$. Let $H \in \Sigma$. We need that R-embeds in R_Σ, so $M = \operatorname{coker} H$ has to be a bound R-module by Remarks 8.44. It has presentation $0 \to R^m \xrightarrow{H} R^n \to M \to 0$. If we tensor it by $\_\otimes_R D$, we get $0 \to D^m \xrightarrow{H\otimes 1_D} D^n \to M \otimes_R D \to 0$ because ${}_RD$ is flat. Then $m \leq n$ because $M \otimes_R 1_D$ is injective. If we apply the same reasoning to $\operatorname{Tr} M$, we get that $m = n$. But then R_Σ is the localization of R at the Ore set $\mathfrak{S} = \{s_{H_1} \cdots s_{H_n} \mid n \in \mathbb{N},\ H_i \in \Sigma,\ s_{H_i} = \det H_i\}$. Indeed, by Remark 8.73, R_Σ is a commutative ring. Since every $H \in \Sigma$ is invertible over R_Σ, we get that $\det H$ is invertible in R_Σ for each $H \in \Sigma$. So there exists a unique morphism of R-rings $R\mathfrak{S}^{-1} \to R_\Sigma$. On the other hand, any $H \in \Sigma$ is invertible over $R\mathfrak{S}^{-1}$ because $R\mathfrak{S}^{-1}$ is commutative and $\det H$ is invertible in $R\mathfrak{S}^{-1}$. Hence there exists a unique morphism of R-rings $R_\Sigma \to R\mathfrak{S}^{-1}$. Now both compositions are the identity by the universal properties of R_Σ and $R_{\mathfrak{S}}^{-1}$. □

6. Tame hereditary algebras

Let R be a ring. Let $G_0(R)$ be the abelian group which has as generators the set of isomorphism classes $[M]$ of finitely presented right R-modules and as relations the expressions $[A] + [C] - [B]$ for each exact sequence $0 \to A \to B \to C \to 0$ of finitely presented right R-modules A, B, C. It is known that if R is a hereditary ring, then $G_0(R) \cong K_0(R)$ via the map $[M] \to [Q] - [P]$ where $0 \to P \to Q \to M \to 0$ is a presentation of M with P and Q finitely generated projective right R-modules. From now on we will identify $G_0(R)$ with $K_0(R)$ in case R is a hereditary ring.

We say that a ring R is *connected* if R has no nontrivial central idempotents.

An *artin algebra* is an algebra over a commutative artinian ring k which, in addition, is a finitely generated k-module.

Let R be a connected hereditary artin k-algebra. Hence R is finitely generated as a module by its center. Then R is a PI-ring and the center of R is a field by [**RS74**, Theorem 4].

From now on we will suppose that R is a connected hereditary artin algebra and k is the center of R.

We denote by $\text{mod-}R$ (respectively $R\text{-mod}$) the full subcategory of $\text{Mod-}R$ ($R\text{-Mod}$) consisting of the finitely generated right (left) R-modules. And by $\text{mod}_{\mathcal{P}}\text{-}R$ we mean the full subcategory of $\text{mod-}R$ consisting of the modules without nonzero projective summands. By $\text{mod}_{\mathcal{I}}\text{-}R$ the full subcategory of $\text{mod-}R$ consisting of the modules without nonzero injective summands.

If there is only a finite number of indecomposable objects up to isomorphisms in $\text{mod-}R$, then R is said to be of *finite representation type*. We say that R is of *infinite representation type* if it is not of finite representation type.

Let $D\colon \text{mod-}R \to R\text{-mod}$ be the duality defined by $M \mapsto \operatorname{Hom}_k(M, k)$. It is known that the correspondence $\tau = D\operatorname{Tr}\colon \text{mod}_{\mathcal{P}}\text{-}R \to \text{mod}_{\mathcal{I}}\text{-}R$, $M \mapsto \tau M = D(\operatorname{Ext}^1_R(M, R))$ is an equivalence of categories with inverse $\tau^- = \operatorname{Tr} D\colon \text{mod}_{\mathcal{I}}\text{-}R \to \text{mod}_{\mathcal{P}}\text{-}R$ defined by $N \mapsto \tau^- N = \operatorname{Ext}^1_R(DM, R)$. The correspondence τ is called the *AR-translation of* R [**ARS95**].

Recall the Auslander-Reiten formulae [**ARS95**]: Let $X \in \text{mod}_{\mathcal{P}}\text{-}R$, $Y \in \text{Mod-}R$, then

$$\operatorname{Hom}_R(Y, \tau X) = D\operatorname{Ext}^1_R(X, Y), \qquad D\operatorname{Hom}_R(X, Y) = \operatorname{Ext}^1_R(Y, \tau X). \tag{90}$$

Any hereditary artin algebra R has associated a valued quiver Γ_R, called the *Auslander-Reiten quiver* of R. Before giving how it is constructed we need some definitions.

Definitions 8.75. Let R be a hereditary artin algebra.

(a) A morphism or right R-modules $g\colon B \to C$ in $\text{mod-}R$ is called *right almost split* if
  (i) g is not a split epimorphism, and
  (ii) if $h\colon X \to C$ is a morphism in $\text{mod-}R$ that is not a split epimorphism, then h factors through g, i.e.

$$\begin{tikzcd} B \arrow[rr, "g"] & & C \\ & X \arrow[ul, "f"] \arrow[ur, "h"'] & \end{tikzcd}$$

(b) $g\colon B \to C$ is called *minimal right almost split* if it is almost split and whenever $X = B$ in the foregoing diagram, then f is an isomorphism.

(c) The definition of a *(minimal) left almost split* map is dual.

(d) An exact sequence in $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ $\text{mod-}R$ is called *almost split* if f is left almost split and g is right almost split.

(e) A morphism $h\colon B \to C$ between indecomposable R-modules B, C is said to be *irreducible* if h is not an isomorphism, and in any commutative diagram

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "\alpha"'] & & C \\ & Z \arrow[ur, "\beta"'] & \end{tikzcd}$$

either α is a split monomorphism or β is a split epimorphism. $\square$

Now the construction of Γ_R is as follows. The vertices of Γ_R are in one to one correspondence with the isomorphism classes of finitely generated indecomposable right R-modules M and are denoted by $[M]$. There is an arrow $[M] \stackrel{(a,b)}{\to} [N]$ if and only if there is an irreducible map $M \to N$. The value (a, b) of the arrow is given since there exist unique $a, b \geq 0$ such that there is a minimal right almost split morphism $M^{(a)} \oplus X \to N$ where M is not a direct summand of X, and a minimal left almost split morphism $M \to N^{(b)} \oplus Y$ where N is not a direct summand of Y. For details see [**ARS95**].

The component $\mathbf{p}$ of Γ_R containing the isomorphism classes of all finitely generated indecomposable projective modules is called the *preprojective component*, and consists of the isomorphism classes of the indecomposable right R-modules M such that there exists a nonnegative integer n such that $\tau^n M$ is a projective module. The component $\mathbf{q}$ of Γ_R containing the isomorphism classes of all finitely generated indecomposable injective modules is called the *preinjective component*, and consists of the isomorphism classes of the indecomposable right R-modules M such that there exists a nonnegative integer n such that $(\tau^-)^n M$ is an injective module. Any other component is said to be a *regular component*.

Let M be a nonzero finitely generated indecomposable right R-module. If $[M] \in \mathbf{p}$, we say that M is a *preprojective module*. If $[M] \in \mathbf{q}$, we say that M is a *preinjective module*. If $[M]$ is in a regular component, we say that M is a *regular module*. Notice that if M is a regular module then $M \in \mathrm{mod}_{\mathcal{P}}\text{-}R$ and $M \in \mathrm{mod}_{\mathcal{I}}\text{-}R$. A *simple regular module* is a regular module of minimal length inside the component they belong to.

It is known how the regular components of Γ_R look like. Recall the following result [**ARS95**, Chapter VIII, Theorem 4.15]:

Theorem 8.76. *Let R be a hereditary artin algebra of infinite representation type, and let $\mathcal{R}$ be a regular component of Γ_R. The following hold.*

(i) *Let S be a simple regular module with $[S] \in \mathcal{R}$. Then there exists an infinite chain of irreducible monomorphisms $S = S[1] \stackrel{f_1}{\to} S[2] \stackrel{f_2}{\to} \cdots S[n] \stackrel{f_n}{\to} \cdots$ with $[S[i]] \in \mathcal{R}$.*

(ii) *For each $n \in \mathbb{Z}$ and $i \geq 1$, there is an almost split sequence*

$$0 \to \tau^{n+1}S[i] \to \tau^{n+1}S[i+1] \oplus \tau^n S[i-1] \to \tau^n S[i] \to 0,$$

where $S[0] = 0$.

(iii) *The set $\{\tau^n S[i] \mid n \in \mathbb{Z},\ i \geq 1\}$ constitutes a complete set of representatives of indecomposable modules in $\mathcal{R}$ up to isomorphism.*

(iv) *If $h\colon \tau^n S[i+1] \to \tau^{n-1}S[i]$ is any irreducible morphism, then* $\ker h \cong \tau^n S[1]$.

(v) *If $\tau^n S[i] \cong S[i]$ for some $n \in \mathbb{Z}$ and $i \geq 1$, then $\tau^n S[j] \cong S[j]$ for all $j \geq 1$.* $\square$

Hence the valuation (a, b) of any arrow equals $(1, 1)$, and a regular component has the following form

$$\begin{tikzcd}
 & & & \vdots & & \vdots & & \vdots \\
 & & & \tau S[3] \arrow[dr] & & S[3] \arrow[dr] & & \tau^{-} S[3] \\
 & \cdots & \tau S[2] \arrow[ur] \arrow[dr] & & S[2] \arrow[ur] \arrow[dr] & & \tau^{-} S[2] \arrow[ur] & \\
\cdots & \tau S \arrow[ur] & & S \arrow[ur] & & \tau^{-} S \arrow[ur] & & \cdots
\end{tikzcd}$$

There is the possibility that $\tau^n S = S$ for some $n \geq 1$. Then $\tau^n S[i] = S[i]$ for all i, and we obtain what is called a *stable tube*. If such an n exists, the smallest one is called the *width* of the stable tube.

Notice that every finitely presented (=generated) right R-module M has finite dimension over k. It is known that $K_0(R)$ is the free abelian group with basis the (finite number of) isomorphism classes of simple right R-modules $\{S_1, \dots, S_n\}$. Then we can define on $K_0(R)$ the bilinear form $B_R\colon K_0(R) \times K_0(R) \to \mathbb{Z}$ given by

$$B_R([M],[N]) = \dim_k \operatorname{Hom}_R(M,N) - \dim_k \operatorname{Ext}^1_R(M,N).$$

In this way we can define the quadratic form $\chi_R\colon \mathbb{Q} \otimes_{\mathbb{Z}} K_0(R) \to \mathbb{Q}$ induced by B_R. It is known that R is of finite representation type if and only if χ_R is positive definite [**ARS95**].

A *tame hereditary algebra* R is a connected hereditary artinian algebra which is of finite dimension over its center k and whose quadratic form χ_R is positive semidefinite but not positive definite.

If R is a tame hereditary algebra, the $\mathbb{Q}$-subspace $N \leq \mathbb{Q} \otimes_{\mathbb{Z}} K_0(R)$ formed by the radical vectors of B_R is one-dimensional and can be generated by a vector v with coordinates $(v_1, \dots, v_n) \in \mathbb{N}^n$ in the basis $\{[S_1], \dots, [S_n]\}$ with at least one component $v_i = 1$, see [**Rin84**]. Hence $\chi_R(v) = 0$ and any other w such that $\chi_R(w) = 0$ is a $\mathbb{Q}$-multiple of v. Then, following [**CB91**, Section 4], we define the faithful rank function $\partial_R\colon K_0(R) \to \mathbb{Q}$ given by

$$\partial_R([M]) = \frac{B_R([M],v)}{B_R([R],v)},$$

which is called the *normalized defect for R*.

The preprojective, preinjective and regular modules are determined numerically. More concretely, let X be a finitely generated indecomposable right R-module. Then X is preprojective if and only if $\partial_R([X]) > 0$, X is preinjective if and only if $\partial_R([X]) < 0$ and X is regular if and only if $\partial_R([X]) = 0$. Moreover, the indecomposable ∂_R-torsion modules coincide with the regular modules and the ∂_R-simple modules with the simple regular modules.

The following key result for us is [**CB91**, Lemma 4.4].

THEOREM 8.77. *Let R be a tame hereditary algebra. Then R_{∂_R} is a simple artinian ring.* □

THEOREM 8.78. *Let R be a tame hereditary algebra. Then, for every set $\mathcal{U}$ of isomorphism classes of simple regular right (left) R-modules, the right (left) R-module $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting right R-module such that $T_{\mathcal{U}}^{\perp} = \mathcal{U}^{\perp}$.*

Proof. By Theorem 8.77, R_{∂_R} is simple artinian. Then, by Theorem 8.50, $R_{\mathcal{U}}/R$ is a directed union of finitely presented modules N_i, $R_{\mathcal{U}}/R = \underset{i\in I}{\cup} N_i$, such that N_i is a finite filtration of elements of $\mathcal{U}$. Since ∂_R is faithful, R embeds in $R_{\mathcal{U}}$. Hence we can apply Theorem 8.49 to obtain the desired result. $\square$

Finally we are ready to give the promised example of a tilting module T of the form $T = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$, constructed from the universal localization of R at a class $\mathcal{U}$ of bound right R-modules, such that $T^{\perp} \neq \mathcal{U}^{\perp}$.

Example 8.79. Let R be a tame hereditary algebra over the field k with a stable tube $\mathbf{t}_{\nu_0}$ of width at least 3. Let S be a simple regular module such that $[S] \in \mathbf{t}_{\nu_0}$. Consider τS, $\tau^{-}S$, $S[2]$ and $\tau S[2]$.

The ∂_R-torsion module $S[2]$ ($\tau S[2]$) is an extension of the ∂_R-simple modules S and $\tau^{-}S$ (τS and S) by Theorem 8.76. Hence we can suppose that $S[2]$, $\tau S[2]$ have finite projective presentations

$$0 \to P_1 \xrightarrow{\alpha} Q_1 \to \tau S[2] \to 0$$
$$0 \to P_2 \xrightarrow{\beta} Q_2 \to S[2] \to 0$$

where α and β are full morphisms such that $\alpha = \delta\gamma$, $\beta = \varepsilon\delta$ and $\gamma, \delta, \varepsilon$ are full atomic morphisms with $\operatorname{coker}\gamma = \tau S$, $\operatorname{coker}\delta = S$ and $\operatorname{coker}\varepsilon = \tau^{-}S$. Indeed, given the commutative diagrams,

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & P_1' \arrow[d, "\alpha'"] & & \\
 & & Q_1' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \tau S \arrow[r] & \tau S[2] \arrow[r] \arrow[d] & S \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & P_2' \arrow[d, "\beta'"] & & \\
 & & Q_2' \arrow[r] \arrow[d] & \tau S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & S \arrow[r] & S[2] \arrow[r] \arrow[d] & \tau S \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

We can complete them taking kernels to obtain

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P_1' \arrow[r, "="] \arrow[d, "\gamma'"] & P_1' \arrow[d, "\alpha'"] & & \\
0 \arrow[r] & K_1 \arrow[r, "\delta_1'"] \arrow[d] & Q_1' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \tau S \arrow[r] \arrow[d] & \tau S[2] \arrow[r] \arrow[d] & S \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P_2' \arrow[r, "="] \arrow[d, "\delta_2'"] & P_2' \arrow[d, "\beta'"] & & \\
0 \arrow[r] & K_2 \arrow[r, "\varepsilon'"] \arrow[d] & Q_2' \arrow[r] \arrow[d] & \tau S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & S \arrow[r] \arrow[d] & S[2] \arrow[r] \arrow[d] & \tau S \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Observe that δ'_1 and δ'_2 are presentations of the same module S. Therefore they are stably associated by Lemma 3.53. So there exist projective modules W and Z such that $K_1 \oplus W \cong P'_2 \oplus Z$, and $Q'_1 \oplus W \cong K_2 \oplus Z$ and the following diagrams are commutative

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P'_1 \oplus W \arrow[r, "="] \arrow[d, "\gamma"] & P'_1 \oplus W \arrow[d, "\alpha"] & & \\
0 \arrow[r] & K_1 \oplus W \arrow[r, "\delta"] \arrow[d] & Q'_1 \oplus W \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \tau S \arrow[r] \arrow[d] & \tau S[2] \arrow[r] \arrow[d] & S \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P'_2 \oplus Z \arrow[r, "="] \arrow[d, "\delta"] & P'_2 \oplus Z \arrow[d, "\beta"] & & \\
0 \arrow[r] & K_2 \oplus Z \arrow[r, "\varepsilon"] \arrow[d] & Q'_2 \oplus Z \arrow[r] \arrow[d] & \tau S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & S \arrow[r] \arrow[d] & S[2] \arrow[r] \arrow[d] & \tau S \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Now put $P_1 = P'_1 \oplus W$, $Q_1 = Q' \oplus W$, $P_2 = P'_2 \oplus Z$ and $Q_2 = Q'_2 \oplus Z$ to obtain the desired result.

Let $\mathcal{V} = \{S[2],\ \tau S[2]\}$ and $\mathcal{U} = \{S,\ \tau S,\ \tau^- S\}$. Then $R_{\mathcal{V}} = R_{\{\alpha,\beta\}} \cong R_{\{\gamma,\delta,\varepsilon\}} = R_{\mathcal{U}}$. Therefore $T_{\mathcal{V}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting right R-module with $T_{\mathcal{V}}^{\perp} = \mathcal{U}^{\perp}$ which is different from $\mathcal{V}^{\perp}$. To prove this, using the AR-formula we have

$$\operatorname{Ext}^1_R(S[2], S) \ \cong \ D\operatorname{Hom}_R(S, \tau S[2]) = 0,$$

because if $f\colon S \to S[2]$ is not zero, since S is ∂_R-simple (i.e. simple in the category of regular modules) and τS is not isomorphic to it, then composing f with the projection $\tau S[2] \to \tau S[2]/\tau S$, gives that $\tau S \to \tau S[2]$ is a split monomorphism, a contradiction. Also

$$\operatorname{Ext}^1_R(\tau S[2], S) \ \cong \ D\operatorname{Hom}_R(S, \tau^2 S[2]) = 0,$$

because S is not a factor of the series with ∂_R-simple factors of $\tau^2 S[2]$. Hence $S \in \mathcal{V}^{\perp}$. But

$$\operatorname{Ext}^1_R(\tau^- S, S) \cong D\operatorname{Hom}_R(S, \tau\tau^- S) \cong D\operatorname{Hom}_R(S, S) \neq 0,$$

that is, $S \notin \mathcal{U}^{\perp}$. $\square$

"It's just the beginning
The beginning, not the end"

Europe, *Just the beginning*

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