Analysis of some diffusive and kinetic models in mathematical biology and physics

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CERTIFICO que la present memòria ha estat realitzada per en Jesús Rosado Linares, sota la direcció de José Antonio Carrillo de la Plata.
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Signat: Dr. José Antonio Carrillo de la Plata

A book, when open, is a talking mind;
closed, is a waiting friend;
forgotten, is a forgiving soul;
destroyed, is a crying heart...
Indian proverb

The outside world is something independent from man,
something absolute, and the quest for the laws
which apply to this absolute appeared to me as
the most sublime scientific pursuit in life.
Max Planck

[...] "All right, The Answer to the Great Question...
Of Life, the Universe and Everything... is..."
"Yes...!...?"
"Forty-two"
"Forty-two! Is that all you have to show for
seven and a half million years' work?"
"That quite definitely is the answer. I think the problem is
that you've never actually known what the question is."[...]
Conversation between Deep Thought and two
hyperintelligent pandimensional beings.

Douglas Adams,
The Hitchhiker guide to the Galaxy.

Contents

Introducció

De vegades es diu que les matemàtiques són el llengutge que va emprar Déu per escriure el món. La gent pot estar-hi d'acord o no, fins i tot amb la premisa més bàsica d'aquesta afirmació, però el fet que les matemàtiques són el *millor* llenguatge que disposem per descriure'l es troba més enllà de qualsevol discussió. Si en algun moment vaig tenir algun dubte que això fos cert, aquest s'ha esvaït completament després d'aquests anys i al llarg de les pàgines que ens aguarden provaré de fer-vos compartir aquesta opinió a través de l'estudi de diferents exemples de modelització.

Alguns d'ells són antics, remuntant-se a models i equacions que foren plantejats fa gairebé un segle, altres en canvi són realment nous, amb poc més d'un parell d'anys de vida i tots ells s'utilitzen a diferents àrees d'interès: física, biologia, estudis del comportament, etc. Però si es miren amb prou atenció, un cop hem aconseguit traduir l'objecte del nostre estudi en termes matemàtics, en forma d'equacions (com si aquesta fos la part fàcil!) veiem que tots ells ens suggereixen les mateixes preguntes bàsiques. En primer lloc, i potser la més important: què és una solució del model per a nosaltres? O més precisament, en quin espai abstracte viurà? Què vol dir en termes del problema original que volíem estudiar? Un cop hem donat resposta a aquestes preguntes, la següent qüestió apareix naturalment: existeix realment una solució així? Sota quines condicions? Durant quan de temps estarà definida? N'hi ha més d'una? Parlava d'una pregunta i n'he plantejat quatre. De fet en podrien haver estat més, totes elles rellevants, però en el fons, per moltes que en fem, totes venen arrossegades per la primera, per matisar-la. Aquesta té un plantejament molt simple, i la resposta que espera no ho podria ésser menys: sí, no o, com ens trobem en la majoria de casos, potser. Sí la resposta resulta ser no, ens ho mirem com ens ho mirem, voldria dir que algun error ha trobat el seu camí a través del filtre de la pregunta inicial. Podria ser que necessitem plantejar-nos la solució de forma diferent a com ho fèiem, que haguem de trobar una nova forma d'interpretar la natura o fins i tot que ens haguem de qüestionar la validesa d'allò que assumim com a cert en el món que ens envolta. En els altres casos, amb totes les precisions que facin falta, només

queda una pregunta esperant-nos: com evolucionarà aquesta solució? Perquè el que realment volem és ser capaços de fer prediccions, no només de descriure fets que ja han passat.
Hem organitzat els models que estudiarem en tres capítols, d'acord amb les respostes que donem a les preguntes que acabem de formular, o amb la forma en què les busquem. Malgrat que a cada model ja ho precisarem adequadament, donem un petit avançament: a tots ells, per a nosaltres, una solució serà una mesura que descrigui la probabilitat de trobar l'objecte que estudiem en un estat determinat. La resposta a la segona pregunta serà, també, comúna per a tots els models: un rotund potser (amb el permís d'Oasis). Finalment, la resposta a l'última pregunta anirà canviant d'un model a un altre. Comentem a continuació què s'ha fet a cada capítol.
Al capítol 1 explorem l'abast de l'equació de Fokker-Planck estudiant dos models que, tot i estar basats a la mateixa equació, s'apliquen a camps aparentment tant llunyans com són la mecànica quàntica i la biologia. El primer d'aquests models descriu la distribució en velocitat de bosons i fermions, mentre que el segon modelitza el fenomen conegut com a quimiotaxis. Malgrat tot, en ambdós casos emprem arguments molt similars per a provar l'existència de solucions i podem recórrer a mètodes d'entropia per tal d'estudiar el seu comportament asimptòtic. Els resultats que s'exposen aquí apareixen en [46, 42, 68].

Les equacions cinètiques per a interacció de partícules s'han introduït a la literatura física en els treballs [81, 107, 109, 108, 156, 82]. Diverses equacions per estudiar distribucions no homogènies en l'espai apareixen com a derivacions formals de l'equació de Boltzmann generalitzada i de les equacions cinètiques d'Uehling-Uhlenbeck, ambdues per a bosons i fermions. Un model per a fermions en aquesta situació ha estat estudiat recentment a [142], on es prova que el comportament asimptòtic de solucions per a aquesta equació en el Tor pot ser descrit per estats d'equilibri homogenis en l'espai, donats per distribucions de Fermi-Dirac, quan la condició inicial no és lluny de l'equilibri en un espai de Sobolev adequat. Aquest resultat es basa en els treballs previs [139, 141]. Altres resultats per a models de tipus Boltzmann relacionats amb aquest han aparegut a [71, 127]. Quan podem assumir que les partícules han assolit l'equilibri a la seva distribució espacial podem plantejar-nos equacions espacialment homogènies. A [106] un model de Fokker-Planck per a bosons i fermions es deriva de l'equació de Kramer. Aquest serà el model que estudiarem a la primera secció d'aquest capítol:

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \mathrm{div}(v\rho(1 + \kappa\rho))$$

on $\kappa = -1$ s'empra quan volem descriure el comportament dels fermions i $\kappa = 1$ en el cas dels bosons. El signe negatiu en el primer cas representa l'aversió d'aquest tipus de partícules a compartir la mateixa fase. Treballarem als espais $\Upsilon = \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{L}^1_1(\mathbb{R}^d) \cap \mathrm{L}^p_m(\mathbb{R}^d)$ i $\Upsilon_T = \mathcal{C}([0,T];\Upsilon)$ amb normes

$$\|f(t)\|_\Upsilon = \max\{\|f(t)\|_\infty, \|f(t)\|_{\mathrm{L}^1_1}, \|f(t)\|_{\mathrm{L}^p_m}\} \quad \text{i} \quad \|f\|_{\Upsilon_T} = \max_{0\le t\le T} \|f\|_\Upsilon$$

per tot $T > 0$. El primer que farem es provar l'existència local de solucions, tant per a fermions com per a bosons. Després, trobarem estimacions a priori que ens permetran demostrar el següent teorema d'existència global per a fermions:

Teorema (Existència Global). *Siguin $f_0 \in \mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > d$, $p \ge 2$, $m \ge 1$ tals que $0 \le f_0 \le 1$. Llavors el problema de Cauchy* (1.1) *amb condició inicial f_0 té una única solució definida a $[0,\infty)$ que pertany a Υ_T per a tot $T > 0$. També tenim que $0 \le f(t,v) \le 1$, per a tot $t \ge 0$ i $v \in \mathbb{R}^N$, i $\|f(t)\|_1 = \|f_0\|_1 = M$ per a tot $t \ge 0$.*

Un cop que l'existència de solucions per a qualsevol temps ha estat establerta podem mirar el seu comportament asimptòtic. Els estadístics de Fermi-Dirac i de Bose-Einstein,

$$F_\beta(v) = \frac{1}{\beta \mathrm{e}^{\frac{|v|^2}{2}} - \kappa}$$

amb $\beta > 0$ són solucions estacionàries. Per a cada valor de la massa inicial M existeix un únic valor de β tal que F_β té massa M. En el cas de bosons tenim que $\beta \ge 1$. Si $d \ge 3$ convergeix cap a una solució singular però integrable a mesura que $\beta \to 1^+$, de manera que ens trobem amb la coneguda massa crítica per a distribucions d'equilibri de Bose-Einstein. Usant el fet que

$$H(g) := \int_{\mathbb{R}^d} s(g(v))\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 g(v)\,\mathrm{d}v,$$

on

$$s(r) := (1-r)\log(1-r) + r\log(r) \le 0, \quad r \in [0,1],$$

és un funcional de Lyapunov, podem emprar una desigualtat d'entropia basada en el treball realitzat a [41] per a demostrar convergència exponencial de les solucions cap als estats estacionaris. Més precisament, podem provar

Teorema (Rati de decaïment de l'entropia). *Assumim que f és una solució del problema de Cauchy* (1.1) *amb condició inicial a $\mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > \max(d,2)$, $m \ge 1$ tal que $0 \le f_0 \le F_{M^*} \le 1$ amb F_{M^*} una distribució de Fermi -Dirac de massa M^*. Llavors la solució en temps global del problema de Cauchy* (1.1) *amb condició inicial f_0 satisfà*

$$H(f) - H(F) \le (H(f_0) - H(F))\mathrm{e}^{-2Ct}$$

i

$$\|f(t) - F\|_{L^1(\mathbb{R}^d)} \le C_2(H(f_0) - H(F))^{1/2}\mathrm{e}^{-Ct}$$

per a tot $t \geq 0$, on C depèn de M^* i F és la distribució de Fermi-Dirac que satisfà $\|F\|_1 = \|f_0\|_1$.

No obstant, demanar a la condició inicial que estigui acotada per un estadístic de Fermi-Dirac és força restrictiu. Així doncs, proposem un resultat intermedi que, si bé no ens permet dir res sobre el rati de convergència cap a l'estat estacionari, sí que garanteix que aquesta es produeix.

Teorema (Convergència d'entropia). *Sigui f una solució del problema de Cauchy* (1.1) *amb dada inicial $f_0 \in \mathrm{L}^1_{\mathrm{mp}}(\mathbb{R}^\mathrm{d})$ tal que existeix una funció radialment simètrica g_0 no creixent i amb segon moment acotat, amb $0 \leq f_0 \leq g_0 \leq 1$. Llavors $H(f) \to H(F_M)$ quan $t \to \infty$, on F_M és la distribució de Fermi-Dirac amb la mateixa massa que f_0.*

Per als bosons també provarem convergència exponencial en una dimensió.
A la segona secció dirigim la nostra atenció a la biologia, estudiant un model per la quimiotaxis. La quimiotaxis és el fenomen pel qual organismes unicel·lulars es mouen sota la influència de substàncies químiques en el seu entorn. Considerem la següent versió parabòlica-parabòlica del model de Keller-Segel:

$$\begin{cases} \rho_t = \varepsilon\Delta\rho - \operatorname{div}(\rho(1-\rho)\nabla S) \\ S_t = \Delta S - S + \rho \end{cases}$$

on ρ modelitza la densitat de cèl·lules i S la concentració de la substància química. El terme $(1-\rho)$ juga el paper del terme de sensibilitat quimiotàctica, χ, que acostuma a ser decisiu en el desenllaç de la competició entre convecció i difusió. En aquest cas, aquest terme prevé que la solució "exploti" atès que les cèl·lules deixen d'agregar-se quan s'assoleix una concentració màxima (que normalitzem a 1 per simplicitat). Podem llegir [146, 97] per els detalls biològics darrera d'aquesta assumpció.
Una forma típica del model de Keller-Segel és la corresponent al cas parabòlic--el·líptic, on la segona equació és substituïda per $0 = \Delta S + \rho - S$ o l'equació de Poisson $-\Delta S = \rho$. En el cas en què $\chi(\rho, S) \equiv$ constant, el sistema anterior ha estat estudiat extensivament. En particular, és ben conegut (cf. [72]) que el sistema de Keller-Segel 2-dimensional

$$\begin{cases} \rho_t = \Delta\rho - \operatorname{div}(\rho\chi\nabla S) \\ 0 = \Delta S + \rho \end{cases}$$

mostra un llindar m^* que depèn de χ per a la massa total, el qual determina l'existència global de solucions (per massa per sota del llindar) o l'"explosió" de les solucions en temps finit (per masses inicials majors que m^*). Resultats relacionats amb aquest es poden trobar a [60, 151, 20, 59] i a [35] per al cas parabòlic. Aquí duem a terme una anàlisi similar a la realitzada en el cas de Fermi-Dirac per provar existència global i unicitat de solucions a l'espai funcional

$$\mathcal{U} := (\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)) \times (\mathrm{W}^{1,1}(\mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d)).$$

A continuació ens fixem en el comportament asimptòtic d'aquesta solució. En un primer pas demostrem que la concentració tant de les cèl·lules com de la substància química decau fins a zero amb velocitat polinomial. Per a fer això, provem que el decaïment té lloc a L^2 i a rel d'aquest el resultat a L^∞. Més precisament provem

Proposició. *Sigui $\varepsilon > \frac{1}{4}$. Sigui (ρ, S) una solució del problema parabòlic* (1.80) *amb condició inicial (ρ_0, S_0) satisfent $\rho_0, \nabla S_0 \in L^2(\mathbb{R}^d)$. Llavors existeix una constant $\lambda > 0$ que depèn de ε i una constant $C > 0$ que depèn de la dimensió d, de la massa inicial de ρ i de ε tal que*

$$\|\rho(t)\|_2 + \lambda \|\nabla S(t)\|_2 \leq C(t+1)^{-\frac{N}{4}}$$

i

Proposició. *Sigui $\varepsilon > \frac{1}{4}$ i $d = 1$. Sigui el parell (ρ, S) una solució de (1.80) amb condició inicial $(\rho_0, S_0) \in \mathcal{U}$ tal que $0 \leq \rho_0 \leq 1$. Llavors $\|\rho\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ i $\|S\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ quan $t \to +\infty$.*

Fixem-nos que demanem a la constant de difusivitat satisfer $\varepsilon > 1/4$ i ens restringim a dimensió 1 en el cas L^∞. Aquestes condicions són tècniques i les comentarem amb més detall a les Observacions 1.33 i 1.34.
Finalment, a la secció 1.2.3 estudiem el comportament asimptòtic autosemblant per mitjà d'un canvi d'escala depenent del temps i provant la convergència cap a solucions estacionàries en les noves variables. El resultat principal que presentem aquí és

Teorema. *Sigui $d = \varepsilon = 1$ i sigui (ρ, S) la solució de* (1.80) *amb condició inicial $(\rho_0, S_0) \in \mathcal{U}$. Sigui $\rho^\infty(t)$ la distribució gaussiana amb la mateixa massa que ρ_0. Siguin (v, σ) les solucions del problema de Cauchy definit pel canvi d'escala (1.104) i v^∞ la solució gaussiana autosemblant traslladada en temps de la equació de la calor. Llavors, per a $\delta > 0$ arbitràriament petit existeix una constant C que depèn de δ i la condició inicial tal que*

$$\|v(\theta) - v^\infty\|_1 \leq C\mathrm{e}^{-(1-\delta)\theta}$$

per a tot $\theta > 0$*, o equivalentment*

$$\|\rho(t) - \rho^{\infty}(t)\|_1 \leq C(t+1)^{-\frac{1-\delta}{2}}$$

per a tot $t > 0$.

Si recordem que l'equació de la calor produeix un rati de convergencia a solucions autosemblants en L^1 de la forma $t^{-1/2}$ en una dimensió, de forma que podem afirmar que el rati de convergència aquí és quasi òptim.

El capítol 2 està dedicat a una altra equació molt versàtil, l'equació d'agregació, que ens ajudarà a connectar els capítols 1 i 3. D'una banda aquesta equació ha estat utilitzada en models de distribució de volocitats de partícules inelàstiques en col·lisió i correspon a un cas particular del model de Keller-Segel que hem vist al capítol 1. D'altra banda l'estudi del comportament col·lectiu d'animals és un camp on proliferen models d'aquest tipus. L'equació d'agregació multidimensional

$$\begin{aligned} \frac{\partial u}{\partial t} + \text{div}\,(uv) &= 0, \\ v &= -\nabla K * u, \\ u(0) &= u_0, \end{aligned}$$

apareix en diversos models d'agregació biològica [24, 29, 30, 87, 116, 138, 136, 163, 164] així com les ciències dels materials [100, 99] i medis granulars [11, 43, 44, 122, 166]. La mateixa equació amb un terme de difusió adicional s'ha tingut en compte en [17, 20, 28, 72, 110, 119, 120, 121] encara que nosaltres no considerarem aquest cas aquí S'ha fet molta recerca sobre la qüestió de l'"explosió" en temps finit de la solució d'equacions d'aquest tipus, malgrat començar amb una condició inicial afitada o prou regular [22, 14, 12, 13]. Un estudi recent demostra que el problema per nuclis semi-convexos està ben posat, quan pensem en les solucions com a mesures. S'ha provat l'existència global (malgrat que no la unicitat) de solucions d'aquest tipus a [125, 75] en dues dimensions quan K és exactament el potencial newtonià. A més, simulacions numèriques d'agregacions que involucren el nucli $K(x) = |x|$ [15], mostren "explosions" en temps finit partint de condicions inicials acotades a les quals la singularitat inicial es manté a L^p, per algun p, enlloc de concentrar la massa a l'instant que comença l'"explosió". Aquests fets evidencien qüestions realment interessants sobre el comportament d'aquestes equacions en general quan considerem dades inicials en L^p, que poden ésser localment no afitades però no involucren concentració de massa. Aquest treball ens proporciona una teoria prou completa del problema en L^p, malgrat que encara queden algunes preguntes interessants sobre exponents crítics p_s per nuclis que viuen precisament a L^{p_s}. Entre aquests és particularment interessant el cas $p_s = d/(d-1)$ pel cas especial del nucli $K(x) = |x|$. El context L^p que adoptem en aquest capítol ens permet fer dos avenços significants en la comprensió de l'equació d'agregació. En primer lloc, aquest ens permet considerar potencials que són més singulars que els considerats fins ara (amb l'excepció de [125, 75], on consideren un potencial newtonià en dues dimensions). En treballs anteriors, s'ha demanat sovint al potencial K que, com a molt, tingués una singularitat de tipus Lipschitz a l'origen, i.e. $K(x) \sim |x|^\alpha$ amb $\alpha \geq 1$ (vegeu [116, 14, 13, 38]). En aquest context és possible considerar potencials amb singularitats a l'origen d'ordre $|x|^\alpha$ amb $\alpha > 2-d$. Aquests potencials poden tenir una cúspide (en 2D) o fins i tot "explotar" (en 3D) a l'origen. Resulta interessant que en dimensió $d \geq 3$, $|x|^{2-d}$ és exactament el potencial newtonià. Així doncs, podem plantejar el nostre resultat d'una altra manera tot dient que provem existència local i unicitat quan la singularitat del potencial és "millor" que la del potencial newtonià.

Teorema. *Considerem $1 < q < \infty$ i p el seu conjugat Hölder. Suposem que $\nabla K \in W^{1,q}(\mathbb{R}^d)$ i $u_0 \in L^p(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ és no negatiu. Llavors existeix un temps $T^* > 0$ i una única funció no negativa*

$$u \in C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$$

tal que

$$\begin{gathered} u'(t) + \operatorname{div}\big(u(t)\, v(t)\big) = 0 \qquad \forall t \in [0,T^*], \\ v(t) = -u(t) * \nabla K \qquad \forall t \in [0,T^*], \\ u(0) = u_0. \end{gathered}$$

També, el segon moment roman acotat i la norma L^1 es conserva. A més, si $\operatorname{ess\,sup} \Delta K < +\infty$, llavors el problema està ben posat globalment.

Per provar unicitat partim de les idees desenvolupades a [126] i emprem arguments de transport òptim per fitar la distància entre dues solucions transportades al llarg de les característiques del sistema. Aquest mètode pot ser utilitzat per provar unicitat de manera relativament simple per a una família força amplia de models i per tant estudiarem el seu ús en general abans de centrar-nos en cas particular de l'equació d'agregació que ens ocupa.
En el cas de potencials que es comporten com potències a l'origen, $K(x) \sim |x|^\alpha$, la situació queda il·lustrada com segueix:

Teorema (Existència i unicitat per potencials que es comporten com potències). *Suposem que ∇K té suport compacte (o decau exponencialment a l'infinit). Suposem, a més, que $K \in C^2(\mathbb{R}^d \backslash \{0\})$ i $K(x) \sim |x|^\alpha$ quan $|x| \to 0$.*

(i) Si $2 - d < \alpha < 2$ llavors l'equació d'agregació està ben plantejada localment a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ per a tot $p > p_s$ i, a més a més, no ho està globalment.

(ii) Si $\alpha \geq 2$ llavors l'equació d'agregació està ben plantejada globalment per a tot $p > 1$ en $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.

Això ens porta al segon resultat important provat en aquest capítol. Aquest té a veure amb el cas concret del potencial $K(x) = |x|$, biològicament molt rellevant. Per aquest potencial, es dóna un concepte de solució com a mesura a [38]. Identifiquem la regularitat crítica que necessitem a la condició inicial per tal de garantir que les solucions seran absolutament contínues respecte a la mesura de Lebesgue, al menys per un temps curt. Més concretament, provem que les solucions amb condició inicial a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ es mantenen a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, si més no, per temps petits si $p > d/(d-1)$. Aquí $\mathcal{P}_2(\mathbb{R}^d)$ denota l'espai de mesures de probabilitat amb segon moment acotat. D'altra banda per a qualsevol $p < d/(d-1)$ som capaços de trobar dades inicials a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ per les quals apareix una delta de Dirac instantàniament a la solució (la solució perd la seva continuïtat absoluta respecte a la mesura de Lebesgue instantàniament). L'últim resultat d'aquest capítol és un criteri per l'existència global de solucions a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.

Teorema (Condició d'Osgood). *Suposem que K és un potencial natural*

(i) Si K és repulsiu en rang curt, llavors l'equació d'agregacó està globalment ben plantejada a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$.

(ii) Si K és estrictament atractiu en rang curt, llavors l'equació d'agregació està globalement ben plantejada a $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$, si i només si

$$r \mapsto \frac{1}{k'(r)} \text{ no és integrable en 0.}$$

Un potencial es diu que és natural si és radialment simètric, regular lluny de l'origen, amb una singularitat com a molt de tipus Lipschitz a l'origen, no mostra oscil·lacions patològiques a l'origen i les seves derivades decauen prou ràpidament a l'infinit. Diem que un potencial natural és repulsiu (o atractiu) si té un màxim (respectivament un mínim) a l'origen. A [13] i [38] s'ha demostrat que si la condició d'Osgood no se satisfà, llavors solucions amb suport compacte col·lapsaran instantàniament en una massa puntual en temps finit. Els resultats que exposem aquí han estat publicats a [45, 16].

Per acabar, al capítol 3 ens dedicarem a estudiar el comportament col·lectiu d'animals a través de diferents perspectives. La descripció del moviment col·lectiu de grups formats per un gran nombre d'individus, i les estructures que en sorgeixen, és un fenomen sorprenent, com es pot veure als exemples proporcionats per ocells, peixos, abelles o formigues. Explicar l'aparició d'aquests moviments coordinats en termes de decisions microscòpiques de cada individu es un tema de recerca en auge a les ciències naturals. [36, 61, 147], robòtica i teoria de control [150], així com a la sociologia i a l'economia. La formació d'eixams o estructures precises de moviment ha estat observada en animlas amb una organització social molt desenvolupada, com és el cas d'insectes (llagostes, abelles, formigues, etc.) [61], peixos [8, 18] i ocells [36, 147] però també en microorganismes com la myxobacteria [111]. A més, el coneixement de la formació natural d'eixams ha estat emprat en el disseny d'operacions amb robots no dirigits per humans i ha jugat papers centrals en el desenvolupament de xarxes de sensors, amb àmplies aplicacions al control del mediambient [104, 55, 25, 150].

La literatura física de matemàtica aplicada ha proliferat en aquesta direcció en els últims anys intentant modelitzar aquest fenomen, basant-se principalment en dues estratègies de descripció: models basats en individus (IBMs per les sigles en anglès) o descripcions de la dinàmica de partícules [170, 147, 118, 36, 137, 89, 61, 73, 64, 63, 123, 128, 159], i models continus basats en EDPs per descriure la densitat o el moment del conjunt de partícules [147, 163, 164, 54, 76, 66]. Tot i que no entrarem en aquesta perspectiva, les mateixes idees s'han utilitzat a les ciències socials per a estudiar comportaments emergents a l'economia [51, 74], o la formació d'eleccions i opinions [84, 162]. La idea clau en tot plegat és que el comportament col·lectiu d'un grup format per un nombre prou gran d'individus (agents) pot ésser descrit per mitjà de les lleis de la mecànica estadística, com succeeix en dinàmica de gasos, per exemple. En particular, mètodes molt potents de teoria cinètica s'han utilitzat amb èxit per construir equacions cinètiques de tipus Boltzmann que descriuen la formació d'estructures universals a través dels seus equilibris, vegeu per exemple [131] i les referències que hi trobareu. El tret que cal explicar és l'aparició d'auto-organització.

En particular, el treball dut a terme recentment per Cucker i Smale [63], connectat amb el comportament observat a remats i bandades d'ocells, ha obtingut un reconeixement notable a la comunitat matemàtica. En analogia amb la física, l'estudi de models idealitzats pot sovint oferir-nos alguna informació sobre patrons i estructures observats al món real si aconsegueixen captar l'essència del problema. En biologia i física l'objectiu de simulacions de remats és el de ser capaços d'interpretar i predir aquest comportament. Una part rellevant dels treballs existent, no obstant, s'ha centrat en modelitzar i fer simulacions [163, 164, 170]. Anàlisis quantitatives [63, 104] dels ratis asimptòtics d'aparició d'aquests comportaments i de la convergència cap a ells, d'altra banda són relativament rares. Els esforços per part dels matemàtics en aquesta àrea multidisciplinària són cada cop més remarcables. Per exemple, en el límit continu s'han produït diferents treballs [163, 164, 38], en els quals patrons globals de formació d'eixams són modelitzats i analitzats a través d'equacions en derivades parcials adequades que involucren tant difusió com interacció entre els agents via potencials d' atracció/repulsió. Els diferents patrons per a models discrets han estat classificats a [73] per a potencials d'interacció típics i estudiats per mitjà d'equacions hidrodinàmiques [54] i cinètiques [39].

Les descripcions dels models de partícules normalment inclouen tres mecanismes bàsics en diferents regions: repulsió a distàncies curtes, atracció a distàncies llargues i una zona intermèdia d'orientació, que ens porta al conegut *model de tres zones*. A més, alguns d'ells incorporen un mecanisme per establir una velocitat asimptòtica fixada, com s'acostuma a observar a la natura. Alguns models només contemplen el vector d'orientació i no la velocitat en la seva versió discreta. La major diferència entre tots aquests models resideix en com aquestes tres interaccions es tenen en compte en cada cas particular.

De fet, com acostuma a passar a la física estadística, hi ha un terreny intermedi entre les descripcions del model per partícules i l'hidrodinàmic, donat per equacions cinètiques mesoscòpiques que descriuen la probablitat de trobar partícules a l'espai de fases. Diferents models cinètics de formació d'eixams s'han proposat recentment a [91, 39, 40] i la connexió entre els sistemes de partícules i els models continus a través de la teoria cinètica s'ha tractat recentment a [90, 39, 32]. L'interés dels models cinètics és el de donar una eina rigorosa per connectar els IBMs i les descripcions hidrodinàmiques així com interpretar alguns patrons com a solució d'un model donat [39]. L'anàlisi numèrica, descripcions numèriques del comportament complex de les solucions d'aquests models cinètics i hidrodinàmics de l'estabilitat dels patrons que segueixen són algunes de les preguntes obertes en aquesta direcció. Aquí treballarem amb dos exemples genèrics als quals s'inclouen diversos efectes dels mencionats abans. Concretament, ens centrarem en el model per partícules autopropulsades amb interaccions d'atracció/repulsió proposat per D'Orsogna et al. a [73] i el model d'alineament introduït per Cucker i Smale [64, 63]. La versió cinètica d'aquests models és la següent:

$$\partial_t f + v \cdot \nabla_x f - (\nabla U * \rho) \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta\,|v|^2) v f) = 0,$$

on U és un potencial d'interacció adequat i ρ representa la *densitat* macroscòpica de f:

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,dv \qquad \text{per } t \geq 0, x \in \mathbb{R}^d,$$

i

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot [\xi[f]\, f]$$

on $\xi[f](x,v,t) = (H * f)\,(x,v,t)$, amb $H(x,v) = w(x)v$ i $*$ representa la convolució en posició i velocitat (x and v).

Estudiarem el correcte plantejament d'aquests models a l'espai de mesures de probabilitat a l'espai de fases i els connectarem a les descripcions a través dels sistemes de partícules. Com comentarem en més detall més endavant, la idea fonamental és pensar en les solucions del sistema de partícules com a solucions en forma de mesures atòmiques de l'equació cinètica, la qual cosa ens permet abordar ambdós casos amb la mateixa perspectiva, la del transport òptim. Usarem les bones propietats de dualitat de les distàncies de Wasserstein per obtenir el resultat principal del capítol, una propietat d'estabilitat de solucions de les equacions de formació d'eixams. A partir d'aquesta estimació derivarem el límit de camp mitjà sense necessitat de recórrer a al jerarquia BBGKY i obtenir la dependència contínua de les solucions respecte a la condició inicial.

Teorema. *Sigui $U \in \mathcal{C}^1(\mathbb{R}^d)$ un potencial tal que ∇U és localment Lipschitz i tal que per alguna constant $C > 0$,*

$$|\nabla U(x)| \leq C(1 + |x|) \quad \textit{per a tot } x \in \mathbb{R}^d,$$

I $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ és de suport compacte. Siguin també mesures f_0, g_0 a $\mathbb{R}^d \times \mathbb{R}^d$ de suport compacte i consideremles solucions f, g de l'equació (3.4) *amb condició inicial f_0 i g_0, respectivament.*
Llavors, existeix una funció regular estrictament creixent $r(t) : [0, \infty) \longrightarrow \mathbb{R}_0^+$ amb $r(0) = 1$ depenent tant sols de la mida del suport de f_0 i g_0, tal que

$$W_1(f_t, g_t) \leq r(t)\, W_1(f_0, g_0), \quad t \geq 0.$$

Hem enunciat el resultat en el cas que considerem els efectes d'atracció i repulsió, però pot generalitzar-se fàcilment per incloure tots els models amb què treballarem. El mètode que emprem per estudiar aquests problemes presenta algunes limitacions. La primera és que, com que treballem amb solucions febles hem de demanar als termes d'interacció que siguin com a mínim Lipschitz. Aquesta és una limitació ben coneguda a la literatura per treballar amb el límit de camp mitjà i solucions que siguin mesures, vegeu [161, 92] i les referències contingudes en ells. Una altra limitació no tan fonamental és que treballarem sempre amb solucions de suport compacte. Segurament hom podria desenvolupar una teoria on se substituís aquesta condició per un control apropiat dels moments i adaptant les estimacions, però en aquest treball no ens proposem fer cap extensió en aquesta direcció. Per acabar, mostrarem com utilitzar aquest resultat per obtenir propietats qualitatives de les solucions de l'equació cinètica a partir de les solucions dels IBMs en el cas particular del model de Cucker-Smale, tot i que les constants a l'estimació d'"estabilitat" depenen del temps. Podem estendre els resultats de [91, 90] i proporcionar un teorema de "formació de ramats" incondicional amb la mateixa estimació de la força vàlida per al model per partícules amb afitacions independents del nombre de partícules, la qual cosa permet estendre'l a les solucions del model cinètic gràcies a la continuïtat respecte la condició inicial que ja hem mencionat.

Teorema. *Donada una mesura* $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ *de suport comacte amb velocitat mitjana zero i*

$$w(x) = \frac{1}{(1+|x|^2)^\gamma}$$

amb $\gamma \leq 1/2$*, llavors l'única solució de* (3.56) *a l'espai de mesures satisfà la següent fita en el seu support:*

$$\textit{supp}\ \mu(t) \subset B(x_c(0), R^x(t)) \times B(0, R^v(t))$$

per a tot $t \geq 0$*, amb*

$$R^x(t) \leq \bar{R} \qquad \textit{i} \qquad R^v(t) \leq R_0\, e^{-\lambda t}$$

amb $\bar{R}^x$ *depenent només del valor inicial de* $R_0 = \max\{R^x(0), R^v(0)\}$ *i* $\lambda = w(2\bar{R})$*.*

Tots aquests resultats s'han publicat a [32, 40, 37].

Introduction

It is sometimes said that mathematics is the language God used to write the world. People may agree or not, even with the very premise of this statement, but it is beyond any discussion that mathematics is the *best* language we know to describe the world. If I ever had any doubt about that, after these years it is gone for good, and in the following pages I will try to make you share this opinion through the study of several examples of modeling.

Some of them are old, having its roots in models and equations posed almost one century ago, some of them very recent with only some years of life, and all of them apply to a different field of interest: physics, biology, crowd behavior... but looking deep enough into them, once we have succeeded in translating the physical phenomena into mathematical words in the form of an equation or system of them (as if it was the easy part!), all of them share the same kind of questions. First one, and maybe the more important: what is a solution of the model for us? In which abstract mathematical space will it live? and what does it mean in terms of the original problem we want to study? After this is answered, the next question comes by itself: do such a solution exist? Under which assumptions? For how long? Are there more than one? I was talking about one question and asked four; they could have been more, and all of them are important and relevant. But it is also true that all of them are there only to tinge the first one. It is simply stated and requires an equally simply answer, yes, not or as it is in most of the cases, maybe. If that answer came to be not, no matter what, it would mean that an error found its way through the first stages. It could be that we need to think about the solution in a different way, that we shall find a different way to interpret nature or even that we have to question about the validity of the assumptions we have made about the world. On the other cases, with all the nuances we need, then there is only one question awaiting for us: how will this solution evolve? Because we want to be able to make predictions, and not just to describe some fact already gone.

We have gathered the models we study in three chapters, according to answers we give to the previous questions, or the way we look for it. Although it will made precise for each model, in all of them a solution for us will be some sort of measure, describing the probability of finding the object of our study in a given state. The answer to the second question will also be common for all of them, a definite maybe, if Oasis may excuse me. Finally, the answer to the last question will vary from one model to the other. Let us next comment briefly what have we done in each chapter.

In Chapter 1 we explore the scope of the Fokker-Plank equation through the study of two models which, albeit both of them are based on the same equation, apply to fields seemingly so far apart from each other as quantum mechanics and biology. The first one models the distribution in velocity of bosonic and fermionic particles whilst the second one gives a description of the chemotaxis phenomena. Nevertheless in both of them we are able to use entropy methods to study its asymptotic behavior. The results exposed here are collected in [46, 42, 68].

Kinetic equations for interacting particles have been introduced in the physics literature in [81, 107, 109, 108, 156]. Spatially inhomogeneous equations appear from formal derivations of generalized Boltzmann equations and Uehling-Uhlenbeck kinetic equations both for fermionic and bosonic particles.
A model for fermions in this situation has been recently studied in [142], where the long time asymptotics of these models in the torus is shown to be described by spatially homogeneous equilibrium given by Fermi-Dirac distributions when the initial data is not far from equilibrium in a suitable Sobolev space. This result is based on techniques developed in previous works [139, 141]. Other related mathematical results for Boltzmann-type models have appeared in [71, 127].
When we can assume that the particles have reached some equilibrium in their distribution in space we can look at spatially homogeneous equations. In [106] a Fokker-Planck model for bosons and fermions is derived from Kramer's equation in this case. This is what we study in the first section of the chapter:

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \mathrm{div}(v\rho(1+\kappa\rho))$$

where $\kappa = -1$ is used for the fermions, the negative sign meaning the aversion of this kind of particles to share the same phase, and $\kappa = 1$ stands for the bosons. We will work in the space $\Upsilon = \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{L}^1_1(\mathbb{R}^d) \cap \mathrm{L}^p_m(\mathbb{R}^d)$ and $\Upsilon_T = \mathcal{C}([0,T];\Upsilon)$ with norms

$$\|f(t)\|_\Upsilon = \max\{\|f(t)\|_\infty, \|f(t)\|_{\mathrm{L}^1_1}, \|f(t)\|_{\mathrm{L}^p_m}\} \quad \text{and} \quad \|f\|_{\Upsilon_T} = \max_{0\le t\le T} \|f\|_\Upsilon$$

for any $T > 0$. The first thing we do is proving local existence of weak solutions, both for fermions and bosons. Then we compute a priori estimates that allow us to show the following global existence theorem for fermions:

Theorem (Global Existence). *Let* $f_0 \in \mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > d$, $p \geq 2$, $m \geq 1$ *such that* $0 \leq f_0 \leq 1$. *Then the Cauchy problem* (1.1) *with initial data* f_0 *has a unique solution defined in* $[0, \infty)$ *belonging to* Υ_T *for all* $T > 0$. *Also, we have* $0 \leq f(t, v) \leq 1$, *for all* $t \geq 0$ *and* $v \in \mathbb{R}^N$, *and* $\|f(t)\|_1 = \|f_0\|_1 = M$ *for all* $t \geq 0$.

Once the existence of solution for any time has been established we can look at its asymptotic behavior. Fermi-Dirac and Bose-Einstein statistics,

$$F_\beta(v) = \frac{1}{\beta \mathrm{e}^{\frac{|v|^2}{2}} - \kappa}$$

with $\beta > 0$ are stationary solutions. For each value of the initial mass M there exists a unique value of β such that F_β has mass M. In the case of bosons we have that $\beta \geq 1$. If $d \geq 3$ the stationary solution converges as $\beta \to 1^+$ to an integrable singular solution, and thus, we have the well-known critical mass for the Bose-Einstein equilibrium distributions.
Using the fact that

$$H(g) := \int_{\mathbb{R}^d} s(g(v))\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 g(v)\,\mathrm{d}v,$$

where

$$s(r) := (1-r)\log(1-r) + r\log(r) \leq 0, \quad r \in [0,1],$$

is a Lyapunov functional, we can use an entropy inequality based upon previous work in [41] to show exponential convergence to the steady state of the solutions. More precisely we show

Theorem (Entropy Decay Rate). *Assume that* f *is a solution to the Cauchy problem* (1.1) *with initial data in* $\mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > \max(d, 2)$, $m \geq 1$ *such that* $0 \leq f_0 \leq F_{M^*} \leq 1$ *with* F_{M^*} *a Fermi-Dirac distribution of mass* M^*. *Then the global in time solution of the Cauchy problem* (1.1) *with initial data* f_0 *satisfies*

$$H(f) - H(F) \leq (H(f_0) - H(F))\mathrm{e}^{-2Ct}$$

and

$$\|f(t) - F\|_{L^1(\mathbb{R}^d)} \leq C_2(H(f_0) - H(F))^{1/2}\mathrm{e}^{-Ct}$$

for all $t \geq 0$, *where* C *depends on* M^* *and* F *being the Fermi-Dirac Distribution such that* $\|F\|_1 = \|f_0\|_1$.

Nevertheless, asking the initial data to be bounded by a Fermi-Dirac statistic is quite restrictive. Thus we present an intermediate result where convergence to the steady state is obtained from more general hypothesis, albeit in this case we cannot say anything about the rate of convergence.

Theorem (Entropy Convergence). *Let f be a solution for the Cauchy problem* (1.1) *with initial data $f_0 \in \mathrm{L}^1_{\mathrm{mp}}(\mathbb{R}^{\mathrm{d}})$ such that there exists a radially symmetric function g_0 with bounded 2-moment and non-increasing, with $0 \leq f_0 \leq g_0 \leq 1$. Then $H(f) \to H(F_M)$ as $t \to \infty$, where F_M is the Fermi-Dirac distribution with the same mass as f_0.*

For the bosons we can also show convergence with exponential rate in one dimension, and so we do afterwards.

In the second section we turn our attention to biology, studying a model for chemotaxis. Chemotaxis is the phenomenon by which cells move under the influence of chemical substances in their environment. We consider the following fully parabolic version of the Keller-Segel model:

$$\begin{cases} \rho_t = \varepsilon \Delta \rho - \mathrm{div}(\rho(1-\rho)\nabla S) \\ S_t = \Delta S - S + \rho \end{cases}$$

where ρ models the density of cells and S the concentration of the chemical substance, known as chemoattractant. The term $(1-\rho)$ plays the role of the chemotactic sensitivity term χ, which usually determines the result of the competition between diffusion and convection. In this case it will prevent blow-up, since cells stop aggregating when a maximum concentration (that we normalize to 1 for simplicity) has been reached. We can look at [146, 97] for biological details of this assumption.
A typical form of the Keller-Segel model is the corresponding to the parabolic-elliptic case, with the second equation replaced by $0 = \Delta S + \rho - S$ or by Poisson's equation $-\Delta S = \rho$. In the case $\chi(\rho, S) \equiv$ constant, the above system has been extensively studied. In particular, it is now common knowledge (cf. [72]) that the 2 dimensional Keller-Segel system

$$\begin{cases} \rho_t = \Delta \rho - \mathrm{div}\,(\rho \chi \nabla S) \\ 0 = \Delta S + \rho \end{cases}$$

features a χ–dependent critical threshold m^* for the total mass of ρ determining finite time blow-up or global existence (blow-up for initial mass larger than m^*, global existence otherwise). Related results are contained in [60, 151, 20, 59] and in [35] for the parabolic case.
Here we perform a similar analysis to the one done for the Fermi-Dirac model to prove global existence and uniqueness of solution in the functional space

$$\mathcal{U} := (\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)) \times (\mathrm{W}^{1,1}(\mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d)).$$

Next we look at the asymptotic behavior of this solution. In a first step we show that the concentration of cells and chemical (ρ and S respectively) decays to zero at polynomial rate as time goes to infinity. This is done by proving that the decay occurs in L^2 and from this we get too the result in L^∞. More precisely we show

Proposition. *Let $\varepsilon > \frac{1}{4}$. Let (ρ, S) be a solution of the parabolic problem* (1.80) *with initial datum (ρ_0, S_0) satisfying $\rho_0, \nabla S_0 \in L^2(\mathbb{R}^d)$, then there exist a constant $\lambda > 0$ depending on ε and a constant $C > 0$ depending on the dimension d, on the initial mass of ρ and on ε such that*

$$\|\rho(t)\|_2 + \lambda\|\nabla S(t)\|_2 \leq C(t+1)^{-\frac{N}{4}}$$

and

Proposition. *Let $\varepsilon > \frac{1}{4}$ and $d = 1$. Let the pair (ρ, S) be solution of (1.80) with initial datum $(\rho_0, S_0) \in \mathcal{U}$ such that $0 \leq \rho_0 \leq 1$. Then $\|\rho\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ and $\|S\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ as $t \to +\infty$.*

Notice that we ask the diffusivity constant to satisfy $\varepsilon > 1/4$ and we restrict ourselves to dimension 1 in the L^∞ case. These are technical conditions and we shall comment them in more detail in Remarks 1.33 and 1.34.
Finally, in section 1.2.3 we study the asymptotic self–similar behavior of the solutions by time dependent scaling and by proving convergence to the steady state in the new variables. The main result we present here is

Theorem. *Let $d = \varepsilon = 1$ and let (ρ, S) be the solution to* (1.80) *with initial condition (ρ_0, S_0) satisfying the assumptions of theorem 1.32 and let $\rho^\infty(t)$ be defined by (1.102). Let (v, σ) be defined by (1.104) and v^∞ be the time translated self-similar gaussian solution of the Heat equation. Then, for any arbitrarily small $\delta > 0$ there exists a constant C depending on δ and on the initial data such that*

$$\|v(\theta) - v^\infty\|_1 \leq C\mathrm{e}^{-(1-\delta)\theta}$$

for all $\theta > 0$, or equivalently

$$\|\rho(t) - \rho^\infty(t)\|_1 \leq C(t+1)^{-\frac{1-\delta}{2}}$$

for all $t > 0$.

If we recall that the Heat equation produces a rate of convergence to self similarity in L^1 of the form $t^{-1/2}$ in 1 space dimension, we can state that the rate of convergence here is “quasi sharp”.

Chapter 2 is devoted to another versatile equation, the aggregation equation which will help us to bridge the gap between Chapters 1 and 3. The multidimensional aggregation equation

$$\frac{\partial u}{\partial t} + \text{div}\,(uv) = 0,$$
$$v = -\nabla K * u,$$
$$u(0) = u_0,$$

arises in a number of models for biological aggregation [24, 29, 30, 87, 116, 138, 136, 163, 164] as well as problems in materials science [100, 99] and granular media [11, 43, 44, 122, 166]. The same equation with additional diffusion has been considered in [17, 20, 28, 72, 110, 119, 120, 121] although we do not consider that case in this chapter. For the inviscid case, much work has been done recently on the question of finite time blow-up in equations of this type, from bounded or smooth initial data [22, 14, 12, 13]. A recent study proves well-posedness of measure solutions for semi-convex kernels. Global existence (but not uniqueness) of measure solutions has been proven in [125, 75] in two space dimension when K is exactly the Newtonian Potential. Moreover, numerical simulations [15], of aggregations involving $K(x) = |x|$, exhibit finite time blow-up from bounded data in which the initial singularity remains in L^p for some p rather than forming a mass concentration at the initial blow-up time. These facts together bring up the very interesting question of how these equations behave in general when we consider initial data in L^p, that may be locally unbounded but does not involve mass concentration. This work serves to provide a fairly complete theory of the problem in L^p, although some interesting questions remain regarding critical exponents p_s for general kernels and for data that lives precisely in L^{p_s}, with $p_s = d/(d-1)$ for the special kernel $K(x) = |x|$. The L^p framework adopted in this chapter allows us to make two significant advances in the understanding of the aggregation equation. First, it allows us to consider potentials which are more singular than the one which have been considered up to now (with the exception of [125, 75], where they consider the Newtonian potential in 2D). In previous works, the potential K was often required to be at worst Lipschitz singular at the origin, i.e. $K(x) \sim |x|^\alpha$ with $\alpha \geq 1$ (see [116, 14, 13, 38]). In our L^p framework it is possible to consider potentials whose singularity at the origin is of order $|x|^\alpha$ with $\alpha > 2 - d$. Such potentials might have a cusp (in 2D) or even blow up (in 3D) at the origin. Interestingly, in dimension $d \geq 3$, $|x|^{2-d}$ is exactly the Newtonian potential. So we can rephrase our result by saying that we prove local existence and uniqueness when the singularity of the potential is “better” than that of the Newtonian potential.

Theorem (well-posedness). *Consider* $1 < q < \infty$ *and* p *its Hölder conjugate. Suppose* $\nabla K \in W^{1,q}(\mathbb{R}^d)$ *and* $u_0 \in L^p(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ *is nonnegative. Then there* *exists a time $T^* > 0$ and a unique nonnegative function*

$$u \in C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$$

such that

$$\begin{gathered} u'(t) + \operatorname{div}\big(\, u(t)\, v(t)\,\big) = 0 \qquad \forall t \in [0,T^*], \\ v(t) = -u(t) * \nabla K \qquad \forall t \in [0,T^*], \\ u(0) = u_0. \end{gathered}$$

Moreover the second moment stays bounded and the L^1 norm is conserved. Furthermore, if $\operatorname{ess\,sup} \Delta K < +\infty$, then we have global well-posedness.

To prove the uniqueness we start from the ideas in [126] and use optimal transport arguments to bound the distance between solutions transported through the characteristics of the system. This method can be used to prove uniqueness in a rather simple way for a quite big family of models, and thus we comment it in general before resorting to the particular case of the aggregation equation.
In the case of power like potentials, $K(x) \sim |x|^\alpha$ at the origin, we obtain the following picture

Theorem (Existence and uniqueness for power potential). *Suppose that ∇K is compactly supported (or decays exponentially fast at infinity). Suppose also that $K \in C^2(\mathbb{R}^d \backslash \{0\})$ and $K(x) \sim |x|^\alpha$ as $|x| \to 0$.*

- *(i) If $2 - d < \alpha < 2$ then the aggregation equation is locally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for every $p > p_s$. Moreover, it is not globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.*
- *(ii) If $\alpha \geq 2$ then the aggregation equation is globally well posed for every $p > 1$ in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.*

This leads to the second important result proven in this chapter. It concerns the specific and biologically relevant potential $K(x) = |x|$. For such a potential, a concept of measure solution is provided in [38]. We identify the critical regularity needed on the initial data in order to guarantee that the solution will stay absolutely continuous with respect to the Lebesgue measure at least for short time. To be more specific, we prove that solutions whose initial data are in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ remain in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ at least for short time if $p > d/(d-1)$. Here $\mathcal{P}_2(\mathbb{R}^d)$ denotes probability measure with bounded second moment. On the other hand for any $p < d/(d-1)$ we are able to exhibit initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for which a delta Dirac appears instantaneously in the solution – the solution loses its absolute continuity with respect to the Lebesgue measure instantaneously.
The last result of this chapter is a criteria for global well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$

Theorem (Osgood condition for global well posedness). *Suppose that K is a natural potential.*

(i) If K is repulsive in the short range, then the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$.

(ii) If K is strictly attractive in the short range, the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$, if and only if

$$r \mapsto \frac{1}{k'(r)} \text{ is not integrable at 0.}$$

A potential is said to be natural if it is radially symmetric potential, smooth away from the origin, with at most Lipschitz singularity at the origin, doesn't exhibit pathological oscillation at the origin, and its derivatives decay fast enough at infinity. We say that it is repulsive (or attractive) if it has a maximum (respectively a minimum) at the origin. In [13] and [38] it was shown that if (2.10) is not satisfied, then compactly supported solutions will collapse into a point mass – and therefore leave L^p – in finite time. The results exposed here have been published in [45, 16].

Finally, in chapter 4 we are concerned with the study of collective animal behavior through different new perspectives. The description of the collective motion (swarming) of multi-agent aggregates resulting into large-scale structures is a striking phenomena, as illustrated by the examples provided by birds, fish, bees or ants. Explaining the emergence of these coordinated movements in terms of microscopic decisions of each individual member of a swarm is a hot matter of research in the natural sciences [36, 61, 147], robotics and control theory [150], as well as sociology and economics. The formation of swarms and milling or flocking patterns have been reported in animals with highly developed social organization like insects (locusts, bees, ants, ...) [61], fishes [8, 18] and birds [36, 147] but also in micro-organisms as myxo-bacteria [111]. Moreover, the understanding of natural swarms has been used as an engineering design principle for unmanned artificial robots operation and have been playing central roles in sensor networking, with broad applications in environmental control [104, 55, 25, 150].
The physics and applied mathematics literature has proliferated and sprung in this direction in the recent years trying to model these phenomena, mainly based on two strategies of description: individual-based models or particle dynamics [170, 147, 118, 36, 137, 89, 61, 73, 64, 63, 123, 128, 159] and continuum models based on PDEs for the density or for the momentum of the particle ensemble [147, 163, 164, 54, 76, 66]. Although we do not explore this approach here, the same ideas have been used applied to social sciences to study emergent economic

behaviors [51, 74], or the formation of choices and opinions [84, 162]. The key idea is that the collective behavior of a group composed by a sufficiently large number of individuals (agents) could be described using the laws of statistical mechanics as it happens in gas dynamics, for instance. In particular, powerful methods borrowed from kinetic theory have been fruitfully employed to construct kinetic equations of Boltzmann type which describe the emergency of universal structures through their equilibria, see [131] and the references therein. The key feature to explain is the emergence of self-organization: flocking, milling, double milling patterns or other coherent behavior.

In particular, the recent mathematical work of Cucker and Smale [63], connected with the emergent behaviors of flocks, obtained a noticeable resonance in the mathematical community. In analogy with physics, the study of idealized models can often shed light on various observed patterns in the real world, if such models can indeed catch the very essence. In biology and physics, the main goal of flocking simulation is to be able to interpret and predict different flocking or multi-agent aggregating behavior. A relevant part of the existing works, however, have been focusing on modeling and simulation [163, 164, 170]. Quantitative analysis [63, 104] on the asymptotic rates of emergence and convergence, on the other hand, are relatively rare. Mathematical efforts are gradually gaining strength in this multidisciplinary area. In the continuum limit, for example, there have been several recent efforts [163, 164, 38], in which global swarming (i.e., with densely populated agents) patterns are modeled and analyzed via suitable partial differential equations involving both diffusion and interaction via pairwise attraction/repulsion potentials. Patterns for discrete models have been classified in [73] for typical interaction potentials and studied by hydrodynamic [54] and kinetic equations [39].

Particle descriptions usually include three basic mechanisms in different regions: short-range repulsion zone, long-range attraction zone and alignment or orientation zone, leading to the so-called *three-zone models*. In addition, some of them incorporate a mechanism for establishing a fixed asymptotic speed/velocity vector of agents, as is usually observed in nature. Some of the models only consider the orientation vector and not the speed in their discrete version. The main differences of all these models reside in how these three interactions are specifically considered.

In fact, as usually done in statistical physics, there is a middle ground in modeling between particle and hydrodynamic descriptions given by the mesoscopic kinetic equations describing the probability of finding particles in phase space. Kinetic models of swarming has recently been proposed [91, 39, 40] and the connection between the particle systems, or Individual Based Models (IBMs) and the continuum models via kinetic theory has been tackled very recently in [90, 39, 32]. The interest of the kinetic theory models is to give a rigorous tool to connect IBMs and hydrodynamic descriptions as well as to interpret certain patterns as solutions of a given model, as in the case of the double mills [39]. The analysis, numerical description of the complex behavior of these kinetic and hydrodynamic models and patterns stability are some of the open questions in this research direction. We will mainly work with two generic examples in which several of the effects above are included, namely the model for self-propelled interacting particles introduced by D'Orsogna et al in [73] and the model of alignment proposed by Cucker and Smale [64, 63]. In their kinetic form they read

$$\partial_t f + v \cdot \nabla_x f - (\nabla U * \rho) \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta |v|^2) v f) = 0,$$

where U is a suitable interaction potential and ρ represents the macroscopic *density* of f:

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\, dv \quad \text{for } t \geq 0, x \in \mathbb{R}^d,$$

and

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot [\xi[f]\, f]$$

where $\xi[f](x,v,t) = (H * f)\,(x,v,t)$, with $H(x,v) = w(x)v$ and $*$ standing for the convolution in both position and velocity (x and v).
We will study the well-posedness of these models in the set of probability measures in phase space and connect them to the description by particles. As we will comment with more detail later, the key idea is to think about solutions to the particle system as atomic-measure solutions for the kinetic equation, which enables us to approach both cases from the perspective of optimal transport. We will use the good duality properties of Wasserstein distances to obtain the main result of the chapter, a stability property of solutions to swarming equations. From this estimate we can derive the mean-field limit without using the BBGKY hierarchy and we obtain a continuous dependence with respect to initial data.

Theorem. *Take a potential $U \in \mathcal{C}^1(\mathbb{R}^d)$ such that ∇U is locally Lipschitz and such that for some $C > 0$,*

$$|\nabla U(x)| \leq C(1 + |x|) \quad \textit{for all } x \in \mathbb{R}^d,$$

and $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ with compact support. Take also f_0, g_0 measures on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support, and consider the solutions f, g to eq. (3.4) *with initial data f_0 and g_0, respectively.*
Then, there exists a strictly increasing smooth function $r(t) : [0, \infty) \longrightarrow \mathbb{R}_0^+$ with $r(0) = 1$ depending only on the size of the support of f_0 and g_0, such that

$$W_1(f_t, g_t) \leq r(t)\, W_1(f_0, g_0), \quad t \geq 0.$$

We have stated here the result in the attraction/repulsion case, but can be easily generalized to include all the models we will deal with.

Let us comment on some limitation of the method we use. The first is that, as we work with solutions in a weak measure sense, we have to require our interaction terms to be locally Lipschitz in order to carry out the theory. This is a well-known limitation in the literature for working with the mean field limit and measure solutions see [161, 92] and the references therein. A less fundamental one is that we always work with compactly supported solutions. One could probably develop a theory substituting this condition by a suitable control on moments of the solution, and then adapting the estimates to this setting: however, int the present work we do not pursue further extensions in this direction.

To conclude, we show how to use this result to get qualitative properties of the solutions to the kinetic equation from the solutions of the IBMs in the particular case of the Cucker-Smale model, even though the constants on the ”stability” estimate are time dependent. We can extend the results in [91, 90] and provide an unconditional flocking theorem with the same strength estimate valid for the finite particle models with estimates independent of the number of particles, which allows us to extend it to the solutions to the kinetic equation due to the continuous dependence with respect to initial data already shown.

Theorem. *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported with zero mean velocity and*

$$w(x) = \frac{1}{(1+|x|^2)^\gamma}$$

with $\gamma \leq 1/2$, then the unique measure-valued solution to (3.56) *satisfies the following bounds on their supports:*

$$\textit{supp}\ \mu(t) \subset B(x_c(0), R^x(t)) \times B(0, R^v(t))$$

for all $t \geq 0$, with

$$R^x(t) \leq \bar{R} \qquad \textit{and} \qquad R^v(t) \leq R_0\, e^{-\lambda t}$$

with $\bar{R}^x$ depending only on the initial value of $R_0 = \max\{R^x(0), R^v(0)\}$ and $\lambda = w(2\bar{R})$.

All these results have been published in [32, 40, 37].

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It has been already some years since I made the first step of the path that will lead to the work you have now in your hands, and now that it is time to write these lines, I regret that I haven't started earlier. There is so many people whose name should be here, for so many things, small and big, that I cannot avoid be afraid of letting someone out. Thus, to be sure that won't happen, let me start by thanking all the people I won't explicitly or implicitly mention.

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Coming back home, my thanks to the people of the PDEs group in UAB and UPC; discussions, both during and after the seminars, helped opening perspectives . I

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Chapter 1

The Fokker-Planck equation

The contents of this chapter appear in:

- *Carrillo, J. A.; Rosado, J.; Salvarani, F. "1D nonlinear Fokker-Planck equations for fermions and bosons". Appl. Math Lett.* **21** *(2008), no. 2, 148-154. [46]*

- *Di Francesco, M.; Rosado, J. "Fully parabolic Keller-Segel model with prevention of overcrowding". Nonlinearity* **21** *(2008), 2715-2730. [68]*

- *Carrillo, J. A.; Laurençot, P.; Rosado, J. "Fermi-Dirac-Fokker-Planck equation: well-posedness and long-time asymptotics". J. Diff. Eq.* **247** *(2009), 2209-2234. [42].*

Sometimes, when we try to model nature we find ourselves dealing with systems composed by an outrageous number of individuals (particles, many times), each of them providing its small contribution to the whole system. In most of this situations though, we are not interested in the particular behavior of all the particles, but rather in the synergetics of the system.
In those cases, when we want to study the macroscopic qualitative changes that arise as a result of the interaction between the individuals which compose the system, or even between whole subsystems, Fokker-Planck equations have shown themselves to provide a good description of the situation based for instance on Langevin equations. An important feature of this model is that it can be applied to system which are far from thermal equilibrium, therefore allowing us to describe not only the stationary properties, but also the dynamics of systems.
It was first used in [80, 152] to describe Brownian motion of particles. In its (simpler) linear form, for $x \in \mathbb{R}^N$ the Fokker-Planck equation reads

$$\frac{df}{dt}(t,x) = \Delta\left[d_1(x) f\right] - \nabla \cdot \left[d_2(x) f\right]$$

where $d_1(x)$ and $d_2(x)$ are respectively the diffusion and the drift coefficient. In the mathematical literature, this presentation of the Fokker-Planck equation is also known as the *forward Kolmogorov equation*. There also other versions of this equation, for particular choices of the drift and diffusion coefficients, which have earned a name by themselves in the literature, as is the case of the *Kramer equation* or the *Smoluchowski equation*.

In this chapter we shall study two models which, despite being used in two completely different frameworks, have in common its basis in the Fokker-Planck equation. In the first section we will deal with the Fermi-Dirac and Bose-Einstein model for quantum particles, fermions and bosons respectively. In the second one, we move from physics to biology and focus our attention in the Keller-Segel model for chemotaxis. In both cases we will be interested in the existence and uniqueness of solution to these models and the asymptotic behavior of these solutions.

These two models are nonlinear versions of the Fokker-Planck equation. For a more comprehensive study of this equation, both its linear and nonlinear versions, and its applications we refer to [82, 155] and the reference therein.

1.1 A Model for bosons and fermions

Kinetic equations for interacting particles, such as bosons or fermions, have been introduced in the physics literature in [81, 106, 107, 109, 108, 156] and the review [82]. Spatially inhomogeneous equations appear from formal derivations of generalized Boltzmann equations and Uehling-Uhlenbeck kinetic equations both for fermionic and bosonic particles. The most relevant questions related to these problems concern their long-time asymptotics and the rate of convergence towards global equilibrium if any.

The spatially inhomogeneous situation in the fermion case has been recently studied in [142]. There, the long time asympotics of these models in the torus is shown to be described by spatially homogeneous Fermi-Dirac distributions when the initial data is not far from equilibrium in a suitable Sobolev space. This nice result is based on techniques developed in previous works [139, 141]. Other related mathematical results for Boltzmann-type models have appeared in [71, 127].

In this section, we focus on the global existence of solutions and the convergence of solutions towards global equilibrium for fermions in the spatially homogeneous case without any smallness assumption on the initial data. The convergence to steady states for the bosons is also shown in the one-dimensional setting, as it was reported in [46]. More precisely, we analyze in detail the following Fokker-Planck equation for fermions, see for instance [82],

$$\frac{\partial f}{\partial t} = \Delta_v f + \mathrm{div}_v[vf(1-f)], \qquad v \in \mathbb{R}^d, t > 0, \tag{1.1}$$

with initial condition $f(0,v) = f_0(v) \in L^1(\mathbb{R}^d)$, $0 \leq f_0 \leq 1$ satisfying suitable moment conditions to be specified below. Here, $f = f(t,v)$ is the density of particles with velocity v at time $t \geq 0$.

This equation has been proposed in order to describe the dynamics of classical interacting particles, obeying the exclusion-inclusion principle in [106]. In the bosons case, where this principle does not apply, we shall consider

$$\frac{\partial f}{\partial t} = \Delta_v f + \mathrm{div}_v[vf(1+f)], \qquad v \in \mathbb{R}^d, t > 0, \tag{1.2}$$

instead. In fact, equations (1.1) and (1.2) are formally equivalent to

$$\frac{\partial f}{\partial t} = \mathrm{div}_v \left[f(1-f) \nabla_v \left(\log\left(\frac{f}{1+\kappa f}\right) + \frac{|v|^2}{2} \right) \right]$$

with $\kappa = 1$ for the bosons and $\kappa = -1$ for the fermions, from which it is easily seen that Bose-Einstein and Fermi-Dirac distributions defined by

$$F_\beta(v) = \frac{1}{\beta \mathrm{e}^{\frac{|v|^2}{2}} - \kappa}$$

with $\beta \geq 0$ are stationary solutions. Moreover, for each value of the initial mass or $M := ||f_0||_1$, there exists a unique $\beta = \beta(M) \geq 0$ such that $F_{\beta(M)}$ has mass M. In the case of bosons $\beta(M) \geq 1$ and, if the dimension is bigger than 2 the stationary solution converges as $\beta \to 1^+$ to an integrable singular solution, and thus, we have the well-known critical mass for the Bose-Einstein equilibrium distributions.

Another striking property of these equations is the existence of a formal Lyapunov functional, related to the standard entropy functional for linear and non-linear Fokker-Planck models [47, 41], given by

$$H(f) = \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 f(v)\,\mathrm{d}v + \int_{\mathbb{R}^d} [f\log(f) - \kappa(1+\kappa f)\log(1+\kappa f)]\,\mathrm{d}v.$$

We will show that this functional plays the same role as the H-functional for the spatially homogeneous Boltzmann equation and will be crucial to characterize long-time asymptotics of (1.1), see for instance [165]. In fact, the entropy method will be the basis of the main results in this work; more precisely by taking the formal time derivative of $H(f)$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) = -\int_{\mathbb{R}^d} f(1-f)\left| v + \nabla_v \log\left(\frac{f}{1-f}\right)\right|^2 \mathrm{d}v \leq 0.$$

Therefore, to show the global equilibration of solutions to (1.1) we need to find the right functional setting to show the entropy dissipation. Furthermore, if we succeed in relating functionally the entropy and the entropy dissipation, we will be able to give decay rates towards equilibrium. These are the main objectives of this work. Let us finally mention that these equations are of interest as typical examples of gradient flows with respect to euclidean Wasserstein distance of entropy functionals with non-linear mobility, see [31, 44] for other examples and related problems.

Now, before going any further into the mathematical analysis of this model, let us dedicate a few paragraphs to present the meaning of the type of particles we study here, bosons and fermions, together with the distributions of their steady states.

Then, in Section 1.1.1 we will show the global existence of solutions for equation (1.1) based on fixed point arguments, estimates involving moment bounds and the conservation of certain properties of the solutions. The right functional setting is reminiscent to the one used in equations sharing a similar structure and technical difficulties as those treated in [78, 85]. The main technical obstacle for the Fermi-Dirac-Fokker-Planck equation (1.1) lies in the control of moments. Next, in section 1.1.2 we show that the entropy is decreasing for the constructed solutions, and from that, we prove the convergence towards global equilibrium without rate. Again, here the uniform-in-time control of the second moment is crucial. To conclude this section, we obtain an exponential rate of convergence towards equilibrium if the initial data are controlled by Fermi-Dirac distributions and relative entropy convergence when controlled by radial solutions, as in [42]. Finally, in section 1.1.3 we study the asymptotic behaviour of bosons in dimension 1 and in section 1.1.5 we present Schafferter-Gummel scheme for our model where its mass preserving property holds, and show some examples where the decay rate theoretically predicted can be seen.

Physical Background of the Model

As we said at the beginning of this chapter, when studying systems composed of a huge number of quantum particles, for which we do not know their microscopic state exactly, statistical mechanics is needed; we try to describe them by their macroscopic properties. A particular macroscopic state corresponds to a whole set of microscopic states, and therefore, its statistical weight is proportional to the number of distinct microscopic states which correspond to it. The system is in its most probable macroscopic state at the thermodynamic equilibrium.

In classical statistical mechanics the N particles of a system are treated as if they were of different nature. The case of having identical particles in the system is studied as the general one; each particle is supposed to move along a well defined trajectory, which enables us to distinguish it from the others.

It is clear that in quantum mechanics the situation is different. Since particles have no definite trajectories, we cannot follow them throughout the evolution of the system. Hence when we detect one particle in a region of space in which the probability of being there is non-zero for both of them, there is no way of knowing which of the particles is the one we have detected. We have to take into account the symmetrization postulate, which states that when a system includes several particles, physical states are, depending on the nature of the identical particles, either completely symmetric or completely antisymmetric with respect to permutation of these particles. There exists an empirical rule, by which all currently known particles are divided in two groups: elemental particles are either bosons or fermions according to their spin (bosons have integer spin and fermions half-integer spin), and composite particles are also classified by the sum of the spin of the elemental particles which compose them. Due to the spin-statistics theorem by Pauli (1940) we can consider this rule to be a consequence of a more general theory, although recently, in 2003 O'Hara derived this classification in bosons and fermions from what he call coupling principle, according to which certain class of rotationally invariant states can only occur in pairs, relating his approach to the one by Pauli (see [149, 145]).

The symmetrization postulate thus limits the state space for a system of identical particles. This is no longer, as it was in the case of particles of different nature, the tensor product of the individual state spaces of the particles constituting the system, but a subspace of it which depends on whether the particles are bosons or fermions.

This difference between bosons and fermions may seem not relevant, but actually due to the sign difference in the symmetry of physical states for a system of identical bosons, with positive sign, the access to individual states is not restricted whereas the negative sign of fermions oblige them to obey Pauli's exclusion principle: two identical fermions cannot occupy the same quantum mechanical state (see [58]). Hence, different statistical properties result: bosons obey Bose-Einstein statistics (introduced in the early 1920s by Satyendra Nath Bose) and fermions, Fermi-Dirac statistics (developed by Enrico Fermi and Paul Dirac in 1926) (see [49]). This is the origin of the terms *bosons* and *fermions*.

1.1.1 Global Existence of Solutions

In this section, we will show the global existence of solutions for the Cauchy problem to (1.1). We start by proving local existence of solutions together with a characterization of the live-span of these solutions. Later, we show further regularity properties of these solutions with the help of estimates on derivatives. Based on these estimates we can derive further properties of the solutions: conservation of mass, positivity, L^∞ bounds, comparison principle, moment estimates and entropy estimates. All of these uniform estimates allow us to show that solutions can be extended.

Local Existence

We will prove the local existence and uniqueness of solution using contraction-principle arguments as in [31, 78] for instance. As a first step, let us note that we can write (1.1) as

$$\frac{\partial f}{\partial t} = \mathrm{div}_v(vf + \nabla_v(vf)) - \mathrm{div}_v(vf^2) \tag{1.3}$$

and, due to Duhamel's formula, we are led to consider the corresponding integral equation

$$f(t,v) = \int_{\mathbb{R}^d} \mathcal{F}(t,v,w) f_0(w) \mathrm{d}w - \int_0^t \mathcal{F}(t-s,v,w) \left(\mathrm{div}_w(wf(s,w)^2)\right) \,\mathrm{d}s \tag{1.4}$$

where $\mathcal{F}(t,v,w)$ is the fundamental solution for the homogeneous Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \mathrm{div}_v(vf + \nabla_v f)$$

given by

$$\mathcal{F}(t,v,w) := e^{dt} M_{\nu(t)}(e^t v - w)$$

with

$$\nu(t) = \mathrm{e}^{2t} - 1 \quad \text{and} \quad M_\lambda(\xi) = (2\pi\lambda)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{2\lambda}}$$

for any $\lambda > 0$. Let us define the operator $\mathcal{F}[g](t,v)$ acting on functions $g : \mathbb{R}^d \to \mathbb{R}$ as:

$$\mathcal{F}[g(w)](t,v) := \int_{\mathbb{R}^d} \mathcal{F}(t,v,w) g(w) \,\mathrm{d}w. \tag{1.5}$$

Note that by integration by parts, the expression $\mathcal{F}[\mathrm{div}_w(wf^2(w))](t,v)$ is equivalent to:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\frac{\mathrm{e}^{dt}}{(2\pi\,(\mathrm{e}^{2t}-1))^{\frac{d}{2}}} \,\mathrm{e}^{-\frac{|\mathrm{e}^t v - w|^2}{2(\mathrm{e}^{2t}-1)}} \right) \mathrm{div}_w(wf(w)^2) \,\mathrm{d}w \\
&\qquad = -\int_{\mathbb{R}^d} \left[\nabla_w \left(\frac{\mathrm{e}^{dt}}{(2\pi\,(\mathrm{e}^{2t}-1))^{\frac{d}{2}}} \,\mathrm{e}^{-\frac{|\mathrm{e}^t v - w|^2}{2(\mathrm{e}^{2t}-1)}} \right) \cdot w \right] f(w)^2 \,\mathrm{d}w \\
&\qquad = -\int_{\mathbb{R}^d} \mathrm{e}^{-t} \left(\nabla_v \mathcal{F}(t,v,w) \cdot w\right) f(w)^2 \,\mathrm{d}w \\
&\qquad =: -\mathrm{e}^{-t}\, \nabla_v \mathcal{F}[wf(w)^2](t,v)
\end{aligned} \tag{1.6}$$

so that (1.4) becomes

$$f(t,v) = \mathcal{F}[f_0(w)](t,v) + \int_0^t \mathrm{e}^{-(t-s)} \nabla_v \mathcal{F}[wf(s,w)^2](t-s,v)\,\mathrm{d}s. \qquad (1.7)$$

We will now define a space in which the functional induced by (1.7)

$$\mathcal{T}[f](t,v) := \mathcal{F}[f_0(w)](t,v) + \int_0^t \mathrm{e}^{-(t-s)} \nabla_v \mathcal{F}[wf(s,w)^2](t-s,v)\,\mathrm{d}s \qquad (1.8)$$

has a fixed point. To this end, we define the spaces $\Upsilon = \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{L}^1_1(\mathbb{R}^d) \cap \mathrm{L}^p_m(\mathbb{R}^d)$ $\Upsilon_T = \mathcal{C}([0,T];\Upsilon)$ with norms

$$\|f(t)\|_\Upsilon = \max\{\|f(t)\|_\infty, \|f(t)\|_{\mathrm{L}^1_1}, \|f(t)\|_{\mathrm{L}^p_m}\} \quad \text{and} \quad \|f\|_{\Upsilon_T} = \max_{0\le t\le T} \|f\|_\Upsilon$$

for any $T>0$, where we omit the d-dimensional euclidean space $\mathbb{R}^d$ for notational convenience and

$$\|f\|_{\mathrm{L}^p_m} := \|(1+|v|^m)f\|_p \quad \text{and} \quad \|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \mathrm{d}v\right)^{\frac{1}{p}}.$$

Thereupon, we can provide a criteria under which (1.8) would be bounded in Υ_T.

Lemma 1.1. *Let $p>d$, $p\ge 2$, and $m\ge 1$. Then we can choose q and r satisfying*

$$\frac{dp}{d+p} < \frac{p}{2} \le r \le \frac{mp}{m+1} < p \qquad \textit{and} \qquad \frac{p}{2} \le q \le p \qquad (1.9)$$

such that $\|\mathcal{T}[f]\|_{\Upsilon_T}$ is bounded by $\|f\|_{\Upsilon_T}$.

Proof.- Let us fix such parameters p, m, r, q and $0\le t\le T$. Due to Proposition 1.29 from appendix, and since $q\le p\le 2q$, we can compute

$$\begin{aligned}
\|\mathcal{T}[f](t)\|_\infty &\le C\mathrm{e}^{dt}\|f_0\|_\infty + \int_0^t C\frac{\mathrm{e}^{d(t-s)}}{\nu(t-s)^{\frac{d}{2q}+\frac{1}{2}}}\, \||w|f^2(s)\|_q\,\mathrm{d}s \\
&\le C\mathrm{e}^{dt}\|f_0\|_\infty + \int_0^t C\frac{\mathrm{e}^{d(t-s)}}{\nu(t-s)^{\frac{d}{2q}+\frac{1}{2}}}\, \|f(s)\|_\infty^{2-\frac{p}{q}} \|f(s)\|_{\mathrm{L}^p_m}^{\frac{p}{q}}\,\mathrm{d}s \\
&\le C\mathrm{e}^{dt}\|f_0\|_\infty + \int_0^t C\frac{\mathrm{e}^{d(t-s)}}{\nu(t-s)^{\frac{d}{2q}+\frac{1}{2}}}\,\mathrm{d}s\, \|f\|_{\Upsilon_T}^2 \\
&\le C\mathrm{e}^{dt}\|f_0\|_\infty + C\,\mathcal{I}_1(t)\,\|f\|_{\Upsilon_T}^2,
\end{aligned}$$

where doing the change of variables $\chi = \mathrm{e}^{-2t}$ we can write

$$\mathcal{I}_1(t) := \int_{\mathrm{e}^{-2t}}^{1} \chi^{-\frac{1}{2}\left(d-\frac{d}{q}-1\right)-1}(1-\chi)^{-\frac{1}{2}\left(\frac{d}{q}+1\right)}\mathrm{d}\chi < \infty$$

by the choice (1.9) of q. In the same way, since r satisfies $(m+1)r \leq mp$ and $2r \geq p$, we get

$$\begin{aligned}
\|\mathcal{T}[f](t)\|_{\mathrm{L}^p_m} &\leq C\mathrm{e}^{\frac{d}{p'}t}\|f_0\|_{\mathrm{L}^p_m} + \int_0^t C\frac{\mathrm{e}^{\frac{d}{p'}(t-s)}}{\nu(t-s)^{\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{1}{2}}}\||w|f^2(s)\|_{\mathrm{L}^r_m}\mathrm{d}s \\
&\leq C\mathrm{e}^{\frac{d}{p'}t}\|f_0\|_{\mathrm{L}^p_m} + \int_0^t C\frac{\mathrm{e}^{\frac{d}{p'}(t-s)}}{\nu(t-s)^{\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{1}{2}}}\|f(s)\|_\infty^{2-\frac{p}{r}}\|f(s)\|_{\mathrm{L}^p_m}^{\frac{p}{r}}\mathrm{d}s \\
&\leq C\mathrm{e}^{\frac{d}{p'}t}\|f_0\|_{\mathrm{L}^p_m} + \int_0^t C\frac{\mathrm{e}^{\frac{d}{p'}(t-s)}}{\nu(t-s)^{\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{1}{2}}}\,\mathrm{d}s\,\|f\|^2_{\Upsilon_T} \\
&\leq C\mathrm{e}^{\frac{d}{p'}t}\|f_0\|_{\mathrm{L}^p_m} + C\,\mathcal{I}_2(t)\,\|f\|^2_{\Upsilon_T},
\end{aligned}$$

where, with the same change as before,

$$\mathcal{I}_2(t) := \int_{\mathrm{e}^{-2t}}^{1} \chi^{-\frac{1}{2}\left[\frac{d}{p'}-\left(d\left(\frac{1}{r}-\frac{1}{p}\right)+1\right)\right]-1}(1-\chi)^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2}}\mathrm{d}\chi < \infty$$

by the choice (1.9) of r. Note that p' is the conjugate of p and therefore $\frac{d}{p'} \leq 2d$. Finally we can estimate

$$\|\mathcal{T}[f](t)\|_{\mathrm{L}^1_1} \leq C\|f_0\|_{\mathrm{L}^1_1} + \int_0^t \frac{C}{\nu(t-s)^{\frac{1}{2}}}\||w|f^2(s)\|_{\mathrm{L}^1_1}\,\mathrm{d}s$$

where by interpolation, we get as $p \geq 2$ and $m \geq 1$

$$\begin{aligned}
\||w|f^2\|_{\mathrm{L}^1_1} &= \int_{\mathbb{R}^d}(1+|w|)|w|f^2\,\mathrm{d}w \leq \int_{\mathbb{R}^d}(1+|w|)^2f^2\,\mathrm{d}w \\
&\leq \left(\int_{\mathbb{R}^d}(1+|w|)f\,\mathrm{d}w\right)^{\frac{p-2}{p-1}}\left(\int_{\mathbb{R}^d}(1+|w|)^pf^p\,\mathrm{d}w\right)^{\frac{1}{p-1}} \\
&\leq \|f\|_{\mathrm{L}^1_1}^{\frac{p-2}{p-1}}\|f\|_{\mathrm{L}^p_m}^{\frac{p}{p-1}}.
\end{aligned} \tag{1.10}$$

Consequently, we get

$$\|\mathcal{T}[f](t)\|_{\mathrm{L}^1_1} \leq C\|f_0\|_{L^1_1} + C\int_{\mathrm{e}^{-2t}}^{1}\chi^{-\frac{3}{2}}(1-\chi)^{-\frac{1}{2}}\mathrm{d}\chi\,\|f\|^2_{\Upsilon_T},$$

as desired. □

We next check the existence of a fixed point of (1.8) in Υ_T. Collecting all the above estimates, we can write

$$\|f_{n+1}(t)\|_\Upsilon \leq C_1(d,t)\|f_0\|_\Upsilon + C_2(d,p,q,r,t)\|f_n\|^2_{\Upsilon_T}$$

for any $0 \leq t \leq T$ and any $T > 0$, with

$$\begin{aligned} C_1(d,t) &:= C\mathrm{e}^{dt} \\ C_2(d,p,q,r,t) &:= C\max\left\{ \mathcal{I}_1(t), \mathcal{I}_2(t), \int_{\mathrm{e}^{-2t}}^1 \chi^{-\frac{3}{2}}(1-\chi)^{-\frac{1}{2}}\mathrm{d}\chi \right\} \end{aligned}$$

which are clearly increasing with t and $C_2(T)$ tends to 0 as T does. Thus, for any $T > 0$

$$\|f_{n+1}\|_{\Upsilon_T} \leq C_1(T)\,\|f_0\|_\Upsilon + C_2(T)\,\|f_n\|^2_{\Upsilon_T}$$

with $C_1(T) = C_1(d,T)$ and $C_2(T) = C_2(d,p,q,r,T)$, both being increasing functions of T. We may also assume that $C_1(T) \geq 1$ without loss of generality. From now on, we will follow the arguments in [117]. We will first show that if T is small enough, the functional $\mathcal{T}$ is bounded in Υ_T, which will in turn imply the convergence. Let us take $T > 0$ which verify

$$0 < \|f_0\|_\Upsilon < \frac{1}{4C_1(T)C_2(T)}.$$

Then, let us prove by induction that $\|f_n\|_{\Upsilon_T} \leq 2C_1(T)\|f_0\|_\Upsilon$ for all n. It is clear that we have $\|f_0\|_\Upsilon \leq C_1(T)\|f_0\|_\Upsilon \leq 2C_1(T)\|f_0\|_\Upsilon$. If we suppose that $\|f_n\|_{\Upsilon_T} \leq 2C_1(T)\|f_0\|_\Upsilon$, we have

$$\|f_{n+1}\|_{\Upsilon_T} < C_1(T)\|f_0\|_\Upsilon + 4C_1^2(T)C_2(T)\|f_0\|^2_\Upsilon \leq 2C_1(T)\|f_0\|_\Upsilon,$$

hence the claim. Now, computing the difference between two consecutive iterations of the functional and proceeding with the same estimates as above, we can see for any $0 \leq t \leq T$ that

$$\begin{aligned} \|f_{n+1} - f_n\|_{\Upsilon_T} &= \left\| \int_0^t \mathrm{e}^{-(t-s)} \nabla_v \mathcal{F}\left[w\left[f_n^2 - f_{n-1}^2 \right] \right](t-s,v)\mathrm{d}s \right\|_{\Upsilon_T} \\ &\leq C_2(T) \sup_{[0,T]} \left\| f_n + f_{n-1} \right\|_\infty \left\| f_n - f_{n-1} \right\|_{\Upsilon_T} \\ &\leq C_2(T)\left(\left\|f_n\right\|_{\Upsilon_T} + \left\|f_{n-1}\right\|_{\Upsilon_T} \right) \left\| f_n - f_{n-1} \right\|_{\Upsilon_T} \\ &\leq 4C_1(T)C_2(T)\|f_0\|_\Upsilon \left\| f_n - f_{n-1} \right\|_{\Upsilon_T} \\ &\leq \left(4C_1(T)C_2(T)\|f_0\|_\Upsilon\right)^n \left\| f_1 - f_0 \right\|_{\Upsilon_T}. \end{aligned}$$

Since $4C_1(T)C_2(T)\|f_0\|_\Upsilon < 1$ we can conclude that there exists a function f_* in Υ_T which is a fixed point for $\mathcal{T}$, and hence a solution to the integral equation (1.4). It is not difficult to check that the solution $f \in \Upsilon_T$ to the integral equation is a solution of (1.1) in the sense of distributions defining our concept of solution. We summarize the results of this subsection in the following result.

Theorem 1.2 (Local Existence). *Let $m \geq 1$, $p > d$, $p \geq 2$, and let $f_0 \in \mathrm{L}^\infty \cap \mathrm{L}^p_m \cap \mathrm{L}^1_1(\mathbb{R}^d)$. Then there exists $T > 0$, depending only on the norm of the initial data in $L^\infty \cap \mathrm{L}^p_m \cap \mathrm{L}^1_1(\mathbb{R}^d)$, such that equation* (1.1) *has a unique solution in $\mathcal{C}([0,T]; \mathrm{L}^\infty \cap \mathrm{L}^p_m \cap \mathrm{L}^1_1(\mathbb{R}^d))$.*

Remark 1.3. *The previous theorem is also valid for $f_0 \in \mathrm{L}^\infty \cap \mathrm{L}^p_m \cap \mathrm{L}^1(\mathbb{R}^d)$, with a solution defined in $\mathcal{C}([0,T]; \mathrm{L}^\infty \cap \mathrm{L}^p_m \cap \mathrm{L}^1(\mathbb{R}^d))$ but we will need to have the first moment of the solution bounded in order to be able to extend it to a global in time solution. We thus include here this additional condition.*

Remark 1.4. *With the same arguments used to prove Theorem* 1.2 *we can prove an equivalent result for the Bose-Einstein equation for Bosons*

$$\frac{\partial f}{\partial t} = \Delta_v f + \mathrm{div}_v[vf(1+f)], \qquad v \in \mathbb{R}^d, t > 0.$$

Estimates on Derivatives

Let us now work on estimates on the derivatives. By taking the gradient in the integral equation, we obtain

$$\nabla_v f(t,v) = \nabla_v \mathcal{F}[f(w)](t,v) - \int_0^t \nabla_v \mathcal{F}[\mathrm{div}_w(wf^2(s,w))](t-s,v)\,\mathrm{d}s. \quad (1.11)$$

where $\nabla_v \mathcal{F}[g](t,v)$ is defined as in (1.6) for the real-valued function g. Here, we will consider a space with suitable weighted norms for the derivatives,

$$X_T := \left\{ f \in \Upsilon_T \mid \nabla_v f \in \mathrm{L}^p_m \cap \mathrm{L}^1_1 \text{ and } \|f\|_{X_T} < \infty \right\}$$

for

$$\|f\|_{X_T} = \max\left\{ \|f\|_{\Upsilon_T}, \sup_{0<t<T} \nu(t)^{\frac{1}{2}} \|\nabla_v f\|_{\mathrm{L}^p_m}, \sup_{0<t<T} \nu(t)^{\frac{1}{2}} \|\nabla_v f\|_{\mathrm{L}^1_1} \right\}$$

where for notational simplicity we refer to $\||\nabla_v f|\|_{\mathrm{L}^p_m}$ as $\|\nabla_v f\|_{\mathrm{L}^p_m}$. Let us estimate the L^p_m and L^1 norms of $\nabla_v f$ using again the results in Proposition 1.29 as follows: for $r \in [1,p)$ to be chosen later

$$
\begin{aligned}
\|\nabla_v f(t)\|_{\mathrm{L}^p_m} &\le C\frac{\mathrm{e}^{\left(\frac{d}{p'}+1\right)t}}{\nu(t)^{\frac{1}{2}}}\|f_0\|_{\mathrm{L}^p_m} + \int_0^t \|\nabla_v \mathcal{F}[2f(w\cdot\nabla_w f)] + Nf^2\|_{\mathrm{L}^p_m}\mathrm{d}s \\
&\le C\frac{\mathrm{e}^{\left(\frac{d}{p'}+1\right)t}}{\nu(t)^{\frac{1}{2}}}\|f_0\|_{\mathrm{L}^p_m} + C\int_0^t \frac{\mathrm{e}^{\left(\frac{d}{p'}+1\right)(t-s)}}{\nu(t-s)^{\frac{1}{2}}}\|f(s)\|_{\mathrm{L}^p_m}\|f(s)\|_\infty \mathrm{d}s \\
&\quad + C\int_0^t \frac{\mathrm{e}^{\left(\frac{d}{p'}+1\right)(t-s)}}{\nu(t-s)^{\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{1}{2}}}\|f(w\cdot\nabla_w f)\|_{\mathrm{L}^r_m}\mathrm{d}s \\
&\le C\frac{\mathrm{e}^{\left(\frac{d}{p'}+1\right)t}}{\nu(t)^{\frac{1}{2}}}\|f_0\|_{\mathrm{L}^p_m} + C\|f\|^2_{\Upsilon_T}\int_{\mathrm{e}^{-2t}}^1 \chi^{-\frac{d+2p'}{2p'}}(1-\chi)^{-\frac{1}{2}}\mathrm{d}s \\
&\quad + C\sup_{0<s<T}\left\{\nu(s)^{1/2}\|f(s)(w\cdot\nabla_w f(s))\|_{\mathrm{L}^r_m}\right\} I(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu(t)^{\frac{1}{2}} I(t) &\le \nu(t)^{\frac{1}{2}}\frac{\mathrm{e}^{-t}}{2}\int_{\mathrm{e}^{-2t}}^t \mathrm{e}^{t\left(\frac{d+2r'}{r'}\right)}(1-\chi)^{-(\frac{d}{2}(\frac{1}{r}-\frac{1}{p})+\frac{1}{2})}(\chi-e^{-2t})^{-\frac{1}{2}}\mathrm{d}\chi \\
\nu(t)^{\frac{1}{2}} &\le \frac{1}{2}\mathrm{e}^{t(\frac{d+r'}{r'})}\left[\int_{\mathrm{e}^{-2t}}^{\frac{1+\mathrm{e}^{-2t}}{2}}\left(\frac{1-\mathrm{e}^{-2t}}{2}\right)^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}}(\chi-\mathrm{e}^{-2t})^{-\frac{1}{2}}\mathrm{d}\chi\right. \\
&\quad \left. + \int_{\frac{1+\mathrm{e}^{-2t}}{2}}^1 (\chi-\mathrm{e}^{-2t})^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2}}\left(\frac{1-\mathrm{e}^{-2t}}{2}\right)^{-\frac{1}{2}}\mathrm{d}\chi\right] \\
&\le C\mathrm{e}^{t\frac{d+r'}{r'}}(1-\mathrm{e}^{-2t})^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})}\nu(t)^{\frac{1}{2}} \\
&\le C\mathrm{e}^{t\left(\frac{d+p'}{p'}\right)}\nu(t)^{\frac{1}{2}-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}
\end{aligned}
$$

Note that the right hand side of the previous inequality is an increasing function of time taking zero at $t=0$ since $p>r>dp/(d+p)$. It remains to estimate $\|f(w\cdot\nabla_w f)\|_{\mathrm{L}^r_m}$:

$$
\|f(w\cdot\nabla_w f)\|_{\mathrm{L}^r_m} \le C\left(\int_{\mathbb{R}^d} f^r|\nabla_w f|^r\mathrm{d}w + \int_{\mathbb{R}^d}|w|^{(m+1)r}f^r|\nabla_w f|^r\mathrm{d}w\right)^{\frac{1}{r}}
$$

Now, we can bound these integrals by using H¨older's inequality, to obtain

$$
\int_{\mathbb{R}^d} f^r|\nabla_w f|^r\mathrm{d}w \le \left(\int_{\mathbb{R}^d} f^{\frac{pr}{p-r}}\mathrm{d}w\right)^{\frac{p-r}{p}}\left(\int_{\mathbb{R}^d}|\nabla_w f|^p\mathrm{d}w\right)^{\frac{r}{p}}
$$

and

$$\int_{\mathbb{R}^d} |w|^{(m+1)r} f^r |\nabla_w f|^r \mathrm{d}w \leq \left(\int_{\mathbb{R}^d} |w|^{\frac{pr}{p-r}} f^{\frac{pr}{p-r}} \mathrm{d}w\right)^{\frac{p-r}{p}} \left(\int_{\mathbb{R}^d} |w|^{mp} |\nabla_w f|^p \mathrm{d}w\right)^{\frac{r}{p}}.$$

Assuming that $p < \frac{pr}{p-r} \leq mp$ or equivalently $\frac{m+1}{m} r \leq p < 2r$, we have for any $0 < t \leq T$

$$\int_{\mathbb{R}^d} f^r |\nabla_w f|^r \mathrm{d}w \leq \|f\|_\infty^{2r-p} \|f\|_p^{p-r} \|\nabla_w f\|_p^r \leq \frac{\|f\|_{X_T}^{2r}}{\nu(t)^{\frac{r}{2}}}$$

and

$$\int_{\mathbb{R}^d} |w|^{(m+1)r} f^r |\nabla_w f|^r \mathrm{d}w \leq \|f\|_\infty^{2r-p} \|f\|_{\mathrm{L}^p_m}^{p-r} \|\nabla_w f\|_{\mathrm{L}^p_m}^r \leq \frac{\|f\|_{X_T}^{2r}}{\nu(t)^{\frac{r}{2}}}.$$

Putting together the above estimates we have shown that,

$$\nu(t)^{1/2} \|f_t(w \cdot \nabla_w f(t))\|_{\mathrm{L}^r_m} \leq C \|f\|_{X_T}^2$$

and

$$\nu(t)^{\frac{1}{2}} \|\nabla_v f(t)\|_{\mathrm{L}^p_m} \leq C_1^1(T,N,p) \|f_0\|_{\mathrm{L}^p_m} + C_2^1(T,N,p,r) \|f\|_{X_T}^2 \tag{1.12}$$

with C_1^1 and C_2^1 increasing functions of T and for any $0 < t \leq T$. Analogously, we reckon

$$\begin{aligned} \|\nabla_v f(t)\|_{\mathrm{L}^1_1} \leq & C \frac{e^t}{\nu(t)^{\frac{1}{2}}} \|f_0\|_{\mathrm{L}^1_1} + C \int_0^t \frac{\mathrm{e}^{t-s}}{\nu(t-s)^{\frac{1}{2}}} \|f(s)\|_\infty \|f(s)\|_{\mathrm{L}^1_1} \, \mathrm{d}s \\ & + C \int_0^t \frac{\mathrm{e}^{(t-s)}}{\nu(t-s)^{\frac{1}{2}}} \|f(w \cdot \nabla_w f)(s)\|_{\mathrm{L}^1_1} \, \mathrm{d}s \end{aligned}$$

where by taking $p \geq 2$ and by interpolation as in (1.10), we have

$$\begin{aligned} \|f(w \cdot \nabla_w f)\|_{\mathrm{L}^1_1} & \leq \||w|^{\frac{1}{2}} f\|_2 \||w|^{\frac{1}{2}} |\nabla_w f|\|_2 \\ & \leq \|f\|_{\mathrm{L}^1_1}^{\frac{p-2}{2(p-1)}} \|f\|_{\mathrm{L}^p_m}^{\frac{p}{2(p-1)}} \|\nabla_w f\|_{\mathrm{L}^1_1}^{\frac{p-2}{2(p-1)}} \|\nabla_w f\|_{\mathrm{L}^p_m}^{\frac{p}{2(p-1)}} \\ & \leq \frac{\|f\|_{X_T}^2}{\nu(t)^{1/2}}. \end{aligned}$$

Putting together the last estimates, we deduce

$$\nu(t)^{\frac{1}{2}} \|\nabla_v f(t)\|_{\mathrm{L}^1_1} \leq C_1^3(T,d,p) \|f_0\|_{\mathrm{L}^1_1} + C_2^3(T,d,p,r) \|f\|_{X_T}^2 \tag{1.13}$$

with C_1^3 and C_2^3 increasing functions of T and for any $0 < t \leq T$. From (1.12) and (1.13) and all the estimates of the previous section, we finally get

$$\|f\|_{X_T} \leq C_1(T,d,p)\|f_0\|_\Upsilon + C_2(T,d,p,r)\|f\|_{X_T}^2$$

for any $T > 0$. From these estimates and proceeding as at the end of the previous section, it is easy to show that we have uniform estimates in X_T of the iteration sequence and the convergence of the iteration sequence in the space X_T. From the uniqueness obtained in the previous section, we conclude that the solution obtained in this new procedure is the same as before and lies in X_T. Summarizing, we have shown:

Theorem 1.5. *Let $m \geq 1$, $p > d$, $p \geq 2$, and let $f_0 \in \Upsilon$. Then there exists $T > 0$, depending only on the norm of the initial condition $f_0 \in \Upsilon$ such that* (1.1) *has a unique solution in $\mathcal{C}([0,T];\Upsilon)$ such that $f(0) = f_0$ and its velocity gradients satisfy that $t \to \nu(t)^{\frac{1}{2}}|\nabla_v f| \in BC((0,T), \mathrm{L}^p_m \cap \mathrm{L}^1(\mathbb{R}^d))$.*

Properties of the solutions

As (1.1) belongs to the general class of convection-diffusion equation, it enjoys several classical properties which we gather in this section. The proofs of these results use classical approximation arguments, see [78, 169] for instance. Even if they are somehow standard we shall include them for completeness. Before going into the properties of the solutions of (1.1) let us introduce some notation in the following remark

Remark 1.6. *We can define a regularized version of the sign function as*

$$\operatorname{sign}_\varepsilon(x) = \begin{cases} -1 & \text{if } x \leq -\varepsilon \\ \eta(x) & \text{if } -\varepsilon \leq x \leq \varepsilon \\ 1 & \text{if } x \geq \varepsilon \end{cases},$$

with $\eta \in C^\infty([-\varepsilon,\varepsilon],\mathbb{R})$ increasing, odd and such that $\operatorname{sign}_\varepsilon$ is C^∞ at $x = \pm\varepsilon$, and, analogously, $sign_\varepsilon^+$ and $sign_\varepsilon^-$. It is clear that we can construct a radial non-increasing function $\zeta \in \mathcal{C}_0^\infty((0,\infty))$ such that $\zeta(r) = 1$ for $0 \leq r \leq 1$ and $\zeta(r) = 0$ for $r \geq 2$. We shall define

$$\zeta_n(x) = \zeta\left(\frac{|x|}{n}\right).$$

Note that $\zeta_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is a cut-off function satisfying $0 \leq \zeta_n \leq 1$, $\zeta_n(v) = 1$ if $|v| \leq n$, $\zeta_n(v) = 0$ if $|v| \geq 2n$ and, since $\nabla\zeta_n(x) = \frac{1}{n}\zeta'(|x|/n)\frac{x}{|x|}$ and $\Delta\zeta_n = \frac{1}{n^2}\zeta''(|x|/n) + \frac{1}{n}\zeta'(|x|/n)\frac{d-1}{|x|}$, then $|\nabla_v\zeta_n| \leq \frac{C}{n}$ and $|\Delta_v\zeta_n| \leq \frac{C}{n^2}$.

Lemma 1.7 (Positivity). *Let $f \in X_T$ be the solution of the Cauchy problem* (1.1) *with initial condition $f_0 \in \Upsilon$. Then, if f_0 is non-negative a.e. in $\mathbb{R}^d$, so is $f(t)$ for any $0 < t \leq T$.*

Proof.- We will obtain this result from the time evolution of $(f)^-_\varepsilon$. Multiplying both sides of equation (1.1) by $\operatorname{sign}^-_\varepsilon(f)$ and integrating over $\mathbb{R}^d$ we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}(f)^-_\varepsilon \,\mathrm{d}v &= \int_{\mathbb{R}^d} \operatorname{sign}^-_\varepsilon(f)\mathrm{div}_v\left[\nabla_v f + vf(1-f)\right]\mathrm{d}v \\ &= -\int_{\mathbb{R}^d} \operatorname{sign}^{-'}_\varepsilon(f)|\nabla_v f|^2\,\mathrm{d}v - \int_{\mathbb{R}^d} \operatorname{sign}^{-'}_\varepsilon(f)\nabla_v f v f(1-f)\,\mathrm{d}v \\ &\leq \int_{\mathbb{R}^d} v \operatorname{sign}^{-'}_\varepsilon(f) f^2 \nabla_v f\,\mathrm{d}v - \int_{\mathbb{R}^d} v \operatorname{sign}^{-'}_\varepsilon(f) f \nabla_v f\,\mathrm{d}v.\end{aligned}$$

It is easy to see that

$$\operatorname{sign}'^{-}_\varepsilon(f) f^2 \nabla_v f = \nabla_v(f^2 \operatorname{sign}^-_\varepsilon(f) - f(f)^-_\varepsilon) - (f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon)\nabla_v f$$

and

$$\operatorname{sign}'^{-}_\varepsilon(f) f \nabla_v f = \nabla_v(f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon),$$

whence, by integration by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}(f)^-_\varepsilon dv \leq N\int_{\mathbb{R}^d}\big[f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon\big]\mathrm{d}v - N\int_{\mathbb{R}^d}\big[f(f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon)\big]\mathrm{d}v \\ - \int_{\mathbb{R}^d} v(f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon)\nabla_v f \mathrm{d}v.\end{aligned}$$

In the limit when $\varepsilon \to 0$ the functions inside both integrals converges almost everywhere to zero and dominated by L^1 functions (indeed, $|f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon| \leq 2f$ and $|f(f \operatorname{sign}^-_\varepsilon(f) - (f)^-_\varepsilon)| \leq 2f^2$ with $f(t) \in L^1 \cap L^\infty(\mathbb{R}^d)$ for any $t > 0$, and $\nabla_v f \in \mathrm{L}^1_1(\mathbb{R}^d)$ for$0 < t < T$) and thus, due to the Dominated Convergence Theorem, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} f^- \,\mathrm{d}v \leq 0 \tag{1.14}$$

which concludes the proof. □

Remark 1.8. *In the previous argument, we have skipped the truncation in v and worked with the integrated version of the differential inequality to be completely rigorous, although these steps are easily done taking into account that $f \in X_T$. We will do in detail these argument in the next property.*

Lemma 1.9 (Boundedness). *Let $f \in X_T$ be the solution of the Cauchy problem* (1.1) *with initial condition $f_0 \in \Upsilon$. Then, if $f_0 \leq 1$, also $f(t) \leq 1$ for any $0 < t \leq T$.*

Proof.- This time we study the time evolution of $|f-1|_\varepsilon^+$. We multiply equation (1.1) by $\zeta_n(v)\,\mathrm{sign}_\varepsilon^+(f-1)$,where $\zeta_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is a cut-off function satisfying $0 \le \zeta_n \le 1$, $\zeta_n(v) = 1$ if $|v| \le n$, $\zeta_n(v) = 0$ if $|v| \ge 2n$, and $|\nabla_v \zeta_n| \le \frac{1}{n}$, and integrate over $\mathbb{R}^d$ to obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} \zeta_n(v)|f-1|_\varepsilon^+ \mathrm{d}v = \\
&- \int_{\mathbb{R}^d} \zeta_n(v)\,\mathrm{sign}_\varepsilon^{+'}(f-1)|\nabla_v(f-1)|^2 \mathrm{d}v \\
&+ \int_{\mathbb{R}^d} \zeta_n(v)\,\mathrm{sign}_\varepsilon^{+'}(f-1)(\nabla_v(f-1)\cdot v)f(f-1)\mathrm{d}v \\
&- \int_{\mathbb{R}^d} \mathrm{sign}_\varepsilon^+(f-1)\nabla_v\zeta_n\cdot(\nabla_v f + vf(1-f))\mathrm{d}v \\
\le &\int_{\mathbb{R}^d} \zeta_n(v)v\,\mathrm{sign}_\varepsilon^{'+}(f-1)(f-1)\nabla_v(f-1)\mathrm{d}v \\
&+ \int_{\mathbb{R}^d} \zeta_n(v)v\,\mathrm{sign}_\varepsilon^{'+}(f-1)(f-1)^2\nabla_v(f-1)\mathrm{d}v \\
&- \int_{\mathbb{R}^d} \mathrm{sign}_\varepsilon^+(f-1)\nabla_v\zeta_n\cdot(\nabla_v f + vf(1-f))\mathrm{d}v \\
= &\int_{\mathbb{R}^d} \zeta_n(v)v\nabla_v((f-1)\,\mathrm{sign}_\varepsilon^+(f-1) - |f-1|_\varepsilon^+)\mathrm{d}v \\
&+ \int_{\mathbb{R}^d} \zeta_n(v)v\nabla_v((f-1)^2\,\mathrm{sign}_\varepsilon^+(f-1) - (f-1)|f-1|_\varepsilon^+)\mathrm{d}v \\
&- \int_{\mathbb{R}^d} \zeta_n(v)v((f-1)\,\mathrm{sign}_\varepsilon^+(f-1) - |f-1|_\varepsilon^+)\nabla_v(f-1)\mathrm{d}v \\
&- \int_{\mathbb{R}^d} \mathrm{sign}_\varepsilon^+(f-1)\nabla_v\zeta_n\cdot(\nabla_v f + vf(1-f))\mathrm{d}v.
\end{aligned}
$$

This last result can be rewritten by integration-by-parts as

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} \zeta_n(v)|f-1|_\varepsilon^+ \mathrm{d}v \le \\
&- \int_{\mathbb{R}^d} \mathrm{div}_v(v\zeta_n(v))((f-1)^2\,\mathrm{sign}_\varepsilon^+(f-1) - (f-1)|f-1|_\varepsilon^+)\mathrm{d}v \\
&- \int_{\mathbb{R}^d} \mathrm{div}_v(v\zeta_n(v))((f-1)\,\mathrm{sign}_\varepsilon^+(f-1) - |f-1|_\varepsilon^+)\mathrm{d}v \\
&- \int_{\mathbb{R}^d} \zeta_n(v)v((f-1)\,\mathrm{sign}_\varepsilon^+(f-1) - |f-1|_\varepsilon^+)\nabla_v(f-1)\mathrm{d}v \\
&+ \frac{1}{n}\int_{\mathbb{R}^d} |\nabla_v(f-1) + vf(1-f)|\,\mathrm{d}v
\end{aligned}
$$

If we consider the limit as $\varepsilon \to 0$ in the first integrals we notice that due to the same arguments we used in the proof of previous lemma they become 0 for every n. Since our solution f belongs to X_T, in particular we get that $\nabla_v(f-1) + vf(1-f) \in L^1(\mathbb{R}^d)$ for any $0 < t \leq T$. Thus, all of them disappear when we do the limits $\epsilon \to 0$ and then, $n \to \infty$ obtaining

$$\frac{d}{dt}\int_{\mathbb{R}^d} (f-1)^+ \,\mathrm{d}v \leq 0 \tag{1.15}$$

from which the lemma follows. □

Similar arguments show the conservation of the mass.

Lemma 1.10 (Mass Conservation). *Let f be a solution of the Cauchy problem* (1.1) *with positive initial data $f_0 \in \Upsilon$, then the L^1-norm of f is conserved, i.e. $\|f(t)\|_1 = \|f_0\|_1$ for all $t \geq 0$.*

Proof.- Due to Lemma 1.7, we know that solutions are positive for positive initial data, then multiplying equation (1.1) by $\zeta_n(v)$ and integrate over $\mathbb{R}^d$, we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \zeta_n(v) f(t,v)\mathrm{d}v - \int_{\mathbb{R}^d} \zeta_n(v) f_0(v)\mathrm{d}v \right| = \\
&\qquad = \left| \int_0^t \int_{\mathbb{R}^d} \nabla_v \zeta_n(v) \cdot (\nabla_v f + vf(1-f))\,\mathrm{d}v\,ds \right| \\
&\qquad \leq \frac{1}{n} \int_0^t \int_{\mathbb{R}^d} |\nabla_v f + vf(1-f)|\,\mathrm{d}v\,ds \\
&\qquad \leq \frac{1}{n}\, C \int_0^t \nu(s)^{-1/2}\,ds
\end{aligned}$$

due to $f \in X_T$ and for any $0 \leq t \leq T$. Now, taking the limit as $n \to \infty$, we obtain the conservation of mass. □

With analogous ideas, we can prove a contraction and comparison property:

Lemma 1.11 (L^1-Contraction and Comparison Principle). *Let $f, g \in X_T$ be the solutions of the Cauchy problem* (1.1) *with non-negative initial condition $f_0, g_0 \in \Upsilon$ respectively. Then*

$$\|f(t) - g(t)\|_1 \leq \|f_0 - g_0\|_1 \tag{1.16}$$

for all $0 < t \leq T$. Furthermore, if $0 \leq f_0 \leq g_0$ then $f \leq g$ for all $0 < t \leq T$.

Proof.- Since f and g solve (1.1),

$$\frac{d}{dt}(f-g) = \Delta(f-g) + \nabla_v(v(f-g)) - \nabla_v(v(f^2-g^2)) \tag{1.17}$$

holds. We want to repeat the argument in previous lemmas, so we multiply this equation by $\zeta_n(v)\,\mathrm{sign}_\varepsilon(f-g)$ and integrate over $\mathbb{R}^d$, so that we can study the time evolution of $|f-g|_\varepsilon$. Thus, we obtain as before

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\zeta_n(v)|f-g|_\varepsilon\,\mathrm{d}v \le\\ &-\int_{\mathbb{R}^d}\zeta_n(v)\,\mathrm{sign}'_\varepsilon(f-g)(v\cdot\nabla_v(f-g))(f-g)\,\mathrm{d}v\\ &+\int_{\mathbb{R}^d}\zeta_n(v)\,\mathrm{sign}'_\varepsilon(f-g)(v\cdot\nabla_v(f-g))(f^2-g^2)\,\mathrm{d}v\\ &-\int_{\mathbb{R}^d}\nabla_v\zeta_n\,\mathrm{sign}_\varepsilon(f-g)(\nabla_v(f-g)+v(f-g-(f^2-g^2)))\,\mathrm{d}v\\ =&-\int_{R^d}\zeta_n(v)(v\cdot\nabla_v((f-g)\,\mathrm{sign}_\varepsilon(f-g)-|f-g|_\varepsilon))\,\mathrm{d}v\\ &+\int_{\mathbb{R}^d}\zeta_n(v)(f+g)(v\cdot\nabla_v((f-g)\,\mathrm{sign}_\varepsilon(f-g)-|f-g|_\varepsilon))\,\mathrm{d}v\\ &-\int_{\mathbb{R}^d}\nabla_v\zeta_n\,\mathrm{sign}_\varepsilon(f-g)(\nabla_v(f-g)+v(f-g-(f^2-g^2)))\,\mathrm{d}v.\end{aligned}$$

Integrating by parts, we finally get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\zeta_n(v)|f-g|_\varepsilon\,\mathrm{d}v \le\\ &\int_{\mathbb{R}^d}\mathrm{div}_v(v\zeta_n(v))((f-g)\,\mathrm{sign}_\varepsilon(f-g)-|f-g|_\varepsilon)\,\mathrm{d}v\\ &-\int_{\mathbb{R}^d}\mathrm{div}_v(\zeta_n(v)v(f+g))((f-g)\,\mathrm{sign}_\varepsilon(f-g)-|f-g|_\varepsilon)\,\mathrm{d}v\\ &+\frac{1}{n}\int_{\mathbb{R}^d}|\nabla_v(f-g)+v(f-g-(f^2-g^2))|\,\mathrm{d}v\end{aligned}$$

As before, for every n, the first two integrals becomes zero as $\varepsilon\to 0$, since f and g are in X_T whence $f(t),g(t)\in L^1_1\cap L^\infty(\mathbb{R}^d)$ and $\nabla_v f(t),\nabla_v g(t)\in L^1_1(\mathbb{R}^d)$ for any $0<t\le T$, allowing for a Lebesgue dominated convergence argument. As before, we have that $\nabla_v f + vf(1-f)\in L^1(\mathbb{R}^d)$ and $\nabla_v g + vg(1-g)\in L^1(\mathbb{R}^d)$ for any $0<t\le T$, and thus the second integral disappears as $n\to\infty$, getting finally

$$\frac{d}{dt}\int_{\mathbb{R}^d}|f-g|\,\mathrm{d}v \le 0 \tag{1.18}$$

which concludes the proof of the first part of the lemma. Now, taking into account Lemma 1.7 and the simple formula

$$(f-g)^- = \frac{|f-g| - f + g}{2},$$

we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^d} (f-g)^- \,\mathrm{d}v \leq 0$$

from which the comparison principle follows. □

Finally, we can also establish time dependent bounds on moments of the solution to (1.1). More precisely, we will show that moments increase at most as a polynomial on t. First, let us note that given $a, b \geq 1$ and $f \in \mathrm{L}^1_{ab}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then

$$\|f\|_{\mathrm{L}^b_a} \leq C \|f\|^{\frac{1}{b}}_{\mathrm{L}^1_{ab}} \|f\|^{1-\frac{1}{b}}_\infty, \tag{1.19}$$

indeed,

$$\begin{aligned} \|f\|_{\mathrm{L}^b_a} &= \left(\int_{\mathbb{R}^d} (1+|v|^a)^b f^b \mathrm{d}v \right)^{\frac{1}{b}} \leq \left(C \int_{\mathbb{R}^d} (1+|v|^{ab}) f^b \mathrm{d}v \right)^{\frac{1}{b}} \\ &\leq \left(C \|f\|^{b-1}_\infty \int_{\mathbb{R}^d} (1+|v|^{ab}) f \mathrm{d}v \right)^{\frac{1}{b}} = C \|f\|^{\frac{1}{b}}_{\mathrm{L}^1_{ab}} \|f\|^{1-\frac{1}{b}}_\infty . \end{aligned}$$

In particular, $\left(L^1_{mp} \cap L^\infty\right)\left(\mathbb{R}^d\right) \subset \Upsilon$.

We next define $\lceil \gamma \rceil$ to be the smallest integer larger or equal than γ.

Lemma 1.12 (Moments Bound). *Let f be a solution of the Cauchy problem* (1.1) *with initial data belonging to $f_0 \in \mathrm{L}^1_{pm} \cap \mathrm{L}^\infty(\mathbb{R}^d)$ with $mp \geq 2$. Then, for $0 \leq t \leq T$ and $1 \leq \gamma \leq \frac{mp}{2}$ the 2γ-moment of f is bounded by a polynomial $P_{\lceil \gamma \rceil}(t)$, of degree $\lceil \gamma \rceil$, which depends only on the moments of f_0.*

Proof.- We will prove it by induction on γ. First, we will see that the second moment is bounded (and therefore all γ_*^{th}-moments with $0 < \gamma_* \leq 2$). Afterwards, we will assume that we can bound the $2(\gamma-1)$-moment and from this induction hypothesis obtain that the 2γ-moment of the solution is bounded.

Let $(\zeta_n)_{n\geq 1}$ be a sequence of smooth cut-off functions as defined in Remark 1.6.

We multiply equation (1.1) by $|v|^2\zeta_n(v)$, and integrate over $\mathbb{R}^d$ to get

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\zeta_n(v)|v|^2 f(t,v)\mathrm{d}v &= \int_{\mathbb{R}^d}\zeta_n(v)|v|^2\Delta_v f\mathrm{d}v + \int_{\mathbb{R}^d}\zeta_n(v)|v|^2\mathrm{div}_v(vf(1-f))\mathrm{d}v \\
&\leq \int_{\mathbb{R}^d}\left[\Delta_v\zeta_n|v|^2 + 4\nabla_v\zeta_n v + 2N\zeta_n\right] f\mathrm{d}v + \int_{\mathbb{R}^d}|\nabla_v\zeta_n||v|^3 f(1-f)\mathrm{d}v \\
&\quad - 2\int_{\mathbb{R}^d}\zeta_n|v|^2 f\mathrm{d}v + 2\int_{\mathbb{R}^d}\zeta_n|v|^2 f^2\mathrm{d}v \\
&\leq 5\int_{n<|v|<2n} f\mathrm{d}v + 2N\int_{\mathbb{R}^d}\zeta_n f\mathrm{d}v + \int_{n<|v|<2n}|v|^2 f\mathrm{d}v
\end{aligned}$$

since $|\nabla_v\zeta_n| \leq \frac{1}{n}$ and $|\Delta_v\zeta_n| \leq \frac{1}{n^2}$ and supported in $[n \leq |v| \leq 2n]$. Now, letting $n \to \infty$ and noticing that $f\mathbb{1}_{\{n<|v|<2n\}}$ and $|v|^2 f\mathbb{1}_{\{n<|v|<2n\}}$ converge pointwise to zero and are bounded by f and $|v|^2 f$ respectively, with $f \in X_T$, whence due to the Dominated convergence theorem, the first and the last integrals become zero. Finally, integrating in time, we get

$$\int_{\mathbb{R}^d}|v|^2 f(t,v)\mathrm{d}v \leq \int_{\mathbb{R}^d}|v|^2 f_0(v)\mathrm{d}v + 2NMt \tag{1.20}$$

for all $0 \leq t \leq T$. By the conservation of mass and this bound, all moments $0 < \gamma < 2$ are bounded.
Now, let us assume that we have the $(2\gamma-2)$-moment of the solution bounded by a polynomial of degree $\gamma-1$. Then, for the moment 2γ we can proceed in the same way

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\zeta_n(v)|v|^{2\gamma} f(t,v)\mathrm{d}v = \\
&\quad \int_{\mathbb{R}^d}\zeta_n(v)|v|^{2\gamma}\Delta_v f\mathrm{d}v + \int_{\mathbb{R}^d}\zeta_n(v)|v|^{2\gamma}\mathrm{div}_v(vf(1-f))\mathrm{d}v \\
&\leq \int_{\mathbb{R}^d}\left[\Delta_v\zeta_n|v|^{2\gamma} + 4\gamma\nabla_v\zeta_n|v|^{2(\gamma-1)}v + 2\gamma(2(\gamma-1)+d)|v|^{2(\gamma-1)}\zeta_n\right] f\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d}|\nabla_v\zeta_n||v|^{2\gamma+1} f(1-f)\mathrm{d}v - 2\gamma\int_{\mathbb{R}^d}\zeta_n|v|^{2\gamma} f\mathrm{d}v + 2\gamma\int_{\mathbb{R}^d}\zeta_n|v|^{2\gamma} f^2\mathrm{d}v \\
&\leq (4\gamma+1)\int_{n<|v|<2n}|v|^{2(\gamma-1)} f\mathrm{d}v + 2\gamma(2(\gamma-1)+d)\int_{\mathbb{R}^d}\zeta_n|v|^{2(\gamma-1)} f\mathrm{d}v \\
&\quad + \int_{n<|v|<2n}|v|^{2\gamma} f\mathrm{d}v
\end{aligned}$$

and again, we let n go to infinity. If $2\gamma \leq mp$, the previous argument ensures that only the second integral remains, and integrating in time, we conclude

$$\int_{\mathbb{R}^d} |v|^{2\gamma} f(t,v)\mathrm{d}v \leq \int_{\mathbb{R}^d} |v|^{2\gamma} f_0(v)\mathrm{d}v + 2\gamma(2(\gamma-1)+d)\int_0^t\int_{\mathbb{R}^d} |v|^{2(\gamma-1)} f(s,v)\mathrm{d}v\,\mathrm{d}s$$

for all $0 \leq t \leq T$. Whence, by induction,

$$\int_{\mathbb{R}^d} |v|^{2\gamma} f(t,v)\mathrm{d}v \leq \int_{\mathbb{R}^d} |v|^{2\gamma} f_0(v)\mathrm{d}v + 2\gamma(2(\gamma-1)+d)\int_0^t P_{[\gamma-1]}(s)\,\mathrm{d}s \quad (1.21)$$

for all $0 \leq t \leq T$, defining by induction the polynomial $P_{\lceil\gamma\rceil}(t)$. □

Remark 1.13. *This lemma could have been stated for $f_0 \in \mathrm{L}^1_\alpha(\mathbb{R}^d)$, with $\alpha > 2$, but we have decided to use this notation to point out that α shall be obtained as a combination of m and p satisfying the conditions of the existence theorem.*

Global existence

Given an initial data $f_0 \in \mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > d$, $p \geq 2$, $m \geq 1$ such that $0 \leq f_0 \leq 1$, we have shown in previous sections, that there exists a unique local solution for (1.1) on an interval $[0,T)$. In fact, we can extend this solution to be global in time. If there exists $T_{max} < \infty$ such that the solution does not exist out of $(0, T_{max})$, then the Υ-norm of it shall go to infinity as t goes to T_{max}; as we will see, that situation cannot happen.
Due to Lemma 1.7, we have that $0 \leq f(t,v) \leq 1$ for any $0 \leq t < T$ and any $v \in \mathbb{R}^d$, and thus a bound for the L^∞-norm of $f(t)$. Also, the conservation of the mass in Lemma 1.10 together with the positivity in Lemma 1.7 provide us with a bound for the L^1-norm. Finally, due to lemma 1.12 the L^p_m-norm is also bounded for finite time.

Theorem 1.14 (Global Existence). *Let $f_0 \in \mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > d$, $p \geq 2$, $m \geq 1$ such that $0 \leq f_0 \leq 1$. Then the Cauchy problem* (1.1) *with initial data f_0 has a unique solution defined in $[0,\infty)$ belonging to X_T for all $T > 0$. Also, we have $0 \leq f(t,v) \leq 1$, for all $t \geq 0$ and $v \in \mathbb{R}^d$ and $\|f(t)\|_1 = \|f_0\|_1 = M$ for all $t \geq 0$.*

Remark 1.15. *Note that for any $K > 0$ we can consider* (1.1) *restricted to the cylinder $C_K := [0,\infty) \times \{|v| \leq K\}$. Then, due to the fact that our solutions are in L^∞, we can show that the solution is indeed $\mathcal{C}^\infty(C_K)$ by applying regularity results in* [115] *for the Cauchy problem for quasilinear parabolic equations.*

Corollary 1.16. *If $f_0 \in \mathrm{L}^1_{mp}(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ is a radially symmetric and non-increasing function (that is, $f_0(v) = \varphi_0(|v|)$ for some non-increasing function φ_0), then so is $f(t)$ for all $t \geq 0$, that is, $f(t,v) = \varphi(t,|v|)$ and $r \to \varphi(t,r)$ is non-increasing for all $t \geq 0$. In addition, φ solves*

$$\frac{\partial}{\partial t}\varphi = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\varphi + r^d\varphi(1-\varphi)\right) \tag{1.22}$$

with $\frac{\partial}{\partial r}\varphi(t,0) = 0$ and $\varphi(0,r) = \varphi_0(r)$

Proof.- The uniqueness part of Theorem 1.14 and the rotational invariance of (1.1) imply that $f(t)$ is radially symmetric for all $t \geq 0$. The other properties are proved by classical arguments, the monotonicity of $r \mapsto \varphi(t,r)$ being a consequence of the comparison principle applied to the equation solved by $\partial\varphi/\partial r$, which is obtained from (1.22) taking the derivative with respect to r. □

1.1.2 Asymptotic Behaviour

Now that we have shown that under the appropriate assumptions equation (1.1) has a unique solution which is global in time, we are interested in how does this solution behave when the time is large. For that we will define an appropriate entropy functional for the solution and study its properties.

Associated Entropy Functional

In this section, we will show that the solutions constructed above satisfy an additional dissipation property, the entropy decay. We define, for $g \in \Upsilon$ such that $0 \leq g \leq 1$, the functional

$$H(g) := S(g) + E(g) \tag{1.23}$$

with the entropy given by

$$S(g) := \int_{\mathbb{R}^d} s(g(v))\,\mathrm{d}v \tag{1.24}$$

where

$$s(r) := (1-r)\log(1-r) + r\log(r) \leq 0, \quad r \in [0,1] \tag{1.25}$$

and the kinetic energy given by

$$E(g) := \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 g(v)\,\mathrm{d}v. \tag{1.26}$$

We first check that $H(g)$ is indeed well defined and a control of the entropy in terms of the kinetic energy.

Lemma 1.17 (Entropy Control). *For* $\varepsilon \in (0,1)$*, there exists a positive constant* C_ε *such that*

$$0 \le -S(g) \le \varepsilon E(g) + C_\varepsilon \tag{1.27}$$

for every $g \in \mathrm{L}^1_2(\mathbb{R}^d)$*, such that* $0 \le g \le 1$*.*

Proof.- For $\varepsilon \in (0,1)$ and $v \in \mathbb{R}^d$, we put $z_\varepsilon := 1/(1+\mathrm{e}^{\varepsilon|v|^2/2})$. The convexity of s ensures that

$$\begin{aligned} s(g(v)) - s(z_\varepsilon(v)) &\ge s'(z_\varepsilon(v))(g(v) - z_\varepsilon(v)) \\ -s(z_\varepsilon(v)) + s(g(v)) &\ge \log\left(\frac{z_\varepsilon(v)}{1-z_\varepsilon(v)}\right)(g(v) - z_\varepsilon(v)) \end{aligned}$$

for $v \in \mathbb{R}^d$. Since $z_\varepsilon(v)/(1-z_\varepsilon(v)) = \mathrm{e}^{-\varepsilon|v|^2/2}$, we end up with

$$\begin{aligned} -s(g(v)) &\le \frac{\varepsilon|v|^2}{2} g(v) - s(z_\varepsilon(v)) - \frac{\varepsilon|v|^2}{2} z_\varepsilon(v) \\ &= \frac{\varepsilon|v|^2}{2} g(v) + (1-z_\varepsilon(v))\log\left(1+\mathrm{e}^{-\varepsilon|v|^2/2}\right) + z_\varepsilon(v)\log\left(1+\mathrm{e}^{-\varepsilon|v|^2/2}\right) \\ &\le \frac{\varepsilon|v|^2}{2} g(v) + \mathrm{e}^{-\varepsilon|v|^2/2} \end{aligned} \tag{1.28}$$

for $v \in \mathbb{R}^d$, where we used $\log(1+a) \le a$ for $a \ge 0$ and $0 \le z_\varepsilon \le 1$. Integrating the previous inequality yields (1.27). $\square$

Next, given f the solution to (1.1) with initial data f_0, with $M := \|f_0\|_1$, we denote by F_M the unique Fermi-Dirac equilibrium state satisfying $\|F_M\|_{L^1} = M$, which is $F_M(v) = 1/(1+\beta\mathrm{e}^{|v|^2/2})$ for some β depending only on $\|f_0\|_{L^1}$ and d (it can be easily checked by an analogous calculus to the one we show in Section 1.1.3 that Fermi-Dirac distribution is indeed the steady state of (1.1)); then we can introduce the next property for H.

Lemma 1.18 (Entropy Monotonicity). *Assume that* f *is a solution to the Cauchy problem* (1.1) *with initial data in* $\mathrm{L}^1_{mp}(\mathbb{R}^d)$ *for some* $p > max(d,2)$ *and* $m \ge 1$ *satisfying* $0 \le f_0 \le 1$*. Then, the function* H *is a non-increasing function of time satisfying for all* $t > 0$ *that*

$$H(f_0) \ge H(f(t)) \ge H(F_M). \tag{1.29}$$

Proof.- First of all, we observe that we can formulate (1.1) as

$$\frac{\partial f}{\partial t} = \mathrm{div}_v\left[f(1-f)\nabla_v\left(s'(f) + \frac{|v|^2}{2}\right)\right].$$

We multiply the previous equation by $s'(f) + |v|^2/2$ and integrate over $\mathbb{R}^d$ to obtain that

$$\frac{d}{dt}H(f) = -\int_{\mathbb{R}^d} f(1-f)|v + \nabla_v s'(f)|^2 \, \mathrm{d}v \leq 0. \tag{1.30}$$

Consequently, the function $t \longrightarrow H(f(t))$ is a non-increasing function of time, whence the first inequality in (1.29). To prove the second inequality, we observe that the convexity of s entails that

$$\begin{aligned} s(f(t,v)) - s(F_M(v)) &\geq s'(F_M(v))(f(t,v) - F_M(v)) \\ s(F_M(v)) - s(f(t,v)) &\leq \left(\log\beta + \frac{|v|^2}{2}\right)(f(t,v) - F_M(v)) \end{aligned}$$

for $(t,v) \in [0,\infty) \times \mathbb{R}^d$. The second inequality in (1.29) now follows from the integration of the previous inequality over $\mathbb{R}^d$ since $\|F\|_{L^1} = \|f(t)\|_{L^1}$.
We shall point out that, in order to justify the previous computations leading to the first inequality, one should first start with an initial datum f_0^ε, $\varepsilon \in (0,1)$, given by

$$f_0^\varepsilon(v) = \max\left\{ min\left\{ f_0(v), \frac{1}{1+\varepsilon e^{|v|^2/2}}\right\}, \frac{\varepsilon}{\varepsilon + e^{|v|^2/2}}\right\} \in \left[\frac{\varepsilon}{\varepsilon + e^{|v|^2/2}}, \frac{1}{1+\varepsilon e^{|v|^2/2}}\right],$$

$v \in \mathbb{R}^d$. Owing to the comparison principle, the corresponding solution f^ε to (1.1) satisfies

$$0 < \frac{\varepsilon}{\varepsilon + e^{|v|^2/2}} \leq f^\varepsilon(t,v) \leq \frac{1}{1+\varepsilon e^{|v|^2/2}} < 1\,, \quad (t,v) \in (0,\infty) \times \mathbb{R}^d\,, \tag{1.31}$$

for which the previous computations can be performed since the solutions are immediately smooth (see remark 1.15) and fast decaying at infinity for all $t > 0$ by previous inequality, and thus $H(f^\varepsilon(t)) \leq H(f_0^\varepsilon)$ for all $t \geq 0$.
Since $f_0^\varepsilon \to f_0$ in Υ and in $L^1_{mp}(\mathbb{R}^d)$ as $\varepsilon \to 0$, it is not difficult to see that redoing all estimates in sections 2.1 and 2.2, we have continuous dependence of solutions with respect to the initial data, and thus, f^ε converges towards f in X_T for any $T > 0$ and moreover, we have uniform in ε bounds of the moments in bounded time intervals using Lemma 1.12. Direct estimates easily show that $H(f_0^\varepsilon) \to H(f_0)$ as $\varepsilon \to 0$.
Let us now prove that $H(f^\varepsilon(t)) \to H(f(t))$ as $\varepsilon \to 0$, at least for a subsequence, that we denote with the same index for simplicity. Let us fix $R > 0$, and consider a "mesa" smooth function φ_R with

$$\varphi_R(v) = \begin{cases} 1 & if\ |v| \leq R \\ 0 & if\ |v| \geq R+1 \end{cases} \quad \text{and } 0 \leq \varphi \leq 1.$$

Since $f^\varepsilon(t) \to f(t)$ in $L^1(\mathbb{R}^d)$ and we have uniform estimates in ε of moments of order $mp > 2$ then

$$\left| \int_{\mathbb{R}^d} |v|^2 (f^\varepsilon(t) - f(t)) \, \mathrm{d}v \right| \leq$$

$$\begin{aligned}
&\leq \int_{|v|\geq R} |v|^2 |f^\varepsilon(t) - f(t)| \, \mathrm{d}v + \left| \int_{\mathbb{R}^d} |v|^2 (f^\varepsilon(t) - f(t)) \varphi_R \, \mathrm{d}v \right| \\
&\leq \frac{1}{R^{mp-2}} \int_{|v|\geq R} |v|^{mp} (f^\varepsilon(t) + f(t)) \, \mathrm{d}v + \left| \int_{\mathbb{R}^d} |v|^2 (f^\varepsilon(t) - f(t)) \varphi_R \, \mathrm{d}v \right| \\
&\leq \frac{C}{R^{mp-2}} + \left| \int_{\mathbb{R}^d} |v|^2 (f^\varepsilon(t) - f(t)) \varphi_R \, \mathrm{d}v \right|.
\end{aligned}$$

Since the above inequality is valid for all $R > 0$, then we conclude that $E(f^\varepsilon(t)) \to E(f(t))$ as $\varepsilon \to 0$. Now, taking into account that $(1 + |v|^2) f^\varepsilon(t) \to (1 + |v|^2) f(t)$ in $L^1(\mathbb{R}^d)$, we deduce that there exists $h \in L^1(\mathbb{R}^d)$ such that $||v|^2 f^\varepsilon(t)| \leq h$ and $f^\varepsilon(t) \to f(t)$ a.e. in $\mathbb{R}^d$, for a subsequence that we denote with the same index. Using inequality (1.28), we deduce that

$$0 \leq -s(f^\varepsilon(t, v)) \leq \frac{1}{4} h(v) + \mathrm{e}^{-|v|^2/4} \in L^1(\mathbb{R}^d)$$

and that $-s(f^\varepsilon(t, v)) \to -s(f(t, v))$ a.e. in $\mathbb{R}^d$. Thus, by the Lebesgue dominated convergence theorem, we finally deduce that $S(f^\varepsilon(t)) \to S(f(t))$ as $\varepsilon \to 0$. The convergence as $\varepsilon \to 0$ is actually true for the whole family (and not only a subsequence) due to the uniqueness of the limit. As a consequence, we showed $H(f^\varepsilon(t)) \to H(f(t))$ as $\varepsilon \to 0$ and passing to the limit $\varepsilon \to 0$ in the inequality $H(f^\varepsilon(t)) \leq H(f_0^\varepsilon)$, we get the desired result. □

Now, it is easy to see the existence of a uniform in time bound for the kinetic energy $E(f(t))$, or equivalently, of the solutions in $\mathrm{L}^1_2(\mathbb{R}^d)$. If we take equations (1.23), (1.27) (with $\varepsilon = 1/2$) and (1.29) we get that for $t \geq 0$

$$E(f(t)) = H(f(t)) - S(f(t)) \leq \frac{1}{2} E(f(t)) + C_{1/2} + H(f_0) \tag{1.32}$$

whence $E(f(t)) \leq 2\big(C_{1/2} + H(f_0)\big)$.

Convergence to the Steady State

Theorem 1.19 (Convergence). *Assume that f is a solution to the Cauchy problem* (1.1) *with initial data in* $\mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > \max(d, 2)$, $m \geq 1$ *with* $0 \leq f_0 \leq 1$. *Let*

F be the Fermi-Dirac Distribution such that $\|F\|_1 = \|f_0\|_1$. Then $\{f(t)\}_{t\geq 0}$ converges strongly in $\mathrm{L}^1(\mathbb{R}^d)$ towards F as $t \to \infty$.

For the proof, we first need a technical lemma.

Lemma 1.20. *Assume that f is a solution to the Cauchy problem* (1.1) *with initial data $f_0 \in \mathrm{L}^1_{mp}(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$, $p > \max(d, 2)$, $m \geq 1$. If A is a measurable subset of $\mathbb{R}^d$, we have*

$$\int_0^\infty \left(\int_A |vf(1-f) + \nabla_v f| \, dv \right)^2 dt \leq H(F) \sup_{t\geq 0} \left\{ \int_A f(t,v) \mathrm{d}v \right\} \tag{1.33}$$

Proof.- Owing to the second inequality in (1.29) and the finiteness of $H(f_0)$, we also infer from (1.30) that $(t,v) \longmapsto f(1-f)\,|v + \nabla_v s'(f)|^2$ belongs to $L^1((0,\infty) \times \mathbb{R}^d)$. Working again with the regularized solutions f^ε, it then follows from Lemma 1.10 and the Cauchy-Schwarz inequality that, if A is a measurable subset of $\mathbb{R}^d$, we can compute

$$\begin{aligned}
&\int_0^\infty \left(\int_A |vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon| \mathrm{d}v \right)^2 \mathrm{d}t \\
&\qquad = \int_0^\infty \left(\int_A \frac{|vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon|}{(f^\varepsilon(1-f^\varepsilon))^{1/2}} \left(f^\varepsilon(1-f^\varepsilon)\right)^{1/2} \mathrm{d}v \right)^2 \mathrm{d}t \\
&\qquad \leq \int_0^\infty \left(\int_A \frac{|vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon|^2}{f^\varepsilon(1-f^\varepsilon)} \mathrm{d}v \right) \left(\int_A f^\varepsilon(1-f^\varepsilon) \mathrm{d}v \right) \mathrm{d}t \\
&\qquad \leq \sup_{t\geq 0} \left\{ \int_A f^\varepsilon(t,v) \mathrm{d}v \right\} \int_0^\infty \int f^\varepsilon(1-f^\varepsilon) \left[v + \nabla_v s'(f^\varepsilon)\right]^2 \mathrm{d}v dt \\
&\qquad \leq H(F_{M^\varepsilon}) \sup_{t\geq 0} \left\{ \int_A f^\varepsilon(t,v) \mathrm{d}v \right\} \leq H(F^\varepsilon) \sup_{t\geq 0} \left\{ \int_A f^\varepsilon(t,v) \mathrm{d}v \right\}.
\end{aligned}$$

Here, $M^\varepsilon := \|f_0^\varepsilon\|_1$ so that F_{M^ε} is the Fermi-Dirac distribution with the mass of the regularized initial data f_0^ε. It is easy to check that $H(F^\varepsilon) \to H(F)$ as $\varepsilon \to 0$ since $\|f^\varepsilon\|_1 \to M$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ as $f^\varepsilon \to f$ in X_T for any $T > 0$, we get the conclusion. □

Proof of Theorem 1.19.- We first establish that

$$\{f(t)\}_{t\geq 0} \quad \text{is bounded in} \quad \mathrm{L}^1_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\,. \tag{1.34}$$

From (1.32) and Theorem 1.14, it is straightforward that $E(f(t))$ is bounded in $[0,\infty)$. Recalling the mass conservation, the boundedness of $\{f(t)\}_{t\geq 0}$ in $L^1_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ follows.

We next turn to the strong compactness of $\{f(t)\}_{t\geq 0}$ in $L^1(\mathbb{R}^d)$. For that purpose, we put $R(t,v) := vf(t,v)(1-f(t,v))$ for $(t,v)\in(0,\infty)\times\mathbb{R}^d$ and deduce from Theorem 1.14 and (1.34) that

$$\sup_{t\geq 0}\left(\|R(t)\|_{L^1} + \|R(t)\|_{L^2}^2\right) \leq 2\ \sup_{t\geq 0}\int_{\mathbb{R}^d}(1+|v|^2)f(t,v)\mathrm{d}v < \infty\,. \tag{1.35}$$

Denoting the linear heat semi-group on $\mathbb{R}^d$ by $(e^{t\Delta})_{t\geq 0}$, it follows from (1.1) that f is given by the Duhamel formula

$$f(t) = e^{t\Delta}f_0 + \int_0^t \nabla_v e^{(t-s)\Delta}R(s)\mathrm{d}s\,, \quad t\geq 0\,. \tag{1.36}$$

It is straightforward by direct Fourier transform techniques to check that

$$\|e^{t\Delta}g\|_{\dot{H}^\alpha} \leq C(\alpha)\ \min\left\{t^{-\alpha/2}\|g\|_{L^2}, t^{-(2\alpha+d)/4}\|g\|_{L^1}\right\}$$

for $t\in(0,\infty)$, $g\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and $\alpha\in[0,2]$ with

$$\|g\|_{\dot{H}^\alpha} := \left(\int_{\mathbb{R}^d}|\xi|^{2\alpha}\,|\widehat{g}(\xi)|^2\,d\xi\right)^{1/2}$$

and $\widehat{g}$ being the Fourier transform of g. Thus, we deduce from (1.36) that, if $t\geq 1$ and $\alpha\in((1-(d/2))^+,1)$, we have

$$\begin{aligned}
\|f(t)\|_{\dot{H}^\alpha} \leq{}& C(\alpha)t^{-\frac{2\alpha+d}{4}}\|f_0\|_{L^1} + C(\alpha+1)\int_0^{t-1}(t-s)^{-\frac{2+2\alpha+d}{4}}\|R(s)\|_{L^1}\mathrm{d}s\\
&+\ C(\alpha+1)\int_{t-1}^t (t-s)^{-\frac{1+\alpha}{2}}\|R(s)\|_{L^2}\mathrm{d}s\\
\leq{}& C\left(1+\int_1^t s^{-\frac{2+2\alpha+d}{4}}\mathrm{d}s + \int_0^1 s^{-(1+\alpha)/2}\mathrm{d}s\right)\\
\leq{}& C\,,
\end{aligned}$$

thanks to the choice of α. Consequently, $\{f(t)\}_{t\geq 1}$ is also bounded in $\dot{H}^\alpha$ for $\alpha\in((1-(d/2))^+,1)$. Owing to the compactness of the embedding of $\dot{H}^\alpha\cap L^1_2(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, we finally conclude that

$$\{f(t)\}_{t\geq 0}\quad\text{is relatively compact in}\quad L^1(\mathbb{R}^d)\,. \tag{1.37}$$

Consider now a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $t_n\to\infty$ as $n\to\infty$. Owing to (1.37), there are a subsequence of $\{t_n\}$ (not relabelled) and $g_\infty\in L^1(\mathbb{R}^d)$ such that $\{f(t_n)\}_{n\in\mathbb{N}}$ converges towards g_∞ in $L^1(\mathbb{R}^d)$ as $n\to\infty$.

Putting $f_n(t) = f(t_n + t)$, $t \in [0,1]$ and denoting by g the unique solution to (1.1) with initial datum g_∞, we infer from the contraction property (1.11) that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \|f_n(t) - g(t)\|_{L^1} = 0\,. \tag{1.38}$$

Next, on one hand, we deduce from the proof of Lemma 1.20 with $A = \mathbb{R}^d$, that $(t,v) \longmapsto vf(t,v)(1-f(t,v)) + \nabla_v f(t,v)$ belongs to $L^2(0,\infty;L^1(\mathbb{R}^d))$. Since

$$\int_0^1 \left(\int_{\mathbb{R}^d} |vf_n(1-f_n) + \nabla_v f_n|\,\mathrm{d}v\right)^2 \mathrm{d}t = \int_{t_n}^{t_n+1} \left(\int_{\mathbb{R}^d} |vf(1-f) + \nabla_v f|\,\mathrm{d}v\right)^2 \mathrm{d}t\,,$$

we end up with

$$\lim_{n\to\infty} \int_0^1 \left(\int_{\mathbb{R}^d} |vf_n(1-f_n) + \nabla_v f_n|\,\mathrm{d}v\right)^2 \mathrm{d}t = 0\,. \tag{1.39}$$

On the other hand, it follows from the mass conservation and (1.33) that, if A is a measurable subset of $\mathbb{R}^d$ with finite measure $|A|$, we have

$$\int_0^1 \left(\int_A |vf_n(1-f_n) + \nabla_v f_n|\,\mathrm{d}v\right)^2 \mathrm{d}t \le H(F)|A|\,,$$

which implies that $\{vf_n(1-f_n) + \nabla_v f_n\}_{n\in\mathbb{N}}$ is weakly relatively compact in $L^1((0,1)\times\mathbb{R}^d)$ by the Dunford-Pettis theorem. Since $\{vf_n(1-f_n)\}_{n\in\mathbb{N}}$ converges strongly towards $vg(1-g)$ in $L^1((0,1)\times\mathbb{R}^d)$ by (1.34) and (1.38), we conclude that $\{\nabla_v f_n\}_{n\ge 0}$ is weakly relatively compact in $L^1((0,1)\times\mathbb{R}^d)$. Upon extracting a further subsequence, we may thus assume that $\{\nabla_v f_n\}_{n\ge 0}$ converges weakly towards $\nabla_v g$ in $L^1((0,1)\times\mathbb{R}^d)$. Consequently,

$$\int_0^1\int_{\mathbb{R}^d} |vg(1-g) + \nabla_v g|\,\mathrm{d}v\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^1\int_{\mathbb{R}^d} |vf_n(1-f_n) + \nabla_v f_n|\,\mathrm{d}v\,\mathrm{d}t = 0$$

by (1.39), from which we readily deduce that $vg(1-g) + \nabla_v g = 0$ a.e. in $(0,1)\times\mathbb{R}^d$. Since $\|g(t)\|_{L^1} = M$ for each $t \in [0,1]$ by Lemma 1.10 and (1.38), standard arguments allow us to conclude that $g(t) = F$ for each $t \in [0,1]$. We have thus proved that F is the only possible cluster point in $L^1(\mathbb{R}^d)$ of $\{f(t)\}_{t\ge 0}$ as $t \to \infty$, which, together with the relative compactness of $\{f(t)\}_{t\ge 0}$ in $L^1(\mathbb{R}^d)$, implies the assertion of Theorem 1.19. □

By now, we have seen that the entropy of the solution of (1.1) with initial condition f_0 tends to the one of the Fermi-Dirac distribution with the same mass as f_0 as $t \to \infty$, but we are also interested in how fast this happens. We will answer that question with the next result, which was already proved in [46] in the one dimensional case, and easily extends to any dimension based on the existence and entropy decay results above.

Theorem 1.21 (Entropy Decay Rate). *Assume that f is a solution to the Cauchy problem* (1.1) *with initial data in* $\mathrm{L}^1_{mp}(\mathbb{R}^d)$, $p > \max(d,2)$, $m \geq 1$ *such that* $0 \leq f_0 \leq F_{M^*} \leq 1$ *with* F_{M^*} *a Fermi-Dirac distribution of mass* M^**. Then the global in time solution of the Cauchy problem* (1.1) *with initial data* f_0 *satisfies*

$$H(f) - H(F) \leq (H(f_0) - H(F))\mathrm{e}^{-2Ct} \tag{1.40}$$

and

$$\|f(t) - F\|_{L^1(\mathbb{R}^d)} \leq C_2(H(f_0) - H(F))^{1/2}\mathrm{e}^{-Ct} \tag{1.41}$$

for all $t \geq 0$*, where* C *depends on* M^* *and* F *being the Fermi-Dirac Distribution such that* $\|F\|_1 = \|f_0\|_1$.

Proof.- Since $0 \leq f_0 \leq F_{M^*}$, then the initial data satisfy all the hypotheses of Theorems 1.14 and Theorem 1.19. In order to show, the exponential convergence, we use the same arguments as in [46]. We first remark that the entropy functional H coincides with the one introduced in [41] for the nonlinear diffusion equation

$$\frac{\partial g}{\partial t} = \mathrm{div}_x \left[g \nabla_x \left(x + h(g) \right) \right] \tag{1.42}$$

for the function $0 \leq g(t,x) \leq 1$, $x \in \mathbb{R}$, $t > 0$, where $h(g) = s'(g) = \log g - \log(1-g)$. Let us point out that the relation between the entropy dissipation for the solutions of the nonlinear diffusion equation (1.42), given by

$$-D_0(g) = \frac{\partial}{\partial t} H(g) = -\int_{\mathbb{R}^d} g \left| x + \frac{\partial}{\partial x} h(g) \right|^2 dx \tag{1.43}$$

and the entropy dissipation for the solution of (1.1), given by (1.30), is the basic idea of this proof. Indeed, $h(f)$ verifies, if we restrict to $f \in (0,1)$, the hypotheses of the Generalised Log-Sobolev Inequality [41, thm 17]. The Generalised Log-Sobolev Inequality asserts then that

$$H(g) - H(F_M) \leq \frac{1}{2} D_0(g) \tag{1.44}$$

for all integrable positive g with mass M for which the right-hand side is well-defined and finite. We can now, by the same regularization argument as before, compare the entropy dissipation $D(f) = -\frac{d}{dt}H(f)$ of equation (1.1) and the one $D_0(f)$ of equation (1.42). Thanks to Lemma 1.11 we have $f(t,v) \leq F_{M^*}(v) \leq (\beta^* + 1)^{-1}$ a.e. in $\mathbb{R}$, and thus

$$D(f) = \int_{\mathbb{R}^d} f(1-f) \left| v + \nabla_v h(f) \right|^2 dv \geq C \int_{\mathbb{R}^d} f \left| v + \nabla_v h(f) \right|^2 dv \tag{1.45}$$

where $C = 1 - (\beta^* + 1)^{-1}$. Applying the Generalised Log-Sobolev Inequality (1.44) to the solution $f(t)$ and taking into account previous estimates, we conclude

$$H(f(t)) - H(F_M) \leq (2C)^{-1} D(f(t)). \tag{1.46}$$

Finally, coming back to the entropy evolution:

$$\frac{d}{dt}\left[H(f(t)) - H(F)\right] = -D(f(t)) \leq -2C\left[H(f(t)) - H(F)\right],$$

and the result follows from Gronwall's lemma. The convergence in L^1 is obtained by a Csiszar-Kullback type inequality proven in [46, Corollary 4.3], whose proof is valid for any dimension and it is a consequence of a direct application of Taylor theorem to the relative entropy $H(f) - H(F)$ obtaining:

$$\|f - F\|^2_{L^1(\mathbb{R}^d)} \leq 2M(H(f) - H(F_M)). \tag{1.47}$$

For the sake of completeness we will sketch it here.
First of all we give an equivalent expression of the relative entropy:

$$H(f|F) := \int_{\mathbb{R}^d} \left[s(f) - s(F) - s'(F)(f - F)\right] \, dv = H(f) - H(F).$$

Then applying Taylor theorem to it, we obtain:

$$H(f|F) \geq \frac{1}{2}\int_{\mathbb{R}^d} s''(\xi(t,v))(f-F)^2 \, dv \geq \frac{1}{2}\int_{S_\infty} s''(\xi(t,v))(f-F)^2 \, dv$$

where $S_\infty = \{v \in \mathbb{R}^d \text{ such that } f(t,v) \leq F(v)\}$ and $\xi(t,v)$ lies on the interval between $f(t,v)$ and $F(v)$. Now, a direct Cauchy-Schwartz inequality gives

$$\begin{aligned}\|f - F\|^2_{L^1(S_\infty)} &\leq \left(\int_{S_\infty} \frac{1}{s''(\xi(t,v))} \, dv\right)\left(\int_{S_\infty} s''(\xi(t,v))(f-F)^2 \, dv\right) \\ &\leq 2\left(\int_{S_\infty} F(v)\, dv\right) H(f|F) \leq 2MH(f|F).\end{aligned} \tag{1.48}$$

Taking into account that $f(t,v)$ and $F(v)$ have equal mass, then

$$\|f - F\|_{L^1(\mathbb{R}^d)} = 2\|f - F\|_{L^1(S_\infty)}. \tag{1.49}$$

Inequality (1.41) is obtained putting together (1.48) and (1.49).

□

Propagation of Moments and Consequences

In this section, we shall see that under more restrictive conditions on the initial data there exists a uniform in time bound for the moments of f, being f the solution of equation (1.1) with initial data f_0. In this new situation, we will be able to prove the convergence of the entropy of f to the one of the steady state without imposing that f_0 is under a Fermi-Dirac, although we are not able to get a decay rate.

Lemma 1.22 (Time independent bound for Moments). *Let $g_0 \in \mathrm{L}^1_{mp}$ with $m \geq 1$, $p > \max(d, 2)$ such that $0 \leq g_0 \leq 1$, and assume further that g_0 is a radially symmetric and non-increasing function, i.e., there is a non-increasing function φ_0 such that $g_0(v) = \varphi_0(|v|)$. Then, the unique solution g in X_T is a radially symmetric non-increasing function and the control of moments propagates in time, i.e.,*

$$\lim_{R\to\infty} \sup_{t\geq 0} \int_{\{|v|\geq R\}} |v|^{mp} g(t,v)\mathrm{d}v = 0. \tag{1.50}$$

Proof.- We have already seen in Corollary 1.16 the existence of g. Furthermore, we have that its moments are given by

$$M := \int_{\mathbb{R}^d} f(t,v)\,\mathrm{d}v = d\omega_d \int_0^\infty r^{d-1}\varphi(t,r)\,\mathrm{d}r \tag{1.51}$$

and

$$\int_{\mathbb{R}^d} |v|^{mp} f(t,v)\,\mathrm{d}v = d\omega_d \int_0^\infty r^{d+mp-1}\varphi(t,r)\,\mathrm{d}r \tag{1.52}$$

for $t \geq 0$, where ω_d denotes the volume of the unit ball of $\mathbb{R}^d$.
Next, since $|v|^{mp} f_0 \in \mathrm{L}^1(\mathbb{R}^d)$, the map $v \to |v|^{mp}$ belongs to $\mathrm{L}^1(\mathbb{R}^d; f_0(v)\,dv)$ and a refined version of De la Vallée-Poussin theorem [67, 52] ensures that there is a non-decreasing, non-negative and convex function $\psi \in \mathcal{C}^\infty([0,\infty))$ such that $\psi(0) = 0$, ψ' is concave,

$$\lim_{r\to\infty} \frac{\psi(r)}{r} = \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \psi(|v|^{mp}) f_0(v)\,\mathrm{d}v < \infty. \tag{1.53}$$

Observe that, since $\psi(0) = 0$ and $\psi'(0) \geq 0$, the convexity of ψ and the concavity of ψ' ensure that for $r \geq 0$

$$r\psi''(r) \leq \psi'(r) \qquad \text{and} \qquad \psi(r) \leq r\psi'(r). \tag{1.54}$$

Then, after integration by parts, it follows from (1.22) that

$$\begin{aligned} &\frac{1}{mp}\frac{d}{dt}\int_0^\infty \psi(r^{mp}) r^{d-1}\varphi\,\mathrm{d}r \\ &\qquad = -\int_0^\infty r^{mp-1}\psi'(r^{mp})\left(r^{d-1}\frac{\partial}{\partial r}\varphi + r^d\varphi(1-\varphi)\right)\mathrm{d}r = I_1 + I_2 \end{aligned} \tag{1.55}$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \varphi \left[(mp+d-2) r^{mp+d-3} \psi'(r^{mp}) + mp r^{2mp+d-3} \psi''(r^{mp}) \right] \mathrm{d}r \\ I_2 &= -\int_0^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi (1-\varphi) \, \mathrm{d}r. \end{aligned}$$

We now fix $R > 0$ such that $\omega_d R^d \geq 4M$ and $R^2 \geq 4(2mp+d-2)$, and note that due to the monotonicity of φ with respect to r and (1.51)-(1.52) the inequality

$$M \geq d\omega_d \int_0^R r^{d-1} \varphi \mathrm{d}r \geq \omega_d R^d \varphi(R) \tag{1.56}$$

holds. Therefore, we first use the monotonicity of ϕ' and φ together with (1.56) to obtain

$$\begin{aligned} I_2 &\leq -\int_R^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi (1-\varphi) \, \mathrm{d}r \\ &\leq (\varphi(R) - 1) \int_R^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r \\ &\leq \left(\frac{M}{\omega_d R^d} - 1 \right) \int_R^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r \\ &\leq -\frac{3}{4} \int_R^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r \\ &\leq \frac{3}{4} \int_0^R r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r - \frac{3}{4} \int_0^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r \\ &\leq \frac{3MR^{mp}\psi'(R^{mp})}{4d\omega_d} - \frac{3}{4} \int_0^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r. \end{aligned}$$

On the other hand, from (1.51),(1.52), (1.54), (1.56) and the monotonicity of ϕ'

$$\begin{aligned} I_1 &\leq (d+2mp-2) \int_0^\infty r^{d+mp-3} \psi'(r^{mp}) \varphi \, \mathrm{d}r \\ &\leq (d+2mp-2) \psi'(R^{mp}) R^{mp-2} \int_0^R r^{d-1} \varphi \, \mathrm{d}r \\ &\quad + \frac{d+2mp-2}{R^2} \int_R^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r \\ &\leq \frac{(d+2mp-2)\psi'(R^{mp}) R^{mp-2} M}{d\omega_d} + \frac{1}{4} \int_R^\infty r^{d+mp-1} \psi'(r^{mp}) \varphi \, \mathrm{d}r. \end{aligned}$$

Inserting these bounds for I_1 and I_2 in (1.55) and using (1.54) we end up with

$$\begin{aligned}&\frac{1}{mp}\frac{d}{dt}\int_0^\infty \psi(r^{mp})r^{d-1}\varphi\,\mathrm{d}r\\ &\leq \frac{\psi'(R^{mp})MR^{mp-2}}{d\omega_d}\left(\frac{3R^2}{4}+d+2mp-2\right)-\frac{1}{2}\int_0^\infty r^{d+mp-1}\psi'(r^{mp})\varphi\,\mathrm{d}r\\ &\leq \frac{\psi'(R^{mp})MR^{mp-2}}{d\omega_d}\left(\frac{3R^2}{4}+d+2mp-2\right)-\frac{1}{2}\int_0^\infty r^{d-1}\psi(r^{mp})\varphi\,\mathrm{d}r.\end{aligned}$$

We then use the Gronwall lemma to conclude that there exists $C>0$ depending on d, M, g_0 and ψ such that

$$\sup_{0\leq t\leq T}\int \psi(|v|^{mp})g(t,v)\mathrm{d}v \leq C$$

from which (1.50) readily follows by (1.53). □

Theorem 1.23 (Entropy Convergence). *Let f be a solution for the Cauchy problem* (1.1) *with initial data $f_0 \in \mathrm{L}^1_{\mathrm{mp}}(\mathbb{R}^{\mathrm{d}})$ such that there exists a radially symmetric function g_0 with bounded 2-moment and non-increasing, with $0 \leq f_0 \leq g_0 \leq 1$. Then $H(f) \to H(F_M)$ as $t\to\infty$, where F_M is the Fermi-Dirac distribution with the same mass as f_0.*

Proof.-
On one hand, we know that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \to F_M$ in L^1, so for any function $\psi \in \mathcal{C}_0^\infty$

$$\int_{\mathbb{R}^d} f_n(t,v)\psi\mathrm{d}v \to \int_{\mathbb{R}^d} F_M(v)\psi\mathrm{d}v.$$

Since also $|v|^2\psi \in \mathcal{C}_0^\infty$ for any $\psi\in\mathcal{C}_0^\infty$ we can ensure that $|v|^2 f(t) \rightharpoonup |v|^2 F_M$. We want actually to show that

$$\int_{\mathbb{R}^d} |v|^2 f_n(t,v)\mathrm{d}v \to \int_{\mathbb{R}^d} |v|^2 F_M(v)\mathrm{d}v.$$

As in the proof of Theorem 1.19, let us fix $R>0$, and consider a “mesa” smooth function with

$$\varphi_R = \begin{cases} 1 & |v|\leq R\\ 0 & |v|\geq R+1\end{cases} \quad \text{and } 0\leq\varphi\leq 1.$$

Then, we can estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^d}|v|^2(f_n-F_M)\mathrm{d}v\right| &= \left|\int_{\mathbb{R}^d}|v|^2(f_n-F_M)(1-\varphi_R+\varphi_R)\mathrm{d}v\right| \\ &\leq \int_{\mathbb{R}^d}|v|^2|f_n-F_M|(1-\varphi_R)\mathrm{d}v+\left|\int_{\mathbb{R}^d}|v|^2(f_n-F_M)\varphi_R\mathrm{d}v\right| \\ &\leq \frac{C}{R^{\gamma-2}}+\left|\int_{\mathbb{R}^d}|v|^2(f_n-F_M)\varphi_R\mathrm{d}v\right|\end{aligned}$$

Since we have

$$\int_{\mathbb{R}^d}|v|^2|f_n-F_M|(1-\varphi_R)\mathrm{d}v \leq \frac{1}{R^{\gamma-2}}\int_{|v|\geq R}|v|^\gamma|f_n-F_M|\mathrm{d}v \leq \frac{C}{R^{\gamma-2}},$$

so $E(f_n)$ tends to $E(F_M)$ as $n\to\infty$.
On the other hand, the entropy functional H is decreasing and bounded from below by $H(F_M)$, whence $E(f_n)+S(f_n)$ converges to $H^*\geq H(F_M)$, which we can view as

$$\lim_{n\to\infty} S(f_n)=H^*-E(F_M)\geq S(F_M).$$

Now, Fatou's lemma tells us that

$$\int_{\mathbb{R}^d}\liminf_{n\to\infty}(-s(f_n))\mathrm{d}v \leq \liminf_{n\to\infty}\int_{\mathbb{R}^d}(-s(f_n))\mathrm{d}v,$$

and since the limits exist we can read this inequality as $S(F_M)\geq \lim_{n\to\infty} S(f_n)$, whence $\lim_{n\to\infty} S(f_n)=S(F_M)$ and

$$\lim_{n\to\infty} H(f_n)=H(F_M).$$

□

1.1.3 Bose-Einstein-Fokker-Planck equation in one Dimension

In this section we focus on the analysis of the large-time behavior of solutions of the Cauchy problem:

$$\frac{\partial f}{\partial t}=\frac{\partial^2}{\partial v^2}f+\frac{\partial}{\partial v}[vf(1+f)], \qquad v\in\mathbb{R}, t>0, \tag{1.57}$$

for bosons in the one-dimensional case, with initial data

$$f(v,0)=f_0(v). \tag{1.58}$$

As we have seen in the previous part of this work, we can assume that we are dealing with smooth positive fast-decaying solutions of equation (1.57). Then, we first check that stationary solutions for equation (1.57) coincide with the Bose-Einstein distribution:

Lemma 1.24. *Let F be an integrable, strictly positive, stationary solution for Equation* (1.57). *Then*

$$B(v) = \frac{1}{\beta e^{\frac{v^2}{2}} - 1}.$$

Moreover, for each value of the mass $M > 0$, there exists a unique $\beta = \beta(M) \geq 1$ such that $F(v)$ has mass M.

Proof.- We consider the stationary version of Equation (1.57):

$$\frac{\partial^2}{\partial v^2} f + \frac{\partial}{\partial v}[vf(1+f)] = 0,$$

that can be written in the form

$$\frac{\partial}{\partial v}\left\{ f(1+f)\left[\frac{1}{f(1+f)}\frac{\partial f}{\partial v} + \frac{\partial}{\partial v}\left(\frac{v^2}{2}\right)\right]\right\} = 0,$$

or equivalently,

$$\frac{\partial}{\partial v}\left\{ f(1+f)\frac{\partial}{\partial v}\left[\log\left(\frac{f}{1+f}\right) + \frac{v^2}{2}\right]\right\} = 0.$$

Since the solution is smooth fast-decaying previous equation implies that

$$\frac{\partial}{\partial v}\left[\log\left(\frac{f}{1+f}\right) + \frac{v^2}{2}\right] = 0$$

from which we analytically obtain the stationary solution to Equation (1.57):

$$B(v) = \frac{1}{\beta e^{\frac{v^2}{2}} - 1}$$

with $\beta \geq 1$. Now, it is easy to check that these stationary solutions are integrable for all $\beta > 1$, and moreover, the map $M(\beta) : \beta \in (1, \infty) \longrightarrow (0, \infty)$ given by:

$$M(\beta) = \int_{\mathbb{R}} \frac{1}{\beta e^{\frac{v^2}{2}} - 1}\, dv$$

is decreasing, surjective and invertible. □

The asymptotic behavior of solutions of (1.57) follows from the same kind of entropy arguments we have used in the fermion case. Thus, we start by defining the entropy of f as in previous chapter by

$$H(f) = \int_{\mathbb{R}} \left[\frac{v^2}{2} f + \Phi(f) \right] dv \tag{1.59}$$

where

$$\Phi(f) = f \log(f) - (1+f) \log(1+f) \tag{1.60}$$

which acts as a Lyapunov functional for the system, namely:

Proposition 1.25 (H-theorem). *The functional H defined on the set of positive integrable functions with given mass M attains its unique minimum at $F(v)$. Moreover, given any solution to* (1.57) *with initial data f_0 of mass M, we have*

$$H(F) \le H(f(t)) \le H(f_0) \tag{1.61}$$

for all $t \ge 0$.

Proof.- We first recall that the entropy functional coincides with (1.42) choosing now $h(g) = \Phi'(g)$. The nonlinear diffusion defining this equation verifies all hypotheses needed in [41, Proposition 5], that implies the first statement of this proposition. Let us remark that the minimizing character of the Bose-Einstein distributions for this entropy is also a consequence of the results in [77, 165]. Concerning the second part, we can compute, as we did in the fermion case, the evolution of the entropy functional along solutions getting

$$-D(f) := \frac{\partial}{\partial t} H(f) = - \int_{\mathbb{R}} f(1+f) \left[v + \frac{\partial}{\partial v} h(f) \right]^2 dv \le 0 \tag{1.62}$$

where $D(f)$ is by definition the entropy dissipation for equation (1.57). □

In one dimension, the a-priori estimates that we have already reckoned for the fermions also holds for bosons. By using them, and recalling that the convergence of the entropy is derived from the relation between (1.43) and (1.62), we can show the next results

Theorem 1.26. *Let f be a solution for* (1.57) *and $F_{\infty,M}$ be the stationary state of the solution with the same mass M. Then*

$$H(f) - H(F) \le (H(f_0) - H(F)) \mathrm{e}^{-2t} \tag{1.63}$$

for all $t \ge 0$.

Proof.- We can see that $h(f)$ given by $\Phi'(f)$ also verifies in one dimension the hypotheses of the Generalized Log-Sobolev Inequality [41, thm 17]. Due to the Generalized Log-Sobolev Inequality (1.44) the right-hand side is well-defined and finite. We can now compare the entropy dissipation $D(f)$ of equation (1.57) and the one $D_0(f)$ of equation (1.42):

$$D(f) = \int_{\mathbb{R}} \left(f + f^2\right) \left[v + \frac{\partial}{\partial v} h(f)\right]^2 dv \geq \int_{\mathbb{R}} f \left[v + \frac{\partial}{\partial v} h(f)\right]^2 dv. \tag{1.64}$$

Applying the Generalized Log-Sobolev Inequality (1.44) to the solution $f(t)$ and taking into account a-priori estimates, we conclude

$$H(f(t)) - H(F) \leq \frac{1}{2} D(f(t)). \tag{1.65}$$

Finally, coming back to the entropy evolution:

$$\frac{d}{dt}\left[H(f(t)) - H(F)\right] = -D(f(t)) \leq -2\left[H(f(t)) - H(F)\right],$$

and the result follows from Gronwall's lemma. □

Now, we can try to give more accurate convergence properties by reckoning rates of decay for the entropy dissipation:

$$D(f) = \int_{\mathbb{R}} f(1+f)\xi^2 \, dv$$

where $\xi = v + \partial_v h(f)$. Computing the evolution of the dissipation of the entropy in time, we deduce

$$DD(f) = \frac{d}{dt} D(f) = \int_{\mathbb{R}} (1+2f) \frac{\partial f}{\partial t} \xi^2 \, dv + 2 \int_{\mathbb{R}} f(1+f)\xi \frac{\partial \xi}{\partial t} \, dv = (I) + (II)$$

Integrating (II) by parts, we obtain that

$$\text{(II)} = -2 \int_{\mathbb{R}} \frac{1}{f(1+f)} \left(\frac{\partial}{\partial v}[f(1+f)\xi]\right)^2 dv$$

Using again integration by parts with (I) and repeating the process for the term with $\frac{\partial}{\partial v}(1+2f)$ we obtain

$$\begin{aligned} \text{(I)} = &- 2 \int_{\mathbb{R}} \left(f + \frac{3}{2} f^2 + f^3\right) \xi^2 \frac{\partial \xi}{\partial v} \, dv \\ = &- 2 \int_{\mathbb{R}} \varphi_1(f) \xi^2 \, dv + 2 \int_{\mathbb{R}} \varphi_2'(f) \left(\frac{\partial f}{\partial v}\right)^2 f(1+f)\xi^2 \, dv \\ &+ 4 \int_{\mathbb{R}} \varphi_2(f) \xi \frac{\partial f}{\partial v} \frac{\partial}{\partial v}[f(1+f)\xi] \, dv \end{aligned}$$

where we have considered

$$\varphi_1(f) = f + \frac{3}{2}f^2 + f^3 \qquad \text{and} \qquad \varphi_2(f) = \frac{\varphi_1(f)}{(f(1+f))^2}.$$

Finally, we have

$$DD(f) = -2\int_{\mathbb{R}} \varphi_1(f)\xi^2\, dv - 2\int_{\mathbb{R}} (B - \varphi_2(f)A)^2 dv + 2\int_{\mathbb{R}} \left[\varphi_2(f)^2 + \varphi_2'(f)\right] A^2$$

where

$$A := \xi \frac{\partial f}{\partial v}\sqrt{f(1+f)} \qquad \text{and} \qquad B := \frac{\frac{\partial}{\partial v}[f(1+f)\xi]}{\sqrt{f(1+f)}}.$$

It is easy to show then, that $[\varphi_2(f)^2 + \varphi_2'(f)] \leq 0$, so the last term in $DD(f)$ is negative, and we get

$$DD(f) \leq -2\int_{\mathbb{R}} \varphi_1(f)\xi^2\, dv \leq -2D_1(f) \tag{1.66}$$

since, we have $\varphi_1(f) \geq f(1+f)$. We conclude:

Proposition 1.27 (Entropy Dissipation Decay for bosons). *Let f be a solution for* (1.57). *Then, for all $t \geq 0$,*

$$D(f(t)) \leq D(f_0)\mathrm{e}^{-2t}.$$

Finally, we will remark that due to mass conservation and positivity of the stationary states F, as a consequence of the entropy convergence we have also a Csiszar-Kullback type inequality for bosons.

1.1.4 $\mathrm{L^p_m}$-bounds of the Fokker-Planck Operator

Here we follow similar arguments in [85] to show some bounds for $\|\partial_\alpha \mathcal{F}[f](t)\|_{\mathrm{L}^p_m}$ which will be useful in the fixed point argument in section 1.1.1.

Proposition 1.28. *Let $1 \leq p \leq \infty$, $m \geq 0$ and $\alpha \in \mathbb{N}^d$. Then for $t > 0$,*

$$\|(\partial_\alpha \mathcal{F}[f])(t)\|_{\mathrm{L}^p_m} \leq C\mathrm{e}^{(\frac{d}{p'}+|\alpha|)t}\nu(t)^{-|\alpha|/2}\|f\|_{\mathrm{L}^p_m} \tag{1.67}$$

with $|\alpha| = max\{\alpha_1, ..., \alpha_N\}$.

Proof.- For all $\alpha \in \mathbb{N}^d$, we have

$$\begin{aligned} \left(\partial_\alpha \mathcal{F}[f]\right)(t) &= \partial_\alpha \int_{\mathbb{R}^d} \left(\frac{\mathrm{e}^{td}}{\left(2\pi\left(\mathrm{e}^{2t}-1\right)\right)^{\frac{d}{2}}} \mathrm{e}^{\frac{|\mathrm{e}^t v - w|^2}{2(\mathrm{e}^{2t}-1)}} \right) f(w)\,\mathrm{d}w \\ &= \partial_\alpha \int_{\mathbb{R}^d} \left(\frac{\mathrm{e}^{2dt}}{\left(2\pi\left(\mathrm{e}^{2t}-1\right)\right)^{\frac{d}{2}}} \mathrm{e}^{\frac{|\mathrm{e}^t (v - w)|^2}{2(\mathrm{e}^{2t}-1)}} \right) f(\mathrm{e}^t w)\,\mathrm{d}w \\ &= \frac{\mathrm{e}^{t(2d+|\alpha|)}}{\nu(t)^{\frac{d+|\alpha|}{2}}} \int_{\mathbb{R}^d} \phi_\alpha \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) f(\mathrm{e}^t w)\,\mathrm{d}w \end{aligned} \tag{1.68}$$

where

$$\phi_\alpha(\chi) = \partial_\chi^\alpha \left(\phi_0\right)(\chi) = \mathcal{P}_{|\alpha|}(\chi)\phi_0(\chi),$$

being $\mathcal{P}_{|\alpha|}(\chi)$ a polynomial of degree $|\alpha|$ which we can recursively reckon by

$$\mathcal{P}_0(\chi) = 1,\, \mathcal{P}_{|\alpha|}(\chi) = \mathcal{P}'_{|\alpha|-1}(\chi) - \chi \mathcal{P}_{|\alpha|-1}(\chi) \text{ and } \phi_0(\chi) = (2\pi)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\chi|^2}{2}}.$$

Since $1 + |v|^m \le C(1 + |v-w|^m)(1 + |w|^m)$, we deduce

$$\begin{aligned} &(1+|v|^m)|(\partial_\alpha \mathcal{F} * f)(t)| \le \\ &\quad \le C \frac{\mathrm{e}^{t(2d+|\alpha|)}}{\nu(t)^{\frac{d+|\alpha|}{2}}} \int_{\mathbb{R}^d} (1 + |v-w|^m) \left| \phi_\alpha \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) \right| (1 + |w|^m) \left| f(\mathrm{e}^t w) \right| \mathrm{d}w. \end{aligned} \tag{1.69}$$

On one hand, we can compute

$$\int_{\mathbb{R}^d} (1+|v-w|^m) \left| \phi_\alpha \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) \right| \mathrm{d}w = C(I + II),$$

with

$$\begin{aligned} I &= \int_{\mathbb{R}^d} \mathcal{P}_{|\alpha|} \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) \phi_0 \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) \mathrm{d}w = \frac{\nu(t)^{d/2}}{\mathrm{e}^{dt}} \int_{\mathbb{R}^d} \mathcal{P}_{|\alpha|}(\chi)\phi_0(\chi)\,\mathrm{d}\chi \\ &= C_1 \frac{\nu(t)^{d/2}}{\mathrm{e}^{dt}} \end{aligned}$$

and

$$\begin{aligned} II &= \int_{\mathbb{R}^d} |v-w|^m \mathcal{P}_{|\alpha|} \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) \phi_0 \left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}} \right) \mathrm{d}w \\ &= \frac{\nu(t)^{(d+m)/2}}{\mathrm{e}^{(d+m)t}} \int_{\mathbb{R}^d} |\chi|^m \mathcal{P}_{|\alpha|}(\chi)\phi_0(\chi)\mathrm{d}\chi = C_2 \frac{\nu(t)^{(d+m)/2}}{\mathrm{e}^{(d+m)t}}. \end{aligned}$$

Now, since $\nu(t)^{\frac{1}{2}}\mathrm{e}^{-t} < 1$ we have $I + II < C_3\nu(t)^{d/2}\mathrm{e}^{-dt}$, and hence

$$\frac{\mathrm{e}^{dt}}{\nu(t)^{d/2}}\int_{\mathbb{R}^d}(1+|v-w|^m)\left|\phi_\alpha\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)\right|\mathrm{d}w \leq C. \tag{1.70}$$

On the other hand, we get

$$\begin{aligned}\left\|(1+|w|^m)\left|f(\mathrm{e}^t w)\right|\right\|_p &= \left(\int(1+|w|^m)^p\left|f(\mathrm{e}^t w)\right|^p\mathrm{d}w\right)^{\frac{1}{p}}\\ &= \left(\int \mathrm{e}^{-dt}(1+|\mathrm{e}^{-t}\chi|^m)^p\left|f(\chi)\right|^p\mathrm{d}w\right)^{\frac{1}{p}}\\ &\leq \mathrm{e}^{-\frac{dt}{p}}\left(\int(1+|\chi|^m)^p\left|f(\chi)\right|^p\mathrm{d}w\right)^{\frac{1}{p}}.\end{aligned} \tag{1.71}$$

Putting together estimates (1.70) and (1.71) and applying Young's inequality with $r = p$ and $q = 1$ in (1.69), we get the desired result. ∎

Analogously, we can obtain the following generalization.

Proposition 1.29. *Let* $1 \leq q \leq p \leq \infty$, $m \geq 0$ *and* $\alpha \in \mathbb{N}^d$. *Then for* $t > 0$,

$$\|(\partial_\alpha\mathcal{F}[f])(t)\|_{\mathrm{L}^p_m} \leq \frac{C\mathrm{e}^{\left(\frac{d}{p'}+|\alpha|\right)t}}{\nu(t)^{\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{|\alpha|}{2}}}\|f\|_{\mathrm{L}^q_m}. \tag{1.72}$$

Proof.- Indeed, we can write

$$\int_{\mathbb{R}^d}(1+|v-w|^m)^r\left|\phi_\alpha\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)\right|^r\mathrm{d}w = C(I+II)$$

with

$$\begin{aligned}I &= \int\mathcal{P}^r_{|\alpha|}\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)\phi_0\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)^r\mathrm{d}w = \frac{\nu(t)^{d/2}}{\mathrm{e}^{dt}}\int\mathcal{P}^r_{|\alpha|}(\chi)\phi_0(\chi)^r\\ &= C_1\frac{\nu(t)^{d/2}}{\mathrm{e}^{dt}}\end{aligned}$$

and

$$\begin{aligned}II &= \int|v-w|^{mr}\mathcal{P}^r_{|\alpha|}\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)\phi_0\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)^r\mathrm{d}w\\ &= \frac{\nu(t)^{(d+mr)/2}}{\mathrm{e}^{(d+mr)t}}\int|\chi|^{mr}\mathcal{P}^r_{|\alpha|}(\chi)\phi_0(\chi)^r = C_2\frac{\nu(t)^{(d+mr)/2}}{\mathrm{e}^{(d+mr)t}},\end{aligned}$$

and likely to (1.70), we conclude

$$\frac{\mathrm{e}^{dt}}{\nu(t)^{d/2}}\int_{\mathbb{R}^d}(1+|v-w|^m)^r\left|\phi_\alpha\left(\frac{v-w}{\mathrm{e}^{-t}\nu(t)^{1/2}}\right)\right|^r \mathrm{d}w \le C. \tag{1.73}$$

Putting (1.73) together with (1.71), we can use Young's inequality in (1.69) as before, since $1 \le q \le p$ with r given by $\frac{1}{p}+1=\frac{1}{r}+\frac{1}{q}$ to get the desired bound. □

1.1.5 Scharfetter-Gummel-type discretization of the Fokker-Planck Equation for Bosons and Fermions

In this section we want to give a numerical scheme to solve equation (1.1) in dimension three. For simplicity we will assume, like in [53], that the solutions we are dealing with are spherically symmetric. Hence, we write (1.1) in its radial version

$$f_t - v^{-2}(v^2(f_v + vf(1+\kappa f)))_v = 0 \quad \forall v \in [0,\infty), k = \pm 1 \tag{1.74}$$

$$f_v|_{v=0} = 0 \quad f(0,v) = f_0(v)\ \forall x \in [0,\infty), \tag{1.75}$$

where we use the simplified notation $f_t = \frac{\partial}{\partial t}f(t,v)$ and a subindex v means that we are taking the derivative with respect to v of the expression by which it is preceded. Also, we have already set $N = 3$. We will discretize it by a semi-implicit scheme as follows:
First we denote $f(t_n, v_i)$, where $t_n = n\Delta t$, and $v_i = i\Delta v$ by f_i^n, and abbreviate $f_i^* = f_i^{n+1}$. Then we rewrite (1.74) as

$$v^2 f_t = (v^2 J)_v,$$

where we have written J for $f_v + vf(1+\kappa f)$, and by finite differences we obtain

$$v_i^2 \frac{f^*(v_i) - f^n(v_i)}{\Delta t} = \frac{1}{\Delta x}\left[v_{i+1/2}^2 J_{i+1/2} - v_{i+1/2}^2 J_{i-1/2}\right].$$

At this point we want to find a discretization for J_l, $l = 1 \pm 1/2$, using an extension of the Schafertter-Gummel type approximation, as suggested in [133] (see also [2, 83]). The main idea is to suppose that J is constant on the interval $\mathbb{I}_l = [v_{l-1/2}, v_{l+1/2})$ and solve for each time step $n+1$

$$J_l = f_v + gf \tag{1.76}$$

explicitly on $\mathbb{I}_l$, with g a numeric approximation of $v(1+\kappa f)$ in terms of the known values f^n. In this way, equation (1.76) can be handled as if it was a linear

ODE, whence in order to solve it we only need to multiply (1.76) by $e^{g(v-v_0)}$ and integrate over $\mathbb{I}_l$. Now, note that we can take both $v_0 = l - 1/2$ and $v_0 = l + 1/2$, each election of v_0 leading us to a different result:

$$J_l^- = \frac{g e^{g\Delta v}}{e^{g\Delta v} - 1} f^*_{l+1/2} - \frac{g}{e^{g\Delta v} - 1} f^*_{l.1/2}.$$

in the first case, and

$$J_l^+ = \frac{g e^{-g\Delta v}}{e^{-g\Delta v} - 1} f^*_{l-1/2} - \frac{g}{e^{-g\Delta v} - 1} f^*_{l+1/2},$$

in the second one. We will take J_l as the mean of these two values (i.e. $J_l = \frac{1}{2}(J_l^- + J_l^+)$). Then, if we define

$$B(z) = \frac{z}{\exp(z) - 1} \qquad \text{and} \qquad h = \Delta v,$$

by giving explicitly the approximation of g we are finally able to present a complete discretization of 1.74:

$$v_i^2 \frac{f_i^* - f_i^n}{\Delta t} = \frac{1}{h}(v^2_{i+1/2} J_{i+1/2} - v^2_{i-1/2} J_{i-1/2}) \qquad 1 \le i \le N \qquad (1.77)$$

$$J_l = \frac{1}{h}\big(B(-hg_l) f^*_{l+1/2} - B(+hg_l) f^*_{l-1/2}\big) \quad l = i \pm 1/2 \qquad (1.78)$$

$$g_l = v_l(1 + \kappa f_l^n), \;\; f_l^n = \frac{1}{2}(f^n_{j+1} + f^n_j) \qquad j = l - 1/2 \qquad (1.79)$$

From this discretization we can extract a tridiagonal $N \times (N+2)$ matrix which relates f^* with the values of f^n at the N interior points of the mesh:

$$\begin{pmatrix} f_1^n \\ \vdots \\ f_N^n \end{pmatrix} = \begin{pmatrix} * & * & * & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & * & * & * \end{pmatrix} \cdot \begin{pmatrix} f_0^* \\ f_1^* \\ \vdots \\ f_N^* \\ f_{N+1}^* \end{pmatrix}$$

From the boundary condition $f_v|_{v=0} = 0$ the restriction $f_1^* - f_0^*$ follows. Thus, using it will provide us with an extra row for our matrix, but we still lack one in order to be able to invert it and calculate f^*. This last one will come from the conservation of the mass.

Since f is a representation of a spherical symmetric function F, we may define the total mass in a ball $B_{v_N}(0)$ numerically by:

$$M_n = \sum_{i=0}^{N} 4\pi v_i^2 f_i^n,$$

then, since we want M_n to be constant in time, the approximation by finite differences of its time derivative will give us

$$\sum_{i=0}^{N+1} \frac{v_i^2(f_i^* - f_i^n)}{\Delta t} = 0$$

and extending the sum, it simplifies to

$$\frac{1}{\Delta t}\left(v_{N+1/2}^2 J_{N+1/2} - v_{1/2}^2 J_{1/2}\right) = 0.$$

Then, we just need to look at (1.78) to obtain the last relation we wanted

$$\frac{1}{\Delta t}\Bigg(B(-hg_{N+1/2})x_{N+1/2}^2 f_{N+1}^* - B(hg_{N+1/2})x_{N+1/2}^2 f_N^* \\ - B(-hg_{1/2})x_{1/2}^2 f_1^* + B(hg_{1/2})x_{1/2}^2 f_0^*\Bigg) = 0$$

from which we see, in addition, that when calculating the mass we have to restrict to points 0 to N, and use the point $N+1$ in the formulation of the mass conservation condition.
Finally, to conclude this first part of the chapter, let us show some examples obtained by running our scheme with different initial data, both for the boson and the fermion equation. Figures 1 to 4 correspond to the evolution of solutions of (1.1) (equation for fermions) with different initial data of respective masses 22.5566 and 1.0294×10^3. First picture of each figure shows the solution at different times so that it can be seen how it approaches its stationary state. Second and third pictures show respectively the difference between the entropies of the solutions and its stationary state, and the L^1-norm of the difference $(f - F)$ for each time step. In both cases the theoretically predicted bounds for these differences is also plotted. As seen in the proof of Theorem 1.21, the predicted convergence rate for fermions depends on the mass of the solutions. It is of order e^{-2Ct} for the entropy and the square root of it for the mass, with $C = 1 - (\beta + 1)^{-1}$. As the mass increases, β decreases to 0 and so does C. In the first examples, C is of order 0.1 while in the second it is of order 10^{-9}. For the bosons, since the numerical scheme is written in $\mathbb{R}^3$, there is a maximum mass for the steady states when β tends to 1: 41.0182. Figure 1.1.5 shows the solution of (1.57) for an initial condition of mass 26.6108, below the critical mass. In the the first picture the convergence of the solution to the steady state can be seen, then we see how the difference between their entropies and L^1-norm of the difference between them stop to decrease when we reach the precision of the scheme. We could improve that with a finer mesh,

although this solution is far from the optimal. In figure 1.1.5, the mass of the initial condition is 44.5218, which is slightly bigger than the critical. In this case we conjectured a convergence to the Bose-Einstein distribution with $\beta = 1$ plus a Dirac distribution on zero. It can be seen in the first picture that there is no stabilization in time and that the value at zero continues to increase up to the computed time. Here we use a logarithmic scale for the y axis in order to be able to appreciate the evolution of the solution, and in the second picture, where we zoom the plot around $t = 0$. Nevertheless, this kind of scheme is not the appropriate one to reproduce blow-up of solutions. This numerical part is still a work in progress where we expect to get better results in the boson case.

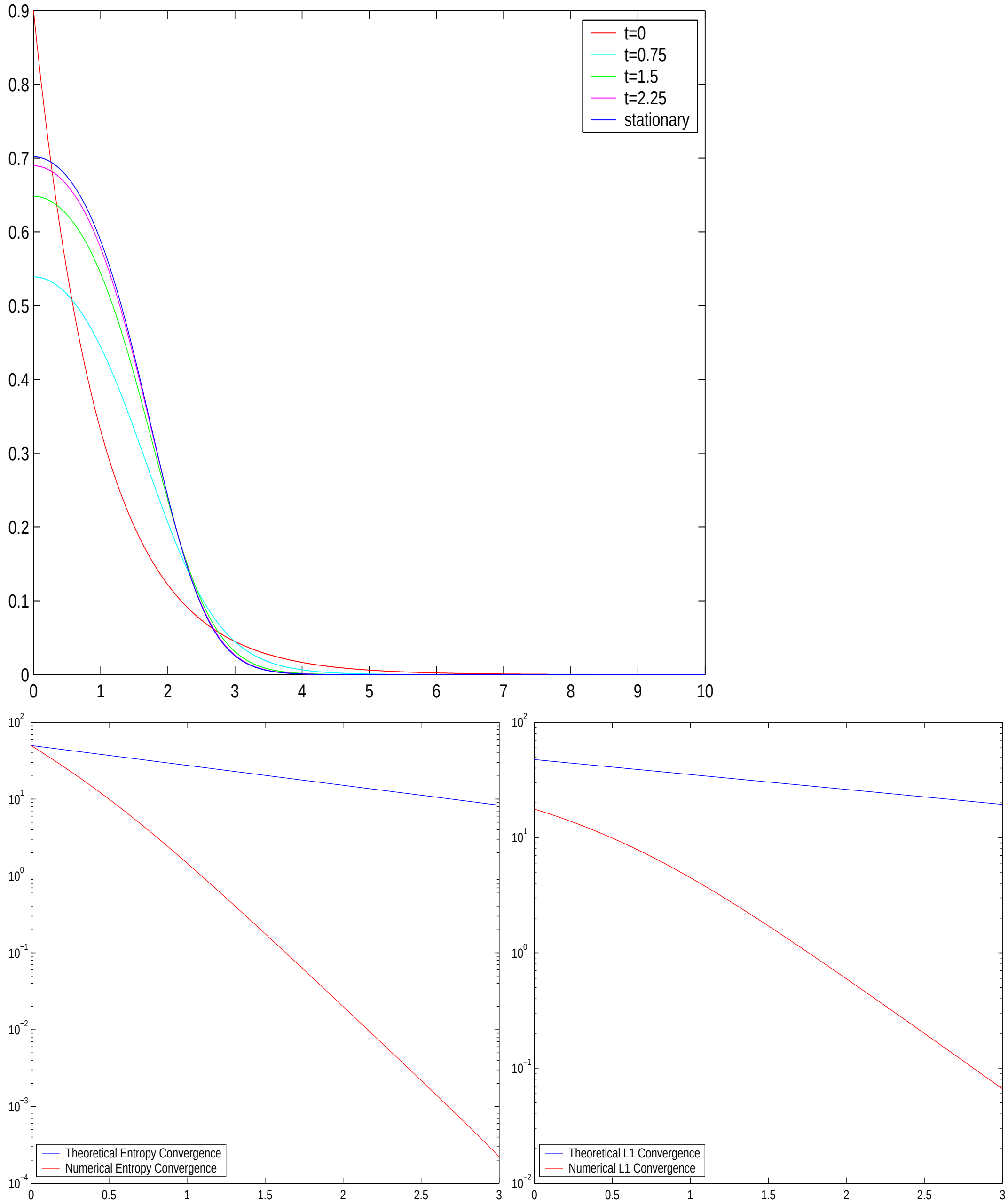

Figure 1.1: Initial data: $0.9e^{-v}$. Up: Some steps of the time evolution of the solution. Down, left: Comparison of the theoretical and numerical rate of convergence to the stationary state for the entropy; right: Comparison of the theoretical and numerical rate of convergence to the stationary state in the L^1-norm

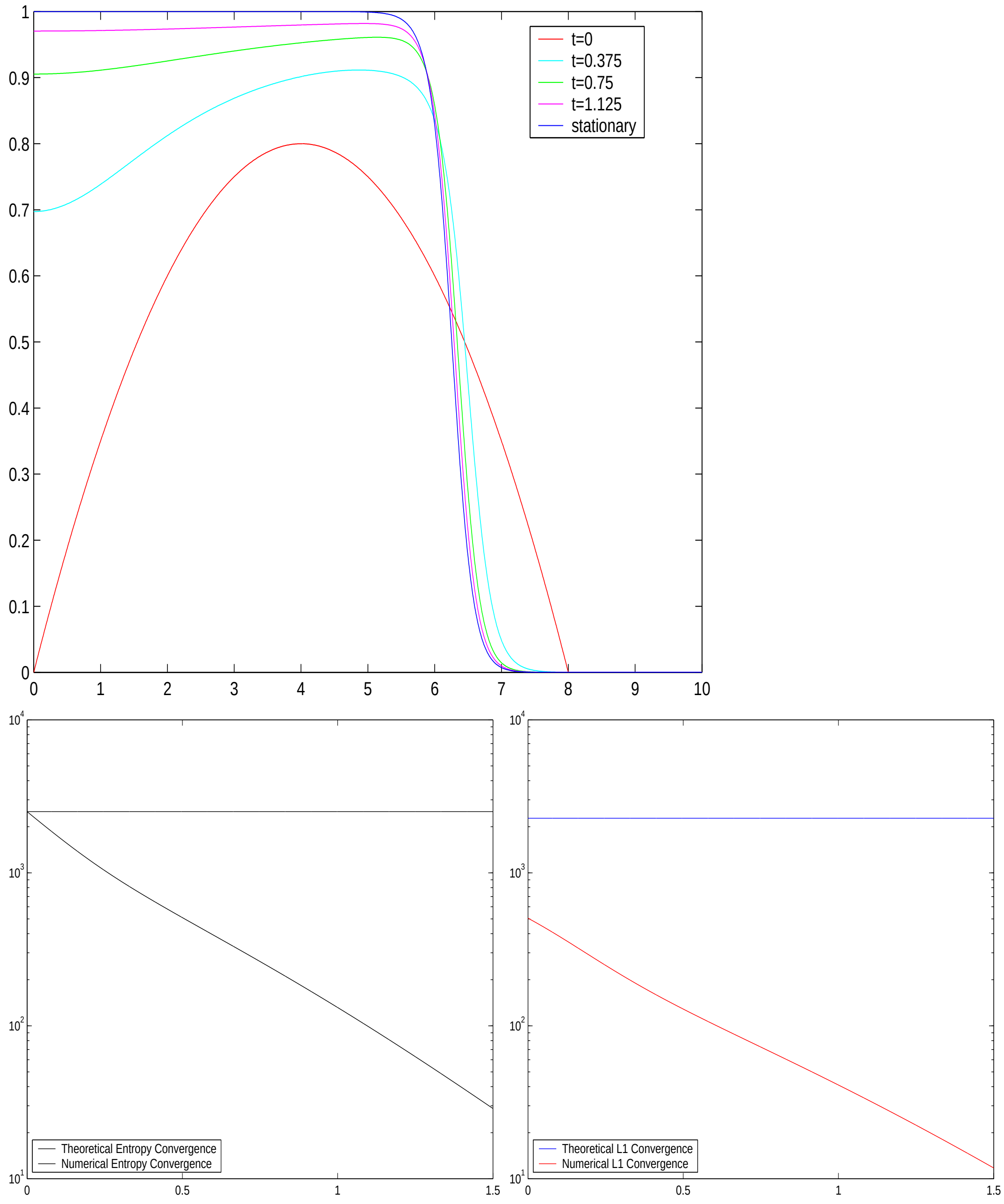

Figure 1.2: Initial data: $\max(0.05(-v^2 + 8v), 0)$. Up: Some steps of the time evolution of the solution. Down, left: Comparison of the theoretical and numerical rate of convergence to the stationary state for the entropy; right: Comparison of the theoretical and numerical rate of convergence to the stationary state in the L^1-norm

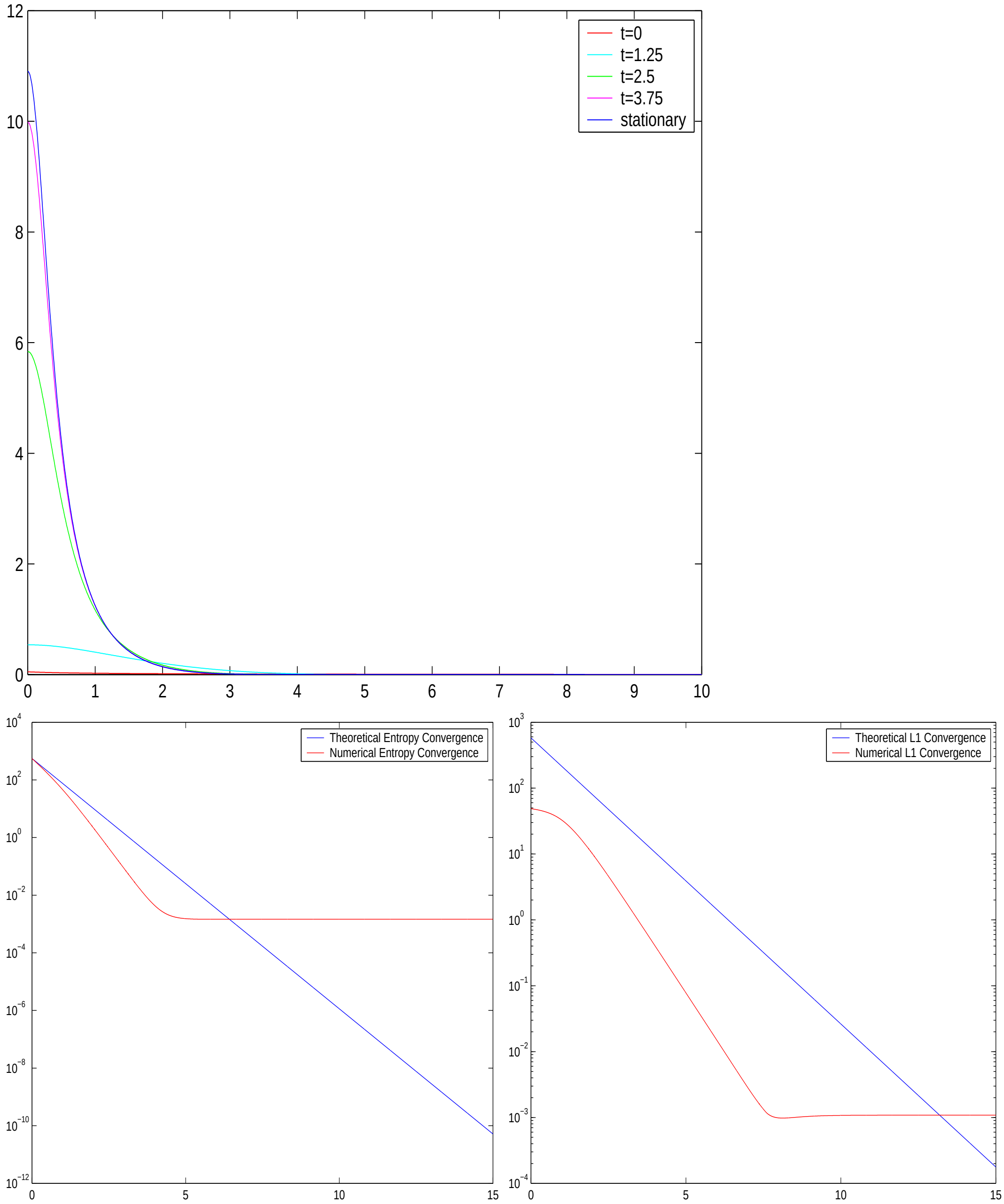

Figure 1.3: Initial data: $(20(v+1))^{-1}$. Up: Some steps of the time evolution of the solution. Down, left: Comparison of the theoretical and numerical rate of convergence to the stationary state for the entropy; right: Comparison of the theoretical and numerical rate of convergence to the stationary state in the L^1-norm

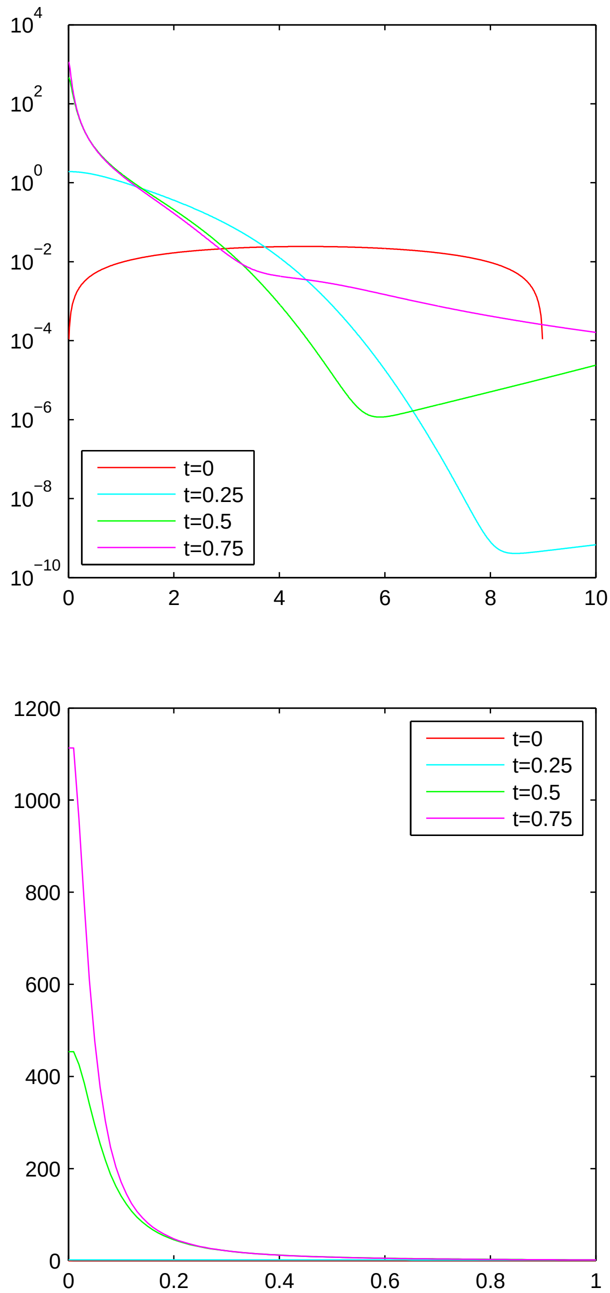

Figure 1.4: Initial data: $(20(v+1))^{-1}$. Some steps of the time evolution of the solution. Up: Plot in logarithmic scale. Down: Plot zoomed around zero.

1.2 The Keller-Segel model for chemotaxis

In this section we shall deal with another model based on the Fokker-Planck equations, but applied to completely different scenario than the previous one. We study the following parabolic system modeling *chemotaxis with prevention of overcrowding*

$$\begin{cases} \rho_t = \varepsilon\Delta\rho - \mathrm{div}(\rho(1-\rho)\nabla S) \\ S_t = \Delta S - S + \rho \end{cases} \tag{1.80}$$

Here ρ models the density of cells, S is the concentration of the chemical substance (chemoattractant). The parameter $\varepsilon > 0$ models the diffusivity of the cells. The present model is posed on the whole space $\mathbb{R}^d$ with $L^1 \cap L^\infty$ initial data for both ρ and S (plus some further assumption on S, see section 1.2.1 for more precise statements). In the sequel we shall present a brief overview of results in the literature concerning chemotaxis models, by justifying the variants included in (1.80).

Chemotaxis is the phenomenon by which cells move under the influence of chemical substances in their environment. It has been known and widely studied since first descriptions were done by T.W. Engelmann and W.F. Pfeffer for bacteria in 1881 and 1884, and H.S. Jennings for ciliates in 1906. First mathematical models based on partial differential equations arose from the works of C.S. Patlak in 1953, who derived similar models with applications to the study of long-chain polymers (cf. [148]) and E.F. Keller and L.A. Segel in 1970, who proposed a macroscopic model for aggregation of cellular slime molds (cf. [110]). Afterwards, several transport phenomena in biological systems have been labeled with the term *chemotaxis*, such as the bacteria *Escherichia coli*, or the amoebae *Dyctiostelium discoideum*, or endothelial cells of the human body responding to angiogenic factors secreted by a tumor. The main feature of these systems (in a very simplified form involving only two species) is the motion of a species ρ being biased by linear diffusion modeling random motion, with a diffusivity $\varepsilon > 0$ and by the gradient of a certain chemical substance S, whereas the flow of S features secretion/degradation mechanism without cross–diffusion. More precisely, one usually deals with solutions to the Cauchy problem on $\mathbb{R}^d$ for the system

$$\begin{cases} \rho_t = \varepsilon\Delta\rho - \mathrm{div}\left(\rho\chi(\rho,S)\nabla S\right) \\ S_t = \Delta S + r(\rho,S). \end{cases} \tag{1.81}$$

In system (1.81), the secretion/degradation mechanisms for S are contained in the term $r(\rho,S)$. A typical form is the linear one $r(\rho,S) = \alpha\rho - \beta S$ with $\alpha,\beta > 0$.

The term $\chi(\rho, S)$, called *chemotactic sensitivity*, is very important in this context. In many situations it turns out that the expression of $\chi(\rho, S)$ determines the final outcome of the competition between diffusion (dispersion of particles) and singular aggregation phenomena (concentration to deltas) at the level of ρ (cf. the works of Jäger–Luckaus [105], Nagai [140], Herrero–Velazquez [95] among others). In the case $\chi(\rho, S) \equiv$ constant, the above system has been extensively studied, especially in its parabolic–elliptic variants with the second equation in (1.81) replaced by $0 = \Delta S + \rho - S$ or by Poisson's equation $-\Delta S = \rho$. In particular, it is well known (cf. [72]) that the 2 dimensional Keller-Segel system

$$\begin{cases} \rho_t = \Delta\rho - \mathrm{div}\,(\rho\chi\nabla S) \\ 0 = \Delta S + \rho \end{cases} \tag{1.82}$$

(with $L^1_+ \cap \mathrm{L}\log\mathrm{L}(\mathbb{R}^2)$ data for ρ) features a χ–dependent critical threshold m^* for the total mass of ρ determining finite time blow–up or global existence (blow–up for initial mass larger than m^*, global existence otherwise). Related results are contained in [60, 151, 20, 59] and in [35] for the parabolic case. For further references about all the several versions of the Keller–Segel system and related models, we refer to the review papers by Horstmann [101, 102] and the references therein.

How to avoid finite time blow–up of cells has been the aim of an extensive research in the last years. This issue is motivated both by the attempt of constructing an "approximate" notion of solution preventing blow up for any initial mass on the one side, and by modeling issues related with *volume filling effects* occurring when the density of cells becomes very large on the other side. There are mainly two ways to prevent blow up of ρ. The first one introduces a volume filling effect at the level of the diffusion of cells, replacing $\Delta\rho$ by a nonlinear diffusion term $\Delta\rho^\gamma$ with $\gamma > 1$. This modification of the model (cf. [112, 34]) allows to define a global solution $\rho(t) \in L^1 \cap L^\infty$ for all $t > 0$ no matter how large the initial mass is. The second way to prevent blow up consists in modifying the chemotactical sensitivity. Among the possible ways to do that (cf. [98, 20]), we mention the one suggested by Hillen and Painter in [146, 97], which considers $\chi(\rho) = \rho_{max} - \rho$ for a certain $\rho_{max} > 0$ representing the maximum allowed density (ρ_{max} can be taken equal to 1 for simplicity). Basically, in this model cells stop aggregating when the density reached a maximum allowed value. An extensive mathematical theory for this model with prevention of overcrowding has been performed first in [70] on bounded domains and then in [31], where also a variant with nonlinear diffusion has been considered in order to stop any mobility mechanism (including diffusion) at a certain density. Both [70] and [31] concern with the parabolic–elliptic model. More recent results on volume filling effect have been also achieved in [65, 56, 57]). In this section we try to generalize some of the results in [31] to

the fully parabolic model (1.80). In particular, we aim to prove large time decay of solutions and the large time self–similar behavior of the density of cells. This last issue is extremely non trivial, because of the strong coupling between the two species. In order to perform this task, we use a diffusive time dependent scaling and a variant of the relative entropy method going back to [4, 41]. The major difficulty with respect to the parabolic–elliptic case treated in [31] is the fact the cells' interaction energy cannot be expressed in the form of a nonlocal interaction potential as in that case, since S has now an evolution of its own. Similar issues have been recently faced too in [35], where a similar model has been endowed with a suitable energy functional involving ρ and S.
Our goal will be to perform the same kind of analysis we have done in the model for fermions in the first part of the chapter, and for that we proceed as follows: section 1.2.1 is devoted to the existence theory. First we prove the existence and uniqueness of solution locally in time for any initial condition, and then we provide some a priori estimates which we will use to prove the existence of a global solution for (1.80). In section 1.2.2 we concern about the long time behavior of the solutions and establish decay rates in $\mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{L}^\infty(\mathbb{R}^d)$ for both the cells density and the chemical. Finally, in section 1.2.3 we study the asymptotic self–similar behavior of the solutions by time dependent scaling and by proving convergence to a stationary state in the new variables.
The results in section 1.2.1, as well as the $\mathrm{L}^2(\mathbb{R}^d)$–decay of the cells' density and the concentration of the chemoattractant in section 1.2.2, are valid in any dimension d. For the decay of the L^∞–norms of ρ and S and for the large time self–similar behavior of ρ, we shall need to restrict ourselves to the case $d = 1$ (see also the Remark 1.34 below).
The main result we present here deals with the self–similar behavior of ρ as $t \to +\infty$. It is contained in the Theorem 1.38. Such result strongly relies on the a–priori decay estimate of the L^∞ norms of ρ and S, which are contained in Proposition 1.36 and which are valid only in the case $d = 1$. However, we remark here that the proof of Theorem 1.38 is also valid in case $d > 1$ provided one can prove that ρ and S satisfy suitable decay estimates for large times (see remark 1.41).
A key technical result about the decay of the L^2 norm of ρ and ∇S is contained in Proposition 1.35. Here (and hence in all the following results), the condition

$$\varepsilon > 1/4$$

on the diffusivity constant is needed. Roughly speaking, this condition ensures that the parabolic cross–diffusion operator on the r.h.s. of system (1.80) is uniformly elliptic with respect to $\rho \in [0,1]$ and $S \geq 0$. Whether this condition is necessary to have a large time decay (and consequently a self–similar be-havior) for ρ is still an open problem even in the (simpler) parabolic–elliptic case described in [31] (cf. Remark 1.33 for further explanations).

1.2.1 Existence and Regularity

Our aim in this section is to prove the existence and uniqueness of solutions for the Cauchy problem for the parabolic system (1.80). We will use a fix point argument to show that a unique solution exists locally in time and then we shall provide some estimates to extend this solution globally in time. For future use, we introduce the functional space

$$\mathcal{U} := (\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)) \times (\mathrm{W}^{1,1}(\mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d)). \tag{1.83}$$

As a first step, for a given $(\rho_0, S_0) \in \mathcal{U}$ we will rewrite the system (1.80) in its integral form

$$\rho(t,x) = \mathcal{G}_\varepsilon(t,x) * \rho_0(x) - \int_0^t \mathcal{G}_\varepsilon(x, t-\theta) * \mathrm{div}(\rho(1-\rho)\nabla S)(x,\theta)\mathrm{d}\theta \tag{1.84}$$

$$S(t,x) = \mathrm{e}^{-t}\mathcal{G}(t,x) * S_0(x) + \int_0^t \mathrm{e}^{-t+\theta}\mathcal{G}(x, t-\theta) * \rho(x,\theta)\mathrm{d}\theta, \tag{1.85}$$

where, the scaled kernel $\mathcal{G}$ for a $\alpha > 0$ is defined by

$$\mathcal{G}_\alpha(t,x) = \frac{1}{(4\pi\alpha t)^{d/2}} \mathrm{e}^{-\frac{|x|^2}{4\alpha t}}$$

and we adopt the notation $\mathcal{G}$ in case $\alpha = 1$.
The relaxed problem (1.84)–(1.85), which is easily obtained by using Duhamel's Formula, leads to the definition of a functional on $\mathcal{U}$ as follows

$$\mathcal{T}[\rho, S] = (\mathcal{T}_1[\rho, S], \mathcal{T}_2[\rho, S]),$$

with

$$\mathcal{T}_1[\rho, S](t,x) = \mathcal{G}_\varepsilon(t,x) * \rho_0(x) - \int_0^t \mathcal{G}_\varepsilon(t-\theta, x) * \mathrm{div}(\rho(1-\rho)\nabla S)(\theta, x)\mathrm{d}\theta \tag{1.86}$$

$$\mathcal{T}_2[\rho, S](t,x) = \mathrm{e}^{-t}\mathcal{G}(t,x) * S_0(x) + \int_0^t \mathrm{e}^{-t+\theta}\mathcal{G}(t-\theta, x) * \rho(\theta, x)\mathrm{d}\theta. \tag{1.87}$$

Let us introduce the notation

$$X_T^R = \{(\rho, S) \in \mathcal{U} \ : \ \|(\rho, S)\|_{X_T} \le R\} \tag{1.88}$$

where

$$\begin{aligned}\|(\rho,S)\|_{X_T} := \sup_{0\le t\le T}\Big\{&\|\rho(t)-\mathcal{G}_\varepsilon(t)*\rho_0\|_1+\|\rho(t)-\mathcal{G}_\varepsilon(t)*\rho_0\|_\infty\\&+\|S(t)-\mathrm{e}^{-t}\mathcal{G}(t)*S_0\|_1+\|S(t)-\mathrm{e}^{-t}\mathcal{G}(t)*S_0\|_\infty\\&+\|\nabla S-\nabla(\mathrm{e}^{-t}\mathcal{G}(t)*S_0)\|_1+\|\nabla S-\nabla(\mathrm{e}^{-t}\mathcal{G}(t)*S_0)\|_\infty\Big\}\end{aligned}$$

We will prove that X_T^R is invariant under the map $\mathcal{T}$ for T sufficiently small. Then we show that $\mathcal{T}$ is a strict contraction on X_T^R, whence we have the following theorem.

Theorem 1.30. *Let* $(\rho_0,S_0)\in\mathcal{U}$*. Then, there exists* $T>0$ *and a pair* $(\rho,S)\in\mathcal{C}([0,T];\mathcal{U})$ *such that* (ρ,S) *solves* (1.84)-(1.85) *in* X_T^R *and it is unique.*

Proof.- In order to prove the invariance of X_T^R we shall provide a suitable bound for each of the quantities we have to take into account to compute $\|\mathcal{T}\|_{X_T}$. Let $(\rho,S)\in X_T^R$. For the sake of completeness, we shall compute the first bound in detail. In the following computations, C will denote a generic positive constant possibly depending on ε (the role of the diffusivity constant ε is not relevant at this stage).

$$\begin{aligned}\|(\mathcal{T}_1[\rho,S](t)-\mathcal{G}_\varepsilon*\rho_0)(t)\|_1 &\le \int_0^t\|\nabla\mathcal{G}_\varepsilon(t-\theta)\|_1\|(\rho(1-\rho)\nabla S)(\theta)\|_1\mathrm{d}\theta\\&\le\int_0^t C(t-\theta)^{-\frac12}\|\nabla S(\theta)\|_1(\|\rho(\theta)\|_\infty+\|\rho(\theta)\|_\infty^2)\mathrm{d}\theta\\&\le\int_0^t C(t-\theta)^{-\frac12}(\|\nabla(\mathrm{e}^{-\theta}\mathcal{G}(\theta)*S_0)\|_1+R)\times\\&\qquad(\|\mathcal{G}(\theta)*\rho_0\|_\infty+R)(\|\mathcal{G}(\theta)*\rho_0\|_\infty+R+1)\mathrm{d}\theta\\&\le C(R,\|\rho_0\|_\infty,\|\nabla S_0\|_1)t^{\frac12}\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}&\|(\mathcal{T}_1[\rho,S](t)-\mathcal{G}_\varepsilon*\rho_0)(t)\|_\infty\le C(R,\|\rho_0\|_\infty,\|\nabla S_0\|_\infty)t^{\frac12}\\&\|(\mathcal{T}_2[\rho,S](t)-\mathrm{e}^{-t}\mathcal{G}*S_0)(t)\|_1\le C(R,\|\rho_0\|_1)(1-e^{-t})\\&\|(\mathcal{T}_2[\rho,S](t)-\mathrm{e}^{-t}\mathcal{G}*S_0)(t)\|_\infty\le C(R,\|\rho_0\|_\infty)(1-e^{-t})\end{aligned}$$

Finally,

$$\begin{aligned}\|\nabla[\mathcal{T}_2(t)-\mathrm{e}^{-t}\mathcal{G}(t)*S_0]\|_1&\le\int_0^t\mathrm{e}^{-t+\theta}\|\nabla\mathcal{G}(t-\theta)*\rho(\theta)\|_1\\&\le C\int_0^t\mathrm{e}^{-t+\theta}(t-\theta)^{-\frac12}\|\rho(\theta)\|_1\mathrm{d}\theta\le C(R,\|\rho_0\|_1)t^{\frac12},\end{aligned}$$

and in the same spirit,

$$\|\nabla[\mathcal{T}_2 - \mathrm{e}^{-t}\mathcal{G} * S_0]\|_\infty \leq C(R, \|\rho_0\|_\infty)t^{\frac{1}{2}}.$$

Therefore, for a given $(\rho, S) \in X_T^R$ we have that

$$\|\mathcal{T}[\rho, s]\|_{X_T} \leq C(R, \|\rho_0\|_1, \|\rho_0\|_\infty, \|\nabla S_0\|_1, \|\nabla S_0\|_\infty)T^{\frac{1}{2}},$$

whence the invariance of X_T^R under $\mathcal{T}$ if T is small enough. Now we want to see that $\mathcal{T}$ is strictly contractive on X_T^R. For that let us consider two pairs (ρ_1, S_1) and (ρ_2, S_2) belonging to X_T^R and look at the norm of the difference of their images by $\mathcal{T}$ in $\mathcal{C}([0,T];\mathcal{U})$.

$$\begin{aligned}
&\|(\mathcal{T}_1[\rho_1, S_1] - \mathcal{T}_1[\rho_2, S_2])(t)\|_1 \leq \\
&\leq \int_0^t \|\nabla\mathcal{G}_\varepsilon(t-\theta) * [\rho_1(1-\rho_1)\nabla S_1 - \rho_2(1-\rho_2)\nabla S_2]\,(\theta)\|_1 \mathrm{d}\theta \\
&\leq C\int_0^t (t-\theta)^{-\frac{1}{2}} \|\,[\rho_1(1-\rho_1)\nabla S_1 - \rho_2(1-\rho_1)\nabla S_1 + \rho_2(1-\rho_1)\nabla S_1 \\
&\quad -\rho_2(1-\rho_1)\nabla S_2 + \rho_2(1-\rho_1)\nabla S_2 - \rho_2(1-\rho_2)\nabla S_2]\,(\theta)\|_1 \mathrm{d}\theta \\
&\leq C\int_0^t (t-\theta)^{-\frac{1}{2}} \Big[\|\nabla S_1\|_\infty \|\rho_1 - \rho_2\|_1 + \|\rho_2\|_\infty \|\nabla S_1 - \nabla S_2\|_1 \\
&\qquad + \|\rho_2\|_\infty \|\nabla S_2\|_\infty \|\rho_1 - \rho_2\|_1\Big](\theta)\mathrm{d}\theta \\
&\leq C(\varepsilon, R, \|\rho_1\|_\infty, \|\rho_2\|_\infty, \|\nabla S_1\|_\infty, \|\nabla S_2\|_\infty)T^{\frac{1}{2}} \times \\
&\qquad \left(\sup_{0\leq t\leq T} \|(\rho_1 - \rho_2)(t)\|_1 + \sup_{0\leq t\leq T} \|(\nabla S_1 - \nabla S_2)(t)\|_1\right)
\end{aligned}$$

In the same way we see that

$$\begin{aligned}
\|(\mathcal{T}_1[\rho_1, S_1] - \mathcal{T}_1[\rho_2, S_2])(t)\|_\infty \;\leq\; & \\
& C(\varepsilon, R, \|\rho_1\|_\infty, \|\rho_2\|_\infty, \|\nabla S_1\|_\infty, \|\nabla S_2\|_\infty)T^{\frac{1}{2}} \times \\
& \left(\sup_{0\leq t\leq T} \|(\rho_1 - \rho_2)(t)\|_\infty + \sup_{0\leq t\leq T} \|(\nabla S_1 - \nabla S_2)(t)\|_\infty\right)
\end{aligned}$$

and

$$\|(\mathcal{T}_2[\rho_1,S_1]-\mathcal{T}_2[\rho_2,S_2])(t)\|_1 \le C(1-\mathrm{e}^{-T})\sup_{0<t<T}\|(\rho_1-\rho_2)(t)\|_1$$
$$\|(\mathcal{T}_2[\rho_1,S_1]-\mathcal{T}_2[\rho_2,S_2])(t)\|_\infty \le C(1-\mathrm{e}^{-T})\sup_{0<t<T}\|(\rho_1-\rho_2)(t)\|_\infty.$$
$$\|\nabla(\mathcal{T}_2[\rho_1,S_1]-\mathcal{T}_2[\rho_2,S_2])(t)\|_1 \le CT^{\frac{1}{2}}\sup_{0<t<T}\|(\rho_1-\rho_2)(t)\|_1$$
$$\|\nabla(\mathcal{T}_2[\rho_1,S_1]-\mathcal{T}_2[\rho_2,S_2])(t)\|_\infty \le CT^{\frac{1}{2}}\sup_{0<t<T}\|(\rho_1-\rho_2)(t)\|_\infty.$$

Hence, if T is small we have $\|\mathcal{T}[\rho_1,S_1]-\mathcal{T}[\rho_2,S_2]\|_{\mathcal{U}} \le \alpha\|(\rho_1,S_1)-(\rho_2,S_2)\|_{\mathcal{U}}$ for $0<\alpha<1$ which implies the contractivity of $\mathcal{T}$ and concludes the proof. □

At this point we shall remark some properties about the solution to (1.80), namely we notice that the mass of ρ is preserved and that the interval $[0,1]$ is an invariant domain for ρ. We collect these properties in the next proposition.

Proposition 1.31. *Let* $\rho_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *such that* $0\le\rho_0\le 1$*; then for any* t*,* $0\le t\le T$

$$\int_{\mathbb{R}^d}\rho(t,x)\mathrm{d}x = \int_{\mathbb{R}^d}\rho_0(t,x)\mathrm{d}x.$$

and $0\le\rho(t)\le 1$ *for* $0\le t\le T$.

Proof.- Concerning the first part of the proposition, we consider a family of non-increasing cut-off functions $\{\theta_n\}$; then multiply (1.80) by θ_n and integrate over $\mathbb{R}^d\times[0,T]$. The result follows from dominated convergence theorem when we let n go to ∞. For the second part, it is easy to see that $\rho\equiv 0$ and $\rho\equiv 1$ are sub- and super-solutions respectively, so if initially ρ belongs to the interval $[0,1]$ it will remain there (see [154]). □

Then we are ready to state the existence of a global and unique solution for (1.80)

Theorem 1.32. *Let* $(\rho_0,S_0)\in\mathcal{U}$*,* $0<\rho_0<1$*. Then there exist a unique weak solution for* (1.80) *defined in* $[0,\infty)$ *which belongs to* $\mathcal{U}$ *for each* $T>0$*.*

Proof.- From theorem 1.30 we know that there exists a unique weak solution for (1.80) defined in $(0,T)$ for some $T>0$. By contradiction, let $T_{\max}>0$ be the maximal time of existence of (ρ,S). Then an easy continuation argument shows that $\|(\rho,S)\|_{X_t}$ should go to ∞ as $t\to T_{\max}$. But the conservation of the mass and the existence of the sub and super–solutions for ρ tell us that ρ is uniformly bounded in $\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ and then global in time. By using the uniform $\mathrm{L}^1 \cap L^\infty$–bound for ρ we can estimate the L^1 and the L^∞ norms of S in the integral expression (1.85) as follows

$$\|S(t)\|_{L^1} \leq e^{-t}\|S_0\|_{L^1} + \int_0^t e^{-t+\theta}\|\rho(\theta)\|_{L^1} d\theta,$$
$$\|S(t)\|_{L^\infty} \leq e^{-t}\|S_0\|_{L^\infty} + \int_0^t e^{-t+\theta}\|\rho(\theta)\|_{L^\infty} d\theta$$

and the above estimate imply that $\|S(t)\|_{L^1}$ and $\|S(t)\|_{L^\infty}$ are finite as $t \nearrow T_{\max}$. In a similar way, we can estimate ∇S as follows

$$\begin{aligned}\|\nabla S(t)\|_{L^1} &\leq e^{-t}\|S_0\|_{L^1} + \int_0^t e^{-t+\theta}\|\nabla\mathcal{G}(t-\theta)\|_{L^1}\|\rho(\theta)\|_{L^1} d\theta \\ &\leq e^{-t}\|S_0\|_{L^1} + C\int_0^t e^{-t+\theta}(t-\theta)^{-1/2}\|\rho(\theta)\|_{L^1} d\theta, \\ \|\nabla S(t)\|_{L^\infty} &\leq e^{-t}\|S_0\|_{L^\infty} + C\int_0^t e^{-t+\theta}(t-\theta)^{-1/2}\|\nabla\mathcal{G}(t-\theta)\|_{L^1}\|\rho(\theta)\|_{L^\infty} d\theta.\end{aligned}$$

Therefore, such a finite $T_{\max}$ cannot exist and the thesis follows. □

1.2.2 Decay Rates for the concentration of cells and the chemical

Here we want to provide a decay rate for the L^∞-norm of the density of cells ρ. Also a bound for the L^∞-norm of the gradient of the concentration of the chemical will be derived. Our first goal will be to find a decay rate for the L^2-norm and the L^∞-norm of ρ so that we can give a bound for the decay of $\|\rho\|_p$ by interpolation. Next proposition provides the decay of L^2-norm of ρ, and with the same effort, we will obtain too an estimate for the decay of the L^2-norm of ∇S.

Remark 1.33 (Moderate diffusivity condition $\varepsilon > 1/4$). From now on, we shall need the assumption on the diffusivity constant

$$\varepsilon > \frac{1}{4}.$$

We remark that the same restriction is present in [70, 31]. As it will be clear from the proof of Proposition 1.35, such a condition appears naturally in order to have the elliptic operator

$$\begin{pmatrix} \varepsilon\Delta\rho - \operatorname{div}(m(\rho)\nabla S) \\ \Delta S \end{pmatrix}, \qquad m(\rho) = \rho(1-\rho),$$

uniformly elliptic with respect to (ρ, S) taking values in the state space $[0,1]\times\mathbb{R}_+$. In case of a general mobility function $m(\rho)$, the generalization of such condition is clearly $\varepsilon > \sup_\rho m(\rho)$ (we observe that $1/4 = \sup_{\rho\in[0,1]}\rho(1-\rho)$). Whether this condition is necessary to achieve a large time decay for ρ (which corresponds to a diffusion dominated behavior for ρ) or not, is still an open problem even in the parabolic–elliptic case (cf. [31]). However, several arguments suggest that ρ decays for any $\varepsilon > 0$, which implies that cells obey to diffusion rather than chemotaxis in the large time no matter how small the diffusivity is:

- Even in case $\varepsilon < \frac{1}{4}$ this model (together with its parabolic–elliptic version) does not feature nontrivial L^1 steady states; therefore, there's no other candidate as asymptotic state than zero.
- Numerical simulations in [31] suggest that ρ decays even in case $\varepsilon < 1/4$.
- The linearization of the elliptic operator above around a constant state $\rho = \overline{\rho} \neq 1/2$ would imply a lowering of the threshold $\varepsilon > 1/4$ to have ellipticity (and therefore local stability of constant states). Therefore, it is reasonable to expect that, at least for 'small' data of the density, ρ decays no matter how small ε is.

Remark 1.34. The main result of this section, namely the L^∞ decay for ρ and S in Proposition 1.36, will be proven only in the one dimensional case $d = 1$. However, the result in the following proposition 1.35 (which is the basis for proving the L^∞ decay) is valid in any space dimension. The (technical) reason of the restriction to the one dimensional case is in the use of Duhamel's formula explained after proposition 1.35: the one dimensional case allows for proving a uniform estimate for ρ by means of one single intermediate estimate of the L^4 norm. In case of higher dimensions, the number of steps cannot be controlled and the proof of the L^∞ decay of ρ seems to the authors an extremely non trivial issue they are not able to address for the moment.

Proposition 1.35. *Let $\varepsilon > \frac{1}{4}$. Let (ρ, S) be a solution of the parabolic problem* (1.80) *with initial datum (ρ_0, S_0) satisfying $\rho_0, \nabla S_0 \in L^2(\mathbb{R}^d)$, then there exist a constant $\lambda > 0$ depending on ε and a constant $C > 0$ depending on the dimension d, on the initial mass of ρ and on ε such that*

$$\|\rho(t)\|_2 + \lambda\|\nabla S(t)\|_2 \leq C(t+1)^{-\frac{d}{4}} \tag{1.89}$$

Proof.-
To get this result we look at the time evolution of $\mathcal{E}[\rho,S] := \frac{1}{2}\left(\|\rho\|_2^2 + \lambda^2\|\nabla S\|_2^2\right)$. Recalling (1.80) we can see that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}\left[\frac{\rho^2}{2} + \lambda\frac{|\nabla S|^2}{2}\right]\mathrm{d}x &= \int_{\mathbb{R}^d}\rho_t\rho\,\mathrm{d}x + \lambda\int_{\mathbb{R}^d}\nabla S\nabla S_t\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\rho(\varepsilon\Delta\rho - \nabla(\rho(1-\rho)\nabla S))\,\mathrm{d}x - \lambda\int_{\mathbb{R}^d}\Delta S(\Delta S - S + \rho)\,\mathrm{d}x \\
&= -\varepsilon\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x + \int_{\mathbb{R}^d}\rho(1-\rho)\nabla S\nabla\rho\mathrm{d}x - \lambda\int_{\mathbb{R}^d}(\Delta S)^2\mathrm{d}x \\
&\quad - \lambda\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x + \lambda\int_{\mathbb{R}^d}\nabla S\nabla\rho\mathrm{d}x \\
&\le -\varepsilon\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x + \left(\frac{1}{4}+\lambda\right)\int_{\mathbb{R}^d}|\nabla S||\nabla\rho|\mathrm{d}x - \lambda\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x. \qquad (1.90)
\end{aligned}
$$

Now, let us introduce the quadratic form

$$
< v, Q(\lambda)v >, \qquad v := \begin{pmatrix}|\nabla\rho| \\ |\nabla S|\end{pmatrix}, \qquad Q(\lambda) := \begin{pmatrix}\varepsilon & -\frac{1}{2}\left(\frac{1}{4}+\lambda\right) \\ -\frac{1}{2}\left(\frac{1}{4}+\lambda\right) & \lambda\end{pmatrix}.
$$

A simple computation shows that

$$
\det(Q(\lambda)) = -\frac{1}{4}\left[\lambda^2 + \left(\frac{1}{2} - 4\varepsilon\right)\lambda + \frac{1}{16}\right]
$$

and therefore the inequality $\det(Q(\lambda)) > 0$ has solutions in an interval $\lambda \in (\lambda_1(\varepsilon), \lambda_2(\varepsilon))$ if and only if

$$
\left(\frac{1}{2} - 4\varepsilon\right)^2 - \frac{1}{4} > 0 \qquad \Leftrightarrow \qquad \varepsilon > \frac{1}{4}.
$$

Such a choice of λ ensures that the quadratic form $< v, Q(\lambda)v >$ is strictly positive definite. Hence, there exist two positive constants a and b, both depending on ε, such that

$$
\begin{aligned}
&-\varepsilon\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x + \left(\frac{1}{4}+\lambda\right)\int_{\mathbb{R}^d}|\nabla S||\nabla\rho|\mathrm{d}x - \lambda\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x \\
&\le -a\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x - b\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x. \qquad (1.91)
\end{aligned}
$$

The above inequality (1.91) and (1.90) imply in particular that $\mathcal{E}(\rho(t), S(t))$ is non–increasing. We want to prove that in fact it is decaying to zero as $t \to +\infty$ with a suitable polynomial rate. (1.90) and (1.91) imply

$$
\mathcal{E}(\rho(t), S(t)) + \int_0^t\left[a\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x + b\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x\right]\mathrm{d}\theta \le \mathcal{E}(\rho_0, S_0). \qquad (1.92)
$$

Then, in one hand, by the following interpolation inequality (which is a direct consequence of the Gagliardo–Nirenberg inequality)

$$\|\rho\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{(d(p-1)+2)p}{d(p-1)}} \le C(p,d)\|\nabla\rho^{\frac{p}{2}}\|_{\mathrm{L}^p(\mathbb{R}^d)}^2\|\rho\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{2p}{d(p-1)}} \tag{1.93}$$

we have that

$$-a\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x \le -a\frac{\|\rho\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\frac{2(d+2)}{d}}}{C(1,d)\|\rho\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{4}{d}}}. \tag{1.94}$$

Now, since (1.92) is valid for all $t \ge 0$, we have

$$\int_0^\infty \left[a\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x + b\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x\right]\mathrm{d}\theta < +\infty$$

Then, there exists a sequence $t_k \to +\infty$ such that

$$\left[a\int_{\mathbb{R}^d}|\nabla\rho|^2\mathrm{d}x + b\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x\right](t_k) \to 0$$

as $k \to +\infty$, which implies

$$\int_{\mathbb{R}^d}|\nabla\rho(t_k,x)|^2\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^d}|\nabla S(t_k,x)|^2\mathrm{d}x \to 0$$

and therefore, using again inequality (1.93),

$$\int_{\mathbb{R}^d}\rho^2(x,t_k)\mathrm{d}x \to 0.$$

Hence, $\mathcal{E}(\rho(t_k),S(t_k)) \to 0$ as $k \to \infty$, but since $\mathcal{E}$ is non-increasing w.r.t time, we get that in fact $\mathcal{E}(\rho(t),S(t)) \to 0$ as $t \to \infty$. In particular, this implies that $\int_{\mathbb{R}^d}|\nabla S(t,x)|^2\mathrm{d}x \to 0$ as $t \to \infty$ and for big enough t

$$-b\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x \le -b\left(\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x\right)^\alpha,$$

for $\alpha > 1$. Thus from (1.90) and (1.91) we have, for a suitably chosen $C > 0$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\left[\frac{\rho^2}{2}+\lambda\frac{\nabla S^2}{2}\right]\mathrm{d}x \le& -C\left[\left(\int_{\mathbb{R}^d}\rho^2\mathrm{d}x\right)^{\frac{(d+2)}{d}} + \left(\lambda\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x\right)^{\frac{(d+2)}{d}}\right].\\ \le& -C\left(\int_{\mathbb{R}^d}\rho^2\mathrm{d}x + \lambda\int_{\mathbb{R}^d}|\nabla S|^2\mathrm{d}x\right)^{\frac{(d+2)}{d}}\end{aligned} \tag{1.95}$$

Now, by time integration of the previous expression the thesis follows. □

At this point, we only need to prove an estimate for the decay of the L^∞ norm of ρ. From now on, due to the integrability problems that higher dimension entails (see remark 1.34), we restrict ourselves to the one–dimensional case. We look at the expression of ρ for a time $2t$ in terms of its value at time t given by Duhamel's formula

$$\rho(x,2t) = \mathcal{G}_\varepsilon(t,x)*\rho(t,x)+\int_0^t \mathcal{G}_{e,x}(x,t-\theta)*((\rho(1-\rho))S_x)(x,t+\theta)d\theta \quad (1.96)$$

so we can estimate

$$\begin{aligned}\|\rho(2t)\|_\infty &\le \|\mathcal{G}_\varepsilon(t)*\rho(t)\|_\infty + \int_0^t \|\mathcal{G}_{\varepsilon,x}(t-\theta)*((\rho(1-\rho))S_x)(t+\theta)\|_\infty \mathrm{d}\theta \\ &\le \|\mathcal{G}_\varepsilon(t)\|_\infty\|\rho(t)\|_1 + \int_0^t \|\mathcal{G}_{\varepsilon,x}(t-\theta)\|_2\|(\rho S_x)(t+\theta)\|_2\mathrm{d}\theta \\ &\le Ct^{-\frac{1}{2}}\|\rho_0\|_1 + \int_0^t C(t-\theta)^{-\frac{3}{4}}\|\rho(t+\theta)\|_4\|S_x(t+\theta)\|_4\mathrm{d}\theta \quad (1.97)\end{aligned}$$

Thus, to get the decay we need to compute an estimate for $\|S_x\|_4$ and $\|\rho\|_4$. Let us start by $\|\rho\|_4$. Similarly as before we can estimate :

$$\begin{aligned}\|\rho(2t)\|_4 &\le \|\mathcal{G}_\varepsilon(t)*\rho(t)\|_4 + \int_0^t \|\mathcal{G}_{\varepsilon,x}(t-\theta)*((\rho(1-\rho))S_x)(t+\theta)\|_4\mathrm{d}\theta \\ &\le \|\mathcal{G}_\varepsilon(t)\|_4\|\rho(t)\|_1 + \int_0^t \|\mathcal{G}_{\varepsilon,x}(t-\theta)\|_4\|(\rho S_x)(t+\theta)\|_1\mathrm{d}\theta \\ &\le Ct^{-\frac{3}{8}}\|\rho_0\|_1 + \int_0^t C(t-\theta)^{-\frac{7}{8}}\|\rho(t+\theta)\|_2\|S_x(t+\theta)\|_2\mathrm{d}\theta \\ &\le Ct^{-\frac{3}{8}}\|\rho_0\|_1 + \int_0^t \hat{C}(t-\theta)^{-\frac{7}{8}}(t+\theta)^{-\frac{1}{2}}\mathrm{d}\theta \\ &\le Ct^{-\frac{3}{8}}\|\rho_0\|_1 + \hat{C}t^{-\frac{3}{8}} = C(M)t^{-\frac{3}{8}} \quad (1.98)\end{aligned}$$

($C(M)$ is a constant depending on the total mass M of ρ) and due to the integral equation (1.85) satisfied by S, we have

$$
\begin{aligned}
\|S_x(2t)\|_4 &\leq \mathrm{e}^{-t}\|\mathcal{G}_x(t)\|_4\|(S_0)_x\|_1 + \int_0^t \mathrm{e}^{-t+\theta}\|\mathcal{G}_x(t-\theta)\|_1\|\rho(t+\theta)\|_4 \mathrm{d}\theta \\
&\leq Ct^{-\frac{3}{8}}\|(S_0)_x\|_1 + \int_0^t \mathrm{e}^{\theta-t}(t-\theta)^{-\frac{1}{2}}(t+\theta)^{-\frac{3}{8}}\mathrm{d}\theta \\
&\leq t^{-\frac{3}{8}}\left(C\|(S_0)_x\|_1 + \int_0^t \mathrm{e}^{\theta-t}(t-\theta)^{-\frac{1}{2}}\right)\mathrm{d}\theta \\
&= C(\|(S_0)_x\|_1)t^{-\frac{3}{8}}. \qquad (1.99)
\end{aligned}
$$

Now we can continue from (1.97) and finish the computation:

$$
\begin{aligned}
\|\rho(2t)\|_\infty &\leq Ct^{-\frac{1}{2}}\|\rho_0\|_1 + \int_0^t C(t-\theta)^{-\frac{3}{4}}C(M,\|(S_0)_x\|_1)(t+\theta)^{-\frac{6}{8}}\mathrm{d}\theta \\
&\leq Ct^{-\frac{1}{2}}\|\rho_0\|_1 + \int_0^t C(t-\theta)^{-\frac{3}{4}}C(M,\|(S_0)_x\|_1)(t)^{-\frac{6}{8}}\mathrm{d}\theta \\
&\leq Ct^{-\frac{1}{2}}\|\rho_0\|_1 + \widetilde{C}t^{-\frac{6}{8}+\frac{1}{4}} = Ct^{-\frac{1}{2}}. \qquad (1.100)
\end{aligned}
$$

and with the same idea we can also see that S is decaying

$$
S(2t) = e^{-t}\mathcal{G}(t) * S(t) + \int_0^t e^{-t+\theta}\mathcal{G}(t-\theta,x) * \rho(t+\theta,x)\mathrm{d}\theta \qquad (1.101)
$$

so

$$
\begin{aligned}
\|S(2t)\|_\infty &\leq C\mathrm{e}^{-t}\|S(t)\|_\infty + C\int_o^t \mathrm{e}^{-t+\theta}(t+\theta)^{-\frac{1}{2}} \\
&\leq C\mathrm{e}^{-t}\|S(t)\|_\infty + Ct^{-\frac{1}{2}}(1-\mathrm{e}^{-t})
\end{aligned}
$$

These results can be sumarized in the next

Proposition 1.36. *Let $\varepsilon > \frac{1}{4}$ and $d = 1$. Let the pair (ρ, S) be solution of (1.80) with initial datum $(\rho_0, S_0) \in \mathcal{U}$ such that $0 \leq \rho_0 \leq 1$. Then $\|\rho\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ and $\|S\|_\infty = \mathrm{O}(t^{-\frac{1}{2}})$ as $t \to +\infty$.*

Remark 1.37. It is clear from last estimate above that the L^∞ assumptions on $(S_0)_x$ could be slightly relaxed. We shall not deal with this issue for the sake of simplicity.

1.2.3 Asymptotic self-similar behavior

Once we know that there exists a time–decaying solution for the fully parabolic problem (1.80) from previous section, in this one we will be concerned about its long-time asymptotics. For simplicity, we will assume ε to be equal to 1, and show by means of a time dependent scaling and entropy dissipation tools that as time grows to infinity the solution of (1.80) converges in L^1 towards the following time translated self-similar gaussian solution of the Heat equation

$$\rho^\infty(t) = \frac{C_M}{(4\pi(2t+1))^{1/2}} e^{-\frac{|x|^2}{2(2t+1)}}. \tag{1.102}$$

For that let us consider the scaling

$$\begin{cases} \rho(t,x) &= (2t+1)^{-\frac{1}{2}} v(\theta, y) \\ S(t,x) &= (2t+1)^{-\frac{1}{2}} \sigma(\theta, y) \\ y(t,x) &= x(2t+1)^{-\frac{1}{2}} \\ \theta(t,x) &= \frac{1}{2}\log(2t+1) \end{cases} \tag{1.103}$$

so that (1.80) becomes

$$\begin{cases} v_\theta &= (yv)_y + v_{yy} - \mathrm{e}^{-\theta}[v(1-\mathrm{e}^{-\theta} v)\sigma_y]_y \\ \sigma_\theta &= (y\sigma)_y + \sigma_{yy} + \mathrm{e}^{2\theta}(v-\sigma) \end{cases} . \tag{1.104}$$

Also, we define the entropy functional for the v variable

$$E(v) = \int_{\mathbb{R}} v \left(\log v + \frac{y^2}{2} \right) \mathrm{d}y. \tag{1.105}$$

This functional admits a unique global minimum v_M^∞ in the space of L_+^1 densities with prescribed mass M. More precisely,

$$v^\infty = C_M \mathrm{e}^{-\frac{y^2}{2}} \tag{1.106}$$

is the scaled gaussian and the constant C_M depends on the total mass M of v. With these settings we are ready to prove the following theorem.

Theorem 1.38. *Let $d = \varepsilon = 1$ and let (ρ, S) be the solution to* (1.80) *with initial condition (ρ_0, S_0) satisfying the assumptions of theorem 1.32 and let $\rho^\infty(t)$ be defined by (1.102). Let (v, σ) be defined by (1.104) and v^∞ as in (1.106). Then, for any arbitrarily small $\delta > 0$ there exist a constant C depending on δ and on the initial data such that*

$$\|v(\theta) - v^\infty\|_1 \le C\mathrm{e}^{-(1-\delta)\theta} \tag{1.107}$$

for all $\theta > 0$, or equivalently

$$\|\rho(t) - \rho^\infty(t)\|_1 \le C(t+1)^{-\frac{1-\delta}{2}} \tag{1.108}$$

for all $t > 0$.

Proof.- First, let us introduce the short notation

$$W = \left(\log v + \frac{y^2}{2}\right)_y = \frac{v_y}{v} + y \tag{1.109}$$

by which we can write the scaled problem as

$$\left\{ \begin{array}{rcl} v_\theta & = & (vW)_y - \mathrm{e}^{-\theta}(v(1-\mathrm{e}^{-\theta}v)\sigma_y)_y \\ \sigma_\theta & = & \sigma_{yy} + (y\sigma)_y + \mathrm{e}^{2\theta}(v-\sigma) \end{array} \right. \tag{1.110}$$

The proof is based on the two following lemmas, which provide us with the dissipation of the entropy functional (1.105) and $\|\sigma_y\|_2^2$.

Lemma 1.39. *For all* $\delta \in (0,1)$ *we have*

$$\frac{d}{d\theta}E(v) \leq -(1-\delta)\int_{\mathbb{R}} vW^2 \mathrm{d}y + \frac{\mathrm{e}^{-2\theta}}{4\delta}\int_{\mathbb{R}} v\sigma_y^2 \mathrm{d}y \tag{1.111}$$

Proof.- We can compute the entropy dissipation by multiplying the first equation in (1.104) by $(\log v + \frac{y^2}{2})$ and integrating by parts to get

$$\begin{aligned} \frac{d}{d\theta}\int_{\mathbb{R}} v\left(\log v + \frac{y^2}{2}\right)\mathrm{d}y = \\ = -\int_{\mathbb{R}} v\left(\log v + \frac{y^2}{2}\right)_y^2 \mathrm{d}y + \mathrm{e}^{-\theta}\int_{\mathbb{R}} v(1-\mathrm{e}^{-\theta}v)\sigma_y\left(\log v + \frac{y^2}{2}\right)_y \mathrm{d}y \\ \leq -(1-\delta)\int_{\mathbb{R}} v\left(\log v + \frac{y^2}{2}\right)_y^2 \mathrm{d}y + \frac{\mathrm{e}^{-2\theta}}{4\delta}\int_{\mathbb{R}} v\sigma_y^2 \mathrm{d}y \end{aligned} \tag{1.112}$$

Using the notation introduced in (1.109) the lemma follows. □

Lemma 1.40. *The following inequality holds*

$$\begin{aligned} \frac{d}{d\theta}\left(\mathrm{e}^{-4\theta}\int_{\mathbb{R}} \sigma_y^2 \mathrm{d}y\right) \leq & \mathrm{e}^{-2\theta}\|v\|_\infty \int_{\mathbb{R}} vW^2 \mathrm{d}y + 2\mathrm{e}^{-2\theta}\|v\|_\infty \|v\|_1 \\ & - \mathrm{e}^{-2\theta}\int_{\mathbb{R}} \sigma_y^2 \mathrm{d}y - \mathrm{e}^{-4\theta}\int_{\mathbb{R}} \sigma_y^2 \mathrm{d}y \end{aligned} \tag{1.113}$$

Proof.- By using the second equation in (1.110) we compute the l.h.s. in (1.113) as follows

$$
\begin{aligned}
\frac{d}{d\theta}\left(\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\right) &= -4\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y + 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y(\sigma_y)_\theta\mathrm{d}y \\
&= -4\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_{yy}\left((y\sigma)_y + \sigma_{yy} + \mathrm{e}^{2\theta}(v-\sigma)\right)\mathrm{d}y \\
&= -4\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y - 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_{yy}\sigma\mathrm{d}y \\
&\quad - \mathrm{e}^{-4\theta}\int_{\mathbb{R}}(2\sigma_{yy}\sigma_y)y\mathrm{d}y - 2\mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_{yy}(v-\sigma)\mathrm{d}y \\
&= -\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y + 2\mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_y v_y\mathrm{d}y - 2\mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\le -\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y + \mathrm{e}^{-2\theta}\int_{\mathbb{R}}\left(\sigma_y^2 + v_y^2\right)\mathrm{d}y \\
&\quad - 2\mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&= -\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - 2\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_{yy}^2\mathrm{d}y + \mathrm{e}^{-2\theta}\int_{\mathbb{R}}v\frac{v_y^2}{v}\mathrm{d}y - \mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\le \mathrm{e}^{-2\theta}\|v\|_\infty\int_{\mathbb{R}}vW^2\mathrm{d}y - \mathrm{e}^{-2\theta}\|v\|_\infty\int_{\mathbb{R}}vy^2\mathrm{d}y - 2\mathrm{e}^{-2\theta}\|v\|_\infty\int_{\mathbb{R}}v_y y\mathrm{d}y \\
&\quad - \mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y - \mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\le \mathrm{e}^{-2\theta}\|v\|_\infty\int_{\mathbb{R}}vW^2\mathrm{d}y + 2\mathrm{e}^{-2\theta}\|v\|_\infty\|v\|_1 - \mathrm{e}^{-2\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \\
&\quad - \mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y,
\end{aligned}
$$

where in the last but one inequality we have used

$$\int_{\mathbb{R}}vW^2\mathrm{d}y = \int_{\mathbb{R}}\frac{v_y^2}{v}\mathrm{d}y + 2\int_{\mathbb{R}}v_y y\mathrm{d}y + \int_{\mathbb{R}}vy^2\mathrm{d}y.$$

□

Now, in view of the uniform decay estimates proven in Proposition 1.36, we are able to find a constant $\mu > 0$ such that $\|v\|_\infty \le \mu$ for all θ. With such μ at hand we introduce the functional

$$\Phi(\theta, v, \sigma) := E(v) - E(v^\infty) + \frac{\mu}{2\delta}\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y \tag{1.114}$$

Next we compute the evolution of Φ with respect to θ to get

$$\begin{aligned}
&\frac{d}{d\theta}\left[E(v)-E(v^\infty)+\frac{\mu}{2\delta}\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\right]\leq\\
&\qquad\leq-(1-\delta)\int_{\mathbb{R}}vW^2\mathrm{d}y+\frac{\mathrm{e}^{-2\theta}}{4\delta}\int_{\mathbb{R}}v\sigma_y^2\mathrm{d}y+\mathrm{e}^{-2\theta}\frac{\mu}{2\delta}\|v\|_\infty\int_{\mathbb{R}}vW^2\mathrm{d}y\\
&\qquad\quad+2\mathrm{e}^{-2\theta}\frac{\mu}{2\delta}\|v\|_\infty\|v\|_1-\mathrm{e}^{-2\theta}\frac{\mu}{2\delta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y-\mathrm{e}^{-4\theta}\frac{\mu}{2\delta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\\
&\qquad\leq-(1-\delta-\mathrm{e}^{-2\theta}\frac{\mu^2}{2\delta})\int_{\mathbb{R}}vW^2\mathrm{d}y-\frac{\mathrm{e}^{-2\theta}}{4\delta}\mu\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y+2\mathrm{e}^{-2\theta}\frac{\mu^2}{2\delta}\|v\|_1\\
&\qquad\quad-\mathrm{e}^{-4\theta}\frac{\mu}{2\delta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\\
&\qquad\leq-(1-\delta-\mathrm{e}^{-2\theta}\frac{\mu^2}{2\delta})\int_{\mathbb{R}}vW^2\mathrm{d}y+2\mathrm{e}^{-2\theta}\frac{\mu^2}{2\delta}\|v\|_1-\mathrm{e}^{-2\theta}\frac{\mu}{2\delta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y
\end{aligned}$$

which implies, for $\theta\geq\theta^*$ with a fixed $\theta^*>0$,

$$\begin{aligned}
&\frac{d}{d\theta}\left[E(v)-E(v^\infty)+\frac{\mu}{2\delta}\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\right]\\
&\qquad\leq-(1-2\delta)\int_{\mathbb{R}}vW^2\mathrm{d}y+O(\mathrm{e}^{-2\theta})-A\mathrm{e}^{-4\theta}\frac{\mu}{2\delta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y
\end{aligned}\tag{1.115}$$

with $A:=e^{2\theta^*}$. Let us recall the following version of the Log–Sobolev inequality (cf. [4])

$$2(E(v)-E(v^\infty))\leq\int_{\mathbb{R}}vW^2\mathrm{d}y.$$

and thus the previous estimate (1.115) reads

$$\begin{aligned}
&\frac{d}{d\theta}\left[E(v)-E(v^\infty)+\frac{\mu}{2\delta}\mathrm{e}^{-4\theta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y\right]\\
&\qquad\leq-(2-4\delta)(E(v)-E(v^\infty))+O(\mathrm{e}^{-2\theta})-A\mathrm{e}^{-4\theta}\frac{\mu}{2\delta}\int_{\mathbb{R}}\sigma_y^2\mathrm{d}y
\end{aligned}$$

for $\theta\geq\theta^*$. Now, we choose θ^* such that $A=2-4\delta$ in order to obtain

$$\frac{d}{d\theta}\Phi(\theta,v,\sigma)\leq-(2-4\delta)\Phi(v,\sigma,\theta)+O(\mathrm{e}^{-2\theta})\tag{1.116}$$

for all $\theta\geq\theta^*$, whence,

$$\Phi(v,\sigma,\theta)\leq C\mathrm{e}^{-(2-4\delta)\theta}.\tag{1.117}$$

Here a suitable constant $C > 0$ (depending on the initial data) can be chosen in such a way that (1.117) is valid for all $\theta > 0$. This can be done by proving that the modified entropy functional $\Phi(v(\theta), \sigma(\theta), \theta)$ is uniformly bounded on all compact intervals $\theta \in [0, t^*]$. In order to prove that, we can combine the time–integrated version of (1.111)

$$E(v(\theta)) + (1-\delta)\int_0^\theta \int_{\mathbb{R}} vW^2 \mathrm{d}y\mathrm{d}\theta' \leq E(v_0) + \int_0^\theta \frac{\mathrm{e}^{-2\theta'}}{4\delta}\int_{\mathbb{R}} v\sigma_y^2 \mathrm{d}y\mathrm{d}\theta'$$

with the following consequence of (1.113)

$$\mathrm{e}^{-4\theta}\int_{\mathbb{R}} \sigma_y^2 \mathrm{d}y \leq \int_{\mathbb{R}} \sigma_y^2(y,0)\mathrm{d}y + \int_0^\theta \mathrm{e}^{-2\theta'}\|v\|_\infty \int_{\mathbb{R}} vW^2\mathrm{d}y\mathrm{d}\theta' + 2\theta\|v\|_\infty\|v\|_1$$

in order to achieve an estimate of the form

$$\mathrm{e}^{-4\theta}\int_{\mathbb{R}} \sigma_y^2 \mathrm{d}y \leq C(1+\theta) + Ce^{2\theta}\mathrm{e}^{-4\theta}\int_{\mathbb{R}} \sigma_y^2\mathrm{d}y$$

where C depends on δ, on μ and on the initial data. The last inequality proves that $\int \sigma_y^2 \mathrm{d}y$ is finite for finite times and this proves the assertion due to (1.111).
The statement (1.107) follows by the Csizar-Kullbak inequality

$$E(v) - E(v^\infty) \geq \|v - v^\infty\|_{L^1}^2$$

(see [5]). By going back to the original variables $\rho = \rho(t,x)$ we also recover (1.108) and the proof is complete. □

Remark 1.41. It is clearly seen from the above procedure that the proof of theorem 1.38 could be easily generalized to the multi– dimensional case provided one has a uniform bound for the re–scaled variables v and σ, which corresponds to a polynomial decay for ρ and S in L^∞ with rate $t^{-d/2}$.

Remark 1.42. It is well known that under similar assumptions on the initial data, the heat equation produces a rate of convergence to self similarity in L^1 of the form $t^{-1/2}$ in 1 space dimension. In this sense, we can state that the rate of convergence here is 'quasi sharp'.

Remark 1.43. By comparing our result with the one of ([31]) concerning with self similar decay, we recover that the decay rate toward self similarity for the density of cells ρ is the same as in the parabolic elliptic model. Whether S features also a self similar behavior for large times is an open problem, which we shall address in the future.

Chapter 2

The Aggregation Equation

The contents of this chapter appear in:

- *Carrillo, J. A.; Rosado, J. "Uniqueness of bounded solutions to Aggregation equations by optimal transport methods". To appear in the Proceedings of the 5th European Congress of Mathematicians. [45].*
- *Bertozzi, A. L.; Laurent, T.; Rosado, J. "L^p theory for the aggregation equation". To appear in Comm. Pur. Appl. Math.. [16].*

This chapter is devoted to the study of the aggregation equation

$$u'(t) + \mathrm{div}\big(\, u(t)\, v(t) \,\big) = 0 \qquad \forall t \in [0, T^*], \tag{2.1}$$

$$v(t) = -u(t) * \nabla K \qquad \forall t \in [0, T^*], \tag{2.2}$$

$$u(0) = u_0. \tag{2.3}$$

This kind of nonlocal interaction equations have been proposed as models for velocity distributions of inelastic colliding particles [11, 10, 166, 43, 44]. Here, typical interaction kernels $K(x)$ are convex and increasing algebraically at infinity. Convexity gives rates of expansion/contraction of distances between solutions, see also [1], and thus uniqueness.
Other source of these models is in the field of collective animal behavior. One of the mathematical problems arising there is the analysis of the long time behavior of a collection of self-interacting individuals via pairwise potentials leading to patterns such as flocks, schools or swarms formed by insects, fishes and birds. The simplest models based on ODE/SDEs systems, for instance [24, 138], led to continuum descriptions [30, 29, 163, 164] for the evolution of densities of individuals. Here, one of the typical potentials used is the Morse potential, which is radial $K(x) = k(|x|)$ and given by

$$k(r) = -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r}, \tag{2.4}$$

with C_a, C_r attractive and repulsive strengths and ℓ_a, ℓ_r their respective length scales. Typically, these interaction potentials are not convex, and they are composed of an attraction part usually in a certain annular region and a repulsive region closed to the origin while the interaction gets asymptotically zero for large distances, see [29]. Global existence and uniqueness of weak solutions in Sobolev spaces when the potential is well-behaved and smooth, say $K \in C^2(\mathbb{R}^d)$ with bounded second derivatives, were established in [163, 116]. Uniqueness results in the smooth potential case also follows from the general theory developed in [1] as used in [30]. One of the interesting mathematical difficulties in these problems relates to the case of only attractive potentials with a Lipschitz point at the origin as the Morse Potential with $C_r = 0$. In this particular case, finite time blow-up for $L^1 - L^\infty$ solutions have been proved for compactly supported initial data, see [116, 14, 12, 13] for a series of results in this direction. In this particular case, a result of uniqueness of $L^1 - L^\infty$ solutions under some additional technical hypotheses was obtained in [12] inspired by ideas from 2D-incompressible Euler equations in fluid mechanics [178].
Finally, another source of problems of this form is the so-called Patlak-Keller-Segel (PKS) model [148, 110] for chemotaxis in the parabolic-elliptic approximation. This equation corresponds to the case in which the potential is the fundamental solution of the operator $-\Delta$ in any dimension. Originally, this model was written in two dimensions with linear diffusion, see [72, 20, 19] for a state of the art in two dimensions and [60] in larger dimensions. Therefore, in the rest we will refer as PKS equation without diffusion and the PKS equation. In the case without diffusion, it is know that bounded solutions will exist locally in time and that smooth fast-decaying solutions cannot exist globally. In the classical PKS system in 2D dimensions, the mass is a critical quantity and thus there are global solutions below a critical mass and local in time solutions that may blow-up in time for mass values larger than the critical one. In more dimensions, this dichotomy is not so well-known and there are criteria for both situations.

2.1 Main results

Below we state the main results of this chapter and how they connect to previous results in the literature.

Theorem 2.1 (well-posedness). *Consider $1 < q < \infty$ and p its Hölder conjugate. Suppose $\nabla K \in W^{1,q}(\mathbb{R}^d)$ and $u_0 \in L^p(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ is nonnegative. Then there exists a time $T^* > 0$ and a unique nonnegative function*

$$u \in C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$$

such that

$$u'(t) + \operatorname{div}\big(u(t)\,v(t)\big) = 0 \qquad \forall t \in [0, T^*], \tag{2.5}$$
$$v(t) = -u(t) * \nabla K \qquad \forall t \in [0, T^*], \tag{2.6}$$
$$u(0) = u_0. \tag{2.7}$$

Moreover the second moment stays bounded and the L^1 norm is conserved. Furthermore, if $\operatorname{ess\,sup} \Delta K < +\infty$*, then we have global well-posedness.*

Theorem 2.1 1 is proved in sections 2.2 and 2.3. The fact that $W^{1,q_1}_{loc}(\mathbb{R}^d) \subset W^{1,q_2}_{loc}(\mathbb{R}^d)$ for $q_1 \leq q_2$ allows us to make the following definition:

Definition 2.2 (critical exponents q_s and p_s). *Suppose $\nabla K(x)$ is compactly supported (or decays exponentially fast as $|x| \to \infty$) and belongs to $W^{1,q}(\mathbb{R}^d)$ for some $q \in (1, +\infty)$. Then there exists an exponent $q_s \in (1, +\infty]$ such that $\nabla K \in W^{1,q}(\mathbb{R}^d)$ for all $q < q_s$ and $\nabla K \notin W^{1,q}(\mathbb{R}^d)$ for all $q > q_s$. The Hölder conjugate of this exponent q_s is denoted p_s.*

The exponent q_s quantifies the singularity of the potential. The more singular the potential, the smaller is q_s. For potentials that behave like a power function at the origin, $K(x) \sim |x|^\alpha$ as $|x| \to 0$, the exponents are easily computed:

$$q_s = \frac{d}{2-\alpha}, \qquad \text{and} \qquad p_s = \frac{d}{d-(2-\alpha)}, \qquad \text{if } 2-d < \alpha < 2, \tag{2.8}$$
$$q_s = +\infty, \qquad \text{and} \qquad p_s = 1, \qquad \text{if } \alpha \geq 2. \tag{2.9}$$

We obtain the following picture for power like potentials:

Theorem 2.3 (Existence and uniqueness for power potential). *Suppose that ∇K is compactly supported (or decays exponentially fast at infinity). Suppose also that $K \in C^2(\mathbb{R}^d \backslash \{0\})$ and $K(x) \sim |x|^\alpha$ as $|x| \to 0$.*

(i) If $2-d < \alpha < 2$ then the aggregation equation is locally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for every $p > p_s$. Moreover, it is not globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.

(ii) If $\alpha \geq 2$ then the aggregation equation is globally well posed for every $p > 1$ in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.

As a consequence we have existence and uniqueness for all potentials which are less singular than the Newtonian potential $K(x) = |x|^{2-d}$ at the origin. In two dimensions this includes potentials with cusp such as $K(x) = |x|^{1/2}$. In three dimensions this includes potentials that blow up such as $K(x) = |x|^{-1/2}$. From [13, 38] we know that the support of compactly supported solutions shrinks to a

point in finite time, proving the second assertion in point *(i)* above. The first part of *(i)* and statement (ii) are direct corollary of Theorem 2.1, Definition 2.2 and the fact that $\alpha \geq 2$ implies ΔK bounded.
In the case where $\alpha = 1$, i.e. $K(x) \sim |x|$ as $|x| \to 0$, the previous Theorem gives local well posedness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all $p > p_s = \frac{d}{d-1}$. The next Theorem shows that it is not possible to obtain local well posedness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $p < p_s = \frac{d}{d-1}$.

Theorem 2.4 (Critical p-exponent to generate instantaneous mass concentration). *Suppose $K(x) = |x|$ in a neighborhood of the origin, and suppose ∇K is compactly supported (or decays exponentially fast at infinity). Then, for any $p < p_s = \frac{d}{d-1}$, there exists initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for which a delta Dirac appears instantaneously in the measure solution.*

In order to make sense of the statement of the previous Theorem, we need a concept of measure solution. The potentials $K(x) = |x|$ is semi-convex, i.e. there exist $\lambda \in \mathbb{R}$ such that $K(x) - \frac{\lambda}{2}|x|^2$ is convex. In [38], Carrillo et al. prove global well-posedness in $\mathcal{P}_2(\mathbb{R}^d)$ of the aggregation equation with semi-convex potentials. The solutions in [38] are weak measure solutions - they are not necessarily absolutely continuous with respect to the Lebesgue measure. Theorems 2.3 and 2.4 give a sharp condition on the initial data in order for the solution to stay absolutely continuous with respect to the Lebesgue measure for short time. Theorem 2.4 is proven in Section 2.4.
Finally, in section 2.5 we consider a class of potential that will be referred to as the class of **natural potentials**. A potential is said to be natural if it satisfies that

a) it is a radially symmetric potential, i.e.: $K(x) = k(|x|)$,

b) it is smooth away from the origin and it's singularity at the origin is not worse than Lipschitz,

c) it doesn't exhibit pathological oscillation at the origin,

d) its derivatives decay fast enough at infinity.

All these conditions will be more rigorously stated later. It will be shown that the gradient of natural potentials automatically belongs to $W^{1,q}$ for $q < d$, therefore, using the results from the sections 2.2 and 2.3, we have local existence and uniqueness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > \frac{d}{d-1}$.
A natural potential is said to be repulsive in the short range if it has a local maximum at the origin and it is said to be attractive in the short range if it has a local minimum at the origin. If the maximum (respectively minimum) is strict, the natural potential is said to be strictly repulsive (respectively strictly attractive) at the origin. The main theorem of section 2.5 is the following:

Theorem 2.5 (Osgood condition for global well posedness). *Suppose K is a natural potential.*

(i) If K is repulsive in the short range, then the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$.

(ii) If K is strictly attractive in the short range, the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$, if and only if

$$r \mapsto \frac{1}{k'(r)} \text{ is not integrable at 0.} \tag{2.10}$$

By globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, we mean that for any initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ the unique solution of the aggregation equation will exist for all time and will stay in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all time. Notice that the exponent $d/(d-1)$ is not sharp in this theorem.
Condition (2.10) will be refered as the Osgood condition. It is easy to understand why the Osgood condition is relevant while studying blow-up: the quantity

$$T(d) = \int_0^d \frac{dr}{k'(r)}$$

can be thought as the amount of time it takes for a particle obeying the ODE $\dot{X} = -\nabla K(X)$ to reach the origin if it starts at a distance d from it. For a potential satisfying the Osgood condition, $T(d) = +\infty$, which means that the particle can not reach the origin in finite time. The Osgood condition was already shown in [13] to be necessary and sufficient for global well posedness of L^∞-solutions. Extension to L^p requires L^p estimates rather than L^∞ estimates. See also [175] for an example of the use of the Osgood condition in the context of the Euler equations for incompressible fluid.
The "only if" part of statement (ii) was proven in [13] and [38]. In these two works it was shown that if (2.10) is not satisfied, then compactly supported solutions will collapse into a point mass – and therefore leave L^p – in finite time. In section 2.5 we prove statement (i) and the "if" part of statement (ii).

2.2 Existence of L^p-solutions

In this section we show that if the interaction potential satisfies

$$\nabla K \in W^{1,q}(\mathbb{R}^d), \quad 1 < q < +\infty, \tag{2.11}$$

and if the initial data is nonnegative and belongs to $L^p(\mathbb{R}^d)$ (p and q are Hölder conjugates) then there exists a solution to the aggregation equation. Moreover, either this solution exists for all times, or its L^p-norm blows up in finite time. The duality between L^p and L^q guarantees enough smoothness in the velocity field $v = -\nabla K * u$ to define characteristics. We use the characteristics to construct a solution. The argument is inspired by the existence of L^∞ solutions of the incompressible 2D Euler equations by Yudovich [178] and of L^∞ solutions of the aggregation equation [12]. To prove the uniqueness we extend the ideas by Loeper [126]. We have moved this proof to Section 2.3, where we show how the method we use is valid for a more general family of equations. We prove in Theorem 2.18 of the present section that if $u_0 \in \mathcal{P}_2$ then the solution stays in $\mathcal{P}_2$, where the uniqueness holds. Finally we prove that if in addition to (2.11), we have

$$\text{ess sup } \Delta K < +\infty, \tag{2.12}$$

then the solution constructed exists for all time.
Most of the section is devoted to the proof of the following theorem:

Theorem 2.6 (Local existence). *Consider $1 < q < \infty$ and p its Hölder conjugate. Suppose $\nabla K \in W^{1,q}(\mathbb{R}^d)$ and suppose $u_0 \in L^p(\mathbb{R}^d)$ is nonnegative. Then there exists a time $T^* > 0$ and a nonnegative function*

$$u \in C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$$

such that

$$u'(t) + \operatorname{div}\big(\, u(t)\, v(t)\, \big) = 0 \qquad \forall t \in [0,T^*], \tag{2.13}$$

$$v(t) = -u(t) * \nabla K \qquad \forall t \in [0,T^*], \tag{2.14}$$

$$u(0) = u_0. \tag{2.15}$$

Moreover the function $t \to \|u(t)\|_{L^p}^p$ is differentiable and satisfies

$$\frac{d}{dt}\left\{\|u(t)\|_{L^p}^p\right\} = -(p-1)\int_{\mathbb{R}^d} u(t,x)^p \,\text{div}\, v(t,x)\, dx \qquad \forall t \in [0,T^*]. \tag{2.16}$$

The choice of the space

$$\mathcal{Y}_p := C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$$

is motivated by the fact that, if $u \in \mathcal{Y}_p$ and $\nabla K \in W^{1,q}$, then the velocity field is automatically C^1 in space and time:

Lemma 2.7. *Consider $1 < q < \infty$ and p its Hölder conjugate. If $\nabla K \in W^{1,q}(\mathbb{R}^d)$ and $u \in \mathcal{Y}_p$ then*

$$u * \nabla K \in C^1\left([0,T^*] \times \mathbb{R}^d\right)$$

and

$$\|u * \nabla K\|_{C^1([0,T^*]\times\mathbb{R}^d)} \leq \|\nabla K\|_{W^{1,q}(\mathbb{R}^d)}\, \|u\|_{\mathcal{Y}_p} \tag{2.17}$$

where the norm $\|\cdot\|_{C^1([0,T^*]\times\mathbb{R}^d)}$ *and* $\|\cdot\|_{\mathcal{Y}_p}$ *are defined by*

$$\|v\|_{C^1([0,T^*]\times\mathbb{R}^d)} = \sup_{[0,T^*]\times\mathbb{R}^d} |v| + \sup_{[0,T^*]\times\mathbb{R}^d} \left|\frac{\partial v}{\partial t}\right| + \sum_{i=1}^{d} \sup_{[0,T^*]\times\mathbb{R}^d} \left|\frac{\partial v}{\partial x_i}\right|, \tag{2.18}$$

$$\|u\|_{\mathcal{Y}_p} = \sup_{t\in[0,T^*]} \|u(t)\|_{L^p(\mathbb{R}^d)} + \sup_{t\in[0,T^*]} \|u'(t)\|_{W^{-1,p}(\mathbb{R}^d)}. \tag{2.19}$$

Proof.- Recall that the convolution between a L^p-function and a L^q-function is continuous and $\sup_{x\in\mathbb{R}^d} |f*g(x)| \leq \|f\|_{L^p}\|g\|_{L^q}$ Therefore, since ∇K and ∇K_{x_i} are in L^q, the mapping

$$f \mapsto \nabla K * f$$

is a bounded linear transformation from $L^p(\mathbb{R}^d)$ to $C^1(\mathbb{R}^d)$, where $C^1(\mathbb{R}^d)$ is endowed with the norm

$$\|f\|_{C^1} = \sup_{x\in\mathbb{R}^d} |f(x)| + \sum_{i=1}^{d} \sup_{x\in\mathbb{R}^d} |\frac{\partial f}{\partial x_i}(x)|.$$

Since $u \in C([0,T^*], L^p)$ it is then clear that $u*\nabla K \in C([0,T^*], C^1)$. In particular $w(t,x) = (u(t) * \nabla K)(x)$ and $\frac{\partial w}{\partial x_i}(t,x)$ are continuous on $[0,T^*]\times\mathbb{R}^d$. Let us now show that $\frac{\partial w}{\partial t}(t,x)$ exists and is continuous on $[0,T^*]\times\mathbb{R}^d$. Since $u'(t) \in C([0,T^*], W^{-1,p})$ and $\nabla K \in W^{1,q}$, we have

$$\frac{\partial w}{\partial t}(t,x) = -(u'(t) * \nabla K)(x) = -\langle u'(t), \tau_x \nabla K\rangle$$

where $\langle\ ,\ \rangle$ denote the pairing between the two dual spaces $W^{-1,p}(\mathbb{R}^d)$ and $W^{1,q}(\mathbb{R}^d)$, and τ_x denote the translation by x. Since $x \mapsto \tau_x\nabla K$ is a continuous mapping from $\mathbb{R}^d$ to $W^{1,q}$ it is clear that $\frac{\partial w}{\partial t}(t,x)$ is continuous with respect to space. The continuity with respect to time come from the continuity of $u'(t)$ with respect to time. Inequality (2.17) is easily obtained. ∎

Remark 2.8. *Let us point out that (2.13) indeed makes sense, when understood as an equality in* $W^{-1,p}$ *. Since* $v \in C([0,T^*], C^1(\mathbb{R}^d))$ *one can easily check that* $uv \in C([0,T^*], L^p(\mathbb{R}^d))$*. Also recall that the injection* $i : L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ *and the differentiation* $\partial_{x_i} : L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ *are bounded linear operators. Therefore it is clear that both* u *and div*(uv) *belong to* $C([0,T^*], W^{-1,p}(\mathbb{R}^d))$*. Equation* (2.13) *has to be understand as an equality in* $W^{-1,p}$*.*

The rest of this section is organized as follows. First we give the basic a priori estimates in subsection 2.2.1 Then, in subsection 2.2.2, we consider a mollified and cut-off version of the aggregation equation for which we have global existence of smooth and compactly supported solutions. In subsection 2.2.3 we show that the characteristics of this approximate problem are uniformly Lipschitz continuous on $[0,T^*]\times\mathbb{R}^d$, where $T^*>0$ is some finite time depending on $\|u_0\|_{L^p}$. In subsection 2.2.4 we pass to the limit in $C([0,T^*],L^p)$. To do this we need the uniform Lipschitz bound on the characteristics together with the fact that the translation by x, $x\mapsto\tau_x u_0$, is a continuous mapping from $\mathbb{R}^d$ to $L^p(\mathbb{R}^d)$. In subsection 2.2.5 we prove three theorems. We first prove continuation of solutions. We then prove that L^p-solutions which start in $\mathcal{P}_2$ stay in $\mathcal{P}_2$ as long as they exist. And finally we prove global existence in the case where ΔK is bounded from above.

2.2.1 A priori estimates

Suppose $u\in C_c^1((0,T)\times\mathbb{R}^d)$ is a nonnegative function which satisfies (2.13)-(2.14) in the classical sense. Suppose also that $K\in C_c^\infty(\mathbb{R}^d)$. Integrating by part, we obtain that for any $p\in(1,+\infty)$:

$$\frac{d}{dt}\int_{\mathbb{R}^d}u(t,x)^p dx=-(p-1)\int_{\mathbb{R}^d}u(t,x)^p\,\text{div}\,v(t,x)dx\quad\forall t\in(0,T).\tag{2.20}$$

As a consequence we have:

$$\frac{d}{dt}\|u(t)\|_{L^p}^p\le(p-1)\|\text{div}\,v(t)\|_{L^\infty}\|u(t)\|_{L^p}^p\quad\forall t\in(0,T),\tag{2.21}$$

and by Hölder's inequality:

$$\frac{d}{dt}\|u(t)\|_{L^p}^p\le(p-1)\|\Delta K\|_{L^q}\|u(t)\|_{L^p}^{p+1}.\tag{2.22}$$

We now derive L^∞ estimates for the velocity field $v=-\nabla K*u$ and its derivatives. Hölder's inequality easily gives

$$|v(t,x)|\le\|u(t)\|_{L^p}\|\nabla K\|_{L^q}\quad\forall(t,x)\in(0,T)\times\mathbb{R}^d,\tag{2.23}$$

$$\left|\frac{\partial v_j}{\partial x_i}(t,x)\right|\le\|u(t)\|_{L^p}\left\|\frac{\partial^2 K}{\partial x_i dx_j}\right\|_{L^q}\quad\forall(t,x)\in(0,T)\times\mathbb{R}^d.\tag{2.24}$$

Since $\frac{\partial v}{\partial t}=-\nabla K*\frac{\partial u}{\partial t}=\nabla K*\text{div}(uv)=\Delta K*uv$ we have

$$\left|\frac{\partial v}{\partial t}(t,x)\right|\le\|u(t)v(t)\|_{L^p}\|\Delta K\|_{L^q}\le\|u(t)\|_{L^p}\|v(t)\|_{L^\infty}\|\Delta K\|_{L^q},$$

which in light of (2.23) gives

$$\left|\frac{\partial v}{\partial t}(t,x)\right|\le\|u(t)\|_{L^p}^2\|\nabla K\|_{L^q}\|\Delta K\|_{L^q}\quad\forall(t,x)\in(0,T)\times\mathbb{R}^d.\tag{2.25}$$

2.2.2 Approximate smooth compactly supported solutions

In this section we deal with a smooth version of equation (2.13)-(2.15). Suppose $u_0 \in L^p(\mathbb{R}^d)$, $1 < p < +\infty$, and $\nabla K \in W^{1,q}(\mathbb{R}^d)$. Consider the approximate problem

$$u_t + \text{div}(uv) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^d, \tag{2.26}$$
$$v = -\nabla K^\epsilon * u \qquad \text{in } (0, +\infty) \times \mathbb{R}^d, \tag{2.27}$$
$$u(0) = u_0^\epsilon, \tag{2.28}$$

where $K^\epsilon = J_\epsilon K,\ u_0^\epsilon = J_\epsilon u_0$ and J_ϵ is an operator which mollifies and cuts-off, $J_\epsilon f = (f M_{R_\epsilon}) * \eta_\epsilon$ where $\eta_\epsilon(x)$ is a standard mollifier:

$$\eta_\epsilon(x) = \frac{1}{\epsilon^d}\eta\left(\frac{x}{\epsilon}\right),\ \eta \in C_c^\infty(\mathbb{R}^d), \eta \geq 0, \int_{\mathbb{R}^d} \eta(x)dx = 1,$$

and $M_{R_\epsilon}(x)$ is a standard cut-off function: $M_{R_\epsilon}(x) = M(\frac{x}{R_\epsilon})$, $R_\epsilon \to \infty$ as $\epsilon \to 0$,

$$M \in C_c^\infty(\mathbb{R}^d), \qquad \begin{cases} M(x) = 1 & \text{if } |x| \leq 1, \\ 0 < M(x) < 1 & \text{if } 1 < |x| < 2, \\ M = 0 & \text{if } 2 \leq |x|. \end{cases}$$

Let τ_x denote the translation by x, i.e.:

$$\tau_x f(y) := f(y - x).$$

It is well known that given a fixed $f \in L^r(\mathbb{R}^d),\ 1 < r < +\infty$, the mapping $x \mapsto \tau_x f$ from $\mathbb{R}^d$ to $L^r(\mathbb{R}^d)$ is uniformly continuous. In (iv) of the next lemma we show a slightly stronger result which will be needed later.

Lemma 2.9 (Properties of J_ϵ). *Suppose $f \in L^r(\mathbb{R}^d),\ 1 < r < +\infty$, then*

(i) $J_\epsilon f \in C_c^\infty(\mathbb{R}^d)$,

(ii) $\|J_\epsilon f\|_{L^r} \leq \|f\|_{L^r}$,

(iii) $\lim_{\epsilon \to 0} \|J_\epsilon f - f\|_{L^r} = 0$,

(iv) The family of mappings $x \mapsto \tau_x J_\epsilon f$ from $\mathbb{R}^d$ to $L^r(\mathbb{R}^d)$ is equicontinuous, i.e.: for each $\delta > 0$, there exist a $\eta > 0$ independent of ϵ such that if $|x - y| \leq \eta$, then$\|\tau_x J_\epsilon f - \tau_y J_\epsilon f\|_{L^r} \leq \delta$.

Proof.- Statements (i) and (ii) are obvious. If f is compactly supported, one can easily prove (iii) by noting that $fM_{R_\epsilon} = f$ for ϵ small enough. If f is not compactly supported, (iii) is obtained by approximating f by a compactly supported function and by using (ii). Let us now turn to the proof of (iv). Using (ii) we obtain

$$\begin{aligned} \|\tau_x J_\epsilon f - J_\epsilon f\|_{L^r} &\leq \|(\tau_x M_{R_\epsilon})(\tau_x f) - M_{R_\epsilon} f)\|_{L^r} \\ &\leq \|\tau_x M_{R_\epsilon} - M_{R_\epsilon}\|_{L^\infty} \|\tau_x f\|_{L^r} + \|M_{R_\epsilon}\|_{L^\infty} \|\tau_x f - f\|_{L^r}. \end{aligned}$$

Because $x \mapsto \tau_x f$ is continuous, the second term can be made as small as we want by choosing $|x|$ small enough. Since $\|\tau_x M_{R_\epsilon} - M_{R_\epsilon}\|_{L^\infty} \leq \|\nabla M_{R_\epsilon}\|_{L^\infty}|x| \leq \frac{1}{R_\epsilon}\|\nabla M\|_{L^\infty}|x|$, the first term can be made as small as we want by choosing $|x|$ small enough and independently of ϵ. ∎

Proposition 2.10 (Global existence of smooth compactly-supported approximates). *Given $\epsilon, T > 0$, there exists a nonnegative function $u \in C^1_c((0,T) \times \mathbb{R}^d)$ which satisfy (2.28) in the classical sense.*

Proof.- Since u_0^ϵ and K^ϵ belong to $C_c^\infty(\mathbb{R}^d)$, we can use [116, Theorem 3, p. 1961] to get the existence of a function u^ϵ satisfying

$$u^\epsilon \in L^\infty(0,T;H^k), u_t^\epsilon \in L^\infty, (0,T;H^{k-1}) \text{ for all } k, \tag{2.29}$$

$$u_t^\epsilon + \text{ div } (u^\epsilon(-\nabla K^\epsilon * u^\epsilon)) = 0 \text{ in } (0,T) \times \mathbb{R}^d, \tag{2.30}$$

$$u^\epsilon(0) = u_0^\epsilon, \tag{2.31}$$

$$u^\epsilon(t,x) \geq 0 \text{ for a.e. } (t,x) \in (0,T), \times\mathbb{R}^d. \tag{2.32}$$

Statement (2.29) implies that $u^\epsilon \in C((0,T);H^{k-1})$. Using the continuous embedding $H^{k-1}(\mathbb{R}^d) \subset C^1(\mathbb{R}^d)$ for k large enough we find that u^ϵ and $u^\epsilon_{x_i}$, $1 \leq i \leq d$, are continuous on $(0,T) \times \mathbb{R}^d$. Finally, (2.30) shows that u_t^ϵ is also continuous on $(0,T) \times \mathbb{R}^d$. We have proven that $u^\epsilon \in C^1((0,T) \times \mathbb{R}^d)$. It is then obvious that $v^\epsilon = -\nabla K^\epsilon * u^\epsilon \in C^1((0,T) \times \mathbb{R}^d)$. Note moreover that

$$|v^\epsilon(t,x)| \leq \|u^\epsilon\|_{L^\infty(0,T;L^2)} \|\nabla K^\epsilon\|_{L^2}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$. This combined with the fact that v^ϵ is in C^1 shows that the characteristics are well defined and propagate with finite speed. This proves that u^ϵ is compactly supported in $(0,T) \times \mathbb{R}^d$ (because u_0^ϵ is compactly supported in $\mathbb{R}^d$). ∎

2.2.3 Study of the velocity field and the induced flow map

Note that K^ϵ and u^ϵ are in the right function spaces so that we can apply to them to the a priori estimates derived in section 2.1. In particular we have:

$$\frac{d}{dt}\|u^\epsilon(t)\|_{L^p}^p \leq (p-1)\|\Delta K\|_{L^q}\|u^\epsilon(t)\|_{L^p}^{p+1},$$
$$\|u^\epsilon(0)\|_{L^p} \leq \|u_0\|_{L^p}.$$

Using Gronwall inequality and the estimate on the supremum norm of the derivatives derived in section 2.1 we obtain:

Lemma 2.11 (uniform bound for the smooth approximates). *There exists a time $T^* > 0$ and a constant $C > 0$, both independent of ϵ, such that*

$$\|u^\epsilon(t)\|_{L^p} \leq C \qquad \forall t \in [0, T^*], \tag{2.33}$$
$$|v^\epsilon(t,x)|,\ |v^\epsilon_{x_i}(t,x)|,\ |v^\epsilon_t(t,x)| \ \leq C \qquad \forall (t,x) \in [0,T^*] \times \mathbb{R}^d. \tag{2.34}$$

From (2.34) it is clear that the family $\{v^\epsilon\}$ is uniformly Lipschitz on $[0,T^*] \times \mathbb{R}^d$, with Lipschitz constant C. We can therefore use the Arzelà-Ascoli Theorem to obtain the existence of a continuous function $v(t,x)$ such that

$$v^\epsilon \to v \text{ uniformly on compact subset of } [0,T^*] \times \mathbb{R}^d. \tag{2.35}$$

It is easy to check that this function v is also Lipschitz continuous with Lipschitz constant C. The Lipschitz and bounded vector field v^ϵ generates a flow map $X_\epsilon(t,\alpha),\ t \in [0,T^*],\ \alpha \in \mathbb{R}^d$:

$$\frac{\partial X_\epsilon(t,\alpha)}{\partial t} = v^\epsilon(t, X_\epsilon(t,\alpha)),$$
$$X_\epsilon(0,\alpha) = \alpha,$$

where we denote by $X_\epsilon^t : \mathbb{R}^d \to \mathbb{R}^d$ the mapping $\alpha \mapsto X_\epsilon(t,\alpha)$ and by X_ϵ^{-t} the inverse of X_ϵ^t.
The uniform Lipschitz bound on the vector field implies uniform Lipschitz bound on the flow map and its inverse (see for example [12, Lemma 4.2,4.3, p. 7] for a proof of this statement) we therefore have:

Lemma 2.12 (uniform Lipschitz bound on X_ϵ^t and X_ϵ^{-t}). *There exists a constant $C > 0$ independent of ϵ such that:*

(i) *for all $t \in [0,T^*]$ and for all $x_1, x_2 \in \mathbb{R}^d$*

$$|X_\epsilon^t(x_1) - X_\epsilon^t(x_2)| \leq C|x_1 - x_2| \quad \textit{and} \quad |X_\epsilon^{-t}(x_1) - X_\epsilon^{-t}(x_2)| \leq C|x_1 - x_2|,$$

(ii) *for all* $t_1, t_2 \in [0, T^*]$ *and for all* $x \in \mathbb{R}^d$

$$|X_\epsilon^{t_1}(x) - X_\epsilon^{t_2}(x)| \leq C|t_1 - t_2| \quad \textit{and} \quad |X_\epsilon^{-t_1}(x) - X_\epsilon^{-t_2}(x)| \leq C|t_1 - t_2|.$$

The Arzela-Ascoli Theorem then implies that there exists mapping X^t and X^{-t} such that

$$\begin{aligned} X_{\epsilon_k}^t(x) \to X^t(x) \qquad &\text{uniformly on compact subset of } [0, T^*] \times \mathbb{R}^d, \\ X_{\epsilon_k}^{-t}(x) \to X^{-t}(x) \qquad &\text{uniformly on compact subset of } [0, T^*] \times \mathbb{R}^d. \end{aligned}$$

Moreover it is easy to check that X^t and X^{-t} inherit the Lipschitz bounds of X_ϵ^t and X_ϵ^{-t}.
Since the mapping $X^t : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous, by Rademacher's Theorem it is differentiable almost everywhere. Therefore it makes sense to consider its Jacobian matrix $DX^t(\alpha)$. Because of Lemma 2.12-(i) we know that there exists a constant C independent of t and ε such that

$$\sup_{\alpha \epsilon \mathbb{R}^d} |\text{det}\, DX^t(\alpha)| \leq C \qquad \text{and} \qquad \sup_{\alpha \in \mathbb{R}^d} |\text{det}\, DX_\epsilon^t(\alpha)| \leq C.$$

By the change of variable we then easily obtain the following Lemma:

Lemma 2.13. *The mappings* $f \mapsto f \circ X^{-t}$ *and* $f \mapsto f \circ X_\epsilon^{-t},\ t \in [0, T^*],\ \epsilon > 0,$ *are bounded linear operators from* $L^p(\mathbb{R}^d)$ *to* $L^p(\mathbb{R}^d)$. *Moreover there exists a constant* C^* *independent of* t *and* ϵ *such that*

$$\|f \circ X^{-t}\|_{L^p} \leq C^* \|f\|_{L^p} \quad \textit{and} \quad \|f \circ X_\epsilon^{-t}\|_{L^p} \leq C^* \|f\|_{L^p} \quad \textit{for all } f \in L^p(\mathbb{R}^d).$$

Note that Lemma 2.12-(ii) implies

$$|X_\epsilon^t(\alpha) - \alpha| \leq Ct \text{ for all } (t, \alpha) \in [0, T^*) \times \mathbb{R}^d,$$

and therefore

$$|X^t(\alpha) - \alpha| \leq Ct \text{ for all } (t, \alpha) \in [0, T^*) \times \mathbb{R}^d.$$

This gives us the following lemma:

Lemma 2.14. *Let* Ω *be a compact subset of* $\mathbb{R}^d$, *then*

$$X_\epsilon^t(\Omega) \subset \Omega + Ct \qquad \textit{and} \qquad X^t(\Omega) \subset \Omega + Ct,$$

where the compact set $\Omega + Ct$ *is defined by*

$$\Omega + Ct := \{x \in \mathbb{R}^d : \textit{ dist } (x, \Omega) \leq Ct\}.$$

2.2.4 Convergence in $C([0,T^*),L^p)$

Since u^ϵ and $v^\epsilon = u^\epsilon * \nabla K$ are C^1 functions which satisfy

$$u_t^\epsilon + v^\epsilon \cdot \nabla u^\epsilon = -(\text{div } v^\epsilon)u^\epsilon \qquad \text{and} \qquad u^\epsilon(0) = u_0^\epsilon \tag{2.36}$$

we have the simple representation formula for $u^\epsilon(t,x),\ t \in [0,T^*),\ x \in \mathbb{R}^d$:

$$u^\epsilon(t,x) = u_0^\epsilon(X_\epsilon^{-t}(x))e^{-\int_0^t \text{div } v^\epsilon(s,X_\epsilon^{-(t-s)}(x))ds} = u_0^\epsilon(X_\epsilon^{-t}(x))\, a^\epsilon(t,x).$$

Lemma 2.15. *There exists a function* $a(t,x) \in C^1([0,T^*) \times \mathbb{R}^d)$ *and a sequence* $\epsilon_k \to 0$ *such that*

$$a^{\epsilon_k}(t,x) \to a(t,x) \text{ uniformly on compact subset of } [0,T^*] \times \mathbb{R}^d. \tag{2.37}$$

Proof.- By the Arzelà-Ascoli Theorem, it is enough to show that the family

$$b^\epsilon(t,x) := \int_0^t \text{div } v^\epsilon(s,X_\epsilon^{-(t-s)}(x))ds$$

is equicontinuous and uniformly bounded. The uniform boundedness simply come from the fact that

$$|\text{div } v^\epsilon| = |u^\epsilon * \Delta K^\epsilon| \le \|u^\epsilon\|_{L^p}\|\Delta K\|_{L^q}.$$

Let us now prove equicontinuity in space, i.e., we want to prove that for each $\delta > 0$, there is $\eta > 0$ independent of ϵ and t such that

$$|b^\epsilon(t,x_1) - b^\epsilon(t,x_2)| \le \delta \text{ if } |x_1 - x_2| \le \eta.$$

First, note that by Hölder's inequality we have

$$|b^\epsilon(t,x_1) - b^\epsilon(t,x_2)| \le \int_0^t \|u^\epsilon(s)\|_{L^p}\|\tau_\xi J_\epsilon \Delta K - \tau_\zeta J_\epsilon \Delta K\|_{L^q}\, ds$$

where ξ stands for $X_\epsilon^{-(t-s)}(x_1)$ and ζ for $X_\epsilon^{-(t-s)}(x_2)$. Then equicontinuity in space is a consequence of Lemma 2.9 (iv) together with the fact that

$$|X_\epsilon^{-(t-s)}(x_1) - X_\epsilon^{-(t-s)}(x_2)| \le C|x_1 - x_2|$$

where C is independent of t, s and ϵ.
Let us finally prove equicontinuity in time. First note that, assuming that $t_1 < t_2$,

$$\begin{aligned} b^\epsilon(t_1,x) - b^\epsilon(t_2,x) &= \int_0^{t_1} \text{div } v^\epsilon(s,X^{-(t_1-s)}(x)) - \text{div } v^\epsilon(s,X^{-(t_2-s)}(x))ds \\ &- \int_{t_1}^{t_2} \text{div } v^\epsilon(s,X_\epsilon^{-(t_2-s)}(x))ds. \end{aligned}$$

Since div v^ϵ is uniformly bounded we clearly have

$$\left| \int_{t_1}^{t_2} \text{div } v^\epsilon(s, X_\epsilon^{-(t_2-s)}(x)) ds \right| \leq C|t_1 - t_2|.$$

The other term can be treated exactly as before, when we proved equicontinuity in space. ∎

Recall that the function

$$u^\epsilon(t,x) = u_0^\epsilon(X_\epsilon^{-t}(x)) \; a^\epsilon(t,x)$$

satisfies the ϵ-problem (2.28). We also have the following convergences:

$$u_0^\epsilon \to u_0 \qquad \text{in } L^p(\mathbb{R}^d), \tag{2.38}$$

$$X_{\epsilon_k}^{-t}(x) \to X^{-t}(x) \qquad \text{unif. on compact subset of } [0,T^*] \times \mathbb{R}^d, \tag{2.39}$$

$$a^{\epsilon_k}(t,x) \to a(t,x) \qquad \text{unif. on compact subset of } [0,T^*] \times \mathbb{R}^d, \tag{2.40}$$

$$v^{\epsilon_k}(t,x) \to v(t,x) \qquad \text{unif. on compact subset of } [0,T^*] \times \mathbb{R}^d. \tag{2.41}$$

Define the function

$$u(t,x) := u_0(X^{-t}(x)) \; a(t,x). \tag{2.42}$$

Convergence (2.38)-(2.41) together with Lemma 2.13 and 2.14 allow us to prove the following proposition.

Proposition 2.16. *: u, u^ϵ, uv and $u^\epsilon v^\epsilon$ all belong to the space $C([0,T^*), L^p(\mathbb{R}^d))$. Moreover we have:*

$$u^{\epsilon_k} \to u \qquad \text{in } C([0,T^*), L^p), \tag{2.43}$$

$$u^{\epsilon_k} v^{\epsilon_k} \to uv \qquad \text{in } C([0,T^*), L^p). \tag{2.44}$$

Proof.- Since all the convergences (2.39)-(2.41) take place on compact sets, one of the key ideas of this proof will be to approximate $u_0 \in L^p$ by a function with compact support and to use the fact that X^t maps compact sets to compact sets (Lemma 2.14).

PART I: We will prove that $u(t,x) = u_0(X^{-t}(x))a(t,x)$ belongs to the space $C([0,T^*), L^p(\mathbb{R}^d))$. Assume first that $u_0 \in C_c(\mathbb{R}^d)$. The function u_0 is then uniformly continuous and, since

$$\sup_{x \in \mathbb{R}^d} |X^{-t}(x) - X^{-s}(x)| \leq C|t-s|,$$

it is clear that the quantity

$$\|u_0(X^{-t}(x)) - u_0(X^s(x))\|_{L^p}$$

can be made as small as we want by choosing $|t-s|$ small enough. We have therefore proven that $u_0(X^{-t}(x)) \in C([0,T^*), L^p(\mathbb{R}^d)$. Assume now that $u_0 \in L^p(\mathbb{R}^d)$. Approximate it by a function $g \in C_c(\mathbb{R}^d)$ and write:

$$\begin{aligned} \|u_0(X^{-t}(x)) - u_0(X^{-s}(x))\|_{L^p} &\leq \|u_0(X^{-t}(x)) - g(X^{-t}(x))\|_{L^p} \\ &\quad +\|g(X^{-t}(x)) - g(X^{-s}(x))\|_{L^p} \\ &\quad +\|g(X^{-s}(x)) - u_0(X^{-s}(x))\|_{L^p} \\ &= I + II + III. \end{aligned}$$

As we have seen above, the second term can be made as small as we want by choosing $|t-s|$ small enough. Using Lemma 2.13 we get

$$I, III \leq C^* \|u_0 - g\|_{L^p},$$

which can be made as small as we want since $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. We have therefore proven that, if $u_0 \in L^p(\mathbb{R}^d)$, then

$$(t,x) \mapsto u_0(X^{-t}(x)) \in C([0,T^*), L^p(\mathbb{R}^d)). \tag{2.45}$$

Let us now consider the function $u_0(X^{-t}(x))a(t,x)$. Recall that $a(t,x)$ is continuous and bounded on $[0,T^*] \times \mathbb{R}^d$. Write

$$\begin{aligned} &\|u_0(X^{-t}(x))a(t,x) - u_0(X^{-s}(x))a(s,x)\|_{L^p} \leq \\ &\quad \|u_0(X^{-t}(x))\{a(t,x) - a(s,x)\}\|_{L^p} \\ &\quad +\|\{u_0(X^{-t}(x)) - u_0(X^{-s}(x))\}a(s,x)\|_{L^p} \\ &\quad = I + II. \end{aligned}$$

Since $a(s,x)$ is bounded, (2.45) implies that II can be made as small as we want by choosing $|t-s|$ small enough. Let us now take care of I. Approximate u_0 by a function $g \in C_c(\mathbb{R}^d)$ and write

$$\begin{aligned} I &\leq \|\{u_0(X^{-t}(x)) - g(X^{-t}(x))\}\{a(t,x) - a(s,x)\}\|_{L^p} \\ &\quad +\|g(X^{-t}(x))\{a(t,x) - a(s,x)\}\|_{L^p} \\ &= A + B. \end{aligned}$$

From Lemma 2.13 we have

$$A \leq 2\, C^* \|u_0 - g\|_{L^p} \sup_{(t,x)\in[0,T^*]\times\mathbb{R}^d} |a(t,x)|,$$

and since $C_c(\mathbb{R}^d)$ is dense in L^p, A can be made as small as we want. Let Ω denote the compact support of g. Using Lemmas 2.13 and 2.14 we obtain:

$$\begin{aligned} B &= \left(\int_{\Omega + Ct} \left| g(X^{-t}(x))\{a(t,x) - a(s,x)\} \right|^p dx \right)^{1/p} \\ &\leq \left(\sup_{x \in \Omega + ct} |a(t,x) - a(s,x)| \right) C^* \|g\|_{L^p}. \end{aligned}$$

Since $a(t,x)$ is uniformly continuous on compact subset of $[0,T^*]\times\mathbb{R}^d$, it is clear that by choosing $|t-s|$ small enough we can make B as small as we want. We have proven that

$$u \in C([0,T^*], L^p(\mathbb{R}^d).$$

PART II: In Part I we have proven that $u \in C([0,T^*], L^p)$. The same proof work to show that u^ϵ, uv and $u^\epsilon v^\epsilon$ belong to $C([0,T^*], L^p)$ ($v(t,x)$ is continuous and bounded, so we can handle it exactly like $a(t,x)$.)

PART III: Let us now prove that $u^\epsilon(t,x) = u_0^\epsilon(X_\epsilon^{-t}(x))a^\epsilon(t,x)$ converges to $u(t,x) = u_0(X^{-t}(x))a(t,x)$. For convenience we write ϵ instead of ϵ_k. To do this, we will successively prove:

$$u_0(X_\epsilon^{-t}(x)) \to u_0(X^{-t}(x)) \qquad \text{in } C([0,T^*),L^p), \tag{2.46}$$
$$u_0^\epsilon(X_\epsilon^{-t}(x)) \to u_0(X^{-t}(x)) \qquad \text{in } C([0,T^*),L^p), \tag{2.47}$$
$$u_0^\epsilon(X_\epsilon^{-t}(x))a^\epsilon(t,x) \to u_0(X^{-t}(x))a(t,x) \qquad \text{in } C([0,T^*),L^p). \tag{2.48}$$

To prove (2.46), approximate $u_0 \in L^p(\mathbb{R}^d)$ by a function $g \in C_c(\mathbb{R}^d)$ and write:

$$\begin{aligned}
\sup_{t\in[0,T^*)} &\|u_0(X_\epsilon^{-t}(x) - u_0(X^{-t}(x))\|_{L^p} \\
&\le \sup_{t\in[0,T^*)} \|u_0(X_\epsilon^{-t}(x)) - g(X_\epsilon^{-t}(x))\|_{L^p} \\
&\quad + \sup_{t\in[0,T^*)} \|g(X_\epsilon^{-t}(x)) - g(X^{-t}(x))\|_{L^p} \\
&\quad + \sup_{t\in[0,T^*)} \|g(X^{-t}(x)) - u_0(X^{-t}(x))\|_{L^p} \\
&= I + II + III.
\end{aligned}$$

From Lemma 2.13 it is clear that I and III can be made as small as we want by choosing an appropriate function g. Using Lemma 2.14, we see that, if Ω is the support of g

$$II = \sup_{t\in[0,T^*} \left(\int_{\Omega+Ct} \left| g(X_\epsilon^{-t}(x)) - g(X^{-t}(x)) \right|^p dx \right)^{1/p}.$$

Using the uniform continuity of g together with the fact that $X_\epsilon^{-t}(x)$ converges uniformly to $X^{-t}(x)$ on $[0,T^*]\times\Omega + CT^*$, we can make II as small as we want by choosing ϵ small enough.

This concludes the proof of (2.46). To prove (2.47), write

$$
\begin{aligned}
\sup_{t\in[0,T^*)} &\|u_0^\epsilon(X_\epsilon^{-t}(x)) - u_0(X^{-t}(x))\|_{L^p} \\
&\le \sup_{t\in[0,T^*)} \|u_0^\epsilon(X_\epsilon^{-t}(x)) - u_0(X_\epsilon^{-t}(x)\|_{L^p} \\
&\quad + \sup_{t\in[0,T^*)} \|u_0(X_\epsilon^{-t}(x)) - u_0(X^{-t}(x))\|_{L^p} \\
&= I + II.
\end{aligned}
$$

From (2.46) we know that II can be made as small as we want by choosing ϵ small enough. Using Lemma 2.13 we obtain

$$
I \le C^* \|u_0^\epsilon - u_0\|_{L^p} \to 0 \qquad \text{as } \epsilon \to 0.
$$

Let us now prove (2.48). Write

$$
\begin{aligned}
\sup_{t\in[0,T^*]} &\|u_0^\epsilon(X_\epsilon^{-t}(x))a^\epsilon(t,x) - u_0(X^{-t}(x))a(t,x)\|_{L^p} \\
&\le \sup_{t\in[0,T^*]} \|\{u_0^\epsilon(X_\epsilon^{-t}(x)) - u_0(X^{-t}(x))\}a^\epsilon(t,x)\|_{L^p} \\
&\quad + \sup_{t\in[0,T^*]} \|u_0(X^{-t}(y))\{a^\epsilon(t,x) - a(t,x)\}\|_{L^p} \\
&= I + II.
\end{aligned}
$$

Since $a^\epsilon(t,x)$ is uniformly bounded on $[0,T^*]\times\mathbb{R}^d$, it is clear from (2.47) that $I\to 0$ as $\epsilon\to 0$. If u_0 is in $C_c(\mathbb{R}^d)$, then it is easy to prove that $II\to 0$ as $\epsilon\to 0$. If $u_0\in L^p(\mathbb{R}^d)$, then approximate it by $g\in C_c(\mathbb{R}^d)$ and proceed as before.

Part IV: To prove that $u^{\epsilon_k}v^{\epsilon_k}$ converges to uv in $C([0,T^*],L^p)$, proceed exactly as in the proof of (2.48). □

We now turn to the proof of the main theorem of this section.

Proof.- [Proof of Theorem 2.6] Let $\phi\in C_c^\infty(0,T^*)$ be a scalar test function. It is obvious that u^ϵ and v^ϵ satisfy:

$$
-\int_0^{T^*} u^\epsilon(t)\,\phi'(t)\,dt + \int_0^{T^*} \mathrm{div}\big(\,u^\epsilon(t)\,v^\epsilon(t)\,\big)\ \phi(t)\ dt = 0, \tag{2.49}
$$

$$
v^\epsilon(t,x) = (u^\epsilon(t) * \nabla K^\epsilon)(x) \qquad \text{for all } (t,x)\in[0,T^*]\times\mathbb{R}^d, \tag{2.50}
$$

$$
u^\epsilon(0) = u_0^\epsilon, \tag{2.51}
$$

where the integrals in (2.49) are the integral of a continuous function from $[0,T^*]$ to the Banach space $W^{-1,p}(\mathbb{R}^d)$. Recall that the injection $i: L^p(\mathbb{R}^d)\to W^{-1,p}(\mathbb{R}^d)$

and the differentiation $\partial_{x_i} : L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ are bounded linear operators. Therefore (2.43) and (2.44) imply

$$u^{\epsilon_k} \to u \qquad \text{in } C([0,T^*], W^{-1,p}(\mathbb{R}^d)),$$
$$\text{div } [u^{\epsilon_k} v^{\epsilon_k}] \to \text{ div } [uv] \qquad \text{in } C([0,T^*], W^{-1,p}(\mathbb{R}^d)), \tag{2.52}$$

which is more than enough to pass to the limit in relation (2.49). To pass to the limit in (2.50), it is enough to note that for all $(t,x) \in [0,T^*] \times \mathbb{R}^d$ we have

$$|(u^\epsilon(t) * \nabla K^\epsilon) - (u(t) * \nabla K)(x)| \leq \|u^\epsilon(t) - u(t)\|_{L^p} \|\nabla K^\epsilon\|_{L^q} + \|u(t)\|_{L^p} \|\nabla K^\epsilon - \nabla K\|_{L^q}, \tag{2.53}$$

and finally it is trivial to pass to the limit in relation (2.51).
Equation (2.49) means that the continuous function $u(t)$ (continuous function with values in $W^{-1,p}(\mathbb{R}^d)$) satisfies (2.13) in the distributional sense. But (2.13) implies that the distributional derivative $u'(t)$ is itself a continuous function with value in $W^{-1,p}(\mathbb{R}^d)$. Therefore $u(t)$ is differentiable in the classical sense, i.e, it belongs to $C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$, and (2.13) is satisfied in the classical sense.
We now turn to the proof of (2.16). The u^ϵ's satisfy (2.20). Integrating over $[0,t],\ t < T^*$, we get

$$\|u^\epsilon(t)\|^p_{L^p} = \|u^\epsilon_0\|^p_{L^p} - (p-1) \int_0^t \int_{\mathbb{R}^d} u^\epsilon(s,x)^p \text{ div } v^\epsilon(s,x)\ dxdt. \tag{2.54}$$

Proposition 2.16 together with the general inequality

$$\||f|^p - |g|^p\|_{L^1} \leq 2p\partial \|f\|^{p-1}_{L^p} + \|g\|^{p-1}_{L^p}\ \|f-g\|_{L^p} \tag{2.55}$$

implies that

$$(u^\epsilon)^p, u^p \in C([0,T^*],\ L^1(\mathbb{R}^d)), \tag{2.56}$$
$$(u^{\epsilon_k})^p \to u^p \in C([0,T^*],\ L^1(\mathbb{R}^d)). \tag{2.57}$$

On the other hand, replacing ∇K by ΔK in (2.53) we see right away that

$$\text{div}\, v,\ \text{div}\, v^\epsilon \in C([0,T^*],\ L^\infty(\mathbb{R}^d)), \tag{2.58}$$
$$\text{div}\, v^{\epsilon_k} \to \text{div}\, v \text{ in } C([0,T^*], L^\infty(\mathbb{R}^d)). \tag{2.59}$$

Combining (2.56)-(2.59) we obtain

$$u^p \text{ div}\, v,\ (u^\epsilon)^p \text{ div}\, v^\epsilon \in C([0,T^*],\ L^1(\mathbb{R}^d)), \tag{2.60}$$
$$(u^{\epsilon_k})^p \text{ div}\, v^{\epsilon_k} \to u^p \text{ div}\, v \text{ in } C([0,T^*], L^1(\mathbb{R}^d)). \tag{2.61}$$

So we can pass to the limit in (2.54) to obtain

$$\|u(t)\|_{L^p}^p = \|u_0\|_{L^p}^p - (p-1)\int_0^t \int_{\mathbb{R}^d} u(s,x)^p \text{ div } v(s,x)\; dxdt.$$

But (2.60) implies that the function $t \to \int_{\mathbb{R}^d} u(t,x)^p$ div $v(t,x)\; dx$ is continuous, therefore the function $t \to \|u(t)\|_{L^p}^p$ is differentiable and satisfies (2.16). □

2.2.5 Continuation and conserved properties

Theorem 2.17 (Continuation of solutions). *The solution provided by Theorem 2.6 can be continued up to a time $T_{\max} \in (0, +\infty]$. If $T_{\max} < +\infty$, then*

$$\lim_{t\to T_{\max}} \sup_{\tau\in[0,t]} \|u(\tau)\|_{L^p} = +\infty$$

Proof.- The proof is standard. It is just needed to use the continuity of the solution with respect to time and the uniqueness proved in Section 2.3. For more details in the method one can read the discussion leading to [14, Theorem 4, p. 728]. □

Theorem 2.18 (Conservation of mass/ second moment). *(i) Under the assumption of Theorem 2.6, and if we assume moreover that $u_0 \in L^1(\mathbb{R}^d)$, then the solution u belongs to $C([0,T^*], L^1(\mathbb{R}^d))$ and satisfies $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$ for all $t \in [0,T^*]$.*
(ii) Under the assumption of Theorem 2.6, and if we assume moreover that u_0 has bounded second moment, then the second moment of $u(t)$ stays bounded for all $t \in [0,T^*]$.

Proof.- We just need to revisit the proof of Proposition 2.16. Since $u_0 \in L^1 \cap L^p$ it is clear that

$$u_0^\epsilon = J_\epsilon u_0 \to u_0 \text{ in } L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d). \tag{2.62}$$

Using convergences (2.62), (2.39), (2.40) and (2.41) we prove that

$$u^{\epsilon_k} \to u \text{ in } C([0,T^*],\; L^1 \cap L^p). \tag{2.63}$$

The proof is exactly the same than the one of Proposition 2.16 . Since the aggregation equation is a conservation law, it is obvious that the smooth approximates satisfy $\|u^\epsilon(t)\|_{L^1} = \|u_0^\epsilon\|_{L^1}$. Using (2.63) we obtain $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$.
We now turn to the proof of (ii). Since the smooth approximates u_ϵ have compact support, their second moment is clearly finite, and the following manipulation are

justified:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|x|^2u_\epsilon(t,x)dx = 2\int_{\mathbb{R}^d}\vec{x}\cdot v_\epsilon\, du_\epsilon(x)$$
$$\leq 2\left(\int_{\mathbb{R}^d}|x|^2u_\epsilon(t,x)dx\right)^{1/2}\left(\int_{\mathbb{R}^d}|v_\epsilon|^2u_\epsilon(t,x)dx\right)^{1/2}$$
$$\leq C\left(\int_{\mathbb{R}^d}|x|^2u_\epsilon(t,x)dx\right)^{1/2}. \qquad (2.64)$$

Assume now that the second moment of u_0 is bounded. A simple computation shows that if η_ϵ is radially symmetric, then $|x|^2*\eta_\epsilon = |x|^2+$second moment of η_ϵ. Therefore

$$\int_{\mathbb{R}^d}|x|^2u_0^\epsilon(x)dx \leq \int_{\mathbb{R}^d}|x|^2*\eta_\epsilon(x)\; u_0(x)dx$$
$$\leq \int_{\mathbb{R}^d}|x|^2u_0(x)dx + \int_{\mathbb{R}^d}|x|^2\eta_\epsilon(x)dx$$
$$\leq \int_{\mathbb{R}^d}|x|^2u_0(x)dx + 1 \quad \text{for } \epsilon \text{ small enough.} \qquad (2.65)$$

Inequality (2.65) come from the fact that the second moment of η_ϵ goes to 0 as ϵ goes to 0. Estimate (2.64) together with (2.65) provide us with a uniform bound of the second moment of the $u^\epsilon(t)$ which only depends the second moment of u_0. Since u^ϵ converges to u in L^1, we obviously have, for a given R and t:

$$\int_{|x|\leq R}|x|^2u(t,x)dx = \lim_{\epsilon\to 0}\int_{|x|\leq R}|x|^2u_\epsilon(t,x)dx \leq \limsup_{\epsilon\to 0}\int_{\mathbb{R}^d}|x|^2u_\epsilon(t,x)dx.$$

Since R is arbitrary, this show that the second moment of $u(t,\cdot)$ is bounded for all t for which the solution exists. □

Combining Theorem 2.17 and 2.18 together with equality (2.16) we get:

Theorem 2.19 (Global existence when ΔK is bounded from above). *Under the assumption of Theorem 2.6, and if we assume moreover that $u_0 \in L^1(\mathbb{R}^d)$ and ess sup $\Delta K < +\infty$, then the solution u exists for all times (i.e.: $T_{\max} = +\infty$).*

Proof.- Equality (2.16) can be written

$$\frac{d}{dt}\{\|u(t)\|_{L^p}^p\} = (p-1)\int_{\mathbb{R}^d}u(t,x)^p(u(t)*\Delta K)(x)\,dx. \qquad (2.66)$$

Since ΔK is bounded from above we have

$$(u(s)*\Delta K)(x) \leq (\text{ess sup }\Delta K)\int_{\mathbb{R}^d}u(s,x)dx = (\text{ess sup }\Delta K)\,\|u_0\|_{L^1}. \qquad (2.67)$$

Combining (2.66), (2.67) and Gronwall inequality gives

$$\|u(t)\|_{L^p}^p \le \|u_0\|_{L^p}^p \, e^{(p-1)(\text{ess sup }\Delta K)\|u_0\|_{L^1} t},$$

so the L^p-norm can not blow-up in finite time which, because of Theorem 2.17, implies global existence. □

2.3 Uniqueness of solutions in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$

In this section we aim to study the uniqueness of solutions to continuity equations evolving a nonnegative density $\rho(t,x)$ at position $x \in \mathbb{R}^d$ and time $t > 0$ by the equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \operatorname{div}\left[u(t,x)v(t,x)\right] = 0 & t > 0\,,\ x \in \mathbb{R}^d\,, \\ v(t,x) := -\nabla K * u(t,x) & t > 0\,,\ x \in \mathbb{R}^d\,, \\ u(0,x) = \rho_0(x) \ge 0 & x \in \mathbb{R}^d\,, \end{cases} \tag{2.68}$$

where $v(t,x) := -\nabla K * u(t,x)$ is the velocity field.
The initial data is assumed to have total finite mass, $u_0 \in L^1(\mathbb{R}^d)$. Moreover, since solutions of (2.68) formally preserves the total mass of the system

$$\int_{\mathbb{R}^d} u(t,x)\,dx = \int_{\mathbb{R}^d} u_0(y)\,dy := M, \tag{2.69}$$

we can assume, without loss of generality, that we work with probability measures, i.e. $M = 1$, by suitable scalings of the equation. A further assumption that will be done through this work is the boundedness of the initial data, i.e., $u_0 \in L^\infty(\mathbb{R}^d)$.
The results we show in the sequel can also be found in [45].
Here, we will essentially work with three type of interaction potentials: bounded second derivatives, pointy potentials and Poisson kernels, to show uniqueness of bounded weak solutions on a given time interval $[0,T]$. The idea is based on G. Loeper's work [126] who showed the uniqueness of bounded weak solutions for the Vlasov-Poisson system and the 2D incompressible Euler equations using as "distance" an estimate on the euclidean optimal transport distance between probability measures. An adaptation of this idea using a coupling method [50, 172] to the case with diffusion by assuming we have an stochastic representation formula can be done without much extra effort. You can see how in Appendix C. Next we handle the limiting case $p = \infty$ and finally, in the last subsection we show how to extend this argument so that the whole range of solutions that we consider in this chapter are included.

2.3.1 Uniqueness for bounded weak solutions

Let us start by working with the continuity equation (2.68) with a given velocity field $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. The continuity equation comes from the assumption that the mass density of individuals in a set is preserved by the flow map or characteristics associated to the ODE system determined by

$$\begin{cases} \dfrac{dX(t,\alpha)}{dt} & = u(t, X(t,\alpha)) \qquad t \geq 0 \,, \\ X(0,\alpha) & = \alpha \qquad \alpha \in \mathbb{R}^d \,. \end{cases}$$

Let us assume that the given velocity field v is such that the solutions to the ODE system are globally defined in $[0,T]$ and unique. Moreover, let us assume that the flow map $X(t) : \mathbb{R}^d \to \mathbb{R}^d$ for all $t \geq 0$ associated to the velocity field $v(t,x)$, $X(t)(\alpha) := X(t,\alpha)$ for all $\alpha \in \mathbb{R}^d$, is a family of homeomorphisms from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Typically in our cases, $u \in C([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ and is either Lipschitz or Log-Lipschitz in space, which implies the above statements on the ODE system, see for instance [129, 130].

Given $u \in C_w([0,T], L^1_+(\mathbb{R}^d))$, we will say that it is a distributional solution to the continuity equation (2.68) with the given velocity field v and initial data $u_0 \in L^1_+(\mathbb{R}^d)$, if it verifies

$$\int_0^T \int_{\mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t}(t,x) + v(t,x) \cdot \nabla \varphi(t,x) \right) u(t,x)\, dx\, dt = \int_{\mathbb{R}^d} \varphi(0,x) u_0(x)\, dx$$

for all $\varphi \in C_0^\infty([0,T) \times \mathbb{R}^d))$. Here, the symbol C_w means continuity with the weak-* topology of measures. Let us point out that under the above hypotheses the term $(v \cdot \nabla\varphi)u$ makes perfect sense as duality $L^1 - L^\infty$.

In fact, the distributional solution of the continuity equation with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ is uniquely characterized by

$$\int_B u(t,x)\, dx = \int_{X(t)^{-1}(B)} u_0(x)\, dx$$

for any measurable set $B \subset \mathbb{R}^d$, see [1]. In the optimal transport terminology, this is equivalent to say that $X(t)$ *transports* the measure u_0 onto $u(t)$ and we denote it by $u(t) = X(t)\#u_0$ defined by

$$\int_{\mathbb{R}^d} \zeta(x) u(t,x)\, dx = \int_{\mathbb{R}^d} \zeta(X(t,x)) u_0(x)\, dx \quad \forall \zeta \in \mathcal{C}_b^0(\mathbb{R}^d) \,. \tag{2.70}$$

With these ingredients, we can define the notion of solution for which we will prove its uniqueness.

Definition 2.20. *A function u is a bounded weak solution of* (2.68) *on [0,T] for a nonnegative initial data $u_0 \in L^1(\mathbb{R}^d)$, if it satisfies*

1. $u \in C_w([0,T], L^1_+(\mathbb{R}^d))$.

2. *The solutions of the ODE system $X'(t,\alpha) = v(t, X(t,\alpha))$ with the velocity field $v(t,x) := -\nabla K * u(t,x)$ are uniquely defined in $[0,T]$ for any initial data $\alpha \in \mathbb{R}^d$.*

3. *$\rho(t) = X(t)\#u_0$ is the unique distributional solution to the continuity equation with given velocity field v.*

4. $u \in L^\infty(0,T; L^\infty(\mathbb{R}^d))$.

Remark 2.21.

1. *Let us point out that even if the solution to the continuity equation with given velocity field v is unique, the uniqueness issue for* (2.68) *is not settled due to the nonlinear coupling through $v = -\nabla K * u$.*

2. *In order to show that bounded weak solutions exist, one usually needs more assumptions on the initial data depending on the particular choices of the kernel K. Typically for initial data $u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$, we will have a time $T > 0$ possibly depending on the initial data and a bounded weak solution on the time interval $[0,T]$.*

Let us assume that u_1 and u_2 are two bounded weak solutions to (2.68), we look at the two characteristics flow maps, X_1 and X_2, such that $u_i = X_i\#u_0$, $i = 1, 2$, and provide a bound for the distance between them at time t in terms of its distance at time $t = 0$.
In the following, we will address the uniqueness with $v = -\nabla K * u$, first providing the details of the computation for a regular smooth kernel $K \in C^2(\mathbb{R}^d)$ and with L^∞-bounded Hessian and then modify it in order to include a more general family of kernels with possibly Lipschitz point at the origin, namely, for kernels with the Hessian bounded in $L^1(\mathbb{R}^d)$. Let us remark that the potential $K(x) = e^{-|x|}$ belongs to this class for $N \geq 2$. Finally, we look at the Keller-Segel model without diffusion, i.e. taking $v = \nabla c = -\nabla\Gamma_N * \rho$. The main theorem is summarized as:

Theorem 2.22. *Let u_1, u_2 be two bounded weak solutions of equation* (2.68) *in the interval $[0,T]$ with initial data $\rho_0 \in L^1_+(\mathbb{R}^d)$ and assume that either:*

- *v is given by $v = -\nabla K * u$, with K such that $K \in C^2(\mathbb{R}^d)$ and $|D^2K| \in L^\infty(\mathbb{R}^d)$.*

- *v is given by $v = -\nabla K * u$, with K such that $\nabla K \in L^2(\mathbb{R}^d)$ and $|D^2 K| \in L^1(\mathbb{R}^d)$.*

- *$v = -\nabla \Gamma_N * u$.*

Then $u_1(t) = u_2(t)$ for all $0 \le t \le T$.

Idea of the proof.- Given the two bounded weak solutions to (2.68), let us define the quantity

$$Q(t) := \frac{1}{2}\int_{\mathbb{R}^d} |X_1(t) - X_2(t)|^2 u_0(x)\,\mathrm{d}x, \tag{2.71}$$

with X_i the flow map associated to each solution, $u_i(t) = X_i \# u_0$, $i = 1, 2$. Taking into account the remarks to the definition of the W_2-distance, we have $W_2^2(u_1(t), u_2(t)) \le 2Q(t)$. It is clear then that $Q(t) \equiv 0$ would imply that $u_1 = u_2$.

1. *Regular kernel case.-* In this case, the velocity field is continuous and Lipschitz in space, therefore the characteristics are globally defined and unique. Now, by taking the derivative of Q w.r.t. time, we get

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(x_1) - v_2(x_2)\rangle u_0(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(x_1) - v_1(x_2)\rangle u_0(x)\mathrm{d}x + \\ &\quad \int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(x_2) - v_2(x_2)\rangle u_0(x)\mathrm{d}x \end{aligned} \tag{2.72}$$

   where the time variable has been omitted for clarity. Now, taking into account the Lipschitz properties of u into the first integral and using Hölder inequality in the second one, we can write

$$\begin{aligned}\frac{\partial Q}{\partial t} &\le CQ(t) + Q(t)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |v_1\left(X_2(t,x)\right) - v_2\left(X_2(t,x)\right)|^2 u_0(x)\mathrm{d}x \right)^{\frac{1}{2}} \\ &= CQ(t) + Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}}. \end{aligned} \tag{2.73}$$

   Now, let us work in the term $I(t)$. By using that the solutions are constructed transporting the initial data through their flow maps, we deduce

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^d} |\nabla K * (u_1 - u_2)\,[X_2(t,x)]\,|^2 u_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \nabla K(X_2(x) - y) u_1(y)\,\mathrm{d}y \\ &\qquad\qquad - \int_{\mathbb{R}^d} \nabla K(X_2(x) - y) u_2(y)\,\mathrm{d}y \Big|^2 u_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} [\nabla K(X_2(x) - X_1(y)) - \nabla K(X_2(x) - X_2(y))] \times \\ &\qquad\qquad u_0(y)\,\mathrm{d}y \Big|^2 u_0(x)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big| \nabla K(X_2(x) - X_1(y)) - \nabla K(X_2(x) - X_2(y)) \Big|^2 \times \\ &\qquad\qquad u_0(y) u_0(x)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

the last step holding due to Jensen's inequality. Using Taylor's theorem, since K is twice differentiable, we deduce

$$\nabla K(A) = \nabla K(B) + \int_0^1 (D^2 K)\,[X_2(x) - X_1(y) + \zeta(X_1(y) - X_2(y))] \times (X_1(y) - X_2(y))\,\mathrm{d}\zeta$$

with $A = X_2(x) - X_2(y)$ and $B = X_2(x) - X_1(y)$, and thus $|\nabla K(A) - \nabla K(B)| \leq C|X_1(y) - X_2(y)|$ since $|D^2 K| \in L^\infty(\mathbb{R}^d)$. This finally gives that $I(t) \leq CQ(t)$. Going back to (2.73), we recover $\frac{\partial Q}{\partial t} \leq CQ(t)$, and hence we can conclude that if $Q(0) = 0$ then $Q(t) \equiv 0$, implying $\rho_1 = \rho_2$.

2. *Kernels allowing Lipschitz singularity.*- Under the assumptions on the kernel K and the properties of bounded weak solutions, it was shown in [12, Lemma 4.2] that the velocity field u is Lipschitz continuous in space and time. Therefore, we can recover exactly the relation (2.73) again. Now, in order to estimate $I(t)$, we write it as:

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^d} |\nabla K * (u_1 - u_2)\,[X_2(t,x)]\,|^2 u_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |\nabla K * (u_1 - u_2)\,(x)|^2 u_2(x)\,\mathrm{d}x \end{aligned}$$

Using Proposition B.12, we deduce that

$$I(t) \leq \|u_2\|_{L^\infty(\mathbb{R}^d)} \max(\|u_1\|_{L^\infty(\mathbb{R}^d)}, \|u_2\|_{L^\infty(\mathbb{R}^d)})\, W_2^2(u_1, u_2) \leq CQ(t),$$

and we can conclude similarly as in the previous case.

3. *PKS model without diffusion.*- Under the assumptions on bounded weak solutions, the velocity field in our case is Log-Lipschitz in space. This is a classical result used in 2D incompressible Euler equations and easily generalized to any dimension [129, 130, 126]. More precisely, $v \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists a constant C depending on the L^1 and L^∞ norms of $u(t)$ such that

$$|v(t,x) - v(t,y)| \leq C|x-y| \log \frac{1}{|x-y|} \text{ when } |x-y| \leq \frac{1}{2}$$

for any $t \in [0,T]$. The flow map under these conditions can be uniquely defined and it is a Hölder homeomorphism.

The uniqueness proof follows estimating the second term in (2.72) as in the previous case. More precisely, we use Proposition B.11 to infer that

$$I(t) \leq \|u_2\|_{L^\infty(\mathbb{R}^d)} \max(\|u_1\|_{L^\infty(\mathbb{R}^d)}, \|u_2\|_{L^\infty(\mathbb{R}^d)})\, W_2^2(u_1,u_2) \leq CQ(t),$$

implying

$$\int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(x_2) - v_2(x_2)\rangle u_0(x) \mathrm{d}x \leq Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}} \leq CQ(t).$$

Now, let us concentrate in the first term of (2.72), we just repeat the standard arguments in [126] to get that by taking T small enough then

$$\int_{\mathbb{R}^d} \langle X_1 - X_2, u_1(x_1) - u_1(x_2)\rangle u_0(x) \mathrm{d}x \leq CQ(t) \log^2(2Q(t))$$

where the log-Lipschitz property of v was used. This finally gives the differential inequality

$$\frac{d}{dt} Q(t) \leq CQ(t) \left(1 + \log \frac{1}{Q(t)}\right),$$

for $0 \leq t \leq T$ with T small enough. Standard Gronwall-like arguments as in [129] imply $Q(t) = 0$, and thus, the uniqueness. □

2.3.2 Uniqueness for L^p solutions

One can easily check that solutions of the aggregation equation constructed in section 2.2 are distribution solutions, i.e. they satisfy

$$\int_0^T \int_{\mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t}(t,x) + v(t,x) \cdot \nabla \varphi(t,x) \right) u(t,x)\, dx\, dt = \int_{\mathbb{R}^d} \varphi(0,x) u_0(x)\, dx \tag{2.74}$$

for all $\varphi \in C_0^\infty([0,T^*) \times \mathbb{R}^d))$.
Thus we can use the previous arguments to show uniqueness of solutions to (2.13)-(2.14).

Theorem 2.23 (Uniqueness). *Let u_1, u_2 be two solutions of equation* (2.13) *in the interval $[0,T^*]$ with initial data $u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $1 < p < \infty$, such that their L^p-norm remains bounded for all $t \in [0,T^*]$. Assume too that v is given by $v = -\nabla K * u$, with K such that $\nabla K \in W^{1,q}(\mathbb{R}^d)$, p and q conjugates. Then $u_1(t) = u_2(t)$ for all $0 \le t \le T^*$.*

Proof.- As we have done before, consider two characteristics flow maps, X_1 and X_2, such that $u_i = X_i \# u_0$, $i = 1,2$ and define the quantity

$$Q(t) := \frac{1}{2}\int_{\mathbb{R}^d} |X_1(t) - X_2(t)|^2 u_0(x)\, dx, \tag{2.75}$$

From Remark B.9, we have $W_2^2(u_1(t), u_2(t)) \le 2Q(t)$ which we now prove is zero for all times, implying that $u_1 = u_2$. Now, to see that $Q(t) \equiv 0$ we compute the derivative of Q with respect to time as in subsection 2.3.1.

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(X_1) - v_2(X_2)\rangle u_0(x) dx \\
&= \int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(X_1) - v_1(X_2)\rangle u_0(x) dx \\
&\quad + \int_{\mathbb{R}^d} \langle X_1 - X_2, v_1(X_2) - v_2(X_2)\rangle u_0(x) dx
\end{aligned}$$

The above argument is justified because, due to Lemma 2.7, the velocity field is C^1 and bounded. Taking into account the Lipschitz properties of v into the first integral and using Hölder inequality in the second one, we can write

$$\begin{aligned}
\frac{\partial Q}{\partial t} &\le CQ(t) + Q(t)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |v_1\left(X_2(t,x)\right) - v_2\left(X_2(t,x)\right)|^2 u_0(x) dx \right)^{\frac{1}{2}} \\
&= CQ(t) + Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}}.
\end{aligned} \tag{2.76}$$

Now, in order to estimate $I(t)$, we use that the solutions are constructed transporting the initial data through their flow maps, so we can write it as

$$\begin{aligned}
I(t) &= \int_{\mathbb{R}^d} |\nabla K * (u_1 - u_2)\left[X_2(t,x)\right]|^2 u_0(x)\, dx \\
&= \int_{\mathbb{R}^d} |\nabla K * (u_1 - u_2)(x)|^2 u_2(x)\, dx.
\end{aligned}$$

Thus, taking an interpolation measure u_θ between u_1 and u_2 and using Hölder inequality and first statement of Theorem B.10 we can get a bound for $I(t)$

$$I(t) \leq \left(\int_{\mathbb{R}^d} \left| \nabla K * \left(\int_1^2 \frac{d}{d\theta} u_\theta \right) \right|^{2q} \right)^{1/q} \|u_2(t)\|_{L^p} \tag{2.77}$$

$$\leq \int_1^2 \|D^2 K * (\nu_\theta u_\theta)\|^2_{L^{2q}} d\theta \ \ \|u_2(t)\|_{L^p}, \tag{2.78}$$

where $\nu_\theta \in L^2(\mathbb{R}^d, u_\theta dx)$ is a vector field, as described in Theorem B.10. Let us work on the first term of the right hand side. Using Young inequality, for α such that $1 + \frac{1}{2q} = 1/q + 1/\alpha$ we obtain

$$\int_1^2 \|D^2 K * (\nu_\theta u_\theta)\|^2_{L^{2q}} d\theta \leq \int_1^2 \|D^2 K\|^2_{L^q} \|\nu_\theta u_\theta\|^2_{L^\alpha} d\theta. \tag{2.79}$$

Note that $q \in (1, +\infty)$ implies $\alpha \in (1, 2)$. Therefore we can use Hölder inequality with conjugate exponents $2/(2-\alpha)$ and $2/\alpha$ to obtain

$$\|\nu_\theta u_\theta\|^2_{L^\alpha} = \left(\int |u_\theta|^{\alpha/2} |u_\theta|^{\alpha/2} |\nu_\theta|^\alpha \right)^{2/\alpha}$$

$$\leq \left(\int |u_\theta|^{\alpha/(2-\alpha)} \right)^{(2-\alpha)/\alpha} \left(\int |u_\theta| \, |\nu_\theta|^2 \right) \tag{2.80}$$

whence, since we can see from simple algebraic manipulations with the exponents that $\frac{\alpha}{2-\alpha} = p$, the conjugate of q,

$$\int_1^2 \|D^2 K * (\nu_\theta u_\theta)\|^2_{L^{2q}} d\theta \leq \|D^2 K\|^2_{L^q} \int_1^2 \|u_\theta\|_{L^p} \left(\int |u_\theta| \, |\nu_\theta|^2 \right) d\theta. \tag{2.81}$$

Therefore, using statements (ii) and (iii) of Theorem B.10 we obtain

$$I(t) \leq \|u_2\|_{L^p} \max\{\|u_1\|_{L^p}, \|u_2\|_{L^p}\} \|D_2 K\|^2_{L^q} W_2^2(u_1, u_2) \leq CQ(t). \tag{2.82}$$

Finally, going back to (2.76) we see that $\frac{dQ}{dt} \leq Q(t)$, whence, since $Q(0) = 0$, we can conclude $Q(t) \equiv 0$ and thus $u_1 = u_2$. The limiting case $p = \infty$ is the one studied in Subsection 2.3.1. ∎

Remark 2.24. *Note that in order to make the above argument rigorous, we need the gradient of the kernel to be at least C^1 when estimating I. It is not the case here, but we can still obtain the estimate using smooth approximations of the potential. let us define*

$$I^\epsilon(t) = \int_{\mathbb{R}^d} |\nabla K^\epsilon * (u_1 - u_2) \, [X_2(t, x)] \, |^2 u_0(x) \, dx$$

where $K^\epsilon = J_\epsilon K$ (see section 2.2). Since ∇K^ϵ converges to ∇K in L^q, it is clear that $\nabla K^\epsilon * (u_1 - u_2)$ converges pointwise to $\nabla K * (u_1 - u_2)$. Using the dominated convergence theorem together with the fact that $\|\nabla K^\epsilon * (u_1 - u_2)\|_{L^\infty}$ is uniformly bounded we get that $I^\epsilon(t)$ converges to $I(t)$ for every $t \in (0, T)$.
On the other hand, due to the definition of u_θ we can write the difference $u_2 - u_1$ as the integral between 1 and 2 of $\partial_\theta u_\theta$ with respect to θ. Now, since the equation $\partial_\theta u_\theta + \mathrm{div}(u_\theta \nu_\theta) = 0$ is satisfied in the sense of distribution, and $\nabla K^\epsilon \in C_c^\infty(\mathbb{R}^d)$, we can replace $\partial_\theta u_\theta$ for $\mathrm{div}(\nu_\theta u_\theta)$ and pass the divergence to the other term of the convolution, so that the equality

$$\int_1^2 (D^2 K^\epsilon * \nu_\theta u_\theta)(x) d\theta = \nabla K^\epsilon * (u_2 - u_1)(x)$$

holds for all $x \in \mathbb{R}^d$. The rest of the manipulations performed above are straightforward with K^ϵ. Passing to the limit in (2.82) *is easy since $D^2 K^\epsilon$ converges to $D^2 K$ in L^q.*

2.4 Instantaneous mass concentration: $K(x) = |x|$

In this section we consider the aggregation equation with an interaction potential equal to $|x|$ in a neighborhood of the origin and whose gradient is compactly supported (or decay exponentially fast at infinity). The Laplacian of this kind of potentials has a $1/|x|$ singularity at the origin, therefore ∇K belongs to $W^{1,q}(\mathbb{R}^d)$ if and only if $q \in [1, d)$. The Hölder conjugate of d is $\frac{d}{d-1}$. Using the theory developed in section 2 and 3 we therefore get local existence and uniqueness of solutions in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all $p > \frac{d}{d-1}$. Here we study the case where the initial data is in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $p < \frac{d}{d-1}$.
Given $p < \frac{d}{d-1}$ we exhibit initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for which the solution instantaneously concentrates mass at the origin (i.e. a delta Dirac at the origin is created instantaneously). This shows that the existence theory developed in section 2 and 3 is in some sense sharp.
This also shows that it is possible for a solution to lose instantaneously its absolute continuity with respect to the Lebesgue measure.
The solutions constructed in this section have compact support, hence we can simply consider $K(x) = |x|$ without changing the behavior of the solution, given that if the solution has a small enough support, it only feels the part of the potential around the origin.
We build on the work developed in [38] on global existence for measure solutions with bounded second moment:

Theorem 2.25 (Existence and uniqueness of measure solutions [38]). *Suppose that $K(x) = |x|$. Given $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a unique weakly continuous family of probability measures $(\mu_t)_{t\in(0,+\infty)}$ satisfying*

$$\partial_t \mu_t + \mathit{div}(\mu_t v_t) = 0 \textit{ in } \mathcal{D}'((0,\infty) \times \mathbb{R}^d), \tag{2.83}$$

$$v_t = -\partial^0 K * \mu_t, \tag{2.84}$$

$$\mu_t \textit{ converges weakly to } \mu_0 \textit{ as } t \to 0. \tag{2.85}$$

Here $\partial^0 K$ is the unique element of minimal norm in the subdifferential of K. Simply speaking, since $K(x) = |x|$ is smooth away from the origin and radially symmetric, we have $\partial^0 K(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\partial^0 K(0) = 0$, and thus:

$$(\partial^0 K * \mu)(x) = \int_{y \neq x} \frac{x - y}{|x - y|} \, d\mu(y). \tag{2.86}$$

Note that, μ_t being a measure, it is important for $\partial^0 K$ to be defined for every $x \in \mathbb{R}^d$ so that (2.84) makes sense. Equation (2.83) means that

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \left(\frac{d\psi}{dt}(t,x) + \nabla\psi(t,x) \cdot v_t(x) \right) d\mu_t(x) \, dt = 0, \tag{2.87}$$

for all $\psi \in C_0^\infty(\mathbb{R}^d \times (0, +\infty))$. From (2.86) it is clear that $|v_t(x)| \leq 1$ for all x and t, therefore the above integral makes sense.
The main Theorem of this section is the following:

Theorem 2.26 (Instantaneous mass concentration). *Consider the initial data*

$$u_0(x) = \begin{cases} \frac{L}{|x|^{d-1+\epsilon}} & \textit{if } |x| < 1, \\ 0 & \textit{otherwise}, \end{cases} \tag{2.88}$$

where $\epsilon \in (0,1)$ and $L := \left(\int_{|x|<1} |x|^{-(d-1+\epsilon)} dx \right)^{-1}$ is a normalizing constant. Note that $u_0 \in L^p(\mathbb{R}^d)$ for all $p \in [1, \frac{d}{d-1+\epsilon})$. Let $(\mu_t)_{t\in(0,+\infty)}$ be the unique measure solution of the aggregation equation with interaction potential $K(x) = |x|$ and with initial data u_0. Then, for every $t > 0$ we have

$$\mu_t(\{0\}) > 0,$$

i.e., mass is concentrated at the origin instantaneously and the solution is no longer continuous with respect to the Lebesgue measure.

Theorem 2.26 is a consequence of the following estimate on the velocity field:

Proposition 2.27. *Let $(\mu_t)_{t\in(0,+\infty)}$ be the unique measure solution of the aggregation equation with interaction potential $K(x) = |x|$ and with initial data* (2.88)*. Then, for all $t \in [0, +\infty)$ the velocity field $v_t = -\partial^0 K * \mu_t$ is focussing and there exists a constant $C > 0$ such that*

$$|v_t(x)| \geq C|x|^{1-\epsilon} \qquad \text{for all } t \in [0, +\infty) \text{ and } x \in B(0, 1). \tag{2.89}$$

By focussing, we mean that the velocity field points inward, i.e. there exists a nonnegative function $\lambda_t : [0, +\infty) \to [0, +\infty)$ such that $v_t(x) = -\lambda_t(|x|)\frac{\vec{x}}{x}$.

2.4.1 Representation formula for radially symmetric measure solutions

In this section, we show that for radially symmetric measure solutions, the characteristics are well defined. As a consequence, the solution to (2.13) can be expressed as the push forward of the initial data by the flow map associated with the ODE defining the characteristics.
In the following the unit sphere $\{x \in \mathbb{R}^d, |x| = 1\}$ is denoted by S_d and its surface area by ω_d.

Definition 2.28. *If $\mu \in \mathcal{P}(\mathbb{R}^d)$ is a radially symmetric probability measure, then we define $\hat{\mu} \in \mathcal{P}([0, +\infty))$ by*

$$\hat{\mu}(I) = \mu(\{x \in \mathbb{R}^d : |x| \in I\})$$

for all $I \in \mathcal{B}([0, +\infty))$.

Remark 2.29. *If a measure μ is radially symmetric, then $\mu(\{x\}) = 0$ for all $x \neq 0$, and therefore*

$$\int_{\mathbb{R}^d\setminus\{x\}} \nabla K(x-y)\, d\mu(y) = \int_{\mathbb{R}^d} \nabla K(x-y)\, d\mu(y) \qquad \text{for all } x \neq 0.$$

In other words, for $x \neq 0$, $(\nabla K * \mu)(x)$ is well defined despite the fact that ∇K is not defined at $x = 0$. As a consequence $(\partial^0 K * \mu)(x) = (\nabla K * \mu)(x)$ if $x \neq 0$ and $(\partial^0 K * \mu)(0) = 0$.

Remark 2.30. *If the radially symmetric measure μ is continuous with respect to the Lebesgue measure and has radially symmetric density $u(x) = \tilde{u}(|x|)$, then $\hat{\mu}$ is also continuous with respect to the Lebesgue measure and has density $\hat{u}$, where*

$$\hat{u}(r) = \omega_d r^{d-1}\tilde{u}(r). \tag{2.90}$$

Lemma 2.31 (Polar coordinate formula for the convolution). *Suppose $\mu \in \mathcal{P}(\mathbb{R}^d)$ is radially symmetric. Let $K(x) = |x|$, then for all $x \neq 0$ we have:*

$$(\mu * \nabla K)(x) = \left(\int_0^{+\infty} \phi \left(\frac{|x|}{\rho} \right) \, d\hat{\mu}(\rho) \right) \frac{x}{|x|} \tag{2.91}$$

where the function $\phi : [0, +\infty) \to [-1, 1]$ is defined by

$$\phi(r) = \frac{1}{\omega_d} \int_{S_d} \frac{re_1 - y}{|re_1 - y|} \cdot e_1 d\sigma(y). \tag{2.92}$$

Proof.- This comes from simple algebraic manipulations. These manipulations are shown in [13]. □

In the next Lemma we state properties of the function ϕ defined in (2.92).

Lemma 2.32 (Properties of the function ϕ).

(i) *ϕ is continuous and non-decreasing on $[0, +\infty)$. Moreover $\phi(0) = 0$, and $\lim_{r\to\infty} \phi(r) = 1$.*

(ii) *$\phi(r)$ is $O(r)$ as $r \to 0$. To be more precise:*

$$\lim_{\substack{r\to 0\\ r>0}} \frac{\phi(r)}{r} = 1 - \frac{1}{\omega_d} \int_{S_d} (y \cdot e_1)^2 \, d\sigma(y). \tag{2.93}$$

Proof.- Consider the function $F : [0, +\infty) \times S_d \to [-1, 1]$ defined by

$$(r, y) \mapsto \frac{re_1 - y}{|re_1 - y|} \cdot e_1. \tag{2.94}$$

Since F is bounded, we have that

$$\phi(r) = \frac{1}{\omega_d} \int_{S_d} F(r, y) d\sigma(y) = \frac{1}{\omega_d} \int_{S_d \setminus \{e_1\}} F(r, y) d\sigma(y).$$

If $y \in S_d \setminus \{e_1\}$ then the function $r \mapsto F(r, y)$ is continuous on $[0, +\infty)$ and C^∞ on $(0, +\infty)$. An explicit computation shows then that

$$\frac{\partial F}{\partial r}(r, y) = \frac{1 - F(r, y)^2}{|re_1 - y|} \geq 0, \tag{2.95}$$

thus ϕ is non-decreasing and, by the Lebesgue dominated convergence, it is easy to see that ϕ is continuous, $\phi(0) = 0$ and $\lim_{r\to\infty} \phi(r) = 1$, which prove (i).

To prove (ii), note that the function $\frac{\partial F}{\partial r}(r,y)$ can be extended by continuity on $[0,+\infty)$. Therefore the right derivative with respect to r of $F(r,y)$ is well defined:

$$\lim_{\substack{r\to 0\\ r>0}} \frac{F(r,y)-F(0,y)}{r} = \frac{1-F(0,y)^2}{|y|} = 1-(y\cdot e_1)^2.$$

and since $\partial F/\partial r$ is bounded on $(0,+\infty)\times S_d\backslash\{e_1\}$, we can now use the Lebesgue dominated convergence theorem to conclude:

$$\lim_{\substack{r\to 0\\ r>0}} \frac{\phi(r)}{r} = \lim_{\substack{r\to 0\\ r>0}} \frac{\phi(r)-\phi(0)}{r} = \lim_{\substack{r\to 0\\ r>0}} \frac{1}{\omega_d}\int_{S_d} \frac{F(r,y)-F(0,y)}{r} d\sigma(y)$$
$$= \frac{1}{\omega_d}\int_{S_d} 1-(y\cdot e_1)^2 d\sigma(y).$$

□

Remark 2.33. *Note that for $r>0$ the function $\rho\mapsto\phi\left(\frac{r}{\rho}\right)$ is non increasing and continuous. Indeed, it is equal to 1 when $\rho=0$ and it decreases to 0 as $\rho\to\infty$. In particular, the integral in* (2.91) *is well defined for any probability measure $\hat{\mu}\in\mathcal{P}([0,+\infty))$.*

Remark 2.34. *In dimension two, it is easy to check that $\lim_{r\to 1}\phi'(r)=+\infty$ which implies that the derivative of the function ϕ has a singularity at $r=1$ and thus, that the function ϕ is not C^1.*

Proposition 2.35 (Characteristic ODE). *Let $K(x)=|x|$ and let $(\mu_t)_{t\in[0,+\infty)}$ be a weakly continuous family of radially symmetric probability measures. Then the velocity field*

$$v(t,x)=\begin{cases} -(\nabla K*\mu_t)(x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases} \tag{2.96}$$

is continuous on $\mathbb{R}^d\backslash\{0\}\times[0+\infty)$. Moreover, for every $x\in\mathbb{R}^d$, there exists an absolutely continuous function $t\to X_t(x)$, $t\in[0,+\infty)$, which satisfies

$$\frac{d}{dt}X_t(x)=v(t,X_t(x)) \qquad \text{for a.e. } t\in(0,+\infty), \tag{2.97}$$
$$X_0(x)=x. \tag{2.98}$$

Proof.- From formula (2.91), Remark 2.33, and the weak continuity of the family $(\mu_t)_{t\in[0,+\infty)}$, we obtain continuity in time. The continuity in space simply comes from the continuity and boundedness of the function ϕ together with the Lebesgue dominated convergence theorem.

Since v is continuous on $\mathbb{R}^d \backslash \{0\} \times [0 + \infty)$ we know from the Peano theorem that given $x \in \mathbb{R}^d \backslash \{0\}$, the initial value problem (2.97)-(2.98) has a C^1 solution at least for short time. We want to see that it is defined for all time. For that, note that by a continuation argument, the interval given by Peano theorem can be extended as long as the solution stays in $\mathbb{R}^d \backslash \{0\}$. Then, if we denote by T_x the maximum time so that the solution exists in $[0, T_x)$ we have that either $T_x = \infty$ and we are done, or $T_x < +\infty$, in which case clearly $\lim_{t \to T_x} X_t(x) = 0$, and we can extend the function $X_t(x)$ on $[0, +\infty)$ by setting $X_t(x) := 0$ for $t \geq T_x$.

The function $t \to X_t(x)$ that we have just constructed is continuous on $[0, +\infty)$, C^1 on $[0, +\infty) \backslash \{T_x\}$ and satisfies (2.97) on $[0, +\infty) \backslash \{T_x\}$. If $x = 0$, we obviously let $X_t(x) = 0$ for all $t \geq 0$. □

Finally, we present the representation formula, by which we express the solution to (2.13) as a push-forward of the initial data. See [1] or [171] for a definition of the push-forward of a measure by a map.

Proposition 2.36 (Representation formula). *Let $(\mu_t)_{t \in [0, +\infty)}$ be a radially symmetric measure solution of the aggregation equation with interaction potential $K(x) = |x|$, and let $X_t : \mathbb{R}^d \to \mathbb{R}^d$ be defined by (2.96),* (2.97) *and (2.98). Then for all $t \geq 0$,*

$$\mu_t = X_t \# \mu_0.$$

Proof.- In this proof, we follow arguments from [1]. Since for a given x the function $t \mapsto X_t(x)$ is continuous, one can easily prove, using the Lebesgue dominated convergence theorem, that $t \mapsto X_t \# \mu_0$ is weakly continuous. Let us now prove that $\mu_t := X_t \# \mu_0$ satisfies (2.87) for all $\psi \in \mathbb{C}_0^\infty(\mathbb{R}^d \times (0, \infty))$. Given that the test function ψ is compactly supported, there exist $T > 0$ such that $\psi(t, x) = 0$ for all $t \geq T$. We therefore have:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \psi(x, T) d\mu_T(x) - \int_{\mathbb{R}^d} \psi(x, 0) d\mu_0(x) \\ &= \int_{\mathbb{R}^d} \Big(\psi(X_T(x), T) - \psi(x, 0) \Big) d\mu_0(x). \end{aligned} \tag{2.99}$$

If we now take into account that from Proposition 2.35 the mapping $t \to \phi(t, X_t(x))$

is absolutely continuous, we can rewrite (2.99) as

$$0 = \int_{\mathbb{R}^d} \int_0^T \Big(\frac{d}{dt}\psi(t, X_t(x)) \Big) dt \; d\mu_0(x) \tag{2.100}$$

$$= \int_{\mathbb{R}^d} \int_0^T \Big(\nabla\psi(t, X_t(x)) \cdot v(t, X_t(x)) + \frac{d\psi}{dt}(t, X_t(x)) \Big) dt \; d\mu_0(x) \tag{2.101}$$

$$= \int_0^T \int_{\mathbb{R}^d} \Big(\nabla\psi(t, X_t(x)) \cdot v(t, X_t(x)) + \frac{d\psi}{dt}(t, X_t(x)) \Big) d\mu_0(x) \; dt \tag{2.102}$$

$$= \int_0^T \int_{\mathbb{R}^d} \Big(\nabla\psi(t, x) \cdot v(t, x) + \frac{d\psi}{dt}(t, x) \Big) d\mu_t(x) \; dt.$$

The step from (2.101) to (2.102) holds because of the fact that $|v(t, x)| \leq 1$, which justifies the use of the Fubini Theorem. ∎

Remark 2.37 (Representation formula in polar coordinates). *Let μ_t and X_t be as in the previous proposition. Let $R_t : [0, +\infty) \to [0, +\infty)$ be the function such that $|X_t(x)| = R_t(|x|)$. Then*

$$\hat{\mu}_t = R_t \# \hat{\mu}_0. \tag{2.103}$$

Remark 2.38. *Since ϕ is nonnegative (Lemma 2.32), from* (2.91), (2.96) *and* (2.97) *we see that the function $t \mapsto |X_t(x)| = R_t(|x|)$ is non increasing.*

2.4.2 Proof of Proposition 2.27 and Theorem 2.26

We are now ready to prove the estimate on the velocity field and the instantaneous concentration result. We start by giving a frozen in time estimate of the velocity field.

Lemma 2.39. *Let $K(x) = |x|$, and let $u_0(x)$ be defined by* (2.88) *for some $\epsilon \in (0, 1)$. Then there exist a constant $C > 0$ such that*

$$|(u_0 * \nabla K)(x)| \geq C|x|^{1-\epsilon} \tag{2.104}$$

for all $x \in B(0, 1)\backslash\{0\}$.

Proof.- Note that if we do the change of variable $s = \frac{|x|}{\rho}$ in equation (2.91), we find that

$$|(u_0 * \nabla K)(x)| = |x| \int_0^{+\infty} \phi(s) \; \hat{u}_0(\frac{|x|}{s}) \frac{ds}{s^2}. \tag{2.105}$$

On the other hand, using (2.90) we see that the $\hat{u}_0(r)$ corresponding to the $u_0(x)$ defined by (2.88) is

$$\hat{u}_0(r) = \begin{cases} \frac{\omega_d}{r^\epsilon} & \text{if } r < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.106}$$

Then, plugging (2.106) in (2.105) we obtain that for all $x \neq 0$

$$|(u_0 * \nabla K)(x)| = \omega_d |x|^{1-\epsilon} \int_{|x|}^{+\infty} \frac{\phi(\rho)}{\rho^{2-\epsilon}} d\rho. \tag{2.107}$$

In light of statement (ii) of Lemma 2.32, we see that the previous integral converges as $|x| \to 0$. Hence $|(u_0 * \nabla K)(x)|$ is $O(|x|^{1-\epsilon})$ as $|x| \to 0$ and (2.104) follows. □

Finally, the last piece we need in order to prove Proposition 2.27 from the previous lemma, is the following comparison principle:

Lemma 2.40 (Temporal monotonicity of the velocity). *Let $(\mu_t)_{t\in(0,+\infty)}$ be a radially symmetric measure solution of the aggregation equation with interaction potential $K(x) = |x|$. Then, for every $x \in \mathbb{R}^d \backslash \{0\}$ the function*

$$t \mapsto |(\nabla K * \mu_t)(x)|$$

is non decreasing.

Proof.- Combining (2.91) and (2.103) we see that

$$|(\mu_t * \nabla K)(x)| = \left(\int_0^{+\infty} \phi(\frac{|x|}{R_t(\rho)})\, d\hat{\mu}_0(\rho) \right). \tag{2.108}$$

Now, by Lemma 2.32, ϕ is non decreasing and due to Remark 2.37, $t \mapsto R_t(\rho)$ is non increasing. Henceforth it is clear that (2.108) is itself non decreasing. □

At this point, Proposition 2.27 follows as a simple consequence of the frozen in time estimate (2.104) together with Lemma 2.40, and we can give an easy proof for the main result we introduced at the beginning of the section.
Proof.- [Proof of Theorem 2.26] Using the representation formula (Proposition 2.36) and the definition of the push forward we get

$$\mu_t(\{0\}) = (X_t \# \mu_0)(\{0\}) = \mu_0(X_t^{-1}(\{0\})).$$

Then, note that the solution of the ODE $\dot{r} = -r^{1-\epsilon}$ reaches zero in finite time. Therefore, from Proposition 2.27 and 2.35 we obtain that for all $t > 0$, there exists $\delta > 0$ such that

$$X_t(x) = 0 \qquad \text{for all } |x| < \delta.$$

In other words, for all $t > 0$, there exists $\delta > 0$ such that $B(0,\delta) \subset X_t^{-1}(\{0\})$. Clearly, given our choice of initial condition, we have that $\mu_0(B(0,\delta)) > 0$ if $\delta > 0$, and therefore $\mu_t(\{0\}) > 0$ if $t > 0$. □

2.5 Osgood condition for global well-posedness

This section considers the global well-posedness of the aggregation equation in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, depending on the potential K. We start by giving a precise definitions of "natural potential", "repulsive in the short range" and "strictly attractive in the short range", and then we prove Theorem 2.5.

Definition 2.41. *A* natural potential *is a radially symmetric potential* $K(x) = k(|x|)$*, where* $k : (0, +\infty) \to \mathbb{R}$ *is a smooth function which satisfies the following conditions:*

(C1) $\sup_{r \in (0,\infty)} |k'(r)| < +\infty$,

(C2) $\exists\, \alpha > d$ *such that* $k'(r)$ *and* $k''(r)$ *are* $O(1/r^\alpha)$ *as* $r \to +\infty$,

(MN1) $\exists\, \delta_1 > 0$ *such that* $k''(r)$ *is monotonic (either increasing or decreasing) in* $(0, \delta_1)$,

(MN2) $\exists\, \delta_2 > 0$ *such that* $rk''(r)$ *is monotonic (either increasing or decreasing) in* $(0, \delta_2)$.

Remark 2.42. *Note that monotonicity condition (MN1) implies that* $k'(r)$ *and* $k(r)$ *are also monotonic in some (different) neighborhood of the origin* $(0, \delta)$*. Also, note that (C1) and (MN1) imply*

(C3) $\lim_{r \to 0^+} k'(r)$ *exists and is finite.*

Remark 2.43. *The far field condition (C2) can be dropped when the data has compact support.*

Definition 2.44. *A natural potential is said to be repulsive in the short range if there exists an interval* $(0, \delta)$ *on which* $k(r)$ *is decreasing. A natural potential is said to be strictly attractive in the short range if there exists an interval* $(0, \delta)$ *on which* $k(r)$ *is strictly increasing.*

We would like to remark that the two monotonicity conditions are not very restrictive as, in order to violate them, a potential would have to exhibit some pathological behavior around the origin, like oscillating faster and faster as $r \to 0$.

2.5.1 Properties of natural potentials

As a last step before proving Theorem 2.5, let us point out some properties of natural potentials, which show the reason behind the choice of this kind of potentials to work with.

Lemma 2.45. *If* $K(x) = k(|x|)$ *is a natural potential, then* $k''(r) = o(1/r)$ *as* $r \to 0$.

Proof.- First, note that since $k(r)$ is smooth away from 0 we have

$$k'(1) - k'(\epsilon) = \int_{\epsilon}^{1} k''(r)dr.$$

Now, because of (MN1) we know that there exists a neighborhood of zero in which k'' doesn't change sign. Therefore, letting $\epsilon \to 0$ and using (C3) we conclude that k'' is integrable around the origin. A simple integration by part, together with (C3) gives then that

$$\int_0^r k''(s)s\ ds = -\int_0^r k'(s)dr + k'(r)r.$$

Dividing both sides by r and letting $r \to 0$ we obtain

$$\lim_{r\to 0} \frac{1}{r}\int_0^r k''(s)s\ ds = 0.$$

which, combined with (MN2) implies $\lim_{r\to 0} k''(r)r = 0$. □

The next lemma shows that the existence theory developed in the previous section applies to this class of potentials.

Lemma 2.46. *If* K *is a natural potential then* $\nabla K \in W^{1,q}(\mathbb{R}^d)$ *for all* $1 \le q < d$. *As a consequence, the critical exponents* p_s *and* q_s *associated to a natural potential satisfy*

$$q_s \ge d \quad \textit{and} \quad p_s \le \frac{d}{d-1}.$$

Proof.- Recall that

$$\nabla K(x) = k'(|x|)\frac{x}{|x|}$$

and (2.109)

$$\frac{\partial^2 K}{\partial x_i \partial x_j}(x) = \left(k''(|x|) - \frac{k'(|x|)}{|x|} \right) \frac{x_i x_j}{|x|^2} + \delta_{ij}\frac{k'(|x|)}{|x|}, \tag{2.110}$$

where δ_{ij} is the Kronecker delta symbol. In order to prove the lemma, it is enough to show that $k'(|x|),\ k''(|x|)$ and $\frac{k'(|x|)}{|x|}$ belong to $L^q(\mathbb{R}^d)$ for all $1 \le q < d$. To do that, observe that the decay condition (C2) implies that they belong to $L^q(B(0,1)^c)$ for all $q \ge 1$. Then, we take into account that (C3) implies that $\frac{k'(r)}{r} = O(1/r)$ as $r \to 0$ and that we have seen in the previous Lemma that

$k''(r) = o(1/r)$ as $r \to 0$. This is enough to conclude, since the function $x \mapsto 1/|x|$ is in $L^q(B(0,1))$ for all $1 \leq q < d$. ∎

The following Lemma together with Theorem 2.19 gives global existence of solutions for natural potentials which are repulsive in the short range, whence part (i) of Theorem 2.5 follows.

Lemma 2.47. *Suppose K is a natural potential which is repulsive in the short range. Then ΔK is bounded from above.*

Proof.-
We will prove that there is a neighborhood of zero on which $\Delta K \leq 0$. This combined with the decay condition (C2) give the desired result. First, recall that $\Delta K(x) = k''(|x|) + (d-1)k'(|x|)|x|^{-1}$. Then, since k is repulsive in the short range, there exists a neighborhood of zero in which $k' \leq 0$. Now, we have two possibilities: on one hand, if $\lim_{r\to 0^+} k'(r) = 0$, then given $r \in (0, +\infty)$ there exists $s \in (0, r)$ such that $\frac{k'(r)}{r} = k''(s)$. Together with (MN1), this implies that k'' is also non-positive in some neighborhood of zero. On the other hand if $\lim_{r\to 0^+} k'(r) < 0$, then the fact that $k''(r) = o(1/r)$ implies that $\frac{rk''(r)+(d-1)k'(r)}{r}$ is negative for r small enough. ∎

Finally, the next Lemma will be needed to prove global existence for natural potentials which are strictly attractive in the short range and satisfy the Osgood criteria.

Lemma 2.48. *Suppose that K is a natural potential which is strictly attractive in the short range and satisfies the Osgood criteria* (2.10). *If moreover $\sup_{x\neq 0} \Delta K(x) = +\infty$ then the following holds*

(Z1) $\lim_{r\to 0^+} k''(r) = +\infty$ *and* $\lim_{r\to 0^+} \frac{k'(r)}{r} = +\infty$,

(Z2) $\exists \delta_1 > 0$ *such that* $k''(r)$ *and* $\frac{k'(r)}{r}$ *are decreasing for* $r \in (0, \delta_1)$,

(Z3) $\exists \delta_2 > 0$ *such that* $k''(r) \leq \frac{k'(r)}{r}$ *for* $r \in (0, \delta_2)$.

Proof.- Let us start by proving by contradiction that

$$\lim_{r\to 0^+} \sup_{p\in(0,r)} \frac{k'(r)}{r} = +\infty. \tag{2.111}$$

If we suppose that

$$\frac{k'(r)}{r} < C \qquad \forall r \in (0,1], \tag{2.112}$$

then given a sequence $r_n \to 0^+$ there will exist another sequence $s_n \to 0^+$, $0 < s_n < r_n$, such that

$$\frac{k'(r_n)}{r_n} = k''(s_n) < C.$$

Since $k''(r)$ is monotonic around zero, this implies that k'' is bounded from above, and combining this with (2.112) we see that ΔK must also be bounded from above, which contradicts our assumption. Now, statements (Z1), (Z2), (Z3) follow easily: First note that if $\lim_{r\to 0^+} k'(r) > 0$, then clearly the Osgood condition (2.10) is not satisfied, whence $\lim_{r\to 0^+} k'(r) = 0$. This implies that for all $r > 0$ there exists $s \in (0, r)$ such that

$$\frac{k'(r)}{r} = k''(s). \tag{2.113}$$

Combining (2.113), (2.111) and the monotonicity of k'' we get that $\lim_{r\to 0^+} k''(r) = +\infty$ and k'' is decreasing on some interval $(0, \delta)$, which corresponds with the first part of (Z1) and (Z2). Now, going back to (2.113) we see that if $0 < s < r < \delta$ then $\frac{k'(r)}{r} = k''(s) \geq k''(r)$ which proves (Z3). This implies

$$\frac{d}{dr}\left\{\frac{k'(r)}{r}\right\} = \frac{1}{r}\left(k''(r) - \frac{k'(r)}{r}\right) \leq 0$$

and therefore $k'(r)/r$ decreases on $(0, \delta)$. Thus, (2.111) implies $\lim_{r\to 0^+} \frac{k'(r)}{r} = +\infty$ and the proof is complete. ∎

2.5.2 Global bound of the L^p-norm using Osgood criteria

We have already proven global existence when the Laplacian of the potential is bounded from above. In this section we prove the following proposition, which allows us to prove global existence for potentials which are attractive in the short range, satisfy the Osgood criteria, and whose Laplacian is not bounded from above. From it, second part of Theorem 2.5 follows readily. This extends prior work on L^∞-solutions [13] to the L^p case.

Proposition 2.49. *Suppose that K is a natural potential which is strictly attractive in the short range, satisfies the Osgood criteria* (2.10) *and whose Laplacian is not bounded from above (i.e. $\sup_{x\neq 0} \Delta K(x) = +\infty$). Let $u(t)$ be the unique solution of the aggregation equation starting with initial data $u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $p > d/(d-1)$. Define the length scale*

$$R(t) = \left(\frac{\|u(t)\|_{L^1}}{\|u(t)\|_{L^p}}\right)^{q/d}.$$

Then there exists positive constants δ and C such that the inequality

$$\frac{dR}{dt} \geq -C\, k'(R) \tag{2.114}$$

holds for all time t for which $R(t) \leq \delta$.

Remark 2.50. *$R(t)$ is the natural length scale associated with blow up of L^p norm. Given that mass is conserved, $R(t) > 0$ means that the L^p norm is bounded. The differential inequality* (2.114) *tell us that $R(t)$ decays slower than the solutions of the ODE $\dot{y} = -Ck'(y)$. Since $k'(r)$ satisfies the Osgood criteria* (2.10)*, solutions of this ODE do not go to zero in finite time. $R(t)$ therefore stays away from zero for all time which provides us with a global upper bound for $\|u(t)\|_{L^p}$.*

Proof.- Equality (2.16) can be written

$$\frac{d}{dt}\left\{\|u(t)\|_{L^p}^p\right\} = (p-1)\int_{\mathbb{R}^d} u(t,x)^p (u(t) * \Delta K)(x)\, dx.$$

Using the chain rule we get

$$\frac{d}{dt}\left\{\|u(t)\|_{L^p}^{-\frac{q}{d}}\right\} = -\frac{(p-1)q}{pd}\|u(t)\|_{L^p}^{-\frac{q}{d}-p}\int_{\mathbb{R}^d} u(t,x)^p (u(t) * \Delta K)(x)\, dx \tag{2.115}$$

$$\geq -\frac{(p-1)q}{pd}\|u(t)\|_{L^p}^{-\frac{q}{d}} \sup_{x\in\mathbb{R}^d}\left\{(u(t) * \Delta K)(x)\right\}. \tag{2.116}$$

To obtain (2.114), we now need to carefully estimate $\sup_{x\in\mathbb{R}^d}\{(u(t) * \Delta K)(x)\}$:

Lemma 2.51 (potential theory estimate). *Suppose that $K(x) = k(|x|)$ satisfies (C2), (Z1), (Z2) and (Z3). Suppose also that $p > \frac{d}{d-1}$. Then there exists positive constants δ and C such that inequality*

$$\sup_{x\in\mathbb{R}^d}(u * \Delta K)(x) \leq C\|u\|_{L^1}\frac{k'(R)}{R} \quad \textit{where} \quad R = \left(\frac{\|u\|_{L^1}}{\|u\|_{L^p}}\right)^{q/d} \tag{2.117}$$

holds for all nonnegative $u \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ satisfying $R \leq \delta$.

Remark 2.52. *By Lemma 2.48, potentials satisfying the conditions of Proposition 2.49 automatically satisfy (C2), (Z1), (Z2) and (Z3).*

Proof.- [Proof of Lemma 2.51] Recall that $\Delta K(x) = k''(|x|) + (d-1)\frac{k'(|x|)}{|x|}$ so that (Z3) implies that $\Delta K(x) < d\; k'(|x|)/|x|$ in a neighborhood of zero. So for ϵ

small enough we have:

$$\int_{|y|<\epsilon} u(x-y)\Delta K(y)dy \le d\int_{|y|<\epsilon} u(x-y)\frac{k'(|y|)}{|y|}dy \tag{2.118}$$

$$\le d\left(\int_{|y|<\epsilon} u(x-y)^p dy\right)^{1/p}\left(\int_{|y|<\epsilon} \partial\frac{k'(|y|)^q}{|y|}dy\right)^{1/q} \tag{2.119}$$

$$\le d\left\|u\right\|_{L^p}\left(\int_0^\epsilon \left(\frac{k'(r)}{r}\right)^q r^{d-1}dr\right)^{1/q} \tag{2.120}$$

$$= d\left\|u\right\|_{L^p}\left(\int_0^\epsilon k'(r)^q r^{d-1-q}dr\right)^{1/q} \tag{2.121}$$

$$\le d\left\|u\right\|_{L^p}k'(\epsilon)\left(\int_0^\epsilon r^{d-1-q}dr\right)^{1/q} \tag{2.122}$$

$$= \frac{d}{(d-q)^{1/q}}\left\|u\right\|_{L^p}\frac{k'(\epsilon)}{\epsilon}\epsilon^{d/q}. \tag{2.123}$$

To go from (2.118) to (2.119) we have used the fact that k' is nonnegative in a neighborhood of zero (because $\lim_{r\to 0^+} k'(r)/r = +\infty$) and that u is also nonnegative. To go from (2.121) to (2.122) we have used the fact that k' is increasing in a neighborhood of zero (because $\lim_{r\to 0^+} k''(r) = +\infty$). Finally, to go from (2.122) to (2.123) we have used the fact that, since $q < d$, $\int_0^\epsilon r^{d-1-q}dr = \epsilon^{d-q}/(d-q)$. Let us now estimate $\int_{|y|\ge\epsilon} u(x-y)\Delta K(y)dy$. On one hand $k''(r)$ and $k'(r)/r$ go to 0 as $r\to+\infty$. On the other hand $k''(r)$ and $k'(r)/r$ go monotonically to $+\infty$ as $r\to 0^+$. Therefore for ϵ small enough we have

$$\sup_{|x|\ge\epsilon}\Delta K(x) = \sup_{r\ge\epsilon}\left\{k''(r)+(d-1)\frac{k'(r)}{r}\right\} \le k''(\epsilon)+(d-1)\frac{k'(\epsilon)}{\epsilon} \le d\frac{k'(\epsilon)}{\epsilon}$$

which gives, since u is nonnegative,

$$\int_{|y|\ge\epsilon} u(x-y)\Delta K(y)dy \le d\left\|u\right\|_{L^1}\frac{k'(\epsilon)}{\epsilon}. \tag{2.124}$$

Combining (2.123) and (2.124) we see that for ϵ small enough we have

$$\sup_{x\in\mathbb{R}^d}\partial u * \Delta K(x) \le c\frac{k'(\epsilon)}{\epsilon}(\epsilon^{d/q}\|u\|_{L^p}+\|u\|_{L^1}) \tag{2.125}$$

where c is a positive constant depending on d and q. Finally, choose $\epsilon = R = \left(\frac{\|u\|_{L^1}}{\|u\|_{L^p}}\right)^{q/d}$. ∎

Combining (2.116) and (2.117) allows us to get (2.114), which concludes the proof of Proposition 2.49. ∎

2.6 Conclusions

This chapter develops refined analysis for well-posedness of the multidimensional aggregation equation for initial data in L^p. To get uniqueness we have to assume that the second moment is bounded, which provides the necessary decay condition. This condition though, is preserved by the dynamics of the equation.
The results we have shown here connect recent theory developed for L^∞ initial data [12, 13, 14] to recent theory for measure solutions [38]. As it was proved in [13], whenever the potential violates the Osgood condition the L^p spaces provide a good understanding of the transition from a regular (bounded) solution to a measure solution. More precisely, in [13] it is shown that for the special case of $K(x) = |x|$, does not exist any 'first kind' similarity solutions to describe the blow-up to a mass concentration for odd space dimensions larger than one. A subsequent numerical study of blow-up for this potenital [15], illustrates that for dimensions larger than two, there is a 'second kind' self-similar blow-up in which the solution stays in L^p for some $p < p_s = d/(d-1)$, rather than showing mass concentration. The asymptotic structure of these solutions is like the example constructed in Section 2.4 of this chapter, thus we can expect that it concentrates mass in a delta after the initial blow-up time.

We remark that these results provide an interesting connection to classical results for Burgers equation. By defining $w(x) = \int_0^x u(y)dy$ [22], our equation reduces to a form of Burgers equation in dimension one for the potential $K = |x|$. Thus, an initial blow-up for the aggregation problem is the same as a a singularity in the slope for Burgers equation. Generically, Burgers singularities form by creating a $|x|^{1/3}$ power singularity in w, which gives a $-2/3$ power blow-up in u. This corresponds to a blow-up in L^p for $p > 3/2$, but does not result in an initial mass concentration. However, as we well know, the Burgers solution forms a shock immediately afterwards, resulting in a delta concentration in u. Thus the scenario described above is a multidimensional analogue of the well-known behavior of how singularities initially form in Burgers equation in one dimension. The delta-concentrations are analogues of shock formation in scalar conservation laws in one dimension.

Section 2.4 constructs an example of a measure solution with initial data in L^p, $p < p_s = \frac{d}{d-1}$, that instantaneously concentrates mass, for the special kernel $K(x) = |x|$ (near the origin). We conjecture that such solutions exist for more general power-law kernels $K(x) = |x|^\alpha$, $2 - d < \alpha < 2$ when the initial data is in L^p, $p < p_s = \frac{d}{d-(2-\alpha)}$. The proof of instantaneous concentration for the special kernel $K(x) = |x|$ uses some monotonicity properties of the convolution operator $\nabla K(x) = \vec{x}/|x|$ which would need to be proved for the more general case.

Several interesting open problems remain, in addition to proving sharpness of the exponent p_s for more general kernels. For instance, local well-posedness for initial data in L^{p_s} is not known yet.

Chapter 3

Models for Collective Behavior

The contents of this chapter appear in:

- *Carrillo, J. A.; Fornasier, M.; Rosado, J.; Toscani, G. "Asymptotic Flocking Dynamics for the kinetic Cucker-Smale model". SIAM J. Math. Anal.* **42** *(2010), 218-236. [40].*

- *Cañizo, J. A.; Carrillo, J. A.; Rosado, J. "A well-posedness theory in measures for some kinetic models of collective motion". To appear in Math. Mod. Meth. Appl. Sci. [32].*

- *Cañizo, J. A.; Carrillo, J. A.; Rosado, J. "Collective Behavior of Animals: Swarming and Complex Patterns". To appear in ARBOR. [37].*

In this chapter, we will be concerned with models which give the evolution of a set of individuals through a certain number of effects. Most of them include alignment or orientation effects, possibly involving some randomness. The self-organization of agents described by the so called Individual Based Models (IBMs) when self-propelling forces and pairwise attractive and repulsive interactions are considered was described in [137, 118, 73]. The classification, in terms of particular strengths of interaction and propulsion, of the morphology of patterns obtained includes: translationally invariant flocks, rotating single and double mills, rings and clumps, patterns that we will show later. Flocking patterns have also been shown in models of alignment or orientation averaging in [64, 63]. All these models share the objective of pinpointing the minimal effects or interactions leading to certain particular type of pattern or collective motion of the agents. These IBMs can be considered also as "particle" models in the optic of treating agents or animals as point particles in a physics-based description as in statistical mechanics. Let us also mention that this issue has also received attention from the control engineering viewpoint trying to reproduce these self-organization patterns with artificial robots or devices, as in the ESA-mission DARWIN, a flotilla of four or five free-flying spacecrafts that will search for Earth-like planets around other stars and analyse their atmospheres for the chemical signature of life. The fundamental problem is to ensure that, with a minimal amount of fuel expenditure, the spacecraft fleet keep remaining in flight (flock), without loosing mutual radio contact, and eventually scattering. See [55, 64, 150] and the references therein, with the aim of controlling unmanned vehicle operation. Finally, the combination of these minimal bricks in the modeling such as interaction, alignment and orientation with other effects is leading to rich complex behavior and detailed dynamical systems models for particular species, see for instance [18, 7, 93, 6, 96, 25, 176].

In some of the applications above, IBMs models are enough to describe the system under reasonable number of individuals/agents N. However, if the number is large as in migration of fish [177] or in myxobacteria [147, 111], the use of continuum models for the evolution of a density of individuals is convenient for numerical simulation, and even necessary. Some continuum models in the literature [163, 164, 29] include attraction-repulsion mechanisms and spatial diffusion to deal with random effects.

Together with particle and continuum models based on macroscopic densities, there has been a very recent trend of mesoscopic models by means of kinetic equations for swarming, describing the probability of finding particles in phase space [91, 39, 90, 40]. In these models one works with a statistical description of the interacting agent system. Let us represent by $x \in \mathbb{R}^d$ the position, where $d \geq 1$ stands for the physical space dimension, and by $v \in \mathbb{R}^d$ the velocity. We are interested in studying the evolution of $f = f(t, x, v)$ representing the probability measure/density of individuals at position x, with velocity v, and at time $t \geq 0$. Given that we cover a variety of these models, we refer the reader to Section 3.2 for a more detailed presentation of the equations.

These kinetic models bridge the particle description of swarming to the hydrodynamic one as already discussed in [91, 39, 90]. The main key idea is that solutions to particle systems are in fact atomic-measure solutions for the kinetic equations, and solutions to the hydrodynamic equations are solutions of a special form to the kinetic equation; see Section 3.6.2 for more details.

In some cases, suitable compactness arguments based on the stability properties in distances between probability measures allow to construct a well-posedness theory for the kinetic equation. Such an approach was done for the Vlasov equation in classical kinetic theory [143, 26, 69, 160] with several nice reviews in [144, 161, 88]. Of these references, [69] uses the Monge-Kantorovich-Rubinstein distance (the one we use in the present paper); the others, as well as the recent work [90] for the kinetic Cucker-Smale model, use an approach based on the bounded Lipschitz distance.

Below we will focus on a review of certain IBMs for swarming proposed in the literature, including some of the basic effects above: attraction, repulsion and orientation. We will describe these particle models and explain some of their features and basic properties in terms of asymptotic patterns. Then, we will present a generic approach to the well-posedness of many of these kinetic models in the set of probability measures in phase space based on the modern theory of optimal transport [171]. In fact, we will use the well-known Monge-Kantorovich-Rubinstein distance between probability measures instead of the bounded Lipschitz distance. Its better duality properties actually make this technical approach easier in terms of estimates leading to one of our crucial results: a stability property of solutions to swarming equations under quite general conditions.

We derive some consequences from this stability estimate. First, we prove the mean-field limit, or convergence of the particle method towards a measure solution of the kinetic equation. This mean field limit is then established without any resorting to the BBGKY hierarchy or the molecular chaos hypothesis [23, 39, 90]. Second, we show the stability for arbitrary times of the hydrodynamic solutions, assuming they exist, although with constants depending on time. Finally, the stability result can be used to obtain qualitative properties of the measure solutions of the kinetic equations, as it has been done in [40] for the kinetic Cucker-Smale model.

This strategy is quite general, and we first demonstrate its use in a particular kinetic model introduced in [39] for dealing with the mesoscopic description and certain patterns not covered by the particle model proposed in [73]. Other models are treated by the same procedure in subsequent sections, as the kinetic Cucker-Smale model proposed in [91] for the original alignment mechanism in [64, 63], the models studied in [123, 128], or any linear combination of these mechanisms. We finally give general conditions on a model that are sufficient for our well-posedness results to be valid.

We will start by addressing particle models for describing mathematically these different effects. Despite their simplicity, these models show how the combination of these simple rules can produce striking phenomena, such as pattern formations: flocks and single or double mills, resembling those observed in nature. Then, we will concentrate in the study of the kinetic version of these models obtained through the mean field limit as we increase the number of particles and show the results mentioned above.

3.1 Individual-Based Models

Most of the basic IBMs of collective behavior or swarming consider the so-called *three-zone models*, meaning that each individual is influenced in a different way by other individuals depending on their relative position, and distinguishing three different zones; see Figure 3.1.

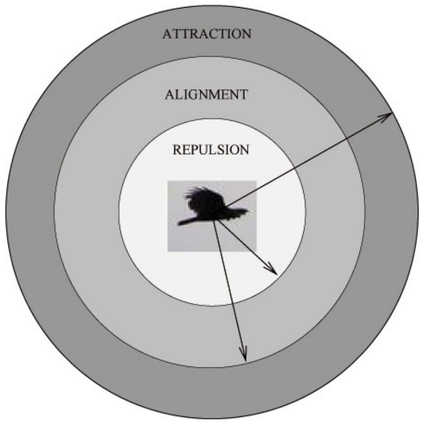

Figure 3.1: Three-zone model.

The inner region, *repulsion zone*, models the “vital space” in which any individual has a tendency to avoid the presence of any other in the swarm, due to collision avoidance or just for sociological reasons. The intermediate region, *orientation or alignment zone*, takes into account the process of mimicking other individual’s behavior by averaging their directions for instance with the others in that region. The outer region models the effect of inherent socialization of the animals since they want to be not too far from other individuals in that region. The strength, particularities and details of these different effects depends on the distance, and on the number of other individuals located instantaneously in the different zones; hence the direction of the individual will be changed according to some *weighted superposition* of these effects. The “attraction”, “repulsion” and “velocity mimicking” are loosely stated here and can take a wide range of forms. This modeling based on *social forces* finds its roots in the works of [3, 103] for fish schooling and it has been widely used, expanded, and improved by theoretical biologists, physicists and applied mathematicians; see [62, 137, 113, 94, 114, 173, 174, 8, 118, 123, 76] and the references therein. This approach has been made much more specific for particular animals and species as in [18, 7, 177, 93] for fish (studying migration patterns for the capelin around Iceland) and in [6, 96] for birds (starlings grouping in Rome).

Here, we concentrate mainly in these three effects as they constitute the main modeling “bricks”, but the reader should be warned, as we mentioned before, that in order to reproduce realistic swarming behavior as in the starlings [6] or the migration of fish [7], one has to include much more involved interactions. For instance, real three dimensional effects due to the *aerodynamics* of birds, in which drag, lift and friction forces are included, or *roosting* forces trying to model their tendency to stay close to certain home area as in [93]; or changes in movement due to currents or temperature changes as in [7]. Other improvements in the modeling include detailed vision zones for the individuals, assuming for instance that birds only see in a certain vision cone which depends on their position and direction of motion.

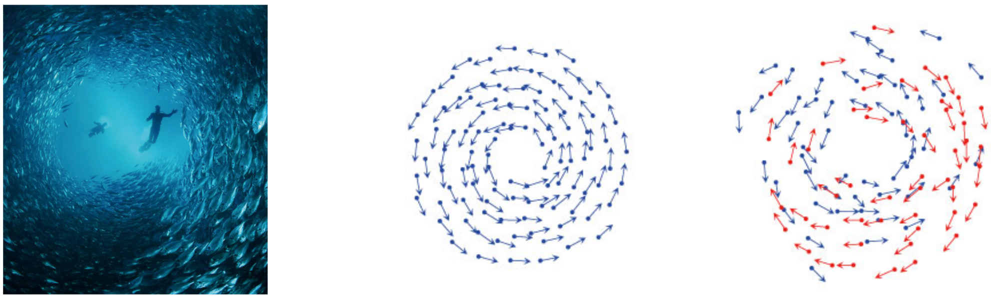

Figure 3.2: Mills in nature and in the IBMs!

3.1.1 Asymptotic speed with attraction-repulsion interaction model

The particle model proposed in [73] reads as:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, & (i = 1, \ldots, N) \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \dfrac{1}{N}\displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|), & (i = 1, \ldots, N). \end{cases} \tag{3.1}$$

where $x_i, v_i \in \mathbb{R}^3$ represent the position and velocity of the i-th individual, for i varying from 1 to the total number N of individuals. Here, α and β are nonnegative parameters and $U : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a given potential encoding the short-range repulsion and long-range attraction typical in these models. Here, the potential has been scaled depending on the number of particles as in [39], as it is convenient in order to study the limit of the system for large N. The term corresponding to α models the self-propulsion of individuals, whereas the term corresponding to β is the friction assumed to follow Rayleigh's law. The balance of these two terms imposes an asymptotic speed to the agent (if other effects are ignored), but does not influence the orientation vector. The main point here is that of having a "preferred" velocity $\sqrt{\alpha/\beta}$, and not the precise shape of the term involving α and β.

On the other hand, a typical choice for U is the *Morse potential*, a radial potential given by

$$U(x) = k(|x|) \qquad \text{with} \qquad k(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R},$$

where C_A, C_R and ℓ_A, ℓ_R are the strengths and the typical lengths of attraction and repulsion, respectively. This potential is only Lipschitz due to its singularity at the origin but the qualitative behavior of the particle system does not heavily depend on this fact [73]. To analyze the limit of large number of particles N, and for simplicity, we scale the amplitude of the potential through a normalization which corresponds to assuming that all particles have mass $1/N$.

The interesting particularity of this system is that it exhibits very different behavior depending on the values of the parameters. Let us have a look at some possibilities. Taking the values $C_R = 50$, $\ell_R = 2$, $C_A = 20$, $\ell_A = 100$, $\beta = .05$, $\alpha = .07$ and solving numerically with some randomly chosen initial conditions, we find that individuals arrange into a sort of swirl or *mill*, see Fig. 3.3, as this type of arrangement has been called in the literature. After some hesitation, they seem to always agree to turn in only one direction.

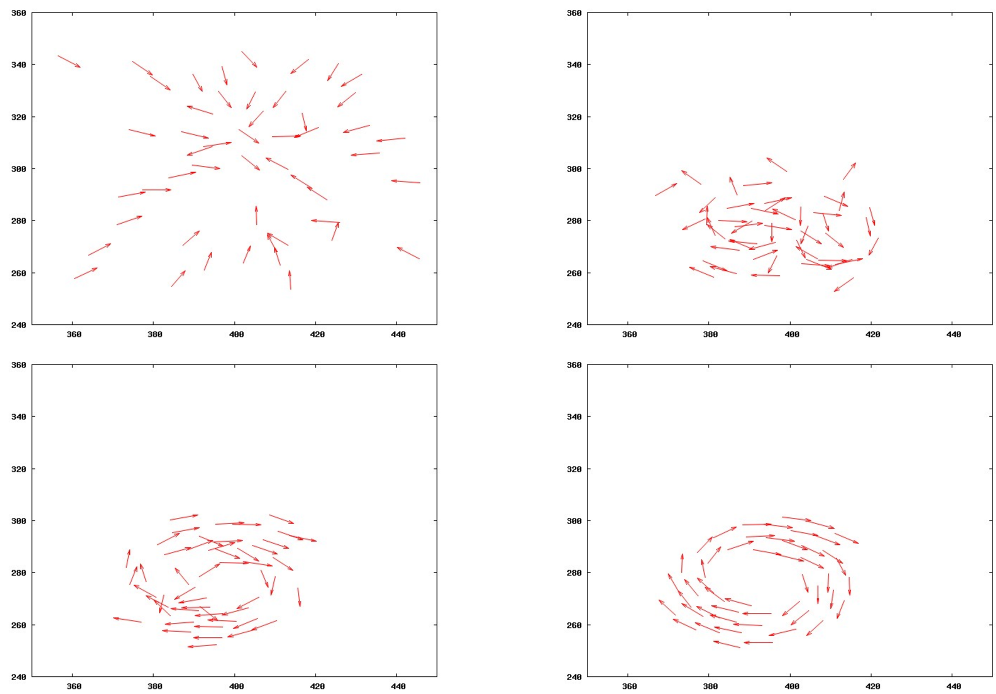

Figure 3.3: Formation of a single mill.

However, consider now $C_R = 50$, $\ell_R = 20$, $C_A = 100$, $\ell_A = 100$, $\beta = .05$, $\alpha = .15$ giving the results in Fig. 3.4. Though the kind of behavior seems initially alike, now we may have individuals turning *in both directions*: clockwise and counterclockwise. As we have reduced the strength of the short-range repulsion, it is no problem now for individuals to cross very close.

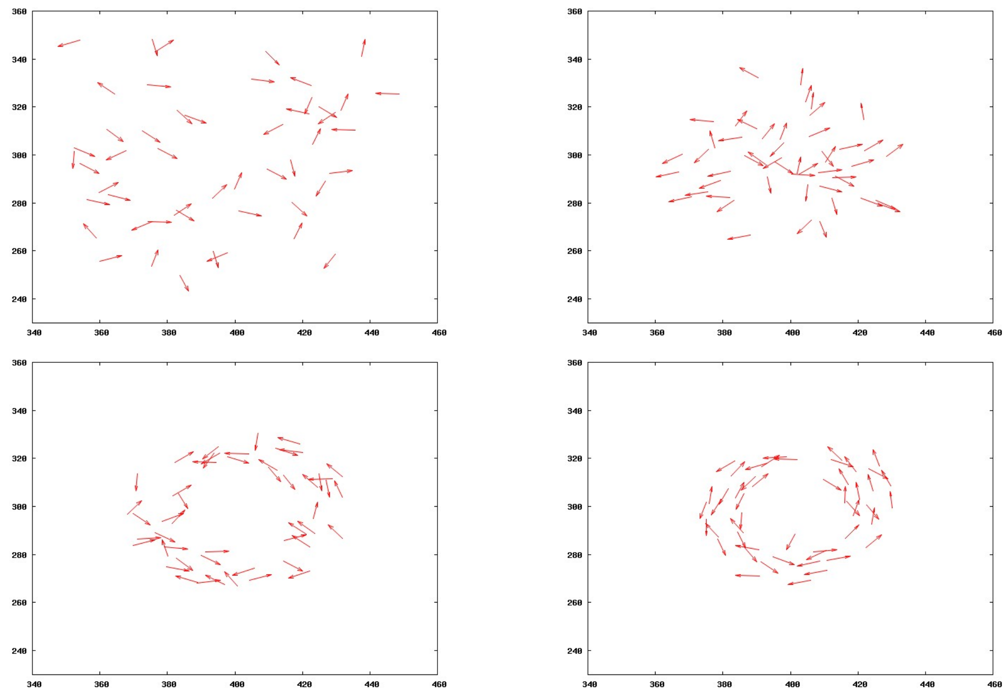

Figure 3.4: Formation of a double mill.

There are yet other possible patterns in which individuals tend to organize, depending on the value of the parameters. For instance, sometimes they organize in a kind of crystalline structure and they move translationally, this pattern is called flock. Identifying the tendency for each parameter seems difficult, and there are even some parameters for which one observes sometimes one type of organization, sometimes another one.

However, part of it can be analyzed, at least heuristically. In the case of the Morse potential, patterns of aggregation depend on the relative amplitudes $C = C_R/C_A$ and $\ell = \ell_R/\ell_A$, and the number of individuals N does not seem to affect the qualitative features of the observed patterns, so we may center the discussion on the values of C and ℓ.

In two dimensions, the different patterns were classified in [73] using the concept of H-*stability* of potentials [157, 161]. The most relevant set of parameters for

biological applications concerns long-range attraction and short-range repulsion leading to $C > 1$ and $\ell < 1$. For these potentials, there exists a unique minimum of the pairwise potential and a typical distance minimizing the potential energy. What they remarked is that the behavior of the system (3.1) depends on whether the parameters are in the so-called *H-unstable* or *catastrophic* region $C\ell^d < 1$, or in the H-stable region $C\ell^d \geq 1$, where d stands for the dimension (the terminology coming from statistical mechanics). Complex behavior such as the formation of mills or double mills seems to happen only in the catastrophic region, while individuals seem to either form a swarm or just disperse in the stable region. There is also a difference between the catastrophic and stable regions in the size of "aggregates" (swarms or mills) as N grows. However, it does not seem easy to decide, in the catastrophic region, which parameters will produce swarms, mills, double mills, rings, or other patterns.
These shapes may contain some clue on the organization of animals in nature: while coherent flocks and single mill states are the most common patterns observed in biological swarms [147, 158], double-mill patterns, as seen in Figure 3.2, are also reported in the biological literature; for instance *M. xanthus* cells show distinct cell subpopulations swarming in two opposite directions during part of their life cycle [111].

3.1.2 The Cucker-Smale model

In the Cucker-Smale model, introduced in [64, 63], the only mechanism taken into account is the reorientation interaction between agents. Each agent in the swarm tries to mimick other individuals by adjusting/averaging their relative velocity with all the others. This averaging is weighted in such a way that closer individuals have a larger influence than further ones. For a system with N individuals the Cucker-Smale model, normalized as before, reads

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} w_{ij} \left(v_j - v_i\right), \end{cases} \tag{3.2}$$

with the *communication rate* $w(x)$ given by:

$$w_{ij} = w(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^\gamma}$$

for some $\gamma \geq 0$. For this model, one would expect that individuals tend to adopt finally the same velocity and move translationally, as they change their velocity to

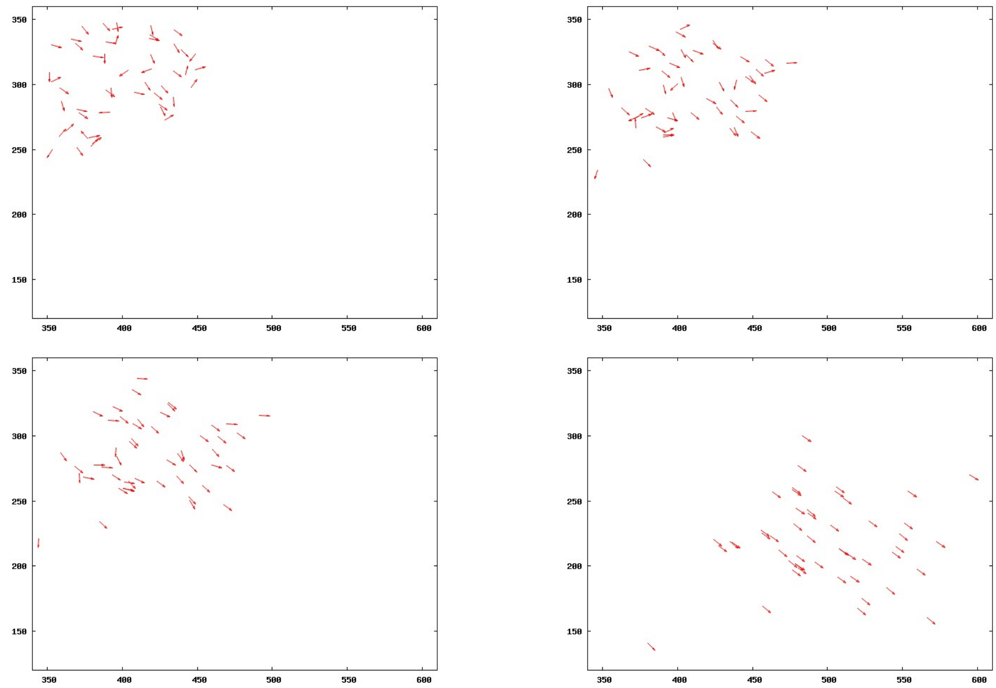

Figure 3.5: Formation of a non-universal flock.

adapt to that of others. This behavior is observed indeed by numerically solving the above equations, see Fig. 3.5, with $\gamma = 0.45$.

After several improvements, it has been shown in [64, 63, 91, 90, 40] that the asymptotic behavior of the system for $\gamma \leq 1/2$ does not depend on N; in this case, called *unconditional non-universal flocking*, the behavior of the population is perfectly specified, all the individuals tend to flock. Flocking means moving with the same mean velocity and to form a group with fixed mutual distances, not necessarily in a crystalline-like pattern, but rather depending on the the initial positions and velocities (actually, a set of individuals moving initially at the same speed will continue to do so indefinitely regardless of their initial positions). On the other hand, in the regime $\gamma > 1/2$, flocking can be expected under certain conditions on the initial configuration but there are counterexamples to the generic flocking [64]. We refer to [64, 63, 40] for further discussion about this model and its qualitative properties, but we give a short proof of the flocking behavior when $\gamma \leq 1/2$.

3.1.3 3-zone model

Let us consider now the model with attraction, repulsion, and Cucker-Smale effects included, which can then be properly called a "three-zone model" with three zones which are not disjoint "at all":

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, & (i = 1, \ldots, N) \\ \dfrac{dv_i}{dt} = -\dfrac{1}{N}\displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|) + \dfrac{1}{N}\sum_{j=1}^{N} w_{ij}\left(v_j - v_i\right), & (i = 1, \ldots, N). \end{cases} \tag{3.3}$$

How does the asymptotic behavior of this model look like? Individuals tend to eventually adopt the same velocity due to the Cucker-Smale effect, as in section 3.1.2. They also tend to arrange into some group, due to the potential interaction. The result is that they tend to an arrangement which should be a local minimum of the potential energy, but this does not seem obvious at all to prove. For instance, if we take the values $C_R = 500$, $\ell_R = 2$, $C_A = 200$, $\ell_A = 100$, $\beta = .1$, $\alpha = .2$, $\gamma = 0.45$ and look at the system with 50 individuals with random initial conditions, we observe numerically that no matter how we start, they seem to end up arranging themselves in a group-like fashion as a crystalline structure, and moving in some direction at the preferred speed $\sqrt{\alpha/\beta}$, see Fig. 3.6.

3.2 Kinetic Models

Unlike the control of a finite number of agents, the numerical simulation of a rather large population of interacting agents can constitute a serious difficulty which stems from the accurate solution of a possibly very large system of ODEs. Borrowing the strategy from the kinetic theory of gases, we may want instead to consider a density distribution of agents, depending on spatial position, velocity, and time evolution, which interact with stochastic influence (corresponding to classical collisional rules in kinetic theory of gases)– in this case the influence is spatially "smeared" since two birds do interact also when they are far apart. Hence, instead of simulating the behavior of each individual agent, we would like to describe the collective behavior encoded by the density distribution whose evolution is governed by one sole mesoscopic partial differential equation.
In this section we will present the kinetic version of the models we have discussed above. This formulation can be obtained from the IBMs by computing the mean field limit as the number of particles increases, as we will show in Section 3.3 and 3.4. To fix ideas, let us look at what the specific particle models that we introduced in the previous section become when we look at the mean field limit for them.

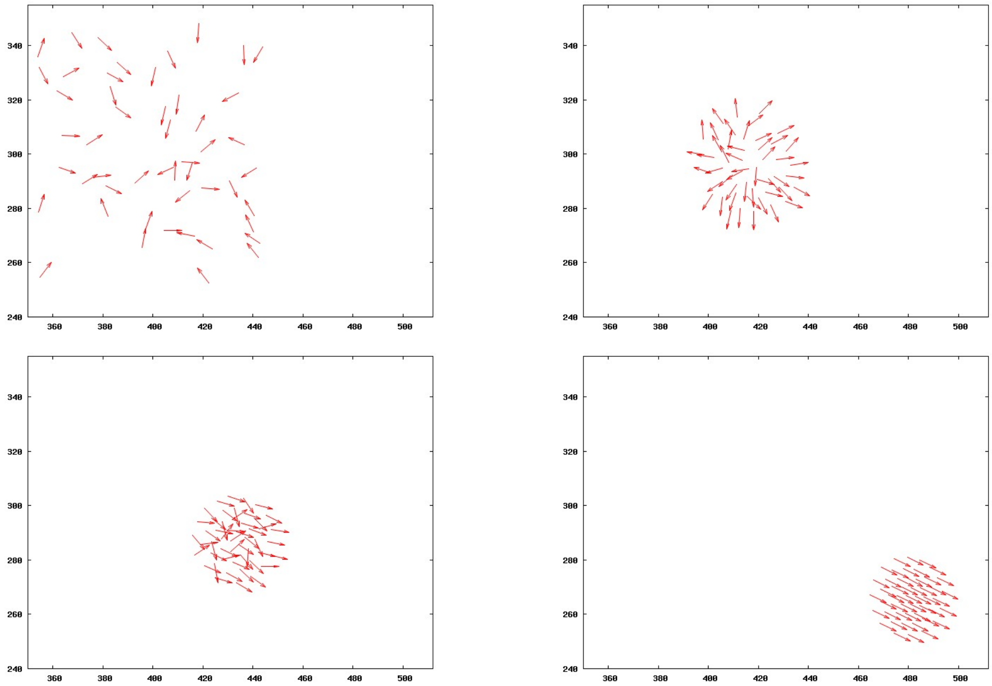

Figure 3.6: Formation of Flocking.

The kinetic equation associated to this particle model as discussed in [39] gives the evolution of $f = f(t, x, v)$ as

$$\partial_t f + v \cdot \nabla_x f - (\nabla U * \rho) \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta\,|v|^2) v f) = 0, \tag{3.4}$$

where ρ represents the macroscopic *density* of f:

$$\rho(t, x) := \int_{\mathbb{R}^d} f(t, x, v)\, dv \qquad \text{for } t \geq 0, x \in \mathbb{R}^d. \tag{3.5}$$

However, the Morse potential described before does not satisfy the smoothness assumption in our main theorems, although the qualitative behavior of the particle system does not depend on this particular fact [73]. In fact, a typical potential satisfying all of our hypotheses is

$$U(x) = -C_A e^{-|x|^2/\ell_A^2} + C_R e^{-|x|^2/\ell_R^2}.$$

In the case of Cucker-Smale, the particle model leads to the following kinetic model [91, 90, 40]:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot [\xi[f]\, f] \tag{3.6}$$

where $\xi[f](t,x,v) = (H * f)(t,x,v)$, with $H(x,v) = w(x)v$ and $*$ standing for the convolution in both position and velocity (x and v). We refer to [64, 63, 40] for further discussion about this model and qualitative properties.

Moreover, quite general models incorporating the three effects previously discussed have been considered in [123, 128]. In particular, they consider that N individuals follow the system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = F_i^A + F_i^I, \end{cases} \tag{3.7}$$

where F_i^A is the self-propulsion autonomously generated by the ith-individual, while F_i^I is due to interaction with the others. The model in Section 3 corresponds to $F_i^A = (\alpha - \beta\,|v_i|^2)v_i$, while the term $F_i^A = -\beta\, v_i$ is considered in [118], and $F_i^A = a_i - \beta\, v_i$ in [123, 128]. Here, a_i is an autonomous self-propulsion force generated by the ith-particle, and may depend on environmental influences and the location of the particle in the school. The interaction with other individuals can be generally modeled as:

$$F_i^I = F_i^{I,x} + F_i^{I,v} = \sum_{j=1}^{N} g_\pm(|x_i - x_j|)\frac{x_j - x_i}{|x_i - x_j|} + \sum_{j=1}^{N} h_\pm(|v_i - v_j|)\frac{v_j - v_i}{|v_i - v_j|}.$$

Here, g_+ and h_+ (g_- and h_-) are chosen when the influence comes from the front (behind), i.e., if $(x_j - x_i)\cdot v_i > 0$ (< 0); choosing $g_+ \neq g_-$ and $h_+ \neq h_-$ means that the forces from particles in front and those from particles behind are different. The sign of the functions $g_\pm(r)$ encodes the short-range repulsion and long-range attraction for particles in front of (+) and behind (-) the ith-particle. Similarly, $h_+ > 0$ (< 0) implies that the velocity-dependent force makes the velocity of particle i get closer to (away from) that of particle j.

In what follows we will be concerned with the well-posedness for measure solutions to (3.4), (3.6) and generalized kinetic equations including the corresponding to the N-individuals model in (3.7).

3.3 Well-posedness for a system with interaction and self-propulsion

Let us start by fixing some notation. We denote by B_R the closed ball with center 0 and radius $R > 0$ in the Euclidean space $\mathbb{R}^d$ of some dimension d. When we need to explicitly indicate the dimension of the space, we will write B_R^d. For a function

$H : \mathbb{R}^d \to \mathbb{R}^m$, we will write $Lip_R(H)$ to denote the Lipschitz constant of H in the ball $B_R \subseteq \mathbb{R}^d$. For $T > 0$ and a function $H : [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$, $H = H(t,x)$, we again write $Lip_R(H)$ to denote the Lipschitz constant *with respect to* x of H in the ball $B_R \subseteq \mathbb{R}^d$; this is, $Lip_R(H)$ is the smallest constant such that

$$|H(t,x_1) - H(t,x_2)| \leq Lip_R(H)\,|x_1 - x_2| \qquad \text{for all } x_1, x_2 \in B_R, t \in [0,T].$$

For any such function H, we will denote the function depending on x at a fixed time t by H_t; this is, $H_t(x) := H(t,x)$.
In this section we consider eq. (3.4). In this model (and in fact, in every model considered in this paper) the total mass is preserved, and by rescaling the equation and adapting the parameters suitably one easily sees that we can normalize the equation and consider only solutions with total mass 1. We will do so and reduce ourselves to work with probability measures.

3.3.1 Notion of solution

In order to motivate our definition of solution to equation (3.4) let us consider for a moment a general field E instead of $-\nabla U * \rho$. Precisely, fix $T > 0$ and a function $E : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that:

Hypothesis 3.3.1 (Conditions on E).

1. *E is continuous on $[0,T] \times \mathbb{R}^d$.*
2. *For some $C_E > 0$,*
$$|E(t,x)| \leq C_E(1 + |x|), \textit{ for all } t, x \in [0,T] \times \mathbb{R}^d. \tag{3.8}$$
3. *E is* locally Lipschitz with respect to x, *i.e., for any compact set $K \subseteq \mathbb{R}^d$ there is some $L_K > 0$ such that*
$$|E(t,x) - E(t,y)| \leq L_K\,|x - y|\,, \qquad t \in [0,T], \quad x, y \in K. \tag{3.9}$$

We consider the equation

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \operatorname{div}_v((\alpha - \beta\,|v|^2) v f) = 0, \tag{3.10}$$

which is a linear first-order equation. The associated characteristic system of ode's is

$$\frac{d}{dt} X = V, \tag{3.11a}$$

$$\frac{d}{dt} V = E(t,X) + V(\alpha - \beta\,|V|^2). \tag{3.11b}$$

Lemma 3.3.2 (Flow Map). *Take a field* $E : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *satisfying Hypothesis 3.3.1. Given* $(X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$ *there exists a unique solution* (X, V) *to equations* (3.11a)-(3.11b) *in* $\mathcal{C}^1([0,T]; \mathbb{R}^d \times \mathbb{R}^d)$ *satisfying* $X(0) = X_0$ *and* $V(0) = V_0$*. In addition, there exists a constant* C *which depends only on* T, $|X_0|$, $|V_0|$, α, β *and the constant* C_E *in eq.* (3.8)*, such that*

$$|(X(t), V(t))| \leq |(X_0, V_0)| \, e^{Ct} \quad \text{for all } t \in [0,T]. \tag{3.12}$$

Proof.- As the field E satisfies the regularity and growth conditions in Hypothesis 3.3.1, standard results in ordinary differential equations show that for each initial condition $(X(0), V(0)) \in \mathbb{R}^d \times \mathbb{R}^d$ this system has a unique solution defined on $[0,T)$ (the only term in the equations which does not grow linearly is $-\beta V \, |V|^2$, and it makes $|V|$ decrease, so the solution is globally defined in time). The bound (3.12) on the solutions follows from direct estimates on the equation, using the linear growth of the field E. □

Calling $P \equiv (X, V)$, the system (3.11a)-(3.11b) can be conveniently written as

$$\frac{d}{dt} P = \Psi_E(t, P), \tag{3.13}$$

where $\Psi_E : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is the right hand side of eqs. (3.11a), (3.11b). When the field E is understood we will just write Ψ instead of Ψ_E. Using this notation, Equation (3.10) can also be rewritten as

$$\frac{\partial f}{\partial t} + \operatorname{div}(\Psi_E f) = 0. \tag{3.14}$$

We can thus consider the flow at time $t \in [0,T)$ of eqs. (3.11),

$$P_E^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d.$$

Again by basic results in ode's, the map $(t, x, v) \mapsto P_E^t(x, v) = (X, V)$ with (X, V) the solution at time t to (3.12) with initial data (x, v), is jointly continuous in (t, x, v). For a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ it is well-known that the function

$$f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \quad t \mapsto f_t := P_E^t \# f_0$$

is a measure solution to eq. (3.10), i.e., a solution in the distributional sense. Here we are using the mass transportation notation of *push-forward*: $f_t = P_E^t \# f_0$ is defined by

$$\int_{\mathbb{R}^{2d}} \zeta(x,v) \, f(t,x,v) \, d(x,v) = \int_{\mathbb{R}^{2d}} \zeta(P_E^t(x,v)) \, f_0(x,v) \, d(x,v), \tag{3.15}$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^{2d})$. Note that in the case where the initial condition f_0 is regular (say, $\mathcal{C}_c^\infty$) this is just a way to rewrite the solution of the equation through the method of characteristics. This motivates the following definition:

Definition 3.3.3 (Notion of Solution). *Take a potential $U \in \mathcal{C}^1(\mathbb{R}^d)$ such that*

$$|\nabla U(x)| \leq C(1 + |x|), \qquad x \in \mathbb{R}^d, \tag{3.16}$$

for some constant $C > 0$. Take also a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, and $T \in (0, \infty]$. We say that a function $f : [0, T] \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ is a solution of the swarming equation (3.4) *with initial condition f_0 when:*

1. *The field $E[f] = -\nabla U * \rho$ satisfies the conditions in Hypothesis 3.3.1.*
2. *It holds $f_t = P^t_{E[f]} \# f_0$.*

Remark 3.3.4. *This definition gives a convenient condition on U so that a measure solution in $\mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ makes sense. One can weaken the requirement on U in this definition as long as the requirements on f are suitably strengthened (e.g., one can allow a faster growth of the potential if one imposes a faster decay of f, or less local regularity of U if one assumes more regularity of f), but we will not consider these modifications in the present work. Since we ask the gradient of the potential to be locally Lipschitz, we cannot consider potentials with a singularity at the origin. This is a strong limitation of the classical theory, and is considered a difficult problem for the mean-field limit. As for the existence theory, if one wants to consider more singular potentials, one can work with functions f which are more regular than just measures, so that $\nabla U * \rho$ becomes locally Lipschitz and a parallel existence theory can be developed.*

3.3.2 Estimates on the characteristics

We gather in this section some estimates on solutions to the characteristic equations (3.11). In this section we fix $T > 0$ and fields $E, E^1, E^2 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ which are assumed to satisfy Hypothesis 3.3.1, and we consider their corresponding characteristic equations (3.11). Recall that Ψ_E is a shorthand for the right hand side of (3.11), as in (3.13).
We first gather some basic regularity results for the function which defines the right hand side of eqs. (3.11a)–(3.11b):

Lemma 3.3.5 (Regularity of the characteristic equations). *Take a field $E : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfies Hypothesis 3.3.1. Consider a number $R > 0$ and the closed ball $B_R \subseteq \mathbb{R}^d \times \mathbb{R}^d$.*

1. *Ψ_E is bounded in compact sets: For $P = (X, V) \in B_R$ and $t \in [0, T]$,*

   $$|\Psi_E(t, P)| \leq C$$

   for some $C > 0$ which depends only on α, β, R, and $\|E\|_{L^\infty([0,T]\times B_R)}$.

2. *Ψ_E is locally Lipschitz with respect to x, v: For all $P_1 = (X_1, V_1)$, $P_2 = (X_2, V_2)$ in B_R, and $t \in [0, T]$,*

$$|\Psi_E(t, P_1) - \Psi_E(t, P_2)| \le C(1 + Lip_R(E_t)) \, |P_1 - P_2| \, ,$$

for some number $C > 0$ which depends only on α and β.

Proof.- This can be obtained by a direct calculation from eqs. (3.11a)–(3.11b). □

Lemma 3.3.6 (Dependence of the characteristic equations on E). *Take two fields $E^1, E^2 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying Hypothesis 3.3.1, and consider the functions Ψ_{E^1}, Ψ_{E^2} which define the characteristic equations* (3.11) *as in eq.* (3.13)*. Then, for any compact (in fact, any measurable) set B,*

$$\|\Psi_{E^1} - \Psi_{E^2}\|_{L^\infty(B)} \le \left\|E^1 - E^2\right\|_{L^\infty(B)} .$$

Proof.- Trivial from the expression of Ψ_{E^1}, Ψ_{E^2}. □

Now we explicitly state some results which give a quantitative bound on the regularity of the flow P_E^t, and its dependence on the field E.

Lemma 3.3.7 (Dependence of characteristics on E). *Take two fields $E^1, E^2 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying Hypothesis 3.3.1, and a point $P^0 \in \mathbb{R}^d \times \mathbb{R}^d$. Take $R > 0$, and assume that*

$$\left|P^t_{E^1}(P^0)\right| \le R, \quad \left|P^t_{E^2}(P^0)\right| \le R \qquad \text{for } t \in [0, T].$$

Then for $t \in [0, T]$ it holds that

$$\left|P^t_{E^1}(P^0) - P^t_{E^2}(P^0)\right| \le \int_0^t e^{C(t-s)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)} ds \tag{3.17}$$

for some constant C which depends only on α, β, R and $\mathrm{Lip}_R(E^1)$. As a consequence,

$$\left|P^t_{E^1}(P^0) - P^t_{E^2}(P^0)\right| \le \frac{e^{Ct} - 1}{C} \sup_{s \in [0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)} . \tag{3.18}$$

Proof.- For ease of notation, write $P_i(t) \equiv P^t_{E_i}(P^0) \equiv (X_i(t), V_i(t))$, for $i = 1, 2$, $t \in [0, T]$. These functions satisfy the characteristic equations (3.11):

$$\frac{d}{dt} P_i = \Psi_{E_i}(t, P_i), \quad P_i(0) = P^0, \quad \text{for } i = 1, 2.$$

Then, for $t \in [0,T]$, and using Lemmas 3.3.5 and 3.3.6,

$$\begin{aligned}
|P_1(t) - P_2(t)| &\leq \int_0^t |\Psi_{E^1}(s, P_1(s)) - \Psi_{E^2}(s, P_2(s))| \, ds \\
&\leq \int_0^t |\Psi_{E^1}(s, P_1(s)) - \Psi_{E^1}(s, P_2(s))| \, ds \\
&\quad + \int_0^t |\Psi_{E^1}(s, P_2(s)) - \Psi_{E^2}(s, P_2(s))| \, ds \\
&\leq C \int_0^t |P_1(s) - P_2(s)| \, ds + \int_0^t \left\| E^1_s - E^2_s \right\|_{L^\infty(B_R)} \, ds
\end{aligned}$$

where C is the constant in point 2 of Lemma 3.3.5, which depends on α, β, R and the Lipschitz constant of E^1 with respect to x in the ball B_R. By Gronwall's Lemma,

$$|P_1(t) - P_2(t)| \leq \int_0^t e^{C(t-s)} \left\| E^1_s - E^2_s \right\|_{L^\infty(B_R)} \, ds.$$

This proves the first part of our result. To prove the second part, continue from above to write

$$\begin{aligned}
|P_1(t) - P_2(t)| &\leq \left(\int_0^t e^{C(t-s)} \, ds \right) \sup_{s \in (0,T)} \left\| E^1_s - E^2_s \right\|_{L^\infty(B_R)} \\
&= \frac{e^{Ct} - 1}{C} \sup_{s \in (0,T)} \left\| E^1_s - E^2_s \right\|_{L^\infty(B_R)},
\end{aligned}$$

which finishes the proof. □

Lemma 3.3.8 (Regularity of characteristics with respect to initial conditions). *Take $T > 0$ and a field $E : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying Hypothesis 3.3.1. Take also $P_1, P_2 \in \mathbb{R}^d \times \mathbb{R}^d$ and $R > 0$, and assume that*

$$\left| P^t_E(P_1) \right| \leq R, \quad \left| P^t_E(P_2) \right| \leq R \qquad t \in [0,T].$$

Then it holds that

$$\left| P^t_E(P_1) - P^t_E(P_2) \right| \leq |P_1 - P_2| \, e^{C \int_0^t (\mathrm{Lip}_R(E_s) + 1) \, ds}, \quad t \in [0,T], \tag{3.19}$$

for some constant C which depends only on R, α and β. Said otherwise, P^t_E is Lipschitz on $B_R \subseteq \mathbb{R}^d \times \mathbb{R}^d$, with constant

$$\mathrm{Lip}_R(P^t_E) \leq e^{C \int_0^t (\mathrm{Lip}_R(E_s) + 1) \, ds}, \quad t \in [0,T].$$

Proof.- Write $P_i(t) \equiv P_E^t(P_i) \equiv (X_i(t), V_i(t))$, for $i = 1, 2$, $t \in [0, T]$. These functions satisfy the characteristic equations (3.11):

$$\frac{d}{dt} P_i = \Psi_E(t, P_i), \quad P_i(0) = P_i, \quad \text{for } i = 1, 2.$$

For $t \in [0, T]$, using Lemma 3.3.5,

$$\begin{aligned} |P_1(t) - P_2(t)| &\leq |P_1 - P_2| + \int_0^t |\Psi_E(s, P_1(s)) - \Psi_E(s, P_2(s))| \, ds \\ &\leq |P_1 - P_2| + C \int_0^t (\mathrm{Lip}_R(E_s) + 1) \, |P_1(s) - P_2(s)| \, ds \end{aligned}$$

We get our result by applying Gronwall's Lemma to this inequality. □

Lemma 3.3.9 (Regularity of characteristics with respect to time). *Take $T > 0$ and a field $E : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying Hypothesis 3.3.1. Take $P^0 \in \mathbb{R}^d \times \mathbb{R}^d$, $R > 0$ and assume that*

$$\left| P_E^t(P^0) \right| \leq R, \qquad t \in [0, T].$$

Then it holds that

$$\left| P_E^t(P^0) - P_E^s(P^0) \right| \leq C \, |t - s| \quad \text{for } s, t \in [0, T], \tag{3.20}$$

for some constant C which depends only on α, β, R and $\|E\|_{L^\infty([0,T] \times B_R)}$.

Proof.- By definition, $\frac{d}{dt} P_E^t(P^0) = \Psi_E(t, P_E^t(P_0))$, and from point 1 of Lemma 3.3.5 we know that

$$\left| \Psi_E(t, P_E^t(P^0)) \right| \leq C \quad \text{for } t \in [0, T],$$

for some number C depending on the allowed quantities, as we are assuming that $P_E^t(P_0)$ remains on a certain compact subset of $\mathbb{R}^d \times \mathbb{R}^d$. The statement directly follows from this. □

3.3.3 Existence and uniqueness

Theorem 3.3.10 (Existence and uniqueness of measure solutions). *Take a potential $U \in \mathcal{C}^1(\mathbb{R}^d)$ such that ∇U is locally Lipschitz and such that for some $C > 0$,*

$$|\nabla U(x)| \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}^d, \tag{3.21}$$

and $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ *with compact support. There exists a solution* f *on* $[0, +\infty)$ *to equation* (3.4) *with initial condition* f_0 *in the sense of Definition 3.3.3. In addition,*

$$f \in \mathcal{C}([0, +\infty); \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)) \tag{3.22}$$

and there is some increasing function $R = R(T)$ *such that for all* $T > 0$*,*

$$\operatorname{supp} f_t \subseteq B_{R(T)} \subseteq \mathbb{R}^d \times \mathbb{R}^d \quad \textit{for all } t \in [0, T]. \tag{3.23}$$

This solution is unique among the family of solutions satisfying (3.22) *and* (3.23)*.*

The rest of this section is dedicated to the proof of this result, for which we will need some previous lemmas. We begin with a general result on the transportation of a measure by two different functions:

Lemma 3.3.11. *Let* $P_1, P_2 : \mathbb{R}^d \to \mathbb{R}^d$ *be two Borel measurable functions. Also, take* $f \in \mathcal{P}_1(\mathbb{R}^d)$*. Then,*

$$W_1(P_1 \# f, P_2 \# f) \leq \|P_1 - P_2\|_{L^\infty(\operatorname{supp} f)}. \tag{3.24}$$

Proof.- We consider a transference plan defined by $\pi := (P_1 \times P_2)\# f$. One can check that this measure has marginals $P_1 \# f$, $P_2 \# f$. Then,

$$\begin{aligned} W_1(P_1 \# f, P_2 \# f) &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \, \pi(x, y) \, dx \, dy \\ &= \int_{\mathbb{R}^d} |P_1(x) - P_2(x)| \, f(x) \, dx \leq \|P_1 - P_2\|_{L^\infty(\operatorname{supp} f)}, \end{aligned}$$

which proves the lemma. □

Lemma 3.3.12 (Continuity with respect to time). *Take* $T > 0$ *and a field* $E : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *in the conditions of Hypothesis 3.3.1. Take also a measure* f *on* $\mathbb{R}^d \times \mathbb{R}^d$ *with compact support contained in the ball* B_R*.*
Then, there exists $C > 0$ *depending only on* α*,* β*,* R *and* $\|E\|_{L^\infty([0,T] \times B_R)}$ *such that*

$$W_1(P_E^s \# f, P_E^t \# f) \leq C \, |t - s| \,, \quad \textit{for any } t, s \in [0, T].$$

Proof.- From Lemma 3.3.11 and the continuity of characteristics with respect to time, Lemma 3.3.9, we get

$$W_1(P_E^s \# f, P_E^t \# f) \leq \|P_E^s - P_E^t\|_{L^\infty(\operatorname{supp} f)} \leq C \, |t - s| \,,$$

for some $C > 0$ which depends only on the quantities in the lemma. □

Lemma 3.3.13. *Take a locally Lipschitz map $P : \mathbb{R}^d \to \mathbb{R}^d$ and $f, g \in \mathcal{P}_1(\mathbb{R}^d)$, both with compact support contained in the ball B_R. Then,*

$$W_1(P\#f, P\#g) \leq L\, W_1(f, g), \tag{3.25}$$

where L is the Lipschitz constant of P on the ball B_R.

Proof.- Set π to be an optimal transportation plan between f and g. The measure $\gamma = (P \times P)\#\pi$ has marginals $P\#f$ and $P\#g$, as can be easily checked, so we can use it to bound $W_1(P\#f, P\#g)$:

$$\begin{aligned} W_1(P\#f, P\#g) &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - w|\ \gamma(z, w)\, dz\, dw \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |P(z) - P(w)|\, \pi(z, w)\, dz\, dw \\ &\leq L \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - w|\, \pi(z, w)\, dz\, dw = L\, W_1(f, g), \end{aligned}$$

using that the support of π is contained in $B_R \times B_R$, as both f and g have support inside B_R. □

Recalling that $E[f] := \nabla U * \rho$, the properties of convolution lead immediately to the following information:

Lemma 3.3.14. *Take a potential $U : \mathbb{R}^d \to \mathbb{R}$ in the conditions of Theorem 3.3.10, and a measure $f \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in a ball B_R. Then,*

$$\|E[f]\|_{L^\infty(B_R)} \leq \|\nabla U\|_{L^\infty(B_{2R})}, \tag{3.26}$$

and

$$\mathrm{Lip}_R(E[f]) \leq \mathrm{Lip}_{2R}(\nabla U). \tag{3.27}$$

Lemma 3.3.15. *For $f, g \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ and $R > 0$ it holds that*

$$\|E[f] - E[g]\|_{L^\infty(B_R)} \leq \mathrm{Lip}_{2R}(\nabla U) W_1(f, g). \tag{3.28}$$

Proof.- Take π to be an optimal transportation plan between the measures f and g. Then, for any $x \in B_R$, using that π has marginals f and g,

$$\begin{aligned} E[f](x)\ - E[g](x) &= \int_{\mathbb{R}^d} (\rho[f](y) - \rho[g](y)) \nabla U(x - y)\, dy \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(y, v) \nabla U(x - y)\, dy\, dv - \int_{\mathbb{R}^d \times \mathbb{R}^d} g(z, w) \nabla U(x - z)\, dz\, dw \\ &= \int_{\mathbb{R}^{4d}} (\nabla U(x - y) - \nabla U(x - z))\, d\pi(y, v, z, w). \end{aligned}$$

Taking absolute value,

$$
\begin{aligned}
|E[f](z) - E[g](z)| &\leq \int_{\mathbb{R}^{4d}} |\nabla U(x-y) - \nabla U(x-z)| \, d\pi(y,v,z,w) \\
&\leq \mathrm{Lip}_{2R}(\nabla U) \int_{\mathbb{R}^{4d}} |y-z| \; d\pi(y,v,z,w) \\
&\leq \mathrm{Lip}_{2R}(\nabla U) W_1(f,g),
\end{aligned}
$$

using that $\pi(y,v,z,w)$ has support on $B_R \times B_R \subseteq \mathbb{R}^{4d}$. □

We can now give the proof of the existence and uniqueness result.

Proof.- [Proof of theorem 3.3.10] Take $f_0 \in \mathcal{P}^1(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in a ball $B_{R^0} \subseteq \mathbb{R}^d \times \mathbb{R}^d$, for some $R^0 > 0$. We will prove local existence and uniqueness of solutions by a contraction argument in the metric space $\mathcal{F}$ formed by all the functions $f \in \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$ such that the support of f_t is contained in B_R for all $t \in [0,T]$, where $R := 2R^0$ and $T > 0$ is a fixed number to be chosen later. Here, we consider the distance in $\mathcal{F}$ given by

$$
\mathcal{W}_1(f,g) := \sup_{t \in [0,T]} W_1(f_t, g_t). \tag{3.29}
$$

Let us define an operator on this space for which a fixed point will be a solution to the swarming equation (3.4). For $f \in \mathcal{F}$, consider $E[f] := \nabla U * \rho[f]$. Then, $E[f]$ satisfies Hypothesis 3.3.1 (because of the above two Lemmas 3.3.14 and 3.3.15, and the bound (3.21)) and we can define

$$
\Gamma[f](t) := P^t_{E[f]} \# f_0. \tag{3.30}
$$

In other words, $\Gamma[f]$ is the solution of the swarming equations obtained through the method of characteristics, with field $E[f]$ assumed known, and with initial condition f_0 at $t = 0$.

Clearly, a fixed point of Γ is a solution to eq. (3.4) on $[0,T]$. In order for Γ to be well defined, we need to prove that $\Gamma[f]$ is again in the space $\mathcal{F}$, for which we need to choose T appropriately. To do this, observe that from eq. (3.26) in Lemma 3.3.14 we have

$$
\|E[f]\|_{L^\infty([0,T]\times B_R)} \leq \|\nabla U\|_{L^\infty(B_{2R})} =: C_1,
$$

and from point 1 in lemma 3.3.5,

$$
\left| \frac{d}{dt} P^t_{E[f]}(P) \right| \leq C_2,
$$

for all $P \in B_{R^0} \subseteq \mathbb{R}^d \times \mathbb{R}^d$, and some $C_2 > 0$ which depends only on α, β, R_0 and C_1. Choosing any $T < R^0/C_2$ one easily sees that $P^t_{E[f]} \# f_0$ has support contained in B_R, for all $t \in [0,T]$ (recall that we set $R := 2R^0$). Then, for each $t \in [0,T]$, $\Gamma[f](t) \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, as follows from mass conservation (by definition, $P^t_{E[f]}\#f_0$ has mass 1), the support of $\Gamma[f](t)$ is contained in B_R (since we just chose T for this to hold), and the function $t \mapsto \Gamma[f](t)$ is continuous, as shown by Lemma 3.3.12. This shows that the map $\Gamma : \mathcal{F} \to \mathcal{F}$ is well defined. Let us prove now that this map is contractive (for which we will have to restrict again the choice of T). Take two functions $f, g \in \mathcal{F}$, and consider $\Gamma[f], \Gamma[g]$; we want to show that

$$\mathcal{W}_1(\Gamma[f], \Gamma[g]) \leq C\, \mathcal{W}_1(f,g) \tag{3.31}$$

for some $0 < C < 1$ which does not depend on f and g. Using (3.29) and (3.30),

$$\mathcal{W}_1(\Gamma[f], \Gamma[g]) = \sup_{t\in[0,T]} W_1(P^t_{E[f]}\#f_0, P^t_{E[g]}\#f_0), \tag{3.32}$$

and hence we need to estimate the above quantity for each $t \in [0,T)$. For $t \in [0,T]$, use lemmas 3.3.11, 3.3.7 and 3.3.15 to write

$$\begin{aligned} W_1(P^t_{E[f]}\#f_0, P^t_{E[g]}\#f_0) &\leq \left\| P^t_{E[f]} - P^t_{E[g]} \right\|_{L^\infty(\operatorname{supp} f_0)} \\ &\leq C(t) \sup_{t\in[0,T]} \|E[f_t] - E[g_t]\|_{L^\infty(B_R)} \\ &\leq C(t)\, L \sup_{t\in[0,T]} W_1(f_t, g_t) = C(t)\, L\, \mathcal{W}_1(f,g), \end{aligned}$$

where $C(t)$ is the function $(e^{C_3 t} - 1)/C_3$ which appears in eq. (3.18), for some constant C_3 which depends only on α, β, R, and the Lipschitz constant L of ∇U on B_{2R} (see eq. (3.27)). Clearly,

$$\lim_{t\to 0} C(t) = 0. \tag{3.33}$$

With (3.32), this finally gives

$$\mathcal{W}_1(\Gamma[f], \Gamma[g]) \leq C(T)\, L\, \mathcal{W}_1(f,g).$$

Taking into account (3.33), we can additionally choose T small enough so that $C(T)L < 1$. For such T, Γ is contractive, and this proves that there is a unique fixed point of Γ in $\mathcal{F}$, and hence a unique solution $f \in \mathcal{F}$ of eq. (3.4).

Finally, as mass is conserved, by usual arguments one can extend this solution as long as the support of the solution remains compact. Since in our case the growth of characteristics is bounded (see Lemma 3.3.2), one can construct a unique global solution satisfying (3.22) and (3.23). □

3.3.4 Stability

Theorem 3.3.16. *Take a potential U in the conditions of Theorem 3.3.10, and f_0, g_0 measures on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support, and consider the solutions f, g to eq.* (3.4) *given by Theorem 3.3.10 with initial data f_0 and g_0, respectively. Then, there exists a strictly increasing smooth function $r(t) : [0, \infty) \longrightarrow \mathbb{R}_0^+$ with $r(0) = 1$ depending only on the size of the support of f_0 and g_0, such that*

$$W_1(f_t, g_t) \leq r(t)\, W_1(f_0, g_0), \quad t \geq 0. \tag{3.34}$$

Proof.- Fix $T > 0$, and take $R > 0$ such that $\operatorname{supp} f_t$ and $\operatorname{supp} g_t$ are contained in B_R for $t \in [0, T]$ (which can be done thanks to theorem 3.3.10). For $t \in [0, T]$, call L_t the Lipschitz constant of $P^t_{E[g]}$ on B_R, and notice that from lemmas 3.3.8 and 3.3.14 we have

$$L_t \leq e^{C_1 t}, \qquad t \in [0, T] \tag{3.35}$$

for some allowed constant $C_1 > 0$. Then we have, using lemmas 3.3.11, 3.3.13, 3.3.7 and 3.3.15,

$$\begin{aligned} W_1(f_t, g_t) &= W_1(P^t_{E[f]}\#f_0, P^t_{E[g]}\#g_0) \\ &\leq W_1(P^t_{E[f]}\#f_0, P^t_{E[g]}\#f_0) + W_1(P^t_{E[g]}\#f_0, P^t_{E[g]}\#g_0) \\ &\leq \left\| P^t_{E[f]} - P^t_{E[g]} \right\|_{L^\infty(\operatorname{supp} f_0)} + L_t\, W_1(f_0, g_0) \\ &\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} \, ds + L_t\, W_1(f_0, g_0) \\ &\leq C_2 \mathrm{Lip}_{2R}(\nabla U) \int_0^t e^{C_2(t-s)} W_1(f_s, g_s)\, ds + e^{C_1 t}\, W_1(f_0, g_0). \end{aligned}$$

Calling $C = \max\{C_1, C_2, C_2 \mathrm{Lip}_{2R}(\nabla U)\}$ and multiplying by e^{-Ct},

$$e^{-Ct} W_1(f_t, g_t) \leq C \int_0^t e^{-Cs} W_1(f_s, g_s)\, ds + W_1(f_0, g_0), \qquad t \in [0, T],$$

where we have taken into account that $L_t \leq e^{Ct} \leq e^{CT}$ by Lemma 3.3.8. By Gronwall's Lemma,

$$e^{-Ct} W_1(f_t, g_t) \leq W_1(f_0, g_0)\, e^{Ct}, \quad t \in [0, T],$$

which proves our result. We point out that the particular rate function $r(t)$ can be obtained by carefully looking at the dependencies on time of the constants above leading to double exponentials. □

Remark 3.3.17 (Possible generalizations). *As in Remark 3.3.4, by assuming more restrictive growth properties at infinity of the potential U, we may weaken the requirements on the support of the initial data allowing f_0 with bounded first moment for instance. We do not follow this strategy in the present work.*

3.3.5 Regularity

If the initial condition for eq. (3.4) is more regular than a general measure on $\mathbb{R}^d \times \mathbb{R}^d$ one can easily prove that the solution f is also more regular. For example, if f_0 is Lipschitz, then f_t is Lipschitz for all $t \geq 0$. We will show this next.

Lemma 3.3.18. *Take an integrable function $f_0 : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$, with compact support, and assume that f_0 is also Lipschitz. Take also a potential $U \in \mathcal{C}^2(\mathbb{R}^D)$.*
Consider the global solution f to eq. (3.4) *with initial condition f_0 given by Theorem 3.3.10. Then, f_t is Lipschitz for all $t \geq 0$.*

Proof.- Solutions obtained from Theorem 3.3.10 have bounded support in velocity for all times $t > 0$, and their fields $E \equiv E(t,x) := -\nabla U * \rho$ are Lipschitz with respect to x. Hence, one can rewrite eq. (3.4) as a general equation of the form

$$\partial_t f + \operatorname{div}(af) = 0,$$

where $a = a(t,x,v)$ is the expression appearing in the equation,

$$a(t,x,v) = (v, E(t,x) + (\alpha - \beta\,|v|^2)v).$$

Then, a is bounded and Lipschitz with respect to x, v on the domain considered as the support in velocity is bounded, and classical results show that f_t is Lipschitz for all $t \geq 0$. □

3.4 Well-posedness and Asymptotic behaviour for the Cucker-Smale system

The same reasoning that we have just presented can be applied to more general models with similar results, as we will see later on this section for the Cucker-Smale model, and in the next section for an even more general models. The kinetic version of this model, 1.47 has been derived recently in the work of Ha and Tadmor [91] and further analysed in [90] by Ha and Liu. In the particle model proposed by Cucker and Smale, the particles influence each other according to a decreasing function of their mutual space distance; summarizing the results in the finite particle model, they prove that all the particles tend exponentially fast to move with their global mean velocity whenever the mutual interaction was strong enough at far distance, independently of the initial conditions. This situation is called *unconditional flocking*. In the work [91], the authors are able to show that the fluctuation of energy is a Lyapunov functional for classical solutions of the kinetic equation and it vanishes sub-exponentially fast in time, however with more

restrictive conditions than for the finite particle model on the strength of the long-range interaction between particles in order to achieve the unconditional flocking. In this section we shall also review these results and then we will improve them and propose further generalizations. Before doing this, though, let us point out that the same kinetic model can be obtained formally from the Boltzmann equation performing a grazing collision limit. We will devote the first part of this section to show how this can be done in the particular case of the Cucker-Smale model. Then we will see how to adapt the mean field limit argument we have just seen to the more general Cucker-Smale model and will extend these latter results and provide an unconditional flocking theorem with the same strength estimates valid for the finite particle models, not only for classical solutions, but also for measure valued solutions.
The way we proceed is first by reformulating the finite particle model in terms of atomic measures and by proving the unconditional flocking theorem in this measure setting. In particular, we show in Proposition 3.4.10 the existence of an atomic measure fully concentrated on the mean velocity to which the atomic measure valued solution of the kinetic equation converges exponentially fast in time in the Wasserstein distance. In particular, the support in space of the measure valued solution keeps bounded all over the process. Our rate of convergence in time and our support estimates do not depend on the number of particles. Therefore, in Theorem 3.4.11, we are able to extend the mentioned properties of the support by an approximation argument to any measure valued solution with initial compactly supported measure datum. There we combine the particle result with a stability property provided in [32] (we report the stability result below in Theorem 3.4.8). As an immediate consequence, we obtain a refinement of the result of Ha and Tadmor, by showing that the kinetic energy vanishes with exponential rate. We complete the picture and we fully extend the results valid for the Cucker and Smale finite particle model, by proving our unconditional flocking Theorem 3.4.14, which states the convergence of any measure valued solution with compactly supported initial datum to a compactly supported measure fully concentrated on the mean velocity.

3.4.1 A Boltzmann type equation for flocking

Our first aim would be to use an analogous methodology to the one used in [131] to describe the work of Cucker and Smale [63] on flocking analysis by means of kinetic equations. Making use of a collisional mechanism between individuals similar to the change in velocity of birds population introduced in [63], we derive a dissipative spatially dependent Boltzmann-type equation which describes the behavior of the flock in terms of a density $f = f(t, x, v)$. This equation is reminiscent of the modification of the Boltzmann equation due to Povzner [153].

Next, in the asymptotic procedure known as *grazing collision* limit, we obtain a simpler equation in divergence form, which retains all properties of the underlying Boltzmann equation, and in addition can be studied in detail.
In [63] Cucker and Smale studied the phenomenon of flocking in a population of birds, whose members are moving in the physical space $\mathbb{E} = \mathbb{R}^3$. The goal was to prove that under certain communication rates between the birds, the state of the flock converges to one in which all birds fly with the same velocity. The main hypothesis justifying the behavior of the population is that every bird adjusts its velocity to a weighted average of the relative velocity with respect to the other birds. That is, given a population of N birds, at time $t_n = n\Delta t$ with $n \in \mathbb{N}$ and $\Delta t > 0$, for the i-th bird,

$$v_i(t_n + \Delta t) - v_i(t_n) = \frac{\lambda \Delta t}{N} \sum_{i=1}^{N} w_{ij} \left(v_j(t_n) - v_i(t_n) \right), \tag{3.36}$$

where the weights w_{ij} quantify the way the birds influence each other, communication rate, independently of their total number N and λ measures the interaction strength.
Condition (3.36) can be fruitfully rephrased in a different way, which will be helpful in the following. Suppose that we have a population composed by two birds, say i and j. Assume that their velocities are modified in time according to the rule

$$v_i(t_n + \Delta t) = (1 - \lambda \Delta t\, w_{ij}) v_i(t_n) + \lambda \Delta t\, w_{ij} v_j(t_n), \tag{3.37a}$$
$$v_j(t_n + \Delta t) = \lambda \Delta t\, w_{ij} v_i(t_n) + (1 - \lambda \Delta t\, w_{ij}) v_j(t_n). \tag{3.37b}$$

Then, the momentum is preserved after the interaction

$$v_i(t_n + \Delta t) + v_j(t_n + \Delta t) = v_i(t_n) + v_j(t_n),$$

while the energy increases or decreases according to the value of λ

$$v_i^2(t_n + \Delta t) + v_j^2(t_n + \Delta t) = - 2\lambda \Delta t w_{ij} \, \left(1 - \lambda \Delta t\, w_{ij}\right) \left(v_i(t_n) - v_j(t_n)\right)^2 \tag{3.38}$$
$$+ v_i^2(t_n) + v_j^2(t_n). \tag{3.39}$$

For $\lambda \Delta t < 1$, the energy is dissipated. Note that in this case that the relative velocity is decreasing, since

$$\begin{aligned} |v_i(t_n + \Delta t) - v_j(t_n + \Delta t)| =& |1 - 2\lambda \Delta t\, w_{ij}| |v_i(t_n) - v_j(t_n)| \\ <& |v_i(t_n) - v_j(t_n)|, \end{aligned} \tag{3.40}$$

and the velocities of the two birds tend towards the mean velocity $(v_i + v_j)/2$. In the case $\lambda\Delta t < 1/2$, the interaction (3.37) is similar to a binary interaction between molecules of a dissipative gas, see [48] and the references therein.
In the general case of a population of N birds, the binary law (3.37) is taken into account together with the assumption that the i-th bird modifies its velocity giving the same weight to all the other velocities. In consequence of this,

$$v_i(t_n + \Delta t) = \frac{1}{N}\sum_{j=1}^{N}\{(1 - \lambda\Delta t\, w_{ij})v_i(t_n) + \lambda\Delta t\, w_{ij}v_j(t_n)\}\,, \tag{3.41}$$

that is a different way to write formula (3.36).
Let $f(t, x, v)$ denote the density of birds in the position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ at time $t \geq 0$, $d \geq 1$. The kinetic model for the evolution of $f = f(t, x, v)$ can be easily derived by standard methods of kinetic theory, considering that the change in time of $f(t, x, v)$ depends both on transport (birds fly freely if they do not interact with others) and interactions with other birds. Discarding other effects, this change in density depends on a balance between the gain and loss of birds with velocity v due to binary interactions. Let us assume that two birds with positions and velocities (x, v) and (y, ω) modify their velocities after the interaction, according to (3.37)

$$v^* = (1 - \gamma w(x-y))v + \gamma w(x-y)\omega, \tag{3.42a}$$

$$\omega* = \gamma w(x-y)v + (1 - \gamma w(x-y))\omega. \tag{3.42b}$$

where now the communication rate function w takes the form

$$w(x) = \frac{1}{(1 + |x|^2)^\gamma}, \quad x \in \mathbb{R}^d, \tag{3.43}$$

and $\gamma < 1/2$. Note that, as usual in collisional kinetic theory, the change in velocities due to binary interactions does not depend on time. This leads to the following integro-differential equation of Boltzmann type,

$$\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right)(t, x, v) = Q(f, f)(t, x, v), \tag{3.44}$$

where

$$Q(f, f)(x, v) = \sigma \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{1}{J} f(x, v_*) f(y, \omega_*) - f(x, v) f(y, \omega)\right) d\omega\, dy. \tag{3.45}$$

In (3.45) (v_*, ω_*) are the pre-collisional velocities that generate the couple (v, ω) after the interaction. $J = (1-2\gamma w)^d$ is the Jacobian of the transformation of (v, ω)

into (v^*, ω^*). Note that, since we fixed $\gamma < 1/2$, the Jacobian J is always positive. The bilinear operator Q in (3.45) is the analogous of the Boltzmann equation for Maxwell molecules [21, 48], where the collision frequency σ is assumed to be constant. In a number of different kinetic equations, the evolution of the density f is driven by collisions and the rate of change is defined through the collision term Q. One of the assumptions in the derivation of the Boltzmann collision operator is that only pair collisions are significant and that each separate collision between two molecules occurs at one point in space. Povzner [153] proposed a modified Boltzmann collision operator considering a smearing process for the pair collisions. This modified Povzner collision operator looks as follows

$$Q_P(f,f)(x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(x-y, v-w)\big(f(x,v_*)f(y,\omega_*) \\ - f(x,v)f(y,\omega)\big)\, d\omega\, dy. \tag{3.46}$$

In (3.46) B is the collision kernel and the post-collision velocities $(v^*, \omega*)$ are given by

$$v^* = (I - A(|x-y|))v + A(|x-y|)\omega, \tag{3.47a}$$
$$\omega* = A(|x-y|)v + (I - A(|x-y|))\omega. \tag{3.47b}$$

where A is a 3×3 matrix and I the identity matrix. These last relations imply the conservation of momentum. We remark that, differently from interactions (3.42), in Povzner's equation the matrix A is such that also the energy is preserved in a collision. It is clear that equation (3.44) can be viewed like a Povzner type equation with dissipative interactions, where the matrix $A(|x-y|) = \gamma w(x-y)I$. It is remarkable that, while the Povzner equation was first introduced for purely mathematical reasons and usually ignored by the physicists, related kinetic equations can be fruitfully introduced to model many agents systems in biology and ecology.

A first important consequence of the interaction mechanism given by (3.42) is that the support of the allowed velocities can not increase. In fact, since $0 \le w \le 1$ and $0 < \gamma < \frac{1}{2}$,

$$|v^*| = |(1-\gamma w(|x-y|))v + \gamma w(|x-y|)\omega| \le (1-\gamma w)|v| + \gamma w|\omega| \le \max\{|v|, |\omega|\}.$$

The presence of the Jacobian in the collision operator (3.45) can be avoided by considering a weak formulation. By a weak solution of the initial value problem for equation (3.44), corresponding to the initial density $f_0(x,v)$, we shall mean any density satisfying the weak form of (3.44)-(3.45) given by

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^{2d}}\phi(x,v)f(t,x,v)\,dv\,dx+\int_{\mathbb{R}^{2d}}(v\cdot\nabla_x\phi(x,v))f(t,x,v)\,dv\,dx=\sigma\int_{\mathbb{R}^{4d}}(\phi(x,v^*)-\phi(x,v))f(t,x,v)f(t,y,\omega)dv\,dx\,d\omega\,dy \tag{3.48}$$

for $t>0$ and all smooth functions ϕ with compact support, and such that

$$\lim_{t\to 0}\int_{\mathbb{R}^{2d}}\phi(x,v)f(t,x,v)\,dx\,dv=\int_{\mathbb{R}^{2d}}\phi(x,v)f_0(x,v)\,dx\,dv. \tag{3.49}$$

The form (3.48) is easier to handle, and it is the starting point to explore the evolution of macroscopic quantities (moments).

While the time-evolution of the population density is described in details by the Boltzmann equation (3.44), a precise description of the phenomenon of flocking is mainly related to the large-time behavior of the solution. On the other hand, this large-time behavior has to depend mainly from the type of collisions (3.42), and not from the size of the parameter γ which determines the *strength* of the interaction itself. In this situation, an accurate description can be furnished as well by resorting to simplified models, which turn out to be valid exactly for large times.

This idea has been first used in dissipative kinetic theory by McNamara and Young [134] to recover from the Boltzmann equation in a suitable asymptotic procedure, simplified models of nonlinear frictions for the evolution of the gas density [11, 167]. Similar asymptotic procedures have been subsequently used to recover Fokker-Planck type equations for wealth distribution [33], or opinion formation [168].

Let us assume that the parameter γ, which measures the intensity of the velocity change in the binary interactions is small ($\gamma \ll 1$). Then, in order that the effect of the collision integral do not vanish, the collision frequency has to be increased consequently. The most interesting case comes out from the choice $\sigma\gamma=\lambda$, where λ is a fixed positive constant. In this case, by expanding $\phi(x,v^*)$ in Taylor's series of v^*-v up to the second order the weak form of the collision integral takes the form

$$\sigma \int_{\mathbb{R}^{4d}} (\phi(x,v^*) - \phi(x,v)) f(t,x,v) f(t,y,\omega)\, dx\, dv\, dy\, dw$$
$$= \gamma\sigma \int_{\mathbb{R}^{4d}} (\nabla_v \phi(x,v)\cdot(\omega - v))\, w(x-y) f(t,x,v)\, f(t,y,\omega)\, dx\, dv\, dy\, d\omega \tag{3.50}$$
$$+ \frac{\gamma^2}{2}\sigma \underbrace{\int_{\mathbb{R}^{4d}} \left[\sum_{i,j=1}^{d} \frac{\partial^2 \phi(x,\tilde{v})}{\partial v_i^2}(\omega_j - v_j)^2\right] w(x-y)^2 f(x,v) f(y,\omega)\, dx\, dv\, dy\, d\omega}_{:=I},$$

with $\tilde{v} = \theta v + (1-\theta)v^*$, $0 \le \theta \le 1$. If the collisions are nearly grazing, $\gamma \ll 1$, while $\gamma\sigma = \lambda$), we can cut the expansion (3.50) after the first–order term. In fact, since the second moment of the solution to the Boltzmann equation is non-increasing

$$\int_{\mathbb{R}^{2d}} |v|^2 f(t,x,v)\, dx\, dv \le \int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\, dx\, dv,$$

and $w(x) \le 1$,

$$|I| \le 2\|\phi(x,v)\|_{C_0^2} \int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\, dx\, dv. \tag{3.51}$$

Hence, we have a uniform in time upper bound for the remainder term of order $\gamma^2\sigma = \lambda\gamma \ll 1$. It follows that, in the regime of small γ and high collision frequency, so that $\sigma\gamma = \lambda$, the Boltzmann collision operator $Q(f,f)$ is approximated by the dissipative operator $I_\gamma(f,f)$, in strong divergence form,

$$I_\gamma(f,f) = \lambda \nabla_v \cdot \{f(t,x,v)\,[(w(x)\nabla_v W(v)) * f]\,(t,x,v)\} \tag{3.52}$$

where $W(v) = \frac{1}{2}|v|^2$ and $*$ is the (x,v)-convolution. Notice that, as remarked by McNamara and Young in their pioneering paper [134], the operator $I_\gamma(f,f)$ maintains the same dissipation properties of the full Boltzmann collision operator.

3.4.2 A review on the nonlinear friction equation

In [63], it is assumed that the communication rate is a function of the distance between birds, namely

$$w_{ij} = \frac{1}{(1 + \|x_i - x_j\|^2)^\gamma} \tag{3.53}$$

for some $\gamma \ge 0$. For $x, v \in \mathbb{E}^N$, denote

$$\Gamma(x) = \frac{1}{2}\sum_{i \ne j} \|x_i - x_j\|^2, \tag{3.54}$$

and

$$\Lambda(v) = \frac{1}{2}\sum_{i\neq j} \|v_i - v_j\|^2. \tag{3.55}$$

Then, under suitable restrictions on γ and λ, and certain initial configurations (see [63] for details), it is proven that there exists a constant B_0 such that $\Gamma(x(t_n)) \leq B_0$ for all $n \in \mathbb{N}$, while $\Lambda(v(t_n))$ converges towards zero as $n \to \infty$, and the vectors $x_i - x_j$ tend to a limit vector $\widehat{x}_{ij}$, for all $i, j \leq N$.
In particular, and rather remarkably, when $\gamma < 1/2$, no restrictions on λ and initial configurations are needed [64, Theorem 1] and [63]. In this case, called unconditional flocking, the behavior of the population of birds is perfectly specified. All birds tend to fly exponentially fast with the same velocity, while their relative distances tend to remain constant. This result has been recently improved in several works [90, 91, 159], where other weights or communication rates are studied and sharper rate of convergence are obtained in certain cases.
For convenience, let us fix $\lambda = 1$ in the rest of this work. This approximated equation corresponds to a nonlinear-type friction equation for the density $f = f(t,x,v)$, which reads

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = \nabla_v\cdot[\xi(f)(t,x,v)f(t,x,v)] \tag{3.56}$$

where

$$\begin{aligned}\xi(f)(t,x,v) &= [(w(x)\nabla_v W(v)) * f]\,(t,x,v)\\ &= \int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(t,y,w)\,dy\,dw.\end{aligned}$$

The same equation has been derived and analyzed by Ha and Tadmor [91] as the mean-field limit of the discrete and finite dimensional model (3.36) by Cucker and Smale. Their main result [91, Theorem 4.3] states that the following fluctuation of energy functional

$$\begin{aligned}\Lambda(f)(t) &= \int_{\mathbb{R}^{2d}} |v-m|^2 f(t,x,v)\,dx\,dv, \text{ where} \qquad (3.57)\\ m(f) &= \int_{\mathbb{R}^{2d}} vf(t,x,v)\,dx\,dv = \int_{\mathbb{R}^{2d}} vf_0(x,v)\,dx\,dv,\end{aligned}$$

is a Lyapunov functional for classical solutions of (3.56), and it converges to zero *sub-exponentially* for $0 \leq \gamma \leq \frac{1}{4}$, i.e.,

$$\Lambda(f)(t) \leq C\Lambda(f_0)\times\begin{cases} e^{-\kappa t^{1-4\gamma}} & , \quad 0\leq\gamma<\frac{1}{4}\\ (1+t)^{-\kappa'} & , \quad \gamma = \frac{1}{4},\end{cases}$$

where the constants C, κ, and κ' are positive and depend on $\gamma > 0$. Classical solutions are constructed for initial data $f_0 \in C^1 \cap W^{1,\infty}(\mathbb{R}^{2d})$, compactly supported in (x, v).
Let us now present our own approach to the Kinetic version of the Cucker-Smale Model through the mean field limit.

3.4.3 Kinetic Cucker-Smale Model

We will prove well-posedness in a slightly more general setting than that of the Cucker-Smale model in section 3.1.2, being less restrictive on the communication rate and the velocity averaging. To be more precise, we shall consider $\xi[f](t, x, v) = [(H(x, v)) * f]\,(t, x, v)$ as in (3.6), but for a general $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, for which we only assume the following hypotheses:

Hypothesis 3.4.1 (Conditions on H).

1. *H is locally Lipschitz.*
2. *For some $C > 0$,*

$$|H(x, v)| \leq C(1 + |x| + |v|) \textit{ for all } x, v \in \mathbb{R}^d. \tag{3.58}$$

Since the procedure to prove the well-posedness results to (3.6) is the same we have already applied in the previous section, we will state some of the results without proof. First of all, fix $T > 0$ and let us introduce the system of ODE's solved by the characteristics of (3.6):

$$\frac{d}{dt} X = V, \tag{3.59a}$$
$$\frac{d}{dt} V = -\xi(t, X, V), \tag{3.59b}$$

where $\xi : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is any function satisfying the following hypothesis:

Hypothesis 3.4.2 (Conditions on ξ).

1. *ξ is continuous on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.*
2. *For some $C > 0$,*

$$|\xi(t, x, v)| \leq C(1 + |x| + |v|), \textit{ for all } t, x, v \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^v. \tag{3.60}$$

3. *ξ is* locally Lipschitz with respect to x and v, *i.e., for any compact set $K \subseteq \mathbb{R}^d \times \mathbb{R}^d$ there is some $L_K > 0$ such that*

$$|\xi(t, P_1) - \xi(t, P_2)| \leq L_K \, |P_1 - P_2| \,, \quad t \in [0, T], \; P_1, P_2 \in K. \tag{3.61}$$

Under these conditions, we may consider the flow map $P_\xi^t = P_\xi^t(x, v)$ associated to (3.59), defined as the solution to the system (3.59) with initial condition (x, v). For ease of notation, we will write the system (3.59) as

$$\frac{dP_\xi^t}{dt} = \Psi_\xi(t, P_\xi^t).$$

Remark 3.4.3. *Under Hypothesis 3.4.1 on H, note that whenever*

$$\tilde{f} \in C([0, T], \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$$

is a given compactly supported measure with $\operatorname{supp}(\tilde{f}_t) \subset B_{R^x} \times B_{R^v}$ for all $t \in [0, T]$, the field $\xi[\tilde{f}] = H * \tilde{f}$ satisfies Hypothesis 3.4.2.

Definition 3.4.4 (Notion of Solution). *Take H satisfying Hypothesis 3.4.1, a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, and $T \in (0, \infty]$. We say that a function $f : [0, T] \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ is a solution of the swarming equation 3.6 with initial condition f_0 when:*

1. *The field $\xi = H * f$ satisfies Hypothesis 3.4.2.*

2. *It holds $f_t = P_\xi^t \# f_0$.*

Now, an analogue to Lemma 3.3.5 can be stated. We shall state this one and the following lemmas for a general ξ satisfying Hypothesis (3.4.2).

Lemma 3.4.5 (Regularity of the characteristic equations). *Take $T > 0$, ξ satisfying Hypothesis* (3.4.2)*, $R > 0$ and $t \in [0, T]$. Then there exist constants C and L_p depending on $\operatorname{Lip}_R(\xi)$ and T such that*

$$|\Psi_\xi(P)| \leq C \quad \textit{for all } P \in B_R \times B_R$$

and

$$|\Psi_\xi(P_1) - \Psi_\xi(P_2)| \leq L_p |P_1 - P_2| \quad \textit{for all } P_1, P_2 \in B_R \times B_R.$$

Lemmas 3.3.6–3.3.9 are valid as they are presented, taking ξ and Hypothesis 3.4.2 to play the role of E and Hypothesis 3.3.1, and making the obvious minor modifications on the dependence of the constants. Now we can look at the existence of solutions:

Theorem 3.4.6 (Existence and uniqueness of measure solutions). *Assume H satisfies Hypothesis 3.4.1, and take $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ compactly supported. Then there exists a unique solution $f \in C([0,T], \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$ to equation* (3.6) *in the sense of Definition 3.4.4 with initial condition f_0. Moreover, the solution remains compactly supported for all $t \in [0,T]$, i.e., there exist R^x and R^v depending on T, H and the support of f_0, such that*

$$\mathrm{supp}(f_t) \subset B_{R^x} \times B_{R^v} \textit{ for all } t \in [0,T].$$

The proof of this result can be done following the same steps as for proving Theorem 3.3.10. Lemmas 3.3.11 to 3.3.13 still hold in this situation, and we recombine Lemmas 3.3.14 and 3.3.15 in the following result:

Lemma 3.4.7. *Take H satisfying Hypothesis 3.4.1, $\tilde{f} \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\mathrm{supp}(\tilde{f}_t) \subset B_{R^x} \times B_{R^v}$, and $\xi := \xi[\tilde{f}] = H * \tilde{f}$. Then, for any $R > 0$*

$$L_R(\xi) \leq L_{R+\hat{R}}(H),$$

with $\hat{R} := \max R^x, R^v$. Furthermore, if $\tilde{g} \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ it holds that

$$\left\| \xi[\tilde{f}] - \xi[\tilde{g}] \right\|_{\mathrm{L}^\infty(B_R)} \leq L_{R+\hat{R}}(H)\, W_1(\tilde{f}, \tilde{g}). \tag{3.62}$$

Proof.- The first part follows directly from the properties of convolution. For the second one, take π to be an optimal transportation plan between the measures $\tilde{f}$ and $\tilde{g}$. Then, for any $x, v \in B_R$, using that π has marginals $\tilde{f}$ and $\tilde{g}$,

$$\begin{aligned}
\xi[\tilde{f}](x,v) \; - \xi[\tilde{g}](x,v) & \\
&= \int_{\mathbb{R}^{2d}} H(x-y, v-u) \tilde{f}(y,u)\, d(y,u) \\
&\quad - \int_{\mathbb{R}^{2d}} H(x-z, v-w) \tilde{g}(z,w)\, d(z,w) \\
&= \int_{\mathbb{R}^{4d}} \left[H(x-y, v-u) - H(x-z, v-w) \right] \, d\pi(y,u,z,w).
\end{aligned}$$

Taking absolute value, and using that the support of π is contained in the ball $B_{\hat{R}} \subseteq \mathbb{R}^{4d}$,

$$\begin{aligned}
|\xi[\tilde{f}](x,v) - \xi[\tilde{g}](x,\, v)| & \\
&\leq \int_{B_{\hat{R}}} |H(x-y, v-u) - H(x-z, v-w)|\, d\pi(y,u,z,w) \\
&\leq \; L_{R+\hat{R}}(H) \int_{\mathbb{R}^{4n}} |(y-z, u-w)| \; d\pi(y,u,z,w) = L_{R+\hat{R}}(H) W_1(\tilde{f}, \tilde{g}).
\end{aligned}$$

□

Finally, a stability result also follows using the same steps as in Theorem 3.3.16.

Theorem 3.4.8 (Stability in W_1). *Assume H satisfies Hypothesis 3.4.1, and take $f_0, g_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ compactly supported. Consider the solutions f, g to eq.* (3.6) *given by Theorem 3.4.6 with initial data f_0 and g_0, respectively. Then, there exists a strictly increasing function $r(t) : [0, \infty) \longrightarrow \mathbb{R}_0^+$ with $r(0) = 1$ depending only on H and the size of the support of f_0 and g_0, such that*

$$W_1(f_t, g_t) \le r(t)\, W_1(f_0, g_0), \quad t \ge 0. \tag{3.63}$$

As it is pointed out by Ha and Tadmor in [91, Remark 1, pag. 482], the results they obtained, which we presented in the previous section, were sub-optimal, in the sense that they do not reproduce at the continuous level the analysis in [63, 64]; no uniform bound of the spatial support of the density is provided, resulting in the sub-optimal estimate for $\gamma \le \frac{1}{4}$, instead of $\gamma \le \frac{1}{2}$ valid for the discrete and finite dimensional model. Moreover, while the functional Λ defined in (3.57) encodes information about the velocities, no dependence on the space is explicitly given.
On the other hand, in [90] the authors also give a well-posedness result of measure-valued solutions for the Cauchy problem to (3.56), based on the bounded Lipschitz distance. Their results show three important consequences: particles can be seen as measure-valued solutions to (3.56), the particle method converges towards a measure solution to (3.56), and finally the constructed solutions are unique.Also, they improved the results in the discrete original Cucker-Smale model for the exponent $\gamma = \frac{1}{2}$ for which they prove unconditional flocking [90, Proposition 4.3]. However, again the results in [90] are suboptimal concerning the asymptotic behavior and the qualitative properties of the constructed measure-valued solutions. They provide increasing in time estimates on the growth of the support of the solutions in position and in velocity [90, Lemma 5.4], which do not reflect the qualitative picture of flocking in the discrete model [63, 64] implying convergence in velocity towards its mean and bounded growth in position variables. Here, we have recovered the same results but based on the stability in the Wasserstein distance W_1, which allows us to obtain sharper constants and rates, as we will see in the next section.
More precisely, we will concentrate on three main goals. We will show an improvement in the estimates of the evolution of the support in (x, v), hence bridging the gap pointed out in [91, Remark 1, pag. 482]. In fact, we will show that the support in velocity shrinks towards its mean velocity exponentially fast while the support in position is bounded around the position of the center of mass that increases linearly due to the constant mean velocity. This result is crucial for the rest and is valid for measure valued solutions.
Based on these improvements on the support evolution, we have two main consequences, the convergence towards flocking behavior for measure solutions and the improved exponential convergence to zero of the Lyapunov functional (3.57) for classical solutions in the whole range $\gamma \le 1/2$.

3.4.4 Exponential in time collapse of the velocity support in the Cucker-Smale model

In this section we will present a sharp bound on the evolution of the support for the kinetic Cucker-Smale equation, improving the more general result stated in Theorem 3.4.6 for this particular case. We shall state rigorously the result and give a proof of it. Because of the previous result about the stability for the kinetic equation, we can address the question of the asymptotic behavior of solutions from the point of view of the particle system, by identifying particles with atomic measure solutions to the kinetic equation (1.47). Let us consider a N-particle system following the dynamics (3.2):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & , \ x_i(0) = x_i^0 \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} m_j w(|x_i - x_j|)\,(v_j - v_i) & , \ v_i(0) = v_i^0, \end{cases} .$$

First of all let us point out some facts and set some notation that will be useful in the following discussion. Since the equation is translational invariant, let us assume without loss of generality that the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$. Then, let us fix any $R_0^x > 0$ and $R_0^v > 0$, such that all the initial velocities lie inside the ball $B(0, R_0^v)$ and all positions inside $B(x_c, R_0^x)$. Now, the solutions to this system are $C^1([0,\infty), \mathbb{R}^{2d})$ for both positions and velocities and for any label i. Let us define the function $R^v(t)$ to be

$$R^v(t) := \max_{i=1,\dots,N} |v_i(t)|.$$

Since the number of particles is finite and the curves are smooth in time, $R^v(t)$ is a Lipschitz function and, therefore, differentiable with respect to time almost everywhere. We can thus look at the right derivative of $R^v(t)$ like in [38]

$$\begin{aligned} \frac{d^+}{dt} R^v(t)^2 &= \max_{i:|v_i(t)|=R^v(t)} \frac{d^+}{dt} |v_i|^2 \\ &= -2 \max_{i:|v_i(t)|=R^v(t)} \sum_{j\neq i} m_j \left[(v_i - v_j)\cdot v_i\right] \, w(|x_i - x_j|) \, . \end{aligned} \tag{3.64}$$

Because of the choice of the label i, we have that $(v_i - v_j) \cdot v_i \geq 0$ for all $j \neq i$ that together with $w \geq 0$ imply that $R^v(t)$ is a non-increasing function. Hence, $R^v(t) \leq R_0^v$ for all $t \geq 0$. Now coming back to the equation for the positions, we deduce that

$$|x_i(t) - x_i^0| \leq R_0^v t \quad \text{for all } t \geq 0 \text{ and all } i = 1, \dots, N.$$

We infer that $|x_i - x_j| \leq 2R_0^x + 2R_0^v t$ and thus

$$w(|x_i - x_j|) \geq \frac{1}{[1 + 4R_0^2(1+t)^2]^\gamma} \text{ for all } t \geq 0 \text{ and all } i, j = 1, \dots, N,$$

with $R_0 = \min(R_0^x, R_0^v)$. Again, we come back to the equation for the maximal velocity (3.64), and we deduce

$$\begin{aligned}
\frac{d^+}{dt} R^v(t)^2 &= -2 \max_{i:|v_i(t)|=R^v(t)} \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i \right] w(|x_i - x_j|) \\
&\leq -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} \max_{i:|v_i(t)|=R^v(t)} \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i \right] \\
&= -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} R^v(t)^2 := -f(t)\, R^v(t)^2,
\end{aligned}$$

from which we obtain by direct integration or Gronwall's lemma:

$$R^v(t) \leq R_0^v \exp\left\{ -\frac{1}{2} \int_0^t f(s)\, ds \right\}.$$

It is immediate to check that

$$\lim_{t \to \infty} t^{2\gamma} f(t) = \left(\frac{1}{2R_0^2} \right)^\gamma,$$

then, for $\gamma \leq 1/2$, the function $f(t)$ is not integrable at ∞ and therefore

$$\lim_{t \to \infty} \int_0^t f(s)\, ds = +\infty$$

and $R^v(t) \to 0$ as $t \to \infty$ giving the convergence to a single point, its mean velocity, of the support for the velocity.
Now, let us estimate the position variables again. We deduce that

$$\int_0^t |v_i(s)|\, ds \leq R_0^v \int_0^t \exp\left\{ -\frac{1}{2} \int_0^s f(\tau)\, d\tau \right\} ds. \tag{3.65}$$

Let us distinguish two cases. It is trivial to check that for $0 < \gamma < 1/2$

$$\lim_{t\to\infty} (1+t)f(t) = +\infty,$$

then for any $C > 0$, there exists $\varepsilon > 0$ such that if $s \geq \varepsilon^{-1}$ then $(1+s)f(s) \geq C$. We conclude that

$$-\int_0^t f(s)\,ds \leq -\int_0^{1/\varepsilon} f(s)\,ds - C\ln(1+t) + C\ln(1+\varepsilon^{-1})$$

and thus,

$$\exp\left\{-\frac{1}{2}\int_0^t f(s)\,ds\right\} \leq A(\varepsilon)\,(1+t)^{-C/2}.$$

Plugging into (3.65), we deduce

$$\int_0^t |v_i(s)|\,ds \leq R_0^v \int_0^t \exp\left\{-\frac{1}{2}\int_0^s f(\tau)\,d\tau\right\} ds \leq R_0^v\,A(\varepsilon) \int_0^t (1+s)^{-C/2}\,ds$$

and since the constant C can be chosen arbitrarily large by choosing ε small, we conclude that the integral is bounded in $t > 0$, and that there exists $R_1^x > 0$ such that

$$|x_i(t) - x_i^0| \leq R_1^x$$

for all $t \geq 0$ and $i = 1, \dots, N$. This implies that $x_i(t) \in B(x_c, \bar{R}^x)$ for all $t \geq 0$ and $i = 1, \dots, N$ with $\bar{R}^x = R_1^x + R_0^x$.
Moreover, coming back again to the velocities, we deduce now that $w(|x_i(t) - x_j(t)|) \geq w(2\bar{R}^x)$. So that from (3.64), we get

$$\begin{aligned}\frac{d^+}{dt}R^v(t)^2 =& -2 \max_{ist|v_i(t)|=R^v(t)} \sum_{j\neq i} m_j\,[(v_i - v_j)\cdot v_i]\; w(|x_i - x_j|)\\ \leq& -2w(2\bar{R}^x) \max_{ist|v_i(t)|=R^v(t)} \sum_{j\neq i} m_j\,[(v_i - v_j)\cdot v_i] = -2w(2\bar{R}^x)R^v(t)^2\end{aligned}$$

from which we finally deduce the exponential decay to zero of $R^v(t)$.
Now again, if we come back to the position variables since the velocity curve is integrable on time due to the exponential decay, we deduce that

$$\lim_{t\to\infty} x_i(t) = x_i^\infty$$

and that the lengths of the curves followed by each of particles is finite once subtracted the translational movement due to the constant mean velocity.

Now, let us come back to the case $\gamma = \frac{1}{2}$. Recalling (3.65), we can now compute the integral explicitly implying that

$$R_0 \int_0^t f(s)\, ds = \ln \left[2R_0(1+t) + \sqrt{1+4R_0^2(1+t)^2} \right] - \ln \left[2R_0 + \sqrt{1+4R_0^2} \right].$$

It is straightforward to check that

$$\int_0^t |v_i(s)|\, ds \le C \int_0^t \frac{1}{1+s}\, ds = C \ln(1+t),$$

with C depending only on R_0. In this case, we do another loop, going to the position variables to deduce as before that

$$\frac{d^+}{dt} R^v(t)^2 \le -\frac{2}{[1+C(1+\ln(1+t))^2]^\gamma} R^v(t)^2 := -g(t) R^v(t)^2.$$

It is obvious that

$$\lim_{t\to\infty} (1+t) g(t) = +\infty$$

and thus we come back to the same situation as for $\gamma < \frac{1}{2}$ and thus, giving the desired result for $\gamma = \frac{1}{2}$.

Remark 3.4.9. *The unconditional flocking result in the whole range* $0 < \gamma \le \frac{1}{2}$ *was already obtained in Section* 3 *and* 4 *of* [90] *with a completely different argument. The authors give an alternative proof of the unconditional flocking of the Cucker-Smale model for any* $\gamma < \frac{1}{2}$ *and prove it for* $\gamma = \frac{1}{2}$ *based on estimates of the dissipation of the dynamical system with Euclidean norms.*
It is important to note that the constants we obtain in the exponential decay rate to zero of the support in velocity of the particles do not depend either on the number of particles nor on their masses. They only depend on the initial values of R_0^v *and* R_0^x*, i.e., on the initial values of the particles with the largest velocity and the furthest away from the center of mass.*
The importance of this presented alternative proof of the unconditional flocking is that the estimates are independent of the number of particles. This was not the case in [90] *since their argument is based on the Euclidean norm of the velocity and position vectors and not in the bounded norm. Some of the above ideas are reminiscent of arguments used for continuum models of swarming in* [55, 13].

To see how the alignment shows up in the Cucker-Smale model, we can also consider a more general model in which the averaging takes into account the strength of the relative speed,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j=1}^N w_{ij} (v_j - v_i) |v_i - v_j|^{p-2}, \end{cases} \tag{3.66}$$

with $p > 0$. This model reduces to the original Cucker-Smale model we have just seen for $p = 2$.
Again, we prove that the concentration of the velocities around its mean value through a series of recurrent steps, in each of which the bounds we have for $R^x(t)$ and $R^v(t)$ are alternatively improved until we find an estimate for the concentration rate of the velocities. The first steps give us quite the same information as in the case $p = 2$, nevertheless, will include them here too in order keeping the argumentative line. As a first step, we compute the right derivative of $R^v(t)^2$ with respect to time, which gives us

$$\frac{d^+}{dt} R^v(t)^2 = -\frac{2}{N} \max_{i:|v_i(t)|=R^v(t)} \sum_{j\neq i} w_{ij}(v_i - v_j) \cdot v_i \, |v_i - v_j|^{p-2} \leq 0,$$

due to the choice of the label i, which ensures that $(v_i - v_j) \cdot v_i \geq 0$, for all j. Hence, $R^v(t)$ is a non-increasing function. As a consequence, the distance between a particle and its original position can only grow linearly in time. Thus,

$$|x_i(t) - x_j(t)| \leq |x_i(t) \pm x_i^0 \pm x_j^0 - x_j(t)| \leq 2R(t+1), \tag{3.67}$$

where $R = \max(R_0^x, R_0^v)$. In turn, this implies that we have a lower bound for the weight:

$$w_{ij} = \frac{1}{(1 + |x_i - x_j|^2)^\gamma} \geq \frac{1}{(1 + |2R(t+1)|^2)^\gamma}. \tag{3.68}$$

Now, using the bound on w_{ij}, we can extract more information from the computation of the time derivative of $R^v(t)^2$:

$$\begin{aligned} \frac{d^+}{dt} R^v(t)^2 &= -\frac{2}{N} \max_{i:|v_i(t)|=R^v(t)} \sum_{j\neq i} w_{ij}(v_i - v_j) \cdot v_i \, |v_i - v_j|^{p-2} \\ &\leq -\frac{2}{N\,(1 + |2R(t+1)|^2)^\gamma} \max_{i:|v_i(t)|=R^v(t)} \sum_{j\neq i} (v_i - v_j) \cdot v_i \, |v_i - v_j|^{p-2}. \end{aligned}$$

At this point we need to be careful with the exponent $p - 2$. Due to the choice of i, $|v_i - v_j| \leq 2|v_i|$, for all j. Then, if $0 < p \leq 2$, we obtain

$$\frac{1}{N} \sum_{j\neq i} (v_i - v_j) \cdot v_i \, |v_i - v_j|^{p-2} \geq \frac{2^{p-2}}{N} \left[\sum_{1\leq j\leq N} |v_i|^p - v_i |v_i|^{p-2} \cdot \sum_{1\leq j\leq N} v_j \right]$$

and since we have set the mean velocity to zero the second term in the right hand side of the equation disappears and we get

$$\frac{d^+}{dt} R^v(t)^2 \leq -2^{p-1} a(t) R^v(t)^p,$$

where we have set $a(t) := (1 + |2R(t+1)|^2)^{-\gamma}$. In case $2 < p < 4$, we can estimate it from below as

$$\sum_{j\neq i}(v_i - v_j)\cdot v_i\,|v_i - v_j|^{p-2} \geq (2|v_i|)^{p-4}\sum_{j\neq i}(v_i - v_j)\cdot v_i\,|v_i - v_j|^2,$$

and expanding $|v_i - v_j|^2$, we get

$$\sum_{j\neq i}(v_i - v_j)\cdot v_i\,|v_i - v_j|^2 \geq \sum_{1\leq j\leq N}|v_i|^4.$$

Summarizing, we have that for any $0 < p < 4$

$$\frac{d^+}{dt}R^v(t)^2 \leq -C_p a(t) R^v(t)^p,$$

and integrating with respect to time we obtain, for $p < 2$ and $2 < p < 4$,

$$R^v(t) \leq \left[(R_0^v)^{2-p} + \frac{p-2}{2}C_p W(t)\right]^{\frac{1}{2-p}} := f(t) \tag{3.69}$$

where $W(t) := \int_0^t a(s)\,ds$. This result can be extended by using a slightly more delicate argument to any $p > 2$, but since the next steps will require a stronger assumption on p than $p < 4$, used at this point, we omit it here. Coming again to (3.69), we notice that $W(t)$ is an increasing function of time and for $\gamma < \frac{1}{2}$ it diverges as $t \to \infty$. Thus, for $p < 2$ there exist $T < \infty$ such that $R^v(T) = 0$. While for $p > 2$, since the exponent $(2-p)^{-1}$ is negative, $R^v(t) \to 0$ as $t \to \infty$ algebraically, whence the concentration in velocity holds at infinite time.

Once a concentration in velocity has been established, we can improve the bound on the positions to show that $|x_i(t) - x_j(t)|$ remains uniformly bounded in time. For $0 < p < 2$, if we define $f(t) \equiv 0$ for $t \geq T$ we have that

$$|x_i(t) - x_i(0)| \leq \int_0^t |f(s)|ds \leq \int_0^\infty |f(s)|ds = \int_0^T |f(s)|ds \leq T R_0^v.$$

On the other hand, for $2 < p < 4$, $f(t)$ behaves like $W(t)^{1/(2-p)}$ as $t \to \infty$. In order for $f(t)$ to be integrable in time up to infinity, we assume further that $2 < p < 3 - 2\gamma < 4$. Proceeding as in (3.67) and (3.68), for $0 < p < 2$ and $2 < p < 3 - 2\gamma$, we have a uniform in time bound on $|x_i(t) - x_j(t)|$ and an uniform bound from below in $w_{ij} \geq W_0 > 0$.

Finally, we use again the computation of the derivative of $R^v(t)^2$ to finally get

$$R^v(t) \leq \left[(R_0^v)^{2-p} + \frac{p-2}{2}C_p\,W_0 t\right]^{\frac{1}{2-p}} \tag{3.70}$$

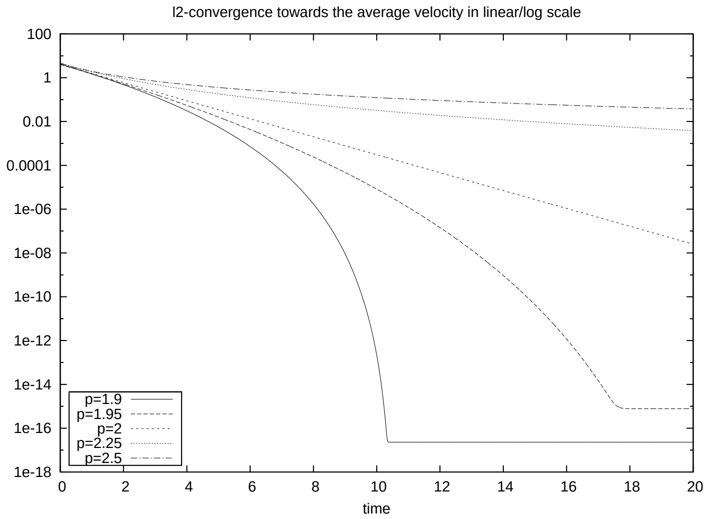

Figure 3.7: Different Convergence rates in Log-scale to the zero mean velocity for nonlinear velocity dependent Cucker-Smale models.

for all $0 < p < 2$ and $2 < p < 3 - 2\gamma$.
Summarizing, we have shown that for this modified version of the Cucker-Smale model, the flocking behavior happens in a finite time for $0 < p < 2$ and $0 < \gamma < 1/2$ and with an algebraic speed if $2 < p < 3 - 2\gamma$ in contrast with the exponential speed obtained for the standard Cucker-Smale model with $p = 2$ and $0 < \gamma < 1/2$. This can be reassured numerically as seen in Fig. 3.7 for different values of p. It is an open problem to check if these limits for the parameter p and γ when $p > 2$ are sharp for the flocking pattern to appear for generic initial data.
The previous discussion can be summarized in the following theorem written in terms of solutions of the equation (3.56).

Proposition 3.4.10. *For any $\tilde\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ composed by finite number of particles, i.e., an atomic measure with N atoms, there exists $\{x_1^\infty, \dots, x_N^\infty\}$ such that the unique measure-valued solution to* (3.56) *with $\gamma \le 1/2$, given by*

$$\tilde\mu(t) = \sum_{i=1}^{N} m_i \,\delta(x - x_i(t))\,\delta(v - v_i(t)),$$

where the curves are given by the ODE system (3.2)*, satisfies that*

$$\lim_{t\to\infty} d\left(\tilde\mu(t), \tilde\mu^\infty\right) = 0$$

with

$$\tilde\mu^\infty = \sum_{i=1}^{N} m_i \,\delta(x - x_i^\infty - mt)\,\delta(v - m)$$

with m the initial mean velocity of the particles. Moreover, given the largest velocity $R^v(t)$ defined as

$$R^v(t) := \max_{i=1,\dots,N_p} |v_i(t) - m|.$$

and the most distant space location $R^x(t)$ with respect to the initial center of mass x_c,

$$R^x(t) := \max_{i=1,\dots,N_p} |x_i(t) - x_c - mt|,$$

then

$$R^x(t) \le \bar{R}^x \qquad \textit{and} \qquad R^v(t) \le R_0^v\, e^{-\lambda t}$$

for all $t \ge 0$, with $\bar{R}^x$ depending only on and $\lambda = w(2\bar{R}^x)$ and the initial value of $R_0 = \max\{R^v(0), R^x(0)\}$.

Let us point out that the proof is trivial noting that the Wasserstein distance between finite atomic measures is bounded by a sum of Euclidean distances for any permutation of the points of one of them, see [171] for instance. We also wrote them as solutions of the partial differential equation instead of the particle system since this will be useful for general measure solutions.
Once we have the control on the support of particles independent of the number of particles given in the previous theorem, it is easy to deduce the main result of this section.

Theorem 3.4.11. *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to* (3.56) *with $\gamma \le 1/2$, satisfies the following bounds on their supports:*

$$\textit{supp}\ \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \ge 0$, with

$$R^x(t) \le \bar{R} \qquad \textit{and} \qquad R^v(t) \le R_0\, e^{-\lambda t}$$

with $\bar{R}^x$ depending only on the initial value of $R_0 = \max\{R^x(0), R^v(0)\}$ and $\lambda = w(2\bar{R})$.

Proof.- As for particles, we can assume $m = 0$ and $x_c(t) = x_c(0)$ for all $t \ge 0$ without loss of generality. Given any compactly supported measure

$$\mu_0 \in B(x_c(0), R^x(0)) \times B(0, R^v(0))$$

and any $\eta > 0$, we can find a number of particles $N = N(\eta)$, set of positions $\{x_1^0, \dots, x_N^0\} \subset B(c_M(0), R^x(0))$, a set of velocities $\{v_1^0, \dots, v_N^0\} \subset B(0, R^v(0))$, and masses $\{m_1, \dots, m_N\}$, such that

$$W_1\left(\mu_0, \sum_{i=1}^{N} m_i \delta(x - x_i^0)\delta(v - v_i^0)\right) \leq \eta.$$

Let us denote by $\mu_\eta(t)$ the particle solution associated to the initial datum

$$\mu_\eta(0) = \sum_{i=1}^{N_p} m_i\, \delta(x - x_i^0)\, \delta(v - v_i^0).$$

Using Proposition 3.4.10, we have that

$$\text{supp}\ \mu_\eta(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

with $R^x(t)$ and $R^v(t)$ verifying the stated properties for all small η since the result was independent of the number of particles.
By the stability result in Theorem 3.4.8 included in [90], we obtain

$$W_1(\mu(t), \mu_\eta(t)) \leq \alpha(t)\, W_1\left(\mu_0, \sum_{i=1}^{N_p} m_i\, \delta(x - x_i^0)\, \delta(v - v_i^0)\right) \leq \alpha(t)\eta.$$

Since t is fixed and η can be arbitrarily small, we conclude that $\mu_\eta(t) \to \mu(t)$ weakly-$*$ as measures when $\eta \to 0$ for all $t \geq 0$. Since the support of a measure is stable under weak-$*$ limits, we conclude the proof. □
Let us remark that the same strategy of proof has been used for continuum models of aggregation [38]. An immediate consequence is to control directly the decay of the Lyapunov functional used by Ha and Tadmor in [91].

Corollary 3.4.12. *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to* (3.56) *with $\gamma \leq 1/2$, satisfies*

$$\Lambda(\mu)(t) \leq R_0^2\, e^{-2\lambda t}$$

with R_0 and λ given in Theorem 3.4.11.

Proof.- Since the solution by Theorem 3.4.11 is supported in velocity in $B(m, R^v(t))$, then

$$\Lambda(\mu)(t) = \int_{\mathbb{R}^{2d}} |v - m|^2\, d\mu(t)(x,v) = \int_{|v-m| \leq R^v(t)} |v - m|^2\, d\mu(t)(x,v) \leq R^v(t)^2$$

concluding the proof. □

Remark 3.4.13. *Let us point out that there is another interesting functional associated to the system*

$$\mathscr{F}(\mu)(t) = \frac{1}{2}\int_{\mathbb{R}^{4d}} \frac{|v-w|^2}{(1+|x-y|^2)^\gamma}\, d\mu(t)(x,v)\, d\mu(t)(y,w). \tag{3.71}$$

This functional comes naturally as a Lyapunov functional for classical solutions of the equation

$$\frac{\partial f}{\partial t} = \nabla_v \cdot [\xi(f)(t,x,v) f(t,x,v)]\,, \tag{3.72}$$

which stems from (3.56) *if the transport term $v \cdot \nabla_x f$ is dropped [43, 44]. As with the other functional it is trivial to check that since the solution by Theorem 3.4.11 is supported in velocity in $B(m, R^v(t))$ and in position in $B(x_c(0)+mt, R^x(t))$, then $\mathscr{F}(\mu)(t) \le R^v(t)^2$, for any weak measure-valued solution μ of* (3.56)*. Moreover, due to the uniform space support bound given in Theorem 3.4.11, the functionals $\Lambda(f)$ and $\mathscr{F}(f)$ are equivalent in the sense that*

$$C_{\mathscr{F}}\Lambda(\mu) \le \mathscr{F}(\mu) \le C^{\mathscr{F}}\Lambda(\mu),$$

for $C_{\mathscr{F}} \le C^{\mathscr{F}}$ positive constants independent of t. Actually, using again Theorem 3.4.11, the solution is supported in velocity in $B(m, R^v(t))$ and in position in $B(x_c(0)+mt, R^x(t))$ and thus

$$\begin{aligned}
\mathscr{F}(\mu)(t) &= \frac{1}{2}\int_{\mathbb{R}^{4d}} \frac{|v-w|^2}{(1+|x-y|^2)^\gamma}\, d\mu(t)(x,v)\, d\mu(t)(y,w) \\
&\ge \frac{1}{2(1+\bar{R}^2)^\gamma}\int_{\mathbb{R}^{4d}} \left(|v-m|^2 + |w-m|^2\right)\, d\mu(t)(x,v)\, d\mu(t)(y,w) \\
&= \frac{2}{(1+\bar{R}^2)^\gamma}\Lambda(\mu)(t)
\end{aligned}$$

and trivially $\mathscr{F}(\mu)(t) \le 2\Lambda(\mu)(t)$.

Finally, let us obtain some more information about the asymptotic limit. Using the characterization of solutions by characteristics in Theorem 3.4.8, then we have that

$$\mu(t,x,v) = (X(t;x,v), V(t;x,v))\#\mu_0$$

where the characteristics $(X(t;x,v), V(t;x,v))$ satisfy (3.59). It is then clear that

$$|V(t;x,v) - m| \le R_0\, e^{-\lambda t} \text{ for all } v \in B(m,R_0)\,,\ x \in B(x_c(0), R_0).$$

But using the equation for the position variables we find

$$\frac{d}{dt}\left[X(t;x,v) - mt\right] = V(t;x,v) - m$$

with a right hand side whose components are exponentially decaying in time and thus, integrable at infinity. As a consequence, for all (x, v) initially in the support of μ_0, we have

$$\lim_{t\to\infty}[X(t;x,v) - mt] = x + \int_0^\infty [V(s;x,v) - m]\,ds.$$

This can be rephrased in terms of the density in position associated to the solutions. We need a bit of notation: given a measure $\mu \in \mathcal{M}(\mathbb{R}^{2d})$, we define its translate μ^h with vector $h \in \mathbb{R}^d$ by:

$$\int_{\mathbb{R}^d} \zeta(x,v)\,\mathrm{d}\mu^h(x,v) = \int_{\mathbb{R}^{2d}} \zeta(x-h,v)\,\mathrm{d}\mu(x,v),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^{2d})$. We will also denote by μ_x the marginal in the position variable, that is,

$$\int_{\mathbb{R}^{2d}} \zeta(x)\,\mathrm{d}\mu(x,v) = \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}\mu_x(x),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$. With this we can write the main conclusion about the asymptotic behavior, i.e., the convergence in relative to the center of mass variables to a fixed density characterized by the initial data and its unique solution.

Theorem 3.4.14. *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to* (3.56) *with $\gamma \le 1/2$, satisfies*

$$\lim_{t\to\infty} W_1(\mu_x^{mt}(t), L_\infty(\mu_0)) = 0,$$

where the measure $L_\infty(\mu_0)$ is defined as

$$\int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x) = \int_{\mathbb{R}^{2d}} \zeta\left(x + \int_0^\infty [V(s;x,v) - m]\,ds\right)\,\mathrm{d}\mu_0(x,v),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$.

Proof.- Given a test function $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$, we compute

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}\mu_x^{mt}(t)(x) - \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \zeta(x)\,\mathrm{d}\mu^{mt}(t)(x,v) - \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \zeta(x-mt)\,\mathrm{d}\mu(t)(x,v) - \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \left[\zeta(X(t;x,v)-mt) - \zeta\left(x + \int_0^\infty [V(s;x,v)-m]\,ds\right)\right]\,\mathrm{d}\mu_0(x,v)\right| \\
&\le \mathrm{Lip}(\zeta) \int_{\mathbb{R}^{2d}} \left|(X(t;x,v)-mt) - \left(x + \int_0^\infty [V(s;x,v)-m]\,ds\right)\right|\,\mathrm{d}\mu_0(x,v).
\end{aligned}$$

An easy application of the Lebesgue dominate convergence theorem gives the result where one uses the uniform in time bound on the characteristics above. □

3.5 Well-posedness for General Models

In this section we want to show that the same results we have obtained in the previous sections are also valid, with suitable modifications, for much more general models than (3.4) or (3.6). With the techniques used in the previous sections one can include quite general kinetic models in the well-posedness theory. In this section we illustrate this by giving a result for a model which includes both the potential interaction and self-propulsion effects of section 3.3, the velocity-averaging effect of section 3.4.3 and the more general models above [123, 128].

Let us introduce some notation for this section: $\mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ denotes the subset of $\mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ consisting of measures of compact support in $\mathbb{R}^d \times \mathbb{R}^d$, and we consider the non-complete metric space $\mathcal{A} := \mathcal{C}([0,T], \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d))$ endowed with the distance $\mathcal{W}_1$. On the other hand, we consider the set of functions $\mathcal{B} := \mathcal{C}([0,T], \mathrm{Lip}_{loc}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d))$, which in particular are locally Lipschitz with respect to (x,v), uniformly in time. We consider an operator $\mathcal{H}[\cdot] : \mathcal{A} \longrightarrow \mathcal{B}$ and assume the following:

Hypothesis 3.5.1 (Hypothesis on a general operator). *Take any $R_0 > 0$ and $f, g \in \mathcal{A}$ such that $\mathrm{supp}(f_t) \cup \mathrm{supp}(g_t) \subseteq B_{R_0}$ for all $t \in [0,T]$. Then for any ball $B_R \subset \mathbb{R}^d \times \mathbb{R}^d$, there exists a constant $C = C(R, R_0)$ such that*

$$\max_{t\in[0,T]} \|\mathcal{H}[f] - \mathcal{H}[g]\|_{L^\infty(B_R)} \le C\, \mathcal{W}_1(f,g),$$
$$\max_{t\in[0,T]} \mathrm{Lip}_R(\mathcal{H}[f]) \le C.$$

Associated to this operator, we can consider the following general equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot [\mathcal{H}[f] f] = 0. \tag{3.73}$$

Remark 3.5.2 (Generalization). *It is not difficult to see that the choices $\mathcal{H}[f] = (\alpha - \gamma |v|^2)v - \nabla U * \rho$ and $\mathcal{H}[f] = H * f$ correspond to* (3.4) *and* (3.6)*, respectively, and that they satisfy Hypothesis 3.5.1 if we assume the hypotheses of Theorems 3.3.10 and 3.4.6 respectively. Moreover, one can cook up an operator of the form:*

$$\mathcal{H}[f] = F_A(x,v) + G(x) * \rho + H(x,v) * f$$

with F_A, G and H given functions satisfying suitable hypotheses, such that the kinetic equation (3.73) *corresponds to the model* (3.7).

We will additionally require the following:

Hypothesis 3.5.3 (Additional constraint on H). *Given $f \in \mathcal{C}([0,T], \mathcal{P}_c(B_{R_0}))$, and for any initial condition $(X^0, V^0) \in \mathbb{R}^d \times \mathbb{R}^d$, the following system of ordinary differential equations has a globally defined solution:*

$$\frac{d}{dt}X = V, \tag{3.74a}$$

$$\frac{d}{dt}V = \mathcal{H}[f](t, X, V), \tag{3.74b}$$

$$X(0) = X^0, \quad V(0) = V^0. \tag{3.74c}$$

Of course, this is a requirement that has to be checked for every particular model, and it is difficult to give useful properties of $\mathcal{H}$ that imply this and are general enough to encompass a range of utile models; therefore, we prefer to give a general condition which reduces the problem of existence and stability to the simpler one of existence of the characteristics.
In the above conditions one can follow a completely analogous argument to that in the proof of Theorems 3.3.10 and 3.3.16, and obtain the following result:

Theorem 3.5.4 (Existence, uniqueness and stability of measure solutions for a general model). *Take an operator $\mathcal{H}[\cdot] : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying Hypotheses 3.5.1 and 3.5.3, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution f on $[0, +\infty)$ to equation* (3.73) *with initial condition f_0. In addition,*

$$f \in \mathcal{C}([0, +\infty); \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)) \tag{3.75}$$

and there is some increasing function $R = R(T)$ such that for all $T > 0$,

$$\operatorname{supp} f_t \subseteq B_{R(T)} \subseteq \mathbb{R}^d \times \mathbb{R}^d \quad \text{for all } t \in [0,T]. \tag{3.76}$$

This solution is unique among the family of solutions satisfying (3.75) *and* (3.76)*. Moreover, given any other initial data $g_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ and g its corresponding solution, then there exists a strictly increasing function $r(t) : [0, \infty) \longrightarrow \mathbb{R}_0^+$ with $r(0) = 1$ depending only on $\mathcal{H}$ and the size of the support of f_0 and g_0, such that*

$$W_1(f_t, g_t) \leq r(t)\, W_1(f_0, g_0), \quad t \geq 0.$$

3.6 Consequences of Stability

3.6.1 N-Particle approximation and the mean-field limit

The stability theorems 3.3.16 and 3.4.8, or the general version 3.5.4, give in particular a justification of the approximation of this family of models by a finite set of particles satisfying a system of ordinary differential equations. We will state results for the general model (3.73), under the conditions on $\mathcal{H}$ from section 3.5. One can easily check that the following holds:

Lemma 3.6.1 (Particle solutions). *Assume $\mathcal{H}$ satisfies the conditions of Theorem 3.5.4. Take N positive numbers $m_1, \dots, m_N$, and consider the following system of differential equations:*

$$\dot{x}_i = v_i, \qquad i = 1, \dots, N, \tag{3.77a}$$

$$\dot{v}_i = \sum_{j \neq i} m_j \mathcal{H}[f](t, x_i, v_i), \qquad i = 1, \dots, N. \tag{3.77b}$$

where $f^N : [0,T] \to \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is the measure defined by

$$f_t^N := \sum_{i=1}^{N} m_i \, \delta_{(x_i(t), v_i(t))} \tag{3.78}$$

If $x_i, v_i : [0,T] \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the system (3.77)*, then the function f^N, is the solution to* (3.73) *with initial condition*

$$f_0^N = \sum_{i=1}^{N} m_i \, \delta_{(x_i(0), v_i(0))}. \tag{3.79}$$

As a consequence of the stability in W_1, we have an alternative method to derive the kinetic equations (3.4), (3.6) or (3.73), based on the convergence of particle approximations, other than the formal BBGKY hierarchy in [23, 39].

Corollary 3.6.2 (Convergence of the particle method). *Given $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ compactly supported and $\mathcal{H}$ satisfying the conditions of Theorem 3.5.4, take a sequence of f_0^N of measures of the form* (3.79) *(with m_i, $x_i(0)$ and $v_i(0)$ possibly varying with N), in such a way that*

$$\lim_{N \to \infty} W_1(f_0^N, f_0) = 0.$$

Consider f_t^N given by (3.78)*, where $x_i(t)$ and $v_i(t)$ are the solution to system* (3.77) *with initial conditions $x_i(0)$, $v_i(0)$. Then,*

$$\lim_{N \to \infty} W_1(f_t^N, f_t) = 0,$$

for all $t \geq 0$, where $f = f(t, x, v)$ is the unique measure solution to eq. (3.73) *with initial data f_0.*

3.6.2 Hydrodynamic limit

We state our hydrodynamic limit result for eq. (3.4). If we look for solutions of (3.4) of the form

$$f(t,x,v) = \rho(t,x)\,\delta(v - u(t,x)) \tag{3.80}$$

for some functions $\rho, u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, one formally obtains that ρ and u should satisfy the following equations:

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0, \tag{3.81a}$$

$$\partial_t u + (u \cdot \nabla) u = u(\alpha - \beta\,|u|^2) - \nabla U * \rho. \tag{3.81b}$$

This is made precise by the following result whose existence part was already obtained in [39]:

Lemma 3.6.3 (Uniqueness for Hydrodynamic Solutions). *Take a potential $U \in \mathcal{C}^2(\mathbb{R}^d)$ and assume that there exists a smooth solution (ρ, u) with initial data (ρ_0, u_0) to the system* (3.81) *defined on the interval $[0,T]$. Then, if we define $f : [0,+\infty) \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ by*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v)\,\phi(x,v)\,dx\,dv = \int_{\mathbb{R}^d} \phi(x, u(t,x))\,\rho(t,x)\,dx \tag{3.82}$$

for any test function $\phi \in \mathcal{C}^0_C(\mathbb{R}^d \times \mathbb{R}^d)$, then f is the unique solution to (3.4) *obtained from Theorem 3.3.10 with initial condition $f_0 = \rho_0 \delta(v - u_0)$.*

As a direct consequence of Lemma 3.6.3 and the stability result in Theorem 3.3.16, we get the following result.

Corollary 3.6.4 (Local-in-time Stability of Hydrodynamics). *Take a potential $U \in \mathcal{C}^2(\mathbb{R}^d)$ and assume that there exists a smooth solution (ρ, u) with initial data (ρ_0, u_0) to the system* (3.81) *defined on the interval $[0,T]$. Let us consider a sequence of initial data $f_0^k \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

$$\lim_{k\to\infty} W_1(f_0^k, \rho_0\,\delta(v - u_0)) = 0.$$

Consider the solution f^k to the swarming eq. (3.4) *with initial data f_0^k. Then,*

$$\lim_{k\to\infty} W_1(f_t^k, f_t) = 0,$$

for all $t \in [0,T]$ with $f(t,x,v) = \rho(t,x)\,\delta(v - u(t,x))$.

3.7 Some open questions

Very few of the observed features of the complex dynamics behavior described in Section 3.1 can be actually proved in a rigorous way. Apart from the asymptotic convergence to the mean velocity of the Cucker-Smale model, we do not have a way to distinguish the values of the parameters in the system (3.1) for which one behavior or other takes place, and we do not know how to prove the convergence to a swarm, a mill or other kind of organization. A feasible way of studying these problems may be the following: one can consider, instead of the system (3.1) or (3.3), a partial differential equation which is obtained as its *mean-field limit*; this is an evolution equation whose solutions are approximated by the evolution of the density of particles of the system (3.1) (or (3.3)) when the number of particles is very large [32]. Then, studying stationary states for this equation may be a simpler task than directly studying (3.3). For example, the partial differential equation obtained as a limit when $N \to \infty$ of the system (3.3) is

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v((\nabla U * \rho) f) + \mathrm{div}_v((H * f) f) = 0, \tag{3.83}$$

where $f = f(t, x, v)$ represents the density of individuals at time t in an infinitesimal region $dx\, dv$, $\rho = \rho(t, x)$ is the macroscopic density of individuals at time t obtained from f by integration on v, the first convolution is in the x variable, and the second one in both the x, v variables [32]. The function H is given by

$$H(x, v) = \frac{v}{(1 + |x|^2)^\gamma} \qquad (x, v \in \mathbb{R}^2).$$

"Flocking" solutions of the system (3.3) (solutions for which every individual moves at the same fixed velocity v_0) correspond to solutions of (3.83) of the form

$$f(t, x, v) = \rho(x - v_0 t)\, \delta(v - v_0), \tag{3.84}$$

where δ is the Dirac delta function. These are solutions which have a constant mass profile ρ, and in which every point moves at the same velocity v_0. Can we find solutions like this?

Particles	250	500	1000	2000	4000
Radius	16.5326	17.1331	17.4068	17.4016	17.5150

Table 3.1: Maximum distance of the particles to the centre of mass with respect to the number of particles in the simulation.

Looking for such a profile ρ turns out not to be a simple problem, and very little is known about it. We may restrict ourselves to looking for solutions with velocity $v_0 = 0$, as all others are just translations in velocity of these. Then, if we look at the limiting shape of flocking solutions of the system (3.3), we see that the "flocks" tend to grow indefinitely with N when the potential is in the H-stable region; hence, in this case, the conjecture is that there are no nice (say, continuous and compactly supported) functions ρ for which (3.84) is a solution of (3.83). On the other hand, when the potential is not H-stable, or "catastrophic", the flocks seem numerically to converge to an asymptotic distribution as we add more and more particles. As shown in Table 3.1 and first plot in Fig. 3.8, the maximum radius of the particles with respect to the center of mass tends to stabilize as $t \to \infty$ and as N gets larger to a fixed value. Moreover, we have computed the normalized cumulative distribution of particles in the radial direction starting from the center of mass as we increase the number of particles N. The second plot in Fig. 3.8 shows numerical evidence of the convergence towards a fixed continuous profile as N gets larger.

The conjecture in this case is that there *are* continuous and compactly supported profiles ρ for which (3.84) is a solution of the kinetic equation, and that there are N-individual flocks whose density converges to ρ in the limit of N going to ∞. These problems are, to our knowledge, open for the moment, and they are also not particular to this field: they are more generally related to the shape of N-particle equilibria for a given potential U.

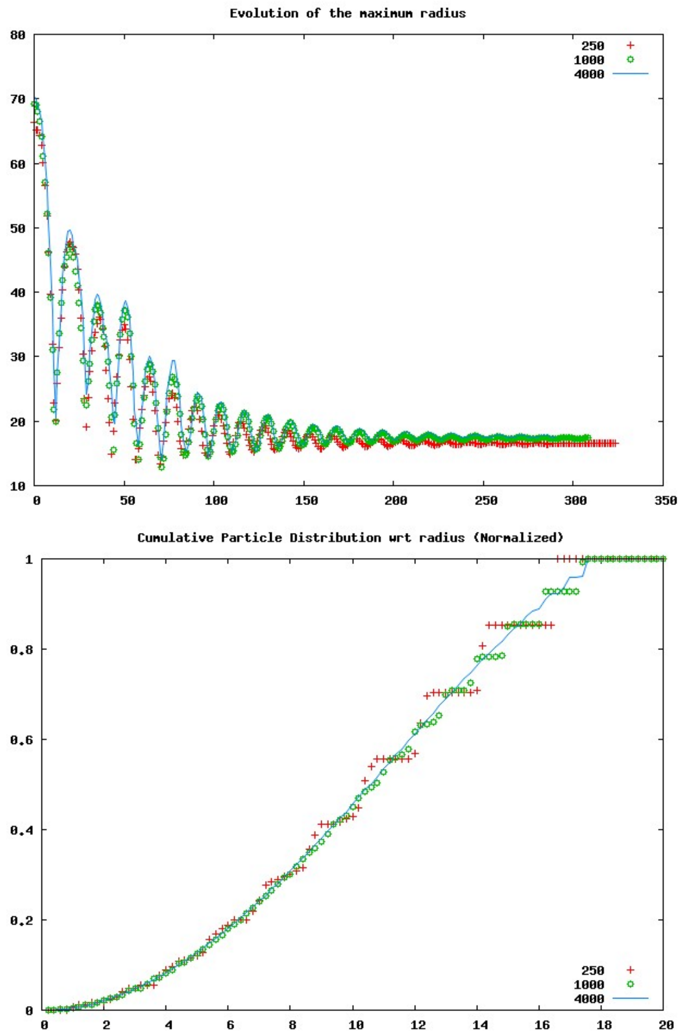

Figure 3.8: Evolution with respect to time of the maximum radius and cumulative distribution of particles with respect to radius computed for 3 different number of particles.

Appendix A

Basic Functional Analysis Results

All throughout the pages of this work we make constant references to some results coming from the functional analysis which we need in order to prove certain convergence properties or to bound some quantity. In this appendix we try to collect them, so that they will be at hand for the reader who wants to recall some detail. Since the proving these results is not the intent of this work, we will just state them. More detailed descriptions can be found in [79, 124], for instance.

Theorem A.1 (Arzelà-Ascoli). *Let X be a compact set and take $\{f_n\} \in \mathcal{C}(X)$ an equicontinuous sequence uniformly bounded. Then $\{f_n\}$ has a subsequence which converges uniformly.*

Theorem A.2 (de la Vallée-Poussin). *let H be a set in L^1. The next properties are equivalent*

- *H is uniformly integrable.*

- *There exists a positive function $\phi(t)$ defined on $\mathbb{R}_+$, such that*

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty \quad \textit{and} \quad \sup_{f\in H} \int_{\mathbb{R}^d} \phi \circ f < \infty$$

Theorem A.3 (Dunford-Pettis). *Let H be a set in L^1. Then the next properties are equivalent:*

- *H is uniformly integrable.*
- *H is relatively compact in L^1 by the weak topology $\sigma(\mathrm{L}^1, \mathrm{L}^\infty)$.*
- *Every sequence in H has a subsequence which converge in the sense of the topology $\sigma(\mathrm{L}^1, \mathrm{L}^\infty)$.*

Theorem A.4 (Gagliardo-Nieremberg-Sobolev inequality). *Assume* $1 \leq p \leq d$ *and define*

$$p^* := \frac{dp}{d-p}.$$

Then, there exists a constant C*, depending only on* p *and* d*, such that*

$$\|u\|_{L^{p^*}\left(\mathbb{R}^d\right)} \leq C\|Du\|_{L^p\left(\mathbb{R}^d\right)}$$

for all $u \in \mathcal{C}_c^1\left(\mathbb{R}^d\right)$.

Theorem A.5 (Hölder's inequality). *Assume* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then if* $u \in \mathrm{L}^p(U)$, $v \in \mathrm{L}^q(U)$*, we have*

$$\int_U |uv| \mathrm{d}x \leq \|u\|_{\mathrm{L}^p(U)} \|v\|_{\mathrm{L}^q(U)}$$

Theorem A.6 (Interpolation inequality for L^p-norms). *Assume* $1 \leq s \leq r \leq t \leq \infty$ *and*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}.$$

Suppose also $u \in \mathrm{L}^s(U) \cap \mathrm{L}^t(U)$. *Then* $u \in \mathrm{L}^r(U)$ *and*

$$\|u\|_{\mathrm{L}^r(U)} \leq \|u\|_{\mathrm{L}^s(U)}^{\theta} \|u\|_{\mathrm{L}^t(U)}^{1-\theta}$$

Theorem A.7 (Young's inequality for convolution). *Let* $f \in \mathrm{L}^p(\mathbb{R}^d)$, $g \in \mathrm{L}^q(\mathbb{R}^d)$ *and* $r > 0$ *with*

$$1 \leq p \leq \infty, \quad 1 \leq q \leq \infty \quad \text{and} \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

Then

$$f * g \in \mathrm{L}^r(\mathbb{R}^d) \quad \text{and} \quad \|f * g\|_r \leq \|f\|_p \|g\|_q$$

Appendix B

Preliminaries on mass transportation

Optimal transport studies the way of minimizing the cost of transporting one mass into another. It was born in the late 18th century but has experienced an important development in the last decades when mathematicians of all areas saw how it linked to their subject, providing new perspectives to approach old problems. Through the pages of this work we make use of it in several occasions and thus we want to recall here some notation and known results. For a more detailed approach, the interested reader can refer to [48, 171, 172].

B.1 Definitions and Notation

Definition B.1 (Bounded Lipschitz). *We denote by $Lip_b(\mathbb{R}^d)$ the set of bounded and Lipschitz functions φ on $\mathbb{R}^d$ endowed with the norm*

$$\|\varphi\|_{Lip_b(\mathbb{R}^d)} := \|\varphi\|_{L^\infty(\mathbb{R}^d)} + Lip(\varphi),$$

For a function $\varphi \in Lip_b(\mathbb{R}^d)$ we shall write $\mathrm{Lip}(\varphi)$ to denote the Lipschitz constant of φ.

Definition B.2 (Bounded Lipschitz Distance). *We define the bounded Lipschitz distance between two measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ as*

$$d_{\mathbb{R}^d}(\mu,\nu) = \sup\left\{ \left| \int_{\mathbb{R}^d} \varphi(z)\, d\mu(z) - \int_{\mathbb{R}^d} \varphi(z)\, d\nu(z) \right| ;\ \|\varphi\|_{Lip_b(\mathbb{R}^d)} \leq 1 \right\}. \quad \text{(B.1)}$$

This distance was classically used in particle limits for the Vlasov equation in [143, 161]. We refer to [171] for comments and relations of these dual distances to optimal transport distances. When think about optimal transport, though, one would typically work in the space of probability measures $\mathcal{P}(\mathbb{R}^d)$.

Definition B.3. *We define $\mathcal{P}_1(\mathbb{R}^d)$ as the space of probability measures consisting of all probability measures on $\mathbb{R}^d$ with finite first moment.*

In $\mathcal{P}_1(\mathbb{R}^d)$ a natural concept of distance to work with is the so-called *Monge-Kantorovich-Rubinstein distance,*

Definition B.4 (Monge-Kantorovich-Rubinstein distance).

$$W_1(f,g) = \sup\left\{\left|\int_{\mathbb{R}^d}\varphi(P)(f(P)-g(P))\,dP\right|, \varphi\in \mathrm{Lip}(\mathbb{R}^d), \mathrm{Lip}(\varphi)\leq 1\right\}. \tag{B.2}$$

We can give a more general definition of this distance:

Definition B.5 (Wasserstein Distance). *Given two probability measures ρ_1 and ρ_2 probability measures with bounded second moment, the Euclidean Wasserstein Distance is defined as*

$$W_2(\rho_1,\rho_2) = \inf_{\Pi\in\Gamma}\left\{\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\,d\Pi(x,y)\right\}^{1/2} \tag{B.3}$$

where Π runs over the set of transference plans Γ, that is, the set of joint probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals ρ_1 and ρ_2, i.e.,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x)\,d\Pi(x,y) = \int_{\mathbb{R}^d}\varphi(x)\,\rho_1(x)\,dx$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(y)\,d\Pi(x,y) = \int_{\mathbb{R}^d}\varphi(y)\,\rho_2(y)\,dy$$

for all $\varphi\in C_b(\mathbb{R}^d)$, the set of continuous and bounded functions on $\mathbb{R}^d$.

Definition B.6. *Throughout these pages we shall denote the integral of a function $\varphi=\varphi(x)$ with respect to a measure μ by $\int\varphi(x)\mu(x)\,dx$, even if the measure is not absolutely continuous with respect to Lebesgue measure, and hence does not have an associated density.*

Definition B.7. *Given a probability measure $f\in\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d)$ we always denote by ρ its first marginal, written as follows by an abuse of notation:*

$$\rho(x) := \int_{\mathbb{R}^d} f(x,v)\,dv. \tag{B.4}$$

To be more precise, ρ is given by its action on a $\mathcal{C}_c^0$ function $\phi:\mathbb{R}^d\to\mathbb{R}$,

$$\int_{\mathbb{R}^d}\rho(x)\phi(x)\,dx = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\phi(x)\,dx\,dv.$$

For $T > 0$ and a function $f : [0, T] \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, it is understood that ρ is the function $\rho : [0, T] \to \mathcal{P}_1(\mathbb{R}^d)$ obtained by taking the first marginal at each time t. Whenever we need to indicate explicitly the dependence of ρ on f, we write $\rho[f]$ instead of just ρ.

Definition B.8 (Interpolation measure). *Let ρ_1 and ρ_2 be two probability measures. Let T be the optimal transportation map beween them due to Brenier's theorem [27] and let $\mathbb{I}_{\mathbb{R}^d}$ be the identity map. We define the displacement interpolation between these measures as*

$$\rho_\theta = ((\theta - 1)T + (2 - \theta)\mathbb{I}_{\mathbb{R}^d})_{\#}\rho_1 \tag{B.5}$$

for $\theta \in [1, 2]$.

B.2 Results

$\mathcal{P}_1(\mathbb{R}^d)$ endowed with the Wasserstein distance is a complete metric space. In the following proposition we recall some of its properties. We refer to [171] for a survey of these basic facts.

Proposition B.2.1. *Denoting by Λ the set of transference plans between the measures f and g, i.e., probability measures in the product space $\mathbb{R}^d \times \mathbb{R}^d$ with first and second marginals f and g respectively, then we have*

$$W_1(f, g) = \inf_{\pi \in \Lambda} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |P_1 - P_2| \, d\pi(P_1, P_2) \right\} \tag{B.6}$$

by Kantorovich duality.

Proposition B.2.2 (W_1-properties). *The following properties of the distance W_1 hold:*

i) **Optimal transference plan:** *The infimum in the definition of the distance W_1 is achieved. Any joint probability measure Π_o satisfying:*

$$W_1(f, g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |P_1 - P_2| \, d\Pi_o(P_1, P_2).$$

is called an optimal transference plan and it is generically non unique for the W_1-distance.

ii) **Convergence of measures:** *Given $\{f_k\}_{k \geq 1}$ and f in $\mathcal{P}_1(\mathbb{R}^d)$, the following three assertions are equivalent:*

a) $W_1(f_k, f)$ *tends to* 0 *as* n *goes to infinity.*

b) f_k *tends to* f *weakly-* as measures as* k *goes to infinity and*

$$\sup_{k\geq 1} \int_{|v|>R} |v|\, f_k(v)\, dv \to 0 \text{ as } R \to +\infty.$$

c) f_k *tends to* f *weakly-* as measures and*

$$\int_{\mathbb{R}^d} |v|\, f_k(v)\, dv \to \int_{\mathbb{R}^d} |v|\, f(v)\, dv \text{ as } n \to +\infty.$$

iii) **Lower semicontinuity:** W_1 *is weakly-* lower semicontinuous in each argument.*

vi) **Convexity:** *Given* f_1, f_2, g_1 *and* g_2 *in* $\mathcal{P}_1(\mathbb{R}^d)$ *and* α *in* $[0, 1]$*, then*

$$W_1(\alpha f_1 + (1-\alpha)f_2, \alpha g_1 + (1-\alpha)g_2) \leq \alpha W_1(f_1, g_1) + (1-\alpha)W_1(f_2, g_2).$$

As a simple consequence, given f, g *and* h *in* $\mathcal{P}_1(\mathbb{R}^d)$*, then*

$$W_1(h * f, h * g) \leq W_1(f, g)$$

where $*$ *stands for the convolution in* $\mathbb{R}^d$*.*

vii) **Additivity with respect to convolution:** *Given* f_1, f_2, g_1 *and* g_2 *in* $\mathcal{P}_2(\mathbb{R}^d)$ *with with equal mean values, then*

$$W_1(f_1 * f_2, g_1 * g_2) \leq W_1(f_1, g_1) + W_1(f_2, g_2).$$

Remark B.9. *Given two maps* $X_1, X_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *and a given probability measure with bounded second moment* ρ_0*, then*

$$W_2^2(X_1\#\rho_0, X_2\#\rho_0) \leq \int_{\mathbb{R}^d} |X_1(x) - X_2(x)|^2 \mathrm{d}\rho_0(x)$$

just by using $\Pi = (X_1 \times X_2)\#\rho_0$ *in the definition of the distance* W_2*.*

Theorem B.10. [9, 132, 86, 1] *Let* ρ_1 *and* ρ_2 *be two probability measures on* $\mathbb{R}^d$*, such that they are absolutely continuous with respect to the Lebesgue measure and* $W_2(\rho_1, \rho_2) < \infty$*. Then there exists a vector field* $\nu_\theta \in L^2(\mathbb{R}^d, \rho_\theta\, dx)$ *such that*

i. $\dfrac{d}{d\theta}\rho_\theta + \mathrm{div}(\rho_\theta \nu_\theta) = 0$ *for all* $\theta \in [1, 2]$*.*

ii. $\displaystyle\int_{\mathbb{R}^d} \rho_\theta |\nu_\theta|^2\, dx = W_2^2(\rho_1, \rho_2)$ *for all* $\theta \in [1, 2]$*.*

iii. We have the L^∞-interpolation estimate

$$\|\rho_\theta\|_{L^\infty(\mathbb{R}^d)} \leq \max\left\{\|\rho_1\|_{L^\infty(\mathbb{R}^d)}, \|\rho_2\|_{L^\infty(\mathbb{R}^d)}\right\}$$

for all $\theta \in [1,2]$. Indeed, this property is also true in L^p for $1 \leq p < \infty$ due to the displacement convexity property of $\int \rho^p$ (See [132])

Proposition B.11. [126] *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^d$ with L^∞ densities with respect to the Lebesgue measure. Let c_i the solution of the Poisson's equation $-\Delta c_i = \rho_i$ in $\mathbb{R}^d$ given by $c_i = \Gamma_N * \rho_i$ with Γ_N the fundamental solution of $-\Delta$ in $\mathbb{R}^d$. Then,*

$$\|\nabla c_1 - \nabla c_2\|_{L^2(\mathbb{R}^d)} \leq \max(\|\rho_1\|_{L^\infty(\mathbb{R}^d)}, \|\rho_2\|_{L^\infty(\mathbb{R}^d)})^{1/2}\, W_2(\rho_1,\rho_2)$$

Proposition B.12. *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^d$ with L^∞ densities with respect to the Lebesgue measure. Let $c_i = K * \rho_i$ with $\nabla K \in L^2(\mathbb{R}^d)$ and $|D^2 K| \in L^1(\mathbb{R}^d)$. Then,*

$$\|\nabla c_1 - \nabla c_2\|_{L^2(\mathbb{R}^d)} \leq \max(\|\rho_1\|_{L^\infty(\mathbb{R}^d)}, \|\rho_2\|_{L^\infty(\mathbb{R}^d)})^{1/2}\, W_2(\rho_1,\rho_2).$$

Appendix C

Uniqueness of the Aggregation equation with diffusion

In this appendix we use the same ideas as in the proof of uniqueness in Chapter 2 to deal with this uniqueness issue for the associated aggregation equations in which a linear diffusion term is added, i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \operatorname{div}\left[\rho(t,x)v(t,x)\right] = \Delta u & t > 0\,,\ x \in \mathbb{R}^d\,, \\ u(0,x) = u_0(x) \geq 0 & x \in \mathbb{R}^d\,. \end{cases} \tag{C.1}$$

We start by defining the concept of solution, we shall work with in this case.

Definition C.1. *A function ρ is a bounded weak solution of* (C.1) *on [0,T] for a nonnegative initial data $\rho_0 \in L^1(\mathbb{R}^d)$, if it satisfies*

1. $\rho \in C_w([0,T], L^1_+(\mathbb{R}^d))$.

2. *The SDE system*

$$dX(t) = u(t, X(t))\,dt + \sqrt{2}\,dW_t$$

with the velocity field $v(t,x) := -\nabla K * u(t,x)$ and initial data $X(0)$ with law $u_0(x)$ has a solution given by a Markov process $X(t)$ of law $u(t,x)$. Here W_t is the standard Wiener process.

3. *$u(t)$ is a distributional solution to* (C.1)*.*

4. $\rho \in L^\infty(0,T; L^\infty(\mathbb{R}^d))$.

We point out that again more additional assumptions on the kernel K and the initial data u_0 are needed to prove the existence of such solutions. Solutions of this form have been obtained for particular cases of K in [135, 50]. Moreover, stability estimates, leading in particular to uniqueness of solutions, are obtained under convexity assumptions on the kernel K in [50]. Here, we will assume the existence of bounded weak solutions for the three models introduced in the previous section with diffusion. The existence theory seems a challenging problem to be tackled in the PKS system. The main theorem for these models with diffusion can be summarized as:

Theorem C.2. *Let u_1, u_2 be two bounded weak solutions of equation* (C.1) *in the interval $[0,T]$ with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ and assume that either:*

- *v is given by $v = -\nabla K * u$, with K such that $K \in C^2(\mathbb{R}^d)$ and $|D^2K| \in L^\infty(\mathbb{R}^d)$.*
- *v is given by $v = -\nabla K * u$, with K such that $\nabla K \in L^2(\mathbb{R}^d)$ and $|D^2K| \in L^1(\mathbb{R}^d)$.*
- *$v = -\nabla \Gamma_N * u$.*

Then $u_1(t) = u_2(t)$ for all $0 \le t \le T$.

Proof.- Given the two bounded weak solutions to (C.1), let us consider that the solutions of the SDE systems:

$$dX_i(t) = v(t, X_i(t))\, dt + \sqrt{2} dW_t$$

with initial data $X_1(0) = X_2(0)$ a random variable with law u_0, are constructed based upon the same Wiener process as in [50], see also [172, Chapter 2]. Then, the stochastic process $X_1(t) - X_2(t)$ follow a deterministic equation:

$$\frac{d}{dt}(X_1(t) - X_2(t)) = v(t, X_1(t)) - v(t, X_2(t)).$$

Therefore, the quantity used in this case will be

$$Q(t) := \frac{1}{2}\,\mathbb{E}\left[|X_1(t) - X_2(t)|^2\right]. \tag{C.2}$$

It is easy also to check that $W_2^2(u_1(t), u_2(t)) \le 2Q(t)$ by defining an admissible plan π transporting $u_1(t)$ to $u_2(t)$ by

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) \mathrm{d}\pi(x,y) = \mathbb{E}\left[\varphi(X_1(t), X_2(t))\right]$$

for all $\varphi \in C_b(\mathbb{R}^d)$. This plan has the right marginals since the law of $X_i(t)$ is given by $u_i(t,x)$ meaning that

$$\int_{\mathbb{R}^d} \varphi(x) u_i(t,x)\,\mathrm{d}x = \mathbb{E}\left[\varphi(X_i(t))\right].$$

It is clear then that $Q(t) \equiv 0$ would imply that $X_1(t) = X_2(t)$, and thus their laws $u_1 = u_2$. With this new quantity the proof now follow exactly the same steps as in Theorem 2.22. We make a quick summary of the new ingredients to consider. We first compute the time derivative of $Q(t)$ as

$$\frac{dQ}{dt} = \mathbb{E}\left[\langle X_1 - X_2, v_1(X_1) - v_1(X_2)\rangle\right] + \mathbb{E}\left[\langle X_1 - X_2, v_1(X_2) - v_2(X_2)\rangle\right],$$

with abuse of notation since an integrated in time version of it would give full rigor. Now, the proof of the smooth case can be really copied directly to this case by replacing integration with respect to the measure u_0 by expectations. The second and third cases can be also adapted by using the following ingredients:

1. The interpolation results in Propositions B.12 and B.11 can be used by realizing:

   $$I(t) = \mathbb{E}\left[|\nabla K * (u_1 - u_2)\left[X_2(t)\right]|^2\right] = \int_{\mathbb{R}^d} |\nabla K * (u_1 - u_2)(x)|^2 u_2(x)\,\mathrm{d}x$$

   since $u_2(t,x)$ is the law of $X_2(t)$.

2. In the case of the PKS system, one of the ingredients used by G. Loeper in his proof in [126] was the continuity in time of the solutions of the ODE system for small time. This step was not detailed in the previous section since the proof coincided with the one in [126]. This needed continuity can be also proved in the present case since being the two SDE systems solved with the same Brownian motion, then

   $$\frac{d}{dt}(X_1(t) - X_2(t)) = u_1(t, X_1(t)) - u_2(t, X_2(t)).$$

   Using that under the assumptions of bounded densities the velocity fields are bounded and Log-Lipschitz and since the initial data is the same, we deduce $|X_1(t) - X_2(t)| \leq Ct$ for all $0 \leq t \leq T$ a.e. in the probability space.

All the rest of the details are left to the interested reader. □

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Bibliography

[1] Ambrosio, L. A.; Gigli, N.; Savarè, G. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics, Birkhäuser, 2005.

[2] Ancona, M. G. Finite-Difference Schemes for the Density-Gradient Equations. *Journal of Computational Electronics* **1** (2002), no. 3, 435–443.

[3] Aoki, I. A Simulation Study on the Schooling Mechanism in Fish. *Bulletin of the Japanese Society of Scientific Fisheries* **48** (1982), no. 8, 1081–1088.

[4] Arnold, A.; Markowich, P.; Toscani, G.; Unterreiter, A. On generalized Csiszár-Kullback inequalities. *Monatsh. Math.* **131** (2000), no. 3, 235–253.

[5] Arnold, A.; Markowich, P.; Toscani, G.; Unterreiter, A. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations* **26** (2001), no. 1-2, 43–100.

[6] Ballerini, M.; Cabibbo, N.; Candelier, R.; Cavagna, A.; Cisbani, E.; Giardina, L.; Lecomte, L.; Orlandi, A.; Parisi, G.; Procaccini, A.; Viale, M.; Zdravkovic, V. Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study. Preprint.

[7] Barbaro, A.; Einarsson, B.; Birnir, B.; Sigurthsson, S.; Valdimarsson, H.; Palsson, O. K.; Sveinbjornsson, S.; Sigurthsson, T. Modelling and simulations of the migration of pelagic fish. *ICES J. Mar. Sci.* **66** (2009), no. 5, 826 – 838.

[8] Barbaro, A.; Taylor, K.; Trethewey, P. F.; Youseff, L.; Birnir, B. Discrete and continuous models of the dynamics of pelagic fish: application to the capelin. Preprint.

[9] Benamou, J. D.; Brenier, Y. A computational Fluid Mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.* **84** (2000), 375–393.

[10] Benedetto, D.; Caglioti, E.; Carrillo, J. A.; Pulvirenti, M. A non-maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys.* **91** (1998), 979–990.

[11] Benedetto, D.; Caglioti, E.; Pulvirenti, M. A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér.* **31** (1997), 615–641.

[12] Bertozzi, A.; Brandman, J. Finite-time blow-up of L^∞-weak solutions of an aggregation equation. *Comm. Math. Sci.* **8** (2010), no. 1, 45–65. Special issue in honor of Andrew Majda's 60th birthday.

[13] Bertozzi, A.; Carrillo, J. A.; Laurent, T. Blowup in multidimensional aggregation equations with mildly singular interaction kernels. *Nonlinearity* **22** (2009), 683–710.

[14] Bertozzi, A.; Laurent, T. Finite-time blow-up of solutions of an aggregation equation in $\mathbb{R}^n$. *Comm. Math. Phys.* **274** (2007), 717–735.

[15] Bertozzi, A. L.; Huang, Y. Self-similar blow-up solutions to an aggregation equation, 2009. Preprint.

[16] Bertozzi, A. L.; Laurent, T.; Rosado, J. L^p theory for the aggregation equation, 2010. Preprint. To appear in Comm. Pur. Appl. Math.

[17] Biler, P.; Woyczyński, A. Global and Expoding Solutions for Nonlocal Quadratic Evolution Problems. *SIAM J. Appl. Math.* **59** (1998), 845–869.

[18] Birnir, B. An ODE model of the motion of pelagic fish. *J. Stat. Phys.* **128** (2007), 535–568.

[19] Blanchet, A.; Carrillo, J. A.; Masmoudi, N. Infinite Time Aggregation for the Critical Patlak-Keller-Segel model in $\mathbb{R}^2$. *Comm. Pure Appl. Math.* **61** (2008), no. 10, 1449–1481.

[20] Blanchet, A.; Dolbeault, J.; Perthame, B. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations* **44** (2006). 32 pp. (electronic).

[21] Bobylev, A. V. The theory of the nonlinear spatially uniform Boltzmann equation for Maxwellian molecules. *Sov. Sci. Rev. C.* **7** (1988), 111–233.

[22] Bodnar, M.; Velázquez, J. J. L. An integro-differential equation arising as a limit of individual cell-based models. *J. Differential Equations* **222** (2006), 341–380.

[23] Bogoliubov, N. N. *Problems of a dynamical theory in statistical physics*, *Studies in Statistical Mechanics*, vol. 1. Interscience, New York, 1962.

[24] Boi, S.; Capasso, V.; Morale, D. Modeling the aggregative behavior of ants of the species *Polyergus rufescens*. *Nonlinear Anal. Real World Appl.* **1** (2000), 163–176. Spatial heterogeneity in ecological models (Alcalá de Henares, 1998).

[25] Bonabeau, E.; Dorigo, M.; Theraulaz, G. *Swarm Intelligence: From Natural to Artificial Systems*. Oxford University Press, New York, 1999.

[26] Braun, W.; Hepp, K. The Vlasov Dynamics and Its Fluctuations in the 1/N Limit of Interacting Classical Particles. *Commun. Math. Phys.* (1977), 101–113.

[27] Brenier, Y. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44** (1991), 375–417.

[28] Brenner, M. P.; Constantin, P.; Kadanoff, L. P.; Schenkel, A.; Venkataramani, S. C. Diffusion, attraction and collapse. *Nonlinearity* **12** (1999), 1071–1098.

[29] Burger, M.; Capasso, V.; Morale, D. On an aggregation model with long and short range interactions. *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal* **8** (2007), 939–958.

[30] Burger, M.; Di Francesco, M. Large time behavior of nonlocal aggregation models with nonlinear diffusion. *Networks and Heterogenous Media* **3** (2008), no. 4, 749–785.

[31] Burger, M.; Di Francesco, M.; Dolak-Struß, Y. The Keller-Segel model for chemotaxis with prevention of overcrowding: linear vs. nonlinear diffusion. *SIAM J. Math. Anal.* **38** (2006), 1288–1315.

[32] Cañizo, J. A.; Carrillo, J. A.; Rosado, J. A well-posedness theory in measures for some kinetic models of collective motion. To appear in Math. Mod. Meth. Appl. Sci.

[33] Cáceres, M. J.; Toscani, G. Kinetic approach to long time behavior of linearized fast diffusion equations. *J. Stat. Phys.* **128** (2007), 883–925.

[34] Calvez, V.; Carrillo, J. A. Volume effects in the Keller-Segel model: energy estimates preventing blow-up. *J. Math. Pures Appl. (9)* **86** (2006), no. 2, 155–175.

[35] Calvez, V.; Corrias, L. The parabolic Keller-Segel model in $\mathbb{R}^2$. *Comm. Math. Sci.* **6** (2008), no. 2, 417–447.

[36] Camazine, S.; Deneubourg, J. L.; Franks, N. R.; Sneyd, J.; Theraulaz, G.; Bonabeau, E. *Self-Organization in Biological Systems*. Princeton University Press, 2003.

[37] Carrillo, J. A.; Cañizo, J. A.; Rosado, J. Collective Behavior of Animals: Swarming and Complex Patterns, 2010. Preprint. To appear in ARBOR.

[38] Carrillo, J. A.; Di Francesco, M.; Figalli, A.; Laurent, T.; Slepčev, D. Global-in-time weak measure solutions, finite-time aggregation and confinement for nonlocal interaction equations. Preprint. To appear in Duke.

[39] Carrillo, J. A.; D'Orsogna, M. R.; Panferov, V. Double milling in self-propelled swarms from kinetic theory. *Kinetic and Related Models* **2** (2009), 363–378.

[40] Carrillo, J. A.; Fornasier, M.; Rosado, J.; Toscani, G. Asymptotic Flocking Dynamics for the kinetic Cucker-Smale model. *SIAM J. Math. Anal.* (2010), 218–236.

[41] Carrillo, J. A.; Jüngel, A.; Markowich, P. A.; Toscani, G.; Unterreiter, A. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.* **133** (2001), no. 1, 1–82.

[42] Carrillo, J. A.; Laurençot, P.; Rosado, J. Fermi-Dirac-Fokker-Planck equation: well-posedness and long-time asymptotics. *J. Diff. Eq.* **247** (2009), 2209–2234.

[43] Carrillo, J. A.; McCann, R. J.; Villani, C. Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates. *Rev. Matematica Iberoamericana* **19** (2003), no. 3, 971–1018.

[44] Carrillo, J. A.; McCann, R. J.; Villani, C. Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Ration. Mech. Anal.* **179** (2006), 217–263.

[45] Carrillo, J. A.; Rosado, J. Uniqueness of bounded solutions to Aggregation equations by optimal transport methods. *Proceedings of the 5th European Congress of Mathematicians* (2009). To appear.

[46] Carrillo, J. A.; Rosado, J.; Salvarani, F. 1D nonlinear Fokker-Planck equations for fermions and bosons. *Appl. Math. Lett.* **21** (2008), no. 2, 148–154.

[47] Carrillo, J. A.; Toscani, G. Exponential convergence toward equilibrium for homogeneus Fokker-Planck-type equations. *Math. Meth. Appl. Sci.* **21** (1998), 1269–1286.

[48] Carrillo, J. A.; Toscani, G. Contractive probability metrics and asymptotic behavior of dissipative kinetic equations. *Riv. Mat. Univ. Parma* **6** (2007), 75–198.

[49] Carter, A. H. *Classical ans Statistical Thermodynamics*. Prentice-Hall, Inc., New Jersey, 2001.

[50] Cattiaux, P.; Guillin, A.; Malrieu, F. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields* **140** (2008), 19–40.

[51] Chakraborti, A. Distributions of money in models of market economy. *Int. J. Modern Phys. C* **13** (2002), 1315–1321.

[52] Châu-Hoàn, L. Etude de la classe des opérateurs m-accrétifs de $L^1(\Omega)$ et accrétifs dans $L^\infty(\Omega)$. Tech. rep., Université de Paris VI, 1977. Thèse de $3^{\text{ème}}$ cycle.

[53] Chavanis, P. H.; Sire, C.; Sopik, J. Dynamics of the Bose-Einstein condensation: analoy with the collapse dynamics of a classical self-graviting Brownian gas. *Phys. Rev. E* **74** (2006), 011 112.1–011 112.15.

[54] Chuang, Y. L.; D'Orsogna, M. R.; Marthaler, D.; L., B. A.; Chayes, L. State transitions and the continuum limit for a 2D interacting, self-propelled particle system. *Physica D* **232** (2007), 33–47.

[55] Chuang, Y. L.; Huang, Y. R.; D'Orsogna, M. R.; Bertozzi, A. L. Multi-vehicle flocking: scalability of cooperative control algorithms using pairwise potentials. *IEEE International Conference on Robotics and Automation* (2007), 2292–2299.

[56] Cieślak, T. Global existence of solutions to a chemotaxis system with volume filling effect. *Colloq. Math.* **111** (2008), no. 1, 117–134.

[57] Cieślak, T.; Winkler, M. Finite-time blow-up in a quasilinear system of chemotaxis. *Nonlinearity* **21** (2008), 1057–1076.

[58] Cohen-Tannoudji, C.; Diu, B.; Laloë, F. *Quantum Mechanics*. Hermann and John Wiley & Sons, Inc., 1977.

[59] Corrias, L.; Perthame, B. Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces. *Mathematical and Computer Modelling* (2008), no. 47, 755–764.

[60] Corrias, L.; Perthame, B.; Zaag, H. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. *Milan J. Math.* **72** (2004), 1–28.

[61] Couzin, I. D.; Krause, J.; Franks, N. R.; Levin, S. A. Effective leadership and decision making in animal groups on the move. *Nature* **433** (2005), 513–516.

[62] Couzin, I. D.; Krause, J.; James, J.; Ruxton, G.; Franks, N. Collective memory and spatial sorting in animal groups. *Journal of Theoretical Biology* **218** (2002), no. 1, 1–11.

[63] Cucker, F.; Smale, S. Emergent behavior in flocks. *IEEE Trans. Automat. Control* **52** (2007), 852–862.

[64] Cucker, F.; Smale, S. On the mathematics of emergence. *Japan. J. Math.* **2** (2007), 197–227.

[65] D., H.; Winkler, M. Boundedness vs. blow-up in a chemotaxis system. *J. Differential Equations* **215** (2005), no. 1, 52–107.

[66] Degond, P.; Motsch, S. Macroscopic limit of self-driven particles with orientation interaction. *C.R. Math. Acad. Sci. Paris* **345** (2007), no. 10, 555–560.

[67] Dellacherie, C.; Meyer, P. A. *Probabilités et Potentiel*. Hermann, Paris, 1975.

[68] Di Francresco, M.; Rosado, J. Fully parabolic Keller-Segel model with prevention of overcrowding. *Nonlinearity* **21** (2008), 2715–2730.

[69] Dobrushin, R. Vlasov equations. *Funct. Anal. Appl.* **13** (1979), 115–123.

[70] Dolak, Y.; Schmeiser, C. The Keller-Segel Model with logistic sensitivity function and small diffusivity. *SIAM J. Appl. Math.* **66** (2005), no. 1, 286–308.

[71] Dolbeault, J. Kinetic models and quantum effects: a modified Boltzmann equation for Fermi-Dirac particles. *Arch. Rational Mech. Anal.* **127** (1994), 101–131.

[72] Dolbeault, J.; Perthame, B. Optimal critical mass in the two-dimensional Keller-Segel model in $\mathbb{R}^2$. *C. R. Math. Acad. Sci. Paris* **339** (2004), no. 9, 611–616.

[73] D'Orsogna, M. R.; Chuang, Y. L.; Bertozzi, A. L.; Chayes, L. Self-propelled particles with soft-core interactions: patterns, stability, and collapse. *Phys. Rev. Lett.* **96** (2006), 104 302–1/4.

[74] Dragulescu, A.; Yakovenko, V. M. Statistical mechanics of money. *Eur. Phys. Jour. B* **17** (2000), 723–729.

[75] Du, Q.; Zhang, P. Existence of weak solutions to some vortex density models. *Siam J. Math. Anal.* **34** (2003), no. 6, 1279–1299.

[76] Eftimie, R.; de Vries, G.; Lewis, M. A. Complex spatial group patterns result from different animal communication mechanisms. *Proceedings of the National Academy of Sciences* **104** (2007), 6974–6979.

[77] Escobedo, M.; Mischler, S. On a quantum Boltzmann equation for a gas of photons. *J. Math. Pures Appl.* **80** (2001), 471–515.

[78] Escobedo, M.; Zuazua, E. Large time behavior for convection-diffusion equations in $\mathbb{R}^N$. *J. Funct. Anal.* **100** (1991), 119–161.

[79] Evans, L. C. *Partial Differential Equations*. Graduate Studies in Mathematics, AMS, 1998.

[80] Fokker, A. D. Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld. *Ann. Physik* **43** (1914), 810–820.

[81] Frank, T. D. Classical Langevin equations for the free electron gas and blackbody radiation. *J. Phys. A* **37** (2004), 3561–3567.

[82] Frank, T. D. *Nonlinear Fokker-Planck equations*. Springer Series in Synergetics, Springer, 2005.

[83] Frensley, W. Scharfetter-Gummel Discretization Scheme for Drift-Diffusion Equations, 2004. Preprint.

[84] Galam, S.; Gefen, Y.; Shapir, Y. Sociophysics: a new approach of sociological collective behavior. *J. Math. Sociology* **9** (1982), 1–13.

[85] Gallay, T.; Wayne, C. E. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$. *Arch. Ration. Mech. Anal.* **163** (2002), 209–258.

[86] Gangbo, W.; McCann, R. J. The geometry of optimal transportation. *Acta Math.* **177** (1996), 113–161.

[87] Gazi, V.; Passino, K. Stability analysis of swarms. *IEEE Trans. Auto. Control* **48** (2003), 692–697.

[88] Golse, F. The Mean-Field Limit for the Dynamics of Large Particle Systems. *Journées équations aux dérivées partielles* **9** (2003), 1–47.

[89] Grégoire, G.; Chaté, H. Onset of collective and cohesive motion. *Phy. Rev. Lett.* **92** (2004).

[90] Ha, S. Y.; Liu, J. G. A simple proof of the Cucker-Smale flocking dynamics and mean-field limit. *Comm. Math. Sci.* **7** (2009), 297–325.

[91] Ha, S. Y.; Tadmor, E. From particle to kinetic and hydrodynamic descriptions of flocking. *Kinetic and Related Models* **1** (2008), 415–435.

[92] Hauray, M.; Jabin, P.-E. N-particles approximation of the Vlasov equations with singular potential. *ARch. Ration. Mech. Anal.* **183** (2007), 489–524.

[93] Hemelrijk, C. K.; Hildenbrandt, H. Self-Organized Shape and Frontal Density of Fish Schools. *Ethology* **114** (2008), 245–254.

[94] Hemelrijk, C. K.; Kunz, H. Density distribution and size sorting in fish schools: an individual-based model. *Behavioral Ecology* **16** (2005), 178–187.

[95] Herrero, M. A.; Velázquez, J. J. L. Singularity patterns in a chemotaxis model. *Math. Ann.* **306** (1996), no. 3, 583–623.

[96] Hildenbrandt, H.; Carere, C.; Hemelrijk, C. K. Self-organised complex aerial displays of thousands of starlings:a model. Preprint.

[97] Hillen, T.; Painter, K. Global existence for a parabolic chemotaxis model with prevention of overcrowding. *Adv. in Appl. Math.* **26** (2001), no. 4, 280–301.

[98] Hillen, T.; Painter, K.; Schmeiser, C. Global existence for chemotaxis with finite sampling radius. *Discrete Contin. Dyn. Syst. Ser. B* **7** (2007), no. 1, 125–144 (electronic).

[99] Holm, D.; Putkaradze, V. Aggregation of finite size particles with variable mobility. *Phys. Rev. Lett.* **95** (2005), no. 22.

[100] Holm, D.; Putkaradze, V. Formation of clumps and patches in self-aggregation of finite-size particles. *Physica D.* **220** (2006), no. 2, 183–196.

[101] Horstmann, D. From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, Part I. *Jahresbericht der DMV* **105** (2003), no. 3, 103–165.

[102] Horstmann, D. From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, Part II. *Jahresbericht der DMV* **106** (2004), no. 2, 51–69.

[103] Huth, A.; Wissel, C. The Simulation of the Movement of Fish Schools. *Journal of Theoretical Biology* **156** (1992), 365–385.

[104] Jadbabaie, A.; Lin, J.; Morse, A. S. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. on Autom. Control* **48** (2003), 988–1001.

[105] Jäger, W.; Luckhaus, S. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.* **329** (1992), no. 2, 819–824.

[106] Kaniadakis, G. Generalized Boltzmann equation describing the dynamics of bosons and fermions. *Physics leters A* **203** (1995), 229–234.

[107] Kaniadakis, G. H-theorem and generalized entropies within the framework of nonlinear kinetics. *Phys. Lett. A* **288** (2001), 283–291.

[108] Kaniadakis, G.; Lapenta, G.; Quarati, P. Stochastic evolution of systems of particles obeying an exclusion principle. *Phys. A* **225** (1996), 323–335.

[109] Kaniadakis, G.; Quarati, P. Kinetic equation for classical particles obeying an exclusion principle. *Phys. Rev. E* **48** (1993), 4263–4270.

[110] Keller, E. F.; Segel, L. A. Initiation of Slime Mold Aggregation Viewed as an Instability. *J. Theor. Biol.* **26** (1970), 399–415.

[111] Koch, A. L.; White, D. The social lifestyle of myxobacteria. *Bioessays* **20** (1998).

[112] Kowalczyk, R. Preventing blow-up in a chemotaxis model. *J. Math. Anal. Appl.* **305** (2005), no. 2, 566–588.

[113] Kunz, H.; Hemelrijk, C. K. Artificial fish schools: collective effects of school size, body size, and body form. *Artificial Life* **9** (2003), no. 3, 237–253.

[114] Kunz, H.; Züblin, T.; Hemelrijk, C. K. On prey grouping and predator confusion in artificial fish schools. *Proceedings of the Tenth International Conference of Artificial Life. MIT Press* (2006). Cambridge, Massachusetts.

[115] Ladyzenscaja, O. A.; Solonnikov, V. A.; Uralceva, N. N. *Linear and quasi-linear equations of parabolic type*. AMS, 1968.

[116] Laurent, T. Local and Global Existence for an Aggregation Equation. *Communications in Partial Differential Equations* **32** (2007), 1941–1964.

[117] Lemarié-Rieusset, P. G. *Recent developments in the Navier-Stokes problem*. Chapman & Hall-CRC, Boca-Raton, 2002.

[118] Levine, H.; Rappel, W.; Cohen, I. Self-organization in systems of self-propelled particles. *Phys. Rev. E* **63** (2000), 017 101–1/4.

[119] Li, D.; Rodrigo, J. Finite-time singularities of an aggregation equation in $\mathbb{R}^n$ with fractional dissipation. *Comm. Math. Phys.* **287** (2009), no. 2, 687–703.

[120] Li, D.; Rodrigo, J. Refined blowup criteria and nonsymmetric blowup of an aggregation equation. *Advances in Mathematics* **220** (2009), no. 1, 1717–1738.

[121] Li, D.; Zhang, X. On a nonlocal aggregation model with nonlinear diffusion, 2008. Preprint.

[122] Li, H.; Toscani, G. Long-time asymptotics of kinetic models of granular flows. *Arch. Ration. Mech. Anal.* **172** (2004), 407–428.

[123] Li, Y. X.; Lukeman, R.; Edelstein-Keshet, L. Minimal mechanisms for school formation in self-propelled particles. *Physica D* **237** (2008).

[124] Lieb, E. H.; Loss, M. *Analysis*. Graduate Studies in Mathematics, AMS, 2001.

[125] Lin, F.; Zhang, P. On the hydrodynamic limit of Ginzburg-Landau vortices. *Discrete Contin. Dynam. Systems* **6** (2000), 121–142.

[126] Loeper, G. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *J. Math. Pures Appl.* **86** (2006), 68–79.

[127] Lu, X.; Wennberg, B. On stability and strong convergence for the spatially homogeneous Boltzmann equation for Fermi-Dirac particles. *Arch. Ration. Mech. Anal.* **168** (2003), 1–34.

[128] Lukeman, R.; Li, Y. X.; Edelstein-Keshet, L. A conceptual model for milling formations in biological aggregates. *Bull Math Biol.* **71** (2008), 352–382.

[129] Majda, A.; Bertozzi, A. L. *Vorticity and Incompressible Flow*. Cambridge Texts in Applied Mathematics, Cambridge University Press, United Kingdom, 2002.

[130] Marchioro, C.; Pulvirenti, M. *Mathematical theory of incompressible nonviscous fluids*, *Applied Mathematical Sciences*, vol. 96. Springer-Verlag, New York, 1994.

[131] Matthes, D.; Toscani, G. Analysis of a model for wealth redistribution. *Kinetic and related Models* **1** (2008), 1–22.

[132] McCann, R. J. A convexity principle for interacting gases. *Adv. Math.* **128** (1997), 153–179.

[133] McCartin, B. J. Seven Deadly Sins of Numerical Computation. *The American Mathematical Monthly* **105** (1998), no. 10, 929–941.

[134] McNamara, S.; Young, W. R. Kinetics of a one-dimensional granular medium in the quasi-elastic limit. *Phys. Fluids A* **5** (1993), 34–45.

[135] Méléard, S. Asymptotic behaviour of some interacting particle systems: McKean-Vlasov and Boltzmann models. *Lecture Notes in Mathematics* **1627** (1995), 42–95. Probabilistic Models for Nonlinear Partial Differential Equations, Talay and Tubaro Editors.

[136] Mogilner, A.; Edelstein-Keshet, L. A non-local model for a swarm. *J. Math. Bio.* **38** (1999), 534–570.

[137] Mogilner, A.; Edelstein-Keshet, L.; Bent, L.; Spiros, A. Mutual interactions, potentials, and individual distance in a social aggregation. *J. Math. Biol.* **47** (2003), 353–389.

[138] Morale, D.; Capassso, V.; Oelschläger, K. An interacting particle system modelling aggregation behavior: from individuals to populations. *J. Math. Biol.* **50** (2005), 49–66.

[139] Mouhot, C.; Neumann, L. Quantitative perturbative study of convergence to equlibrium for collisional kinetic models in the torus. *Nonlinearity* **19** (2006), 969–998.

[140] Nagai, T. Blow-up of radially symmetric solutions to a chemotaxis system. *Adv. Math. Sci. Appl.* **5** (1995), 581–601.

[141] Neumann, L.; Schmeiser, C. Convergence to global equilibrium for a kinetic model for fermions. *SIAM J. Math. Anal.* **36** (2005), 1652–1663.

[142] Neumann, L.; Sparber, C. Stability of Steady States in kinetic Fokker-Planck equations for bosons and fermions. *Comm. Math. Sci.* **5** (2007), no. 4, 765–777.

[143] Neunzert, H. The Vlasov equation as a limit of Hamiltonian classical mechanical systems of interacting particles. *Trans. Fluid Dynamics* **18** (1977), 663–678.

[144] Neunzert, H. *An introduction to the nonlinear Boltzmann-Vlasov equation, Lecture Notes in Math.*, vol. 1048. Springer, Berlin, 1984. **????????**Kinetic theories and the Boltzmann equation, (Montecatini, 1981), pp. 60-110.

[145] O'Hara, P. Rotational Invariance and the Spin-Statistics Theorem. *Foun. Phys.* **33** (2003), 1349–1368.

[146] Painter, K.; Hillen, T. Volume-Filling and Quorum Sensing in Models for Chemosensitive Movement. *Canadian Applied Mathematics Quaterly* **10** (2003), no. 4, 280–301.

[147] Parrish, J.; Edelstein-Keshet, L. Complexity, pattern, and evolutionary trade-offs in animal aggregation. *Science* **294** (1999), 99–101.

[148] Patlak, C. S. Random walk with persistence and external bias. *Bull. of Math. Biophys.* **15** (1953), 311–338.

[149] Pauli, W. The Connection Between Spin and Statistics. *Phys. Rev.* **58** (1940), 716–722.

[150] Perea, L.; Gómez, G.; Elosegui, P. Extension of the Cucker-Smale control law to space flight formations. *AIAA Journal of Guidance, Control, and Dynamics* **32** (2009), 527–537.

[151] Perthame, B. PDE models for chemotactic movements: parabolic, hyperbolic and kinetic. *Appl. Math.* **49** (2004), no. 6, 539–564.

[152] Planck, M. *Sitzber. Preuss. Akad. Wiss.* (1917), 324–.

[153] Povzner, A. Y. The Boltzmann equation in kinetic theory of gases. *Amer. Math. Soc. Transl. Ser. 2* **47** (1962), 193–216.

[154] Protter, M. H.; Weinberger, H. F. *Maximum Principles in Differential Equations*. Spinger, New York, 1984.

[155] Risken, H. *The Fokker-Planck Equation: Methods of Solution and Applications*. Springer-Verlag, 1984.

[156] Rossani, A.; Kaniadakis, G. A generalized quasi-classical Boltzmann equation. *Physica A* **277** (2000), 349–358.

[157] Ruelle, D. *Statistical mechanics: Rigorous results*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[158] Schneirla, T. C. *Army Ants: A Study in Social Organization*. W.H. Freeman, 1971.

[159] Shen, J. Cucker-Smale Flocking under Hierarchical Leadership. *SIAM J. Appl. Math.* **68** (2008), 694–719.

[160] Spohn, H. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.* **52** (1980), 569–615.

[161] Spohn, H. *Large scale dynamics of interacting particles*. Texts and Monographs in Physics, Springer, 1991.

[162] Sznajd-Weron, K.; Sznajd, J. Opinion evolution in closed community. *Int. J. Mod. Phys. C* **11** (2000), 1157–1165.

[163] Topaz, C. M.; Bertozzi, A. L. Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM J. Appl. Math.* **65** (2004), 152–174.

[164] Topaz, C. M.; Bertozzi, A. L.; Lewis, M. A. A nonlocal continuum model for biological aggregation. *Bulletin of Mathematical Biology* **68** (2006), no. 7, 1601–1623.

[165] Toscani, G. Remarks on entropy and equilibrium states. *Appl. Math. Lett.* **12** (1999), 19–25.

[166] Toscani, G. One-dimensional kinetic models of granular flows. *RAIRO Modél. Math. Anal. Numér.* **34** (2000), no. 6, 1277–1291.

[167] Toscani, G. Kinetic and hydrodynamic models of nearly elastic granular flow. *Monatsh. Math.* **142** (2004), 179–192.

[168] Toscani, G. Kinetic models of opinion formation. *Comm. Math. Sci.* **4** (2006), no. 3, 481–496.

[169] Vázquez, J. L. *The Porous Medium Equation: Mathematical Theory*. Oxford Science Publications, New York, 2007.

[170] Vicsek, T.; Cziròk, A.; Ben-Jacob, E.; Cohen, I.; Shochet, O. Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.* **75** (1995), 1226–1229.

[171] Villani, C. *Topics in optimal transportation*, *Graduate Studies in Mathematics*, vol. 58. American Mathematical Society, Providence, RI, 2003.

[172] Villani, C. *Optimal transport, old and new*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 338. Springer-Verlag, Berlin, 2009.

[173] Viscido, S. V.; Parrish, J. K.; Grünbaum, D. Individual behavior and emergent properties of fish schools: a comparison of observation and theory. *Marine Ecology Progress Series* **273** (2004), 239–249.

[174] Viscido, S. V.; Parrish, J. K.; Grünbaum, D. The effect of population size and number of influential neighbors on the emergent properties of fish schools. *Ecological Modelling* **183** (2005), 347–363.

[175] Vishik, M. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. scient. Éc. Norm. Sup.* (1999), 769–812. 4^e série, t. 32.

[176] Yates, C.; Erban, R.; Escudero, C.; Couzin, L.; Buhl, J.; Kevrekidis, L.; Maini, P.; Sumpter, D. Inherent noise can facilitate coherence in collective swarm motion. *Proceedings of the National Academy of Sciences* (2009), no. 14, 5464–5469.

[177] Youseff, L. M.; Barbaro, A. B. T.; Trethewey, P. F.; Birnir, B.; Gilbert, J. Parallel modeling of fish interaction. *11th IEEE International Conference on Computational Science and Engineering* (2008), 234–241.

[178] Yudovich, V. I. Non-stationary flow of an incompressible liquid. *Zh. Vychisl. Mat. Mat. Fiz* **3** (1963), 1032–1066.