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Smooth sets and two problems in the Dirichlet space

Author:
Daniel SECO FORSNACKE

Advisor:
Dr. Artur NICOLAU NOS

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Declaration

Daniel Seco Forsnacke wrote this thesis for the award of Doctor of Philosophy under the supervision of Artur Nicolau Nos.

Daniel Seco Forsnacke | Dr. Artur Nicolau Nos

"It's not the mountain we conquer, but ourselves."

Edmund Hillary

To the memory of my brother, Pablo, who made me do this.

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Contents

Chapter 1

Introduction

This thesis consists of two clearly separate parts. The first deals with so called smooth sets and the second with certain properties of sampling sequences and cyclic vectors in Dirichlet-type spaces.

Throughout the text, the *density* of a set A in a set Q in $\mathbb{R}^n$ will refer to the quotient $D(Q) = |A \cap Q|/|Q|$, where $|\cdot|$ is the Lebesgue measure. In Chapter 2, we will study smoothness of sets, in the sense inherent to the work by Kahane and formally introduced by Hungerford:

Definition 1.0.1. A measurable set $A \subset \mathbb{R}^n$ is called *smooth* (in $\mathbb{R}^n$) if

$$\lim_{\delta \to 0} \sup |D(Q) - D(Q')| = 0$$

where the supremum is taken over all pairs of cubes Q, Q' whose sides are parallel to the axes with $l(Q) = l(Q') \leq \delta$ and with one face in common.

Sets A with $|A| = 0$ or $|\mathbb{R} \backslash A| = 0$ are trivially smooth but, as shown by Kahane (see [24]), there are examples of other nontrivial smooth sets. In dimension $n = 1$, this notion was studied by Hungerford, who, in his PhD Thesis (see [22], see also [26]) proved the following result:

Theorem 1.0.2 (Hungerford, 1988.). *If $A \subset \mathbb{R}$ is a nontrivial smooth set the Hausdorff* *dimension of its boundary is 1.*

Chapter 2 is based on our paper [27]. With the notation that $Q(x,h)$ is the cube centered at x and of sidelength h, the main result of the chapter is the following sharpening and generalization of Hungerford's result:

Theorem 1.0.3. *Let A be a smooth set in $\mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \backslash A| > 0$. Fix $0 < \alpha < 1$. Then the set*

$$E(A,\alpha) = \left\{ x \in \mathbb{R}^n : \lim_{h \to 0} D(Q(x,h)) = \alpha \right\}$$

has Hausdorff dimension n.

In its proof, we use a dyadic decomposition of the ambient space and stopping-time techniques, to construct a Cantor-type set, contained in $E(A,\alpha)$. The good averaging properties of the density are used to estimate the dimension of the Cantor set.

We also show that bilipschitz mappings with uniformly continuous Jacobian preserve the smoothness of a set and that this is not true without the assumption on the Jacobian. As a consequence of this preservation, the definition of smoothness can be based on other families of cubes.

In chapter 3 we study sampling sequences from the Dirichlet space onto a space of sequences ℓ^2_Δ, defined ad hoc. Sampling and interpolation phenomena have been studied broadly, becoming a classical problem in analytic function space analysis. For other spaces, a good reference is [34]. The Dirichlet space D is the space of all analytic functions f on the unit disk $\mathbb{D}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ whose Taylor coefficients satisfy

$$\|f\|_D^2 = \sum_{k=0}^{\infty} (k+1)|a_k|^2 < \infty.$$

Consider a space of sequences ℓ and a sequence Z of points in $\mathbb{D}$ such that for all $f \in D$, $R_Z(f) = \{f(z_n)\}_{n \in \mathbb{N}}$ is an element of ℓ. Then the operator $R_Z : D \to \ell$ is well defined. We call R_Z the *restriction operator* induced by Z from D into ℓ. We say that a sequence $Z \subset \mathbb{D}$ is a *sequence of interpolation* (from D into ℓ) if the restriction operator is bounded and surjective. Sequences of interpolation from D into a certain weighted space of sequences $\ell^2(\omega)$ have been studied and characterized in the work by Marshall and Sundberg ([25]), Bishop ([5]) and Boe ([7] and [8]).

Analogously, we say that $Z \subset \mathbb{D}$ is a *sampling sequence* (from D into ℓ) if there exist positive constants C_1 and C_2 such that, for all $f \in D$ we have:

$$C_1 ||f||_D^2 \leq ||R_Z(f)||_\ell^2 \leq C_2 ||f||_D^2 \tag{1.0.1}$$

It is well known (see, for instance, [32]) that the Dirichlet norm can also be defined by:

$$||f||_D^2 = |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{z}w|^4} dA(z) dA(w)$$

Our aim in this chapter is to study sampling sequences from D into a space, ℓ_Δ^2, that is constructed from a discrete version of this expression of the norm. ℓ_Δ^2 is defined as the space of those sequences $W = \{w_n\}_{n \in \mathbb{N}}$ of complex numbers with:

$$||\{w_n\}||_{\ell_\Delta^2}^2 = |w_0|^2 + \sum_{n,k} |w_n - w_k|^2 (1 - \rho(z_n, z_k)^2)^2 < \infty$$

Here we study the conditions for a sequence to be sampling. Denote $D_H(z, R)$ the hyperbolic disk centered at z of radius R.

Theorem 1.0.4. *(a) Let $Z = \{z_n\}$ be a separated sequence. If Z is sampling then there exists a radius $R > 0$ such that, for all $z \in \mathbb{D}$, we have $D_H(z, R) \cap Z \neq \emptyset$.*

(b) There exists a constant $\varepsilon > 0$, with the property that if Z is a sequence of points in the unit disk and there exists $0 < R < \varepsilon$ such that for all $z \in \mathbb{D}$, we have $D_H(z, R) \cap Z \neq \emptyset$, then Z is a sampling sequence.

We do not arrive at a characterization. Another sufficient condition is given in terms of harmonic measure in champagne-type domains, following ideas introduced by Ortega-Cerdà and Seip in related contexts (see [28]). See also [1], [16] and [30]. A good reference about harmonic measure is [18].

Finally, chapter 4 is devoted to the joint work with Bénéteau, Condori, Liaw and Sola studying the phenomenom of cyclicity in Dirichlet-type spaces ([4]).

For $-\infty < \alpha < \infty$, D_α is the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $z \in \mathbb{D}$, whose Taylor coefficients satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty. \tag{1.0.2}$$

Three values of α correspond to spaces that have been studied extensively: $\alpha = -1$ corresponds with the Bergman space; $\alpha = 0$ with the Hardy space H^2; and $\alpha = 1$ is the standard Dirichlet space D, as defined above.

Definition 1.0.5. A function $f \in D_\alpha$ is called *cyclic* (in D_α) if $[f] = D_\alpha$, where $[f] = \overline{\text{span}\{z^k f \colon k = 0, 1, 2, \ldots\}}$, with closure in D_α norm.

It is well known that the cyclicity of a function f is equivalent with the existence of a sequence of polynomials $\{p_n\}$ satisfying:

$$\|p_n f - 1\|_\alpha^2 \to 0, \quad \text{as } n \to \infty.$$

Hence, cyclicity in D_α is a stronger condition than cyclicity in D_β for all $\beta < \alpha$.

Cyclic functions in the D_α spaces have been studied by numerous authors, but a characterization is known only for the Hardy case and for the cases $\alpha > 1$. Beurling proved that a function is cyclic in the Hardy space if and only if it is outer (see [17]). For Bergman spaces ($\alpha < 0$), stronger sufficient conditions are known but they are not sharp (see, for instance, [19]). For $\alpha > 1$, a function is cyclic in D_α if and only if it is bounded away from zero, as proved by Brown and Shields (see [11]). In the case of the Dirichlet space, the problem is more complicated. Brown and Shields proved a necessary condition, that the set of zeros of the function in the boundary, defined in a suitable manner, is of logarithmic capacity zero (see [11]). They conjectured that cyclic functions in D were exactly outer functions satisfying this condition. This problem has been studied by several authors (see [9], [10], [11], [14], [15], [20], [31] and the survey [33]), but it remains open.

It is easy to show that analytic functions in the closed unit disk without zeros in the open unit disk are cyclic in D_α for all $\alpha \leq 1$.

We are interested in the *optimal norm*

$$N_{n,\alpha}(f) = \inf ||pf - 1||_\alpha^2$$

where the infimum is taken over all polynomials p of degree less or equal to n. For $\alpha < 1$, we set $\varphi_\alpha(t) = t^{1-\alpha}$, $t \in \mathbb{N}$. In the case $\alpha = 1$, we take $\varphi_1(t) = \log(t)$, $t \in \mathbb{N}$. With this notation, the main result in this chapter is the following:

Theorem 1.0.6. *Fix $\alpha \leq 1$ and suppose $f \in D_\alpha$ can be extended analytically to a neighborhood of the closed unit disc. Suppose also that f does not vanish in $\mathbb{D}$. Then there exists a constant $C_0 = C_0(\alpha, f)$, such that the optimal norm satifies*

$$N_{n,\alpha}(f) \leq \frac{C_0}{\varphi_\alpha(n+1)}.$$

Moreover, if f has at least one zero on the boundary of $\mathbb{D}$, then there is a constant $C_1 = C_1(\alpha, f)$ such that

$$\frac{C_1}{\varphi_\alpha(n+1)} \leq N_{n,\alpha}(f).$$

To prove the first part of this result (the upper bound for $N_{n,\alpha}$), we start by considering the case of a polynomial f with simple zeros on $\partial\mathbb{D}$. We use generalized Riesz means to provide good candidates to polynomials p_n close to the sharp bounds for their degree. Then we make use of convolution and cancelation properties of the Taylor coefficients of a function f and its reciprocal $1/f$, to achieve a control of the norm of $p_n f - 1$. To extend the result to functions f analytic on the closed unit disk, we show a uniform control of the Wiener algebra norm of $p_n f$. For the sharpness, we use the Hilbert space structure of the D_α spaces to give necessary and sufficient conditions for the optimality of a polynomial in the sense of the definition of $N_{n,\alpha}$. We also construct explicitly the polynomials satisfying such conditions for the particular function $f(z) = 1 - z$, using generalized Riesz means, allowing to compute the optimal bounds and infere from these bounds, the sharpness for other functions.

The theorem is applicable to a small class of functions and it is therefore very far from the state of the art around conditions for cyclicity in Dirichlet-type spaces, but it provides insight about certain functions already known to be cyclic, in terms of control on the decay of $N_{n,\alpha}$.

In addition, we prove some results on sufficient conditions for cyclicity in D_α, studying the problem of whether $f \in D_\alpha$ will be cyclic whenever $\log f \in D_\alpha$. We observe that this is well known in the cases $\alpha > 1$ and $\alpha = 0$ and prove it for $\alpha = 1$. The remaining cases are solved only partially with a stronger hypothesis.

Chapter 2

Smooth sets in Euclidean spaces

2.1 Introduction

The Lebesgue Density Theorem tells us that the density of a measurable set approximates the characteristic function of the set at almost every point. We are going to study sets whose densities at small scales vary uniformly.

In this chapter, a cube will mean a cube in the Euclidean space $\mathbb{R}^n$ with sides parallel to the axes. Two cubes $Q, Q' \subset \mathbb{R}^n$ with the same sidelength $l(Q) = l(Q')$ are called consecutive if the intersection of their closures is one of their faces. Given a measurable set $A \subset \mathbb{R}^n$, let $|A|$ denote its Lebesgue measure and $D(Q)$ its density in a cube $Q \subset \mathbb{R}^n$, that is, $D(Q) = |A \cap Q|/|Q|$. A measurable set $A \subset \mathbb{R}^n$ is called *smooth* (in $\mathbb{R}^n$) if

$$\lim_{\delta \to 0} \sup |D(Q) - D(Q')| = 0$$

where the supremum is taken over all pairs of consecutive cubes Q, Q' with $l(Q) = l(Q') \leq \delta$. In dimension $n = 1$, this notion was introduced by Hungerford ([22]) in relation to the small Zygmund class. Actually, a set $A \subset \mathbb{R}$ is smooth if and only if its distribution function $g(t) = |A \cap (-\infty, t)|$ is in the small Zygmund class, or equivalently, the restriction of the Lebesgue measure to the set A is a smooth measure in the sense of Kahane ([24]).

Sets $A \subset \mathbb{R}^n$ with $|A| = 0$ or $|\mathbb{R}^n \backslash A| = 0$ are trivially smooth but a method of Kahane allowed Hungerford to provide non trivial examples. In fact, one can consider a sharper notion. Given a continuous increasing function $\omega : [0,1] \to [0,\infty)$ with $\omega(0) = 0$, a set $A \subset \mathbb{R}^n$ is called *ω-smooth* (in $\mathbb{R}^n$) if

$$|D(Q) - D(Q')| \leq \omega(l(Q))$$

for any pair of consecutive cubes $Q, Q' \subset \mathbb{R}^n$ of sidelength $l(Q) = l(Q')$. Notice that if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are ω-smooth, then $A \times B$ is ω-smooth in $\mathbb{R}^{n+m}$. The existence of nontrivial ω-smooth sets was discussed in [2]. Let $\omega : [0,1] \to [0,\infty)$ be a continuous increasing function with $\omega(0) = 0$. Assume that there exists $\varepsilon > 0$ such that the function

$$\omega(t)/t^{1-\varepsilon} \quad \text{is decreasing.} \tag{2.1.1}$$

Then there exists an ω-smooth set $A \subset \mathbb{R}$, with $|A| > 0$ and $|\mathbb{R}\backslash A| > 0$ if and only if the following quadratic condition holds:

$$\int_0 \omega^2(t)\frac{dt}{t} = \infty \tag{2.1.2}$$

See [2]. In Section 2, we will discuss a slight variant of this result. As in [2], the necessity of condition (2.1.2) follows easily from a standard martingale argument and does not use the hypothesis (2.1.1). Our proof of the sufficiency of (2.1.2) is based on the examples of smooth sets produced by Hungerford, who made use of a nice recursive construction by Kahane. Actually, in our approach, we substitute the hypothesis (2.1.1) by the related condition $\omega(4t) \leq 3\omega(t)/2, \quad t \in [0,1/4]$.

In dimension $n = 1$, Hungerford proved that the boundary of a nontrivial smooth set has full Hausdorff dimension ([22], see also [26]). His argument shows that if A is a smooth set in $\mathbb{R}$ with $|A| > 0$ and $|\mathbb{R}\backslash A| > 0$, then the set of points $x \in \mathbb{R}$ for which there exists a sequence of intervals $\{I_j\}$ containing x such that

$$\lim_{j\to\infty} D(I_j) = 1/2,$$

with $|I_j| \to 0$ as $j \to \infty$, still has Hausdorff dimension 1. The main goal of this paper is to sharpen this result and to extend it to higher dimensions. It is worth mentioning that Hungerford arguments cannot be extended to several dimensions since it is used that the image under a nontrivial linear mapping of an interval is still an interval, or more generally, that an open connected subset of the real line is an interval and this obviously does not hold for cubes in $\mathbb{R}^n$, for $n > 1$. Given a point $x \in \mathbb{R}^n$ and $h > 0$, let $Q(x,h)$ denote the cube centered at x of sidelength h. With this notation, our main result is the following:

Theorem 2.1.1. *Let A be a smooth set in $\mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \backslash A| > 0$. Fix $0 < \alpha < 1$. Then the set*

$$E(A,\alpha) = \left\{ x \in \mathbb{R}^n : \lim_{h \to 0} D(Q(x,h)) = \alpha \right\}$$

has Hausdorff dimension n.

Our result is local, meaning, given a cube $Q \subset \mathbb{R}^n$ with $0 < |A \cap Q| < |Q|$, we have that $E(A,\alpha) \cap Q$ has full Hausdorff dimension. As a consequence the Hausdorff dimension of $\partial A \cap Q$ is n.

Section 3 contains a proof of Theorem 2.1.1. A Cantor type subset of $E(A,\alpha)$ will be constructed and its dimension will be computed using a standard result. The generations of the Cantor set will be defined recursively by means of a stopping time argument. The good averaging properties of the density are used to estimate the dimension of the Cantor set.

The definition of smooth set concerns the behavior of the density of the set on the grid of cubes in $\mathbb{R}^n$ with sides parallel to the axes. We consider two natural questions arising from this definition. First, we study how the definition depends on the grid of cubes, that is, if other natural grids, such as dyadic cubes or general parallelepipeds would lead to the same notion. Second, we consider whether the class of smooth sets is preserved by regular mappings. It turns out that these questions are related and in Section 4 we provide a positive answer to both of them. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is bilipschitz if there exists a constant $C \geq 1$ such that $C^{-1}||x-y|| \leq ||\phi(x) - \phi(y)|| \leq C||x-y||$ for any $x, y \in \mathbb{R}^n$.

Theorem 2.1.2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz $\mathcal{C}^1$ mapping with uniformly continuous Jacobian. Let $A \subset \mathbb{R}^n$ be a measurable set. Then the following are equivalent:*

(a) *A is a smooth set*

(b) *$\phi^{-1}(A)$ is a smooth set*

(c) *A verifies the smoothness condition taking, instead of the grid of cubes, their images through ϕ, that is,*

$$\lim_{|Q|\to 0} \frac{|A \cap \phi(Q)|}{|\phi(Q)|} - \frac{|A \cap \phi(Q')|}{|\phi(Q')|} = 0$$

As part (c) states, one could replace in the definition of smooth set, the grid of cubes by other grids such as the grid of dyadic cubes or the grid of general parallelepipeds with bounded eccentricity whose sides are not necessarily parallel to the axes or even the pullback by ϕ of the grid of cubes. One can combine Theorems 2.1.1 and 2.1.2 to obtain the following:

Corollary 2.1.3. *Let A be a smooth set in $\mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \backslash A| > 0$. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz $\mathcal{C}^1$ mapping with uniformly continuous jacobian. Fix $0 < \alpha < 1$. Then the set*

$$\left\{ x \in \mathbb{R}^n : \lim_{h\to 0} \frac{|A \cap \phi(Q(x,h))|}{|\phi(Q(x,h))|} = \alpha \right\}$$

has Hausdorff dimension n.

To conclude, we show that the assumption on the Jacobian of the function in Theorem 2.1.2 is relevant, answering a question of X. Tolsa:

Theorem 2.1.4. *There exists a bilipschitz function from $\mathbb{R}$ to $\mathbb{R}$ that does not preserve smoothness.*

2.2 ω-smoothness

The main purpose of this section is to discuss the following result:

Theorem 2.2.1. *Let $\omega : [0, 1] \to [0, \infty)$ be a continuous increasing function with $\omega(0) = 0$.*

(a) Let there be an ω-smooth set A in $\mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \backslash A| > 0$. Then

$$\int_0 \omega^2(t)\frac{dt}{t} = \infty \tag{2.2.1}$$

(b) Assume $\omega(4t) \leq 3\omega(t)/2$ for $0 < t < 1/4$ and

$$\int_0 \omega^2(t)\frac{dt}{t} = \infty \tag{2.2.2}$$

Then there exists an ω-smooth set $A \subset \mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \backslash A| > 0$.

We begin with a description of the basic tools. Given a measurable set A contained in the unit cube of $\mathbb{R}^n$, let $D(Q)$ denote its density on the cube Q, that is, $D(Q) = |Q \cap A|/|Q|$. For $j = 1, 2...$ let $\mathfrak{D}_j$ be the family of the pairwise disjoint dyadic cubes Q of the form $Q = [k_1 2^{-j}, (k_1 + 1)2^{-j}) \times ... \times [k_n 2^{-j}, (k_n + 1)2^{-j})$ where $k_1, ..., k_n$ are integers. Let $Q_j(x)$ denote the unique dyadic cube in $\mathfrak{D}_j$ containing the point $x \in \mathbb{R}^n$. Fix a dyadic cube $Q \in \mathfrak{D}_{j_0}$ and consider the function f_j defined as $f_j(x) = D(Q_{j+1}(x)) - D(Q_j(x))$ for $j \geq j_0$, choose $x \in Q$ and define the square function

$$< D >_Q^2 (x) = \sum_{j \geq j_0} f_j^2(x), \quad x \in Q.$$

Observe that for any $j_0 \leq k \leq j$ and any $Q_k \in \mathfrak{D}_k$ we have the cancelation property $\int_{Q_k} f_j = 0$. We will need the following well known identity:

Lemma 2.2.2. ***(L^2 identity)*** *Let Q be a dyadic cube. Then*

$$\int_Q (\mathbb{1}_A(x) - D(Q))^2 dx = \int_Q < D >_Q^2 (x) dx$$

Proof. Since $\sum_{j \geq j_0} f_j = \mathbb{1}_A - D(Q)$ one only needs to check the orthogonality relation

$$\int_Q f_j f_k = 0, k \neq j \tag{2.2.3}$$

To see this, one may assume that $k < j$. Let $R \in \mathfrak{D}_{k+1}$. Since f_k is constant on R and $\int_R f_j = 0$ we deduce $\int_R f_j f_k = 0$ and (2.2.3) follows. □

Now we may proceed with the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. **(a)** We can assume that $0 < |A \cap [0,1]^n| < 1$. Take a large positive integer j_0, so that we can find cubes $Q_1, Q_2 \in \mathfrak{D}_{j_0}$ with $D(Q_1) < 1/4$ and $D(Q_2) > 3/4$ and such that $\omega(2^{-j_0}) \leq 1/2$. Since A is ω-smooth there exists a dyadic cube $Q \in \mathfrak{D}_{j_0}$ with $1/4 < D(Q) < 3/4$. Then

$$\int_Q (\mathbb{1}_A(x) - D(Q))^2 dx \geq \frac{|Q|}{16}$$

so the L^2 identity tells us that

$$\frac{|Q|}{16} \leq \int_Q < D >_Q^2 (x) dx \leq n|Q| \sum_{j \geq j_0} \omega^2(2^{-j})$$

Thus,

$$\sum_{j \geq j_0} \omega^2(2^{-j}) \geq \frac{1}{16n}$$

Since one can take j_0 as large as desired, one deduces $\sum \omega^2(2^{-j}) = \infty$ and condition (2.2.1) is satisfied.

(b) Assuming that the integral condition (2.2.2) holds, we want to construct a non trivial ω-smooth set. We start with the one dimensional case. The ω-smooth set $A \subset \mathbb{R}$ will be constructed using a method introduced by Hungerford in [22], based on a nice previous example by Kahane ([24]). For $j = 1, 2...$ consider the family $\mathfrak{Q}_j$ formed by the pairwise disjoint intervals $\{I_k\}$ of the form $I_k = [k4^{-j}, (k+1)4^{-j})$ where k is an integer. These intervals will be called the *quadriadic intervals of generation* j.

1. Construction of the set.

We can assume that $\omega(1) < 1/2$. Denote $\omega_j = \omega(4^{-j})/14$, $j = 1, 2, ...$ and observe that

$$\omega_{j+1} \geq 2\omega_j/3 \tag{2.2.4}$$

We will construct the ω-smooth set A via its density $D_j(I)$ on each quadriadic interval I of the generation j, meaning by density the proportion of the length of the interval lying on the set A. To do this, we define these densities D_j inductively. Pick, firstly, $D_0 \equiv 1/2$. Assume that for any quadriadic interval I of the j-th generation, the quantity $D_j(I)$ has been defined, in such a way that $0 \leq D_j(I) \leq 1$. We will define the next function D_{j+1}. Let $I \in \mathfrak{Q}_j$ and let $\{I_i : i = 1, ..., 4\}$ be the four intervals in $\mathfrak{Q}_{j+1}$ which are contained in I, ordered from left to right. In addition, let I' and I'' be the two intervals in $\mathfrak{Q}_j$ adjacent to I on the left and right hand side respectively. We first define a sign function ε_{j+1} which will be used later. Take $\varepsilon_{j+1}(I_1)$ to be the sign of $D_j(I') - D_j(I)$ and $\varepsilon_{j+1}(I_2) = -\varepsilon_{j+1}(I_1)$. Similarly, take $\varepsilon_{j+1}(I_4)$ to be the sign of $D_j(I'') - D_j(I)$ and $\varepsilon_{j+1}(I_3) = -\varepsilon_{j+1}(I_4)$. Notice that

$$\sum_{k=1}^{4} \varepsilon_{j+1}(I_k) = 0 \tag{2.2.5}$$

For $k = 1, ..., 4$ define the density on the interval I_k as

$$D_{j+1}(I_k) = D_j(I) + \min\{D_j(I), 1 - D_j(I), \omega_{j+1}\} \cdot \varepsilon_{j+1}(I_k) \tag{2.2.6}$$

Observe that, if $D_j(I)$ is zero or one, then $D_{j+1}(I_k) = D_j(I)$ for any $k = 1, ..., 4$. Also, whenever $D_j(I) \leq \omega_{j+1}$ (respectively, $D_j(I) \geq 1 - \omega_{j+1}$), two of the four intervals $\{I_k : k = 1, ..., 4\}$ will have density zero (respectively, one). Consider the step functions $D_j = \sum D_j(I)\mathbb{1}_I$ where the sum is taken over all intervals $I \in \mathfrak{Q}_j$. Condition (2.2.5) tells us that $\{D_j\}$ is a quadriadic martingale, that is, a discrete martingale with respect to the filtration generated by the quadriadic intervals. We will use the following Theorem proved by Doob:

$$\begin{aligned}\{x \in \mathbb{R} : \exists \lim_{j\to\infty} D_j(x)\} &\overset{a.e.}{=} \{x \in \mathbb{R} : \sup_j |D_j(x)| < \infty\} \overset{a.e.}{=} \\ &\overset{a.e.}{=} \{x \in \mathbb{R} : < D > (x) < \infty\}\end{aligned}$$

Here $A \overset{a.e.}{=} B$ means that the sets A and B coincide except for at most a set of Lebesgue measure zero. See, for instance, [35] (p. 65). Since $0 \leq D_j(x) \leq 1$, $x \in \mathbb{R}$, the martingale converges at almost every point, that is, for almost every $x \in \mathbb{R}$, the limit $\lim D_j(x)$ exists when $j \to \infty$. We next show that at almost every $x \in \mathbb{R}$, one has either $\lim D_j(x) = 0$ or $\lim D_j(x) = 1$ when $j \to \infty$. Actually, consider $B = \left\{ x \in \mathbb{R} : \lim_{j\to\infty} D_j(x)(1 - D_j(x)) \neq 0 \right\}$. Observe that, since ω_j tends to 0, for any $x \in B$ we have $|D_{j+1}(x) - D_j(x)| = \omega_{j+1}$ for j sufficiently large. Then the quadratic condition (2.2.2) tells us that $< D >^2 (x) = \infty$ for $x \in B$. Since $\{D_j(x)\}$ converges at almost every point $x \in \mathbb{R}$, we have $|B| = 0$. We define the set A to be

$$A = \left\{ x \in \mathbb{R} : \lim_{j\to\infty} D_j(x) = 1 \right\}$$

2. A is not trivial.

Next we check that $0 < |A \cap [0, 1]| < 1$. Let $I \in \mathfrak{Q}_j$ and observe that

$$\frac{|I \cap A|}{|I|} = \frac{1}{|I|} \lim_{k\to\infty} \int_I D_k(x)dx$$

The cancelation property (2.2.5) yields $\int_I D_k(x)dx = \int_I D_j(x)dx$ for any $k \geq j$. Hence

$$\frac{|I \cap A|}{|I|} = D_j(I)$$

and so we have $|A \cap [0, 1]| = D_0([0, 1]) \equiv 1/2$.

3. A is ω-smooth.

Finally, we prove the remainder, that is, A is ω-smooth. We start by computing the difference of densities on quadriadic intervals of the same generation and prove that for any pair of consecutive intervals $I, I' \in \mathfrak{Q}_j$, one has

$$\left|D_j(I) - D_j(I')\right| \leq 14\omega_j \leq w(4^{-j}) \tag{2.2.7}$$

If I and I' are contained in the same quadriadic interval of generation $j - 1$, (2.2.7) follows from the definition (2.2.6) of the density function. Otherwise let $\widetilde{I}, \widetilde{I}'$ be two different intervals in $\mathfrak{Q}_{j-1}$ with $I \subset \widetilde{I}, I' \subset \widetilde{I}'$. If $|D_{j-1}(\widetilde{I}) - D_{j-1}(\widetilde{I}')| \leq \omega_j$ the only way for I, I' to increase their difference in density compared to their parents $\widetilde{I}, \widetilde{I'}$ is for $D_{j-1}(\widetilde{I}) - D_{j-1}(\widetilde{I'})$ and $D_j(I) - D_j(I')$ to have different signs. In this case the difference is smaller than the sum of the two jumps from $D_{j-1}(\widetilde{I})$ to $D_j(I)$ and from $D_{j-1}(\widetilde{I'})$ to $D_j(I')$. Hence, (2.2.7) follows. If $|D_{j-1}(\widetilde{I}) - D_{j-1}(\widetilde{I'})| > \omega_j$ then either $D_{j-1}(\widetilde{I})$ or $D_{j-1}(\widetilde{I'})$ lies on the central band $(\omega_j, 1-\omega_j)$ and hence by the choice of the sign function ε_j one deduces that $|D_j(I) - D_j(I')| \le |D_{j-1}(\widetilde{I}) - D_{j-1}(\widetilde{I'})| - \omega_j$. Arguing by induction we can assume that $|D_{j-1}(\widetilde{I}) - D_{j-1}(\widetilde{I'})| \le 2\omega_{j-1}$ and together with (2.2.4) we get $|D_j(I) - D_j(I')| \le 2\omega_{j-1} - \omega_j \le 2\omega_j$ which is exactly (2.2.7).

We next apply (2.2.7) to estimate the difference of densities between arbitrary quadriadic intervals to obtain that if $I \in \mathfrak{Q}_{j+k}, I' \in \mathfrak{Q}_{j-1}$ with $\overline{I} \cap \overline{I'} \neq \emptyset$, then

$$|D_{j+k}(I) - D_{j-1}(I')| \le 2\omega_{j-1} + \sum_{l=j}^{j+k} \omega_l \le (4+k)\omega_j \tag{2.2.8}$$

Let I be an interval such that $4^{-j} \le |I| < 4^{-j+1}$ for some positive integer j. Decompose I into pairwise disjoint maximal quadriadic intervals. So, $I = \bigcup_{k\ge 0} A_{j+k}$, where A_{j+k} is a union of at most six intervals of $\mathfrak{Q}_{j+k}$. Fix $I' \in \mathfrak{Q}_{j-1}$ with $\overline{I} \cap \overline{I'} \neq \emptyset$. Then,

$$\big||I \cap A| - |I| D_{j-1}(I')\big| = \left| \sum_{k\ge 0} \sum_{J \in A_{j+k}} |J| (D_{j+k}(J) - D_{j-1}(I')) \right|$$

Applying now (2.2.8) we deduce

$$\big||I \cap A| - |I| D_{j-1}(I')\big| \le \omega_j \sum_{k\ge 0} \sum_{J \in A_{j+k}} |J| (4+k)$$

As we can deduce from the decomposition of I in terms of the intervals $J \in A_k$, this implies that

$$\big||I \cap A| - |I| D_{j-1}(I')\big| \le 7\omega_j |I| \tag{2.2.9}$$

Now, let I and I^* be two consecutive intervals of the same size $|I| = |I'| \le 1$. Pick a positive integer j such that $4^{-j} \le |I| < 4^{-j+1}$ and let $I' \in \mathfrak{Q}_{j-1}$ such that $I' \cap \overline{I} \neq \emptyset$ and

$I' \cap \overline{I^*} \neq \emptyset$. Applying (2.2.9) to both intervals, one gets

$$\big||I \cap A| - |I' \cap A|\big| \leq 14\omega_j |I|$$

Since $14\omega_j \leq \omega(|I|)$, A is an ω-smooth set in dimension $n = 1$. If A is an ω-smooth set in $\mathbb{R}$, then both $\mathbb{R}^{n-1} \times A$ and A^n are ω-smooth sets in $\mathbb{R}^n$. □

2.3 Proof of Theorem 2.1.1

2.3.1 Preliminary results

We begin with a preliminary result on the Hausdorff dimension of certain Cantor type sets which will be used in the proof of Theorem 2.1.1. In dimension $n = 1$, the result was given by Hungerford in [22]. See also Theorem 10.5 in [29]. The proof in the higher dimensional case only requires minor adjustments and it will be omitted.

Lemma 2.3.1. *For $s = 0, 1, 2, \ldots$ let $G(s)$ be a collection of closed dyadic cubes in $\mathbb{R}^n$ with pairwise disjoint interiors. Assume that the families are nested, that is, for every $Q \in G(s+1)$ there is $\widetilde{Q} \in G(s)$ with $Q \subset \widetilde{Q}$. Suppose that there exist two positive constants $0 < P < C < 1$ such that the following two conditions hold:*

(a) *For any cube $Q \in G(s+1)$ with $Q \subset \widetilde{Q} \in G(s)$ one has $|Q| \leq P|\widetilde{Q}|$.*

(b) *For any $\widetilde{Q} \in G(s)$ one has*

$$\sum |Q| \geq C|\widetilde{Q}|$$

where the sum is taken over all cubes $Q \in G(s+1)$ contained in $\widetilde{Q}$.

Let $E(s) = \bigcup Q$, where the union is taken over all cubes in $G(s)$ and $E \equiv \bigcap_{s=0}^{\infty} E(s)$. Then $\dim E \geq n(1 - \log_P C)$.

The next auxiliary result is the building block of the Cantor set on which the set has a fixed density.

Lemma 2.3.2. *Let A be an ω-smooth set of $\mathbb{R}^n$ with $0 < |A \cap [0,1]^n| < 1$. Let Q be a dyadic cube. Fix a constant $\varepsilon > 0$ such that $n\omega(l(Q)) < \varepsilon < \min\{D(Q), 1 - D(Q)\}$. Let $\mathfrak{A}(Q)$ be the family of maximal dyadic cubes Q_k contained in Q such that*

$$|D(Q_k) - D(Q)| \geq \varepsilon \tag{2.3.1}$$

Then:

(a) *for any $Q_k \in \mathfrak{A}(Q)$ one has*

$$|Q_k| \leq 2^{-\varepsilon/\omega(l(Q))}|Q|$$

(b) *let $\mathfrak{A}^+(Q)$ (respectively $\mathfrak{A}^-(Q)$) be the subfamily of $\mathfrak{A}(Q)$ formed by those cubes $Q_k \in \mathfrak{A}(Q)$ for which $D(Q_k) - D(Q) \geq \varepsilon$ (respectively $D(Q) - D(Q_k) \geq \varepsilon$); then*

$$\sum |Q_k| \geq |Q|/4$$

where the sum is taken over all the cubes $Q_k \in \mathfrak{A}^+(Q)$ (respectively $Q_k \in \mathfrak{A}^-(Q)$).

Proof. If $Q_1 \subset Q_2 \subset Q$ are two dyadic cubes with $l(Q_1) = l(Q_2)/2$ then $|D(Q_1) - D(Q_2)| \leq n\omega(l(Q))$. So if (2.3.1) holds, one deduces that $\log_2 l(Q_k)^{-1} \geq \log_2 l(Q)^{-1} + \varepsilon/n\omega(l(Q))$. Therefore, (a) is proved.

To prove (b), we observe first that, by Lebesgue Density Theorem, one has $\sum_{\mathfrak{A}(Q)} |Q_k| = |Q|$. Also,

$$\sum_{\mathfrak{A}(Q)} (D(Q_k) - D(Q))|Q_k| = 0 \tag{2.3.2}$$

We argue by contradiction. Assume that $\sum_{Q_k \in \mathfrak{A}^+(Q)} |Q_k| < |Q|/4$ and hence $\sum_{Q_k \in \mathfrak{A}^-(Q)} |Q_k| \geq 3|Q|/4$, which gives us

$$\sum_{Q_k \in \mathfrak{A}^-(Q)} (D(Q_k) - D(Q))|Q_k| \leq -3\varepsilon|Q|/4$$

The maximality of Q_k tells us that $|D(Q_k) - D(Q)| \leq \varepsilon + n\omega(l(Q)) < 2\varepsilon$. Therefore

$$\sum_{Q_k \in \mathfrak{A}^+(Q)} (D(Q_k) - D(Q))|Q_k| \leq \varepsilon|Q|/2$$

which contradicts (2.3.2). The same argument works for $\mathfrak{A}^-(Q)$. □

2.3.2 The dyadic case

Our next goal is to prove a dyadic version of Theorem 2.1.1, which already contains its core. Let $Q_k(x)$ be the dyadic cube of generation k which contains the point $x \in \mathbb{R}^n$.

Proposition 2.3.3. *Let A be a smooth set in $\mathbb{R}^n$ with $0 < |A \cap [0,1]^n| < 1$. For $0 < \alpha < 1$ consider the set $E_1(A,\alpha) = \left\{x \in [0,1]^n : \lim_{k\to\infty} D(Q_k(x)) = \alpha\right\}$. Then $\dim E_1(A,\alpha) = n$.*

Proof. Fix $0 < \alpha < 1$. A Cantor type set contained in $E_1(A,\alpha)$ will be constructed and Lemma 2.3.1 will be used to calculate its Hausdorff dimension. The Cantor type set will be constructed using generations $G(s)$ which will be defined using Lemma 2.3.2, yielding the estimates appearing in Lemma 2.3.1.

Given the smooth set $A \subset \mathbb{R}^n$ consider the function

$$\omega(t) = \sup |D(Q) - D(Q')|, \quad 0 < t \leq 1$$

where the supremum is taken over all pair of consecutive cubes Q and Q' of the same sidelength $l(Q) = l(Q') \leq t$. Observe that $\lim \omega(t) = 0$ as $t \to 0$. Pick a positive integer k_0 such that $\omega(2^{-k_0}) < \min\{\alpha, 1-\alpha\}/20$. Define an increasing sequence $\{c_k\}$ with $c_k \to \infty$ as $k \to \infty$ and $c_k \geq 2n$ for any k, satisfying $\varepsilon_k = c_k\omega(2^{-k-k_0}) \to 0$ as $k \to \infty$. We can also assume $\varepsilon_k < \min\{\alpha, 1-\alpha\}/10$ for any $k = 1, 2...$ Since $0 < |A \cap [0,1]^n| < 1$, there are some small dyadic cubes in $[0,1]^n$ with density close to 0 and others with density close to 1. Since A is smooth, we can choose a dyadic cube Q_1 with $l(Q_1) \leq 2^{-k_0-1}$ and $|D(Q_1) - \alpha| < \varepsilon_1/2$. Then define the first generation $G(1) = \{Q_1\}$. The next generations are constructed inductively as follows. Assume that the k-th generation $G(k)$ has been defined so that the following two conditions are satisfied: $l(Q) \leq 2^{-k-k_0}$ and

$|D(Q) - \alpha| < \varepsilon_k/2$ for any cube $Q \in G(k)$. The generation $G(k+1)$ is constructed in two steps. Roughly speaking, we first find cubes whose density is far away from α and later we find subcubes with density close to α. For $Q \in G(k)$ consider the family $\mathfrak{R}(Q)$ of maximal dyadic cubes $R \subset Q$ such that $|D(R) - D(Q)| \geq \varepsilon_k$. Observe that, by Lebesgue Density Theorem, $\sum |R| = |Q|$, where the sum is taken over all cubes $R \in \mathfrak{R}(Q)$. Fix $R \in \mathfrak{R}(Q)$. Since the set A is ω-smooth, the difference of densities between two dyadic cubes $Q_1 \subset Q_2 \subset Q$, with $l(Q_2) = 2l(Q_1)$, is smaller than $n\omega(l(Q))$. Hence to achieve such a cube R we need to go through at least $\varepsilon_k/n\omega(2^{-k-k_0}) = c_k/n$ dyadic steps. Hence

$$|R| \leq 2^{-c_k}|Q| \tag{2.3.3}$$

The maximality and the estimate $l(Q) \leq 2^{-k-k_0}$ give that $|D(R) - D(Q)| \leq \varepsilon_k + n\omega(2^{-k-k_0-1})$. Since

$$|D(R) - \alpha| > \varepsilon_k/2 > n\omega(2^{-k-k_0}) \geq n\omega(l(R))$$

one can apply Lemma 2.3.2 with the parameter $\varepsilon = |D(R) - \alpha|$. In this way, one obtains two families $\mathfrak{A}^-(R)$ and $\mathfrak{A}^+(R)$ of dyadic cubes contained in R, according to whether their densities are smaller or bigger than $D(R)$, but we will only be interested on one of them which will be called $G_{k+1}(R)$. If $D(R) > \alpha$ we choose $G_{k+1}(R) = \mathfrak{A}^-(R)$. Otherwise, take $G_{k+1}(R) = \mathfrak{A}^+(R)$. Fix, now, $Q^* \in G_{k+1}(R)$. The maximality gives that $|D(Q^*) - \alpha| \leq n\omega(l(R))$. Since $l(R) \leq 2^{-k-k_0-1}$ we deduce that $|D(Q^*) - \alpha| < \varepsilon_{k+1}/2$. Also, $l(Q^*) < l(R)/2 \leq 2^{-k-k_0-1}$. Notice that any dyadic cube $\widetilde{Q}$ with $Q^* \subset \widetilde{Q} \subset Q$ satisfies

$$|D(\widetilde{Q}) - \alpha| \leq 6\varepsilon_k \tag{2.3.4}$$

The generation $G(k+1)$ is defined as

$$G(k+1) = \bigcup_{Q \in G(k)} \bigcup_{R \in \mathfrak{R}(Q)} G_{k+1}(R)$$

Next we will compute the constants appearing in Lemma 2.3.1. Let $Q \in G(k)$ and $R \in \mathfrak{R}(Q)$. Part (b) of Lemma 2.3.2 says that $\sum |Q_j| \geq |R|/4$ where the sum is taken over all cubes $Q_j \in G_{k+1}(R)$. Since $\sum_{\mathfrak{R}(Q)} |R| = |Q|$ one deduces that

$$\sum |Q_j| \geq |Q|/4 \tag{2.3.5}$$

where the sum is taken over all cubes $Q_j \in G(k+1), Q_j \subset Q$. Also, if $Q_j \in G(k+1)$ and $Q_j \subset Q \in G(k)$, estimate (2.3.3) guarantees that

$$|Q_j| \leq 2^{-c_k}|Q| \tag{2.3.6}$$

For $k = 1, 2...$ let $E(k)$ be the union of the cubes of the family $G(k)$ and let $E = \bigcap E(k)$ be the corresponding Cantor type set.

Next we show that E is contained in $E_1(A, \alpha)$. To do this, fix $x \in E$ and for any $k = 1, 2, ...$ pick the cube $Q_k \in G(k)$ containing x. Let Q be a dyadic cube which contains x, and k the integer for which $Q_{k+1} \subset Q \subset Q_k$. Observe that $k \to \infty$ as $l(Q) \to 0$. By (2.3.4) one deduces that $|D(Q) - \alpha| \leq 6\varepsilon_k$ and therefore $x \in E_1(A, \alpha)$.

Finally, we apply Lemma 2.3.1 to show that the dimension of E is n. Actually (2.3.5) and (2.3.6) give that one can take $C = 1/4$ and $P = 2^{-c_k}$ in Lemma 2.3.1. Hence, the dimension of E is bigger than $n(1 - (2/c_k))$. Since $c_k \to \infty$ as $k \to \infty$, we deduce that $\dim E = n$. □

2.3.3 Affine control and Proof of Theorem 2.1.1

We want to study the density of a smooth set in non dyadic cubes. The proof of Theorem 2.1.1 will be based on Proposition 2.3.3 and on the following auxiliary result on the behavior of the densities with respect to affine perturbations.

Lemma 2.3.4. *Let A be a smooth set in $\mathbb{R}^n$. Consider the function*

$$\omega(t) = \sup |D(Q) - D(Q')|$$

where the supremum is taken over all pairs of consecutive cubes $Q, Q' \subset \mathbb{R}^n$*, of sidelength* $l(Q) = l(Q') \leq t$*.*

(a) *Let* $Q, \widetilde{Q}$ *be two cubes in* $\mathbb{R}^n$ *with non empty intersection such that* $l(Q) = l(\widetilde{Q})$*. Then* $|D(Q) - D(\widetilde{Q})| \leq 3n^2\omega(l(Q))$

(b) *Let* Q *be a cube in* $\mathbb{R}^n$ *and let* tQ *denote the cube with the same center as* Q *and sidelength* $tl(Q)$*. Then for any* $1 \leq t \leq 2$ *one has*

$$|D(Q) - D(tQ)| \leq c(n)\omega(l(Q))$$

Here $c(n)$ *is a constant which only depends on the dimension.*

Proof. To prove (a), suppose, without loss of generality, that $Q = [0,1]^n$. Let $Q' = [x, 1+x] \times [0,1]^{n-1}$ where $-1 < x < 1$. We will show that

$$|D(Q) - D(Q')| \leq 3n\omega(1) \tag{2.3.7}$$

Since any Q' intersecting Q is of the form $Q' = [x_1, 1+x_1] \times ... \times [x_n, 1+x_n]$ part (a) follows using (2.3.7) n times.

To show (2.3.7), decompose $[x, 1+x]$ into dyadic intervals, that is, $[x, 1+x] = \bigcup I_k$, where I_k is a dyadic interval of length 2^{-k} for $k = 1, 2, ...$ Consider the parallelepiped $R_k = I_k \times [0,1]^{n-1}$ and the density $D(R_k)$ of the set A on R_k, meaning, $D(R_k) = |R_k \cap A|/|R_k|$, $k = 1, 2....$ The set R_k can be split into a family $\mathfrak{F}_k$ of $2^{k(n-1)}$ pairwise disjoint cubes S of sidelength 2^{-k}. Since $|D(S) - D(Q)| \leq n\omega(1)(k+1)$ for any $S \in \mathfrak{F}_k$ and $D(R_k)$ is the mean of $D(S), S \in \mathfrak{F}_k$, we deduce that $|D(R_k) - D(Q)| \leq n\omega(1)(k+1)$. Since

$$D(Q') = \sum_{k=1}^{\infty} 2^{-k} D(R_k)$$

we deduce that $|D(Q') - D(Q)| \leq 3n\omega(1)$ which is (2.3.7).

We turn now to (b). We can assume $Q = [-1/2, 1/2)^n$. Consider the binary decomposition of t, that is, $t = 1 + \sum_{k=1}^{\infty} t_k 2^{-k}$, with $t_k \in \{0, 1\}$. For $m = 1, 2, ...$ let Q_m be the cube with the same center as Q but with sidelength $l(Q_m) = 1 + \sum_{k=1}^{m} t_k 2^{-k}$. Then $tQ = \bigcup_{m=0}^{\infty} R_m$ where $R_0 \equiv Q_0 \equiv Q$ and $R_m = Q_m \backslash Q_{m-1}$ for $m \geq 1$. So for $m \geq 1, R_m$ is empty whenever $t_m = 0$ and otherwise we will estimate $|D(R_m) - D(Q)|$. With this aim assume that $t_m = 1$ and split R_m into a family $\mathfrak{F}_m$ of pairwise disjoint cubes S of sidelength 2^{-m-1}. Thus, $|D(S) - D(Q)| \leq (n(m+1)+1)\omega(l(Q))$ for any $S \in \mathfrak{F}_m$. Since $D(R_m)$ is the mean of $D(S), S \in \mathfrak{F}_m$, this implies that

$$|D(R_m) - D(Q)| \leq (n(m+1)+1)\omega(l(Q)) \quad m = 1, 2... \tag{2.3.8}$$

As we also have

$$D(tQ) = \sum_{m=0}^{\infty} \frac{|R_m|}{|tQ|} D(R_m)$$

using (2.3.8) we obtain

$$|D(tQ) - D(Q)| \leq \omega(l(Q)) \sum_{m=0}^{\infty} \frac{|R_m|}{|tQ|}(n(m+1)+1)$$

Since $|R_m| \leq C(n)2^{-m}$ the sum is convergent and the proof is complete. □

We are now ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Applying Proposition 2.3.3, we only need to check that

$$\lim_{h \to 0} D(Q(x,h)) = \alpha$$

for any $x \in E_1(A, \alpha)$. Given $0 < h < 1/2$, let k be the unique integer such that $2^{-k} \leq h < 2^{-k+1}$. Consider the cube $h2^k Q_k(x)$ and apply Lemma 2.3.4 to deduce that

$$\lim_{h \to 0} |D(Q(x,h)) - D(Q_k(x))| = 0$$

□

2.4 Equivalent definitions and invariance

The definition of a smooth set involves the density of the set on the grid of all cubes in $\mathbb{R}^n$ with sides parallel to the axes. The main purpose of this section is to study the situation for perturbations of this grid of cubes. The first step consists of considering linear deformations of the family of cubes, obtaining a certain grid of parallelepipeds in $\mathbb{R}^n$. Afterwards, we will consider the more general case of the grid arising from a bilipschitz image of the family of cubes.

Proposition 2.4.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping, and let $A \subset \mathbb{R}^n$ be a smooth set. Then $\phi(A)$ is smooth.*

Proof. We can assume that ϕ is a linear isomorphism. Given the smooth set A consider

$$\omega(t) = \sup |D(Q) - D(Q')|, \quad 0 < t \leq 1$$

where the supremum is taken over all pairs of consecutive cubes Q and Q' of the same sidelength $l(Q) = l(Q') \leq t$. Since $|\phi(A) \cap Q| = c(\phi)|A \cap \phi^{-1}(Q)|$, where $c(\phi)$ is a constant which only depends on ϕ, it is sufficient to show the smoothness condition, taking, instead of cubes, their preimages through ϕ, that is

$$\lim_{|Q| \to 0} \frac{|A \cap \phi^{-1}(Q)| - |A \cap \phi^{-1}(Q')|}{|Q|} = 0 \tag{2.4.1}$$

Apply the Singular Value Decomposition (see, for instance, Theorem 7.3.5 on [21]) to ϕ, which allows us to write $\phi = V\Sigma W$ where V, W are orthogonal mappings and Σ is a dilation, that is, the matrix of Σ is diagonal. Moreover the elements of the diagonal of Σ are the positive square roots of the eigenvalues of $\phi\phi^*$. It will prove useful later that the elements $\lambda \in \mathbb{R}$ of the diagonal of Σ verify $||\phi^{-1}|| \leq |\lambda| \leq ||\phi||$, where $||\phi||$ denotes the norm of ϕ as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. So, we proceed to prove the Proposition for these two cases: (a) ϕ is an orthogonal mapping and (b) ϕ is a dilation.

We study first case (a). As any orthogonal application is either a rotation or the composition of a rotation and a reflection by a subspace parallel to the axes (leaving invariant the grid of cubes), we can reduce the orthogonal case to rotations. Furthermore, we can assume that ϕ is the identity on a subspace of dimension $n-2$ generated by elements of the canonical basis of $\mathbb{R}^n$ and a rotation of angle $\alpha \in [\pi/6, \pi/4]$ on its orthogonal complement. Actually, any rotation can be written as the composition of at most $3n(n-1)$ rotations of this form (see, for instance, [21]). Let $\widetilde{Q}$ and $\widetilde{Q'}$ be the cubes centered at the centers of $\phi^{-1}(Q)$ and $\phi^{-1}(Q')$,respectively, of sidelength $l(Q)$ with sides parallel to the axes. Observe that $\widetilde{Q'}$ is a translation of $\widetilde{Q}$ by a vector of norm less than $nl(Q)$, so Lemma 2.3.4 implies that $|D(\widetilde{Q}) - D(\widetilde{Q'})| \leq 3n^3\omega(l(Q))$. Hence, to show (2.4.1) it is enough to prove that

$$\lim_{|Q|\to 0} \frac{|A \cap \phi^{-1}(Q)| - |A \cap \widetilde{Q}|}{|Q|} = 0 \tag{2.4.2}$$

We study first the case $n = 2$. We are going to decompose $\phi^{-1}(Q)$ into squares as follows. Let Q_0 be the maximal square with sides parallel to the axes contained in $\phi^{-1}(Q)$ and write

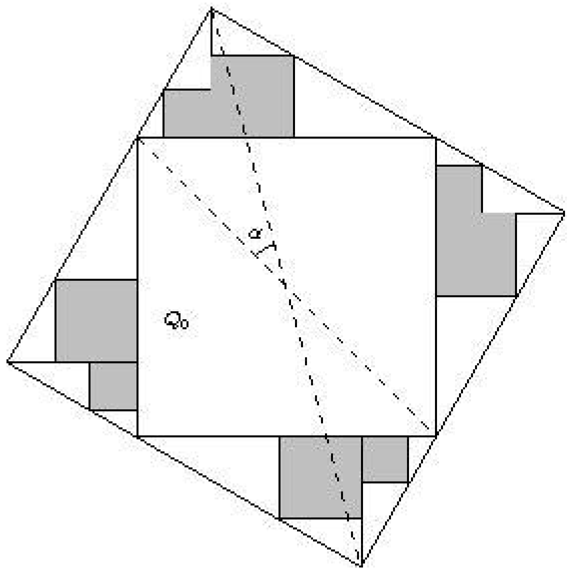

Figure 2.4.1: The shaded squares are the eight elements of $\mathfrak{F}_1$ for $\alpha = \pi/6$.

$\mathfrak{F}_0 = \{Q_0\}$. Observe that the ratio of the area of Q_0 to that of $\phi^{-1}(Q)$ is $C = 1/(1+\sin(2\alpha))$ and that $0.5 \leq C \leq 4 - 2\sqrt{3}$. Then $\phi^{-1}(Q)\backslash Q_0$ is the union of eight right-angled triangles whose hypothenuse is contained in $\partial\phi^{-1}(Q)$. Take again the maximal square with sides parallel to the axes contained in each triangle, obtaining a family $\mathfrak{F}_1$ of eight squares of total area $(1-C)^2|Q|$. See Figure 2.4.1. Thus $\phi^{-1}(Q)\backslash(\mathfrak{F}_0 \cup \mathfrak{F}_1)$ is the union of 16 right-angled triangles and we continue inductively, constructing, for $k = 1, 2...$, a family $\mathfrak{F}_k$ of 2^{k+2} squares of total area $C^{k-1}(1-C)^2|Q|$. Observe that

$$D(\phi^{-1}(Q)) - D(\widetilde{Q}) = \sum_k \sum_{R \in \mathfrak{F}_k} \frac{|R|}{|Q|}(D(R) - D(\widetilde{Q}))$$

Let R be a square in $\mathfrak{F}_k$. Since A is smooth, there exists a constant $C_1 > 0$ such that $|D(R) - D(\widetilde{Q})| \le C_1 k\omega(l(Q))$. We deduce that

$$|D(\phi^{-1}(Q)) - D(\widetilde{Q})| \le C_1\omega(l(Q)) \sum_k k \frac{\left|\bigcup_{\mathfrak{F}_k} R\right|}{|\phi^{-1}(Q)|} \le$$
$$\le C_1(1-C)^2 C^{-1}\omega(l(Q)) \sum_k kC^k = C_1\omega(l(Q))$$

This implies (2.4.2) and finishes the proof in dimension 2 when ϕ is a rotation.

We now study the higher dimensional case $n > 2$. Recall that ϕ^{-1} is a rotation on a two dimensional subspace E and ϕ^{-1} is the identity on its orthogonal complement. Without loss of generality, we may assume that E is generated by the two first vectors of the canonical basis of $\mathbb{R}^n$. Consider the orthogonal projection Π of $\mathbb{R}^n$ onto E and decompose $\Pi(\phi^{-1}(Q))$ as in the two dimensional case, that is $\Pi(\phi^{-1}(Q)) = \bigcup_{k=0}^{\infty} \mathfrak{F}_k$, where $\mathfrak{F}_k$ is, as before, the union of 2^{k+2} (2 dimensional) squares with sides parallel to the axes of total area $C^{k-1}(1-C)^2 l(Q)^2$. Since $\phi^{-1}(Q) = \Pi(\phi^{-1}(Q)) \times B$ where B is a cube in $\mathbb{R}^{n-2}$ with sides parallel to the axes, we have $\phi^{-1}(Q) = \bigcup_{k=0}^{\infty} G_k$ where $G_k = \bigcup_{R \in \mathfrak{F}_k} R \times B$. Using the smoothness condition one can show that there exists a constant $C_2 > 0$ such that $|D(R \times B) - D(\widetilde{Q})| \le C_2 k\omega(l(Q))$ for any square $R \in \mathfrak{F}_k$. Then

$$|D(\phi^{-1}(Q)) - D(\widetilde{Q})| \le C_2\omega(l(Q)) \sum_{k=0}^{\infty} k \frac{|G_k|}{|Q|}$$

Since $|G_k| = (1-C)^2 C^{k-1}|Q|$ with $0.5 < C < 4 - 2\sqrt{3}$ we deduce that $|D(\phi^{-1}(Q)) -$

$D(\widetilde{Q})| \leq C_2\omega(l(Q))$ and thus (2.4.1) is satisfied, completing the proof in the case that ϕ is a rotation.

Let us now assume that ϕ is a dilation, that is, ϕ has a diagonal matrix. Without loss of generality we can assume $\phi^{-1}(x_1, ..., x_n) = (\lambda x_1, x_2, ..., x_n)$ for some $\lambda \in \mathbb{R}$. Assume $\lambda > 1$. To prove (2.4.1) it is sufficient to find a constant $C(\lambda) > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and any cube $\widetilde{Q} \subset \phi^{-1}(Q)$ with $l(\widetilde{Q}) = l(Q)$ we have

$$|D(\phi^{-1}(Q)) - D(\widetilde{Q})| \leq C(\lambda)\omega(l(Q)) \tag{2.4.3}$$

The proof of (2.4.3) resembles that of part (a) of Lemma 2.3.4. We can assume that $\widetilde{Q}$ is the unit cube. Let $[\lambda]$ denote the integer part of λ and write the interval $[[\lambda], \lambda)$ as a union of maximal dyadic intervals $\{I_k\}$ with $|I_k| = 2^{-k}$, that is, $[[\lambda], \lambda) = \bigcup I_k$. Consider $R_k = I_k \times [0, 1]^{n-1}$, $k = 1, 2, ...$ Observe that the variation in density in $|D(\phi^{-1}(Q)) - D(\widetilde{Q})|$ is equal to

$$\sum_{j=0}^{[\lambda]-1} \frac{1}{\lambda} \left(D([j, j+1) \times [0, 1]^{n-1}) - D(\widetilde{Q}) \right) + \sum_{k=1}^{\infty} \frac{2^{-k}}{\lambda} (D(R_k) - D(\widetilde{Q}))$$

For $j = 0, 1, ..., [\lambda] - 1$, we have $|D([j, j+1) \times [0, 1]^{n-1}) - D(\widetilde{Q})| \leq \lambda\omega(1)$. Since R_k can be split into a family of dyadic cubes of generation k, we deduce that $|D(R_k) - D(\widetilde{Q})| \leq (\lambda + 1 + k)\omega(1)$. Therefore

$$|D(\phi^{-1}(Q)) - D(\widetilde{Q})| \leq (\lambda + 1 + 3/\lambda)\omega(1)$$

which proves (2.4.3). An analogous argument can be used if $\lambda < 1$. □

Remark 2.4.2. The first part of the proof shows that there exists a constant $C = C(n) > 0$ such that for any rotation ϕ in $\mathbb{R}^n$ and any ω-smooth set $A \subset \mathbb{R}^n$, its image $\phi(A)$ is $C\omega$-smooth. When ϕ is a dilation in a single direction with parameter $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}^n$ is an ω-smooth set, the proof shows that $\phi(A)$ is $C(\lambda)\omega$-smooth, with $C(\lambda) \leq 4(\lambda + 1/\lambda)$.

Remark 2.4.3. Let $\{T_i\}$ be a countable family of linear isomorphisms in $\mathbb{R}^n$ for which there exists a constant $M > 0$ such that $M^{-1}||x|| \leq ||T_i(x)|| \leq M||x||$ for any $x \in \mathbb{R}^n$ and $i = 1, 2...$ Then there exists a constant $C = C(M, n) > 0$ such that for any ω-smooth set A and any i one has

$$\frac{||T_i(Q) \cap A| - |T_i(Q') \cap A||}{|Q|} \leq C\omega(l(Q))$$

Remark 2.4.4. Proposition 2.4.1 and part (a) of Lemma 2.3.4 give that affine mappings preserve smooth sets.

We could have defined smooth sets using the grid of dyadic cubes or, in the opposite direction, using the grid of all cubes, even without taking them parallel to the axes. The previous results imply that both grids would lead to equivalent definitions.

Corollary 2.4.5. *Let A be a measurable set in $\mathbb{R}^n$. The following are equivalent:*

(a) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|D(Q) - D(Q')| \leq \varepsilon$ for any pair of consecutive dyadic cubes Q, Q', of the same side length $l(Q) = l(Q') < \delta$.*

(b) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|D(Q) - D(Q')| \leq \varepsilon$ for any pair of consecutive cubes Q, Q' with sides non necessarily parallel to the axes, of the same side length $l(Q) = l(Q') < \delta$.*

(c) *The measurable set A is a smooth set.*

Proof. Since any cube in $\mathbb{R}^n$ is the affine image of a dyadic cube, Lemma 2.3.4 shows that (a) implies (b). The other implications are obvious. □

Applying locally Proposition 2.4.1, we can extend it to certain diffeomorphisms, but we need extra assumptions to guarantee that the local bounds that we obtain are satisfied uniformly, and this is the main idea behind the proof of Theorem 2.1.2, to which we proceed now.

Proof of Theorem 2.1.2. Since ϕ is bilipschitz, $|\phi(Q)|$ is comparable to $|Q|$. Also, $|J\phi|$ is uniformly bounded from above and below. Therefore, $J\phi^{-1}$ is uniformly continuous as well. We will also need that

$$\lim_{|Q|\to 0} \frac{|\phi(Q)| - |\phi(Q')|}{|Q|} = 0 \tag{2.4.4}$$

To show this, observe that this quantity is

$$\frac{1}{|Q|}\left(\int_Q J\phi - \int_{Q'} J\phi\right)$$

which tends to 0 uniformly when $l(Q) \to 0$ because of the uniform continuity of $J\phi$.

We first show that (b) is equivalent to (c). A change of variables gives that

$$|\phi^{-1}(A) \cap Q| - |\phi^{-1}(A) \cap Q'| = \int J\phi^{-1}(x)\left(\mathbb{1}_{A\cap\phi(Q)}(x) - \mathbb{1}_{A\cap\phi(Q')}(x)\right) dx$$

Let $p(Q)$ be a point in $\overline{\phi(Q)} \cap \overline{\phi(Q')}$. Given $\varepsilon > 0$, if $l(Q)$ is sufficiently small one has $||J\phi^{-1}(x) - J\phi^{-1}(p(Q))|| < \varepsilon$ for any $x \in \phi(Q)$. Hence, the uniform continuity of $J\phi^{-1}$ gives us that

$$\lim_{|Q|\to 0} \frac{|\phi^{-1}(A) \cap Q| - |\phi^{-1}(A) \cap Q'|}{|Q|}$$

is equal to

$$\lim_{|Q|\to 0} \frac{(|A \cap \phi(Q)| - |A \cap \phi(Q')|)\, J\phi^{-1}(p(Q))}{|Q|}$$

Let $D(\phi(Q))$ be the density of A in $\phi(Q)$, that is, $D(\phi(Q)) = |A\cap\phi(Q)|/|\phi(Q)|$. Applying (2.4.4) we have that this last limit is equal to

$$\lim_{|Q|\to 0} \left(D(\phi(Q)) - D(\phi(Q'))\right) J\phi^{-1}(p(Q))$$

Since $J\phi^{-1}$ is uniformly bounded both from above and below, we deduce that (b) and (c) are equivalent.

We now show that (a) implies (c). Observe that, applying (2.4.4), it is sufficient to show

$$\lim_{|Q|\to 0} \frac{|A \cap \phi(Q)| - |A \cap \phi(Q')|}{|Q|} = 0 \tag{2.4.5}$$

Let $z(Q)$ be a point in Q with dyadic coordinates. Let $T = T(Q)$ be the affine mapping defined by $T(x) = \phi(z(Q)) + D\phi(z(Q))(x - z(Q))$, for any $x \in \mathbb{R}^n$, where $D\phi$ denotes the differential of ϕ. Given $\varepsilon > 0$, the uniform continuity of $J\phi$ tells that $|\phi(x) - T(x)| \leq \varepsilon l(Q)$, for any $x \in Q \cup Q'$ if $l(Q)$ is sufficiently small. There thus exists a constant $C_1(n) > 0$ such that if $l(Q)$ is sufficiently small then

$$| (\phi(Q) \backslash T(Q)) \cup (T(Q) \backslash \phi(Q)) | \leq C_1(n) \varepsilon |Q|$$

and similarly for Q'. So we deduce that (2.4.5) is equivalent to

$$\lim_{|Q| \to 0} \frac{|A \cap T(Q)| - |A \cap T(Q')|}{|Q|} = 0 \tag{2.4.6}$$

Now (2.4.6) follows from Remark 2.4.3 because, since ϕ is bilipschitz, there exists a constant $M > 0$ such that $M^{-1}||x|| \leq ||D\phi(z(Q))(x))|| \leq M||x||$ for any $x \in \mathbb{R}^n$ and any cube Q in $\mathbb{R}^n$. This finishes the proof that (a) implies (c). The proof that (b) implies (a) follows applying the previous part to ϕ^{-1}. □

To end this chapter we give, now, the proof of Theorem 2.1.4.

Proof. Let $A \subset \mathbb{R}$ be a nontrivial smooth set and $0 < \alpha < 1$. Then, by Theorem 2.1.1, there exists a point $x_0 \in E(A, \alpha)$. We are going to define a bilipschitz function $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $\phi(A)$ is not smooth.

To do so, we choose ϕ as the only continuous function with $\phi(x_0) = x_0$ satisfying:

$$\phi'(x) \overset{a.e.}{=} 1 + \mathbb{1}_{(-\infty, x_0) \backslash A}(x) \tag{2.4.7}$$

Essentially, if $x \in A \cup (x_0, \infty)$ then $\phi'(x) = 1$, and otherwise $\phi'(x) = 2$. As a result, ϕ is bilipschitz. Also, $\phi_{|(x_0, \infty)} = Id$.

We need to see that $\phi(A)$ is not smooth. Denote $I = [x_0 - h, x_0]$ and $I' = [x_0, x_0 + h]$.

By changing variables it is clear that:

$$\frac{|\phi(A)\cap I|}{|I|} - \frac{|\phi(A)\cap I'|}{|I'|} = \frac{1}{|I|}\left(\int_{A\cap\phi^{-1}(I)} \phi'(t)dt - \int_{A\cap\phi^{-1}(I')} \phi'(t)dt\right) \tag{2.4.8}$$

That is, as ϕ' is 1 on almost every point of A, the same as:

$$\frac{|A\cap\phi^{-1}(I)|}{|I|} - \frac{|A\cap\phi^{-1}(I')|}{|I'|}$$

As ϕ is the identity over (x_0,∞), so it is over I', and hence the previous quantity is equal to:

$$D(\phi^{-1}(I))\cdot\frac{|\phi^{-1}(I)|}{|I|} - D(I')$$

If the size of I is small enough, then $D(\phi^{-1}(I))$ and $D(I)$ are close to each other, and to $D(I')$. As $x_0 \in E(A,\alpha)$, these values will be close to α.

Hence, if $\phi(A)$ is smooth, we must have that $\frac{|\phi^{-1}(I)|}{|I|}$ converges uniformly to 1 as $h = |I|$ tends to 0.

It remains to show this contradicts $0 < \alpha < 1$. By the Change of Variable Theorem, we have:

$$\frac{|\phi^{-1}(I)|}{|I|} = \frac{|I\cap\phi(A)|}{|I|} + \frac{1}{2}\frac{|I\backslash\phi(A)|}{|I|}$$

It is easy to see the right hand side is equal to:

$$\frac{1}{2} + \frac{1}{2}\frac{|I\cap\phi(A)|}{|I|}$$

Applying again the Change of Variable Theorem we get:

$$\frac{|\phi^{-1}(I)|}{|I|} = \frac{1}{2} + \frac{1}{2}\frac{|A\cap\phi^{-1}(I)|}{|I|}$$

An algebraic manipulation and taking limits as $|I|$ tends to 0, gives $\alpha = 1$. □

Chapter 3

Sampling in the Dirichlet space

3.1 Introduction

3.1.1 Sampling and interpolation in the Dirichlet space

The Dirichlet space is the space of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $z \in \mathbb{D}$, whose Taylor coefficients satisfy

$$\|f\|_{\alpha}^{2} = \sum_{k=0}^{\infty}(k+1)|a_k|^2 < \infty. \tag{3.1.1}$$

It corresponds with the space of functions whose derivatives have finite area integral:

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

Recent surveys concerning the Dirichlet space D include [3] and [33].

Given a sequence Z of points in $\mathbb{D}$, consider a space of sequences ℓ, such that for all $f \in D$, $R_Z(f) = \{f(z_n)\}$ is an element of ℓ. Then the operator $R_Z : D \to \ell$ is well defined. We call R_Z the *restriction operator* induced by Z from D into ℓ. We say that a sequence $Z \subset \mathbb{D}$ is a *sequence of interpolation* (from D onto ℓ) if the restriction operator is bounded and surjective. Analogously, we say that $Z \subset \mathbb{D}$ is a *sampling sequence* (from D into ℓ) if there exist positive constants C_1 and C_2 such that, for all $f \in D$ we have:

$$C_1||f||_D^2 \leq ||R_Z(f)||_\ell^2 \leq C_2||f||_D^2 \tag{3.1.2}$$

3.1.2 Variational approach to the Dirichlet space

Marshall and Sundberg ([25]) characterized sequences of interpolation, from the Dirichlet space into ℓ^2 with a certain weight. The same result was proved by Bishop ([5]), and in two different ways by Boe ([7] and [8]). In order to study sampling sequences, it may be of interest to search for a space of sequences which is more adequate to D, a space that mimics some properties of the Dirichlet space.

Our focus will be on a space of sequences ℓ with the property that the restriction operator R_Z induced by any separated sequence Z from D into ℓ is bounded.

For this, we concentrate on the double integral description of the Dirichlet norm. It is well known that for any function f in the Dirichlet space D with $f(0) = 0$ we have the following identity of norms:

$$||f||_D^2 = \frac{1}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{z}w|^4} dA(z)dA(w) \tag{3.1.3}$$

For the sake of simplicity, during this chapter, we will deliberately ignore the term of the norm that comes from the value of functions at $z = 0$. This assumption is equivalent with working only with those functions f such that $f(0) = 0$.

It is by looking at a discrete version of (3.1.3) that we will construct the space of sequences where the restriction operator will arrive. For this we need to introduce a last basic concept which preserves the notions of distance and separation that are characteristic of this environment. The pseudohyperbolic distance ρ between z and w in $\mathbb{D}$ is given by:

$$\rho(z, w) = \frac{|z - w|}{|1 - \overline{z}w|}$$

A transformation of this quantity is the hyperbolic distance β:

$$\beta(z,w)=\frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}$$

Throughout the text we will denote by $D_H(z,R)$ the hyperbolic disk centered at z of radius R, i.e., those points $w\in\mathbb{D}$ with $\beta(z,w)<R$. The most relevant fact that we will use about such disks is that automorphisms of the unit disk take hyperbolic disks to hyperbolic disks of the same radius.

We define a sequence $Z=\{z_n\}_{n\in\mathbb{N}}$ to be *separated* if there exists some $R>0$ such that, for any two different n and m in $\mathbb{N}$, $\beta(z_n,z_m)>R$. If R is such a constant we say that Z is *R-separated.* The optimal such R is the *radius of separation.* Given $d>0$, we say that a sequence $Z=\{z_n\}$ is *d-dense* if all points $z\in\mathbb{D}$ satisfy $\beta(z,Z)<d$. The optimal such d is the *radius of density.*

Fix a separated sequence Z. Define $\ell=\ell^2_\Delta$ to be the space of those sequences $W=\{w_n\}_{n\in\mathbb{N}}\subset\mathbb{C}$ with:

$$\sum_{n,k}|w_n-w_k|^2(1-\rho(z_n,z_k)^2)^2<\infty$$

Remark 3.1.1. There is a more general statement than (3.1.3) in the context of Dirichlet-type spaces. See [32] or [37]. All that we are going to do may, from there, be done also for these spaces, leading to similar results, but we concentrate only on the Dirichlet space.

In Section 2 we start with some basics about sampling and interpolation between our pair of spaces. We will firstly show that separated sequences generate bounded restriction operators between D and ℓ^2_Δ. This justifies the choice of space of values, ℓ^2_Δ. We also show that there are no infinite separated sequences of interpolation. To conclude the section, we prove that a sampling sequence needs to be d-dense for some value of d and that there exists a small enough radius d such that d-density is also sufficient for sampling.

To conclude, in Section 3, we deepen into sampling sequences, by analyzing properties of harmonic measure in certain domains Ω associated to a sequence Z. For a separated sequence $Z=\{z_n\}_{n\in\mathbb{N}}$, consider the domain $\Omega=\mathbb{D}\setminus\cup_n D_n$, where $D_n=D_H(z_n,d)$ and let ω_z be the harmonic measure in Ω. We will prove the main result in this chapter, saying that a fast enough decay of the averages, over $z \in \Omega$, of $\omega_z(\partial D_n)$ as n goes to infinity, is sufficient for the sequence Z to be sampling.

By C, we will denote a constant whose value may change from line to line.

3.2 Basic results

Given a function $f \in D$, define the *Z-norm* of f as:

$$||f||_Z^2 = \sum_{n,k} \frac{|f(z_n) - f(z_k)|^2 (1 - |z_n|^2)^2 (1 - |z_k|^2)^2}{|1 - \overline{z_n} z_k|^4} =$$

$$= \sum_{n,k} |f(z_n) - f(z_k)|^2 (1 - \rho(z_n, z_k)^2)^2$$

With the notation that we have introduced, Z is a sampling sequence if and only if there exists positive constants C_0, C_1, so that for all $f \in D$ we have:

$$C_0 ||f||_D^2 \leq ||f||_Z^2 \leq C_1 ||f||_D^2$$

It is worth mentioning that this notion is conformally invariant, that is, Z is a sampling sequence if and only if so is $\tau(Z)$, for any automorphism τ of the unit disc. Moreover, the sampling constants C_1 and C_2 remain unchanged. The restriction operator boundedness, needed also for interpolation, is the second of these inequalities.

Proposition 3.2.1. *For any separated sequence Z the restriction operator R_Z from D into ℓ^2_Δ, is bounded.*

Proof. The sequence $Z = \{z_n\}_{n \in \mathbb{N}}$ is separated and so, for some $C > 0$, we have:

$$\inf_{n \neq m} \beta(z_n, z_m) > C$$

Thus, we can define a sequence of disjoint disks $D_n = \{z : \beta(z, z_n) \leq C/2\}$, satisfying that for all f in D:

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{z}w|^4} dA(z)dA(w) \geq \sum_{n,m} \int_{D_n}\int_{D_m} \frac{|f(z)-f(w)|^2}{|1-\overline{z}w|^4} dA(z)dA(w)$$

Using the subharmonicity of the argument of the integrals we can see that, for some constants $C_0, C_1 > 0$, the right hand side in the previous inequality is greater or equal to:

$$C_0 \sum_{n,m} \frac{(diam D_n)^2 \cdot (diam D_m)^2 \cdot |f(z_n)-f(z_m)|^2}{|1-\overline{z_n}z_m|^4} \geq C_1 ||f||_Z^2 \tag{3.2.1}$$

□

Hence, from now on, unless otherwise stated, we will assume for sequences to be separated.

Let us mention, before going further, a well known property of the Dirichlet space (see, for instance, [38]) that we will make use of. Functions f in D satisfy a hyperbolic version of a Lipschitz-1/2 condition:

Lemma 3.2.2. *There exists a universal constant $C > 0$ such that, for all $f \in D$, and all $z, w \in \mathbb{D}$, the following inequality holds:*

$$|f(z)-f(w)|^2 \leq C||f||_D^2 \beta(z,w)$$

Now we will prove another basic fact:

Proposition 3.2.3. *There are no sequences of interpolation from D into ℓ^2_Δ.*

Proof. Suppose Z is a sequence of interpolation. By an automorphism of the disk, we can suppose the first point of Z, z_1 is 0.

Now, define $W = \{w_1, w_2\}$ with $w_1 = 0$ and $w_2 = \frac{1}{1-|z_2|}$. As the norm squared of W in ℓ^2_Δ is at least 2 and Z is of interpolation, there exists a function $f \in D$, such that $||f||_D^2 \leq C$, for some $C > 0$, and moreover $f(z_1) = 0$ and $f(z_2) = w_2$.

Hence, from (3.2.2) we see:

$$\frac{1}{(1-|z_2|)^2} \leq C\beta(0, z_2)$$

This implies that the points of Z have to be at a positive fixed distance from the boundary of $\mathbb{D}$ and it is a separated sequence. Hence Z must be finite. $\square$

Finally, now we will show a necessary and a sufficient condition for sampling. We don't know whether any of the two conditions is equivalent with sampling.

Proposition 3.2.4. *(a) Let $Z = \{z_n\}$ be a separated sequence. If Z is sampling then it is d-dense, for some $d > 0$.*

(b) There exists a constant $\varepsilon > 0$, with the property that if Z is a sequence for which there exists $0 < R < \varepsilon$ such that Z is R-dense, then Z is sampling.

Before we start with the proof of the proposition, we need to define notation. Given a point $z \in \mathbb{D}\backslash\{0\}$, consider the Carleson square, $Q(z)$ given by:

$$Q(z) = \{re^{i\theta} : r \geq 1 - |z|, |\theta - Arg(z)| < 2\pi(1 - |z|)\}$$

For $0 < \lambda < 1/(1-|z|)$, $\lambda Q(z)$ means the Carleson square $\lambda Q(z) = Q(z_\lambda)$ where $z_\lambda = \frac{z}{|z|}(1 - \lambda(1 - |z|))$.

Proof. **Part (a).** Suppose there is a sampling sequence Z that is not d-dense for any d. This means that there exists a sequence of points in the unit disk $\{\xi_k\}_{k\in\mathbb{N}}$, such that $D_H(\xi_k, k) \cap Z = \emptyset$. Let τ_k be an automorphism of the disk such that $\tau_k(\xi_k) = 0$. The image of $D_H(\xi_k, k) \cap Z$ under τ_k is $D_H(0, k) \cap \{\tau_k(z_n)\}$ which is, therefore, empty.

Now take, for instance, the function $f(z) = z$. One can check $||f||_D^2 = \pi$.

First, as the definition of sampling sequence is conformally invariant, we have:

$$||\tau_k(z)||_D^2 = ||z||_D^2 \leq C \sum_{n,m} |\tau_k(z_n) - \tau_k(z_m)|^2 (1 - \rho^2(z_n, z_m))^2$$

Just applying the definition of ρ we get that the right hand side above equals:

$$C\sum_n (1-|\tau_k(z_n)|^2)^2 \sum_m \frac{|\tau_k(z_n)-\tau_k(z_m)|^2(1-|\tau_k(z_m)|^2)^2}{|1-\overline{\tau_k(z_n)}\tau_k(z_m)|^4}$$

The inner sum (the one in m) can be decomposed in the sum over those points z_m of the sequence whose image under τ falls on each dyadic expansion, $Q_j = 2^j Q_0$, of the square $Q_0 = Q(\tau_k(z_n))$. This inner sum is bounded by:

$$C \sum_{j=0}^{\log\frac{1}{1-|\tau_k(z_n)|}} \sum_{z_m\in Q_j} \frac{2^{2j}(1-|\tau_k(z_n)|^2)^2(1-|\tau_k(z_m)|^2)^2}{2^{4j}(1-|\tau_k(z_n)|^2)^4}$$

For $z_m \in Q_j$, since Z is a separated sequence, there exists a constant C, such that one has $\sum_{z_m\in Q_j}(1-|\tau_k(z_m)|^2)^2 \le C2^{2j}(1-|\tau_k(z_n)|^2)^2$. We see that the overall sum in j is bounded by some constant times the upper limit of summation, $\log\frac{1}{1-|\tau_k(z_n)|}$.

What we have, altogether, is:

$$\pi = ||z||_D^2 \le C\sum_n (1-|\tau_k(z_n)|^2)^2 \log\frac{1}{1-|\tau_k(z_n)|}$$

Here, as $\{\tau_k(z_n)\}$ is separated and at a hyperbolic distance away from 0 larger than k, we get:

$$\pi \le C\int_{\mathbb{D}\setminus D_H(0,k)} \log\frac{1}{1-|w|}dA(w)$$

As the function is integrable over the domain, we obtain a contradiction.

Part (b). We are going to work with the expression of the Dirichlet norm in terms of a double integral:

$$||f||_D^2 = \frac{1}{\pi}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{z}w|^4}dA(z)dA(w), \quad f\in D, \quad f(0)=0$$

Decompose the unit disk $\mathbb{D}$ in a sequence of subsets $\{Q_k\}$ with pairwise disjoint interiors so that $\mathbb{D} = \bigcup_k Q_k$ and so that the hyperbolic diameter of Q_k is smaller or equal to R but

$Q_k \cap Z \neq \emptyset$. Pick $z_k \in Q_k \cap Z$. We have the identity:

$$||f||_D^2 = \frac{1}{\pi} \sum_{n,k} \int_{Q_n} \int_{Q_k} \frac{|f(z) - f(w)|^2}{|1 - \overline{z}w|^4} dA(z)dA(w)$$

We can add and subtract the same quantity inside the absolute value, obtaining:

$$\frac{1}{\pi} \sum_{n,k} \int_{Q_n} \int_{Q_k} \frac{|f(z_n) - f(z_k) + f(z) - f(z_n) + f(z_k) - f(w)|^2}{|1 - \overline{z}w|^4} dA(z)dA(w)$$

We can separate this integral in 3 pieces, pairing the elements in the modulus. The triangular inequality yields that, for some constant $C_1 > 0$:

$$||f||_D^2 \leq \frac{1}{\pi C_1} ||f||_Z^2 + \frac{2}{\pi} \sum_{n,k} \int_{Q_n} \int_{Q_k} \frac{|f(z) - f(z_k)|^2}{|1 - \overline{z}w|^4} dA(z)dA(w) \tag{3.2.2}$$

Let us see that, when R is smaller than some fixed constant, the second term in the right hand side is essentially irrelevant. As the Q_k have disjoint interiors and cover $\mathbb{D}$, for some universal constant $C_2 > 0$, we have:

$$\sum_{n,k} \int_{Q_n} \int_{Q_k} \frac{|f(z) - f(z_k)|^2}{|1 - \overline{z}w|^4} dA(z)dA(w) \leq C_2 \sum_{k} \int_{Q_k} \frac{|f(z) - f(z_k)|^2}{(1 - |z|^2)^2} dA(z) \tag{3.2.3}$$

To bound the argument of this last integral, first we see that there is a constant $C_3 > 0$ such that, for any $z \in Q_k$, we have:

$$|z - z_k| \leq C_3 R(1 - |z_k|) \tag{3.2.4}$$

This allows us to control the variation of f. By the Fundamental Theorem of Calculus, we know:

$$|f(z) - f(z_k)| \leq \int_{z_k}^{z} |f'(w)||dw| \tag{3.2.5}$$

Now, we want to use the subharmonicity of f', and for this, we need to extend Q_k from its center by a fixed number $a > 1$. Choose $\widetilde{Q_k} = \{z \in \mathbb{D} \colon \beta(z, Q_k) \leq a - 1\}$ Then there exists $c = c(a)$ such that for all $w \in Q_k$ we have:

$$|f'(w)|^2 \leq c \frac{\int_{\widetilde{Q_k}} |f'(z)|^2}{\int_{\widetilde{Q_k}} dA(z)}$$

Every point in $\mathbb{D}$ is inside at least 1, and at most 4, squares of the form $\widetilde{Q_k}$, $k \in \mathbb{N}$.

From (3.2.5), we obtain that:

$$|f(z) - f(z_k)| \leq c \int_{z_k}^{z} \left(\frac{\int_{\widetilde{Q_k}} |f'(z)|^2 dA(z)}{\int_{\widetilde{Q_k}} dA(z)} \right)^{1/2} |dw| \tag{3.2.6}$$

Now, the arguments of the integrals inside the parenthesis are independent of w. Putting together (3.2.4) and (3.2.6), we control the variation of f:

$$|f(z) - f(z_k)| \leq CC_3 R(1 - |z_k|) \left(\frac{\int_{\widetilde{Q_k}} |f'(z)|^2 dA(z)}{\int_{\widetilde{Q_k}} dA(z)} \right)^{1/2} \tag{3.2.7}$$

If we go back to (3.2.3), when we integrate over the union of $\{\widetilde{Q_k}\}_{k \in \mathbb{N}}$, as $a > 1$, for some constant $C_4 > 0$ we have:

$$\frac{2}{\pi} \sum_{n,k} \int_{Q_n} \int_{Q_k} \frac{|f(z) - f(z_k)|^2}{|1 - \bar{z}w|^4} dA(z) dA(w) \leq C_4 R^2 ||f||_D^2 \tag{3.2.8}$$

What we have seen, altogether is that:

$$||f||_D^2 \leq \frac{1}{\pi C_1} ||f||_Z^2 + C_4 R^2 ||f||_D^2$$

If $R < 1/\sqrt{C_4}$, then we can divide by $(1 - C_4 R^2)$:

$$||f||_D^2 \leq \frac{1}{\pi C_1 (1 - C_4 R^2)} ||f||_Z^2$$

□

3.3 Harmonic measure and sampling inequality

Remember a sequence Z is d-dense, if any point of $\mathbb{D}$ is at a hyperbolic distance at most d of Z. In the previous section we've seen that a d-dense sequence, with d close to 0, is a sampling sequence.

Now we want to find a more quantitative property that can be explicitly measured.

Consider $Z = \{z_n\}$, a separated sequence and $d > 0$, with $\bigcup_n D_H(z_n, d) = \mathbb{D}$. Because of the equivalence of the norms in Remark 3.1.1, what we want to see is whether, for a $C > 0$ depending only on the sequence and for any function $f \in D$, the following holds:

$$\sum_{n,m} |f(z_n) - f(z_m)|^2 (1 - \rho(z_n, z_m)^2)^2 \geq C \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} dA(z) dA(w)$$

Call $D_n = D_H(z_n, \epsilon)$ for a certain fixed and small $\epsilon > 0$, so for the sequence Z to be 2ϵ-separated (and hence, for the disks D_n to be disjoint).

Define now a domain Ω formed by the unit disk from which we remove the disks $\{D_n\}$ forming a champagne-type domain (see [1], [16], [28] and [30]):

$$\Omega = \mathbb{D} \setminus \left(\bigcup_{n \in \mathbb{N}} D_n \right) \tag{3.3.1}$$

Now, as Ω satisfies the so called exterior cone condition, it is well known that the Dirichlet problem is solvable in Ω, and so, given $f \in \mathcal{C}(\partial\Omega)$, there exists a unique harmonic function u on the domain, such that $u_{|\partial\Omega} \equiv f$. Moreover, by Radon's Representation Theorem, for any $z \in \Omega$ there exists a probability measure (called *harmonic measure*) ω with the following property:

$$u(z) = \int_{\partial\Omega} f(\xi) d\omega(z, \xi, \Omega)$$

Furthermore, given a subset E of the boundary, $\omega(z, E, \Omega)$ coincides with the probability that a brownian motion starting from z leaves Ω for the first time through the set E.

Remark 3.3.1. This measure has been broadly studied, and in particular, in [1], necessary and sufficient conditions are established for the domain Ω to satisfy that the harmonic measure of the unit circle is zero.

For that, it is sufficient, for instance, that, the hyperbolic radius of the disks is a constant, although sharper results are known (see [1] and [28]).

Consider the measure mu in $\partial\Omega$ defined by $\mu(E) = \int_\Omega \omega(z, E, \Omega) dA(z)$ for $E \subset \partial\Omega$.

Let us see that we can use this measure to prove the central result in this section. We will establish the sufficiency of the density with an additional hypothesis for the obtention of the sampling inequality.

For a separated sequence Z, denote $C(Z)$ the optimal constant $C > 0$ so that for any $\xi \in \mathbb{D}$ we have:

$$\sum_m \frac{(1-|z_m|^2)^2}{|1-\overline{\xi} z_m|^4} \leq \frac{C}{(1-|\xi|^2)^2}$$

Clearly $0 < C(Z) < \infty$, but its actual numerical value is relevant to the statement that we prove.

Theorem 3.3.2. *Let $Z = \{z_n\} \subset \mathbb{D}$ be a separated sequence. Suppose there exists a constant $d > 0$ such that Z is d-dense and there exists $c < \sqrt{\frac{\pi}{16C(Z)}}$, such that for all $m \in \mathbb{N}$, the following holds:*

$$\mu(\partial D_m) \leq c(1-|z_m|^2)^2 \tag{3.3.2}$$

Then Z is a sampling sequence.

Let Z be a separated sequence with radius of density d and radius of separation R. The decay of μ in (3.3.2) can be shown to hold if both d and d/R are sufficiently small. The proof is too technical to be included here.

The proof of Theorem 3.3.2 will build on that of Proposition 3.2.4.

Proof. First, define Ω as in (3.3.1) and observe that the function $h(z, w) = \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^4}$ is subharmonic in both parameters $z, w \in \mathbb{D}$ and hence, in $\Omega \times \Omega$.

From there, fixing $z, w \in \Omega$, we have that:

$$h(z,w) = \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^4} \leq \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(\xi)-f(\eta)|^2}{|1-\overline{\eta}\xi|^4} d\omega(z,\xi,\Omega)d\omega(w,\eta,\Omega)$$

Now, by the definition of μ, and integrating on both sides of the previous inequality, we obtain:

$$\int_{\Omega}\int_{\Omega} \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^4} dA(z)dA(w) \leq \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(\xi)-f(\eta)|^2}{|1-\overline{\eta}\xi|^4} d\mu(\xi)d\mu(\eta)$$

In this moment, we apply Remark 3.1.1 to see that $\omega(z,\partial\mathbb{D},\Omega) = 0$ for any $z \in \Omega$. As a result, the right hand side in the previous inequality is an integral over those components of the boundary of Ω lying inside the unit disk:

$$\int_{\Omega}\int_{\Omega} \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^4} dA(z)dA(w) \leq \sum_{n,m}\int_{\partial D_n}\int_{\partial D_m} \frac{|f(\xi)-f(\eta)|^2}{|1-\overline{\eta}\xi|^4} d\mu(\xi)d\mu(\eta)$$

Adding and subtracting $f(z_n) - f(z_m)$ inside the modulus in the argument of the integral, the right hand side in the previous inequality is also equal to:

$$\sum_{n,m}\int_{\partial D_n}\int_{\partial D_m} \frac{|f(\xi)-f(z_n)+f(z_n)-f(z_m)+f(z_m)-f(\eta)|^2}{|1-\overline{\eta}\xi|^4} d\mu(\xi)d\mu(\eta)$$

Separate that integral, as in the proof of Theorem 3.2.4 part b), in 3 pieces, two of which are symmetric (in terms of the roles played by the variables) and the other is controlled by a constant C times the norm $||f||_Z^2$ (the constant depends now on c, as we substitute the element of area by $d\mu$).

The remainder will be short once we prove an auxiliar lemma:

Lemma 3.3.3. *Suppose that there exists $c < \sqrt{\frac{\pi}{16C(Z)}}$ such that for all $m \in \mathbb{N}$ we have:*

$$\mu(\partial D_m) \leq c(1-|z_m|^2)^2$$

Then there exists a $t = t(c, Z)$*, such that* $0 < t < 1$ *and:*

$$\sum_{n,m} \int_{\partial D_n} \int_{\partial D_m} \frac{|f(\xi) - f(z_n)|^2}{|1 - \overline{\eta}\xi|^4} d\mu(\xi) d\mu(\eta) \leq t ||f||_D^2$$

Proof of lemma. In the first place, we take out, from the inner integral, the term which does not depend on m:

$$\sum_{n,m} \int_{\partial D_n} \int_{\partial D_m} \frac{|f(\xi) - f(z_n)|^2}{|1 - \overline{\eta}\xi|^4} d\mu(\xi) d\mu(\eta) =$$

$$= \sum_{n} \int_{\partial D_n} |f(\xi) - f(z_n)|^2 \left(\sum_m \int_{\partial D_m} \frac{d\mu(\eta)}{|1 - \overline{\xi}\eta|^4} \right) d\mu(\xi) \tag{3.3.3}$$

Consider the variation appearing inside the outer integral. By the Fundamental Theorem of Calculus, between ξ and z_n we can see that:

$$|f(\xi) - f(z_n)| \leq \int_{z_n}^{\xi} |f'(s)| ds$$

Now, as a consequence of subharmonicity, taking the maximum when s runs over all the points in the segment joining ξ and z_n, it is clear that, for some constant $C_1(\epsilon)$ that tends to 1 when ϵ tends to 0, we have:

$$|f(\xi) - f(z_n)| \leq \frac{4C_1(\epsilon)}{\sqrt{\pi}} \max_s \left(\int_{D_H(s,\epsilon)} |f'(w)|^2 dA(w) \right)^{\frac{1}{2}}$$

The disk in which we are integrating is contained in that centered at z_n with hyperbolic radius 2ϵ and so we know:

$$|f(\xi) - f(z_n)|^2 \leq \frac{16(C_1(\epsilon))^2}{\pi} \int_{D_H(z_n,2\epsilon)} |f'(w)|^2 dA(w) \tag{3.3.4}$$

On the other hand, each term in the inner sum of integrals in (3.3.3) is controlled by the value of its argument at $\eta = z_m$ multiplied by the area of the disk:

$$\sum_m \int_{\partial D_m} \frac{d\mu(\eta)}{|1-\overline{\xi}\eta|^4} \leq C_2(\epsilon) \sum_m \frac{\mu(\partial D_m)}{|1-\overline{\xi}z_m|^4}$$

Here, $C_2(\epsilon)$ tends to 1 when ϵ tends to 0.

Applying our hypothesis we have:

$$\sum_m \int_{\partial D_m} \frac{d\mu(\eta)}{|1-\overline{\xi}\eta|^4} \leq cC_2(\epsilon) \sum_m \frac{(1-|z_m|^2)^2}{|1-\overline{\xi}z_m|^4}$$

The right hand side is controlled by the definition of $C(Z)$. Now, use this inequality together with (3.3.4) in the sum of integrals we want to bound, (3.3.3). First, we have just arrived to:

$$\begin{aligned} &\sum_{n,m} \int_{\partial D_n} \int_{\partial D_m} \frac{|f(\xi)-f(z_n)|^2}{|1-\overline{\eta}\xi|^4} d\mu(\xi) d\mu(\eta) \leq \\ \leq \quad & cC_2(\epsilon)C(Z) \sum_n \int_{\partial D_n} \frac{|f(\xi)-f(z_n)|^2}{(1-|\xi|^2)^2} d\mu(\xi) \end{aligned}$$

And now, using our initial estimate (3.3.4), we have that this is less or equal to:

$$\frac{16c(C_1(\epsilon))^2 C_2(\epsilon) C(Z)}{\pi} \sum_n \left(\int_{D_H(z_n, 2\epsilon)} |f'(w)|^2 dA(w) \right) \frac{\mu(\partial D_n)}{(1-|z_n|^2)^2}$$

Applying once more the hypothesis on $\mu(\partial D_n)$, we can bound this by:

$$\frac{16c^2(C_1(\epsilon))^2 C_2(\epsilon) C(Z)}{\pi} \int_{\mathbb{D}} |f'(w)|^2 dA(w)$$

So one can take $t = \frac{16c^2(C_1(\epsilon))^2 C_2(\epsilon) C(Z)}{\pi}$, for $\epsilon > 0$ small enough. □

Now, we have all the ingredients to end the proof of our Theorem. Until now we had seen that, for some $0 < t < 1$, and some constant C:

$$||f||_D^2 \leq C||f||_Z^2 + t||f||_D^2$$

From here, we can conclude that:

$$||f||_D^2 \leq \frac{C}{(1-t)}||f||_Z^2$$

This means, the sampling inequality holds.

□

Chapter 4

Cyclicity in Dirichlet-type spaces

4.1 Introduction

For $-\infty < \alpha < \infty$, the *Dirichlet-type space of order* α, D_α, consists of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

The spaces D_α become smaller as α increases, and $f \in D_\alpha$ if and only if the derivative $f' \in D_{\alpha-2}$. Three values of α correspond to spaces that have been studied extensively, and are often defined in terms of integrability:

- $\alpha = -1$ corresponds to *the Bergman space* B, consisting of functions with

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi},$$

- $\alpha = 0$ to *the Hardy space* H^2, consisting of functions with

$$\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$

- and $\alpha = 1$ to the usual *Dirichlet space* D of functions f with

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

A description like that of the Dirichlet space, in terms of an integral is possible for the D_α spaces for $\alpha < 2$. Indeed, when $\alpha < 2$, then $f \in D_\alpha$ if and only if

$$D_\alpha(f) = \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z) < \infty. \tag{4.1.1}$$

This expression may be used to define an equivalent norm for $f \in D_\alpha$, which we use in Section 4.2. We refer the reader to the books [12], [13] and [19] for in-depth treatments of Hardy and Bergman spaces; recent surveys concerning the Dirichlet space D include [3] and [33].

A function $f \in D_\alpha$ is said to be *cyclic* in D_α if the closed subspace generated by polynomial multiples of f,

$$[f] = \overline{\mathrm{span}\{z^k f \colon k = 0, 1, 2, \ldots\}},$$

coincides with D_α. The *multiplier space* $M(D_\alpha)$ consists of the analytic functions ψ with induced operator $M_\psi \colon f \mapsto \psi f$ maps D_α into itself; such a function ψ is called a *multiplier*. Thus cyclic functions are precisely those that are cyclic with respect to the operator M_z. Since the polynomials themselves are dense in the D_α spaces, $[1] = D_\alpha$. It is well known (see [11]) that an equivalent (and more useful) condition for the cyclicity of f is that there exist a sequence of polynomials $\{p_n\}_{n=1}^\infty$ with

$$\|p_n f - 1\|_\alpha \to 0, \quad \text{as } n \to \infty.$$

We note that for certain values of α, the multiplier spaces of D_α are relatively easy to determine. For $\alpha \leq 0$, we have $M(D_\alpha) = H^\infty$, and when $\alpha > 1$ the multiplier space coincides with D_α itself (see [11, p.273]).

For a general α, it is not an easy problem to characterize cyclic functions in D_α. However, for H^2 (the case $\alpha = 0$), a complete answer to the cyclicity problem is given by a theorem of Beurling (see [12, Chapter 7]): f is cyclic if and only if f is an outer function. In particular, a cyclic function $f \in H^2$ cannot vanish in $\mathbb{D}$; the additional condition is that its singular factor must be trivial. In the Bergman space, the situation is considerably more complicated (see [19, Chapter 7]). A common feature of all D_α is that cyclic functions have to be non-vanishing in $\mathbb{D}$. If $\alpha > 1$, being non-vanishing in the *closed* unit disk, or equivalently,

$$|f(z)| > c > 0, \quad z \in \mathbb{D},$$

is a necessary and sufficient condition (see [11]), but when $\alpha \leq 1$, functions may still be cyclic and have zeros on the unit circle $\mathbb{T}$. Here, we define the zero set in an appropriate sense via, for instance, non-tangential limits.

In [11], L. Brown and A.L. Shields studied the phenomenon of cyclicity in the Dirichlet space. In particular, they established the following equivalent condition for cyclicity: f is cyclic for D_α if and only if there exists a sequence of polynomials $\{p_n\}$ such that

$$\sup_n \|p_n f - 1\|_\alpha < \infty \tag{4.1.2}$$

and, pointwise as $n \to \infty$,

$$p_n(z)f(z) \to 1, \quad z \in \mathbb{D}. \tag{4.1.3}$$

Brown and Shields also obtained a number of partial results towards a characterization of cyclic vectors in the Dirichlet space D. Their starting point was a result of Beurling, stating that, for any $f \in D$, the non-tangential limit $f^*(\zeta) = \lim_{z\to\zeta} f(z)$ exists *quasi-everywhere*, that is, for every $\zeta \in \partial\mathbb{D}$ except, possibly, a set of logarithmic capacity zero. Brown and Shields proved that if the zeros of f^*,

$$\mathcal{Z}(f^*) = \{\zeta \in \mathbb{T} \colon f^*(\zeta) = 0\},$$

form a set of positive logarithmic capacity, then f cannot be cyclic. On the other hand, Brown and Shields proved that $(1-z)^\beta$ is cyclic for any $\beta > 0$, and they also showed that any polynomial without zeros in $\mathbb{D}$ is cyclic. Hence, they asked if being outer and having $\mathrm{cap}(\mathcal{Z}(f^*)) = 0$ is enough for f to be cyclic. This problem remains open and is usually referred to as the *Brown-Shields conjecture*; see however [15] for recent progress by El-Fallah, Kellay, and Ransford, and for background material. Subsequently, Brown and Cohn showed (see [10]) that sets of logarithmic capacity zero do support zeros of cyclic functions, and later Brown (see [9]) proved that if $f \in D$ is invertible, that is $1/f \in D$, then f is cyclic. Nevertheless, there exist functions that are cyclic but have $1/f \notin D$. An example is the function $f(z) = 1 - z$. The cyclicity properties of functions of this type is one of the themes of this chapter.

The problem of cyclicity in D has been addressed in many papers. An incomplete list includes [20], where sufficient conditions for cyclicity are given in terms of Bergman-Smirnov exceptional sets; the paper [14], where these ideas are developed further, and examples of uncountable Bergman-Smirnov exceptional sets are found; and [31] where multipliers and invariant subspaces are discussed, including, also, a proof that non-vanishing univalent functions in the Dirichlet space are cyclic.

4.1.1 Plan of the chapter

Instead of addressing the Brown-Shields conjecture and general cyclicity problems directly, we set for ourselves the more modest goal of understanding cyclicity better by studying certain classes of cyclic functions in detail. Most of the results in this chapter are a variation of the following themes: Suppose $f \in D_\alpha$ is cyclic. Can we obtain an explicit sequence of polynomials $\{p_n\}$ such that

$$\|p_n f - 1\|_\alpha \overset{n\to\infty}{\longrightarrow} 0?$$

Can we give an estimate on the rate of decay of these norms as $n \to \infty$? What can we say about the approximating polynomials?

A natural first guess is to take $\{p_n\}$ as the Taylor polynomials of the function $1/f$. Then, since $1/f$ is analytic in $\mathbb{D}$ by the cyclicity assumption, we have $p_n \to 1/f$ pointwise, and hence (4.1.3) is satisfied. However, it may be the case that norm boundedness fails: this is certainly true for the Taylor polynomials of $1/f$, that we denote $T_n(1/f)$, in the case $f(z) = 1 - z$. Clearly, $1/f \notin B \supset H^2 \supset D$, and a computation shows that

$$\|T_n(1/f)f - 1\|_D^2 = \|z^{n+1}\|_D^2 = n + 2.$$

Much of the development that follows is motivated by our goal of finding concrete substitutes for the Taylor polynomials of $1/f$.

Definition 4.1.1. Let $f \in D_\alpha$. We say that a polynomial p_n of degree at most n is an *optimal approximant* of order n to $1/f$ if p_n minimizes $\|pf - 1\|_\alpha$, among all polynomials p of degree at most n.

In other words, p_n is an optimal polynomial of order n to $1/f$ if

$$\|p_n f - 1\|_\alpha = \mathrm{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n),$$

where $\mathcal{P}_n$ denotes the space of polynomials of degree at most n and

$$\mathrm{dist}_X(x, A) = \inf\{\|x - a\|_X : a \in A\}$$

for any normed space X, $A \subseteq X$ and $x \in X$.

Notice that, given $f \in D_\alpha \setminus \{0\}$, the existence and uniqueness of an optimal approximant of order n to $1/f$ follows immediately from the fact that $f \cdot \mathcal{P}_n$ is a finite dimensional subspace of the Hilbert space D_α. Thus, f is cyclic if and only if the optimal approximants p_n of order n to $1/f$ satisfy $\|p_n f - 1\|_\alpha \to 0$ as $n \to \infty$. Furthermore, since $\|p_n f - 1\|_\alpha \leq \|f - 1\|_\alpha$, it follows from (4.1.2) and (4.1.3) that f is cyclic if and only if the sequence of optimal approximants $\{p_n\}_{n=1}^\infty$ converges pointwise to $1/f$.

In Section 4.2, we describe a constructive approach for computing the coefficients of the optimal approximant of order n to $1/f$ for a general function f. In particular, Theorem 4.2.1 below states that the coefficients of the optimal approximants can be computed as ratios of determinants of matrices whose entries can be explicitly computed via the moments of the derivative of f. When f itself is a polynomial, these matrices are banded (see Proposition 4.2.2). As a simple but fundamental example, we compute optimal approximants to the function $1/f$ when $f(z) = 1 - z$.

We are also interested in the rate of convergence of these optimal approximants:

Definition 4.1.2. Let $f \in D_\alpha$. The *optimal norm* of degree n associated with f is

$$N_{n,\alpha}(f) = \|p_n f - 1\|_\alpha^2,$$

where p_n is the optimal approximant of $1/f$ of degree n.

Note that we use the word "norm" by abuse of terminology, even though we are referring to the square of a norm. The optimal norm $N_{n,\alpha}(f)$ will decay exponentially for any function f such that $1/f$ is analytic in the closed unit disk. Therefore, functions that have zeros on the unit circle are of particular interest. In Section 4.3, we examine the question of whether all functions with no zeros in the open unit disk but with zeros on the boundary, admitting an analytic continuation to a bigger disk, have optimal norm achieving a similar speed of decay. In other words, are all such functions "equally good" for the purpose of cyclicity? In Theorem 4.3.7, we prove that this is, indeed, the case by giving bounds for the optimal norms. We conclude the section with considerations about the further extension of these results to a larger class of functions.

In Section 4.4, we deal with a generalization to all D_α of a subproblem of the Brown-Shields conjecture. We ask the question whether a function f satisfying $f \in D_\alpha$ and $\log f \in D_\alpha$, must be cyclic in D_α. We note that this is true in the simple cases of $\alpha = 0$ or $\alpha > 1$. In Theorem 4.4.4 we are able to solve in the affirmative the case $\alpha = 1$. Then, Theorem 4.4.5 shows that for the case $\alpha < 1, \alpha \neq 0$, the same holds with an additional technical condition. We do not know if this condition is necessary; however, it is satisfied by a large class of examples, namely, all the functions constructed in Brown-Cohn ([10]).

We conclude, in Section 5, with open questions and basic computations around the zero sets $\mathcal{Z}(p_n)$ of the optimal approximants p_n of $1/f$ for cyclic functions f.

4.2 Construction of optimal approximants

The optimal approximants p_n of order n to $1/f$ are determined by the fact that $p_n f$ is the orthogonal projection of 1 onto the space $f \cdot \mathcal{P}_n$, and hence, in principle, if $f \in D_\alpha \setminus \{0\}$, they can be computed using the Gram-Schmidt process. More precisely, once a basis for $f \cdot \mathcal{P}_n$ is chosen, one can construct an orthonormal basis for $f \cdot \mathcal{P}_n$ to compute the coefficients of p_n with respect to this orthonormal basis.

In this section, we present a simple method which yields the optimal approximants p_n without the use of the Gram-Schmidt process, for $\alpha < 2$. To that end, we make use of the integral norm of D_α defined in (4.1.1), namely,

$$\|f\|_\alpha^2 = |f(0)|^2 + D_\alpha(f).$$

Recall that we seek an explicit solution to

Problem 1. Let $n \in \mathbb{N}$. Given $f \in D_\alpha \setminus \{0\}$,

$$\text{minimize} \quad \|pf - 1\|_\alpha \ \text{ over } p \in \mathcal{P}_n\,.$$

As mentioned in Section 4.1.1, there is a unique optimal approximant $p_n \in \mathcal{P}_n$ of order n to $1/f$ that solves Problem 1, that is,

$$\|p_n f - 1\|_\alpha = \operatorname{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n).$$

Observe that for any polynomial $p(z) = \sum_{k=0}^{n} c_k z^k \in \mathcal{P}_n$,

$$\begin{aligned}\|pf - 1\|_\alpha^2 &= |p(0)f(0) - 1|^2 + \int_{\mathbb{D}} |(pf)'|^2 d\mu_\alpha \\ &= |p(0)f(0) - 1|^2 + \int_{\mathbb{D}} \left| \sum_{k=0}^{n} c_k (z^k f)' \right|^2 d\mu_\alpha,\end{aligned}$$

where $d\mu_\alpha(z) = (1 - |z|^2)^{1-\alpha}\, dA(z)$. Thus, if the optimal approximant of order n to $1/f$ vanishes at the origin, then $\|pf - 1\|_\alpha^2$ is minimal if and only if $c_0 = c_1 = \ldots = c_n = 0$. Since we will be dealing with a cyclic f, we may assume $p_n(0) \neq 0$. By replacing f with $p_n(0)f$, we may also assume that $p_n(0) = 1$. Hence, under this latter assumption, $p_n(z) = 1 + \sum_{k=1}^{n} c_n^* z^k$ is the optimal approximant of order n to $1/f$ if and only if $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to

Problem 2. Let $n \in \mathbb{N}$. Given $f \in D_\alpha \setminus \{0\}$,

$$\text{minimize} \quad \int_{\mathbb{D}} \left| f' + \sum_{k=1}^{n} c_k (z^k f)' \right|^2 d\mu_\alpha \text{ over } (c_1, \ldots, c_n) \in \mathbb{C}^n.$$

It is evident that $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to Problem 2 if and only if

$$g = \sum_{k=1}^{n} c_k^* (z^k f)' \;\text{ satisfies }\; \|f' + g\|_{L^2(\mu_\alpha)} = \mathrm{dist}_{L^2(\mu_\alpha)}(f', Y),$$

where $Y = \mathrm{span}\{(z^k f)' : 1 \leq k \leq n\}$. Equivalently, $f' + g$ is orthogonal to Y with respect to the $L^2(\mu_\alpha)$ inner product; that is, for each j, $1 \leq j \leq n$,

$$\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \langle g, (z^j f)' \rangle_{L^2(\mu_\alpha)}.$$

Hence, $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to Problem 2 if and only if it is the solution to the non-homogeneous system of linear equations

$$\sum_{k=1}^{n} c_k \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}, \quad 1 \leq j \leq n, \tag{4.2.1}$$

with $(c_1, \dots, c_n) \in \mathbb{C}^n$.

Theorem 4.2.1. *Let $n \in \mathbb{N}$ and $f \in D_\alpha \setminus \{0\}$. Let M denote the $n \times n$ matrix with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$. Then the unique $p_n \in \mathcal{P}_n$ satisfying*

$$\|p_n f - 1\|_\alpha = \operatorname{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$$

is given by

$$p_n(z) = p_n(0) \left(1 + \sum_{k=1}^{n} \frac{\det M^{(k)}}{\det M} z^k \right), \tag{4.2.2}$$

where $M^{(k)}$ denotes the $n \times n$ matrix obtained from M by replacing the kth column of M by the column with entries $\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \le j \le n$.

Proof. As mentioned before, if p_n is the optimal approximant of order n to f and $p_n(0) \neq 0$, then the optimal approximant of order n to $1/f_n$ is $[p_n(0)]^{-1} p_n$, where $f_n = p_n(0) f$. If $[p_n(0)]^{-1} p_n = 1 + \sum_{k=1}^{n} c_k^* z^k$, then $(c_1^*, \dots, c_n^*) \in \mathbb{C}^n$ is the unique solution to the system in (4.2.1) because

$$\langle (z^k f_n)', (z^j f_n)' \rangle_{L^2(\mu_\alpha)} = |p_n(0)|^2 \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$$

for $0 \le k \le n$ and $1 \le j \le n$. It follows now that the $n \times n$ matrix M with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$ has non-zero determinant and thus

$$c_k^* = \frac{\det M^{(k)}}{\det M}, \quad 1 \le k \le n,$$

by Cramer's rule, where $M^{(k)}$ denotes the $n \times n$ matrix obtained from M by replacing the kth column of M by the column with entries $\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \le j \le n$. Hence p_n is given by (4.2.2). □

If f is a polynomial, then the computation of the determinants appearing in Theorem 4.2.1 can be simplified in view of the following proposition.

Proposition 4.2.2. *Suppose f is a polynomial of degree t. Then the matrix M in Theorem 4.2.1 is banded and has bandwidth at most $2t+1$.*

Proof. The orthogonality of z^l and z^m for $l \neq m$ (under the $L^2(\mu_\alpha)$ inner product) implies that the (j,k)-entry of M equals 0 if the degree of $(z^k f)'$ is strictly less than $j-1$, that is, $k+t-1 < j-1$, or if the degree of $(z^j f)'$ is strictly less than $k-1$, that is, $j+t-1 < k-1$. Therefore, the only entries of M that do not necessarily vanish are the ones whose indices j and k satisfy $-t \leq j-k \leq t$. Thus, M is banded and has bandwidth at most $2t+1$. $\square$

4.2.1 An explicit example of optimal approximants

Now, we calculate explicitly optimal approximants to $1/f$, where $f(z) = 1 - z$. This example is already interesting because f is cyclic in D_α for $\alpha \leq 1$, even though it is not invertible for any $\alpha \geq -1$.

We begin with some general computations. Let $\beta = 1 - \alpha$. Then

$$\|z^m\|^2_{L^2(\mu_\alpha)} = \int_0^1 u^m (1-u)^{1-\alpha}\, du = \frac{-1}{m+2-\alpha} \prod_{\ell=1}^{m} \frac{\ell}{\ell+\beta}$$

holds for any non-negative integer m. Therefore, if $f = \sum_{i=0}^{t} a_i z^i$, we have, under the usual convention that $a_i = 0$ for any integer $i < 0$ or $i > t$,

$$\begin{aligned} \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)} &= \sum_{i=0}^{t} \sum_{\ell=0}^{t} a_i \bar{a}_\ell (i+k)(\ell+j) \langle z^{i+k-1}, z^{\ell+j-1} \rangle_{L^2(\mu_\alpha)} \\ &= \sum_{i=0}^{t} a_i \bar{a}_{i+k-j} (i+k)^2 \|z^{i+k-1}\|^2_{L^2(\mu_\alpha)} \\ &= \sum_{i=0}^{t} a_i \bar{a}_{i+k-j} (i+k) \prod_{\ell=1}^{i+k} \frac{\ell}{\ell+\beta} \end{aligned} \tag{4.2.3}$$

as z^l and z^m are orthogonal for $l \neq m$ under the $L^2(\mu_\alpha)$ inner product. Since $p_n(0)$ is non-zero, we conclude that the optimal approximant $p_n(z) = p_n(0)\left(1 + \sum_{i=1}^{n} c_i z^i\right)$ of order n

to $1/f$ satisfies the system

$$\sum_{k=1}^{n} c_k \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}, \quad 1 \le j \le n. \tag{4.2.4}$$

Let $n \in \mathbb{N}$ and $f(z) = 1 - z$. We proceed to compute optimal approximants $p_n(z) = \sum_{i=0}^{n} c_i z^i$ of order n to $1/f$. Let $M_n = M$ and $M_n^{(k)} = M^{(k)}$ be the $n \times n$ matrices corresponding to f as in Theorem 4.2.1. By Proposition 4.2.2, the matrix M_n is tridiagonal and so it suffices to compute the coefficients above and below each entry of its main diagonal. We simplify notation by calling, for $k \in \mathbb{N}$,

$$\Lambda_\beta(k) = k \prod_{\ell=1}^{k} \frac{\ell}{\ell + \beta}$$

Since $a_0 = 1$ and $a_1 = -1$, it follows from (4.2.3) that

$$\begin{aligned} \langle (z^k f)', (z^{k-1} f)' \rangle_{L^2(\mu_\alpha)} &= -\Lambda_\beta(k), \\ \langle (z^k f)', (z^k f)' \rangle_{L^2(\mu_\alpha)} &= \Lambda_\beta(k) + \Lambda_\beta(k+1), \\ \langle (z^k f)', (z^{k+1} f)' \rangle_{L^2(\mu_\alpha)} &= -\Lambda_\beta(k+1), \quad \text{and} \\ \langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)} &= \begin{cases} \Lambda_\beta(1) & \text{if } j = 1 \\ 0 & \text{if } j \ge 2 \end{cases}. \end{aligned}$$

Thus, in view of (4.2.4), the coefficients of p_n satisfy the sytem of equations

$$\begin{aligned} c_1 \left[\Lambda_\beta(1) + \Lambda_\beta(2)\right] - c_2 \left[\Lambda_\beta(2)\right] &= \Lambda_\beta(1) \\ -c_{j-1} \left[\Lambda_\beta(j)\right] + c_j \left[\Lambda_\beta(j) + \Lambda_\beta(j+1)\right] - c_{j+1} \left[\Lambda_\beta(j+1)\right] &= 0 \\ -c_{n-1} \left[\Lambda_\beta(n)\right] + c_n \left[\Lambda_\beta(n) + \Lambda_\beta(n+1)\right] &= 0 \end{aligned}$$

or, interpreting $c_{n+1} = 0$, equivalently, for all $2 \le j \le n+1$:

$$\Lambda_\beta(j)(c_j - c_{j-1}) = \Lambda_\beta(1)(c_1 - 1)$$

For fixed k, $2 \leq k \leq n+1$, by a repeated use of the previous identity, we obtain the following:

$$c_k = \left[\Lambda_\beta(1)\sum_{j=1}^{k}\frac{1}{\Lambda_\beta(j)}\right](c_1-1)+1 \tag{4.2.5}$$

On the other hand, we also have

$$\Lambda_\beta(1)(c_1-1) = -\Lambda_\beta(n+1)c_n$$

and so, we can recover the value of c_1,

$$c_1 - 1 = -\frac{1}{\Lambda_\beta(1)\sum_{j=1}^{n+1}\frac{1}{\Lambda_\beta(j)}}.$$

Finally, we obtain the explicit solution, which can be expressed as follows, for $1 \leq k \leq n$:

$$c_k = \left[\sum_{j=k+1}^{n+1}\frac{1}{j}\prod_{\ell=2}^{j}\left(1+\frac{\beta}{\ell}\right)\right]\left[\sum_{j=1}^{n+1}\frac{1}{j}\prod_{\ell=2}^{j}\left(1+\frac{\beta}{\ell}\right)\right]^{-1} \tag{4.2.6}$$

Whenever the value of β is a natural number, the product in the previous equation, $\prod_{l=2}^{j}(1+\beta/l)$, has a large proportion of cancelations allowing to compute exactly the polynomials general formula. In the particular case of the Dirichlet space, for any integer n, the optimal approximant is an example of a generalized Riesz mean polynomial: more specifically, defining $H_n = \sum_{j=1}^{n}\frac{1}{j}$ and $H_0 = 0$,

$$p_n(z) = p_n(0)\left(\sum_{k=0}^{n}\left(1-\frac{H_k}{H_{n+1}}\right)z^k\right).$$

If we look at the case $\beta = 1$, the Hardy space, the optimal approximant is a modified Cesàro mean polynomial,

$$p_n(z) = p_n(0)\left(\sum_{k=0}^{n}\left(1-\frac{k+H_k}{n+1+H_{n+1}}\right)z^k\right),$$

and for the Bergman space, $\beta = 2$, the optimal approximants are

$$p_n(z) = p_n(0)\left(1 + \sum_{k=1}^{n}\left(1 - \frac{k(k+7) + 4H_k}{(n+1)(n+8) + 4H_{n+1}}\right)z^k\right).$$

We will return to these polynomials in Section 4.3.

4.3 Rate of decay of the optimal norms

In this section, we obtain estimates for $\operatorname{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$ as $n \to \infty$, when f is an analytic function in the closed unit disk without zeros in $\mathbb{D}$. First, we study further the example for which we computed the optimal polynomials at the end of the previous section. As we will see, it is useful as a model for the general case.

To simplify notation, define the auxiliary function φ_α on $[0, \infty)$ to be

$$\varphi_\alpha(s) = \begin{cases} s^{1-\alpha}, & \text{if } \alpha < 1 \\ \log^+(s), & \text{if } \alpha = 1. \end{cases}$$

Lemma 4.3.1. *If $f(z) = \zeta - z$, for $\zeta \in \mathbb{T}$, then $\operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_n)$ is comparable to $1/\varphi_\alpha(n+1)$.*

Proof. First notice that if p_n are polynomials such that $p_n(z)(1-z) - 1 \to 0$ in D_α, then, since rotation by $\zeta \in \mathbb{T}$ is an isometry in D_α, the polynomials $q_n(z) := \overline{\zeta} p_n(\overline{\zeta} z)$ are approximants for $\zeta - z$ with exactly the same growth rate. Therefore, it is enough to consider $f(z) = 1 - z$.

Now, recall that by (4.2.6), if $f(z) = 1 - z$, the optimal approximant of order n to $1/f$ is

$$p_n(z) = p_n(0)\sum_{k=0}^{n} c_k z^k$$

where

$$c_k = \left[\sum_{j=k+1}^{n+1} \frac{1}{j}\prod_{\ell=2}^{j}\left(1 + \frac{\beta}{\ell}\right)\right]\left[\sum_{j=1}^{n+1} \frac{1}{j}\prod_{\ell=2}^{j}\left(1 + \frac{\beta}{\ell}\right)\right]^{-1}, \; 0 \le k \le n,$$

and $\beta = 1 - \alpha$. We claim that $\|p_n f - 1\|_\alpha^2$ is comparable to $1/\varphi_\alpha(n+1)$.

First of all, notice that

$$p_n(z)f(z) - 1 = p_n(0) - 1 + p_n(0)\left[\sum_{k=1}^{n}(c_k - c_{k-1})z^k - c_n z^{n+1}\right].$$

To simplify notation, define for $1 \leq k \leq n$

$$d_k = c_k - c_{k-1} = -\left[\frac{1}{k}\prod_{\ell=2}^{k}\left(1+\frac{\beta}{\ell}\right)\right]\left[\sum_{j=1}^{n+1}\frac{1}{j}\prod_{\ell=2}^{j}\left(1+\frac{\beta}{\ell}\right)\right]^{-1}$$

and $d_{n+1} = -c_n$. Then

$$\sum_{k=1}^{n} k^\alpha |d_k|^2 = \left[\sum_{j=1}^{n+1}\frac{1}{j}\prod_{\ell=2}^{j}\left(1+\frac{\beta}{\ell}\right)\right]^{-2}\sum_{k=1}^{n} k^{\alpha-2}\left[\prod_{\ell=2}^{k}\left(1+\frac{\beta}{\ell}\right)\right]^2. \tag{4.3.1}$$

Recalling that $x/2 \leq \log(1+x) \leq x$ holds for all $x \in [0,1]$, the product

$$\prod_{\ell=2}^{k}\left(1+\frac{\beta}{\ell}\right) = \exp\left[\sum_{\ell=2}^{k}\log\left(1+\frac{\beta}{\ell}\right)\right]$$

is comparable to

$$\exp\left[\beta\sum_{\ell=2}^{k}\frac{1}{\ell}\right]$$

and so, comparable to k^β. Thus, the sum in (4.3.1) is comparable to:

$$\left[\sum_{j=1}^{n+1} j^{\beta-1}\right]^{-2}\sum_{k=1}^{n} k^{2\beta+\alpha-2} = \left[\sum_{j=1}^{n+1}\frac{1}{j^\alpha}\right]^{-2}\sum_{k=1}^{n}\frac{1}{k^\alpha}$$

Moreover,

$$(n+1)^\alpha |d_{n+1}|^2 = (n+1)^\alpha\left[\frac{1}{n+1}\prod_{\ell=2}^{n+1}\left(1+\frac{\beta}{\ell}\right)\right]^2\left[\sum_{j=1}^{n+1}\frac{1}{j}\prod_{\ell=2}^{j}\left(1+\frac{\beta}{\ell}\right)\right]^{-2}$$

is comparable to $(n+1)^{\alpha-2}$

Putting all together, since $\sum_{j=1}^{n} j^{-\alpha}$ is comparable to $\varphi_\alpha(n+1)$, the sum

$$\sum_{k=1}^{n+1} k^\alpha |d_k|^2$$

is comparable to $1/\varphi_\alpha(n+1)$. This proves the lemma. □

We wonder whether such estimates also hold for more general functions and this is the motivation for our following study. First, we will generalize the previous Lemma to the case when f is a polynomial whose zeros are simple and lie in $\mathbb{C} \setminus \mathbb{D}$.

To begin, let us first introduce some notation. Let $A(\mathbb{T})$ denote the *Wiener algebra*, that is, $A(\mathbb{T})$ consists of functions f, defined on $\mathbb{T}$, whose Fourier coefficients are absolutely sumable, and is equipped with the norm

$$\|f\|_{A(\mathbb{T})} = \sum_{k=-\infty}^{\infty} |a_k|.$$

The positive Wiener algebra consists of analytic functions whose Fourier coefficients satisfy $\sum_{k=0}^{\infty} |a_k| < \infty$; in particular, these functions belong to H^∞, the space of bounded analytic functions in $\mathbb{D}$ and $\|f\|_{H^\infty} \le \|f\|_{A(\mathbb{T})}$ holds for all $f \in H^\infty$, where $\|f\|_{H^\infty} = \sup\{|f(z)| \colon z \in \mathbb{D}\}$.

Proposition 4.3.2. *Let $\alpha \le 1$, $t \in \mathbb{N}$ and f be a polynomial of degree t. If the zeros of f are simple and lie in $\mathbb{C}\setminus\mathbb{D}$, then for each $n > t$ there is $p_n \in \mathcal{P}_n$ such that $(p_n f)(0) = 1$ and*

$$\|p_n f - 1\|_\alpha^2 \le \frac{C}{\varphi_\alpha(n+1)}, \tag{4.3.2}$$

holds for some constant C that depends on f and α but not on n, and such that the sequence $\{p_n f\}_{n>t}$ is uniformly bounded in $A(\mathbb{T})-$norm.

Proof. Suppose f has simple zeros $z_1, \dots, z_t \in \mathbb{C} \setminus \mathbb{D}$. Then there are constants $d_1, \dots, d_t$ such that

$$\frac{1}{f(z)} = \sum_{j=1}^{t} \frac{d_j}{z_j - z} = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{t} \frac{d_j}{z_j^{k+1}} \right) z^k.$$

Define $b_k = \sum_{j=1}^{t} d_j z_j^{-(k+1)}$ for $k \geq 0$. It follows that the sequence $\{b_k\}_{k=0}^{\infty}$ is bounded in modulus by $\sum_{j=1}^{t} |d_j|$, and the Taylor series of $1/f$ is of the form

$$\frac{1}{f(z)} = \sum_{k=0}^{\infty} b_k z^k.$$

Let $f(z) = \sum_{k=0}^{t} a_k z^k$. Set $a_k = 0$ for $k > t$. Consequently, $f \cdot 1/f - 1 = 0$ translates into

$$\sum_{j=0}^{k} b_j a_{k-j} = 0 \quad \text{for } k \in \mathbb{N}\backslash\{0\}. \tag{4.3.3}$$

Consider the polynomial $p_n(z) = \sum_{k=0}^{n} c_k z^k$ with coefficients

$$c_0 = a_0^{-1} \quad \text{and} \quad c_k = \left(1 - \frac{\varphi_\alpha(k)}{\varphi_\alpha(n+1)}\right) b_k \text{ for } 1 \leq k \leq n.$$

For convenience of notation, we will consider $c_k = 0$ if $k > n$. Evidently, $(p_n f)(0) = 1$. Let us prove (4.3.2). To estimate $\|p_n f - 1\|_\alpha^2$, we consider separately the norms of

$$\mathbf{mp} = \sum_{k=t+1}^{n+t} \left(\sum_{i=0}^{k} c_i a_{k-i}\right) z^k, \quad \text{and}$$
$$\mathbf{sp} = \sum_{k=1}^{t} \left(\sum_{i=0}^{k} c_i a_{k-i}\right) z^k,$$

and note that

$$\|p_n f - 1\|_\alpha^2 = \|\mathbf{mp}\|_\alpha^2 + \|\mathbf{sp}\|_\alpha^2 \tag{4.3.4}$$

and

$$\sum_{i=0}^{k} c_i a_{k-i} = \frac{-1}{\varphi_\alpha(n+1)} \sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i} \tag{4.3.5}$$

by (4.3.3). To estimate the norm of $\mathbf{mp}$, we need the following result.

Lemma 4.3.3. *Under the assumptions of Proposition 4.3.2, if $k > t$, there is a constant $C = C(\alpha, f)$ such that*

$$\left|\sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i}\right| \leq \frac{C}{(k+1)^\alpha}.$$

We finish the proof of Proposition 4.3.2 before proving the Lemma.

By (4.3.5) and the Lemma 4.3.3,

$$\begin{aligned}\|\mathbf{mp}\|_\alpha^2 &= \sum_{k=t+1}^{n+t}\left|\sum_{i=0}^{k} c_i a_{k-i}\right|^2 (k+1)^\alpha \\ &\leq \frac{C_1}{\varphi_\alpha^2(n+1)}\sum_{k=t+1}^{n+t}\frac{1}{(k+1)^\alpha}\end{aligned}$$

for some constant $C_1 = C_1(\alpha, f)$. It follows now that, for $t \leq n$ there is a constant $C_2 = C_2(\alpha, f)$ such that

$$\sum_{k=t+1}^{n+t}\frac{1}{(k+1)^\alpha} \leq C_2\varphi_\alpha(n+1) \tag{4.3.6}$$

and so

$$\|\mathbf{mp}\|_\alpha^2 \leq \frac{C_1C_2}{\varphi_\alpha(n+1)}. \tag{4.3.7}$$

Next, we estimate the norm of **sp**. Recalling (4.3.5), we see that

$$\|\mathbf{sp}\|_\alpha^2 = \frac{1}{\varphi_\alpha^2(n+1)}\sum_{k=1}^{t}\left|\sum_{i=0}^{k}\varphi_\alpha(i)b_i a_{k-i}\right|^2 (k+1)^\alpha.$$

By the Triangle inequality and since φ is increasing, if $1 \leq k \leq t$, then

$$\left|\sum_{i=0}^{k}\varphi_\alpha(i)b_i a_{k-i}\right| \leq \|b\|_{\ell^\infty}\|a\|_{\ell^\infty}(t+1)\varphi_\alpha(t), \tag{4.3.8}$$

where $a = \{a_k\}_{k=0}^\infty$ and $b = \{b_k\}_{k=0}^\infty$. Thus,

$$\|\mathbf{sp}\|_\alpha^2 \leq \frac{1}{\varphi_\alpha^2(n+1)}\|b\|_{\ell^\infty}^2\|a\|_{\ell^\infty}^2(t+1)^2\varphi_\alpha^2(t)\sum_{k=1}^{t}(k+1)^\alpha$$

and so, for some constant $C_3 = C_3(\alpha, f)$,

$$\|\mathbf{sp}\|_\alpha^2 \leq \frac{C_3}{\varphi_\alpha^2(n+1)}. \tag{4.3.9}$$

Hence, (4.3.2) follows from (4.3.4), (4.3.7) and (4.3.9).

Finally, we show that the sequence $\{p_n f\}_{n>t}$ is bounded in $A(\mathbb{T})$. Notice that, for $1 \leq k \leq n+t$, (4.3.5) and Lemma 4.3.3 imply

$$\left|\sum_{i=0}^{k} c_i a_{k-i}\right| \leq \frac{C}{(k+1)^\alpha \varphi_\alpha(n+1)} \tag{4.3.10}$$

for some constant $C = C(\alpha, f)$. Therefore, by (4.3.6),

$$\|p_n f\|_{A(\mathbb{T})} = \sum_{k=1}^{n+t}\left|\sum_{i=0}^{k} c_i a_{k-i}\right| \leq \frac{C}{\varphi_\alpha(n+1)} \sum_{k=1}^{n+t} \frac{1}{(k+1)^\alpha} \leq C(\alpha, f).$$

□

We now proceed to prove Lemma 4.3.3.

Proof of Lemma 4.3.3. For $k-t \leq s \leq k$, for $k \geq t+1$ and for any α we have

$$\varphi_\alpha'(s) \leq C(\alpha, t)(k+1)^{-\alpha} \tag{4.3.11}$$

and hence, for $k \geq i$

$$\varphi_\alpha(k) - \varphi_\alpha(i) \leq C(\alpha, t)(k-i)(k+1)^{-\alpha}.$$

Recalling (4.3.3) and that $a_i = 0$ for $i > t$, we obtain

$$\begin{aligned}\left|\sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i}\right| &= \left|\sum_{i=0}^{k} [\varphi_\alpha(k) - \varphi_\alpha(i)] b_i a_{k-i}\right| \\ &\leq \sum_{i=k-t}^{k} |\varphi_\alpha(k) - \varphi_\alpha(i)| \cdot |b_i a_{k-i}| \\ &\leq \|a\|_{\ell^\infty} \|b\|_{\ell^\infty} C(k+1)^{-\alpha} \sum_{i=k-t}^{k} (k-i),\end{aligned}$$

where $a = \{a_i\}_{i=0}^\infty$ and $b = \{b_i\}_{i=0}^\infty$. □

It seems natural to ask whether the growth rate given in Proposition 4.3.2 is sharp, and whether the proof can be extended to polynomials f whose zeros are not necessarily simple. Regarding the second question, however, even in the simple case of $f(z) = (1 - z)^2$, the coefficients of the Taylor series representation centered at the origin of $1/f$ are not bounded; consequently, the proof of Proposition 4.3.2 cannot be extended directly because the boundedness of these coefficients is needed. Nevertheless, if f is an arbitrary polynomial, we will use Proposition 4.3.2 to obtain an estimate for $\text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$.

Now let's show that the conclusion of Proposition 4.3.2 holds for any polynomial, and that the growth rate is sharp.

Theorem 4.3.4. *Let $\alpha \leq 1$. If f is a polynomial whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that*

$$\text{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m) \leq \frac{C}{\varphi_\alpha(m+1)} \tag{4.3.12}$$

holds for all m. Moreover, this estimate is sharp, in the sense that if such a polynomial f has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\frac{\tilde{C}}{\varphi_\alpha(m+1)} \leq \text{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m).$$

Proof. Suppose f has factorization

$$f(z) = K \prod_{k=1}^{s} (z - z_k)^{r_k}$$

with $r_1, \ldots, r_s \in \mathbb{N}$, $z_1, \ldots, z_s \in \mathbb{C} \setminus \mathbb{D}$ are distinct, and $K \in \mathbb{C} \setminus \{0\}$. Define

$$g(z) = \prod_{k=1}^{s} (z - z_k) \quad \text{and} \quad h(z) = K^{-1} \prod_{k=1}^{s} (z - z_k)^{\gamma - r_k},$$

where $\gamma = \max\{r_1, \ldots, r_s\}$, and let d equal the degree of h. Then $fh = g^\gamma$,

$$\text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_{n+d}) \leq \text{dist}_{D_\alpha}(1, fh \cdot \mathcal{P}_n) \quad \text{for } n \in \mathbb{N}. \tag{4.3.13}$$

Since the zeros of g are simple and lie in $\mathbb{C}\setminus\mathbb{D}$, by Proposition 4.3.2, for $n > s$, we can choose $q_n \in \mathcal{P}_n$ such that $(q_n g)(0) = 1$ and

$$\|q_n g - 1\|_\alpha^2 \le \frac{C_1}{\varphi_\alpha(n+1)} \tag{4.3.14}$$

holds for some $C_1 = C_1(\alpha, g)$, and such that the sequence $\{q_n g\}_{n>s}$ is bounded in $A(\mathbb{T})$.

Let $d\mu_\alpha(z) = (1-|z|^2)^{1-\alpha}\, dA(z)$. Recalling that $\|p\|_\alpha^2$ is comparable to $|p(0)|^2 + D_\alpha(p) = |p(0)|^2 + \|p'\|^2_{L^2(\mu_\alpha)}$ for all $p \in D_\alpha$, we obtain

$$\begin{aligned}\|q_n^\gamma g^\gamma - 1\|_\alpha^2 &\le C_2 \|(q_n^\gamma g^\gamma)'\|^2_{L^2(\mu_\alpha)} \\ &\le C_2 \gamma^2 \|q_n g\|_{H^\infty}^{2\gamma-2} \|q_n' g + q_n g'\|^2_{L^2(\mu_\alpha)} \\ &\le C_3 \gamma^2 \|q_n g\|_{A(\mathbb{T})}^{2\gamma-2} \|q_n g - 1\|_\alpha^2 \end{aligned} \tag{4.3.15}$$

for some constants $C_2 = C_2(\alpha)$ and $C_3 = C_3(\alpha)$, as $(q_n g)(0) = 1$. Therefore, (4.3.14) and (4.3.15) imply that there is a constant $C_4 = C_4(\alpha, \gamma, g)$ such that

$$\operatorname{dist}^2_{D_\alpha}(1, g^\gamma \cdot \mathcal{P}_{n\gamma}) \le \frac{C_4}{\varphi_\alpha(n+1)}$$

because $q_n^\gamma \in \mathcal{P}_{n\gamma}$ and $\{q_n g\}_{n>s}$ is bounded in $A(\mathbb{T})$. Thus, by (4.3.13),

$$\operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_{n\gamma+d}) \le \frac{C_4}{\varphi_\alpha(n+1)}. \tag{4.3.16}$$

For fixed a and b, $\varphi_\alpha(n+1)$ is comparable to $\varphi_\alpha(am+b)$. Hence, (4.3.12) holds.

Let us now show that the inequality is sharp. If f is any polynomial with zeros outside $\mathbb{D}$ that has at least one zero on $\mathbb{T}$, then $f(z) = h(z)(\zeta - z)$ for some polynomial h of degree say d. Then for any polynomial p_m of degree at most m,

$$\|p_m(z)h(z)(\zeta - z) - 1\|_\alpha^2 \ge \operatorname{dist}^2_{D_\alpha}(1, (\zeta - z) \cdot \mathcal{P}_{m+d}).$$

By Lemma 4.3.1, there exists a constant $C_1 = C_1(\alpha)$ such that

$$\text{dist}^2_{D_\alpha}(1, (\zeta - z) \cdot \mathcal{P}_{m+d}) \geq \frac{C_1}{\varphi_\alpha(m+d+1)}.$$

Now, we can choose a constant $C_2 = C_2(\alpha, d)$ such that

$$\frac{1}{\varphi_\alpha(m+d+1)} \geq \frac{C_2}{\varphi_\alpha(m+1)}.$$

Finally, letting $\tilde{C} = C_1 C_2$ and noting that the polynomial p_m was arbitrary, we obtain the desired result that

$$\text{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m) \geq \frac{\tilde{C}}{\varphi_\alpha(m+1)}.$$

□

In fact, we are going to show that the rates in Theorem 4.3.4 hold for more general functions f, namely functions that have an analytic continuation to the closed unit disk. Since such functions can be factored as $f(z) = h(z)g(z)$, where h is a polynomial with a finite number of zeros on the circle and g is a function analytic in the closed disk with no zeros there, the estimates in Theorem 4.3.4 hold for h. Moreover, we can obtain estimates on g that will allow us to give upper bounds on the product $h(z)g(z)$. The estimates needed for g are contained in the following lemma.

Lemma 4.3.5. *Let g be analytic in the closed unit disk. If $T_n(g)$ is the Taylor polynomials of g of degree n, then*

$$\|g - T_n(g)\|^2_\alpha = O(S^{-n}),$$

for some $S > 1$. Moreover, there exists a constant $C = C(\alpha)$ such that

$$\|T_n(g)\|_{M(D_\alpha)} \leq C.$$

Proof. Suppose $g(z) = \sum_{k=0}^\infty d_k z^k$ is convergent in a disk of radius bigger than $R > 1$.

Then there exists $C_1 > 0$ such that $|d_k| \leq C_1 R^{-k}$. Therefore

$$\|g - T_n(g)\|_\alpha^2 = \sum_{k=n+1}^{\infty} (k+1)^\alpha |d_k|^2 \leq C_1^2 \sum_{k=n+1}^{\infty} (k+1)^\alpha R^{-2k}.$$

By factoring out R^{-2n} and relabeling the index of summation, this last term is equal to

$$C_1^2 R^{-2n} \sum_{j=1}^{\infty} (j+n+1)^\alpha R^{-2j} \leq C_2 R^{-2n} (n+1)^\alpha,$$

where $C_2 = C_2(\alpha, R) < \infty$. Therefore, we have that for all $\alpha \leq 1$,

$$\|g - T_n(g)\|_\alpha^2 \leq C_2 R^{-2n} (n+1)^\alpha = O(S^{-n})$$

for some $S > 1$.

The same type of argument can be used to show that the Taylor polynomials $T_n(g)$ have uniformly bounded multiplier norms. Indeed, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in D_\alpha$, then in a manner similar as above, one can easily show that for every integer k,

$$\|d_k z^k \cdot f(z)\|_\alpha \leq R^{-k} C_3 \|f\|_\alpha,$$

where $C_3 = C_3(k, \alpha)$ is of polynomial type on k. Therefore,

$$\|T_n(g) \cdot f\|_\alpha \leq \sum_{k=0}^{n} \|d_k z^k \cdot f(z)\|_\alpha \leq \left(\sum_{k=0}^{n} C_3 R^{-k} \right) \|f\|_\alpha.$$

Writing $C = \sum_{k=0}^{\infty} C_3 R^{-k}$, we obtain

$$\|T_n(g)\|_{M(D_\alpha)} \leq C.$$

□

Remark 4.3.6. Furthermore, in the previous Lemma, one can show that the exponential decay on $||g - T_n(g)||_\alpha^2$ also holds for $||g - T_n(g)||_{M(D_\alpha)}^2$. For the case when $\alpha > 1$, D_α

is an algebra, so our statement is obvious. This already implies that, for all $\alpha \leq 1$, $||g - T_n(g)||^2_{M(D_\alpha)}$ decays exponentially as the norm $||\cdot||_{M(D_\alpha)}$ is controlled by the norm in D_2.

Theorem 4.3.7. *Let $\alpha \leq 1$. If f is a function admitting an analytic continuation to a neighborhood of the closed unit disk and whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that*

$$\text{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m) \leq \frac{C}{\varphi_\alpha(m+1)}$$

holds for all sufficiently large m. Moreover, this estimate is sharp, since if such a function f has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\frac{\tilde{C}}{\varphi_\alpha(m+1)} \leq \text{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m).$$

Proof. Let us first examine the upper bound. As f is not identically 0, it can only have a finite number of zeros on the unit circle $\mathbb{T}$. Write $f(z) = h(z)g(z)$, where h is the polynomial formed from the zeros of f that lie on $\mathbb{T}$, and g is analytic in the closed disk with no zeros there. Therefore, $1/g$ is also analytic in the closed unit disk (and obviously has no zeros there), and hence Lemma 4.3.5 applies to $1/g$. Notice also that g and g' are bounded in the disk, and therefore g is a multiplier for D_α.

Now let q_m be the optimal approximant of order m to $1/h$, and define $p_m = q_m T_m(1/g)$. Then

$$\|p_m f - 1\|^2_\alpha = \|T_m(1/g) g q_m h - 1\|^2_\alpha.$$

Applying the triangle inequality after subtracting and adding $T_m(1/g)g$, we obtain

$$\|p_m f - 1\|_\alpha \leq \|T_m(1/g) g(q_m h - 1)\|_\alpha + \|T_m(1/g) g - 1\|_\alpha.$$

We know that g is a multiplier for D_α, that q_m is optimal for h, and that all $T_m(1/g)$ are uniformly bounded in multiplier norm by Lemma 4.3.5. Hence, the square of the first term on the right hand side is dominated by $C/\varphi_\alpha(m+1)$, for some constant C independent of m. On the other hand, by the second part of Lemma 4.3.5, the square of the second term is $o(1/\varphi_\alpha(m+1))$, and thus is negligeable by comparison. Therefore,

$$\mathrm{dist}^2_{D_\alpha}(1, f\cdot\mathcal{P}_m) \leq \frac{C}{\varphi_\alpha(m+1)}$$

for some constant $C = C(\alpha, f)$, as desired.

Let us now address the lower bound for such functions f. Notice first that if the lower bound holds for functions of the form $(\zeta - z)g(z)$, where g is analytic and without zeros in the closed unit disk, then the conclusion holds for f. Moreover, as in the proof of Lemma 4.3.1, it is enough to consider $\zeta = 1$. Therefore, we write $f(z) = h(z)g(z)$, where $h(z) = 1 - z$ and g as above. Again, since g is analytic and has no zeros in the closed disk, note that both g and $1/g$ are multipliers for D_α. Therefore, if p_m is any polynomial of degree less than or equal to m,

$$\|p_m f - 1\|_\alpha \leq \|g\|_{M(D_\alpha)} \|p_m h - 1/g\|_\alpha \leq \|g\|_{M(D_\alpha)} \|1/g\|_{M(D_\alpha)} \|p_m f - 1\|_\alpha.$$

Therefore, $\|p_m f - 1\|^2_\alpha$ and $\|p_m h - 1/g\|^2_\alpha$ have the same rate of decay as $m \to \infty$. Now, let's choose p_m to be the optimal polynomials of degree less than or equal to m for f. Then by the above discussion, we can assume $p_m h - 1/g \to 0$ in D_α, and in particular, the norms $\|p_m h\|_\alpha$ are bounded. We thus obtain

$$\begin{aligned} \|p_m f - 1\|_\alpha &= \|p_m h(g - T_m(g) + T_m(g)) - 1\|_\alpha \\ &\geq \|p_m h T_m(g) - 1\|_\alpha - \|p_m h(g - T_m(g))\|_\alpha \end{aligned}$$

Now, by Lemma 4.3.5, $\|p_m h T_m(g) - 1\|^2_\alpha$ is, at least, comparable to $1/\varphi_\alpha(2m+1)$, which in turn is comparable to $1/\varphi_\alpha(m+1)$. On the other hand,

$$\|p_m h(g - T_m(g))\|_\alpha \leq \|p_m h\|_\alpha \|g - T_m(g)\|_{M(D_\alpha)},$$

so by the Remark 4.3.6 and since the norms of $\|p_m h\|_\alpha$ are bounded, this term decays much faster than the other. Therefore, there exist constants C_1 and C_2 such that

$$\text{dist}^2_{D_\alpha}(1, f\cdot\mathcal{P}_m) = \|p_m f - 1\|^2_\alpha \geq C_1\|p_m h T_m(g) - 1\|_\alpha \geq \frac{C_2}{\varphi_\alpha(m+1)}.$$

□

Remark 4.3.8. The methods used in the proofs of Theorems 4.3.4 and 4.3.7 can be used to produce an independent proof of the upper bound for the optimal norm in the Dirichlet space (the case $\alpha = 1$), valid for a class of functions with the property that the Taylor coefficients of f and of $1/f$ exhibit simultaneously rapid decay. More specifically, if $\{a_j\}$ denotes the sequence of Taylor coefficients of a function $f \in D$, and $\{b_k\}$ denotes the coefficients of $1/f$, we say that f is a *strongly invertible function* if f has no zeros in $\mathbb{D}$ and if for all j and k, we have $|a_j| \leq \frac{C_1}{(j+1)^3}$, and $|b_k| \leq \frac{C_2}{k+1}$, for some constants C_1 and C_2. For example, one can show that if f is strongly invertible, then $1/f$ is in the Dirichlet space. In fact, much more is true, $1/f \in D_2$. That is, strongly invertible implies invertible in D_2, and such functions are known to be cyclic (see [11], p. 274). Now, by defining polynomials analogous to those at the end of Section 2, namely,

$$P_n(z) = \sum_{k=0}^{n}\left(1 - \frac{H_k}{H_{n+1}}\right) b_k z^k,$$

one can use the stronger condition on the decay of the coefficients of $1/f$ to prove a version of the Lemma 4.3.3 with these coefficients H_k and then one can obtain the conclusion of Theorem 4.3.7 for these strongly invertible functions. In particular, we obtain:

Corollary 4.3.9. *Let f be a strongly invertible function, $\gamma \in \mathbb{N}$ and $g = f^\gamma$. Then there exist polynomials q_n of degree n, for which $\|q_n g - 1\|^2_D \leq C/\log(n+2)$.*

It would be natural to investigate whether these Riesz-type polynomials provide close to optimal approximants for more general functions, in particular functions of the form $f_\beta(z) = (1-z)^\beta$, when $\beta < 1$. Another interesting question would be whether the rate of decay that we have observed for functions admitting an analytic continuation to the closed disk holds for all functions that vanish only on a finite set.

4.4 Logarithmic conditions

It is well-known that if f is invertible in the Hardy or Dirichlet space, then f is cyclic in that space. In addition, it is easy to see that if both f and $1/f$ are in D_α and f is bounded then $\log f \in D_\alpha$, but the converse does not hold. The condition that $\log f \in D_\alpha$ can be thought of as an intermediate between $f \in D_\alpha$ and $1/f \in D_\alpha$. Indeed, $\log f \in D_\alpha$ is equivalent to f'/f being a $D_{\alpha-2}$ function. On the other hand, $f \in D_\alpha$ if and only if $f' \in D_{\alpha-2}$, while $1/f \in D_\alpha$ if and only if $f'/f^2 \in D_{\alpha-2}$. We therefore want to study the following question:

Problem 4.4.1. Is any function $f \in D_\alpha$, with logarithm $q = \log f \in D_\alpha$, cyclic in D_α?

In several cases the statement is true: If $\alpha > 1$ or $\alpha = 0$. Indeed, for $\alpha > 1$, $\log f \in D_\alpha$ implies $1/f \in H^\infty$, which is equivalent to the cyclicity of f (as explained on p. 274 of [11]). For $\alpha = 0$, it is easy to see that $\log f \in H^1$ is enough for a function to be outer, that is, cyclic in H^2. Moreover, the logarithmic condition implies the following interpolation result, valid for all $\alpha < 2$.

Lemma 4.4.2. *Suppose $f \in D_\alpha$ and $\log f \in D_\alpha$. Then, for any $\tau \in (0,1]$, we have*

$$D_\alpha(f^\tau) \leq \tau^2 \left(D_\alpha(f) + D_\alpha(\log f)\right),$$

and consequently, $f^\tau \in D_\alpha$.

Proof. It suffices to establish the bound on $D_\alpha(f^\tau)$. To this end, we write

$$D_\alpha(f^\tau) = \int_{\mathbb{D}} |(f^\tau)'(z)|^2 d\mu_\alpha(z) = \tau^2 \int_{\mathbb{D}} \left|\frac{f'(z)}{f(z)}\right|^2 |f(z)|^{2\tau} d\mu_\alpha(z)$$

Splitting the unit disk between $A = \{z \in \mathbb{D} \colon |f(z)| < 1\}$ and $\mathbb{D} \backslash A$ we have the following control:

$$D_\alpha(f^\tau) \le \tau^2 \left(\int_A \left| \frac{f'(z)}{f(z)} \right|^2 d\mu_\alpha(z) + \int_{\mathbb{D}\setminus A} |f'(z)|^2 d\mu_\alpha(z) \right)$$

and the resulting integrals can be bounded respectively in terms of $D_\alpha(f)$ and $D_\alpha(\log f)$, as claimed. □

This lemma allows us to show that for a function f in the Dirichlet space D, corresponding to the case $\alpha = 1$, the condition $\log f \in D$ does imply the cyclicity of f. The proof relies on the following theorem due to Richter and Sundberg (see [31, Theorem 4.3] and let μ be Lebesgue measure).

Theorem 4.4.3 (Richter and Sundberg, 1992)**.** *If $f \in D$ is an outer function, and if $\tau > 0$ is such that $f^\tau \in D$, then $[f] = [f^\tau]$.*

In [31], Richter and Sundberg applied this theorem by showing that if f is univalent and non-vanishing, then $f^\tau \in D$, and hence f is cyclic. In what follows, we do not require univalence.

Theorem 4.4.4. *Suppose $f \in D$ and $\log f \in D$. Then f is cyclic in the Dirichlet space.*

Proof. As was pointed out before, the logarithmic condition $\log f \in D$ implies that f is outer. Next, by Lemma 4.4.2, $f^\tau \in D$ for all $\tau > 0$, and so $[f] = [f^\tau]$ for each τ. Since the Lemma also implies $f^\tau \to 1$ in D as $\tau \to 0$, we have $[f] = [1]$, and the assertion follows. □

The following is the main result for the remaining cases $\alpha < 0$ and $0 < \alpha < 1$.

Theorem 4.4.5. *Let $f \in H^\infty$ and $q = \log f \in D_\alpha$. Suppose there is a sequence of polynomials $\{q_n\}$ that approach q in D_α norm with*

$$2 \sup_{z\in\mathbb{D}} \mathrm{Re}(q(z) - q_n(z)) + \log(\|q - q_n\|_\alpha^2) \le C$$

for some constant $C > 0$. Then f is cyclic in D_α.

Remark 4.4.6. An immediate consequence of Theorem 4.4.5 is that if $q = \log f$ can be approximated in D_α by polynomials $\{q_n\}$ with $\sup_{z\in\mathbb{D}} \mathrm{Re}(q(z) - q_n(z)) < C$, then f is cyclic. Brown and Cohn proved (see [10, Theorem B]) that for any closed set of logarithmic capacity zero $E \subset \partial\mathbb{D}$, there exists a cyclic function f in D such that $\mathcal{Z}(f^*) = E$. The functions they build satisfy this hypothesis on q_n, and therefore, we understand Brown and Cohn probably found this fact in the case $\alpha = 1$, although they do not make any such statement in their article [10], leaving the proof of cyclicity of certain functions, as an exercise.

Proof of Theorem 4.4.5. We can assume $\alpha \leq 1$, because otherwise the statement is immediate. As f is a multiplier of $D_{\alpha-2}$ and $f'/f \in D_{\alpha-2}$, the function f is in D_α. The function f is cyclic, if there exists a sequence of polynomials $\{p_n\}$ such that $\|p_n f - 1\|^2_\alpha$ remains bounded as n goes to infinity while p_n converge pointwise to $1/f$.

Applying the triangle inequality, we obtain:

$$\|p_n f - 1\|_\alpha \leq \|p_n f - e^{-q_n} f\|_\alpha + \|e^{-q_n} f - 1\|_\alpha. \tag{4.4.1}$$

The first term on the right hand side can be bounded by:

$$\|(p_n - e^{-q_n})f\|_\alpha \leq \|p_n - e^{-q_n}\|_{M(D_\alpha)} \|f\|_\alpha.$$

For $\alpha \leq 1$, we can see that the multiplier norm of a function is controlled by the H^∞ norm of the derivative (see [11, Prop.3]).

Hence, a good choice of approximating polynomials is to select $\{p_n\}$ so that $p_n(0) = e^{-q_n(0)}$ and $\|p'_n + q'_n e^{-q_n}\|_{H^\infty} \leq 1/n$, which is possible as e^{-q_n} is entire. The polynomials p_n converge pointwise to $1/f$. Hence, for the cyclicity of f, it is sufficient for the norm of $p_n f - 1$ to stay bounded.

Now, we have that the first term of the right hand side in (4.4.1) is negligible as $n \to \infty$.

So what remains is to show that, as n goes to infinity, $\|e^{-q_n} f - 1\|^2_\alpha$ is uniformly bounded. To evaluate this expression for large n, we use the norm in terms of the derivative:

$$\|e^{-q_n} f - 1\|_\alpha^2 \approx \| - q_n' e^{-q_n} f + e^{-q_n} f'\|_{\alpha-2}^2 + |e^{-q_n(0)} f(0) - 1|^2.$$

The last term tends to 0 since q_n approaches q pointwise.

In the other term of the right hand side, taking out a common factor, we see that

$$\| - q_n' e^{-q_n} f + e^{-q_n} f'\|_{\alpha-2}^2 \leq \|e^{q-q_n}\|_{H^\infty}^2 \left\| \frac{f'}{f} - q_n' \right\|_{\alpha-2}^2.$$

Therefore, we have:

$$\| - q_n' e^{-q_n} f + e^{-q_n} f'\|_{\alpha-2}^2 \leq e^{2 \sup \mathrm{Re}(q-q_n)} \|q - q_n\|_\alpha^2.$$

With our assumptions, the right hand side is less than a constant. This concludes the proof. □

It would be interesting to determine whether the required approximation property of the polynomials q_n in Theorem 4.4.5 is a consequence of the other hypotheses.

4.5 Asymptotic zero distributions for approximating polynomials

In this paper we have primarily been interested in functions $f \in D_\alpha$ that are cyclic and have $f^*(\zeta) = 0$ for at least one $\zeta \in \mathbb{T}$. Prime examples of such a function are

$$f_\beta(z) = (1 - z)^\beta, \quad \beta \in [0, \infty),$$

which we have examined closely in this paper for β a natural number.

In these cases, numerical experiments, described below, suggest that a study of the zero sets $\mathcal{Z}(p_n)$ of approximating polynomials may be interesting from the point of view of cyclicity. It seems that the rate at which zeros approach the circle is related to the extent to which the corresponding polynomials are adequate approximants in D_α. For instance, we have compared the zero sets associated with the Taylor polynomials of $1/f_\beta$ with those of Riesz-type polynomials,

$$\mathcal{R}_n\left(\frac{1}{f_\beta}\right)(z) = \sum_{k=0}^{n}\left(1 - \frac{H_k}{H_{n+1}}\right) b_k z^k, \quad n \geq 1. \tag{4.5.1}$$

Intuitively, since $1/f_\beta$ has a pole at $z = 1$, we should expect the approximating polynomials p_n to be "large" in the intersection of disks of the form $B(1, r)$ with the unit disk. On the other hand, the remainder functions $p_n f - 1$ have to tend to zero in norm (and hence pointwise). We note that since $1/f_\beta$ has a pole on $\mathbb{T}$, the Taylor series of $1/f_\beta$ cannot have radius of convergence greater than 1. It therefore follows from Jentzsch's theorem that every point on $\mathbb{T}$ is a limit point of the zeros of the sequence $\{T_n(1/f_\beta)\}_{n=1}^{\infty}$. See [23]. A more current reference is [6]. We also refer the reader to [36] for background material concerning Taylor (and other) polynomials, and for useful computer code.

We start with the simplest case $f_1(z) = 1 - z$. The zeros of the Taylor polynomials $T_n(1/f)$, the Cesàro polynomials $C_n(1/f)$, and the Riesz polynomials $R_n(1/f)$, for $n = 1, \ldots, 50$, can be found in Figure 4.5.1. All the zeros of these polynomials are located outside the unit disk, and inside a certain cardioid-like curve. In the case of the Taylor polynomials, the explicit formula

$$T_n(1/f_1)(z) = \frac{1 - z^{n+1}}{1 - z}$$

holds, and so $\mathcal{Z}(T_n)$ simply consists of the n-th roots of unity, minus the point $\zeta = 1$. Replacing Taylor polynomials by Cesàro polynomials has the effect of repelling zeros away from the unit circle, and into the exterior of the disk. This effect is even more pronounced for the Riesz polynomials (4.5.1), where it appears that convergence of roots to the unit circumference, and the roots of unity in particular, is somewhat slower. Note also the relative absence of zeros close to the pole of $1/f_1$, and the somewhat tangential approach region at $\zeta = 1$.

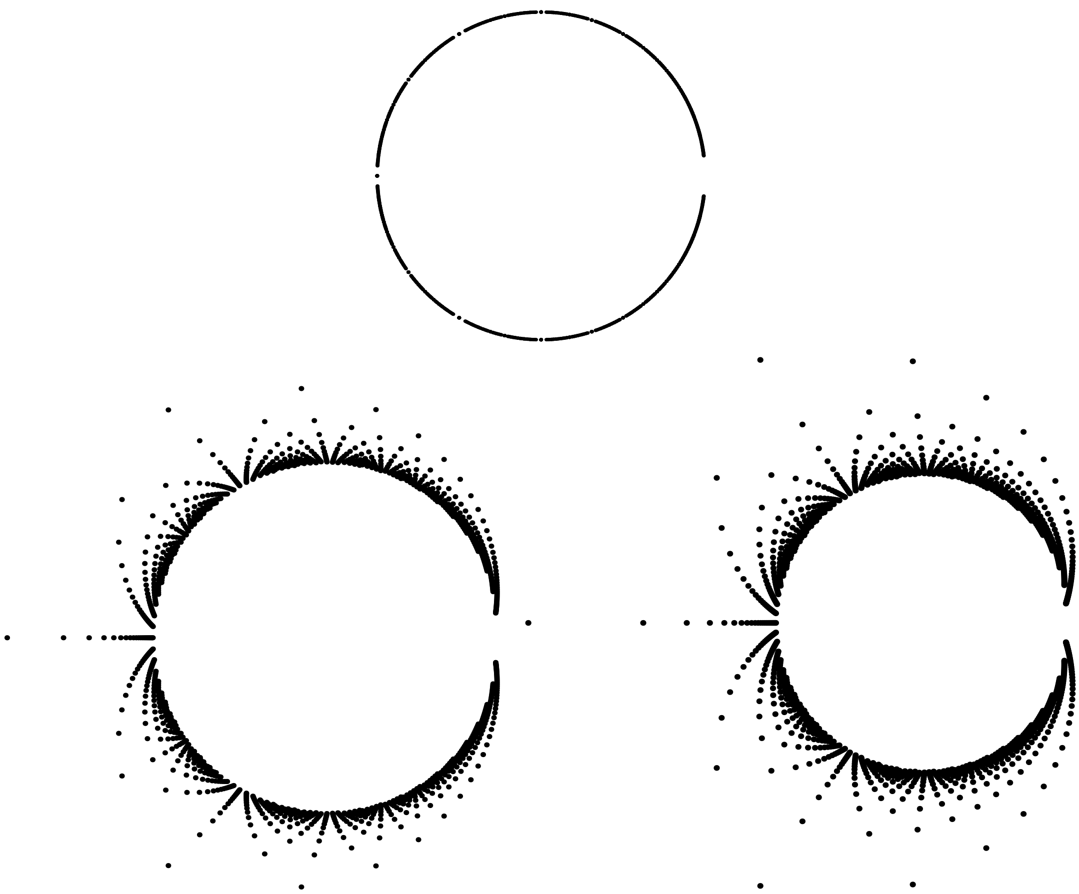

Figure 4.5.1: Left to right: Successive zero sets $\mathcal{Z}(T_n)$, $\mathcal{Z}(C_n)$, and $\mathcal{Z}(R_n)$, for $f_1 = 1 - z$, and $n = 1, \ldots, 50$.

Next, we turn to a function with two simple zeros on $\mathbb{T}$, namely

$$f = 1 - z + z^2.$$

Plots of zeros of successive approximating polynomials are displayed in Figure 4.5.2. While $\mathcal{Z}(T_n)$ is more complicated, the general features of Figure 4.5.1 persist. We again note a relative absence of zeros close to the two poles of $1/f$, and the zeros of the Cesàro and Riesz polynomials are again located in the exterior disk, and seem to tend to $\mathbb{T}$ more slowly. We observe approach regions with vertices at the symmetrically placed poles, and the angle at these vertices seems to decrease as we move from Taylor polynomials through

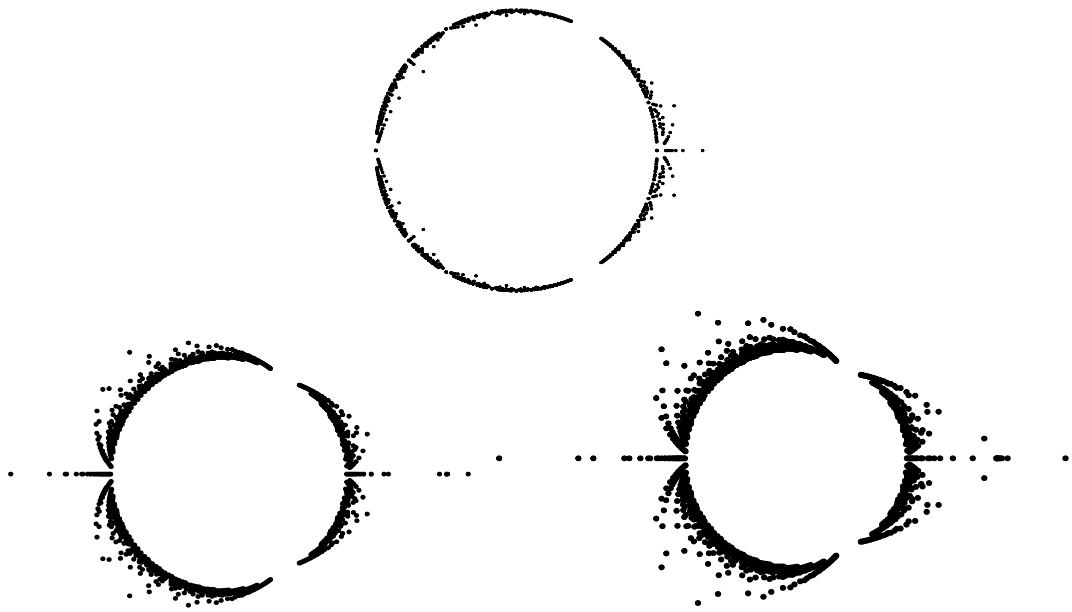

Figure 4.5.2: Left to right: Successive zero sets $\mathcal{Z}(T_n)$, $\mathcal{Z}(C_n)$, and $\mathcal{Z}(R_n)$, for $f_1 = 1 - z + z^2$, and $n = 1, \ldots, 50$.

Cesàro polynomials to the polynomials in (4.5.1).

It seems natural to suspect that locally the picture would be similar for a polynomial f with a large number of zeros on the unit circle.

It would be interesting to investigate whether there is a relationship between zeros of approximating polynomials, the region of convergence of the Taylor series of $1/f$, and the cyclicity of f in future work.

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