Universitat Autònoma de Barcelona

Dynamical Classification of some Birational Maps of $\mathbb{C}^2$

Sundus Zafar

Supervised by Dr. Anna Cima

Universitat Autònoma de Barcelona

Department of Mathematics

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by Sundus Zafar

Supervised by Dra. Anna Cima

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Dra. Anna Cima (Adviser)

To desire, passion, love and happiness

Abstract

This dissertation addresses three different problems in the study of discrete dynamical systems. Firstly, this work dynamically classifies a $9-$parametric family of planar birational maps $f : \mathbb{C}^2 \quad \mathbb{C}^2$ that is

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y} \right),$$

where the parameters are complex numbers. This is done by finding the dynamical degree δ for the degenerate and non degenerate cases of F that is the extended map of f in projective space. The dynamical degree δ defined as

$$\delta(F) := \lim_{n} \, (\deg(F^n))^{\frac{1}{n}},$$

indicates the subfamilies which are chaotic, that is when $\delta > 1$, and otherwise. The study of the sequence of degrees d_n of F shows the degree growth rate of all the subfamilies of f. This gives the families which have bounded growth , or they grow linearly, quadratically or grow exponentially. The family f includes the birational maps studied by Bedford and Kim in [BK04] as one of its subfamily.

The second problem includes the study of the subfamilies of f with zero entropy that is for $\delta = 1$. These includes the families with bounded (in particular periodic), linear or quadratic growth rate. Two transverse fibrations are found for the families with bounded growth. In the periodic case the period of the families is indicated. It is observed that there exist infinite periodic subfamilies of f, depending on the parameter region. The families with linear growth rate preserve rational fibration and the quadratic growth rate families preserve elliptic fibration that is unique depending on the parameters. In all the cases with

zero entropy all the mappings are found up to affine conjugacy.

Thirdly, it deals with non-autonomous Lyness type recurrences of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $a_n \ \ _n$ is a k-periodic sequence of complex numbers with minimal period k. We treat such non-autonomous recurrences via the autonomous dynamical system generated by the birational mapping $F_{a_k} \circ F_{a_{k-1}} \circ \cdots \circ F_{a_1}$ where F_a is defined by

$$F_a(x, y) = (y, \frac{a + y}{x}).$$

For the cases $k \quad 1, 2, 3, 6$ the corresponding mappings have a rational first integral. By calculating the dynamical degree we show that for $k = 4$ and for $k = 5$ generically the dynamical system is no longer rationally integrable. We also prove that the only values of k for which the corresponding dynamical system is rationally integrable for all the values of the involved parameters, are $k \quad 1, 2, 3, 6$.

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I played with you to live inside
We traveled through the unknown world
You took my hand and said let it go
Breathing in the living fantasy
Places where I could have never been
The spirit, the passion you blown into me
Pictures of reality you showed to me
By giving me the wings to fly around the universe
You opened the doors to the hidden world
By giving the consciousness of my energies
You showed me to live, to be part of world's flow
The reason that I am looking around me
You are the reason that I am breathing
The way you have defined things for me
My path is more clear and my goal is more near
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Contents

Mathematics, as much as music or any other art, is one of the means by which we rise to a complete self-consciousness. The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds - John William Navin Sullivan

Chapter 1

Introduction

1.1 Motivation and Significance

The subject of studying the dynamics of rational maps arose in early part of 20^{th} century. The global study of iteration of analytic functions which falls in the area of studying holomorphic dynamics was first introduced by Pierre Fatou in 1917 Many basic results were found independently in parallel by Gaston Julia and Samuel Lattes in 1918. This initiated the interest of studying the complex dynamics. The computer generated pictures appeared showing ordered and disordered chaos underlying in this theory stimulated great interest not only in mathematicians but also outside of the mathematical community. The studies of Henri Poincaré on planetary motion stared the interest of iterative dynamical systems in mathematics and physics community. These iterations naturally appear in the beautiful attractive picture showing complex dynamics of different phenomenons. Therefore in last two decades an increased interest is observed to study the difference equations that model different phenomenons of life. The population dynamics, economic and financial dynamics, ecological and biological aspects of life are widely studied now by using the tools from discrete dynamical systems [Bla09, Gan97, EK78, Mic90, CGM06b]. The difference equations which are linear or nonlinear rational recurrences are of much interest [KLP00, LGK02, ZEM08, CLV03, Bed, CGM08b, San90, CZ14, FJL96].

In recent years the study of holomorphic dynamics of complex manifolds has received much attention [DF01, BK06, BK04]. One of the main objective is to study the dynamics of rational mappings of $P\mathbb{C}^n$ These can be studied as the systems of difference equation. The intention is to classify these mappings with respect to the involved parameters that are closely related to the life phenomenon under study. This is done in order to know the local and global behavior of these iterative mappings in real and complex space [KLMR03, DS05, ZEM08, Lad95, BC, FS92, FS45, DF01, BK04, CGM06a, CGM06b, KL93, BPvdV84]. In these works several authors have devoted their attention to the Cremona transformations [Hud27, Des12, Dil11, Bla09]. The birational maps in $P\mathbb{C}^2$ form the Cremona group **Bir**($P\mathbb{C}^2$), see [Des12, Bla09, Dil96]. From a dynamical point of view the Cremona transformations are considered to define a dynamical system. Much efforts have been made in [BK06, BK04, Dil11, BC, BD12, Bla09, Dil11, dMV06, CLV03] to understand the dynamics and global behavior of such birational maps. In this work we consider the family of fractional maps $f : \mathbb{C}^2 \quad \mathbb{C}^2$ of the form:

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y\ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right) \tag{1.1}$$

where the parameters are complex numbers.

The study of family f in (1.1) can be considered as a continuation of this exploration. This is a subfamily of Cremona group with nine different unknown parameters.

In [BB00, McM07, BF00, Jac05, Fri91, FS45, Fri91, Gro03, CGM06a] authors have studied birational maps by using different tools from algebraic geometry, dynamical systems, (see [KH95, Sha94, Har77, Sil86, San90, Aga92, Ela96, Can99]) and by utilizing several techniques including blowups in projective space (see [Har85, BK04, CZ14]). In past decade the study of degree growth of birational maps and its estimates are studied in different aspects (see [dMV06, Des12]), the quantity dynamical degree is of most interest, (see [Bel99, DF01, BK06, CZ14, BC, BD12, BV99, Yom87, Sil]).

In those, Diller and Favre [DF01] gave a classification of bimeromorphic maps of complex surfaces and discussed their dynamics by utilizing the quantity dynamical degree associated to these maps in $P\mathbb{C}^n$ The degree growth rate was studied and results were produced for the four cases which are bounded, linear, quadratic or exponential growth of the sequence of degrees. The conditions for the mapping to be an automorphisms were addressed and for what values of dynamical degree one can expect to have an integrable mapping, or a mapping that preserves fibrations was also discussed.

Many other authors including (see [CGM08a, CGM06a, CGM07, CGM08b, KL93, May75, CZ14]) have studied the difference equation related to birational maps. Ladas, Camouzis and Zayed in [CLV03, Lad95, ZEM08] have intended to study some birational maps, which includes family (1.1)

as a subfamily, in real space with respect to the involved parameters. The intention to classify these maps has resulted in providing thousands of cases which are tedious to study and inefficient to work with.
In [BK06], Bedford and Kim have studied a subfamily of mapping 1 1 They have classified their family of mappings for all the values of parameters and studied their dynamics by using the notion of dynamical degree as used in [DF01].
The objective of this work is to dynamically classify the family f in (1.1) into it s subclasses and indicate their independent behavior for all the values of unknown parameters. We do this by finding the dynamical degree, for all the existing sub cases, which is the associated quantity with growth sequence of these maps in $P\mathbb{C}^2$ This work provides families as examples of the classification done by Diller and Favre in [DF01]. This also shows that the methodology used to achieve the results is more efficient and provides more concrete and concise results. This family 1.1 includes the family of Bedford and Kim in [BK06] as its subfamily for the values of parameters $\alpha_0 = 0 \ \alpha_1 = 0$ and $\alpha_2 = 1$

1.2 Introduction of the problem

In this work we consider the family of fractional maps $f : \mathbb{C}^2 \quad \mathbb{C}^2$ of the form:

$$f(x \ y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y \ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right) \tag{1.2}$$

where the parameters are complex numbers.

This family of maps can be extended to the projective plane $P\mathbb{C}^2$ by considering the embedding $(x_1 \ x_2) \quad \mathbb{C}^2 \quad [1 : x_1 : x_2] \quad P\mathbb{C}^2$ into projective space. The induced map $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ has three components $F_i[x_0 : x_1 : x_2] \ \ i = 1 \ 2 \ 3$ which are homogeneous polynomials of degree two. For general values of the parameters the three components have no a common factor: we say that these maps have degree two. Similarly we can define the degree of $F^n = F \circ \cdots \circ F$ for each $n \quad \mathbb{N}$ It can be seen (see Proposition (6) or [DF01]) that if $f(x_1 \ x_2)$ is a birational map, then the sequence of its degrees satisfies a homogeneous linear recurrence with constant coefficients.

In order to determine the behavior of iterates $F^n = F \circ \cdots \circ F$ we begin by studying their degree growth rate. In particular we are going to determine

$$\delta(F) := \lim_{n} \ (\deg(F^n))^{\frac{1}{n}} \tag{1.3}$$

which is known as the *dynamical degree* and the logarithm of this quantity which is called the *algebraic entropy*.

It is also known (see [Yom87]) that the algebraic entropy is an upper bound of the *topological entropy*, which in turn is a dynamic measure of the complexity of the mapping. For this reason the results that we get can be seen as a dynamical classification of family (1.1). Furthermore, they generalize the results obtained in [BK06], in which a similar problem is treated for the case in which $\alpha_0 = 0 \quad \alpha_1 = 0$ and $\alpha_2 = 1$ that is, for Linear Fractional Recurrences.

Birational mappings $F : P\mathbb{C}^2 \qquad P\mathbb{C}^2$ have an indeterminacy set $\mathcal{I}(F)$ of points where F is ill-defined as a continuous map. This set is given by:

$$\begin{aligned} \mathcal{I}(F) \quad &= \quad [x_0 : x_1 : x_2] \quad P\mathbb{C}^2 : F_1[x_0 : x_1 : x_2] = 0 \\ &\qquad F_2[x_0 : x_1 : x_2] = 0 \;\; F_3[x_0 : x_1 : x_2] = 0] \end{aligned}$$

On the other hand, if we consider one irreducible component V of the determinant of the Jacobian of F, it is known (see [DF01]) that $F(V)$ reduces to a point, which belongs to the indeterminacy set of the inverse of F The set of these curves which are sent to a single points is called the *exceptional locus* of F and it is denoted by $\mathcal{E}(F)$

It is known that the dynamical degree depends on the orbits of the indeterminacy points of the inverse of F under the action of F (see Proposition 5 of Chapter 2). Indeed, the key point is whether the iterates of such a points coincide with any of the indeterminacy points of F

If this happens we do a series of blow ups to remove the indeterminacies. As a result we get an expanded space X where our map is algebraically stable AS. We discuss all this process of blowups and stability of map in more detail in Chapter 2.

For an AS map f we can consider the induced map $f^* : Pic(X) \qquad Pic(X)$ According to [DF01] for an AS map we have $(f^n)^* = (f^*)^n$ where $\delta(F)$ is the spectral radius of f^*

We are later interested to know if $\delta(F) \geq 1$ in order to make a classification of the zero entropy ($\delta(F) = 1$) cases and the ones which have dynamical degree greater than one. We later study the zero entropy families.

The aim of this work is to study the dynamical consequences. For instance finding the mappings which are globally periodic, this is to look for the ones whose sequence of degrees are periodic. To know the mappings which preserve rational or elliptic fibrations is to look for the ones whose sequence of degrees grow linearly or quadratically (see Theorem 7 in Chapter 2). Also the study of the complete dynamics of f by using symbolic dynamics requires to look for the models who have same dynamical degree as f

1.3 Thesis layout

This work discusses the classification of birational maps f of $P\mathbb{C}^2$ given in $(1\ 1)$ An introduction to the family, scheme and methods to tackle the problem are discussed in the introduction and classification section of chapter 1. The subsection of classification provides the conditions for the classification of maps into two larger subfamilies, which provides the information for the existence of families discussed in later chapters $3-6$
The family of birational maps $(1\ 1)$ are bimeromorphic maps as discussed in chapter 2. We are using several results proved in [DF01] to study the complex dynamics of such maps in compact complex spaces. To get an AS map the method of regularization of the map f is discussed in detail in chapter 2. The orbit structure and the collision of orbits is also discussed in this chapter.
Chapter 3 discusses the families of maps which are degenerate that is all the parameters satisfy either the condition $(\alpha\gamma)_{12}=0$ or $(\beta\gamma)_{12}=0$ on the parameters of the map 1.1. The results in both of these cases are given in two different theorems which classify the families later into further subfamilies by finding the dynamical degree and then the dynamics of families existing inside these cases which have dynamical degree one or zero entropy is discussed in the end of this chapter.
In chapters 4 to 6 discussed the families of maps which are non degenerate that is all the parameters satisfy either the condition $(\alpha\gamma)_{12}=0$ and $(\beta\gamma)_{12}=0$ on the parameters of the map (1.1). The results for all the cases are given in the form of different theorems which classify the families later into further subfamilies by finding the dynamical degree. The dynamics of families existing inside these cases which have dynamical degree one or zero entropy is discussed in the end of this chapter.
Chapter 7 discusses the non integrability of $k-$periodic Lyness recurrences globally. This work serves as one of the applications of dynamical degree associated to birational maps of $P\mathbb{C}^n$ The result is presented as a theorem in the end of chapter that uses the four propositions described before in the same chapter to prove its result. These propositions are the non integrable cases for four different values of k These four cases suffice the required result globally.
In chapter 9 We also discuss as the continuity of this work the possible aspects of future work and its significance as a contribution in the present work in the study of complex dynamics of birational maps.
In 8 we give the calculation done for finding the exact values for different orbit lengths of the indeterminacy points A_1 and A_2 of the inverse of the birational map f

In the end we give the bibliography.

1.3.1 Quick review

This thesis work is organized as follows: The introduction to the family is given in chapter 1. The required preliminaries which include basic settings of the work, some background of birational maps, Picard group, orbit and list structure of orbits are discussed in chapter 2. From chapter 3 to 6 the classification and zero entropy families are discussed for both non degenerate and degenerate cases of f Chapter 7 gives the non integrability result for Lyness $k-$periodic recurrences. In chapter 9 we the future work scope. Then in the appendix we give required calculation used in the work. We end with mentioning all the references utilized to complete this work.

Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere, Archimedes did it in a bathtub - Richard J. Trudeau, Dots and Lines.

Chapter 2

Preliminaries

For the better understanding of the subject and problem of this research thesis work, in this chapter we provide some necessary background definitions, the tools that we are going to use in our work and the methodologies we follow in order to get regularized maps. We introduce the general family of maps to study in this work in the section of settings and provide a first classification of it in two major cases, the degenerate and non degenerate. To achieve the results in degenerate case, that we discuss in Chapter 3, the tools and methodology that we use are described in detail in this chapter, that is to perform a series of blowups, get an extended space and then find the characteristic polynomial by using Picard group in order to find the dynamical degree. These are the same tools that we use to achieve AS maps in Chapter 7. However to get the results in non degenerate case that we discuss in Chapters 4, 5, 6, we use a theorem of [BK06]. We describe the basic tools for the understanding of this theorem in the last section of this chapter.

2.1 Background knowledge

2.1.1 Projective space $P\mathbb{C}^2$

The **projective space** of dimension two is defined as the quotient of $\mathbb{C}^3 \quad 0$ by the relation identifying p with λp for any $p \quad \mathbb{C}^3 \quad 0$ and any $\lambda \quad \mathbb{C}$ That is, complex lines through origin in $\mathbb{C}^3$ become points in $P\mathbb{C}^2$ The equivalence class of $p = (x_0 \;\; x_1 \;\; x_2) \quad \mathbb{C}^3$ is denoted by $[x_0 : x_1 : x_2]$ called the homogeneous coordinates of p Note that **homogeneous coordinates** are unique only up to multiplication by non zero complex numbers. Considering the natural inclusion

$$i : \mathbb{C}^2 \quad P\mathbb{C}^2 \qquad (x_1 \ \ x_2) \quad [1 : x_1 : x_2]$$

we see that i is injective and that

$$P\mathbb{C}^2 \quad i(\mathbb{C}^2) = \ \ [0 : x_1 : x_2] \ \ = \ \ x_0 = 0$$

We refer to this set as the line at infinity.

An **algebraic curve** $V \quad P\mathbb{C}^2$ is a set of the form $V = \ \ x \quad P\mathbb{C}^2 : P(x) = 0$ where P is a non constant homogeneous polynomial. If P has the smallest possible degree among defining polynomials for V then we say that P is minimal and that the **algebraic degree** $deg(V)$ of V is $deg(P)$ If P is a minimal defining polynomial for V and P cannot be written as $P_1 \cdot P_2$ for any pair of non constant homogeneous polynomial, then we say that V **is irreducible**.

2.1.2 Rational maps of $P\mathbb{C}^2$ and the dynamical degree

Let f be a map of $\mathbb{C}^2$ with rational coordinates functions. This map can be extended to the projective plane $P\mathbb{C}^2$ by considering the embedding $(x_1 \ \ x_2) \quad \mathbb{C}^2 \qquad [1 : x_1 : x_2] \quad P\mathbb{C}^2$ into projective space. The induced map $F : P\mathbb{C}^2 \qquad P\mathbb{C}^2$ has three components $F_i[x_0 : x_1 : x_2]$ $i = 1 \ \ 2 \ \ 3$ which are homogeneous polynomials of the same degree.

Notice that a rational map f of $\mathbb{C}^2$ in general is not defined in all $\mathbb{C}^2$ In fact we have points on which the denominator vanishes. We manage this difficulty with the extension of F on $P\mathbb{C}^2$ For a point of $\mathbb{C}^2$ where the numerator and denominator of some component of f both vanish, at this point F is ill-defined, that is, all components of F vanish at this point. The set of points where F is ill-defined is called the **indeterminacy set of** F and it is denoted by $\mathcal{I}(F)$ such that

$$\mathcal{I}(F) = \ \ x \quad P\mathbb{C}^2 : F_1(x) = F_2(x) = F_3(x) = 0$$

The set $\mathcal{I}(F)$ is an algebraic variety. If $\mathcal{I}(F)$ has a component V of positive dimension and Q is the minimal defining polynomial of V then Q necessarily has to divide the three components of F By dropping this common factor we get a map which induces the same map F on $P\mathbb{C}^2 \quad V$ After reducing F by all possible polynomial factors, we obtain a minimal homogeneous map inducing F The indeterminacy set will then be at most a finite set of points in $P\mathbb{C}^2$

If F is minimal, degree d homogeneous map defined on $P\mathbb{C}^2$ $\mathcal{I}(F)$ we say that F is a **degree** d **rational map** of $P\mathbb{C}^2$ Despite the fact that F is ill-defined on $\mathcal{I}(F)$ we write $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ for simplicity.

Let F and G be two minimal homogeneous maps. Then $F \circ G$ is not necessarily a minimal map for the composition of F and G because by finding $F \circ G$ a common factor M in the components of $F \circ G$ can appear. If this is the case, then the curve $V = \quad M = 0$ satisfies that $G(V) \subset \mathcal{I}(F)$ Clearly the converse is also true. Hence we have the following results (see proposition 2 3 of [DF01] and [FS92]):

Proposition 1 *([DF01, FS92]) Let F and G be two minimal homogeneous maps. Then $F \circ G$ is a minimal representative of the composition if and only if there exists no algebraic curve V such that $G(V) \subset \mathcal{I}(F)$ In particular,*

$$deg(F \circ G) \leq (deg\, F) \cdot (deg\, G) \tag{2.1}$$

with equality holding if and only if no such V exists.

Given a rational map F and an algebraic curve V it is said that V is a **degree lowering curve** for F if $F^k(V) \subset \mathcal{I}(F)$ for some integer k Thus F has maximal degree $deg(F)$ if and only if F has no degree lowering curves. According to the previous proposition, the degree of F^n will be $(deg\, F)^n$ for all n if and only if F has no degree lowering curves. Calling $a_n = log\, deg\, F^n$ inequality (2.1) implies that $a_{n+m} \leq a_n + a_m$ and it is well known that is if the sequence a_n satisfies this subadditivity condition then the limit $\frac{a_n}{n}$ exists. From that observation and the definition of degree lowering curve the following proposition can be easily proved (see corollary 2 4 of [DF01]).

Proposition 2 *([DF01]) Given a rational map F*

$$\delta(F) = \lim_{n} (deg(F^n))^{\frac{1}{n}} \tag{2.2}$$

always exists. Furthermore $\delta(F) \leq deg(F)$ with equality if and only if F has no degree lowering curves.

The number $\delta(F)$ is called the **dynamical degree of** F and the logarithm of $\delta(F)$ is called the **algebraic entropy of** F see [BK04, Bed, BV99].

The degree of F is not a birational invariant because for H being birational, $deg(H\,F\,H^{-1}) = deg(F)$ But the dynamical degree is invariant because

$$\frac{1}{C} deg(F^n) \leq deg(H\,F^n\,H^{-1}) \leq deg(F^n)\,C$$

where $C = deg(H)^2$ Taking the power $\frac{1}{n}$ and the limit we get that $\delta(F) = \delta(H\,F\,H^{-1})$ Here we have used that the $deg(H) = deg(H^{-1})$

2.1.3 Birational map of $P\mathbb{C}^2$

A rational map $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ is birational if there exists another rational map G and an algebraic curve V such that $F \circ G = G \circ F = id$ on $P\mathbb{C}^2 \; V$ As usual we call $G = F^{-1}$ It is known that for F being birational, the degree of F coincides with the degree of F^{-1} It is a consequence of the structure of birational maps on $P\mathbb{C}^2$ see Theorem 3 2 of [DF01] for the proof.

The exceptional locus of F denoted by $\mathcal{E}(F)$ is defined as follows:

$$\mathcal{E}(F) = \quad x \quad P\mathbb{C}^2 : Det(D\,F)_x = 0$$

The following statements are true for birational maps on $P\mathbb{C}^2$(see proposition 3 3 of [DF01]).

Proposition 3 *([DF01]) Let $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ be a birational map and let F^{-1} be its inverse. Then:*

1. *Given any irreducible curve $V \quad \mathcal{E}(F) \;\; F(V)$ is a single point in $\mathcal{I}(F^{-1})$ For any point $p \quad \mathcal{I}(F^{-1})$ the preimage of p for F is an element of $\mathcal{E}(F)$*
2. *$\mathcal{I}(F) \subset \mathcal{E}(F)$ and every irreducible element of $\mathcal{E}(F)$ contains a point of $\mathcal{I}(F)$*
3. *$F : P\mathbb{C}^2 \;\; \mathcal{E}(F) \quad P\mathbb{C}^2 \;\; \mathcal{E}(F^{-1})$ is a bimeromorphic map.*

Recall that for a complex Eucledean space $\mathbb{C}^n$ a mapping $f(x_1 \quad x_n) = (f_1 \quad f_n) : D_1 \quad D_2$ for $D_1 \;\; D_2 \subset \mathbb{C}^n$ is **holomorphic** if each of the coordinates is holomorphic. That is it is complex differentiable in the neighborhood of every point in its domain. This implies that it is infinitely differentiable and coincides with its own Taylor series. *Analytic function*, *regular function* and *conformal mappings* are the terms often used in the text as an interchange of the term holomorphic functions. If f is a bijective map then its inverse $f^{-1} : D_2 \quad D_1$ exists. For f^{-1} to be holomorphic we obtain a **biholomorphic map**.

The mapping f is **meromorphic** if it is holomorphic except for a finite number of points. These points are the singularities of f on which it goes to infinity like a polynomial. These singularities are the poles of f Therefore f can be expressed as a ratio of two holomorphic functions g h such that $f = \frac{g}{h}$ Then a pole of f is a zero of the the function h For f^{-1} to be meromorphic we obtain a **bimeromorphic map**.

Hence from 3 we see that F maps each exceptional irreducible curve to a single point. If V is not exceptional then the following holds: (see corollary 3 7 of [DF01], also [Dil96]).

Proposition 4 *([DF01, Dil96]) Let* $V \subset P\mathbb{C}^2$ *be an irreducible algebraic curve such that* V $\mathcal{E}(F)$ *Then* $F(V)$ *is an irreducible algebraic curve and*

$$deg(F(V)) \leq deg(F) \cdot deg(V)$$

with equality holding if and only if V *has no indeterminacy points of* F

Returning to the notion of dynamical degree for birational maps, in the light of proposition 3, we have the following result:

Proposition 5 *A birational map* F *has a degree lowering curve* V *for* F *if and only if there exists an exceptional curve* S $\mathcal{E}(F)$ *such that* $F^n(S)$ $\mathcal{I}(F)$ *for some* n $\mathbb{N}$

Proof. Let V be a degree lowering curve for F and let k $\mathbb{N}$ such that $F^k(V)$ $\mathcal{I}(F)$ Then from Proposition 4, there exists j 0 1 $k-1$ such that $F^j(V)$ $\mathcal{E}(F)$

We can assume that $F^j(V)$ is irreducible (if not we take one of their components). Then from Proposition 3, $F^{j+1}(V)$ is a single point q $\mathcal{I}(F^{-1})$ Since $F^k(V) \subset \mathcal{I}(F)$ we see that $F^{k-(j+1)}(q)$ $\mathcal{I}(F)$ The result follows considering $S = F^j(V)$ and $n = k-(j+1)$ The converse is trivially satisfied. ∎

Hence in order to determine the dynamical degree we have to follow the orbit of the points in $\mathcal{I}(F^{-1})$ If the orbit of each v_j $\mathcal{I}(F^{-1})$ never reaches some point in $\mathcal{I}(F)$ then no degree lowering curve for F exist and $\delta(F) = deg(F)$

2.1.4 The blow-up technique and the Picard group

When there is a point in $\mathcal{I}(F^{-1})$ such that after some iterates lands in $\mathcal{I}(F)$ we blow-up such a point and all its iterates including the last one that is in $\mathcal{I}(F)$ Given a point $p \quad \mathbb{C}^2$ we are going to consider $(X \ \ \pi)$ the blow up of $\mathbb{C}^2$ at the point p where π is the holomorphic map from $X \qquad \mathbb{C}^2$ The If $p = (0 \ \ 0) \quad \mathbb{C}^2$ (if not we do a translation) then

$$X = \quad ((x \ \ y) \ \ [u : v]) \quad \mathbb{C}^2 \times P\mathbb{C}^1 \ : \ xv = yu$$

and

$$\pi : X - \quad \mathbb{C}^2$$

is the projection on the first component:

$$\pi \left((x \ \ y) \ \ [u : v] \right) = (x \ \ y)$$

We notice that

$$\pi^{-1} p = \pi^{-1}(0 \ \ 0) = \quad ((0 \ \ 0) \ \ [u : v]) \quad := E_p \quad P\mathbb{C}^1$$

and $\pi : X - E_p \qquad \mathbb{C}^2 - \quad (0 \ \ 0)$ is a biholomorphic map. Given the point $((0 \ \ 0) \ \ [u : v]) \qquad E_p$ (resp. $((x \ \ y) \ \ [x : y])$) we are going to represent it by $[u : v]_{E_p}$(resp. by $(x \ \ y) \qquad \mathbb{C}^2$ or by $[1 : x : y] \qquad P\mathbb{C}^2$ if it is convenient). After every blow up we have the new expanded space X and the induced map $\tilde{F} : \pi^{-1} \circ F \circ \pi : X \qquad X$ As usual, given an irreducible curve C in $\mathbb{C}^2$ the *proper transform* of C is an irreducible curve in X that is the closure of $\pi^{-1}(C \qquad p \ )$ in the Zariski topology. This pullback of C by π is denoted as $\hat{C}$

We are going to consider the **Picard group of** X, denoted by $\mathcal{P}ic(X)$ The set $Div(X)$ of the divisors of X is formed by formal sums $D = \sum d_i \, D_i$ where $d_i \quad \mathbb{Z}$ and $D_i \ _i \quad \mathbb{N}$ a locally finite sequence of irreducible hypersurfaces on X (locally finite means that every point has a neighbourhood which meets only finitely many D_i s). Then $\mathcal{P}ic(X)$ is the set $Div(X) \quad \sim$ modulo linear equivalence. That is, a divisor D is linearly equivalent to zero if $D = div(h)$ where h denotes a rational (or meromorphic) function h on X and $div(h) = Zeros(h) - Poles(h)$ is the associated divisor (see [Bed, BPvdV84]). Then in $\mathcal{P}ic(P\mathbb{C}^2)$ two divisors are equal if and only if they have the same degree and if D has degree m then D is equivalent to m times the class of L where L is a generic line in $P\mathbb{C}^2$ Hence $\mathcal{P}ic(P\mathbb{C}^2)$ is generated by the class of L

We are dealing with the complex manifolds X obtained after performing a finite sequence of blowing-ups. If the base points of the blow-ups are $p_1\ p_2\ \ \ \ p_k\ \subset P\mathbb{C}^2$ and $E_i := \pi^{-1}\ p_i$ are the exceptional fibres at these base points then the $\mathcal{P}ic(X)$ is generated by $<\hat{L}\ E_1\ E_2\ \ \ \ E_k>$ called a basis $\mathcal{B}$ for $\mathcal{P}ic(X)$ (see [Bed, BK04]) where $\hat{L} = \pi^{-1}(L)$ and L is disjoint from all the p_i s. Furthermore $\pi : X -\ \ P\mathbb{C}^2$ induces a morphism of groups $\pi^* : \mathcal{P}ic(P\mathbb{C}^2) -\ \ \mathcal{P}ic(X)$ with the property that for any complex curve $C \subset P\mathbb{C}^2$

$$\pi^*(C) = \hat{C} + \sum m_i\, E_i \tag{2.3}$$

where m_i is the algebraic multiplicity of C at p_i and if $p_i\ \ C$ then $m_i = 0$ (see [Bea78]).

The induced map $\tilde{F}$ induces a morphism of groups, $\tilde{F}^* : \mathcal{P}ic(X)\ \ \ \mathcal{P}ic(X)$ This is a pullback map that is by taking classes of preimages, and it is a well defined linear map on $\mathcal{P}ic(X)$ The interesting thing here is that

$$\tilde{F}^*(\hat{L}) = d\,\hat{L} + \sum_{i=1}^{k} c_i\, E_i \qquad c_i\ \ \ \mathbb{Z}$$

where d is the degree of F The above formula plays a key role in this theory because of the appearance of the degree of F By iterating F we get the corresponding formula by changing F by F^n and d by d_n In order to deduce the behavior of the sequence d_n it is convenient to deal with maps $\tilde{F}$ such that

$$(\tilde{F}^n)^* = (\tilde{F}^*)^n \tag{2.4}$$

Maps $\tilde{F}$ satisfying condition (2.4) are called *Algebraically Stable maps* (**AS** for short), (see [DF01]). We call this extension the **regularization of** F on $P\mathbb{C}^2$

It is known (see Theorem 0 1 of [DF01]) that one can always arrange for a birational map to be AS by performing a finite number of blowups. In order to get AS maps we will use the following useful result showed by Fornaess and Sibony in [FS92] (see also Theorem 1.14 of [DF01]):

$$\tilde{F} \text{ is } AS \text{ if and only if for every exceptional curve } C \text{ and all } n \geq 0 \quad \tilde{F}^n(C)\ \ \ \mathcal{I}(\tilde{F}) \tag{2.5}$$

Notice that this result is related to Proposition 5.

Proposition 6 *Let F be a birational map of $P\mathbb{C}^2$ and let d_n be the degree of F^n Then d_n satisfies a homogeneous linear recurrence with constant coefficients.*

Proof. Consider $\tilde{F}^* = (m_{ij}) = M$ the matrix with integer entries with respect to the basis of $\mathcal{P}ic(X)$ that is $\mathcal{B} =< \hat{L}\ E_1\ E_2 \quad E_k >$. As after the blowup process we get an AS map satisfying 2.4 therefore M^n represents the matrix for $(\tilde{F^n})^*$ with respect to the basis $\mathcal{B}$ Let $\mathcal{X}(\lambda) = \lambda^k + c_{k-1}\,\lambda^{k-1} + c_{k-2}\,\lambda^{k-2} + \cdots + c_1\,\lambda + c_0$ be the characteristic of M Now note that the degree of F $d = d_1 = M_{1,1}$ the first entry of the matrix M Since $\tilde{F}$ is AS also $d_n = (\tilde{F^n})^*_{1,1} = M^n_{1,1}$ Since $\mathcal{X}(M) = 0$ we get that d_n satisfies the recursion formula

$$d_k = -(c_0 + c_1 d_1 + c_2 d_2 + \cdots + c_{k-1} d_{k-1})$$

i. e., the sequence d_n satisfies a homogeneous linear recurrence with constant coefficients. ∎

The following result is quiet useful in our work. It is a direct consequence of Theorem 0 2 of [DF01]. Given a birational map F of $P\mathbb{C}^2$ Let $\tilde{F}$ be its regularized map so that the induced map $\tilde{F}^* : \mathcal{P}ic(X) \quad \mathcal{P}ic(X)$ satisfies $(\tilde{F^n})^* = (\tilde{F}^*)^n$ Then

Theorem 7 *Let $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ be a birational map and let $d_n = deg(F^n)$ Then exactly one of the following holds:*

- *The sequence d_n grows quadratically and F preserves an elliptic fibration. In this case there exists a regularization $\tilde{F}$ of F such that $\tilde{F}$ is an automorphism.*

- *The sequence d_n grows linearly and F preserves a rational fibration and there does not exists any regularization of F being an automorphism.*

- *The sequence d_n is bounded and there exists a regularization $\tilde{F}$ of F such that $\tilde{F}$ is an automorphism.*

- *The sequence d_n grows exponentially.*

In the first three cases $\delta(F) = 1$ while in the last one $\delta(F) > 1$ Furthermore in the first and second, the invariant fibrations are unique.

Remark 2.1.1 *Note that when $\delta(F) = 1$ and F has two generically transverse fibrations then d_n is bounded.*

2.2 Settings

Consider the family of fractional maps $f : \mathbb{C}^2 \quad \mathbb{C}^2$:

$$f(x \ y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y \ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right) \tag{2.6}$$

where the parameters are complex numbers. We call $F[x_0 : x_1 : x_2] = [F_1[x_0 : x_1 : x_2] : F_2[x_0 : x_1 : x_2] : F_3[x_0 : x_1 : x_2]]$ the extension of $f(x \ y)$ to the projective plane. Then

$$\begin{aligned}
F_1[x_0 : x_1 : x_2] &= x_0(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2) \\
F_2[x_0 : x_1 : x_2] &= (\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2)(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2) \\
F_3[x_0 : x_1 : x_2] &= x_0(\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2)
\end{aligned}$$

The indeterminacy set of $F[x_0 : x_1 : x_2]$ is

$$\mathcal{I}(F) = \quad O_1 \ O_2 \ O_3$$

where

$$\begin{aligned}
O_0 &= [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01}] \\
O_1 &= [0 : \alpha_2 : -\alpha_1] \\
O_2 &= [0 : \gamma_2 : -\gamma_1]
\end{aligned}$$

and

$$(\beta\gamma)_{ij} := \beta_i \gamma_j - \gamma_j \beta_i$$

for $i \ \ j = 0 \ \ 1 \ \ 2$

The exceptional locus of $F[x_0 : x_1 : x_2]$ is

$$\mathcal{E}(F) = \quad S_0 \ S_1 \ S_2$$

where

$$\begin{aligned}
S_0 = &\quad x_0 = 0 \\
S_1 = &\quad \gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0 \\
S_2 = &\quad (\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01}) \ x_0 + \alpha_1(\beta\gamma)_{12} + \alpha_2(\beta\gamma)_{12} x_2 = 0
\end{aligned}$$

By calling $f^{-1}(x \ y)$ the inverse of $f(x \ y)$ and by $F^{-1}[x_0 : x_1 : x_2]$ its extension on $P\mathbb{C}^2$ also a indeterminacy set $\mathcal{I}(F^{-1})$ exists. In fact:

$$\mathcal{I}(F^{-1}) = \{A_1, A_2, A_3\},$$

where

$$\begin{aligned} &A_0 = [0:1:0],\\ &A_1 = [0:0:1],\\ &A_2 = [(\beta\gamma)_{12}\,(\alpha\gamma)_{12} : (\alpha_0\,(\beta\gamma)_{12} - \alpha_1\,(\beta\gamma)_{02} + \alpha_2\,(\beta\gamma)_{01})\,(\alpha\gamma)_{12} : (\alpha\beta)_{12}\,(\beta\gamma)_{12}], \end{aligned}$$

and the exceptional locus of $F^{-1}[x_0 : x_1 : x_2]$ is

$$\mathcal{E}(F^{-1}) = \{T_0, T_1, T_2\},$$

where

$$\begin{aligned} &T_0 = \{(\gamma_0(\alpha\beta)_{12} - \gamma_1(\alpha\beta)_{02} + \gamma_2(\alpha\beta)_{01})\,x_0 - (\beta\gamma)_{12}x_1 = 0\}\,,\\ &T_1 = \{(\alpha\beta)_{12}x_0 - (\alpha\gamma)_{12}x_2 = 0\},\\ &T_2 = \{x_0 = 0\}. \end{aligned}$$

For the birational map 2.6 the exceptional curves of F and F^{-1} collapse to a single point that are the indeterminacy points of F^{-1} and F that is $S_i \twoheadrightarrow A_i,\ T_j \twoheadrightarrow O_j$ for $i, j = 0, 1, 2$. From now on we will use the notation $S_i \twoheadrightarrow A_i$ to indicate that F maps the set S_i to a single point $A_i \in \mathcal{E}(F^{-1})$. The following figure shows how the exceptional curves and indeterminacy points of F and F^{-1} behave under the action of F and F^{-1}.

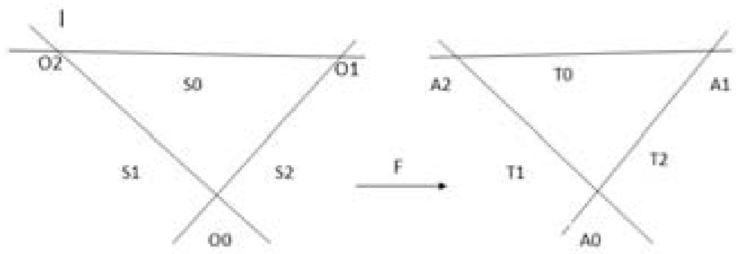

Figure 2.1: Behavior of exceptional curves and indeterminacy points of F and F^{-1}

We are interested in the mappings (2.6) which are birational mappings for all the values of parameters and, to avoid the already studied mappings, we will consider that $(\gamma_1 \;\gamma_2) = (0 \;0)$ Next lemma states the conditions on the parameters of f in (2.6) to be birational also, to distinguish the degenerate (non degenerate case resp) of f that is when the map F has exactly two distinct exceptional curves (has exactly three distinct exceptional curves resp).

Lemma 8 *Consider the mappings*

$$f(x_1 \;x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \;\; \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right) \quad (\gamma_1 \;\gamma_2) = (0 \;0) = (\alpha_1 \;\alpha_2)$$

Then:

(a) The mapping f is birational if and only if the vectors $(\beta_0 \;\beta_1 \;\beta_2)$ $(\gamma_0 \;\gamma_1 \;\gamma_2)$ are linearly independent and $((\alpha\beta)_{12} \;(\alpha\gamma)_{12}) = (0 \;0)$ $((\alpha\gamma)_{12} \;(\beta\gamma)_{12}) = (0 \;0)$ and either $((\alpha\beta)_{12} \;(\beta\gamma)_{12}) = (0 \;0)$ or $(\beta_1 \;\beta_2) = (0 \;0)$

(b) The mapping f has two distinct exceptional curves if and only if $(\beta\gamma)_{12} = 0$ or $(\alpha\gamma)_{12} = 0$

Proof. The conditions in *(a)* are necessary for f to be invertible as if the vectors $(\beta_0 \;\beta_1 \;\beta_2)$ $(\gamma_0 \;\gamma_1 \;\gamma_2)$ are linearly dependent then the second component of f is a constant, also if $((\alpha\beta)_{12} \;(\alpha\gamma)_{12}) = (0 \;0)$ or $((\alpha\gamma)_{12} \;(\beta\gamma)_{12}) = (0 \;0)$ then f only depends on $\alpha_1 x_1 + \alpha_2 x_2$ or on $\gamma_1 x_1 + \gamma_2 x_2$ If $((\alpha\beta)_{12} \;(\beta\gamma)_{12}) = (0 \;0)$ and $(\beta_1 \;\beta_2) = (0 \;0)$ the f only depends on $\beta_1 x_1 + \beta_2 x_2$.

Now assume that conditions (a) are satisfied. Then the inverse of f which formally is

$$f^{-1}(x \;y) = \left(\frac{-(\alpha\beta)_{02} + \beta_2 x + (\alpha\gamma)_{02} y - \gamma_2 xy}{(\alpha\beta)_{12} - (\alpha\gamma)_{12} y} \;\; \frac{(\alpha\beta)_{01} - \beta_1 x + (\alpha\gamma)_{10} y + \gamma_1 xy}{(\alpha\beta)_{12} - (\alpha\gamma)_{12} y}\right)$$

is well defined. Furthermore the numerators of the determinants of the Jacobian of f and f^{-1} are

$$\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01} + \alpha_1(\beta\gamma)_{12} x + \alpha_2(\beta\gamma)_{12} y \tag{2.7}$$

and

$$\alpha_0(\beta\gamma)_{12} - \alpha_1(\beta\gamma)_{02} + \alpha_2(\beta\gamma)_{01} - (\beta\gamma)_{12} y \tag{2.8}$$

respectively. It is easily seen that conditions (a) imply that both (2.7) and (2.8) are not identically zero. Hence, $f \circ f^{-1} = f^{-1} \circ f = id$ in $\mathbb{C}^2 \quad V$ where V is the algebraic curve determined by the common zeros of (2.7) and (2.8).

To see *(c)* we know that since S_i maps to A_i this implies that the points $A_0 \;A_1 \;A_2$ are not all distinct. Since $A_0 = A_1$ we have two possibilities: $A_0 = A_2$ or $A_1 = A_2$ Condition $A_0 = A_2$

writes as $(\beta\gamma)_{12}\,(\alpha\gamma)_{12} = 0$ and $(\alpha\beta)_{12}(\beta\gamma)_{12} = 0$ From *(a)*, the vector $((\alpha\beta)_{12} \;\; (\alpha\gamma)_{12}) = (0 \;\; 0)$ Hence $(\beta\gamma)_{12}$ must be zero. In a similar way it is seen that $A_1 = A_2$ if and only if $(\alpha\gamma)_{12} = 0$

Remark 2.2.1 *Note that the conditions* (a) *in Lemma 8 also imply that* $(\alpha_1 \;\; \beta_1 \;\; \gamma_1) = (0 \;\; 0 \;\; 0)$ *and* $(\alpha_2 \;\; \beta_2 \;\; \gamma_2) = (0 \;\; 0 \;\; 0)$

Proposition 9 *Let* $f(x \;\; y)$ *be the map 2.6 and let* F *be its extension to the projective space. Let* $V = \quad P = 0$ *be an irreducible algebraic curve such that* $V \quad \mathcal{E}(F)$ *Then*

$$deg(F(V)) = 2\,deg(V) - \sum_{p \quad \mathcal{I}(F)} m_p(P) \tag{2.9}$$

where $m_p(P)$ *is the algebraic multiplicity of* P *at* p

Proof. Let P be a minimal polynomial of V and consider $Q = P \circ F^{-1}$ Clearly $deg(Q) = 2\,deg(V)$ and if we apply Q at the points $q \quad F(V)$ we have

$$Q(q) = (P \circ F^{-1})\,(F(p)) = P(p) = 0$$

Assume that there exists some indeterminacy point of F which belongs to V Then by Proposition 3 there is some $T \quad \mathcal{E}(F^{-1})$ such that F^{-1} maps T to p Hence $Q(T(x)) = (P \circ F^{-1})\,(T(x)) = P(p) = 0$ Since $F(V)$ is an irreducible curve, T cannot be a factor of $F(V)$ Furthermore since each $T \quad \mathcal{E}(F^{-1})$ has degree one, if $m_p(P) = 1$ then $deg(F(V)) \leq 2\,deg(V) - 1$ Furthermore if $m_p(P) = k$ then $deg(F(V)) \leq 2\,deg(V) - k$ If V contains several indeterminacy points then the inequality

$$deg(F(V)) \leq 2\,deg(V) - \sum_{p \quad \mathcal{I}(F)} m_p(P)$$

Now the equality (2.9) holds. Because if $P(F^{-1}) = 0$ then $F^{-1}(x) \quad V$ and either $F^{-1}(x) \quad \mathcal{I}(F)$ and $x \quad F(x)$ or $F^{-1}(x) = p \quad \mathcal{I}(F)$ and $x \quad T$ ∎

We will not use these settings for the families studied in Chapter 7 of Lyness $k-$periodic recurrences. As those families of maps are different from rest of the mappings considered to study in previous chapters, hence we prefer to introduce all the required settings in the same chapter.

2.2.1 Classification

We first classify our family 2.6 into two large subfamilies. These are the degenerate mappings or non degenerate mappings. For general values of the parameters, the mappings in our family have three indeterminacy points and the exceptional locus is formed by three straight lines. But there is a subfamily such that the exceptional locus only has two straight lines. We call these mappings **degenerate mappings**, this happens when $(\alpha\gamma)_{12} = 0$ or $(\beta\gamma)_{12} = 0$ When all three lines are distinct we call such maps as **non degenerate mappings**, this happens when $(\alpha\gamma)_{12} = 0$ and $(\beta\gamma)_{12} = 0$ We study these cases separately for all the values of parameters satisfying these condition of existing in one or the other group of mappings. To study the behavior of the sequence of degrees, we treat differently the degenerate case of the non degenerate. In the non degenerate context we use a Theorem of Bedford and Kim (see [BK06] or Theorem 10 in chapter 2) to find the characteristic polynomial that gives the recurrence which follow the sequence of degrees. In the degenerate case we explicitly compute the action of F in the corresponding Picard group.

2.3 Lists of orbits

We derive our results in the non-degenerate case by using Theorem 10 discussed in the end of this subsection, established and proved in [BK06, BK04] with the same tools explained in the previous subsection. In order to determine the matrix of the extended map in the Picard group, it is necessary to distinguish between different behaviors of the iterates of the map on the indeterminacy points of its inverse.

The result in Theorem 10 is for a general family G of quadratic maps of the form $G = L \circ J$ The maps of family (2.6), when the triangle is non-degenerate, are linearly conjugated to such maps. Here L is an invertible linear map and J is the involution in $P\mathbb{C}^2$ as follows:

$$J[x_0 : x_1 : x_2] = [x_0^{-1} : x_1^{-1} : x_2^{-1}] = [x_1x_2 : x_0x_2 : x_0x_1]$$

We find that the involution J has an indeterminacy locus $\mathcal{I} = \ \epsilon_0\ \epsilon_1\ \epsilon_2$ and a set of exceptional curves $\mathcal{E} = \ \Sigma_0\ \Sigma_1\ \Sigma_2$, where $\Sigma_i = \ x_i = 0$ for $i = 0\ 1\ 2$ and $\epsilon_i = \Sigma_j \bigcap \Sigma_k$ with $i\ j\ k = 0\ 1\ 2$ Let $\mathcal{I}(G^{-1}) := \ a_0\ a_1\ a_2$ the elements of this set are determined by $a_i := G(\Sigma_i - \mathcal{I}(J)) = L\,\epsilon_i$ for $i = 0\ 1\ 2$; see [BK06]. To follow the orbits of the points of $\mathcal{I}(G^{-1})$ we need to understand the following definitions and construction of orbits and lists in order to apply the result of Theorem 10.

We assemble the orbit of a point $p \quad P\mathbb{C}^2$ under the map G as follows. For a point p $\mathcal{E}(G) \quad \mathcal{I}(G)$ we say that the orbit $\mathcal{O}(p) = \ p\ $. Now consider that there exits a $p \quad P\mathbb{C}^2$ such that its $n^{th}-$ iterate belongs to $\mathcal{E}(G) \quad \mathcal{I}(G)$ for some n, whereas all the other $n-1$ iterates of p under G are never in $\mathcal{E}(G) \quad \mathcal{I}(G)$. This is to say that for some n the orbit of p reaches an exceptional curve of G or an indeterminacy point of G We thus define the orbit of p as $\mathcal{O}(p) = \ p \ G(p) \quad G^n(p)$ and we call it a **singular orbit**. If for some $p \quad P\mathbb{C}^2$ in turns out that p and all of its iterates under G are never in $\mathcal{E}(G) \quad \mathcal{I}(G)$ for all n we set as $\mathcal{O}(p) = \ p \ G(p) \ G^2(p)$ and $\mathcal{O}(p)$ is **non singular orbit**. We now make another characterization of these orbits. Consider that a singular orbit reaches an indeterminacy point of G, this is to say that $G^n(p) \quad \mathcal{I}(G)$ but its not in $\mathcal{E}(G)$ We call such orbits as **singular elementary orbits** and we refer them as **SE**-orbits.

We are now proceeding towards the main preliminary result of this section which is Theorem 10. We will use this theorem to find the characteristic polynomial in our following work. To apply Theorem 10 we need to organize our SE orbits into lists in the following way.

Two orbits $\mathcal{O}_1 = \ a_1 \quad \epsilon_{j_1}$ and $\mathcal{O}_2 = \ a_2 \quad \epsilon_{j_2}$ are in the same list if either $j_1 = 2$ or $j_2 = 1$ that is, if the ending index of one orbit is the same as the beginning index of the other. We have the following possibilities:

- *Case 1: One SE-orbit,* $\mathcal{O}_i = \ a_i \quad \epsilon_{\tau(i)}$ Then we have the list $\mathcal{L} = \ \mathcal{O}_i = \ a_i \quad \epsilon_{\tau(i)}$ If $\tau(i) = i$ we say that $\mathcal{L}$ is a closed list. Otherwise it is an open list.

- *Case 2: Two SE-orbits,* $\mathcal{O}_i = \ a_i \quad \epsilon_{\tau(i)}$ and $\mathcal{O}_j = \ a_i \quad \epsilon_{\tau(j)}$ In this case we can have either two closed lists,

$$\mathcal{L}_1 = \ \mathcal{O}_i = \ a_i \quad \epsilon_i \qquad \text{and} \qquad \mathcal{L}_2 = \ \mathcal{O}_j = \ a_j \quad \epsilon_j \qquad \text{with} \quad i = j$$

or one open and one closed list

$$\begin{aligned}\mathcal{L}_1 &= \ \mathcal{O}_i = \ a_i \quad \epsilon_i \qquad \text{and}\\ \mathcal{L}_2 &= \ \mathcal{O}_j = \ a_j \quad \epsilon_k \qquad \text{with} \quad i = j \ \ j = k \ \ k = i\end{aligned}$$

or a single list

$$\mathcal{L} = \ \mathcal{O}_i = \ a_i \quad \epsilon_j \ \ \mathcal{O}_j = \ a_j \quad \epsilon_{\tau(j)} \qquad \text{with} \quad i = j$$

which is closed if $\tau(j) = i$ and an open list otherwise.

Notice that we cannot have two open lists because there are at most three SE-orbits.

- *Case 3: Three SE orbits*: In this case we can have either *three closed lists*

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{O}_0 = a_0 \quad \epsilon_0 \quad , \mathcal{L}_2 = \mathcal{O}_1 = a_1 \quad \epsilon_1 \quad \text{and} \\ \mathcal{L}_3 &= \mathcal{O}_2 = a_2 \quad \epsilon_2 \end{aligned}$$

or *two closed lists*

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{O}_i = a_i \quad \epsilon_j \quad \mathcal{O}_j = a_j \quad \epsilon_i \quad \text{and} \\ \mathcal{L}_2 &= \mathcal{O}_k = a_k \quad \epsilon_k \quad \text{with} \quad i = k = j \quad \text{and} \quad i = j \end{aligned}$$

or *one closed list*

$$\mathcal{L} = \mathcal{O}_0 = a_0 \quad \epsilon_1 \quad \mathcal{O}_1 = a_1 \quad \epsilon_2 \quad \mathcal{O}_2 = a_2 \quad \epsilon_0$$

We now define two polynomials $\mathcal{T}_\mathcal{L}$ and $\mathcal{S}_\mathcal{L}$ which we will use to apply theorem 10. Let n_i denote the sum of the number of elements of an orbit $\mathcal{O}_i$ and let $\mathcal{N}_\mathcal{L} = n_u + \quad + n_{u+\mu}$ denote the sum of the numbers of elements of each list $\mathcal{L}$ If $\mathcal{L}$ is closed then $\mathcal{T}_\mathcal{L} = x^{\mathcal{N}_\mathcal{L}} - 1$ and if $\mathcal{L}$ is open then $\mathcal{T}_\mathcal{L} = x^{\mathcal{N}_\mathcal{L}}$ Now we define $\mathcal{S}_\mathcal{L}$ for different lists as follows:

$$\mathcal{S}_\mathcal{L}(x) = \begin{cases} 1 & \text{if } \mathcal{L} = n_1 \\ x^{n_1} + x^{n_2} + 2 & \text{if } \mathcal{L} \text{ is closed and } \mathcal{L} = n_1 \; n_2 \\ x^{n_1} + x^{n_2} + 1 & \text{if } \mathcal{L} \text{ is open and } \mathcal{L} = n_1 \; n_2 \\ \sum_{i=1}^{3} \left[x^{\mathcal{N}_\mathcal{L} - n_i} + x^{n_i}\right] + 3 & \text{if } \mathcal{L} \text{ is closed and } \mathcal{L} = n_1 \; n_2 \; n_3 \\ \sum_{i=1}^{3} x^{\mathcal{N}_\mathcal{L} - n_i} + \sum_{i=2} x^{n_i} + 1 & \text{if } \mathcal{L} \text{ is open and } \mathcal{L} = n_1 \; n_2 \; n_3 \end{cases}$$

Theorem 10 *If $G = L \circ J$, then the dynamical degree $\delta(G)$ is the largest real zero of the polynomial*

$$\mathcal{X}(x) = (x-2) \prod_{\mathcal{L} \quad \mathcal{L}^c \quad \mathcal{L}^o} \mathcal{T}_\mathcal{L}(x) + (x-1) \sum_{\mathcal{L} \quad \mathcal{L}^c \quad \mathcal{L}^o} S_L(x) \prod_{\mathcal{L}' = \mathcal{L}} \mathcal{T}_{\mathcal{L}'}(x)$$

Here $\mathcal{L}$ runs over all the orbit lists.

2.3.1 Orbit collision

The study of the orbits of the indeterminacy points of G^{-1} is important for the understanding of the mappings of the form $G = L \circ J$ as discussed above. To do this we first locate the singular orbits. These can be the ones which reach an indeterminacy point of G or an exceptional curve of G The first case corresponds to the elementary case where we get an SE orbit. The lists for such orbits are discussed above. The later case is a non elementary one in which the orbit of a point a_i reaches some $\bar{\sigma}_j \quad \mathcal{E}(G)$ If the orbit $\mathcal{O}_i$ of a point a_i joins the orbit $\mathcal{O}_j$ of some other point a_j then this shows that $\mathcal{O}_i$ collides with the $\mathcal{O}_j$. This is when of **orbit collision** happens. There can be more than one orbit collision. In this case every time when we blow up the space X, as discussed earlier, we check for the possible collisions and the new indeterminacy points for the induced map on X It can happen that the orbit of a_i after colliding with the orbit of a_j reaches an indeterminacy point that exists in $\mathcal{O}_j$ To understand this, in the family 6 2 of chapter 6, there has appeared an example of this case. This is discussed in Theorem 42. But it is also possible that $\mathcal{O}_j$ does not contain any point that is still indeterminate for new induced map on X In this case if $\mathcal{O}_i$ collides with $\mathcal{O}_j$ then $\mathcal{O}_i$ cannot be SE. This can be understood by looking at Theorem 28 in this work. The collision of orbits is discussed in detail in [BK06].

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of that beauty that strikes the senses, the beauty of qualities and appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp - Henri Poincaré.

Chapter 3

Degenerate Case

This chapter classifies the mappings in the degenerate case. Two different cases depending on $(\beta\gamma)_{12} = 0$ or $(\alpha\gamma)_{12} = 0$ are discussed. We identify the families with dynamical degree one and study their dynamics.

We refer to degenerate case and call that the mappings f in (2.6) are degenerate when we only have two exceptional curves in $\mathcal{E}(F)$ where F is the extension of f in projective space. Since S_i maps to A_i this implies that the points A_0 A_1 A_2 are not all distinct. Since $A_0 = A_1$ we have two possibilities: $A_0 = A_2$ or $A_1 = A_2$ Condition $A_0 = A_2$ writes as $(\beta\gamma)_{12}\,(\alpha\gamma)_{12} = 0$ and $(\alpha\beta)_{12}(\beta\gamma)_{12} = 0$ From *(a)*, the vector $((\alpha\beta)_{12}\ \ (\alpha\gamma)_{12}) = (0\ \ 0)$ Hence $(\beta\gamma)_{12}$ must be zero. In a similar way it is seen that $A_1 = A_2$ if and only if $(\alpha\gamma)_{12} = 0$

To prove the result in this chapter for both cases $(\beta\gamma)_{12} = 0$ and $(\alpha\gamma)_{12} = 0$ in the first two sections we first calculate the dynamical degree depending on the coefficients using the methodologies explained in Chapter 2. That is to perform the necessary blow-up s in order to extend the map F to a space X getting a map $\tilde{F}$ which is AS. This will be the case if (2.5) is satisfied. We then study the induced map $\tilde{F}$ on the corresponding Picard group to find the associated characteristic polynomial which gives the sequence of degrees of F

In the third and last section of this chapter we focus our attention to detect the families with zero entropy that is with dynamical degree one. They appear to be able to separate in two different sets of mappings, i.e. the maps whose sequence of degrees grows linearly and the ones whose sequence of degrees is bounded. For the maps in the first set we find explicitly the rational invariant fibration which assures Lemma $4\ 2$ in [DF01]. In almost all the cases the fibration is trivial. In the second set of mappings when the sequence of degrees is bounded we distinguish those mappings that have periodic sequence of degrees, and hence we show that the mappings indeed are periodic. But the mappings (5) required more work. As there appear a family that is $2\,k$ periodic for all k but this family is too much singular, this is to say that it requires several multiple blowups to get an AS map. However the family is quiet simple and to deduce the behavior of the degrees we have used the well known results on Möbius one dimensional maps in [CGM06a]. For the other mappings with bounded sequence of degrees we explicitly find two fibrations generically transverse.

We begin by the case $(\beta\gamma)_{12} = \beta_1\,\gamma_2 - \beta_2\,\gamma_1 = 0$ This condition implies that $S_0 = S_2 = \; x_0 = 0$ and $O_0 = O_2 = [0 : -\gamma_2 : \gamma_1]$. Thus $\mathcal{E}(F) = \; S_0 \; S_1$ and $\mathcal{I}(F) = \; O_1 \; O_2$ where $S_1 = \; \gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0$ and $O_1 = [0 : -\alpha_2 : \alpha_1]$ The two exceptional curves $S_0 \; S_1$ collapses to $A_0 \; A_1$ respectively where $A_0 = [0 : 1 : 0]$ and $A_1 = [0 : 0 : 1]$ We notice that F maps $P\mathbb{C}^2 \quad \mathcal{E}(F)$ to $P\mathbb{C}^2 \quad \mathcal{E}(F^{-1})$ in a bijective way, where F^{-1} is the inverse of F It is seen that in this case $\mathcal{E}(F^{-1}) = \; T_0 \; T_1$ where $T_0 = \; x_0 = 0$ and $T_1 = \; (\alpha\beta)_{12}x_0 + (\alpha\gamma)_{21}x_2 = 0$

We now study these two cases separately in the following sections:

3.1 Mappings with $(\beta\gamma)_{12} = 0$

Theorem 11 *Assume that $(\beta\gamma)_{12} = \beta_1\,\gamma_2 - \beta_2\,\gamma_1 = 0$ Then the following hold:*

1. *If $\alpha_1 \cdot \gamma_1 = 0 = \alpha_2 \cdot \gamma_2$ then the dynamical degree of F is $\delta(F) = 2$*

2. *If $\alpha_1 \cdot \gamma_1 = 0$ and $\alpha_2 = 0$ then $\alpha_1 = 0 = \gamma_2$ and the dynamical degree is $\delta(F) = 1 \quad d_n = 1 + n$ for all $n \quad \mathbb{N}$*

3. *If $\alpha_1 \cdot \gamma_1 = 0$ and $\gamma_2 = 0$ then $\beta_2 = 0 \quad \alpha_2 = 0 = \gamma_1$ and the dynamical degree is $\delta(F) = \frac{1+\ 5}{2} \quad d_{n+2} = d_{n+1} + d_n$ for all $n \quad \mathbb{N}$*

4. *If $\alpha_1 = 0 \quad \gamma_2 = 0$ then $\gamma_1 = 0 = \alpha_2$ and the dynamical degree is $\delta(F) = \frac{1+\ 5}{2} \quad d_{n+2} = d_{n+1} + d_n$ for all $n \quad \mathbb{N}$*

5. *If $\gamma_1 = 0$ then $\gamma_2 = 0 = \alpha_1 \quad \beta_1 = 0$ and the dynamical degree of F is $\delta(F) = 1$ Furthermore the sequence of degrees is $d_n = 1 + n$ or $d_n = 2$ for all $n \quad \mathbb{N}$ depending on $\alpha_2 = 0$ or $\alpha_2 = 0$ respectively.*

6. *If $\alpha_1 = 0 = \gamma_2$ then $\gamma_1 = 0 = \alpha_2 \quad \beta_2 = 0$ and the dynamical degree is $\delta(F) = 1$ This family is $2\,k$ periodic if $\left(\frac{\lambda_1}{\lambda_2}\right)^k = 1$ for some $k > 1 \quad \mathbb{N}$ where λ_1 and λ_2 are the roots of*

$x^2 - \gamma_0\, x - \beta_0 = 0$ *with* $\gamma_0^2 + 4\,\beta_0 = 0$ *If* $\gamma_0^2 + 4\,\beta_0 = 0$ *then* f *is not periodic and* $d_n = 2$ *for all* n $\mathbb{N}$

Proof.

1. The exceptional curves $S_0 \twoheadrightarrow A_0$ and $S_1 \twoheadrightarrow A_1$ Consider that $(\beta\gamma)_{12} = \beta_1\,\gamma_2 - \beta_2\,\gamma_1 = 0$ and $\alpha_1 \cdot \gamma_1 = 0 = \alpha_2 \cdot \gamma_2$ Then we observe that $F(A_0) = [0 : \alpha_1\gamma_1 : 0] = A_0$ $\mathcal{I}(F)$ and $F(A_1) = [0 : \alpha_2\gamma_2 : 0] = A_0$ $\mathcal{I}(F)$ Thus using (2.5) we see that F is AS, which implies that $d_n = 2^n$ and consequently $\delta(F) = 2$.

2. Assume that $\alpha_1 \cdot \gamma_1 = 0$ and $\alpha_2 = 0$ Then from lemma 8 we see that for f to be birational in this case we have conditions $(\alpha\,\beta)_{12} = 0 = (\alpha\,\gamma)_{12}$ with $(\alpha_1\ \ \alpha_2) = (0\ \ 0)$ $(\gamma_1\ \ \gamma_2) = (0\ \ 0)$ but $(\beta_1\ \ \beta_2)$ can be simultaneously zero. Also as we know that $(\beta\,\gamma)_{12} = 0$ thus we find that to satisfy these conditions we must have α_1 and γ_2 non zero. Observe that S_0 collapses to $A_0 = [0 : 1 : 0]$ S_0 and that $F[0 : 1 : 0] = [0 : 1 : 0]$ which is not equal to any of the indeterminacy points O_1 O_2 Hence A_0 is a fixed point.

   Now $S_1 \twoheadrightarrow A_1 = O_1 = [0 : 0 : \alpha_1]$ $\mathcal{I}(F)$ Hence blow-up A_1 to obtain E_1. We identify E_1 with $P\mathbb{C}^1$ in the following way. For $[u : v]_{E_1}$ $P\mathbb{C}^1$, we associate the point

$$[u : v]_{E_1} := \lim_{t\quad 0} \pi^{-1}[tu : tv : 1] \quad E_1$$

   We may now determine the map $\widetilde{F}$ on S_1 For

$$x = \left[x_0 : x_1 : \frac{-(\gamma_0\, x_0 + \gamma_1\, x_1)}{\gamma_2}\right] = \lim_{t\quad 0}\left[x_0 : x_1 : \frac{t - (\gamma_0\, x_0 + \gamma_1\, x_1)}{\gamma_2}\right] \quad S_1$$

   we assign

$$\widetilde{F}(x) := \lim_{t\quad 0} F\left[x_0 : x_1 : \frac{t - (\gamma_0\, x_0 + \gamma_1\, x_1)}{\gamma_2}\right] \quad X$$

   That is

$$F\left[x_0 : x_1 : \tfrac{-(\gamma_0\, x_0 + \gamma_1\, x_1)}{\gamma_2}\right] = [t\, x_0 : (\alpha_0\, x_0 + \alpha_1\, x_1)\, t : \tfrac{x_0\,(\beta_0\,\gamma_2\, x_0 + \gamma_2\,\beta_1\, x_1 + \beta_2\, t - x_0\,\beta_2\,\gamma_0 - \beta_2\,\gamma_1\, x_1)}{\gamma_2}]$$

   and thus taking limit as t 0 yields

$$\widetilde{F}[0 : x_1 : x_2] = [x_0 : \alpha_0 x_0 + \alpha_1 x_1]_{E_1}$$

Now we make similar computations for a point $[u : v]_{E_1}$ in the fibre E_0 over the indeterminacy point A_1. We set $x = [tu : tv : 1]$ so that

$$\begin{aligned} F(x) &= [tu(\gamma_0\, t\, u + \gamma_1\, t\, v + \gamma_2) : t(\alpha_0 u + \alpha_1 v)(\gamma_0\, t\, u + \gamma_1\, t\, v + \gamma_2) : \\ &\quad t\, u\, (\beta_0\, t\, u + \beta_1\, t\, v + \beta_2)] \end{aligned}$$

Dropping t from three components and taking the limit as $t \quad 0$ we get

$$\tilde{F}[u : v]_{E_1} \quad [u : \alpha_0\, u + \alpha_1\, v : u] \quad T_1$$

The above equation shows that $\widetilde{F}$ maps E_1 to T_1 biholomorphically and the assumptions on the coefficients imply that no points on E_1 are undetermined for $\widetilde{F}$

Let X be the manifold obtained after the blow-up. We have extended F on X getting $\tilde{F}$ We see that now $\mathcal{E}(\tilde{F}) = \quad S_0$ and that $\mathcal{I}(\tilde{F}) = \quad O_2$ To know if $\tilde{F}$ is AS we have to assure that the orbit of A_0 under $\tilde{F}$ never reaches O_2 Notice that A_0 is also fixed under $\tilde{F}$ and $A_0 = O_2$ Thus using (2.5) we see that F is AS.

In order to find the matrix of the induced mapping $\widetilde{F}^*$ on $Pic(X)$ since $Pic(X) =< \hat{L} \ E_1 >$ and $\widetilde{F}^*$ acts taking preimages, the above calculations show that:

$$\widetilde{F}^*(E_1) = \hat{S}_1$$

From (2.3) we know that $\pi^*(S_1) = \hat{S}_1$ and since S_1 is equivalent to L in $Pic(P\mathbb{C}^2)$ and $\pi^*(L) = \hat{L}$ we get that $\pi^*(S_1) = \hat{L}$ Hence,

$$\widetilde{F}^*(E_1) = \hat{L}$$

It remains to calculate $\widetilde{F}^*(\widehat{L})$ To this end, we observe that again from (2.3) and since A_1 $F^{-1}(L)$ we get:

$$\pi^*(F^{-1}(L)) = \widehat{F^{-1}(L)} + E_1 \tag{3.1}$$

and since $F^{-1}(L) \equiv 2L$ in $Pic(X)$ it implies that $\widehat{F^{-1}(L)} = 2\widehat{L} - E_1$ and hence that

$$\widetilde{F}^*(\widehat{L}) = \widehat{F^{-1}(L)} = 2\widehat{L} - E_1 \tag{3.2}$$

Now we are ready to write the matrix of $\widetilde{F}^*$:

$$\widetilde{F}^* = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3}$$

The characteristic polynomial of $(\widetilde{F}^*)$ is $(z-1)^2$ and hence the dynamical degree is 1 By finding the degree of first two iterates of F i.e. $d_1 = 2$ $d_2 = 3$ we can write the sequence of degrees as $d_n = c_0 + c_1\, n = 1 + n$ for all n $\mathbb{N}$ Hence proved.

3. Assume that $\alpha_1 \cdot \gamma_1 = 0$ and $\gamma_2 = 0$ Then from lemma 8 we see that for f to be birational we have conditions $(\alpha\,\beta)_{12} = 0 = (\alpha\,\gamma)_{12}$ with $(\alpha_1\ \alpha_2) = (0\ 0)$ $(\gamma_1\ \gamma_2) = (0\ 0)$ but $(\beta_1\ \beta_2)$ can be simultaneously zero. Also as we know that $(\beta\,\gamma)_{12} = 0$ thus we find that to satisfy these conditions we must have $\beta_2 = 0$ and $\alpha_2 = 0 = \gamma_1$

   Observe that now we have $S_1 \twoheadrightarrow A_1 = O_2 = [0:0:1]$ and $F(A_0) = [0 : \alpha_1\gamma_1 : 0] = A_0$ $\mathcal{I}(F)$ So we have to blow-up $A_1 = [0:0:1]$ getting E_1 The extension of F on S_1 and on E_1 acts as follows:

$$\tilde{F}\left[x_0 : \frac{-\gamma_0}{\gamma_1}x_0 : x_2\right] = [\gamma_1 x_0 : (\alpha\gamma)_{01}x_0 + \alpha_2\gamma_1 x_2]_{E_1} \text{ and } \tilde{F}[u:v]_{E_1} = [0:1:0]$$

   On the other hand since A_0 is a fixed point, we see that $\tilde{F}$ is AS and the computations give:

$$\tilde{F}^*(\hat{L}) = 2\,\hat{L} - E_1$$
$$\tilde{F}^*(E_1) = \hat{L} - E_1$$

   The matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L}\ E_1 >$ is:

$$\widetilde{F}^* = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{3.4}$$

   with characteristic polynomial $z^2 - z - 1$ Hence the dynamical degree is $\delta(F) = \frac{1+\ 5}{2}$ and $d_{n+2} = 2\,d_{n+1} + d_n$ for all n $\mathbb{N}$

4. Assume that $\alpha_1 = 0$ and $\gamma_2 = 0$ Then from lemma 8 we see that for f to be a birational map in this case we have conditions $(\alpha\,\beta)_{12} = 0 = (\alpha\,\gamma)_{12}$ with $(\alpha_1\ \alpha_2) = (0\ 0)$ $(\gamma_1\ \gamma_2) = (0\ 0)$ but $(\beta_1\ \beta_2)$ can be simultaneously zero. Also as we know that $(\beta\,\gamma)_{12} = 0$ thus we find that to satisfy these conditions we must have $\gamma_1 = 0 = \alpha_2$

In this case $S_0 \twoheadrightarrow A_0 = O_1 \quad \mathcal{I}(F)$ Hence we blow-up A_0 to obtain E_0. So we have to blow-up A_0 to get E_0 The extension of F on S_0 and on E_0 acts as follows:

$$\tilde{F}\,[0 : x_1 : x_2] = [\gamma_1\, x_1 + \gamma_2\, x_2 : \beta_1\, x_1 + \beta_2\, x_2]_{E_0}$$

and

$$\tilde{F}[u : v]_{E_0} = [\gamma_1\, u : \gamma_1(\alpha_0\, u + \alpha_2\, v) : \beta_1\, u] \quad T_1$$

As $A_1 \quad S_0$ is not an indeterminacy point of F we find that $\tilde{F}(A_1) = [\gamma_2 : \beta_2]_{E_0}$ As we have not created new points of indeterminacy after the blow-up process then $\mathcal{I}(\tilde{F}) = \quad O_2$ and $O_2 \quad S_0 = T_2$ We know that the only points on T_2 which have preimages are A_0 and A_1 which implies that if the iterates of A_1 reaches O_2 for some iterate of F then O_2 should be equal to either A_0 or A_1 But as $\gamma_1 = 0 = \gamma_2$ therefore $A_0 = O_2 = A_1$ This implies that O_2 has no preimages hence no the iterates of A_1 cannot reach O_2 Hence we see that $\tilde{F}$ is AS. As described in the previous case we can write the following equations:

$$\tilde{F}^*(\hat{L}) = 2\,\hat{L} - E_0$$
$$\tilde{F}^*(E_0) = \hat{L} - E_0$$

The matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L} \;\; E_0 >$ is:

$$\widetilde{F^*} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{3.5}$$

with characteristic polynomial $z^2 - z - 1$ Hence the dynamical degree is $\delta(F) = \frac{1+\ \ 5}{2}$ and $d_{n+2} = d_{n+1} + d_n$ for all $n \quad \mathbb{N}$

5. Assume that $\gamma_1 = 0$ then from lemma 8 we see that for f to be a birational map we have conditions $(\alpha\,\beta)_{12} = 0 = (\alpha\,\gamma)_{12}$ with $(\alpha_1 \;\; \alpha_2) = (0 \;\; 0) \;\; (\gamma_1 \;\; \gamma_2) = (0 \;\; 0)$ but $(\beta_1 \;\; \beta_2)$ can be simultaneously zero. Also as we know that $(\beta\,\gamma)_{12} = 0$ thus we find that $\gamma_2 = 0 = \alpha_1$ and $\beta_1 = 0$

   In this case $S_0 \twoheadrightarrow A_0 = O_2 \quad \mathcal{I}(F)$ Hence we blow-up A_0 to obtain E_0. So we have to blow-up A_0 to get E_0 The extension of F on S_0 and on E_0 acts as follows:

$$\tilde{F}\,[0 : x_1 : x_2] = [\gamma_2 : \beta_2]_{E_0} \text{ and } \tilde{F}[u : v]_{E_0} = [\gamma_0\, u + \gamma_2\, v : \beta_0\, u + \beta_2\, v]_{E_0}$$

As A_1 S_0 is not an indeterminacy point of F we find that $\tilde{F}(A_1) = [\gamma_2 : \beta_2]_{E_0}$ As we have not created new points of indeterminacy after the blow-up process then $\mathcal{I}(\tilde{F}) =$ O_1 and O_1 $S_0 = T_2$ We know that the only points on T_2 which have preimages are A_0 and A_1 which implies that if the iterates of A_1 reaches O_1 for some iterate of F then O_1 should be equal to either A_0 or A_1 But as $\alpha_1 = 0$ therefore $A_0 = O_1$ For $\alpha_2 = 0$ $A_1 = O_1$ and for $\alpha_2 = 0$ $A_1 = O_1$ We first consider the case when $\alpha_2 = 0$ In this case $\tilde{F}$ is AS. Then we find that

$$\tilde{F}^*(\hat{L}) = 2\,\hat{L} - E_0$$
$$\pi^*(S_0) = \hat{S}_0 + E_0 \text{ then } \tilde{F}^*(E_0) = \hat{L}$$

The matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L}\ \ E_0 >$ is:

$$\widetilde{F}^* = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.6}$$

with characteristic polynomial $(z-1)^2$ Hence the dynamical degree is $\delta(F) = 1$ We observe that the sequence of the degrees d_n exactly satisfies the recurrence

$$d_{n+2} = 2d_{n+1} - d_n$$

Then $d_1 = 2$ $d_2 = 3$ and $d_n = 1 + n$

Now consider the case when $\alpha_2 = 0$ then $A_1 = O_1$ We then blow up A_1 to get E_1 Then under the action of $\tilde{F}$ we find that S_1 E_1 in the following way:

$$\tilde{F}[x_0 : x_1 : -\frac{\gamma_0\,x_0 - t}{\gamma_2}] = [x_0 : \alpha_0\,x_0 + \alpha_1\,x_1]_{E_1}$$

and

$$\tilde{F}[u : v]_{E_1} = [\gamma_2\,u : \gamma_2\,(\alpha_0\,u + \alpha_1\,v) : \beta_2\,u] \quad T_1$$

Hence $\tilde{F}$ is AS. Then we find that

$$\tilde{F}^*(\hat{L}) = 2\,\hat{L} - E_0 - E_1$$
$$\tilde{F}^*(E_0) = \hat{L} - E_1$$
$$\tilde{F}^*(E_1) = \hat{L} - E_0$$

The matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L} \;\; E_0 \;\; E_1 >$ is:

$$\widetilde{F}^* = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \tag{3.7}$$

with characteristic polynomial $z\,(z-1)^2$ Hence the dynamical degree is $\delta(F) = 1$ We observe that the sequence of the degrees d_n exactly satisfies the recurrence

$$d_{n+3} = 2\,d_{n+2} - d_{n+1}$$

Then $d_1 = 2 \;\; d_2 = 2$ and $d_n = 2$

6. This family is very degenerate and needs multiple blowups in order to find the dynamical degree. We therefore use other methodology to provide results in this case and to show that these families of maps have dynamical degree one. If $\alpha_1 = \gamma_2 = 0$ Then from lemma 8 we see that for f to be a birational map we have conditions $(\alpha\,\beta)_{12} = 0 = (\alpha\,\gamma)_{12}$ with $(\alpha_1 \;\; \alpha_2) = (0 \;\; 0)$ $(\gamma_1 \;\; \gamma_2) = (0 \;\; 0)$ but $(\beta_1 \;\; \beta_2)$ can be simultaneously zero. As $(\beta\,\gamma)_{12} = 0$ we get that $\alpha_2 = 0 = \gamma_1$ and $\beta_2 = 0$

   For $\xi_1 \;\; \xi_2 \;\; \lambda_1 \;\; \lambda_2 \quad \mathbb{C}$ let $\gamma : (x \;\; y) \qquad (\xi_1 x + \lambda_1 \;\; \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$\begin{aligned}(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \qquad & \Big(\frac{\alpha_0 + \lambda_2\,\alpha_2 - \lambda_1}{\xi_1} \;\; 0 \;\; \frac{\alpha_2\xi_2}{\xi_1} \;\; \beta_0 - \lambda_2\,\gamma_0 + \beta_1\lambda_1 - \\ & \lambda_1\,\lambda_2\,\gamma_1 \;\; \beta_1\,\xi_1 - \gamma_1\lambda_2\,\xi_1 \;\; 0 \;\; (\gamma_0 + \gamma_1\lambda_1)\xi_2 \;\; \gamma_1\xi_1\xi_2 \;\; 0\Big)\end{aligned}$$

   By choosing

$$(\xi_1 \;\; \xi_2 \;\; \lambda_1 \;\; \lambda_2) = \left(\frac{1}{\gamma_1} \;\; \frac{1}{\gamma_1\alpha_2} \;\; \frac{\beta_1\alpha_2 + \alpha_0\gamma_1}{\gamma_1} \;\; \frac{\beta_1}{\gamma_1}\right)$$

   the new parameters are

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \qquad (0 \;\; 0 \;\; 1 \;\; \tilde{\beta}_0 \;\; 0 \;\; 0 \;\; \tilde{\gamma}_0 \;\; 1 \;\; 0)$$

   By renaming the parameters we see that the map after conjugation in this case is the following map f

$$f(x\ \ y) = \left(y\ \ \frac{\beta_0}{\gamma_0 + x}\right) \qquad \beta_0 = 0 \tag{3.8}$$

Now observe that $f^{2\,k}(x\ \ y) = (h^k(x)\ \ h^k(y))$ where $h(\eta) = \frac{\beta_0}{\gamma_0+\eta}$ and $f^{2\,k+1}(x\ \ y) = (h^k(y)\ \ h^{k+1}(x))$ This mapping f in general satisfies $d_n = 2$ for all $n \quad \mathbb{N}$ Notice that $h(\eta)$ is a Möbius transformation $\frac{a\,z+b}{c\,z+d}$ with $a = 0\ \ b = \beta_0\ \ c = 1\ \ d = \gamma_0$ It is well known that any iterate of a Möbius transformation is again a Möbius if and only if the coefficient of z in the denominator is non zero. We claim that f is $2\,k-$periodic if and only if $\left(\frac{\lambda_2}{\lambda_1}\right)^k = 1$ where $\lambda_1\ \ \lambda_2$ are the roots of $\lambda^2 - \gamma_0\,\lambda - \beta_0 = 0$ To prove our claim we will use the results found in [CGM06a]. It is easy to see that the numerator of h^k can be written as $N_{k+1} = b\,D_k$ where D_k is the denominator of h^k Then $D_{k+1} = c\,D_k + N_k$ This implies that $D_{k+2} - c\,D_{k+1} - b\,D_k = 0$ for $D_k = \alpha_k + \beta_k\,x$ for some $\alpha_k\ \ \beta_k \quad \mathbb{C}$ Then we can write that the equation $\beta_{k+2} - c\,\beta_{k+1} - b\,\beta_k = 0$ Then the associated characteristic polynomial is $\lambda^2 - \gamma_0\,\lambda - \beta_0 = 0$ this has the roots $\lambda_1 = \frac{\gamma_0 + \ \overline{\Delta}}{2}\ \ \lambda_2 = \frac{\gamma_0 - \ \overline{\Delta}}{2}$ for $\Delta = \gamma_0^2 + 4\,\beta_0$ We know that when $\Delta = 0$ then $\lambda_1 = \lambda_2$ and for $\Delta = 0$ we have $\lambda_1 = \lambda_2$ We first consider the case when $\Delta = 0$ Then $\beta_k = c_1\,\lambda_1^k + c_2\,\lambda_2^k$ Now if $\beta_k = 0$ then this implies that $\frac{c_1}{c_2} = -\left(\frac{\lambda_2}{\lambda_1}\right)^k$ As $\beta_1 = 1 = c_1\,\lambda_1 + c_2\,\lambda_2\ \ \beta_2 = \gamma_0 = c_1\,\lambda_1^2 + c_2\,\lambda_2^2$ we get that $\frac{c_1}{c_2} = -1$ This implies that $\left(\frac{\lambda_2}{\lambda_1}\right)^k = 1$ To prove the periodicity we now use the results established in Corollary 3 in [CGM06a] for periodic solutions. We see that for $\Delta = 0$ then $H(h(x)) = \xi\,H(x)$ where H is a non autonomous invariant with multiplier ξ Also $H(h^k(x)) = \xi^k\,H(x)$ We observe that $\xi^k = 1$ if and only if $\left(\frac{\lambda_2}{\lambda_1}\right)^k = 1$ This gives that $h^k(x) = x$ This shows that $f^{2\,k}(x\ \ y) = (h^k(x)\ \ h^k(y)) = (x\ \ y)$ for some $k > 1 \quad \mathbb{N}$ such that $\left(\frac{\lambda_2}{\lambda_1}\right)^k = 1$

We now consider the case when $\Delta = 0$ then $\lambda_1 = \lambda_2 = \lambda$ which implies that $\beta_0 = -\frac{\gamma_0}{4}$ and $h(\eta) = -\frac{\gamma_0^2}{4(\gamma_0+\eta)}$ Then $\beta_k = (c_1 + c_2\,k)\left(\frac{c}{2}\right)^k$ as $\lambda = \frac{\gamma_0}{2}$ Now we observe that $\beta_1 = 4$ and $\beta_2 = 16\,\gamma_0$ which gives that $\beta_k = (-48 + 56\,k)\left(\frac{c^{k-1}}{2^k}\right)$ As for $k \geq 1\ \ -48 + 56\,k > 0$ therefore $\beta_k = 0$ for all $k \geq 1\ \ k \quad \mathbb{N}$ This implies that the denominator keeps the coefficient of η which for all $2\,k+1$ iterates gives a Möbius. Therefore the degree remains the same for all $2\,k+1$ iterates that is $d_n = 2$ for all $n \quad \mathbb{N}$

3.2 Mappings with $(\alpha\gamma)_{12} = 0$

Consider the case $(\alpha\gamma)_{12} = \beta_1\,\gamma_2 - \beta_2\,\gamma_1 = 0$

Next proposition gives the dynamical degree of F in this case, by specifying the behavior of the sequence of the degrees of F in the cases that the dynamical degree equals one.

Theorem 12 *Assume that* $(\alpha\gamma)_{12} = \alpha_1\,\gamma_2 - \alpha_2\,\gamma_1 = 0$ *Then the following hold:*

(i) If $\alpha_1 = 0 \ \ \alpha_2 = 0 \ \ \gamma_1 = 0$ *and* $\gamma_2 = 0$ *then the dynamical degree is* $\delta(F) = 2$

(ii) If any of $\alpha_1 \ \ \alpha_2 \ \ \gamma_1 \ \ \gamma_2$ *is zero, then the dynamical degree is* $\delta(F) = 1$ *or* $\delta(F) = \frac{1+\ \ 5}{2}$ *More precisely:*

Case 1. *Assume that* $\alpha_1 = \gamma_1 = 0$ *Then* $\alpha_2 = 0 \ \ \gamma_2 = 0 \ \ \beta_1 = 0$ *and the dynamical degree of F is* $\delta(F) = \frac{1+\ \ 5}{2}$

Case 2 *Assume that* $\alpha_2 = \gamma_2 = 0$ *Then* $\alpha_1 = 0 \ \ \gamma_1 = 0 \ \ \beta_2 = 0$ *and the dynamical degree of F is* $\delta(F) = 1$ *Furthermore the sequence of degrees is* $d_n = 1 + n$

Proof. From hypothesis $(\alpha\gamma)_{12} = 0$ and this implies that $S_0 = S_2 = \ \ x_0 = 0$ and $O_0 = O_2 = [0 : \gamma_2 : -\gamma_1]$ Thus $\mathcal{E}(F) = \ \ S_0 \ \ S_1$ and $\mathcal{I}(F) = \ \ O_1 \ \ O_2$ with $S_1 = \ \ \gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0$ and $O_1 = [0 : \alpha_2 : -\alpha_1]$ We also know that S_0 collapses to $A_0 = [0 : 1 : 0]$ and S_1 collapses to $A_1 = [0 : 0 : 1]$

In general when $\alpha_1 \ \ \alpha_2 \ \ \gamma_1$ and γ_2 are non zero we observe that $F(A_0) = [0 : \alpha_1\gamma_1 : 0] = A_0$ $\mathcal{I}(F)$ and $F(A_1) = [0 : \gamma_2\alpha_2 : 0] = A_0 \quad \mathcal{I}(F)$ Thus F is AS, which implies that $d_n = 2^n$ and consequently $\delta(F) = 2$.

From now on we separate the proof in two cases: In case 1 we assume that $\alpha_1 = 0$ Then from Lemma 8 $\alpha_2 = 0$ Since $(\alpha\gamma)_{12} = 0$ we get that γ_1 also must be zero. Furthermore $\beta_1 = 0$ because $\beta_1 = 0$ implies that $f(x \ \ y)$ does not depend on x and it is not birational. In a similar way, in case 2 we assume that $\alpha_2 = 0$ what implies that $\alpha_1 = 0 \ \ \gamma_2 = 0 \ \ \gamma_1 = 0$ and $\beta_2 = 0$

- **Case 1**: $\alpha_1 = 0 \ \ \gamma_1 = 0$ with $\alpha_2 = 0 \ \ \gamma_2 = 0$ and $\beta_1 = 0$

  In this case $O_1 = [0 : 1 : 0] = A_0$ and we do the blow-up at this point, getting E_0 The calculations give:

$$\tilde{F}[0 : x_1 : x_2] = [\gamma_2 x_2 : \beta_1 x_1 + \beta_2 x_2]_{E_0} \text{ and } \tilde{F}[u : v :]_{E_0} = [0 : 0 : 1] \quad S_0$$

  Furthermore $\tilde{F}[0 : 0 : 1] = [\gamma_2 : \beta_2]_{E_0}$ which is sent to $[0 : 0 : 1]$ This shows that A_1 is fixed under $\tilde{F}^2$. Hence, the orbit of $A_1 = [0 : 0 : 1]$ never reaches indeterminacy point O_2 and $\tilde{F} : X \quad X$ is AS.

Hence, $\tilde{F}^*(E_0) = \hat{S}_0$ which can be written as $\hat{S}_0 = \hat{L} - E_0$ and $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$ The matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L} \;\; E_0 >$:

$$\widetilde{F}^* = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{3.9}$$

with characteristic polynomial $z^2 - z - 1$ Hence the dynamical degree is $\delta = \frac{1+\sqrt{5}}{2}$ Furthermore, since $d_1 = 2$ and $d_2 = 3$ we get that the sequence of the degrees of F is $2 \;\, 3 \;\, 5 \;\, 8 \;\, 13 \;\, 21$

- **Case 2**: $\alpha_2 = 0 \quad \gamma_2 = 0$ with $\alpha_1 = 0 \quad \gamma_1 = 0$ and $\beta_2 = 0$

  In this case $O_1 = [0 : 0 : 1] = A_1$ and we do the blow-up at this point, getting E_1 Now $S_1 = \;\; \gamma_0 x_0 + \gamma_1 x_1 = 0$ The extension of F at $\hat{S}_1$ and E_1 is the following:

  $$\tilde{F}\left[x_0 : \frac{-\gamma_0 x_0}{\gamma_1} : x_2\right] = [\gamma_1 : (\alpha\gamma)_{01}]_{E_1} \text{ and } \tilde{F}[u : v]_{E_1} = [u : \alpha_0 u + \alpha_1 v]_{E_1}$$

  The second equality tells us that E_1 is $\tilde{F}$-invariant and the assumptions on the parameters let us to know that no points on E_1 are indeterminacy points of $\tilde{F}$

  On the other hand, S_0 is mapped to $A_0 = [0 : 1 : 0]$ and $F(A_0) = [0 : \alpha_1\gamma_1 : 0] = A_0$ Hence, A_0 is a fixed point. All together implies that $\tilde{F} : X \qquad X$ is AS. Then,

  $$\tilde{F}^*(E_1) = \hat{S}_1 + E_1 = \hat{L} \text{ and } \tilde{F}^*(\hat{L}) = 2\hat{L} - E_1$$

  The matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L} \;\; E_1 >$:

  $$\widetilde{F}^* = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.10}$$

  with characteristic polynomial $(z-1)^2$ Hence the dynamical degree is $\delta = 1$ Furthermore, since $d_1 = 2$ and $d_2 = 3$ we get that $d_n = 1 + n$

The next section discusses the dynamics of the families with zero entropy classified in this chapter in previously discussed theorems. The results stated provides the examples for the results in [DF01] and Theorem 7.

3.3 Zero entropy

In this section the following theorem identifies and state the particular families in the degenerate case when the dynamical degree $\delta = 1$ It is as follows:

Theorem 13 *Let $f(x\ \ y)$ be the mappings which satisfy the hypothesis of Theorems 11 and 12.*

- *If it has entropy zero and the sequence of degrees d_n grows linearly in n then after an affine change of coordinates it can be written in one of the following ways:*

  (1) $f(x\ \ y) = \left(\alpha_0 + \alpha_1\, x\ \ \frac{\beta_0+\beta_1\, x+y}{\gamma_0+\gamma_1\, x+\gamma_2\, y}\right) \qquad \alpha_1 = 0 = \gamma_2$

  (2) $f(x\ \ y) = \left(\alpha_0 + \alpha_1\, x + y\ \ \frac{\beta_0}{\gamma_0+y}\right) \qquad \alpha_1 = 0 = \beta_0$

  (3) $f(x\ \ y) = \left(\alpha_0 + \alpha_1\, x\ \ \frac{\beta_0+y}{x}\right) \qquad \alpha_1 = 0$ *Moreover the maps* $1\ \ 3$ *preserve the fibration* $V_1(x\ \ y) = x$ *such that* $V_1(f(x\ \ y)) = \alpha_0 + \alpha_1 V_1(x\ \ y)$ i $\quad (1)\ (3)\ (4)$ *For* $\alpha_0 = 0$ *and* $\alpha_1 = 1$ *these mappings* f *are integrable. Also the mapping* (2) *preserve the fibration* $V_2(x\ \ y) = y$ *such that* $V_2(f(x\ \ y)) = \frac{\beta_0}{\gamma_0+V_2(x,y)}$

- *If it has entropy zero and the sequence of degrees d_n is bounded, then after an affine change of coordinates it can be written as follows:*

  (4)
  $$f(x\ \ y) = \left(\alpha_0 + \alpha_1\, x\ \ \frac{\beta_0}{\gamma_0 + y}\right) \qquad \alpha_1 = 0 = \beta_0$$

  (5)
  $$f(x\ \ y) = \left(y\ \ \frac{\beta_0}{\gamma_0 + x}\right) \qquad \beta_0 = 0 = \gamma_0$$

  and the mapping (4) *preserve both above given fibrations* V_1 *and* V_2 *in a similar manner.*

  The mapping (5) *preserves the following fibrations*

  $$H_1(x\ \ y) = \frac{(x-m_1)\,(y-m_1)}{(x-m_2)\,(y-m_2)} \qquad H_2(x\ \ y) = \frac{x\,y + a\,x + (\gamma_0 - a)\,y - \beta_0}{x\,y - a\,x + (\gamma_0 + a)\,y - \beta_0}$$

  if $\gamma_0^2 + 4\,\beta_0 = 0$ *where* $m_1\ \ m_2$ *are the roots of* $m^2 + \gamma_0\, m - \beta_0 = 0$ *and* $a^2 = -\beta_0$ *If* $\gamma_0^2 + 4\,\beta_0 = 0$ *then it preserves the fibrations*

  $$W_1(x\ \ y) = \frac{2\,(x+y+\gamma_0)}{(2\,x+\gamma_0)\,(2\,y+\gamma_0)} \qquad W_2(x\ \ y) = \frac{2\,(y-x)}{(2\,x+\gamma_0)\,(2\,y+\gamma_0)}$$

Furthermore $H_1(f(x\ \ y)) = \frac{m_1}{m_2} H_1(x\ \ y)\ \ \ H_2(f(x\ \ y)) = -H_2(x\ \ y)$ *and* $W_1(f(x\ \ y)) = W_1(x\ \ y) + \frac{1}{\gamma_0}\ \ \ W_2(f(x\ \ y)) = -W_2(x\ \ y) + \frac{1}{\gamma_0}$

Proof. From Theorem 11 we know that the families with zero entropy are the following: For $(\beta\gamma)_{12} = 0$

1. Assume that $\alpha_1 \cdot \gamma_1 = 0$ and $\alpha_2 = 0$ Then from lemma 8 we see that for f to be birational in this case we have conditions $(\alpha\,\beta)_{12} = 0 = (\alpha\,\gamma)_{12}$ Also as we know that $(\beta\,\gamma)_{12} = 0$ thus we find that to satisfy these conditions we must have $\alpha_1\ \ \beta_2$ and γ_2 all non zero. Then for $\xi_2\ \ \lambda_2\ \ \ \ \mathbb{C}$ let $\gamma : (x\ \ y)\ \ \ \ \ (x\ \ \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ for $\boldsymbol{\alpha} = (\alpha_0\ \ \alpha_1\ \ \alpha_2)\ \ \boldsymbol{\beta} = (\beta_0\ \ \beta_1\ \ \beta_2)\ \ \boldsymbol{\gamma} = (\gamma_0\ \ \gamma_1\ \ \gamma_2)$ we have

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \qquad \Big(\alpha_0\ \ \alpha_1\ \ 0\ \ \beta_0 - \lambda_2\,\gamma_0 + \beta_2\lambda_2 - \gamma_2\,\lambda_2^2$$
$$\beta_1 - \gamma_1\lambda_2\ \ \beta_2\xi_2 - \lambda_2\gamma_2\xi_2\ \ (\gamma_0 + \gamma_2\lambda_2)\xi_2\ \ \gamma_1\xi_2\ \ \gamma_2\xi_2^2\Big)$$

We find that by choosing

$$(\xi_2\ \ \lambda_2) = \left(\xi_2\ \ -\frac{\beta_2\,\xi_2 - 1}{\gamma_2\,\xi_2}\right)$$

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \qquad (\tilde{\alpha_0}\ \ \alpha_1\ \ 0\ \ \tilde{\beta_0}\ \ \tilde{\beta_1}\ \ 1\ \ \tilde{\gamma_0}\ \ \tilde{\gamma_1}\ \ \tilde{\gamma_2})$$

By renaming the parameters we see that the map after conjugation in this case is the following map f

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1\,x\ \ \frac{\beta_0 + \beta_1\,x + y}{\gamma_0 + \gamma_1\,x + \gamma_2\,y}\right) \qquad \alpha_1 = 0 = \gamma_2 \tag{3.11}$$

which gives us the required map $f_1(x\ \ y)$ Observe that the mapping (1) has the first component $\alpha_0 + \alpha_1 x$ This gives us the scaled translation in x This implies that this mapping preserves a fibration $V_1(x\ \ y) = x$ with the property that $V_1(f(x\ \ y)) = \alpha_0 + \alpha_1 V_1(x\ \ y)$ For $\alpha_0 = 0$ and $\alpha_1 = 1$ we have $V_1(f(x\ \ y)) = V_1(x\ \ y)$ which gives the following integrable subfamily of f :

$$f(x\ \ y) = \left(x\ \ \frac{\beta_0 + \beta_1\,x + y}{\gamma_0 + \gamma_1\,x + \gamma_2\,y}\right) \qquad \gamma_2 = 0 \tag{3.12}$$

2. If $\beta_1 = \gamma_1 = 0$ Then $\alpha_1 = 0$ and $\gamma_2 = 0$ Moreover when $\alpha_2 = 0$ then the dynamical degree of F is $\delta(F) = 1$ and the sequence of degrees is $d_n = 1 + n$ for all $n \quad \mathbb{N}$ For $\xi_1 \ \ \xi_2 \ \ \lambda_2 \quad \mathbb{C}$ let $\gamma : (x \ \ y) \qquad (\xi_1 x \ \ \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \qquad \Bigg(\frac{\alpha_0 + \lambda_2\,\alpha_2}{\xi_1} \ \ \alpha_1 \ \ \frac{\alpha_2\xi_2}{\xi_1} \ \ (\beta_0 - \lambda_2^2\,\gamma_2 + \beta_2\lambda_2 - \lambda_2\,\gamma_0)$$
$$0 \ \ -\lambda_2\gamma_2\xi_2 + \beta_2\,\xi_2 \ \ (\gamma_0 + \gamma_2\lambda_2)\xi_2 \ \ 0 \ \ \gamma_2\xi_2^2\Bigg)$$

By choosing

$$(\xi_1 \ \ \xi_2 \ \ \lambda_2) = \left(\frac{\alpha_2}{\gamma_2} \ \ \frac{1}{\gamma_2} \ \ \frac{\beta_2}{\gamma_2}\right)$$

then the new parameters are

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \qquad (\tilde{\alpha_0} \ \ \alpha_1 \ \ 1 \ \ \tilde{\beta_0} \ \ 0 \ \ 0 \ \ \tilde{\gamma_0} \ \ 0 \ \ 1)$$

By renaming the parameters we see that the map after conjugation in this case is the following map f

$$f(x \ \ y) = \left(\alpha_0 + \alpha_1\,x + y \ \ \frac{\beta_0}{\gamma_0 + y}\right) \qquad \alpha_1 = 0 = \beta_0 \tag{3.13}$$

which gives us the required map (2) Observe that the mappings (2) have the second component $\frac{\beta_0}{\gamma_0+y}$ This implies that these mappings preserve a fibration $V_2(x \ \ y) = y$ with the property that $V_2(f(x \ \ y)) = \frac{\beta_0}{\gamma_0+V_2(x,y)}$ Therefore the curve y at level C passes to the the curve at level $\frac{\beta_0}{\gamma_0+C}$

We now discuss the family with zero entropy that satisfies the hypothesis of second Theorem 12 of degenerate case. The sequence of degrees for this family grows linearly in n as discussed follows:

3. From Theorem 12 we know that we have only one family with zero entropy that is the following: For $(\alpha\gamma)_{12} = 0$ if $\alpha_2 = \gamma_2 = 0$ then $\gamma_1 = 0 = \alpha_1$ and also $\beta_2 = 0$ then the dynamical degree $\delta(F) = 1$ also the sequence of degrees $d_n = 1 + n$ Then for $\xi_1 \ \ \xi_2 \ \ \lambda_1 \ \ \lambda_2 \quad \mathbb{C}$ let $\gamma : (x \ \ y) \qquad (\xi_1 x + \lambda_1 \ \ \xi_2 y + \lambda_1)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space

under the action of γ we have

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad \Big(\frac{\alpha_0+\lambda_1\,\alpha_1-\lambda_1}{\xi_1}\ \ \alpha_1\ \ 0\ \ \beta_0-\lambda_2\,\gamma_0+\beta_1\lambda_1-\lambda_1\,\lambda_2\,\gamma_1+$$
$$\beta_2\,\lambda_2\ \ \beta_1\xi_1-\lambda_2\gamma_1\xi_1\ \ \beta_2\xi_2\ \ (\gamma_0+\gamma_1\lambda_1)\xi_2\ \ \gamma_1\xi_1\xi_2\ \ 0\Big)$$

By choosing

$$(\xi_1\ \ \xi_2\ \ \lambda_1\ \ \lambda_2)=\left(\frac{\beta_2}{\gamma_1}\ \ \frac{1}{\beta_2}\ \ -\frac{\gamma_0}{\gamma_1}\ \ \frac{\beta_1}{\gamma_1}\right)$$

the new parameters are

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad (\tilde{\alpha_0}\ \ \alpha_1\ \ 0\ \ \tilde{\beta_0}\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0)$$

By renaming the parameters we see that the map after conjugation in this case is the following map f

$$f(x\ \ y)=\left(\alpha_0+\alpha_1\,x\ \ \frac{\beta_0+y}{x}\right) \qquad \alpha_1=0 \tag{3.14}$$

which gives us the required map (3) This preserves V_1 as discussed above for f in (1)

In the following we discuss the dynamics of the families which have bounded growth rate and they satisfy the hypothesis of first Theorem 11 of degenerate case.

4. If $\beta_1=\gamma_1=0$ Then $\alpha_1=0$ and $\gamma_2=0$ Moreover when $\alpha_2=0$ then the dynamical degree of F is $\delta(F)=1$ and the sequence of degrees is $d_n=2$ for all $n\quad\mathbb{N}$ For $\xi_1\ \ \xi_2\quad\mathbb{C}$ let $\gamma:(x\ \ y)\quad(x\ \ \xi_1 y+\xi_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1}\circ f\circ\gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad \Big(\alpha_0\ \ \alpha_1\ \ 0\ \ \beta_0-\xi_2^2\,\gamma_2+\beta_2\xi_2-\xi_2\,\gamma_0\ \ 0\ \ -\xi_2\gamma_2\xi_1+$$
$$\beta_2\,\xi_1\ \ (\gamma_0+\gamma_2\xi_2)\xi_1\ \ 0\ \ \gamma_2\xi_1^2\Big)$$

By choosing

$$(\xi_1\ \ \xi_2)=\left(\frac{1}{\gamma_2}\ \ \frac{\beta_2}{\gamma_2}\right)$$

then the new parameters are

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad (\alpha_0 \ \ \alpha_1 \ \ 0 \ \ \tilde{\beta}_0 \ \ 0 \ \ 0 \ \ \tilde{\gamma}_0 \ \ 0 \ \ 1)$$

By renaming the parameters we see that the map after conjugation in this case is the following map f

$$f(x \ \ y) = \left(\alpha_0 + \alpha_1\, x \ \ \frac{\beta_0}{\gamma_0 + y}\right) \qquad \alpha_1 = 0 = \beta_0 \tag{3.15}$$

which gives us the required map (4) Observe that the mappings (4) have their first component $\alpha_0 + \alpha_1 x$ and the second component $\frac{\beta_0}{\gamma_0+y}$ This implies that these mappings preserve two fibrations generically transversal. The first one $V_1(x \ \ y) = x$ with $V_1(f(x \ \ y)) = \alpha_0 + \alpha_1 V_1(x \ \ y)$ and the other $V_2(x \ \ y) = y$ with $V_2(f(x \ \ y)) = \frac{\beta_0}{\gamma_0+V_2(x,y)}$ For $\alpha_0 = 0$ and $\alpha_1 = 1$ we have $V_1(f(x \ \ y)) = V_1(x \ \ y)$ gives the integrable subfamily of map (4) which is the following

$$f(x \ \ y) = \left(x \ \ \frac{\beta_0}{\gamma_0 + y}\right) \qquad \beta_0 = 0 \tag{3.16}$$

5. From Theorem 7 we know that this mapping should preserve an elliptic fibration. This implies that f sends a curve of genus one to another curve of same genus. We search for a rational fibration such that $V = \frac{P}{Q} = k$ for some polynomials $P(x \ \ y)$ $Q(x \ \ y)$ with no common factors, goes to $\gamma(k) = \frac{w_1\, k+w_2}{w_3\, k+w_4}$ for some $w_1 \ \ w_2 \ \ w_3 \ \ w_4 \ \ k \quad \mathbb{C}$ Therefore $V(f) = \frac{w_1\, V+w_2}{w_3\, V+w_4}$ for some rational function $V = \frac{P}{Q}$ Then in particular we will have the following cases:

   (a) $V(f) = V$ the integrable case,

   (b) $V(f) = w_1\, V$ the scaled fibration case,

   (c) $V(f) = w_1\, V + w_2$ the scaled translated fibration case.

   Note that in case a the functions P and Q are invariant under f as they satisfy the equation $P \cdot Q(f) = Q \cdot P(f)$ Similarly for case b it follows. In case c only Q is invariant as it satisfies the equation $P(f) \cdot Q = Q(f) \cdot (w_1\, P + w_2\, Q)$

   To find the invariant curves we start by considering the curve C to be the curve of minimum degree 3 Then f must send this curve to some other curve of same degree. As we know that degree of $f(C)$ is maximum equal to $deg(f) \cdot deg(C)$ This implies that $f(C)$ will be equal to 6 for a curve C of degree 3 The degree of $f(C)$ will decrease if C contains points of indeterminacy of F which is the extension of f in projective space. This happens because of the fact that curve C which is an algebraic curve is sent to another curve that is given by $C(F^{-1})$ see Proposition 9 in Chapter (2). This curve is a product of the exceptional curves of F^{-1} that collapse to the indeterminacy points of F contained in C and another curve $C^{'}$ of degree less than the $deg(C)\,deg(F)$ Observe that same should be true for F^{-1} If F sends a curve C of degree 3 to another curve $C^{'}$ of degree 3 then F^{-1} must send $C^{'}$ C which implies that $C^{'}$ must contain the three points of indeterminacy of F^{-1} so that the degree of the curve $F(C^{'})$ decreases by 3 Note that this curve can be reducible. In this case the curve C will be a product of either a line and a conic that is $C = L \cdot conic$ or a product of three lines $L_1 \cdot L_2 \cdot L_3 = C$

Then by considering the curve C in affine plane $\mathbb{C}^2$ (in this case it simplifies the calculations however one can do the calculation in $P\mathbb{C}^2$ as well) as f preserves a fibration of curves of genus 1 Hence we start by looking for one invariant cubic curve C under f This implies that $C = 0$ implies that $C(f) = 0$ For the choice of C we use the above discussed method, and consider some general cubic curve C that contains the points of indeterminacy of F We find as stated below that $C = L_1 \cdot L_2 \cdot L_3$ Also L_1 L_2 L_3 L_1 which implies that the curve C goes to some curve $C^{'}$ which is the same curve C for this mapping f. Therefore the curve C also contains all the points of indeterminacy of F^{-1} as well.

Therefore the invariant fibrations for f can be obtained by searching for invariant curves. Since $f(x\ \ y) = \left(y\ \ \frac{\beta_0}{\gamma_0+x}\right)$, it can be seen that $x = m$ $y = \beta_0\gamma_0 + m$ By choosing m such that $y = m$ we get that m is the root of the $m(\gamma_0 + m) - \beta_0 = 0$ Then we find that the product $(x - m)\,(y - m)$ is an invariant conic for f Assume that $\gamma_0^2 + 4\,\beta_0 = 0$ Then there are two different roots m_1 m_2 of $m^2 + \gamma_0\,m - \beta_0 = 0$ This gives two invariant conics $(x - m_1)\,(y - m_1)$ and $(x - m_1)\,(y - m_1)$ Then by taking their fraction we find that for $H_1(x\ \ y) = \frac{(x-m_1)\,(y-m_1)}{(x-m_2)\,(y-m_2)}$ we get $H_1(f(x\ \ y)) = \frac{m_1}{m_2}\,H_1(x\ \ y)$

To find the other invariant curve consider a general conic that is a degree two polynomial $Q(x\ \ y)$ such that $Q(x\ \ y) = 0$ passes through the two indeterminacy points O_0 O_1 of F Then it follows that $Q(x\ \ y) = q_0\,x\,y + q_1\,x + q_2\,y + q_3$ Now the image of $Q(x\ \ y) = 0$ is also a degree two curve (see Proposition 9). The calculations give that

$$Q(x\ \ y) = 0 \qquad (q_2 - \beta_0\,q_0)\,x\,y + \beta_0\,q_0\,x + (q_3 - \gamma_0\,q_1)\,y + \beta_0\,q_1 = 0$$

Imposing that the two curves agree we find that $Q_1(x\ \ y) = x\,y + a\,x + (\gamma_0 - a)\,y - \beta_0 = 0$

and $Q_2(x\ \ y) = x\,y - a\,x + (\gamma_0 + a)\,y - \beta_0 = 0$ are two invariant curves for f Calling $H_2(x\ \ y) = \frac{Q_1(x,y)}{Q_2(x,y)}$ after some computations we find that $H_2(f(x\ \ y)) = -H_2(x\ \ y)$ In the case $\gamma_0^2 + 4\,\beta_0 = 0$ $H_1(x\ \ y)$ is identically 1 But $(x - m)\,(y - m)$ is still an invariant curve for f where $m = -\frac{\gamma_0}{2}$ is the unique root of $m^2 + \gamma_0\,m - \beta_0 = 0$ Now we search a degree one curve that is a line $l_0 + l_1\,x + l_2\,y = 0$ such that $W(x\ \ y) = \frac{l_0 + l_1\,x + l_2\,y}{(2\,x+\gamma_0)\,(2\,y+\gamma_0)}$ satisfies $W(f(x\ \ y)) = w_1\,W(x\ \ y) + w_2$ and we find the following:

$$W_1(x\ \ y) = \frac{2\,(x + y + \gamma_0)}{(2\,x + \gamma_0)\,(2\,y + \gamma_0)} \qquad W_2(x\ \ y) = \frac{2\,(y - x)}{(2\,x + \gamma_0)\,(2\,y + \gamma_0)}$$

which satisfy $W_1(f(x\ \ y)) = W_1(x\ \ y) + \frac{1}{\gamma_0}$ $W_2(f(x\ \ y)) = -W_2(x\ \ y) + \frac{1}{\gamma_0}$

Mathematics is the language with which God has written the universe - Galileo
The laws of nature are but the mathematical thoughts of God - Euclid
Nature speaks the language of beauty and beauty has letters of Mathematics - Sundus

Chapter 4

Non Degenerate Case $\gamma_2 = 0$

In this chapter we study the family of mappings 2.6 when they are non degenerate for $\gamma_2 = 0$ This implies that all the three exceptional curves of F are distinct. Two different cases depending on α_1 is zero are discussed. We classify them into subfamilies with dynamical degree one and greater than one. In the end we study the mappings with dynamical degree one. Note that the case $\alpha_1 = 0$ gives us the mappings studied in [BK06]. For the sake of completeness of results we study this case in this chapter in detail again.

To find the dynamical degree in both cases we use the result for the linear fractional maps established in [BK06]. This result is given in Chapter 2 in Theorem 10. We first look for the singular and elementary orbits of the indeterminacy points of F^{-1}. Then to calculate the dynamical degree depending on the coefficients we organize these orbits into lists in order to apply the Theorem 10. This gives the associated characteristic polynomial of F which gives us the sequence of degrees of F and dynamical degree of F The methodologies are explained in Chapter 2 in detail.

In the third and last section of this chapter we detect the families with zero entropy that is with dynamical degree one. We separate them in three different sets of mappings, i.e. the maps whose sequence of degrees is periodic, grows quadratically or grows linearly. For the maps in the first set we illustrate exactly what is the period of the mappings and hence we show that the mappings indeed are periodic. For quadratic growth mappings we have illustrated the techniques we use to find the elliptic fibrations which assures Theorem 4 3 and Proposition 4 4 in [DF01], also revised in this chapter. For the mappings with linear growth rate we demonstrate that they preserve rational invariant fibrations that agrees with the results of Lemma 4 2 in [DF01].

We now start by introducing our family of mappings. Consider the birational mapping $f : \mathbb{C}^2$

$\mathbb{C}^2$ of the form

$$f(x \;\; y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y \;\; \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right) \tag{4.1}$$

for complex numbers $\alpha_i \;\; \beta_i \;\; \gamma_i \;\; i = 0 \;\; 1 \;\; 2$ We discuss the general result for this family in the following theorem.

Theorem 14 *Suppose that the triangle is non degenerate and $\alpha_1 \;\; \alpha_2 \;\; \gamma_1 \;\; \gamma_2$ are all non zero. Then either,*

(i) If it exists $p \quad \mathbb{N}$ such that $F^p(A_2) = O_0$ then the dynamical degree of F is given by the largest root of the polynomial

$$x^{p+2} - 2x^{p+1} + x - 1$$

(ii) If no such p exists then dynamical degree of F is 2.

Notice that Proposition 14 says us that the family (**??**) generically has dynamical degree equal 2.

Proof. The conditions on the parameters imply that $F(A_0) = A_0$ with $A_0 \quad \mathcal{I}(F)$ and $F(A_1) = A_0$ Since $A_0 \;\; A_1 \quad S_0 \quad \mathcal{E}(F)$, thus we find that their orbits are $\mathcal{O}_0 = \;\; A_0$ and $\mathcal{O}_1 = \;\; A_1$ which are singular but not elementary. Now it remains to analyze the behavior of iterates of A_2. We claim that $\nexists p \quad \mathbb{N} : F^p(A_2) = O_1$ and $\nexists p \quad \mathbb{N} : F^p(A_2) = O_2$ It is so because if $F^p(A_2) = O_1$ then, since $O_1 \quad S_0 = T_2$ it would imply $O_1 = A_0$ or $O_1 = A_1$ that is $\alpha_1 = 0$ or $\alpha_2 = 0$ Similarly, $F^p(A_2) = O_2$ implies $\gamma_1 = 0$ or $\gamma_2 = 0$ Therefore the only possibility is that any iterate of A_2 reaches O_0 Thus we assume the following cases:

1. Assume that $F^p(A_2) = O_0$ for all $p \quad \mathbf{N}$ Then the map F is itself AS. Hence, $\delta(f) = 2$

2. Now assume that $F^p(A_2) = O_0$ for some $p \quad \mathbb{N}$ Thus we have a SE orbit of A_2 which is as follows:

$$\mathcal{O}_2 = \;\; A_2 \;\; F(A_2) \qquad F^p(A_2) = O_0$$

In this case we have only one list $\mathcal{L}_o$ which is open. It is as follows:

$$\mathcal{L}_o = \;\; \mathcal{O}_2 = \;\; A_2 \;\; F(A_2) \qquad F^p(A_2) = O_0$$

To find the characteristic polynomial we use Theorem **??**. We find that $\mathcal{N}_{\mathcal{L}_o} = p$, $\mathcal{T}_{\mathcal{L}_o} = x^p$ and $\mathcal{S}_{\mathcal{L}_o} = 1$ Then the dynamical degree of f is the largest root of the polynomial $x^{p+2} - 2x^{p+1} + x - 1$

We now study the case when $\gamma_2 = 0$ Consider the birational mapping $f : \mathbb{C}^2 \quad \mathbb{C}^2$ of the form

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y\ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x}\right) \tag{4.2}$$

for complex numbers $\alpha_i\ \ \beta_i\ \ i = 0\ \ 1\ \ 2$ and $\gamma_i\ \ i = 0\ \ 1$ From lemma 8 we know that f is *non degenerate* i.e. all S_i are distinct if and only if $(\beta\gamma)_{12} = 0 = (\alpha\gamma)_{12}$ For $\gamma_2 = 0$ this implies that $\beta_2\gamma_1 = 0 = \alpha_2\gamma_1$ Therefore $\alpha_2\ \ \beta_2\ \ \gamma_1$ are all non zero. However the parameters $\alpha_0\ \ \alpha_1\ \ \beta_0\ \ \beta_1\ \ \gamma_0$ can be zero.

We consider the imbedding $(x\ \ y) \quad [1 : x : y] \quad P\mathbb{C}^2$ into projective space and consider the induced map $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ given by

$$F[x_0 : x_1 : x_2] = [x_0(\gamma_0 x_0 + \gamma_1 x_1) : (\alpha \cdot x)(\gamma_0 x_0 + \gamma_1 x_1) : x_0(\beta \cdot x)]$$

where $\boldsymbol{\alpha} \cdot x = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$ The indeterminacy locus of F is $\mathcal{I}(F) = \ \ O_0\ \ O_1\ \ O_2$ where

$$O_0 = [-\beta_2\gamma_1 : \beta_2\gamma_0 : (\beta\gamma)_{01}] \qquad O_1 = [0 : \alpha_2 : -\alpha_1] \qquad O_2 = [0 : 0 : 1]$$

and the indeterminacy locus of F^{-1} is $\mathcal{I}(F^{-1}) = \ \ A_0\ \ A_1\ \ A_2$ where

$$\begin{aligned} A_0 &= [0 : 1 : 0] \qquad A_1 = [0 : 0 : 1] \\ A_2 &= [-\alpha_2\beta_2\gamma_1 : (-\alpha_0\beta_2\gamma_1 + \alpha_2(\beta\gamma)_{01} + \gamma_0\alpha_1\beta_2)\alpha_1 : \beta_2(\alpha\beta)_{12}] \end{aligned}$$

where $(\alpha\beta)_{12} = \alpha_1\beta_2 - \alpha_2\beta_1$.

We see that $A_1 = O_2$ Also $\alpha_2 = 0$ this implies that for $\alpha_1 = 0$ we get $A_0 = O_1$ For $\alpha_1 = 0$ we have $A_0 = O_1$ We therefore separately study these two cases.

4.1 Mappings with $\alpha_1 = 0$

For $\xi_1\ \ \xi_2\ \ \lambda_1\ \ \lambda_2 \quad \mathbb{C}$ let $\gamma : (x\ \ y) \quad (\xi_1 x + \lambda_1\ \ \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad \Big(\frac{\alpha_0 - \lambda_1 + \alpha_2\lambda_2}{\xi_1} \;\; 0 \;\; \frac{\alpha_2\xi_2}{\xi_1} \;\; (\beta_2\lambda_2 - \gamma_0\lambda_2 - \gamma_1\lambda_1\lambda_2 + \beta_0 + \beta_1\lambda_1)$$
$$(\beta_1 - \gamma_1\lambda_2)\xi_1 \;\; \beta_2\xi_2 \;\; (\gamma_1\lambda_1 + \gamma_0)\xi_2 \;\; \gamma_1\xi_1\xi_2 \;\; 0\Big)$$

By choosing

$$(\xi_1 \;\; \xi_2 \;\; \lambda_1 \;\; \lambda_2) = \left(\frac{\alpha_2}{\beta_2} \;\; \frac{1}{\beta_2} \;\; \frac{(\alpha_0\gamma_1 + \alpha_2\beta_1)}{\gamma_1} \;\; \frac{\beta_1}{\gamma_1}\right)$$

the new parameters are

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad (0 \;\; 0 \;\; 1 \;\; \tilde{\beta}_0 \;\; 0 \;\; 1 \;\; \tilde{\gamma}_0 \;\; \tilde{\gamma}_1 \;\; 0)$$

For $\kappa_1 \;\; \kappa_2 \quad \mathbb{C}$ let $\quad : (x \;\; y) \quad (\kappa_1 x \;\; \kappa_2 y)$ be the scaling map. By the group action of for $\kappa_1 = \frac{1}{\gamma_1} = \kappa_2$ we get the following parameters

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad (0 \;\; 0 \;\; 1 \;\; \tilde{\tilde{\beta}}_0 \;\; 0 \;\; 1 \;\; \tilde{\tilde{\gamma}}_0 \;\; 1 \;\; 0)$$

The dynamical property that an exceptional curve is mapped to a point of indeterminacy is preserved under the above given two conjugations. By renaming the parameters we see that the map $f(x \;\; y)$ in (4.2) is conjugated to the following map f

$$f(x \;\; y) = \left(y \;\; \frac{\beta_0 + y}{\gamma_0 + x}\right) \tag{4.3}$$

We consider the induced map in projective space as follows: $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ given by

$$F[x_0 : x_1 : x_2] = [x_0(\gamma_0 x_0 + x_1) : x_2\,(\gamma_0 x_0 + x_1) : x_0(x_2 + \beta x_0)] \tag{4.4}$$

The set of indeterminacy of F and F^{-1} are $\mathcal{I}(F) = \quad O_0 \;\; O_1 \;\; O_2$ where

$$O_0 = [1 : -\gamma_0 : -\beta_0] \qquad O_1 = [0 : 1 : 0] \qquad O_2 = [0 : 0 : 1]$$

and $\mathcal{I}(F^{-1}) = \quad A_0 \;\; A_1 \;\; A_2$ where

$$A_0 = [0 : 1 : 0] \qquad A_1 = [0 : 0 : 1] \qquad A_2 = [1 : -\beta_0 : 0]$$

Furthermore the exceptional curves of F and F^{-1} are as follows

$$\begin{array}{lllll} S_0 & = & x_0 = 0 & S_1 = \gamma_0 x_0 + x_1 = 0 & S_2 = \beta_0 x_0 + x_2 = 0 \\ T_0 & = & \beta_0 x_0 + x_1 = 0 & T_1 = x_2 = 0 & T_2 = x_0 = 0 \end{array}$$

- **Orbits of A_0 and A_1**

Observe that $S_0 \twoheadrightarrow A_0 = O_1$ and $S_1 \twoheadrightarrow A_1 = O_2$ Therefore we need to blow up the points $A_1 = O_2$ and $A_0 = O_1$ Let X be the new space we get after blowing up the points A_0 A_1 and let E_0 E_1 be the exceptional fibre at these points. We recognize the induced map after the blow up process as $\tilde{F} : X \quad X$ For the points $x = [0 : x_1 : x_2] \quad S_0$ we find that the map $\tilde{F}$ sends the curve S_0 to E_0 in the following way:

$$\tilde{F}[0 : x_1 : x_2] \quad S_0 \qquad [x_1 : x_2]_{E_0}$$

Moreover it sends all the points $x = [tu : 1 : tv] \quad E_0$ as $t \quad 0$ to T_1 in a way such that

$$\tilde{F}[u : v]_{E_0} \qquad [u : v : 0] \quad T_1$$

Also $\tilde{F}$ sends the curve S_1 to E_1 and E_1 to $T_2 = S_0$ as follows:

$$\tilde{F}[x_0 : -\gamma_0 x_0 : x_2] \quad S_1 \qquad [x_0 : x_2]_{E_1} \qquad \tilde{F}[u : v]_{E_1} \qquad [0 : \gamma_0 u + v : u] \quad S_0 = T_2$$

Since we have not created new indeterminacy points hence the indeterminacy set is $\mathcal{I}(\tilde{F}) = \quad O_0$ and the exceptional set is $\mathcal{E}(\tilde{F}) = \quad S_2$

- **Orbit of A_2**

The exceptional curve $S_2 \twoheadrightarrow A_2$ After the observations we see that the only way that $\mathcal{O}_2$ is an SE orbit is that $\tilde{F}^p(A_2) = O_0$ for some $p \quad \mathbb{N}$

If no such p exists, then $\tilde{F} : X \quad X$ is an AS mapping. Notice that in this case, when we apply $\tilde{F}$ S_2 still collapses to A_2

Assume that there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ Then we have to blow-up all the points A_2 $\tilde{F}(A_2)$ $\tilde{F}^2(A_2) \quad \tilde{F}^p(A_2) = O_0$ Let E_i $i = 2$ $3 \quad p + 2$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1 \quad X_1$ be the induced map. Then under the action of $\tilde{F}_1$ we have that:

$$E_2 \quad E_3 \quad \cdots \quad E_{p+2}$$

As $T_0 \twoheadrightarrow O_0$ therefore this sequence can be completed in the following manner:

$$S_2 \quad E_2 \quad E_3 \quad \cdots \quad E_{p+2} \quad T_0$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_2 and on E_{p+2} is:

$$\tilde{F}_1[x_0 : x_1 : -\beta_0 x_0] = [\gamma_0 x_0 + \ x_1 : x_0]_{E_2}$$

and

$$\tilde{F}_1[u : v]_{E_{p+2}} = [u : -\beta_0 u : v] \quad T_0$$

Observe that now $\tilde{F}_1 : X_1 \qquad X_1$ is an AS map. Furthermore:

$$\begin{array}{lllllll} S_0 & E_0 & T_2 & & & & \\ S_1 & E_1 & S_0 = T_1 & & & & \\ S_2 & E_2 & E_3 & \cdots & E_{p+1} & E_{p+2} & T_0 \end{array}$$

Hence all the S_i which were collapsing to a single point i.e. they were exceptional, are no more exceptional because under the action of $\tilde{F}_1$ all the points on S_i now have their images defined on some other new curve i.e. the blown up fibres. Moreover previously the only points on T_i with pre-images were $A_j \quad A_k$ with $j = i = k$ But now under the action of $\tilde{F}_1$ all the points on the curves T_i have pre-images i.e. we have found the curves which map to the whole curve T_i and do not collapse only to a single point of T_i We can therefore say that $\tilde{F}_1$ is an automorphism.

The result is the following:

Theorem 15 *Let F be the map (4.4) and let $\tilde{F}$ be the induced map after blowing up the points $A_0 \quad A_1$ If there exists some $p \qquad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ then the characteristic polynomial associated with F is given by*

$$\mathcal{Y}_p = x^{p+1}(x^3 - x - 1) + (x^3 + x^2 - 1)$$

If no such p *exists then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Y} = x^3 - x - 1$$

Proof. From above given calculations we observe that now we don t have any exceptional curves which can reach any indeterminacy point of $\tilde{F}_1$ Hence from 2.5 we see that $\tilde{F}_1$ is an AS or regularized map now. We can now organize our orbits into lists. We have the following orbits:

$$\begin{aligned} \mathcal{O}_0 &= \quad A_0 = O_1 \qquad \mathcal{O}_1 = \; A_1 = O_2 \\ \mathcal{O}_2 &= \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0 \end{aligned}$$

and we have one closed list as follows

$$\mathcal{L}_c = \;\; \mathcal{O}_0 = \;\; A_0 = O_1 \qquad \mathcal{O}_1 = \;\; A_1 = O_2 \qquad \mathcal{O}_2 = \;\; A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

Then for $N_{L_c} = p+3$ we have $T_{L_c} = (x^{p+3} - 1)$ and $S_{L_c} = 2x^{p+2} + x^{p+1} + x^2 + 2x + 3$ Utilizing Theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{Y}_p = x^{p+1}(x^3 - x - 1) + (x^3 + x^2 - 1)$

Now suppose that no p exists such that $\tilde{F}^p(A_2) = O_0$ In this case we have one open list as follows:

$$\mathcal{L}_o = \;\; \mathcal{O}_0 = \;\; A_0 = O_1 \quad \mathcal{O}_1 = \;\; A_1 = O_2$$

Then for $N_{L_o} = 2$ we have $T_{L_o} = x^2$ and $S_{L_o} = 2x + 1$ By using the result from Theorem 10 we find that in this case the dynamical degree of f is given by the polynomial $\mathcal{Y}(x) = (x-2)T_{L_o} + (x-1)S_{L_o} = x^3 - x - 1.$ ∎

Proposition 16 *Consider the mappings which satisfy the hypothesis of Theorem 15. Then for* $p \leq 5$ *the sequence of degrees of map* F *is periodic. For* $p = 6$ *either the sequence of degrees grows quadratically or it is periodic. For* $p > 6$ *it grows exponentially.*

Proof. The characteristic polynomial $\mathcal{Y}_p$ is similar to the one discussed in Theorem $(4\ 3)$ in [BK06]. For giving a complete account of results we study $\mathcal{Y}_p$ in this work too.

For $p \quad \mathbb{N}$ we have the following polynomials:

$$\begin{aligned}
\mathcal{Y}_0(x) &= (x^2-1)\,(x^2+x+1)\\
\mathcal{Y}_1(x) &= (x-1)\,(x^4+x^3+x^2+x+1)\\
\mathcal{Y}_2(x) &= (x^2-1)\,(x^4+1)\\
\mathcal{Y}_3(x) &= (x-1)\,(x^2+x+1)\,(x^4-x^2+1)\\
\mathcal{Y}_4(x) &= (x^2-1)\,(x^6-x^3+1)\\
\mathcal{Y}_5(x) &= (x-1)\,(x^8+x^7-x^5-x^4-x^3+x+1)\\
\mathcal{Y}_6(x) &= (x-1)^3\,(x+1)\,(x^2+x+1)\,(x^4+x^3+x^2+x+1)
\end{aligned}$$

For $p=0$ the polynomial $(x^2-1)\,(x^2+x+1)$ has roots $x = 1 \;\; -1 \;\; \lambda_1 \;\; \lambda_2$ where $\lambda_1 = \lambda_2$ and $\lambda_1^3 = 1 \;\; \lambda_2^3 = 1$ This implies that the sequence of degrees

$$d_n = c_1 + c_2\,(-1)^n + c_3(\lambda_1)^n + c_4(\lambda_2)^n$$

satisfies $d_{n+6} = d_n$ i.e, it s periodic of period 6

For $p=1$ the polynomial $(x-1)\,(x^4+x^3+x^2+x+1)$ has roots $x = 1 \;\; \lambda_1 \;\; \lambda_2\,\lambda_3 \;\; \lambda_4$ where all λ_i s are distinct and $1 \;\; \lambda_i$ s are all roots of unity of period 5 This implies that the sequence of degrees

$$d_n = c_1 + c_2\,(\lambda_1)^n + c_3(\lambda_2)^n + c_4(\lambda_3)^n + c_5(\lambda_4)^n$$

satisfies $d_{n+5} = d_n$ i.e, it s periodic of period 5

For $p=2$ the polynomial $(x^2-1)\,(x^4+1)$ has roots of unity of period 8 hence $d_{n+8} = d_n$

For $p=3$ the polynomial $(x-1)\,(x^2+x+1)\,(x^4-x^2+1)$ has roots of unity of period 12 and we have $d_{n+12} = d_n$

For $p=4$ the polynomial $(x^2-1)\,(x^6-x^3+1)$ has roots of unity of period 18 hence $d_{n+18} = d_n$

For $p=5$ the polynomial $(x-1)\,(x^8+x^7-x^5-x^4-x^3+x+1)$ has roots of unity of period 30 and $d_{n+30} = d_p$

For $p=6$ the polynomial $(x-1)^3\,(x+1)\,(x^2+x+1)\,(x^4+x^3+x^2+x+1)$ has roots $x = 1 \;\; 1 \;\; 1 \;\; -1 \;\; \lambda_1 \;\; \lambda_2\,\lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6$ where all λ_i are distinct and $1 \;\; -1 \;\; \lambda_i$ s are all the roots of unity. This implies that the sequence of degrees is

$$d_n = c_1 + c_2\,n + c_3\,n^2 + c_4\,(-1)^n + c_5\,(\lambda_1)^n + c_6(\lambda_2)^n + c_7(\lambda_3)^n + c_8(\lambda_4)^n + c_9(\lambda_5)^n + c_{10}(\lambda_6)^n$$

As $\tilde{F}_1$ is an automorphism, we know that the sequence of degrees does not grow linearly. Then either, $c_3 = 0$ and d_n grows quadratically or $c_2 = 0 = c_3$ and d_n is periodic.

Now observe that for $p > 6$ $\mathcal{Y}_p(1) = 0$ $\acute{\mathcal{Y}}_p(1) = 6 - p < 0$ $\lim_x \; + \; \mathcal{Y}_p = +$ therefore there always exists a root $\lambda > 1$ such that $\mathcal{Y}_p(\lambda) = 0$ Therefore the sequence of degrees grows exponentially.

The result can be seen observed in the following figure 4.1

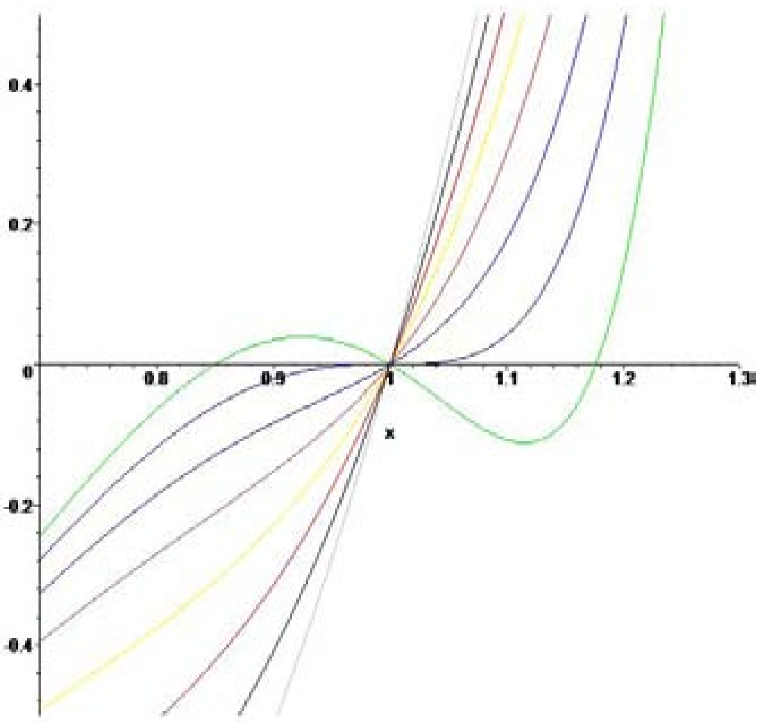

Figure 4.1: Behavior of Y_p for $p = 0 , 1 , 2 , 3 , 4 , 5 , 6$.

∎

4.2 Mappings with $\alpha_1 = 0$

We know from the start of this chapter that the parameters α_1 α_2 β_2 γ_1 are all non zero. However the parameters α_0 β_0 β_1 γ_0 can be zero. For ξ_1 ξ_2 λ_1 λ_2 $\mathbb{C}$ let $\gamma : (x \; y)$ $(\xi_1 x + \lambda_1 \; \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad \Bigg(\frac{\alpha_0 - \lambda_1 + \alpha_1\lambda_1 + \alpha_2\lambda_2}{\xi_1} \;\; 0 \;\; \frac{\alpha_2\xi_2}{\xi_1} \;\; (\beta_2\lambda_2 - \gamma_0\lambda_2 - \gamma_1\lambda_1\lambda_2 + \beta_0 + \beta_1\lambda_1) \;\; (\beta_1 - \gamma_1\lambda_2)\xi_1 \;\; \beta_2\xi_2 \;\; (\gamma_1\lambda_1 + \gamma_0)\xi_2 \;\; \gamma_1\xi_1\xi_2 \;\; 0\Bigg)$$

By choosing

$$(\xi_1 \;\; \xi_2 \;\; \lambda_1 \;\; \lambda_2) = \left(\frac{\alpha_2}{\beta_2} \;\; \frac{1}{\beta_2} \;\; \frac{-\gamma_0}{\gamma_1} \;\; \frac{\beta_1}{\gamma_1}\right)$$

the new parameters are

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad (\tilde{\alpha_0} \;\; \alpha_1 \;\; 1 \;\; \tilde{\beta_0} \;\; 0 \;\; 1 \;\; 0 \;\; \tilde{\gamma_1} \;\; 0)$$

For κ_1 $\mathbb{C}$ let : $(x \;\; y)$ $(\kappa_1 x \;\; \kappa_2 y)$ be the scaling map in x. By the group action of for the choice of $\kappa_1 = 1 \;\; \gamma_1 = \kappa_2$ we get the following parameters

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad (\tilde{\tilde{\alpha_0}} \;\; \alpha_1 \;\; 1 \;\; \tilde{\tilde{\beta_0}} \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; 0)$$

Hence our map f in (4.2) is conjugated to the following map f

$$f(x \;\; y) = \left(\alpha_0 + \alpha_1 x + y \;\; \frac{\beta_0 + y}{x}\right) \qquad \alpha_1 = 0 \tag{4.5}$$

We consider the induced map in projective space as follows: $F : P\mathbb{C}^2$ $P\mathbb{C}^2$ given by

$$F[x_0 : x_1 : x_2] = [x_0 x_1 \; : (\alpha_0 x_0 + \alpha_1 x_1 + x_2)x_1 \; : x_0(x_2 + \beta_0 x_0)] \qquad \alpha_1 = 0 \tag{4.6}$$

The indeterminacy sets of F and F^{-1} are $\mathcal{I}(F) = \;\; O_0 \;\; O_1 \;\; O_2$ where

$$O_0 = [1 : 0 : -\beta] \qquad O_1 = [0 : 1 : -\alpha_1] \qquad O_2 = [0 : 0 : 1]$$

and $\mathcal{I}(F^{-1}) = \;\; A_0 \;\; A_1 \;\; A_2$ where

$$A_0 = [0 : 1 : 0] \qquad A_1 = [0 : 0 : 1] \qquad A_2 = [1 : (\beta_0 - \alpha_0) : -\alpha_1]$$

Furthermore the exceptional curves of F and F^{-1} are as follows

$$S_0 = \quad x_0 = 0 \qquad S_1 = \quad x_1 = 0 \qquad S_2 = \quad \beta_0 x_0 + \alpha_1 x_1 + x_2 = 0$$

$$T_0 = \quad (\beta_0 - \alpha_0)x_0 + x_1 = 0 \qquad T_1 = \quad \alpha_1 x_0 + x_2 = 0 \qquad T_2 = \quad x_0 = 0$$

- **Orbit of** A_0

Observe that $S_0 \twoheadrightarrow A_0 = O_i$ for any $i \quad 0 \ 1 \ 2$ also $A_0 \quad S_0$ Hence A_0 is a fixed point under F Observe that the orbit of A_0 that is $\mathcal{O}_0 = \quad A_0$ is not an SE orbit.

- **Orbit of** A_1

Now $S_1 \twoheadrightarrow A_1 = O_2$ therefore we need to blow up the point $A_1 = O_2$ Let X be the new space we get after blowing up the point O_2 and let E_1 be the exceptional fibre at this point. We recognize the induced map after the blow up process as $\tilde{F} : X \quad X$ We find that the map $\tilde{F}$ sends the curve S_1 to E_1 and E_1 to $T_2 = S_0$ in the following way:

$$\tilde{F}[x_0 : 0 : x_2] \quad S_1 \qquad [x_0 : \alpha_0 x_0 + x_2]_{E_1} \qquad \tilde{F}[u : v]_{E_1} \qquad [0 : v : u] \quad T_2 = S_0 = \quad x_0 = 0$$

Hence the indeterminacy set is now $\mathcal{I}(\tilde{F}) = \quad O_0 \ \ O_1$ and the exceptional set is $\mathcal{E}(\tilde{F}) = \quad S_0 \ \ S_2$ Therefore the orbit of A_1 is SE.

- **Orbit of** A_2

The exceptional curve $S_2 \twoheadrightarrow A_2$ Following the orbit of A_2 we need to know if any of it s iterates reach any indeterminacy point of $\tilde{F}$ Now $\mathcal{I}(\tilde{F}) = \quad O_0 \ \ O_1$ and $O_1 \quad S_0 = T_2$ We know that the only points on T_2 which have preimages are A_0 and A_1 which implies that if the orbit of A_2 through F reaches O_1 at some iterate of F then O_1 should be equal to either A_0 or A_1 Now as $\alpha_1 = 0$ hence $A_0 = O_1 = A_1$ This implies that $F^p(A_2) = O_1$ for all p but it is possible that $\tilde{F}^p(A_2) = O_1$ Then in general there are two possibilities that for some $p \quad \mathbb{N}$ either $\tilde{F}^p(A_2) = O_0$ or for some $q \quad \mathbb{N} \quad \tilde{F}^q(A_2) = O_1$ In both cases the orbit of A_2 is SE.

Now if there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ then we have to blow-up all the points $A_2 \ \tilde{F}(A_2) \ \tilde{F}^2(A_2) \qquad \tilde{F}^p(A_2) = O_0$ Let $E_i \ \ i = 2 \ \ 3 \qquad p + 2$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1 \quad X_1$ be the induced map. But the action of $\tilde{F}_1$ is same as the action of F on the orbit of A_2 Then under the action of $\tilde{F}_1$:

$$E_2 \qquad E_3 \qquad \cdots \qquad E_{p+2}$$

This sequence can be completed in the following manner:

$$S_2 \quad E_2 \quad E_3 \quad \cdots \quad E_{p+2} \quad T_0$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_2 and on E_{p+2} is:

$$\tilde{F}_1[x_0 : x_1 : -\beta_0 x_0 - \alpha_1 x_1] = [x_1 : x_0]_{E_2}$$

and

$$\tilde{F}_1[u : v]_{E_{p+2}} = [u : (\alpha_0 - \beta_0)u : v] \quad T_0$$

Observe that now $\tilde{F}_1 : X_1 \quad X_1$ is an AS map.

Also if there exists some $q \quad \mathbb{N}$ such that $\tilde{F}^q(A_2) = O_1$ then we have to blow-up all the points $A_2 \ \tilde{F}(A_2) \ \tilde{F}^2(A_2) \qquad \tilde{F}^q(A_2) = O_1$ Let $G_i \ i = 2 \ 3 \qquad q + 2$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1 \quad X_1$ be the induced map. We see that under the action of $\tilde{F}_2$:

$$G_2 \quad G_3 \quad \cdots \quad G_{q+2}$$

This sequence can be completed in the following manner:

$$S_2 \quad G_2 \quad G_3 \quad \cdots \quad G_{q+2} \quad T_1$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_2 and on G_{q+2} is:

$$\tilde{F}_1[x_0 : x_1 : (-\beta_0 x_0 - \alpha_1 x_1)] = [\, x_1 : x_0]_{G_2}$$

and

$$\tilde{F}_1[u : v]_{G_{q+2}} = [u : (\alpha_0 u + v) : -\alpha_1 u] \quad T_1$$

Observe that now $\tilde{F}_1 : X_1 \quad X_1$ is an AS map. In these cases $\tilde{F}_1$ is not an automorphism as S_0 still collapses.

Following are the results:

Theorem 17 *Let F be the map (4.6) and let $\tilde{F}$ be the induced map after blowing up the point A_1 If there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ then the characteristic polynomial associated* *with F is given by*

$$\mathcal{Y}_p = x^{p+1}(x^2 - x - 1) + x^2 - 1$$

Proof. From above given calculations we observe that now we don t have any exceptional curves which can reach any indeterminacy point of $\tilde{F}_1$ Hence $\tilde{F}_1$ is an AS or regularized map now. We can now organize our orbits in to lists. We have the following orbits:

$$\mathcal{O}_1 = \quad A_1 = O_2 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

and we have one open list as follows

$$\mathcal{L}_0 = \quad \mathcal{O}_1 = \quad A_1 = O_2 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

Then for $N_{L_o} = p + 2$ we have $T_{L_o} = x^{p+2}$ and $S_{L_o} = x^{p+1} + x + 1$ Utilizing Theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{Y}_p = x^{p+1}(x^2 - x - 1) + x^2 - 1$

■

Proposition 18 *Consider the mappings which satisfy the hypothesis of Theorem 17. Then for all* $p \quad \mathbb{N}$ *the sequence of degrees has exponential growth rate.*

Proof. We observe that for all the values of $p \quad \mathbb{N}$ the polynomial $\mathcal{Y}_p = x^{p+1}(x^2 - x - 1) + x^2 - 1$ has always the largest root $\lambda > 1$ This is because $\mathcal{Y}_p(1) = -1 < 0$ and $\lim_{x \quad +} \mathcal{Y}_p = +$ therefore there always exists a root $\lambda > 1$ such that $\mathcal{Y}_p(\lambda) = 0$ It can be seen in figure 4.2. We thus conclude that as all the families with associated characteristic polynomial $\mathcal{Y}_p$ have $\delta = \lambda > 1$ for all $p \quad \mathbb{N}$ Therefore the sequence of degrees has exponential growth rate. ■

Theorem 19 *Let F be the map (4.6) and let $\tilde{F}$ be the induced map after blowing up the point A_1 If there exists some $q \quad \mathbb{N}$ such that $\tilde{F}^q(A_2) = O_1$ then the characteristic polynomial associated with F is given by*

$$\mathcal{Y}_q = x^{q+1}(x^2 - x - 1) + x^2$$

Proof. From above given calculations we observe that $\tilde{F}_1$ is an AS map now and we have the following orbits:

$$\mathcal{O}_1 = \quad A_1 = O_2 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^q(A_2) = O_1$$

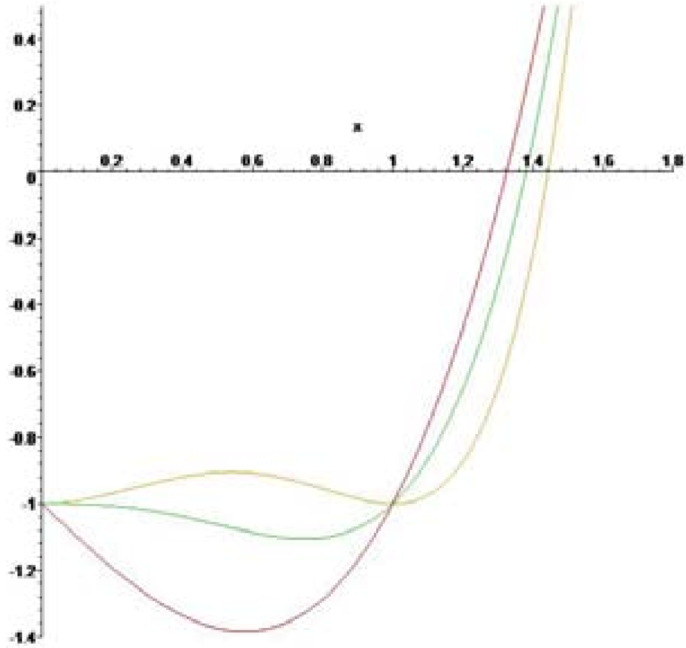

Figure 4.2: Behavior of Y_p for $p = 0\,, 1\,, 2$.

and we have one closed list as follows:

$$\mathcal{L}_c = \quad \mathcal{O}_1 = \quad A_1 = O_2 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^q(A_2) = O_1$$

Then for $N_{L_c} = q+2$ we have $T_{L_c} = x^{q+2} - 1$ and $S_{L_c} = x^{q+1} + x + 2$ Utilizing Theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}_1^*$ is $\mathcal{Y}_q = x^{q+1}(x^2 - x - 1) + x^2$ The proof of the second polynomial is discussed in the result of Theorem 17. ∎

We observe the characteristic polynomial of this theorem for the values of $q \quad \mathbb{N}$ in the following proposition.

Proposition 20 *Consider the mappings which satisfy the hypothesis of Theorem 19. Then*

(i) For $q = \quad 0 \;\; 1$ *there are no such mappings.*

(ii) The sequence of the degrees is bounded or grows linearly when $q = 2$

(iii) The sequence of the degrees grows exponentially when $q > 2$

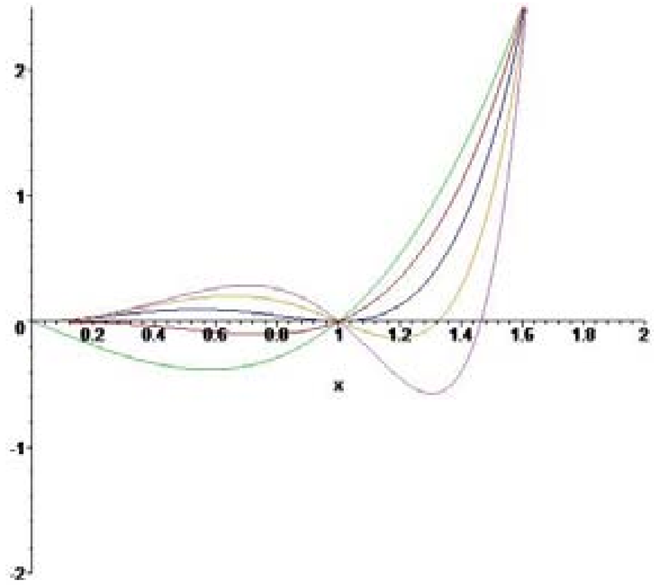

Figure 4.3: Behavior of $\mathcal{Y}_q$ for $q = 1, 2, 3, 4, 5.$

Proof. For $q = 0$ the above Theorem 19 gives us the condition $A_2 = O_1$ But we observe that $O_1[1] = 0$ and $A_2[1] = 1$ which implies that $A_2 = O_1$ For $q = 1$ we have two possibilities when $A_2 \quad S_1$ then the condition $\tilde{F}(A_2) = F(A_2) = O_1$ which shows that the orbit of A_2 can never reach O_1 because $O_1 \quad T_2$ and the only point other than A_1 on T_2 which has preimage is A_0 where $O_1 = A_0$ Now if $A_2 \quad S_1$ then we have the condition $\tilde{F}(A_2) = O_1$ But in this case $\tilde{F}(A_2) \quad E_1$ and it is clear that $O_1 \quad E_1$ therefore $q = 1$ is not possible. Hence there are no mappings for $q \quad 0 \; 1$

We know that $\mathcal{Y}_q = x^{q+1}(x^2 - x - 1) + x^2$ For $q = 2$ we get the polynomial $\mathcal{X}_2 = x^5 - x^4 - x^3 + x^2 = x^2(x+1)(x-1)^2$ This polynomial has roots $x = 0 \;\; 0 \;\; -1 \;\; 1 \;\; 1$ This implies that the sequence of degrees

$$d_n = c_2\,(-1)^n + c_3 + c_4\,n$$

Then $c_4 = 0$ (resp. $c_4 = 0$) gives that d_n grows linearly (resp.d_n is bounded).

For $q > 2$ we observe that $\mathcal{Y}_q(1) = 0 \;\; \mathcal{Y}\,_q(1) = 2 - q < 0$ and $\lim_{x \quad +} \; \mathcal{Y}_q(x) = +$ Hence $\mathcal{Y}_q$ always has a root $\lambda > 1$ and the result follows. This can be observed in the following figure.

∎

Theorem 21 *Let F be the map (4.6) and let $\tilde{F}$ be the induced map after blowing up the point A_1 If $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^p(A_2) = O_1$ for all p $\mathbb{N}$ then the characteristic polynomial associated with F is given by*

$$\mathcal{Y}_p = x^2 - x - 1$$

and the sequence of degrees grows exponentially.

Proof. Now suppose that no p exists such that $F^p(A_2) = O_0$ and $F^p(A_2) = O_1$ In this case we have one open list as follows:

$$\mathcal{L}_o = \quad \mathcal{O}_1 = \quad A_1 = O_2$$

Then for $N_{L_o} = 1$ we have $T_{L_o} = x$ and $S_{L_o} = 1$ By using the result from Theorem 10 we find that in this case the dynamical degree of f is given by the polynomial $Y(x) = (x-2)T_{L_o} + (x-1)S_{L_o} = x^2 - x - 1$

The next section discusses the dynamics of the families with zero entropy classified in this chapter in previously discussed theorems. The following families of mappings provide examples for the results stated and proved in [DF01, Can99, Bel99], also given in Theorem (7).

4.3 Zero entropy

In this section the following theorem identifies and state the particular families in the degenerate case when the dynamical degree $\delta = 1$ It is as follows:

Theorem 22 *Let $f(x\ \ y)$ be the map*

$$f(x\ \ y) = \left(y\ \ \frac{\beta_0 + y}{\gamma_0 + x}\right)$$

If $f(x\ \ y)$ has zero entropy then either, the sequence of degrees is periodic or it grows quadratically. Furthermore:

- *If the sequence of degrees is periodic then $f(x\ \ y)$ is one of the following:*

  (1) $f(x\ \ y) = \left(y\ \ \frac{y}{x}\right)$ and $f(x\ \ y)$ is 6-periodic.

(2) $f(x\ \ y) = \left(y\ \ \frac{1+y}{x}\right)$ *and* $f(x\ \ y)$ *is 5-periodic.*

(3) $f(x\ \ y) = \left(y\ \ \frac{\beta_0+y}{\gamma_0+x}\right)$ *with* $\gamma_0^2 + 1 = 0\ \ \ 2\beta_0 = 1 + \gamma_0$ *and* $f(x\ \ y)$ *is 8-periodic.*

(4) $f(x\ \ y) = \left(y\ \ \frac{\beta_0+y}{\gamma_0+x}\right)$ *with* $\gamma_0^2 + 1 = 0\ \ \ \beta_0^2 - (\gamma_0 + 2)\beta_0 + \gamma_0 = 0$ *and* $f(x\ \ y)$ *is 12-periodic.*

(5) $f(x\ \ y) = \left(y\ \ \frac{\beta_0+y}{\gamma_0+x}\right)$ *with* $3\gamma_0^6 + 9\gamma_0^4 + 6\gamma_0^2 + 1 = 0\ \ \ 2\beta_0 = 3 + \gamma_0 + 7\gamma_0^2 + 3\gamma_0^4$ *and* $f(x\ \ y)$ *is 18-periodic.*

(6) $f(x\ \ y) = \left(y\ \ \frac{\beta_0+y}{\gamma_0+x}\right)$ *with* $\gamma_0^8 + 8\gamma_0^6 + 14\gamma_0^4 + 7\gamma_0^2 + 1 = 0\ \ \ 2\beta_0 = -5 + \gamma_0 - 22\gamma_0^2 - 15\gamma_0^4 - 2\gamma_0^6$ *and* $f(x\ \ y)$ *is 30-periodic.*

- *If the sequence of degrees grows quadratically then* $f(x\ \ y)$ *is one of the following:*

(7) $f(x\ \ y) = \left(y\ \ \frac{\beta_0+y}{x}\right)$ *with* $\beta_0 = 0$

(8) $f(x\ \ y) = \left(y\ \ \frac{\beta_0+y}{\gamma_0+x}\right)$ *with* $\gamma_0^4 + 5\gamma_0^2 + 5 = 0\ \ \ 2\beta_0 = -1 + \gamma_0 - \gamma_0^2$

The map (7) *is integrable being*

$$V_1(x\ \ y) = \frac{(x+1)(y+1)(x+y+\beta_0)}{xy}$$

a first integral. The maps (8) *preserve the elliptic fibration*

$$V_2(x\ \ y) = \frac{LQ}{L_1\, L_2\, L_3}$$

where

$$\begin{aligned} L &:= (2 - 3\gamma_0 + \gamma_0^2 - \gamma_0^3)x - (3 + \gamma_0 + \gamma_0^2)y + 2 \\ Q &:= (1 - \gamma_0 + \gamma_0^2) - 2x - (2 + \gamma_0 + \gamma_0^2 + \gamma_0^3)y + (-2 + 3\gamma_0 - \gamma_0^2 + \gamma_0^3)xy \end{aligned}$$

and

$$\begin{aligned} L_1 &:= 2y - (3 - \gamma_0 + \gamma_0^2)x - (1 + 2\gamma_0 + \gamma_0^3) \\ L_2 &:= 2y - (3 - \gamma_0 + \gamma_0^2) \\ L_3 &:= 2x - (3 - \gamma_0 + \gamma_0^2) \end{aligned}$$

with $V_2(f(x\ \ y)) = \quad V(x\ \ y)$ *where* $\quad = 1 - \frac{3}{2}\gamma_0 + \frac{1}{2}\gamma_0^2 - \frac{1}{2}\gamma_0^3$ *and* $\ ^5 = 1$

Proof. From Proposition 7 we see that all the maps of family 4.1 which have zero entropy are the ones which $\tilde{F}^p(A_2) = O_0$ for $p = 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6$ Looking at each one of these values of p it is easy to see that $\tilde{F}^p(A_2) = O_0$ is equivalent to $F^p(A_2) = O_0$ as the orbit of A_2 does not collide with the orbit of A_0 and A_1 for these values of p Hence the conditions on the parameters $\beta_0\ \ \gamma_0$ are given in the appendix in subsection non-degenerate case $\gamma_2 = 0 = \alpha_1$. This gives us the required families for $p = 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6$

We are going to prove that for $p = 1\ \ 2\ \ 3\ \ 4\ \ 5$ the map is indeed periodic. For $p = 0$ the proof is trivial. We call n_p the period of the sequence of the degrees of the map F which satisfies $\tilde{F}^p(A_2) = F^p(A_2) = O_0$ i.e., $n_0 = 6\ \ n_1 = 5$ and so on.

If $\tilde{F}_1$ is the regularization of F then we know that in this case $\tilde{F}_1$ acts in the following way:

$$S_1 \qquad E_1 \qquad T_2 \qquad E_0 \qquad T_1$$

where $T_2 = S_0$

$$S_2 \qquad E_2 \qquad E_3 \qquad \cdots \qquad E_{p+2} \qquad T_0$$

and that $\tilde{F}_1^* : Pic(X) \qquad Pic(X)$ where $Pic(X) =< \hat{L}\ \ E_0\ \ E_1\ \ E_2 \qquad E_{p+2} >$ Since $(\tilde{F}_1^*)^{n_p} = Id$ it is a linear map so we know that $(\tilde{F}_1^*)^{n_p}(E_i) = E_i$ for each $i = 0\ \ 1 \qquad p+2$ It implies that F^{n_p} sends the base point of each E_i to itself. On the other hand F^{n_p} is a map of degree one, because of the periodicity of the sequence of degrees. Hence we have

$$F^{n_p}[x_0 : x_1 : x_2] = [p_0x_0 + p_1x_1 + p_2x_2 : q_0x_0 + q_1x_1 + q_2x_2 : r_0x_0 + r_1x_1 + r_2x_2]$$

Now we see that:

$$\begin{aligned} F^{n_p}(A_0) = A_0 &\quad = \quad p_1 = r_1 = 0 \quad q_1 = 0 \\ F^{n_p}(A_1) = A_1 &\quad = \quad p_2 = q_2 = 0 \quad r_2 = 0 \\ F^{n_p}(A_2) = A_2 &\quad = \quad r_0 = 0 \quad q_0 = \beta_0 q_1 - p_0\beta_0 \\ F^{n_p}(F(A_2)) = F(A_2) &\quad = \quad q_0 = 0 \quad p_0 = r_2 \end{aligned}$$

All together implies that F^{n_p} is the identity.

Now we deal with the case $p = 6$ When $\gamma_0 = 0$ we find the celebrated Lyness map which is very well known that it is integrable. For mapping (8) we are going to explain the way to find the invariant fibration.

To find the invariant curve we use the same methodology defined for family 5 in last section of

Chapter 3. We begin by taking all the lines which pass through $O_0 = [1 : -\gamma_0 : -\beta_0]$ that is the lines $L_1 : y = m(\gamma_0 + x) - \beta_0$ We find the following:

$$\begin{array}{llllll} L_1 := & y = m(\gamma_0 + x) - \beta_0 & L_2 := & y = m & L_3 := & x = m \\ L_4 := & y = \frac{1}{\gamma_0+m}x + \frac{\beta_0}{\gamma_0+m} & & & & \end{array}$$

We see that a straight line passing through O_0 gives a straight line passing through O_1 and hence their image also will be a straight line. Also a straight line passing through O_1 gives a straight line passing through O_2

Now we look for m such that the last line coincide whit the first one. It gives us a couple of equations:

$$m^2 + m\gamma_0 = 1 \text{ and } \gamma_0 m^2 + (\gamma_0^2 - \beta_0)m - \beta_0 - \gamma_0\beta_0 = 0$$

It is easily seen that these two equations are compatible if and only if $\gamma_0 = 0$ or $\gamma_0 - 2\beta_0 - \gamma_0\beta_0 + \beta_0^2 = 0$ and in this case $m = \frac{\gamma_0 - \beta_0 - \gamma_0\beta_0}{\beta_0}$ If the condition $\tilde{F}^p(A_2) = O_0$ of the mapping (8) is satisfied then $\gamma_0 - 2\beta_0 - \gamma_0\beta_0 + \beta_0^2 = \frac{1}{4}\gamma_0(\gamma_0^4 + 5\gamma_0^2 + 5)$ which gives the condition on γ_0 which shows that $L_4 = L_1$ In this way we find that $L_1\, L_2\, L_3$ is an invariant curve for f

In a similar manner we search cubics being the product of a conic and a line. We take a general conic passing through O_0 O_1 O_2 in such a way that its image is a straight line. And we impose that the image of this straight line coincides with the conic. In this way we find Q and L After, a calculation gives that $V_2(x\;\; y) = \frac{L\,Q}{L_1\,L_2\,L_3}$ satisfies $V_2(f(x\;\; y)) = \;\; V_2(x\;\; y)$ We want to point out that all the calculations have been done taking into account the only parameter γ_0 and in each step of the calculation considering the remainder of the expressions after dividing by $\gamma_0^4 + 5\gamma_0^2 + 5 = 0$

Theorem 23 *If a map in 4.5 has zero entropy then it can be written as:*

$$f(x\;\; y) = \left(\quad + \alpha_1 x + y \quad \frac{\quad + y}{x} \right) \qquad = \frac{\alpha_1^2 - 1}{\alpha_1} \quad \alpha_1 = 0$$

Moreover the corresponding sequence of degrees grows linearly and f preserves the rational fibration

$$V(x\;\; y) = \frac{(\alpha_1 x + 1)(\alpha_1 x + y + \alpha_1)}{x\,\alpha_1\,(\alpha_1 + 1)}$$

with $V(f(x\;\; y)) = \alpha_1\, V(x\;\; y)$ If $\alpha_1 = 1$ then f is integrable. If $\alpha_1^n = 1$ for some $n \quad \mathbb{N}$ then f^n is integrable.

Proof. From the results discussed in Propositions 18 and 20 we see that there is only one family which has zero entropy. This mapping satisfies the hypothesis of Theorem 19 for $q = 2$ We see that for $d_2 = 3 \;\; d_3 = 5 \;\; d_4 = 7$ we can write that $d_n = -1 + 2\,n \;\; n > 1$ which implies that d_n grows linearly in n

We study the condition in this case that is $\tilde{F}^2(A_2) = O_1$ We see that we have two possibilities, with or without collision of orbits. When $A_2 \quad S_1$ then the condition $\tilde{F}^2(A_2) = F^2(A_2) = O_1$ which shows that the orbit of A_2 can never reach O_1 It is because $O_1 \quad S_0 = T_2$ and the only points on T_2 which have preimage by F are A_0 and A_1 and we know that $A_0 = O_1 = A_1$

Now if $A_2 \quad S_1$ then we have the condition $\tilde{F}^2(A_2) = O_1$ This implies that $\tilde{F}^2(A_2) \quad S_0$ as $O_1 \quad S_0$ Now as

$$S_1 \quad E_1 \quad S_0 = T_2$$

This shows that there is a collision of orbits of A_2 with the orbit of A_1 We observe that $A_2 \quad S_1$ implies that $\alpha_0 = \beta_0 = \alpha_1$ Hence the orbit of A_2 is as follows:

$$A_2 = [1 : 0 : -\alpha_1] \quad [1 : \alpha_0 - \alpha_1]_{E_1} \quad [0 : \alpha_0 - \alpha_1 : 1] = [0 : 1 : -\alpha_1] = O_1$$

From the above we find that the condition $\tilde{F}^2(A_2) = O_1$ is satisfied for $\alpha_0 = \frac{\alpha_1^2 - 1}{\alpha_1} = \beta_0$ This gives us our required mapping f

We find that the line $L_1 := \;\; x = -\frac{1}{\alpha_1} \qquad L_2 := \;\; y = -\alpha_1\,(x+1) \qquad L_1$ Now by considering a general line $L_3 = l_0 + l_1\,x + l_2\,y$ and taking $V(x \;\; y) = \frac{L_1\,L_2}{L_3}$ then after an easy computation we find that the function $V(x \;\; y)$ gives a rational fibration for this mapping f where

$$V(x \;\; y) = \frac{(\alpha_1 x + 1)(\alpha_1 x + y + \alpha_1)}{x\,\alpha_1\,(\alpha_1 + 1)}$$

with $V(f(x \;\; y)) = \alpha_1\,V(x \;\; y)$ Note that for $\alpha_1 = 1$ we have $V(f(x \;\; y)) = V(x \;\; y)$ which shows that the map f is integrable. Note that $V(f^n(x \;\; y)) = \alpha_1^n\,V(x \;\; y)$ therefore if $\alpha_1^n = 1$ for some $n \quad \mathbb{N}$ then f^n is integrable. This completes the proof. ∎

It is not enough to have a good mind. The main thing is to use it well - Rene Descartes
Millions saw the apple fall, but Newton asked why - Bernard Baruch
If there's no struggle, there's no progress - Frederick Douglass

Chapter 5

Non Degenerate Case $\gamma_1 = 0 = \gamma_2$, ($\alpha_1 = 0$ or $\alpha_2 = 0$)

In this chapter we classify the family of maps in the the non degenerate case when $\gamma_1 = 0 = \gamma_2$ We study the dynamical degree of all these mappings and discuss these families in two different cases depending on $\alpha_1 = 0$ or $\alpha_2 = 0$. The families with dynamical degree one are dynamically studied.

To prove the result in this chapter for both cases we use a Theorem 10 for the linear fractional maps which is given in Chapter 2. We first calculate the dynamical degree depending on the coefficients using the methodology explained in Chapter 2, that is to find the SE orbits and organize them into lists in order to apply the Theorem 10 to find the associated characteristic polynomial of F which gives us the sequence of degrees of F and dynamical degree of F

We find that in mappings $\alpha_1 = 0$ there exist the case when the orbits of indeterminacy points of F^{-1} can collide. This is discussed in Theorem 28. The theory of collision of orbits of the indeterminacy points of F^{-1} is discussed in detail in the Chapter 2.

In the third and last section of this chapter we detect the two families of mappings with zero entropy that is with dynamical degree one. We find that these mappings have linear growth hence we illustrate that they preserve a rational invariant fibrations.

We now start by introducing our family of mappings we consider to study in this chapter.

Given complex numbers α_i β_i γ_i $i = 0$ 1 2 Consider the birational mapping $f : \mathbb{C}^2 \quad \mathbb{C}^2$ of the form

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y\ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right)$$

From lemma 8 we know that f is *non degenerate* i.e. all S_i are distinct if and only if $(\beta\gamma)_{12} = 0 = (\alpha\gamma)_{12}$ This implies that $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$ Therefore we consider that the parameters $\alpha_0\ \ \beta_0\ \ \beta_1\ \ \beta_2\ \ \gamma_0$ and one of $\alpha_1\ \ \alpha_2$ at the same moment can be zero.

5.1 Mappings with $\alpha_1 = 0$

Now the birational map f has the following form

$$f(x\ \ y) = \left(\alpha_0 + \alpha_2 y\ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right) \tag{5.1}$$

For $\xi_1\ \ \xi_2\ \ \lambda_1\ \ \lambda_2 \quad \mathcal{C}$ let $\gamma : (x\ \ y) \quad (\xi_1 x + \lambda_1\ \ \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$\begin{aligned}(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad \Bigg(&\frac{\alpha_0 - \lambda_1 + \alpha_2\lambda_2}{\xi_1}\ \ 0\ \ \frac{\alpha_2\xi_2}{\xi_1}\ \ (\beta_2\lambda_2 - \gamma_0\lambda_2 - \gamma_1\lambda_1\lambda_2 + \beta_0 + \beta_1\lambda_1)\\ &(\beta_1 - \gamma_1\lambda_2)\xi_1\ \ \beta_2\xi_2 - \gamma_2\lambda_2\ \ (\gamma_1\lambda_1 + \gamma_0)\xi_2\ \ \gamma_1\xi_1\xi_2\ \ \gamma_2\xi_2\Bigg)\end{aligned}$$

By choosing

$$(\xi_1\ \ \xi_2\ \ \lambda_1\ \ \lambda_2) = \left(\frac{\gamma_2}{\gamma_1}\ \ \frac{1}{\gamma_2}\ \ \alpha_0 + \frac{\alpha_2\beta_1}{\gamma_1}\ \ \frac{\beta_1}{\gamma_1}\right)$$

the new parameters are

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \quad (0\ \ 0\ \ \tilde{\alpha_2}\ \ \tilde{\beta_0}\ \ 0\ \ \tilde{\beta_2}\ \ \tilde{\gamma_0}\ \ 1\ \ 1)$$

By Renaming the coefficients we see that the map $f(x\ \ y)$ in (5.1) is conjugated to the following map

$$f(x\ \ y) = \left(\alpha_2 y\ \ \frac{\beta_0 + \beta_2 y}{\gamma_0 + x + y}\right) \qquad \alpha_2 = 0 = \beta_2 \tag{5.2}$$

Note that the parameter $\beta_2 = -(\beta\gamma)_{12} = 0$ We consider the imbedding $(x\ \ y) \quad [1 : x : y] \quad P\mathbb{C}^2$ into projective space and consider the induced map $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ in projective space as follows:

$$F[x_0 : x_1 : x_2] = [x_0(\gamma_0 x_0 + x_1 + x_2)\ : \alpha_2 x_2(\gamma_0 x_0 + x_1 + x_2)\ : x_0(\beta_0 x_0 + \beta_2 x_2)]\ \ \alpha_2 = 0 = \beta_2 \tag{5.3}$$

The indeterminacy sets of F and F^{-1} are $\mathcal{I}(F) = \quad O_0 \;\; O_1 \;\; O_2$ where

$$O_0 = [-\beta_2 : (\gamma_0\beta_2 - \beta_0) : \beta_0] \qquad O_1 = [0 : 1 : 0] \qquad O_2 = [0 : -1 : 1]$$

and $\mathcal{I}(F^{-1}) = \quad A_0 \;\; A_1 \;\; A_2$ where

$$A_0 = [0 : 1 : 0] \qquad A_1 = [0 : 0 : 1] \qquad A_2 = [-\beta_2 : \alpha_2\beta_0 : 0]$$

Furthermore the exceptional curves of F and F^{-1} are as follows

$$S_0 = \quad x_0 = 0 \qquad S_1 = \quad \gamma_0 x_0 + x_1 + x_2 = 0 \qquad S_2 = \quad \beta_0 x_0 + \beta_2 x_2 = 0$$

$$T_0 = \quad \alpha_2\beta_0 x_0 + \beta_2 x_1 = 0 \qquad T_1 = \quad x_2 = 0 \qquad T_2 = \quad x_0 = 0$$

- **Orbit of** A_0

Observe that $S_0 \twoheadrightarrow A_0 = O_1$ We need to blow up $A_0 = O_1$ Let X be the new space we get after blowing up the point O_1 and let E_0 be the exceptional fibre at this point. We recognize the induced map after the blow up process as $\tilde{F} : X \qquad X$ For the points $x = [0 : x_1 : x_2] \quad S_0$ we find that the map $\tilde{F}$ sends the curve S_0 to E_0 as follows:

$$[0 : x_1 : x_2] \quad S_0 \qquad [x_1 + x_2 : \beta_2 x_2]_{E_0}$$

Moreover $\tilde{F}$ sends all the points $x = [tu : 1 : tv] \quad E_0$ as $t \quad 0$ to T_1 in a way such that

$$\tilde{F}[u : v]_{E_0} \qquad [u : \alpha_2 v : 0] \quad T_1$$

Hence the new indeterminacy set is $\mathcal{I}(\tilde{F}) = \quad O_0 \;\; O_2$ and the new exceptional set is $\mathcal{E}(\tilde{F}) = \quad S_1 \;\; S_2$

- **Orbit of** A_1 **and** A_2

We see that $S_1 \twoheadrightarrow A_1 = O_i$ for $i \quad 0 \;\; 1 \;\; 2$ Also as $A_1 \quad S_0$ this implies that the orbit of A_1 collides with the orbit of A_0 therefore $\tilde{F}$ sends the curve

$$S_1 \twoheadrightarrow A_1 \qquad [1 : \beta_2]_{E_0} \qquad [1 : \alpha_2\beta_2 : 0] \quad T_1$$

Following the orbit of A_1 we need to know if any of it s iterates reach any indeterminacy point of $\tilde{F}$ As $\mathcal{I}(\tilde{F}) = \quad O_0 \ O_2$ and $O_2 \quad S_0 = T_2$ We know that the only points on T_2 which have preimages are A_0 and A_1 which implies that if the orbit of A_2 reaches O_2 at some iterate then O_2 should be equal to either A_0 or A_1 But $A_0 = O_2 = A_1$ This implies that $\tilde{F}^p(A_1) = O_2$ for all p but it is possible that $\tilde{F}^p(A_1) = O_0$ for some $p \quad \mathbb{N}$ In this case the orbit of A_1 is SE. Moreover if no such p exists, then the orbit of A_1 is not SE

Assume that there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_1) = O_0$ Then we have to blow-up all the points $A_1 \ \tilde{F}(A_1) \ \tilde{F}^2(A_1) \qquad \tilde{F}^p(A_1) = O_0$ Let $E_i \ i = 1 \ 2 \qquad p+1$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1 \qquad X_1$ be the induced map. Hence we see that under the action of $\tilde{F}_1$:

$$E_1 \qquad E_2 \qquad \cdots \qquad E_{p+1}$$

This sequence can be completed in the following manner:

$$S_1 \qquad E_1 \qquad E_2 \qquad \cdots \qquad E_{p+1} \qquad T_0$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_1 and on E_{p+1} is:

$$\tilde{F}_1[x_0 : -(\gamma_0 x_0 + x_2) : x_2] = [x_0 : \alpha_2 x_2]_{E_1}$$

and

$$\tilde{F}_1[u : v]_{E_{p+1}} = [\beta_2(u+v) : -\alpha_2\beta_0(u+v) : \beta_2^2 v] \quad T_0$$

Since $\alpha_2 = 0 = \beta_2$ we see that we have not created any new points of indeterminacy, hence $\mathcal{I}(\tilde{F}_1) = \quad O_2$ and the exceptional set is $\mathcal{E}(\tilde{F}_1) = \quad S_2$

The exceptional curve $S_2 \twoheadrightarrow A_2$ Following the orbit of A_2 we need to know if any of it s iterates reach any indeterminacy point of $\tilde{F}_1$ As $\mathcal{I}(\tilde{F}_1) = \quad O_2$ We notice that if O_2 has a preimage via $\tilde{F}_1$ then since $O_2 \qquad T_0 \ T_1$ then O_2 has a preimage F But $O_2 \quad S_0 = T_2$ we know that the only points on T_2 which have preimages are A_0 and A_1 which implies that if the orbit of A_2 reaches O_2 at some iterate then O_2 should be equal to either A_0 or A_1 But $A_0 = O_2 = A_1$ Therefore for no $q \quad \mathbb{N}$ we can get $\tilde{F}^q(A_2) = O_2$ Thus the orbit of A_2 is not SE.

Now assume that no such p exists such that $\tilde{F}^p(A_1) = O_0$ Then the orbit of A_1 is not SE as it cannot reach any indeterminacy point of $\tilde{F}$ Thus for some q it is possible that $\tilde{F}^q(A_2) = O_0$ In this case we have to blow-up all the points A_2 $\tilde{F}(A_2)$ $\tilde{F}^2(A_2)$ $\quad$ $\tilde{F}^q(A_2) = O_0$ Let E_i $i =$ 1 2 $\quad$ $q+1$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1 \quad X_1$ be the induced map. Then under the action of $\tilde{F}_1$ we have that:

$$E_1 \quad E_2 \quad \cdots \quad E_{q+1}$$

This sequence can be completed in the following manner:

$$S_2 \quad E_1 \quad E_2 \quad \cdots \quad E_{q+1} \quad T_0$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_2 and on E_{q+1} is:

$$\tilde{F}_1[x_0 : x_1 : -\frac{\beta_0}{\beta_2}x_0] = [\alpha_2(\beta_2\gamma_0 x_0 + \beta_2 x_1 - \beta_0 x_0) : x_0\beta_2^2]_{E_1}$$

and

$$\tilde{F}_1[u : v]_{E_{q+1}} = [\beta_2(u+v) : -\alpha_2\beta_0(u+v) : \beta_2^2 v] \quad T_0$$

Observe that now $\tilde{F}_1 : X_1 \quad X_1$ is an AS map and that $\tilde{F}_1$ is not an automorphism as S_1 still collapses. The results are as follows:

Theorem 24 *Let F be the map (5.3) and let $\tilde{F}$ be the induced map after blowing up the point A_0 If there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_1) = O_0$ and $\tilde{F}^q(A_2) = O_0$ for all $q \quad \mathbb{N}$ then the characteristic polynomial associated with F is given by*

$$\mathcal{Z}_p = x^{p+1}(x^2 - x - 1) + x^2$$

Proof. From above given calculations we observe that $\tilde{F}_1$ is an AS map now and we have the following orbits:

$$\mathcal{O}_0 = \quad A_0 = O_1 \qquad \mathcal{O}_1 = \quad A_1 \ \ \tilde{F}(A_1) \qquad \tilde{F}^p(A_1) = O_0$$

and we have one closed list as follows:

$$\mathcal{L}_c = \quad \mathcal{O}_0 = \quad A_0 = O_1 \qquad \mathcal{O}_1 = \quad A_1 \ \ \tilde{F}(A_1) \qquad \tilde{F}^p(A_1) = O_0$$

Then for $N_{L_c} = p+2$ we have $T_{L_c} = x^{p+2} - 1$ and $S_{L_c} = x^{p+1} + x + 2$ Utilizing Theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{Z}_p = x^{p+1}(x^2 - x - 1) + x^2$ ∎

We observe the characteristic polynomial of this theorem for the values of p $\mathbb{N}$ in the following proposition.

Proposition 25 *Consider the mappings which satisfy the hypothesis of Theorem 24. Then for* $p \geq 2$*:*

(i) The sequence of the degrees is bounded or grows linearly for $p = 2$*.*

(ii) The sequence of the degrees grows exponentially for $p > 2$*.*

For $p = \; 0 \; 1$ *there are no such mappings.*

Proof. We know that $\mathcal{Z}_p = x^{p+1}(x^2 - x - 1) + x^2$

For $p = 2$ we get the polynomial $\mathcal{X}_2 = x^5 - x^4 - x^3 + x^2 = x^2(x+1)(x-1)^2$ This polynomial has roots $x = 0 \; 0 \; -1 \; 1 \; 1$ This implies that the sequence of degrees

$$d_n = c_2\,(-1)^n + c_3 + c_4\,n$$

Then depending on $c_4 = 0$ (resp. $c_4 = 0$) d_n is bounded (resp. grows linearly in n).

For $p > 2$ we observe that $\mathcal{Z}_p(1) = 0$ $\mathcal{Z}\,_p(1) = 2 - q < 0$ and $\lim_{x \quad +} \mathcal{Z}_p(x) = +$ Hence $\mathcal{Z}_p$ always has a root $\lambda > 1$ and the result follows. This can be observed in figure 5.1.

Note that for $p = 0$ the above Theorem 24 gives us the condition $A_1 = O_0$ But we observe that $O_0[1] = 0$ and $A_1[1] = 0$ which implies that $A_1 = O_0$ Also for $p = 1$ we get the condition $\tilde{F}(A_1) = O_0$ which implies that $\tilde{F}(A_1) = [1 : \beta_2]_{E_0}$ As O_0 E_0 therefore $\tilde{F}(A_1)$ cannot be equal to O_0 Hence there are no families with $p = \; 0 \; 1$

∎

We now discuss the other possibility when there exist some q $\mathbb{N}$ such that $\tilde{F}^q(A_2) = O_0$ In this case $\tilde{F}^p(A_1) = O_0$ for all p $\mathbb{N}$

Theorem 26 *Let* F *be the map (5.3) and let* $\tilde{F}$ *be the induced map after blowing up the point* A_0 *If there exists some* q $\mathbb{N}$ *such that* $\tilde{F}^q(A_2) = O_0$ *and* $\tilde{F}^p(A_1) = O_0$ *for all* p $\mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Z}_q = x^{q+1}(x^2 - x - 1) + x^2 - 1$$

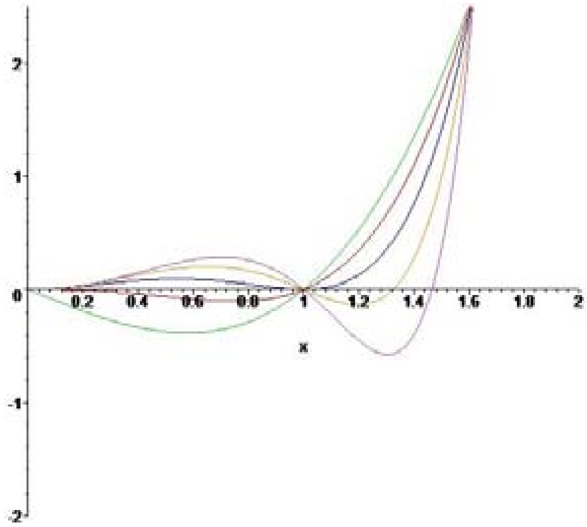

Figure 5.1: Behavior of $\mathcal{Z}_p$ for $p = 1, 2, 3, 4, 5.$

Proof. As we don t have any exceptional curves which can reach any indeterminacy point of $\tilde{F}_1$ Hence $\tilde{F}_1$ is an AS or regularized map now. We can now organize our orbits in to lists. We have the following orbits:

$$\mathcal{O}_0 = \quad A_0 = O_1$$

$$\mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^q(A_2) = O_0$$

and we have one open list as follows

$$\mathcal{L}_o = \quad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^q(A_2) = O_0 \qquad \mathcal{O}_0 = \quad A_0 = O_1$$

Then for $N_{L_o} = q + 2$ we have $T_{L_o} = x^{q+2}$ and $S_{L_o} = x^{q+1} + x + 1$ Utilizing Theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{Z}_q = x^{q+1}(x^2 - x - 1) + x^2 - 1$ ∎

Proposition 27 *Consider the mappings which satisfy the hypothesis of Theorem 26. Then for all* $q \quad \mathbb{N}$ *the sequence of degree has exponential growth rate.*

Proof. We observe that for all the values of $q \quad \mathbb{N}$ the polynomial $\mathcal{Z}_q = x^{q+1}(x^2 - x - 1) + x^2 - 1$

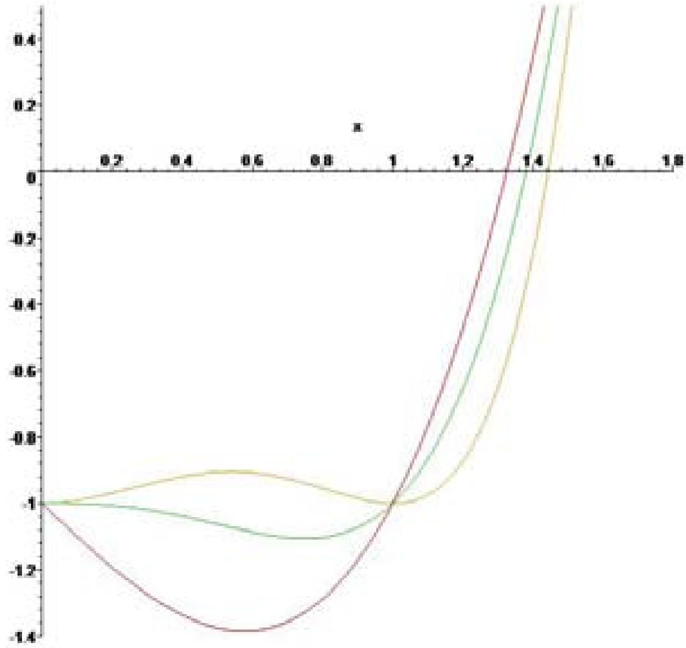

Figure 5.2: Behavior of Z_q for $q = 0\,, 1\,, 2.$

has always the largest root $\lambda > 1$ This is because $\mathcal{Z}_q(1) = -1 < 0$ and $\lim_{x \quad +} \quad \mathcal{Z}_q = +$ therefore there always exists a root $\lambda > 1$ such that $\mathcal{Z}_q(\lambda) = 0$

This can be observed in the figure 5.1. We thus conclude that all the families with associated characteristic polynomial $\mathcal{Z}_q$ have $D = \lambda > 1$ for all $q \quad \mathbb{N}$ therefore they have exponential growth rate. ∎

Theorem 28 *Let F be the map (5.3) and let $\tilde{F}$ be the induced map after blowing up the point O_1 If $\tilde{F}^p(A_1) = O_0$ and $\tilde{F}^q(A_2) = O_0$ for some $p \quad q \quad \mathbb{N}$ then $p = q$ and*

- *for $p > q$ the characteristic polynomial associated with F is $\mathcal{Z}_q$*
- *for $p < q$ the characteristic polynomial associated with F is $\mathcal{Z}_p$*

Proof. We know that the birational map $\tilde{F}$ is well defined and bijective near all the points except the points of the exceptional curves $S_1 \ S_2$ that collapse to the single points $A_1 \ A_2$ of $T_0 \ T_1$ This implies that if for some $k \quad \mathbb{N}$ any iterate of A_1 or A_2 belongs to any $T_0 \ T_1$ then it can have multiple preimages and not otherwise.

We claim that p must be different from q Assume that $\tilde{F}^k(A_1) = A_2$ and $\tilde{F}^j(A_2) = A_1$ for any $0 < k < p$ and $0 < j < q$ Because otherwise these points can have multiple preimages. Then $p = q$ gives the condition that $\tilde{F}^p(A_1) = \tilde{F}^p(A_2) = O_0$ implies that $A_1 = A_2$ which is a contradiction because we are in the non degenerate case and all A_i s must be different. Now if there exists some k or j such that $\tilde{F}^k(A_1) = A_2$ or $\tilde{F}^j(A_2) = A_1$ then either A_1 collides with the orbit of A_2 or vice versa. This gives that $k > 0$ which shows that $p = q$

Now consider that $q > p$ In this case the orbit of A_1 cannot collide with the orbit of A_2 as if this happens then the orbit of A_1 cannot reach O_0 But the orbit of A_2 collides with the orbit of A_1 Because if not then this claims that $\tilde{F}^k(A_2) = A_1$ for all $0 < k < q$ As $q > p$ then for some $j > 0$ we can write $q = p + j$ This gives $\tilde{F}^q(A_2) = \tilde{F}^{j+p}(A_2) = O_0 = \tilde{F}^p(A_1)$ As O_0 has unique preimage and there is no collision this implies that no orbit enters any T_1 or T_2 Therefore the points $\tilde{F}^{j+p}(A_2)$ and $\tilde{F}^p(A_1)$ have unique preimages. Then for $\tilde{F}^{j+p}(A_2) = O_0 = \tilde{F}^p(A_1)$ we can find the preimages by iterating p times with $\tilde{F}^{-1}$ This gives us $\tilde{F}^{-p}(\tilde{F}^{j+p}(A_2)) = \tilde{F}^{-p}(\tilde{F}^p(A_1))$ which implies that $\tilde{F}^j(A_2) = A_1$ for some $0 < j < q$ which gives contradiction to our claim. This implies that in the case when $q > p$ or $q < p$ we always have collision of orbits of A_2 with A_1 or A_1 with A_2

Now for $q > p$ we must have $\tilde{F}^k(A_2) = A_1$ for some $0 < j < q$ Then we have the following situation:

$$S_2 \twoheadrightarrow A_2 \quad \tilde{F}(A_2) \quad \cdots \quad \tilde{F}^{q-p}(A_2) = A_1 \quad \tilde{F}(A_1) \quad \cdots \quad \tilde{F}^p(A_1) = O_0$$

and

$$S_1 \twoheadrightarrow A_1 \quad \tilde{F}(A_1) \quad \cdots \quad \tilde{F}^p(A_1) = O_0$$

This implies that the orbit of A_2 is no more SE. Hence in this case we have two SE orbits that are the orbits of A_0 and A_1 This leads us to the case of two SE orbits discussed above hence the characteristic polynomial is Z_p in this case. Similarly, in the second case the characteristic polynomial is Z_q

Theorem 29 *Let F be the map (5.3) and let $\tilde{F}$ be the induced map after blowing up the point O_1 If for all p q $\mathbb{N}$ $\tilde{F}^p(A_1) = O_0$ and $\tilde{F}^q(A_2) = O_0$ then the characteristic polynomial associated with F is given by $Z = x^2 - x - 1$ and the sequence of degree grows exponentially.*

Proof. From hypothesis we know that for no p q $\mathbb{N}$ $\tilde{F}^p(A_1) = O_0$ or $\tilde{F}^q(A_2) = O_0$ Thus in this case we have one open list as follows:

$$\mathcal{L}_o = \quad \mathcal{O}_0 = \quad A_0 = O_1$$

Then for $N_{L_o} = 1$ we have $T_{L_o} = x$ and $S_{L_o} = 1$ By using the result from theorem 10 we find that in this case the dynamical degree of f is given by the polynomial $\mathcal{Z}(x) = (x-2)T_{L_o} + (x-1)S_{L_o} = x^2 - x - 1$ ∎

5.2 Mappings with $\alpha_2 = 0$

We know from the start of this chapter that the parameters $\alpha_1 \ \ \gamma_1 \ \ \gamma_2$ are all non zero. However the parameters $\alpha_0 \ \ \beta_0 \ \ \beta_1 \ \ \beta_2 \ \ \gamma_0$ can be zero. Now the birational map f has the following form

$$f(x \ \ y) = \left(\alpha_0 + \alpha_1 x \ \ \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right) \tag{5.4}$$

For $\xi_1 \ \ \xi_2 \ \ \lambda_1 \ \ \lambda_2 \quad \mathcal{C}$ let $\gamma : (x \ \ y) \quad (\xi_1 x + \lambda_1 \ \ \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$\begin{aligned}(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \quad & \left(\frac{\alpha_0 - \lambda_1 + \alpha_2\lambda_2}{\xi_1} \ \ \alpha_1 \ \ 0 \ \ (\beta_2\lambda_2 - \gamma_0\lambda_2 - \gamma_1\lambda_1\lambda_2 + \beta_0 + \beta_1\lambda_1)\right. \\ & \left.(\beta_1 - \gamma_1\lambda_2)\xi_1 \ \ \beta_2\xi_2 - \gamma_2\lambda_2 \ \ (\gamma_1\lambda_1 + \gamma_0)\xi_2 \ \ \gamma_1\xi_1\xi_2 \ \ \gamma_2\xi_2\right)\end{aligned}$$

By choosing

$$(\xi_1 \ \ \xi_2 \ \ \lambda_1 \ \ \lambda_2) = \left(\frac{\gamma_2}{\gamma_1} \ \ \frac{1}{\gamma_2} \ \ -\frac{\gamma_0}{\gamma_1} \ \ \frac{\beta_1}{\gamma_1}\right)$$

the new parameters are

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \quad (\tilde{\alpha_0} \ \ \alpha_1 \ \ 0 \ \ \tilde{\beta_0} \ \ 0 \ \ \tilde{\beta_2} \ \ 0 \ \ 1 \ \ 1)$$

Since f is non degenerate we have $(\beta\gamma)_{12} = -\beta_2\gamma_1 = 0$ this implies that the parameter $\beta_2 = 0$ For $\kappa_1 \ \ \kappa_2 \quad \mathbb{C}$ let $\quad : (x \ \ y) \quad (\kappa_1 x \ \ \kappa_2 y)$ be the scaling map. By the group action of for $\kappa_1 = \beta_2 = \kappa_2$ we get the following parameters

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \quad (\tilde{\tilde{\alpha_0}} \ \ \alpha_1 \ \ 0 \ \ \tilde{\tilde{\beta_0}} \ \ 0 \ \ 1 \ \ 0 \ \ 1 \ \ 1)$$

Renaming the coefficients we see that the map $f(x\ \ y)$ in (5.4) is conjugated to the map f of the following form:

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x\ \ \frac{\beta_0 + y}{x + y}\right) \qquad \alpha_1 = 0 \tag{5.5}$$

We consider the induced map in projective space as follows: $F : P\mathbb{C}^2 \qquad P\mathbb{C}^2$ given by

$$F[x_0 : x_1 : x_2] = [x_0(x_1 + x_2)\ : (\alpha_0 x_0 + \alpha_1 x_1)(x_1 + x_2)\ : x_0(\beta_0 x_0 + x_2)] \tag{5.6}$$

The indeterminacy sets of F and F^{-1} are the following: $\mathcal{I}(F) = \quad O_0\ \ O_1\ \ O_2$ where

$$O_0 = [1 : \beta_0 : -\beta_0] \qquad O_1 = [0 : 0 : 1] \qquad O_2 = [0 : 1 : -1]$$

and $\mathcal{I}(F^{-1}) = \quad A_0\ \ A_1\ \ A_2$ where

$$A_0 = [0 : 1 : 0] \qquad A_1 = [0 : 0 : 1] \qquad A_2 = [1 : (\alpha_0 + \alpha_1\beta_0) : 1]$$

Furthermore the exceptional curves of F and F^{-1} are as follows

$$S_0 = \quad x_0 = 0 \qquad S_1 = \quad x_1 + x_2 = 0 \qquad S_2 = \quad \beta_0 x_0 - x_1 = 0$$

$$T_0 = \quad (\alpha_0 + \alpha_1\beta_0)x_0 - x_1 = 0 \qquad T_1 = \quad x_0 - x_2 = 0 \qquad T_2 = \quad x_0 = 0$$

- **Orbit of** A_0

Observe that $S_0 \twoheadrightarrow A_0 = O_i$ for any $i \qquad 0\ \ 1\ \ 2$ also $A_0 \qquad S_0$ hence A_0 is a fixed point under F This implies that the orbit of A_0 is not SE.

- **Orbit of** A_1

Now $S_1 \twoheadrightarrow A_1 = O_1$ In order to get a defined trajectory of the points of S_1 we need to blow up $A_1 = O_1$ Let X be the new space we get after blowing up the point O_1 and let E_1 be the exceptional fibre at this point. We recognize the induced map after the blow up process as $\tilde{F} : X \qquad X$ We find that the map $\tilde{F}$ sends the curve S_1 to E_1 in the following way:

$$[x_0 : -x_2 : x_2] \quad S_1 \qquad [x_0 : \alpha_0 x_0 - \alpha_1 x_2]_{E_1} \qquad \tilde{F}[u : v]_{E_1} \qquad [u : \alpha_0 u + \alpha_1 v : u] \quad T_1$$

Now the indeterminacy set is $\mathcal{I}(\tilde{F}) = \quad O_0\ \ O_2$ and the exceptional set is $\mathcal{E}(\tilde{F}) = \quad S_0\ \ S_2$

- **Orbit of** A_2

The exceptional curve $S_2 \twoheadrightarrow A_2$ Following the orbit of A_2 we need to know if any of it s iterates reach any indeterminacy point of $\tilde{F}$ Now $\mathcal{I}(\tilde{F}) = \ \ O_0 \ O_2$ and $O_2 \quad S_0 = T_2$ We know that the only points on T_2 which have preimages through F are A_0 and A_1 which implies that if the orbit of A_2 reaches O_2 at some iterate of F then O_2 should be equal to either A_0 or A_1 But $A_0 = O_2 = A_1$ hence we get that $F^p(A_2) = O_2$ for all $p \quad \mathbb{N}$ Furthermore since $O_2 \quad T_1$ we infer that also $\tilde{F}^p(A_2) = O_2$ for all $p \quad \mathbb{N}$
Then in general there is a possibility that for some $p \quad \mathbb{N}$ we have $\tilde{F}^p(A_2) = O_0$ In this case the orbit of A_2 is SE.

If no such p exists, then $\tilde{F} : X \qquad X$ is an AS mapping. Notice that when we apply $\tilde{F}$ S_2 still collapses to A_2

If there exists $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ then we have to blow-up all the points A_2 $\tilde{F}(A_2)$, $\tilde{F}^2(A_2) \qquad \tilde{F}^p(A_2) = O_0$ Let E_i $i = 2 \ 3 \qquad p+2$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1 \qquad X_1$ be the induced map. Then under the action of $\tilde{F}_1$ we have that:

$$E_2 \qquad E_3 \qquad \cdots \qquad E_{p+2}$$

This sequence can be completed in the following manner:

$$S_2 \qquad E_2 \qquad E_3 \qquad \cdots \qquad E_{p+2} \qquad T_0$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_2 and on E_{p+2} is:

$$\tilde{F}_1[x_0 : -\beta_0 x_0 : x_2] = [-\alpha_1(\beta_0 x_0 + x_2 : x_0]_{E_2}$$

and

$$\tilde{F}_1[u : v]_{E_{p+2}} = [(u+v) : (\alpha_0 + \alpha_1\beta_0)(u+v) : v] \quad T_0$$

Observe that now $\tilde{F}_1 : X_1 \qquad X_1$ is an AS map. Since we have not created any new points of indeterminacy of $\tilde{F}_1$ we get that $\mathcal{I}(\tilde{F}) = \ \ O_2$ and the exceptional set is $\mathcal{E}(\tilde{F}) = \ \ S_0$

Note that $\tilde{F}_1$ is not an automorphism as S_0 still collapses. The result is as follows:

Theorem 30 *Let F be the map 5.6 and let $\tilde{F}$ be the induced map after blowing up the point A_1 If there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ then the characteristic polynomial associated* *with F is given by*

$$\mathcal{Z}_p = (x^{p+1} + 1)(x - 1)^2$$

If no such p exists then the characteristic polynomial associated with F is given by $\mathcal{Z} = (x - 1)^2$

Proof. We see that now there is no exceptional curves which can reach any indeterminacy point of $\tilde{F}_1$ Hence $\tilde{F}_1$ is an AS or regularized map now. We can now organize our orbits in to lists. We have the following orbits:

$$\mathcal{O}_1 = \quad A_1 = O_1 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

and we have one closed list and one open list as follows

$$\mathcal{L}_c = \quad \mathcal{O}_1 = \quad A_1 = O_1 \qquad \mathcal{L}_o = \quad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

Then for $N_{L_c} = 1 \;\; N_{L_o} = p + 1$ we have $T_{L_c} = 1 \;\; T_{L_o} = x^{p+1}$ and $S_{L_c} = 1 \;\; S_{L_o} = 1$ Utilizing theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{Z}_p = (x^{p+1} + 1)(x - 1)^2$

Now suppose that no p exists such that $\tilde{F}^p(A_2) = O_0$ In this case we have one open list as follows:

$$\mathcal{L}_c = \quad \mathcal{O}_1 = \quad A_1 = O_1$$

Then for $N_{L_c} = 1$ we have $T_{L_c} = x - 1$ and $S_{L_c} = 1$ By using the result from theorem [9] we find that in this case the dynamical degree of f is given by the polynomial $\mathcal{Z}(x) = (x - 2)T_{L_o} + (x - 1)S_{L_o} = (x - 1)^2$. ∎

Proposition 31 *Consider the mappings which satisfy the hypothesis of Theorem 30. Then for all $p \quad \mathbb{N}$ the sequence of degrees d_n grows linearly.*

Proof. We observe that for all the values of $p \quad \mathbb{N}$ the polynomial $\mathcal{Z}_p = (x^{p+1} + 1)(x - 1)^2$ The polynomial $\mathcal{Z}_p$ has roots for the equations $(x - 1)^2 = 0 \;\; (x^{p+1} + 1) = 0$ Then sequence of degrees is

$$d_n = c_0 + c_1\, n + c_2\, \lambda_1^n + c_3\, \lambda_2^n + \quad + c_{p+2}\, \lambda_{p+1}^n$$

where $\lambda_1 \;\; \lambda_2 \qquad \lambda_{p+1}$ are the roots of polynomial $(x^{p+1} + 1)$ i.e. they all are the roots of unity. Now we claim that $c_1 = 0$ From (5.5) we can write $f(x \;\; y) = (\alpha_0 + \alpha_1\, x \;\; \frac{N_1}{D_1})$ where $N_1 = \beta_0 + y$

and $D_1 = x + y$ Then

$$f^2(x\ \ y) = \left(\alpha_0\,(1+\alpha_1) + \alpha_1^2\,x\ \ \ \frac{\beta_0 + \frac{N_1}{D_1}}{\alpha_0 + \alpha_1\,x + \frac{N_1}{D_1}}\right)$$

which implies that $N_2 = \beta_0\,D_1 + N_1\ \ D_2 = (\alpha_0 + \alpha_1\,x)D_1 + N_1$ We find that the first component of some iterate f^{i+1} can be written as $\alpha_0\,(1 + \alpha_1 + \cdots + \alpha_1^{i+1}) + \alpha_1^{i+2}\,x$ and the numerator and denominator of the second component of f^{i+1} can be written as $N_{i+1} = \beta_0\,D_1 + N_i\ \ D_{i+1} = (\alpha_0 + \alpha_1\,x)D_i + N_i$ By looking at the degrees $(\eta\ \ \kappa)$ of numerator and denominator of the second component, we find that they follow the sequence $(1\ \ 1)\quad (1\ \ 2)\quad (2\ \ 3)\quad (3\ \ 4)\quad \cdots$ This implies that for some iterate $2\,p+2$ the degrees $(\eta_{2\,p+2}\ \ \kappa_{2\,p+2}) = (2\,p+1\ \ 2\,p+2)$ Now without simplifications we consider the homogenized numerator and denominator as N^h and D^h Then we can write

$$F^{2\,p+2} = [D^h_{2\,p+2}\,x_0 : (\alpha_0\sum_{i=0}^{2\,p+1}\alpha_1^i\,x_0 + \alpha_1^{2\,p+2}\,x_1)\,D^h_{2\,p+2} : N^h_{2\,p+2}\,x_0^2]$$

Now if $c_1 = 0$ this implies that d_n is periodic of period $2\,p+2$ Then $d_{2\,p+2} = d_0 = 1$ that is, $F^{2\,p+2}$ is of degree 1 i.e. is linear. This implies that inspite of all the other possible cancellations, $D^h_{2\,p+2}$ must have a factor x_0 that can be dropped as a common factor from all the three components of $F^{2\,p+2}$ But $D_{2\,p+2}(x\ \ y)$ has degree $2\,p+2$ which implies that if $D^h_{2\,p+2}$ has a factor x_0 then after dropping this factor $D_{2\,p+2}(x\ \ y)$ has degree $2\,p+1$ This gives contradiction. Hence $c_1 = 0$ and sequence of degrees grows linearly in n ∎

In the coming section we are going to discuss the families with dynamical degree one. We separately indicate the families of mappings that have appeared in previous sections of this chapter with zero entropy and discuss their behavior.

5.3 Zero entropy

Theorem 32 *If a map of family 5.2 has zero entropy then it can be written as:*

$$f(x\ \ y) = \left(\alpha_2 y\ \ \ \frac{\beta_2 y}{-\alpha_2\beta_2 + x + y}\right)\quad \alpha_2 = 0 = \beta_2$$

Moreover the corresponding sequence of degrees grows linearly and f preserves the fibration

$$V(x\ y) = \frac{(\beta_2 - y)(\alpha_2\beta_2 - x)}{y}$$

with $V(f(x\ y)) = -\alpha_2 V(x\ y)$ If $\alpha_2^n = (-1)^n$ then f^n is integrable. In particular for $\alpha_2 = -1$ f is integrable.

Proof. From the results discussed in Propositions 25 and 27 we see that there is only one mapping in Family 3 1 which has zero entropy. This mapping satisfies the hypothesis of Theorem 24 for $p = 2$ We find that $d_2 = 3$ $d_3 = 5$ $d_4 = 7$ therefore we can write that $d_n = -1 + 2n$ $n > 1$ which implies that d_n grows linearly in n

We now study the condition in this case that is $\tilde{F}^2(A_1) = O_0$ The condition $\tilde{F}^2(A_1) = O_0$ implies that

$$[1 : \alpha_2\beta_2 : 0] = \left[1 : \frac{(\beta_0 - \gamma_0\beta_2)}{\beta_2} : -\frac{\beta_0}{\beta_2}\right]$$

This gives us that $\beta_0 = 0$ which further implies that $\gamma_0 = -\alpha_2\beta_2$ which shows that for $p = 2$ the condition $\tilde{F}^2(A_1) = O_0$ is satisfied. This gives us our required mapping f

Now we first look for an invariant curve. We find that the straight line $L_1 :\ \ y = 0$ because $f(x\ 0) = (0\ 0)\quad L_1$ Also we observe that the line

$$L_2 :\ y = c \qquad L_3 =\ \ x = \alpha_2 c \quad \text{and } L_4 :\ x = k \qquad L_5 = \left\{ y = \frac{\beta_2 x}{-\alpha_2^2\beta_2 + k\,\alpha_2 + x} \right\}$$

Then by taking $k = \alpha_2\,\beta_2$ $c = \beta_2$ we get that $y = \beta_2$ $\qquad x = \alpha_2\,\beta_2$ $\qquad y = \beta_2$ Then we consider

$$V(x\ y) = \frac{(\beta_2 - y)(\alpha_2\beta_2 - x)}{y}$$

which satisfies $V(f(x\ y)) = -\alpha_2 V(x\ y)$ Note that for $\alpha_2 = -1$ we have $V(f(x\ y)) = V(x\ y)$ which shows that the map f is integrable. Since $V(f(x\ y)) = -\alpha_2 V(x\ y)$ for all $n \quad \mathbb{N}$ then $V(f^n(x\ y)) = (-1)^n\alpha_2^n V(x\ y)$ and when n is even, if $\alpha_2^n = 1$ for some $n \quad \mathbb{N}$ then f^n is integrable. Also for n odd, if $\alpha_2^n = -1$ for some $n \quad \mathbb{N}$ then f^n is integrable.

∎

Theorem 33 *All the mappings f such that*

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x\ \ \frac{\beta_0 + y}{x + y}\right)\ \ \alpha_1 = 0$$

have zero entropy and the sequence of degrees grows linearly. They all preserve the fibration

$$V(x\ \ y) = x$$

with $V(f(x\ \ y)) = \alpha_0 + \alpha_1 V(x\ \ y)$ For $\alpha_0 = 0$ and $\alpha_1 = 1$ these mappings f are integrable.

Proof. From the result discussed in proposition (31) we see that all the mappings in equation (5.5) satisfy the hypothesis of Theorem 30 and they all have zero entropy as the sequence of degrees grow linearly.
Observe that these mappings f in equation 5 5 have first component $\alpha_0 + \alpha_1 x$ This gives us the scaled translation in x This implies that these mappings preserve a fibration $V(x\ \ y) = x$ with $V(f(x\ \ y)) = \alpha_0 + \alpha_1 V(x\ \ y)$ Where for $\alpha_0 = 0$ and $\alpha_1 = 1$ we have $V(f(x\ \ y)) = V(x\ \ y)$ gives the integrable mappings f

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas - Godfrey Harold Hardy
The measure of our intellectual capacity is the capacity to feel less and less satisfied with our answers to better and better problems - C.W. Churchmann

Chapter 6

Non Degenerate Case $\gamma_1 = 0$

This family of mappings is the most largest and dynamically rich families we have studied so far in this work. It first divides into two different subfamilies for the zero and nonzero values of α_2 and from there it spreads vastly into infinite number of subfamilies existing inside this small family of maps. This happens because of the collisions of orbits happening inside the maps with $\alpha_2 = 0$ This opens the door to a large number of families which were not clearly visible at first without the consideration of collisions of orbits. The more interesting and beautiful thing to observe in this Chapter is that these families have shown a variety of different dynamical behavior and it covers completely the dynamics of the families discussed in [DF01]. Also there are found families with quadratic growth rate which preserve elliptic fibrations. As far as this is the most largest family of maps we have found so far, on the same hand it includes several families with the coefficients which create difficulty once you start iterating the family in order to get information about its growth and dynamics. Although here the dynamical degree has well paid to provide us the information about these families, however it yet remains to look for elliptic fibrations for few families because of unreachable tedious computer calculations.

In this chapter we again use Theorem 10 to calculate the dynamical degree. To find the SE orbits and organize them into lists we use the methodologies discussed in Chapter 2. We then find the associated characteristic polynomial of F which gives us the sequence of degrees of F and the dynamical degree of F

In the last section we study the zero entropy families. These happen to have the sequence of degrees that is bounded, periodic, grows quadratically. The maps with bounded growth are found to preserve two fibrations whereas for the maps in the second set we illustrate exactly what is the period of the mappings and hence we show that the mappings indeed are periodic. In this there appears a family of mappings whose sequence of degrees is periodic but the mappings appears to be linear for that period. They include infinite families as they satisfy a certain condition on $k \quad \mathbb{N}$ as discussed in this chapter. In the quadratic growth mappings appear some families with elliptic fibration and invariant elliptic curve which assures the results of [DF01] stated in Chapter 4. However there are few families whose calculation is too large and tedious to compute. Moreover they are not found to preserve the invariant cubics of the types that are discussed in the proof of Proposition 46. The family of mapping we consider to study in this chapter also includes in the non degenerate case as all the exceptional curves of F are distinct as discussed in the following work, it is as follows:

We consider the birational mapping $f : \mathbb{C}^2 \quad \mathbb{C}^2$ of the form

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y \quad \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_2 y}\right) \tag{6.1}$$

for the complex numbers $\alpha_i\ \ \beta_i\ \ i = 0\ \ 1\ \ 2$ and $\gamma_i\ \ i = 0\ \ 2$ From lemma 8 we know that h is *non degenerate* i.e. all S_i are distinct if and only if $(\beta\gamma)_{12} = 0 = (\alpha\gamma)_{12}$ For $\gamma_1 = 0$ this implies that $\beta_1\gamma_2 = 0 = \alpha_1\gamma_2$ Therefore $\alpha_1\ \ \beta_1\ \ \gamma_2$ are all non zero. However the parameters $\alpha_0\ \ \alpha_2\ \ \beta_0\ \ \beta_2\ \ \gamma_0$ can be zero.

We consider the imbedding $(x\ \ y) \quad [1 : x : y] \quad P\mathbb{C}^2$ into projective space and consider the induced map $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ given by

$$F[x_0 : x_1 : x_2] = [x_0(\gamma_0 x_0 + \gamma_2 x_2) : (\boldsymbol{\alpha} \cdot \boldsymbol{x})(\gamma_0 x_0 + \gamma_2 x_2) : x_0(\boldsymbol{\beta} \cdot \boldsymbol{x})]$$

where $\boldsymbol{\alpha} \cdot \boldsymbol{x} = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$ The indeterminacy locus of F is $\mathcal{I}(F) = \quad O_0\ \ O_1\ \ O_2 \quad$ where

$$O_0 = [\beta_1\gamma_2 : -(\beta\gamma)_{02} : -\beta_1\gamma_0] \qquad O_1 = [0 : \alpha_2 : -\alpha_1] \qquad O_2 = [0 : 1 : 0]$$

and the indeterminacy locus of F^{-1} is $\mathcal{I}(F^{-1}) = \quad A_0\ \ A_1\ \ A_2 \quad$ where

$$\begin{aligned} A_0 &= [0 : 1 : 0] \qquad A_1 = [0 : 0 : 1] \\ A_2 &= [\alpha_1\beta_1\gamma_2 : (\alpha_0\beta_1\gamma_2 - \alpha_1(\beta\gamma)_{02} - \gamma_0\alpha_1\beta_1)\alpha_1 : \beta_1(\alpha\beta)_{12}] \end{aligned}$$

and $(\alpha\beta)_{12} = \alpha_1\beta_2 - \alpha_2\beta_1$.

We see that $A_0 = O_2$ Also $\alpha_1 = 0$ this implies that for $\alpha_2 = 0$ we get $A_1 = O_1$ For $\alpha_2 = 0$ we have $A_1 = O_1$ We therefore separately study these two cases.

6.1 Mappings with $\alpha_2 = 0$

For $\lambda_1\ \ \lambda_2 \quad \mathbb{C}$ let $\gamma : (x\ \ y) \quad (x + \lambda_1\ \ y + \lambda_2)$ be the linear translation map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad \Big(\alpha_0 + (\alpha_1 - 1)\lambda_1\ \ \alpha_1\ \ 0\ \ -\gamma_2\lambda_2^2 + (\beta_2 - \gamma_0)\lambda_2 + \beta_0 + \beta_1\lambda_1$$
$$\beta_1\ \ -\gamma_2\lambda_2 + \beta_2\ \ \gamma_2\lambda_2 + \gamma_0\ \ 0\ \ \gamma_2\Big)$$

By different choices of λ_1 and λ_2 we can make some parameters zero in the following way. For $\lambda_2 = -\gamma_0\ \ \gamma_2$ and $\lambda_1 = (\beta\gamma)_{20}\ \ \beta_1\gamma_2$ we get $\gamma_2\lambda_2 + \gamma_0 = 0$ and $-\gamma_2^2\lambda_2^2 + (\beta_2 - \gamma_0)\lambda_2 + \beta_0 + \beta_1\lambda_1 = 0$ The new parameters are

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad (\bar{\alpha_0}\ \ \alpha_1\ \ 0\ \ 0\ \ \bar{\beta_1}\ \ \bar{\beta_2}\ \ 0\ \ 0\ \ \bar{\gamma_2})$$

For $\kappa_1\ \ \kappa_2 \quad \mathbb{C}$ let $\quad : (x\ \ y) \quad (\kappa_1 x\ \ \kappa_2 y)$ be the scaling map. By the group action of we have

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad (\bar{\alpha_0}\ \ \alpha_1\kappa_1\ \ 0\ \ 0\ \ \bar{\beta_1}\kappa_1\ \ \bar{\beta_2}\kappa_2\ \ 0\ \ 0\ \ \bar{\gamma_2}\kappa_2^2)$$

For the choice of $\kappa_1 = 1\ \ \beta_1$ and $\kappa_2^2 = 1\ \ \gamma_2$ we get the following parameters

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) \qquad (\bar{\bar{\alpha_0}}\ \ \alpha_1\ \ 0\ \ 0\ \ 1\ \ \bar{\bar{\beta_2}}\ \ 0\ \ 0\ \ 1)$$

Hence by renaming the parameters our map f in (6.1) can be written as map f and has the following form

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x\ \ \frac{x + \beta_2 y}{y}\right) = \left(\alpha_0 + \alpha_1 x\ \ \beta + \frac{x}{y}\right) \text{ with } \alpha_1 = 0 \tag{6.2}$$

The second component of f has only one parameter namely β_2 so we recognize it as β We consider the induced map in projective space as follows: $F : P\mathbb{C}^2 \quad P\mathbb{C}^2$ given by

$$F[x_0 : x_1 : x_2] = [x_0\, x_2\ : (\alpha_0 x_0 + \alpha_1 x_1)x_2\ : x_0(x_1 + \beta x_2)] \tag{6.3}$$

The indeterminacy sets of F and F^{-1} are $\mathcal{I}(F) = \quad O_0\ \ O_1\ \ O_2 \quad$ where

$$O_0 = [1 : 0 : 0] \qquad O_1 = [0 : 0 : 1] \qquad O_2 = [0 : 1 : 0]$$

and $\mathcal{I}(F^{-1}) = \quad A_0 \;\; A_1 \;\; A_2$ where

$$A_0 = [0:1:0] \qquad A_1 = [0:0:1] \qquad A_2 = [1:\alpha_0:\beta]$$

Furthermore the exceptional curves of F and F^{-1} are as following:

$$S_0 = \quad x_0 = 0 \qquad S_1 = \quad x_2 = 0 \qquad S_2 = \quad x_1 = 0$$

$$T_0 = \quad \alpha_0 x_0 - x_1 = 0 \qquad T_1 = \quad \beta x_0 - x_2 = 0 \qquad T_2 = \quad x_0 = 0$$

- **Orbits of A_0 and A_1**

We observe that $S_0 \twoheadrightarrow A_0 = O_2$ and $S_1 \twoheadrightarrow A_1 = O_1$ In order to get a defined trajectory of the points of $S_0 \;\; S_1$ we need to blow up $A_0 = O_2$ and $A_1 = O_1$ Let X be the new space we get after blowing up the points $A_0 \;\; A_1$ and let $E_0 \;\; E_1$ be the exceptional fibre at these points. We recognize the induced map after the blow up process as $\tilde{F} : X \quad X$ For the points $x = [0:x_1:x_2] \quad S_0$ we find that the map $\tilde{F}$ sends the curve S_0 to E_0 as follows:

$$\tilde{F} : [0:x_1:x_2] \qquad [x_2 : x_1 + \beta x_2]_{E_0}$$

Moreover $\tilde{F}$ sends all the points $x = [tu:1:tv] \quad E_0$ as $t \quad 0$ back to S_0 in a way such that

$$\tilde{F} : [u:v]_{E_0} \qquad [0:\alpha_1 v:u] \quad T_2 = S_0$$

In a similar way $\tilde{F}$ sends S_1 to E_1 and E_1 to T_1 as follows:

$$\tilde{F} : [x_0:x_1:0] \qquad [x_0 : \alpha_0 x_0 + \alpha_1 x_1]_{E_1} \qquad \tilde{F} : [u:v]_{E_1} \qquad [u : \alpha_0 u + \alpha_1 v : u\beta] \quad T_1$$

As $\alpha_1 = 0$ hence we have not created any new point of indeterminacy therefore $\mathcal{I}(\tilde{F}) = \quad O_0$ and $\mathcal{E}(\tilde{F}) = \quad S_2$

- **Orbit of A_2**

After the above observations we see that the only way that $\mathcal{O}_2$ is an SE orbit is that $\tilde{F}^p(A_2) = O_0$ for some $p \quad \mathbb{N}$

If no such p exists, then $\tilde{F} : X \quad X$ is an AS mapping. Notice that when we apply $\tilde{F} \;\; S_2$ still collapses to A_2

If it exists p $\mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ then we have to blow-up also all the points A_2 $\tilde{F}(A_2)$ $\tilde{F}^2(A_2)$ $\tilde{F}^p(A_2) = O_0$ Let E_i $i = 2$ 3 $p+2$ be the exceptional fibres we get after the blow up process. Let X_1 be the expanded space and let $\tilde{F}_1 : X_1$ X_1 be the induced map. Then under the action of $\tilde{F}_1$ we have that:

$$E_2 \quad E_3 \quad \cdots \quad E_{p+2}$$

This sequence can be completed in the following manner:

$$S_2 \quad E_2 \quad E_3 \quad \cdots \quad E_{p+2} \quad T_0$$

where after some calculations we find that the action of $\tilde{F}_1$ on S_2 and on E_{p+2} is:

$$\tilde{F}_1[x_0 : 0 : x_2] = [a_1\, x_1 : x_0]_{E_2} \tag{6.4}$$

and

$$\tilde{F}_1[u : v]_{E_{p+2}} = [v : a_0\, v : u + \beta\, v] \quad T_0 \tag{6.5}$$

Observe that now $\tilde{F}_1 : X_1$ X_1 is an AS map. Furthermore:

$$\begin{array}{lllllll} S_0 & E_0 & S_0 = T_2 & & & & \\ S_1 & E_1 & T_1 & & & & \\ S_2 & E_2 & E_3 & \cdots & E_{p+1} & E_{p+2} & T_0 \end{array}$$

Hence all the S_i which were collapsing to a single point i.e. they were exceptional, are no more exceptional because under the action of $\tilde{F}_1$ all the points on S_i now have their images defined on some other new curve i.e. the blown up fibres. Moreover previously the only points on T_i with pre-images were A_j A_k with $j = i = k$ But now under the action of $\tilde{F}_1$ all the points on the curves T_i have pre-images i.e. we have found the curves which map to the whole curve T_i and do not collapse only to a single point of T_i We can therefore say that $\tilde{F}_1$ is an automorphism.

The result is the following:

Theorem 34 *Let F be the mapping (6.3) and let $\tilde{F}$ be the extension of F after blowing-up the points A_0 A_1 Assume that there exists some p $\mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ Then the characteristic* *polynomial associated with F is given by*

$$\mathcal{X}_p = (x^{p+1} + 1)(x-1)^2(x+1)$$

If no such p exists then the characteristic polynomial associated with F is

$$\mathcal{X} \; = \; (x-1)^2\,(x+1)$$

Proof. Assume that there exists $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ and consider the extension $\tilde{F}_1$ which satisfies (6.4) and (6.5) and which is an AS map. We can now organize our orbits in to lists. We have the following orbits:

$$\begin{aligned} \mathcal{O}_0 \quad &= \quad A_0 = O_2 \qquad\quad \mathcal{O}_1 = \;\; A_1 = O_1 \\ \mathcal{O}_2 \quad &= \quad A_2 \;\; \tilde{F}(A_2) \qquad\quad \tilde{F}^p(A_2) = O_0 \end{aligned}$$

and we have two closed lists as follows

$$\begin{aligned} \mathcal{L}_{c_1} \quad &= \quad \mathcal{O}_0 = \;\; A_0 = O_2 \qquad \mathcal{O}_2 = \;\; A_2 \;\; \tilde{F}(A_2) \qquad\quad \tilde{F}^p(A_2) = O_0 \\ \mathcal{L}_{c_2} \quad &= \quad \mathcal{O}_1 = \;\; A_1 = O_1 \end{aligned}$$

Then for $N_{L_{c_1}} = p+2 \;\; N_{L_{c_2}} = 1$ we have $T_{L_{c_1}} = (x^{p+2} - 1) \;\; T_{L_{c_2}} = x - 1$ and $S_{L_{c_1}} = x^{p+1} + x + 2 \;\; S_{L_{c_2}} = 1$ Utilizing theorem 10 of Bedford by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{X} = (x^{p+1} + 1)(x-1)^2(x+1)$ If $\tilde{F}^p(A_2) = O_0$ for all $p \quad \mathbb{N}$ then we have two lists which are open and closed as follows:

$$\mathcal{L}_o = \;\; \mathcal{O}_0 = \;\; A_0 = O_2 \qquad\qquad \mathcal{L}_c = \;\; \mathcal{O}_1 = \;\; A_1 = O_1$$

Then for $N_{L_o} = 1 \;\; N_{L_c} = 1$ we have $T_{L_o} = x \;\; T_{L_c} = x - 1$ and $S_{L_o} = 1 \;\; S_{L_c} = 1$ By using the result from theorem we find that in this case the dynamical degree of f is given by the polynomial $X(x) = (x-2)T_{L_o}T_{L_c} + (x-1)(S_{L_o}T_{L_c} + S_{L_c}T_{L_o}) = (x-1)^2(x+1)$. ∎

Proposition 35 *Consider the mappings which satisfy the hypothesis of Theorem 34. Then the sequence of degrees d_n of F is periodic with period $2p+2$*

If no such p exists then the sequence of degrees d_n grows linearly.

Proof. In the first case we know that the characteristic polynomial associated to F is

$$\mathcal{X} = (x^{p+1} + 1)(x - 1)^2(x + 1)$$

If p is even then $x^{p+1} + 1$ has the factor $x + 1$ and $\mathcal{X} = (x - 1)^2\,(x + 1)^2\,(x^p - x^{p-1} + \cdots - x + 1)$ Hence the sequence of degrees is $d_n = c_0 + c_1\,n + c_2\,(-1)^n + c_3\,n\,(-1)^n + c_4\,\lambda_1^n + c_5\,\lambda_2^n + \cdots + c_{p+3}\,\lambda_p^n$ where c_i are constants and $\lambda_1 \quad \lambda_2 \qquad \lambda_p$ are the roots of polynomial $x^p - x^{p-1} + \cdots - x + 1$ We see that $\lambda_1 \quad \lambda_2 \qquad \lambda_p$ are roots of unity. By looking at d_n we observe that in our family we do not have quadratic or exponential growth. Also we cannot have linear growth as our map $\tilde{F}_1$ in this case is an automorphism. Hence the sequence of degrees must be periodic. then we have $c_1 = c_3 = 0$ This implies that $d_{2p+2+n} = d_n$ i.e. the sequence of degrees is periodic with period $2p + 2$

In a similar manner, if p is odd then $d_n = c_0 + c_1\,n + c_2\,(-1)^n + c_3\lambda_1^n + c_4\,\lambda_2^n + \quad + c_{p+3}\,\lambda_{p+1}^n$ where c_i are constants and $\lambda_1 \quad \lambda_2 \qquad \lambda_{p+1}$ are the roots of the polynomial $x^{p+1} + 1$ Hence c_1 must be zero and d_n is periodic with period $2p + 2$

If $\tilde{F}^p(A_2) = O_0$ for all p then we know that the characteristic polynomial associated to F is $(x - 1)^2\,(x + 1)$ By finding the sequence of degrees of $F^2 \quad F^3$ we see that $d_2 = 2$ and $d_3 = 3$ which implies that $d_n = \frac{5}{4} + \frac{1}{2}\,n - \frac{1}{4}\,(-1)^n$ hence d_n grows linearly.

It is clear that the condition $\tilde{F}^p(A_2) = O_0$ is satisfied when $F^j(A_2) \quad S_0 \quad S_1$ for all $j = 0 \ 1 \qquad p - 1$ and $F^p(A_2) = O_0$ But there is another possibility, with collision of orbits. Assume that it exists some $l \quad \mathbb{N}$ with $F^l(A_2) \quad S_1 \quad O_0$ Then $\tilde{F}^2(F^l(A_2)) \quad T_1$ and perhaps iterating A_2 under $\tilde{F}$ we can meet the point O_0 There is a very special case in this setting, which we want to remark: this is the case $l = 0$ that is, when $A_2 \quad S_1 \quad O_0$ It happens when the parameter $\beta = 0$ And we suppose that $\alpha_0 = 0$ if not $A_2 = O_0$ which corresponds to the situation $F^p(A_2) = O_0$ with $p = 0$ Notice that in this case $T_1 = S_1$

The orbit of A_2 under the action of $\tilde{F}$ is as follows:

$$A_2 = [1 : \alpha_0 : 0] \qquad [1 : \alpha_0(1 + \alpha_1)]_{E_1} \qquad [1 : \alpha_0(1 + \alpha_1 + \alpha_1^2) : 0] \quad S_1 \qquad \cdots$$

After some iterates we can write the expression of $\tilde{F}^{2q}(A_2)$ for all $q > 0$:

$$\tilde{F}^{2q}(A_2) = [1 : \alpha_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{2q}) : 0] \quad S_1$$

Since $O_0 \quad S_1$ it is possible that $\tilde{F}^{2q}(A_2) = O_0$ This happens when the following condition is satisfied for any q We recognize it as *condition* q

$$1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{2q} = 0$$

The result is as follows:

Corollary 36 *Let F be the mapping (6.3) and assume that $\beta = 0 \quad \alpha_0 = 0$ If condition q is satisfied, then the characteristic polynomial associated with F is given by*

$$\mathcal{X}_q = (x^{2q+1} + 1)(x^2 - 1)(x - 1)$$

and the sequence of degrees is periodic with period $4q + 2$

6.2 Mappings with $\alpha_2 = 0$

From the start of this chapter we know that the parameters $\alpha_1 \ \beta_1 \ \gamma_2$ are all non zero and the parameters $\alpha_0 \ \alpha_2 \ \beta_0 \ \beta_2 \ \gamma_0$ can be zero but in this case we have $\alpha_2 = 0$ For $\xi_1 \ \xi_2 \ \lambda_1 \ \lambda_2$ $\mathbb{C}$ let $\gamma : (x \ y) \quad (\xi_1 x + \lambda_1 \ \xi_2 y + \lambda_2)$ be the linear translated scaling map. We consider the conjugation $\gamma^{-1} \circ f \circ \gamma$ in affine coordinates. By the natural group action on parameter space under the action of γ we have

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \quad \Bigg(\frac{(\alpha_0 - \lambda_1 + \alpha_1\lambda_1 + \alpha_2\lambda_2)}{\xi_1} \quad \alpha_1 \quad \frac{\alpha_2\xi_2}{\xi_1} \quad (\beta_2\lambda_2 - \gamma_0\lambda_2 + \beta_0 + \beta_1\lambda_1 + \gamma_2\lambda_2^2)$$
$$(\beta_1)\xi_1 \quad (\beta_2 - \gamma_2\lambda_2)\xi_2 \quad (\gamma_2\lambda_2 + \gamma_0)\xi_2 \quad 0 \quad \gamma_2\xi_2^2\Bigg)$$

By choosing

$$(\xi_1 \quad \xi_2 \quad \lambda_1 \quad \lambda_2) = \left(\frac{1}{\beta_1} \quad \frac{1}{\beta_1\alpha_2} \quad \frac{(\beta\gamma)_{02}}{\beta_1\gamma_2} \quad \frac{\beta_2}{\gamma_2}\right)$$

the new parameters are

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \quad (\bar{\alpha}_0 \quad \alpha_1 \quad 1 \quad 0 \quad 1 \quad 0 \quad \bar{\gamma}_0 \quad 0 \quad \bar{\gamma}_2)$$

For $\kappa_1 \ \kappa_2$ $\mathbb{C}$ let $: (x \ y) \quad (\kappa_1 x \ \kappa_2 y)$ be the scaling map. By the group action of for $\kappa_1 = \frac{1}{\gamma_2} = \kappa_2$ we get the following parameters

$$(\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) \quad (\bar{\bar{\alpha}}_0 \quad \alpha_1 \quad 1 \quad 0 \quad 1 \quad 0 \quad \bar{\bar{\gamma}}_0 \quad 0 \quad 1)$$

By renaming the parameters we can write our map f in (6.1) in the following form:

$$f(x\ y) = \left(\alpha_0 + \alpha_1 x + y\ \ \frac{x}{\gamma_0 + y}\right) \text{ with } \alpha_1 = 0 \tag{6.6}$$

We consider the imbedding $(x\ y)\quad [1 : x : y]\quad P\mathbb{C}^2$ into projective space and consider the induced map $F : P\mathbb{C}^2\quad P\mathbb{C}^2$ given by:

$$F[x_0 : x_1 : x_2] = [x_0(\gamma_0 x_0 + x_2) : (\alpha_0 x_0 + \alpha_1 x_1 + x_2)(\gamma_0 x_0 + x_2) : x_0 x_1] \tag{6.7}$$

The indeterminacy locus of F is $\mathcal{I}(F) = \ O_0\ O_1\ O_2$ where

$$O_0 = [1 : 0 : -\gamma_0] \quad O_1 = [0 : 1 : -\alpha_1] \quad O_2 = [0 : 1 : 0]$$

and the indeterminacy locus of F^{-1} is $\mathcal{I}(F^{-1}) = \ A_0\ A_1\ A_2$ where

$$A_0 = [0 : 1 : 0] \quad A_1 = [0 : 0 : 1] \quad A_2 = [-\alpha_1 : -\alpha_1(\alpha_0 - \gamma_0) : 1]$$

The set of exceptional curves is given as $\mathcal{E}(F) = \ S_0\ S_1\ S_2$ where

$$S_0 = \ x_0 = 0 \qquad S_1 = \ \gamma_0 x_0 + x_2 = 0 \qquad S_2 = \ \gamma_0 x_0 + x_2 + \alpha_1 x_1 = 0$$

and the set of exceptional curves of F^{-1} is given as $\mathcal{E}(F^{-1}) = \ T_0\ T_1\ T_2$ where

$$T_0 = \ (\alpha_0 - \gamma_0)x_0 - x_1 = 0 \qquad T_1 = \ x_0 - \alpha_1 x_2 \qquad T_2 = \ x_0 = 0$$

- ***Orbit of*** A_0

Observe that $S_0 \twoheadrightarrow A_0 = O_2$ and $T_2 = S_0$ Hence we need to blow up the point $A_0 = O_2$ Let X be the new space we get after blowing up the point O_2 and let E_0 be the exceptional fibre at this point. We recognize the induced map after the blow up process as $\tilde{F} : X \quad X$ For the points $\mathbf{x} = [0 : x_1 : x_2]\quad S_0$ we find that the map $\tilde{F}$ satisfies:

$$S_0 \quad E_0 \quad S_0$$

with

$$\tilde{F}[0 : x_1 : x_2] = [x_2 : x_1]_{E_0} \tag{6.8}$$

and

$$\tilde{F}[u : v]_{E_0} = [0 : \alpha_1(\gamma_0 u + v) : u] \quad T_2 = S_0 \tag{6.9}$$

Hence the indeterminacy set is $\mathcal{I}(\tilde{F}) = \quad O_0 \;\; O_1$ and the exceptional set is $\mathcal{E}(\tilde{F}) = \quad S_1 \;\; S_2$

- ***Orbit of*** A_1

We now consider the orbit of S_1 The exceptional curve $S_1 \twoheadrightarrow A_1 \quad S_0$ Since $O_1[2] = 0$ we get that $A_1 = O_1$ But under the action of F

$$S_1 \twoheadrightarrow A_1 \quad S_0 \twoheadrightarrow A_0$$

we observe that we have a collision of orbits here. Now under the action of $\tilde{F}$

$$S_1 \twoheadrightarrow A_1 \quad \tilde{F}(A_1) \quad E_0 \quad \tilde{F}^2(A_1) \quad S_0$$

As $O_1 \quad S_0$ it is possible that some iterate of A_1 under $\tilde{F}$ reaches the indeterminacy point O_1 Note that A_1 or any iterate of it under $\tilde{F}$ can never reach O_0 This happens because S_0 is invariant under $\tilde{F}^2$ and $O_0 \quad S_0$ Hence all the points belonging to S_0 will come back to S_0 after every two iterates.

The orbit of A_1 under $\tilde{F}$ is as follows:

$$S_1 \twoheadrightarrow A_1 \quad [1 : 0]_{E_0} \quad [0 : \alpha_1\gamma_0 : 1] \quad S_0 \quad \cdots$$

After some iterates we can write the expression of $\tilde{F}^{2k}(A_1)$ for all $k > 0 \quad \mathbb{N}$ as follows:

$$\tilde{F}^{2k}(A_1) = [0 : \alpha_1\gamma_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : 1] \quad S_0$$

Observe that for some value of $k \quad \mathbb{N}$ it is possible that $\tilde{F}^{2k}(A_1) = O_1$ This happens when the following condition is satisfied for any k We recognize it as *condition* k

$$\alpha_1^2\gamma_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) + 1 = 0 \tag{6.10}$$

If it is the case then $\mathcal{O}_1 = \quad A_1 \;\; \tilde{F}(A_1) \;\; \tilde{F}^2(A_1) \qquad \tilde{F}^{2k}(A_1) = O_1$ is a singular elementary orbit. We will blow-up all the points on $\mathcal{O}_1$ We call X_1 the new space after this process and $\tilde{F}_1 : X_1 \quad X_1$ the induced map. Let $G_i \;\; i = 0 \;\; 1 \;\; 2 \qquad 2k$ be the exceptional fibres we get after blowing up. Then under the action of $\tilde{F}_1$ we have

$$G_0 \quad G_1 \quad G_2 \quad \cdots \quad G_{2k-1} \quad G_{2k}$$

where the corresponding sequence of base points is

$$A_1 \quad \tilde{F}(A_1) \quad \tilde{F}^2(A_1) \qquad\qquad \tilde{F}^{2k}(A_1) = O_1$$

Notice that the sequence (6.2) is naturally extended in the following manner:

$$S_1 \quad G_0 \quad G_1 \quad G_2 \quad \cdots \quad G_{2k-1} \quad G_{2k} \quad T_1$$

In fact, the calculations show that $\tilde{F}_1 : S_1 \quad G_0$ through

$$\tilde{F}_1[x_0 : x_1 : -\gamma_0 x_0] = [x_0 : (\alpha_0 - \gamma_0)\, x_0 + \alpha_1\, x_1]_{G_0}$$

and $\tilde{F}_1 : G_{2k} \quad T_1$ through

$$\tilde{F}_1[u : v]_{G_{2k}} = \lim_{t \quad 0} F[tu : 1 : -\alpha_1 + tv] = [\alpha_1 u : (\alpha_0\, u + v)\, \alpha_1 : -u] \quad T_1$$

Note that $\mathcal{I}(\tilde{F}_1) = \quad O_0$ and $\mathcal{E}(\tilde{F}_1) = \quad S_2$

If condition k is not satisfied then the orbit of A_1 is no more singular.

- ***Orbit of*** A_2

Following the orbit of A_2 we need to know if any of it s iterates reach any indeterminacy point of the map. But we have to consider $\tilde{F}$ or $\tilde{F}_1$ depending that condition k is satisfied or not.

If the *condition* k is satisfied then $\mathcal{I}(\tilde{F}_1) = \quad O_0$ and $\mathcal{E}(\tilde{F}_1) = \quad S_2$ In order to have a third singular elementary orbit it must happen that $\tilde{F}_1^p(A_2) = O_0$ for some $p \quad \mathbb{N}$ Then the orbit of A_2 is SE.

Now assume that $\nexists\, k$ such that *condition* k is satisfied. In this case, we have to take into account the first extension of $F \quad \tilde{F}$ which satisfies (6.8) and (6.9). Since $\mathcal{I}(\tilde{F}) = \quad O_0 \; O_1$ in principle we have two options: $\tilde{F}^p(A_2) = O_0$ or $\tilde{F}^p(A_2) = O_1$

We claim that for all $p \quad \mathbb{N} \;\; \tilde{F}^p(A_2) = O_1$ Assume that $\tilde{F}^p(A_2) = O_1$ and assume that $F^j(A_2) \quad S_0$ for $j = 1 \; 2 \qquad p-1$ Then $\tilde{F}^p(A_2) = F^p(A_2) = O_1$ Since $O_1 \quad S_0$ and $A_2 \quad S_0$ if $F^p(A_2) = O_1$ then p would be greater than zero and since $S_0 = T_2$ it would imply that $O_1 = A_1$

or $O_1 = A_2$ which is not the case (recall that the only points in T_2 which have a preimage are A_1 and A_2).

Contrarily, if it exists some $l \quad \mathbb{N} \ l < p$ such that $F^j(A_2) \quad S_0$ for $j = 1 \ 2 \qquad l-1$ but $F^l(A_2) \quad S_0 \quad O_1$ then $F^l(A_2)$ must be equal to A_1 that is, $\tilde{F}^p(A_2) = \tilde{F}^{p-l}(F^l(A_2)) = \tilde{F}^{p-l}(A_1) = O_1$ which implies that $p = l + 2r$ and $\tilde{F}^{2r}(A_1) = O_1$ Hence condition k must be satisfied for $k = r$ and it is a contradiction as $k \quad \mathbb{N}$

It implies that the only available possibility for $\mathcal{O}_2$ to be singular and elementary is to have that for some p $\tilde{F}^p(A_2) = O_0$ or $\tilde{F}_1^p(A_2) = O_0$

If $\tilde{F}^p(A_2) = O_0$ then it will be necessary to blow-up all the points on $\mathcal{O}_2$ in order to get an AS map. Let E_i $i = 1 \ 2 \qquad p+1$ be the exceptional fibre at the points A_2 $\tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$ Then

$$S_2 \quad E_1 \quad E_2 \quad \cdots \quad E_p \quad E_{p+1} \quad T_0$$

where

$$\tilde{F}_1 : [x_0 : x_1 : -\gamma_0 \, x_0 - \alpha_1 \, x_1] \quad S_2 - \quad [x_0 : (\alpha_0 - \gamma_0) \, x_0 - \alpha_1 \, x_1]_{E_1} \tag{6.11}$$

and

$$\tilde{F}_1 : [u : v]_{E_{p+1}} - \quad [v : (\alpha_0 - \gamma_0) v : u] \quad T_0 \tag{6.12}$$

This leaves us with the following possible cases.

Case 1 : One SE orbits i.e. $A_0 = O_2$ with the conditions that $\tilde{F}^{2k}(A_1) = O_1$ and $\tilde{F}^p(A_2) = O_0$ for all $k \ p \quad \mathbb{N}$

Case 2 : Two SE orbits i.e. $(a) : A_0 = O_2 \ \ \tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) = O_1$ for all $k \quad \mathbb{N}$ or $(b) : A_0 = O_2 \ \ \tilde{F}^{2k}(A_1) = O_1$ and ${\tilde{F}_1}^p(A_2) = O_0$ for all $p \quad \mathbb{N}$

Case 3 : Three SE orbits i.e. $A_0 = O_2 \ \ \tilde{F}^{2k}(A_1) = O_1 \ \ \tilde{F}_1^p(A_2) = O_0$ for a certain $p \ k \quad \mathbb{N}$

where SE describes a singular and elementary orbit. Next theorem deals with Case 1.

Theorem 37 *Let F be the mapping (6.7) and let $\tilde{F}$ be the induced map after blowing up the point A_0 Assume that $\tilde{F}^{2k}(A_1) = O_1$ and $\tilde{F}^p(A_2) = O_0$ for all $k \ p \quad \mathbb{N}$ Then the characteristic polynomial associated with $\tilde{F}^*$ is given by*

$$X(x) = x^2 - x - 1$$

Proof. By using Theorem 10 we see that in this case we have only one list $\mathcal{L}_o$ which is open. It is as follows:

$$\mathcal{L}_o = \quad \mathcal{O}_0 = \quad A_0 = O_2$$

Then for $N_{L_o} = 1$ we have $T_{L_o} = x$ and $S_{L_o} = 1$ By using the result from theorem 10 we find that in this case the dynamical degree of F is given by the greatest root of the polynomial $X(x) = x^2 - x - 1$.

Next theorem deals with (b) of Case 2.

Theorem 38 *Let* F *be the mapping (6.7), let* $\tilde{F}$ *be the induced map after blow-up the point* $A_0 = [0 : 1 : 0]$ *and assume that there exists some* $k \quad \mathbb{N}$ *such that* $\tilde{F}^{2k}(A_1) = O_1$ *Let* $\tilde{F}_1$ *be the induced map after we blow-up the points* $A_0 \ \ A_1 \ \ \tilde{F}(A_1) \ \cdots \ \ \tilde{F}^{2k}(A_1) = O_1$ *If for each* $p \quad \mathbb{N}$ $\tilde{F}_1^p(A_2) = O_0$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{X}_k = x^{2k+1}(x^2 - x - 1) + 1 \tag{6.13}$$

Proof. From the above discussion it is clear that after the mentioned series of blow-up s, $\mathcal{E}(\tilde{F}_1) = \quad S_2 \quad \mathcal{I}(\tilde{F}_1) = \quad O_0$ and that the orbit of A_2 is no more singular. So we have the following orbits

$$\mathcal{O}_0 = \quad A_0 = O_2 \qquad\qquad \mathcal{O}_1 = \quad A_1 \ \ \tilde{F}(A_1) \qquad \tilde{F}^{2k}(A_1) = O_1$$

and we have one open and one closed list as follows

$$\mathcal{L}_o = \quad \mathcal{O}_0 = \quad A_0 = O_2 \qquad\qquad \mathcal{L}_c = \quad \mathcal{O}_1 = \quad A_1 \ \ \tilde{F}(A_1) \qquad \tilde{F}^{2k}(A_1) = O_1$$

Then for $N_{L_o} = 1 \quad N_{L_c} = 2k+1$ we have $T_{L_o} = x \quad T_{L_c} = x^{2k+1} - 1$ and $S_{L_o} = 1 \quad S_{L_c} = 1$ By utilizing theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{X}_k = x^{2k+1}(x^2 - x - 1) + 1$

Proposition 39 *The sequence of the degrees of the iterates of all the maps satisfying the hypothesis of Theorem 38 has exponential growth rate.*

Proof. We observe that for all the values of $k \quad \mathbb{N} \quad k \geq 1$ the polynomial $\mathcal{X}_k = x^{2k+1}(x^2 - x - 1) + 1$ has always one root $\lambda > 1$ It is so because if $k > 0$ then $\mathcal{X}_k(1) = 0 \quad \mathcal{X}\ _k(1) = -2k < 0$ and $\lim_{x \quad +} \ \mathcal{X}_k(x) = +$

Now consider that F satisfies the hypothesis of (a) of Case 2.

Theorem 40 *Let F be the mapping (6.7) and let $\tilde{F}$ be the induced map after blowing up the point A_0 If there exists some $p \quad \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$ and there does not exists any $k \quad \mathbb{N}$ such that $\tilde{F}^{2k}(A_1) = O_1$ then the characteristic polynomial associated with F is given by*

$$\mathcal{X}_p = x^{p+1}(x^2 - x - 1) + x^2$$

Proof. Since $\mathcal{O}_0$ and $\mathcal{O}_2$ are SE orbits, we have to blow-up all the points in these orbits. Let X_1 be the extended space after this process and let $\tilde{F}_1 : X_1 \quad X_1$ be the corresponding extended map. Then $\mathcal{E}(\tilde{F}_1) = S_1$ and $\mathcal{I}(\tilde{F}_1) = \quad O_1$ Since condition k is not satisfied then the orbit of A_1 is contained in S_0 but it does not reach the point O_1 Hence $\tilde{F}_1$ is an AS or regularized map.

We can now organize our orbits into lists. We have the following orbits:

$$\mathcal{O}_0 = \quad A_0 = O_2 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

and we have one closed list as follows:

$$\mathcal{L}_c = \quad \mathcal{O}_0 = \quad A_0 = O_2 \qquad \mathcal{O}_2 = \quad A_2 \;\; \tilde{F}(A_2) \qquad \tilde{F}^p(A_2) = O_0$$

Then for $N_{L_c} = p + 2$ we have $T_{L_c} = x^{p+2} - 1$ and $S_{L_c} = x^{p+1} + x + 2$ Utilizing theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{X}_p = x^{p+1}(x^2 - x - 1) + x^2$

Proposition 41 *Consider the mappings which satisfy the hypothesis of Theorem 40. Then:*

(i) For $p = 0$ and $p = 1$ the sequence of the degrees is bounded.

(ii) For $p = 2$ the sequence of the degrees is bounded or grows linearly.

(iii) For $p > 2$ the sequence of the degrees grows exponentially.

Proof. We know that $\mathcal{X}_p = x^{p+1}(x^2 - x - 1) + x^2$

For $p = 0$ and $p = 1$ we deal with the polynomials $x^3 - x$ and $x^4 - x^3$ respectively. This implies that the sequence of degrees satisfies $d_{n+3} = d_n$ and $d_{n+4} = d_{n+3}$ respectively.

For $p = 2$ we get the polynomial $\mathcal{X}_2 = x^5 - x^4 - x^3 + x^2 = x^2(x+1)(x-1)^2$ This polynomial has roots $x = 0 \ \ 0 \ \ -1 \ \ 1 \ \ 1$ This implies that the sequence of degrees

$$d_n = c_2\,(-1)^n + c_3 + c_4\,n$$

Hence d_n grows linearly or d_n is bounded depending on $c_4 = 0$ or $c_4 = 0$

For $p > 2$ we observe that $\mathcal{X}_p(1) = 0$ $\mathcal{X}\ _p(1) = 2 - p < 0$ and $\lim_{x \quad +} \ \mathcal{X}_p(x) = +$ Hence $\mathcal{X}_p$ always has a root $\lambda > 1$ and the result follows.

Theorem 42 *Let F be the mapping (6.7) and let $\tilde{F}$ be the induced map after blowing up the point A_0 If there exist some $k \quad \mathbb{N}$ such that $\tilde{F}^{2k}(A_1) = O_1$ then let $\tilde{F}_1$ be the induced map after blowing up the points of the orbit of A_1 If there exist some $p \quad \mathbb{N}$ such that $\tilde{F}_1^p(A_2) = O_0$ then the characteristic polynomial associated with F is given by*

$$\mathcal{X}_{(k,p)} = x^{p+1}(x^{2k+3} - x^{2k+2} - x^{2k+1} + 1) + x^{2k+3} - x^2 - x + 1$$

Proof. This theorem states the result for *Case* 3 mentioned above. Since $\mathcal{O}_0 \quad \mathcal{O}_1$ and $\mathcal{O}_2$ are SE orbits, we have to blow-up all the points in these orbits. Let X_2 be the extended space after this process and let $\tilde{F}_2 : X_2 \qquad X_2$ be the corresponding extended map. Then $\mathcal{E}(\tilde{F}_2) =$ and $\mathcal{I}(\tilde{F}_2) =$ Hence $\tilde{F}_2$ is an AS or regularized map.

We can now organize our orbits into lists. We have the following orbits under $\tilde{F}$:

$$\mathcal{O}_0 = \ \ A_0 = O_2 \qquad \mathcal{O}_1 = \ \ A_1 \qquad \tilde{F}^{2k}(A_1) = O_1 \qquad \mathcal{O}_2 = \ \ A_2 \qquad \tilde{F}_1^p(A_2) = O_0$$

and we have two closed lists as follows

$$\mathcal{L}_c = \ \ \mathcal{O}_0 = \ \ A_0 = O_2 \qquad \mathcal{O}_2 = \ \ A_2 \ \ \tilde{F}_1(A_2) \qquad \tilde{F}_1^p(A_2) = O_0$$

$$\mathcal{L}_c = \ \ \mathcal{O}_1 = \ \ A_1 \ \ \tilde{F}(A_1)_{E_0} \qquad \tilde{F}^{2k}(A_1)_{S_0} = O_1$$

Then for $N_{L_{c_1}} = p + 2$ $N_{L_{c_2}} = 2k + 1$ we have $T_{L_{c_1}} = x^{p+2} - 1$ $T_{L_{c_2}} = x^{2k+1} - 1$ and $S_{L_{c_1}} = x^{p+1} + x + 2$ $S_{L_{c_2}} = 1$ Utilizing theorem 10 and by using the previous information we find that the characteristic polynomial associated to $\tilde{F}^*$ is $\mathcal{X}_{(k,p)} = x^{p+1}(x^{2k+3} - x^{2k+2} - x^{2k+1} + 1) + x^{2k+3} - x^2 - x + 1$

Let $F_{(k,p)}$ be the family we get in *case* 3 with associated characteristic polynomial

$$\mathcal{X}_{(k,p)} = x^{p+1}(x^{2k+3} - x^{2k+2} - x^{2k+1} + 1) + x^{2k+3} - x^2 - x + 1$$

Lemma 43 *If $p > \frac{2(1+k)}{k}$ then the polynomial $\mathcal{X}_{(k,p)}$ has a root greater than one. For $p \leq \frac{2(1+k)}{k}$ $k \geq 1$ all the roots of $\mathcal{X}_{(k,p)}$ have modulus one.*

Proof. We observe the behavior of $\mathcal{X}_{(k,p)}$ around $x = 1$ For it we consider it s Taylor expansion near $x = 1$:

$$\mathcal{X}_{(k,p)}(x) = 2(2 - k\,p + 2k)(x-1)^2 + O(\ x-1\ ^3)$$

Thus the function $\mathcal{X}_{(k,p)}$ vanishes at $x = 1$ and has a maximum at $x = 1$ when $2(2 - kp + 2k) < 0$ i. e., when $p > \frac{2(1+k)}{k}$ Since $\lim_{x \quad +} \ \mathcal{X}_{(k,p)}(x) = +$ always exists a root greater than one.

If $p \leq \frac{2(1+k)}{k}$ $k \geq 1$ then it is easy to see that the pairs $(k\ \ p)$ with this property are the ones in the following set:

$$A_{(k,p)} = \ \ ((k \geq 1)\ \ 0)\ \ ((k \geq 1)\ \ 1)\ \ ((k \geq 1)\ \ 2)\ \ (1\ \ 3)\ \ (2\ \ 3)\ \ (1\ \ 4)$$

For $(k\ \ p)$ $\ \ A_{(k,p)}$ we have the following polynomials:

$$\begin{aligned}
\mathcal{X}_{(k,0)}(x) &= (x^2-1)\,(x^{2k+2}-1) \\
\mathcal{X}_{(k,1)}(x) &= (x-1)\,(x^{2k+4}-1) \\
\mathcal{X}_{(k,2)}(x) &= (x-1)^2\,(x+1)\,(x^{2k+3}+1) \\
\mathcal{X}_{(1,3)}(x) &= (x-1)^2\,(x+1)\,(x^6-x^3+1) \\
\mathcal{X}_{(2,3)}(x) &= (x-1)^4\,(x+1)\,(x^2+x+1)\,(x^4+x^3+x^2+x+1) \\
\mathcal{X}_{(1,4)}(x) &= (x-1)^4\,(x+1)^2\,(x^2+1)\,(x^2+x+1)
\end{aligned}$$

And all the roots of these polynomials are roots of unity.

Proposition 44 *Let F be the mapping (6.7) and assume that condition k is satisfied and $\tilde{F}^p(A_2) = O_0$ Then*

(i) For the values $(k\ \ p)$ $\ \ A_{(k,p)}$: $(k\ \ p)$ $\ \ (2\ \ 3)\ \ (1\ \ 4)$ the sequence of degrees d_n associated to F is periodic.

(ii) For $(k\ \ p)$ $\ \ (2\ \ 3)\ \ (1\ \ 4)$ the sequence of degrees d_n is either periodic or it grows quadratically.

Proof. As we noted before in this case we have to blow-up all the points in $\mathcal{O}_0$ $\mathcal{O}_1$ and $\mathcal{O}_2$ Let X_2 be the extended space after this process and let $\tilde{F}_2 : X_2 \quad X_2$ be the corresponding extended map. Then $\tilde{F}_2$ acts in the following manner:

$$S_0 \quad E_0 \quad S_0 = T_2$$

$$S_1 \quad G_0 \quad G_1 \quad G_2 \quad \cdots \quad G_{2k-1} \quad G_{2k} \quad T_1$$

and

$$S_2 \quad E_1 \quad E_2 \quad \cdots \quad E_p \quad E_{p+1} \quad T_0$$

where E_i $i = 1 \; 2 \qquad p+1$ is the exceptional fibre at the points A_2 $F(A_2) \qquad F^p(A_2) = O_0$ In fact, the calculations show that $\tilde{F}_3 : S_2 \quad E_1$ through

$$\tilde{F}_2 \left[x_0 : \frac{-\gamma_0 \, x_0}{\alpha_1} - \frac{x_2}{\alpha_1} : x_2 \right] = [\alpha_1 \gamma_0 x_0 + x_2 : x_0]_{E_1} \tag{6.14}$$

and $\tilde{F}_2 : E_{p+1} \quad T_0$ trough

$$\tilde{F}_2 [u : v]_{E_{p+1}} = [v : -\gamma_0 v : (u+v)] \quad T_0 \tag{6.15}$$

Hence, it is clear that the map $\tilde{F}_2$ is an automorphism for all the values $(k \;\; p)$ According to Diller and Favre in [DF01] the growth of degrees of iterates of an automorphism could be bounded, quadratic or exponential but it cannot be linear as in such a case the map is never an automorphism as discussed in chapter 4. We do the analysis case by case.

- For the values $(k \;\; 0)$

$$\begin{aligned} \mathcal{X}_{(k,0)} = & \; (x^2-1)\,(x^{2k+2}-1) = (x^2-1)\,(x^{k+1}-1)\,(x^{k+1}+1) = \\ = & \; (x-1)^2\,(x+1)\,(x^k + x^{k-1} + \cdots + x + 1)\,(x^{k+1}+1) \end{aligned}$$

  If k is even then $x^{k+1}+1$ has the factor $x+1$ and the sequence of degrees is

$$d_n = c_0 + c_1 n + c_2\,(-1)^n + c_3\,(-1)^n\,n + c_4\,\lambda_1^n + c_5\,\lambda_2^n + \qquad + c_{2k+3}\,\lambda_{2k}^n$$

  where c_i are constants and $\lambda_1 \;\; \lambda_2 \qquad \lambda_{2k}$ are the roots of polynomial $x^{2k+2} = 1$ different from ± 1 If k is odd then

$$d_n = l_0 + l_1 n + l_2\,(-1)^n + l_3\,\mu_1^n + l_4\,\mu_2^n + \cdots + l_{2k+3}\,\mu_{2k+1}^n$$

where l_i are constants and $\mu_1 \ \ \mu_2 \qquad \mu_{2k+1}$ are the roots of polynomial $(x^{k+1} - 1)\,(x^k + x^{k-1} + \cdots + x + 1)$

Since $\tilde{F}_2$ is an automorphism for all $(k \ \ p)$ using [DF01] we have $c_1 = 0 = c_3$ and also $l_1 = 0$ This implies that $d_{2k+2+n} = d_n$ i. e., the sequence of degrees is periodic with period $2k + 2$

- For the values $(k \ \ 1)$

$$\mathcal{X}_{(k,1)} = (x^{2k+4} - 1)(x - 1)$$

Since all the roots of $\mathcal{X}_{(k,1)}$ are roots of $x^{2k+4} - 1 = 0$ the same argument as before shows that the sequence of degrees is periodic with period $2k + 4$

- For the values $(k \ \ 2)$

$$\mathcal{X}_{(k,2)} = (x^{2k+3} + 1)(x - 1)^2(x + 1)$$

Since all the roots of $\mathcal{X}_{(k,2)}$ are roots of $x^{4k+6} - 1 = 0$ the same argument as before shows that the sequence of degrees is periodic with period $4k + 6$

- For the value $(1 \ \ 3)$

$$\mathcal{X}_{(1,3)} = (x^6 - x^3 + 1)(x - 1)^2(x + 1)$$

and we can assert that the sequence of degrees is periodic with period 18

- When $(k \ \ p) = (2 \ \ 3)$

$$\mathcal{X}_{(2,3)} = (x - 1)^3(x + 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)$$

The sequence of degrees is then

$$d_n = c_0\,(-1)^n + c_1 + c_2\,n + c_3\,n^2 + c_4\,\lambda_1^n + c_5\,\lambda_2^n + \cdots + c_9\,\lambda_6^n$$

where $\lambda_1 \ \ \lambda_2 \qquad \lambda_6$ are the roots of $(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)$ Since our map is an automorphism we see that either $c_3 = c_2 = 0$ and d_n is 30-periodic or $c_3 = 0$ and d_n has quadratic growth.

- If $(k \;\; p) = (1 \;\; 4)$ then

$$\mathcal{X}_{(1,4)} = (x-1)^4(x+1)^2(x^2+1)(x^2+x+1)$$

and the same arguments as before let us to say that either, d_n is periodic with period 12 or d_n grows quadratically.

Next result states the behavior of the sequence of the degrees when the collision has appeared.

Proposition 45 *Let F be the mapping (6.7), let $\tilde{F}$ be the induced map after blow-up the point $A_0 = [0:1:0]$ and assume that there exists some $k \quad \mathbb{N}$ such that $\tilde{F}^{2k}(A_1) = O_1$ Let $\tilde{F}_1$ be the induced map after we blow-up the points $A_0 \;\; A_1 \;\; \tilde{F}(A_1) \qquad \tilde{F}^{2k}(A_1) = O_1$ Assume that there exist $l \;\; p \quad \mathbb{N}$ such that $F_1^l(A_2) \quad S_1 \quad O_0$ and $\tilde{F}_1^p(A_2) = O_0 \;\; p > l$ Then:*

(i) If $l > 0$ then the sequence of degrees grows exponentially.

(ii) If $l = 0$ and $k > 1$ then the sequence of degrees grows exponentially. If $l = 0$ and $k = 1$ then the sequence of degrees is either periodic with period 12 or it grows quadratically.

Proof. By using the hypothesis we can write the following

$$S_0 \qquad E_0 \qquad S_0$$

$$S_1 \qquad G_0 \qquad G_1 \qquad G_2 \qquad \cdots \qquad G_{2k-1} \qquad G_{2k} \qquad T_1$$

Now assume that $l > 0$ Then p satisfies

$$p \geq l + 2k + 2 > 2(k+1) > \frac{2(k+1)}{k}$$

From the above lemma we know that the characteristic polynomial associated to $\tilde{F}_2^*$ has a root greater than one.

Assume now that $l = 0$ that is, $A_2 \quad S_1 \quad O_0$ It happens if

$$\gamma_0\alpha_1 - 1 = (\gamma_0\alpha_1 - 1) = 0$$

that is $A_2 \quad S_1 \quad O_0$ if and only if

$$\gamma_0\alpha_1 = 1$$

It turns out that whit these conditions on the parameters $T_1 = S_1$ and hence we have the following cycle:

$$S_1 \quad G_0 \quad G_1 \quad G_2 \quad \cdots \quad G_{2k-1} \quad G_{2k} \quad S_1$$

So the condition $\tilde{F_1}^p(A_2) = O_0$ with $A_1 = O_0$ can happen when $p = m(2k+2)$ with $m \quad \mathbb{N} \quad m \geq 1$ Then

$$p = m(2k+2) \geq 2(k+1) \text{ and } 2(k+1) > \frac{2(k+1)}{k}$$

if and only if $k > 1$ From the above Lemma, when $k > 1$ the polynomial $\mathcal{X}_{(k,p)}$ has some root greater than one. If $k = 1$ then $p = m(2k+2) = 4m$ satisfies $p \leq \frac{2k+2}{k} = 4$ only if $m \leq 1$ Then when $k = 1 \quad m = 1$ the characteristic polynomial is

$$\mathcal{X}_{(1,4)}(x) = \; (x-1)^4 \, (x+1)^2 \, (x^2+1) \, (x^2+x+1)$$

Since the extended map $\tilde{F}_2 : X_2 \qquad X_2$ introduced in the proof of Theorem 42 is an automorphism and all the roots of $\mathcal{X}_{(1,4)}$ are roots of unity, from the results in Theorem 7 the statement follows.

In the following section we discuss the families with dynamical degree one that is zero algebraic entropy. We give the families for all the values of $k \;\; p \quad \mathbb{N}$ and discuss their growth and find their preserves fibrations, also periods for periodic mappings.

6.3 Zero entropy

Theorem 46 *Let $f(x \;\, y)$ be a mapping such that the corresponding F satisfies the hypothesis of Theorem 40 and it has zero entropy. Then f can be written in one of the following way:*

(1)

$$f(x \;\, y) = \left(\frac{1}{\alpha_1} + \alpha_1 x + y \quad \frac{x}{\frac{1}{\alpha_1} + y} \right)$$

with $\alpha_1 = 0$ and $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k = 0$ for all $k \quad \mathbb{N} \;\; d_n = 2$ for all $n \quad \mathbb{N}$ and it preserves two rational fibrations $V_1 = \frac{L_1}{Q_1} \;\; V_2 = \frac{Q_2}{L_1}$ such that $V_1(f(x \;\, y)) = \frac{1}{\alpha_1} V_1(x \;\, y)$ and $V_2(f(x \;\, y)) = \alpha_1 \, V_2(x \;\, y) + 1$ where

$$\begin{aligned} L_1 &= 1 + \alpha_1 \, y \\ Q_1 &= ((\alpha_1 - 1) \, y + 1) \, (\alpha_1 \, (\alpha_1 - 1) \, x + \alpha_1 \, y + 1) \\ Q_2 &= \alpha_1 \, (\alpha_1 - 1) \, x \, y + \alpha_1 \, y^2 + \alpha_1 \, x + y \end{aligned}$$

(2)

$$f(x\ \ y) = \left(\frac{\alpha_1^2 + \alpha_1 + 1}{\alpha_1\,(\alpha_1 + 1)^2} + \alpha_1\,x + y \ \ \frac{x}{\frac{1}{\alpha_1 + 1} + y} \right)$$

with $\alpha_1 \quad\ 0\ \ -1$ *and* $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k = 0$ *for all* $k \quad \mathbb{N}\ \ d_1 = 2\ \ d_2 = 3\ \ d_n = 4$ *for all* $n \geq 3 \quad \mathbb{N}$ *and* f *preserves two rational fibrations* $H_i = \frac{Q_3(x,y,\omega_i)}{Q_4(x,y)}$ *such that* $H_i(f(x\ \ y)) = \ _i\,H_i(x\ \ y) + \alpha\ \ i = 1\ \ 2$ *where* $\ _i$ *is the root of* $\alpha_1\,x^2 = 1\ \ \alpha = -(\ _i\,\alpha_1 + \ _i - 1)$ *where*

$$\begin{aligned}
Q_3(x\ \ y\ \ W_i) &= A_3\,x\,y + A_2\,x - A_1\,y - A_0 \quad \textit{for} \\
A_3 &= W_i\,\alpha_1^6 + 3\,W_i\,\alpha_1^5 + (3\,W_i + 1)\,\alpha_1^4 + (W_i + 3)\,\alpha_1^3 + 3\,\alpha_1^2 + \alpha_1 \\
A_2 &= W_i\,\alpha_1^5 + (3\,W_i - 1)\,\alpha_1^4 + (3\,W_i - 2)\,\alpha_1^3 + (W_i - 1)\,\alpha_1^2 \\
A_1 &= \alpha_1^3 + 2\,\alpha_1^2 + 2\,\alpha_1 + 1 \\
A_0 &= \alpha_1^2 + \alpha_1 + 1 \\
\textit{and } Q_4(x\ \ y) &= B_4\,y^2 + B_3\,x\,y + B_2\,x + B_1\,y + B_0 \quad \textit{for} \\
B_4 &= \alpha_1^2\,(\alpha_1^2 + 2\,\alpha_1 + 1) \\
B_3 &= \alpha_1\,(\alpha_1^4 + 2\,\alpha_1^3 - 2\,\alpha_1 - 1) \\
B_2 &= \alpha_1^2\,(\alpha_1^2 + 2\,\alpha_1 + 1) \\
B_1 &= 2\,\alpha_1^3 + 3\,\alpha_1^2 + 2\,\alpha_1 + 1 \\
B_0 &= \alpha_1^2 + \alpha_1 + 1
\end{aligned}$$

(3)

$$f(x\ \ y) = \left(\frac{\ ^3 \mp \ ^2 \pm 1}{(\ \ \pm 1)\,(\ ^2 \mp \ \ + 1)^2} + \ \ ^2x + y \ \ \frac{x}{\frac{(\omega \mp 1)}{\omega(\omega^2 \mp \omega + 1)} + y} \right)$$

with $\ \ = 0 \quad \ \pm 1 = 0 \quad \ ^2 \mp \ \ + 1 = 0\ \ 1 \mp \ \ + \ ^2 \mp \ ^3 \cdots \mp \ ^{2k+1} + \ ^{2k+2} = 0$ *for all* $k \quad \mathbb{N}\ \ d_n$ *grows linearly in* n *and* f *preserves the rational fibration* $H_3 = \frac{Q_5(x,y)}{Q_6(x,y)}$ *such that* $H_3(f(x\ \ y)) = \ _1\,H_3(x\ \ y) + \ _2$ *for* $\ _1 = - \quad \ _2 = -\frac{(\omega - 1)^2}{\omega^2 - 2\omega + 1}$ *where*

$$\begin{aligned}
Q_5(x\ \ y) &= (A_3\,y + A_2)\,x + A_1\,y + A_0 \quad \textit{for} \\
A_3 &= \ ^9 - 2\ ^8 + 3\ ^7 - \ ^6 - \ ^5 + 3\ ^4 - 2\ ^3 + \ ^2 \\
A_2 &= \ ^4 + \ ^7 - \ ^6 + \ ^5 - \ ^3 + \ ^2 \\
A_1 &= 2\ ^2 - \ \ - 2\ ^3 + \ ^4 \\
A_0 &= \ ^2 - 2\ \ + 1
\end{aligned}$$

and

$$\begin{aligned} Q_6(x\ \ y) &= B_3\, y^2 + B_2\, y + B_1\, x + B_0 \quad \textit{for} \\ B_3 &= {}^{3} + {}^{8} - {}^{7} + {}^{5} + {}^{6} - {}^{4} \\ B_2 &= 2\ {}^{6} + 2\ {}^{4} - 3\ {}^{5} + \quad + \ {}^{3} - 2\ {}^{2} \\ B_1 &= {}^{3} - {}^{7} - {}^{5} - {}^{4} + {}^{6} - {}^{2} \\ B_0 &= {}^{4} - 2\ {}^{3} + {}^{2} + \quad - 1 \end{aligned}$$

Proof.

(1) From Proposition 41 we know that if $\tilde{F}^p(A_2) = O_0$ and $p > 2$ then the dynamical degree of F is greater than 1. Hence for the cases of $p \leq 2$ we have dynamical degree 1 Also as for $p = 0$ $d_{n+3} = d_n$ we find that $d_n = 2$ for all $n \quad \mathbb{N}$ Now from $p = 0$ we mean, when $A_2 = O_0$ from Proposition 1 of the appendix we know that the coefficients of f have to satisfy conditions $(C1)$ and we obtain the maps (1) of the theorem.

To find the fibrations for this family we use the same methodology we utilize to get our results in Chapter 3. By considering a general cubic C, we impose that it passes through all the three points of indeterminacy of F After some calculations we find the cubic coefficients of C are zero and that f preserves a conic Q_1 We then consider another general conic Q and consider a rational function of two conic to find our required fibration. It results that Q reduces to a line L_1 Hence giving our first rational fibration V_1 such that it satisfies $V_1(f) = \frac{1}{\alpha_1} V_1$. In a similar context to find the other fibration we consider a function $\frac{Q}{L_1}$ for a general conic Q With some easy computations we find that $Q = Q_2$ we therefore get V_2 such that it satisfies $V_2(f) = \alpha_1 V_2 + 1$

(2) For $p = 1$ we observe that $\tilde{F}(A_2) = O_0$ is equivalent to $F(A_2) = O_0$ because the point $A_2 \quad E_0$ From Proposition 2 of the appendix again we know that the coefficients of f have to satisfy condition $(C2)$ and we obtain the map (2) of the theorem. Now observe that for $p = 1$ $d_{n+4} = d_{n+3}$ we find that $d_n = 4$ for all $n \geq 3 \quad \mathbb{N}$ Hence both of these families have bounded growth rate. Now we first intend to look for an invariant cubic curve C for f by considering that the cubic passes through all the points of indeterminacy and then looking for the image of such a curve we find that this results in providing us with the invariant conic Q_4 Then by considering a rational function of Q_4 with another general conic Q we find that the function $\frac{Q}{Q_4}$ gives us the required fibrations H_i with $Q = Q_3$ and it satisfies $H_i(f(x\ \ y)) = \ _i\, H_i(x\ \ y) + \alpha \ \ i = 1\ \ 2$ for $_i$ the root of $\alpha_1\, x^2 = 1 \quad \alpha = -(\ _i\,\alpha_1 + \ _i - 1)$ and $H_i = \frac{Q_3}{Q_4}$

(3) From Proposition 41 we know that if $\tilde{F}^p(A_2) = O_0$ and $p \leq 2$ we have dynamical degree equal to 1 We see that for $p = 2$ this mapping has linear growth which is given by $d_n = c_2\,(-1)^n + c_3 + c_4\,n$ which for $d_1 = 2$ $d_2 = 3$ $d_3 = 5$ $d_4 = 7$ we can write that $d_n = -1 + 2\,n$ $n > 1$ which implies that d_n grows linearly in n

For $p = 2$ we have the condition $\tilde{F}^2(A_2) = O_0$ which is equivalent to $F^2(A_2) = O_0$ because the point A_2 S_0 Looking at Proposition 3 of the appendix we find the conditions on the coefficients of f have to satisfy condition $(C3)$ and we obtain the required map, hence the result follows. Now if the condition $F^{2k}(A_1) = O_1$ for all k $\mathbb{N}$ then

$$F^{2k}(A_1) = [0 : \alpha_1\gamma_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : 1] = O_1 = [0 : \frac{-1}{\alpha_1} : 1]$$

By substituting $\gamma_0 = \frac{(\omega\mp1)}{\omega(\omega^2\mp\omega+1)}$ from $(C3)$ for $= 0$ $^2 \mp + 1 = 0$ we can write the above equation as following:

$$F^{2k}(A_1) = [0 : {}^2\left(\frac{(\ \mp 1)}{(\ {}^2\mp \ +1)}\right)(1+ {}^2+ {}^4+\cdots+ {}^{2k-2}) : 1] = O_1 = [0 : \frac{-1}{{}^2} : 1]$$

then we have

$$(\ \mp 1)(\ {}^3+ {}^5+ {}^7+\cdots+ {}^{2k+1}) + (\ {}^2\mp \ +1) = 0 \tag{6.16}$$

which can be written as the following two conditions

$$1 \mp \ + {}^2\mp {}^3\cdots\mp {}^{2k+1} + {}^{2k+2} = 0 \tag{6.17}$$

for all k $\mathbb{N}$ This gives the required condition on the parameters. Now observe that for $p = 1$ $d_{n+4} = d_{n+3}$ we find that $d_n = 4$ for all $n \geq 3$ $\mathbb{N}$ Hence both of these families have bounded growth rate. We now look for an invariant cubic curve C such that it passes through all the points of indeterminacy of f We find that this results in providing us with the invariant conic Q_6 We then consider a rational function $\frac{Q}{Q_6}$ such that it gives us the required fibrations H_3 then we get that $Q = Q_5$ and it satisfies $H_3(f(x\ \ y)) = {}_1 H_3(x\ \ y) + {}_2$ $i = 1$ 2 for ${}_1 = -$ ${}_2 = -\frac{(\omega-1)^2}{\omega^2-2\omega+1}$ and $H_3 = \frac{Q_5}{Q_6}$

∎

Remark 6.3.1 *Note that f in (1) in Theorem 46 is birationally conjugate to a linear map. We* *observe that for the rational map* $h(x\ \ y) = \left(\frac{a_1\,x}{1+a_1\,y}\ \ y\right)$ *whose inverse is given by* $h^{-1}(x\ \ y) = \left(\frac{x\,(1+y\,a_1)}{a_1}\ \ y\right)$ *we can write the equation* $h(f(x\ \ y)) = g(h(x\ \ y))$ *for* $g(x\ \ y) = (1 + a_1\,y\ \ x)$ *This implies that* g *is the birationally conjugate to* f

Theorem 47 *Let* $f(x\ \ y)$ *be a mappings such that the corresponding* F *satisfies the hypothesis of Theorem 42 and it has zero entropy. If the sequence of degrees of* F *is periodic then* f *is one of following:*

(1)

$$f(x\ \ y) = \left(\frac{1}{\alpha_1} + \alpha_1 x + y\ \ \frac{x}{\frac{1}{\alpha_1} + y}\right)$$

and f *is* $2k+2$ *periodic for some* $k\ \ \mathbb{N}$ *such that* $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k = 0$

(2)

$$f(x\ \ y) = \left(\frac{\alpha_1^2 + \alpha_1 + 1}{\alpha_1\,(\alpha_1+1)^2} + \alpha_1\,x + y\ \ \frac{x}{\frac{1}{\alpha_1+1} + y}\right)$$

and f *is* $2k+4$ *periodic for some* $k\ \ \mathbb{N}$ *such that* $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k = 0$

(3)

$$f(x\ \ y) = \left(\frac{{}^3 \mp {}^2 \pm 1}{(\ \ \pm 1)\,(\ {}^2 \mp\ \ + 1)^2} +\ {}^2 x + y\ \ \frac{x}{\frac{(\omega\mp 1)}{\omega(\omega^2\mp\omega+1)} + y}\right)$$

with $\ \ = 0\ \ \ {}^2 \mp\ \ + 1 = 0\ \ \ \mp 1 = 0$ *and* f *is* $4\,k+6$ *periodic for some* $k\ \ \mathbb{N}$ *such that* $1 \mp\ \ +\ {}^2 \mp\ {}^3 \cdots \mp\ {}^{2k+1} +\ {}^{2k+2} = 0$

(4)

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + y\ \ \frac{x}{\gamma_0 + y}\right)$$

with $\alpha_0 = -2\alpha_1^5 + \alpha_1^3 - \alpha_1^2 - \alpha_1\ \ \ \gamma_0 = -\frac{1}{\alpha_1^2}$ *and* f *is* $18-$*periodic such that* $\alpha_1^6 + \alpha_1^3 + 1 = 0$

Proof. From Proposition 44 we know that if $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) = O_1$ then the for the values of $(k\ \ p)\ \ \ A_{(k,p)}$ the dynamical degree of F is equal to 1. We first consider the case $(k\ \ p) = (k\ \ 0)$ then

(1) For $p = 0$ we mean, when $A_2 = O_0$ from Proposition of the appendix we know that the coefficients of f have to satisfy conditions $(C1)$ and we obtain the maps (1) of the theorem.

As this family satisfies the hypothesis of theorem 42 for $p = 0$ and for all k Therefore we know that the condition $\tilde{F}^{2k}(A_1) = O_1$ should be satisfied. By using the same condition $(C1)$ of appendix we can thus write the condition $F^{2k}(A_1) = O_1$ as following

$$F^{2k}(A_1) = [0 : \alpha_1\gamma_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : 1] = O_1 = \left[0 : \frac{-1}{\alpha_1} : 1\right]$$

By substituting $\alpha_0 = \gamma_0 = \frac{1}{\alpha_1}$ from $(C1)$ we can write the above equation as following:

$$F^{2k}(A_1) = [0 : 1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1} : 1] = O_1 = \left[0 : \frac{-1}{\alpha_1} : 1\right]$$

which can be written as the following condition

$$1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1} + \alpha_1^k = 0 \qquad \alpha_1^{k+1} = 1 \tag{6.18}$$

Now consider the mapping 1 of the above theorem that is

$$f(x\ \ y) = \left(\frac{1}{\alpha_1} + \alpha_1 x + y \quad \frac{x}{\frac{1}{\alpha_1} + y}\right)$$

with $\alpha_1 = 0$ and $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k = 0$ for some $k \quad \mathbb{N}$

By iterating f and using a simple induction process we find that the expression of $f^{2r}(x\ \ y)$ for any $r \quad \mathbb{N}$ can be written as follows:

$$f^{2r}(x\ \ y) = \left(\tfrac{(1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^r+\alpha_1^{r+1}y)((1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^{r-1})+\alpha_1 y(1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^{r-1})+\alpha_1^{r+1}x)}{\alpha_1(1+\alpha_1 y)} \quad 1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k + \alpha_1^r y\right)$$

Therefore we can write the expression of $f^{2k+2}(x\ \ y)$ as follows:

$$f^{2k+2}(x\ \ y) = \left(\tfrac{(1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^{k+1}+\alpha_1^{k+2}y)((1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^k)+\alpha_1 y(1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^k)+\alpha_1^{k+2}x)}{\alpha_1(1+\alpha_1 y)} \quad 1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k + \alpha_1^{k+1} y\right)$$

As we know from above calculations that $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k = 0$ which implies that

$\alpha_1^{k+1} = 1$ and $\alpha_1^{k+2} = \alpha_1$ By substituting these values in the above expression of f^{2k+2} we find that $f^{2k+2}(x\ \ y) = (x\ \ y)$ i.e. it is the identity.

(2) We now consider the case $(k\ \ p) = (k\ \ 1)$ then for $p = 1$ we observe that $\tilde{F}(A_2) = O_0$ is equivalent to $F(A_2) = O_0$ because the point $A_2 \quad E_0$ From Proposition of the Appendix again we know that the coefficients of f have to satisfy condition $(C2)$ and we obtain the map (2) of the theorem. This family satisfies the hypothesis of theorem 42 for $p = 1$ and for all k Therefore we know that the condition $\tilde{F}^{2k}(A_1) = O_1$ should be satisfied. By using the same condition $(C2)$ of appendix we can thus write the condition $F^{2k}(A_1) = O_1$ as following

$$F^{2k}(A_1) = [0 : \alpha_1\gamma_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : 1] = O_1 = \left[0 : \frac{-1}{\alpha_1} : 1\right]$$

By substituting $\gamma_0 = \frac{1}{\alpha_1+1}$ from $(C2)$ for $\alpha_1 \quad 0\ \ -1$ we can write the above equation as following:

$$F^{2k}(A_1) = \left[0 : \frac{\alpha_1}{\alpha_1 + 1}(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : 1\right] = O_1 = \left[0 : \frac{-1}{\alpha_1} : 1\right]$$

which can be written as the following condition

$$1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^k + \alpha_1^{k+1} = 0 \tag{6.19}$$

which implies that $\alpha_1^{k+2} = 1$ Now consider the mapping (2) of the above theorem that is

$$f(x\ \ y) = \left(\frac{\alpha_1^2 + \alpha_1 + 1}{\alpha_1(\alpha_1 + 1)^2} + \alpha_1 x + y \quad \frac{x}{\frac{1}{\alpha_1+1} + y}\right)$$

from the above calculations we see that we can write the following conditions on the parameters: $\alpha_1 \quad 0\ \ -1$ and $1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k+1} = 0$ for some $k \quad \mathbb{N}$

From Proposition 44 we know that for the case $(k\ \ p) = (k\ \ 1)$ the sequence of degrees d_n is $2k + 4$ periodic. This implies that $d_{2k+4} = d_0 = 1$ Therefore the function F^{2k+4} is linear. Hence we can consider that for some constants $r_i\ \ p_i\ \ q_i \quad \mathbb{R}$ the function F^{2k+4} has the following form:

$$F^{2k+4}[x_0 : x_1 : x_2] = [r_0\, x_0 + r_1\, x_1 + r_2\, x_2 : p_0\, x_0 + p_1\, x_1 + p_2\, x_2 : q_0\, x_0 + q_1\, x_1 + q_2\, x_2]$$

We know that S_0 is invariant under the action F^2 therefore it is invariant under the action of F^{2k+4} as well. This implies that

$$F^{2k+4}[0 : x_1 : x_2] = [0 : x_1 : x_2]$$

which further implies that $r_1\, x_1 + r_2\, x_2 = 0$ for all complex numbers $x_1 \;\; x_2$ This is only possible if $r_1 = r_2 = 0$ Then we can write

$$F^{2k+4}[x_0 : x_1 : x_2] = \left[x_0 : \frac{p_0}{r_0}\, x_0 + \frac{p_1}{r_0}\, x_1 + \frac{p_2}{r_0}\, x_2 : \frac{q_0}{r_0}\, x_0 + \frac{q_1}{r_0}\, x_1 + \frac{q_2}{r_0}\, x_2\right]$$

which in affine plane by taking $x_0 = 1$ and rewriting the parameters, as new parameters, the function F^{2k+4} can be written as following:

$$f^{2k+4}(x \;\; y) = (p_0 + p_1\, x + p_2\, y \;\; q_0 + q_1\, x + q_2\, y) \tag{6.20}$$

for any $p_0 \;\; p_1 \;\; p_2 \;\; q_0 \;\; q_1 \;\; q_2 \quad \mathbb{R}$

We find that the following curve C is a curve fixed by f^2 This implies that it is also fixed by f^{2k+4}

$$\begin{aligned} C = \quad & (\alpha_1^2 + \alpha_1 + 1) + \alpha_1^2(\alpha_1 + 1)^2\, x + (1 + 2\,\alpha_1^3 + 2\,\alpha_1 + 3\,\alpha_1^2)\, y + \\ & (\alpha_1^2 - 1)\,(a1 + 1)^2\,\alpha_1\, x\, y + (\alpha_1^4 + 2\,\alpha_1^3 + \alpha_1^2)\, y^2 = 0 \end{aligned}$$

We take any three arbitrary points on this curve namely;

$$\begin{aligned} fix_1 \;\; &= \left(-\tfrac{\alpha_1^2+\alpha_1+1}{\alpha_1^2\,(\alpha_1+1)^2} \;\; 0\right) \qquad fix_2 = \left(0 \;\; -\tfrac{1}{\alpha_1+1}\right) \\ fix_3 \;\; &= \left(\tfrac{1}{\alpha_1\,(\alpha_1+1)^2} \;\; -\tfrac{1}{\alpha_1}\right) \end{aligned}$$

As these points are fixed by f^2 so they are also fixed points of f^{2k+4} Then finding the images of $fix_1 \;\; fix_2$ and fix_3 under the action of f^{2k+4} using 6.20 we get a system of following equations:

$$\begin{aligned}
p_0 + p_1\left(-\tfrac{\alpha_1^2+\alpha_1+1}{\alpha_1^2\,(\alpha_1+1)^2}\right) &= -\tfrac{\alpha_1^2+\alpha_1+1}{\alpha_1^2\,(\alpha_1+1)^2}\\
p_0 + p_2\left(-\tfrac{1}{\alpha_1+1}\right) &= 0\\
p_0 + p_1\left(\tfrac{1}{\alpha_1\,(\alpha_1+1)^2}\right) + p_2\left(-\tfrac{1}{\alpha_1}\right) &= \tfrac{1}{\alpha_1\,(\alpha_1+1)^2}\\
q_0 + q_1\left(-\tfrac{\alpha_1^2+\alpha_1+1}{\alpha_1^2\,(\alpha_1+1)^2}\right) &= -\tfrac{\alpha_1^2+\alpha_1+1}{\alpha_1^2\,(\alpha_1+1)^2}\\
q_0 + q_2\left(-\tfrac{1}{\alpha_1+1}\right) &= 0\\
q_0 + q_1\left(\tfrac{1}{\alpha_1\,(\alpha_1+1)^2}\right) + q_2\left(-\tfrac{1}{\alpha_1}\right) &= \tfrac{1}{\alpha_1\,(\alpha_1+1)^2}
\end{aligned}$$

By solving the above system of equation for the values of $p_0\ \ p_1\ \ p_2\ \ q_0\ \ q_1\ \ q_2$ we find that $(p_0\ \ p_1\ \ p_2\ \ q_0\ \ q_1\ \ q_2) = (0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1)$ which implies that $f^{2k+4}(x\ \ y) = (x\ \ y)$

(3) From Proposition 44 we know that if $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) = O_1$ then the for the values of $(k\ \ p)\quad A_{(k,p)}$ the dynamical degree of F is equal to 1. We consider the case $(k\ \ p) = (k\ \ 2)$ then for $p = 2$ we have that $\tilde{F}^2(A_2) = O_0$ is equivalent to $F^2(A_2) = O_0$ because the point $A_2\quad S_0$ Looking at Proposition of the Appendix we find the conditions on the coefficients of f have to satisfy condition $(C3)$ and we obtain the required map.

This family satisfies the hypothesis of theorem 42 for $p = 2$ and for all k Therefore we know that the condition $\tilde{F}^{2k}(A_1) = O_1$ should be satisfied. By using the same condition $(C3)$ of appendix we can thus write the condition $F^{2k}(A_1) = O_1$ as following

$$F^{2k}(A_1) = [0 : \alpha_1\gamma_0(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : 1] = O_1 = \left[0 : \frac{-1}{\alpha_1} : 1\right]$$

By substituting $\gamma_0 = \frac{(\omega\mp1)}{\omega(\omega^2\mp\omega+1)}$ from $(C3)$ for $\quad = 0\quad {}^2 \mp \quad + 1 = 0$ we can write the above equation as following:

$$\begin{aligned}
F^{2k}(A_1) &= \left[0 :\ {}^2\left(\tfrac{\omega\mp1}{\omega(\omega^2\mp\omega+1)}\right)(1 + \ {}^2 + \ {}^4 + \cdots + \ {}^{2k-2}) : 1\right] = O_1\\
&= [0 : \tfrac{-1}{\omega^2} : 1]
\end{aligned}$$

then we have

$$(\quad\mp 1)(\ {}^3 + \ {}^5 + \ {}^7 + \cdots + \ {}^{2k+1}) + (\ {}^2 \mp \quad + 1) = 0 \tag{6.21}$$

which can be written as the following two conditions

$$1 \mp \quad + \ {}^2 \mp \ {}^3 \cdots \mp \ {}^{2k+1} + \ {}^{2k+2} = 0 \tag{6.22}$$

The above two equations give us the required condition k on the parameter α_1 for this family, that is $\alpha_1^{2\,k+3} = (\ \ ^2)^{2\,k+3} = \ \ ^{4\,k+6} = (\ \ ^{2\,k+3})^2 = 1$ for some $k \quad \mathbb{N}$

From proposition 44 we know that for the case $(k\ \ p) = (k\ \ 2)$ the sequence of degrees d_n is $4k + 6$ periodic. This implies that $d_{4k+6} = d_0 = 1$ Therefore the function F^{4k+6} is linear. Then F^{4k+6} has the following form:

$$F^{4k+6}[x_0 : x_1 : x_2] = [r_0\,x_0 + r_1\,x_1 + r_2\,x_2 : p_0\,x_0 + p_1\,x_1 + p_2\,x_2 : q_0\,x_0 + q_1\,x_1 + q_2\,x_2]$$

for some constants $r_i\ \ p_i\ \ q_i \quad \mathbb{R}$ the function. As S_0 is invariant under the action F^2 therefore it is invariant under the action of F^{4k+6} as well. This implies that we can write

$$f^{4k+6}(x\ \ y) = (p_0 + p_1\,x + p_2\,y\ \ q_0 + q_1\,x + q_2\,y) \tag{6.23}$$

for any $p_0\ \ p_1\ \ p_2\ \ q_0\ \ q_1\ \ q_2 \quad \mathbb{N}$

We find that the following two are the fixed points of f and the third one is fixed by f^2

$$\begin{aligned}
fix_1 &= \left(\frac{1}{(\omega^2-\omega+1)\,(\omega+1)\,(\omega^3+1)} \quad -\frac{\omega}{\omega^3+1} \right) \\
fix_2 &= \left(\frac{\omega^3-\omega^2+1}{\omega\,(\omega^2-\omega+1)\,(\omega^2-1)\,(\omega^4-\omega^3+\omega-1)} \quad -\frac{\omega^3-\omega^2+1}{\omega\,(\omega^4-\omega^3+\omega-1)} \right) \\
fix_3 &= \left(\frac{1}{\omega^6+2\,\omega^3+1} \quad -\frac{\omega}{\omega^3+1} \right)
\end{aligned}$$

Now these points must also be fixed by f^{4k+6} Then by finding the images of $fix_1\ \ fix_2$ and fix_3 under the action of f^{4k+6} using 6.23 such that $f^{4k+6}(fix_1) = (fix_1)\ \ f^{4k+6}(fix_2) = (fix_2)\ \ f^{4k+6}(fix_3) = (fix_3)$ Also as the sequence of degrees is periodic of period $4\,k+6$ this implies that the matrix $(\tilde{F}_1^*)^{4k+6}$ fixes the elements in the basis of Picard group. This implies that $(\tilde{F}_1^*)^{4k+6}$ also fixes E_1 that is the blown up fibre at A_2 Then F^{4k+6} fixes the base point A_2 in $P\mathbb{C}^2$ By utilizing this information and then solving this system of four equations for the values of $p_0\ \ p_1\ \ p_2\ \ q_0\ \ q_1\ \ q_2$ we find that $(p_0\ \ p_1\ \ p_2\ \ q_0\ \ q_1\ \ q_2) = (0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1)$ which implies that $f^{4k+6}(x\ \ y) = (x\ \ y)$

(4) From Proposition 44 we know that if $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) = O_1$ then the for the values of $(k\ \ p) \quad A_{(k,p)}$ the dynamical degree of F is equal to 1. We now consider the case $(k\ \ p) = (1\ \ 3)$ then we have that $\tilde{F}^2(A_1) = O_1$ and $\tilde{F}^3(A_2) = O_0$ which is equivalent to $F^3(A_2) = O_0$ This is because the point $A_2 \qquad S_0\ \ S_1$ and by looking at the orbits

of A_0 and A_1 we find by easy calculations that the condition $\tilde{F}^3(A_2) = O_0$ implies that $F(A_2) \quad S_0 \ S_1$ and $F^2(A_2) \quad S_0 \ S_1$ Looking at Proposition of the Appendix we find the conditions on the coefficients of f have to satisfy condition $(C4)$ and we get our required family. This family satisfies the hypothesis of Theorem 42 for $p = 3$ and $k = 1$

From proposition 44 we know that for the case $(k \ p) = (1 \ 3)$ the sequence of degrees d_n is 18 periodic. This implies that $d_{18} = d_0 = 1$ Therefore the function F^{18} is linear. Hence we can consider that for some constants $r_i \ p_i \ q_i \quad \mathbb{R}$ the function F^{18} has the following form:

$$F^{18}[x_0 : x_1 : x_2] = [r_0\, x_0 + r_1\, x_1 + r_2\, x_2 : p_0\, x_0 + p_1\, x_1 + p_2\, x_2 : q_0\, x_0 + q_1\, x_1 + q_2\, x_2]$$

As S_0 is invariant then as discussed above we can write the function F^{18} as follows:

$$f^{18}(x \ y) = (p_0 + p_1\, x + p_2\, y \ \ q_0 + q_1\, x + q_2\, y) \tag{6.24}$$

for any $p_0 \ p_1 \ p_2 \ q_0 \ q_1 \ q_2 \quad \mathbb{R}$

As α_1 satisfy the equation $\alpha_1^6 + \alpha_1^3 + 1$ then by choosing any three roots of this equation for the values of α_1 we find that the f has three fixed points $fix_1 = (x_1 \ y_1) \ fix_2 = (x_2 \ y_2) \ fix_3 = (x_3 \ y_3)$ As these points are fixed by f so they are also fixed points of f^{18} Then finding the images of $fix_1 \ fix_2$ and fix_3 under the action of f^{18} by using (6.24). Then by solving the system of equations for the values of $p_0 \ p_1 \ p_2 \ q_0 \ q_1 \ q_2$ we find that $(p_0 \ p_1 \ p_2 \ q_0 \ q_1 \ q_2) = (0 \ 1 \ 0 \ 0 \ 0 \ 1)$ which implies that $f^{18}(x \ y) = (x \ y)$ This completes the proof. ∎

Theorem 48 *Let $f(x \ y)$ be a mappings such that the corresponding F satisfies the hypothesis of Theorem 42 and it has zero entropy. If the sequence of degrees is $12-$periodic or grows quadratically then f can be written in one of the following way:*

(1)
$$f(x \ y) = \left(\alpha_0 - x + y \ \ \frac{x}{-1+y}\right)$$

then f has quadratic growth rate and preserves the following elliptic curve:

$$C(x \ y) = \frac{y-1}{x\,y^2 - x^2\,y + \alpha_0\,x\,y + y - 1}$$

with $C(f(x \ y)) = C(x \ y)$ Hence f is integrable.

(2)

$$f(x\ \ y) = \left(x + y\ \ \frac{x}{-1+y}\right)$$

then f has quadratic growth rate and it preserves the elliptic fibration

$$C(x\ \ y) = \frac{(-2\,y^2 + 2\,x + y + 1)}{x\,y\,(x+y)}$$

with $C(f(x\ \ y)) = -C(x\ \ y)$ Hence f^2 is integrable.

(3)

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + y\ \ \frac{x}{\gamma_0 + y}\right)$$

with one of the following conditions on the parameters

(a) $\alpha_1^2 + 1 = 0\ \ \alpha_0 = 0\ \ \ \gamma_0 = 1$ *then f has quadratic growth rate and it preserves the elliptic fibration*

$$C(x\ \ y) = \frac{C'}{x\,y\,(\alpha_1\,x + y)}$$

with $C(f(x\ \ y)) = -\alpha_1\,C(x\ \ y) + 2\,(1+I)$ *where* $C' = -\alpha_1 + (1-\alpha_1)\,x + (1 - 2\,\alpha_1)\,y + 2\,x\,y + (1-\alpha_1)\,y^2 + 2\,\alpha_1\,x^2\,y + 2\,x\,y^2$ *or*

(b) $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0\ \ \ \alpha_0 = -\frac{1}{5}(3\alpha_1^3 + \alpha_1^2 + 4\alpha_1 + 2)$ *and* $\gamma_0 = -\frac{1}{\alpha_1^2}$ *or*

(c) $\alpha_1^4 + 1 = 0\ \ \ \alpha_0 = 1 - \alpha_1^3$ *and* $\gamma_0 = -\frac{1}{\alpha_1^2}$

Proof. From Proposition 44 we know that if $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) = O_1$ then the for the values of $(k\ \ p)\quad A_{(k,p)}$ the dynamical degree of F is equal to 1. Consider the case $(k\ \ p) = (1\ \ 4)$ then we have $\tilde{F}^2(A_1) = O_1$ and $\tilde{F}^4(A_2) = O_0$ which for $A_2\quad S_1$ by looking at Proposition 6 of Appendix we find that condition on the coefficients of f have to satisfy condition $(C7)$ which give us the family (1) Also the condition $\tilde{F}^4(A_2) = O_0$ is equivalent to $F^4(A_2) = O_0$ when the point $A_2\quad S_0\ \ S_1$ Looking at the same Proposition of the Appendix we find that the conditions on the coefficients of f have to satisfy conditions $(C8\ \ C9\ \ C10\ \ C11)$ which give us the required families $(2\ \ 3a\ \ 3b\ \ 3c)$ of theorem.

- We see that the sequence of degrees d_n for this case can be written as follows:

$$d_n = c_0\,(-1)^n + c_1\,n\,(-1)^n + c_2 + c_3\,n + c_4\,n^2 + c_5\,n^3 + c_6\,\lambda_1^n + c_7\,\lambda_2^n + c_8\,\lambda_3^n + c_9\,\lambda_4^n$$

where $\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4$ are the roots of $(x^2+1)(x^2+x+1)$ Since our map is an automorphisms we know it cannot have linear growth rate. Therefore d_n is either 12-periodic or $c_4 = 0$ and d_n has quadratic growth. But we find that the sequence of degrees for f grows quadratically, we do this by finding the values of the constants c_i Utilizing the information from the sequence of degrees of f for the families $(1 \ 2 \ 3(a) \ 3(c))$ that is:

$$2 \ \ 3 \ \ 5 \ \ 7 \ \ 11 \ \ 15 \ \ 20 \ \ 25 \ \ 32 \ \ 39$$

we find that the sequence of degrees d_n can be written as follows:

$$d_n = \frac{1}{8}\left(-\frac{3}{2}(-1)^n - \iota^n - (-\iota)^n + \frac{23}{2} + 3\,n^2\right)$$

which shows that f has quadratic growth.

By using the methodology defined in the proof of Theorem 46, we start by looking for one invariant cubic curve C_1 under f We first impose that this curve passes through the three indeterminacy points of F We then look for the curve $C(F^{-1})$ This curve is a product of the exceptional curve of F^{-1} and another distinct curve C'. We impose that C' becomes equal to C_1 such that it becomes invariant under F We then we find that $C_1 = x\,y^2 - x^2\,y + \alpha_0\,x\,y + y - 1$ We then consider another general cubic C_2 such that the function $C = \frac{C_2}{C_1}$ satisfies $C(f(x \ y)) = w_1\,C(x \ y) + w_2$ for some $w_1 \ \ w_2 \quad \mathbb{C}$ We then find that $C(x \ y) = \frac{y-1}{x\,y^2 - x^2\,y + \alpha_0\,x\,y + y - 1}$ satisfying $C(f(x \ y)) = C$ which shows that $w_1 = 1$ and $w_2 = 0$ for this family and hence f is integrable.

- The proof is similar as explained in the proof of above given result with $L_3 = \ x + y = 0$ Moreover in this case we find that $w_1 = -1$ and $w_2 = 0$

- The proof is similar as explained in the proof of of above given result. Moreover in this case we find that $w_1 = -\alpha_1$ and $w_2 = 2\,(1 + I)$

Hence proved. ∎

Theorem 49 *Let $f(x \ y)$ be a mappings such that the corresponding F satisfies the hypothesis of Theorem 42 and it has zero entropy. If the sequence of degrees is $30-$periodic or grows quadratically then f can be written in one of the following way:*

(1)

$$f(x\ \ y) = \left(\frac{1}{4} + x + y\ \ \frac{x}{-\frac{1}{2} + y}\right)$$

and d_n grows quadratically.

(2)

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + y\ \ \frac{x}{\gamma_0 + y}\right)$$

with $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0\ \ \gamma_0 = -\frac{1}{\alpha_1^2(1+\alpha_1)}\ \ \alpha_0 = -(\alpha_1^3 + 2\alpha_1^2 + \alpha_1 + 2)$

Proof. From Proposition 44 we know that if $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) = O_1$ then the for the values of $(k\ \ p)$ $A_{(k,p)}$ the dynamical degree of F is equal to 1. Now consider the case $(k\ \ p) = (2\ \ 3)$ then we have that $\tilde{F}^4(A_1) = O_1$ and $\tilde{F}^3(A_2) = O_0$ which is equivalent to $F^3(A_2) = O_0$ because the point A_2 $S_0\ \ S_1$ Looking at Proposition of the Appendix we find the conditions on the coefficients of f have to satisfy conditions $(C5)$ and $(C6)$ we get our required families (1) and (2) We claim that the map (1) is not periodic. We know that d_n is either periodic or period 30 or it grows quadratically. Hence if f is periodic then $f^r(x\ \ y) = (x\ \ y)$ for r a multiple of 30 Now we notice that $f^5(0\ \ y) = (0\ \ \frac{-4y}{4y-3})$ Calling $h(y) = \frac{-4y}{4y-3}$ from the results of [CGM] we know that $h(y)$ is not periodic. It implies that $f^r = Id$ for any r multiple of 30

Do not falter or shrink. Just think out your work, and work out your think - Nixon Waterman

Each problem that I solved became a rule which served afterwards to solve other problems - Rene Descartes

Chapter 7

Lyness K periodic

This chapter discusses the non-autonomous Lyness type recurrences with parameters as k-periodic sequence of complex numbers with minimal period k. Such non-autonomous recurrences are treated by the autonomous dynamical system generated by the birational mapping. The dynamical degree of such mappings is studied for four different values of k and then by using these results the main results on the non-integrability of these systems is established.

7.1 Introduction

Consider the non-autonomous Lyness difference equations of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n} \tag{7.1}$$

where $a_n \ {}_n$ is a k-periodic sequence of real numbers. Such recurrences have been studied in [CGM12, dA05, FJL96, JKN07, KN04, CGM08a, dA05], and recently in [CGM].

In discrete integrability community there is a rising interest in the study of integrability of the systems of difference equations with periodic coefficients. The integrability of some maps coming from periodic difference equations have been studied in [CGM08b, FJL96, GRT11a, GRT11b, CGM07, CGM06b, CGM06a] and [RGW11]. However the investigation in the area of non integrability of periodic difference equations is rare. The work in this article makes an evident contribution in the area of non integrable periodic difference equations by providing large number of classes of non integrable difference equation with periodic coefficients.

From a mathematical biology point of view, it is interesting to study these non autonomous difference equations because they can be used to model different biological systems which possess the change in time. These biological systems can possess certain behavior due to some changes in their environment. We consider those changes which are cyclic. That is to say that the parameters of the biological system are periodic.

In our study we consider the more general case when the parameters and the variables belong to the complex space.

For each k, the *composition mappings* are

$$F_{a_k,\ldots,a_2,a_1} := F_{a_k} \circ \cdots \circ F_{a_2} \circ F_{a_1} \tag{7.2}$$

where each F_{a_i} is defined by

$$F_{a_i}(x \;\; y) = \left(y \;\; \frac{a_i + y}{x} \right)$$

and $a_1 \;\; a_2 \qquad a_k$ are the k elements of the cycle.

For the sake of shortness, we also will use the notation $F_{[k]} := F_{a_k,\ldots,a_2,a_1}$

For instance, when $k = 2$ by setting

$$a_n = \begin{cases} a & \text{for} \quad n = 2 \;\; + 1 \\ b & \text{for} \quad n = 2 \end{cases} \tag{7.3}$$

we get

$$F_{b,a}(x \;\; y) = F_b \circ F_a(x \;\; y) = \left(\frac{a+y}{x} \;\; \frac{a+bx+y}{xy} \right)$$

and when $k = 3$

$$a_n = \begin{cases} a & \text{for} \quad n = 3 \;\; + 1 \\ b & \text{for} \quad n = 3 \;\; + 2 \\ c & \text{for} \quad n = 3 \end{cases} \tag{7.4}$$

and

$$F_{c,b,a}(x \;\; y) = F_c \circ F_b \circ F_a(x \;\; y) = \left(\frac{a+bx+y}{xy} \;\; \frac{a+bx+y+cxy}{y\,(a+y)} \right)$$

Clearly the study of the dynamics of the recurrences given by (7.1) can be deduced from the dynamics generated by the composition mappings (7.2). It is known that for the cases $k \qquad 1 \;\; 2 \;\; 3 \;\; 6$ and for all values of the parameters, the mappings $F_a \;\; F_{b,a} \;\; F_{c,b,a}$ and $F_{f,e,d,c,b,a}$ have a rational first integral :

$$
\begin{aligned}
V_a(x\ \ y) &= \frac{a+(a+1)x+(a+1)y+x^2+y^2+x^2y+xy^2}{xy} \\
V_{b,a}(x\ \ y) &= \frac{ab+(a+b^2)x+(b+a^2)y+bx^2+ay^2+ax^2y+bxy^2}{xy} \\
V_{c,b,a}(x\ \ y) &= \frac{ac+(a+bc)x+(c+ab)y+bx^2+by^2+cx^2y+axy^2}{xy} \\
V_{f,e,d,c,b,a}(x\ \ y) &= \frac{af+(a+bf)x+(f+ae)y+bx^2+ey^2+cx^2y+dxy^2}{xy}
\end{aligned}
$$

In this work we prove that for $k \quad 1\ 2\ 3\ 6$ the corresponding mapping $F_{[k]}$ does not have a rational first integral for some values of the involved parameters. This result has also been stated in [CGM] from a numerical point of view. Here we give an analytical proof.

7.2 Global non-integrability

From the results discussed in Chapter 2 we known that the existence of a foliation of the space by algebraic invariant curves implies that the dynamical degree is one, see [Bel99] for instance, also [DF01]. In order to prove our results about the non-integrability of the mapping 7.1, we use the method of calculating the dynamical degree.

We want to emphasize that the method that we implement allows us to know the sequence d_n for the mappings under consideration. When the growth of this sequence is exponential the calculation of the degrees of the iterates quickly becomes unfeasible.

The main result of this chapter is the following:

Theorem 50 *For $k \quad 1\ 2\ 3\ 6$ there are some values of the parameters for which the mapping $F_{[k]}$ does not have a rational first integral.*

To prove Theorem 50 we begin by studying the cases $k = 4$ and $k = 5$ We calculate the dynamical degree of $F_{[4]}$ and $F_{[5]}$ for general values of the parameters, see Proposition 51 and 52. As usual we say that a set of k parameters $a_1\ a_2 \quad a_k$ is generic if $(a_1\ a_2 \quad a_k) \quad \mathbb{C}^k$ is an open and dense subset of $\mathbb{C}^k$ with the usual topology. We then consider one parametric families for $k = 7$ and $k = 11$ in sections 7 2 3 and 7 2 4 In both case we prove that for almost all the values of the parameter the algebraic entropy is positive. In the last section in 7 2 5 we prove Theorem 50.

7.2.1 Dynamical degree of $F_{d\ c\ b\ a}$

In order to compute the dynamical degree we look for the exceptional and indeterminacy set of the map.

It is easy to prove (see [FS92] and [Dil96, DF01]) that if there is no exceptional curve whose orbit lands on a point of indeterminacy, then the degree of f is multiplicative, i.e., $deg(f^n) = (deg(f))^n$ It easily implies that the dynamical degree coincides with the degree of the map.

If there is an exceptional curve which lands on a point of indeterminacy then we have to perform a series of blow ups and extend the map f at the new space X until we get an AS map $\tilde{f}$

Once we have such AS map we compute the action $\tilde{f}^*$ of $\tilde{f}$ on the $\mathcal{P}ic(X)$ of X We then consider the associated matrix of $\tilde{f}^*$ and we calculate it s characteristic polynomial. Finally the dynamical degree is the spectral radius of the matrix.

Proposition 51 *For a generic set of the values of the parameters $a\ b\ c\ d$ $\mathbb{C}$ the dynamical degree of the mapping $F_{d,c,b,a} = F_d \circ F_c \circ F_b \circ F_a$ is the largest root of the polynomial $z^3 - 2\,z^2 - 3\,z - 1$ which approximately is* 3 079595625

Proof. We consider the family of mappings $F_{d,c,b,a}(x\ \ y) := F_d \circ F_c \circ F_b \circ F_a(x\ \ y)$ which has the following expression:

$$F_{d,c,b,a}(x\ \ y) = \left(\frac{cxy + bx + a + y}{y\,(a+y)}\quad \frac{x\left(dya + dy^2 + cxy + bx + a + y\right)}{(a+y)\,(bx + a + y)} \right)$$

By extending it to $P\mathbb{C}^2$ we get the mapping $f[x_0 : x_1 : x_2]$ with components

$$\begin{aligned}
f_1[x_0 : x_1 : x_2] &= x_0x_2(ax_0 + x_2)(ax_0 + bx_1 + x_2)\\
f_2[x_0 : x_1 : x_2] &= x_0(ax_0 + bx_1 + x_2)(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2)\\
f_3[x_0 : x_1 : x_2] &= x_1x_2(ax_0^2 + bx_0x_1 + (1 + ad)x_0x_2 + cx_1x_2 + dx_2^2)
\end{aligned}$$

In order to find the exceptional locus of f we calculate the determinant of the jacobian of f which we call j_f and it is given by

$$\begin{aligned}
&j_f = 4x_0x_2(ax_0 + x_2)(ax_0 + bx_1 + x_2)^2(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2)\\
&(ax_0^2 + bx_0x_1 + (1 + ad)x_0x_2 + cx_1x_2 + dx_2^2)
\end{aligned}$$

For $i = 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5$ let S_i be defined by $g_i[x_0 \;\; x_1 \;\; x_2] = 0$ with:

$$
\begin{aligned}
S_0 :=& \quad x_0 = 0 \\
S_1 :=& \quad x_2 = 0 \\
S_2 :=& \quad ax_0 + x_2 = 0 \\
S_3 :=& \quad ax_0 + bx_1 + x_2 = 0 \\
S_4 :=& \quad ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2 = 0 \\
S_5 :=& \quad ax_0^2 + bx_0x_1 + (1 + ad)x_0x_2 + cx_1x_2 + dx_2^2 = 0
\end{aligned}
$$

We see that for generic values of the parameters the curves $g_i[x_0 \;\; x_1 \;\; x_2] = 0$ are irreducible and distinct.

Applying the mapping we see that each S_i collapses to A_i, where

$$A_0 := [0 : 0 : 1] \;\; A_1 := [0 : 1 : 0] \;\; A_2 := [0 : b : -a] \;\; A_4 := [c : 0 : -d] \;\; A_5 := [1 : -d : 0]$$

and S_3 collapses to A_0 too.

On the other hand the indeterminacy set of f is given by the five points:

$$O_0 := [0 : 0 : 1] \;\; O_1 := [0 : 1 : 0] \;\; O_2 := [0 : -d : c] \;\; O_3 := [b : -a : 0] \;\; O_4 := [1 : 0 : -a]$$

We observe that $A_0 = O_0 \quad A_1 = O_1$ and that for generic values of the parameters $A_i = O_j$ for $i = 2 \;\; 4 \;\; 5 \quad j = 2 \;\; 3 \;\; 4$

Since $S_0 \quad S_1$ collapse to $O_0 \quad O_1$ respectively, the dynamical degree is less than two. Therefore we begin by blowing up the points O_0 and O_1

Let X be the space we get after blowing up the two points $O_0 \;\; O_1$ Now we are going to extend the mapping f to X on a continuous way. To this end we identify $E_0 := \pi^{-1}(O_0)$ with $P\mathbb{C}^1$ in the following way: given $[u : v] \quad P\mathbb{C}^1$ we associate the point

$$[u : v]_{E_0} := \lim_{t \quad 0} \pi^{-1}[tu : tv : 1] \quad E_0 \tag{7.5}$$

From now on we are going to identify a set $S \subset P\mathbb{C}^2$ with the set $\pi^{-1}(S) \subset X$

To determine the mapping on $S_0 = \quad x_0 = 0 \quad$ let $x = [0 : x_1 : x_2] = \lim_{t \quad 0}[t : x_1 : x_2] \quad S_0$ We assign:

$$\tilde{f}(x) = \lim_{t \quad 0} f[t : x_1 : x_2] \; = \; \lim_{t \quad 0}[tx_2^2(bx_1 + x_2) : tcx_1x_2(bx_1 + x_2) : x_1x_2^2(cx_1 + dx_2)]$$

Assuming $x_1x_2(cx_1 + dx_2) = 0$ we get

$$\tilde{f}(x) = \lim_{t \quad 0} \left[t\,\frac{bx_1 + x_2}{x_1(cx_1 + dx_2)} : t\,\frac{c(bx_1 + x_2)}{x_2(cx_1 + dx_2)} : 1 \right]$$

and we identify this point with:

$$\left[\frac{bx_1 + x_2}{x_1(cx_1 + dx_2)} : \frac{c(bx_1 + x_2)}{x_2(cx_1 + dx_2)} \right]_{E_0} \equiv [x_2 : cx_1]_{E_0}$$

If $x_1x_2(cx_1 + dx_2) = 0$ we have the points $O_0 = [0 : 0 : 1]$ $O_1 = [0 : 1 : 0]$ and $O_2 = [0 : -d : c]$ Then we have that

$$\tilde{f} : S_0 \quad \mathcal{I}(f) - \quad E_0$$

is defined trough

$$\tilde{f}\,[0 : x_1 : x_2] = [x_2 : cx_1]_{E_0} \tag{7.6}$$

To determine the mapping $\tilde{f}$ on E_0 we consider a point $[u : v]_{E_0}$ in the fibre E_0 as shown in (7.5). We need to evaluate $f[tu : tv : 1]$ Its three components are given by

$$\begin{aligned}
&t\,u\,(dtu + 1)\,(dtu + ctv + 1)\\
&tu\,\left(at^2u^2 + bt^2vu + tu + ctv\right)(atu + btv + 1)\\
&tv\,\left(ctv + bt^2vu + at^2u^2 + d + tuda + tu\right)
\end{aligned}$$

Hence, $\lim_{t \quad 0} f[tu : tv : 1] = [u : 0 : dv]$ Calling $T_1 = \quad x_1 = 0 \quad$ we have that

$$\tilde{f} : E_0 - \quad T_1$$

is given by

$$\tilde{f}[u : v]_{E_0} = [u : 0 : dv] \tag{7.7}$$

On the other hand we notice that the action of f on T_1 is given by

$$f[x_0 : 0 : x_2] = [x_0x_2(ax_0 + x_2)^2 : x_0^2(ax_0 + x_2)^2 : 0]$$

If $x_0(ax_0+x_2) = 0$ we get the points $[0 : 0 : 1] = O_0$ and $[1 : 0 : -a] = O_4$ And if $x_0(ax_0+x_2) =$

0 then $f[x_0 : 0 : x_2] = [x_2 : x_0 : 0]$ Hence,

$$\tilde{f} : T_1 \quad \mathcal{I}(f) - \quad S_1$$

is given by

$$\tilde{f}[x_0 : 0 : x_2] = [x_2 : x_0 : 0] \tag{7.8}$$

The same type of arguments and computations allow us to extend f to S_1 and $E_1 := \pi^{-1}(O_1)$ We get that:

$$\tilde{f} : S_1 \quad \mathcal{I}(f) - \quad E_1$$

is defined by:

$$\tilde{f}[x_0 : x_1 : 0] = [ax_0 : x_1]_{E_1} \tag{7.9}$$

and

$$\tilde{f} : E_1 - \quad S_0$$

is given by

$$\tilde{f}[u : v]_{E_1} = [0 : bu : v] \tag{7.10}$$

Hence, from (7.6), (7.7), (7.8), (7.9) and (7.10) we see that we get a cycle between these complex 1-dimensional manifolds:

$$S_0 - \quad E_0 - \quad T_1 - \quad S_1 - \quad E_1 - \quad S_0 \tag{7.11}$$

On the other hand, since f maps S_3 to A_0 we have to extend f to S_3 The result that we get is that S_3 still collapses:

$$\tilde{f} : S_3 \quad \mathcal{I}(f) - \quad [b : -c]_{E_0} \tag{7.12}$$

Hence, S_3 is still exceptional for $\tilde{f}$

From the above calculations we have that the indeterminacy and the exceptional sets of $\tilde{f}$ are:

$$\mathcal{I}(\tilde{f}) = \quad O_2 \;\, O_3 \;\, O_4 \qquad \text{and} \qquad \mathcal{E}(\tilde{f}) = \quad S_2 \;\, S_3 \;\, S_4 \;\, S_5 \tag{7.13}$$

where we again identify sets and points on $P\mathbb{C}^2$ with the equivalent points in X

We claim that, for generic values of the parameters, $\tilde{f} : X \qquad X$ is AS To prove this claim we use Theorem 2.5. So we have to follow the orbits of S_i for $i = 2 \;\, 3 \;\, 4 \;\, 5$ and verify that $\tilde{f}^n(S_i)$

$\mathcal{I}(\tilde{f})$ for all n From (7.11) we observe that each one of the manifolds S_0 E_0 T_1 S_1 and E_1 are invariant by $\tilde{f}^5$ The calculations are the following:

- **The orbit of $\mathbf{S}_2$**

  Since $f(S_2) = A_2 = [0 : b : -a]$ we have that generically:

$$\begin{aligned}
&\tilde{f}^{5k}(A_2) = \tilde{f}^{5k}[0 : b : -a] = [0 : a^{k-1}b^{k+1}c^k d^k : -1] \quad S_0 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+1}(A_2) = \tilde{f}^{5k}[-a : bc]_{E_0} = [-1 : a^{k-1}b^{k+1}c^{k+1}d^k]_{E_0} \quad E_0 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+2}(A_2) = \tilde{f}^{5k}[-a : 0 : bcd] = [-1 : 0 : a^{k-1}b^{k+1}c^{k+1}d^{k+1}] \quad T_1 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+3}(A_2) = \tilde{f}^{5k}[bcd : -a : 0] = [a^{k-1}b^{k+1}c^{k+1}d^{k+1} : -1 : 0] \quad S_1 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+4}(A_2) = \tilde{f}^{5k}[bdc : -1]_{E_1} = [a^k b^{k+1}c^{k+1}d^{k+1} : -1]_{E_1} \quad E_1 \quad \mathcal{I}(\tilde{f})
\end{aligned}$$

- **The orbit of $\mathbf{S}_4$** Since $f(S_4) = A_4 = [-b : 0 : a]$ we have that:

$$\begin{aligned}
&\tilde{f}^{5k}(A_4) = \tilde{f}^{5k}[c : 0 : -d] = [1 : 0 : -a^k b^k c^{k-1}d^{k+1}] \quad T_1 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+1}(A_4) = \tilde{f}^{5k}[d : -c : 0] = [: a^k b^k c^{k-1}d^{k+1} : -1 : 0] \quad S_1 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+2}(A_4) = \tilde{f}^{5k}[-ad : c]_{E_1} = [-a^{k+1}b^k c^{k-1}d^{k+1} : -1]_{E_1} \quad E_1 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+3}(A_4) = \tilde{f}^{5k}[0 : -abd : c] = [0 : -a^{k+1}b^{k+1}c^{k-1}d^{k+1} : 1] \quad S_0 \quad \mathcal{I}(\tilde{f})\\
&\tilde{f}^{5k+4}(A_4) = \tilde{f}^{5k}[c : -abcd]_{E_0} = [c : -(abcd)^{k+1}]_{E_0} \quad E_0 \quad \mathcal{I}(\tilde{f})
\end{aligned}$$

Very similar computations show that generically $\tilde{f}^n(S_i)$ $\mathcal{I}(\tilde{f})$ for $i = 3$ 5

Hence, using 2.5 we see that $\tilde{f} : X -$ X is AS.

Finally we have to compute the matrix of $\tilde{f}^* : Pic(X) -$ $Pic(X)$ To this end we take into account the action of

$$\pi^* : Pic(P\mathbb{C}^2) =< L >- \quad Pic(X) =< \hat{L} \ E_0 \ E_1 >$$

Since O_0 O_1 S_0 and $\deg(S_0) = 1$ then we have that

$$\pi^*(S_0) = \hat{S}_0 + E_0 + E_1 \tag{7.14}$$

Similarly

$$\pi^*(S_1) = \hat{S}_1 + E_1 \tag{7.15}$$

and

$$\pi^*(S_3) = \hat{S}_3 \tag{7.16}$$

But in $Pic(P\mathbb{C}^2)$ all the curves of degree one are equivalent to L i. e., $\pi^*(S_i) = \pi^*(L) = \hat{L}$ for $i = 0\ 1\ 3$ It implies that

$$\hat{S}_0 = \hat{L} - E_0 - E_1 \quad \hat{S}_1 = \hat{L} - E_1 \text{ and } \hat{S}_3 = \hat{L}$$

Now we are ready to calculate the matrix of $\tilde{f}^*$ From (7.6), (7.12), (7.14) and (7.16) we deduce that:

$$\tilde{f}^*(E_0) = \hat{S}_0 + \hat{S}_3 = 2\hat{L} - E_0 - E_1$$

where $\tilde{f}^*$ acts by taking preimages. Similarly,

$$\tilde{f}^*(E_1) = \hat{S}_1 = \hat{L} - E_1$$

It only remains to determine $\tilde{f}^*(\hat{L})$ which we recall that $L = \quad px_0 + qx_1 + rx_2 = 0$ where the parameters $p\ q\ r$ are generic. Considering the algebraic curve of degree 4:

$$f^{-1}(L) = \quad [x_0 : x_1 : x_2] \quad P\mathbb{C}^2 : pf_1[x_0 : x_1 : x_2] + qf_2[x_0 : x_1 : x_2] + rf_3[x_0 : x_1 : x_2] = 0$$

we have that $\pi^*(f^{-1}(L))$ equals to its strict transform plus $m_1E_0 + m_2E_1$ where $m_1\ m_2$ are the multiplicities of $f^{-1}(L)$ at $O_0\ O_1$ respectively. It can easily be seen that $m_1 = 1$ and $m_2 = 2$ Hence,

$$\tilde{f}^*(\hat{L}) = 4\hat{L} - E_0 - 2E_1$$

and the matrix of $\tilde{f}^*$ is given by:

$$(\tilde{f}^*) = \begin{pmatrix} 4 & 2 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^3 - 2\lambda^2 - 3\lambda - 1$ which has a unique real root

$$\lambda_1 = \frac{1}{6}\sqrt[3]{388 + 12\quad 69} + \frac{26}{3}\,\frac{1}{\sqrt[3]{388 + 12\quad 69}} + \frac{2}{3} \approx 3\ 079595625$$

and two complex conjugated roots with modulus less than λ_1 So, Proposition 51 is proved. ∎

We observe that the sequence of the degrees d_n exactly satisfies the recurrence

$$d_{n+3} = 2d_{n+2} + 3d_{n+1} + d_n$$

and since $d_1 = 4 \ d_2 = 12$ and $d_3 = 37$ the sequence of the degrees is

$$4 \quad 12 \quad 37 \quad 114 \quad 351 \quad 1081 \quad 4059 \quad 11712$$

We emphasize that the above result is valid for generic values of the parameters and that for other values of the parameters the entropy can be changed. For example, if we take $a = \alpha \ b = \alpha\beta \ c = \beta \ d = 1 \ \alpha$ we get a mapping that is topologically conjugated to F_{1/ac^2} and so it is rationally integrable with zero entropy. Also the case $a = \alpha^2 \ b = \alpha \ c = 1 \ \alpha \ d = 1 \ \alpha^2$ has zero entropy: the corresponding mapping is 5-periodic (see [CGM]).

Next result deals with the case $k = 5$ The case k multiple of 5 is distinguished from others. This is because in this case, generically, F is well defined on the axes $x = 0$ and $y = 0$ these axes are invariant and $(0 \ 0)$ is a fixed point. Furthermore, when we extend the mapping to the projective space also the line at infinity is invariant. For general values of the parameters the entropy is approximately $\ln(4\ 079595625) \approx 1\ 405997872$ as shown in the next proposition.

7.2.2 Dynamical degree of $F_{e \ d \ c \ b \ a}$

Proposition 52 *For a generic set of the values of the parameters $a \ b \ c \ d \ e \quad \mathbb{C}$ the dynamical degree of the mapping $F_{e,d,c,b,a} = F_e \circ F_d \circ F_c \circ F_b \circ F_a$ is the largest root of the polynomial $z^3 - 5\,z^2 + 4\,z - 1$ which approximately is* $4\ 079595625$

Proof. In order to prove Proposition 52 we follow the same procedure as described above. Consider the composition mapping $F_{e,d,c,b,a} = F_e \circ F_d \circ F_c \circ F_b \circ F_a$ which has the following two components:

$$\frac{ax + bx^2 + (ad+1)xy + cx^2y + dxy^2}{(a+y)(a+bx+y)}$$

$$\frac{ea^2y + a\,(eb+1)\,xy + 2\,eay^2 + bx^2y + (eb+1+da)\,xy^2 + ey^3 + cx^2y^2 + dxy^3}{(a+bx+y+cxy)(a+bx+y)}$$

Let f be the extension of the above mapping to $P\mathbb{C}^2$. Then the homogeneous components of

$f[x_0\ \ x_1\ \ x_2] = [f_1 : f_2 : f_3]$ are the following:

$$
\begin{aligned}
f_1 &= x_0(ax_0 + x_2)(ax_0 + bx_1 + x_2)(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2) \\
f_2 &= x_1(ax_0^2 + bx_0x_1 + (1 + ad)x_0x_2 + cx_1x_2 + dx_2^2)(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2) \\
f_3 &= x_2(ax_0 + x_2)(a^2ex_0^3 + a(1 + be)x_0^2x_1 + 2aex_0^2x_2 + bx_0x_1^2 + (1 + be + ad)x_0x_1x_2 \\
&\quad +ex_0x_2^2 + cx_1^2x_2 + dx_1x_2^2)
\end{aligned}
$$

Since the jacobian of this mapping is:

$$
\begin{aligned}
j_f &= 5(ax_0 + x_2)^2(ax_0 + bx_1 + x_2)(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2)^2(ax_0^2 + bx_0x_1 + \\
&\quad (1 + ad)x_0x_2 + cx_1x_2 + dx_2^2)(a^2ex_0^3 + a(1 + be)x_0^2x_1 + 2aex_0^2x_2 + bx_0x_1^2 \\
&\quad +(1 + be + ad)x_0x_1x_2 + ex_0x_2^2 + cx_1^2x_2 + dx_1x_2^2)
\end{aligned}
$$

we have the following five exceptional curves S_i where we define S_i as $g_i[x_0\ \ x_1\ \ x_2] = 0$ for $i\quad 1\ 2\ 3\ 4\ 5\ :$

$$
\begin{aligned}
S_1 &:= ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2 = 0 \\
S_2 &:= ax_0 + x_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
S_3 &:= ax_0 + bx_1 + x_2 = 0 \\
S_4 &:= ax_0^2 + bx_0x_1 + (1 + ad)x_0x_2 + cx_1x_2 + dx_2^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
S_5 &:= a^2ex_0^3 + a(1 + be)x_0^2x_1 + 2aex_0^2x_2 + bx_0x_1^2 + (1 + be + ad)x_0x_1x_2 + ex_0x_2^2 \\
&\quad +cx_1^2x_2 + dx_1x_2^2 = 0
\end{aligned}
$$

Each S_i collapses to A_i where $A_i \quad \mathcal{I}(f^{-1})$ and they are as follows:

$$A_1 := [0:0:1] \;\; A_2 := [0:1:0] \;\; A_3 := [0:-c:b] \;\; A_4 := [-d:0:e] \;\; A_5 := [1:-e:0]$$

We can see that for generic values of the parameters all these g_is are distinct and irreducible. The indeterminacy points of f are:

$$O_1 := [0:0:1] \;\; O_2 := [0:1:0] \;\; O_3 := [0:-d:c] \;\; O_4 := [-b:a:0] \;\; O_5 := [1:0:-a]$$

We observe that $O_1 = A_1 \; O_2 = A_2$ and that generically $O_i = A_j$ for all $i \; j = 3 \; 4 \; 5$ To regularize our f so that $\tilde{f}$ is AS we need to follow the orbit of each A_i so that we can see if it reaches an indeterminacy point of f. In our case we see that $S_1 \quad A_1 = O_1$ and $S_2 \quad A_2 = O_2$, while $S_3 \quad A_3 \quad x_0 = 0$, $S_4 \quad A_4 \quad x_1 = 0$ and $S_5 \quad A_5 \quad x_2 = 0$. Also we observe that the straight lines $x_0 = 0 \quad x_1 = 0$ and $x_2 = 0$ are invariant under f. More precisely, for all $k \quad \mathbb{N}$:

$$\begin{aligned} f^k[0:x_1:x_2] &= [0:c^k x_1:x_2] \\ f^k[x_0:0:x_2] &= [x_0:0:e^k x_2] \\ f^k[x_0:x_1:0] &= [a^k x_0:x_1:0] \end{aligned}$$

Therefore, for all $k \quad \mathbb{N}$:

$$\begin{aligned} f^k(A_3) &= [0:c^{k+1}:-b] \\ f^k(A_4) &= [-d:0:e^{k+1}] \\ f^k(A_5) &= [a^k:-e:0] \end{aligned}$$

We observe that for generic values of the parameters the orbits of $A_3 \; A_4$ and A_5 will never reach any indeterminacy point of f. Hence we have only two points O_1 and O_2 which we need to blow up. Let X be the space we get after blowing up O_1 and O_2 and let E_1 and E_2 be the exceptional fibres correspondingly. Let $\tilde{f}$ be the corresponding extension of f to X. To determine $\tilde{f}$ on $S_1 \; S_2 \; E_1$

and E_2 after some computations we see that:

$$\tilde{f} : S_1 \quad \mathcal{I}(f) \qquad [c : -d]_{E_1} \qquad\qquad \tilde{f} : S_2 \quad \mathcal{I}(f) \qquad [b : -a]_{E_2}$$

and

$$\tilde{f}[u : v]_{E_1} = [u : dv]_{E_1} \qquad\qquad \tilde{f}[u : v]_{E_2} = [bu : v]_{E_2}$$

This means that S_1 and S_2 are still exceptional for $\tilde{f}$ and that $E_1 \;\; E_2$ are invariant for $\tilde{f}$ But it is easy to see that $\tilde{f}^k(S_1) = [c : -d^k]_{E_1}$ and $\tilde{f}^k(S_2) = [b^k : -a]_{E_2}$ hence for generic values of parameters the points $[c : d^k]_{E_1}$ and $[b^k : -a]_{E_2}$ for all k can never reach any indeterminacy point of $\tilde{f}$. It is now clear that $\tilde{f}$ is AS.

In order to compute the matrix $\tilde{f}^* : Pic(X) - \quad Pic(X)$ we first take into account

$$\pi^* : Pic(P\mathbb{C}^2) = < L >- \quad Pic(X) = < \hat{L} \;\; E_1 \;\; E_2 >$$

Since $\deg(S_1) = 2$ and $O_1 \;\; O_2 \quad S_1$ with multiplicity 1, we have that

$$\pi^*(S_1) = \hat{S}_1 + E_1 + E_2 \tag{7.17}$$

Similarly

$$\pi^*(S_2) = \hat{S}_2 + E_2 \tag{7.18}$$

and

$$\pi^*(f^{-1}(L)) = \widehat{f^{-1}(L)} + E_1 + 2E_2 \tag{7.19}$$

because O_2 has multiplicity 2 in the components of $f[x_0 : x_1 : x_2]$ Now in $Pic(P\mathbb{C}^2)$

$$\hat{S}_1 = 2\hat{L} - E_1 - E_2 \quad \text{ and } \quad \hat{S}_2 = \hat{L} - E_2$$

Therefore by using the above equations we have:

$$\begin{aligned}
\tilde{f}^*(\hat{L}) &= 5\hat{L} - E_1 - 2E_2 \\
\tilde{f}^*(E_1) &= \hat{S}_1 + E_1 = 2\hat{L} - E_2 \\
\tilde{f}^*(E_2) &= \hat{S}_2 + E_2 = \hat{L}
\end{aligned}$$

The matrix of $\tilde{f}^*$ is:

$$(\tilde{f}^*) = \begin{pmatrix} 5 & 2 & 1 \\ -1 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^3 - 5\lambda^2 + 4\lambda - 1$ which has the unique real root

$$\lambda_1 \cong 4\ 079595625$$

and two complex conjugated roots with modulus less than λ_1 Hence proved. ∎

We can see that the sequence of the degrees d_n satisfies the recurrence

$$d_{n+3} = 5d_{n+2} - 4d_{n+1} + d_n$$

Since $d_1 = 5\ \ d_2 = 21$ and $d_3 = 86$ the sequence of the degrees is

$$5\quad 21\quad 86\quad 351\quad 1432\quad 5842\quad 23833$$

Now let $a\ \ b\ \ c\ \ d$ such that the mapping $F_{d,c,b,a}$ has dynamical degree equal to the largest root of the polynomial $z^3 - 2\,z^2 - 3\,z - 1$ and consider $F_{d,c,b,a,d,c,b,a} = F_{d,c,b,a} \circ F_{d,c,b,a}$ Then the dynamical degree of such a mapping is the square of the dynamical degree of $F_{d,c,b,a}$ (see Lemma 55 in section 7.2.5), and so it is also greater than 1 This remark proves the theorem for the case $k = 8$ To prove the theorem for the case $k = 9$ we recall that the Lyness mapping with $a = 1$ is five periodic and we take into account $F_{1,1,1,1,1,d,c,b,a} = F_{1,1,1,1,1} \circ F_{d,c,b,a}$ In fact this mapping actually is $F_{d,c,b,a}$ itself, but it proves that there are values of the parameters when $k = 9$ such that the corresponding mapping has dynamical degree greater than one. Thus the algebraic entropy is greater than zero.

7.2.3 Dynamical degree of $F_{a\ 1\ a\ 1\ a\ 1\ a}$

In order to cover all the cases we also need some families with positive entropy when $k = 7$ and $k = 11$ We do not do the general case because the computations are too large. Hence we give two $1-$parametric families which are some subfamilies of the general $F_{[7]}$ and $F_{[11]}$ families. The general cases are avoided as they involve tedious and much larger calculations. We will prove that in general $F_{[7]}$ and $F_{[11]}$ are not integrable by giving a proof for the $\delta > 1$ for these two $1-$parametric subfamilies.

For the case $k = 7$ we calculate the entropy of $F_{a,1,a,1,a,1,a}$ In general these mappings have two exceptional curves which have degrees 4 and 5, but they have genus zero. As it is well known, the curves of genus zero have a rational parametrization. The existence of such a parameterizations has been useful to deduce the behavior of the induced mapping in the Picard group.

Proposition 53 *For all* $a \quad \mathbb{C} \quad a = 0 \quad a^p = 1$ *for all* $p \quad \mathbb{N}$ *the dynamical degree of the one-parametric family of mappings* $F_{a,1,a,1,a,1,a}$ *is the largest root of the polynomial* $z^3 - 3z^2 + z - 1$ *which approximately is* 2 769292354

Proof. For the case $k = 7$ we consider the one parametric family

$$F_{a,1,a,1,a,1,a} = F_a \circ F_1 \circ F_a \circ F_1 \circ F_a \circ F_1 \circ F_a$$

The extension f of the mapping $F_{a,1,a,1,a,1,a}$ in $P\mathbb{C}^2$ has the form:

$$f[x_0 \;\; x_1 \;\; x_2] = [x_1\, x_2\, g_1\, g_2\, g_3 : x_2\, g_3\, g_4\, g_5 : x_0\, g_1\, g_6\, g_7]$$

where

$$\begin{aligned} g_1 &= a x_0^2 + x_0 x_1 + x_0 x_2 + a x_1 x_2 \\ g_2 &= a x_0^2 + x_0 x_1 + (1+a) x_0 x_2 + a x_1 x_2 + x_2^2 \end{aligned}$$

$$g_3 = a^3 x_0^3 + a(a+1) x_0^2 x_1 + 2a^2 x_0^2 x_2 + x_0 x_1^2 + a x_0 x_2^2 + a x_1^2 x_2 + x_1 x_2^2 + (1+2a) x_0 x_1 x_2$$

$$\begin{aligned} g_4 = \;& a^2 x_0^4 + 2a x_0^3 x_1 + a(a^2+2) x_0^3 x_2 + x_0^2 x_1^2 + (2a^2+1) x_0^2 x_2^2 + (2+a+2a^2) x_0^2 x_1 x_2 + \\ & (1+a) x_0 x_1^2 x_2 + (3a+1) x_0 x_1 x_2^2 + a x_0 x_2^3 + a x_1^2 x_2^2 + x_1 x_2^3 \end{aligned}$$

$$\begin{aligned} g_5 &= ax_0 + x_2 \\ g_6 &= ax_0 + x_1 + x_2 \end{aligned}$$

$$\begin{aligned} g_7 = {} & a^3x_0^5 + a^2(2+a)x_0^4x_1 + a^2(3+a^2)x_0^4x_2 + a(a+2)x_0^3x_1^2 + 3a(1+a^2)x_0^3x_2^2 + \\ & a(4+3a+3a^2)x_0^3x_1x_2 + (1+3a+2a^2+2a^3)x_0^2x_1^2x_2 + (2+3a+7a^2)x_2^2x_1x_2^2 + \\ & ax_0^2x_1^3 + (3a^2+1)x_0^2x_2^3 + 2a^2x_0x_1^3x_2 + (1+2a+3a^2+a^3)x_0x_1^2x_2^2 + \\ & (1+5a)x_0x_1x_2^3 + ax_0x_2^4 + a^3x_1^3x_2^2 + a(a+1)x_1^2x_2^3 + x_1x_2^4 \end{aligned}$$

The jacobian of f is $j_f = 9\, x_2\, {g_1}^2\, g_2\, {g_3}^2\, g_4\, g_5\, g_6\, g_7$ So, we have the following eight exceptional curves S_i for $i = 1\ 2 \quad 8$:

$$S_1 := \ x_2 = 0 \quad S_2 := \ g_5 = 0 \quad S_3 := \ g_6 = 0 \quad S_4 := \ g_2 = 0$$
$$S_5 := \ g_4 = 0 \quad S_6 := \ g_7 = 0 \quad S_7 := \ g_1 = 0 \quad S_8 := \ g_3 = 0$$

Each S_i collapses to A_i where $A_i \quad \mathcal{I}(f^{-1})$ for $i = 1\ 2 \quad 7$ and S_8 collapses to A_1 too, where:

$$A_1 := [0:0:1] \quad A_2 := [-1:0:1] \quad A_3 := [a:-1:0] \quad A_4 := [0:a:-1]$$

$$A_5 := [1:0:-a] \quad A_6 := [1:-a:0] \quad A_7 := [0:1:0]$$

The indeterminacy points of f are

$$O_1 := [0:0:1] \quad O_2 := [1:0:-a] \quad O_3 := [-a:0:1] \quad O_4 := [0:1:0]$$

$$O_5 := [0:1:-a] \quad O_6 := [-1:1:0] \quad O_7 := [1:-a:0]$$

In order to regularize our f we see that in this family we have the following situation:

$$\begin{aligned} &S_1 \quad A_1 = O_1 \\ &S_2 \quad A_2 \quad S_4 \end{aligned}$$

$$S_3 \quad A_3 \quad S_1$$
$$S_4 \quad A_4 \quad e_1 \quad S_1$$
$$S_5 \quad A_5 = O_2$$

$$S_6 \quad A_6 = O_7$$
$$S_7 \quad A_7 = O_4$$
$$S_8 \quad A_1 = O_1$$

where $e_1 = [-a^2 : 1 : 0]$. Therefore we need to blow up A_1 A_5 A_6 and A_7. Let X be the space we get after blowing up the points A_1 A_5 A_6 and A_7 and let E_1 E_2 E_3 and E_4 be the corresponding exceptional fibres. Let $\tilde{f}$ be the extension of f on X.

We begin by determining $\tilde{f}$ on the curves S_1 S_7 and S_8 After some calculations we see that $\tilde{f}$ sends $S_1 = \quad x_2 = 0 \quad$ to the fibre E_1 in the following way:

$$[x_0 : x_1 : 0] \quad S_1 \quad [x_1 : a^2 x_0]_{E_1}$$

and that S_7 and S_8 are still exceptional for $\tilde{f}$, because

$$S_7 \quad [1 : -a]_{E_4} \quad \text{and} \quad S_8 \quad [a : -1]_{E_1}$$

On the other hand, on E_1 and E_4 $\tilde{f}$ acts as:

$$\tilde{f}[u : v]_{E_1} = [v : u]_{E_4} \quad \text{and} \quad \tilde{f}[u : v]_{E_4} = [v : 0 : a\,u] \quad x_1 = 0 \quad := T_1 \quad \mathcal{E}(f^{-1})$$

The mapping $\tilde{f}$ also sends E_2 and E_3 on some exceptional curves of f^{-1} which we call T_2 and T_3 respectively.

To determine $\tilde{f}$ on S_5 and S_6 we observe that the algebraic curves which define S_5 and S_6 have genus $g = 0$ In fact, $S_5 = \quad [x_0 \ x_1 \ x_2] : g_4 = 0 \quad$ can be parametrized trough $\quad (t)$ where

$$\begin{aligned}(t) \quad = \quad & [(t+a^2)(t+t^2+a+ta) : -1+3\,a+t+t^2-4\,a^3+a^5+ta^4+ta-2\,t^2a \\ & -2\,ta^3-t^3a-4\,ta^2-2\,t^2a^2 : -(a^2+2\,a+2\,ta-1+t+t^2)(t+t^2+a+ta)]\end{aligned}$$

and $S_6 = \quad [x_0 \ x_1 \ x_2] : g_7 = 0 \quad$ can be parametrized through $\gamma(t)$ where the three components of $\gamma(t)$ are:

$$(a-1)\,((a-1)\,t+a^4-a^3-a^2+a-1)((a-1)\,t-a-a^2+a^3)\big((2\,a-a^2-1)\,t^2+(a^2-1)\,t+a^5-2\,a^4+2\,a^2-2\,a\big)$$

$$(1-a)\left(p_3(a)t^3+p_2(a)t^2+p_1(a)t+p_0(a)\right)\left((a^2-a)t+a^3-a^2-2a+1\right)$$

and

$$(-1+(a-1)\,t)\left((2\,a-a^2-1)t^2+(-a^3-a+3\,a^2-1)\right)\left(t+a^5-3\,a^4+3\,a^2+2\,a^3+1-5\,a\right)$$

where

$$\begin{aligned} p_3(a): &= -a^3+3\,a^2-3\,a+1 \\ p_2(a): &= a^3-3\,a+2 \\ p_1(a): &= a^7-a^6-5\,a^5+9\,a^4-a^3-9\,a^2+6\,a \\ p_0(a): &= a^9-3\,a^8+a^7+4\,a^6-5\,a^5+4\,a^4-a^3-4\,a^2+5\,a-1 \end{aligned}$$

From these parameterizations it is very easy to see that f maps S_5 to $A_5=[1:0:-a]$ and S_6 to $A_6=[1:-a:0]$

On the other hand to determine $\tilde{f}$ on S_5, we consider the following perturbation:

$$\lim_{s\ \ 0}[\ \ {}_1(t)+sh_1(s)\quad {}_2(t)+sh_2(s)\quad {}_3(t)+sh_3(s)]$$

where $h_1\ \ h_2\ \ h_3$ are analytic functions on s near $s=0$ Applying F we get:

$$\tilde{f}(\ \ (t))=[1:(1+t)s+o(2):-a+(a^2-1)(t+a+1)s+o(2)]\equiv[(1+t):(a^2-1)(t+a+1)]_{E_2}$$

where $o(2)$ means terms of order two in s

In a similar way, the expression of $\tilde{f}$ on S_6 is

$$\tilde{f}(\gamma(t))=[\left(a^2-1\right)(-1+(a-1)t):(a-1)t+a^4-a^3-a^2+a-1]_{E_3}$$

From the above calculations we have the following indeterminacy points of $\tilde{f}$:

$$\mathcal{I}(\tilde{f})=\ \ O_3\ \ O_5\ \ O_6$$

And the exceptional locus of $\tilde{f}$ is

$$\mathcal{E}(\tilde{f})=\ \ S_2\ \ S_3\ \ S_4\ \ S_7\ \ S_8$$

Hence we observe the following:

$$\begin{array}{ccccc} S_2 & A_2 & A_4 & e_1 & S_1 \\ & & S_3 & A_3 & S_1 \\ & S_4 & A_4 & e_1 & S_1 \end{array}$$

Also S_7 collapses to the single point $[1 : -a]_{E_4}$ and S_8 collapses to $[-a : 1]_{E_1}$ We observe that

$$S_1 \qquad E_1 \qquad E_4 \qquad x_1 = 0 \qquad x_0 = 0 \qquad S_1 = \quad x_2 = 0 \tag{7.20}$$

where $\tilde{f}[0 : x_1 : x_2] = [ax_1 : x_2 : 0]$ and $\tilde{f}[x_0 : 0 : x_2] = [0 : ax_2 : x_0]$ Hence we get that for all $k \quad \mathbb{N}$:

$$\begin{aligned} \tilde{f}^{5k}[0 : x_1 : x_2] &= [0 : a^{4k}x_1 : x_2] \\ \tilde{f}^{5k}[x_0 : 0 : x_2] &= [x_0 : 0 : a^{4k}x_2] \\ \tilde{f}^{5k}[x_0 : x_1 : 0] &= [a^{4k}x_0 : x_1 : 0] \\ \tilde{f}^{5k}[u : v]_{E_1} &= [u : a^{4k}v]_{E_1} \\ \tilde{f}^{5k}[u : v]_{E_4} &= [a^{4k}u : v]_{E_4} \end{aligned}$$

Therefore by using the cycle $(7\ 20)$ and by the help of the above equations we have that for all $k \quad \mathbb{N}$:

$$\begin{aligned} \tilde{f}^{5k}(A_3) &= [-a^{4k+1} : 1 : 0] \\ \tilde{f}^{5k}(e_1) &= [-a^{4k+2} : 1 : 0] \\ \tilde{f}^{5k}[-a : 1]_{E_1} &= [1 : -a^{4k-1}]_{E_1} \\ \tilde{f}^{5k}[1 : -a]_{E_4} &= [-a^{4k-1} : 1]_{E_4} \end{aligned}$$

Above equations show that if $a = 0$ and $a^p = 1$ for all $p \quad \mathbb{N}$ then the excepcional curves S_2 S_3 S_4 S_7 and S_8 can never reach any indeterminacy point of $\tilde{f}$. It is now clear that $\tilde{f}$ is AS.

Thus we have the following situation:

$$\begin{array}{lll} S_1 & E_1 & E_4 \\ S_5 & E_2 & T_1 \\ S_6 & E_3 & T_2 \\ S_7 & E_4 & T_3 \\ S_8 & E_1 & \end{array}$$

where $T_1\ T_2\ T_3\ \subset \mathcal{E}(f^{-1})$

To compute the matrix of $\tilde{f}^* : Pic(X) -\quad Pic(X)$ we first compute

$$\pi^* : Pic(P\mathbb{C}^2) = < L > - \quad Pic(X) = < \hat{L}\ E_1\ E_2\ E_3\ E_4 >$$

We can see the degrees of the exceptional curves and the multiplicities of the base points passing through them in the following table:

Table 7.1: Multiplicities at Indeterminacy points of Exceptional curves and $f^{-1}(L)$.

Curves	degree	O_1	O_2	O_4	O_7
S_1	1	0	0	1	1
S_5	4	1	2	2	2
S_6	5	1	3	2	2
S_7	2	1	1	1	1
S_8	3	1	2	1	1
$f^{-1}(L)$	9	2	5	4	4

From the values of table 7 1 we have the following:

$$\begin{aligned}
&\tilde{f}^*(\hat{L}) = 9\hat{L} - 2E_1 - 5E_2 - 4E_3 - 4E_4 \\
&\tilde{f}^*(E_1) = \hat{S}_1 + \hat{S}_8 = 4\hat{L} - E_1 - 2E_2 - 2E_3 - 2E_4 \\
&\tilde{f}^*(E_2) = \hat{S}_5 = 4\hat{L} - E_1 - 2E_2 - 2E_3 - 2E_4 \\
&\tilde{f}^*(E_3) = \hat{S}_6 = 5\hat{L} - E_1 - 3E_2 - 2E_3 - 2E_4 \\
&\tilde{f}^*(E_4) = \hat{S}_7 + E_1 = 2\hat{L} - E_2 - E_3 - E_4
\end{aligned}$$

Hence the matrix $\tilde{f}^*$ can be seen as:

$$(\tilde{f}^*) = \begin{pmatrix} 9 & 4 & 4 & 5 & 2 \\ -2 & -1 & -1 & -1 & 0 \\ -5 & -2 & -2 & -3 & -1 \\ -4 & -2 & -2 & -2 & -1 \\ -4 & -2 & -2 & -2 & -1 \end{pmatrix}$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^2 \left(\lambda^3 - 3\lambda^2 + \lambda - 1\right)$ which has the unique real non-zero root

$$\lambda_1 \cong 2\ 769292354$$

and the other roots have modulus less than λ_1 Hence proposition 4 is proved. ∎

We can see that the sequence of the degrees d_n exactly satisfies the recurrence

$$d_{n+3} = 3d_{n+2} - d_{n+1} + d_n$$

Since $d_1 = 9\ \ d_2 = 25$ and $d_3 = 67$ the sequence of the degrees is

$$9 \quad 25 \quad 67 \quad 185 \quad 513 \quad 1093 \quad 2951$$

We now discuss the case when $k = 11$ by considering a particular family of it in the following result.

7.2.4 Dynamical degree of $F_{a\ 1\ a\ a\ 1\ a\ 1\ a\ a\ 1\ a}$

Proposition 54 *For all* $a \quad \mathbb{C} \quad a \qquad -1\ \ 0\ \ 1$ *the dynamical degree of the one-parametric family of mappings* $F_{a,1,a,a,1,a,1,a,a,1,a}$ *is the largest root of the polynomial* $z^3 - 2z^2 - 3z - 1$ *which approximately is* 3 079595625

Proof. We now illustrate an example for the case $k = 11$. Consider the 1- parametric family of mappings $F_{a,1,a,a,1,a,1,a,a,1,a}$:

$$F_{a,1,a,a,1,a,1,a,a,1,a}(x\ \ y) = \left(y \quad \frac{(a+y)^3}{x(ay+1)^2}\right)$$

Let f be the extension of the above mapping to $P\mathbb{C}^2$. Then the homogeneous components of $f[x_0 \;\; x_1 \;\; x_2] = [f_1 : f_2 : f_3]$ are the following:

$$
\begin{aligned}
f_1 &= x_0(x_0^2 x_1 + 2ax_0x_1x_2 + a^2x_1x_2^2)\\
f_2 &= x_1x_2(x_0 + ax_2)^2\\
f_3 &= x_0(ax_0 + x_2)^3
\end{aligned}
$$

Since the jacobian of this mapping is:

$$j_f = 4x_1x_0(ax_0 + x_2)^3(x_0 + ax_2)^4$$

we have the following four exceptional curves S_i where we define S_i as $g_i[x_0 \;\; x_1 \;\; x_2] = 0$ for $i \quad 1 \;\; 2 \;\; 3$:

$$
\begin{aligned}
S_1 &= \quad x_1 = 0\\
S_2 &= \quad x_0 = 0
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= \quad ax_0 + x_2 = 0\\
S_4 &= \quad x_0 + ax_2 = 0
\end{aligned}
$$

Each S_i for $i \quad 1 \;\; 2 \;\; 3$ collapses to A_i where $A_i \quad \mathcal{I}(f^{-1})$ and they are as follows:

$$A_1 := [0 : 0 : 1] \;\; A_2 := [0 : 1 : 0] \;\; A_3 := [1 : -a : 0]$$

and S_4 goes to A_1 too. We observe that all these g_is are distinct and irreducible for the required values of a The indeterminacy points of f are

$$O_1 := [0 : 0 : 1] \;\; O_2 := [0 : 1 : 0] \;\; O_3 := [1 : 0 : -a]$$

In order to regularize our f we see that in this family we have the following situation:

$$
\begin{aligned}
S_1 &\quad A_1 = O_1\\
S_2 &\quad A_2 = O_2
\end{aligned}
$$

$$
\begin{aligned}
S_3 &\quad A_3 \quad\quad x_2 = 0 \quad\quad \mathcal{E}(f^{-1})\\
S_4 &\quad A_1 = O_1
\end{aligned}
$$

We see that $x_2 = 0 \qquad [x_1 : 0 : x_0] \quad S_1$ Therefore we need to blow up $A_1 \;\; A_2$ and then also follow the orbits of S_3 S_4 to see if they reach any indeterminacy point of f. Let X be the space we get after blowing up the points A_1 A_2 and let E_1 E_2 be the corresponding exceptional fibres. Let $\tilde{f}$ be the extension of f on X. To determine the $\tilde{f}$ on S_1 S_2 E_1 E_2 after some calculations, we see that under the action of $\tilde{f}$:

$$\begin{aligned} &S_1 \quad [x_0 : x_2]_{E_1} \\ &S_2 \quad [a^2 x_1 : x_2]_{E_2} \end{aligned}$$

$$\begin{aligned} \tilde{f}[u:v]_{E_1} &= [0 : a^2 v : u] \quad S_2 \\ \tilde{f}[u:v]_{E_2} &= [u : v : 0] \quad\ \ x_2 = 0 \end{aligned}$$

From the above calculations we have the following indeterminacy point of $\tilde{f}$:

$$\mathcal{I}(\tilde{f}) = \ \pi^{-1}(O_3)$$

And the exceptional locus of $\tilde{f}$ is

$$\mathcal{E}(\tilde{f}) = \ \pi^{-1}(S_3)\ \pi^{-1}(S_4)$$

We observe that $S_3 \quad A_3 \quad [a : 0 : -1] \quad [a : -1]_{E_1}$ Now we are going to perform the calculations on $S_4 = \ x_0 + a\,x_2 = 0$ By calling $t = x_0 + a\,x_2$ the points of S_4 can be described as $\lim_{t \quad 0} [t - a\,x_2 : x_1 : x_2]$ And

$$F[t - a\,x_2 : x_1 : x_2] = [-a\,x_1\,t^2 + o(t^3) : x_1\,t^2 + o(t^3) : (a^2-1)^6\,a\,x_2^3 + o(t)] \equiv [a : -1]_{E_1} \quad (7.21)$$

Hence $\tilde{f}$ mappings $S_4 \quad [a : -1]_{E_1}$ and we observe that it does so with multiplicity 2 if a $-1\ 0\ 1$

Thus we now need to follow the orbit of the point $[a : -1]_{E_1}$ to see if S_3 and S_4 can reach any indeterminacy point of $\tilde{f}$ By doing the same calculations as we did in the case $k = 4$ we observe the following:

$$\tilde{f}^{5k}[a : -1]_{E_1} = [-1 : a^{4k-1}]_{E_1} \quad \text{for all} k \quad \mathbb{N}$$

From the previous calculation it is clear that for of $a \quad -1\ 0\ 1$ the curves $\pi^{-1}(S_3)$ and $\pi^{-1}(S_4)$ can never reach any indeterminacy point of $\tilde{f}$. It is now clear that $\tilde{f}$ is AS. Thus we have the following situation:

$$\begin{array}{ll} S_1 & E_1 \\ S_2 & E_2 \\ S_4 & E_1 \end{array}$$

To compute the matrix of $\tilde{f}^* : Pic(X) - \quad Pic(X)$ we first compute

$$\pi^* : Pic(P\mathbb{C}^2) = < L > - \quad Pic(X) = < \hat{L} \ \ E_1 \ \ E_2 >$$

Now $A_1 \quad S_1$, $A_1 \ A_2 \quad S_2$ and $A_2 \quad S_4$. As we said before S_4 reaches the point $[a : -1]_{E_1}$ with multiplicity 2. Whereas the point $A_2 \quad f^{-1}(L)$ has multiplicity 3. Thus we have the following set of equations:

$$\begin{aligned} &\tilde{f}^*(\hat{L}) = 4\hat{L} - E_1 - 3E_2 \\ &\tilde{f}^*(E_1) = \hat{S}_1 + 2\hat{S}_4 = 3\hat{L} - E_1 - 2E_2 \\ &\tilde{f}^*(E_2) = \hat{S}_2 = \hat{L} - E_1 - E_2 \end{aligned}$$

The matrix of $\tilde{f}^*$ thus can be seen as:

$$(\tilde{f}^*) = \begin{pmatrix} 4 & 3 & 1 \\ -1 & -1 & -1 \\ -3 & -2 & -1 \end{pmatrix}$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^3 - 2\lambda^2 - 3\lambda - 1$ which has the unique real root

$$\lambda_1 \cong 3 \ 079595625$$

and other roots with modulus less than λ_1 Hence proposition 5 is proved. ∎

In this case the sequence of the degrees d_n satisfies the recurrence

$$d_{n+3} = 2d_{n+2} + 3d_{n+1} + d_n$$

and since $d_1 = 4 \ \ d_2 = 10$ and $d_3 = 33$ the sequence of the degrees is

$$4 \quad 10 \quad 33 \quad 100 \quad 309 \quad 951 \quad 2929$$

We now give the proof of the main non-integrability result as follows:

7.2.5 Proof of main result

Lemma 55 *Let $\delta(F)$ be the dynamical degree of the birational mapping F Then $\delta(F^2) = \delta(F)^2$*

Proof. The dynamical degree of F is given as

$$\delta(F) = \lim_{n} \ (d_n)^{\frac{1}{n}}$$

Let K_n be the degree of mapping $(F^2)^n$ Then the dynamical degree of the mapping F^2 is

$$\delta(F^2) = \lim_{n} \ (K_n)^{\frac{1}{n}}$$

Observe that the sequence $K_n^{1/n} = d_{2n}^{1/n} = \left(d_{2n}^{1/2n}\right)^2$ Applying limits on both sides as n tends to infinity we get the result. ∎

We now give the proof of the main non-integrability Theorem 50 as follows:

Proof. Consider $F^{[4]} := F_{d,c,b,a}$ $F^{[5]} := F_{e,d,c,b,a}$ with generic values of a b c d e $\mathbb{C}$ and $F^{[7]} := F_{a,1,a,1,a,1,a}$ $F^{[11]} := F_{a,1,a,a,1,a,1,a,a,1,a}$ with generic values of a $\mathbb{C}$

From Propositions 51, 52, 53 and 54 we know that the dynamical degree of $F^{[4]}$ $F^{[5]}$ $F^{[7]}$ $F^{[11]}$ is greater than one. Hence Theorem 50 is proved for k 4 7 8 11 In order to find examples of mappings for others values of k we recall that the Lyness mapping with $a = 1$ is five-periodic and we consider

$$\begin{aligned}
&F^{[5m]} = \left(F^{[5]}\right)^m \quad m \geq 1\\
&F^{[5m+1]} = F^{m-2}_{1,1,1,1,1} \circ F^{[11]} \quad m \geq 2\\
&F^{[5m+2]} = F^{m-1}_{1,1,1,1,1} \circ F^{[7]} \quad m \geq 1\\
&F^{[5m+3]} = F^{m-1}_{1,1,1,1,1} \circ F^{[4]} \circ F^{[4]} \quad m \geq 1\\
&F^{[5m+4]} = F^{m}_{1,1,1,1,1} \circ F^{[4]} \quad m \geq 0
\end{aligned}$$

Using lemma 55 we conclude that the above mappings have dynamical degree greater than one. ∎

Only if I go deep and deep inside mathematics I realize that behind every open door there is a new one to open - Sundus

Chapter 8

Appendix

8.1 Parameter region

8.1.1 Introduction

This chapter is devoted to explain in detail the calculation done to find the zero entropy families in Chapter 4 and 6. In Chapter 4 in the first section we deal with the mappings

$$f(x\ y) = \left(y\ \frac{\beta_0 + y}{\gamma_0 x}\right)$$

We want to know the conditions on $\beta_0\ \gamma_0$ such that the condition $F^p(A_2) = O_0$ is satisfied for $p = 0 \quad 6$ We find that for $p = 0 \quad 3$ the calculations are easy, whereas for $p = 4\ 5\ 6$ it is necessary to solve some system of polynomial equations. We first find the resultant of such system in order to drop the previous known cases, we then use Grobner basis to find the solutions of the system, which gives an equivalent system of equations easier to solve. We give explicit solutions when required. We give these solutions in the second section of this Chapter.

In the third section we consider the family

$$f(x\ y) = \left(\alpha_0 + \alpha_1 x + y\ \frac{x}{\gamma_0 + y}\right)$$

This family appears in the second section of Chapter 6. Here we find the values of the parameters $\alpha_0\ \alpha_1\ \gamma_0$ for which the conditions $\tilde{F}^k(A_1) = O_1$ and $\tilde{F}_1^p(A_2) = O_0$ are satisfied. We use the same methodology to find the required solutions as discussed above.

8.1.2 Non-degenerate case $\gamma_2 = 0 = \alpha_1$.

If $f(x\ y)$ is a non-degenerate mapping with $\gamma_2 = 0 = \alpha_1$ we know that, after an affine change of coordinates, we can write

$$f(x\ y) = \left(y\ \frac{\beta_0 + y}{\gamma_0 + x}\right) \tag{8.1}$$

Proposition 56 *Assume that $A_2 = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con1) $\beta_0 = 0 = \gamma_0$

Proof. We know that $A_2 = [-1 : \beta_0 : 0]$ and O_0 is

$$O_0 = [-1 : \gamma_0 : \beta_0]$$

Then $A_2 = O_0$ implies that $\beta_0 = \gamma_0 = 0$ and the result follows. ∎

Proposition 57 *Let $\tilde{F}$ be the induced map of F we get after blowing up $A_0 = O_1$ and $A_1 = O_2$ Assume that $\tilde{F}(A_2) = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con2) $\beta_0 = 1\ \ \gamma_0 = 0$

Proof. We know that $F(A_2) = [-\beta_0 + \gamma_0 : 0 : \beta_0]$ and O_0 is

$$O_0 = [-1 : \gamma_0 : \beta_0]$$

Then $F(A_2) = O_0$ implies that $\gamma_0 = 0$ and $\beta_0\,(\beta_0 - 1) = 0$ but $\beta_0 = 0$ gives C1 therefore $\beta_0 = 0$ and $\beta_0 = 1$ and the result follows. ∎

Proposition 58 *Assume that $F^2(A_2) = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con3) $\beta_0 = \frac{1 \pm I}{2}\ \ \gamma_0 = \pm I$

Proof. The expression for $F^2(A_2)$ is the following:

$$[\gamma_0\,(\beta_0 - \gamma_0) : \beta_0\,\gamma_0 : \beta_0\,(-1 + \beta_0 - \gamma_0)]$$

Assume that $F^2(A_2) = O_0 = [-1 : \gamma_0 : \beta_0]$ Then the coefficients have to satisfy the following conditions:

$$\begin{aligned} Q_1 &:= -\beta_0 - \gamma_0^2 + \gamma_0\,\beta_0 = 0 \\ Q_2 &:= \beta_0\,(-\gamma_0 + \beta_0 - 1 - \gamma_0^2 + \gamma_0\,\beta_0) = 0 \end{aligned}$$

We get the above equations by considering the numerators of the equations

$$\frac{F^2(A_2)[2]}{F^2(A_2)[1]} - \frac{O_0[2]}{O_0[1]} = 0 \quad \frac{F^2(A_2)[3]}{F^2(A_2)[1]} - \frac{O_0[3]}{O_0[1]} = 0$$

To find the simultaneous solutions of the above polynomials we are going to take into account some resultants. The resultant of the polynomials Q_1 Q_2 with respect to β_0 and γ_0 factorizes as:

$$R_1(Q_1\;\, Q_2\;\, \beta_0) = \gamma_0^2\,(\gamma_0^2 + 1) \qquad R_2(Q_1\;\, Q_2\;\, \gamma_0) = \beta_0^2\,(-2\,\beta_0 + 1 + 2\,\beta_0^2)$$

We see that these two resultants have all the information about our solutions. To find a solution for Q_1 Q_2 we will see that for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we get our required solutions. Thus we have

$$P_1 = \gamma_0^2 \qquad P_2 = (\gamma_0^2 + 1)$$

$$P_3 = \beta_0^2 \qquad P_4 = -2\,\beta_0 + 1 + 2\,\beta_0^2$$

We now check them one by one. Solving $P_1 = 0$ implies that $\gamma_0 = 0$ By substituting this in Q_1 and Q_2 we find that they both are satisfied for $\beta_0 = 0$ This gives us the conditions (Con1). Therefore we consider that $\gamma_0 = 0$ We get the same result for $P_3 = 0$ this means that $\beta_0 = 0$ implies $\gamma_0 = 0$ Hence we consider $\beta_0 = 0$ as well.

We then take $P_2 = 0$ i.e. $\gamma_0^2 + 1 = 0$ We find $\bar{Q}_1$ $\bar{Q}_2$ by finding the remainder of Q_1 and Q_2 by $\gamma_0^2 + 1$ w.r.t γ_0 We see that $\bar{Q}_1$ $\bar{Q}_2$ both are satisfied for $P_4 = 0$ Hence proved. ∎

Proposition 59 *Assume that $F^3(A_2) = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con4) β_0 *and* γ_0 *are the roots of the polynomials* $\beta_0^2 - \beta_0\,\gamma_0 - 2\,\beta_0 + \gamma_0$ *and* $\gamma_0^2 + 1$

Proof. The expression for $F^3(A_2)$ is the following:

$$\begin{aligned}&[\gamma_0\,(\beta_0 - \gamma_0)\,(-\beta_0 + \gamma_0\,\beta_0 - \gamma_0^2) : \beta_0\,(-1 + \beta_0 - \gamma_0)\,(-\beta_0 + \gamma_0\,\beta_0 - \gamma_0^2) : \\ &\qquad\qquad \beta_0\,(\beta_0 - \gamma_0)\,(-1 + \beta_0 - \gamma_0 + \gamma_0\,\beta_0 - \gamma_0^2)]\end{aligned}$$

Assume that $F^3(A_2) = O_0 = [-1 : \gamma_0 : \beta_0]$ Then the coefficients have to satisfy the following conditions:

$$\begin{aligned} Q_1 &:= -\gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - \gamma_0^3 + \gamma_0^2\,\beta_0 = 0 \\ Q_2 &:= \beta_0\,(\gamma_0^2 + 1)\,(-\gamma_0 + \beta_0 - 1) = 0 \end{aligned}$$

To find the simultaneous solutions of the above polynomials we are going to take into account some resultants. The resultant of the polynomials Q_1 Q_2 with respect to β_0 and γ_0 factorizes as:

$$R_1(Q_1\;\; Q_2\;\; \beta_0) = -\gamma_0^5\,(\gamma_0^2+1)^2 \qquad R_2(Q_1\;\; Q_2\;\; \gamma_0) = -\beta_0^4\,(\beta_0-1)^2\,(1-2\,\beta_0+5\,\beta_0^2-4\,\beta_0^3+\beta_0^4)$$

To find a solution for Q_1 Q_2 we utilize the information we get from above resultants. That is for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we see that we get our required solutions. Thus we have

$$P_1 = \gamma_0 \qquad P_2 = (\gamma_0^2 + 1)$$

$$P_3 = \beta_0 \qquad P_4 = 5\,\beta_0^2 - 2\,\beta_0 + 1 + \beta_0^4 - 4\,\beta_0^3 \qquad P_5 = (\beta_0 - 1)$$

By checking them all, we first consider $P_1 = 0$ that is $\gamma_0 = 0$ By substituting this in Q_1 and Q_2 we find that they both are satisfied for $\beta_0 = 0$ and $\beta_0 = 1$ that is for $P_3 = 0$ and for $P_5 = 0$ This gives us the conditions (Con1) and (Con2). Therefore we consider that $P_1 = 0$ We get the same result for $P_3 = 0$ and for $P_5 = 0$ therefore $\beta_0 = 0$ and $\beta_0 = 1$

We then consider $P_2 = 0$ i.e. $\gamma_0^2 + 1 = 0$ Then we find that the equations Q_1 and Q_2 are satisfied when γ_0 and β_0 are roots of the two polynomials P_2 and P_4 By using Grobner basis we see that the system of equations $Q_1 = 0$ $Q_2 = 0$ $P_2 = 0$ and $Q_1 = 0$ $Q_2 = 0$ $P_4 = 0$ is equivalent to the system of equations $P_2 = 0$ $\beta_0^2 - \beta_0\,\gamma_0 - 2\,\beta_0 + \gamma_0 = 0$ this gives us our required result. Hence proved. ∎

Proposition 60 *Assume that $F^4(A_2) = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con5) $3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1 = 0$ *and* $\beta_0 = \frac{3+\gamma_0+7\,\gamma_0^2+3\,\gamma_0^4}{2}$

Proof. The expression for $F^4(A_2)$ is the following:

$$\begin{aligned}[\gamma_0\,(-\beta_0 - \gamma_0^2 + \gamma_0\,\beta_0)\,(-\gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - \gamma_0^3 + \gamma_0^2\,\beta_0) : \beta_0\,(-\gamma_0 + \beta_0 - 1 - \gamma_0^2 + \gamma_0\,\beta_0) \\ (-\gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - \gamma_0^3 + \gamma_0^2\,\beta_0) : \gamma_0\,(-\gamma_0 + \beta_0)\,\beta_0\,(\gamma_0^2 + 1)\,(-\gamma_0 + \beta_0 - 1)]\end{aligned}$$

Assume that $F^4(A_2) = O_0 = [-1 : \gamma_0 : \beta_0]$ Then the coefficients have to satisfy the following conditions:

$$\begin{aligned} Q_1 &:= -\gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - 2\,\gamma_0^2\,\beta_0 + \gamma_0\,\beta_0^2 - \gamma_0^4 + \gamma_0^3\,\beta_0 = 0 \\ Q_2 &:= \beta_0\,(\gamma_0^4 - 2\,\gamma_0^2\,\beta_0^2 + \gamma_0^3 + \gamma_0^2 - 2\,\gamma_0\,\beta_0 + \gamma_0 + 2\,\beta_0^2 - \\ &\quad \beta_0 - \beta_0^3 + \gamma_0^5 - 2\,\beta_0\,\gamma_0^4 + \gamma_0\,\beta_0^3 + \gamma_0^3\,\beta_0^2) = 0 \end{aligned}$$

To find the simultaneous solutions of the above polynomials we are going to find their resultants. The resultant of the polynomials Q_1 Q_2 with respect to β_0 and γ_0 factorizes as:

$$\begin{aligned} R_1(Q_1\ Q_2\ \beta_0) &= \gamma_0^7\,(\gamma_0^2+1)^2\,(3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1) \\ R_2(Q_1\ Q_2\ \gamma_0) &= -\beta_0^5\,(2\,\beta_0^2 - 2\,\beta_0 + 1)\,(9\,\beta_0^6 - 27\,\beta_0^5 + 36\,\beta_0^4 - 24\,\beta_0^3 + 9\,\beta_0^2 - 3\,\beta_0 + 1 \\ &\quad (\beta_0 - 1)^2 \end{aligned}$$

We see that these two resultants have all the information about our solutions. To find a solution for Q_1 Q_2 we will see that for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we get our required solutions. Thus we have

$$P_1 = \gamma_0 \qquad P_2 = \gamma_0^2 + 1 \qquad P_3 = 3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1$$

$$\begin{aligned} P_4 &= \beta_0 \qquad P_5 = \beta_0^2 + 1 \qquad P_6 = 2\,\beta_0^2 - 2\,\beta_0 + 1 \\ P_7 &= 9\,\beta_0^6 - 27\,\beta_0^5 + 36\,\beta_0^4 - 24\,\beta_0^3 + 9\,\beta_0^2 - 3\,\beta_0 + 1 \qquad P_8 = (\beta_0 - 1) \end{aligned}$$

We now check them one by one. Solving $P_1 = 0$ implies that $\gamma_0 = 0$ By substituting this in Q_1 and Q_2 we find that it satisfies Q_2 and satisfy Q_1 for $\beta_0 = 0$ and $\beta_0 = 1$ that is for $P_4 = 0$ and for $P_8 = 0$ This gives us the conditions (Con1) and (Con2). Therefore we consider that $P_1 = 0$ We get the same result for $P_4 = 0$ and for $P_8 = 0$ therefore we also consider that $\beta_0 = 0$ and $\beta_0 = 1$

Now let $P_2 = 0$ i.e. $\gamma_0^2 + 1 = 0$ By looking at the resultant of Q_1 and Q_2 by $\gamma_0^2 + 1$ w.r.t γ_0 we find that both Q_1 and Q_2 are satisfied for $P_5 = 0 = P_6$ But $P_2 = 0 = P_6$ gives (Con3). Then let $P_5 = 0$ Notice that $P_2 = P_5 = 0$ implies that $\gamma_0^2 = \beta_0^2 = -1$ Which implies that $\beta_0^2 - \gamma_0^2 = 0$ this gives that $\tilde{F}^4(A_2)[3] = 0$ as it includes the factor $(\beta_0 - \gamma_0)$ that is zero as $\beta_0^2 - \gamma_0^2 = 0$ This means that if $\tilde{F}^4(A_2) = O_0$ then $O_0[3] = 0$ which implies that $\beta_0 = 0$ but $\beta_0 = 0$ hence we find that $P_2 = P_5 = 0$ is not a feasible solution so we will not consider it.

Hence we consider $P_3 = 0$ that is $3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1 = 0$ Then we find that the equations Q_1 and Q_2 are satisfied when γ_0 and β_0 are roots of the two polynomials P_3 and P_7 By using Grobner basis we see that the system of equations $Q_1 = 0$ $Q_2 = 0$ $P_3 = 0$ and $Q_1 = 0$ $Q_2 = 0$ $P_7 = 0$

is equivalent to the system of equations $P_3 = 0 \;\; 3 + \gamma_0 + 7\,\gamma_0^2 + 3\,\gamma_0^4 - 2\,\beta_0 = 0$ this gives us our required result. Hence proved. ∎

Proposition 61 *Assume that $F^5(A_2) = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con6) $\gamma_0^8 + 8\,\gamma_0^6 + 14\,\gamma_0^4 + 7\,\gamma_0^2 + 1 = 0$ *and* $\beta_0 = \frac{-5+\gamma_0-22\,\gamma_0^2-15\,\gamma_0^4-2\,\gamma_0^6}{2}$

Proof. The expression for $F^5(A_2)$ is the following:

$$[(-\beta_0 - \gamma_0^2 + \gamma_0\,\beta_0)\,(-\gamma_0^3 + \beta_0\,\gamma_0^2 - \gamma_0\,\beta_0 + \beta_0^2 - \beta_0)\,(-2\,\beta_0\,\gamma_0^2 + \gamma_0\,\beta_0^2 - \gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - \gamma_0^4 + \gamma_0^3\,\beta_0) : (-\gamma_0 + \beta_0)\,\beta_0\,(\gamma_0^2 + 1)\,(\beta_0 - \gamma_0 - 1)\,(-2\,\beta_0\,\gamma_0^2 + \gamma_0\,\beta_0^2 - \gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - \gamma_0^4 + \gamma_0^3\,\beta_0) : \gamma_0\,(-\beta_0 - \gamma_0^2 + \gamma_0\,\beta_0)\,\beta_0\,(\gamma_0^4 - 2\,\gamma_0^2\,\beta_0^2 + \gamma_0^3 + \gamma_0^2 - 2\,\gamma_0\,\beta_0 + \gamma_0 + 2\,\beta_0^2 - \beta_0 - \beta_0^3 + \gamma_0^5 - 2\,\beta_0\,\gamma_0^4 + \gamma_0^3\,\beta_0^2 + \gamma_0\,\beta_0^3)]$$

Assume that $F^5(A_2) = O_0 = [-1 : \gamma_0 : \beta_0]$ Then the coefficients have to satisfy the following conditions:

$$\begin{aligned} Q_1 &:= (3\,\beta_0\,\gamma_0^4 - 5\,\gamma_0^3\,\beta_0^2 + 2\,\gamma_0^2\,\beta_0^3 - \gamma_0^2\,\beta_0^2 + 2\,\gamma_0^3\,\beta_0 + \beta_0\,\gamma_0^2 - \\ &\quad \gamma_0\,\beta_0^2 + \gamma_0\,\beta_0 + \beta_0^3 - \beta_0^2 - \gamma_0\,\beta_0^3 + \gamma_0^6 - 2\,\beta_0\,\gamma_0^5 + \gamma_0^4\,\beta_0^2) = 0 \\ Q_2 &:= \beta_0\,(\gamma_0\,\beta_0 + \beta_0 - 1 - \gamma_0 - \gamma_0^2)\,(\beta_0 - \gamma_0 - 1)\,(\beta_0^2 + 2\,\beta_0\,\gamma_0^2 - \\ &\quad \gamma_0\,\beta_0 + \gamma_0^4 - \gamma_0^3 + \gamma_0^2) = 0 \end{aligned}$$

To find the simultaneous solutions of the above polynomials we are going to take into account the resultants. The resultant of the polynomials $Q_1 \;\; Q_2$ with respect to β_0 and γ_0 factorizes as:

$$\begin{aligned} R_1(Q_1\;\; Q_2\;\; \beta_0) &= -\gamma_0^{14}\,(\gamma_0^2 + 1)^4\,(\gamma_0^8 + 8\,\gamma_0^6 + 14\,\gamma_0^4 + 7\,\gamma_0^2 + 1) \\ R_2(Q_1\;\; Q_2\;\; \gamma_0) &= \beta_0^{10}\,(2\,\beta_0^2 - 2\,\beta_0 + 1)^2\,(\beta_0^4 - 4\,\beta_0^3 + 5\,\beta_0^2 - 2\,\beta_0 + 1)\,(\beta_0 - 1)^4 \\ &\quad (\beta_0^8 - 10\,\beta_0^7 + 37\,\beta_0^6 - 60\,\beta_0^5 + 49\,\beta_0^4 - 20\,\beta_0^3 + 3\,\beta_0^2 + 1) \end{aligned}$$

To find the solution for $Q_1 \;\; Q_2$ we find that for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we get our required solutions as follows:

$$P_1 = \gamma_0 \qquad P_2 = (\gamma_0^2 + 1) \qquad P_3 = \gamma_0^8 + 8\,\gamma_0^6 + 14\,\gamma_0^4 + 7\,\gamma_0^2 + 1$$

$$
\begin{aligned}
P_4 &= \beta_0 \quad P_5 = 2\,\beta_0^2 - 2\,\beta_0 + 1 \quad P_6 = \beta_0^4 - 4\,\beta_0^3 + 5\,\beta_0^2 - 2\,\beta_0 + 1 \\
P_7 &= \beta_0^8 - 10\,\beta_0^7 + 37\,\beta_0^6 - 60\,\beta_0^5 + 49\,\beta_0^4 - 20\,\beta_0^3 + 3\,\beta_0^2 + 1 \quad P_8 = (\beta_0 - 1)
\end{aligned}
$$

We now check them one by one. Solving $P_1 = 0$ implies that $\gamma_0 = 0$ By substituting this in Q_1 and Q_2 we find that it satisfies Q_2 and Q_1 for $\beta_0 = 0$ and $\beta_0 = 1$ that is for $P_4 = 0$ and for $P_8 = 0$ This gives us the conditions (Con1) and (Con2). Thus we take $P_1 = 0$ We get the same conditions if we take $P_4 = 0$ and $P_8 = 0$ therefore we consider $\beta_0 = 0$ and $\beta_0 = 1$

We then consider $P_2 = 0$ that is $\gamma_0^2 + 1 = 0$ By looking at the resultant of Q_1 and Q_2 by $\gamma_0^2 + 1$ w.r.t γ_0 we find that both Q_1 and Q_2 are satisfied for $P_5 = 0 = P_6$ But $P_2 = 0 = P_6$ gives (Con3). This implies that we need to consider $P_5 = 0$ But notice that $P_2 = P_5 = 0$ gives (Con4). Therefore we cannot consider these polynomials to be equal to zero.

The we must have $P_3 = 0$ that is $\gamma_0^8 + 8\,\gamma_0^6 + 14\,\gamma_0^4 + 7\,\gamma_0^2 + 1 = 0$ Then we find that the equations Q_1 and Q_2 are satisfied when γ_0 and β_0 are the roots of polynomials P_3 and P_7 By using Grobner basis we see that the system of equations $Q_1 = 0$ $Q_2 = 0$ $P_3 = 0$ and $Q_1 = 0$ $Q_2 = 0$ $P_7 = 0$ is equivalent to the system of equations $p_3 = 0$ $-5 + \gamma_0 - 22\,\gamma_0^2 - 15\,\gamma_0^4 - 2\,\gamma_0^6 - 2\,\beta_0 = 0$ this gives us our required result. Hence proved. ∎

Proposition 62 *Assume that $F^6(A_2) = O_0$ Then the coefficients of f have to satisfy the following condition:*

(Con7) $\beta_0 = \beta_0$ $\gamma_0 = 0$

(Con8) $\gamma_0^4 + 5\,\gamma_0^2 + 5 = 0$ *and* $\beta_0 = \frac{-1+\gamma_0-\gamma_0^2}{2}$

Proof. The expression for $\tilde{F}^6(A_2) = [F_1 : F_2 : F_3]$ where F_1 F_2 F_3 are the following:

$$
\begin{aligned}
F_1 &= (-2\,\beta_0\,\gamma_0^2 + c0\,\beta_0^2 - \gamma_0\,\beta_0 + \beta_0^2 - \beta_0 - \gamma_0^4 + \gamma_0^3\,\beta_0)\,(-\gamma_0^3 + \beta_0\,\gamma_0^2 \\
&\quad - c0\,\beta_0 + \beta_0^2 - \beta_0)(3\,\beta_0\,\gamma_0^4 - 5\,\gamma_0^3\,\beta_0^2 + 2\,\gamma_0^2\,\beta_0^3 - \gamma_0^2\,\beta_0^2 + 2\,\gamma_0^3\,\beta_0 + \\
&\quad \beta_0\,\gamma_0^2 - \gamma_0\,\beta_0^2 + \gamma_0\,\beta_0 + \beta_0^3 - \beta_0^2 - \gamma_0\beta_0^3 + \gamma_0^6 - 2\,\beta_0\,\gamma_0^5 + \gamma_0^4\,\beta_0^2)
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= \gamma_0\,\beta_0\,(\gamma_0^4 - 2\,\gamma_0^2\,\beta_0^2 + \gamma_0^3 + \gamma_0^2 - 2\,\gamma_0\,\beta_0 + \gamma_0 + 2\,\beta_0^2 - \beta_0 - \beta_0^3 + \gamma_0^5 - \\
&\quad 2\,\beta_0\,\gamma_0^4 + \gamma_0^3\,\beta_0^2 + \gamma_0\,\beta_0^3)\,(3\,\beta_0\,\gamma_0^4 - 5\,\gamma_0^3\,\beta_0^2 + 2\,\gamma_0^2\,\beta_0^3 - \\
&\quad \gamma_0^2\,\beta_0^2 + 2\,\gamma_0^3\,\beta_0 + \beta_0\,\gamma_0^2 - \gamma_0\,\beta_0^2 + \gamma_0\,\beta_0 + \beta_0^3 - \beta_0^2 - \gamma_0\,\beta_0^3 + \\
&\quad \gamma_0^6 - 2\,\beta_0\,\gamma_0^5 + \gamma_0^4\,\beta_0^2)
\end{aligned}
$$

$$
\begin{aligned}
F_3 \;&= (-\beta_0 - \gamma_0^2 + \gamma_0\,\beta_0) \\
&(-\gamma_0^3 + \beta_0\,\gamma_0^2 - \gamma_0\,\beta_0 + \beta_0^2 - \beta_0)\,\beta_0\,(\beta_0 - \gamma_0 - 1)(\gamma_0\,\beta_0 + \beta_0 - 1 - \gamma_0 - \gamma_0^2) \\
&(\beta_0^2 + 2\,\beta_0\,\gamma_0^2 - \gamma_0\,\beta_0 + \gamma_0^4 - \gamma_0^3 + \gamma_0^2)
\end{aligned}
$$

Assume that $\tilde{F}^6(A_2) = O_0 = [-1 : \gamma_0 : \beta_0]$ Then the coefficients have to satisfy the following conditions:

$$
\begin{aligned}
Q_1 \;&:= \gamma_0^2\,(\beta_0 + \gamma_0\,\beta_0 + 2\,\beta_0\,\gamma_0^2 + 2\,\gamma_0\,\beta_0^2 + 3\,\gamma_0^3\,\beta_0 - \gamma_0^2\,\beta_0^2 + 4\,\beta_0\,\gamma_0^4 - \\
&7\,\gamma_0^3\,\beta_0^2 - 4\,\gamma_0\,\beta_0^3 + \gamma_0^6 - 3\,\beta_0^3 + 2\,\beta_0^4 + 3\,\gamma_0^2\,\beta_0^3 - 2\,\beta_0\,\gamma_0^5 + \gamma_0^4\,\beta_0^2) = 0
\end{aligned}
$$

$$
\begin{aligned}
Q_2 \;&:= \gamma_0^2\,\beta_0\,(-\beta_0 - \gamma_0^2 + \gamma_0\,\beta_0 + 2\,\beta_0\,\gamma_0^2 - 12\,\gamma_0\,\beta_0^2 - 2\,\beta_0^2 - 2\,\gamma_0^3\,\beta_0 - 8\,\gamma_0^2\,\beta_0^2 - \\
&4\,\beta_0\,\gamma_0^4 + \gamma_0^3\,\beta_0^2 + 19\,\gamma_0\,\beta_0^3 - \gamma_0^4 - \gamma_0^3 - \gamma_0^5 - \gamma_0^6 + 10\,\beta_0^3 - 9\,\beta_0^4 - \gamma_0^7 + \\
&2\,\beta_0^5 + 11\,\gamma_0^2\,\beta_0^3 - \gamma_0^8 - 4\,\beta_0\,\gamma_0^5 + 7\,\gamma_0^4\,\beta_0^2 - 9\,\gamma_0\,\beta_0^4 + 3\,\gamma_0^7\,\beta_0 - \\
&3\,\gamma_0^6\,\beta_0^2 + \gamma_0^5\,\beta_0^3 - 2\,\gamma_0^6\,\beta_0 + 10\,\gamma_0^5\,\beta_0^2 - 10\,\gamma_0^4\,\beta_0^3 + 3\,\gamma_0^3\,\beta_0^4 + \\
&3\,\gamma_0^3\,\beta_0^3 - 7\,\gamma_0^2\,\beta_0^4 + 2\,\beta_0^5\,\gamma_0) = 0
\end{aligned}
$$

Observe that both equations are satisfied for the value $\gamma_0 = 0$ which gives the required (Con7).Now by dropping γ_0^2 from both Q_1 and Q_2 We have new equations $\bar{Q}_1$ and $\bar{Q}_2$ We consider the resultants to find the simultaneous solutions of $\bar{Q}_1$ and $\bar{Q}_2$ The resultant of the polynomials $\bar{Q}_1$ $\bar{Q}_2$ with respect to β_0 and γ_0 factorizes as:

$$
R_1(\bar{Q}_1\;\bar{Q}_2\;\beta_0) \;= -4\,\gamma_0^9\,(\gamma_0^2 + 1)^6\,(\gamma_0^4 + 5\,\gamma_0^2 + 5)\,(3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1)
$$

$$
\begin{aligned}
R_2(\bar{Q}_1\;\bar{Q}_2\;\gamma_0) \;&= -\beta_0^8\,(\beta_0^4 - 4\,\beta_0^3 + 5\,\beta_0^2 - 2\,\beta_0 + 1) \\
&(\beta_0 - 1)\,(\beta_0^4 - 3\,\beta_0^3 + 4\,\beta_0^2 - 2\,\beta_0 + 1)\,(9\,\beta_0^6 - 27\,\beta_0^5 + 36\,\beta_0^4 - \\
&24\,\beta_0^3 + 9\,\beta_0^2 - 3\,\beta_0 + 1)\,(2\,\beta_0^2 - 2\,\beta_0 + 1)^4
\end{aligned}
$$

To find a solution for $\bar{Q}_1$ $\bar{Q}_2$ we observe that for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we get our required solutions. Thus we have

$$
P_1 = \gamma_0 \quad P_2 = \gamma_0^2 + 1 \quad P_3 = \gamma_0^4 + 5\,\gamma_0^2 + 5 \quad P_4 = 3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1
$$

$$
P_5 \;= \beta_0 \quad P_6 = (\beta_0^4 - 4\,\beta_0^3 + 5\,\beta_0^2 - 2\,\beta_0 + 1) \quad P_7 = \beta_0^4 - 3\,\beta_0^3 + 4\,\beta_0^2 - 2\,\beta_0 + 1
$$

$$
\begin{aligned}
P_8 \;&= 9\,\beta_0^6 - 27\,\beta_0^5 + 36\,\beta_0^4 - 24\,\beta_0^3 + 9\,\beta_0^2 - 3\,\beta_0 + 1 \quad P_9 = (\beta_0 - 1) \\
&P_{10} = (2\,\beta_0^2 - 2\,\beta_0 + 1)
\end{aligned}
$$

We now check them one by one. Solving $P_1 = 0$ implies that $\gamma_0 = 0$ As discussed above it satisfies both $\bar{Q}_1$ and $\bar{Q}_2$ and hence gives the first solution. Now solving $P_2 = \gamma_0^2 + 1 = 0$ we find that it satisfies $\bar{Q}_2$ and satisfy $\bar{Q}_1$ for $P_6 = 0$ and for $P_{10} = 0$ This gives us the conditions (Con3) and (Con4). Therefore we do not consider these polynomials to be equal to zero.

Now consider that $P_4 = 0$ that is $3\,\gamma_0^6 + 9\,\gamma_0^4 + 6\,\gamma_0^2 + 1 = 0$ Finding the resultant of $\bar{Q}_1$ and $\bar{Q}_2$ by P_4 w.r.t γ_0 we find that both $\bar{Q}_1$ and $\bar{Q}_2$ are satisfied for $P_8 = 0$ This gives the (con5). Therefore we consider these factors to be non zero.

We then consider $P_3 = 0$ this implies that $\gamma_0^4 + 5\,\gamma_0^2 + 5 = 0$ Then we find that the equations $\bar{Q}_1$ and $\bar{Q}_2$ are satisfied when γ_0 and β_0 are the roots of polynomials P_3 and P_7 By using Grobner basis we see that the system of equations $\bar{Q}_1 = 0$ $\bar{Q}_2 = 0$ $P_3 = 0$ and $\bar{Q}_1 = 0$ $\bar{Q}_2 = 0$ $P_7 = 0$ is equivalent to the system of equations $P_3 = 0$ $-1 + \gamma_0 - \gamma_0^2 - 2\,\beta_0 = 0$ this gives us our required solution (Con8). Hence proved. ∎

8.1.3 Non-degenerate case $\gamma_1 = 0\,,\ \alpha_2 = 0.$

Consider the following family of maps

$$f(x\ \ y) = \left(\alpha_0 + \alpha_1 x + y\ \ \frac{x}{\gamma_0 + y}\right) \qquad \alpha_1 = 0 \tag{8.2}$$

Proposition 63 *Assume that $A_2 = O_0$ Then the coefficients of f have to satisfy the following condition:*

(C1) $\alpha_0 = \frac{1}{\alpha_1} = \gamma_0$ *with* $\alpha_1 = 0$

Proof. Since $\alpha_1 = 0$ we can write A_2 as:

$$A_2 = \left[1 : (\alpha_0 - \gamma_0) : -\frac{1}{\alpha_1}\right]$$

and O_0 is

$$O_0 = [1 : 0 : -\gamma_0]$$

Then $A_2 = O_0$ implies that $\alpha_0 - \gamma_0 = 0$ $\frac{1}{\alpha_1} = \gamma_0$ and the result follows. ∎

Proposition 64 *Assume that $F(A_2) = O_0$ Then the coefficients of f satisfies the following condition:*

(C2) $\alpha_0 = \frac{\alpha_1^2+\alpha_1+1}{\alpha_1(\alpha_1+1)^2}$ $\gamma_0 = \frac{1}{\alpha_1+1}$ *with* α_1 $\quad 0\ \ -1$

Proof. The image of A_2 is:

$$[\alpha_1(\gamma_0\alpha_1 - 1) : (\gamma_0\alpha_1 - 1)(\alpha_1^2(\alpha_0 - \gamma_0) + \alpha_1\alpha_0 - 1) : \alpha_1^2\,(\alpha_0 - \gamma_0)]$$

If $F(A_2) = O_0 = [1 : 0 : -\gamma_0]$ then $(\gamma_0\alpha_1 - 1) = 0$ and $\alpha_0 = \gamma_0$ So, $F(A_2)$ can be written as:

$$F(A_2) \;=\; \left[1 : (\alpha_1(\alpha_0 - \gamma_0) + \alpha_0 - \frac{1}{\alpha_1}) : \frac{\alpha_1\,(\alpha_0 - \gamma_0)}{(\gamma_0\alpha_1 - 1)}\right]$$

and O_0 is

$$O_0 \;=\; [1 : 0 : -\gamma_0]$$

Hence the parameters have to satisfy the following equations:

$$P_1 := \alpha_1^2(\alpha_0 - \gamma_0) + \alpha_0\alpha_1 - 1 = 0$$

and

$$P_2 := \alpha_1\gamma_0(\gamma_0 - 1) + \alpha_0\alpha_1 - \gamma_0 = 0$$

We are going to calculate the resultant of the polynomials P_1 and P_2 in respect to α_0 and we call it R We know that if these polynomials have a common root in respect to α_0 then the resultant must be zero. The introduction of this resultant will make easier our computations.

$$R = \alpha_1\,(\alpha_1\,\gamma_0 + \gamma_0 - 1)\,(\gamma_0\alpha_1 - 1)$$

Since $\alpha_1 = 0$ and $\gamma_0\alpha_1 - 1 = 0$ we have that $\alpha_1\,\gamma_0 + \gamma_0 - 1 = 0$ which implies that

$$\gamma_0 = \frac{1}{\alpha_1 + 1}$$

Note that $(\alpha_1\,\gamma_0 + \gamma_0 - 1) = 0$ implies that $\gamma_0 = \;\; 0 \;\; 1 \quad \alpha_1 = -1$

We now want that for this value of γ_0 the polynomials $P_1 \;\; P_2$ are zero. By imposing this we get that $P_1 = 0 = P_2$ if and only if the polynomial $\alpha_1^2 + \alpha_1 + 1 = \alpha_0\alpha_1(1 + \alpha_1)^2$ By solving for α_0 the result follows.

∎

Proposition 65 *Assume that $F^2(A_2) = O_0$ Then the coefficients of f satisfies the following conditions:*

(C3) $\alpha_1 = \ ^2$ $\gamma_0 = \frac{\omega\mp 1}{\omega(\omega^2\mp\omega+1)}$ *and* $\alpha_0 = \frac{\omega^3\mp\omega^2\pm 1}{(\omega\pm 1)\,(\omega^2\mp\omega+1)^2}$ *with* $\ = 0 \quad \pm 1 = 0 \quad ^2\mp \quad +1 = 0$

Proof. For the simplification of calculations we consider $\alpha_1 = \ ^2$ for this proof. Therefore the expression of $F^2(A_2)$ is:

$$\begin{aligned}[\ ^2(\gamma_0^2 \ ^2 - \gamma_0 + \alpha_0 \ ^2 - \gamma_0 \ ^2)\,(\gamma_0 \ ^2 - 1) :& \\ ^2(\gamma_0^2 \ ^2 - \gamma_0 + \alpha_0 \ ^2 - \gamma_0 \ ^2)p_1 :& \\ (\alpha_0 \ ^2 + \ ^4\alpha_0 - \ ^4\gamma_0 - 1)\,(\gamma_0 \ ^2 - 1)^2]&\end{aligned}$$

where

$$p_1 = \alpha_0\gamma_0 \ ^2 - \alpha_0 + \alpha_0\gamma_0 \ ^4 + \ ^6\alpha_0\gamma_0 - \alpha_0 \ ^4 - \ ^6\gamma_0^2 + \ ^4\gamma_0 - 2\gamma_0 \ ^2 + 1$$

Assume that $F^2(A_2) = O_0 = [1 : 0 : -\gamma_0]$ then $\alpha_1 = \ ^2 = 0 \ \gamma_0 \ ^2 - 1 = 0$ as it gives $A_2 = O_0$ also $\gamma_0^2 \ ^2 - \gamma_0 + \alpha_0 \ ^2 - \gamma_0 \ ^2 = 0$ Hence we have

$$F^2(A_2) = \left[1 : \frac{p_1}{(\gamma_0 \ ^2 - 1)} : \frac{(\alpha_0 \ ^2 + \ ^4\alpha_0 - \ ^4\gamma_0 - 1)(\gamma_0 \ ^2 - 1)}{^2\,(\gamma_0^2 \ ^2 - \gamma_0 + \alpha_0 \ ^2 - \gamma_0 \ ^2)}\right]$$

Then the coefficients have to satisfy the following conditions:

$$\begin{aligned} Q_1 &:= \ ^2 p_1 = 0 \\ Q_2 &:= 2\alpha_0\gamma_0 \ ^4 - \alpha_0 \ ^2 + \alpha_0\gamma_0 \ ^6 - \ ^4\alpha_0 - \ ^6\gamma_0^2 + \\ & \quad ^4\gamma_0 - \gamma_0 \ ^2 + 1 + \gamma_0^3 \ ^4 - \gamma_0^2 \ ^2 - \gamma_0^2 \ ^4 = 0 \end{aligned}$$

As done in the above proposition here also we are going to take into account some resultants. The resultant of the polynomials $Q_1 \ Q_2$ in respect to $\alpha_0 \ R(Q_1 \ Q_2 \ \alpha_0)$ factorizes as:

$$R(Q_1 \ Q_2 \ \alpha_0) = (\gamma_0 \ ^3 - \gamma_0 \ ^2 + \gamma_0 \ - \ + 1)(\gamma_0 \ ^3 + \gamma_0 \ ^2 + \gamma_0 \ - \ - 1)(\gamma_0 \ ^2 - 1)^2$$

Then equations $Q_1 = 0 \ Q_2 = 0$ imply that

$$P_1 = \gamma_0 \ ^3 \mp \gamma_0 \ ^2 + \gamma_0 \ - \ \pm 1 = 0$$

Solving P_1 for γ_0 we get

$$\gamma_0 = \frac{\mp 1}{(\ \ {}^2 \mp \quad + 1)}$$

Observe that when $\ {}^2 \mp \quad + 1 = 0$ we get $\quad \mp 1 = 0$ but this is incompatible with $P_1 = 0$ By substituting this value of γ_0 in $Q_1 \ \ Q_2$ we find that the following equation satisfies both $Q_1 \ \ Q_2$ that is

$$\alpha_0 \quad {}^5 \mp \alpha_0 \quad {}^4 + \alpha_0 \quad {}^3 \pm \alpha_0 \quad {}^2 - \alpha_0 \quad - \quad {}^3 \pm \quad {}^2 \mp 1 = 0 \tag{8.3}$$

Solving this equation for α_0 we get

$$\alpha_0 = \frac{\ {}^3 \mp \quad {}^2 \pm 1}{(\quad \pm 1)(\ \ {}^2 \mp \quad + 1)^2}$$

Note that when $(\quad \pm 1)(\ \ {}^2 \mp \quad + 1)^2 = 0$ then to satisfy the equation (8.3) we must also have $\ {}^3 \mp \quad {}^2 \pm 1 = 0$ but the resultant of these two equations is non zero. Hence proved. ∎

Proposition 66 *Let $\tilde{F}$ and $\tilde{F}_1$ be the induced map of F we get after blowing up A_0 and the points of $\mathcal{O}_1$ Assume that $\tilde{F}^2(A_1) = O_1$ and $\tilde{F}_1^3(A_2) = O_0$ for $k = 1 \ \ p = 3$ respectively. Then the coefficients of f satisfy the following conditions.*

(C4) $\alpha_1^6 + \alpha_1^3 + 1 = 0 \quad \alpha_0 = -2\alpha_1^5 + \alpha_1^3 - \alpha_1^2 - \alpha_1$ *and* $\gamma_0 = -\frac{1}{\alpha_1^2}$

Proof. The imposed condition $\tilde{F}^2(A_1) = O_1$ implies that *condition* k is satisfied for $k = 1$ This can be written as $\alpha_1^2\gamma_0 + 1 = 0$ which implies that

$$\gamma_0 = -\frac{1}{\alpha_1^2}$$

Observe that if any of its iterates belong to $S_1 = \quad \gamma_0 x_0 + x_2 = 0$ then we cannot impose the condition $\tilde{F}_1^3(A_2) = O_0$ because $O_0 \quad T_1$ and for $k = 1$ we have

$$S_1 \quad G_0 \quad G_1 \quad G_2 \quad T_1$$

and $O_0 \quad T_1$ This shows that the orbit of A_2 if collides with the orbit of A_1 then after three iterates it can not reach O_0 Also if $A_2 \quad S_0 \ \ E_0$ still it s orbit cannot reach O_0 Therefore we read the condition $\tilde{F}_1^3(A_2) = O_0$ as $F^3(A_2) = O_0$ To find such families in the following computations we consider that neither A_2 nor any of its iterate belong to S_1 i.e. there is no collision of orbits.

Then the expression of $F^3(A_2)$ is:

$$[-(1+\alpha_1)\,(1+\alpha_1+\alpha_1^2+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)\,(1+\alpha_1+\alpha_1^2+\alpha_1^4\alpha_0):\\ -(1+\alpha_1+\alpha_1^2+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)p_1:\\ (1+\alpha_1+\alpha_1^2+\alpha_1^4\alpha_0)^2\,(-1+\alpha_0+2\alpha_1\alpha_0+\alpha_1^2\alpha_0+\alpha_1^3\alpha_0)\,\alpha_1^2]$$

where

$$p_1=-(\alpha_0+3\alpha_0\alpha_1+4\alpha_0\alpha_1^2+\alpha_0^2\alpha_1^4+2\alpha_0^2\alpha_1^5+2\alpha_0\alpha_1^3+2\alpha_0^2\alpha_1^6+\alpha_0\alpha_1^4+\\ \alpha_0^2\alpha_1^7+\alpha_0^2\alpha_1^8+\alpha_0\alpha_1^6-\alpha_1-\alpha_1^2-\alpha_1^3)$$

Assume that $F^3(A_2)=O_0=\left[1:0:\frac{1}{\alpha_1^2}\right]$ Since $O_0[1]=0$ we impose that the first component of $F^3(A_2)$ is also non zero, and hence:

$$F^3(A_2)=\left[1:\frac{p_1}{(1+\alpha_1)\,(1+\alpha_1+\alpha_1^2+\alpha_1^4\alpha_0)}:\right.\\ \left.-\frac{(-1+\alpha_0+2\alpha_1\alpha_0+\alpha_1^2\alpha_0+\alpha_1^3\alpha_0)\,\alpha_1^2(1+\alpha_1+\alpha_1^2+\alpha_1^4\alpha_0)}{(1+\alpha_1)\,(1+\alpha_1+\alpha_1^2+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)}\right]$$

Then the coefficients have to satisfy the following conditions:

$$Q_1:=p_1=0$$

$$Q_2:=-\alpha_0^2(\alpha_1^{11}+\alpha_1^{10}+2\alpha_1^9+\alpha_1^8)-\alpha_0(\alpha_1^9+\alpha_1^8+5\alpha_1^7+7\alpha_1^6\\ +7\alpha_1^5+3\alpha_1^4)+(\alpha_1^5+\alpha_1^4+\alpha_1^6-\alpha_1^3-2\alpha_1^2-2\alpha_1-1)=0$$

To find a common solution of $Q_1\;\;Q_2$ we find their resultant with respect to α_0 and α_1. That is $R_1(Q_1\;\;Q_2\;\;\alpha_0)$ factorizes as:

$$R_1(Q_1\;\;Q_2\;\;\alpha_0)=-\alpha_0^{17}(\alpha_0^6+3\alpha_0^5+15\alpha_0^4+6\alpha_0^3-9\alpha_0+3)(\alpha_0+1)^7$$

and

$$R_2(Q_1\;\;Q_2\;\;\alpha_1)=-\alpha_1^8(\alpha_1^6+\alpha_1^3+1)(\alpha_1^2+\alpha_1+1)^3(\alpha_1+1)^7$$

We find that these two resultants have all the information about our solutions. To find a solution for

Q_1 Q_2 we will see that for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we get our required solutions. We know that $\alpha_1 = 0$ We have

$$P_1 = \alpha_1 + 1 \qquad P_2 = \alpha_1^2 + \alpha_1 + 1 \qquad P_3 = \alpha_1^6 + \alpha_1^3 + 1$$

$$P_4 = \alpha_0 + 1 \qquad P_5 = \alpha_0 \qquad P_6 = \alpha_0^6 + 3\alpha_0^5 + 15\alpha_0^4 + 6\alpha_0^3 - 9\alpha_0 + 3$$

We now check them one by one. Solving $P_1 = 0$ we get $\alpha_1 = -1$ By substituting this in Q_1 Q_2 we get $P_4 = 0$ But this gives $\alpha_0 = \gamma_0 = -1$ which implies that $A_2 = O_0$ This gives a contradiction.

By considering $P_2 = 0$ we get $\alpha_1^2 + \alpha_1 + 1 = 0$ and by substituting this in Q_1 Q_2 we see that to satisfy both Q_1 Q_2 we must have $P_5 = \alpha_0 = 0$ But this gives that $F(A_2) = O_0$ Hence gives contradiction.

Now consider that $P_3 = 0$ this implies that $\alpha_1^6 + \alpha_1^3 + 1 = 0$ We find $\bar{Q}_1$ $\bar{Q}_2$ by finding the remainder of Q_1 and Q_2 by $\alpha_1^6 + \alpha_1^3 + 1$ w.r.t α_1 We see that $\bar{Q}_1 = A_1\alpha_0^2 + B_1\alpha_1 + C_1$ and $\bar{Q}_2 = A_2\alpha_0^2 + B_2\alpha_1 + C_2$ where A_i B_i and C_i for i 1 2 are functions of α_1 Now to satisfy Q_1 and Q_2 we must have $\bar{Q}_1 = 0 = \bar{Q}_2$ This gives us that

$$\alpha_0 = \frac{A_1\,C_2 - A_2\,C_1}{A_2\,B_1 - A_1\,B_2}$$

This α_0 is a rational function in α_1 We want this to be equivalent to polynomial in α_1 We call it $h(\alpha_1)$ If the remainder of this function with respect to P_2 is zero then $\bar{Q}_1$ and $\bar{Q}_2$ will be satisfied. We therefore consider that $h(\alpha_1) = h_1\,\alpha_1^5 + h_2\,\alpha_1^4 + h_3\,\alpha_1^3 + h_4\,\alpha_1^2 + h_5\,\alpha_1 + h_6$ to be the desired polynomial for arbitrary values of h_i i 1 6 . It is a degree lower than six because the value for α_1^6 is already determined. We therefore impose the following:

$$\alpha_0 = \frac{A_1\,C_2 - A_2\,C_1}{A_2\,B_1 - A_1\,B_2} \equiv h(\alpha_1) \bmod \alpha_1^6 + \alpha_1^3 + 1 = 0$$

This is to say that we want that the remainder of $(A_1\,C_2 - A_2\,C_1) - h(A_2\,B_1 - A_1\,B_2)$ with respect to $\alpha_1^6 + \alpha_1^3 + 1$ becomes equal to zero. This gives us an equation E in α_1 whose coefficients depend on the constants h_1 h_2 h_6 As the values of α_1 are already determined therefore to make this equation equal to zero we cannot solve this equation for α_1 and cannot look for the new values of α_1 Therefore we solve the coefficients of E and look for the solutions of h_1 h_2 h_6 We find that the only solution for which $E = 0$ is that $\alpha_0 \equiv h(\alpha_1) = -2\alpha_1^5 + \alpha_1^3 - \alpha_1^2 - \alpha_1$ This gives us the relation between α_0 and α_1 which satisfies the equations Q_1 and Q_2 for the polynomial equations $\alpha_1^6 + \alpha_1^3 + 1 = 0$ and $\alpha_0^6 + 3\alpha_0^5 + 15\alpha_0^4 + 6\alpha_0^3 - 9\alpha_0 + 3 = 0$ We observe that we get the same results by using Grobner basis for this system.

Hence proved. ∎

Proposition 67 *Let $\tilde{F}$ and $\tilde{F}_1$ be the induced map of F we get after blowing up A_0 and the points of $\mathcal{O}_1$ Assume that $\tilde{F}^4(A_1) = O_1$ and $\tilde{F}_1^3(A_2) = O_0$ for $k = 2$ $p = 3$ respectively. Then the coefficients of f satisfy the following condition.*

(C5) $\alpha_1 = 1$ $\gamma_0 = -\frac{1}{2}$ *and* $\alpha_0 = \frac{1}{4}$

(C6) $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0$ $\gamma_0 = -\frac{1}{\alpha_1^2(1+\alpha_1)}$ *and* $\alpha_0 = -(\alpha_1^3 + 2\alpha_1^2 + \alpha_1 + 2)$

Proof. The imposed condition $\tilde{F}^4(A_1) = O_1$ implies that *condition k* is satisfied for $k = 2$ This can be written as $\alpha_1^2(1 + \alpha_1)\gamma_0 + 1 = 0$ which implies that

$$\gamma_0 = -\frac{1}{\alpha_1^2(1 + \alpha_1)}$$

Note that $\alpha_1 = 0$ and $1 + \alpha_1 = 0$

Observe that A_2 or any of it s iterates belong to $S_1 \quad S_0$ then we cannot impose the condition $\tilde{F}^3(A_2) = O_0$ because $O_0 \quad T_1$ and for $k = 2$ we have

$$S_1 \qquad G_0 \qquad G_1 \qquad G_2 \qquad G_3 \qquad G_4 \qquad T_1$$

and $O_0 \quad T_1$ This shows that the orbit of A_2 if collides with the orbit of A_1 then after three iterates it can not reach O_0 Also $O_0 \quad S_0 \; E_0$ and the orbit of A_2 can never reach O_0 if it collides with the orbit of A_0 Therefore we read the condition $\tilde{F}^3(A_2) = O_0$ as $F^3(A_2) = O_0$ To find such families from now we consider that neither A_2 nor any of its iterate belong to $S_1 \quad S_0$ i.e. there is no collision of orbits.

Then the expression of $F^3(A_2)$ is:

$$\begin{aligned}
&[-(1+\alpha_1)(1+\alpha_1+\alpha_1^2)(1+\alpha_1+2\alpha_1^2+\alpha_1^3+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)p_1 : \\
&\qquad\qquad -p_1 p_2 : \\
&\qquad \alpha_1^2(1+\alpha_1+2\alpha_1^2+\alpha_1^3+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)^2(\alpha_1+1)p_3]
\end{aligned}$$

where

$$\begin{aligned}
p_1 = 1 + \alpha_1 + 2\alpha_1^2 + 2\alpha_0\alpha_1^4 + 6\alpha_0\alpha_1^5 + 8\alpha_0\alpha_1^6 + \alpha_1^3 + 7\alpha_0\alpha_1^7 + 4\alpha_0\alpha_1^8 + \\
\alpha_0\alpha_1^9 - \alpha_1^4 - 2\alpha_1^5 - 2\alpha_1^6 - \alpha_1^7
\end{aligned}$$

$$p_2 = \alpha_0 - \alpha_1 - 2\alpha_1^2 + 9\alpha_0\alpha_1^2 + 4\alpha_0\alpha_1 + 12\alpha_0\alpha_1^3 + 9\alpha_1^4\alpha_0 + \alpha_0\alpha_1^5 - 8\alpha_0\alpha_1^6 - 10\alpha_0\alpha_1^7 - 6\alpha_0\alpha_1^8 - 3\alpha_0\alpha_1^9 + \alpha_0^2\alpha_1^4 + 5\alpha_0^2\alpha_1^5 + 12\alpha_0^2\alpha_1^6 + 18\alpha_0^2\alpha_1^7 + 18\alpha_0^2\alpha_1^8 + 13\alpha_0^2\alpha_1^9 + 8\alpha_0^2\alpha_1^{10} + 4\alpha_0^2\alpha_1^{11} - \alpha_0\alpha_1^{10} + \alpha_0^2\alpha_1^{12} - 2\alpha_1^3 - 2\alpha_1^4$$

$$p_3 = \alpha_0 + 3\alpha_0\alpha_1 + 4\alpha_0\alpha_1^2 + 3\alpha_0\alpha_1^3 + 2\alpha_0\alpha_1^4 + \alpha_0\alpha_1^5 - 2\alpha_1 - \alpha_1^2 - \alpha_1^3 - 1$$

Assume that $F^3(A_2) = O_0 = \left[1 : 0 : \frac{1}{\alpha_1^2(1+\alpha_1)}\right]$ Since $O_0[1] = 0$ we impose that the first component of $F^3(A_2)$ is also non zero, and hence:

$$F^3(A_2) = \left[1 : \frac{p_2}{(1+\alpha_1)(1+\alpha_1+\alpha_1^2)(1+\alpha_1+2\alpha_1^2+\alpha_1^3+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)} : \frac{-p_3\,\alpha_1^2\,(1+\alpha_1+2\alpha_1^2+\alpha_1^3+(\alpha_1^6+2\alpha_1^5+2\alpha_1^4)\alpha_0)}{(1+\alpha_1+\alpha_1^2)\,p_1}\right]$$

Then the coefficients have to satisfy the following conditions for some polynomials q_1 q_2 q_3 r_1 r_2 r_3 depending on α_1 we have

$$Q_1 := q_1\alpha_0^2 + q_2\alpha_0 + q_3 \qquad Q_2 = r_1\alpha_0^2 + r_2\alpha_0 + r_3$$

To find a common solution of Q_1 Q_2 we find their resultant with respect to α_0 and α_1. That is $R_1(Q_1\ Q_2\ \alpha_0)$ factorizes as:

$$R_1(Q_1\ Q_2\ \alpha_0) = -\alpha_0^{11}(\alpha_0^4 + 4\alpha_0^3 + 6\alpha_0^2 + 9\alpha_0 + 11)(4\alpha_0 - 1)^2\,(4\alpha_0^2 + 1)^3\,(\alpha_0^2 + \alpha_0 + 1)^8$$

and

$$R_2(Q_1\ Q_2\ \alpha_1) = -\alpha_1^8(\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1)\,(\alpha_1^2 + \alpha_1 + 1)^8(\alpha_1 + 1)^9\,(\alpha_1 - 1)^2\,(\alpha_1^2 + 1)^3$$

We find that these two resultants have all the information about our solutions. To find a solution for Q_1 Q_2 we will see that for a particular combinations of pairs of the factors of these resultants R_1 and R_2 we get our required solutions. We know that $\alpha_1 = 0$ We have

$$P_1 = \alpha_1 - 1 \quad P_2 = \alpha_1^2 + \alpha_1 + 1 \quad P_3 = \alpha_1^2 + 1 \quad P_4 = \alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1$$

$$P_5 = 4\alpha_0 - 1 \quad P_6 = \alpha_0^4 + 4\alpha_0^3 + 6\alpha_0^2 + 9\alpha_0 + 11 \quad P_7 = 4\alpha_0^2 + 1 \quad P_8 = \alpha_0^2 + \alpha_0 + 1$$

Now we check all the P_i s to find our required families.
Solving P_1 we get $\alpha_1 = 1$ By substituting this in Q_1 Q_2 we get $P_5 = 0$ that is $\alpha_0 = \frac{1}{4}$ This gives us that $F^3(A_2) = O_0 = [1 : 0 : \frac{1}{2}]$. Thus we have our required family.

By considering $P_2 = 0$ we get $\alpha_1^2 + \alpha_1 + 1 = 0$ and by substituting this in Q_1 Q_2 we see that to satisfy both Q_1 Q_2 we must have $P_8 = \alpha_0^2 + \alpha_0 + 1 = 0$ But we see that this satisfies $F(A_2) = O_0$ Therefore it gives contradiction. Now consider that $P_3 = 0$ this implies that $\alpha_1^2 + 1 = 0$ By substituting this in Q_1 Q_2 we see that to satisfy both Q_1 Q_2 we must have $4\alpha_0^2 + 1 = 0$ But we see that this satisfies $F(A_2) = O_0$ Therefore it gives contradiction.

By taking $P_4 = 0$ it implies that $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0$ By using the same method as discussed above in the proof of $C4$ of previous proposition 66 we find that to satisfy both Q_1 and Q_2 we must have $\alpha_0 = -(\alpha_1^3 + 2\alpha_1^2 + \alpha_1 + 2)$ also $\alpha_1^2 + \alpha_1 + 1 = 0$ or $\alpha_0^4 + 4\alpha_0^3 + 6\alpha_0^2 + 9\alpha_0 + 11 = 0$ We find that by using the Grobner basis one also gets the same solutions for this system of equations. Hence proved. ∎

Proposition 68 *Let $\tilde{F}$ and $\tilde{F}_1$ be the induced map of F we get after blowing up A_0 and the points of $\mathcal{O}_1$ Assume that $\tilde{F}^2(A_1) = O_1$ and $\tilde{F}_1^4(A_2) = O_0$ for $k = 1$ $p = 4$ respectively. Then the coefficients of f satisfy the following conditions:*

(C7) $\alpha_1 = -1$ $\gamma_0 = -1$

(C8) $\alpha_1 = 1$ $\alpha_0 = 0$ $\gamma_0 = -1$

(C9) $\alpha_1^2 + 1 = 0$ $\alpha_0 = 0$ $\gamma_0 = 1$

(C10) $\alpha_0 = -\frac{1}{5}(3\alpha_1^3 + \alpha_1^2 + 4\alpha_1 + 2)$ $\gamma_0 = -\frac{1}{\alpha_1^2}$ *and* $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0$

(C11) $\alpha_0 = 1 - \alpha_1^3$ $\gamma_0 = -\frac{1}{\alpha_1^2}$ *and* $\alpha_1^4 + 1 = 0$

Proof. The imposed condition $\tilde{F}^2(A_1) = O_1$ implies that *condition* k is satisfied for $k = 1$ This can be written as $\alpha_1^2\gamma_0 + 1 = 0$ which implies that

$$\gamma_0 = -\frac{1}{\alpha_1^2}$$

we know that $\alpha_1 = 0$

Observe that A_2 S_0 and if any of its iterates belong to S_0 then we cannot impose the condition $\tilde{F}_1^4(A_2) = O_0$ because O_0 S_0 E_0 Now we know that for $k = 1$ we have

$$S_1 \quad G_0 \quad G_1 \quad G_2 \quad T_1$$

and $O_0 \quad T_1$ Then if $A_2 \quad S_1$ we see that the condition $\tilde{F}^4(A_2) = O_0$ can be fulfilled as after four iterates the orbit of A_2 can reach $O_0 \quad T_1$ But if $F^i(A_2) \quad S_1$ for $i > 0$ then it can not reach O_0 because of the condition $\tilde{F}^4(A_2) = O_0 \quad T_1$ and that $O_0 \quad G_j \;\; j = \;\; 0 \;\; 1 \;\; 2$ Now if A_2 or any of its iterate is not in $\;\; S_1 \;\; S_0 \;\;$ then we read the condition $\tilde{F}^4(A_2) = O_0$ as $F^4(A_2) = O_0$ Hence we study two different cases:

(i) $A_2 \quad S_1$ i.e. the orbit of A_2 collides with the orbit of A_1

(ii) Neither A_2 nor any of its iterate belong to $\;\; S_1 \;\; S_0 \;\;$ i.e. there is no collision of orbit of A_2

We first study case (i) that is $A_2 \quad S_1$ This implies that $\alpha_1 = -1$ which gives $\gamma_0 = -1$ To find $\tilde{F}^4(A_2) = O_0$ we need to follow the orbit $S_1 \quad G_0 \quad G_1 \quad G_2 \quad T_1$ Hence under the action of $\tilde{F}$ we see that

$$S_1 \qquad [x_0 : (\alpha_0 + 1)x_0 - x_1]_{G_0} \qquad \tilde{F}^2[u : v]_{G_0} \qquad [-u : \alpha_0\, u + v]_{G_2}$$

Then the orbit of A_2 under the action of $\tilde{F}$ is

$$A_2 \qquad [1 : 0]_{G_0}$$

also under the action of $\tilde{F}^2$ which takes G_0 directly to G_2 then we find that

$$\tilde{F}^2 : [1 : 0]_{G_0} \qquad [1 : -\alpha_0]_{G_2}$$

Now

$$\tilde{F}[1 : -\alpha_0]_{G_2} = [1 : 0 : 1] = \tilde{F}^4(A_2) = O_0$$

This is true for all values of α_0 hence we get our first solution.

We now study case (ii) when there is no collisions of orbits of A_2 Thus we consider that $\alpha_1 + 1 = 0$ The expression of $F^4(A_2)$ is:

$$[-(1 + \alpha_1 + \alpha_1^4\alpha_0 + \alpha_1^2)\, p_1\, (1 + \alpha_1 + 2\alpha_1^4\alpha_0 + \alpha_1^2 + 2\alpha_1^5\alpha_0 + \alpha_1^6\alpha_0) :$$
$$-p_1\, p_2\, \alpha_0\, (1 + \alpha_1)^2 : \alpha_1^2\, (1 + \alpha_1 + 2\alpha_1^4\alpha_0 + \alpha_1^2 + 2\alpha_1^5\alpha_0 + \alpha_1^6\alpha_0)^2\, p_3]$$

where

$$p_1 = 1 + 2\alpha_1 + 3\alpha_1^4\alpha_0 + 2\alpha_1^2 + 7\alpha_0\alpha_1^5 + 7\alpha_0\alpha_1^6 + \alpha_1^3 + 5\alpha_0\alpha_1^7 - \alpha_1^4 + \\ 8\alpha_0\alpha_1^8 - \alpha_1^5 + \alpha_0^2\alpha_1^8 + 2\alpha_0^2\alpha_1^9 + \alpha_1^{10}\alpha_0^2 + \alpha_1^{11}\alpha_0^2 + \alpha_0\alpha_1^9 - \alpha_1^6$$

$$p_2 = \alpha_0^2\alpha_1^{12} + 2\alpha_1^{10}\alpha_0^2 + \alpha_0\alpha_1^{10} - \alpha_0\alpha_1^9 + 2\alpha_0^2\alpha_1^8 + 2\alpha_1^8\alpha_0 - \alpha_1^7 - \\ \alpha_0\alpha_1^7 + 5\alpha_0\alpha_1^6 + 2\alpha_0\alpha_1^5 - 2\alpha_1^5 + 3\alpha_0\alpha_1^4 - 2\alpha_1^2 + \alpha_1 + 1$$

$$p_3 = \alpha_0 + 3\alpha_0\alpha_1 + 4\alpha_0\alpha_1^2 + \alpha_0^2\alpha_1^4 + 2\alpha_0^2\alpha_1^5 + 2\alpha_0\alpha_1^3 + \\ \alpha_1^4\alpha_0 - \alpha_1 - \alpha_1^2 + 2\alpha_0^2\alpha_1^6 + \alpha_0^2\alpha_1^7 + \alpha_0^2\alpha_1^8 + \alpha_0\alpha_1^6 - \alpha_1^3$$

Assume that $F^4(A_2) = O_0 = \left[1 : 0 : \frac{1}{\alpha_1^2}\right]$ Since $O_0[1] = 0$ we impose that the first component of $F^4(A_2)$ is also non zero, and hence:

$$F^3(A_2) = \Bigg[1 : \frac{\alpha_0\, p_2\,(1+\alpha_1)^2}{(1+\alpha_1+\alpha_1^4\alpha_0+\alpha_1^2)\,(1+\alpha_1+2\alpha_1^4\alpha_0+\alpha_1^2+2\alpha_1^5\alpha_0+\alpha_1^6\alpha_0)} : \\ -\frac{p_3\,\alpha_1^2\,(1+\alpha_1+2\alpha_1^4\alpha_0+\alpha_1^2+2\alpha_1^5\alpha_0+\alpha_1^6\alpha_0)}{-(1+\alpha_1+\alpha_1^4\alpha_0+\alpha_1^2)\,p_1}\Bigg]$$

Then the coefficients have to satisfy the following conditions for some polynomials $q_1 \;\; q_2 \;\; q_3 \;\; r_1 \;\; r_2 \;\; r_3 \;\; r_4$ depending on α_1 we have

$$Q_1 := (q_1\alpha_0^2 + q_2\alpha_0 + q_3)\alpha_0 \qquad Q_2 = r_1\alpha_0^3 + r_2\alpha_0^2 + r_3\alpha_0 + r_4$$

To find a common solution of $Q_1 \;\; Q_2$ we find their resultant with respect to α_0. That is $R(Q_1 \;\; Q_2 \;\; \alpha_0)$ factorizes as:

$$R_1(Q_1 \;\; Q_2 \;\; \alpha_0) = \alpha_0^{45}(5\alpha_0^4 + 5\alpha_0^2 + 1)(\alpha_0^4 - 4\alpha_0^3 + 6\alpha_0^2 - 4\alpha_0 + 2)(\alpha_0 + 1)^5$$

and

$$R_2(Q_1 \;\; Q_2 \;\; \alpha_1) = -\alpha_1^{26}(\alpha_1^4+\alpha_1^3+\alpha_1^2+\alpha_1+1)(\alpha_1^2+\alpha_1+1)^4(\alpha_1+1)^5(\alpha_1-1)(\alpha_1^4+1)(\alpha_1^2+1)$$

We know that $\alpha_1 = 0$ so to find a common solution for $Q_1 \ \ Q_2$ we see that for a particular combination of the following polynomial equations must be zero.

$$P_1 = \alpha_1 - 1 \quad P_2 = \alpha_1^2 + \alpha_1 + 1 \quad P_3 = \alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1$$
$$P_4 = \alpha_1^4 + 1 \quad P_5 = \alpha_1^2 + 1$$

$$P_6 = \alpha_0 \quad P_7 = 5\alpha_0^4 + 5\alpha_0^2 + 1 \quad P_8 = \alpha_0^4 - 4\alpha_0^3 + 6\alpha_0^2 - 4\alpha_0 + 2 \quad P_9 = \alpha_0 + 1$$

Solving $P_6 = 0$ we get $\alpha_0 = 0$ By substituting this in $Q_1 \ \ Q_2$ we find that to satisfy both $Q_1 \ \ Q_2$ equal to zero we must have $P_1 = 0$ or $P_5 = 0$ or $P_2 = 0$ We find that $P_6 = P_i = 0$ for any $i = 1 \ 5$ is sufficient to have $F^4(A_2) = O_0$ whereas $P_2 = 0 = P_6$ gives contradiction as it gives $F(A_2) = O_0$

By taking $P_3 = 0$ it implies that $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0$ By substituting this in $Q_1 \ \ Q_2$ we see that to satisfy both $Q_1 \ \ Q_2$ we use the same methodology as used in the proof of C_4 of proposition 66. This gives us that $\alpha_0 = -\frac{1}{5}(3\alpha_1^3 + \alpha_1^2 + 4\alpha_1 + 2)$ We find that with this value of α_0 and with the conditions $\alpha_1^4 + \alpha_1^3 + \alpha_1^2 + \alpha_1 + 1 = 0$ or $5\alpha_0^4 + 5\alpha_0^2 + 1 = 0$ the equations Q_1 and Q_2 are satisfied. Hence it gives us the required solution.

Now taking $P_4 = 0$ it implies that $\alpha_1^4 + 1 = 0$ By substituting this in $Q_1 \ \ Q_2$ we see that to satisfy both $Q_1 \ \ Q_2$ we use the same methodology as used in the proof of C_4 of proposition 66. This gives us that $\alpha_0 = 1 - \alpha_1^3$ We find that with this value of α_0 and with the conditions $\alpha_1^4 + 1 = 0$ or $\alpha_0^4 - 4\alpha_0^3 + 6\alpha_0^2 - 4\alpha_0 + 2 = 0$ the equations Q_1 and Q_2 are satisfied. This gives us the required solution. By using the Grobner basis one gets the same solutions.

Hence proved. ∎

Chapter 9

Future work

9.1 The problem

The natural continuation of this dissertation is to study the dynamics of the birational maps inside family of mappings (1.1) with dynamical degree $\delta > 1$ These maps have frequently appeared in this work. In broader aspect the study of two different cases is made; the degenerate case and the non degenerate case. The first step could be to consider the birational maps which appear in degenerate case and have dynamical degree $\delta > 1$ These are the following families:

Theorem 69 *For a degenerate case the subfamilies of 1.1 with $\delta > 1$ are the following: Let $(\beta\gamma)_{12} = \beta_1 \gamma_2 - \beta_2 \gamma_1 = 0$ then*

1. *If $\alpha_1 \cdot \gamma_1 = 0 = \alpha_2 \cdot \gamma_2$ then the dynamical degree of F is $\delta(F) = 2$*
2. *If $\alpha_2 \cdot \gamma_2 = 0$ and $\gamma_2 = 0$ then $\beta_2 = 0$ $\alpha_2 = 0$ and the dynamical degree is $\delta(F) = \frac{1+\sqrt{5}}{2}$ $d_{n+2} = d_{n+1} + d_n$ for all n $\mathbb{N}$*
3. *If $\alpha_1 = 0$ then $\gamma_1 = 0 = \alpha_2$ and the dynamical degree is $\delta(F) = \frac{1+\sqrt{5}}{2}$ $d_{n+2} = d_{n+1} + d_n$ for all n $\mathbb{N}$*

Now let $(\alpha\gamma)_{12} = \alpha_1 \gamma_2 - \alpha_2 \gamma_1 = 0$ then the families are the following:

1. *and $\alpha_1 = 0$ $\alpha_2 = 0$ $\gamma_1 = 0$ and $\gamma_2 = 0$ then the dynamical degree is $\delta(F) = 2$*
2. *Assume that $\alpha_1 = \gamma_1 = 0$ Then $\alpha_2 = 0$ $\gamma_2 = 0$ $\beta_1 = 0$ and the dynamical degree of F is $\delta(F) = \frac{1+\sqrt{5}}{2}$*

Diller and Favre [DF01] illustrated the results for such birational maps of plane in general that they can be automorphism and there exist maps which are not automorphisms. The families which are automorphisms are expected to have different dynamics than the non automorphism families. These two classes by studying their dynamics separately invites to investigate the possible periodic orbits, attracting orbits, stable and unstable regions, fixed points and invariant curves, Julia set, Fatou set, ordered chaos etc. It will therefore classify and explain several different types of exponential growth inside the birational maps (1.1). Hence the objective is to investigate the above given families is:

1. To detect the families which are automorphisms and the ones which are not automorphisms.
2. Study the dynamics of the families which are automorphism and find the possible conjugation classes. Also make a parameter space study to find if the dynamical degree δ is invariant for the any choice of parameters this is to say that if it becomes one for any choice of parameters then how the dynamics will change.
3. Study the dynamics of the families which are not automorphism and make a parameter analysis. Also find the possible conjugation classes inside as well.

The second step could be to further study the dynamics of the families in the non degenerate case with dynamical degree greater than one.

9.2 Significance of proposed work

This problem will provide complete the information about the degenerate case of family 1.1 and all of its subfamilies for all the values of parameters. This family as a subfamily of Cremona maps of $P\mathbb{C}^2$ generalizes the results given by Bedford and Kim in [BK06], [BK04] in 2006, provides examples of the theoretical contributions made by Diller and Favre in [DF01] in 2001 and Diller in [Dil96] in 1996 also generalizes the work done by Cima and Zafar in [CZ].

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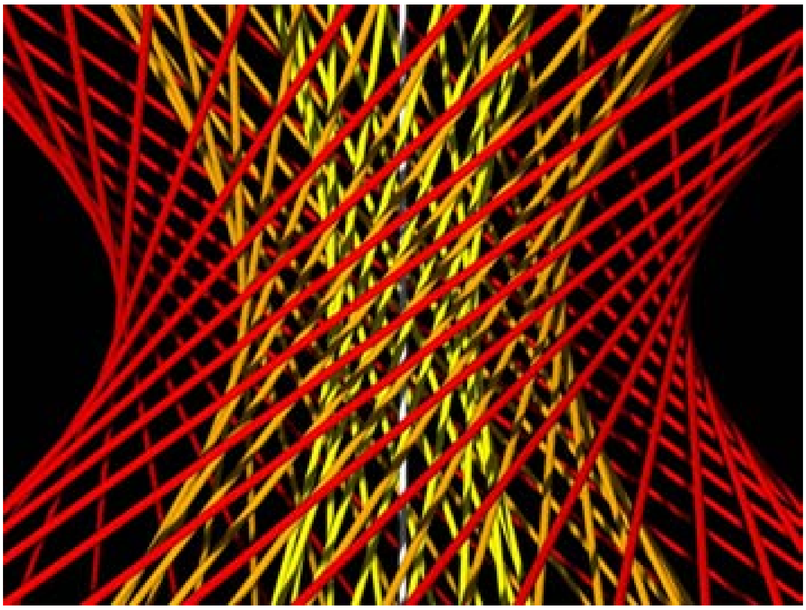
Sundus Zafar

Bellaterra, June, 2014

Barcelona, Spain



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