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Universitat de Barcelona

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Jordi Canela Sánchez

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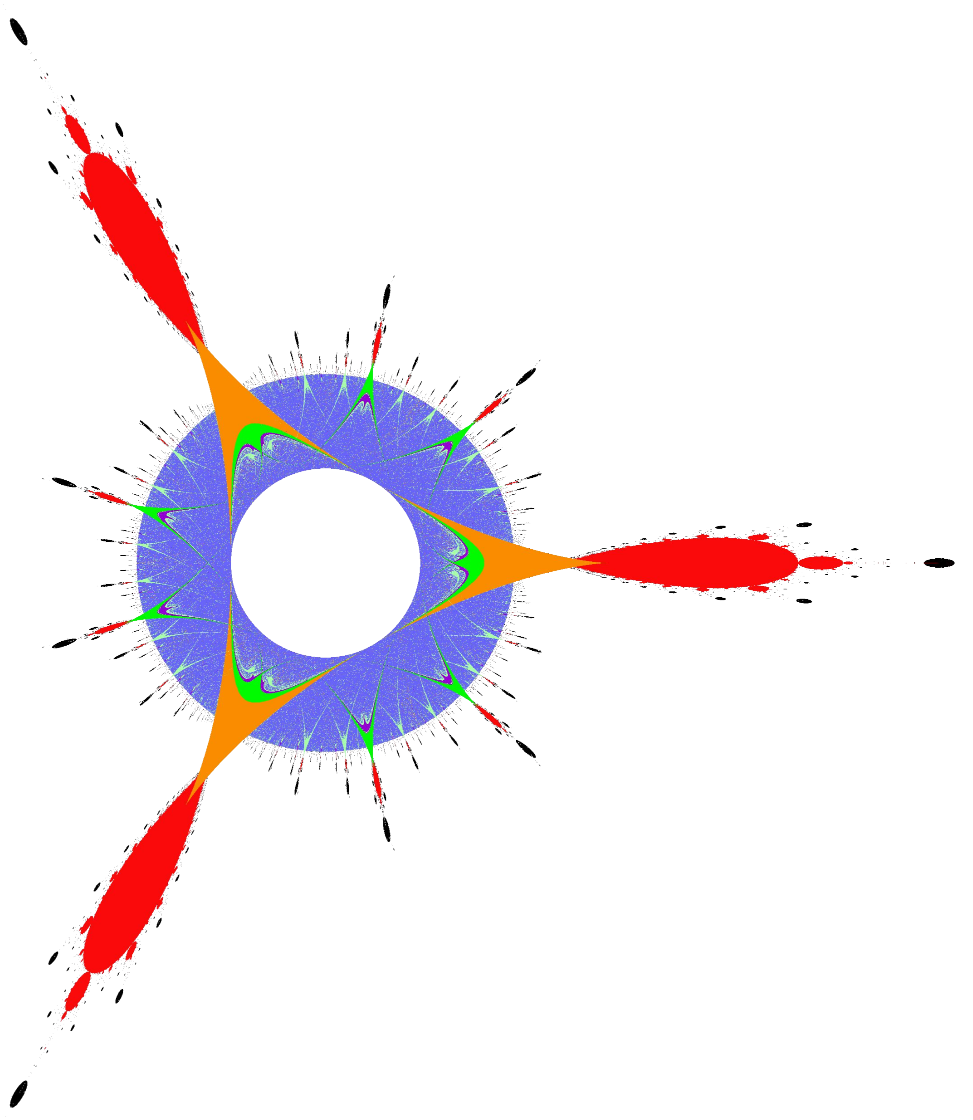

Jordi Canela Sánchez

PhD Advisors: **Núria Fagella Rabionet** and **Antonio Garijo Real**

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Universitat de Barcelona
Programa de doctorat en Matemàtiques

Tesi presentada per en Jordi Canela Sánchez per optar al grau de Doctor en Matemàtiques

Directors: Núria Fagella Rabionet i Antonio Garijo Real

Departament de Matemàtica Aplicada i Anàlisi
Facultat de Matemàtiques
Març de 2015

Certifico que la següent tesi
ha estat realitzada per en
Jordi Canela Sánchez
sota la meva codirecció.

Barcelona, març de 2015.

Núria Fagella Rabionet

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Barcelona, març de 2015.

Antonio Garijo Real

Als meus pares

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List of symbols

$\mathcal{J}(f)$	Julia set of a holomorphic map f.
$\mathcal{F}(f)$	Fatou set of a holomorphic map f.
$< w_0 >$	The p-cycle $\{w_0, \cdots, f^{p-1}(w_0) = w_{p-1}\}$ of a holomorphic map f, where w_0 denotes the marked point of the cycle.
$A(< w_0 >)$	The basin of attraction of the p-cycle $< w_0 >$.
$A^*(w_q)$	The connected component of $A(< w_0 >)$ containing the point w_q of $< w_0 >$
$A^*(< w_0 >)$	The immediate basin of attraction of $< w_0 >$ given by $\bigcup_{n=0}^{p-1} A^*(w_n)$.
$\mathbb{D}$	The unit disk.
$\mathbb{D}^*$	The punctured disk $\mathbb{D} \setminus \{0\}$.
$\mathbb{S}^1$	The unit circle.
$\mathcal{I}$	The reflection with respect to the unit circle $z \to 1/\overline{z}$.
$\mathbb{R}$	The real line.
$\widetilde{\mathcal{I}}$	The reflection with respect to the real line $z \to \overline{z}$.
$\mathbb{C}$	The complex plane.
$\mathbb{C}^*$	The punctured plane $\mathbb{C} \setminus \{0\}$.
$\widehat{\mathbb{C}}$	The Riemann sphere $\mathbb{C} \cup \{\infty\}$.
σ_0	The standard complex structure.
μ_0	The Beltrami coefficient of the standard complex structure.
T_τ	The tongue of type τ.
r_τ	The root of the tongue T_τ.
a_τ	The tip of the tongue T_τ.
ET_τ	The extended tongue of type τ.

Contents

Resum en Català

Aquesta tesi doctoral pertany a l'àmbit dels sistemes dinàmics discrets al pla complex, és a dir, la iteració de funcions analítiques en una variable complexa. Hi ha una llarga llista de resultats importants obtinguts gràcies a la complexificació de l'aplicació logística i d'altres funcions unimodals i bimodals provinents de, per exemple, models discrets biològics i econòmics o algorismes de cerca d'arrels, entre d'altres. Les eines complexes ens permeten veure fenòmens que no són visibles a la recta real. Els comportaments caòtics que poden ser vistos per a aquests models (cascades de bifurcació, etc.) adquireixen una nova dimensió quan són observats al pla complex, on poden ser entesos molt millor.

Aquesta àrea de les matemàtiques va néixer a principis del segle XX com a conseqüència de les investigacions sobre el mètode de Newton, el ben conegut algorisme de cerca d'arrels, al pla complex. Fins aleshores només s'havien dut a terme estudis locals, però P. Fatou i G. Julia van encarar el problema des d'un punt de vista més global. Van classificar les possibles òrbites estables, en termes de normalitat. La frontera entre regions estables, coneguda actualment com a conjunt de Julià, és un objecte invariant d'una gran bellesa i complexitat que fou descrit per Fatou i Julia de forma acurada tot i no comptar amb l'ajut d'ordinadors.

Les bases de la teoria foren establertes durant aquests començaments, arribant tant lluny com era possible amb les eines de que es disposava. Després vingueren uns quants anys de relativa poca activitat que duraren fins a mitjans dels anys 80, quan la l'àrea va renéixer degut a dos factors diferents. D'una banda, D. Sullivan [Sul85] va provar una de les principals conjectures deixades obertes per Fatou i Julia, la no existència de conjunts errants, per mitjà d'eines quasiconformes. Aquestes eines, provinents de l'àrea de la geometria analítica, han estat clau per a molts resultats posteriors i són fortament usades en aquesta tesi. D'altre banda, l'arribada dels primers ordinadors va permetre a B. Mandelbrot dibuixar la primera imatge del que es coneix actualment com a conjunt de Mandelbrot, l'espai de connectivitat del pla de paràmetres de la família quadràtica $z^2 + c$. La visualització dels primers conjunts fractals, juntament amb les noves eines disponibles per encarar els problemes oberts deixats per Fatou i Julia, van atreure l'interès de molts matemàtics. Com a conseqüència, es van fer avanços significatius, alguns mereixedors del més alt reconeixement reconeixements en matemàtiques (com ara J. C. Yoccoz i C. McMullen, medallistes Fields en 1994 i 1998, respectivament).

Donada una funció racional $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denota l'esfera de Riemann, considerem el sistema dinàmic donat pels iterats de f. L'esfera de Riemann es divideix en dos conjunts completament invariants per f: el *conjunt de Fatou* $\mathcal{F}(f)$, definit com el conjunt de punts $z \in \widehat{\mathbb{C}}$ on la família $\{f^n, n \in \mathbb{N}\}$ és normal en algun entorn de z, i el seu complement, el *conjunt de Julià* $\mathcal{J}(f)$. La dinàmica de les òrbites de $\mathcal{F}(f)$ és estable en el sentit de normalitat o equicontinuitat mentre que la dinàmica a $\mathcal{J}(f)$ presenta un caràcter caòtic. El conjunt de Fatou $\mathcal{F}(f)$ és obert i, per tant, el conjunt de Julià $\mathcal{J}(f)$ és tancat. Si a més el grau de la funció racional f és major o igual que 2, aleshores el conjunt de Julià $\mathcal{J}(f)$ és la clausura del conjunt de punts periòdics repulsors de f, sempre no buida.

Les components connexes de $\mathcal{F}(f)$, anomenades *components de Fatou*, són enviades les unes a les altres per la funció f. Sullivan [Sul85] va provar que tota component de Fatou d'una aplicació racional és o bé periòdica o bé preperiòdica. Se segueix del Teorema de Classificació (vegeu Teorema 1.1.18), que tota component de Fatou periòdica d'una funció racional és o bé la conca d'atracció d'un cicle atractor o parabòlic, o bé un domini de rotació simplement connex (un disc de Siegel), o bé un domini de rotació doblement connex (un anell de Herman). A més, qualsevol d'aquestes components periòdiques està relacionada amb algun *punt crític*, un punt $z \in \widehat{\mathbb{C}}$ tal que $f'(z) = 0$. Més concretament, les conques d'atracció dels punts periòdics attractors o parabòlics contenen almenys un punt crític mentre que els discs de Siegel i els anells de Herman tenen orbites crítiques acumulant-se a les seves fronteres. Vegeu el capítol 1 per a una introducció a la dinàmica de funcions racionals.

Tota aplicació holomorfa de grau finit del disc unitat $\mathbb{D}$ en sí mateix és un producte d'automorfismes de $\mathbb{D}$, i.e., un producte de Blaschke finit i, per tant, està definit (per reflexió) a tota l'esfera de Riemann. Els productes de Blaschke han estat usats de forma repetida com a models en dinàmica complexa. Per exemple, els productes $B_{r,\alpha}(z) = e^{2\pi i\alpha}z^2(z-r)/(1-rz)$, on $\alpha, r \in \mathbb{R}$ i $r > 3$, van ser utilitzats per M. R. Herman [Her79] per tal de provar l'existència dels anells de Herman (vegeu Figura 1.4 (d)) i, amb paràmetres complexos, per X. Buff et al [BFGH05] per estudiar les deformacions quasiconformes d'aquests objectes. També van ser usats, prenent $r = 3$, per provar el celebrat resultat que ens diu que les fronteres dels discs de Siegel amb número de rotació de tipus constant dels polinomis de grau 2 són corbes de Jordan (vegeu els treballs de E. Ghys [Ghy84], M. R. Herman [Her86, Her87], A. Douady [Dou87] i G. Świątek [Świ88]). Aquest darrer resultat va ser generalitzat posteriorment per C. Petersen i S. Zakeri [PZ04].

La família de Blaschke

L'objectiu d'aquesta tesi és estudiar la dinàmica dels productes de Blaschke de grau 4 donats per

$$B_a(z) = z^3 \frac{z-a}{1-\bar{a}z}, \tag{1}$$

on $a, z \in \mathbb{C}$. Per tot valor de $a \in \mathbb{C}$, els punts $z = 0$ i $z = \infty$ són punts fixos superattractors de grau local 3 (c.f. [CFG15]). Denotem per $A(0)$ i $A(\infty)$ les seves conques d'atracció i per $A^*(0)$ i $A^*(\infty)$ les seves conques immediates d'atracció, i.e., les components connexes de $A(0)$ i $A(\infty)$ que contenen $z = 0$ i $z = \infty$, respectivament. Si $|a| \leq 1$ aquestes són les úniques components de Fatou, separades pel conjunt de Julià que és necessàriament $\mathbb{S}^1$. Per qualsevol altre paràmetre, B_a té dos punts crítics lliures (diferents si $|a| \neq 2$) que poden donar lloc a l'existència d'altres components de Fatou diferents de $A(0)$ i $A(\infty)$. Tanmateix, si $|a| > 2$, la família és essencialment unicrítica per efecte de la simetria respecte de $\mathbb{S}^1$ que, en aquest cas, lliga en cert sentit les òrbites dels punts crítics.

Volem remarcar que els productes de Blaschke B_a són pertorbacions racionals del *doubling map* $R_2(z) = z^2$ (donat de forma equivalent per $\theta \to 2\theta \pmod 1$). De fet, els

B_a convergeixen de forma uniforme en subconjunts compactes del pla complex punxat $\mathbb{C}^*$ a $e^{4\pi i \mathrm{Arg}(a)} z^2$ quan a tendeix a ∞.

Els B_a són un cas particular d'una família de Blaschke de grau $m+2$ que, tal i com veurem al capítol 7, comparteix moltes de les seves propietats. Ve donada per

$$B_{a;m}(z) = z^{m+1} \frac{z-a}{1-\bar{a}z},$$

on $a, z \in \mathbb{C}$ i $m \geq 1$ és un nombre natural. Aquestes funcions són gairebé bicrítiques en el sentit que els $B_{a;m}$ tenen $z=0$ i $z=\infty$ com a punts fixos superatractors de grau $m+1$ i dos punts crítics lliures que controlen l'existència de qualsevol component de Fatou periòdica altre que $A(0)$ i $A(\infty)$. Fixat $m \geq 1$, $B_{a;m}$ és una família racional de pertorbacions de l'aplicació del cercle $R_m(z) = z^m$. Si $|a| \geq (m+2)/m$, $B_{a;m}|_{\mathbb{S}^1}$ és un cobriment de grau m del cercle (vegeu la secció 7.1). En particular, si $|a| > 3$, aleshores $B_{a;1}|_{\mathbb{S}^1}$ és un homeomorfisme del cercle i la dinàmica de $B_{a;1}$ és coneguda. De fet, la família $B_{a;1}$ és una reparametrització dels $B_{r,\alpha}$ usats per Herman [Her79] per tal de provar l'existència dels anells homònims (vegeu Lema 7.1.1). Conseqüentment, ens centrem en l'estudi de la dinàmica de $B_{a;m}$ per $m \geq 2$, que ve determinada per la posició dels dos punts crítics lliures i el pol respecte del cercle unitat (vegeu la secció 7.1). Tanmateix, les possibles configuracions són independents de m i, per tant, les diferents dinàmiques que apareixen per als $B_{a;m}$, $m > 2$, també es donen per als $B_a = B_{a;2}$. Aquest és el motiu pel qual ens centrem en l'estudi de la família B_a i discutim després com els resultats obtinguts són generalitzats per als productes de Blaschke $B_{a;m}$ amb $m \geq 2$.

La connectivitat del conjunt de Julià és una propietat topològica tot sovint molt relacionada amb la dinàmica de la funció (vegeu e.g. [Shi87], [Prz89], [Pil96] and [DR13]). És equivalent a que totes les components de Fatou siguin simplement connexes. Donat un polinomi P, el seu conjunt de Julià $\mathcal{J}(P)$ és connex si i només si l'òrbita del punt crític no és capturada per la conca d'atracció d'infinit (c.f. [Mil06]). Tanmateix, aquesta classificació no és vàlida per a funcions racionals generals. A diferència dels polinomis, aquestes poden tenir anells de Herman que clarament desconnecten el conjunt de Julià, que pot consistir fins i tot en un conjunt de Cantor de corbes de Jordan (vegeu [McM88]). Això no obstant, la família B_a comparteix moltes de les propietats dels polinomis en aquest aspecte, com ara la no existència d'anells de Herman (vegeu Proposició 3.2.3). També provem el següent criteri de connectivitat del conjunt de Julià que, per als paràmetres a tals que $|a| \geq 2$, és similar al dels polinomis. Es basa en la posició del punt crític $c_+ \in \mathbb{C} \setminus \mathbb{D}$ respecte de la conca immediata d'atracció d'infinit $A^*(\infty)$.

Teorema 3.2.1. *Donat un producte de Blaschke B_a com a (1), tenim que:*

(a) Si $|a| \leq 1$, aleshores $\mathcal{J}(B_a) = \mathbb{S}^1$.

(b) Si $|a| > 1$, aleshores les components connexes de $A(\infty)$ i $A(0)$ són simplement connexes si i només si $c_+ \notin A^*(\infty)$.

(c) Si $|a| \geq 2$, tota component de Fatou U tal que $U \cap A(\infty) = \emptyset$ i $U \cap A(0) = \emptyset$ és simplement connexa.

Conseqüentment, si $|a| \geq 2$, aleshores $\mathcal{J}(B_a)$ és connex si i només si $c_+ \notin A^*(\infty)$.

Els productes de Blaschke B_a amb $|a| > 2$ que no tenen cap cicle atractor o parabòlic a $\mathbb{S}^1$ poden ser relacionats amb els polinomis cúbics. Com veurem al capítol 4, per tals paràmetres $B_a|_{\mathbb{S}^1}$ és quasisimètricament conjugat al doubling map i es pot realitzar una cirurgia quasiconforme que els relaciona amb la família $M_b(z) = bz^2(z-1)$ on $b \in \mathbb{C}$. Aquesta cirurgia estableix una conjugació conforme entre M_b i B_a sobre el conjunt de punts que no cauen mai a $\mathbb{D}$ sota iteració de B_a i els punts que no són atrets a $z = 0$ sota iteració de M_b. En particular, si B_a te un cicle atractor o parabòlic contingut a $\mathbb{C} \setminus \overline{\mathbb{D}}$, aquesta cirurgia conjuga B_a amb M_b conformement a la seva conca d'atracció. Aquests polinomis cúbics amb un punt fix superatractor han estat l'objecte d'estudi de diversos articles. Per exemple, J. Milnor va introduir, en una versió preliminar de [Mil09], l'estudi dels polinomis cúbics amb un cicle superatractor de període p. Més endavant P. Roesch [Roe07] va investigar el tall $\mathcal{S}_1$ de polinomis cúbics amb un punt fix superatractor tot provant algunes de les conjectures plantejades per Milnor. També van ser usats per Tan L. [Tan97] en l'estudi de l'espai de paràmetres de les aplicacions de Newton N_P provinents de polinomis cúbics P, per mitjà dels anomenats matings.

Si B_a té un cicle amb punts dins i fora de $\mathbb{D}$ aleshores la situació és diferent. Tot i que la cirurgia descrita anteriorment encara és possible, molta informació és perduda donat que, sota la nova aplicació, el punt crític sempre rau a la conca d'atracció de $z = 0$. Els paràmetres per als quals l'òrbita de $c_+ \in \mathbb{C} \setminus \overline{\mathbb{D}}$ entra com a mínim un cop a $\mathbb{D}$ són anomenats *paràmetres d'intercanvi* i les components connexes d'aquest conjunt de punts són anomenades *regions d'intercanvi*. Dins d'aquestes regions, la dependència no holomorfa dels B_a respecte del paràmetre a dona lloc al que semblen ser petites "còpies" del Tricorni, l'espai de connectivitat dels antipolinomis $p_c(z) = \overline{z}^2 + c$ (vegeu [CHRSC89] i Figura 5.4 (a)). Milnor [Mil92] va mostrar que una situació similar es dona per als polinomis cúbics amb coeficients reals tot introduint el concepte d'aplicació antipolynomial-like. Distingim dos tipus de cicles atractors per als paràmetres situats a les regions d'intercanvi. Diem que un cicle és *bitransitiu* si la conca immediata d'atracció conté els dos punts crítics, cadascun d'ells en una component connexa diferent. Diem que el cicle és *disjunt* si hi ha dos cicles atractors diferents dels de zero i infinit. La dinàmica que es dóna a aquestes regions d'intercanvi ens permet construir una aplicació polynomial-like de grau 2 o 4 en un entorn de cada paràmetre d'intercanvi bitransitiu o disjunt. Si l'aplicació polynomial-like és de grau 4 fem servir les aplicacions antipolynomial-like introduïdes per Milnor per tal de veure que el polinomi de grau 4 a el qual és híbridament equivalent és de la forma p_c^2. L'equivalència híbrida és un tipus de conjugació més forta que la topològica (i fins i tot que la quasiconforme). Més concretament, provem el següent teorema.

Teorema 5.3.4. *Sigui a_0 un paràmetre d'intercanvi amb un cicle atractor o parabòlic de període $p > 1$. Aleshores hi ha un obert W que conté a_0 i un $p_0 > 1$ divisor de p tals que, per cada $a \in W$, hi ha dos oberts U i V amb $c_+ \in U$ tals que $(B_a^{p_0}; U, V)$ és una aplicació polynomial-like. A més,*

(a) Si a_0 és bitransitiu, $(B_a^{p_0}; U, V)$ és híbridament equivalent a un polinomi de la forma $p_c^2(z) = (z^2 + \overline{c})^2 + c$.

(b) Si a_0 és disjunt, $(B_a^{p_0}; U, V)$ és híbridament equivalent a un polinomi de la forma $p_c^2(z) = (z^2 + \overline{c})^2 + c$ o de la forma $z^2 + c$.

Se sap que la frontera de tota component de Fatou acotada d'un polinomi, amb l'excepció dels discs de Siegel, està acotada per una corba de Jordan. Tanmateix, això no és cert per a funcions racionals sense condicions adicionals (e.g. finitud postcrítica [Pil96]). En el nostre cas, com a conseqüència de les dues construccions prèviament esmentades, sabem que la frontera de tota component connexa de la conca d'atracció d'un cicle atractor o parabòlic no contingut a $\mathbb{S}^1$ i altre que $z=0$ i $z=\infty$ és una corba de Jordan (vegeu Proposició 5.4.1). De fet, si B_a té tal cicle, obtenim una conjugació entre B_a i un polinomi que envia la conca immediata d'atracció del cicle de B_a a la conca immediata d'atracció d'un cicle acotat d'un polinomi. En aquest sentit podem dir que, si $|a|>2$ i $B_a|_{\mathbb{S}^1}$ no té cap cicle atractor o parabòlic, la dinàmica de B_a és d'alguna forma polinomial.

Espai de paràmetres de la família de Blaschke

Després d'investigar el pla dinàmic dels productes de Blaschke B_a estudiem el seu espai de paràmetres (vegeu la coberta i Figura 5.1). Una funció racional és *hiperbòlica* si totes les seves òrbites crítiques s'acumulen en cicles atractors. Una *component hiperbòlica* és una component connexa del conjunt obert $\mathcal{H}=\{a \mid B_a \text{ és hiperbòlic}\}$. La parametrització de components hiperbòliques és ben coneguda si les funcions depenen de forma holomorfa en els paràmetres (vegeu [DH85a], c.f. [BF14]). Altrament, apareixen dificultats afegides. S. Nakane i D. Schleicher [NS03] estudien la parametrització de components hiperbòliques amb cicles de període parell de la família d'antipolinomis $p_{c,d}(z)=\overline{z}^d+c$. Nosaltres ens centrem en la parametrització de components hiperbòliques amb paràmetres de tipus disjunt fent servir eines diferents de les emprades a [NS03]. Noteu que, degut a la simetria dels B_a, els cicles disjunts són simètrics respecte de $\mathbb{S}^1$ i, per tant, tenen el mateix període i multiplicadors conjugats (vegeu Teorema 5.2.2). Conseqüentment, donada una component hiperbòlica Ω amb paràmetres disjunts, té sentit definir l'aplicació multiplicador $\Lambda:\Omega\to\mathbb{D}$ que envia cada $a\in\Omega$ al multiplicador del cicle atractor tal que la seva conca immediata d'atracció conté c_+.

Teorema 5.4.2. *Sigui Ω una component hiperbòlica disjunta, $\Omega\subset\{a\in\mathbb{C};|a|>2\}$. Aleshores l'aplicació multiplicador és un homeomorfisme entre Ω i el disc unitat.*

Al teorema previ fem servir l'aplicació multiplicador per a veure que tota components hiperbòlica disjunta de paràmetres tals que $|a|>2$ és homeomorfa al disc unitat. Donat que el multiplicador de qualsevol cicle bitransitiu és un nombre real no negatiu (vegeu Proposició 5.4.3), el resultat previ no s'aplica a components bitransitives. Aquest fenomen ja va ésser detectat a [NS03] per als polinomis $p_{c,d}^2$.

Donada una família h_a, $a\in\Delta$, d'homeomorfismes del cercle que preserven l'orientació, podem assignar un número de rotació a cadascun dels seus membres que descriu la mitjana de rotació asimptòtica dels punts del cercle. Les llengües (racionals) dels h_a es defineixen com els conjunts de paràmetres $a\in\Delta$ tals que h_a té nombre de rotació $p/q\in\mathbb{Q}$. En aquest cas, h_a té un cicle atractor o parabòlic de període q a $\mathbb{S}^1$. El concepte de llengua va ser introduït per V. Arnold [Arn61] per la família estàndard de pertorbacions de la rotació rígida

$$\theta \to \theta + \alpha + (\beta/2\pi)\sin(2\pi\theta) \pmod 1$$

on $0 \leq \theta < 1$, $0 \leq \alpha \leq 1$ i $0 \leq \beta \leq 1$. Les llengües han estat estudiades per diversos autors, tant per la família estàndard (vegeu e.g. [Boy86], [WBJ91], [EKT95] o [dlLL11]) com per altres famílies d'homeomorfismes del cercle com ara els productes de Blaschke de grau 3 $B_{r,\alpha}$ introduïts per Herman [Her79]. Si les funcions h_a no són homeomorfismes sinó que són cobriments de grau 2 de $\mathbb{S}^1$, no els hi podem assignar un número. Tanmateix, les llengües encara poden ser definides com els conjunts de paràmetres per als que h_a té un cicle atractor a $\mathbb{S}^1$. Podem associar un tipus $\tau(a)$ a cada h_a, on $\tau(a)$ és un punt periodic del doubling map que descriu com el cicle atractor de h_a rota asimptòticament. En aquest escenari, la llengua T_τ és definida com els conjunts de paràmetres $a \in \Delta$ de tipus τ. Aquestes llengües van ser estudiades per M. Misiurewicz i A. Rodrigues [MR07, MR08] per la família estàndard doble de pertorbacions del doubling map

$$\theta \to 2\theta + \alpha + (\beta/\pi)\sin(2\pi\theta) \pmod 1$$

on $0 \leq \theta < 1$, $0 \leq \alpha \leq 1$ i $0 \leq \beta \leq 1$. Més endavant, A. Dezotti [Dez10] va usar l'extensió complexa de la família estàndard doble al pla complex punxat, donada per $z \to e^{i\alpha} z^2 e^{\beta/2(z-1/z)}$, per tal de provar la connectivitat de les llengües. Aquesta família també va ser estudiada per R. de la Llave, M. Shub i C. Simó [dlLSS08]. Més específicament, fixat un nombre natural $k \geq 2$, ells van estudiar l'entropia de la família estàndard k-èsima $\theta \to k\theta + \alpha + \epsilon \sin(2\pi\theta) \pmod 1$ per ϵ petit.

Donat que els productes de Blaschke B_a són pertorbacions racionals del doubling map, $B_a|_{\mathbb{S}^1}$ pot ser considerat l'anàleg racional de la família estàndard doble. Tot i que no hi ha una expressió simple de la restricció de B_a a $\mathbb{S}^1$, la seva dinàmica global és més senzilla que en el cas transcendent. Si $|a| \geq 2$, els $B_a|_{\mathbb{S}^1}$ són cobriments de grau 2 del cercle unitat i les llengües estan ben definides. Inspirats pels mencionats treballs de Misiurewicz, Rodriguez i Dezotti, veiem que aquestes llengües són connexes i simplement connexes. Més precisament, provem el següent teorema.

Teorema 6.2.1. *Donat un punt periòdic τ del doubling map, tenim que:*

(a) La llengua T_τ és no buida i consisteix en tres components connexes (només una si considerem les simetries a l'espai de paràmetres donades per les arrels terceres de la unitat).

(b) Cada component connexa de T_τ conté un únic paràmetre r_τ, anomenat l'arrel de la llengua*, tal que B_{r_τ} té un cicle superatractor a $\mathbb{S}^1$. L'arrel r_τ satisfà $|r_\tau| = 2$.*

(c) Cada component connexa de T_τ és simplement connexa.

(d) La frontera de cada component connexa de T_τ consisteix en dues corbes que són funcions continues respecte de $|a|$ i s'intersequen en un únic paràmetre a_τ anomenat la punta de la llengua.

La fontera d'una llengua T_τ de període p és doncs la unió de dues corbes de paràmetres que s'intersequen a la punta a_τ de la llengua per als quals $B_a|_{\mathbb{S}^1}$ té un cicle parabòlic de període p i multiplicador 1. Al llarg de $\partial T_\tau \setminus a_\tau$ hi ha una bifurcació sella-node persistent, dos cicles de període p col·lisionen a $\mathbb{S}^1$ i en surten. Cal remarcar que aquest tipus de bifurcacions no es poden donar per famílies holomorfes uniparamètriques que depenen holomorfament del paràmetre. La bifurcació sella-node a la recta real va ser estudiada per M. Misiurewicz i R. A. Pérez [MP08] des d'un punt de vista complex. Ells van caracteritzar, depenent del signe de la derivada Schwarziana, si el cicles de periode p que surten del cercle unitat (o la recta real) són atractors o repulsors. Crowe et al [CHRSC89] van mostrar que aquest tipus de bifurcacions també es donen al Tricorni, l'espai de connexitat dels antipolinomis $p_c(z) = \overline{z}^2 + c$. El seu resultat va ser generalitzat més endavant per J. H. Hubbard i D. Schleicher [HS12]. Ells van estudiar aquestes bifurcacions als Multicornis, l'espai de bifurcacions dels antipolinomis $p_{d,a} = \overline{z}^d + a$, tot fent servir l'índex holomorf dels punts fixos. Usant tècniques similars, nosaltres provem el següent teorema.

Teorema 6.3.2. *Sigui a_τ la punta d'una llengua T_τ de període p. Aleshores, hi ha un entorn obert U de a_τ tal que si $a \in U$ aleshores o bé $a \in T_\tau$, o bé $a \in \partial T_\tau$ o bé a pertany a una component hiperbòlica disjunta.*

Si a pertany a l'anell obert $\mathbb{A}_{1,2}$ de radi interior 1 i radi exterior 2, aleshores els productes de Blaschke $B_a|_{\mathbb{S}^1}$ no són cobriments de grau 2 de cercle unitat. Tot i això, el cicle atractor associat a una certa llengua pot ser continuat dins de $\mathbb{A}_{1,2}$. Aquest fet dona lloc a les anomenades *llengües esteses* ET_τ, conjunts oberts de paràmetres a, $|a| > 1$, per als que $B_a|_{\mathbb{S}^1}$ té un cicle atractor que pot ser continuat real analíticament fins al cicle atractor d'una llengua T_τ. Tanmateix, les llengües esteses no són disjuntes. De fet, si $a \in \mathbb{A}_{1,2}$, aleshores els dos punts crítics $c_\pm$ de B_a rauen al cercle unitat i les seves òrbites no estan relacionades per simetria. Conseqüentment, aquestes es poden acumular en cicles atractors diferents tot permetent a un paràmetre pertànyer a dues llengües esteses simultàniament. Nosaltres ens centrem en l'estudi de la llengua estesa fixa ET_0 i provem el següent resultat.

Teorema 6.4.3. *Donades dues components connexes de la llengua estesa fixa T_0, la intersecció de les seves extensions a $\mathbb{A}_{1,2}$ és buida. La frontera de cada component connexa de la llengua estesa ET_0 consisteix en dues components connexes. La component exterior consisteix en paràmetres per als quals hi ha un punt parabòlic fix de multiplicador 1. La component interior consisteix en paràmetres per als quals hi ha un punt parabòlic fix de multiplicador -1. A més, hi ha una bifurcació de doblament de període que té lloc al llarg de la corba de paràmetres interiors.*

La tesi s'estructura com segueix. Al capítol 1 fem un repàs dels resultats preliminars usats al llarg del text. Primer expliquem els conceptes bàsics de la dinàmica de les funcions racionals. Després fem un repàs de les aplicacions del cercle, tot introduint els conceptes de producte de Blaschke i llengües. Finalment, presentem la fórmula de Riemann-Hurwitz i com s'aplica a la dinàmica de funcions racionals.

Al capítol 2 donem una introducció a la cirurgia quasiconforme. Primer de tot definim els conceptes d'aplicació quasiconforme, estructures quasiconformes i pullback sota funcions que preserven l'orientació i introduïm el Teorema Mesurable de Riemann. Tot seguit mostrem com els conceptes previs són generalitzats per a funcions que giren l'orientació i veiem com això s'aplica a aplicacions que són simètriques respecte del cercle unitat. Finalment introduïm els conceptes d'aplicació polynomial-like i antipolynomial-like.

Al capítol 3 donem una visió general del pla dinàmic dels productes de Blaschke B_a. Comencem estudiant les seves propietats bàsiques. Tot seguit mostrem que les funcions B_a no poden tenir anells de Herman (Proposició 3.2.3) i provem un criteri de connectivitat del conjunt de Julià $\mathcal{J}(B_a)$ (Teorema 3.2.1).

Al capítol 4 introduïm la família M_b de polinomis cúbics amb un punt fix superatractor. A continuació veiem com construir polinomis M_b a partir de productes de Blaschke B_a sempre que $B_a|_{\mathbb{S}^1}$ sigui quasisimètricament conjugat al doubling map, tot obtenint una aplicació Γ que envia un subconjunt de l'espai de paràmetres de B_a a l'espai de paràmetres dels polinomis M_b. També provem que l'aplicació Γ és continua i és un homeomorfisme restringida a cada component hiperbòlica disjunta.

Al capítol 5 estudiem l'espai de paràmetres dels productes de Blaschke B_a. Primer de tot en descrivim les simetries. A continuació classifiquem els diferents tipus de comportaments hiperbòlics que es poden donar i veiem a quines regions de l'espai de paràmetres poden aparèixer. Tot seguit construïm una aplicació polynomial-like al voltant de tot paràmetre de no escapament contingut en una regió d'intercanvi que, sota certes condicions, pot relacionar la dinàmica de B_a amb la dels antipolinomis $p_c(z) = \overline{z}^2 + c$ (Teorema 5.3.4). Finalment parametritzem tota component hiperbòlica disjunta els cicles atractors de la qual estan continguts a $\mathbb{C}^* \setminus \mathbb{S}^1$ (Teorema 5.4.2).

Al capítol 6 estudiem les llengües dels productes de Blaschke B_a. Inicialment provem algunes de les seves propietats topològiques bàsiques com ara la seva connectivitat mòdul simetria, la seva connectivitat simple i l'existència d'una única punta per a cada llengua (Teorema 6.2.1). Tot seguit mostrem com es produeixen les bifurcacions en un entorn de la punta de cada llengua (Teorema 6.3.2). Finalment estudiem com les llengües s'estenen per a paràmetres a tals que $1 < |a| < 2$.

Al capítol 7 estudiem els productes de Baschke $B_{a;m}$ i com es generalitzen els resultats provats al llarg de la tesi.

Introduction

This PhD thesis belongs to the area of discrete dynamical systems in the complex plane, i.e., the iteration of analytic functions in one complex variable. These systems appear naturally in the study of analytic dynamics on the real line or the interval for which a complex point of view has proven to be quite useful. Indeed, there is a large list of important results obtained thanks to the complexification of the logistic map and other unimodal and bimodal functions coming from, for instance, discrete biological and economical models or root finding numerical methods among others. Complex tools allow us to notice phenomena which are not visible from the real line. The chaotic behaviors which can be observed for these models (bifurcation cascades, etc.) acquire a new dimension when seen in the complex plane, where they can be much better understood.

This area of mathematics was born at the beginning of the 20th century as a consequence of the investigation of Netwon's method, the well known root finding algorithm, on the complex plane. Until then, only local studies existed but P. Fatou and G. Julia faced the problem from a more global point of view. They classified the possible stable orbit behaviors, in the sense of normality. The boundary between stable regions, nowadays known as the Julia set, is an invariant object of great beauty and complexity, which Fatou and Julia described quite accurately, remarkably without the aid of computers.

During these very beginnings, the basis of the theory was established, getting as far as possible with the available tools. Afterwards came a few years of relatively low activity until the rebirth of the subject during the 80's, due to two different factors. On the one hand, D. Sullivan [Sul85] proved one of the main conjectures left by Fatou and Julia, the non existence of wandering domains, by means of the use of quasiconformal tools. These tools, which came from the area of geometric analysis, have been the key for many later results and they are heavily used in this thesis. On the other hand, the advent of the first computers allowed B. Mandelbrot to draw the first image of what is nowadays known as the Mandelbrot set, the connectedness locus in the parameter plane of the quadratic family z^2+c. The visualization of the first fractal sets, together with the new tools available to face the open questions that Fatou and Julia had left unsolved, awoke the interests of many mathematicians. As a consequence, many new results were proven, some of which provided their authors with the highest prizes in mathematics (e.g., J. C. Yoccoz, C. McMullen, Fields medalists in 1994 and 1998 respectively).

Given a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the compactified plane or the Riemann sphere, we consider the dynamical system given by the iterates of f. The Riemann sphere splits into two totally f-invariant subsets: the *Fatou set* $\mathcal{F}(f)$, which is defined to be the set of points $z \in \widehat{\mathbb{C}}$ where the family $\{f^n, n \in \mathbb{N}\}$ is normal in some neighborhood of z, and its complement, the *Julia set* $\mathcal{J}(f)$. The dynamics of the points in $\mathcal{F}(f)$ are stable in the sense of normality or equicontinuity whereas the dynamics in $\mathcal{J}(f)$ present chaotic behavior. The Fatou set $\mathcal{F}(f)$ is open and therefore $\mathcal{J}(f)$ is closed. Moreover, if the degree of the rational map f is greater or equal than 2, then the Julia set $\mathcal{J}(f)$ is not empty and is the closure of the set of repelling periodic points of f.

The connected components of $\mathcal{F}(f)$, called *Fatou components*, are mapped under f among themselves. Sullivan [Sul85] proved that any Fatou component of a rational map is either periodic or preperiodic. By means of the Classification Theorem (see Theorem 1.1.18), any periodic Fatou component of a rational map is either the basin of attraction of an attracting or parabolic cycle, or a simply connected rotation domain (a Siegel disk), or a doubly connected rotation domain (a Herman Ring). Moreover, any such component is somehow related to a *critical point*, i.e., a point $z \in \widehat{\mathbb{C}}$ such that $f'(z) = 0$. Indeed, the basin of attraction of an attracting or parabolic cycle contains, at least, a critical point whereas Siegel disks and Herman rings have critical orbits accumulating on their boundaries. See Chapter 1 for an introduction to the dynamics of rational functions.

Every holomorphic self-map of $\mathbb{D}$ of finite degree is a finite product of automorphisms of the disk, i.e., a finite Blaschke product, and therefore defined (by reflection) on the whole Riemann sphere. Blaschke products have been used extensively as model maps in complex dynamics. For instance, the products $B_{r,\alpha}(z) = e^{2\pi i\alpha} z^2 (z-r)/(1-rz)$, where $\alpha, r \in \mathbb{R}$ and $r > 3$, were used by M. R. Herman [Her79] to prove the existence of Herman rings (see Figure 1.4 (d)) and, with complex parameters, by X. Buff et al [BFGH05] to study quasiconformal deformations of such objects. They were also used, taking $r = 3$, to prove the celebrated result that the boundaries of Siegel disks of bounded type of quadratic polynomials are Jordan curves (see the works of E. Ghys [Ghy84], M. R. Herman [Her86, Her87], A. Douady [Dou87] and G. Świątek [Świ88]). This result was later generalized by C. Petersen and S. Zakeri [PZ04].

The Blaschke family

The aim of this thesis is to study the dynamics of the degree 4 Blaschke products given by

$$B_a(z) = z^3 \frac{z-a}{1-\bar{a}z}, \tag{1}$$

where $a, z \in \mathbb{C}$. For all values of $a \in \mathbb{C}$, the points $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3 (c.f. [CFG15]). We denote by $A(0)$ and $A(\infty)$ their basins of attraction and by $A^*(0)$ and $A^*(\infty)$ their immediate basins of attraction, i.e., the connected components of $A(0)$ and $A(\infty)$ which contain $z = 0$ and $z = \infty$, respectively. If $|a| \leq 1$ these are the only Fatou components, separated by the Julia set which is necessarily $\mathbb{S}^1$. But for every other parameter, there are two free critical points $c_\pm$ (distinct unless $|a| = 2$) which may lead to the existence of stable components different from $A(0)$ and $A(\infty)$. If $|a| \geq 2$, however, the Blaschke family is essentially unicritical due to the symmetry with respect to $\mathbb{S}^1$ which, in this case, ties the two critical orbits together in a certain sense.

We notice that the Blaschke products B_a are rational perturbations of the doubling map of the circle $R_2(z) = z^2$ (equivalently given by $\theta \to 2\theta \pmod 1$). Indeed, the B_a converge uniformly over compact subsets of the punctured plane $\mathbb{C}^*$ to $e^{4\pi i \mathrm{Arg}(a)} z^2$ as a tends to ∞.

The B_a are a particular case of a more general degree $m+2$ Blaschke family which, as we shall see in Chapter 7, shares most of their properties. It is given by

$$B_{a;m}(z) = z^{m+1}\frac{z-a}{1-\bar{a}z},$$

where $a, z \in \mathbb{C}$ and $m \geq 1$ is a natural number. They are almost bicritical rational maps in the sense that the $B_{a;m}$ have $z = 0$ and $z = \infty$ as superattracting fixed points of local degree $m+1$ and have exactly two free critical points which control the existence of any periodic Fatou component other than the attracting basins of $z = 0$ and $z = \infty$. For fixed $m \geq 1$, $B_{a;m}$ is a family of rational perturbations of the mth map of the circle $R_m(z) = z^m$ (equivalently given by $\theta \to m\theta \pmod 1$). If $|a| \geq (m+2)/m$, $B_{a;m}|_{\mathbb{S}^1}$ is a degree m covering map of the circle (see Section 7.1). In particular, if $|a| \geq 3$, then $B_{a;1}|_{\mathbb{S}^1}$ is a homeomorphism of the unit circle and the dynamics of the $B_{a;1}$ is well understood. Indeed, the family $B_{a;1}$ is a reparametrization of the $B_{r,\alpha}$ used by Herman [Her79] to prove the existence of Herman rings (see Lemma 7.1.1). Therefore, we focus on the study of the dynamics of $B_{a;m}$ for $m \geq 2$, which are determined by the position of the free critical points and its pole with respect to the unit circle (see Section 7.1). However, the possible configurations are independent of m and, therefore, all different dynamics appearing for $B_{a;m}$, $m > 2$, already appear within the family $B_a = B_{a;2}$. This is the reason why we restrict to the study of the family B_a and later on we explain how the results generalize for the Blaschke products $B_{a;m}$ with $m \geq 2$.

The connectivity of the Julia set is a topological property often very related to the dynamics of the map (see e.g. [Shi87], [Prz89], [Pil96] and [DR13]). It is equivalent to the simple connectivity of every Fatou component. Given a polynomial P, its Julia set $\mathcal{J}(P)$ is connected if and only if it has no free critical point captured by the basin of attraction of infinity (c.f. [Mil06]). However, such a classification does not exist for general rational maps. Unlike polynomials, they may have Herman Rings which obviously disconnect the Julia set, which may even consist of a Cantor set of Jordan curves (see [McM88]). However, the family B_a shares some of the features of polynomials in this respect such as the non existence of Herman rings (see Proposition 3.2.3). We also prove the following criterion of connectivity of the Julia set which, for parameters a such that $|a| \geq 2$, is similar to the one of polynomials. It is based on the position of the critical point $c_+ \in \mathbb{C} \setminus \mathbb{D}$ with respect to the immediate basin of attraction of infinity $A^*(\infty)$.

Theorem 3.2.1. *Given a Blaschke product B_a as in* (1)*, the following statements hold:*

(a) If $|a| \leq 1$, then $\mathcal{J}(B_a) = \mathbb{S}^1$.

(b) If $|a| > 1$, then the connected components of $A(\infty)$ and $A(0)$ are simply connected if and only if $c_+ \notin A^*(\infty)$.

(c) If $|a| \geq 2$, then every Fatou component U such that $U \cap A(\infty) = \emptyset$ and $U \cap A(0) = \emptyset$ is simply connected.

Consequently, if $|a| \geq 2$, then $\mathcal{J}(B_a)$ is connected if and only if $c_+ \notin A^*(\infty)$.

The Blaschke products B_a with $|a| > 2$ which have no attracting or parabolic cycle in $\mathbb{S}^1$ can be related to cubic polynomials. For such parameters, $B_a|_{\mathbb{S}^1}$ is quasisymmetrycally conjugate to the doubling map and a quasiconformal surgery can be performed obtaining cubic polynomials of the form $M_b(z) = bz^2(z-1)$ with $b \in \mathbb{C}$ (see Chapter 4, c.f. [Pet07]). This surgery establishes a conformal conjugacy between M_b and B_a on the set of points which never enter $\mathbb{D}$ under iteration of B_a and the points which are not attracted to $z = 0$ under iteration of M_b. In particular, if B_a has an attracting or parabolic cycle contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$, this surgery conjugates B_a with M_b conformally in its basin of attraction. These cubic polynomials with a superattracting fixed point have been the object of research of several papers. For instance, J. Milnor introduced, in a preliminary version of [Mil09], the study of cubic polynomials with period p superattracting cycles. Later on P. Roesch [Roe07] investigated the slice $\mathcal{S}_1$ of cubic polynomials with a superatracting fixed point proving some of the conjectures raised by Milnor. They were also used by Tan L. [Tan97] in the study of the parameter plane of Newtons maps N_P coming from cubic polynomials P, by means of the so called matings.

If B_a has a periodic cycle with points both inside and outside $\mathbb{D}$ the situation is different. Although the previous surgery construction is still possible, a lot of information is lost since, under the new map, the critical point is always captured by the basin of $z = 0$. Parameters for which the orbit of $c_+ \in \mathbb{C} \setminus \overline{\mathbb{D}}$ enters the unit disk at least once are called *swapping parameters* and connected components of the set of swapping parameters are called *swapping regions*. Inside these regions, the non holomorphic dependence of B_a on the parameter a gives rise to what appear to be small "copies" of the Tricorn, the connectedness locus of the antipolynomials $p_c(z) = \overline{z}^2 + c$ (see [CHRSC89] and Figure 5.4 (a)). Milnor [Mil92] showed that a similar situation takes place for real cubic polynomials introducing the concept of antipolynomial-like mapping. We distinguish between two types of attracting cycles for swapping parameters. We say that a parameter is *bitransitive* if it has a cycle whose basin of attraction contains the two free critical points. We say that a parameter is *disjoint* if there are two different attracting cycles other than zero or infinity. The very special dynamics taking place for swapping parameters allows us to build a polynomial-like mapping of degree 2 or 4 in a neighborhood of every bitransitive or disjoint swapping parameter. If the degree of the polynomial-like map is 4 we use the antipolynomial-like mappings introduced by Milnor to prove that it is hybrid equivalent to a degree 4 polynomial of the form p_c^2. Hybrid equivalence is a type of conjugacy stronger than topological (and even quasiconformal) conjugacy. More precisely, we prove the following theorem.

Theorem 5.3.4. *Let a_0 be a swapping parameter with an attracting or parabolic cycle of period $p > 1$. Then, there is an open set W containing a_0 and $p_0 > 1$ dividing p such that, for every $a \in W$, there exist two open sets U and V with $c_+ \in U$ such that $(B_a^{p_0}; U, V)$ is a polynomial-like map. Moreover,*

(a) If a_0 is bitransitive, $(B_a^{p_0}; U, V)$ is hybrid equivalent to a polynomial of the form $p_c^2(z) = (z^2 + \overline{c})^2 + c$.

(b) If a_0 is disjoint, $(B_a^{p_0}; U, V)$ is hybrid equivalent to a polynomial of the form $p_c^2(z) = (z^2 + \overline{c})^2 + c$ or of the form $z^2 + c$.

It is known that the boundary of every bounded Fatou component of a polynomial, with the exception of Siegel disks, is a Jordan curve [RY08]. However, this is not true for rational functions without additional conditions (e.g., postcritically finite [Pil96]).

In our case, as a consequence of the two previous constructions, we know that the boundary of every connected component of the basin of attraction of an attracting or parabolic cycle of B_a not contained in $\mathbb{S}^1$ and other than $z=0$ and $z=\infty$ is a Jordan curve (see Proposition 5.4.1). Indeed, if B_a has such a cycle, we obtain a conjugacy between B_a and a polynomial which sends the immediate basin of attraction of the cycle of B_a to the immediate basin of attraction of a bounded cycle of the polynomial. In this sense we can say that, if $|a|>2$ and $B_a|_{\mathbb{S}^1}$ has no non-repelling cycle, the dynamics of the B_a are somehow polynomial.

Parameter plane of the Blaschke family

After investigating the dynamical plane of the Blaschke products B_a we study its parameter plane (see cover and Figure 5.1). A rational map is *hyperbolic* if all its critical orbits accumulate on attracting cycles. A *hyperbolic component* is a connected component of the open set $\mathcal{H}=\{a \mid B_a \text{ is hyperbolic}\}$. The parametrization of hyperbolic components of rational functions which depend holomorphically on their parameters is well known (see [DH85a], c.f. [BF14]). If the family of functions does not depend holomorphically on parameters, some extra difficulties appear. S. Nakane and D. Schleicher [NS03] studied the parametrization of hyperbolic components with cycles of even period for the family of antipolynomials $p_{c,d}(z)=\overline{z}^d+c$. We focus on the parametrization of hyperbolic components with disjoint parameters using different methods than the ones of [NS03]. Notice that, due to the symmetry of B_a, disjoint cycles are symmetric with respect to $\mathbb{S}^1$ and therefore have the same period and conjugate multiplier (see Theorem 5.2.2). Hence, given a hyperbolic component Ω with disjoint parameters, it makes sense to define the multiplier map $\Lambda:\Omega\to\mathbb{D}$ as the map which sends every $a\in\Omega$ to the multiplier of the attracting cycle whose basin captures the critical orbit of c_+.

Theorem 5.4.2. *Let Ω be a disjoint hyperbolic component, $\Omega\subset\{a\in\mathbb{C}; |a|>2\}$. Then, the multiplier map is a homeomorphism between Ω and the unit disk.*

In the previous theorem we use the multiplier map in order to prove that every hyperbolic component of disjoint parameters with $|a|>2$ is homeomorphic to the unit disk. Since the multiplier of any bitransitive cycle is a non-negative real number (see Proposition 5.4.3), the previous result does not hold for bitransitive components. This phenomena had already been noticed in [NS03] for the polynomials $p_{c,d}^2$.

Given a family of h_a, $a\in\Delta$, of orientation preserving homeomorphisms of the unit circle, one can assign a rotation number to each of its members which gives the average asymptotic rate of rotation of points in the circle. The (rational) tongues of the h_a are defined as the sets of parameters $a\in\Delta$ such that h_a has rotation number $p/q\in\mathbb{Q}$. In this case, h_a has a period q attracting or parabolic cycle in $\mathbb{S}^1$. The concept of tongues was introduced by V. Arnold [Arn61] for the standard family of perturbations of the rigid rotation

$$\theta\to\theta+\alpha+(\beta/2\pi)\sin(2\pi\theta) \pmod 1$$

where $0\le\theta<1$, $0\le\alpha\le1$ and $0\le\beta\le1$. Tongues have been studied by many authors, both for the Arnold standard maps (see e.g. [Boy86], [WBJ91], [EKT95] or [dlLL11]) and for other families of homeomorphisms of the unit circle such as the degree 3 Blaschke products $B_{r,\alpha}$ introduced by Herman [Her79]. If the maps h_a are not homeomorphisms but degree 2 covers of $\mathbb{S}^1$, we cannot assign a number to them. However, tongues can still be defined as sets of parameters for which h_a has an attracting cycle in $\mathbb{S}^1$. We can associate a type $\tau(a)$ to every such h_a, where $\tau(a)$ is a periodic point of the doubling map and describes how the attracting cycle of h_a asymptotically rotates. In this setting, a tongue T_τ is defined as the open set of parameters $a \in \Delta$ of type τ. They were studied by M. Misiurewicz and A. Rodrigues [MR07, MR08] for the double standard family of perturbations of the doubling map

$$\theta \to 2\theta + \alpha + (\beta/\pi)\sin(2\pi\theta) \pmod 1$$

where $0 \leq \theta < 1$, $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Later on, A. Dezotti [Dez10] used the complex extension of the double standard maps on the punctured plane, given by $z \to e^{i\alpha}z^2 e^{\beta/2(z-1/z)}$, in order to prove the connectivity of the tongues. This family was also studied by R. de la Llave, M. Shub and C. Simó [dlLSS08]. More specifically, for a fixed $k \geq 2$, they studied the entropy for the k-th standard maps $\theta \to k\theta + \alpha + \epsilon\sin(2\pi\theta) \pmod 1$ for ϵ small.

Given that the Blaschke products B_a are rational perturbations of the doubling map, $B_a|_{\mathbb{S}^1}$ may be considered as the rational analogue of the double standard family. Although there is no explicit simple expression for the restriction of B_a to $\mathbb{S}^1$, the global dynamics are simpler than in the transcendental case. If $|a| \geq 2$, the $B_a|_{\mathbb{S}^1}$ are degree 2 coverings of the unit circle and the tongues are well defined. Inspired by the mentioned works of Misiurewicz, Rodriguez and Dezotti, we show that they are connected and simply connected. More precisely, we prove the following theorem.

Theorem 6.2.1. *Given any periodic point τ of the doubling map the following results hold.*

(a) The tongue T_τ is not empty and consists of three connected components (only one connected component if we consider the parameter plane modulo the symmetries given by the third roots of the unity).

(b) Each connected component of T_τ contains a unique parameter r_τ, called the root of the tongue*, such that B_{r_τ} has a superattracting cycle in $\mathbb{S}^1$. The root r_τ satisfies $|r_\tau| = 2$.*

(c) Every connected component of T_τ is simply connected.

(d) The boundary of every connected component of T_τ consists of two curves which are continuous graphs as function of $|a|$ and intersect each other in a unique parameter a_τ called the tip of the tongue.

The boundary of a tongue T_τ of period p is the union of two curves of parameters which intersect at the tip a_τ of the tongue and have a persistent parabolic cycle of period p and multiplier 1. Along $\partial T_\tau \setminus a_\tau$ there is a persistent saddle-node bifurcation taking place, two period p cycles collide in $\mathbb{S}^1$ and exit it. Notice that these bifurcation along curves cannot happen for a uniparametric family of holomorphic maps which depend holomorphically in the parameter. The real saddle-node bifurcation was studied by M. Misiurewicz and R. A. Pérez [MP08] from a complex point of view. They characterized, depending on the sign of the Schwarzian derivative, whether the period p cycles exiting the unit circle (or the real line) are attracting or repelling. Crowe et al [CHRSC89] showed that this type of bifurcations also occurs in the Tricorn, the connectedness locus of the antipolynomials $p_c(z) = \overline{z}^2 + c$. Their result was later generalized by J. H. Hubbard and D. Schleicher [HS12]. They studied these bifurcations in the Multicorns, the bifurcation loci of the antipolynomials $p_{d,a} = \overline{z}^d + a$, by using the holomorphic index of the fixed points. Using similar techniques, we prove the following theorem.

Theorem 6.3.2. *Let a_τ be the tip of a tongue T_τ of period p. Then, there exists a neighborhood U of a_τ such that if $a \in U$ then, either $a \in T_\tau$ or $a \in \partial T_\tau$ or a belongs to a disjoint hyperbolic component.*

If a belongs to the open annulus $\mathbb{A}_{1,2}$ of inner radius 1 and outer radius 2, then the Blaschke products $B_a|_{\mathbb{S}^1}$ are no longer degree two coverings of the unit circle. Despite that, the attracting cycle associated to a given tongue may be continued for parameters within $\mathbb{A}_{1,2}$. This leads to the concept of *extended tongues* ET_τ, open connected sets of parameters a, $|a| > 1$, for which $B_a|_{\mathbb{S}^1}$ has an attracting cycle which can be real analytically continued to the attracting cycle of a tongue T_τ. However, extended tongues are not disjoint. Indeed, if $a \in \mathbb{A}_{1,2}$, then the two critical points $c_\pm$ of B_a lie on the unit circle and their orbits are not related by symmetry. Therefore, they may accumulate on different attracting cycles, allowing a parameter to belong to two different extended tongues simultaneously. We focus on the study of the extended fixed tongue ET_0 and prove the following theorem.

Theorem 6.4.3. *Given two connected components of the fixed tongue T_0, the intersection of their extensions in $\mathbb{A}_{1,2}$ is empty. The boundary of every connected component of the extended fixed tongue ET_0 consists of two disjoint connected components. The exterior component consists of parameters for which there is a parabolic fixed point of multiplier* 1. *The interior component consists of parameters for which there is a parabolic fixed point of multiplier* -1. *Moreover, there is a period doubling bifurcation taking place throughout the curve of interior boundary parameters.*

The thesis is structured as follows. In Chapter 1 we give an overview on the preliminary results used throughout the thesis. First we explain the basics on the dynamics of rational functions. Afterwards we give an overview on circle mappings, introduce the concept of Blaschke product and define the concept of tongue. Finally we present the Riemann-Hurwitz formula and how it applies to the dynamics of rational functions.

In Chapter 2 we give an introduction to quasiconformal surgery. First we define the concepts of quasiconformal mappings, almost complex structures and pullback by orientation preserving maps and state the Measurable Riemann Mapping Theorem. Afterwards we show how to generalize the previous results to orientation reversing maps and see how this applies to functions which are symmetric with respect the unit circle. Finally we introduce the concept of polynomial and antipolynomial-like mappings.

In Chapter 3 we give an overview of the dynamical plane of the Blaschke products B_a. We begin by studying their basic properties. Afterwards we show that the maps B_a cannot have Herman rings (Proposition 3.2.3) and prove a criterion of connectivity of the Julia set $\mathcal{J}(B_a)$ (Theorem 3.2.1).

In Chapter 4 we introduce the family M_b of cubic polynomials with a superattracting fixed point. Then we show how to build polynomials M_b from Blaschke products B_a provided that $B_a|_{\mathbb{S}^1}$ is quasisymmetrically conjugate to the doubling map, obtaining a map Γ from a subset of the parameter plane of the B_a to the parameter plane of the polynomials M_b. We also prove that the map Γ is continuous and restricts to a homeomorphism on every disjoint hyperbolic component.

In Chapter 5 we study the parameter plane of the Blaschke products B_a. We first describe the symmetries in the parameter plane. Then we classify the different hyperbolic dynamics which may take place and the sets of parameters for which they may happen. Afterwards we build a polynomial-like map for all non-escaping parameters contained in swapping regions which, under certain conditions, may relate the dynamics of B_a with the one of the antipolynomials $p_c(z) = \overline{z}^2 + c$ (Theorem 5.3.4). Finally we parametrize all disjoint hyperbolic components whose disjoint cycles are contained in $\mathbb{C}^* \setminus \mathbb{S}^1$ (Theorem 5.4.2).

In Chapter 6 we study the tongues of the Blaschke products B_a. We first prove some of their topological properties such as their connectivity modulo symmetry, their simple connectivity and the existence of a unique tip for every tongue (Theorem 6.2.1). Then we show how bifurcations take place along curves in a neighborhood of every tongue (Theorem 6.3.2). Finally we study how tongues extend in the annulus of parameters a such that $1 < |a| < 2$.

In Chapter 7 we study the Blaschke products $B_{a;m}$ and how the results proved throughout the thesis are generalized for them.

Chapter One 1

Preliminaries 1

The dynamics of rational functions has been a subject of interest for over a century.The basis of the theory are now well established and we make intensive use of the main results all over this thesis.

The aim of this chapter is to provide an overview of some notions and results which are used in the following chapters. In Section 1.1 we introduce the basics of the dynamics of rational functions in one complex variable. In Section 1.2 we give an overview of circle mappings as well as some generalities about Blaschke products. Finally, in Section 1.3 we state the Riemann-Hurwitz formula, an important tool in our work.

1.1 Dynamics of rational functions

A rational map is a function of the form $Q(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are non zero complex polynomials with no common roots. We also require that at least one of them is not constant. Rational maps can be extended to holomorphic maps of the Riemann Sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in the natural way. Vice versa, any holomorphic map of $\widehat{\mathbb{C}}$ is a rational map.

The degree of a rational map is defined as the maximum of the degrees of the polynomials p and q. It is important to point out that the degree, say d, of a rational map Q coincides with its topological degree in the sense that every point $w \in \widehat{\mathbb{C}}$ has exactly d preimages under Q counted with multiplicity. Therefore, a rational map is a degree d cover of the Riemann sphere onto itself ramified over a number of branch points which can be shown to be $2d - 2$ (see Corollary 1.3.2 below). These are called critical points and are precisely the points were the derivative of the rational map vanishes, i.e., $Q'(z) = 0$. Its multiplicity is given by their multiplicity as zeros of Q'.

In this section we give an introduction to holomorphic dynamics in one complex variable. We refer to [Bea91], [CG93], [Mil06] and [MNTU00] for general background and proofs of the results.

We now proceed to give a short overview of normal families.

Definition 1.1.1. Let $U \subset \widehat{\mathbb{C}}$ be a domain. A family of holomorphic functions $\mathscr{F}$ from U to $\widehat{\mathbb{C}}$ is *normal* if any sequence of functions $\{f_n\}_{n\in\mathbb{N}} \in \mathscr{F}$ has a subsequence which converges uniformly on compact sets to a limit map.

The concept of normality can be related to the one of equicontinuity by means of Arzelà-Ascoli Theorem. We first recall the definition of local equicontinuity.

Definition 1.1.2. Let $U \subset \widehat{\mathbb{C}}$ be a domain. A family of holomorphic functions $\mathscr{F}$ from U to $\widehat{\mathbb{C}}$ is *locally equicontinuous* if for every $z \in U$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that, for all $f \in \mathscr{F}$, if $z_1, z_2 \in U$ belong to the spherical ball of center z and radius δ, then $d_{\widehat{\mathbb{C}}}(f(z_1), f(z_2)) < \epsilon$, where $d_{\widehat{\mathbb{C}}}(\cdot, \cdot)$ denotes the spherical distance.

Due to compactness of the Riemann sphere, Arzelà-Ascoli Theorem can be stated in the particular case of $\widehat{\mathbb{C}}$ as follows.

Theorem 1.1.3 (Arzelà-Ascoli). *Let $U \subset \widehat{\mathbb{C}}$ be a domain. A family of holomorphic functions $\mathscr{F}$ from U to $\widehat{\mathbb{C}}$ is normal if and only if it is locally equicontinuous.*

We finish this short introduction to normal families stating Montel's Theorem, which gives criteria of normality.

Theorem 1.1.4. *Let $U \subset \widehat{\mathbb{C}}$ be a domain and $\mathscr{F}$ be a family of holomorphic functions from U to $\widehat{\mathbb{C}}$. If there exists three different points w_1, w_2 and w_3 such that $f(U) \subset \widehat{\mathbb{C}} \setminus \{w_1, w_2, w_3\}$ for all $f \in \mathscr{F}$, then the family $\mathscr{F}$ is normal in U.*

Holomorphic maps induce a dynamical partition of the phase space. Given a holomorphic map f in S, where $S = \widehat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{C}^*$, the application of the concept of normality to the family of iterates $\{f^n\}_{n \geq 0}$ leads to a classification of the points in S. The orbits of the points of the connected domains where the family is normal behave in the same dynamical fashion. This fact leads to the concepts of Fatou and Julia sets.

Definition 1.1.5. Given a holomorphic function $f : S \to S$, where $S = \widehat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{C}^*$, its *Fatou set* $\mathcal{F}(f)$ is defined to be the set of points z_0 where the family $\{f^n, n \in \mathbb{N}\}$ is normal in some neighborhood of z_0. Its complement $\mathcal{J}(f) := S \setminus \mathcal{F}(f)$ is called the *Julia set*.

The dynamics within the Fatou set are considered to be stable whilst the ones within the Julia set present chaotic behavior. See Figure 1.1 for a first example of Fatou and Julia sets.

Periodic points play an important role when studying the Fatou and Julia sets of a map f. We denote by $< z_0 >:= \{z_0, z_1, \cdots, z_{p-1}\}$, where $f(z_i) = z_{i+1}$ and subindexes are taken modulus p, a p-cycle of f and refer to z_0 as the *marked point* of the cycle. We define the *multiplier* of the cycle as

$$\lambda(z_0) = (f^p)'(z_0) = f'(z_0) \cdot \ldots \cdot f'(z_{p-1}).$$

Definition 1.1.6. We say that a cycle $< z_0 >$ of a holomorphic map f is

- *attracting* if $|\lambda| < 1$ and *superattracting* if $\lambda = 0$,
- *neutral* (or *indifferent*) if $|\lambda| = 1$,
- repelling if $|\lambda| > 1$.

Moreover, a neutral cycle is

- *parabolic* if $\lambda = e^{2\pi i p/q}$ with $p/q \in \mathbb{Q}$, or
- *irrational* if $\lambda = e^{2\pi i \theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

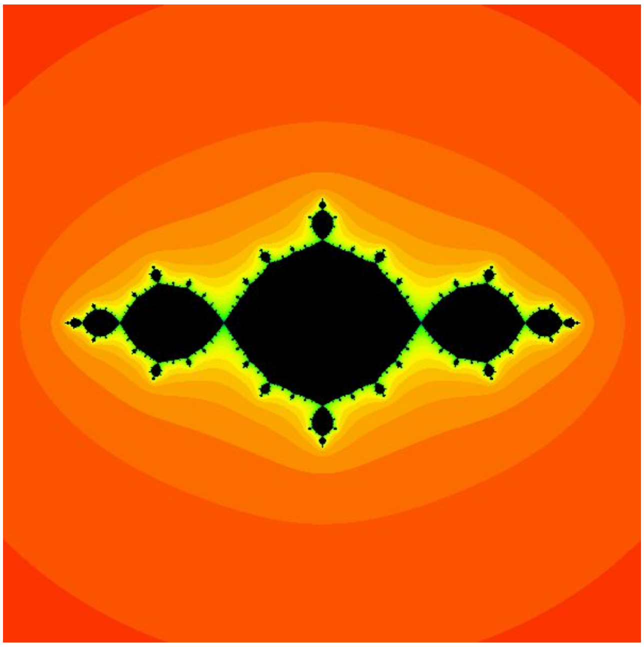


Figure 1.1: The dynamical plane of $p_{-1}(z) = z^2 - 1$. The black components correspond to the basin of attraction of the 2-cycle $\{-1, 0\}$. The scaling from green to orange corresponds to points which escape to infinity. The boundary between these two regions is the Julia set $\mathcal{J}(p_{-1})$.

In Section 1.1.1 we shall discuss the local dynamics around each of the types above. But first we state some properties from the global point of view.

Proposition 1.1.7. *Repelling cycles are topologically repelling, i.e., given a point z_0 of a repelling p-cycle, there is a neighborhood U of z_0 such that for all $z \neq z_0$ in U there is an $n > 0$ such that $f^{np}(z) \notin U$.*

The following concepts of grand orbit and exceptional set will be important when describing the properties of the Fatou and Julia sets.

Definition 1.1.8. Given a map f as before, the *grand orbit* of a point w in $\mathcal{S}$ is defined as

$$\mathrm{GO}(w) = \{z \in S \mid f^k(w) = f^l(z), k, l \in \mathbb{N}\}.$$

If $\mathrm{GO}(w)$ is finite, we say that w belongs to the *exceptional set* $\mathcal{E}(f)$.

The following proposition shows some of the basic properties of the Fatou and Julia sets.

Proposition 1.1.9. *Let $f : S \to S$, where $S = \widehat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{C}^*$, be a holomorphic map. Then the following statements hold.*

(a) The Fatou set $\mathcal{F}(f)$ is open while $\mathcal{J}(f)$ is closed. Both sets are completely invariant, i.e., $\mathcal{F}(f) = \mathcal{F}(f^k)$ for all $k > 0$.

(b) The Julia set $\mathcal{J}(f)$ is not empty. Moreover, either it has empty interior or it coincides with S.

(c) *If f is a rational map and* $\deg(f) \geq 2$*, then* $\mathcal{J}(f)$ *contains a repelling fixed point or a parabolic fixed point with multiplier 1.*

(d) *Every repelling or parabolic cycle of f belongs to* $\mathcal{J}(f)$*.*

(e) *The set of repelling cycles of f are dense in* $\mathcal{J}(f)$*.*

(f) *Given* $w \in S \setminus \mathcal{E}(f)$*, then* $\mathcal{J}(f) \subset \overline{\bigcup_{n\geq 0} f^{-n}(w)}$*.*

(g) *The Julia set* $\mathcal{J}(f)$ *has no isolated points. Moreover, it is either connected or has uncountably many components.*

(h) *If U is an open set containing a point in* $\mathcal{J}(f)$ *then* $\mathcal{S} \setminus \mathcal{E}(f) \subset \bigcup_{n\geq 0} f^n(U)$*. If f is a rational map, by compactness of* $\widehat{\mathbb{C}}$*, there exists $N > 0$ such that, for all $n > N$,* $f^n(U \cap \mathcal{J}(f)) = \mathcal{J}(f)$*.*

Note that the Fatou set of a holomorphic map f may be empty. For instance, there are known examples of rational maps (c.f. [Bea91, Sect. 4.3]) and entire trancentental maps (see [Dev84]) for which $\mathcal{J}(f) = \mathbb{C}$. We also want to remark that there are examples of Julia sets which are not the whole phase space S but have positive Lebesgue measure. Indeed, C. McMullen [McM87] proved that the Julia set of the entire transcendental family $f_{a,b}(z) = ae^z + be^{-z}$ has always positive Lebesgue measure. Later on X. Buff and A. Cheritat [BC12] gave examples of polynomials whose Julia set have positive Lebesgue measure.

1.1.1 Local dynamics

In this subsection we introduce some results about the linearization of holomorphic germs around attracting, superattracting, parabolic or irrational cycles as in Definition 1.1.6. For simplicity we assume that the cycle consists of a single fixed point. Recall that the multiplier of a fixed point z_0 under a holomorphic germ f is given by $\lambda = f'(z_0)$. We refer to [Mil06] for a more general introduction to the problem.

The attracting case: $|\lambda| = 1$, $\lambda \neq 0$

The following result is known as Kœnigs Linearization Theorem (see [Mil06, Thm. 8.2]).

Theorem 1.1.10 (Kœnigs Linearization)**.** *With λ as before, there exists a local holomorphic change of coordinates $w = \phi(z)$, with $\phi(z_0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \to \lambda w$ for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a non-zero constant.*

Let $A(z_0)$ denote the basin of attraction of z_0 and let $A^*(z_0)$ denote the immediate basin of attraction of z_0, i.e., the connected component of $A(z_0)$ containing z_0. Then, the map ϕ can be extended to a unique holomorphic map such that $\phi(f(z)) = \lambda\phi(z)$ for all $z \in A(z_0)$ (see [Mil06, Cor. 8.4]). The map ϕ is conformal in a neighborhood of the attracting fixed point z_0. Hence, we can consider, for ϵ small enough, the inverse map $\varphi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_\epsilon \subset A^*(z_0)$. The following result shows the existence of a maximal domain where the linearization is possible which has at least a critical point of f on its boundary (see [Mil06, Lem. 8.5]).

Lemma 1.1.11. *This local inverse $\varphi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_\epsilon$ extends, by analytic continuation, to some maximal open disk $\mathbb{D}_r$ about the origin in $\mathbb{C}$. This yields a uniquely defined holomorphic map $\varphi : \mathbb{D}_r \to \mathcal{A}_r = \mathcal{A}$ with $\varphi(0) = z_0$ and $\phi(\varphi(w)) \equiv w$. Furthermore, φ extends homeomorphically over the boundary circle $\partial\mathbb{D}_r$, and the image $\varphi(\partial\mathbb{D}_r) \subset A^*(z_0)$ necessarily contains at least one critical point of f.*

Throughout the thesis we refer to $\mathcal{A}_\epsilon$ as a *linearizing domain* or a *domain of linearization*. We refer to the maximal set $\mathcal{A}$ given by Lemma 1.1.11 as the *maximal domain of linearization*.

The superattracting case: $\lambda = 0$

When $\lambda = 0$, f can be expressed as

$$f(z) = z_0 + a(z - z_0)^n + \mathcal{O}(|z - z_0|^{n+1})$$

on a neighborhood of z_0 with $a \in \mathbb{C}$, $a \neq 0$, and where $n \geq 2$ is the *local degree* of the superattracting fixed point. We begin with a result due to L. E. Böttcher (see [Mil06, Thm. 9.1]).

Theorem 1.1.12 (Böttcher's theorem)**.** *Let f be a rational map with a superattracting fixed point z_0 of local degree n. Then, there exists a local holomorphic change of coordinates $w = \phi(z)$ with $\phi(z_0) = 0$ which conjugates f with the nth power map $w \to w^n$ on some neighborhood of z_0. Furthermore, ϕ is unique up to multiplication by $(n-1)$st root of unity.*

The *Böttcher coordinate* ϕ is conformal and defined in a neighborhood U of the superattracting fixed point $z_0 \in A^*(z_0)$. As before, we can consider, for ϵ small enough, the inverse map $\varphi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_\epsilon \subset A^*(z_0)$. The following result tells us that φ_ϵ can be extended to $\mathbb{D}$ unless there is an additional critical point in $A^*(z_0)$ (see [Mil06, Thm. 9.3]).

Theorem 1.1.13. *There exists a unique open disk $\mathbb{D}_r$ of maximal radius $0 < r \leq 1$ such that φ_ϵ extends to a map $\varphi : \mathbb{D}_r \to \mathcal{A} \subset A^*(z_0)$. If $r = 1$, then φ maps $\mathbb{D}$ biholomorphically onto $A^*(z_0)$ and z_0 is the only critical point of the immediate basin of attraction. If $0 < r < 1$, then there is at least one additional critical point in $A^*(z_0)$ which lies on the boundary of $\varphi(\mathbb{D}_r)$.*

The parabolic case: $\lambda = e^{2\pi i p/q}$, $p/q \in \mathbb{Q}$

Let g be a rational map with a parabolic fixed point z_0 with multiplier $\lambda = e^{2\pi i p/q}$, where $p, q \in \mathbb{N}$ are coprime. The point z_0 is also fixed under $f := g^q$ and has multiplier $\lambda^q = 1$. The map f can be expressed in a neighborhood of z_0 as

$$f(z) = z + a(z - z_0)^{n+1} + \mathcal{O}(|z - z_0|^{n+2}),$$

where $a \in \mathbb{C}$ is not zero and $n + 1 \geq 2$ is the *multiplicity* of the parabolic fixed point. The main result on parabolic fixed points is known as Leau-Fatou Flower Theorem (see [Mil06, Thm. 10.7]). It was proven in a preliminary way by L. Leau and improved afterwards by Julia and Fatou. Before stating it we introduce the concept of petals.

Definition 1.1.14. Let z_0 be a parabolic fixed point of multiplicity $n+1 \geq 2$ of a holomorphic germ f which is defined and univalent on a neighborhood U of z_0. An open connected set $\mathcal{P} \subset U$ is called an *attracting petal* for f if

(a) $f(\overline{\mathcal{P}}) \subset \mathcal{P} \cup \{z_0\}$, and

(b) $\bigcap_n f^n(\overline{\mathcal{P}}) = z_0$.

An open set $\mathcal{P} \subset f(U)$ is called a *repelling petal* for f if it is an attracting petal for $f^{-1} : f(U) \to U$, where f^{-1} denotes the branch of the inverse of f fixing z_0.

Theorem 1.1.15 (Leau-Fatou Flower Theorem)**.** *If z_0 is a fixed point of multiplicity $n+1 \geq 2$, then within any neighborhood of z_0 there exist simply connected petals $\mathcal{P}_j$, where $j \in \mathbb{Z}/2n\mathbb{Z}$ and where $\mathcal{P}_j$ is either attracting or repelling according to whether j is even or odd. Moreover, they can be chosen so that $\{z_0\} \cup \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{2n-1}$ is an open neighborhood of z_0. When $n > 1$, each $\mathcal{P}_j$ intersects $\mathcal{P}_{j\pm1}$ in simply connected sets $\mathcal{P}_j \cap \mathcal{P}_{j\pm1}$ and is disjoint from the remaining $\mathcal{P}_k$.*

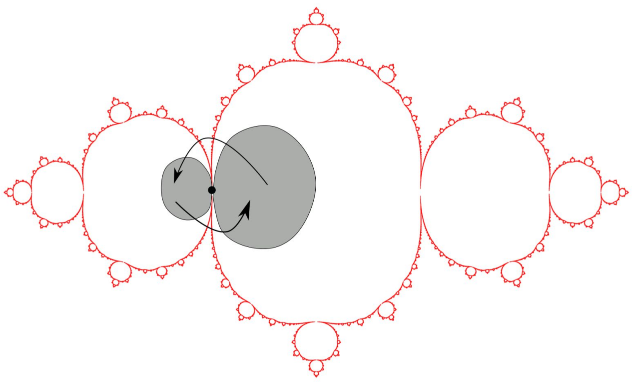


Figure 1.2: Dynamical plane of a quadratic polynomial with a parabolic fixed point with multiplier $\lambda = e^{2\pi i/2} = -1$. It has a 2-cycle of attracting petals. When considering the germ g^2 the situation becomes the one of Theorem 1.1.15, i.e., the fixed point has multiplier 1 and two invariant attracting petals.

The following result explains the existence of a maximal domain within any connected component of the immediate basin of attraction $A^*(z_0)$ of a parabolic fixed point z_0 in which the dynamics is conjugate to the translation $z \to z+1$ (see [Mil06, Thm. 10.9 and Thm. 10.15]). It also states the existence at least a critical point on every connected component of $A^*(z_0)$.

Theorem 1.1.16. *For any attracting or repelling petal $\mathcal{P}$ there is a conformal embedding $\alpha : \mathcal{P} \to \mathbb{C}$ which is unique up to composition with a translation of $\mathbb{C}$ and which satisfies the Abel functional equation $\alpha(f(z)) = 1 + \alpha(z)$ for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$.*

Every immediate basin of attraction of z_0 contains at least one critical point of f. Furthermore, each of them contains one and only one attracting petal $\mathcal{P}_{max}$ which is mapped univalently onto some right half plane under α and which is maximal with respect to this property. This preferred petal $\mathcal{P}_{max}$ always has one or more critical points on its boundary.

The irrational case

The solution of the linearization problem around an irrational fixed point depends strongly on how well the irrational number can be approximated by rational numbers. A formal conjugation can always be defined around it. However, the radius of convergence of such conjugation may be zero. This leads to the so called problem of small divisors studied initially by C. L. Siegel [Sie42] and A. Bryuno [Bry65]. They gave conditions on the irrational number θ so that any holomorphic germ could be linearized around a fixed point of multiplier $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ (c.f. [Mil06, Chap. 11]).

Definition 1.1.17. Let f be a holomorphic germ and let z_0 be an irrational fixed point of f. We say that z_0 is a *Siegel point* if the germ is linearizable around z_0. The maximal domain of linearization is known as the *Siegel disk* around z_0. If the germ is not linearizable around z_0 we say that z_0 is a *Cremer point*.

Siegel disks are simply connected domains foliated with invariant curves were the dynamics are conjugated to the rotation $z \to \lambda z$ (see Figure 1.3).

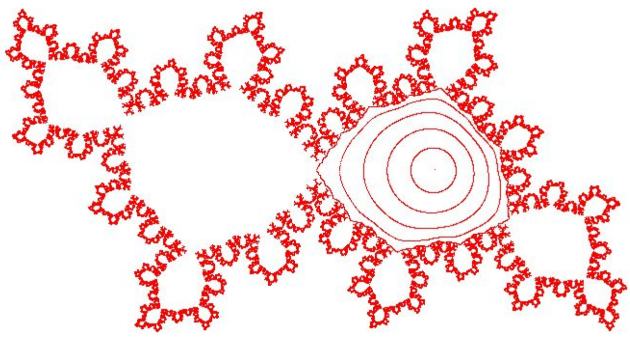


Figure 1.3: The dynamical plane of a degree 2 polynomial with a Siegel disk.

1.1.2 Fatou components

The Fatou set is open and it typically consists of infinitely many connected components, known as *Fatou components*. They are mapped among each other under iteration of f and, a priori, may be either periodic, preperiodic (i.e., eventually mapped to a periodic one) or wandering. In the previous section we already described three different types of Fatou components: basins of attraction of attracting or parabolic cycles and Siegel disks. The following theorem gives a complete classification of periodic Fatou components of rational maps. It is mainly due to H. Cremer [Cre32] and P. Fatou [Fat20] (c.f. [Mil06]).

Theorem 1.1.18 (Classification Theorem)**.** *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map and let U be a periodic Fatou component of period $p \geq 1$. Then, one of the following holds.*

- *U contains a periodic attracting point z_0 of period p such that $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$. We say that U is an* attracting Fatou component *(see Figure 1.4 (a)).*

- *∂U contains a parabolic point z_0 such that $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$. We say that U is a* parabolic Fatou component *(see Figure 1.4 (b)).*

- *U is simply connected and f^p is conformally conjugate to a rigid rotation $R_\theta(z) = e^{2\pi i\theta}z$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The only fixed point under f^p is an irrational fixed point of multiplier $e^{2\pi i\theta}$ and is called the* centre *of the Siegel disk (see Figures 1.4 (c) and 1.3). Respectively, the irrational number θ is called the* rotation number *of U .*

- *U is doubly connected and f^p is conformally conjugate to a rigid rotation $R_\theta(z) = e^{2\pi i\theta}z$ in a round annulus for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then U is called a* Herman ring *(see Figures 1.4 (d) and 1.5) and the irrational number θ is called the* rotation number *of the Herman ring.*

Sullivan [Sul85] proved that rational maps have no wandering Fatou components. His celebrated result makes intensive use of the techniques of quasiconformal surgery (see Section 2.1).

Theorem 1.1.19 (Sullivan's Non-Wandering Theorem)**.** *Let f be a degree $d \geq 2$ rational map. Then $\mathcal{F}(f)$ has no wandering components.*

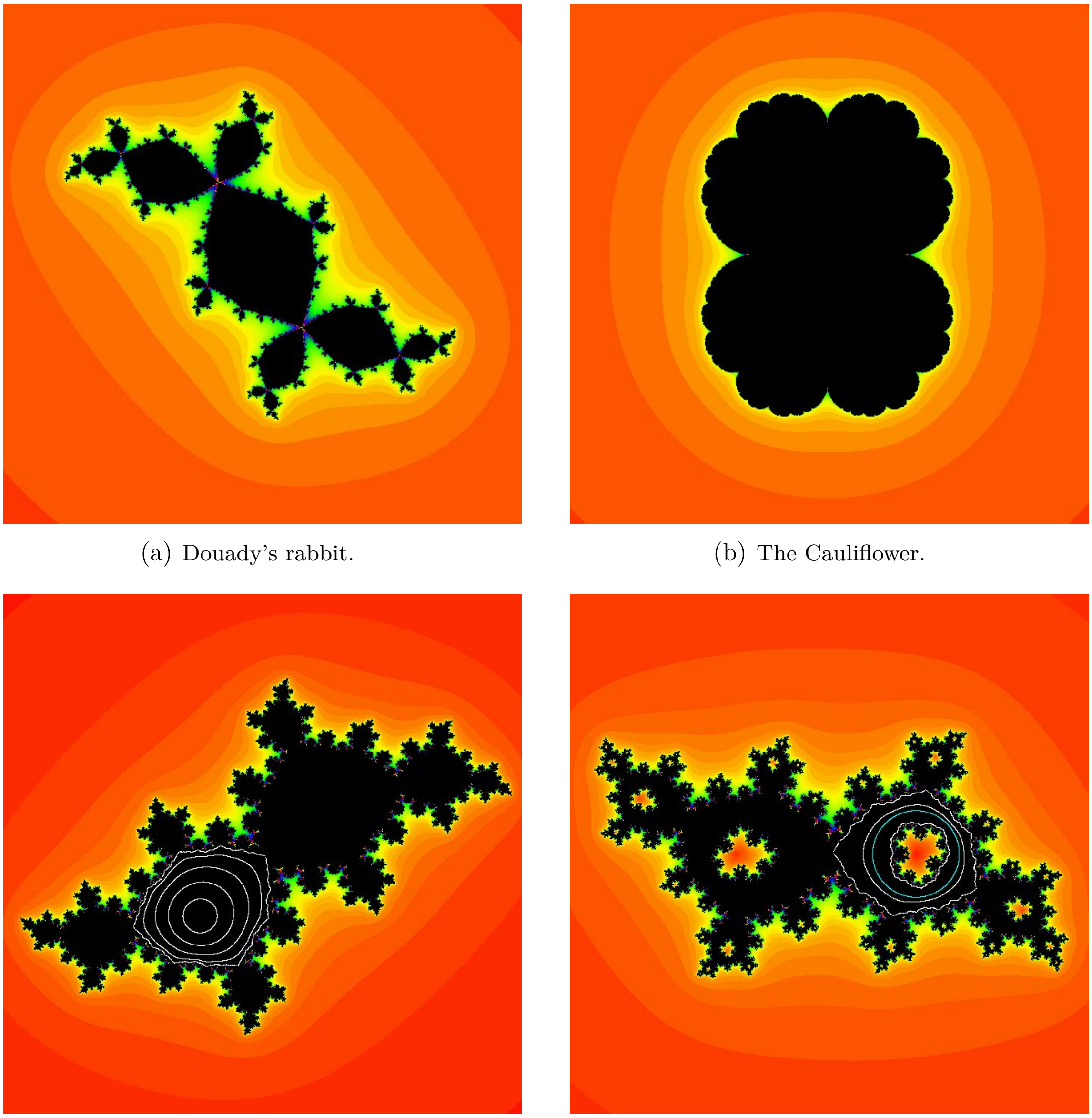
The existence of Herman rings was proven by Herman [Her79] by using a family of degree 3 Blaschke products (see Section 1.2) which restricts, for some values, to a family of diffeomorphisms of $\mathbb{S}^1$. He proved that, for certain parameters, the circle diffeomorphism is analytically conjugate to a rotation $R_\theta(z) = e^{2\pi i\theta}z$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. That analytic conjugation extends to an open neighborhood of the circle, leading to the existence of Herman rings. Later on, M. Shishikura [Shi87] showed how to build rational maps with Herman rings out of other rational maps with Siegel disks by means of a quasiconformal surgery. In the same article Shishikura proved the following result.

Theorem 1.1.20 (Fatou-Shishikura inequality)**.** *Let f be a degree $d \geq 2$ rational map. Let $\mathscr{A}$ denote the number of attracting cycles of f, $\mathscr{P}$ denote the number of cycles of attracting parabolic petals, $\mathscr{I}$ denote the number of irrational cycles and $\mathscr{H}$ denote the number of cycles of Herman Rings. Then, the following inequality holds.*

$$\mathscr{A} + \mathscr{P} + \mathscr{I} + 2\mathscr{H} \leq 2d - 2.$$

Moreover, $\mathscr{H} \leq d - 2$.

The result is based on the fact that rational maps have at most $2d - 2$ different critical points (see Corollary 1.3.2). While the basin of attraction of every attracting or parabolic cycle contains at least a critical point, it is also the case that every Siegel disk and Herman ring needs at least a critical point whose orbit accumulates on its boundary since, otherwise, the domain could be extended further. Shishikura proved that these critical points cannot be shared, that a critical orbit accumulates on at most a connected component of the boundary of a rotation domain.



(a) Douady's rabbit. (b) The Cauliflower.

(c) A Siegel disk. (d) A Herman ring.

Figure 1.4: Figure (a) shows the dynamical plane of $p(z) = z^2 - 0.123 + 0.745i$, also known as Douady's rabbit, which has an attracting cycle of period 3. Figure (b) shows the dynamical plane of $p(z) = z^2 + 0.25$, also known as the Cauliflower, for which $z = 0.5$ is a parabolic fixed point. Figure (c) shows the dynamical plane of $p_\theta(z) = z^2 + e^{2\pi i\theta}z$ where θ is the golden number and p_θ has a Siegel disk. Figure (d) shows the dynamical plane of $q_{\lambda,a} = \lambda z^2(z-a)/(1-az)$, where $|\lambda| = 1$ and $a > 3$ are chosen so that $q_{\lambda,a}$ has a Herman ring. In all figures we plot the points whose orbit escapes to infinity with a scaling from green to orange. In Figure (d) we use the same colors if the orbit converges to the superattracting fixed point $z = 0$.

Figure 1.5: An invariant connected component of a Herman ring of a Blaschke product $q_{\lambda,a} = \lambda z^2(z-a)/(1-az)$ whith $|\lambda| = 1$ and $a > 3$ (see Figure 1.4 (d)). We see how it is foliated with invariant curves.

1.1.3 Hyperbolic rational maps

In this subsection we introduce the notion of hyperbolic rational map as well as some of their main properties. We also give a classification of the dynamics of rational maps with two free critical points. We refer to [Mil06], [CG93] and [Bea91] for proofs of the results.

Definition 1.1.21. A rational map f is said to be *expansive* in a compact set $\mathcal{C} \subset \widehat{\mathbb{C}}$ if there are an open neighborhood U of $\mathcal{C}$, a conformal metric μ defined on U and a constant $k > 1$ such that $||Df_z||_\mu \geq k$ for all z in U.

A rational map f is said to be *hyperbolic* if it is expanding on its Julia set $\mathcal{J}(f)$.

Notice that the Julia set $\mathcal{J}(f)$ of a rational map is compact since it is a closed subset of a compact space. The following theorem provides a characterization of the set of hyperbolic rational maps.

Theorem 1.1.22. *A rational map f is hyperbolic if and only if the forward orbit of all its critical points accumulate on attracting or superattracting cycles.*

It follows from the previous theorem that the dynamics of hyperbolic rational maps is stable in the sense that if f is hyperbolic, any nearby map is also hyperbolic. Moreover, $\mathcal{J}(f)$ has measure zero and depends continuously on f under hyperbolic perturbations of it (see [MSS83] and [Lyu83, Lyu84]).

We finish this subsection giving a classification of the hyperbolic dynamics within a family of rational maps with two free critical orbits. We first introduce a rigorous definition of this concept.

Definition 1.1.23. Let Q_a denote a family of rational maps depending continuously on a parameter a. We say that the family Q_a is *almost bicritical* if the maps Q_a have only two free critical points counted with multiplicity. By this we mean that all but two critical points $c_\pm$ of Q_a (which may collapse in a single one for certain values of a) are permanently captured by (super)attracting cycles which do not depend on a. If the number of critical points is two, we say that the family is *bicritical*.

The Blaschke family B_a (Equation (1)) with $a \in \mathbb{C}^* \setminus \mathbb{S}^1$ is an example of an almost bicritical family (see Section 3.1). Following [Ree90] and [Mil92], we classify the hyperbolic parameters of an almost bicritical family Q_a as follows.

Definition 1.1.24. We say that a hyperbolic map which belongs to an almost bicritical family is

(a) *adjacent* if the free critical points belong to the same component of the immediate basin of attraction of an attracting cycle (see Figure 5.3 (left)),

(b) *bitransitive* if the free critical points belong to different components of the same immediate basin of attraction of an attracting cycle (see Figure 3.2 (b)),

(c) *capture* if one of the free critical points belongs to the immediate basin of attraction of an attracting cycle and the other one belongs to a preperiodic preimage of it (see Figure 5.3 (right)),

(d) *disjoint* if the free critical points belong to the immediate basin of attraction of two different attracting cycles (see Figure 3.2 (c) and (d), Figure 3.4 (right) and Figure 5.6 (right)),

(e) *escaping* if at least one of the free critical orbits is captured by attracting cycles dominated by non free critical orbits. If both critical orbits escape we say that the parameter is *fully escaping* (see Figures 3.4 (left) and 3.5).

Remark 1.1.25. If a hyperbolic parameter a is such that both free critical points of Q_a collapse, then it can only be escaping or adjacent.

1.2 Circle mappings

In this section we introduce some preliminaries on circle mappings which are used along the thesis. We refer to [ALM00] and [dMvS93] for general background and proofs of the results.

Definition 1.2.1. We say that a continuous map $F : \mathbb{R} \to \mathbb{R}$ is the *lift* of a continuous circle map f if the following diagram commutes.

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{f} & \mathbb{S}^1 \\ {\scriptstyle e^{2\pi i x}}\uparrow & & \uparrow{\scriptstyle e^{2\pi i x}} \\ \mathbb{R} & \xrightarrow{F} & \mathbb{R}. \end{array}$$

Notice that every continuous circle map has infinitely many lifts. Indeed, the following result holds (see [ALM00, Prop. 3.1.6]).

Proposition 1.2.2. *Let f be a continuous circle map. Then, two different lifts F_1 and F_2 of f differ in an integer constant. Moreover, there exits $d \in \mathbb{Z}$ such that $F(x+1) = F(x)+d$ for all F lift of f and all $x \in \mathbb{R}$. The integer d is called the* degree *of f.*

Notice that the notion of degree of a circle map that we introduced in the previous proposition is not related with the notion of topological degree that we used previously. A degree 2 circle map may have points with more than two preimages in $\mathbb{S}^1$. However, the two different notions of degree do coincide when the circle map f is a positively oriented degree d covering of $\mathbb{S}^1$. In particular, this is the case if F is strictly increasing.

The following result deals with families of lifts of degree 2 orientation preserving coverings of the circle which are continuously parametrized. It tells us that all lifts of such covering maps are semiconjugate to the doubling map $x \to 2x$ and that the semiconjugacy depends continuously on the same parameters than the lifts (see [MR07, Lemmas 3.1 and 3.3]).

Lemma 1.2.3. *Let $F_a : \mathbb{R} \to \mathbb{R}$ be a continuous and increasing map depending continuously on a. Suppose that $F_a(x + k) = F_a(x) + 2k$ for any integer k and for all $x \in \mathbb{R}$. Then, the limit*

$$H_a(x) = \lim_{n\to\infty} \frac{F_a^n(x)}{2^n} \tag{1.1}$$

exists uniformly on x. This map H_a is increasing, continuous, depends continuously on a and satisfies $H_a(x + k) = H_a(x) + k$ for any integer k and for all $x \in \mathbb{R}$. Moreover, H_a semiconjugates F_a with the multiplication by 2, i.e., $H_a(F_a(x)) = 2H_a(x)$ for any real x. Furthermore, if F_a is increasing with respect to a, then H_a is also increasing with respect to a.

Now let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a degree 2 orientation preserving map of the circle. Since its lift F satisfies the conditions of Lemma 1.2.3, f is semiconjugate to the doubling map $R_2(z) = z^2$ (equivalently given by $\theta \to 2\theta \pmod 1$) by a degree 1 map of the unit circle h whose lift H is given by Lemma 1.2.3. The following standard lemmas deal with the semiconjugacy h. The first one tells us that the semiconjugating map is unique (c.f. Lemma 7.3.2 and [Boy06]).

Lemma 1.2.4. *Let f be a degree 2 orientation preserving map of the circle. Then there exists a unique degree 1 map h of the circle which semiconjugates f with the doubling map R_2.*

The next lemma tells us that the semiconjugacy sends periodic points to periodic points of the same period (see [MR07, Lem. 3.2]). We add the proof for the sake of completeness.

Lemma 1.2.5. *The semiconjugating map h sends points of period k to points of period k.*

Proof. Let p be a periodic point of period k. Then, $f^k(p) = p$. Since h semiconjugates f with the doubling map R_2, we have that $R_2^k(h(p)) = h(p)$. Hence, $h(p)$ is a periodic point of period dividing k. Assume that its period is not k. Then, there are two points x and y of the cycle of p which are mapped to the same point. Therefore, there is an arc of points γ joining x and y which is mapped to the same point by h. Moreover, for all $j \in \mathbb{N}$, we have that $f^j(\gamma)$ is mapped under h to a point. Hence, we have that the full circle is mapped onto a single point, which is a contradiction. □

Under certain conditions this semiconjugacy is a conjugacy.

Definition 1.2.6. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a $\mathcal{C}^1$ map. We say that f is *expanding* if there exist constants $C > 0$ and $\lambda > 1$ such that

$$|Df^n(x)| > C\lambda^n$$

for all $n \in \mathbb{N}$ and all $x \in \mathbb{S}^1$.

Remark 1.2.7. An expanding map is monotone. Hence, if it has topological degree 2, its lift F satisfies the conditions of Lemma 1.2.4. It follows from it that an expanding map of the circle of topological degree 2 is semiconjugate to the doubling map.

Before stating the next result, we introduce the concept of quasisymmetry for circle homeomorphisms. We shall revisit this concept in Section 2.1.

Definition 1.2.8. An homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ is *quasisymmetric* if there exists a constant $M > 0$ such that, for all $z_1, z_2, z_3 \in \mathbb{S}^1$,

$$|z_1 - z_2| = |z_2 - z_3| \Rightarrow \frac{1}{M} \leq \frac{|h(z_1) - h(z_2)|}{|h(z_2) - h(z_3)|} \leq M.$$

Proposition 1.2.9. *Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be an expanding map of topological degree 2. Then, g is quasisymmetrically conjugate to the doubling map R_2, i.e., the map $h : \mathbb{S}^1 \to \mathbb{S}^1$ whose lift H is given by Lemma 1.2.3 is a quasisymmetric map such that*

$$g = h^{-1} \circ R_2 \circ h.$$

The following proposition gives conditions which ensure expansivity.

Proposition 1.2.10. *Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be a $\mathcal{C}^2$ covering map of degree d without attracting or parabolic points and having no critical points. Then g is expanding.*

1.2.1 Blaschke products

We dedicate this subsection to define what Blaschke products are and introduce some of their properties. These products are circle maps which extend to $\widehat{\mathbb{C}}$ as rational maps.

Definition 1.2.11. A degree d (finite) *Blaschke product* is a rational function of the form

$$B(z) = e^{2\pi i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z},$$

where $\alpha \in [0, 1)$ and $|a_i| < 1$ for all i. We say that B is a degree d *generalized Blaschke product* if $\alpha \in [0, 1)$ and $a_i \in \widehat{\mathbb{C}} \setminus \mathbb{S}^1$.

The following proposition contains some properties of finite Blaschke products.

Proposition 1.2.12. *Let B be a Blaschke product and let $\mathcal{I}(z) = 1/\overline{z}$ denote the inversion with respect to the unit circle. Then, the following hold.*

(a) The product B preserves $\mathbb{S}^1$ and, therefore, is symmetric with respect to $\mathbb{S}^1$, i.e., $B(z) = \mathcal{I} \circ B \circ \mathcal{I}(z)$.

(b) The restriction to the unit disk $B|_{\mathbb{D}}$ is a degree d branched cover of $\mathbb{D}$.

(c) The circle map $B|_{\mathbb{S}^1}$ is a degree d covering of $\mathbb{S}^1$.

(d) If $\widetilde{B} : \mathbb{D} \to \mathbb{D}$ is a proper holomorphic map then $\widetilde{B}$ is a Blaschke product of (finite) degree $d \geq 1$ (c.f. Lemma 1.2.13).

(e) Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain. If $f : U \to U$ is a proper holomorphic map then f is conformally conjugate to a Blaschke product.

For the generalized Blaschke products, only property (a) of the previous proposition is preserved. There may appear new preimages of the unit circle as well as preimages of infinity in $\mathbb{D}$. Even if the topological degree of a generalized Blaschke product B is d, the degree of $B|_{\mathbb{S}^1}$ as circle map may be smaller than d.

Generalized Blaschke products are often used as model maps in complex dynamics. For instance, the products $B_{\alpha,a} = e^{2\pi i\alpha}z^2(z-a)/(1-\overline{a}z)$ where used by Herman [Her79] to prove the existence of Herman rings (see Figure 1.4 (d)). They were also used to prove that the boundary of a Siegel disk of a quadratic polynomial with a bounded rotation number is a Jordan curve (see [Ghy84, Dou87, Her86, Świ88, Her87], c.f. [BF14, Sect. 7.2.2]). The following lemma tells us that all rational maps which preserve the unit circle are generalized Blaschke products (see [Mil06, Lem. 15.5]).

Lemma 1.2.13. *Let R be a degree d rational map and assume that $R(\mathbb{S}^1) = \mathbb{S}^1$. Then R can be written as a degree d generalized Blaschke product.*

We finish this subsection presenting a result due to C. Petersen [Pet07] which gives weaker conditions than expansivity so that a generalized Blaschke product restricted to the unit circle is quasisymmetrically conjugate to the doubling map. Recall that the ω-limit set $\omega(z)$ of a point z is defined to be the set of accumulation points of the orbit of z and that z is said to be recurrent if $z \in \omega(z)$.

Theorem 1.2.14. *Let $B : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a generalized Blaschke product with poles in $\mathbb{D}$ such that the restriction $B : \mathbb{S}^1 \to \mathbb{S}^1$ is a (positively oriented) degree $d \geq 2$ covering and such that*

(a) $\omega(c) \cap \mathbb{S}^1 = \emptyset$ for every recurrent critical point c,

(b) $\mathbb{S}^1$ contains no non repelling periodic point.

Then, $B : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetrically conjugate to $R_d(z) = z^d$.

We want to remark that one should not expect to get a much better result than the previous one for maps of the circle having critical points. Indeed, H. Bruin [Bru96] proved that given any family of degree d covering maps of the circle with critical points, there exists a total measure subset of elements of the family on which does not exist a quasisymmetric conjugacy.

1.2.2 Tongues

In this subsection we introduce the concept of tongue for almost bicritical families f_a (see Definition 1.1.23) which leave the unit circle invariant and such that $f_a|_{\mathbb{S}^1}$ is strictly increasing of degree 2.

Given a family of f_a, $a \in \Delta$, of orientation preserving homeomorphisms of the unit circle, one can assign a rotation number to each of its members which gives the average asymptotic rate of rotation of points in the circle. More precisely, if F_a is a lift of f_a, the rotation number of f_a is given by

$$\rho(f_a) = \lim_{n\to\infty} \frac{H_a^n(x)}{n}, \tag{1.2}$$

where $x \in \mathbb{R}$. It is well known that the limit exists and it is independent of x. Moreover, since any two lifts of f_a differ by an integer constant, $\rho(f_a)$ is a well defined number in $\mathbb{R}/\mathbb{Z}$. Hence, we may assume that $\rho(f_a) \in [0,1)$. Moreover, $\rho(f_a) = p/q \in \mathbb{Q}$ if and only if f_a has an attracting or parabolic q-cycle in $\mathbb{S}^1$. The (rational) tongues of the f_a are defined as the sets of parameters $a \in \Delta$ such that f_a has rotation number $p/q \in \mathbb{Q}$. The concept of tongues was introduced by Arnold [Arn61] for the standard family of perturbations of the rigid rotation

$$\theta \to \theta + \alpha + (\beta/2\pi)\sin(2\pi\theta) \pmod 1$$

where $0 \le \theta < 1$, $0 \le \alpha \le 1$ and $0 \le \beta \le 1$ (see Figure 1.6) , but were investigated later on by many authors, for this family or other related ones (see e.g. [Her79], [Boy86], [WBJ91], [EKT95] and [BFGH05]).

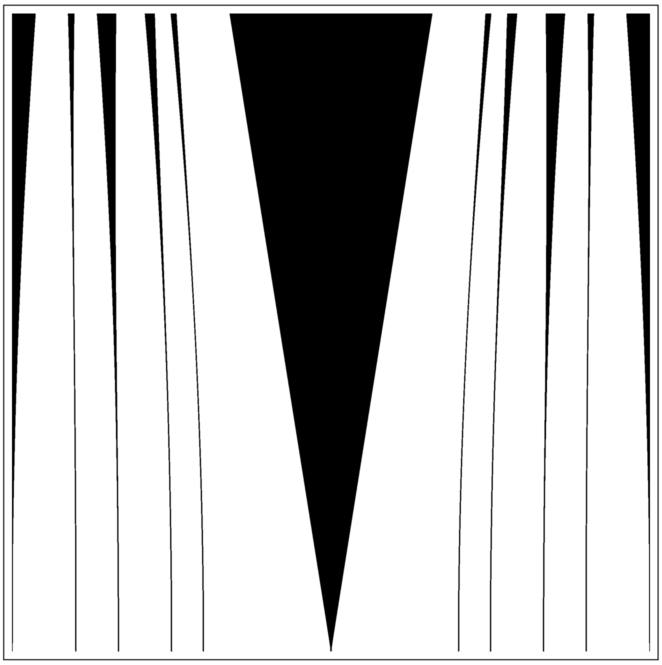


Figure 1.6: The tongues of the standard family. The x-axis shows $\alpha \in [-1/2, 1/2]$ so that the tongue T_0 appears centered. The y-axis shows the parameter $\beta \in [0,1]$. The figure shows, in order, the tongues with rotation numbers $1/2, 3/5, 2/3, 3/4, 4/5, 0, 1/5, 1/4, 1/3, 2/5, 1/2$.

On the other hand, if the maps f_a are not homeomorphisms but degree 2 covers of $\mathbb{S}^1$, we cannot assign rotation number to them. Indeed, the limit

$$H_a(x) = \lim_{n\to\infty} \frac{F_a^n(x)}{2^n}$$

depends continuously on x and provides a semiconjugacy between f_a and the doubling map $R_2(z) = z^2$ (see Lemmas 1.2.3 and 1.2.4, c.f. Equation (1.1)). Nevertheless, tongues can still be defined as sets of parameters for which f_a has an attracting cycle in $\mathbb{S}^1$. Before formally defining them we need to introduce an auxiliary lemma.

Assume that $f_a : S \to S$, where $S = \widehat{\mathbb{C}}$ or $\mathbb{C}^*$ and $a \in \Delta$, is an almost bicritical family of holomorphic maps such that, for all $a \in \Delta$, $f_a|_{\mathbb{S}^1}$ is an increasing degree 2 cover of the unit circle and that either the free critical points are not in $\mathbb{S}^1$ or they collapse in a unique critical point c. Then, the following result holds.

Lemma 1.2.15. *Let f_a as above. Then, $f_a|_{\mathbb{S}^1}$ has at most one attracting cycle $< x_0 >$ in the unit circle. If there is such a cycle, the two free critical points lie in the same connected component of $A^*(< x_0 >)$.*

Proof. It follows from the Schwarz reflection principle that if the two free critical points do not lie in the unit circle, then their orbits are symmetric with respect to $\mathbb{S}^1$. Hence, if one of the critical orbits accumulates on an attracting cycle $< x_0 >$ in $\mathbb{S}^1$, so does the other one. Given that any attracting cycle has a critical point in its basin of attraction, this proves that there can be at most one attracting cycle in the unit circle. Moreover, every connected component of $A^*(< x_0 >)$ intersects $\mathbb{S}^1$ and is symmetric. Hence, both critical points lie in the same connected component. $\square$

Definition 1.2.16. Let f_a be as above with an attracting cycle $< x_0 >$ in the unit circle. The point $x_j \in < x_0 >$ such that the critical points lie in $A^*(x_j)$ is called the *marked point* of the cycle.

We generally rename the cycle so that x_0 denotes the marked point. Now we can formalize the concept of tongue for degree two covers of the circle. Let H_a be the continuous map given by Lemma 1.2.3 which semiconjugates the lift F_a of $f_a|_{\mathbb{S}^1}$ to the doubling map. Then tongues for degree 2 families of coverings of the unit circle are defined as follows.

Definition 1.2.17. Let f_a as above. We say that a parameter $a \in \Delta$, is of *type* τ if $f_a|_{\mathbb{S}^1}$ has an attracting cycle $< x_0 >$ and $H_a(x_0) = \tau$, where x_0 is the marked point point of the cycle. The *tongue* T_τ is defined as the set of parameters $a \in \Delta$, such that a is of type τ.

The type $\tau(a)$ is a well defined number of $\mathbb{R}/\mathbb{Z}$ by Lemma 1.2.4. Hence, we may assume that $\tau(a) \in [0,1)$. Notice that in the previous definition we use an abuse of notation on the definition, naming x_0 both the point in the unit circle and its lifted equivalent in the real line.

It follows from Lemma 1.2.15 that the tongues are disjoint. Indeed, if two different tongues would intersect, we would have sets of parameters with two different attracting cycles, which is not possible.

Given that H_a sends periodic points to periodic points (see Lemma 1.2.5), any realizable type $\tau \in \mathbb{S}^1$ is a periodic point of the doubling map. It also follows from this and the continuity of H_a with respect to parameters that tongues are open subsets of Δ. Hence, a parameter $a \in \partial\Delta$, such that f_a has an attracting cycle in $\mathbb{S}^1$ of type τ is not in the boundary of T_τ.

Misiurewicz and Rodrigues [MR07, MR08] studied the tongues of the double standard family of perturbations of the doubling map

$$\theta \to 2\theta + \alpha + (\beta/\pi)\sin(2\pi\theta) \pmod 1$$

where $0 \leq \theta < 1$, $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ (see Figure 1.7). Its complexification is the family of bicritical entire transcendental maps of the punctured plane $e^{i\alpha} z^2 e^{\beta(z-1/z)}$.

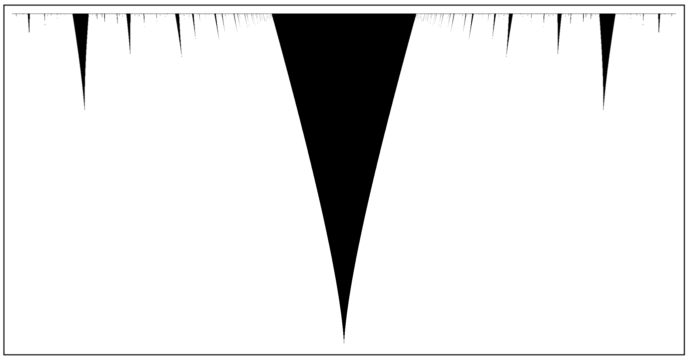


Figure 1.7: The tongues of the double standard family. The x-axis shows the parameter $\alpha \in [-1/2, 1/2]$ so that the tongue T_0 appears centered. The y-axis shows the parameter $\beta \in [0.5, 1]$.

The family $B_a(z) = z^3(z-a)/(1-\overline{a}z)$, where a is such that $|a| \geq 2$, satisfies the required conditions for the existence of tongues. In Chapter 6 we shall study them in detail (see Figure 6.1).

1.3 The Riemann-Hurwitz formula

The Riemann-Hurwitz formula is one of the main results used in the study of dynamical systems in one complex variable. It relates the genus of two surfaces U and V if there is ramified covering $f : U \to V$. The following is a simplified statement which is enough for our purposes (c.f. [Bea91], [Ste93]). It is stated using the connectivity of the open domains instead of the Euler-characteristic. Recall that the connectivity m_U of an open domain $U \in \widehat{\mathbb{C}}$ is defined as the number of connected components of $\widehat{\mathbb{C}} \setminus U$. In particular, $m_{\widehat{\mathbb{C}}} = 0$.

Theorem 1.3.1 (Riemann-Hurwitz Formula)**.** *Let U and V be two connected domains of $\widehat{\mathbb{C}}$ of finite connectivity m_U and m_V and let $f : U \to V$ be a degree k proper map branched over r critical points counted with multiplicity. Then*

$$m_U - 2 = k(m_V - 2) + r.$$

This formula has several important corollaries which apply to complex dynamics. We finish this section presenting two of them. The first of them gives us the number of critical points of a rational map. Its proof is straightforward.

Corollary 1.3.2. *Any rational map of degree d has $2d-2$ critical points.*

The second corollary tells us that two or more different critical points are required to map a multiply connected domain onto a simply connected domain by a proper map. This result is quite useful when trying to prove that a Fatou component is simply connected.

Corollary 1.3.3. *Let f be a rational map and let V be a simply connected domain. Let U be a connected component of $f^{-1}(V)$. If U contains at most one critical point (of arbitrary multiplicity), then U is simply connected.*

Proof. By construction, $f|_U : U \to V$ is proper. Let r be the multiplicity of the critical point. Then, $f|_U$ has at least degree $r+1$. By The Riemann-Hurwitz formula, since $m_V = 1$, we have $m_U - 2 \leq -(r+1) + r = -1$. Since m_U is at least 1, we conclude that it is indeed 1 and U is simply connected. $\square$

Chapter Two 2

Preliminaries 2: Quasiconformal Surgery

Quasiconformal surgery was first used in the setting of complex dynamics by Sullivan [Sul85] in his celebrated proof of non existence of wandering domains for rational functions. Since then, it has become one of the main tools in the study of dynamical systems in one complex variable. We use it in the proof of several results such as Theorem 5.3.4, Theorem 5.4.2 and Theorem 6.2.1.

The aim of this chapter is to give an overview of the results used in quasiconformal surgery. In section 2.1 we describe the basic properties of quasiconformal mappings. In section 2.2 we generalize the previous results to orientation reversing maps. In section 2.3 we introduce the theory of polynomial-like mappings, which makes intensive use of quasiconformal tools.

2.1 Quasiconformal maps and almost complex structures

In this section we introduce the basic definitions and results used on quasiconformal surgery. For a more complete introduction and proofs of the results we refer to [Ahl06], [AIM09], [Hub06] and [BF14].

Quasiconformal and quasisymmetric maps

Whereas conformal mappings cannot modify angles, quasiconformal maps can, but only in a bounded fashion. We introduce two equivalent definitions of them, an analytic one and a geometric one.

Definition 2.1.1 (Analytic definition of quasiconformal map)**.** Let U and V be two domains of $\mathbb{C}$. Given $K \geq 1$ we say that the map $\psi : U \to V$ is K-*quasiconformal* if and only if the following conditions hold:

(a) ψ is a homeomorphism.

(b) The partial derivatives $\partial_z \psi$ and $\partial_{\bar{z}} \psi$ exists in the sense of distributions and are locally square integrable (i.e., belong to L^2_{loc}).

(c) If $k := \frac{K-1}{K+1}$, then $|\partial_{\bar{z}} \psi| \leq k |\partial_z \psi|$ in L^2_{loc}, i.e., almost everywhere.

A map is said to be *quasiconformal* if it is K-*quasiconformal* for some K.

Notice that the third condition of the definition implies that quasiconformal maps are orientation preserving.

Before introducing the geometric definition of quasiconformal maps, we recall the definition of modulus of an annulus. An open annulus A in $\mathbb{C}$ is a doubly connected domain of $\hat{\mathbb{C}}$. It can be mapped conformally to an standard or round annulus $\mathbb{A}_{r,R} = \{z \in \mathbb{C}; 0 \leq r < |z| < R \leq +\infty\}$, which is unique up to multiplication by a real constant.

Definition 2.1.2. Let A be an open annulus in $\mathbb{C}$. Then, its *conformal modulus* is defined as

$$\mathrm{mod}(A) := \mathrm{mod}(\mathbb{A}_{r,R}) := \begin{cases} \frac{1}{2\pi}\log\frac{R}{r} & \text{if } r > 0 \text{ and } R < +\infty \\ \infty & \text{if } r = 0 \text{ or } R = +\infty. \end{cases}$$

Quasiconformal mappings modify the modulus of annuli in a bounded fashion. This property may be used to define geometrically the concept of quasiconformal mappings.

Definition 2.1.3 (Geometric definition of quasiconformal map)**.** Let U and V be two open domains in $\mathbb{C}$ and let $K \geq 1$. We say that a map $\psi : U \to V$ is K*-quasiconformal* if and only if ψ is an orientation preserving homeomorphism so that, for any annulus A compactly contained in U, the following inequality is satisfied

$$\frac{1}{K}\mathrm{mod}(A) \leq \mathrm{mod}(\psi(A)) \leq K\mathrm{mod}(A).$$

The fact that these definitions are actually equivalent is one of the main results of the theory. The following proposition states some of the most important properties of these mappings.

Proposition 2.1.4. *The following statements hold.*

(a) If ψ is K-quasiconformal, then ψ^{-1} is K-quasiconformal.

(b) If ψ is K-quasiconformal, it remains being K-quasiconformal after precomposing or postcomposing with conformal mappings.

(c) The composition of a K_1 and a K_2 quasiconformal maps is a K_1K_2-quasiconformal mapping.

(d) A homeomorphism ψ is K-quasiconformal if, and only if, ψ is locally K-quasiconformal.

(e) Quasiconformal maps send sets of measure zero to sets of measure zero.

Notice that properties (a), (b) and (c) follow easily from the geometric definition and properties (d) and (e) follow from the analytic definition.

Definition 2.1.5. Let $U \subset \mathbb{C}$ be an open set and $K < \infty$. A mapping $f : U \to \mathbb{C}$ is K*-quasiregular* if f is locally K-quasiconformal except at a discrete set of points in U. A map is said to be *quasiregular* if there exists $K \geq 1$ so that it is K-quasiregular.

The discrete set of points where a quasiregular map fails to be a local homeomorphism corresponds to the critical points of a holomorphic map. Indeed, the following result holds.

Proposition 2.1.6. *Let U and V be two open sets of $\mathbb{C}$ and let $\psi : U \to V$ be a quasiregular map. Then, there exist an open set W of $\mathbb{C}$, a holomorphic map $f : U \to W$ and a quasiconformal map $\phi : W \to V$ such that $\psi = \phi \circ f$.*

One of the main differences between holomorphic and quasiregular mappings is that the last ones can be defined piecewise, which is a key property while performing the surgeries. To properly state how this can be done, we introduce the following concept.

Definition 2.1.7. A Jordan arc γ is said to be a *quasiarc* if there exists $C > 0$ so that

$$\operatorname{diam}(\gamma(z_1, z_2)) < C|z_1 - z_2| \text{ for all } z_1 \text{ and } z_2 \text{ in } \gamma,$$

where $\gamma(z_1, z_2)$ is taken to be the arc of smaller diameter in γ joining z_1 and z_2. We say that a quasiarc γ is a *quasicircle*, if it is a Jordan curve.

Notice that points, lines and smooth arcs are quasiarcs. The following proposition relates quasicircles and quasiconformal mappings.

Proposition 2.1.8. *Let γ be a quasicircle. Then, there exists a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ so that $\psi(\mathbb{S}^1) = \gamma$.*

It follows from the next theorem that quasiconformal and quasiregular maps can be defined piecewise.

Theorem 2.1.9. *If $\Gamma \subset U$ is a quasiarc and $\psi : U \to V$ is a homeomorphism that is K-quasiconformal on $U \setminus \Gamma$, then ψ is K-quasiconformal in U. We then say that Γ is* quasiconformally removable*.*

Suppose that a domain U is separated into two domains U_1 and U_2 by a quasiarc γ. Then, if the map $\psi : U \to V$ is a homeomorphism such that $\psi|_{U_1}$ is K_1-quasiconformal and $\psi|_{U_2}$ is K_2-quasiconformal, it follows from the previous proposition that ψ is K-quasiconformal with $K = \max\{K_1, K_2\}$.

The following lemma, known as Rickman Lemma or Bers Sewing Lemma, is also used to build quasiconformal maps by gluing procedures. We shall use it in Chapter 5.

Lemma 2.1.10 (Rickman Lemma)**.** *Let $U \subset \mathbb{C}$ be open, $\mathcal{C} \subset U$ compact and let ψ and ϕ two mappings $U \to \mathbb{C}$ which are homeomorphisms onto their images. Suppose that ψ is quasiconformal on U, that ϕ is quasiconformal on $U \setminus \mathcal{C}$, and that $\psi = \phi$ on $\mathcal{C}$. Then ϕ is quasiconformal on U and $\partial_{\bar{z}}\psi = \partial_{\bar{z}}\phi$ almost everywhere on $\mathcal{C}$.*

When studying boundary properties of quasiconformal mappings, quasisymmetric maps arise in a natural way.

Definition 2.1.11. A map $h : \mathbb{S}^1 \to \mathbb{C}$ is *quasisymmetric* if h is injective and there exists a constant $M > 0$ such that, for all $z_1, z_2, z_3 \in \mathbb{S}^1$,

$$|z_1 - z_2| = |z_2 - z_3| \Rightarrow \frac{1}{M} \leq \frac{|h(z_1) - h(z_2)|}{|h(z_2) - h(z_3)|} \leq M.$$

The following theorem relates the concept of quasisymmetric maps and quasicircles.

Proposition 2.1.12. *Let $h : \mathbb{S}^1 \to \mathbb{C}$ be a quasisymmetric map. Then, $h(\mathbb{S}^1)$ is a quasicircle.*

We continue by giving a notion of what it means to talk about quasisymmetric maps between quasicircles.

Definition 2.1.13. Let γ_j, $j = 1, 2$, be quasicircles. Let G_j be the Jordan domains bounded by γ_j and $R_j : \mathbb{D} \to G_j$ be two Riemann mappings. By Charatheodory's theorem, R_j extend continuously to the boundary giving parametrizations $\hat{R}_j : \mathbb{S}^1 \to \gamma_j$, $j = 1, 2$. Then, we say that an orientation preserving homeomorphism $h : \gamma_1 \to \gamma_2$ is *quasisymmetric* if $\hat{R}_2^{-1} \circ h \circ \hat{R}_1 : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric.

It follows easily from the definitions that the composition of two quasisymmetric maps is quasisymmetric. The following results provide some examples of how quasisymmetric maps can by related with quasiconformal maps via the boundary problems (see [DE86], [BA56]).

Proposition 2.1.14. *Let $\psi : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map. Then, ψ extends continuously to a map $\hat{\psi} : \hat{\mathbb{D}} \to \hat{\mathbb{D}}$ such that $\hat{\psi}|_{\mathbb{S}^1}$ is a quasisymmetric map. Conversely, any quasisymmetric map $h : \mathbb{S}^1 \to \mathbb{S}^1$ extends to a quasiconformal map $H : \mathbb{D} \to \mathbb{D}$.*

We remark that the quasiconformal extension of a quasisymmetric map h of the unit circle to the unit disk is not unique.

Proposition 2.1.15. *Let A_1 and A_2 be two annulus bounded by the quasicircles γ_j^i and γ_j^o, $j = 1, 2$. Let $f : \partial A_1 \to \partial A_2$ be so that $f|_{\gamma_1^i} : \gamma_1^i \to \gamma_2^i$ and $f|_{\gamma_1^o} : \gamma_1^o \to \gamma_2^o$ are quasisymmetric. Then, f can be extented quasiconformally to the whole annulus.*

The interpolation can also be done if the boundary maps on the previous proposition are not homeomorphisms. In that case the interpolating map is no longer quasiconformal but quasiregular. The following proposition, which corresponds to Exercise 2.3.3 in [BF14], tells us that this can be done depending continuously on a parameter if one of the curves of the target annulus depends continuously on the parameter

Proposition 2.1.16. *Let γ_j^o for $j = 1, 2$, and γ_1^i and $\gamma_2^i(\lambda)$ be $\mathcal{C}^2$ curves with $\gamma_2^i(\lambda)$ depending continuously on λ and being outer and inner boundaries of two annuli A_1 and A_2^λ in $\mathbb{C}$. Suppose that there are two orientation preserving $\mathcal{C}^1$-maps of degree n given by $f_\lambda^i : \gamma_1^i \to \gamma_2^i(\lambda)$ and $f^o : \gamma_1^o \to \gamma_2^o$ such that the map $\lambda \to f_\lambda^i(z)$ is continuous for any fixed $z \in \gamma_1^i$. Then, there exists a $\mathcal{C}^1$-extension $f_\lambda : A_1 \to A_2^\lambda$ which is a covering map of degree n and such that, for any fixed $z \in A_1$, the map $\lambda \to f_\lambda(z)$ is continuous.*

Almost complex structures and the Measurable Riemann Mapping Theorem

Before stating the main result necessary for surgery, we introduce a geometric approach to the concept of quasiconformal mappings. We first introduce the concept of almost complex structure.

Definition 2.1.17. Given an ellipse E, its *dilatation* K_E and *Beltrami coefficient* $\mu(E)$ are given by the formulas

$$K_E = \frac{\text{Major axis}}{\text{Minor axis}} = \frac{M}{m} > 1, \qquad \mu(E) = \frac{M-m}{M+m} e^{2\pi i\theta} \in \mathbb{D},$$

where $\theta \in [0, \pi)$ is the argument of the direction of the minor axis.

Let $U \subset \mathbb{C}$ be a domain and let $TU = \bigcup_{u\in U} T_uU$ be its tangent bundle. Then, we define an almost complex structure as follows.

Definition 2.1.18. An *almost complex structure* σ on U is a measurable field of infinitesimal ellipses in TU. By this we mean that we associate an ellipse $E_u \subset T_uU$ defined up to scaling to almost every $u \in U$. Moreover, we require the map $u \to \mu(E_u)$ from U to $\mathbb{D}$ to be measurable in the Lebesgue sense. The measurable map μ is called the *Beltrami coefficient* of σ. We define the *dilatation* of σ as

$$K(\sigma) = \text{ess} \sup_{u\in U} K(u), \quad \text{where} \quad K(u) = K(E_u) = \frac{1+|\mu(u)|}{1-|\mu(u)|}.$$

The *standard complex structure* σ_0 is given by circles at each $u \in U$, i.e., $\mu(u) = 0$ $\forall u \in U$.

The concept of Beltrami coefficient in $\mathbb{C}$ can be generalized to Riemann surfaces (c.f. [BF14]). It is common to refer to Beltrami coefficients as *Beltrami forms* when working in a Riemann Surface.

Almost complex structures may be "pulled back" by quasiregular maps. Let $\psi : U \to V$ be quasiregular and suppose that we have an almost complex structure σ in V. Then we can define a new almost complex structure $\psi^*\sigma$ in U as follows. To almost each point u in U, we associate a new ellipse E'_u given by

$$E'_u = (D_u\psi)^{-1}E_{\psi(u)},$$

where $D_u\psi$ denotes the differential of ψ at the point u. This pullback structure has some interesting properties.

Proposition 2.1.19. *Let $\psi : U \to V$ be L-quasiregular and let σ be an almost complex structure in V. Then,*

(a) $K(\psi^*\sigma) \leq L \cdot K(\sigma)$.

(b) The almost complex structure $\psi^*\sigma_0$ is given by the measurable Beltrami form $\mu_\psi = \partial_{\bar{z}}\psi/\partial_z\psi$.

It follows from the previous proposition and the fact that holomorphic maps are 1-quasiregular that pulling back an almost complex structure by a holomorphic map does not modify its dilatation. Indeed, the following holds.

Proposition 2.1.20. *Let $f : U \to V$ be holomorphic and let σ be an almost complex structure in V. Then, $K(f^*\sigma) = K(\sigma)$. Moreover, if μ denotes the Beltrami coefficient of the almost complex structure σ, then the following formula holds.*

$$f^*\mu(u) = \mu(f(u))\frac{\overline{\partial_z f(u)}}{\partial_z f(u)}. \tag{2.1}$$

It is important to know whether an almost complex structure is preserved under the iteration of a quasiregular map. The following definition introduces formally this concept.

Definition 2.1.21. Let σ be a complex structure in U and let $\psi : U \to U$ be a quasiregular map. We say that σ is *ψ-invariant* if $\psi^*\sigma = \sigma$.

The following result, due to H. Weyl, tells us that quasiconformal maps which leave invariant the standard complex structure are conformal. It is a particular case of Weyl's Lemma, which states that an L^1_{loc} solution of the Laplace equation in the distributional sense is a smooth function. For a proof of this particular case see [Ahl06, p. 16].

Theorem 2.1.22 (Weyl's Lemma)**.** *If ψ is 1-quasiconformal, then ψ is conformal. In other words, a quasiconformal map ψ is conformal if and only if $\psi^*\sigma_0 = \sigma_0$. Likewise, a map ψ is holomorphic if and only if it is* 1*-quasiregular.*

We now introduce the main theorem in quasiconformal surgery. It is due to C. Morrey [Mor38], B. Bojarski [Boy57], L. Ahlfors and L. Bers [AB60]. It gives conditions under which the Beltrami equation $\partial_z\psi(z)\mu(z) = \partial_{\bar{z}}\psi(z)$ has a quasiconformal solution ψ.

Theorem 2.1.23 (Measurable Riemann Mapping Theorem)**.** *Let $U \subset S$, where $S = \mathbb{C}$ or $\widehat{\mathbb{C}}$, be an open simply connected set (resp. $U = S$). Let σ be an almost complex structure on U with Beltrami coefficient μ. Suppose that the dilatation of σ is uniformly bounded, that is $K(\sigma) < \infty$, or equivalently that the essential supremum of $|\mu|$ on U is bounded away by one, i.e., $||\mu||_\infty := k < 1$. Then μ is integrable, i.e., there exists a quasiconformal map $\psi : U \to \mathbb{D}$ (resp. $\psi : S \to S$), which solves the Beltrami equation, i.e., such that*

$$\partial_z\psi(z)\,\mu(z) = \partial_{\bar{z}}\psi(z)$$

for almost every $z \in U$. Moreover, ψ is unique up to post-composition with automorphisms of $\mathbb{D}$ (resp. automorphisms of S).

The previous theorem may be used to obtain holomorphic maps from quasiregular ones which preserve almost complex structures as follows. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a quasiregular map and σ be a φ-invariant bounded almost complex structure. Let ψ be the integrating map of σ given by the previous theorem. Then, the quasiregular map $\psi \circ \varphi \circ \psi^{-1}$ preserves the standard complex structure σ_0. Therefore, it is holomorphic by means of Weyl's Lemma.

The following result tells us about the dependence on parameters of the integrating map ψ.

Theorem 2.1.24 (Dependence on parameters)**.** *Let Λ be an open subset of $\mathbb{C}^n$, $n \geq 1$. Let S be a Riemann surface isomorphic to $\mathbb{C}$ (or $\hat{\mathbb{C}}$), and let $(\mu_\rho)_{\rho\in\Lambda}$ be a family of measurable Beltrami forms on S. Suppose that $\rho \to \mu_\rho(s)$ is holomorphic (respectively continuous, differentiable, real analytic) in ρ for each fixed $s \in S$ (whenever defined) and that there is a $k < 1$ such that $||\mu_\rho||_\infty \leq k$ for all $\rho \in \Lambda$. Let $\psi_\rho : S \to \mathbb{C}$ (or $\hat{\mathbb{C}}$) be the unique quasiconformal homeomorphism, normalized appropriately, which integrates μ_ρ . Then for any fixed $s \in S$ the map $\rho \to \psi_\rho(s)$ is holomorphic (respectively continuous, differentiable, real analytic) in ρ.*

Equivalent conditions for the dependence on parameters may be found in [AIM09, Lem. 5.3.5].

2.2 Quasiconformal surgery in orientation reversing maps

In this thesis we work with a rational family which is symmetric with respect to the unit circle in the sense that, given any map B of the family, $B(z) = \mathcal{I} \circ B \circ \mathcal{I}(z)$ for all $z \in \mathbb{C}$, where $\mathcal{I}(z) = 1/\overline{z}$ denotes the reflection with respect $\mathbb{S}^1$. The symmetry with respect to the unit circle $\mathcal{I}$ is an orientation reversing homeomorphism of the complex plane onto itself. Therefore, it is important for us to know how symmetries are preserved in almost complex structures.

In this section we introduce the so called antiquasiconformal maps, which are the orientation reversing analogous of quasiconformal maps. Afterwards we explain how to pull back almost complex structures by antiquasiconformal maps. We finally describe using orientation reversing maps how to obtain quasiconformal maps which are symmetric with respect to the unit circle or the real line.

Antiquasiconformal maps

Let $\widetilde{\mathcal{I}}(z) = \overline{z}$ be the reflection with respect to $\mathbb{R}$. Given a function $f : U \to V$, we denote by $\overline{f}$ the function $\overline{f} : U \to \widetilde{\mathcal{I}}(V)$ so that $\overline{f}(z) = \overline{f(z)}$, where $\widetilde{\mathcal{I}}(V) = \{z \mid \overline{z} \in V\}$. If not specified, the domains of definition of a function are taken so that it is well defined.

Definition 2.2.1 (First definition of antiquasiconformal map)**.** Let $K \geq 1$ and let U and V be two domains. We say that a map $\psi : U \to V$ is K-*antiquasiconformal* if $\overline{\psi} : U \to \widetilde{\mathcal{I}}(V)$ is K-quasiconformal. We say that a function is *antiquasiconformal* if it is K-*antiquasiconformal* for some $K \geq 1$.

Comparing to the analytic and geometric definitions of quasiconformal mappings (Definitions 2.1.1 and 2.1.3), it is not difficult to see that the previous definition is equivalent to the following two ones.

Definition 2.2.2 (Analytic definition of antiquasiconformal map)**.** Let $K \geq 1$ and let U and V be two domains. We say that a map $\psi : U \to V$ is K-*antiquasiconformal* if the following conditions hold.

(a) ψ is a homeomorphism.

(b) The partial derivatives $\partial_z \psi$ and $\partial_{\overline{z}} \psi$ exists in the sense of distributions and are locally square integrable (i.e., belong to L^2_{loc}).

(c) If $k := \frac{K-1}{K+1}$, then $|\partial_z \psi| \leq k |\partial_{\overline{z}} \psi|$ in L^2_{loc}, i.e., almost everywhere.

Definition 2.2.3 (Geometric definition of antiquasiconformal map)**.** Let U and V be two domains in $\mathbb{C}$ and let $K \geq 1$. We say that a map ψ is K-*antiquasiconformal* if and only if ψ is an orientation reversing homeomorphism so that, for any annulus A compactly contained in U, the following inequality is satisfied

$$\frac{1}{K}\mathrm{mod}(A) \leq \mathrm{mod}(\psi(A)) \leq K\mathrm{mod}(A).$$

This last definition is obtained from Definition 2.1.3 by noticing that given an annulus A, $\operatorname{mod}(A) = \operatorname{mod}(\widetilde{\mathcal{I}}(A))$. The next proposition follows easily from the geometric definition of quasiconformal and antiquasiconformal maps. It mainly tells us that the composition of antiquasiconformal maps and quasiconformal maps works as one would expect.

Proposition 2.2.4. *Let $U, V, W \subset \mathbb{C}$ be open domains and let $K_1, K_2 \geq 1$. Consider $\psi_1 : U \to V$ and $\psi_2 : V \to W$. Then,*

(a) If ψ_1 is K_1-antiquasiconformal and ψ_2 is K_2-antiquasiconformal, then $\psi_2 \circ \psi_1$ is K_1K_2-quasiconformal.

(b) If ψ_1 is K_1-antiquasiconformal and ψ_2 is K_2-quasiconformal, then $\psi_2 \circ \psi_1$ is K_1K_2-antiquasiconformal.

(c) If ψ_1 is K_1-quasiconformal and ψ_2 is K_2-antiquasiconformal, then $\psi_2 \circ \psi_1$ is K_1K_2-antiquasiconformal.

As a corollary of this proposition, we obtain the following equivalent definition of antiquasiconformal map. We use the fact that the map $\widetilde{\mathcal{I}}(z) = \overline{z}$ is anticonformal and, hence, 1-antiquasiconformal.

Definition 2.2.5 (Fourth definition of antiquasiconformal map)**.** Let $K \geq 1$ and let U and V be two domains. We say a map that $\psi : U \to V$ is K-*antiquasiconformal* if $\widetilde{\psi} \circ \widetilde{\mathcal{I}} : \widetilde{\mathcal{I}}(U) \to V$, given by $\widetilde{\psi}(z) = \psi(\overline{z})$, is K-quasiconformal.

We now define the concept of antiquasiregular map.

Definition 2.2.6. We say that a map $\psi : U \to V$ is K-*antiquasiregular* if $\overline{\psi}$ is K-quasiregular or, equivalently, if ψ is K-antiquasiconformal everywhere in U except in a discrete set of points.

Pullback by orientation reversing maps

We now describe how to pull back almost complex structures under orientation reversing maps. If $f : U \to V$ is antiquasiconformal, then its differential $D_u f : T_u U \to T_{f(u)} V$ acts on the tangent bundles of U and V. This differential is used to pull back an almost complex structure σ in V obtaining an new almost complex structure $f^{\circledast}\sigma$ in U. As in the orientation preserving case (see Section 2.1), to almost every point u in U, we associate a new ellipse E'_u given by

$$E'_u = (D_u f)^{-1} (E_{f(u)}),$$

where $D_u f$ denotes the differential of f at the point u. We denote it with $\circledast$ instead of $*$ to remark that we are pulling back by orientation reversing maps. They are conceptually the same and we will not differentiate them throughout the thesis but we keep this notation in this introductory section.

The following proposition describes how the pullback under an antiquasiconformal map modifies a Beltrami coefficient of an almost complex structure. It is based on the fact that $(D_u f)^{-1} (E_{f(u)}) = \left(D_u \overline{f}\right)^{-1} (\overline{E_{f(u)}})$ (c.f. [BF14, Exercises 1.1.4, 1.1.5, 1.2.1 and 1.2.2]).

Proposition 2.2.7. *Let $\psi : U \to V$ be an antiquasiconformal map and let μ be a Beltrami coefficient defined in V. Then $\psi^{\circledast}\mu$ satisfies*

$$\psi^{\circledast}\mu = \overline{\psi}^{\,*}\overline{\mu}, \tag{2.2}$$

where $\overline{\mu}(z) = \overline{\mu(z)}$.

The next result follows from the previous proposition.

Proposition 2.2.8. *Let $\psi : U \to V$ be an antiholomorphic map and let σ be an almost complex structure defined in V with dilatation K_σ. Then $K_{\psi^{\circledast}\sigma} = K_\sigma$.*

Proof. Let μ be the Beltrami coefficient of σ. It is enough to notice that $\overline{\psi}$ is holomorphic and that $K_\mu = K_{\overline{\mu}}$. Then, the result follows from Proposition 2.2.7. □

Proposition 2.2.9 (Weyl's Lemma for antiquasiregular maps)**.** *Let $\psi : U \to V$ be an antiquasiregular map and let σ_0 denote the standard complex structure in V. If $\psi^{\circledast}\sigma_0 = \sigma_0$, then ψ is antiholomorphic.*

Proof. Let μ_0 denote the Beltrami coefficient of σ_0. By Proposition 2.2.7, we have

$$\mu_0 = \psi^{\circledast}\mu_0 = \overline{\psi}^{\,*}\overline{\mu_0} = \overline{\psi}^{\,*}\mu_0.$$

Hence, $\overline{\psi}^{\,*}\mu_0 = \mu_0$. Since $\overline{\psi}$ is quasiregular we conclude by Weyl's Lemma (Theorem 2.1.22) that $\overline{\psi}$ is holomorphic. Therefore, ψ is antiholomorphic.

□

Symmetries

We denote by $\mathcal{I}(z) = 1/\overline{z}$ and $\widetilde{\mathcal{I}}(z) = \overline{z}$ the reflections with respect to the unit circle and the real line, respectively. A holomorphic map $f : U \to V$, is said to be symmetric with respect to the unit circle (resp. the real line) if $\mathcal{I}(U) = U$ (resp. $\widetilde{\mathcal{I}}(U) = U$) and $f(z) = \mathcal{I} \circ f \circ \mathcal{I}(z)$ (resp. $f(z) = \widetilde{\mathcal{I}} \circ f \circ \widetilde{\mathcal{I}}(z)$) for all z in U. Throughout this subsection we assume that all domains of definition of the maps are symmetric in the above sense.

Definition 2.2.10. An almost complex structure σ is said to be *symmetric* with respect to the unit circle (resp. the real line) if and only if $\mathcal{I}^{\circledast}\sigma = \sigma$ (resp. $\widetilde{\mathcal{I}}^{\circledast}\sigma = \sigma$).

Sometimes it is useful to check explicitly if an almost complex structure σ is symmetric using its Beltrami coefficient μ. The following proposition provides a necessary and sufficient condition so that an almost complex structure is symmetric.

Proposition 2.2.11. *An Almost complex structure σ defined in an open set U with Beltrami coefficient μ is symmetric with respect to the unit circle if and only if the equality*

$$\mu(z) = \overline{\mu(1/\overline{z})}\frac{z^2}{\overline{z}^2}$$

holds for almost every $z \in U$. Respectively, σ is symmetric with respect to the real line if and only if $\mu(z) = \overline{\mu(\overline{z})}$ for almost every $z \in U$.

Proof. The proof follows directly from Equation (2.1) and Definition 2.2.7. □

The following lemma tells us that if we pull back a symmetric almost complex structure by a symmetric map we obtain another symmetric almost complex structure. By "symmetric" we mean symmetric with respect to $\mathbb{S}^1$ or $\mathbb{R}$.

Lemma 2.2.12. *Let $f : U \to V$ be a quasiconformal map which is symmetric with respect to the unit circle (resp. the real line) and let σ be a symmetric almost complex structure defined in V. Then the almost complex structure $f^*\sigma$ is also symmetric with respect to the unit circle (resp. the real line).*

Proof. Assume f and σ are symmetric with respect to $\mathbb{S}^1$. Then $f(z) = \mathcal{I} \circ f \circ \mathcal{I}(z)$ for all $z \in U$. Therefore, $f^*\sigma = (\mathcal{I} \circ f \circ \mathcal{I})^*\sigma = \mathcal{I}^{\circledast} f^* \mathcal{I}^{\circledast}\sigma = \mathcal{I}^{\circledast} f^*\sigma$ as we wanted to prove. The symmetric case with respect to $\mathbb{R}$ is analogous.

□

Finally we introduce a lemma which shows that given a symmetric almost complex structure σ with bounded dilatation whose domain of definition is the whole Riemann sphere, then the quasiconformal map given by the Measurable Riemann Mapping Theorem (Theorem 2.1.23) can also be chosen to be symmetric.

Lemma 2.2.13. *Let σ be an almost complex structure with bounded dilatation defined in $\widehat{\mathbb{C}}$ and symmetric with respect to the unit circle. Let ϕ be the integrating map given by the Measurable Riemann Mapping Theorem (Theorem 2.1.23) and normalized so that it fixes 0 and ∞ and so that $\phi(x_0) = x_1$, where $x_0, x_1 \in \mathbb{S}^1$. Then, ϕ is symmetric with respect to the unit circle.*

In the above setting, if σ is symmetric with respect to the real line, ϕ fixes two points z_1 and $z_2 = \widetilde{\mathcal{I}}(z_1) \notin \mathbb{R}$ and is so that $\phi(x_0) = x_1$ with $x_0, x_1 \in \mathbb{R}$, then ϕ is symmetric with respect to the real line.

Proof. We give the proof for the circular case, being the real one analogous. Consider the quasiconformal map $\widehat{\phi}$ given by $\mathcal{I} \circ \phi \circ \mathcal{I}$. It fixes 0 and ∞ and maps x_0 to x_1. If we see that it also integrates σ then we are done by uniqueness of the integrating map up to composition with Möbius transformations. Indeed,

$$\widehat{\phi}^*\sigma_0 = (\mathcal{I} \circ \phi \circ \mathcal{I})^*\sigma_0 = \mathcal{I}^{\circledast}\phi^*\mathcal{I}^{\circledast}\sigma_0 = \mathcal{I}^{\circledast}\phi^*\sigma_0 = \mathcal{I}^{\circledast}\sigma = \sigma.$$

□

2.3 Polynomial and antipolynomial-like maps

In the dynamical plane of many holomorphic maps, we sometimes find certain subsets that look like the filled Julia set of a polynomial (see Figure 2.1). Recall that the filled Julia set of a polynomial P is defined as the set of orbits which do not escape to infinity under iteration of P. The theory of polynomial-like mappings, developed by Douady and Hubbard in [DH85b], describes this phenomenon in a rigorous way. It explains how an arbitrary holomorphic map $f : S \to S$ may act locally like a polynomial.

Definition 2.3.1. A triple $(f; U, V)$ is called a *polynomial-like* (resp. *antipolynomial-like*) *mapping* of degree d if U and V are bounded simply connected subsets of the plane isomorphic to discs, $\overline{U} \subset V$ and $f : U \to V$ is holomorphic (resp. antiholomorphic) and proper of degree d.

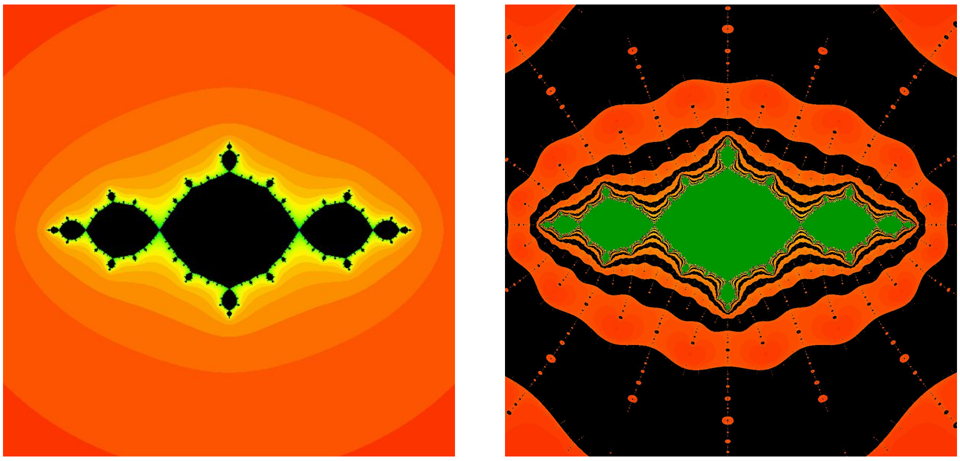


Figure 2.1: The left figure shows the dynamical plane of the quadratic polynomial $z^2 - 1$. The right figure shows a zoom in the dynamical plane of a Blaschke product of the form (1).

We remark that, for every (anti)polynomial P of degree d, there exists an $R > 0$ so that $(P; \mathbb{D}_R, P(\mathbb{D}_R))$ is a (anti)polynomial-like mapping (see Figure 2.2).

Definition 2.3.2. We define the *filled Julia set* of a (anti)polynomial-like map $(f; U, V)$ as

$$\mathcal{K}_f = \bigcap_{n>0} f^{-n}(V) = \{z \in U | \, f^n(z) \in U \ \ \forall n \geq 0\}.$$

Definition 2.3.3. Two (anti)polynomial-like maps $(f; U, V)$ and $(f'; U', V')$ are said to be *hybrid equivalent* if there exist neighborhoods U_f and $U_{f'}$ of $\mathcal{K}_f$ and $\mathcal{K}_{f'}$ respectively, and a quasiconformal conjugacy $\phi : U_f \to U_{f'}$ between f and f' such that $\partial_{\bar{z}}\phi = 0$ almost everywhere in $\mathcal{K}_f$.

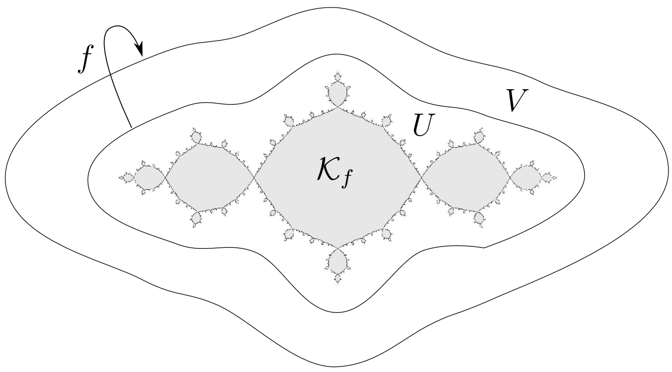


Figure 2.2: A polynomial-like map.

Hybrid equivalence is the strongest type of conjugacy that one can define for polynomials having connected filled Julia sets. Indeed, the following result holds.

Theorem 2.3.4. *Let P and Q be two polynomials with connected Julia sets. Then, P and Q are hybrid equivalent if, and only if, they are affinely conjugate.*

The following theorem is the main result of polynomial-like theory. It tells us that all polynomial-like maps are hybrid equivalent to a polynomial of the same degree.

Theorem 2.3.5 (The Straightening Theorem)**.** *Every degree d polynomial-like mapping $(f; U, V)$ is hybrid equivalent to a polynomial P of degree d. If $\mathcal{K}_f$ is connected, P is unique up to affine conjugation.*

The antipolynomial-like theory was first introduced by Milnor [Mil09] in order to study why small "copies" of the Tricorn appear in the parameter plane of real cubic polynomials. Hubbard and Schleicher [HS12] used this theory afterwards in the study of the Multicorns, the connectedness locus of the antipolynomial maps $p_{c,d}(z) = \overline{z}^d + c$. They stated the Antiholomorphic Straightening Theorem. Its proof is analogous to the one of the Straightening Theorem. We prove, for the sake of completeness, the existence of the hybrid equivalency.

Theorem 2.3.6 (Antiholomorphic Straightening Theorem)**.** *Every antipolynomial-like mapping $(f; U, V)$ of degree d is hybrid equivalent to an antipolynomial P of degree d. If $\mathcal{K}_f$ is connected, then P is unique up to affine conjugation.*

Proof. The proof of the theorem is analogous to the one of the usual Straightening Theorem (Theorem 2.3.5). It mainly consists in gluing $z \to \overline{z}^d$ to f.

We may assume, without loss of generality, that the sets U and V are bounded by analytic curves. If this is not the case, we can define a new antipolynomial-like map $(f; \widetilde{U}, \widetilde{V})$ with $\widetilde{U} \subset U \subset \widetilde{V} \subset V$ which has the same filled Julia set than the original one and is therefore hybrid equivalent to it.

Pick $\rho > 1$. Let $\mathcal{R} : \hat{\mathbb{C}} \setminus \overline{V} \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_{\rho^d}$ be a Riemann map fixing infinity. Then, $\mathcal{R}$ extends to the boundary as an analytic map. Indeed, it extends continuously to the boundary by Charatheodory's Theorem. Since ∂V is an analytic curve, this continuation is analytic (see [BF14, Thm. 2.9]). Let $\psi_1 : \partial V \to \mathbb{S}^1_{\rho^d}$ be the extension map. Since $\overline{\psi_1 \circ f} : \partial U \to \mathbb{S}^1_{\rho^d}$ is a real analytic map of degree d, we can choose a C^1 (or real analytic) lift $\psi_2 : \partial U \to \mathbb{S}^1_{\rho}$ so that $\psi_1(f(z)) = \overline{\psi_2(z)^d}$.

Now consider the annuli $A_0 = V \setminus \overline{U}$ and $\mathbb{A}_{\rho,\rho^d} = \{\eta;\, \rho < |\eta| < \rho^d\}$. By Proposition 2.1.15, there exists $\psi : \partial A_0 \to \partial \mathbb{A}_{\rho,\rho^d}$ quasiconformal such that $\psi|_{\partial U} = \psi_2$ and $\psi|_{\partial V} = \psi_1$. Now we define our model map F as

$$F(z) = \begin{cases} f(z) & \text{for} \quad z \in U \\ \mathcal{R}^{-1}\left(\overline{\psi(z)^d}\right) & \text{for} \quad z \in V \setminus U \\ \mathcal{R}^{-1}\left(\overline{\mathcal{R}(z)^d}\right) & \text{for} \quad z \in \mathbb{C} \setminus V. \end{cases}$$

The map $F(z)$ is antiquasiregular and has topological degree d by construction. We continue by defining an F-invariant Beltrami coefficient μ. Observe that F is an antiholomorphic map everywhere except on $A_0 = V \setminus U$. Notice also that the orbit of a point z can go at most once through A_0. Denote $A_n = \{z \mid f^n(z) \in A_0\}$. Thus, it is enough to define

$$\mu(z) = \begin{cases} \overline{\psi}^{\circledast}\mu_0(z) & \text{for} \quad z \in A_0 \\ (f^n)^{\circledast}\mu(z) & \text{for} \quad z \in A_n \\ \mu_0(z) & \quad\quad elsewhere. \end{cases}$$

By construction, $F^{\circledast}\mu = \mu$. Moreover, since f is an antiholomorphic function and hence preserves dilatation, we have that $K_\mu = K_{\overline{\psi}^{\circledast}\mu_0} = K_{\psi^*\mu_0}$. Thus, μ has bounded dilatation. Let ϕ be an integrating map given by the Measurable Riemann Mapping Theorem (Theorem 2.1.23) fixing infinity. Then, $\phi^*\mu_0 = \mu$. Finally, define $P = \phi \circ F \circ \phi^{-1}$. By construction, $P^{\circledast}\mu_0 = \mu_0$. Hence, $\overline{P}^*\mu_0 = \mu_0$. Then, by Weyl's Lemma (Theorem 2.1.22), $\overline{P}$ is an entire map of topological degree d and, hence, a polynomial of degree d.

The proof finishes by observing that $\mu \equiv 0$ in K_f and, hence $\phi|_{K_f}$ is conformal.

□

Chapter Three 3

Dynamical Plane

We consider the family of degree 4 products $B_a(z) = z^3(z-a)/(1-\overline{a}z)$. As all Blaschke products, B_a leave the unit circle invariant. This fact has some relevant consequences on the dynamics. They have two free critical points. If they lie in the unit circle their orbits are not related and may lead to different stable dynamics. On the other hand, if they do not lie in the unit circle, they have symmetric orbits and therefore their asymptotic dynamics must also be symmetric.

The aim of this chapter is to give an overview of the dynamical plane of the Blaschke products B_a. In Section 3.1 we introduce the basic dynamical properties of these maps. In Section 3.2 we prove that the Fatou set of B_a cannot have Herman rings and give a characterization of the connectivity of the Julia set $\mathcal{J}(B_a)$ for $|a| \geq 2$.

3.1 The Blaschke family

In more generality, we consider the degree 4 Blaschke products of the form

$$B_{a,t}(z) = e^{2\pi i t} z^3 \frac{z-a}{1-\bar{a}z}, \tag{3.1}$$

where $a \in \mathbb{C}$ and $t \in \mathbb{R}/\mathbb{Z}$. Since $B_{a,t}$ leaves invariant the unit circle, it is symmetric with respect $\mathbb{S}^1$, i.e., $B_{a,t}(z) = \mathcal{I} \circ B_{a,t} \circ \mathcal{I}(z)$ where $\mathcal{I}(z) = 1/\bar{z}$.

The next lemma tells us that, for the purpose of classification, we can get rid of the parameter t. The proof is straightforward.

Lemma 3.1.1. *Let $\alpha \in \mathbb{R}$ and let $\eta(z) = e^{-2\pi i \alpha} z$. Then η conjugates the maps $B_{a,t}$ and $B_{ae^{-2\pi i \alpha}, t+3\alpha}$. In particular, $B_{a,t}$ is conjugate to $B_{ae^{\frac{2\pi i t}{3}},0}$.*

Hence, we focus on the study of the family

$$B_a(z) = z^3 \frac{z-a}{1-\bar{a}z} \tag{1}$$

for values $a, z \in \mathbb{C}$. We first give a few comments on $B_a|_{\mathbb{S}^1}$. For $|a| > 1$, the circle map $B_a|_{\mathbb{S}^1}$ has degree 2 in the sense that its lift F_a satisfies $F_a(x+1) = F_a(x) + 2$ for all $x \in \mathbb{R}$. Indeed, it is given by

$$F_a(x) = 3x + \frac{1}{2\pi i} \log\left(\frac{e^{2\pi i x} - a}{1 - \bar{a}e^{2\pi i x}}\right), \tag{3.2}$$

with $x \in \mathbb{R}$. The logarithmic part of the formula corresponds to the lift l_a of the Möbius map $L_a(z) = (z-a)/(1-\overline{a}z)$ which restricts to a homeomorphism of $\mathbb{S}^1$ and, hence,

$l_a(x+1) = l_a(x) \pm 1$. Since L_a has a pole in $\mathbb{D}$, it maps $\mathbb{D}$ to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Therefore $l_a(x+1) = l_a(x) - 1$ and $F_a(x+1) = F_a(x) + 2$. If $|a| = 1$ then $L_a(z) \equiv -a$ is constant and $B_a|_{\mathbb{S}^1}$ has degree 3. If $|a| < 1$ then $B_a|_{\mathbb{S}^1}$ has degree 4 since $l_a(x+1) = l_a(x) + 1$. Therefore, $B_a|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ cannot be a degree 1 covering (and hence conjugate to an irrational rotation). Consequently, every point $z \in \mathbb{S}^1$ either belongs to the Julia set or to the basin of attraction of an attracting or parabolic cycle $\{z_0, ..., z_{p-1}\} \in \mathbb{S}^1$. For further details on the dynamics of $B_a|_{\mathbb{S}^1}$ see Chapter 6.

To have an idea of which stable dynamics the maps B_a may have, we should control the critical orbits. Since these rational maps have degree 4, they have 6 critical points counted with multiplicity (see Corollary 1.3.2). The fixed points $z = 0$ and $z = \infty$ are critical points of multiplicity 2 and hence superattracting fixed points of local degree 3. The other two critical points, denoted by $c_\pm$, are given by

$$c_\pm := c_\pm(a) := a \cdot \frac{1}{3|a|^2}\left(2 + |a|^2 \pm \sqrt{(|a|^2-4)(|a|^2-1)}\right). \tag{3.3}$$

If $1 < |a| < 2$, then the critical points satisfy $c_+ = a \cdot k$ and $c_- = a \cdot \overline{k}$, where

$$k = \frac{1}{3|a|^2}\left(2 + |a|^2 + i\sqrt{(4-|a|^2)(|a|^2-1)}\right) \in \mathbb{C}.$$

In this case c_+ and c_- are not symmetric with respect to the unit circle and it follows that $|c_+| = |c_-| = 1$ since otherwise its symmetric points would lead to two extra critical points. On the other hand, if $|a| > 2$ or $|a| < 1$ the critical points c_+ and c_- are free and satisfy $|c_+| \geq 1$, $|c_-| \leq 1$ and $c_- = 1/\overline{c_+}$. Consequently, their orbits are symmetric with respect to $\mathbb{S}^1$. The following lemma shows that the function may be reparametrized in terms of the position of the critical points if $|a| \geq 2$ or $|a| < 1$.

Lemma 3.1.2. *Given a Blaschke product $B_{a,t}$ as in (3.1) with $|a| \geq 2$ or $|a| < 1$, the parameter a is continuously determined by the critical points $c_\pm$. Moreover, if the image $B_{a,t}(z_0) \neq \{0, \infty\}$ of a point $z_0 \in \mathbb{C}^*$ is fixed, then t depends continuously on a.*

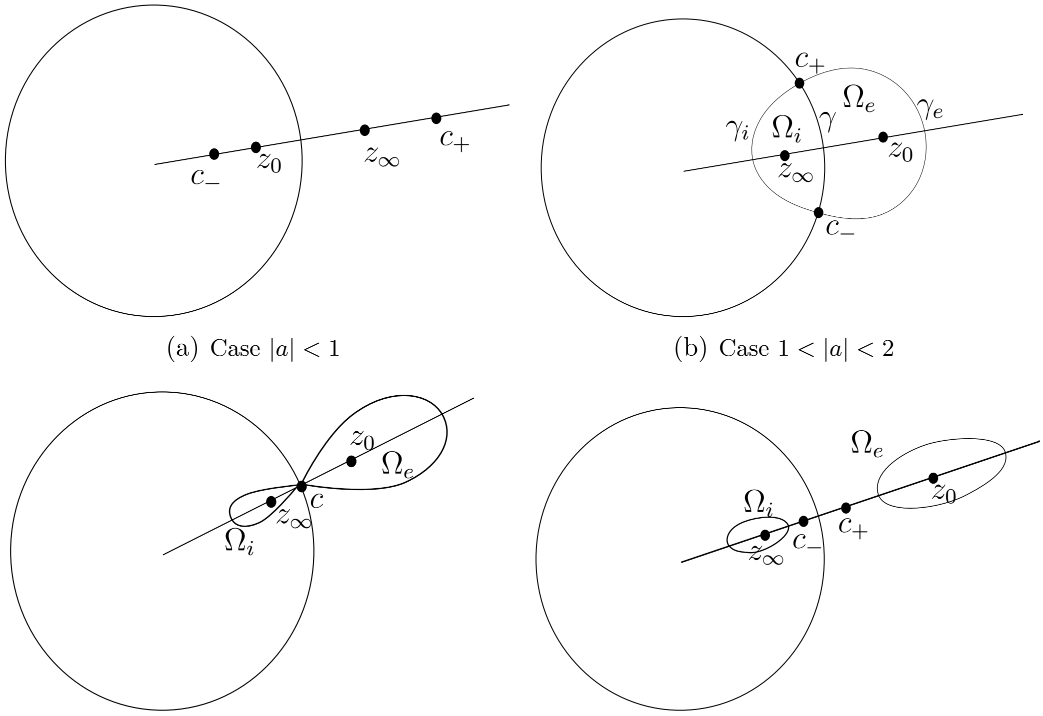
Proof. The continuous dependence of t with respect to a is clear. Let $a = r_a e^{2\pi i\alpha}$, where $\alpha \in \mathbb{R}/\mathbb{Z}$ and $r_a \geq 2$ (resp. $r_a < 1$). It follows from Equation (3.3) that the critical points c_+ and c_- have the same argument α as a. It is left to see that r_a depends continuously on $|c_+| = r_c$. It follows from symmetry that $|c_-| = 1/r_c$. Consider $R(r_c) = r_c + 1/r_c$. For $r_c \geq 1$, R is a strictly increasing function which satisfies $R(1) = 2$. Using Equation (3.3) we have

$$R(r_c)e^{2\pi i\alpha} = c_+ + c_- = \frac{2a}{3|a|^2}(2 + |a|^2) = \frac{2}{3}\frac{r_a e^{2\pi i\alpha}}{r_a^2}(2 + r_a^2),$$

and, therefore, $r_a \cdot R(r_c) = 2(2 + r_a^2)/3$. This quadratic equation yields two solutions $r_{a_\pm} = (3R \pm \sqrt{9R^2 - 32})/4$. The solution $r_{a_+}(R)$ takes the value 2 for $R = 2$ and is strictly increasing and tends to infinity when R tends to infinity. The solution $r_{a_-}(R)$ takes the value 1 for $R = 2$ and is strictly decreasing and tends to zero when R tends to infinity. Therefore, each critical point $c_+ \in \mathbb{C}$, $|c_+| \geq 1$ (resp. $|c_+| > 1$), determines continuously a unique parameter a such that $|a| \geq 2$ (resp. $|a| < 1$).

□

For completeness we describe some features of the dynamics of B_a which depend on the modulus of a. The first thing to consider is whether there is a preimage of ∞ in $\mathbb{D}$. This family has a unique pole at $z_\infty = 1/\overline{a}$ and a unique zero $z_0 = a$. Their position, together with the positions of $c_\pm$, influence the global dynamics of B_a. We proceed to describe the situation depending on $|a|$ (see Figure 3.1).



(a) Case $|a| < 1$

(b) Case $1 < |a| < 2$

(c) Case $|a| = 2$

(d) Case $|a| > 2$

Figure 3.1: Different configurations of the critical points and the preimages of zero and infinity depending on $|a|$.

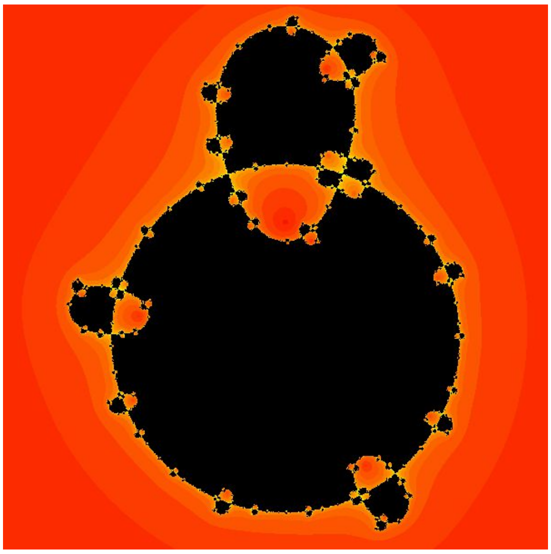
When $|a| < 1$ we have that both critical points $c_\pm$ lie on the half ray containing a. Moreover, $|c_-| < 1$ and $|c_+| > 1$. The only pole, $z_\infty = 1/\overline{a}$ has modulus greater than one. Hence, $B_a : \mathbb{D} \to \mathbb{D}$ is a holomorphic self map of $\mathbb{D}$ having $z = 0$ as a superattracting fixed point. Since, by symmetry, there is no preimage of the unit disk outside the unit circle, $B_a|_{\mathbb{D}}$ is a degree 4 branched covering. By Schwarz Lemma we have that $z = 0$ is the only attracting point of B_a in $\mathbb{D}$ and attracts all orbits in $\mathbb{D}$. Summarizing, we have:

Lemma 3.1.3. *If $|a| < 1$, $A_a(0) = A_a^*(0) = \mathbb{D}$ and by symmetry $A_a(\infty) = A_a^*(\infty) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Hence, $\mathcal{J}(B_a) = \mathbb{S}^1$.*

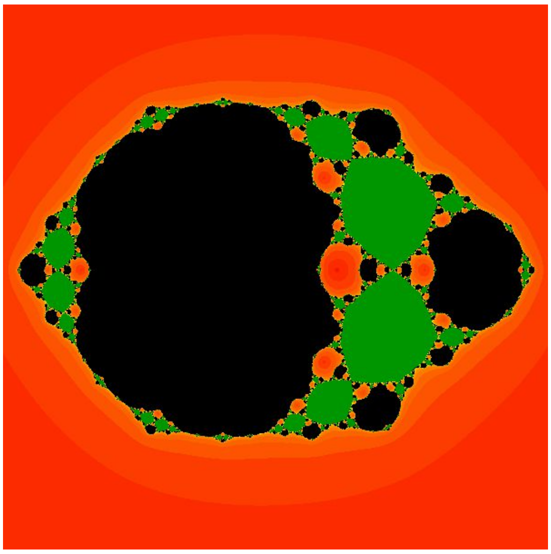
When $|a| = 1$ both critical points and the preimages of 0 and ∞ collapse at the point $z = a$, where the function is not formally defined. Everywhere else we have the equality:

$$B_a(z) = z^3 \frac{z - a}{1 - z/a} = -az^3.$$

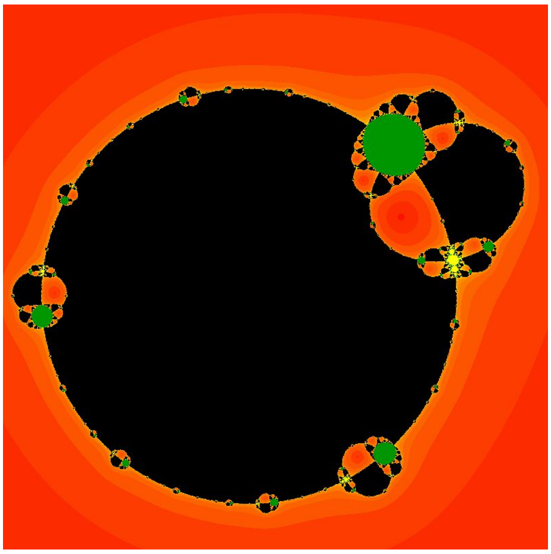
When $1 < |a| < 2$ the two critical points lie in the unit circle, i.e., $|c_\pm| = 1$ (see Figure 3.1 (b)). Consequently, the critical orbits lie in $\mathbb{S}^1$ and are not related to each other by symmetry. The circle map $B_a|_{\mathbb{S}^1}$ has no topological degree defined. Indeed, it can be proven that some points in $\mathbb{S}^1$ have 2 preimages under $B_a|_{\mathbb{S}^1}$ whereas other points have 4 preimages (see Lemma 3.1.4 below). In Figure 3.2 We show the dynamical planes of three maps B_a with $1 < |a| < 2$.



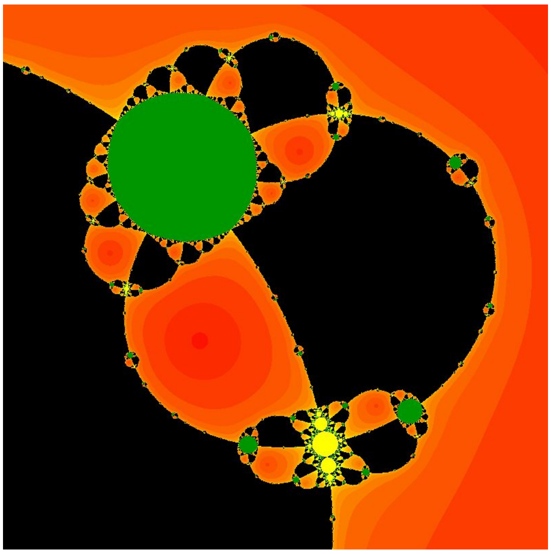
(a) Dynamical plane of $B_{3/2i}$



(b) Dynamical plane of $B_{3/2}$



(c) Dynamical plane of $B_{1.07398+0.5579i}$



(d) Zoom in (c)

Figure 3.2: Dynamical planes of three Blaschke products B_a with $1 < |a| < 2$. The colors are as follows: a scaling of red if the orbit tends to infinity, black if it tends to zero, green if the orbit accumulates on the cycle $< z_0 >$ such that c_+ lies in $A^*(< z_0 >)$ and yellow if the orbit accumulates on a cycle $< w_0 > \neq < z_0 >$ such that $c_- \in A^*(< w_0 >)$. In case (a) there are no other Fatou components than the basins of zero and infinity. In case (b) both free critical orbits accumulate on a period 2 cycle. In Figures (c) and (d) the critical orbits accumulate on two different cycles of period 1 (green) and period 4 (yellow), respectively.

Lemma 3.1.4. *Let a satisfy $1 < |a| < 2$. Then, there exists a unique preimage Ω_e of $\mathbb{D}$ not contained in $\mathbb{D}$ and, by symmetry, a unique preimage Ω_i of $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ not contained in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Moreover, $\partial\Omega_e = \gamma_e \cup \gamma$ and $\partial\Omega_i = \gamma_i \cup \gamma$, where $\gamma \subset \mathbb{S}^1$ is a curve joining the two critical points. The curve γ is mapped univalently to another semiarc $\Gamma \subset \mathbb{S}^1$ whereas the curves γ_e and γ_i are mapped univalently to $\mathbb{S}^1 \setminus \Gamma$. Consequently, the points in $\Gamma \subset \mathbb{S}^1$ have 4 preimages in $\mathbb{S}^1$ whilst the points in $\mathbb{S}^1 \setminus \Gamma$ have 2 preimages in $\mathbb{S}^1$.*

Proof. For $1 < |a| < 2$, the only preimages of the superattracting fixed points $z = 0$ and $z = \infty$ are $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $z_\infty \in \mathbb{D}$ (see Figure 3.1 (b)). Hence, there is a unique open set $\Omega_e \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ containing z_0 which is mapped conformally under B_a onto $\mathbb{D}$. Analogously, there is a unique open set $\Omega_i \subset \mathbb{D}$ containing z_∞ which is mapped conformally under B_a onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Due to symmetry, we have that $\Omega_e = \mathcal{I}(\Omega_i)$.

Since the critical points c_+ and c_- are in $\mathbb{S}^1$, there are arcs of preimages of points in $\mathbb{S}^1$ attached to the critical points. These subarcs of preimages are contained in $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$. Notice also that, due to symmetry and the fact that the critical points are simple, there is a unique subarc of preimages $\gamma_{i,+}$ attached to c_+ and contained in $\mathbb{D}$. Analogously, there is a unique subarc of preimages $\gamma_{i,-}$ attached to c_- and contained in $\mathbb{D}$. Due to the fact that $\mathbb{D} \setminus \Omega_i$ is a degree 3 branched covering of $\mathbb{D}$, we conclude that $\gamma_{i,\pm}$ are contained in $\partial\Omega_i$. Therefore, $\partial\Omega_i$ consists of the union of an arc $\gamma_i \subset \mathbb{D}$ which contains the subarcs $\gamma_{i,\pm}$ and a curve $\gamma \subset \mathbb{S}^1$. By symmetry, $\partial\Omega_e$ consists of the union of an arc $\gamma_e \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ and the curve γ. By Carathéodory's theorem $B_{a|\Omega_{e,i}}$ extends to homeomorphisms from $\partial\Omega_e$ and $\partial\Omega_i$ to $\mathbb{S}^1$ and therefore $B_a|_\gamma$ is univalent.

□

When $|a| = 2$ there is a unique critical point $c = a/2$ of multiplicity 2 in the unit circle. There are two preimages of $\mathbb{S}^1$ which meet at c (see Figure 3.1 (c)). There may or may not be an attracting or parabolic cycle in $\mathbb{S}^1$ when $|a| = 2$. The parameter might be, for example, of Misiurewicz type (i.e., the free critical point is preperiodic). In this situation the only Fatou components of B_a are the basins of $z = 0$ and $z = \infty$. We also remark for further use that the map $B_a|_{\mathbb{S}^1}$ is 2-to-1. In Figure 3.3 we show the dynamical planes of two maps B_a with $|a| = 2$.

When $|a| > 2$, as is the case when $|a| < 1$, both critical points $c_\pm$ lie on the half ray containing a and are symmetric with respect to $\mathbb{S}^1$. In this case there are two disjoint preimages of the unit circle: one of them inside $\mathbb{D}$, surrounding the pole z_∞, and the symmetric one outside surrounding the zero $z_0 = a$ (see Figure 3.1 (d)). As in the case $|a| = 2$, $B_a|_{\mathbb{S}^1}$ is 2-to-1. In Figures 3.4 and 5.3 (a) we show the dynamical planes of three maps B_a with $|a| > 2$.

3.2 Connectivity of the Julia set

The goal of this section is to prove the following theorem, which gives a characterization of the connectivity of $\mathcal{J}(B_a)$ if $|a| \geq 2$. Notice that statement (a) has already been proven in Lemma 3.1.3.

Theorem 3.2.1. *Given a Blaschke product B_a as in* (1)*, the following statements hold.*

(a) If $|a| \leq 1$, then $\mathcal{J}(B_a) = \mathbb{S}^1$.

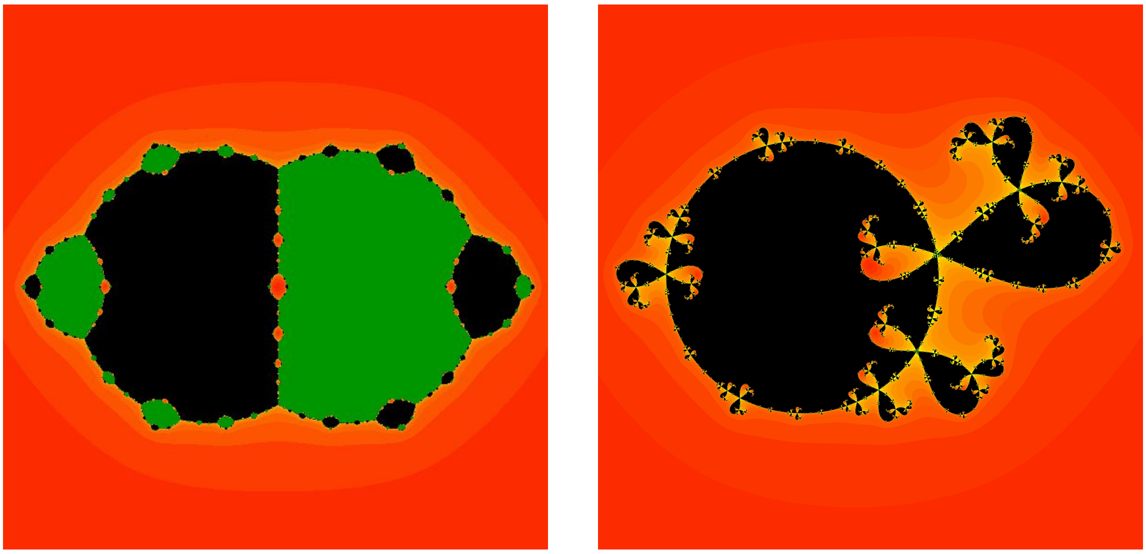


Figure 3.3: Dynamical planes of B_2 (left) and B_{a_0}, where $a_0 = 1.971917+0.333982i$, (right). In the left case the point $z = 1$ is a superattracting fixed point. The parameter a_0 has been chosen numerically so that $B_{a_0}|_{\mathbb{S}^1}$ has no attracting cycle. The colors are as in Figure 3.2.

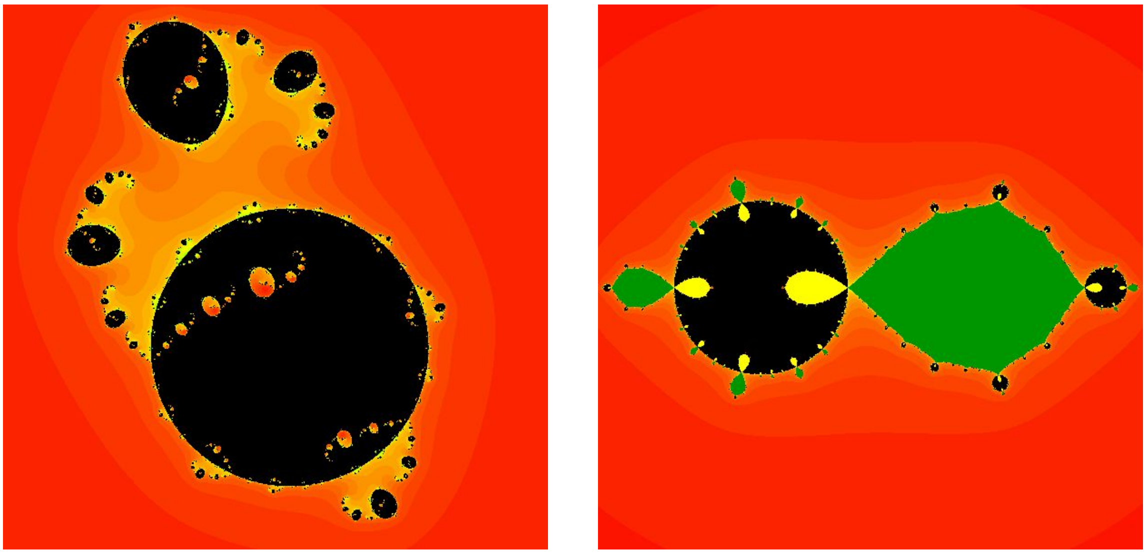


Figure 3.4: Dynamical planes of B_{a_0}, where $a_0 = -0.87 + 2.05333i$, (left) and B_4 (right). In the left case the critical point c_+ belongs to $A^*(\infty)$ and the Julia set is disconnected. In the right case each free critical orbit accumulates on a different basin of attraction. The colors are as in Figure 3.2.

(b) If $|a| > 1$*, then the connected components of* $A(\infty)$ *and* $A(0)$ *are simply connected if and only if* $c_+ \notin A^*(\infty)$*.*

(c) If $|a| \geq 2$*, then every Fatou component* U *such that* $U \cap A(\infty) = \emptyset$ *and* $U \cap A(0) = \emptyset$ *is simply connected.*

Consequently, if $|a| \geq 2$*, then* $\mathcal{J}(B_a)$ *is connected if and only if* $c_+ \notin A^*(\infty)$*.*

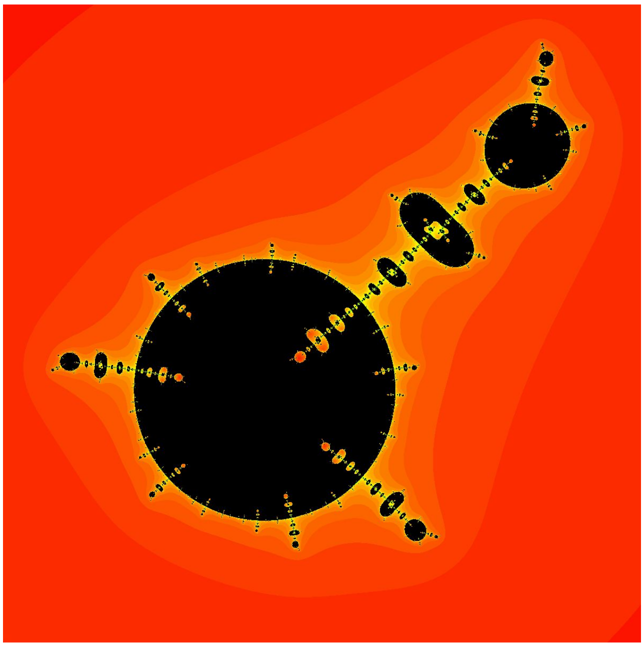


Figure 3.5: Dynamical plane of the Blaschke product B_{a_0}, $a_0 = 2.06547 + 1.91801i$, for which the critical orbit $\mathcal{O}(c_+)$ enters in $\mathbb{D}$ before escaping to infinity, so that $c_+ \in A(\infty) \setminus A^*(\infty)$. By means of Theorem 3.2.1 its Julia set is connected. Color black denotes the basin of attraction of $z = 0$ whilst the scaling from green to orange denotes the basin of attraction of $z = \infty$.

In Figure 3.5 we show an example of a Blaschke product for which c_+ lies in $A(\infty)\setminus A^*(\infty)$ and has connected Julia set by means of statement (b) of Theorem 3.2.1. The proof of Theorem 3.2.1 splits into the following three propositions, which occupy the remainder of the section.

Proposition 3.2.2. *Let B_a be as in (1) and suppose $|a| > 1$. Then, the connected components of $A(\infty)$ and $A(0)$ are simply connected if and only if $c_+ \notin A^*(\infty)$.*

Proof. By symmetry, the connected components of $A(0)$ are simply connected if and only if the ones of $A(\infty)$ are. Therefore, we focus on the simple connectivity of $A(\infty)$. By means of the Riemann-Hurwitz formula (Theorem 1.3.1) and invariance of $\mathbb{S}^1$, the connected components of $A(\infty) \setminus A^*(\infty)$ are simply connected if and only if $A^*(\infty)$ is simply connected since any connected component of $A(\infty) \setminus A^*(\infty)$ can have at most one critical point (see Corollary 1.3.3 and Figure 3.5). Therefore, it is sufficient to prove that $A^*(\infty)$ is simply connected if and only if $c_+ \notin A^*(\infty)$. Let us consider the Böttcher coordinate of the superattracting fixed point $z = \infty$ (see Theorem 1.1.12). If there is no extra critical point in $A^*(\infty)$, the Böttcher coordinate can be extended until it reaches $\partial A^*(\infty)$ and $A^*(\infty)$ is simply connected. If it does contain an extra critical point, the Böttcher coordinate can only be extended until it reaches the critical point (see Theorem 1.1.13). Let U be the maximal domain of definition of the Böttcher coordinate at ∞. Then, either ∂U consists of the union of two topological circles, say $\gamma_\pm$, which are joined in a unique point which is the critical point (see Figure 3.6 (left)), or ∂U is a topological circle containing the critical point (see Figure 3.6 (right)). If it is the last case, there is an extra preimage of $B_a(U)$ attached to the critical point. Hence,

$A^*(\infty)$ would be mapped 4 to 1 onto itself. This is not possible since B_a is of degree 4 and the only pole z_∞ is inside the unit disk and hence does not belong to $A^*(\infty)$. Let V_+ and V_- be the disjoint simply connected regions bounded by γ_+ and γ_-. The result follows by noticing that both $B_a(V_+)$ and $B_a(V_-)$ contain $\widehat{\mathbb{C}} \setminus B_a(U)$ and, hence, both of them contain the Julia set since $\mathcal{J}(B_a)$ is not empty and U is contained in $\mathcal{F}(B_a)$. $\square$

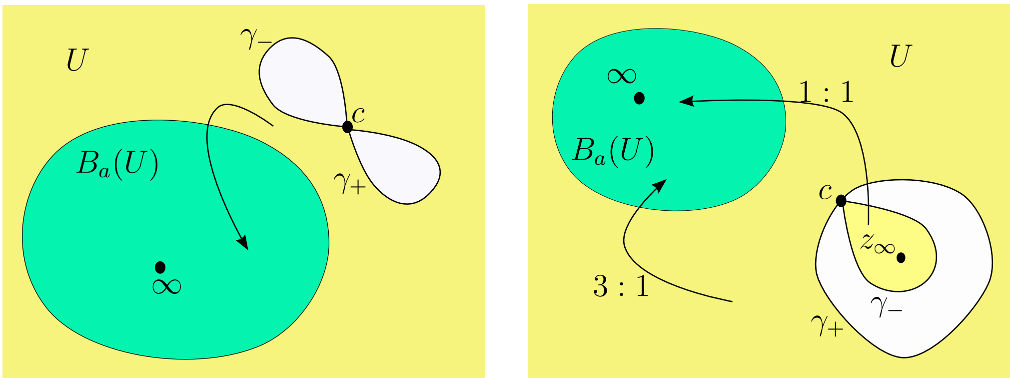


Figure 3.6: Possible positions of the critical point c on the boundary of the maximal domain of definition U of the Böttcher coordinate of $z = \infty$.

This concludes the proof of statement (b). We now begin the proof of statement (c). In propositions 3.2.3 and 3.2.4 we prove that all periodic Fatou components other than $A^*(0)$ or $A^*(\infty)$ are simply connected.

Proposition 3.2.3. *Let B_a be as in (1). Then B_a has no Herman Rings.*

Proof. Shishikura [Shi87] proved that if a rational map has a Herman ring, then it has two different critical points whose orbits accumulate on the two different components of its boundary. It follows that B_a can have at most one cycle of Herman rings. If $|a| \leq 1$, the Julia set satisfies $\mathcal{J}(B_a) = \mathbb{S}^1$ (see Lemma 3.1.3), so B_a cannot have Herman rings. If $1 < |a| \leq 2$, the two critical orbits lie in $\mathbb{S}^1$ and, hence, there can be no Herman rings.

We focus now on the case $|a| > 2$. Notice that no Herman ring can intersect $\mathbb{S}^1$ since $B_a|_{\mathbb{S}^1}$ is not a homeomorphism (see Section 3.1). Hence, by Shishikura's result and symmetry, the cycle of Herman rings would have components both inside and outside the unit disk. Thus, it would have at least one component in the preimage of the unit disk $\Omega_e = B_a^{-1}(\mathbb{D}) \setminus \mathbb{D}$ and another one in the preimage of the complement of the unit disk $\Omega_i = B_a^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$ (see Figure 3.1 (d)). Recall that Ω_e is a simply connected set disjoint from $\mathbb{S}^1$. Moreover, all its preimages are bounded, none of them can intersect the unit circle, and all of them are simply connected by Corollary 1.3.3. Every component of the cycle of Herman rings is contained in a preimage of Ω_e of some order $n \geq 0$. We claim that such a cycle must have a component which surrounds either the unit disk or Ω_e. If this is so, this component cannot be contained in a simply connected preimage of Ω_e, which leads to a contradiction.

Let $\mathcal{I}(z) = 1/\overline{z}$ be the reflection with respect to $\mathbb{S}^1$. To prove the claim observe that, due to symmetry, if A is a component of the cycle of Herman rings, then so is $\mathcal{I}(A)$. Moreover, since infinity is a superattracting fixed point, all components are bounded and at least one of them, say A', surrounds the pole z_∞ (by the Maximum Modulus Principle). Recall that z_∞ is contained in Ω_i and that, again by symmetry, $\mathcal{I}(\Omega_i) = \Omega_e$. Then, either A' surrounds the unit disk or surrounds Ω_i or is contained in Ω_i. In the first case we are done. In the second case $\mathcal{I}(A')$ surrounds Ω_e and we are also done. In the third case, $B_a(A')$ separates infinity and the unit disk and, hence, surrounds the unit disk. This finishes the proof.

□

Proposition 3.2.4. *Let B_a be as in (1) with $|a| \geq 2$. Let $< z_0 >$ be an attracting, superattracting or parabolic p-cycle of B_a other than $\{0\}$ or $\{\infty\}$. Then $A^*(< z_0 >)$ is simply connected.*

Proof. Case 1: First we consider the case in which each connected component of the immediate basin of attraction contains at most one critical point (counted without multiplicity). For the attracting case consider a linearizing domain $\mathcal{A}$ coming from Kœnigs linearization around z_0 (see Theorem 1.1.10). The subsequent preimages U_n defined as the components of $B_a^{-n}(\mathcal{A})$ such that $z_{-n} \in U_n$, contain at most one critical point and are hence simply connected by Corollary 1.3.3. The result follows since the nested subsequence of preimages $\{U_{np}\}$ covers $A^*(z_0)$. The parabolic case follows similarly by taking a petal instead of a linearizing domain (see Theorem 1.1.15) whereas in the superattracting case we may use a Bötcher domain (see Theorem 1.1.12).

Case 2: Now we consider the case in which one connected component, say $A^*(z_0)$, of the immediate basin of attraction contains the two different free critical points. This excludes the case $|a| = 2$ (see Section 3.1). Without loss of generality we assume that z_0 is a fixed point. Indeed, the first return map from $A^*(z_0)$ onto itself has no other critical points since the other components of the immediate basin of attraction contain none.

Due to symmetry of the critical orbits, the fixed point z_0 lies in $\mathbb{S}^1$. Hence, $A^*(z_0)$ intersects $\mathbb{S}^1$, which is invariant. If z_0 is attracting, take the maximal domain $\mathcal{A}$ of the Kœnigs linearization (see Lemma 1.1.11). Its boundary $\partial\mathcal{A}$ contains, due to symmetry, the two critical points. Each critical point has a different simply connected preimage of $B_a(\mathcal{A})$ attached to it. Now consider $V = B_a^{-1}(\mathcal{A})$. The map $B_a|_V : V \to \mathcal{A}$ is proper and of degree 3 since z_0 has three different preimages. Given that V contains exactly 2 critical points and $B_a|_V$ is of degree 3, it follows from the Riemann-Hurwitz formula (see Theorem 1.3.1) that V is simply connected. Using the same reasoning all of its preimages are simply connected. Finally, since $A^*(z_0)$ is covered by the nested sequence of simply connected preimages of $\mathcal{A}$, we conclude that $A^*(z_0)$ is simply connected. The parabolic case is done similarly by taking $\mathcal{P}$ to be the maximal invariant petal (see Theorem 1.1.16). Notice that, due to symmetry, for $|a| > 2$ there cannot be a superattracting cycle of local degree 2 with an extra critical point in $A^*(z_0)$.

□

We now finish the proof of statement (c). Assume that there exists a periodic Fatou component other than $A^*(0)$ and $A^*(\infty)$. Then, such a periodic Fatou component has a critical point related to it. Indeed, if it is a Siegel disk, there is critical point whose orbit accumulates on its boundary (see [Shi87]). If it is the basin of attraction of an attracting, superattracting or parabolic cycle $< z_0 >$, there is a critical point in

$c \in A^*(< z_0 >)$ (see Theorems 1.1.11 and 1.1.16). Therefore, there is at most one unoccupied critical point. Hence, by means of the Corollary 1.3.3 of the Riemann-Hurwitz formula, any preperiodic Fatou component that is eventually mapped to a periodic component other that $A^*(\infty)$ or $A^*(0)$ is also simply connected.

The final statement of the theorem holds since the Julia set of a rational map is connected if and only if all connected Fatou components are simply connected. We conjecture that statement (c) on Theorem 3.2.1 also holds for $1 < |a| < 2$.

Conjecture 3.2.5. *If $|a| > 1$, then the Julia set of a Blaschke product B_a is connected if and only if $c_+ \notin A^*(\infty)$.*

4

Chapter Four

The Blaschke Family and a Family of Cubic Polynomials

When $|a| > 2$ the Blaschke products B_a may present locally polynomial dynamics (see Figure 4.1). In this section we introduce the relation of the Blaschke products B_a with the family of cubic polynomials with a superattracting fixed point $M_b(z) = bz^2(z-1)$ with $b \in \mathbb{C}$. This relation is done by means of a cut and paste quasiconformal surgery procedure (c.f. [BF14] and [Pet07]). Even if this surgery is interesting in itself, it has a direct application to the study of the boundaries of the basin of attraction of disjoint attracting cycles (see Proposition 5.4.1).

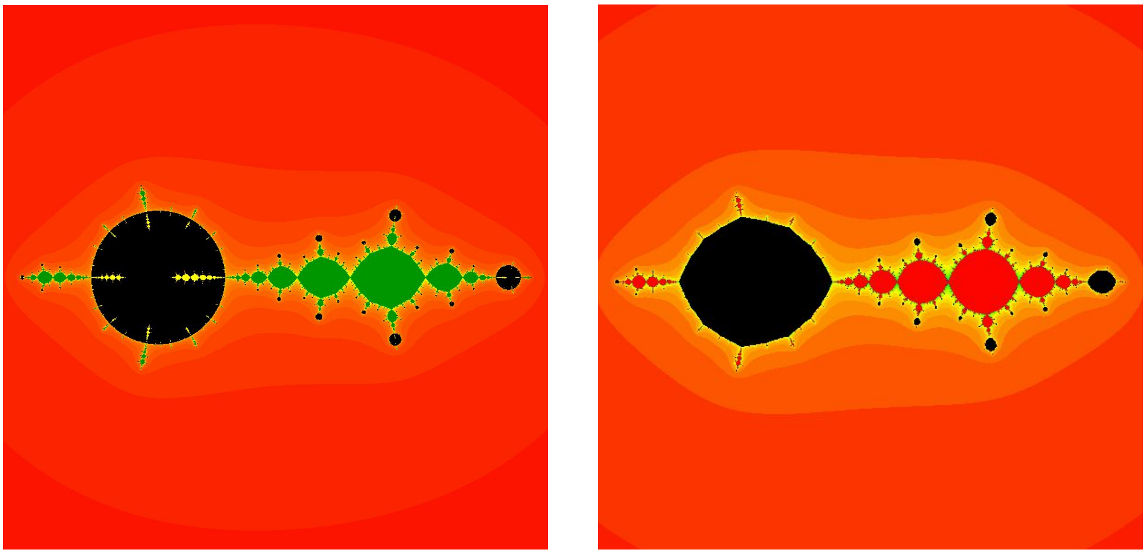


Figure 4.1: Dynamical planes of the Blaschke product $B_{5.25}$ (left) and the cubic polynomial $M_{-5.5}$ (right). The black regions of both figures correspond to the basins of attraction of the superattracting fixed points $z = 0$. For the cubic polynomial we see in red the basin of attraction of a period two attracting cycle. The Blaschke product has two different attracting cycles of period two. One outside the unit disk (green) and the other one inside (yellow).

The goal of this chapter is to introduce the mathematical procedure relating the dynamics of a Blaschke product B_a with the ones of a cubic polynomials M_b. In Section 4.1 we give a first overview on the cubic family M_b. In Section 4.2 we study quasiconformal surgery which relates the Blaschke family B_a and the cubic polynomials M_b.

4.1 Cubic polynomials with a superattracting fixed point

In this section we introduce the family of cubic polynomials $M_b(z) = bz^2(z-1)$ with $b \in \mathbb{C}$. They have the point $z = 0$ as a superattracting fixed point. This one dimensional slice of cubic polynomials (or a cover thereof) was introduced by Milnor in 1991 in a preliminary version of [Mil09]. Since then, these polynomials have been the object of several studies. For instance, Roesch [Roe07] studied its bifurcation locus solving some of the conjectures raised by Milnor. Tan [Tan97] used another parametrization of this family in the description, by means of the so called matings, of the parameter plane of the set of Newton maps N_P coming from degree 3 polynomials P. In Figure 4.2 we show the parameter plane of the family M_b.

This family is in some sense complete, since every cubic polynomial with a superattracting fixed point can be conformally conjugate to one of its members. Moreover, the slice contains only one representative of each conformal conjugacy class. In Figure 4.2 we show the parameter plane, drawn by computing the asymptotic behavior of the orbit of the critical point $c = 2/3$.

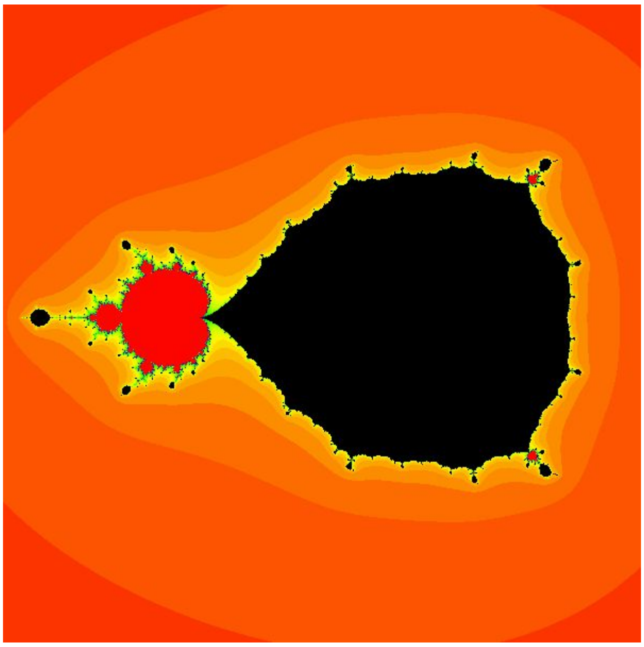


Figure 4.2: Parameter plane of the family of cubic polynomials $M_b(z) = bz^2(z-1)$. The colors are as follows. Black if the free critical orbit tends to the superattracting cycle $z = 0$ and red if it tends neither to $z = 0$ nor to $z = \infty$. The scaling from green to orange corresponds to parameters for which the critical orbit tends to $z = \infty$.

The connected components of parameters of the family M_b for which the critical orbit tends to $z = 0$ are called *capture components*. They are plotted in black in Figure 4.2. The large capture component surrounding the parameter $b = 0$ is called the *main capture component* $\mathcal{C}_M^0$. It corresponds to the set of parameters such that the critical point c belongs in the immediate basin of attraction $A^*(0)$. Given a parameter $b \in \mathcal{C}_M^0$, we have that the Julia set $\mathcal{J}(M_b)$ consists of the common boundary of the

Fatou components $A^*(0)$ and $A^*(\infty)$, which is a quasicircle. There is an analogous set of parameters for the Blashcke family B_a given by $|a| < 1$. The only difference is that, given a Blashcke product B_a, $\mathcal{J}(B_a) = \mathbb{S}^1$, while $\mathcal{J}(M_b)$ is only a Jordan curve.

4.2 Surgery between Blaschke products and cubic polynomials

We proceed to introduce the quasiconformal surgery which relates the Blaschke products B_a to the cubic polynomials M_b (c.f. [Pet07]). We refer to Chapter 2 for an introduction to the tools used in quasiconformal surgery. The idea of the surgery is to "glue" the map $R_2(z) = z^2$ inside $\mathbb{D}$ keeping the dynamics of B_a outside $\mathbb{D}$ whenever the parameter a is such that $B_a|_{\mathbb{S}^1}$ is quasisymmetrically conjugate to the doubling map $R_2|_{\mathbb{S}^1}$.

More precisely, we restrict to the set of parameters a such that $|a| \geq 2$. For these parameters, $B_a|_{\mathbb{S}^1}$ is a degree 2 covering of the unit circle (see Section 3.1) and is hence semiconjugate to the doubling map by a unique non decreasing continuous map h_a, not necessarily surjective, which depends continuously on a (see Lemmas 1.2.3 and 1.2.4). By Theorem 1.2.14 we have that, if $|a| > 2$ and the circle map $B_a|_{\mathbb{S}^1}$ has neither attracting nor parabolic cycles, then h_a is a quasisymmetric homeomorphism of the circle.

Definition 4.2.1. We define $\mathcal{X}$ to be the set of parameters a, $|a| \geq 2$, such that h_a is a quasisymmetric conjugacy between $B_a|_{\mathbb{S}^1}$ and the doubling map $R_2|_{\mathbb{S}^1}$.

Let $a \in \mathcal{X}$. Since the conjugacy h_a is quasisymmetric, it extends to a quasiconformal map $H_a : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. We may choose the quasiconformal extension by Douady and Earle [DE86], which depends continuously on h_a. We define the model map as

$$F_a(z) = \begin{cases} B_a(z) & \text{for} \quad |z| > 1 \\ H_a^{-1} \circ R_2 \circ H_a(z) & \text{for} \quad |z| \leq 1. \end{cases}$$

Proposition 4.2.2. *Let $a \in \mathcal{X}$. Then, there exists $b \in \mathbb{C}$ and a quasiconformal map $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\psi \circ F_a \circ \psi^{-1} = M_b$, where $M_b(z) = b^2 z(z-1)$. Moreover, $b = b(a)$ depends continuously on a.*

Proof. The map F_a is quasiregular since it is continuous in $\widehat{\mathbb{C}}$, holomorphic outside $\overline{\mathbb{D}}$ and locally quasiconformal in $\mathbb{D} \setminus \{0\}$. Moreover, the map F_a depends continuously on a. Its topological degree is 3 since gluing the map $z \to z^2$ in $\overline{\mathbb{D}}$ decreases the degree of B_a by 1. Indeed, recall from Section 3.1 that B_a has three preimages of $\mathbb{D}$ and one preimage of $\mathbb{C} \setminus \mathbb{D}$ in $\mathbb{D}$. Instead, F has only two preimages of $\mathbb{D}$ and none of $\mathbb{C} \setminus \mathbb{D}$ in $\mathbb{D}$.

We now define an F_a-invariant almost complex structure σ_a, i.e., an almost complex structure such that $F_a^* \sigma_a = \sigma_a$, as

$$\sigma_a = \begin{cases} H_a^* \sigma_0 & \text{on} & \mathbb{D} \\ (F_a^m)^* (H_a^* \sigma_0) & \text{on} & F_a^{-m}(\mathbb{D}) \setminus F_a^{-m+1}(\mathbb{D}), \text{ for } m \geq 1 \\ \sigma_0 & \text{otherwise,} & \end{cases}$$

where σ_0 denotes the standard complex structure and * denotes the pullback operation (see Section 2.1). By construction, σ_a has bounded dilatation. Indeed, $\sigma_a|_{\mathbb{D}}$ is the pull back of σ_0 by a quasiconformal map. Everywhere else either we pull back $\sigma_a|_{\mathbb{D}}$ by a holomorphic map (so we do not increase the dilatation) or we use the standard complex structure. Moreover, it depends continuously on a since H_a, F_a and the preimages of the unit circle depend continuously on a.

Let ψ_a be the integrating map of σ_a given by the Measurable Riemann Mapping Theorem 2.1.23 such that $\psi_a(H_a^{-1}(0)) = 0$, $\psi_a(\infty) = \infty$ and $\psi_a(c_+) = 2/3$. Then, the following diagram commutes.

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \sigma_a) & \xrightarrow{F_a} & (\widehat{\mathbb{C}}, \sigma_a) \\ \Big\downarrow{\scriptstyle \psi_a} & & \Big\downarrow{\scriptstyle \psi_a} \\ (\widehat{\mathbb{C}}, \sigma_0) & \xrightarrow{\psi_a \circ F_a \circ \psi_a^{-1}} & (\widehat{\mathbb{C}}, \sigma_0). \end{array}$$

The composition $\psi_a \circ F_a \circ \psi_a^{-1}$ is a quasiregular map preserving the standard complex structure and therefore, by Weyl's Lemma (Theorem 2.1.22), $\psi_a \circ F_a \circ \psi_a^{-1}$ is a holomorphic map of $\widehat{\mathbb{C}}$. Since this map has topological degree 3 and no poles it is a cubic polynomial. By the chosen normalization, $z = 0$ is a superattracting fixed point and $z = 2/3$ is a critical point. Hence, $\psi_a \circ F_a \circ \psi_a^{-1} = M_b$ for some $b \in \mathbb{C}$.

Finally, since σ_a depends continuously on a and we have chosen the normalization of ψ_a to depend continuously on a (the critical point c_+ and $H_a^{-1}(0)$ depend continuously on a), we have by Threorem 2.1.24 that the map $a \to \psi_a(z)$ depends continuously on a for all $z \in \widehat{\mathbb{C}}$. Applying it to $v_+ = F_a(c_+)$, which also depends continuously on a, we conclude that $\psi_a(v_+)$ depends continuously on a. Since ψ_a preserves orbits and therefore $\psi_a(v_+) = M_b(2/3) = -4b/27$, we conclude that b depends continuously on a. □

The surgery described above above defines a map $\Gamma : \mathcal{X} \to \mathcal{Y}$ between the subset $\mathcal{X}$ of the parameter plane of B_a (see Figure 5.1) and a subset $\mathcal{Y}$ of the parameter plane of M_b (see Figure 4.2). It follows from Theorem 1.2.14 and the fact that every parameter a, $|a| > 2$, such that $B_a|_{\mathbb{S}^1}$ has a parabolic cycle belongs to the boundary of a tongue (see Corollary 6.3.5) that, if $|a| > 2$ and a is not in any tongue or its boundary, then $a \in \mathcal{X}$.

The next lemma tells us that the image set $\mathcal{Y}$ does not include any parameter b in the main capture component (i.e., the set of parameters for which the basin of $z = 0$ contains the critical point $c = 2/3$).

Lemma 4.2.3. *Let M_b be a polynomial obtained with the construction of Section 4.2. Then $b \notin \mathcal{C}_M^0$.*

Proof. Given M_b obtained by the construction of Section 4.2, we have that, in the immediate basin of attraction of $z = 0$, M_b is quasiconformally conjugate to $R_2(z) = z^2$. Since $R_2(z)$ has no extra critical point, M_b cannot have the critical point $c = 2/3$ in $A^*(0)$. □

We conjecture that Γ is a degree 3 cover between $\mathcal{X}$ and $\mathcal{Y}$. Indeed, the following results hold.

Proposition 4.2.4. *If $a_1 \in \mathcal{X}$ and $a_2 = \xi a_1$, where $\xi^3 = 1$, then $\Gamma(a_1) = \Gamma(a_2)$.*

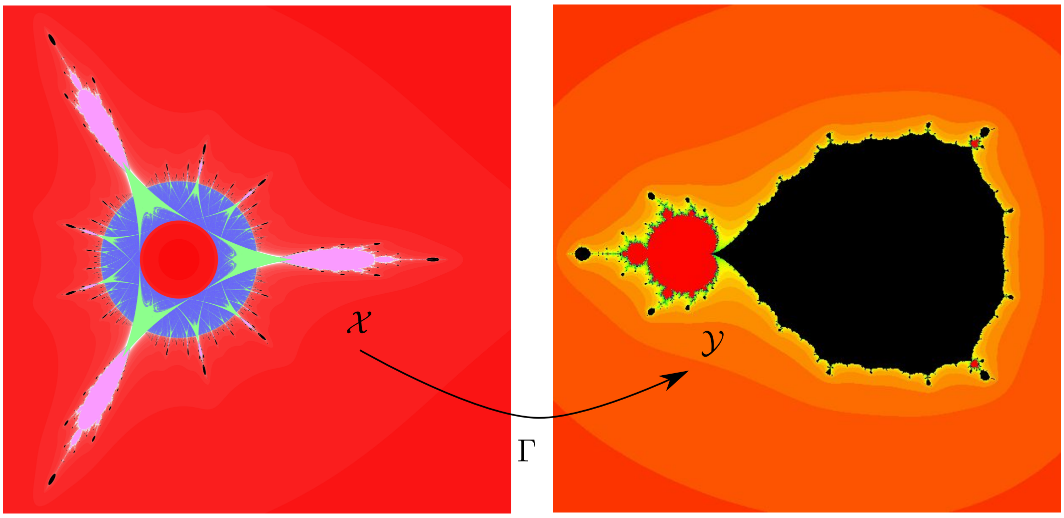


Figure 4.3: By performing the construction, we obtain a map Γ from the exterior of the components in the blue and green zone in the parameter plane of B_a (see also Figure 5.1) to the exterior of the main capture component in the parameter plane of M_b.

The proof of this result uses the following Lemma, which is a special property of the Douady Earle extension [DE86].

Lemma 4.2.5. *Let P_1 and P_2 be two degree 1 Blaschke products as in Definition 1.2.11, h be a quasisymmetric map of the unit circle and H denote the quasiconformal extension of h given by Douady and Earle. Then the Douady-Earle quasiconformal extension of $P_1 \circ h \circ P_2$ is given by $P_1 \circ H \circ P_2$.*

Proof of Proposition 4.2.4. The Blaschke products B_{a_1} and B_{a_2} are conformally conjugate by the rotation $R_\xi(z) = \xi z$ (c.f. Lemma 5.1.1 below), i.e., $B_{a_2}(z) = R_\xi \circ B_{a_1} \circ R_\xi^{-1}(z)$. Using Lemma 4.2.5 we conclude that $H_{a_2}(z) = R_\xi \circ H_{a_1} \circ R_\xi^{-1}(z)$ and, therefore, $F_{a_2}(z) = R_\xi \circ F_{a_1} \circ R_\xi^{-1}(z)$ on $\mathbb{D}$. Since B_{a_1} and B_{a_2} are conjugate by R_ξ we conclude that $F_{a_2}(z) = R_\xi \circ F_{a_1} \circ R_\xi^{-1}(z)$ on $\widehat{\mathbb{C}}$. Moreover, by construction, $\sigma_{a_1} = R_\xi^* \sigma_{a_2}$. Hence, the following diagram commutes.

$$\begin{tikzcd}
(\mathbb{C}, \sigma_0) \arrow[d, "M_{\Gamma(a_1)}"] & (\mathbb{C}, \sigma_{a_1}) \arrow[l, "\psi_{a_1}"'] \arrow[r, "R_\xi"] \arrow[d, "F_{a_1}"] & (\mathbb{C}, \sigma_{a_2}) \arrow[r, "\psi_{a_2}"] \arrow[d, "F_{a_2}"] & (\mathbb{C}, \sigma_0) \arrow[d, "M_{\Gamma(a_2)}"] \\
(\mathbb{C}, \sigma_0) & (\mathbb{C}, \sigma_{a_1}) \arrow[l, "\psi_{a_1}"'] \arrow[r, "R_\xi"] & (\mathbb{C}, \sigma_{a_2}) \arrow[r, "\psi_{a_2}"] & (\mathbb{C}, \sigma_0).
\end{tikzcd}$$

Therefore, the quasiconformal map $\psi_{a_2} \circ R_\xi \circ \psi_{a_1}^{-1}$ leaves the standard complex structure σ_0 invariant and is conformal by Weyl's Lemma (Theorem 2.1.22). Then $M_{\Gamma(a_1)}$ are $M_{\Gamma(a_2)}$ conformally conjugate. We conclude that $\Gamma(a_1) = \Gamma(a_2)$ since the cubic polynomials M_b have a unique representative of each conformal conjugacy class. $\square$

Proposition 4.2.6. *Let Ω be a hyperbolic component with an attracting cycle contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Then the map $\Gamma|_\Omega : \Omega \to \Gamma(\Omega)$ is a homeomorphism.*

Proof. Let Ω be a hyperbolic component of B_a with an attracting cycle contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Notice that the map Γ preserves the multiplier of the cycles of B_a contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$ since ψ_a is conformal in $\widehat{\mathbb{C}} \setminus \bigcup_{n \geq 0} B_a^{-n}(\mathbb{D})$. We shall see in Theorem 5.4.2 that the multiplier map, which maps every $a \in \Omega$ to the multiplier of its attracting cycle in $\mathbb{C} \setminus \overline{\mathbb{D}}$, is a homeomorphism between Ω and the unit disk. The result holds since the multiplier map is also a homeomorphism between every hyperbolic component of the cubic family M_b with an attracting cycle in $\mathbb{C}^*$ and the unit disk (c.f. [DH85a]). $\square$

Since $F = B_a$ outside $\mathbb{D}$, it follows that ψ is a quasiconformal conjugacy between $M_{b(a)}$ and B_a on this region, and so are all iterates on orbits which never enter $\mathbb{D}$. Therefore, for all parameters $a \in \mathcal{X}$ such that the orbit of the exterior critical point $\mathcal{O}(c_+)$ never enters $\overline{\mathbb{D}}$ all relevant dynamics are preserved (see Figure 4.1). The study of the parameters $a \in \mathcal{X}$ such that $\mathcal{O}(c_+)$ meets $\overline{\mathbb{D}}$ is done in Section 5.3. Another application of this surgery construction is explained in Section 5.4 (see Proposition 5.4.1).

5

Chapter Five

Parameter Plane

The non-holomorphic parametrization of the family B_a allows the presence of some features which do not appear in holomorphicaly parametrized uniparametric families families such as the existence of tongues or bifurcations along curves (see Chapter 6). Furthermore, small "copies" of the Tricorn, the connectedness locus of the antiholomorphic polynomials $p_c(z) = \overline{z}^2 + c$, seem to be observed numerically. Despite that, the usual parametrization of hyperbolic components by means of the multiplier map can be adapted to work in many of the hyperbolic components.

The aim of this chapter is to study the parameter plane of the Blaschke family B_a. In Section 5.1 we briefly explain the symmetries of the parameter plane. In Section 5.2 we explain the different hyperbolic behaviors which may take place and describe in which regions of the parameter plane these occur. In Section 5.3 we describe the dynamics for parameters within the so called swapping regions. In Section 5.4 we give a parametrization of all disjoint hyperbolic components with an attracting cycle in $\mathbb{C}^* \setminus \mathbb{S}^1$.

5.1 Preliminaries on the parameter plane

Figure 5.1 shows the parameter plane of the family B_a. The plot shows the result of iterating the critical point c_+. Since the two critical orbits of B_a are related by symmetry unless $1 < |a| < 2$, this information suffices also for c_- everywhere else. If $1 < |a| < 2$, the critical orbits may have completely independent asymptotic behavior (see Figure 3.2 (c), (d)).

There are some symmetries which seem to be observed when looking at Figure 5.1. Indeed, the parameter plane appears to be symmetric with respect to complex conjugation. It also appears to be preserved with respect to rotation with respect to a third root of the unity. The next lemma explains the observed symmetries.

Lemma 5.1.1. *Let $a, b \in \mathbb{C} \setminus \mathbb{S}^1$. Then B_a and B_b are conformally conjugate if and only if $b = \xi a$ or $b = \xi \overline{a}$, where ξ is a third root of the unity.*

Proof. Assume that B_a is conformally conjugate to another Blaschke product B_b. Then, there exists a Möbius transformation $\eta(z) = (cz-d)/(ez-f)$ such that $B_b = \eta^{-1} \circ B_a \circ \eta$, where $c, d, e, f \in \mathbb{C}$ and $cf - de \neq 0$. Since η maps superattracting fixed points to superattracting fixed points preserving the local degree, we conclude that either $\eta(0) = 0$ and $\eta(\infty) = \infty$ or $\eta(0) = \infty$ and $\eta(\infty) = 0$. In the first case we conclude that $\eta(z) = \xi z$ with $\xi \in \mathbb{C}^*$. In the second case we conclude that $\eta(z) = \xi / z$ with $\xi \in \mathbb{C}^*$. Assume that $\eta(z) = \xi z$. Then

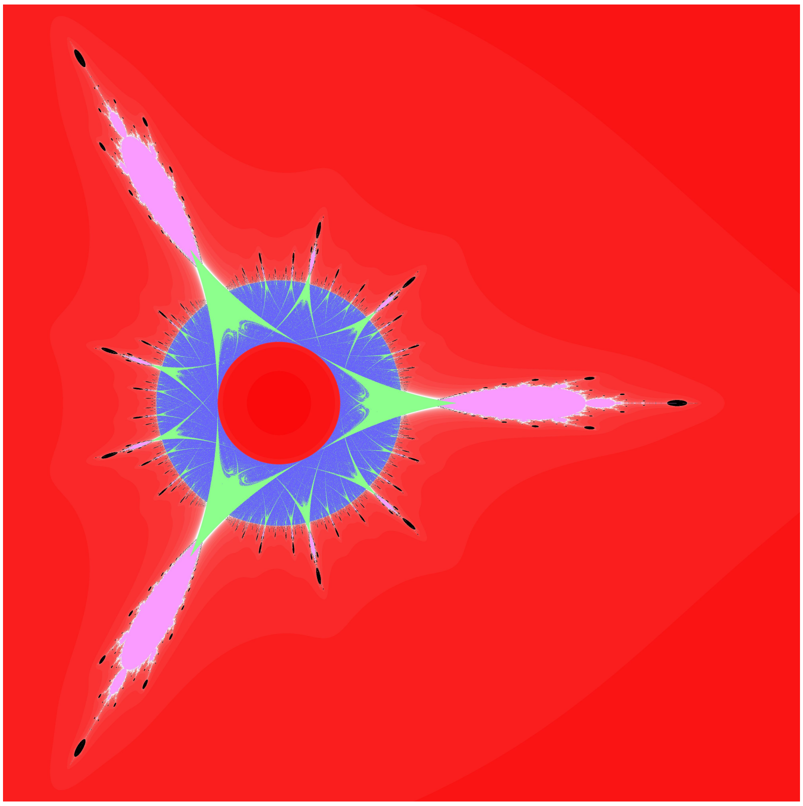


Figure 5.1: Parameter plane of the Blaschke family B_a. The colors are as follows: red if $c_+ \in A(\infty)$, black if $c_+ \in A(0)$, green if $O^+(c_+)$ accumulates on a periodic orbit in $\mathbb{S}^1$, pink if $O^+(c_+)$ accumulates in a periodic orbit not in $\mathbb{S}^1$ and blue in any other case. The inner red disk corresponds to the unit disk.

$$\eta^{-1} \circ B_a \circ \eta(z) = \xi^3 z^3 \frac{z - a\xi^{-1}}{1 - \overline{a}\xi z}.$$

If $|\xi| \neq 1$, the previous map does not provide a generalized Blaschke product (see Definition 1.2.11 and Lemma 1.2.13). It also follows from the previous equation and the definition of Blaschke products that $b = a\xi^{-1} = \overline{\overline{a}\xi}$ and, therefore, $\xi^{-1} = \overline{\xi}$. Finally, to ensure that the obtained Blaschke product belongs to the family B_a, we need to require that $\xi^3 = 1$. On the other hand, if we assume that $\eta(z) = \xi/z$. Then

$$\eta^{-1} \circ B_a \circ \eta(z) = \xi^{-2} z^3 \frac{1 - \overline{a}\xi/z}{\xi/z - a} = \xi^{-2} z^3 \frac{z - \overline{a}\xi}{\xi(1 - a\xi^{-1}z)} = \xi^{-3} z^3 \frac{z - \overline{a}\xi}{1 - a\xi^{-1}z}.$$

As before, we conclude that $|\xi| = 1$, that $\xi^3 = 1$ and that $b = \overline{a}\xi$.

$\square$

As explained before, the parameter plane shown in Figure 5.1 is somehow incomplete in the annulus of inner radius 1 and outer radius 2. Another approach that one can make to understand the parameter plane in this region consists in drawing the bifurcation diagrams of both critical points simultaneously. These diagrams show where the orbits of the critical points accumulate for a certain range of parameters a such that $|a| = \lambda$ and a certain λ. Figure 5.2 shows a computer drawing of this bifurcation diagrams for $\lambda = |1,07398 + 0,5579i|$ and $a = \lambda e^{2\pi i Arg(a)}$. The x-axis shows $Arg(a) \in (0.074, 0.079)$. The y-axis shows the different arguments in $(-0.5, 0.5]$ of the iterates of c_+ and c_-. The upper figure is done iterating c_+ whilst the lower one is done iterating c_-. This choice of parameters has been done since, for $a_0 = 1.07398 + 0.5579i$, B_{a_0} has two disjoint attracting cycles (see Figure 3.2 (c) and (d)). When $Arg(a) \in (0.074, 0.0075)$, c_+ lies in the immediate basin of attraction of an attracting fixed point x_0 whilst c_- either lies in a preperiodic component of the basin of x_0 or $\mathcal{O}^+(c_-)$ accumulates in another attracting cycle ($Arg(a) \in (0.0755, 0.0768)$) or c_- lies in the Julia set.

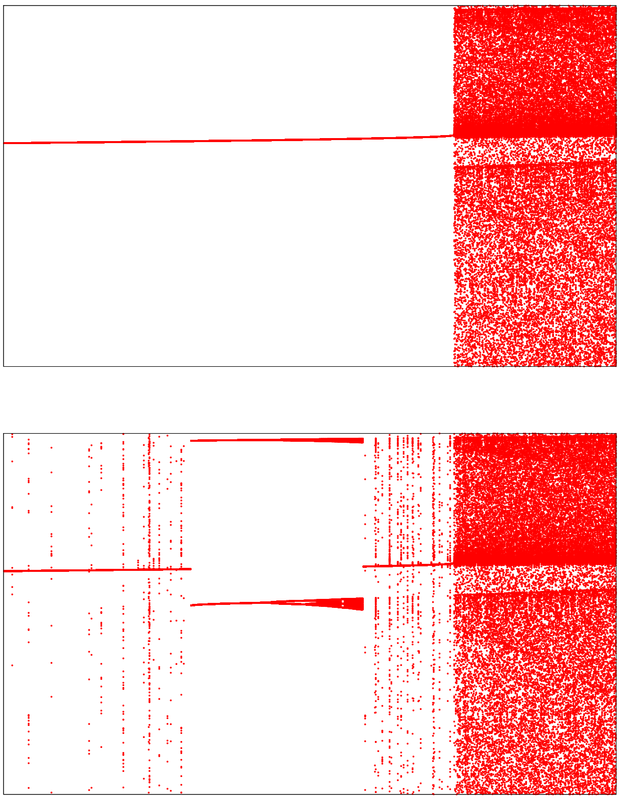


Figure 5.2: Bifurcation diagrams for $\lambda = |a_0|$, $a_0 = 1.07398 + 0.5579i$, near the parameter a_0.

5.2 Hyperbolic parameters

We proceed to study the different types of hyperbolic dynamics which may take place for the Blaschke family B_a. Recall that a rational map Q is hyperbolic if all critical orbits are captured by the basins of attraction of attracting or superattracting cycles (see Section 1.1.3). When restricting to parameters $a \in \mathbb{C} \setminus \mathbb{S}^1$, the Blaschke family B_a is almost bicritical in the sense of Definition 1.1.23. Indeed, the Blaschke products B_a

have only two free critical points $c_\pm$ and two other non-free critical points (0 and ∞) which are persistently superattracting.

Recall from Section 3.1 that the two free critical orbits are symmetric unless $1 < |a| \leq 2$, in which case they belong to $\mathbb{S}^1$. Hence, if one critical orbit accumulates on the superattracting fixed point $z = 0$ (resp. $z = \infty$) the other one accumulates on $z = \infty$ (resp. $z = 0$). Parameters for which this happens are called, following Definition 1.1.24, *escaping parameters*. We shall denote by $\mathcal{E}$ the set of such points, i.e.,

$$\mathcal{E} = \{a \in \mathbb{C} \mid B_a^n(c_+) \to \infty \text{ or } B_a^n(c_+) \to 0 \text{ as } n \to \infty\}.$$

Its complement,

$$\mathcal{B} = \widehat{\mathbb{C}} \setminus \mathcal{E},$$

is the set of *non-escaping parameters*. Observe that $\mathcal{B}$ is not the connectedness locus of B_a. Indeed, in view of Theorem 3.2.1, parameters in $\mathcal{E}$ may have a connected Julia set if the critical points belong to $A(0)\backslash A^*(0)$ or $A(\infty)\backslash A^*(\infty)$ (see Figure 3.5). On the other hand, some Julia sets for $1 < |a| < 2$ (which belong to $\mathcal{B}$) may, a priori, be disconnected (even though we conjecture that this is never the case (see Conjecture 3.2.5)). Notice also that, if $|a| = 1$, the Blaschke products B_a degenerate to degree 3 polynomials without free critical points in which case the map B_a is not almost bicritical and we shall say that a is neither escaping nor non-escaping.

Lemma 5.2.1. *If $|a| < 1$ then $a \in \mathcal{E}$. If $1 < |a| \leq 2$ then $a \in \mathcal{B}$. The non-escaping set $\mathcal{B}$ is bounded.*

Proof. The first two statements have already been proven. To prove the third one we have to show that, if $|a|$ is large enough, then the parameter a is escaping. First we prove that, if $|z| > \lambda(|a| + 1)$ with $\lambda \geq 1$, then $|B_a(z)| > \lambda|z|$. Indeed, one can check that $|z - a| > \lambda$ and that

$$|z|^2 > |z|(|a| + 1) = |za| + |z| > |za| + 1 > |1 - \bar{a}z|.$$

Therefore, we have

$$|B_a(z)| = |z|^3 \frac{|z - a|}{|1 - \bar{a}z|} > |z|^3 \frac{\lambda}{|z|^2} = \lambda|z|.$$

To finish the proof notice that, as $|a|$ tends to infinity, the critical point $c_+(a)$ tends to $2a/3$. Consequently, the modulus of the critical value $v_+ = B_a(c_+(a))$ grows as $M|a|^2$ for some $M > 0$ and, for $|a|$ large enough, $|v_+| > \lambda(|a| + 1)$ with $\lambda > 1$. Therefore, $|B_a^n(v_+)| \to \infty$ when $n \to \infty$. This concludes the proof.

□

Before presenting the main result of the section we want to remark that the Blaschke family satisfies the necessary conditions presented in Section 1.2.2 for the existence of tongues (see Definition 1.2.17). These tongues appear for $|a| \geq 2$ as the sets of parameters for which the circle map $B_a|_{\mathbb{S}^1}$ has an attracting (or superattracting) cycle. If a parameter a belongs to a tongue, due to symmetry, both critical points belong in the same component of the immediate attracting basin of the cycle. Therefore, parameters belonging to tongues are always hyperbolic. We study them more deeply in Chapter 6.

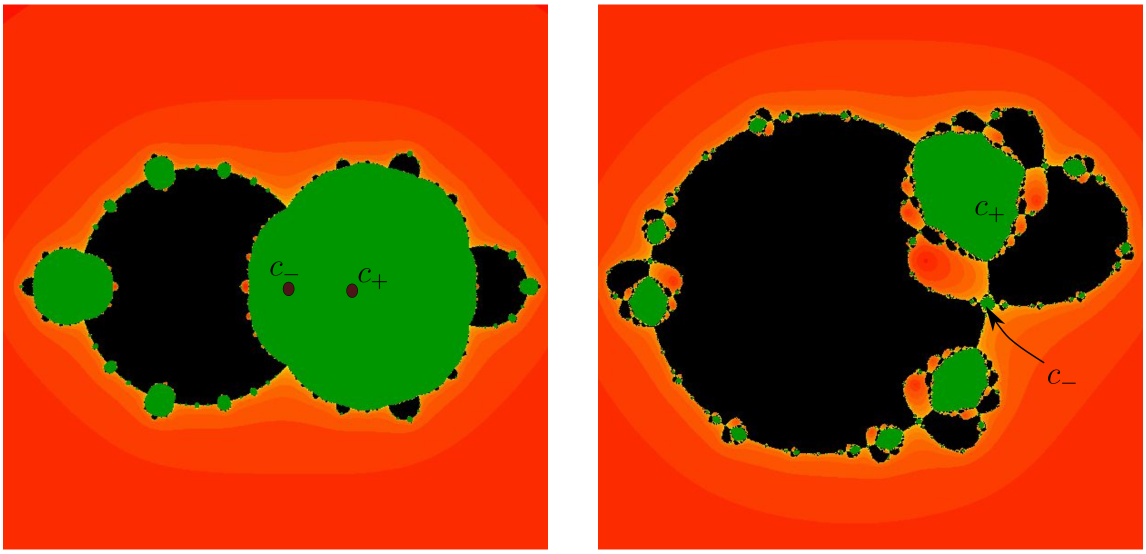


Figure 5.3: Dynamical planes of the Blaschke products $B_{2.5}$ (left) and $B_{1.52+0.325i}$ (right). The colors are as in Figure 3.2. The left case corresponds to an adjacent parameter a in a tongue (see Definition 1.2.17). The right case corresponds to a capture parameter.

Following Definition 1.1.24, non-escaping hyperbolic parameters of the products B_a can only be adjacent, bitransitive, capture or disjoint. We proceed to describe in which regions of the parameter plane of the Blaschke family the previous four possibilities may take place.

Theorem 5.2.2. *Let $a \in \mathcal{B}$. Then, the following hold.*

(a) If a is an adjacent parameter, either $1 < |a| < 2$ or it belongs to a tongue. Conversely, any parameter a belonging to a tongue is adjacent.

(b) If a is a bitransitive parameter, then either $1 < |a| < 2$ or c_+ and c_- enter and exit the unit disk infinitely many times under iteration of B_a.

(c) If a is a capture parameter, then $1 < |a| < 2$.

(d) If a is a disjoint parameter, then either $1 < |a| < 2$ or $|a| > 2$. In the latter case the orbits of the two attracting cycles are symmetric with respect to the unit circle and, hence, have the same period. Moreover, if the multiplier of one attracting cycle is λ, the multiplier of the other attracting cycle is $\overline{\lambda}$.

Proof. We begin with statement (a). If a is an adjacent parameter, then both critical points belong to the same component of the immediate basin of attraction of a periodic cycle. Then, either $1 < |a| < 2$ or, if $|a| \geq 2$, then both critical points are attracted to an attracting cycle in $\mathbb{S}^1$. In this last case, a belongs, by definition, to a tongue. The converse holds by symmetry.

To prove statement (b) notice that, in the bitransitive case, the immediate basin of attraction of the cycle on which the critical orbits accumulate has at least two different connected components. If $|a| > 2$, by symmetry, at least one is contained in the unit disk and another one is in its complement. Thus, the critical orbits enter and exit $\mathbb{D}$ infinitely many times.

Statement (c) follows directly from the fact that, for $|a| \geq 2$, the critical orbits are symmetric.

The first part of (d) follows from symmetry. In order to see that the attracting cycles have conjugate multipliers, we conjugate B_a via a Möbius transformation M to a rational map $\widetilde{B}_a$ that fixes the real line. The result follows then from the fact that $\widetilde{B}'_a(\bar{z}) = \overline{\widetilde{B}'_a(z)}$ and that M preserves the multiplier of the periodic cycles.

□

5.3 Swapping regions

We begin this section by introducing the sets of the parameter plane which we shall call swapping regions.

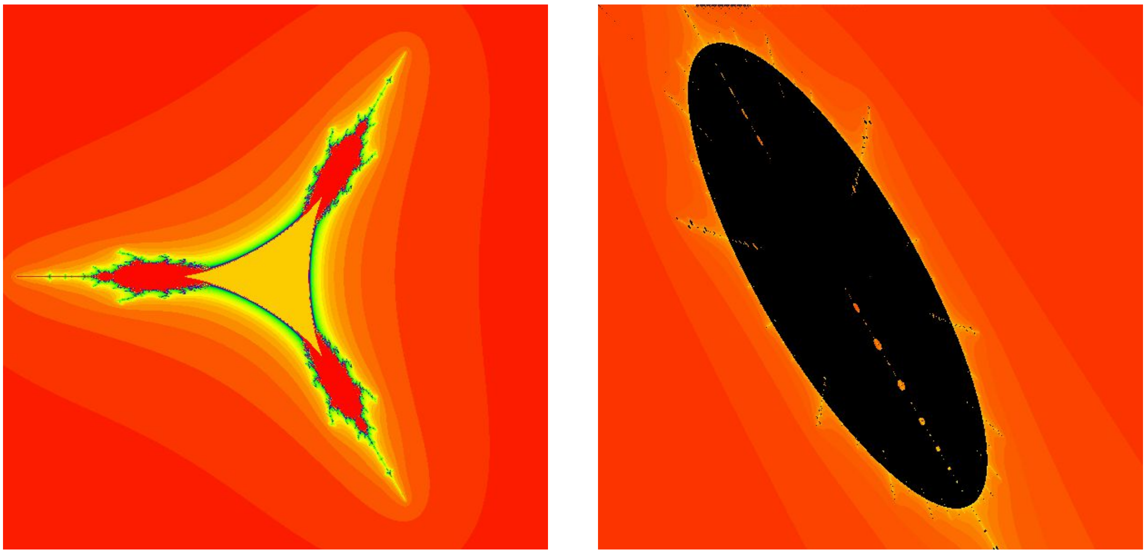
Definition 5.3.1. We say that a parameter a, $|a| > 2$, is a *swapping parameter* if the exterior critical point c_+ eventually falls under iteration in $\mathbb{D}$ (or equivalently if c_- eventually falls in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$). A maximal open connected set of swapping parameters is called a *swapping region*.

The goal of this section is to describe the dynamics which may take place for swapping parameters (see Figure 5.4 (b)). Exploratory work shows that small "copies" of the Tricorn, the connectedness locus of the antipolynomials $p_c(z) = \overline{z}^2 + c$ (see [CHRSC89], [NS03] and Figure 5.4 (a)), and the Mandelbrot set seem to appear embedded inside swapping regions (see Figures 5.4 (c) and (d)). We should no expect these "copies" to be actual homeomorphic copies of the Tricorn (c.f. [IM14]). These Tricorn and Mandelbrot-like sets appear as the accumulation set of parameters for which $\mathcal{O}(c_+)$ enters and exits the unit disk more and more times. These parameters are observed in Figures 5.4 (c) and (d) as the red and black annuli surrounding the the Tricorn and Mandelbrot-like sets. In the limit we may have parameters with attracting cycles which enter and exit the unit disk (see Figure 5.4 (c) and (d)). In this situation, we build, in Theorem 5.3.4, a polynomial-like map of degree either 2 or 4. Then we use antipolynomial-like mappings (see Section 2.3) to prove that when the polynomial-like map built in Theorem 5.3.4 is of degree 4, it is hybrid equivalent to a polynomial of the form $p_c^2(z) = \left(\overline{z^2 + c}\right)^2 + c = (z^2 + \overline{c})^2 + c$. A similar phenomena was described by Milnor [Mil92] for the cubic polynomials with real parameters.

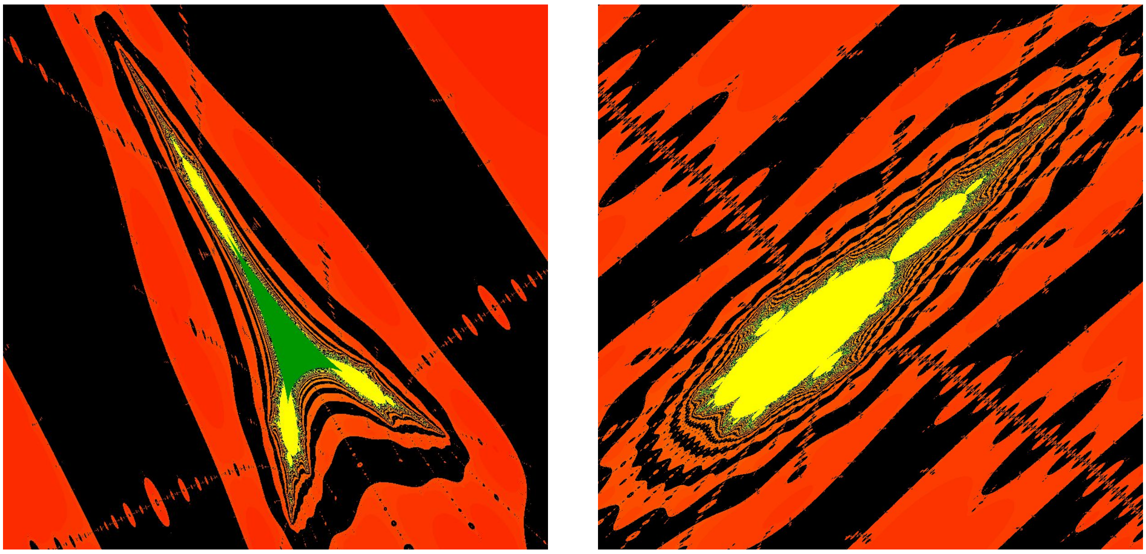
The following lemma tells us that swapping regions are disjoint from tongues (see Defintion 1.2.17).

Lemma 5.3.2. *A parameter a with $|a| > 2$ such that B_a has an attracting or parabolic cycle in $\mathbb{S}^1$ cannot be swapping.*

Proof. Assume that B_a has an attracting cycle in $\mathbb{S}^1$. Let $\mathcal{A}$ be the maximal domain of the Kœnigs linearization of the cycle (see Theorem 1.1.10 and Lemma 1.1.11). By symmetry, $c_\pm \in \partial\mathcal{A}$. Moreover, $\mathcal{A} \cap \mathbb{D}$ is mapped into $\mathcal{A} \cap \mathbb{D}$ under B_a since $B_a|_{\mathbb{S}^1}$ is orientation preserving and the linearizer is injective. Therefore, $\mathcal{O}(c_-)$ cannot exit the unit disk and the parameter is not swapping. The parabolic case is derived similarly taking $\mathcal{P}$ to be the maximal petal having the critical points on its boundary (see Theorems 1.1.15 and 1.1.16). $\square$



(a) The Tricorn. (b) A swapping region.



(c) A Tricorn-like set. (d) A Mandelbrot-like set.

Figure 5.4: Figure (a) shows the Tricorn. Figure (b) shows a swapping region within $(-3.39603, -3.05761) \times (5.45471, 5.79312)$. It corresponds to the parameters bounded by the big black component. Figure (c) shows a zoom of (b) for which a Tricorn-like set can be observed ($a \in (-3.22295, -3.22249) \times (5.58172, 5.58218)$). Figure (d) shows a Mandelbrot-like set inside another swapping region within $(2.080306, 2.080311) \times (1.9339165, 1.9339215)$. In Figures (b), (c) and (d) red points correspond to parameters so that $\mathcal{O}(c_+) \to \infty$ whereas black points correspond to parameters for which $\mathcal{O}(c_+) \to 0$. Green points correspond to bitransitive parameters (see Figure 5.6 (left)), whereas yellow points correspond to disjoint parameters (see Figure 5.6 (right)).

We are interested in the hyperbolic components contained in swapping regions. We shall study them using the theories of polynomial and antipolynomial-like mappings. We focus in the bitransitive parameters which, for $|a| > 2$, are necessarily inside swapping regions (see Theorem 5.2.2 (b)).

We recall some notation from Section 3.1. For $|a| > 2$ the unit circle has two preimages different from itself and not intersecting $\mathbb{S}^1$, say $\gamma_i \subset \mathbb{D}$ and $\gamma_e \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ (see Figure 3.1 (d)). The map B_a sends γ_i and γ_e bijectively to $\mathbb{S}^1$. Let Ω_i be the region bounded by γ_i and contained in $\mathbb{D}$ and let Ω_e be the region bounded by γ_e and contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Then, the maps $B_a|_{\Omega_i} : \Omega_i \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $B_a|_{\Omega_e} : \Omega_e \to \mathbb{D}$ are proper. They have degree 1 since there is only one preimage of $z = \infty$ in $\mathbb{D}$ and one preimage of $z = 0$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Therefore, $B_a|_{\Omega_i}$ and $B_a|_{\Omega_e}$ are conformal and $c_\pm \notin \Omega_i \cup \Omega_e$.

We now prove a lemma which describes the possible periods of attracting and parabolic cycles for parameters inside swapping regions.

Lemma 5.3.3. *Let a, $|a| > 2$, be a parameter inside a swapping region. If B_a has an attracting or parabolic cycle, then its period is at least 3. Moreover, if a is bitransitive, the period is even.*

Proof. First of all notice that, from Lemma 5.3.2 and invariance of $\mathbb{S}^1$, no component of the basin of attraction of the cycle can intersect neither γ_i nor γ_e. A parabolic or attracting cycle needs to have a critical point in its immediate basin of attraction. The component in which the critical point lies is contained neither in Ω_i nor in Ω_e since $B_a|_{\Omega_{e,i}}$ is conformal. Moreover, since the periodic cycle needs to enter and exit the unit disk, the immediate basin of attraction of the cycle has a component in Ω_i and another one in Ω_e. Then, the immediate basin of attraction has at least three different components and, hence, the cycle has at least period three.

Now assume that a is bitransitive. Suppose without loss of generality that the component which contains c_+ is mapped under $k > 0$ iterates to the component which contains c_-. Because of symmetry, the first return map from the component of c_- to component of c_+ also takes k iterates. Hence, the period of the attracting cycle is $2k$. $\square$

The following theorem is the main result of the section.

Theorem 5.3.4. *Let a_0 be a swapping parameter with an attracting or parabolic cycle of period $p > 1$. Then, there is an open set W containing a_0 and $p_0 > 1$ dividing p such that, for every $a \in W$, there exist two open sets U and V with $c_+ \in U$ such that $(B_a^{p_0}; U, V)$ is a polynomial-like map. Moreover,*

(a) If a_0 is bitransitive, $(B_a^{p_0}; U, V)$ is hybrid equivalent to a polynomial of the form $p_c^2(z) = (z^2 + \overline{c})^2 + c$.

(b) If a_0 is disjoint, $(B_a^{p_0}; U, V)$ is hybrid equivalent to a polynomial of the form $p_c^2(z) = (z^2 + \overline{c})^2 + c$ or of the form $z^2 + c$.

Proof. First of all notice that, due to Lemma 5.3.2, $A^*(< z_0 >)$ neither intersects γ_e nor γ_i. Since the cycle enters and exits the unit disk, $A^*(< z_0 >)$ has at least one connected component entirely contained in Ω_e. Let $A^*(z_0)$ be the connected component of $A^*(< z_0 >)$ containing c_+. Let $n_0 \in \mathbb{N}$ be minimal such that $B_{a_0}^{n_0}(z_0) = z_{n_0} \in \Omega_e$. Let $\mathcal{S}_0$ be the connected component of $B_{a_0}^{-n_0}(\Omega_e)$, containing c_+ (and hence $A^*(z_0)$). The set $\mathcal{S}_0$ is simply connected by Corollary 1.3.3 since Ω_e is simply connected and $B_{a_0}^{n_0}|_{\mathcal{S}_0}$ has a unique critical point. Recursively define $\mathcal{S}_n$ to be the connected component of $B_{a_0}^{-1}(\mathcal{S}_{n-1})$ containing the point z_{-n} of the cycle (recall that the subindexes of the cycle are taken in $\mathbb{Z}/p\mathbb{Z}$). Again by Corollary 1.3.3, the components $\mathcal{S}_n$ are simply connected for all $n > 0$. Let $p_0 \in \mathbb{N}$ be the minimal such that $c_+ \in \mathcal{S}_{p_0}$. Since $\partial\Omega_e \cap \mathbb{S}^1 = \emptyset$, we have that $\partial\Omega_e \cap \partial\mathcal{S}_{p_0-n_0} = \emptyset$ and that $\overline{\mathcal{S}_{p_0-n_0}} \subset \Omega_e$. Therefore, we have $\overline{\mathcal{S}_{p_0}} \subset \mathcal{S}_0$. Notice that p_0 is a divisor of p.

The map $B_{a_0}|_{\mathcal{S}_n} : \mathcal{S}_n \to \mathcal{S}_{n-1}$ is conformal if $\mathcal{S}_n$ contains no critical point and 2-to-1 if it contains c_+ or c_- (it cannot contain both critical points at the same time since $\mathcal{S}_n \cap \mathbb{S}^1 = \emptyset$). Hence, the triple $(B_{a_0}^{p_0}; \mathcal{S}_{p_0}, \mathcal{S}_0)$ is a polynomial-like map of degree 4 or 2 depending on whether there is some $\mathcal{S}_{q_0}$ containing c_- or not. As in Lemma 5.3.3, if such q_0 exists, $p_0 = 2q_0$. Notice that, if the parameter is bitransitive, this q_0 exists and, therefore, the degree is 4. Since the condition $c_+(a) \in \mathcal{S}_{p_0}(a) \subset \mathcal{S}_0(a)$ is open, the polynomial-like map can be defined for an open set of parameters W around a_0. From now on we consider $a \in W$.

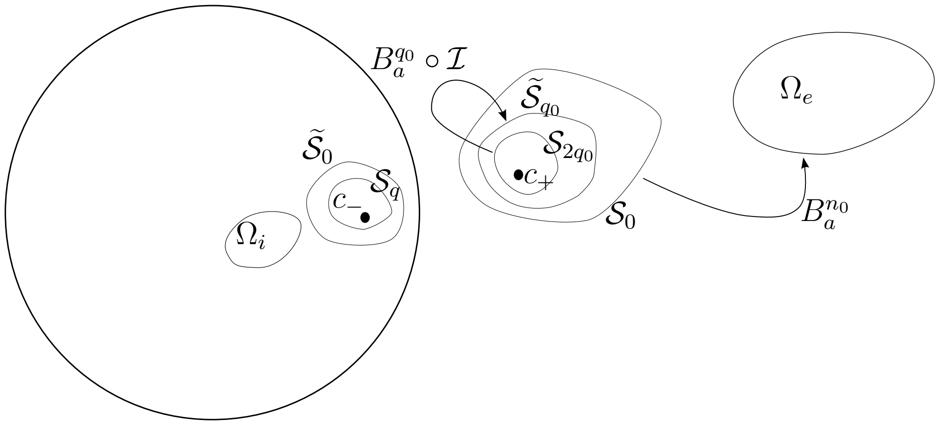


Figure 5.5: Sketch of the situation described in Theorem 5.3.4 for the degree 4 case.

We now use antipolynomial-like mappings to see that, in the case of a degree 4 polynomial-like mapping, the degree 4 polynomial to which $(B_a^{2q_0}; \mathcal{S}_{2q_0}, \mathcal{S}_0)$ is hybrid equivalent can be taken of the form $p_c^2(z) = \left(z^2 + \overline{c}\right)^2 + c$. See Section 2.2 for an introduction to antipolynomial-like mappings. We proceed to construct an antipolynomial-like map $(f; \mathcal{S}_{2q_0}, \mathcal{I}(\mathcal{S}_{q_0}))$ of degree 2, where $\mathcal{I}(z) = 1/\overline{z}$ denotes the reflection with respect to $\mathbb{S}^1$. This antipolynomial-like map is hybrid equivalent to an antipolynomial of the form $\overline{z}^2 + c$. The result then follows if $f(\mathcal{I}(\mathcal{S}_{q_0})) = \mathcal{S}_0$ and $(f^2; \mathcal{S}_{2q_0}, \mathcal{S}_0) = (B_a^{2q_0}; \mathcal{S}_{2q_0}, \mathcal{S}_0)$.

Define $\widetilde{\mathcal{S}}_q = \mathcal{I}(\mathcal{S}_q)$, where $q \in \mathbb{N}$. It is easy to see that $\overline{\mathcal{S}_{q_0}} \subset \widetilde{\mathcal{S}}_0$. Indeed, taking n_0 as in the definition of $\mathcal{S}_0$, by symmetry, $B_a^{n_0}(\widetilde{\mathcal{S}}_0) = \Omega_i$. Since $B_a^{n_0}(\overline{\mathcal{S}_{q_0}})$ is contained in Ω_i, we conclude that $\overline{\mathcal{S}_{q_0}} \subset \widetilde{\mathcal{S}}_0$ (see Figure 5.5). From $\overline{\mathcal{S}_{q_0}} \subset \widetilde{\mathcal{S}}_0$ we can deduce that $\overline{\mathcal{S}_{2q_0}} \subset \widetilde{A}_{q_0}$. Finally, take $f = \mathcal{I} \circ B_a^{q_0}$. Since $B_a = \mathcal{I} \circ B_a \circ \mathcal{I}$ we have that $f^2 = B_a^{2q_0}$. Then, the antipolynomial-like map $(\mathcal{I} \circ B_a^{q_0}; \mathcal{S}_{2q_0}, \widetilde{\mathcal{S}}_{q_0} = \mathcal{I}(\mathcal{S}_{q_0}))$ satisfies the desired conditions.

□

Theorem 5.3.4 tells us that all bitransitive parameters contained in swapping regions can be related to the dynamics of $p_c^2(z)$, where $p_c(z) = \overline{z}^2 + c$, since the polynomial-like map has degree 4. However, notice that if an antipolynomial $p_c(z)$ has an attracting cycle of even period $2q$, then $p_c^2(z)$ has two disjoint attracting cycles of period q. Therefore, there are also disjoint parameters in the Tricorn-like sets in the parameter plane of the Blaschke family (see Figure 5.4 (c) and Figure 5.6 (right)). These disjoint parameters also lead to degree 4 polynomial-like maps. Finally, the polynomial-like maps of degree 2 obtained from the other disjoint parameters are hybrid equivalent to quadratic polynomials z^2+c. These parameters correspond to the ones inside the small Mandelbrot-like sets observed by means of numerical computations (see Figure 5.4 (d)).

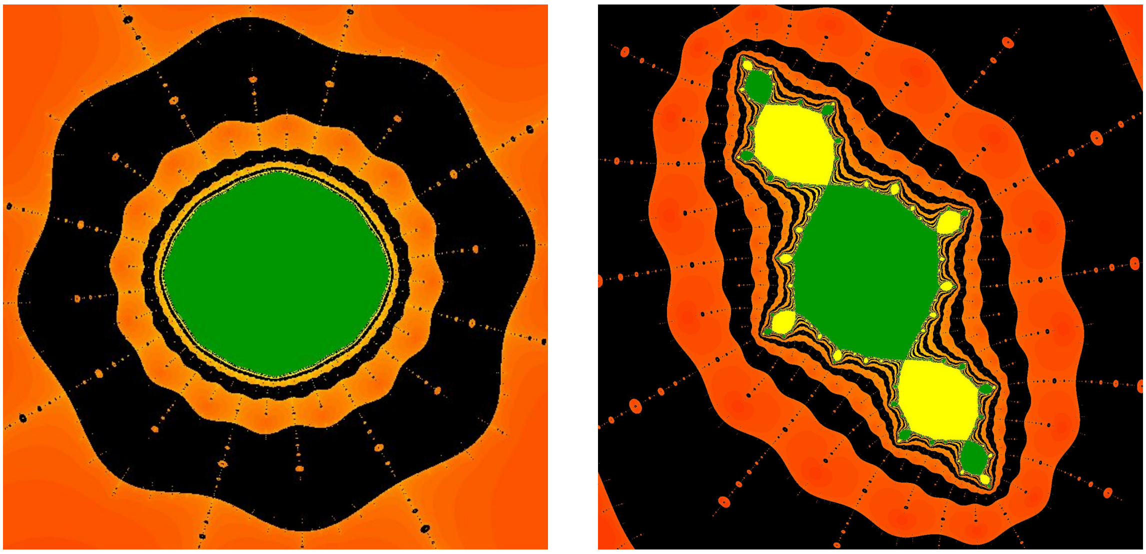


Figure 5.6: The left figure shows a connected component of the bitansitive cycle of B_{a_1}, where $a_1 = -3.22271+5.58189i$. The right figure shows a zoom in the dynamical plane of B_{a_2}, where $a_2 = -3.22278+5.58202i$ is a disjoint swapping parameter. The colors are as in Figure 3.2. Notice that, surrounding the basin of attraction, appear some black and red annuli. These annuli correspond to orbits which enter and exit $\mathbb{D}$ a finite number of times before entering $A^*(0)$ or $A^*(\infty)$.

The next corollary follows from the proof of Theorem 5.3.4.

Corollary 5.3.5. *Let a be a disjoint parameter contained in a swapping region which can be related by polynomial-like theory to a polynomial of the from $p(z) = \left(\overline{\overline{z}^2 + c}\right)^2 + c$. Then, the period of the disjoint cycles is even.*

5.4 Parametrization of hyperbolic components

By means of Theorem 5.2.2, a non-escaping parameter a such that $|a| > 2$ either is disjoint, or is bitransitive, or is adjacent and it belongs to a tongue. We shall study the tongues of the B_a in Chapter 6. The aim of this section is to study the multiplier map of the bitransitive and disjoint hyperbolic components of the Blaschke family B_a for parameters a such that $|a| > 2$. Recall that a hyperbolic component is a connected component of the set of parameters for which B_a is hyperbolic and that the multiplier map Λ sends every disjoint or bitransitive parameter a to the multiplier $\lambda(< z_0 >)$ of the attracting cycle $< z_0 >$ of B_a whose immediate basin of attraction contains the critical point c_+. Even if the parametrization of hyperbolic components of holomorphically parametrized maps is well known (c.f. [DH85a]), the non-holomorphic dependence on a of the family B_a adds some extra difficulties to the procedure.

This section is structured as follows: first we prove a proposition that is useful later on, then we see that the multiplier map is a homeomorphism between any disjoint hyperbolic component and the unit disk proving Theorem 5.4.2, which is the main result of the section, and finally we study the bitransitive case.

The following proposition tells us that, given B_a with $|a| > 2$, the boundaries of the connected components of every attracting cycle contained in $\mathbb{C}^* \setminus \mathbb{S}^1$ are Jordan curves. The result is a direct consequence of the relation of the family B_a with polynomials which has been described in Proposition 4.2.2 and Theorem 5.3.4, respectively.

Proposition 5.4.1. *Assume that B_a has an attracting cycle $< z_0 >$ which is contained in $\mathbb{C}^* \setminus \mathbb{S}^1$. Then, the boundaries of the connected components of the basin of attraction $A(< z_0 >)$ are Jordan curves.*

Proof. It follows from the hypothesis of the proposition that $|a| > 2$ since for $1 < |a| \leq 2$ any attracting cycle other than $z = 0$ or $z = \infty$ is contained in $\mathbb{S}^1$ and for $|a| < 1$ there are no attracting cycles in $\mathbb{C}^*$. It follows from Proposition 4.2.2 and Theorem 5.3.4 that the closure of every connected component of $A^*(< z_0 >)$ is homeomorphic to the closure of a connected component of a bounded attracting cycle of a polynomial. Since the boundary of every bounded Fatou component of a polynomial other than a Siegel disk is a Jordan curve (see [RY08]), the boundary of every connected component of $A^*(< z_0 >)$ is also a Jordan curve. Finally, since all critical points are contained in the immediate basins of attraction of attracting cycles, the closure of every connected component U of $A(< z_0 >) \setminus A^*(< z_0 >)$ is mapped homeomorphically to the closure of a connected component of $A^*(< z_0 >)$ and, therefore, ∂U is a Jordan curve too. □

We now prove Theorem 5.4.2, which states that the multiplier map is a homeomorphism between any hyperbolic component with an attracting cycle in $\mathbb{C}^* \setminus \mathbb{S}^1$ and the unit disk. Notice that, since the family B_a does not depend holomorphically on a, we should not expect this homeomorphism to be conformal. Indeed, it may not even extend to the boundary of the hyperbolic component (c.f. Theorem 6.3.2). The main idea of the proof is to build a local inverse of the multiplier map Λ around every $\Lambda(a_0) \in \mathbb{D}$. It is done performing a cut and paste surgery to change the multiplier of the attracting cycle using a degree 2 Blaschke product with an attracting cycle of the desired multiplier as a model (see [BF14, Chapter 4.2]).

Theorem 5.4.2. *Let Ω be a disjoint hyperbolic component, $\Omega \subset \{a \in \mathbb{C}; |a| > 2\}$. Then, the multiplier map is a homeomorphism between Ω and the unit disk.*

Proof. Consider the family of degree 2 Blaschke products

$$b_\lambda(z) = z\frac{z+\lambda}{1+\overline{\lambda}z},$$

where $\lambda \in \mathbb{D}$. They have 0 and ∞ as attracting fixed points of multipliers λ and $\overline{\lambda}$, respectively. The only other fixed point $\frac{1-\lambda}{1-\overline{\lambda}} \in \mathbb{S}^1$ is repelling. The multiplier λ and the repelling fixed point in $\mathbb{S}^1$ determine univalently the map b_λ since any holomorphic self-map of degree 2 of $\mathbb{D}$ has the previous form. Its Julia set satisfies $\mathcal{J}(b_\lambda) = \mathbb{S}^1$. Furthermore, for every r such that $|\lambda| < r < 1$, $D_\lambda = b_\lambda^{-1}(\mathbb{D}_r)$ is a simply connected open set which compactly contains the disk of radius r, $\mathbb{D}_r = \{z, |z| < r\}$, whereas $b_\lambda(\mathbb{D}_r)$ is compactly contained in $\mathbb{D}_r$.

Let $a_0 \in \Omega$ and let λ_0 be the multiplier of the attracting cycle $< z_0 >$ of period p of B_{a_0} such that $c_+ \in A^*(z_0)$. Since there is no other critical point in $\overline{A^*(< z_0 >) \setminus A^*(z_0)}$ and $\partial A^*(z_0)$ is a Jordan curve (Proposition 5.4.1), the map $B_{a_0}^p : \overline{A^*(z_0)} \to \overline{A^*(z_0)}$ has degree 2 and a unique fixed point z_0' in $\partial A^*(z_0)$. Let $\mathcal{R} : A^*(z_0) \to \mathbb{D}$ be the Riemann map sending z_0 to 0 and z_0' to $\frac{1-\lambda_0}{1-\overline{\lambda}_0}$. The map $\mathcal{R} \circ B_{a_0}^p \circ \mathcal{R}^{-1}$ is, by construction, the restriction to $\mathbb{D}$ of the Blaschke product $b_{\lambda_0}(z)$. Fix r' and r so that $|\lambda_0| < r' < r < 1$. We proceed now to perform a surgery to the product b_{λ_0} which changes the multiplier of the attracting fixed point 0 to λ for any $|\lambda| < r'$.

Let $D_{\lambda_0} = b_{\lambda_0}^{-1}(\mathbb{D}_r)$ and let A_{λ_0} denote the annulus $D_{\lambda_0} \setminus \mathbb{D}_r$. Define $g_\lambda : \mathbb{D} \to \mathbb{D}$ as

$$g_\lambda = \begin{cases} b_{\lambda_0} & \text{on} \quad \mathbb{D} \setminus D_{\lambda_0} \\ b_\lambda & \text{on} \quad \overline{\mathbb{D}}_r \\ h_\lambda & \text{on} \quad A_{\lambda_0}, \end{cases}$$

where h_λ is chosen to be a quasiconformal map which interpolates b_λ and b_{λ_0} depending continuously on λ. Such a interpolating map can be taken since the boundary maps $g_\lambda|_{\partial A_{\lambda_0}}$ are degree 2 analytic maps on analytic curves. The inner boundary map depends continuously on λ whereas the outer map is independent of it. Therefore, the map $h_\lambda : A_{\lambda_0} \to A_\lambda$, where A_λ denotes the annulus $\mathbb{D}_r \setminus b_\lambda(\mathbb{D}_r)$, can be chosen to be a quasiconformal covering map of degree 2 which depends continuously on λ (see Proposition 2.1.16). We define recursively a g_λ-invariant almost complex structure $\tilde{\sigma}_\lambda$ as

$$\widetilde{\sigma}_\lambda = \begin{cases} \sigma_0 & \text{on} \quad \mathbb{D}_r \\ h_\lambda^* \sigma_0 & \text{on} \quad A_{\lambda_0} \\ (b_{\lambda_0}^n)^* \widetilde{\sigma}_\lambda & \text{on} \quad b_{\lambda_0}^{-n}(A_{\lambda_0}), \end{cases}$$

where σ_0 denotes the standard complex structure. Notice that, since any $z \in \mathbb{D}$ can go at most once trough A_{λ_0}, $\widetilde{\sigma}_\lambda$ has bounded dilatation. Indeed, $||h_\lambda^* \sigma_0||_\infty := k(\lambda) < 1$ since h_λ is quasiconformal and the pull backs $(b_{\lambda_0}^n)^*$ do not increase the dilatation. Moreover, $\lambda \to \widetilde{\sigma}_\lambda(z)$ varies continuously with λ for all $z \in \mathbb{D}$ since h_λ depends continuously on λ and A_{λ_0} does not depend on λ. Notice also that the almost complex structures have dilatation uniformly bounded by $k := max_{|\lambda| \leq r'} k(\lambda) < 1$ for all $\lambda \in \mathbb{D}_{r'}$.

Once we have performed the multiplier surgery in the degree 2 Blaschke model, we glue it in B_{a_0}. This is done preserving the symmetry of the family, i.e., the new map is preserved under pre and post composition by $\mathcal{I}(z) = 1/\overline{z}$. Define the model map F_λ as

$$F_\lambda = \begin{cases} (B_{a_0}^{-1})^{(p-1)} \circ \mathcal{R}^{-1} \circ g_\lambda \circ \mathcal{R} & \text{on} \quad A^*(z_0) \\ \mathcal{I} \circ F_\lambda \circ \mathcal{I} & \text{on} \quad \mathcal{I}(A^*(z_0)) \\ B_{\lambda_0} & \text{elsewhere}, \end{cases}$$

where $(B_{a_0}^{-1})^{(p-1)}$ denotes $B_{a_0}^{-1} \circ \overset{p-1}{\cdots} \circ B_{a_0}^{-1}$. It is well defined since $B_{a_0} : A^*(z_i) \to A^*(z_{i+1})$ is conformal for every $i \neq 0$. The map F_λ depends continuously on λ, is symmetric with respect to $\mathbb{S}^1$ and holomorphic everywhere except in $\mathcal{R}^{-1}(A_{\lambda_0}) \cup \mathcal{I}(\mathcal{R}^{-1}(A_{\lambda_0}))$. Notice also that the periodic cycle $< z_0 >$ of F_λ has multiplier λ and that F_λ has a unique critical point in $\mathbb{C} \setminus \mathbb{D}$ which depends continuously on λ. We define recursively an F_λ-invariant almost complex structure σ_λ as

$$\sigma_\lambda = \begin{cases} \mathcal{R}^* \widetilde{\sigma}_\lambda & \text{on} \quad A^*(z_0) \\ \mathcal{I}^* \sigma_\lambda & \text{on} \quad \mathcal{I}(A^*(z_0)) \\ (B^n_{a_0})^* \sigma_\lambda & \text{on} \quad B^{-n}_{a_0}(A^*(z_0)) \setminus A^*(z_0) \\ (B^n_{a_0})^* \sigma_\lambda & \text{on} \quad B^{-n}_{a_0}(\mathcal{I}(A^*(z_0))) \setminus \mathcal{I}(A^*(z_0)) \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

Notice that we are pulling back the almost complex structure σ_λ by the antiholomorphic map $\mathcal{I}(z)$ (see Section 1.2.1 of [BF14] for an introduction to pull backs under orientation reversing maps) and that, since we only pull back under holomorphic and antiholomorphic maps, $||\widetilde{\sigma}_\lambda||_\infty = ||\sigma_\lambda||_\infty$. By Theorem 1.2.14, $B_{a_0}|_{\mathbb{S}^1}$ is conjugate to the doubling map and, therefore, has a unique fixed point $x_0 \in \mathbb{S}^1$. Let $\phi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the integrating map of σ_λ obtained from the Measurable Riemann Mapping Theorem (see Theorem 2.1.23) normalized so that it fixes 0, x_0 and ∞. Since $\sigma_\lambda(z)$ depends continuously on λ for all $z \in \hat{\mathbb{C}}$ with dilatation which is uniformly bounded away from 1, the map $\phi_\lambda(z)$ depends continuously on λ for each $z \in \hat{\mathbb{C}}$. It follows from the uniqueness of the integrating map and the symmetry of σ_λ with respect to $\mathbb{S}^1$ (i.e., $\sigma_\lambda = \mathcal{I}^* \sigma_\lambda$) that ϕ_λ is also symmetric with respect to $\mathbb{S}^1$. Therefore, $\tilde{B}_\lambda = \phi_\lambda \circ F_\lambda \circ \phi_\lambda^{-1}$ is a degree 4 holomorphic map of $\hat{\mathbb{C}}$ symmetric with respect to $\mathbb{S}^1$ which has $z = 0$ and $z = \infty$ as superattracting cycles of local degree 3. Therefore, $\tilde{B}_\lambda$ is a Blaschke product of the form $B_{\tilde{a}(\lambda), \tilde{t}(\lambda)}$ (3.1). Since F_λ has a unique critical point in $\mathbb{C} \setminus \mathbb{D}$ which depends continuously on λ and $\phi_\lambda(z)$ depends continuously on λ for each $z \in \hat{\mathbb{C}}$, $B_{\tilde{a}, \tilde{t}}$ has a unique critical point $c_+(\lambda) \in \mathbb{C} \setminus \mathbb{D}$ which depends continuously on λ. Therefore, since $B_{\tilde{a}(\lambda), \tilde{t}(\lambda)}$ fixes $x_0 \in \mathbb{S}^1$, we have by Lemma 3.1.2 that $\tilde{a}(\lambda)$ and $\tilde{t}(\lambda)$ depend continuously on λ. Finally, by Lemma 3.1.1, $B_{\tilde{a}(\lambda), \tilde{t}(\lambda)}$ is conjugate to a Blaschke product $B_{a(\lambda)}$ (1), where $a(\lambda) = \tilde{a}(\lambda) e^{\frac{2\pi i \tilde{t}(\lambda)}{3}}$ depends continuously on λ.

To finish the proof we check that $a(\lambda_0) = a_0$ and, therefore, every $a(\lambda)$ belongs to the same hyperbolic component Ω as a_0. We have not justified that the quasiconformal interpolating map h_{λ_0} equals b_{λ_0} and, hence, B_{a_0} and $B_{\tilde{a}(\lambda_0), \tilde{t}_0}$ might be distinct. However, the integrating map ϕ_{λ_0} is a conformal conjugacy between them in $\widehat{\mathbb{C}} \setminus \overline{A(< z_0 >) \cup A(< \mathcal{I}(z_0) >)}$ and is a quasiconformal conjugacy in a neighborhood of their Julia sets. Define $\widetilde{\phi}_{\lambda_0}$ to be the conformal map from $A(< z_0 >) \cup A(< \mathcal{I}(z_0) >)$ to $A(< \phi_{\lambda_0}(z_0) >) \cup A(< \mathcal{I}(\phi_{\lambda_0}(z_0)) >)$ such that, restricted to every connected component, coincides with the Riemann map normalized so that the attracting cycles $< z_0 >$, $< \mathcal{I}(z_0) >$, $< z_0' >$ and $< \mathcal{I}(z_0') >$ are mapped to $< \phi_{\lambda_0}(z_0) >$, $< \mathcal{I}(\phi_{\lambda_0}(z_0)) >$, $< \phi_{\lambda_0}(z_0') >$ and $< \mathcal{I}(\phi_{\lambda_0}(z_0')) >$ and their preimages are in correspondence. Since $B^p_{a_0}$ is conjugate to b_{λ_0} (resp. $b_{\overline{\lambda_0}}$) in $A^*(z_0)$ (resp. $A^*(\mathcal{I}(z_0))$) and so is $B^p_{\tilde{a}(\lambda_0), \tilde{t}_0}$ in $A^*(\phi_{\lambda_0}(z_0))$ (resp. $A^*(\mathcal{I}(\phi_{\lambda_0}(z_0)))$), the conformal map $\widetilde{\phi}_{\lambda_0}$ is a conjugacy. Moreover, it extends to the boundary of every connected component of the basins of attraction since they are Jordan domains by Proposition 5.4.1. Given that ϕ_{λ_0} and $\widetilde{\phi}_{\lambda_0}$ conjugate B_{a_0} and $B_{\tilde{a}(\lambda_0), \tilde{t}_0}$ in $\partial A(< z_0 >) \cup \partial A(< \mathcal{I}(z_0) >) \subset \mathcal{J}(B_{a_0})$ they coincide since they map periodic points to periodic points. Consequently, the map φ_{λ_0} defined as ϕ_{λ_0} in $\hat{\mathbb{C}} \setminus (A(< z_0 >) \cup A(< \mathcal{I}(z_0) >))$ and $\widetilde{\phi}_{\lambda_0}$ in $A(< z_0 >) \cup A(< \mathcal{I}(z_0) >)$ is a global conjugacy. Moreover, since φ_{λ_0} is quasiconformal in $\widehat{\mathbb{C}} \setminus (\partial A(< z_0 >) \cup \partial A(< \mathcal{I}(z_0) >))$, coincides with ϕ_{λ_0} in $\mathcal{J}(B_{\lambda_0})$ and ϕ_{λ_0} is quasiconformal in a neighborhood of $\mathcal{J}(B_{\lambda_0})$, φ_{λ_0} is quasiconformal by the Rickman Lemma (Lemma 2.1.10). Since φ_{λ_0} is conformal a.e. in $\widehat{\mathbb{C}}$, it is 1-quasiconformal and therefore a conformal map of $\widehat{\mathbb{C}}$ by Weyl's Lemma (Theorem 2.1.22). Since φ_{λ_0} fixes 0 and ∞, leaves $\mathbb{S}^1$ invariant and fixes $x_0 \in \mathbb{S}^1$, we conclude that φ_{λ_0} is the identity and $B_{\tilde{a}(\lambda_0),\tilde{t}_0} = B_{a_0}$.

For every $a_0 \in \Omega$ we have constructed a continuous local inverse to the multiplier map $\Lambda : \Omega \to \mathbb{D}$. Therefore, Λ is a homeomorphism. □

We finish this section giving some ideas of what happens with bitransitive parameters (see Figure 5.4 (c)). It follows from Theorem 5.3.4 that these parameters are strongly related to the quadratic antiholomorphic polynomials $p_c(z) = \overline{z}^2 + c$. Indeed, the polynomial-like map constructed in Theorem 5.3.4 is hybrid equivalent to a degree 4 polynomial of the form $p_c^2(z)$ with a bitransitive attracting cycle. Therefore, the polynomial $p_c(z)$ also has an attracting cycle of odd period since, otherwise, $p_c^2(z)$ would have two disjoint attracting cycles. Nakane and Schleicher [NS03] studied the parameter plane of the antipolynomials $p_{c,d} = \overline{z}^d + c$ and, in particular, $p_{c,2}(z) = p_c(z)$. If the period of the cycles of a hyperbolic component was even, they proved a result analogous to Theorem 5.4.2. They also showed that the multiplier map is not a good model for the odd period hyperbolic components. The reason why the multiplier map is not good for this case is the fact that the antiholomorphic multiplier $\frac{\partial}{\partial \overline{z}} f^k(z_0)$ of a cycle $< z_0 >$ of odd period k of an antiholomorphic map $f(z)$ is not a conformal invariant, only its absolute value is. They proved that the multiplier of the period k cycle $< z_0 >$ of the holomorphic map $f^2(z)$ equals the square of the absolute value of the previous antiholomorphic multiplier. Given a bitransitive hyperbolic component Ω of $p_c^2(z)$, it also follows from their work that the set of parameters $c \in \Omega$ for which the attracting cycle has multiplier $\lambda \in (0, 1)$ is a Jordan curve and that Ω contains a unique parameter c_0 for which the cycle is superattracting. We expect a similar result for bitransitive hyperbolic components of the Blaschke family B_a, but we only prove, for the sake of completeness, the following result.

Proposition 5.4.3. *Let $< z_0 >$ be a bitransitive cycle of a Blaschke product B_a as in (1) with $|a| > 2$. Then $< z_0 >$ has non-negative real multiplier.*

Proof. By Lemma 5.3.3, the cycle $< z_0 >$ has even period $2q$. Let $\mathcal{I}(z) = 1/\overline{z}$. By symmetry, $\mathcal{I}(B_a^q(z_0)) = z_0$. Hence, z_0 is a fixed point of the antiholomorphic rational map $f = \mathcal{I} \circ B_a^q$. Moreover, $B_a^{2q} = f^2$. Therefore, the multiplier of the cycle is given by

$$\frac{\partial}{\partial z} B_a^{2q}(z_0) = \frac{\partial}{\partial \overline{z}} f(z_0) \cdot \overline{\frac{\partial}{\partial \overline{z}} f(z_0)} = \left| \frac{\partial}{\partial \overline{z}} f(z_0) \right|^2 .$$

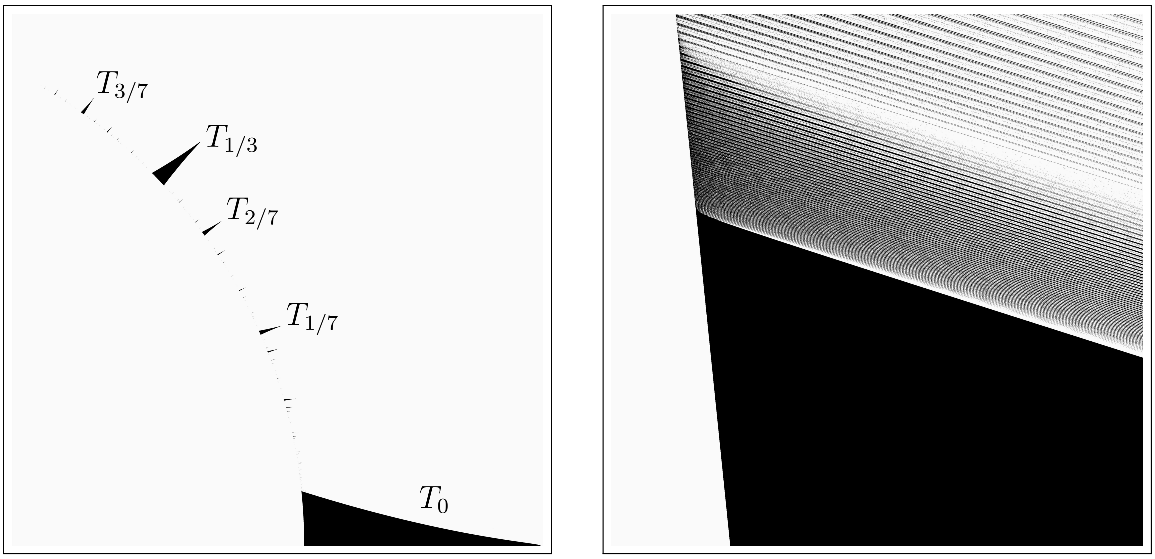
□

Chapter Six

Tongues in the parameter plane

In this chapter we study the tongues in the parameter plane of the Blaschke family B_a as defined in Section 1.2.2. If $|a| \geq 2$, the rational maps B_a restrict to increasing degree 2 covers of the unit circle (see Section 3.1) and hence the tongues are well defined as open sets of parameters such that B_a has an attracting cycle on $\mathbb{S}^1$ (see Figure 6.1). We recall the concept of tongues (c.f. Definition 1.2.17). Let H_a be the continuous map given by Lemma 1.2.3 which semiconjugates the lift of $B_a|_{\mathbb{S}^1}$ to the doubling map $x \to 2x$ and let x_0 be the marked point of the attracting cycle $< x_0 >$ of $B_a|_{\mathbb{S}^1}$, i.e., the point of the cycle such that $A^*(x_0)$ contains both free critical points. Then, tongues are defined as follows.

Definition. We say that a parameter a, $|a| \geq 2$, is of *type* τ if $B_a|_{\mathbb{S}^1}$ has an attracting cycle $< x_0 >$ and $H_a(x_0) = \tau$, where x_0 is the marked point point of the cycle. The *tongue* T_τ is defined as the set of parameters a, $|a| \geq 2$, such that a is of type τ.



(a) Tongues (b) Zoom in the tongues

Figure 6.1: In figure (a) we show the tongues of the Blaschke family for $a = re^{2\pi i\alpha}$ such that $0 < \alpha < 1/6$. Notice that we know, from the symmetries explained in Lemma 5.1.1, that these parameters give complete information about the family. In figure (b) we zoom near the boundary of T_0. We can see how smaller tongues accumulate on it.

In Section 6.1 we introduce an alternative parametrization of the Blaschke family which we use later. In Section 6.2 we prove that all tongues are simply connected and connected modulo symmetry and that every tongue has a unique tip (Theorem 6.2.1). In Section 6.3 we study how bifurcations take place on the boundary of the tip of every tongue. Finally, in Section 6.4 we study how the tongues may be extended within the annulus of parameters given by $1 < |a| < 2$.

6.1 Preliminaries: reparametrizing the Blaschke family

In this section we describe some alternative parametrizations of B_a. They are particularly useful when we restrict to the unit circle. Let $a = re^{2\pi i\alpha}$ with $r \in (1, \infty)$ and $\alpha \in [0, 1/3)$ (or $\alpha \in \mathbb{R}/\frac{1}{3}\mathbb{Z}$). From Lemma 3.1.1 we know that $B_a = B_{a,0}$ (Equation (1)) is conformally conjugate to $B_{r,3\alpha}$ (Equation (3.1)). It is enough to restrict to parameters $\alpha \in [0, 1/3)$ due to symmetry (see Lemma 5.1.1). Notice that if $\alpha \in [0, 1/3)$ we have a one to one correspondence between the parameters a for the family B_a and the parameters (r, α) of $g_{r,\alpha} := B_{r,3\alpha}|_{\mathbb{S}^1}$. Summarizing, we consider the circle maps

$$g_{r,\alpha}(e^{2\pi ix}) = e^{6\pi ix}e^{6\pi i\alpha}\frac{e^{2\pi ix} - r}{1 - re^{2\pi ix}}, \tag{6.1}$$

where $r \in (1, \infty)$ and $\alpha \in [0, 1/3)$. Its lift has the form

$$h_{r,\alpha}(x) = 3x + 3\alpha + \frac{1}{2\pi i}\log\left(\frac{e^{2\pi ix} - r}{1 - re^{2\pi ix}}\right). \tag{6.2}$$

We shall often use $g_{r,\alpha}$ instead of $B_a|_{\mathbb{S}^1}$ given that its lift is somehow simpler. Indeed, it follows directly from its expression that $h_{r,\alpha}$ is strictly increasing with respect to α.

Lemma 6.1.1. *Let $r \geq 2$. Then, the lift $h_{r,\alpha}(x)$ satisfies that $\frac{\partial}{\partial x}h_{r,\alpha}(x)$ is non-negative for all x. Moreover, for any $p \in \mathbb{N}$, the mapping $\alpha \to h^p_{r,\alpha}(x) \in \mathbb{S}^1$ is strictly increasing and, if $r \geq 3$, then $h^p_{r,\alpha}(x) \geq 1$ for all $x, \alpha \in \mathbb{R}$.*

Proof. We prove that $\frac{\partial}{\partial x}h_{r,\alpha}(x)$ is non-negative for all x, and hence so is $\frac{\partial}{\partial x}h^p_{r,\alpha}(x)$ for all p. Then, strict monotonicity with respect to α for all p follows from the fact that we have it for $p = 1$. We also prove that $\frac{\partial}{\partial x}h_{r,\alpha}(x) \geq 1$ if $r \geq 3$. Notice that $\frac{\partial}{\partial x}h_{r,\alpha}(x)$ is given by the formula

$$\frac{\partial}{\partial x}h_{r,\alpha}(x) = 3 + \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)}. \tag{6.3}$$

It can easily be seen that this expression is non-negative for $r \geq 2$. Indeed, the minimum of this function is taken whenever $x = 0$, and

$$\frac{\partial}{\partial x}h_{r,\alpha}(0) = 3 + \frac{1 - r^2}{1 + r^2 - 2r} = 3 + \frac{(1 + r)}{(1 - r)}.$$

For $r > 1$ this is an increasing function which is equal to zero for $r = 2$. Moreover, it is greater than 1 when $r \geq 3$. $\square$

It will also be useful to consider the circle maps $g_{r,\alpha}$ as restrictions of the rational maps

$$G_{a,b}(z) = bz^3 \frac{z-a}{1-az}, \tag{6.4}$$

where $a, b \in \mathbb{C}$. The $G_{a,b}$ define a degree 4 family of almost bicritical maps unless $a = \pm 1$, when they degenerate to the degree 3 polynomials $\mp bz^3$. Notice that the $G_{a,b}$ are not symmetric with respect to the unit circle. Similarly to the Blaschke products B_a, the points $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3 of the $G_{a,b}$. The two free critical points are given by

$$c_\pm := c_\pm(a) := \frac{1}{3a}\left(2 + a^2 \pm \sqrt{(a^2-4)(a^2-1)}\right)$$

and are the solutions of $3az^2 - 2(a^2-2)z + 3a = 0$. In particular we have that $c_+ \cdot c_- = 1$ and none of them is equal to zero if $a \neq 0$ and $a \neq \infty$.

Recall from Definition 1.1.24 that a parameter (a, b) is said to be escaping if the orbit of any of the critical points accumulates on $z = 0$ or $z = \infty$. We finish this section studying the non-escaping set of the family $G_{a,b}$, i.e., the set of parameters (a, b) for which none of the critical orbits accumulates on $z = 0$ or $z = \infty$.

Lemma 6.1.2. *The non-escaping set of the family $G_{a,b}$ is bounded in the a-parameter, i.e., there exists a constant $\mathcal{C} > 0$ such that if $|a| > \mathcal{C}$ then the parameter (a, b) is escaping.*

Proof. The proof is similar to the one of Lemma 5.2.1. However, in this family the critical orbits are not symmetric as is the case for the B_a. We will prove that, if $|a|$ is big enough then one of the critical orbits accumulates on $z = \infty$ or on $z = 0$. Notice that the other critical orbit may accumulate on a bounded attracting cycle even if $|a|$ tends to infinity.

We distinguish two cases. Assume first that $|b| \geq 1$. Using a similar approach as in Lemma 5.2.1 we have that, if $|z| > \lambda(|a| + 1)$ with $\lambda \geq 1$, then $|B_a(z)| > \lambda|b||z|$. As $|a|$ tends to infinity the critical point $c_+(a)$ tends to $2a/3$ and $c_-(a)$ tends to $3/(2a)$. Consequently, the modulus of the critical value $v_+ = G_{a,b}(c_+(a))$ grows as $M|a|^2$ for some $M > 0$ and, for $|a|$ large enough, $|v_+| > \lambda(|a| + 1)$ with $\lambda > 1$. We conclude that $|G^n_{a,b}(v_+)| \to \infty$ when $n \to \infty$. Therefore, if $|a|$ is large enough and $|b| \geq 1$ then the parameter (a, b) is escaping.

Consider now the case $|b| < 1$. First we prove that, if $|a| > 1$ and $|z| < 1/(2|a|)$ then $|G_{a,b}(z)| < 3|b||z|/4$. From these hypothesis we conclude that $|z| < 1/2$ and obtain the inequalities

$$|z - a| < |z| + |a| < \frac{1}{2|a|} + |a| < \frac{|a|}{2} + |a| = 3|a|/2$$

and

$$|1 - az| > 1 - |az| > 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore, we have

$$|G_{a,b}(z)| = |b||z|^3 \frac{|z-a|}{|1-az|} < |b||z|^3 3|a| < \frac{3}{2}|b||z|^2 < \frac{3|b||z|}{4}.$$

Since $c_-(a)$ converges to $3/(2a)$ as a tends to infinity, we conclude that the modulus of the critical value $v_- = G_{a,b}(c_-)$ decreases as $M/|a|^2$ for some $M > 0$. Hence, for $|a|$ large enough, $|v_-| < 1/(2|a|) < 1/3$ and $|G^n_{a,b}(v_-)| \to 0$ when $n \to \infty$. Therefore, if $|a|$ is large enough and $|b| < 1$ then the parameter (a, b) is escaping.

$\square$

Lemma 6.1.3. *For fixed $a_0 \in \mathbb{C}$, $a_0 \neq \pm 1$, the non-escaping set of $G_{a_0,b}$ is bounded with respect to the parameter b, i.e., there exists a constant $\mathcal{C}(a_0) > 0$ such that if $|b| > \mathcal{C}(a_0)$ then the parameter (a_0, b) is escaping.*

Proof. It is enough to prove that if $|b|$ is large enough then the orbit of one of the critical points accumulates on infinity. The critical values of $G_{a_0,b}$ are given by

$$v_\pm = G_{a_0,b}(c_\pm) = bc_\pm^3 \frac{c_\pm - a_0}{1 - c_\pm a_0},$$

where the critical points $c_\pm$ do not depend on b. Moreover, if $a \neq \pm 1$, at least one of the critical values, say v, is different from zero. Indeed, the rational maps $G_{a,b}$ have a unique preimage of zero at $z = a$. The claim holds since the critical points collide only if $a = \pm 2$ and $c_+(\pm 2) = c_-(\pm 2) = \pm 1$. If $|b|$ is large enough the critical value v satisfies $|v| > \lambda(|a_0| + 1)$ with $\lambda > 1$. As in the proof of Lemma 6.1.2, we conclude that the orbit of v accumulates on infinity and, therefore, for $|b|$ large enough the parameter (a_0, b) is escaping.

$\square$

6.2 Topological properties of the tongues

The goal of this section is to prove the following theorem, which states the main topological properties of the tongues of the family B_a.

Theorem 6.2.1. *Given any periodic point τ of the doubling map the following results hold.*

(a) The tongue T_τ is not empty and consists of three connected components (only one connected component if we consider the parameter plane modulo the symmetries given by the third roots of the unity).

(b) Each connected component of T_τ contains a unique parameter r_τ, called the root of the tongue*, such that B_{r_τ} has a superattracting cycle in $\mathbb{S}^1$. The root r_τ satisfies $|r_\tau| = 2$.*

(c) Every connected component of T_τ is simply connected.

(d) The boundary of every connected component of T_τ consists of two curves which are continuous graphs as function of $|a|$ and intersect each other in a unique parameter a_τ called the tip of the tongue.

The proof of Theorem 6.2.1 splits in the next two subsections.

6.2.1 Connectivity of the tongues: proof of statements (a) and (b) of Theorem 6.2.1

In this subsection we prove statements (a) and (b) of Theorem 6.2.1. The proof is inspired by Dezotti [Dez10] and consists in performing a continuous change of the multiplier of the attracting cycle $< x_0 > \subset \mathbb{S}^1$. Given a parameter a of type τ, with multiplier $\lambda \neq 0$, we make a quasiconformal modification of the function that changes the multiplier to $\rho \in (0,1)$ while leaving the rest of the dynamics unchanged, obtaining a new parameter $a(\rho)$. With this modification we obtain a path $\gamma \subset T_\tau$ landing on a parameter $r_\tau \in T_\tau$ having a supperattracting fixed point, which is unique in T_τ.

We begin by changing the multiplier. The main steps of this quasiconformal construction are the following.

(a) First we consider a linearising map ϕ of B_a^p around the attracting periodic point x_0, which has period p and multiplier $\lambda \in (0,1)$.

(b) Then we define a quasiconformal conjugacy $\mathcal{X}$ between the maps $z \to \lambda z$ and $z \to \rho z$, where $\rho \in (0,1)$.

(c) We continue by defining a B_a-invariant Beltrami form μ_ρ. Around x_0, it is defined by pulling back the standard Beltrami coefficient $\mu_0 \equiv 0$ by the quasiconformal homeomorphism $\mathcal{X} \circ \phi$. Then we spread it by the dynamics of B_a.

(d) Finally we consider the map $\varphi_\rho \circ B_a \circ \varphi_\rho^{-1}$, where φ_ρ is the integrating map of μ_ρ which fixes zero, infinity and x_0 (see Theorem 2.1.23). This map is holomorphic and linearly conjugate to a member $B_{a(\rho)}$ of the Blaschke family.

We now proceed to make the construction precise. Let $a \in T_\tau$. Recall from the definition of tongues (see Section 1.2.2) that then $B_a|_{\mathbb{S}^1}$ has an attracting cycle $< x_0 >= \{x_0, ..., x_{p-1}\} \subset \mathbb{S}^1$ of multiplier λ. We assume that x_0 lies in the component of the immediate basin which contains the critical points. Notice that $\lambda \in \mathbb{R}$ since $B_a|_{\mathbb{S}^1}$ is an endomorphism of the unit circle. Let $\mathbb{D}_R = \{z, |z| < R\}$ and let $\phi : \mathcal{A} \twoheadrightarrow \mathbb{D}_R$ be the Kœnigs linearizer of B_a^p around x_0 (see Theorem 1.1.10) normalized as in the following lemma.

Lemma 6.2.2. *The map $\phi : \mathcal{A} \to \mathbb{D}_R$ may be chosen to satisfy $\phi(\mathcal{I}(z)) = \overline{\phi(z)}$, where $\mathcal{I}(z) = 1/\overline{z}$. Moreover, $\mathcal{A} = \mathcal{I}(\mathcal{A})$*

Proof. The map ϕ sends invariant curves of $B_a|_{\mathcal{A}}$ to invariant curves of $z \to \lambda z$, which are straight lines going through $z = 0$ since λ is real. Hence, we may assume that $\phi(\mathcal{A} \cap \mathbb{S}^1) \subset \mathbb{R}$ postcomposing ϕ with a rotation. With the previous normalization, notice that the holomorphic map $\widehat{\phi}(z) = \overline{\phi(1/\overline{z})}$ coincides with ϕ on $\mathcal{A} \cap \mathbb{S}^1$ and, therefore, it equals ϕ. The symmetry of $\mathcal{A}$ follows from the symmetry of ϕ.

□

We now introduce a quasiconformal map $\mathcal{X}$ which is used to change the multiplier of an attracting cycle (c.f. [Dez10] and [BF14]).

Lemma 6.2.3. *Let*

$$\mathcal{X} : \mathbb{C}^* \to \mathbb{C}^*$$
$$z \to |z|^\alpha z,$$

where $\alpha \in (-1, \infty)$ *and let* $0 < R < 1$ *and* $0 < r < 1$*. Then the following hold.*

(a) The Beltrami form $\mu_{\mathcal{X}} = \mathcal{X}^* \mu_0$*, where* μ_0 *denotes the Beltrami form of the standard complex structure, satisfies*

$$\mu_{\mathcal{X}} = \frac{\partial \mathcal{X}/\partial \overline{z}}{\partial \mathcal{X}/\partial z} = \frac{\alpha}{2+\alpha}\frac{z}{\overline{z}},$$

$$||\mu_{\mathcal{X}}||_\infty = \left|\frac{\alpha}{2+\alpha}\right|.$$

(b) $\mathcal{X}$ *is invertible and satisfies* $\mathcal{X}(re^{2\pi i\theta}) = \xi(r)e^{2\pi i\theta}$*, where* $\xi(r) = r^{\alpha+1}$*.*

(c) Let $\alpha = \frac{\log r}{\log R} - 1$*. Then* $\mathcal{X}$ *sends the disk* $\mathbb{D}_R$ *of radius* R *to the disk* $\mathbb{D}_r$ *of radius* r *and, moreover,*

$$||\mu_{\mathcal{X}}||_\infty = \frac{|1 - \log r/\log R|}{1 + \log r/\log R} < 1.$$

(d) Let $\lambda \in (0,1)$*. Let* $\rho = \mathcal{X}(\lambda) = \lambda^{1+\alpha}$*. Then* $\mathcal{X}$ *conjugates the map multiplication by* λ *with the map multiplication by* ρ*, i.e., the following diagram is commutative*

$$\begin{tikzcd}
\mathbb{D}_R \arrow[r, "z \to \lambda z"] \arrow[d, "\mathcal{X}"'] & \mathbb{D}_{\lambda R} \arrow[d, "\mathcal{X}"] \\
\mathbb{D}_r \arrow[r, "z \to \rho z"] & \mathbb{D}_{\rho r}.
\end{tikzcd}$$

Proof. By definition, $\mathcal{X}(z) = |z|^\alpha z = z^{\alpha/2+1}\overline{z}^{\alpha/2}$. Then, we have

$$\frac{\partial \mathcal{X}}{\partial \overline{z}} = \frac{\alpha}{2} z^{\alpha/2}\overline{z}^{(\alpha/2-1)} z,$$

$$\frac{\partial \mathcal{X}}{\partial z} = (\frac{\alpha}{2} + 1) z^{\alpha/2}\overline{z}^{(\alpha/2-1)}\overline{z}.$$

We obtain, by Proposition 2.1.19,

$$\mu_{\mathcal{X}} = \frac{\partial \mathcal{X}/\partial \overline{z}}{\partial \mathcal{X}/\partial z} = \frac{\alpha}{2+\alpha}\frac{z}{\overline{z}}.$$

From $\mathcal{X}(z) = |z|^\alpha z$ we also have that

$$\mathcal{X}(re^{2\pi i\theta}) = r^{\alpha+1}e^{2\pi i\theta} = \xi(r)e^{2\pi i\theta}.$$

The commutativity of the diagram follows from (b). Finally, for $\alpha = \frac{\log r}{\log R} - 1$,

$$\xi(R) = R^{\alpha+1} = R^{\frac{\log r}{\log R}} = r,$$

so $\mathcal{X}$ sends $\mathbb{D}_R$ to $\mathbb{D}_r$.

∎

Notice that, since $\lambda \in (0,1)$ and $\alpha \in (-1,\infty)$, we have that $0<\rho<1$.

Lemma 6.2.4. *The Beltrami form given by $\mu_\rho = \mu_{\mathcal{X}\circ\phi} = \phi^*\mu_{\mathcal{X}}$ depends analytically on ρ and is invariant under B_a^p. Moreover, $\mathcal{I}^*\mu_\rho = \mu_\rho$ on $\mathcal{A}$.*

Proof. The analytic dependence with respect to ρ is obtained from the explicit expression of $\mu_{\mathcal{X}}$ and the fact that $\alpha = \frac{\log\rho}{\log\lambda} - 1$. Invariance under B_a^p follows from the next commutative diagram.

$$\begin{tikzcd}
(\mathcal{A}, \mu_{\mathcal{X}\circ\phi}) \arrow[r, "\phi"] \arrow[d, "B_a^p"'] & (\mathbb{D}_R, \mu_{\mathcal{X}}) \arrow[r, "\mathcal{X}"] \arrow[d, "z\to\lambda z"'] & (\mathbb{D}_r, \mu_0) \arrow[d, "z\to\rho z"] \\
(B_a^p(\mathcal{A}), \mu_{\mathcal{X}\circ\phi}) \arrow[r, "\phi"] & (\mathbb{D}_{\lambda R}, \mu_{\mathcal{X}}) \arrow[r, "\mathcal{X}"] & (\mathbb{D}_{\rho r}, \mu_0).
\end{tikzcd}$$

To see that $\mathcal{I}^*\mu_\rho = \mu_\rho$, we have to check that $\mu_\rho(z) = \overline{\mu_\rho(1/\overline{z})} z^2/\overline{z}^2$ (see Proposition 2.2.11). From the explicit expression of the Beltrami form that we get from the pullback (see Proposition 2.1.20), we have

$$\mu_\rho(z) = \frac{\alpha/2}{1+\alpha/2}\frac{\phi(z)}{\overline{\phi(z)}} \cdot \frac{\overline{\phi'(z)}}{\phi'(z)}.$$

The result follows since $\phi(1/\bar{z}) = \overline{\phi(z)}$ and $\phi'(z) = -\overline{\phi'(1/\overline{z})}/z^2$.

$\square$

Once we have this Beltrami form given by μ_ρ in $\mathcal{A}$, we spread it to $\overline{\mathbb{C}}$ by defining:

$$\mu_\rho = \begin{cases} \mu_\rho & \text{on} \quad \mathcal{A} \\ (B_a^n)^*\mu_\rho & \text{on} \quad B_a^{-n}(\mathcal{A}) \setminus B_a^{-n+1}(\mathcal{A}), \text{ for } n>1 \\ \mu_0 & \text{otherwise,} \end{cases}$$

where $\mu_0 \equiv 0$. Then μ_ρ is well defined since it is B_a^p-invariant. It depends analytically on ρ since we are pulling back by a holomorphic map an almost complex structure which depends analytically on ρ. Furthermore, since $\mu_\rho|_{\mathcal{A}}$ is symmetric with respect to $\mathbb{S}^1$ and μ_ρ is defined recursively by pulling back by B_a, which is also symmetric with respect to $\mathbb{S}^1$, μ_ρ also satisfies $\mathcal{I}^*\mu_\rho(z) = \mu_\rho(z)$ for all $z \in \mathbb{C}$ (see Lemma 2.2.12).

Since μ_ρ is built by pulling back $\mu_{\mathcal{X}}$ by holomorphic mappings and we have that $||\mu_{\mathcal{X}}||_\infty < 1$, we also have that $||\mu_\rho||_\infty < 1$. Let φ_ρ be the integrating map obtained by applying the Measurable Riemann Mapping Theorem (see Theorem 2.1.23) such that it fixes 0, x_0 and ∞. The next lemma follows directly from Lemma 2.2.13 since μ_ρ is symmetric with respect to the unit circle.

Lemma 6.2.5. *The integrating map φ_ρ is symmetric with respect to $\mathbb{S}^1$.*

Once we have φ_ρ, we can build our new rational map.

Proposition 6.2.6. *The map $\varphi_\rho \circ B_a \circ \varphi_\rho^{-1}$ is a rational map of degree 4 of the form B_{a_ρ,t_ρ} (3.1), where $t_\rho \in \mathbb{R}$ and $a_\rho \in \mathbb{C}$. The parameters a_ρ and t_ρ depend continuously on ρ. Moreover, the attracting fixed point x_0 of B_{a_ρ,t_ρ}^p has multiplier ρ.*

Proof. By construction, the quasiregular map $\varphi_\rho \circ B_a \circ \varphi_\rho^{-1}$ preserves the standard complex structure. Consequently, it is a holomorphic map of the Riemann sphere onto itself by Weyl's Lemma (see Theorem 2.1.22). It is of the form B_{a_ρ,t_ρ} since it has local degree 3 around 0 and ∞, it has global degree 4 and it is symmetric with respect to the unit circle (see Lemma 1.2.13).

We now check the dependence on parameters. Recall that B_a has a unique critical point $c_+ \in \mathbb{C} \setminus \mathbb{D}$ (see Section 3.1). Since μ_ρ depends real analytically on ρ, the integrating map $\varphi_\rho(z)$ depends real analytically on ρ for all $z \in \mathbb{C}$ by the analytic dependence on parameters of the Measurable Riemann Mapping Theorem (Theorem 2.1.24). Therefore, the critical point $\varphi_\rho(c_+) \in \mathbb{C} \setminus \mathbb{D}$ of B_{a_ρ,t_ρ} depends real analytically on ρ. We conclude by Lemma 3.1.2 that a_ρ depends continuously on ρ. The parameter t_ρ is continuously determined by the parameter a_ρ together with the image of a point $z_0 \neq 0, \infty$. Given that the $B_{a_\rho,t_\rho}(x_0) = \varphi_\rho \circ B_a(x_0)$ depends real analytically on ρ, we conclude that t_ρ also depends continuously on ρ.

Finally, let $\widetilde{\phi}(z) = \mathcal{X} \circ \phi \circ \varphi_\rho^{-1}(z)$, where $z \in \varphi_\rho(\mathcal{A})$. By construction $\widetilde{\phi}$ is a quasiconformal map which conjugates $B^p_{a_\rho,t_\rho}$ around x_0 to the map $z \to \rho z$. Since it preserves the standard complex structure, it is a conformal map by Weyl's Lemma. Hence, it is the linearizing function and ρ is the multiplier of the new cycle.

□

We know from Lemma 3.1.1 that B_{a_ρ,t_ρ} is conjugate to $B_{a_\rho e^{-it_\rho/3},0} = B_{a_\rho e^{-it_\rho/3}}$ by a conjugacy $\mathcal{L}_\rho$. This gives us a continuous curve in the set of parameters of our Blaschke family. Indeed, we have a (real analytic) curve $\gamma : (0,1) \to \mathbb{C} \setminus \mathbb{D}_2$ defined as $\gamma(\rho) = a_\rho e^{-it_\rho/3} = a(\rho)$.

Lemma 6.2.7. *If the parameter a has type τ, then, for all $\rho \in (0,1)$, the parameter $a(\rho)$ has type τ.*

Proof. We have already seen that $B_{a(\rho)}$ has an attracting cycle, so it has type τ'. It is easy to check that the map $\varrho(z) = \mathcal{L}_\rho \circ \varphi_\rho(z)$ conjugates $B_{a(\rho)}$ and B_a, sending the marked periodic point x_0 to $\varrho(x_0)$. Moreover, the continuous map $H_a \circ \varrho^{-1}$ semiconjugates $B_{a(\rho)}$ with the doubling map. Therefore, we have that

$$\tau' = H_a \circ \varrho^{-1}(\varrho(x_0)) = H_a(x_0) = \tau.$$

□

Now we have a path $\gamma(\rho) = a(\rho)$ defined for $\rho \in (0,1)$ which gives, for each ρ, a parameter $a(\rho) \in T_\tau$. We want to prove that this path lands at a single point when $\rho \to 0$. Note that $|a(\rho)| \to 2$ when $\rho \to 0$ since $|B'_a|_{\mathbb{S}^1}| > \mathcal{C} > 0$ when $|a| > 2+\epsilon$, where $\epsilon > 0$ and $\mathcal{C} = \mathcal{C}(\epsilon)$ is a constant. It follows from the continuous dependence on a of the semiconjugacy H_a (see Lemma 1.2.3) that any limit point of γ has a superattracting fixed point of period p.

Let ω be the limit set of $\gamma(\rho)$ when $\rho \to 0$. Since $\omega = \bigcap_n \overline{\gamma((0,1/n))}$ is a decreasing intersection of connected compact sets, we conclude that it is a connected set of parameters a such that $|a| = 2$.

We restrict now to parameters a such that $|a| = 2$. Let $a = 2e^{2\pi i\alpha}$. Throughout the rest of the proof it will be convenient to work with $B_{2,3\alpha}$ (Equation (3.1)) as in Section 6.1 so as to use Lemma 6.1.1. This map is conformally conjugate to $B_a = B_{a,0}$

(Equation (1)) by the rotation $\mathcal{L}(z) = e^{-2\pi i\alpha}z$ (see Lemma 3.1.1). For $\alpha \in [0, 1/3)$ we have a one to one correspondence between the parameters $2e^{2\pi i\alpha}$ and the parameters $(2, 3\alpha \pmod 1)$. Notice that, for all α, $B_{2,3\alpha}$ has a unique critical point at $c = 1$. Indeed, for $a = 2e^{2\pi i\alpha}$, the two critical points collapse in $c(\alpha) = e^{2\pi i\alpha}$, which is sent to $c = 1$ by the conjugacy $\mathcal{L}(z)$.

Assume that ω is not a single parameter. Then, we have an interval of parameters with a superattracting periodic cycle. Therefore, the critical point $c = 1$ is periodic in this interval of parameters, i.e., $B^p_{2,3\alpha}(1) = 1$ for all parameters $(2, 3\alpha) \in \omega$. This is impossible since $B_{2,\alpha}$ is strictly increasing with respect to α (see Lemma 6.1.1). Hence, ω is a single parameter $r_\tau = \lim_{\rho\to 0} a(\rho)$. Notice also that this parameter has type τ. Indeed, given the fact that it has a superattracting periodic point, it belongs to a tongue of type τ'. Since tongues are open sets in $\mathbb{C}\setminus\mathbb{D}_2 = \{a \text{ s.t. } |a| \geq 2\}$, we conclude that any curve of parameters contained in $\mathbb{C}\setminus\mathbb{D}_2$ and landing in ω necessarily intersects $T_{\tau'}$. We conclude that $\tau' = \tau$ since $a(\rho)$ have type τ for all $\rho \in (0, 1)$ by Lemma 6.2.7.

In order to finish the proof of statements (a) and (b) of Theorem 6.2.1 we have to show that the limit does not depend on the initial parameter $a \in T_\tau$. We use the following lemma.

Lemma 6.2.8. *Let $g_\alpha(x) := B_{2,3\alpha}|_{\mathbb{S}^1}(x)$, $x \in \mathbb{S}^1$. Then, for any $p \in \mathbb{N}$, the mapping $\alpha \to g^p_\alpha(1) \in \mathbb{S}^1$, $\alpha \in [0, 1/3)$, is strictly increasing and of degree $2^p - 1$.*

Proof. The map g_α is increassing by Lemma 6.1.1. We only have to prove that g_α has degree $2^p - 1$. The lift of g_α is given by

$$h_\alpha(x) = 3x + 3\alpha + \frac{1}{2\pi i}\log\left(\frac{e^{2\pi ix} - 2}{1 - 2e^{2\pi ix}}\right).$$

The result is true for $p = 1$ in the sense that $h_{\alpha+1/3}(x) = h_\alpha(x) + 1$. By induction over p and using that $h^p_\alpha(x + 1) = h^p_\alpha(x) + 2^p$ (g_α has degre two as a circle map) we have

$$h^{(p+1)}_{\alpha+1/3}(x) = h^p_{\alpha+1/3}(h_{\alpha+1/3}(x)) = h^p_{\alpha+1/3}(h_\alpha(x) + 1) = h^p_{\alpha+1/3}(h_\alpha(x)) + 2^p$$

$$= h^p_\alpha(h_\alpha(x)) + 2^p - 1 + 2^p = h^{p+1}_\alpha(x) + 2^{p+1} - 1.$$

□

It follows from this lemma that g^p_α has exactly $2^p - 1$ parameters α, $\alpha \in [0, 1/3)$, such that the critical point is periodic of period dividing p. Indeed, for every natural $k \in \{0, 1, ..., 2^p - 2\}$, there exists a unique $\alpha_{p,k} \in [0, 1/3)$ such that $h^p_{\alpha_{p,k}}(0) = 0 + k$. It can be computed using the expression of the semiconjugacy H_a (see Lemma 1.2.3) that a parameter $a_{k,p} = 2e^{2\pi i\alpha_{k,p}}$ has type $\tau(a_{k,p}) = k/(2^p - 1)$ (c.f. [Dez10, Lem. 4.2]). Since the expression in the form $k/(2^p - 1)$ of a periodic point τ of period p of the doubling map is unique, we conclude that, for a fixed a type τ, there exists a unique parameter $\alpha_{p,k} \in [0, 1/3)$ which has a superattracting cycle of type τ. Hence, we can also conclude that no tongue T_τ is empty. It also follows from this that the limit ω is unique up to conjugacy since, for a fixed a type τ, there exists a unique possible limit. This finishes the proof of statements (a) and (b) of Theorem 6.2.1.

6.2.2 Boundary of the tongues: proof of statements (c) and (d) of Theorem 6.2.1

The goal of this subsection is to prove statements (c) and (d) of Theorem 6.2.1. The proof is inspired by Misiurewicz and Rodrigues [MR07, MR08]. In order to do so we describe the boundary of the tongues and the parameters therein. We first show that the boundaries of the tongues are bounded. This is a direct consequence of Lemma 6.1.1, which states that there cannot be any attracting periodic cycle in $\mathbb{S}^1$ if $|a| \geq 3$.

Proposition 6.2.9. *If $a \in T_\tau$ then $|a| < 3$. Consequently, tongues are bounded.*

We now prove some preliminary results. The next lemma deals with parabolic cycles of B_a. It is an extension of Lemma 1.2.15.

Lemma 6.2.10. *Choose a such that $|a| \geq 2$. Then, $B_a|_{\mathbb{S}^1}$ has at most one attracting or parabolic cycle $< x_0 >$ in the unit circle, which has a real multiplier. If the cycle is attracting, the two critical points lie in the same connected component of $A^*(< x_0 >)$. Moreover, if the cycle is parabolic, then it has multiplier $\lambda = 1$ and either every point x_n of the cycle lies in the boundary of a unique connected component of $A^*(< x_0 >)$ or there are two such components which are symmetric and which do not intersect the unit circle (see Figure 6.2).*

Proof. If $B_a|_{\mathbb{S}^1}$ has an attracting cycle then the proof is similar to the one in Lemma 1.2.15. The multiplier is real since $\mathbb{S}^1$ is invariant under B_a. Now suppose that $< x_0 >$ is a parabolic cycle. Then it has multiplier $\lambda = 1$. Indeed, since λ is real, $\lambda = \pm 1$, but it is positive since $B_a|_{\mathbb{S}^1}$ is increasing. Hence, all the connected components of $A^*(< x_0 >)$ have the same period p as has the cycle $< x_0 >$. Since there are at most two free critical points, there can be at most two cycles of such components. Assume that one of them intersects $\mathbb{S}^1$. In that case, the component is symmetric and so are all other domains of its cycle $\mathcal{L}$. Therefore, if $U \in \mathcal{L}$ contains a critical point then it contains both and there can be no other cycle.

□

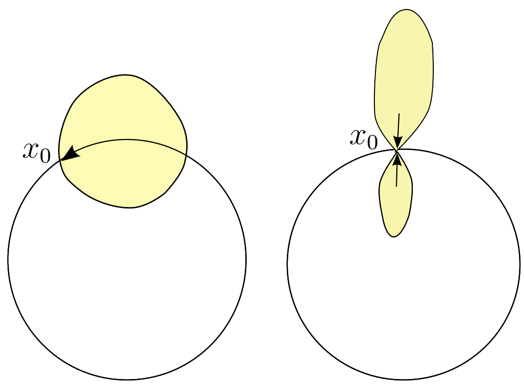


Figure 6.2: The left figure corresponds to the case of a parabolic fixed point $x_0 \in \mathbb{S}^1$ whose attracting basin intersects $\mathbb{S}^1$. The right figure corresponds to the case of a parabolic fixed point $x_0 \in \mathbb{S}^1$ which has two disjoint attracting basins, none of them intersecting $\mathbb{S}^1$.

The following corollary is a result of the fact that a parabolic cycle of $B_a|_{\mathbb{S}^1}$ can have at most one periodic petal intersecting $\mathbb{S}^1$.

Corollary 6.2.11. *A parabolic cycle* $< x_0 > \in \mathbb{S}^1$ *of* B_a *cannot be attracting from both sides in* $\mathbb{S}^1$*. More precisely, if* x_0 *is a parabolic fixed point of* B_a^p*, there cannot exist a neighborhood* U *of* x_0 *in* $\mathbb{S}^1$ *such that all the points* U *are attracted to* x_0 *under iterations of* B_a^p*.*

The next lemma states that parameters on the boundary of tongues have parabolic cycles.

Lemma 6.2.12. *If* a *belongs to the boundary of a tongue, then* B_a *has a parabolic cycle of multiplier* 1*.*

Proof. Let $a_0 \in \partial T_\tau$. Then, there exists a sequence of parameters $a_n \in T_\tau$, $n \in \mathbb{N}$, such that a_n accumulate on a_0. Let x_n be the attracting periodic point of B_{a_n} having the critical points in its immediate basin of attraction. Since $\mathbb{S}^1$ is compact, we may assume that x_n converge to a point $x_0 \in \mathbb{S}^1$. Since B_a depends continuously on a, we conclude that x_0 is a periodic point of B_{a_0}. The multiplier of x_0 has to be 1. Indeed, it is real since $x_0 \in \mathbb{S}^1$, it is positive since $B_a|_{\mathbb{S}^1}$ is increasing and it cannot be smaller than 1 because otherwise would belong to the interior of a tongue $T_{\tau'}$, which is impossible since tongues are disjoint.

□

From now on it will be convenient to work with the alternative parametrization of the Blaschke family $g_{r,\alpha} := B_{r,3\alpha}|_{\mathbb{S}^1}$ as in Section 6.1. We consider the parameter space (r, α) with $r \geq 2$ and $\alpha \in \mathbb{R}/\frac{1}{3}\mathbb{Z}$ instead of $a \in \mathbb{C}$ with $|a| \geq 2$. The main reason to use this alternative parametrization is to use the monotonicity with respect to α given by Lemma 6.1.1.

Definition 6.2.13. We say that a parameter (r_0, α_0), $r_0 > 2$, in a boundary of a tongue T_τ of period p belongs to the *left boundary* of the tongue if there exists an $\epsilon > 0$ such that, for all $0 < \alpha < \epsilon$, $(r_0, \alpha_0 + \alpha)$ belongs to the tongue and $(r_0, \alpha_0 - \alpha)$ does not belong to it. Conversely, we say that it belongs to the *right boundary* if there exists an $\epsilon > 0$ such that, for all $0 < \alpha < \epsilon$, $(r_0, \alpha_0 - \alpha)$ belongs to the tongue and $(r_0, \alpha_0 + \alpha)$ does not belong to it. Finally, we say that it belongs to a *tip of the tongue* if there exists an $\epsilon > 0$ such that, for all $\alpha \in (-\epsilon, 0) \cup (0, \epsilon)$, $(r_0, \alpha_0 + \alpha)$ does not belong to the tongue.

Using Lemma 6.1.1, we have the following result (c.f. [MR07, Lem. 4.1]).

Lemma 6.2.14. *Let* x_0 *be an attracting or parabolic periodic point of* g_{r_0,α_0} *of period* p *and let* H_{r_0,α_0} *be the semiconjugacy between* g_{r_0,α_0} *and the doubling map given by Lemma 1.2.3. Let* J *be the set of points* $x \in \mathbb{S}^1$ *which are sent by* H_{r_0,α_0} *to the same point as* x_0*, i.e.,* $J = \{x \mid H_{r_0,\alpha_0}(x) = H_{r_0,\alpha_0}(x_0)\}$*. Then, either* J *is a connected closed interval or it consists of a single point. Moreover,* $g_{r_0,\alpha_0}|_J$ *is a homeomorphism, the endpoints of* J *are fixed points of* $g^p_{r_0,\alpha_0}$*, and one of the following holds (see Figure 6.3).*

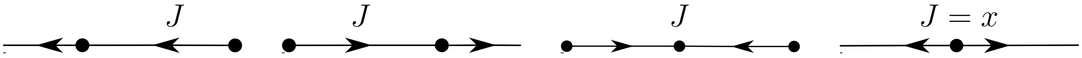
(a) J *is an interval. The left endpoint of* J *is parabolic, topologically attracting from the right and repelling from the left, the right endpoint is repelling and there are no*

other fixed points of $g^p_{r_0,\alpha_0}$ *in* J*. In this case* (r_0, α_0) *belongs to the left boundary of the tongue.*

(b) J *is an interval. The right endpoint of* J *is parabolic, topologically attracting from the left and repelling from the right, the left endpoint is repelling and there are no other fixed points of* $g^p_{r_0,\alpha_0}$ *in* J*. In this case* (r_0, α_0) *belongs to the right boundary of the tongue.*

(c) J *is an interval. Both endpoints of* J *are repelling, there is an attracting fixed point of* $g^p_{r_0,\alpha_0}$ *in* J *and there are no other fixed points of* $g^p_{r_0,\alpha_0}$ *in* J*. this case* (r_0, α_0) *belongs to the interior of the tongue.*

(d) J *consists of a parabolic periodic point which is topologically repelling in* $\mathbb{S}^1$*.*



(a) Case (a) (b) Case (b) (c) Case (c) (d) Case (d)

Figure 6.3: The four different behaviors which may occur in Lemma 6.2.14.

Proof. Throughout the proof we consider the points of $\mathbb{S}^1$ oriented anticlockwise. The fact that J is either a single point or a closed interval follows since H_{r_0,α_0} is an increasing continuous function (see Lemma 1.2.3). Since $g_{r_0,\alpha_0}|_{\mathbb{S}^1}$ is an strictly increasing function (see Lemma 6.1.1), we have that $g_{r_0,\alpha_0}|_{\mathbb{S}^1}$ is a local homeomorphism around each point. It follows that $g_{r_0,\alpha_0}|_J$ is a homeomorphism and its endpoints are fixed points of $g^p_{r_0,\alpha_0}$.

It follows from Lemma 6.2.10 that only these four cases can occur. It states that $g_{r_0,\alpha_0}|_{\mathbb{S}^1}$ can have at most one attracting or parabolic cycle and that a parabolic cycle cannot be topologically attracting from both sides.

It is left to see that case (a) corresponds to the left boundary of the tongue whereas case (b) corresponds to the right boundary. We prove it for case (a). Case (b) is analogous.

Assume that (r_0, α_0) satisfy the hypothesis of case (a). Then, there exists a periodic point of period p which is repelling from the left and attracting from the right. Recall that, from Lemma 6.1.1, we have that $g^p_{r,\alpha}$ is strictly increasing with respect to α for any $p \in \mathbb{N}$.

We first prove that there exists an $\epsilon > 0$ such that if $0 < \alpha < \epsilon$, then $g_{r_0,\alpha_0+\alpha}$ has an attracting cycle of period p. Let x be a point of the parabolic cycle and let y be a point in the immediate basin of attraction of x. Then $x < y$ and $g^p_{r_0,\alpha_0}(y) < y$. Since $g^p_{r,\alpha}$ is strictly increasing with respect to α, there exists an $\epsilon > 0$ such that if $0 < t < \epsilon$, then $g^p_{r_0,\alpha_0+t}(x) > x$ and $g^p_{r_0,\alpha_0+\alpha}(y) < y$. Hence, there has to be a topologically attracting periodic point of period p between x and y. Since, by Lemma 6.2.10, a parabolic periodic point cannot be attracting on both sides, we get that this topological attractor located between x and y is an attractor.

Now we have to see that there exists an $\epsilon > 0$ such that if $0 < \alpha < \epsilon$, then $g_{b_0,\alpha_0-\alpha}$ has no periodic attracting cycles of period p. Since x is repelling from the left and attracting from the right, there exists a $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$, then $g^p_{r_0,\alpha_0}(y) \leq y$. Using that $g_{r_0,\alpha_0+\alpha}$ is strictly increasing with respect to α, we can take

$\epsilon_1 > 0$ such that if $0 < \alpha < \epsilon_1$, then $g^p_{r_0,\alpha_0}(y) < y$ for all $y \in (x - \delta, x + \delta)$. Doing the same around all of the points x_m of the parabolic cycle we obtain $\delta' > 0$ and $\epsilon_2 > 0$ such that if $0 < \alpha < \epsilon_2$, then $g^p_{r_0,\alpha_0}(y) < y$ for all $y \in (x_m - \delta', x_m + \delta')$. Hence, we have erased the periodic points of period p in a δ'-neighborhood U of our cycle. Since $g^p_{r_0,\alpha_0}$ has finitely many fixed points in $\mathbb{S}^1 \setminus U$, all of them repelling, we can take $\epsilon_3 < \epsilon_2$ such that if $0 < \alpha < \epsilon_3$, then $g^p_{r_0,\alpha_0-\alpha}$ has no attracting fixed point at all.

□

We want to prove that case (d) of Lemma 6.2.14 corresponds to a tip of the tongue (see Proposition 6.2.17). We first introduce some auxiliary lemmas. The following lemma corresponds to Lemma 3.1 in [MR08].

Lemma 6.2.15. *Let U be a neighborhood of the origin in $\mathbb{R}^2$ and let $F : U \to \mathbb{R}$ be a real analytic function. Set $f_t(x) = F(t, x)$. Assume that f_0 has a topologically repelling fixed point at $x = 0$ and that*

$$\frac{\partial F}{\partial t}(0, 0) \neq 0,$$

Then there are open intervals I, J containing 0 such that $I \times J \subset U$ and for every $t \in I$ the map f_t has exactly one fixed point $x \in J$. Moreover, if $t \neq 0$, then the fixed point has multiplier $\lambda > 1$.

We use the previous technical lemma in the following result.

Lemma 6.2.16. *Consider a one parameter subfamily $f_t = g_{r(t),\alpha(t)}$ of the Blaschke family such that $r(t)$ and $\alpha(t)$ depend analytically on t. Assume that $f^p_{t_0}$ has a topologically repelling parabolic fixed point x_0 and $\frac{\partial G}{\partial t}(t_0, x_0) \neq 0$, where $G(t, x) = f^p_t(x)$. Then there exists $\epsilon > 0$ such that if $t \in (-\epsilon, 0) \cup (0, \epsilon)$ then $f^p_{t_0+t}$ has no attracting or parabolic fixed point.*

Proof. By Lemma 6.2.15, there exists $\epsilon_1 > 0$ and a neighborhood U_1 of x_0 such that if $t - t_0 \in (-\epsilon_1, 0) \cup (0, \epsilon_1)$, then f^p_t has no attracting or parabolic fixed point in U_1. Now, as in proof of Lemma 6.2.14, we can perform the same argument around the other $p - 1$ points of the parabolic cycle, obtaining an $\epsilon_2 > 0$ and a neighborhood U of $\{x_0, ..., x_{p-1}\}$ such that if $t - t_0 \in (-\epsilon_2, 0) \cup (0, \epsilon_2)$ then f^p_t has no attracting or parabolic fixed point in U. Since f^p_t has only finitely many fixed points in $\mathbb{S}^1 \setminus U$, all of them repelling, we can take $\epsilon_3 < \epsilon_2$ such that if $t - t_0 \in (-\epsilon_3, 0) \cup (0, \epsilon_3)$ then f^p_t has no attracting or parabolic fixed point at all. □

This result gives us directly the following proposition.

Proposition 6.2.17. *A parameter (r, α) in a boundary of a tongue for which case (d) of Lemma 6.2.14 occurs, is a tip of that tongue.*

From Lemma 6.2.14 and Theorem 6.2.17 we obtain the next corollary.

Corollary 6.2.18. *Any parameter (r_0, α_0) of the boundary of a tongue T_τ either belongs to the right or the left boundary of the tongue or is a tip of the tongue.*

We now prove the remaining statements of Theorem 6.2.1. The following theorem proves statement (c).

Theorem 6.2.19. *Given $r_0 \geq 2$, the intersection of any connected component of a tongue T_τ with the parameter circle $|a| = r_0$ is connected. In particular, every connected component of a tongue is simply connected.*

Proof. Assume that the intersection of a connected component of a tongue T_τ with the parameter circle $|a| = r_0$ is not connected. Then, there exists a parameter (r_0, α_0) and $\epsilon > 0$ such that, for any $\alpha \in (-\epsilon, 0) \cup (0, \epsilon)$, the parameter $(r_0, \alpha_0 + \alpha)$ belongs to the tongue. That would imply that (r_0, α_0) is a parameter in the boundary of T_τ which does not belong to the right or left boundary and which is not a tip of the tongue, contradicting Corollary 6.2.18.

□

Finally, we prove statement (d) of Theorem 6.2.1.

Proof of statement (d) of Theorem 6.2.1. It follows from Theorem 6.2.19 that the left and right boundaries are well defined curves. Both the left and the right boundary begin in two different parameters a_- and a_+ with $|a_-| = 2 = |a_+|$. Both boundaries are bounded by Proposition 6.2.9 and hence they have to end at a point where they intersect, which is a tip. The only thing which is left to see is that the boundary of a tongue cannot be flat for any r_0, i.e., we have to see that the intersection of the boundary of T_τ with any parameter circle $|a| = r_0$ does not contain an interval of parameters. Notice that, by Theorem 6.2.19, neither the left nor the right boundaries can have local maximums. The points of such interval cannot be of the left boundary or the right boundary by definition. Hence, the parameters of this open interval are tips of the tongue. Therefore, we would have an $\epsilon > 0$ such that $B_{r_0,\alpha_0+\alpha}$ has a topologically repelling parabolic fixed point for all $|\alpha| < \epsilon$ and for some α_0 and r_0. However, this would contradict Lemma 6.2.16.

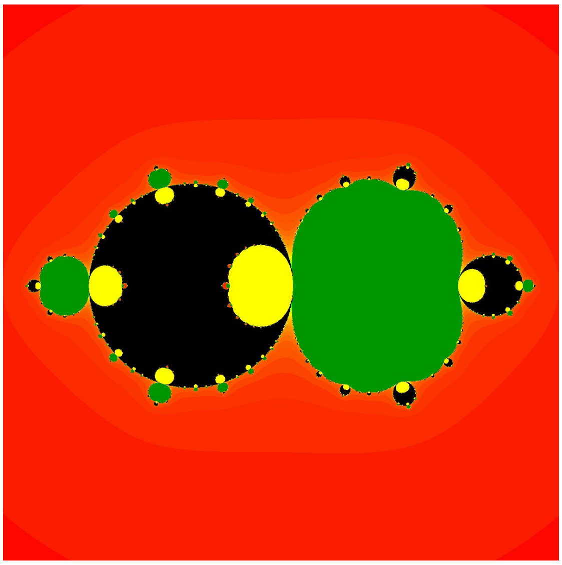
□

6.3 Bifurcations around the tip of the tongues

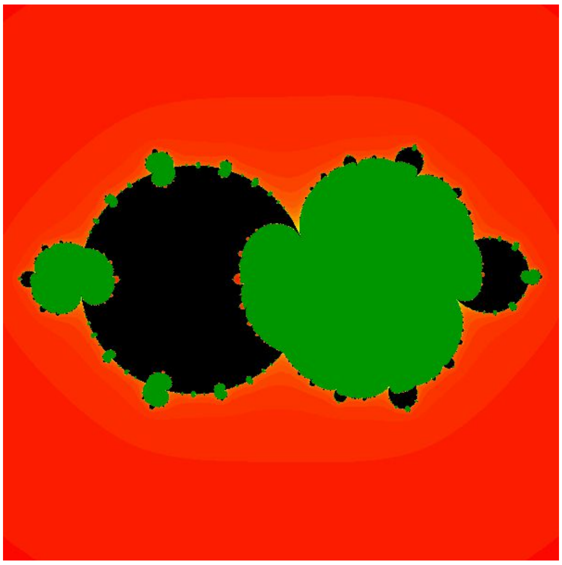
In this section we study the bifurcations which occur throughout the boundaries of the tongues. Given a tongue T_τ, there is a persistent saddle-node bifurcation which takes place along $\partial T_\tau \setminus a_\tau$, two cycles collide in $\mathbb{S}^1$ and exit it (see Figure 6.4). The goal of the section is to prove Theorem 6.3.2. We study the bifurcations in a neighbourhood of the tip of the tongues and see that, if the parameter is close enough to a tip, then the two cycles leaving the unit circle are attracting. We will need the following lemma, which makes use of algebraic geometry. For an introduction to the topic we refer to [Har77] and [Sha13].

Lemma 6.3.1. *For fixed $n > 0$, there is only a finite number of parameters $a \in \mathbb{C}$ for which the Blaschke product B_a has a parabolic cycle of exact period n, multiplier 1 and multiplicity* 3.

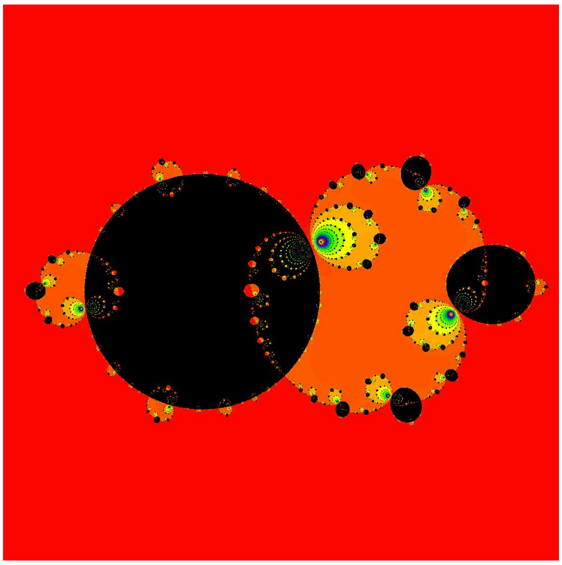
Proof. It will be convenient to work with the alternative parametrizations of the Blaschke products B_a presented in Section 6.1. Recall that, if $a = re^{2\pi i\alpha}$ with $r > 0$ and $\alpha \in \mathbb{R}$, B_a is conjugate with $B_{r,3\alpha}$ (Equation (3.1)). The Blaschke products $B_{r,3\alpha}$ are embedded within the family $G_{a,b}$ (Equation (6.4)), where $a, b \in \mathbb{C}$. We



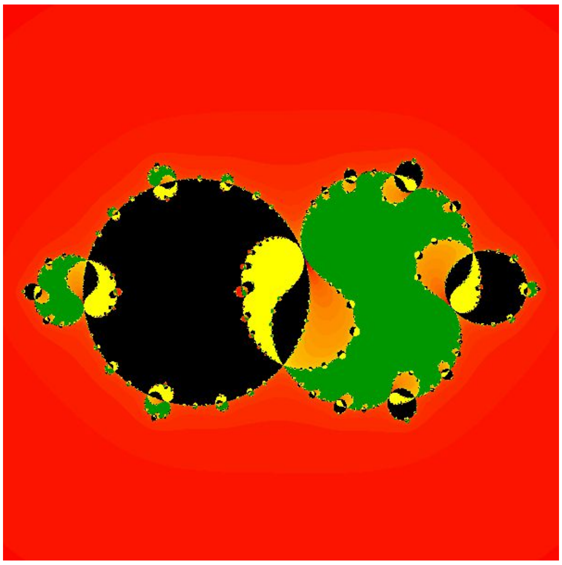
(a) Dynamical plane of B_3.



(b) Dynamical plane of $B_{2.65675+0.0389604i}$.



(c) Dynamical plane of $B_{2.55309+0.063042i}$.



(d) Dynamical plane of $B_{2.64732+0.0421017i}$.

Figure 6.4: Figure (a) shows the dynamical plane of the tip $a_0 = 3$ of the fixed tongue T_0. Figure (b) shows the dynamical plane of B_a, where $a = 2.65675+0.0389604i$ is in $\partial T_0 \backslash a_0$. Figures (c) and (d) show the parameter plane of two Blaschke products with parameters near the boundary of the fixed tongue T_0. In Figure (c) we have $a = 2.55309 + 0.063042i$ and the parabolic fixed point has bifurcated into two repelling points while in Figure (d) we have $a = 2.64732 + 0.0421017i$ and the parabolic fixed point has bifurcated into two attracting points. The colors are as follows: green if the point belongs to a basin of attraction which contains the critical point c_+, yellow if the point belongs to a basin of attraction which contains c_- and not c_+, black if the orbit accumulates on $z = 0$ and a scaling from blue to red if the orbit accumulates on $z = \infty$.

will prove that, for fixed $n > 0$, there is only a finite number of parameters (a, b), where $a, b \in \mathbb{C}$, for which $G_{a,b}$ has a parabolic cycle of exact period n, multiplier 1 and multiplicity 3.

We first show that the immediate basin of attraction of such a cycle contains both free critical points. Indeed, a parabolic cycle $< z_0 >$ of exact period n, multiplier 1 and multiplicity 3 has two disjoint cycles of maximal attracting petals attached to it (see Theorem 1.1.15 and Figure 6.4 (a)). Each of these cycles of maximal petals has at least one critical point on the boundary of one of its components (see Theorem 1.1.16). Therefore, the immediate basin of attraction of $< z_0 >$ contains both free critical points of $G_{a,b}$. Notice also that it follows from this assumption that the multiplicity cannot be greater than 3 since the rational maps $G_{a,b}$ only have two free critical points.

Parameters which satisfy the hypothesis are solutions of the system of rational equations

$$\begin{cases} G^n_{a,b}(z) = z, \\ (\frac{\partial}{\partial z} G^n_{a,b})(z) = 1, \\ (\frac{\partial^2}{\partial z^2} G^n_{a,b})(z) = 0. \end{cases} \tag{6.5}$$

Take the polynomials $p_1(z, a, b)$, $p_2(z, a, b)$, $p_3(z, a, b)$ and $q(z, a, b)$ so that the previous system reduces to

$$\begin{cases} p_1(z, a, b)/q(z, a, b) = z, \\ p_2(z, a, b)/q(z, a, b)^2 = 1, \\ p_3(z, a, b)/q(z, a, b)^3 = 0. \end{cases}$$

Notice that p_2 and p_3 are combinations of p_1, q and their derivatives. We obtain the polynomial system of equations

$$\begin{cases} p_1(z, a, b) - zq(z, a, b) = 0, \\ p_2(z, a, b) - q(z, a, b)^2 = 0, \\ p_3(z, a, b) = 0. \end{cases} \tag{6.6}$$

The solutions of (6.5) also solve (6.6). However, we have added solutions. They come from points (z, a, b) on which either the numerator and the denominator vanish simultaneously or are both equal to infinity. They can be equal to infinity if and only if $z = \infty$ or $a = \infty$ or $b = \infty$. Such points are not solutions of the original system. The point $z = \infty$ is a permanent superattracting fixed point (unless $b = 0$ or $b = \infty$) and, therefore, does not satisfy the equations of a parabolic point. If $a = \infty$ then $G_{a,b}(z)$ degenerates to bz^2, which does not have parabolic cycles. If $b = \infty$ then $G_{a,b}$ is constant and therefore does not have any parabolic cycle. The points for which the numerator and the denominator vanish simultaneously come from the system

$$\begin{cases} p_1(z, a, b) = 0, \\ q(z, a, b) = 0. \end{cases} \tag{6.7}$$

Notice that, if $q(z, a, b) = 0$ but $p_1(z, a, b) \neq 0$ then the first equation in (6.6) is not satisfied and, therefore the point (z, a, b) is not a solution. Assume that (z, a, b) solves (6.7). Then there is a z_0 such that the numerator and the denominator of

$$bG_{a,b}^{n-1}(z_0)^3\frac{G_{a,b}^{n-1}(z_0)-a}{1-aG_{a,b}^{n-1}(z_0)}$$

vanish simultaneously. This can only happen if $b=0$ or $a=\pm 1$. If $b=0$ the $G_{a,b}$ is constant and therefore (6.5) has no solution. If $a=\pm 1$ the family $G_{a,b}(z)$ degenerates to the polynomials $\mp bz^3$ and the system (6.5) has no solution.

We also assumed that the parabolic cycle has exact period n. Thus, the parameters which satisfy the hypothesis of the lemma are such that the equality

$$G_{a,b}^{m}(z)=z \tag{6.8}$$

is not satisfied for any $m<n$. If $G_{a,b}^{m}(z)=\tilde{p}_m(z,a,b)/\tilde{q}_m(z,a,b)$, the set of points which satisfy the previous equality, are solutions of the polynomial equation

$$\tilde{p}_m(z,a,b)-z\tilde{q}_m(z,a,b)=0. \tag{6.9}$$

The set of solutions of (6.6) is an algebraic variety, say Y. Each point of Y is either solution of (6.5) or corresponds to any of the degeneracy situations already described. The set of solutions of (6.9) consists of the solutions of (6.8) and exactly the same degeneracy solutions described for (6.6). Let Y' be the quasiprojective variety obtained by intersecting Y with the open set of the Zariski topology given by $b\neq 0$, $b\neq\infty$, $a\neq\pm 1$, $a\neq\infty$, $z\neq\infty$ and $\tilde{p}_m(z,a,b)-z\tilde{q}_m(z,a,b)\neq 0$ for all $m<n$. If (z,a,b) belongs to Y' then it is a solution of (6.5) and does not solve (6.8). Therefore, $<z>$ is a parabolic cycle of $G_{a,b}$ of period exactly n, multiplier 1 and multiplicity 3 whose immediate basin of attraction contains both free critical orbits. Since by Lemma 6.1.2 the non-escaping set is bounded in a, we conclude that the projection of Y' over the variable a is bounded. We use now that the projection of a quasiprojective variety over a variable is either dense or finite to conclude that the projection of Y' over a is finite. This last assertion follows from Chevalley's Theorem (see [Gro67], c.f. [Har77, Exercise 3.19]) which states that any morphism of quasiprojective varieties sends constructible sets to constructible sets. In particular, the image over $\mathbb{C}$ of a quasiprojective variety under a regular map is either finite or dense in $\mathbb{C}$. Summarizing we have that there are finitely many a for which (6.5) has solution.

Finally, consider the previous equation systems with $a_0\neq\pm 1$ fixed. Let Y' be the quasiprojective variety of points (z,a_0,b) which solve (6.5), do not solve (6.9) for any $m<n$ and the degeneracy conditions are not satisfied. By Lemma 6.1.3 we know that, for fixed $a_0\neq\pm 1$, the non-escaping set is bounded on b. As before we conclude that Y' projects onto a finite number of b, which finishes the proof.

□

Notice that the condition of having exactly period n on the previous lemma is necessary. Indeed, the family B_a has curves of parabolic parameters whose parabolic cycle $<z_0>$ of period n has multiplier -1 (see Theorem 6.4.3). The points z_0 are also parabolic fixed points of B_a^{2n} of multiplier 1 and multiplicity 3. Therefore, if we do not require exact period n then we may obtain infinitely many solutions of (6.5).

We now prove Theorem 6.3.2, which tells us that there is a neighborhood U of the tip of any tongue such that if $a\in U$ then either a belongs to the tongue, or to its boundary, or B_a has two disjoint attracting cycles (see Figure 6.5 and Figure 6.4 (a), (b) and (d)).

Theorem 6.3.2. *Let a_τ be the tip of a tongue T_τ of period p. Then, there exists a neighborhood U of a_τ such that if $a \in U$ then, either $a \in T_\tau$ or $a \in \partial T_\tau$ or a belongs to a disjoint hyperbolic component.*

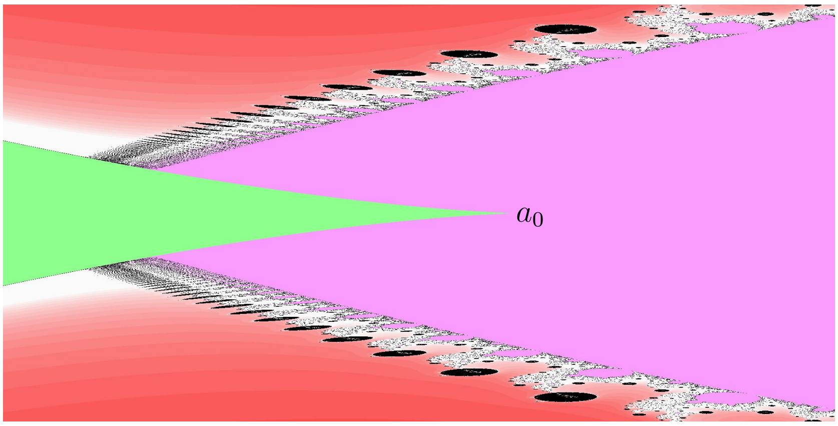


Figure 6.5: A zoom in a neighborhood of the tip a_0 of the tongue T_o. The colors are as in Figure 5.1. We see in green the fixed tongue T_0 and in pink a disjoint hyperbolic component which partially shares the boundary with T_0.

Proof of Theorem 6.3.2. The main ingredient for the proof is the holomorphic index. Given a fixed point z_0 of a holomorphic function f, the holomorphic fixed point index of z_0, $i(z_0)$, is defined to be the residue of $1/(z - f(z))$ around z_0. If the fixed point has multiplier $\rho \neq 1$, then $i(z_0) = 1/(1-\rho)$ (see [Mil06]). Moreover, when n different fixed points collide in a parabolic point z_0 of multiplier 1, their indexes tend to infinity, even if the sum of their indexes tends to the finite index $i(z_0)$ of the parabolic point.

Let $< w_0 >$ be the parabolic cycle of B_{a_τ}. Then, w_0 is a parabolic periodic point of multiplier 1, multiplicity 3 and exact period p of B_{a_τ}. Since, by Lemma 6.3.1, there is a finite number of such parameters, it follows that there is an open neighborhood of the parameter a_τ which contains no other parameter a for which B_a has a parabolic cycle of multiplier 1, multiplicity 3 and the same period than $< w_0 >$. Take a parameter a close to a_τ. The map B_a^p has three fixed points, say z_0, z_+ and z_-, which tend to w_0 when a tends to a_τ. By symmetry and continuity of the semiconjugacy $H_a(x)$ with respect to a and x, at least one of the fixed points lies in $\mathbb{S}^1$, say z_0, and satisfies $H_a(z_0) = \tau$. Also by symmetry, if more than one fixed point lies in $\mathbb{S}^1$, the three of them do. In that later case, since $B_a|_{\mathbb{S}^1}^p$ is strictly increasing, one of them is either parabolic or attracting and satisfies $H_a(z) = \tau$ by continuity of H_a, so either belongs to the tongue T_τ or its boundary. Assume that only z_0 lies in $\mathbb{S}^1$ and is repelling (if it was attracting it would belong to T_τ again by continuity of H_a). Then z_0 has real multiplier $\eta > 1$ (compare Lemma 6.2.10). Due to the symmetry, the other two fixed points, $z_\pm$, are symmetric. Moreover, their multipliers are complex conjugate say ρ and $\overline{\rho}$ (c.f. Theorem 5.2.2).

Consider the sum $\mathcal{S}$ of the indexes of the three periodic points.

$$\mathcal{S} = i(z_0) + i(z_+) + i(z_-) = \frac{1}{1-\eta} + \frac{1}{1-\rho} + \frac{1}{1-\overline{\rho}}.$$

The number $\mathcal{S}$ is a real quantity which tends to the index of the parabolic cycle of the tip of the tongue whenever a tends to a_τ. Moreover, $i(z_0)$ tends to minus infinity when a tends to a_τ. Hence, there is an open neighborhood U of a_0 such that if $a \in U$ then $\tilde{\mathcal{S}} = \mathcal{S} - i(z_0) > 1$. Write $\rho = 1 + \epsilon = 1 + \epsilon_r + i\epsilon_i$, $\epsilon_r, \epsilon_i \in \mathbb{R}$. Then, if $a \in U$, we have

$$\tilde{\mathcal{S}} = \frac{1}{1 - 1 - \epsilon} + \frac{1}{1 - 1 - \overline{\epsilon}} = -\frac{2\epsilon_r}{|\epsilon|^2}.$$

It follows from this equation that, if $a \in U$, then $\epsilon_r < 0$. Finally, using that $|\epsilon|^2 = -2\epsilon_r/\tilde{\mathcal{S}}$, we have

$$|\rho|^2 = (1 + \epsilon_r)^2 + \epsilon_i^2 = 1 + 2\epsilon_r - \frac{2\epsilon_r}{\tilde{\mathcal{S}}} = 1 + 2\epsilon_r(1 - \frac{1}{\tilde{\mathcal{S}}}).$$

Since $\epsilon_r \lesssim 0$ and $\tilde{\mathcal{S}} > 1$ we conclude that $|\rho| < 1$, which finishes the proof. □

We finish the section showing some consequences of the construction presented in the proof of Theorem 6.3.2. The next corollary follows from the previous theorem and Theorem 6.2.19, which states that the boundary of any tongue corresponds to the union of two arcs which intersect at the tip of the tongue. These two arcs can be parametrized univalently with respect to the modulus of the parameter.

Corollary 6.3.3. *Given a tongue T_τ, there exists a hyperbolic component of B_a of disjoint type sharing part of its boundary with T_τ in a neighborhood of the tip a_τ.*

The following proposition tells us that all parameters with a, $|a| > 2$, such that B_a has a parabolic cycle which is topologically repelling in the unit circle are tips of tongues.

Proposition 6.3.4. *If $|a_0| > 2$ and B_{a_0} has a parabolic cycle of multiplicity 3 and exact period n in the unit circle then a_0 is the tip of a tongue of period n.*

Proof. Assume that a_0 is not a tip of a tongue of period n. Then the same perturbation done in the proof of Theorem 6.3.2 can be performed, obtaining a disjoint hyperbolic component of parameters which surrounds a_0. Indeed, since a_0 is not in the boundary of a tongue of period n and there is a finite number of parameters with a parabolic cycle of multiplier 1, multiplicity 3 and exact period n by Lemma 6.3.1, the perturbation presented in the proof of Theorem 6.3.2 gives us an open neighborhood U of a_0 such that if $a \in U$, $a \neq a_0$, then the Blaschke product B_a has two disjoint attracting cycles other than $z = 0$ and $z = \infty$. Therefore, the set of parameters U would be contained in a multiply connected disjoint hyperbolic component whose attracting cycles are not in the unit circle, which is impossible by Theorem 5.4.2. □

Corollary 6.3.5. *If $|a| > 2$ and B_a has a parabolic cycle on the unit circle, then a belongs to the boundary of a tongue.*

Proof. If the parabolic cycle of B_{a_0} has multiplicity 2 then it is on the boundary of a period n tongue by Lemma 6.2.14. If it has multiplicity 3 it is on the tip of a fixed tongue by Proposition 6.3.4.

□

6.4 Extended Tongues

The goal of this section is to give an idea of the dynamics that may take place for parameters within the open annulus $\mathbb{A}_{1,2}$ of inner radius 1 and outer radius 2. The section is structured as follows. We first notice that the tongues studied up to this point may be extended within this annulus. Then we describe more precisely how the fixed tongue extends. Finally we do some numerical computations to obtain an idea of other phenomena which may take place.

The definition of tongues only makes sense for parameters a such that $|a| \geq 2$. However, given a tongue T_τ, its attracting cycle $< z_0 >$ can be analytically continued for parameters $1 < |a| < 2$ (see Figure 6.6), parameters for which $B_a|_{\mathbb{S}^1}$ is not a degree 2 cover of the unit circle (see Section 3.1) and, therefore, is not semiconjugate to the doubling map. We proceed to formally define the concept of extended tongue using the analytic continuation of the attracting cycle.

Definition 6.4.1. An *extended tongue* ET_τ is defined to be the set of parameters for which the attracting cycle of T_τ can be continued analytically. More precisely, we say that a parameter a belongs to the *extended tongue* ET_τ (of period p) if $1 < |a| < 2$, $B_a|_{\mathbb{S}^1}$ has an attracting periodic point of period p and there exists a curve of parameters $\gamma(t)$ such that $\gamma(0) = a$, $\gamma(1) \in T_\tau$ and $B_{\gamma(t)}|_{\mathbb{S}^1}$ has an attracting fixed point of period p for all $t \in (0, 1)$ which depends continuously on t.

Given that the set of hyperbolic parameters is open in $\mathbb{C}$ and the roots of the tongues (parameters for which the cycle is superattracting) have modulus equal to 2 (see Theorem 6.2.1) we conclude that $ET_\tau \cap \mathbb{A}_{1,2}$ is not empty for any periodic point τ of the doubling map. Notice that, with the previous definition, parameters $a \in T_\tau$ also belong to ET_τ. The following lemma describes the parameters on the boundary of the extended tongues. Its proof is analogous to the one of Lemma 6.2.12.

Lemma 6.4.2. *If a belongs to the boundary of an extended tongue and $|a| \neq 1$, then B_a has a parabolic periodic point of multiplier ± 1.*

Notice that the intersection of two different extended tongues might be a non empty open set. Indeed, the critical orbits are not symmetric if $a \in \mathbb{A}_{1,2}$. Because of that, for $1 < |a| < 2$, $B_a|_{\mathbb{S}^1}$ might have two different attracting cycles (see Figure 3.2 (c) and (d)), in which case a might belong to two different tongues (see Figures 6.6 and 6.8).

The extended fixed tongue

We now proceed to study the extended fixed tongue ET_0 (see Figure 6.7). The following theorem is the main result of this section. It describes the shape of the connected components of the extended fixed tongue.

Theorem 6.4.3. *Given two connected components of the fixed tongue T_0, the intersection of their extensions in $\mathbb{A}_{1,2}$ is empty. The boundary of every connected component of the extended fixed tongue ET_0 consists of two disjoint connected components. The exterior component consists of parameters for which there is a parabolic fixed point of multiplier 1. The interior component consists of parameters for which there is a parabolic fixed point of multiplier -1. Moreover, there is a period doubling bifurcation taking place throughout the curve of interior boundary parameters.*

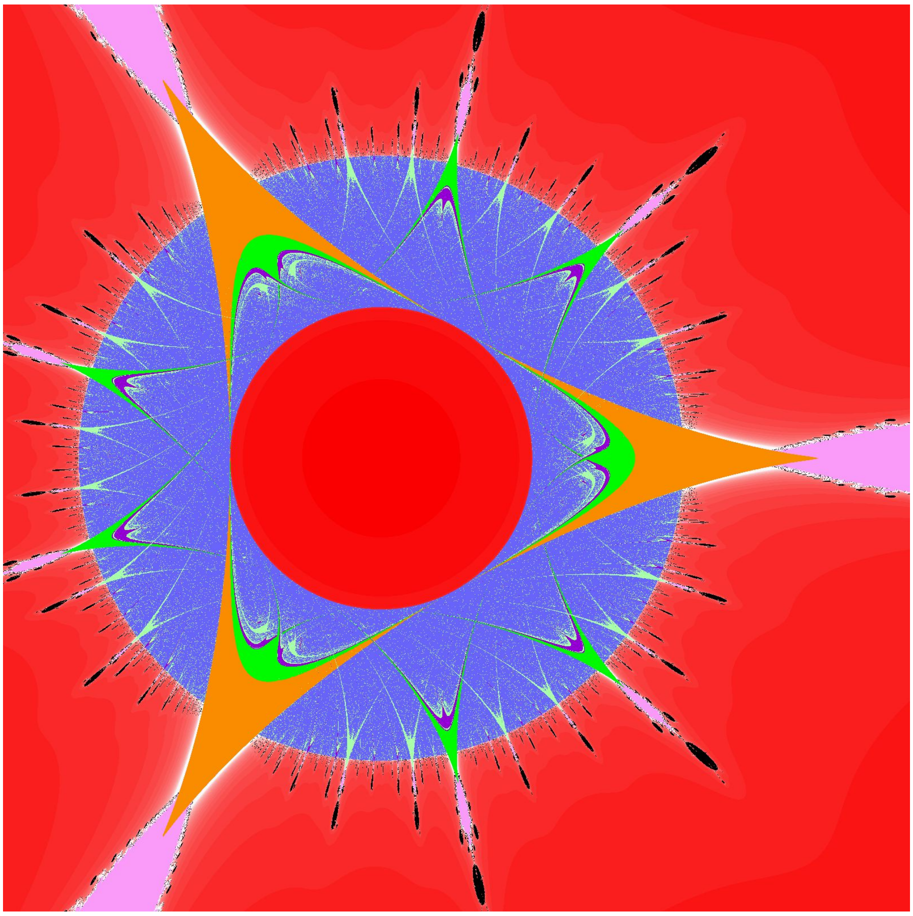


Figure 6.6: Parameter plane of the Blaschke family. The parameters correspond to $-2.5 < \mathrm{Re}(a) < 3.5$ and $-3 < \mathrm{Im}(a) < 3$. We plot in orange the parameters for which there is an attracting fixed point in $\mathbb{S}^1$. These parameters correspond to an extended fixed tongue. Strong green corresponds to parameters having a period 2 attracting cycle in the unit circle, whereas violet corresponds to period 4 cycles. These parameters may belong to extended tongues of period 2 or 4, or to other kinds of components. The rest of the colors are as follows: red for $c_+ \in A(\infty)$, black for $c_+ \in A(0)$, pallid green if $O^+(c_+)$ accumulates on a periodic orbit in $\mathbb{S}^1$, pink if $O^+(c_+)$ accumulates in a periodic orbit not in $\mathbb{S}^1$ and blue in every other case.

We need an auxiliary proposition to prove the previous theorem. The fixed tongue T_0 has three connected components, only one modulo symmetry (see Theorem 6.2.1). Therefore, when studying the extended fixed tongue we restrict to the connected component which intersects the real line. It is convenient to consider the parameter plane given by (r, α), where $a = re^{2\pi i\alpha}$, $1 < r < 2$ and $\alpha \in [0, 1/3)$. We then use the alternative parametrization $g_{r,\alpha} = B_{r,3\alpha}|_{\mathbb{S}^1}$ of $B_a|_{\mathbb{S}^1}$ (see Section 6.1). We denote by $h_{r,\alpha}$ the lift of $g_{r,\alpha}$ (see Equation 6.2). We want to remark that for $r = 1$ and $x = 0$ the function is not well defined (see Section 3.1). Indeed, for $r = 1$, the two critical points and the preimages of 0 and ∞ collapse at the point $x = 0$ and the function becomes a degree 3 polynomial. The following proposition gives us the main properties of ET_0.

Proposition 6.4.4. *Let ET_0 denote the extended fixed tongue which intersects the real line. Then, ET_0 satisfies the following properties:*

(a) ET_0 is symmetric with respect to the real line.

(b) For fixed r_0, $1 < r_0 < 2$, $ET_0 \cap \{\alpha \geq 0\} \cap \{r = r_0\}$ is a connected set on which the multiplier is strictly increasing with respect to α and takes values in $(b, 1)$, where $-1 \leq b < 0$. Moreover, $b = -1$ if and only if $r_0 \leq 5/3$.

(c) If $(r, \alpha) \in ET_0$, then $-1/6 < \alpha < 1/6$.

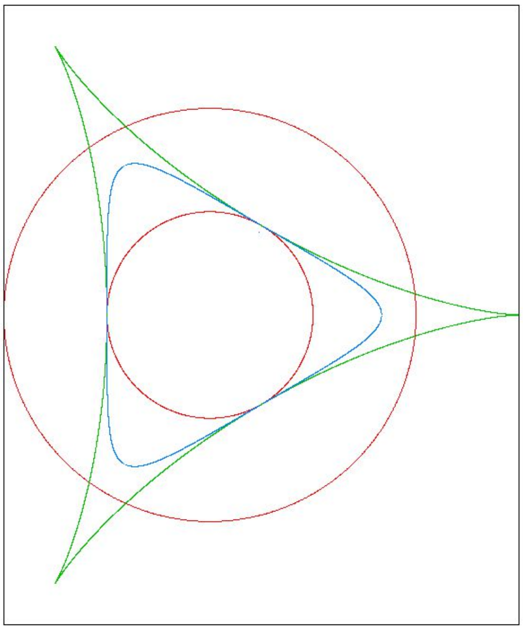


Figure 6.7: Boundaries of the three symmetric extended fixed tongues. The green curves correspond to parameters for which a fixed point has multiplier 1. The blue curves correspond to parameters for which a fixed point has multiplier -1. We also plot in red the two circles of parameters $|a| = 1$ and $|a| = 2$.

Proof. For statement (a) we notice that the extended tongue is symmetric with respect to the real line due to the symmetry of the parameter plane $a \to \bar{a}$ (see Lemma 5.1.1). It is because of that property that we study it for $\alpha \geq 0$. We now use the lift $h_{r,\alpha}$ of $g_{r,\alpha}$ (see Equation (6.2)) to prove (b) and (c). Recall that it is given by

$$h_{r,\alpha}(x) = 3x + 3\alpha + \frac{1}{2\pi i} \log\left(\frac{e^{2\pi i x} - r}{1 - r e^{2\pi i x}}\right).$$

Since we want to study the extended fixed tongue ET_0, we need to investigate for which parameters we have an attracting fixed point which can be continued to the superattracting fixed point $x = 0$ of $h_{2,0}$. Notice that $(2, 0)$ corresponds to the root $2 = 2e^{2\pi i 0}$ of the fixed tongue T_0.

The point $x_{r,0} = 0$ is a fixed point of $h_{r,0}$ for all $r \in (1, 2]$. We denote by $x_{r,\alpha}$ the fixed point of $h_{r,\alpha}$ which is a continuation of $x_{r,0} = 0$. This point $x_{r,\alpha}$ is well defined as long as we do not reach a parameter for which it has multiplier 1. However, this is not an obstruction since it is already a parameter in the boundary of the extended tongue. Notice also that $x_{r,\alpha}$ is strictly decreasing with respect to α since $h_{r,\alpha}$ is strictly increasing with respect to α.

Now we need to study the multipliers of these fixed points. Recall from Equation (6.3) that the derivative of the previous lift is given by the following expression.

$$h'_r(x) := \frac{\partial}{\partial x} h_{r,\alpha}(x) = 3 + \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)},$$

Notice that it does not depend on α. Moreover, h'_r is strictly increasing when x decreases from 0 to $-1/2$. Since for all $1 < r \leq 2$ we have that

$$h'_r(-1/2) = 3 + \frac{1 - r^2}{1 + r^2 + 2r} > 1,$$

we conclude that no parameter (r, α) with $x_{r,\alpha} = -1/2$ can belong to the extended fixed tongue or its boundary. Combining this last result with the fact that $x_{r,\alpha}$ is strictly decreasing with respect to α we conclude that, for fixed $1 < r < 2$, the multiplier $\lambda(x_{r,\alpha})$ of $x_{r,\alpha}$ is strictly increasing with respect to α.

We now prove that for all $r \in (1, 2]$, there exists an $\alpha_1(r)$ which depends continuously on r such that $\lambda(x_{r,\alpha_1}) = 1$ and $0 < \alpha_1 < 1/6$. First of all we notice that, given a parameter $(r, \alpha_1(r))$ with a fixed point x_1 of multiplier 1, the map $h_{r,\alpha_1(r)}$ can be written as

$$h_{r,\alpha_1(r)}(x) = x + \eta(x - x_1)^2 + \mathcal{O}((x - x_1)^3),$$

where $\eta \in \mathbb{R}$. We conclude that $\alpha_1(r)$ is a local graph with respect to r unless $\eta = 0$. However, η is zero if $\partial^2/\partial x^2 h_r(x_1) = 0$, which can only happen if $x_1 = 0$ or $x_1 = -1/2$. Since $h'_r(-1/2) = 3 + (1 - r^2)/(1 + r^2 + 2r) > 1$, $x = -1/2$ cannot be a parabolic fixed point. Moreover, the point $x = 0$ can neither be a parabolic fixed point with multiplier 1 for $1 < r \leq 2$. Indeed, we have that $h'_r(0) = 3 + (1 + r)/(1 - r)$. For $r = 2$ we have $h'_2(0) = 0$. We also have that $h'_r(0)$ is strictly decreasing from 0 to $-\infty$ as r decreases from 2 to 1.

Next we prove that $\alpha_1(r) < 1/6$ by contradiction. Assume that there is an r for which this is not the case. Then, by continuity, there would be an $\tilde{r}$ so that $\lambda(x_{\tilde{r},1/6}) = 1$. Because of the symmetries in the parameter plane, this parameter would give us the intersection between the boundaries of the extension of two different connected components of the fixed tongue. Therefore, at this parameter we would have two different fixed points of multiplier 1. However, each of these parabolic points is a fixed point of multiplicity two and, therefore, this situation would require at least 4 fixed points. This would contradict the fact that the Blaschke products B_a can have at most 3 fixed points other than $z = 0$ and $z = \infty$.

Summarizing we have proven that, for all $r \in (1, 2)$, the fixed point $x_{r,0} = 0$ of $h_{r,0}$ can be monotonously continued to a fixed point $x_{r,\alpha}$ of $h_{r,\alpha}$ as long as $\alpha < \alpha_1(r) < 1/6$, where $\alpha_1(r)$ is a continuous function with respect to r. Moreover, for $0 \leq \alpha < \alpha_1(r)$ the multiplier of $x_{r,\alpha}$ is strictly increasing and takes values in $[h'_r(0), 1)$, where $h'_r(0) \leq -1$ if and only if $r \leq 5/3$. This finishes the proof of the proposition.

□

Using the previous proposition we can prove Theorem 6.4.3.

Proof of Theorem 6.4.4. The fact that two extensions of connected components of the extended fixed tongue cannot intersect follows from statement (c) of Proposition 6.4.4 using the symmetries of the parameter plane and that all connected components of the fixed tongue are symmetric with respect to the rotations given by the third roots of the unity (see Theorem 6.2.1).

The boundary of the connected components of ET_0 is the union of a exterior boundary with parameters of multiplier 1 and a interior boundary with parameters of multiplier -1 by statement (b) of Proposition 6.4.4.

We finally prove that there is a period doubling bifurcation taking place throughout the interior curve. Let (r, α_{-1}), $r < 5/3$, be a parameter of the interior curve. Then $h_{r,\alpha_{-1}}$ has a parabolic fixed point of multiplier -1, say x_-. Hence, x_- is a parabolic fixed point of multiplier 1 of $h^2_{r,\alpha_{-1}}$. Using that $h_{r,\alpha_{-1}}(x_-) = x_-$ and $h'_{r,\alpha_{-1}}(x_-) = -1$ it is not difficult to prove that $\partial^2/\partial x^2 h^2_{r,\alpha_{-1}}(x_-) = 0$ and, therefore, x_- is a parabolic fixed point of multiplicity 3 of $h^2_{r,\alpha_{-1}}$. Hence, x_- has two attracting petals which intersect the unit circle since all critical points lie in $\mathbb{S}^1$. Consequently, x_- is topologically attracting on the unit circle. Using that $h^2_{r,\alpha_{-1}}$ is monotonously decreasing with respect to α in an open neighborhood of x_- and performing the same perturbations as in Lemma 6.2.14 we conclude that, if $\alpha \in (\alpha_{-1} - \epsilon, \alpha_{-1}) \cup (\alpha_{-1}, \alpha_{-1} + \epsilon)$ with $\epsilon > 0$ small enough, then $h^2_{r,\alpha}$ has a topologically attracting fixed point. It follows from the monotonicity of the multipliers of the fixed points of $h_{r,\alpha}$ with respect to α shown in Proposition 6.4.4 that either the parameters (r, α) with $\alpha < \alpha_{-1}$ or the (r, α) with $\alpha > \alpha_{-1}$ are such that $h_{r,\alpha}$ has a period two attracting cycle. □

Other hyperbolic dynamics

We finish this section giving some ideas about the dynamics that take place on $\mathbb{A}_{1,2}$ other than the ones given by the extended fixed tongue. Numerical studies suggest that the properties described for the extended fixed tongue ET_0 are common to all extended tongues. Indeed, we conjecture that all extended tongues have a similar structure than the one presented in Theorem 6.4.4.

Conjecture 6.4.5. *Given an extended tongue ET_τ of period $p > 1$, its connected components are disjoint. The boundary of every connected component of the extended tongue ET_τ consists of two disjoint connected components. The exterior component consists of parameters for which there is a parabolic cycle of period p and multiplier 1. The interior component consists of parameters for which there is a parabolic cycle of period p and multiplier -1. Moreover, there is a period doubling bifurcation taking place throughout the curve of interior boundary parameters.*

We also want to focus on how extended tongues, and more generally hyperbolic components, accumulate on the unit circle. If we observe Figure 6.7 we see that the boundaries of the extended fixed tongue accumulate tangentially to the unit circle onto isolate points. Moreover they do it in pairs. By this we mean that given an accumulation point l, $|l| = 1$, the boundaries of two different connected components of the extended fixed tongue land on it. This seems to happen for hyperbolic regions of arbitrary period (see Figure 6.8).

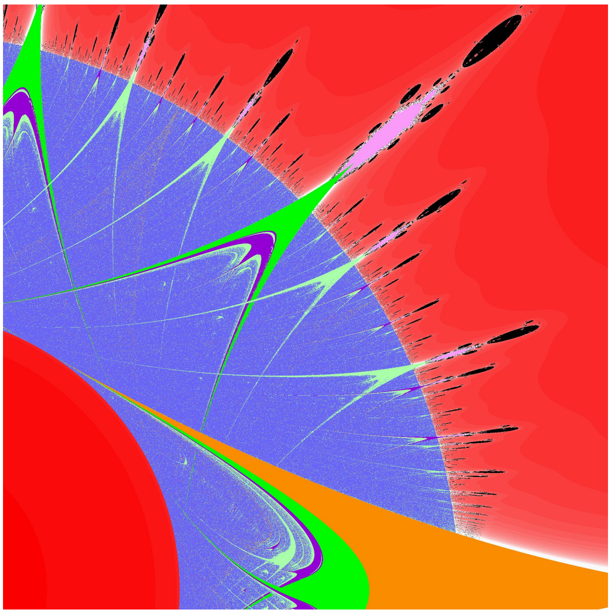


Figure 6.8: Zoom in the parameter plane of the Blaschke family. We see in green how the boundary of an extended tongue of period 2 accumulates onto a parameter l, $|l| = 1$, together with another hyperbolic region of period 2 not coming from the extension of a tongue. The colors are as in Figure 6.6.

Conjecture 6.4.6. *Parabolic curves accumulate on the unit circle on isolate points and tangentially to it. Moreover, if a parabolic curve accumulates on a parameter l, $|l| = 1$, then there is another parabolic curve of the same period landing on l from the opposite side.*

Even though we do not go much further in the study of the parameter plane in the annulus $\mathbb{A}_{1,2}$, we want to point out that the observed bifurcation structures are similar to the ones described for more general maps of $\mathbb{R}^2$ (see [BST98, CMB$^+$91, GSV13]). Indeed, the observed structure around extended tongues is very similar to the one of spring-areas associated to homoclinic tangencies. In that later case, it is proven that there are cascades of period doubling bifurcations which are also observed for the family B_a. In Figure 6.6 we can see these period doublings to period 2 and 4. In Figure 6.9 we show a zoom in the bifurcation diagram in a neighbourhood of the extended fixed tongue ET_0 for $|a| = 1.30$ where the mentioned cascade of bifurcations may be observed. These same cascades of bifurcations may also be observed for a period two extended tongue in Figure 5.2 (down). Other structures of bifurcations described in these papers such as Cross-roads also seem to appear for the Blaschke family B_a when $1 < |a| < 2$ (see Figure 6.10).

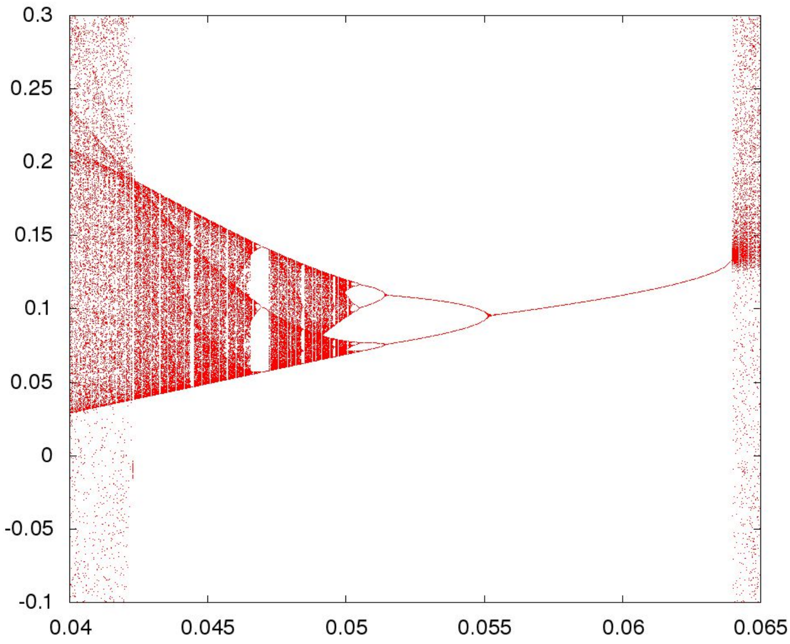


Figure 6.9: Zoom in the bifurcation diagram for $|a| = 1.3$ in a neighbourhood of the extended fixed tongue ET_0 (c.f. Figure 5.2). For $\alpha \in [0.04, 0.065]$ we set $a = 1.3e^{2\pi i\alpha}$ and compute the accumulation set of $c_+ \in \mathbb{S}^1$ under iteration of $B_a|_{\mathbb{S}^1}$. In the x-axis we plot α and in the y-axis we plot the accumulation points of $\mathcal{O}(c_+) \in \mathbb{S}^1$. When drawing the accumulation points we consider their arguments in $(-0.5, 0.5]$ and plot the ones with arguments in $[-0.1, 0.3]$. When $\alpha \in (0.056, 0.064)$ the parameter a belongs to the extended fixed tongue ET_0 and there is one attracting fixed point on which $\mathcal{O}(c_+)$ accumulates. We observe a cascade of bifurcations when $\alpha < 0.056$.

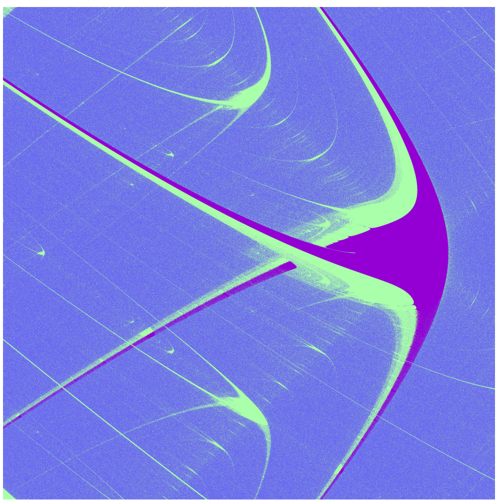


Figure 6.10: Zoom in Figure 6.6. A period 4 Cross-road can be observed in violet. Parameters a are taken so that $1.2 < \mathrm{Re}(a) < 1.24$ and $-0.02 < \mathrm{Im}(a) < 0.02$.

Chapter Seven

7

Generalization of the Blaschke Family

The Blaschke family B_a may be viewed as a rational perturbation of z^2. It is a natural generalization to consider the same type of perturbation applied to z^m for $m \geq 2$.

The goal of these chapter is to present the degree $m+2$ Blaschke products $B_{a;m}$. In Section 7.1 we introduce their dynamical properties. In Section 7.2 we study their parameter plane. Finally, in Section 7.3 we investigate their tongues.

7.1 Dynamical plane of the degree $m+2$ Blaschke families

In analogy to the Blaschke products B_a, for a fixed $m \geq 2$, we consider the degree $m+2$ Blaschke products which have $z=0$ and $z=\infty$ as superattracting fixed points of local degree $m+1$. They are given by the formula

$$B_{a,t;m}(z) = e^{2\pi i t} z^{m+1} \frac{z-a}{1-\bar{a}z}, \tag{7.1}$$

where $a \in \mathbb{C}$ and $t \in \mathbb{R}/\mathbb{Z}$. The next lemma tells us that, for the purpose of classification, we can get rid of the parameter t. The proof is straightforward.

Lemma 7.1.1. *Let $\alpha \in \mathbb{R}$ and let $\eta(z) = e^{-2\pi i\alpha}z$. Then η conjugates the maps $B_{a,t;m}$ and $B_{b,t+(m+1)\alpha;m}$, where $b = e^{-2\pi i\alpha}a$. In particular, $B_{a,t;m}$ is conjugate to $B_{b,0;m}$, where $b = ae^{2\pi it/(m+1)}$.*

Hence, we focus on the study of the family

$$B_{a;m}(z) = z^{m+1} \frac{z-a}{1-\bar{a}z} \tag{7.2}$$

for values $a, z \in \mathbb{C}$ and $m \geq 2$. For $|a| > 1$, the circle maps $B_{a;m}|_{\mathbb{S}^1}$ have degree m in the sense that their lifts $F_{a;m}$ satisfy $F_{a;m}(x+1) = F_a(x) + m\ \forall x \in \mathbb{R}$ (c.f. Section 3.1). These Blaschke products are, indeed, rational perturbations of the map $R_m(z) = z^m$ (alternatively given by $\theta \to m\theta \pmod 1$). As $|a|$ tends to infinity, the $B_{a;m}(z)$ tend to $e^{4\pi i \mathrm{Arg}(a)} z^m$ uniformly on compact sets of $\mathbb{C}^*$. On the other hand, if $|a| < 1$, the circle maps $B_{a;m}|_{\mathbb{S}^1}$ have degree $m+2$.

In order to have an idea of which stable dynamics the maps $B_{a;m}$ may have, we should control the critical points. These rational maps have degree $m+2$ and, hence, have $2(m+2)-2 = 2m+2$ critical points counted with multiplicity (see Corollary 1.3.2).

Since the fixed points $z = 0$ and $z = \infty$ are supperatracting of local degree $m+1$ and, therefore, have multiplicity m as critical points there are only 2 free critical points counted with multiplicity. Hence, the Blaschke families $B_{a;m}$ are almost bicritical. The two free critical points, denoted by $c_\pm(a;m)$, are given by the following formula.

$$c_\pm := c_\pm(a;m) := a \cdot \frac{m+2+m|a|^2 \pm \sqrt{(m^2|a|^2-(m+2)^2)(|a|^2-1)}}{2(m+1)|a|^2}. \tag{7.3}$$

As in the case $m = 2$, it is useful to study for which parameters the critical points lie on the unit circle. For the Blaschke products $B_{a;m}$, if $1 < |a| < (m+2)/m$ then the two critical points lie on $\mathbb{S}^1$. If $|a| = (m+2)/m$ the two critical points collide in a single one in $\mathbb{S}^1$. If $|a| > (m+2)/m$ or $|a| < 1$ then the critical points do not lie on the unit circle and are symmetric with respect to it. Together with the fact that $B_{a;m}$ have a single pole $z_\infty = 1/\overline{a}$ and a unique zero $z_0 = a$, this leads to the exact same structure of preimages of the unit disk as in the case $m = 2$ (see Section 3.1 and Figure 7.1). Summarizing, we have the following.

(a) If $|a| < 1$ the unit disk coincides with the basin of attraction of $z = 0$ and the Julia set $\mathcal{J}(B_{a;m})$ equals the unit circle.

(b) If $|a| = 1$ the Blaschke products $B_{a;m}$ degenerate to a family of degree $m+1$ polynomials.

(c) If $1 < |a| < (m+2)/m$ then the critical points $c_\pm$ lie in the unit circle and the circle map $B_{a;m}|_{\mathbb{S}^1}$ has points with m and $m+2$ preimages.

(d) If $|a| = (m+2)/m$ the two critical points collide in a single one lying on the unit circle and there are two open sets Ω_e and Ω_i which are contained in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$, respectively, whose boundary meets the unit circle at the critical point c and are mapped conformally onto the unit disk and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively. The circle map $B_{a;m}|_{\mathbb{S}^1}$ is a degree m covering of the unit circle.

(e) If $|a| > (m+2)/m$ the two free critical points do not lie on the unit circle and their orbits are symmetric with respect to $\mathbb{S}^1$. For such parameters the circle map $B_{a;m}|_{\mathbb{S}^1}$ is a degree m covering of the unit circle. Moreover, there are two open sets Ω_e and Ω_i which are contained in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$, respectively, whose boundaries do not meet the unit circle and are mapped conformally onto the unit disk and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively.

Using the location of the preimages of the unit disk and its complement that we just described (see Figure 7.1) and which coincides exactly with the situation exposed in Section 3.1 for $B_a = B_{a;2}$, the proof that we gave for Theorem 3.2.1 gives us the following criterion for the connectivity of the Julia set $\mathcal{J}(B_{a;m})$.

Theorem 7.1.2. *Given a Blaschke product $B_{a;m}$ as in (7.2), $m \geq 2$, the following statements hold.*

(a) If $|a| \leq 1$, then $\mathcal{J}(B_{a;m}) = \mathbb{S}^1$.

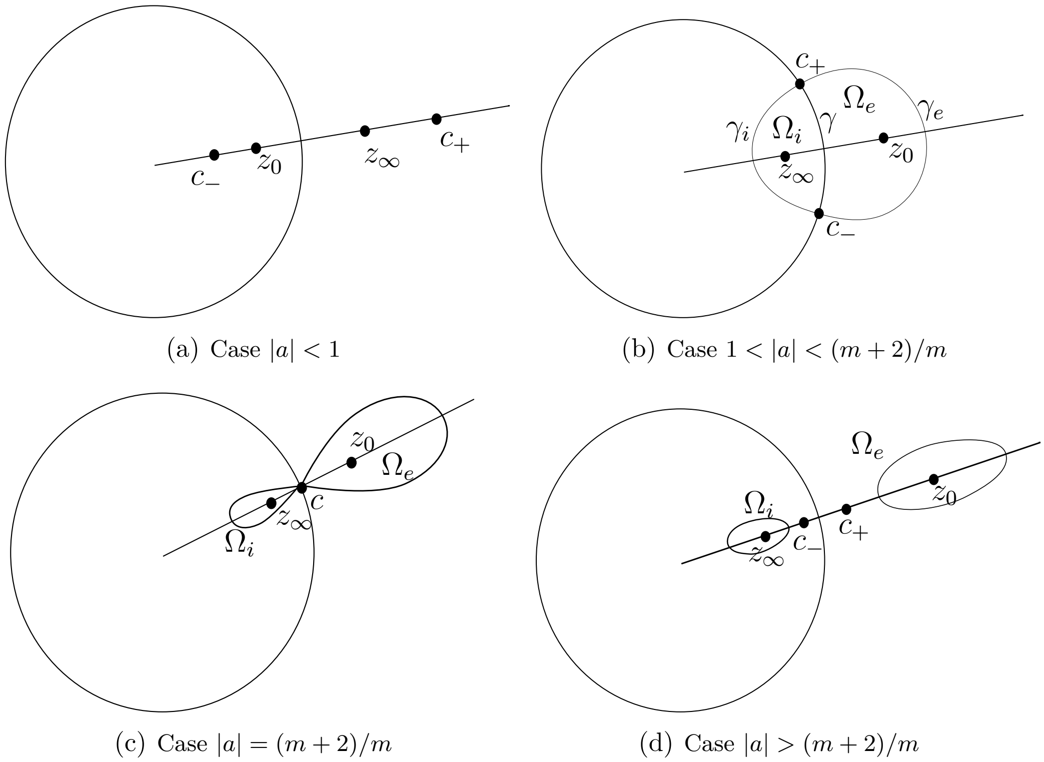


Figure 7.1: Different configurations of the critical points and the preimages of zero and infinity depending on $|a|$.

(b) If $|a| > 1$, then the connected components of $A(\infty)$ and $A(0)$ are simply connected if and only if $c_+ \notin A^*(\infty)$.

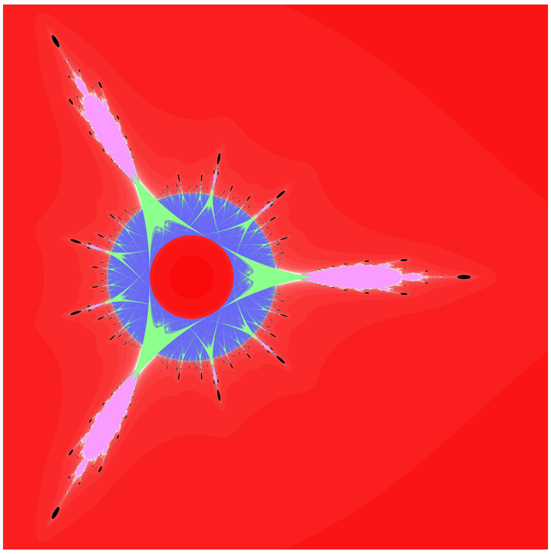
(c) If $|a| \geq (m+2)/m$, then every Fatou component U such that $U \cap A(\infty) = \emptyset$ and $U \cap A(0) = \emptyset$ is simply connected.

Consequently, if $|a| \geq (m+2)/m$, then $\mathcal{J}(B_{a;m})$ is connected if and only if $c_+ \notin A^*(\infty)$.

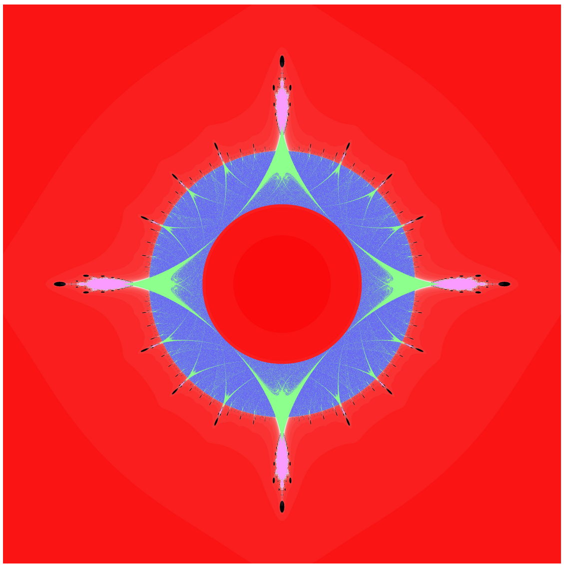
Notice that this Theorem shows that the case $m = 1$ is intrinsically different than the case $m \geq 2$. Indeed, the Blaschke products $B_{a;1}$ may have Herman rings (see Figure 1.4 (d)) and, therefore, its Julia set may not be connected.

7.2 Parameter plane of the degree $m+2$ Blaschke families

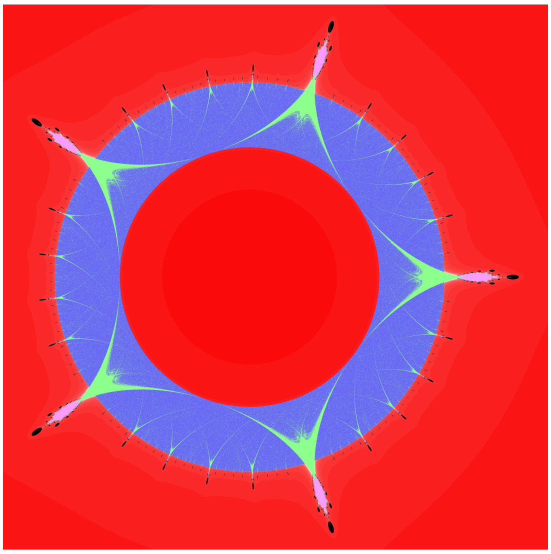
In Figure 7.2 we show the parameter planes of the Blaschke products $B_{a;m}$ for $m = 2, 3, 4$ and 5. In all the cases we observe an inner red disk which corresponds to the unit disk. We see in blue how the annulus of parameters for which both critical points lie on the unit circle becomes narrower as m grows (recall from Section 7.1 that it is given by $1 < |a| < (m+2)/m$). We also observe how the tongues are smaller when m is larger. Indeed, it follows from Corollary 7.3.8 below that if a parameter $(a; m)$ belongs to a tongue of $B_{a;m}$ then $|a| < (m+1)/(m-1)$.



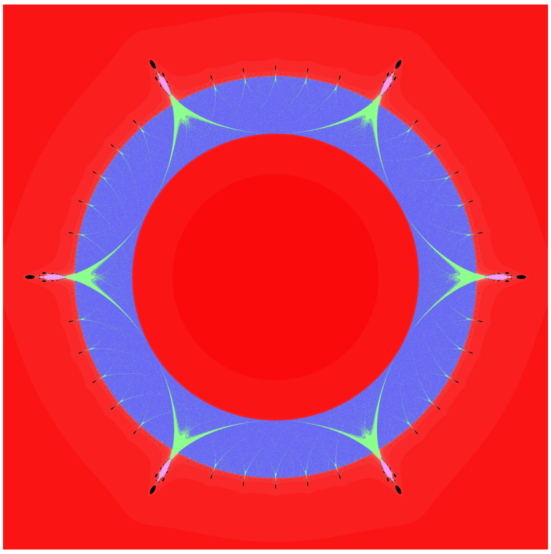
(a) Parameter plane of the family $B_{a;2} = B_a$.



(b) Parameter plane of the family $B_{a;3}$.



(c) Parameter plane of $B_{a;4}$.



(d) Parameter plane of $B_{a;5}$.

Figure 7.2: Parameter plane of the Blaschke family $B_{a;m}$ for $m = 2, 3, 4, 5$. The colors are as follows: red if $c_+ \in A(\infty)$, black if $c_+ \in A(0)$, green if $O^+(c_+)$ accumulates on a periodic orbit in $\mathbb{S}^1$, pink if $O^+(c_+)$ accumulates in a periodic orbit not in $\mathbb{S}^1$ and blue in any other case. The inner red disks correspond to the unit disk.

The following lemma explains the symmetries that are observed in the parameter planes of the Blaschke products $B_{a;m}$. Its proof coincides with the proof of Lemma 5.1.1, which is the analogous result for the family $B_{a;2}$.

Lemma 7.2.1. *Let $a, b \in \mathbb{C} \setminus \mathbb{S}^1$. Then $B_{a;m}$ and $B_{b;m}$ are conformally conjugate if and only if $b = \xi a$ or $b = \xi \overline{a}$, where ξ is an $(m+1)$st root of the unity.*

For fixed $m \geq 2$, we denote by $\mathcal{E}_m$ the set of escaping parameters, i.e., the set of parameters a such that the orbit of the critical point c_+ tends to $z = \infty$ or $z = 0$

under iteration of $B_{a;m}$. We denote its complement, the set of non-escaping parameters, by $\mathcal{B}_m$. The following lemma tells as that the set $\mathcal{B}_m$ of non-escaping parameters is bounded for all $m \geq 2$.

Lemma 7.2.2. *Let $m \geq 2$. Then the following hold.*

(a) If $|a| < 1$ then $a \in \mathcal{E}_m$.

(a) If $1 < |a| \leq (m+2)/m$ then $a \in \mathcal{B}_m$.

(a) The non-escaping set $\mathcal{B}_m$ is bounded.

Proof. Statements (a) and (b) have already been proven. To prove statement (c) we have to show that, if $|a|$ is big enough, then the parameter a is escaping. First we prove that, if $|z| > \lambda(|a|+1)$ with $\lambda \geq 1$, then $|B_a(z)| > \lambda |z|^{m-1}$. It follows from the previous hypothesis that $|z-a| > \lambda$ and that $|z|^2 > |z|(|a|+1) > |1-\overline{a}z|$. Therefore, we have

$$|B_a(z)| = |z|^{m+1}\frac{|z-a|}{|1-\overline{a}z|} > |z|^3\frac{\lambda}{|z|^2} = \lambda|z|^{m-1}.$$

To finish the proof notice that, as $|a|$ tends to infinity, the critical point $c_+(a)$ tends to $ma/(m+1)$. Consequently, it is easy to check that the modulus of the critical value $v_+ = B_{a;m}(c_+(a))$ grows as $M|a|^m$ for some $M > 0$ and, for $|a|$ big enough, $|v_+| > \lambda(|a|+1)$ with $\lambda > 1$. We conclude that $|B^n_{a;m}(v_+)| \to \infty$ when $n \to \infty$. Therefore, for $|a|$ big enough, $a \in \mathcal{E}_m$.

□

Notice that this proof fails for $m = 1$. It may be proven that the family $B_{a;1}$ has attracting cycles in the unit circle for parameters with modulus as big as desired which capture both free critical points (see Figure 7.6). Indeed, if $r \in \mathbb{R}$ and $r \geq 3$, the Blaschke products $B_{r;1}$ have $z = 1$ as a permanent attracting fixed point.

As in the case $m = 2$, the dynamics of the attracting cycles of the Blaschke products $B_{a;m}$ are related to the ones of polynomials if $|a| > (m+2)/m$. On the one hand, it follows from Theorem 1.2.14 that the surgery presented in Chapter 4 can be performed if $B_{a;m}|_{\mathbb{S}^1}$ has no attracting or parabolic cycle. In that case, instead of gluing the squared map $R_2(z) = z^2$ in the unit disk, we glue de mth power map $R_m(z) = z^m$, obtaining a degree $m+1$ polynomial $M_{b;m}(z) = z^{m+1} - \frac{m+1}{m} z^m$ with $z = 0$ as a superattracting fixed point of local degree m. Therefore, if we have an attracting or parabolic cycle of $B_{a;m}$ contained in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, the previous surgery conjugates its dynamics with the ones of a degree $m+1$ polynomial (see Figure 7.3).

On the other hand, if a parameter a is swapping, i.e., the orbit critical point c_+ enters at least once in the unit disk under iteration of $B_{a;m}$, the previous surgery erases all relevant dynamics. In that case, all the constructions build in Section 5.3 apply to the Blaschke products $B_{a;m}$, obtaining the following theorem, which corresponds to Theorem 5.3.4 for $m = 2$.

Theorem 7.2.3. *Let a_0 be a swapping parameter of a Blaschke product $B_{a;m}$ with an attracting or parabolic cycle of period $p > 1$. Then, there is an open set W containing a_0 and $p_0 > 1$ dividing p such that, for every $a \in W$, there exist two open sets U and V with $c_+ \in U$ such that $\left(B^{p_0}_{a;m}; U, V\right)$ is a polynomial-like map. Moreover,*

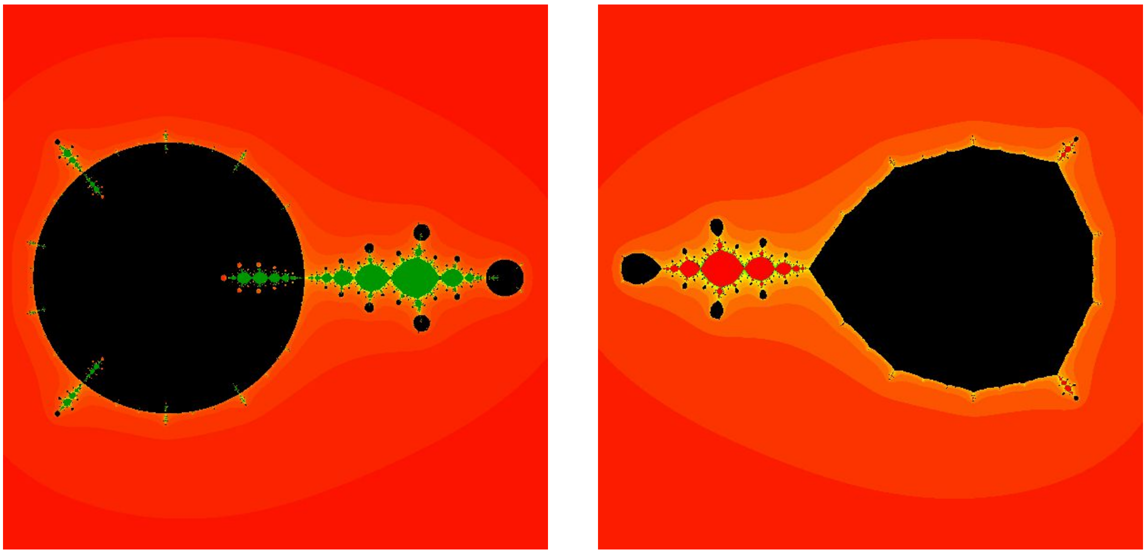


Figure 7.3: Dynamical planes of the Blaschke product $B_{2.5;3}$ (left) and the degree 4 polynomial $M_{-1.5;3}$ (right). The black regions of both figures correspond to the basins of attraction of the superattracting fixed points $z = 0$. For the polynomial we see in red the basin of attraction of a period two attracting cycle. The Blaschke product has two different attracting cycles of period two (2.5 is a disjoint parameter), one outside the unit disk and the other one inside.

(a) If a_0 is bitransitive, $\left(B^{p_0}_{a;m}; U, V\right)$ is hybrid equivalent to a polynomial of the form $p^2_c(z) = \left(z^2 + \overline{c}\right)^2 + c$.

(b) If a_0 is disjoint, $\left(B^{p_0}_{a;m}; U, V\right)$ is hybrid equivalent to a polynomial of the form $p^2_c(z) = \left(z^2 + \overline{c}\right)^2 + c$ or of the form $z^2 + c$.

The previous theorem explains why there are also “copies” of the Mandelbrot set and the Tricorn appearing inside swapping regions of the Blaschke products $B_{a;m}$ (see Figure 7.4).

We want to finish this section remarking that the constructions build in Section 5.4 also generalize to $m > 2$. Therefore, we obtain the following theorem, analogous to Theorem 5.4.2 for $m = 2$, which tells us that the multiplier map is a homeomorphism between any disjoint hyperbolic component of $B_{a;m}$ whose bounded attracting cycles are not in the unit circle and the unit disk. Recall that, given a disjoint hyperbolic component Ω, the multiplier map associates to every parameter $(a; m) \in \Omega$ the multiplier of the attracting cycle $< z_0 >$ of $B_{a;m}$ whose basin of attraction contains the orbit of the critical point c_+.

Theorem 7.2.4. *Let Ω be a disjoint hyperbolic component such that, if $(a; m) \in \Omega$, then $B_{a;m}$ has an attracting cycle in $\mathbb{C}^* \setminus \mathbb{S}^1$. Then, the multiplier map is a homeomorphism between Ω and the unit disk.*

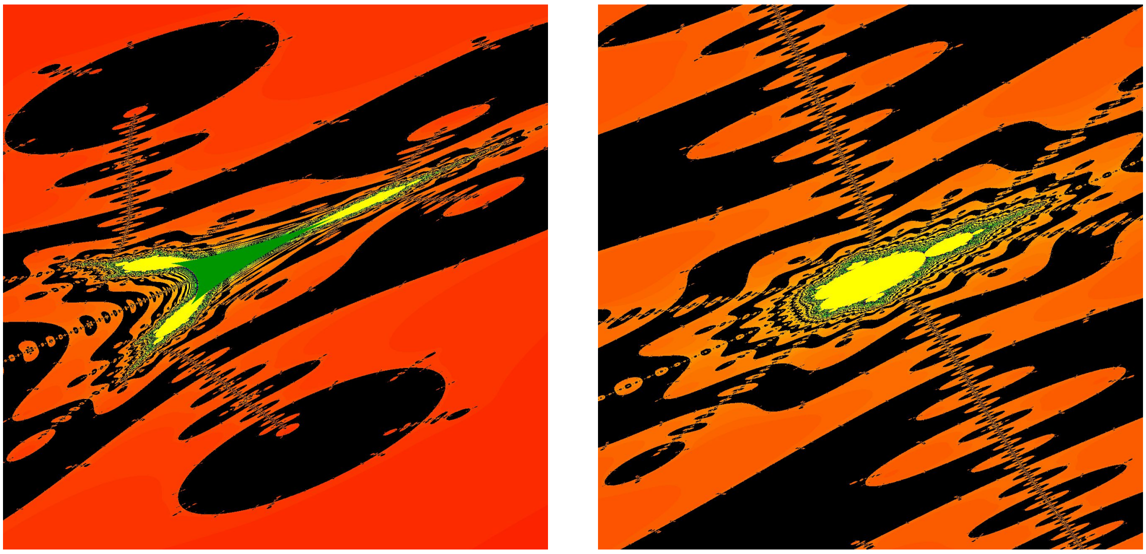


Figure 7.4: Zooms in the parameter plane of the Blaschke family $B_{a;3}$. The left figure shows a Tricorn-like set inside a swapping region $(a \in (1.68561, 1.68702) \times (0.808671, 0.810079))$. The right figure shows a Mandelbrot-like set also inside a swapping region $(a \in (1.693487, 1.693498) \times (0.812587, 0.812598))$. Red points correspond to parameters for which $\mathcal{O}(c_+) \to \infty$ whereas black points correspond to parameters for which $\mathcal{O}(c_+) \to 0$. Green points correspond to bitransitive parameters whereas yellow points correspond to disjoint parameters.

7.3 Tongues of the degree $m+2$ Blaschke products

The goal of this section is to show how to define the tongues for the Blaschke products $B_{a;m}$ and see that they share some properties with the tongues of the family B_a. We first introduce some technical lemmas.

The following lemma provides us with a semiconjugacy between any orientation preserving covering of degree m of the unit circle and the mth map $\theta \to m\theta$ (mod 1) (equivalently given by $R_m(z) = z^m$) (c.f. [MR07, Lemmas 3.1 and 3.3]).

Lemma 7.3.1. *Let $F_a : \mathbb{R} \to \mathbb{R}$ be a continuous and increasing map depending continuously on a. Suppose that $F_a(x+k) = F_a(x) + mk$ for any integer k and for all $x \in \mathbb{R}$. Then, the limit*

$$H_a(x) = \lim_{n\to\infty} \frac{F_a^n(x)}{m^n}$$

exists uniformly on x. This map H_a is increasing, continuous, depends continuously on a and satisfies $H_a(x+k) = H_a(x) + k$ for any integer k and for all $x \in \mathbb{R}$. Moreover, H_a semiconjugates F_a with the multiplication by m, i.e., $H_a(F_a(x)) = mH_a(x)$ for any real x. Furthermore, if F_a is increasing with respect to a, then H_a is also increasing with respect to a.

For $m = 2$ the semiconjugacy h_a obtained from H_a is unique (see Lemma 1.2.4). However, for $m > 2$ it is not. The lift H_a of h_a depends on the lift F_a of the covering map that we may choose. Indeed, the following standard result holds (c.f. [IPRX14]).

Lemma 7.3.2. *Let f be a degree m covering of the unit circle, $m \geq 2$, and let h_1 and h_2 be two semiconjugating maps. Then $h_1 = \xi h_2$ where ξ is a $(m-1)$st root of the unity.*

Proof. Let F be a lift of f. Consider the operator $T_F(H)(x) = H(F(x))/m$. This operator acts in the space $\mathcal{H}$ of non decreasing continuous maps $H : \mathbb{R} \to \mathbb{R}$ such that $H(x+1) = H(x)+1$ provided with the metric $d_{\mathcal{H}}(H_1, H_2) = \sup_{\mathbb{R}}|H_1(x)-H_2(x)|$. Since $\mathcal{H}$ is a complete metric space and the operator T_F is contracting $(d_{\mathcal{H}}(T_F(H_1), T_F(H_2)) = d_{\mathcal{H}}(H_1, H_2)/m)$, there exists a unique fixed point of T_F in $\mathcal{H}$. This fixed point satisfies $H(F) = mH$ and therefore semiconjugates F with the mth map.

Let $F_j(x) = F(x) + j$, $j \in \mathbb{Z}$, be any other lift of f and let H be the fixed point of T_F. Then, it is not difficult to show that the fixed point H_j of T_{F_j} is given by $H_j(x) = H(x) - j/(m-1)$. Therefore, H and H_j are lifts of semiconjugacies h and h_j such that $h = \xi h_j$ where ξ is a $(m-1)$st root of the unity.

To finish the proof it is enough to notice that given any lift H of a semiconjugacy h of f there exist a lift F of f such that $H(F) = mH$ and, therefore, H is the fixed point of the operator T_F. □

The following lemma is analogous to Lemma 1.2.5.

Lemma 7.3.3. *Given a covering f of degree m of the unit circle, a semiconjugating map h sends points of period d of f to points of period d of the mth map.*

We can now define the tongues. Due to symmetry, the Blaschke products $B_{a;m}$ can have at most one attracting cycle (c.f. Lemma 1.2.15). In that case, we call the point x_0 of the cycle lying in the same connected component of the basin of attraction than the critical points the marked point of the cycle. We would like to define the tongues using the lifts of the semiconjugacy coming from Lemma 7.3.1. However, we cannot do it as easily as in the case $m = 2$. For $m = 2$ we said that a parameter a has type $H_a(x_0) = \tau \in \mathbb{R}/\mathbb{Z}$ (see Definition 1.2.17). However, it follows from Lemma 7.3.2, that if $H_{a;m}$ is a lift of a semiconjugacy between $B_{a;m}|_{\mathbb{S}^1}$ and the mth map $\theta \to m\theta \pmod 1$, then $H_{a;m} + 1/(m-1)$ is the lift of another semiconjugacy between them. Therefore, it is convenient to take the type of a parameter $(a; m)$ modulus $1/(m-1)$, i.e., $\tau = H_{a;m}(x_0) \in \mathbb{R}/\left(\frac{1}{m-1}\right)\mathbb{Z}$. Summarizing, we define the tongues as follows.

Definition 7.3.4. Let $B_{a;m}$ (Equation (7.2)) be a Blaschke product and let $H_{a;m}$ be the lift of a semiconjugacy given by Lemma 7.3.1. We say that a parameter a, $|a| \geq (m+2)/m$, is of *type* $\tau \in \mathbb{R}/\left(\frac{1}{m-1}\right)\mathbb{Z}$ if $B_{a;m}|_{\mathbb{S}^1}$ has an attracting cycle $< x_0 >$ and $\tau = H_{a;m}(x_0) \pmod{\frac{1}{m-1}}$, where x_0 is the marked point point of the cycle. The *tongue* $T_\tau = T_{\tau;m}$ is defined as the set of parameters $(a; m)$ such that a is of type τ.

As in the case $m = 2$, every type τ is a periodic point of the mth map (see Lemma 7.3.3). The following theorem is equivalent to Theorem 6.2.1 for $m = 2$. Its proof is analogous.

Theorem 7.3.5. *Given any periodic point τ of the mth map the following results hold.*

(a) The tongue $T_{\tau;m}$ is not empty and consists of $m+1$ connected components (only one connected component if we consider the parameter plane modulo the symmetries given by the (m+1)st roots of the unity).

(b) Each connected component of $T_{\tau;m}$ *contains a unique parameter* r_τ*, called the* root of the tongue*, such that* $B_{r_\tau;m}$ *has a superattracting cycle in* $\mathbb{S}^1$*. The root* r_τ *satisfies* $|r_\tau| = (m+2)/m$*.*

(c) Every connected component of $T_{\tau;m}$ *is simply connected.*

(d) The boundary of every connected component of $T_{\tau;m}$ *consists of two curves which are continuous graphs as function of* $|a|$ *and intersect each other in a unique parameter* a_τ *called the* tip of the tongue*.*

The bifurcations which take place in a neighborhood of the tip of every tongue for $m = 2$ also appear for $m > 2$ (see Figure 7.5). The following theorem explains it. Its proof is analogous to the one of Theorem 6.3.2.

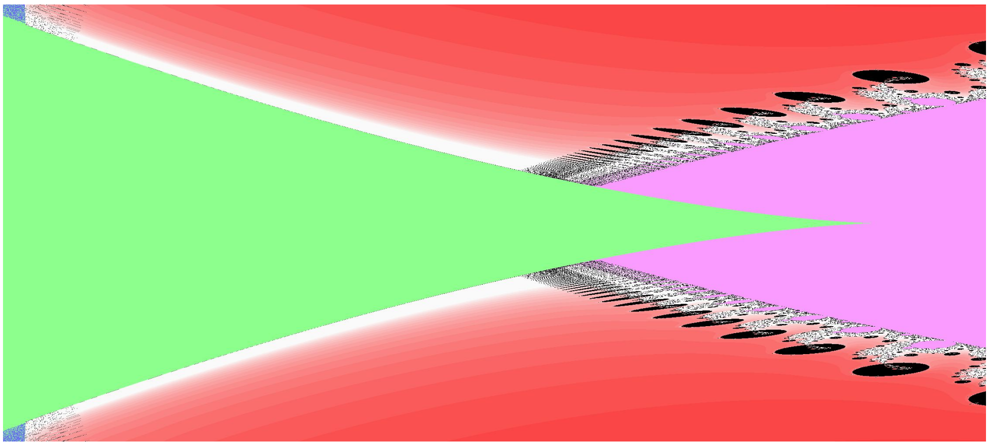


Figure 7.5: A zoom in the parameter plane of the Blaschke family $B_{a;3}$ in a neighborhood of the tongue $T_{0;3}$.

Theorem 7.3.6. *Let* a_τ *be the tip of a tongue* $T_{\tau;m}$ *of period* p*. Then, there exists a neighborhood* U *of* a_τ *such that if* $a \in U$ *then, either* $a \in T_{\tau;m}$ *or* $a \in \partial T_{\tau;m}$ *or* a *belongs to a disjoint hyperbolic component.*

We finish the section proving that as m increases the tongues of $B_{a;m}$ decrease. As in Section 6.1, it is convenient to work with another parametrization of the Blaschle products. It follows from Lemma 7.1.1 that any Blaschke product $B_{a;m}$, where $a = re^{2\pi i\alpha}$, is conjugate to $B_{r,(m+1)\alpha;m}$ (Equation (7.1)). Let $g_{r,\alpha;m} := B_{r,(m+1)\alpha;m}|_{\mathbb{S}^1}$. Then we have

$$g_{r,\alpha;m}(e^{2\pi ix}) = e^{2(m+1)\pi ix}e^{2(m+1)\pi i\alpha}\frac{e^{2\pi ix}-r}{1-re^{2\pi ix}},$$

where $r \in [(m+2)/m, \infty)$ and $\alpha \in [0, 1/(m+1))$. Its lift has the form

$$h_{r,\alpha;m}(x) = (m+1)x + (m+1)\alpha + \frac{1}{2\pi i}\log\left(\frac{e^{2\pi ix}-r}{1-re^{2\pi ix}}\right).$$

Lemma 7.3.7. *Let* $r \geq (m+2)/m$*. Then, the lift* $h_{r,\alpha;m}(x)$ *satisfies that* $\frac{\partial}{\partial x}h_{r,\alpha;m}(x)$ *is non-negative for all* x*. Moreover, for any* $p \in \mathbb{N}$*, the mapping* $\alpha \to h^p_{r,\alpha;m}(x) \in \mathbb{S}^1$ *is strictly increasing and, if* $r \geq (m+1)/(m-1)$*, then* $h^p_{r,\alpha;m}(x) \geq 1$ *for all* $x, \alpha \in \mathbb{R}$*.*

Proof. We prove that $\frac{\partial}{\partial x} h_{r,\alpha;m}(x)$ is non-negative for all x, and hence so is $\frac{\partial}{\partial x} h^p_{r,\alpha;m}(x)$ for all p. Then, strict monotonicity with respect to α for all p follows from the fact that we have it for $p = 1$. We also prove that $\frac{\partial}{\partial x} h_{r,\alpha;m}(x) \geq 1$ if $r \geq (m+1)/(m-1)$. Notice that $\frac{\partial}{\partial x} h_{r,\alpha;m}(x)$ is given by the formula

$$\frac{\partial}{\partial x} h_{r,\alpha;m}(x) = m + 1 + \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)}.$$

It can easily be seen that this expression is non-negative for $r \geq (m+2)/m$. Indeed, the minimum of this function is taken whenever $x = 0$, and

$$\frac{\partial}{\partial x} h_{r,\alpha;m}(0) = m + 1 + \frac{1 - r^2}{1 + r^2 - 2r} = m + 1 + \frac{(1+r)}{(1-r)}.$$

For $r > 1$ this is an increasing function which is equal to zero for $r = (m+2)/m$. Moreover, it is greater than 1 when $r \geq (m+1)/(m-1)$. □

It follows from the previous lemma that, if $r \geq (m+1)/(m-1)$, there cannot be any attracting periodic point in $\mathbb{S}^1$. Therefore, the following result holds.

Corollary 7.3.8. *Let $m \geq 2$. If $a \in T_{\tau;m}$ then $|a| < (m+1)/(m-1)$, i.e., the tongues are bounded for all $m \geq 2$.*

We finish this section noticing that for $m = 1$ all tongues are unbounded (see Figure 7.6). If $r \geq 3$, then $g_{r,\alpha;1}$ is a homeomorphism and, given a rational number p/q, the tongue $T_{p/q;1}$ is defined to be the set of parameters (r, α) such that $g_{r,\alpha;1}$ has rotation number p/q. Moreover, all tongues are unbounded since, for any fixed $r \geq 3$, as we move α from 0 to $1/2$ we take all possible rotation numbers (c.f. [Her79]).

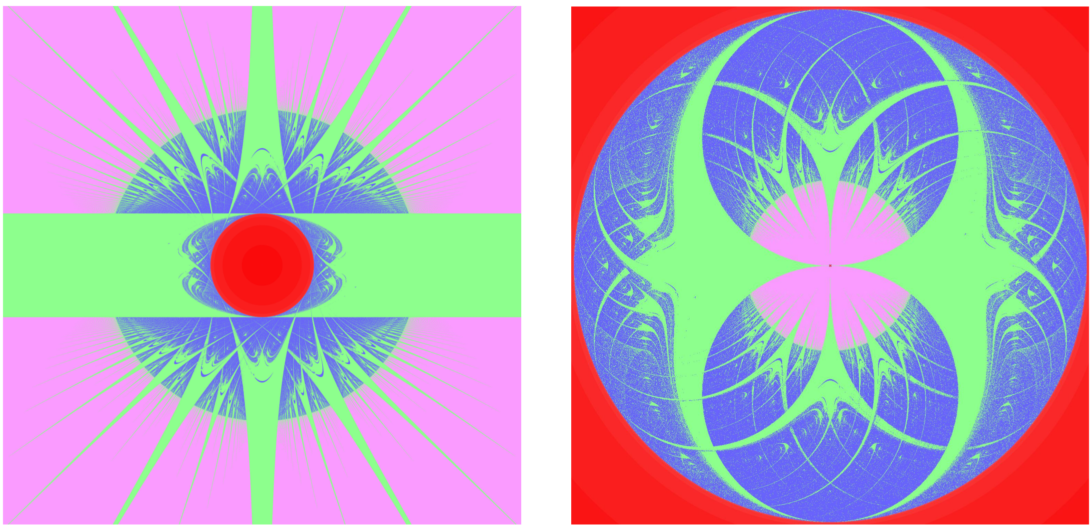


Figure 7.6: The left figure shows the parameter plane of the Blaschke family $B_{a;1}$, where we observe that all tongues are unbounded. The right figure shows the parameter plane of the same family taking $\tilde{a} = 1/a$. With this second parametrization, all tongues tend to zero. The colors are as follows: red if $c_+ \in A(\infty)$, green if $O^+(c_+)$ accumulates on a periodic orbit in $\mathbb{S}^1$, pink if $c_+ \notin \mathbb{S}^1$ and $O^+(c_+)$ is not captured by an attracting cycle and blue otherwise.

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