Universitat Autònoma de Barcelona

Essays in Inflation Expectations, Monetary Economics, and Asset Pricing

Alberto Sánchez Martín

ADVERTIMENT. L'accés als continguts d'aquesta tesi queda condicionat a l'acceptació de les condicions d'ús establertes per la següent llicència Creative Commons: http://cat.creativecommons.org/?page_id=184

ADVERTENCIA. El acceso a los contenidos de esta tesis queda condicionado a la aceptación de las condiciones de uso establecidas por la siguiente licencia Creative Commons: http://es.creativecommons.org/blog/licencias/

WARNING. The access to the contents of this doctoral thesis it is limited to the acceptance of the use conditions set by the following Creative Commons license: https://creativecommons.org/licenses/?lang=en

Ph.D. Thesis: Essays in Inflation Expectations, Monetary Economics, and Asset Pricing

Alberto Sánchez Martín

International Doctorate in Economic Analysis (IDEA)
Departament d'Economia i d'Historia Econòmica
Facultat d'Economia i Empresa

Supervisor: Prof. Albert Marcet Torrens

Cerdanyola del Vallès, June 2016

A mis padres y mi hermano.

Acknowledgements

Foremost, I would like to thank my supervisor Albert Marcet for his inspirational suggestions and guidance during the elaboration of this thesis. I learned from him the value of research and how to understand complex problems in the most simple way.

I am thankful to Juan Pablo Nicolini. He welcomed me at the Federal Reserve Bank of Minneapolis and helped me with outstanding comments. Furthermore, I am grateful to faculty members of the International Doctorate in Economic Analysis and the Universitat Autònoma de Barcelona from which I learned a lot. In particular, I would like to acknowledge to Luca Gambetti because of his support.

I also want to acknowledge to Luis Puch because of his comments and suggestions. Moreover, I also thank my student-mates in IDEA. In special, to Francesco Cerigioni, Francesco Gallio, Edgardo Lara, Isabel Melguizo, Dilan Okcuoglu, Javier Rodríguez, Benjamín Tello, and Alexis Vázquez as the best partners one could have had.

Finally, I am grateful to the Ministerio de Economía y Competitividad for the FPI scholarship linked to the project SEV-2011-0075-01.

I am indebted to my family because of their comprehension, affection, and exceptional support.

Sonia, thank you for being by my side during these years and for helping me out with your care.

Index

General Introduction

This thesis is focused on the role of expectations in the economy. It is well-known that expectations play a prominent role in decision making and are a crucial feature in macroeconomics. Typically, in macroeconomic modeling it is assumed that agents are endowed with rational expectations. This basically means that agents in the model perfectly know the general equilibrium functions of the model and behave optimally. Many times this assumption is appropriate because it simplifies modeling choices, as it avoids having to make separate assumptions about agents' expectations. Also, rational expectations avoids Lucas' Critique as agents in the model incorporate policy changes into their expectations. However, it sometimes attaches features that the data does not reproduce. In those cases, it is worthy to slightly deviate from rational expectations and assume agents does not perfectly know equilibrium functions (but know the form of the function) and still behave optimally. This is what is called adaptive learning expectations. This thesis examines some scenarios where the rational expectations hypothesis fails to reproduce the behavior of the variables in the data and looks for answers using the adaptive learning expectations hypothesis instead.

The first chapter, entitled *Money, Inflation, and Inflation Expectations*, documents a significant negative relation between money and inflation expectations for the 1990-2007 period in the US which is new to the literature. During the same period, there existed a weak relation between money growth and inflation consistent with the literature. I call Inflation Expectations and Money Puzzle (IEMP) to the difficulty of a rational expectations model to reproduce both facts. A simple Money-in-the-Utility (MIU) function model with a money demand shock and exogenous endowment accounts for both facts when the representative agent learns about the true inflation process. If money supply adjusts endogenously to keep inflation rate close to a target and smooth, money varies whereas inflation keeps stable. Because the agent does not know the true inflation process, inflation expectations might respond positively to past shocks whereas money supply decreases to offset their potential effects on inflation. This disentangles the IEMP.

The second chapter, entitled *Long-Run Behavior from an Endogenous Monetary Policy Perspective*, looks to the long-run relation between inflation, nominal interest rate, and money growth. Recently, it has been noted the long-run relation between these variables have changed. Whereas in the period before the Volcker disinflation they had a one-to-one relation, during the last two decades there were no significant relation. This study documents the long-run relation of those variables focusing on a long sample, 1960-2007, and a short-sample, 1990-2007, and questions whether the endogenous monetary policy model of the previous chapter could match the change in the long-run relation of this variables. It concludes neither a rational expectations nor a learning expectations version of the model is able to replicate this behavior and highlights the importance of some feature that is not attached to the model.

The third chapter, entitled *Asset Pricing in an Heterogeneous Expectations Model*, studies how can be introduced a consumption-based asset pricing model with rational agents and agents that learn about the market outcomes. It evaluates the performance of this model estimating the relevant parameters using the U.S. asset pricing data through the method of simulated moments technique. The model suggests that, in the one hand, persistence of stock returns and, in the other hand, volatility of stock returns and price dividends ratio provide a trade-off in which the proportion of learners plays a key role.

Chapter 1

Money, Inflation, and Inflation Expectations

> Undoubtedly, the state of inflation expectations greatly influences actual inflation and thus the central bank's ability to achieve price stability. (Bernanke (2007))

In this chapter I document a significant negative relation between money and inflation expectations for the 1990-2007 period in the US which is new to the literature. However, it is a consensus in the literature that, empirically, there is no relation between money growth and current inflation at high frequency data. This is striking given that a rational expectations model would suggests inflation expectations and inflation must covary similarly with money growth. I argue endogenous monetary policy in an economy populated by an agent who updates expectations about inflation using past data is able to explain both facts.

Inflation expectations are crucial to understand the behavior of monetary policy because they might affect price stability objective as denoted in the opening quote above. I look for the relation between inflation expectations and M2 annual growth using three measures of current expectations about one-year-ahead inflation: the Survey of Professional Forecasters (SPF), the Cleveland Fed Inflation Expectations (CFIE), and the University of Michigan Inflation Expectations (UMIE). I document there is a very significant negative relation between inflation expectations and money growth.

It is typically thought that a given change in the rate of money growth must induce an equal change in the rate of inflation. However, it is a consensus in the literature that empirically there is no relation between money growth and current inflation at high-frequency data (see Lucas (1980) and McCandless and Weber (1995)).

I define the Inflation Expectations and Money Puzzle (IEMP) to the difficulty for a rational expectations model to match a negative relation between money growth and inflation expectations as well as a weak relation between money growth and inflation as found in the data. I formally provide evidence of the difference between the estimated coefficients in the data using Adam, Beutel, and Marcet (2015) test.

Standard New Keynesian model do not provide a role for money in the determination of equilibrium allocation. Presumably, the reason is the weak relation between money aggregates and inflation. However, in the recent years, the development of policies with the aim of increasing nominal balances have posed a particular interest to have a theory that allows us to think in the role of money in the economy. Since inflation and inflation expectations are at the core of the conduction of monetary policy, I focus on providing a theory that is able to generate empirical evidence regarding money, inflation, and inflation expectations.

It is documented by Díaz-Giménez and Kirkby (2014) that standard models in monetary economics generate a proportional proportional relation between money and inflation[1]. Here, I highlight the importance of money in price determination building in an endogenous monetary policy framework as in Nicolini (2014) and Navarro and Nicolini (2012). In this framework, money endogenously adjusts to keep inflation rate stable hence, providing a source to account for a weak relation between money growth and inflation. Differently from these studies, I consider a general class of standard utility functions that allow money demand to depend on nominal interest rates what explicitly accounts for the effects of inflation expectations on inflation.

In particular, I present a Money-in-the-Utility (MIU) function model with a money demand shock. In this setting, there exists a central bank who minimizes an *ad-hoc* welfare loss function by using money endogenously to keep inflation rate close to a target and smooth. Hence, money growth endogenously adjusts to offset potential effects on inflation so that inflation remains stable whereas money growth variates. At the same time, I assume the representative agent in

[1] In particular, these models are versions of the Cash-in Advance, New Keynesian and Search-Money. All of them have attached a version of the exchange equation relating money, prices, real output, and velocity. Hence, the only way prices and money are not proportional in these frameworks depend on how velocity and real output respond to changes in inflation and money.

the economy updates expectations about inflation using past data. In this way, a negative shock to inflation might reduce inflation expectations because the agent does not incorporate the future money growth adjustment. Hence, inflation expectations increase whereas money growth adjusts negatively to offset the increase in inflation expectations. This explains the negative relation between inflation expectations and money growth and disentangles IEMP.

The main contribution of this chapter is twofold. First, it documents a negative relation between money growth and inflation expectations as a new fact in the literature which adds to the weak relation between money growth and inflation consistent with the findings of the literature. Second, it provides a theory to account for the mechanisms of endogenous monetary policy and inflation expectations that explain both relations.

An important attempt to reconcile low relation between inflation and money growth at high-frequency data with theory is the one in Alvarez, Atkeson, and Edmond (2009). They use a Baumol-Tobin type model with heterogeneous agents and explain low correlation between inflation and money is due to an endogenous change in velocity induced by the increase in money supply. That is because agents with less velocity are the most prone to absorb changes in money supply. However, this model would not be able to replicate a negative relation between money growth and inflation since agents are endowed with rational expectations.

There is also literature related in optimal monetary policy when agents are learning. Molnár and Santoro (2014) study optimal monetary policy under discretion in a New Keynesian model where private agents follow adaptive learning and the CB uses their expectations as an additional mechanism to get inflation and output gap closer to zero target. Mele, Molnár, and Santoro (2011) study the same framework when there is monetary policy under commitment. These studies however lie on standard New Keynesian theory were the policy tool is nominal interest rates and do not look relations between money growth, inflation, and inflation expectations.

The rest of the chapter is organized as follows. Section 1.1 includes a description of the inflation expectations measures used. Section 1.2 presents the facts documenting empirical evidence on the relation of inflation and inflation expectations with money growth. This section formally tests rational expectations using inflation expectations measures. Section 1.3 introduces a general framework and evaluates their goodness in matching the empirical evidence when representative agent is endowed with rational expectations and learning expectations. Section 1.4 presents conclusions.

1.1 Inflation Expectations Data

This section briefly introduces three measures of inflation expectations data: Survey of Professional Forecasters (SPF), Cleveland Fed Inflation Expectations (CFIE), and University of Michigan Inflation Expectations (UMIE). The idea is to provide robust results across different sources of inflation expectations measures. Each of these measures is devoted to represent a measure-type of inflation expectations. In particular, a professional forecast survey measure, a model-implied measure of non-arbitrage term structures, and a consumer survey measure.

1.1.1 Survey of Professional Forecasters

The SPF began in 1968 conducted by the American Statistical Association (ASA) and the National Bureau of Economic Research (NBER). Since 1990, the survey is conducted by the Federal Reserve Bank of Philadelphia (Phil FRB). It includes quarterly surveyed forecasts from professional forecasters for many macroeconomic variables that includes GDP, employment, industrial production, housing, and CPI, among others. Answers are available in mean, median, cross-sectional dispersion and individual responses and it includes forecasts for the current quarter, nowcast, up to one year ahead. Number of respondents oscillates from around 30 to 50. I use data on one-year ahead forecasts mean and median of CPI annual inflation rate. Data is taken from the Phil FRB webpage at http://www.phil.frb.org/research-and-data/real-time-center/survey-of-professional-forecasters.

1.1.2 Cleveland Fed Inflation Expectations

The CFIE is a model-based measure, released by the Federal Reserve Bank of Cleveland and estimated using the model in Haubrich, Pennacchi, and Ritchken (2011), that tries to address the shortcomings of existing raw inflation expectations measures. This model-implied inflation expectations combine information about term structures on nominal Treasury Bills, TIPS, inflation swap rates and SPF. I use the 1-year ahead inflation expectations data available from January 1982 at https://www.clevelandfed.org/en/our-research/indicators-and-data/inflation-expectations.aspx.

1.1.3 University of Michigan Inflation Expectations

The UMIE is an inflation expectations measure coming from University of Michigan Surveys of Consumers. These surveys are available since 1977 and include explicit questioning about consumer expectations on price evolution. In particular, consumers are asked about how much they expect prices to go up over the next year and over the next five to ten years. I use mean and median on expected changes in price over one-year-ahead. The data is available at http://www.sca.isr.umich.edu.

1.2 Empirical Evidence

This section is devoted to show empirical evidence regarding inflation, inflation, money growth, and inflation expectations in the US. I use quarterly data for the 1990 to 2007 period. In this way, I exclude disturbances in the data from the so-called Volcker Disinflation and from financial crisis. I use annual growth rates of M2 growth as money growth, and annual CPI growth as inflation rate from the Federal Reserve of St. Louis database (FRED) at http://research.stlouisfed.org/fred2 as well as inflation expectations measures introduced in the previous section. In section 1.2.1, I present three empirical facts. In section 1.2.2, I provide evidence about the difficulty of a rational expectations model to generate the relations of inflation and inflation expectations with money growth. I call this fact the Inflation Expectations and Money Puzzle (IEMP).

1.2.1 Facts

This section explains empirical relations between inflation, money growth and inflation expectations and how they relate with existing literature. I document three facts. First, a low relation between inflation and money growth. Second, a negative relation between inflation expectations and money growth as a new fact in the literature. Third, I document a highly positive inflation autocorrelation which is smaller than in previous decades.

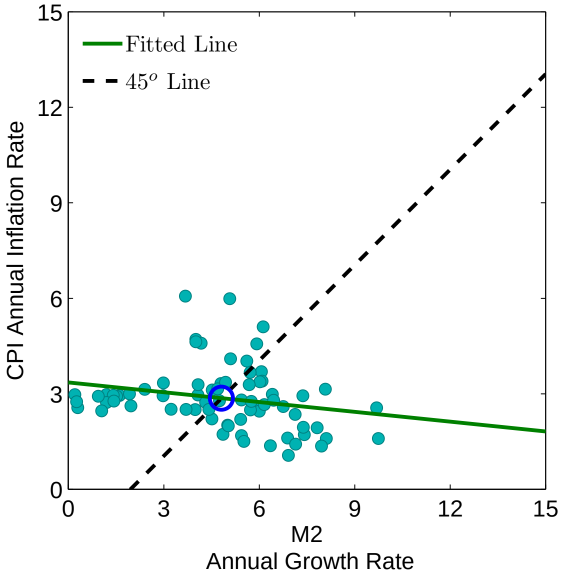

FIGURE 1.1. High-Frequency Relation Between Money and Nominal GDP Growth in the United States from Q1:1990 to Q4:2007. White circle is the grand mean of the series.

	SPF Mean	SPF Median	CFIE	UMIE Mean	UMIE Median
M2 Annual Growth Rate	−0.17 (0.000)	−0.15 (0.000)	−0.11 (0.000)	−0.16 (0.000)	−0.06 (0.005)

TABLE 1.1. OLS coefficient estimate of inflation expectations on money growth. Sample Q1:1990-Q4:2007. P-value reported in parenthesis, Newey-West corrected.

Lucas (1980) suggests that one of the main implications of the Quantity Theory of Money is that a change in the quantity of money must induce an equal change in the rate of price inflation. Then, Lucas used the 1955-1975 period to plot quarterly data on annual inflation against annual money growth and showed there was no clear relationship[2]. Figure 1.1 reproduces the scatter plot between CPI annual inflation against M2 annual growth. Although fitted line has a negative slope, it seems there is not clear pattern on the relation between money and inflation. The slope regression is estimated to be -0.10 with significance

[2]At low frequency data, however, there exists consensus on a unitary relation between inflation and money growth as claimed by Lucas (1980) and McCandless and Weber (1995). Recently, there has been some discussion about the failure of this fact during the last two decades (see Sargent and Surico (2011) and Díaz-Giménez and Kirkby (2014)). In chapter 2, I deeply discuss this fact.

test p-value of 0.043[3]. I confirm the Quantity Theory of Money does not hold at high-frequency data and report it as **Fact 1**[4]. This fact is robust when one uses M1 or MZM as money aggregate.

State of inflation expectations is key to understand inflation determination as well as the ability of a central bank to achieve price determination. It is crucial to look how inflation expectations are related to money growth. To that aim I measure their relation by using OLS regression coefficient of each of the inflation expectations measures presented in the previous section on M2 growth. Inflation expectations measures are interpreted as current expectation about one-year-ahead inflation. Table 1.1 presents estimated slope coefficients. All coefficients are negative and very significant. I conclude there is evidence of a negative relation between inflation expectations and money growth and report it as **Fact 2**.

Finally, very important controversy take place in the literature regarding output volatility, inflation volatility and persistence. Stock and Watson (2003), and Cogley and Sargent (2005) among others find a decline in volatility and inflation persistence took place since early 1980. Here, I take into account these important facts about inflation persistence[5]. I measure persistence as the first order autocorrelation of CPI annual inflation. In the 1990-2007 period, persistence is estimated to be 0.78 with 95% confidence band defined on the interval $(0.67, 0.86)$[6]. I include this feature as **Fact 3**.

1.2.2 The Inflation Expectations and Money Puzzle

In this section I formally raise the Inflation Expectations and Money Puzzle (IEMP). I call the IEMP to the difficulty of a model populated by agents endowed with rational expectations to match the empirical negative relation between inflation expectations and money growth together with the empirical weak relation between inflation and money growth. To observe this, let t be time period mea-

[3]Except when explicitly specified, results on tests provided along this study are obtained by using Newey-West variance-covariance matrix estimator.

[4]Díaz-Giménez and Kirkby (2014) argue that strictly speaking, Quantity Theory of Money predicts that a given change in money growth must induce an equal change in the rate of price inflation plus real output growth. This would not be meaningless if one considers the period after the Volcker Disinflation because output growth to inflation ratio kept relatively high. I find, Fact 1 is robust if one takes into account real output growth.

[5]Since in the theory developed afterward law of motion for inflation turns out to be an AR(1) process, a decrease in inflation persistence induces a decrease in inflation volatility.

[6]In the 1960-1975 period autocorrelation is 0.98 what documents a decrease in inflation persistence.

sured in quarters, let π_{t+4} be the annual inflation rate at period $t+4$, Δm_t the annual money growth rate at period t and $E_t \pi_{t+4}$ expectations about annual inflation at period $t+4$ at period t, i.e., regarding information up to period t. Under rational expectations, the prediction error is defined by $e_{t+4} = \pi_{t+4} - E_t \pi_{t+4}$. For any t, rational expectations implies that e_{t+4} is orthogonal to any variable included in the information set t. That is to say, $Cov(e_{t+4}, x_t) = 0$ where x_t is any variable contained in the information set available at period t. Then, covariance of money growth and one-year ahead inflation is

$$Cov(\Delta m_t, \pi_{t+4}) = Cov(\Delta m_t, E_t \pi_{t+4} + e_{t+4}) = Cov(\Delta m_t, E_t \pi_{t+4}).$$

Therefore, a model where agent is assumed to be endowed with rational expectations seems unable to generate negative covariance between expected inflation and money growth documented by Fact 2 together with a weak relation between inflation and money growth if inflation is sufficiently persistent.

I rely on Adam, Beutel, and Marcet (2015) to provide formal evidence about the IEMP. I test significance of the difference between the OLS slope coefficients of the regression of inflation expectations on money growth and the regression of one-year-ahead inflation on money growth using method. First, write the regression

$$\xi_t = a + b\Delta m_t + u_t + \mu_t, \quad \text{with} \quad E\left(x_t(u_t + \mu_t)\right) = 0 \tag{1.1}$$

for $x_t' = (1, \Delta m_t)$, and where $\xi_t = E_t^{\mathscr{P}} \pi_{t+4} + \mu_t$ denotes an observed measure of the unobservable true inflation expectations $E_t^{\mathscr{P}} \pi_{t+4}$ which includes a measurement error μ_t. The regression residual u_t includes variation in inflation expectations not explained by money growth. Denote $\tilde{b}$ the OLS estimator of equation (1.1).

Under the hypothesis of rational expectations, $E_t = E_t^{\mathscr{P}}$, the above equation implies

$$\pi_{t+4} = a + b\Delta m_t + e_{t+4} + u_t, \quad \text{with} \quad E\left(x_t(u_t + e_t)\right) = 0 \tag{1.2}$$

where $e_{t+4} = \pi_{t+4} - E_t \pi_{t+4}$ is the prediction error that is orthogonal to all the observations before date $t+4$. Denote $\hat{b}$ the OLS estimator of equation 1.2. As discussed above, under the hypothesis of rational expectations the covariance between one-year-ahead inflation and money growth must equal covariance

	M2 Annual Growth Rate	p-value H_0: $\tilde{b} = \hat{b}$
SPF Mean	-0.17 (0.000)	0.008
CPI Annual Inflation	-0.03 (0.292)	
SPF Median	-0.15 (0.000)	0.019
CPI Annual Inflation	-0.03 (0.293)	
CFIE	-0.11 (0.000)	0.153
CPI Annual Inflation	-0.03 (0.315)	
UMIE Mean	-0.16 (0.000)	0.006
CPI Annual Inflation	-0.03 (0.324)	
UMIE Median	-0.06 (0.005)	0.326
CPI Annual Inflation	-0.03 (0.291)	

TABLE 1.2. Rational expectations test in Q1:1990-Q4:2007. Individual significance p-values in parenthesis. Variance-Covariance matrix of coefficients estimated by Newey-West.

between expectations about this future inflation and money growth, and hence both regression coefficients must be equal. Then, one just needs to formally test the null hypothesis, $\tilde{b} = \hat{b}$, what in fact is to test the null hypothesis of rational expectations. For that, one estimates both coefficients together using SUR representation.

Table 1.2 shows the results on the estimation of equations (1.1) and (1.2)[7]. There are five panels, each of which is devoted to one of the inflation expectations measures. First panel, for example, specifies in the first column dependent variable is SPF mean in equation (1.1) and CPI annual inflation in (1.2). Second column presents slope coefficients with individual significance p-values provided in parenthesis aside each estimate. Third column provides the p-value on the equality of the coefficients. When one uses SPF mean, SPF median, and UMIE mean as the measure of inflation expectations in equation (1.1) are statistically different from the estimated coefficient in (1.2). If one uses CFIE as the measure of inflation expectations in (1.1) one cannot reject the null of equality of the coefficients at a 15% significance level. Finally, when one uses UMIE median there is strong evidence on no difference between estimated coefficients. I conclude there is formal evidence confirming difference between the relation of inflation expectations and money growth and the relation of one-year-ahead inflation and money growth for the 1990-2007 period. This formally documents the IEMP.

[7]Estimated slope coefficients are those of Fact 2 and Fact 3 while p-values change because Newey-West variance-covariance matrix is estimated jointly for equations (1.1) and (1.2).

1.3 Theoretical Model of Money Demand

In this section I use a model to explain the facts presented in section 1.2 together with the IEMP. To that aim, in section 1.3.1 I use a simple Money-in-the-Utility (MIU) function model with a representative consumer, exogenous endowment and money demand shock. I explicitly derive a money demand equation relating money, inflation, inflation expectations and the money demand shock. In this setting, a benevolent Central Bank (CB) endowed with rational expectations is assumed to set endogenously money supply to minimize an *ad-hoc* welfare loss function. Welfare loss function induces the CB a double objective: to keep inflation close to a target, and make inflation smooth. Section 1.3.2 assumes representative consumer is endowed with rational expectations and the CB sets monetary policy under commitment. Then, I evaluate whether the model is able to match Fact 1 - Fact 3. In Section 1.3.3 introduces simplest version of the model when the consumer is *learning*. In particular, I assume representative consumer knows the true inflation process form but does not know the true parameter values. Hence, representative consumer optimally estimates the parameter values in the true inflation process using past data, that is, the agent *learns* about the true inflation process. In this version, the welfare loss function just takes into account deviations from target. In this way, model keeps simple and I can easily illustrate how learning helps to explain Fact 1, Fact 2, and hence, the IEMP. Finally, section 1.3.4 evaluates Fact 1 - Fact 3 in the complete model when representative agent learns about true inflation process and the CB cares about inflation targeting and inflation smoothing.

1.3.1 General Setting Model

In this section I introduce the general setting of the money in the utility function model. In the first part I define the model economy and derive a standard money demand equation. In the second part I introduce the CB setting.

Money Demand Equation

I consider an economy populated by an infinitely-lived agent where production is exogenously given by

$$Y_t = e^{\gamma}, \ \forall \ t \geq 0, \ \gamma \in (-\infty, \infty) \tag{1.3}$$

that is, production is constant and I assume it to be deterministically known by the representative agent in the model.

Let t be the current period measured in quarters. Representative agent chooses sequences of quantity of nominal bonds, B_t/P_t, real bonds, b_t, consumption good, C_t, and how much real balances to hold into next period M_t/P_t, with P_t being the price level. Agent has the expected utility function

$$E_0^{\mathscr{P}} \sum_{t=0}^{\infty} \delta^t U\left(C_t, M_t/P_t; \xi_t\right) = E_0^{\mathscr{P}} \sum_{t=0}^{\infty} \delta^t \left(\frac{(C_t)^{1-\sigma}}{1-\sigma} + \xi_t^{\nu} \frac{\left(\frac{M_t}{P_t}\right)^{1-\nu}}{1-\nu} \right) \tag{1.4}$$

where $\sigma \in (0,\infty)$, $\nu \in (0,\infty)$, and $\{\xi_t\}_{t=0}^{\infty}$ is a sequence of money demand shocks. Expectation is taken using a probability measure $\mathscr{P}$ which might be defined by the objective probability measure of the model or not. The exogenous sequence of money demand shocks is defined by the process

$$\begin{aligned} log\xi_t &= log\xi_{t-1} + \omega_t, \quad t \geq 0 \\ \omega_t &= \rho_\omega \omega_{t-1} + \varepsilon_t^{\omega}, \quad \varepsilon_t^{\omega} iidN(0, \sigma_\omega^2), \quad t \geq 0 \end{aligned} \tag{1.5}$$

$\rho_\omega \in (0,1)$, and with $log\xi_{-1}$ and ω_{-1} given.

Agent problem is summarized by

$$\max_{\{C_t, B_t/P_t, b_t, M_t/P_t\}_{t=0}^{\infty}} E_0^{\mathscr{P}} \sum_{t=0}^{\infty} \delta^t U\left(C_t, M_t/P_t; \xi_t\right)$$

$$\begin{aligned} \text{s.t.} \quad C_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} + b_t &= \frac{M_{t-1}}{P_t} + (1+i_{t-1})\frac{B_{t-1}}{P_t} + (1+r_{t-1})\,b_{t-1} \\ &+ T_t + Y_t, \quad t \geq 0 \end{aligned} \tag{1.6}$$

$$B_{-1},\ b_{-1},\ M_{-1},\ i_{-1} \text{ and } r_{-1} \text{ given.} \tag{1.7}$$

with T_t being a lump sum transfer, i_{t-1} and r_{t-1} being nominal interest rate and the real interest rate, respectively.

For all $t \geq 0$ first order conditions are given by the money demand equation, the real bond pricing equation and the Fisher equation, respectively,

$$\left(\frac{M_t}{P_t}\right)^{-\nu} \xi_t^{\nu} = (C_t)^{-\sigma} \frac{i_t}{1+i_t} \tag{1.8}$$

$$\frac{1}{1+r_t} = \delta E_t^{\mathscr{P}} \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \tag{1.9}$$

$$i_t = r_t + E_t^{\mathscr{P}} \pi_{t+1} \tag{1.10}$$

together with the budget constraint (1.6). Using market clearing condition for consumption good, together with the assumption that the representative agent know the deterministic production process and applying logs in equation (1.9) one obtains the equilibrium real interest rate is

$$r_t = \rho \tag{1.11}$$

with $\rho = -log\delta$.

After log-linearizing equation (1.8), using market clearing condition for consumption good, real interest rate and the Fisher equation

$$p_t = m_t - \alpha - log\xi_t + \eta E_t^{\mathscr{P}} \pi_{t+1}$$

where α being a constant term, m_t is log money balances in period t, p_t is log price level in period t, $\pi_t = p_t - p_{t-1}$ is the quarterly inflation rate, $\eta = 1/(\nu i^{ss}(1+i^{ss}))$ is the interest rate semi-elasticity of money demand, $i^{ss} = \rho + \pi^{ss}$ is the steady state nominal interest rate, and π^{ss} is the steady state inflation rate. Adding and subtracting p_{t-1} one obtains the money demand equation

$$\pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta E_t^{\mathscr{P}} \pi_{t+1}, \quad t \geq 0 \tag{1.12}$$

with p_{-1} given.

Central Bank

Assume there exists a Central Bank (CB) endowed with rational expectations that chooses a sequence of money balances under commitment to minimize the *ad-hoc* welfare function,

$$E_0 \sum_{t=0}^{\infty} \delta^t \left(\lambda (\pi_t - \bar{\pi})^2 + (1-\lambda)(\pi_t - \pi_{t-1})^2 \right) \tag{1.13}$$

with $\lambda \in (0,1)$ being the weight given to maintain inflation close to the target $\bar{\pi} \geq 0$. To that aim, the CB chooses a sequence of money supply with a period lag subject to the money demand equation (1.12) and the exogenous process $log\xi_t$ (1.5).

1.3.2 Rational Expectations

Assume the representative agent in the economy is endowed with rational expectations. This means to substitute in expression (1.12) the operator $E_t^{\mathscr{P}}$ by the mathematical expectations operator E_t. The problem of the CB is then defined by,

$$\begin{aligned}
\min_{\{p_t, m_{t+1}, \pi_t\}_{t=0}^{\infty}} \; & E_0 \sum_{t=0}^{\infty} \delta^t \left(\lambda (\pi_t - \bar{\pi})^2 + (1-\lambda)(\pi_t - \pi_{t-1})^2 \right) \\
\text{s.t.} \quad & \pi_t = p_t - p_{t-1}, \quad t \geq 0 \\
& \pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta E_t \pi_{t+1}, \quad t \geq 0 \qquad (1.14) \\
& log\xi_t = log\xi_{t-1} + \omega_t, \quad t \geq 0 \\
& \omega_t = \rho_\omega \omega_{t-1} + \varepsilon_t^\omega, \quad \varepsilon_t^\omega iidN(0, \sigma_\omega^2), \quad t \geq 0 \qquad (1.5) \\
& log\xi_{-1}, \; w_{-1}, \; m_0, \; p_{-1}, \; \pi_{-1} \; \text{given.}
\end{aligned}$$

Let $\mu_{1,t}$ be the Lagrange multiplier of (1.14), then optimality conditions are summarized by

$$E_t \mu_{1,t+1} = 0 \tag{1.15}$$

$$\begin{aligned}
\phi^{-1} \pi_t - \lambda \bar{\pi} - (1-\lambda)\pi_{t-1} - \delta(1-\lambda) E_t \pi_{t+1} & \\
+ \mu_{0,t} - \theta \mu_{1,t-1} + \mu_{1,t} = 0 &
\end{aligned} \tag{1.16}$$

for all $t \geq 0$ with $\mu_{1,-1} = 0$, $\phi = (1 + \delta(1-\lambda))^{-1}$, $\theta = \eta \delta^{-1}$ and together with the money demand equation (1.14). After using (1.16) in (1.15) and substituting

π_t with time t money demand (1.14), one obtains money is given by

$$\begin{aligned} m_{t+1} = \phi(\lambda\bar{\pi} + (1-\lambda)\pi_t + \delta(1-\lambda)E_t\pi_{t+2} + \theta\mu_{1,t}) \\ + p_t + \alpha - \eta E_t\pi_{t+2} + log\xi_t + \rho^{\omega}\omega_t. \end{aligned} \tag{1.17}$$

Hence, by substituting money in (1.14), induced inflation must hold

$$\begin{aligned} \pi_{t+1} =& \phi(\theta\mu_{1,t} + \lambda\bar{\pi} + (1-\lambda)\pi_t + \delta(1-\lambda)E_t\pi_{t+2}) \\ &+ \eta(E_{t+1}\pi_{t+2} - E_t\pi_{t+2}) - \varepsilon^{\omega}_{t+1} \end{aligned} \tag{1.18}$$

and substituting inflation in (1.16) one obtains the expression for the Lagrange multiplier

$$\mu_{1,t+1} = -(\eta\phi^{-1} - \delta(1-\lambda))(E_{t+1}\pi_{t+2} - E_t\pi_{t+2}) + \phi^{-1}\varepsilon^{\omega}_{t+1}. \tag{1.19}$$

From (1.18) and (1.19) define the equilibrium system to solve. Due to its simplicity I look for a closed-form solution using the method of undetermined coefficients. Solution for inflation can be summarized by

$$\pi_t = \psi_0^{RE} + \psi_1^{RE}\pi_{t-1} + \psi_2^{RE}\varepsilon^{\omega}_t + \psi_3^{RE}\varepsilon^{\omega}_{t-1} \tag{1.20}$$

$$\pi_0 = \psi_0^{RE} + \psi_1^{RE}\pi_{-1} + \psi_2^{RE}\varepsilon^{\omega}_0 \tag{1.21}$$

where ψ_0^{RE} is a function of the parameters $(\delta,\lambda,\bar{\pi})$, ψ_1^{RE} is a function of (δ,λ), ψ_2^{RE} is a function of (δ,λ,η), and ψ_3^{RE} is a function of (δ,λ,η). It can be easily proved that $\psi_0^{RE} \to \bar{\pi}$ and $\psi_1^{RE} \to 0$ as $\lambda \to 1$. This basically means that when CB cares about targeting inflation ($\lambda \to 1$), inflation moves around target π.

Next I evaluate the goodness of the model to explain the empirical evidence. Baseline calibration is set to reproduce quarterly moments of the data in the 1990-2007 period. I set target inflation to $\bar{\pi} = 0.005$ which defines 2% annual inflation target. The discount factor is fixed as $\delta = 0.995$, which provides 1% annual real interest rate in steady state. I calibrate the nominal interest rate semi-elasticity of money demand η to match the slope of the regression of 3-month Treasury Bill rate on real balances defined by log(M2) minus log(CPI). Then $\eta = 0.06$ and significantly different from zero. Then, I make one difference of the OLS residual of this regression and estimate a AR(1) model which calibrates the money demand shock persistence parameter to be $\rho_\omega = 0.62$ and significant and the variance of the *iid* term to be $\sigma_\omega = 0.004$. I set $\sigma = 1$, so that utility function is log with respect to consumption and arbitrarily set production by fixing $\gamma = 1$. I provide 72 periods moments mean across 20,000 simulations. I present results for $\lambda > 0.75$ so that the CB cares more about inflation targeting.

Figure 1.2 provides results. Upper-left panel provides the model OLS slope coefficient of the regression of annual inflation on annual money growth, that is the model counterpart of Fact 1[8]. The moment in the model matches qualitatively the OLS slope coefficient estimated from the data, though it does not clearly get into data 95% bands for most of the λ values showed. Upper-right panel shows the OLS slope coefficient of the regression of expected inflation on annual money growth, the model counterpart of Fact 2. This coefficient statistically zero far from the negative coefficient illustrated in the data. Finally, lower panel presents estimated first order autocorrelation of annual inflation, model counterpart of Fact 3. When $\lambda \to 1$, CB just cares about inflation targeting, and even though the model is still able to replicate very well the autocorrelation coefficient showed in the data.

In conclusion, for values of $\lambda > 0.75$ the model is able to qualitatively and quantitatively Fact 3, and reproduces qualitatively Fact 1. However, it is far from replicating FactDashed 2 and thus, cannot explain the IEMP[9]. In particular, when the CB uses money to stabilize inflation close to a target it offsets all possible effects of the money demand shock on inflation. In this way, inflation keeps stable where money variates raising weak relation between them. However, the rational expectations consumer perfectly understands the effect of a given shock on inflation. Then, a given money demand shock affects in the same way inflation expectations as it affects inflation. This explains a weak relation between inflation expectations and money growth too, and it makes the model to fail in obtaining Fact 2.

1.3.3 CG Learning Expectations. No Smoothing Inflation

In this section I illustrate how learning can help to explain Fact 2 and the IEMP. For simplicity, assume $\lambda = 1$. In this simple case, there is no smoothing objective in the objective of the CB or, equivalently, the CB objective is just to keep inflation close to a target. This assumption makes rational expectations model

[8]Notice that production in the model is constant and so output growth is zero. Hence, this coefficient is equivalent to that in Fact 1.

[9]Results are robust to changes in η, ρ, and are independent to changes in σ_ω and $\bar{\pi}$.

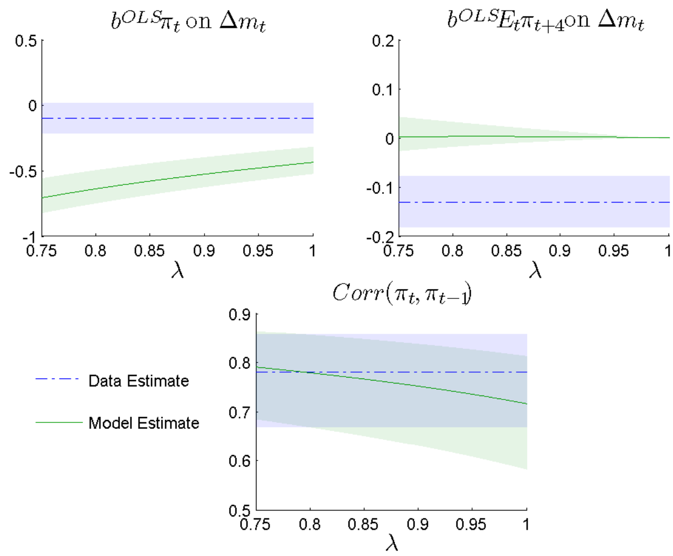

FIGURE 1.2. Rational Expectations Model moments compared with data. Model moments are the mean across simulations. Simulation length is 72 and number of simulations is 20,000. Shaded areas illustrate 95% confidence interval.

solution for inflation being of the form[10]

$$\pi_t = \psi_0^{RE} + \psi_2^{RE}\varepsilon_t^{\omega} + \psi_3^{RE}\varepsilon_{t-1}^{\omega}.$$

I assume representative agent is endowed with *learning expectations*. That is, the agent in the economy learns about the true inflation process. In particular, the agent do not know the exact value of the parameters in the true process of inflation and recursively estimate using observables by updating the past expectation corrected by past forecast error weighted with a *constant gain* (CG) parameter[11]. As it is usual in the adaptive learning literature, one assumes the Perceived Law of Motion of the agent (PLM) to be consistent with the rational expectations solution. That is, assume the PLM is given by

$$\pi_t = \tilde{\psi}_0 + \tilde{\psi}_2\varepsilon_t^{w} + \tilde{\psi}_3\varepsilon_{t-1}^{\omega}.$$

[10]I disregard $t = 0$ solution and assume $t \geq 1$. In this way, inflation process is time-invariant.

[11]Marcet and Sargent (1989) started modern literature on learning by studying convergence of learning algorithms. Evans and Honkapohja (2001) provide an extensive review of adaptive learning algorithms and applications.

Assume representative agent updates expectations at period t with information up to period $t-1$. This assumption is standard in the learning literature to avoid simultaneity problems arising from the joint determination of inflation and the expected inflation. Hence, expected inflation is defined by $E_t^{\mathscr{P}} \pi_{t+1} = \psi_{0,t}$ with coefficient updated according to the learning rule

$$\psi_{0,t+1} = \psi_{0,t} + g(\pi_t - \psi_{0,t}), \quad t \geq 0$$

with $g \in (0,1)$ being the gain and $\psi_{0,0}$ given. As in Gaspar, Smets, and Vestin (2006) and Molnár and Santoro (2014), the problem of the CB incorporates the constraint on how inflation expectations of the agent are formed. That is, inflation expectations enter in the CB problem as an additional state.

Given the definition of inflation expectations above, money demand equation (1.12) becomes,

$$\pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta \psi_{0,t}, \quad t \geq 0.$$

Hence, the CB's problem becomes[12]

$$\min_{\{p_t, m_t, \pi_t, \psi_{0,t+1}\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \delta^t (\pi_t - \bar{\pi})^2$$

$$\text{s.t.} \quad \pi_t = p_t - p_{t-1}, \quad t \geq 1$$

$$\pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta \psi_{0,t}, \quad t \geq 1 \tag{1.22}$$

$$\psi_{0,t+1} = \psi_{0,t} + g(\pi_t - \psi_{0,t}), \quad t \geq 1 \tag{1.23}$$

$$\begin{aligned} log\xi_t &= log\xi_{t-1} + \omega_t, \quad t \geq 1 \\ \omega_t &= \rho_\omega \omega_{t-1} + \varepsilon_t^\omega, \quad \varepsilon_t^\omega iidN(0, \sigma_\omega^2), \quad t \geq 1 \end{aligned} \tag{1.5}$$

$$log\xi_0, \;\; w_0, \;\; p_0, \;\; \pi_0, \;\; \psi_{0,1} \;\; \text{given.}$$

[12] In this setting, given $\psi_{0,0}$, p_{-1}, and m_0 exogenously determine π_0. That is, differently form rational expectations model, π_0 is not a control variable. In the same way, $\psi_{0,1} = \psi_{0,0} + g(\pi_0 - \psi_{0,0})$, and $p_0 = \pi_0 + p_{-1}$ are also determined.

Let $\mu_{1,t}/2$ and $\mu_{2,t}/2$ be the Lagrange multipliers of equation (1.22) and (1.23), respectively. Optimality conditions are summarized by

$$E_{t-1}\mu_{1,t} = 0 \tag{1.24}$$

$$\pi_t - \bar{\pi} + \mu_{1,t} - g\mu_{2,t} = 0 \tag{1.25}$$

$$\mu_{2,t} - \delta\eta E_t\mu_{1,t+1} - \delta(1-g)E_t\mu_{2,t+1} = 0 \tag{1.26}$$

for all $t \geq 1$ and together with money demand (1.22). From (1.24) and (1.26), one obtains $\mu_{2,t} = 0$ for all $t \geq 1$. This basically means that the CB is able to match optimal policy without caring about how inflation expectations are updated. Using this result and substituting (1.25) in (1.24)

$$E_{t-1}\pi_t = \bar{\pi}.$$

Using (1.22) one obtains optimal money supply is given by

$$m_t = \alpha + \bar{\pi} + p_{t-1} + log\xi_{t-1} + \rho_\omega\omega_{t-1} - \eta\,\psi_{0,t} \tag{1.27}$$

and substituting back to equation (1.22), equilibrium law of motion of inflation is given by,

$$\pi_t = \bar{\pi} - \varepsilon_t^\omega. \tag{1.28}$$

The intuition for the solution is that first best policy is to induce inflation to be equal to the target $\bar{\pi}$ in every period. However, since ε_t is not known by the CB when monetary policy m_t is set, the CB cannot offset this element from the money demand equation and it has to set an inflation rate equal to target plus an unpredictable noise. The CB can do so without taking into account how inflation expectations are formed because they are included in its information set when monetary policy is set. Hence, CB can fully offset any effect of inflation expectations in the money supply.

But how this model can generate the IEMP? Using (1.27) one can obtain money growth

$$\Delta m_t = \pi_{t-1} + \omega_{t-1} + \rho_\omega\omega_{t-1} - \rho_\omega\omega_{t-2} - \eta\left(\psi_{0,t} - \psi_{0,t-1}\right)$$

and plugging $t-1$ inflation from (1.28),

$$\Delta m_t = \bar{\pi} + \rho_\omega\omega_{t-1} - \eta\left(\psi_{0,t} - \psi_{0,t-1}\right), \quad t \geq 1. \tag{1.29}$$

Proposition 1. Under the learning model with $\lambda = 1$, the regression slope coefficient, $\tilde{b}_0^T$, of the regression of time length $T \geq 1$,
$E_t^{\mathscr{P}} \pi_{t+1} = \tilde{a}_0^T + \tilde{b}_0^T \Delta m_t + \varepsilon_t$
is *strictly negative* for any $g \in (0,1)$, $\delta \in (0,1)$, $\rho_w \in (0,1)$, $\sigma_w \in (0,\infty)$ and $\bar{\pi} > 0$.

Proof. It is only needed to show that $Cov(\Delta m_T, \psi_{0,T}) < 0$. First, using the solution for the inflation process (1.28) it is possible to write learning rule for $T \geq 1$ as

$$\psi_{0,T} = (1-g)\psi_{0,T-1} - g\varepsilon^w_{T-1} + g\bar{\pi} \tag{1.30}$$

Second, since money growth rate in (1.29) is the sum of target inflation, a term that depends on the past money demand shock, $\rho_w \omega_{T-1}$, and a third term that depends on the difference on current inflation expectations and past inflation expectations of the agent, $-\eta(\psi_{0,T} - \psi_{0,T-1})$. It is just needed to show that each of these two terms have negative or zero covariance with expected inflation $\psi_{0,T}$.

Covariance of the first term and inflation expectations is,

$$\begin{aligned} Cov(\rho_w \omega_{T-1}, \psi_{0,T}) &= Cov\left(\rho_w \sum_{i=0}^{T-1} \rho_w^i \varepsilon^{\omega}_{T-1-i}, -g \sum_{i=0}^{T-1} (1-g)^i \varepsilon^{\omega}_{T-1-i}\right) \\ &= -g\rho_w \sigma^2_{\omega} \sum_{i=0}^{T-1} (\rho_w(1-g))^i = -g\rho_w\sigma^2_{\omega} \frac{1-(\rho_w(1-g))^T}{1-(\rho_w(1-g))} < 0. \end{aligned}$$

$\forall\ g \in (0,1)$, $\delta \in (0,1)$, $\rho_w \in (0,1)$, $\sigma_w \in (0,\infty)$, $\bar{\pi} > 0$, and $T \geq 1$ [13]. First equality comes from recursive substitution in (1.5) and (1.30) up to the initial condition.

Finally, the covariance between the second term and inflation expectations is,

$$\begin{aligned} Cov(-\eta(\psi_{0,T} - \psi_{0,T-1}), \psi_{0,T}) &= -\eta\sigma^2_{\psi_{0,T}} + \eta Cov(\psi_{0,T-1}, (1-g)\psi_{0,T-1}) \\ &= -\eta\sigma^2_{\psi_{0,T}} + \eta(1-g)\sigma^2_{\psi_{0,T-1}} \\ &= -g^2\eta\left[\frac{1-(1-g)^{2T}}{1-(1-g)^2} - (1-g)\frac{1-(1-g)^{2(T-1)}}{1-(1-g)^2}\right]\sigma^2_{\omega} < 0 \end{aligned}$$

[13] Remember that in this model $\nu = (\eta(-log\delta + \bar{\pi})(1 - log\delta + \bar{\pi}))^{-1}$.

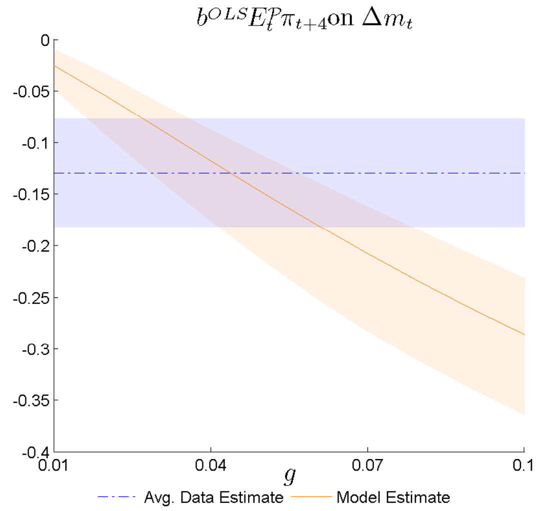

FIGURE 1.3. Learning Expectations with $\lambda = 1$ OLS coefficient expected annual inflation on annual money growth using 72 periods simulations. Mean across 20,000 simulations. Shaded areas illustrates 95% confidence interval. Data estimate is the mean across expectations measures.

for any $g \in (0,1)$, $\delta \in (0,1)$, $\rho_w \in (0,1)$, $\sigma_w \in (0,\infty)$, $\bar{\pi} > 0$, and $T \geq 1$. Third equality comes from computing T inflation expectations variance and $T-1$ inflation expectations variance using (1.30)[14], and last expression is negative because the term in the brackets is positive for $g \in (0,1)$ and $T \geq 1$.

Since the covariance between inflation expectations and each of the terms that adds up to money growth are negative, then $Cov(\Delta m_T, \psi_{0,T}) < 0$. □

The intuition is the following. Assume a past and positive money demand shock, $\downarrow \varepsilon_{t-1}^w$, happens. From the learning rule, and the true inflation process,

$$\psi_{0,t} = \psi_{0,t-1} + g(\bar{\pi} - \varepsilon_{t-1}^w - \psi_{0,t-1})$$

agent revises expectations upwards, $\uparrow \psi_{0,t}$. At the same time money supply adjusts negatively to offset the direct effect of this shock and the effect of the increase in inflation expectations on the money demand, $\downarrow \Delta m_t$. That is, since agent slightly deviates from the true inflation process they are not able to foresight the effect of a shock in monetary policy. In this way, inflation expectations increases while money balances decreases. This effect explains Fact 2. Notice that at the same time, inflation is the sum of the target plus current iid

[14]In particular, $Var(\psi_{0,j}) = g^2 Var(\sum_{i=0}^{j-1}(1-g)^i \varepsilon_{j-1-i}^{\omega}) = g^2 \frac{1-(1-g)^{2j}}{1-(1-g)^2}\sigma_\omega^2$ for $j = 1,2,\ldots,T-1,T,\ldots$

shock whereas money growth depends on past shocks orthogonal to the current shock, what makes covariance equal to zero and explains Fact 3. Hence, IEMP is explained by this simple model.

For evaluation purposes about the quantitative goodness in matching Fact 2. I estimate the OLS coefficient of the regression of expected one-year-ahead annual inflation on annual money growth. I use same baseline calibration as in the rational expectations model explained above. I let the new parameter g to be in the interval $(0.01, 0.10)$. Figure 1.3 presents results. Data estimate is the mean OLS coefficient across inflation expectations measures. The model generates a negative coefficient, independently of the values g in the interval. The model is quantitatively able to match average data estimate for values of g from around 0.025 to 0.07. I conclude, the model with learning expectations and CG is able to match Fact 3 when CB just cares about inflation targeting.

1.3.4 CG Learning with Inflation Targeting and Smoothing

This section is devoted to develop the learning model with $\lambda \in (0,1)$. That is, in this case, the objective of the CB is to keep inflation close to a target whereas keeping inflation smooth. Here, rational expectations solution is of the form

$$\pi_t = \psi_0^{RE} + \psi_1^{RE}\pi_{t-1} + \psi_2^{RE}\varepsilon_t^{\omega} + \psi_3^{RE}\varepsilon_{t-1}^{\omega}.$$

Similarly to previous section I assume agent knows the form of the true inflation process but needs to recursively estimate the parameters on it. I also assume, as before, that agent updates its expectations in period t with information up to period $t-1$. Hence, agent's expected inflation is defined by $E_t^{\mathscr{P}}\pi_{t+1} = \psi_{0,t}(1+\psi_{1,t}) + \psi_{1,t}^2\pi_{t-1}$ with updating learning rules[15]

$$\psi_{0,t} = \psi_{0,t-1} + g(\pi_{t-1} - \psi_{0,t-1} - \psi_{1,t-1}\pi_{t-2})$$
$$\psi_{1,t} = \psi_{1,t-1} + g\pi_{t-2}(\pi_{t-1} - \psi_{0,t-1} - \psi_{1,t-1}\pi_{t-2})$$

with $g \in (0,1)$.

[15]This rule is called Stochastic Gradient. It just ignores to weight the forecast error by the conditional estimate of the forecast error variance. In this way, representative agent does not have to use an extra rule to estimate this second moment and it keeps problem simpler.

Given the definition of inflation expectations above, money demand equation (1.12) becomes,

$$\pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta(\psi_{0,t}(1+\psi_{1,t}) + \psi_{1,t}^2\pi_{t-1}), \quad t \geq 0. \tag{1.31}$$

Hence, the CB's optimization problem is now defined by[16]

$$\min_{\{p_t, m_t, \pi_t, \psi_{0,t+1}\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \delta^t \{\lambda(\pi_t - \bar{\pi})^2 + (1-\lambda)\lambda(\pi_t - \pi_{t-1})^2\}$$

$$\text{s.t.} \quad \pi_t = p_t - p_{t-1}, \quad t \geq 1$$

$$\pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta(\psi_{0,t}(1+\psi_{1,t}) + \psi_{1,t}^2\pi_{t-1}) \quad t \geq 1 \tag{1.32}$$

$$\psi_{0,t+1} = \psi_{0,t} + g(\pi_t - \psi_{0,t} - \psi_{1,t}\pi_{t-1}), \quad t \geq 1 \tag{1.33}$$

$$\psi_{1,t+1} = \psi_{1,t} + g\pi_{t-1}(\pi_t - \psi_{0,t} - \psi_{1,t}\pi_{t-1}), \quad t \geq 1 \tag{1.34}$$

$$\begin{aligned} &log\xi_t = log\xi_{t-1} + \omega_t, \quad t \geq 1 \\ &\omega_t = \rho_\omega \omega_{t-1} + \varepsilon_t^\omega, \quad \varepsilon_t^\omega iidN(0, \sigma_\omega^2), \quad t \geq 1 \end{aligned} \tag{1.5}$$

$$log\xi_0, \ w_0, \ p_0, \ \pi_0, \ \psi_{0,1} \ \text{given.}$$

This optimization problem implies non-linear system of first order conditions to solve. However, in the same way as in previous section when $\lambda = 1$, one can guess that the CB is able to use optimal monetary policy without taking into account how inflation expectations are updated, (1.33) and (1.34). Hence, one can solve a different problem without the learning rules restrictions and then check whether this solution is feasible under the restricted problem. Then I solve

$$\min_{\{p_t, m_t, \pi_t, \psi_{0,t+1}\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \delta^t \{\lambda(\pi_t - \bar{\pi})^2 + (1-\lambda)\lambda(\pi_t - \pi_{t-1})^2\}$$

$$\text{s.t.} \quad \pi_t = p_t - p_{t-1}, \quad t \geq 1$$

$$\pi_t = m_t - p_{t-1} - \alpha - log\xi_t + \eta(\psi_{0,t}(1+\psi_{1,t}) + \psi_{1,t}^2\pi_{t-1}) \tag{32}$$

$$\begin{aligned} &log\xi_t = log\xi_{t-1} + \omega_t, \quad t \geq 1 \\ &\omega_t = \rho_\omega \omega_{t-1} + \varepsilon_t^\omega, \quad \varepsilon_t^\omega iidN(0, \sigma_\omega^2), \quad t \geq 1 \end{aligned} \tag{5}$$

$$log\xi_0, \ w_0, \ p_0, \ \pi_0, \ \psi_{0,1} \ \text{given.}$$

[16]Except for the fact that $\lambda \in (0,1)$ and that this makes agent to use a new updating rule, the model structure is equivalent to the model where $\lambda = 1$.

Let $\mu_{1,t}/2$ be the Lagrange multipliers of equation (1.32). For $t \geq 1$, optimality conditions are summarized by

$$E_{t-1}\mu_{1,t} = 0 \tag{1.35}$$

$$\phi^{-1}\pi_t - \lambda\bar{\pi} - (1-\lambda)\pi_{t-1} - \delta(1-\lambda)E_t\pi_{t+1} + \mu_{1,t} = 0 \tag{1.36}$$

with $\phi = (1+\delta(1-\lambda))^{-1}$ and together with money demand (1.32). So that, the solution of the unrestricted problem for inflation and money is

$$\begin{aligned} m_t =& \phi(\lambda\bar{\pi} + (1-\lambda)\pi_{t-1} + \delta(1-\lambda)E_{t-1}\pi_{t+1}) \\ &+ p_{t-1} + log\xi_{t-1} + \rho_w\omega_{t-1} + \alpha - \eta(\psi_{0,t}(1+\psi_{1,t}) + \psi_{1,t}^2\pi_{t-1}) \end{aligned} \tag{1.37}$$

and inflation,

$$\pi_t = \phi(\lambda\bar{\pi} + (1-\lambda)\pi_{t-1} + \delta(1-\lambda)E_{t-1}\pi_{t+1}) - \varepsilon_t^{\omega}. \tag{1.38}$$

It is easy to check that (1.37) and (1.38) are also feasible when learning rules restrict the problem. Hence, (1.37) and (1.38) also solve the restricted problem. By method of undetermined coefficients one obtains solution for (1.38)

$$\pi_t = \psi_0^{LE} + \psi_1^{LE}\pi_{t-1} - \varepsilon_t^{\omega}. \tag{1.39}$$

where ψ_0^{LE} is a function of $(\delta, \lambda, \bar{\pi})$ and ψ_1^{LE} is a function of (δ, λ)[17].

Next I evaluate the goodness of the model in generating empirical evidence. I use baseline calibration specified in section 1.3.2 to evaluate the model with respect to the empirical evidence. I set the gain parameter $g = 0.02$. Figure 1.4 presents results. Upper-left panel shows the OLS coefficient of the regression of annual inflation on annual money growth, which evaluates Fact 1. The upper-right panel shows OLS coefficient of the regression of one-year-ahead inflation on money. Lower-left panel shows first order autocorrelation coefficient of annual inflation rate, Fact 3. This learning expectations model is able to match Fact 1, and Fact 3 in the same way rational expectations model does. The reasoning is the same as in the rational expectations model. Even though agent's expectations are computed differently form the rational expectations model, the

[17]In the same way as in the rational expectations model, it is easy to check that the constant of the inflation law of motion and the persistence coefficients converge to $\bar{\pi}$ and 0, as $\lambda \to 1$.

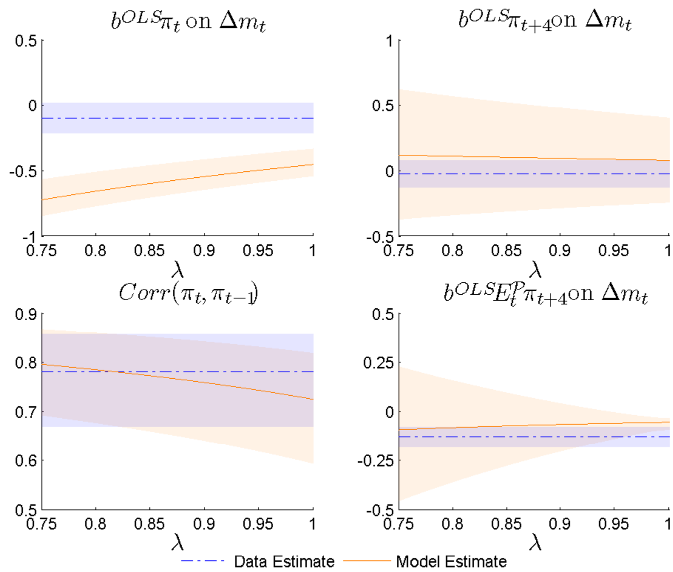

FIGURE 1.4. Learning expectations model with $\lambda \in (0,1)$ model moments using 72 periods simulations compared with data moments. Model moments are the mean across 20,000 simulations. Shaded areas illustrates 95% confidence interval.

CB is still able to make inflation sufficiently stable as λ approaches 1. At the same time, money growth variates and this makes money growth to have low relation with inflation, as well as with one-year-ahead inflation and reproducing Fact 1. Moreover, inflation autocorrelation keeps close to the data inflation autocorrelation as in the rational expectations version of the model[18]. However, differently from the rational expectations model, in the lower-right panel one can see this learning expectations model replicates the negative coefficient of the regression of inflation expectations on money growth, replicating Fact 2. Figure 1.5 takes a closer look to IEMP with the estimated moments of the model in the upper-right and lower-right panels of figure 1.4. One can see point estimates of OLS regression coefficients replicates the IEMP.

In summary, this learning expectations model reproduces Fact 1, and Fact 3 in the same way the rational expectations model does. However, it is able to replicate fairly well Fact 2 where the rational expectations model has no option. Hence, the model is able to explain the IEMP.

[18]Notice that when λ approaches 1, inflation becomes $\pi_t = \bar{\pi} - \varepsilon_t^{\omega}$ and so annual inflation is the sum of four last iid shocks. Hence, first order autocorrelation of annual inflation approaches 0.75.

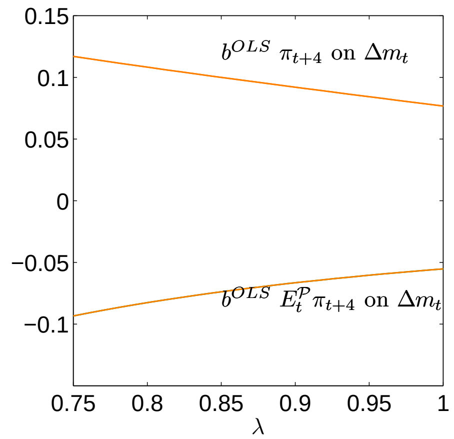

FIGURE 1.5. Learning expectations model with $\lambda \in (0,1)$ difference between the OLS regression of one-year-ahead inflation on money growth and the OLS regression coefficient of expected inflation on money growth. Right panel draws the estimated p-value of the difference between OLS coefficients using Adam, Beutel, and Marcet (2015) and Shaded areas the p-value confidence interval.

1.4 Conclusions

This chapter documents a significant negative relation between money and inflation expectations for the 1990-2007 period in the US which is new to the literature. However, it is a consensus in the literature that empirically there is no relation between money growth and current inflation at high frequency data. I define the Inflation Expectations and Money Puzzle (IEMP) to the difficulty for a rational expectations model to match a negative relation between money growth and inflation expectations as well as a weak relation between money growth and inflation as found in the data. That is because under rational expectations covariance between inflation expectations and money growth must be equal to the covariance between one-period-ahead inflation and money growth. I use Adam, Beutel, and Marcet (2015) test to provide formal evidence in the difference between the slope coefficient of the regression of inflation expectations on money growth and the slope coefficient of the regression one-year-ahead inflation on money growth.

I present a Money-in-the-Utility (MIU) function model with a money demand shock. In this setting, there exists a central bank who minimizes an *ad-hoc* welfare loss function by using money endogenously to keep inflation rate close to a target and smooth. Hence, money growth endogenously adjusts to offset potential effects on inflation so that inflation remains stable whereas money growth variates. At the same time, I assume the representative agent in the economy updates expectations about inflation using past data. In this way, a negative shock to inflation might reduce inflation expectations because the agent does not incorporate the future money growth adjustment. Hence, inflation expectations increase whereas money growth adjusts negatively to offset the increase in inflation expectations. This explains the negative relation between inflation expectations and money growth and disentangles IEMP.

Because this model is developed in a standard simple setting one can attach these features in a large-scale model. Further research would be devoted to implement these features in models with endogenous production in order to be able to research for their importance in those settings.

The theory proposed in this chapter is particularly relevant to account for empirical facts regarding money, inflation, and inflation expectations. I have discussed how effectively monetary authority uses information on inflation expectations from near-rational agents to implement a stable inflation rate. It is of interest to include such deviations for further research in monetary economics literature.

References

Adam, K., J. Beutel, and A. Marcet (2015). Stock price booms and expected capital gains. Working Paper.

Alvarez, F., A. Atkeson, and C. Edmond (2009). Sluggish responses of prices and inflation to monetary shocks in an inventory model of money demand. *The Quarterly Journal of Economics 124*(3), 911–967.

Bernanke, B. S. (2007, July). Inflation expectations and inflation forecasting. Remarks by Chairman Ben S. Bernanke at the Monetary Economics Workshop of the National Bureau of Economic Research Summer Institute, Cambridge, Massachusetts.

Cogley, T. and T. J. Sargent (2005). Drifts and volatilities: monetary policies and outcomes in the post wwii us. *Review of Economic dynamics 8*(2), 262–302.

Díaz-Giménez, J. and R. Kirkby (2014). How to model money? racing monetary frameworks against the quantity theory of money. Working Paper.

Evans, G. W. and S. Honkapohja (2001). *Learning and expectations in macroeconomics*. Princeton University Press.

Gaspar, V., F. Smets, and D. Vestin (2006). Adaptive learning, persistence, and optimal monetary policy. *Journal of the European Economic Association 4*(2-3), 376–385.

Haubrich, J. G., G. Pennacchi, and P. Ritchken (2011). Inflation expectations, real rates, and risk premia: evidence from inflation swaps. Working Paper 1107, Federal Reserve Bank of Cleveland.

Lucas, Robert E, J. (1980). Two illustrations of the quantity theory of money. *American Economic Review 70*(5), 1005–14.

Marcet, A. and T. J. Sargent (1989). Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic theory 48*(2), 337–368.

McCandless, G. T. and W. E. Weber (1995). Some monetary facts. *Federal Reserve Bank of Minneapolis Quarterly Review 19*(3), 2–11.

Mele, A., K. Molnár, and S. Santoro (2011). The suboptimality of commitment equilibrium when agents are learning. *Unpublished paper, University of Oxford*.

Molnár, K. and S. Santoro (2014). Optimal monetary policy when agents are learning. *European Economic Review 66*, 39–62.

Navarro, G. and J. P. Nicolini (2012). Endogenous money supply and the quantity theory of money. *Mimeo*.

Nicolini, J. P. (2014). Monetary policy and the quantity theory of money. mimeo.

Sargent, T. J. and P. Surico (2011). Two illustrations of the quantity theory of money: Breakdowns and revivals. *American Economic Review 101*(1), 109–28.

Stock, J. H. and M. W. Watson (2003). Has the business cycle changed and why? In *NBER Macroeconomics Annual 2002, Volume 17*, pp. 159–230. MIT press.

Chapter 2

Long-Run Behavior from an Endogenous Monetary Policy Perspective

This chapter is devoted to investigate the Quantity Theory of Money and the Fisher equation at low-frequency data. In the long-run, the Quantity Theory of Money predicts a given change in the rate of the average money growth must induce an equal change on average inflation rate as long as velocity is constant. If this is true, average changes in money growth must induce equal average changes in nominal interest rates. Lucas (1980), for example, confirmed these two relations held for the 1955-1975 period. However, recently Sargent and Surico (2011) find evidence of no relation between inflation and money growth if one uses data after the Volcker disinflation, that is, after mid-80's.

Along this study I have provided a model framework in which endogenous monetary policy is able to closely match the failure of the Quantity Theory of Money at high-frequency data. The objective of this chapter is to evaluate whether the models developed in the previous chapter are able to match the Quantity Theory of money at low-frequency data.

To that aim, I consider the low-frequency relation between annual CPI inflation and M2 annual growth and between 3-month Treasury Bills and M2 annual growth in the US for the 1990-2007 period and the 1960-2007 period. In the first part, I filter out trends using Lucas (1980) procedure and the low-frequency relation between inflation and money growth in both periods. In the second part, I simulate rational expectations model in section 1.3.2 and the learning expectations model in section 1.3.4 using the derived empirical trend on inflation as the inflation target used by the CB. Then, I compute the trends of inflation and money growth in the models using Lucas (1980) and evaluate whether these models are able to match low-frequency relation found in the data for the 1990-2007 and 1960-2007 periods. I confirm both models are able to reproduce a one-to-one relation between money growth, inflation, and nominal interest rate for 1960-2007 but are not able to match the low relation showed in the data for the 1990-2007 period. I conclude, there must be anything the models do not capture and that might reproduce the long-run relations for the last period.

This chapter closely relates to Lucas (1980) He computes the trend of inflation and of money growth using a two sided moving average filter and used a scatter plot inflation trend against money growth trend[1] and it showed a clear one-to-one pattern. Recently, Sargent and Surico (2011) use another methodology to measure and argue there does not exist a one-to-one relation after the 1980's. They assumed a VAR structure and computed spectral densities evaluated at zero frequency. They claim that when one uses data after the Volcker Disinflation the one-to-one relation breaks. One critique to their procedure is that they estimate trend relation using too short sample interval. Finally, Díaz-Giménez and Kirkby (2014) use the Lucas (1980) filter in the sample 1960-2009 and confirm a unitary relation between nominal output growth and money growth[2].

The rest of the chapter is organized as follows. Section 2.1 includes a description of the data and the methods used to measure long-run relations together with the empirical estimates of the long-run relations. Section 2.2 presents the estimates of the long-run relations in the models using the methods explained and the comparison with the ones from the data. Section 2.3 concludes.

2.1 Data and Evidence

In this section, I present the data and the empirical evidence on low-frequency relations from two estimation methodologies. The objective of this chapter is to estimate the long-run relation of money growth with inflation and nominal

[1] As Lucas (1980) claimed, this methodology is theory-free as it does not assume any model structure for the series.

[2] Díaz-Giménez and Kirkby (2014) method by exactly following the textbook definition of quantity theory of money. That is, I estimate a one-to-one relation between inflation plus output growth on money growth which is meaningful if one uses data after Volcker Disinflation. Results are robust using nominal output growth instead of inflation.

interest rates. The first method is developed in Lucas (1980) and consists in using a low-frequency filter on the variables and then, estimate their relation by OLS in a simple univariate regression of their filtered values. Originally, Lucas (1980) just provides two-dimensional scatter plots of the filtered variables and eyeballs a unitary relation of money growth with inflation and nominal interest rates. Herein, I present Lucas scatter plots and a its fitted line. The second method is based on Sargent and Surico (2011) and tries to directly estimate the long-run relation between the variables using the estimated spectral densities from a simple VAR model including money, inflation and inflation expectations.

2.1.1 Data

I use quarterly US data from the Federal Reserve of St. Louis database (FRED) at http://research.stlouisfed.org/fred2. I use M2 annual growth rate, which is available from Q1:1960, as money growth. Inflation rate is coming from CPI annual growth rate available from Q1:1949. Finally, 3-month Treasury Bill rates are taken as a nominal short-term interest rate from Q1:1960. Because of the availability of M2 growth the starting point of the sample is Q1:1960 for all the variables[3]. I avoid any disturbances from the financial crisis by fixing the ending of the sample in Q4:2007. I set the beginning of the sample in Q1:1960 which is close to the one in Lucas (1980). Because the end of the chapter is to understand long-run behavior of the variables I let the end of the sample to be fixed in Q4:2007 to short-term disturbances coming from the financial crisis. Finally, I set a break in the sample in Q1:1990 to look for differences in the relations for the first part of the sample and the sample after Volcker disinflation.

Figure 2.1 presents the plot of the series over time. One can see that, after 1990, inflation and nominal interest rate become more stable around a mean. This is the reason why one can wondered if the Quantity Theory of Money and the Fisher equation do not hold anymore after that time.

[3]When using Lucas (1980) filter I extend the data sample on money growth, inflation and nominal interest rate by using data from Balke and Gordon (1986) on M2 growth and 6-month Commercial Paper Rate for the period 1900-1960, and on GDP price deflator for the 1900-1948 period. Results are robust when I do not use this data. After that I use the filtered data for the sample 1960-2007.

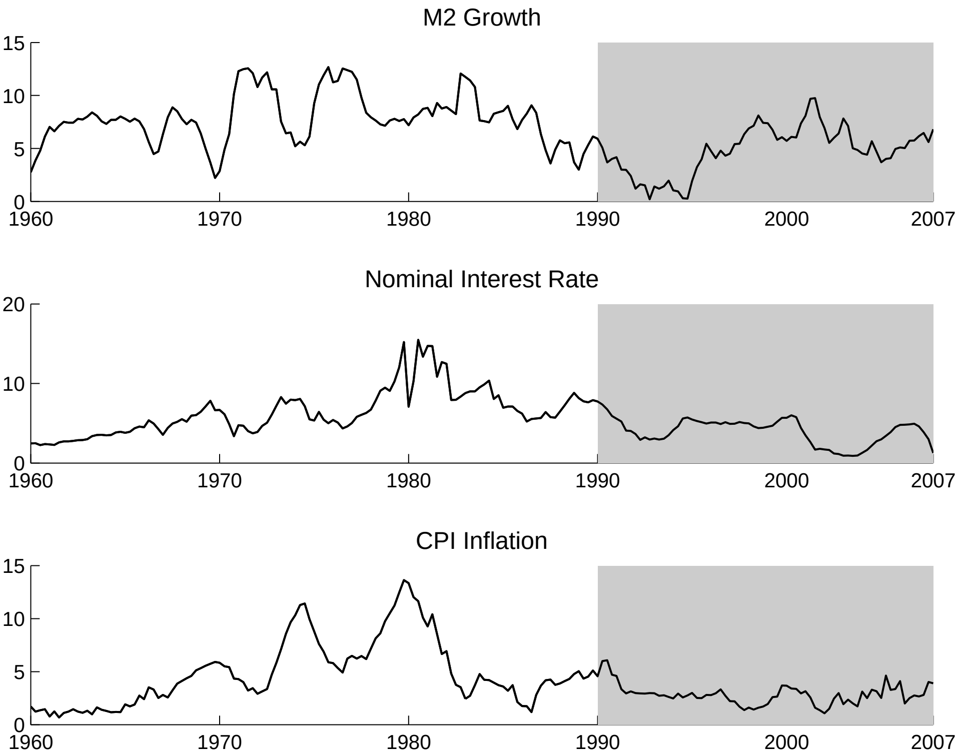

FIGURE 2.1. Quarterly Data from 1960 to 2007 of M2 growth rate, 3-Month Treasury Bill rate and CPI inflation. Shaded regions represent the sample interval 1990-2007.

2.1.2 Lucas Low-Frequency Filter

In this section I present the Lucas (1980) methodology. The stationary solution of many macroeconomic models exhibit a one-to-one relation between money growth rate and the inflation rate, as well as a one-to-one relation between money growth and nominal interest rate[4]. He tests empirically the two relations. First, he uses a low-pass filter in the form of a two-sided infinite moving average with exponentially decreasing weights in the form

$$x_t^i(\beta) = \alpha \sum_{j=-K}^{K} \beta^{|j|} x_{t-j}^i \quad \text{for} \quad i = 0, 1, 2.$$
$$\text{with} \quad \alpha = \frac{1-\beta}{1+\beta-2\beta^K}, \text{ and} \quad \beta \in [0, 1) \tag{2.1}$$

where β is the weight, K defines the length of the bands, x_t^0 is money growth, x_t^1 is inflation rate, and x_t^2 is the nominal interest rate. He claims that, if each one

[4]For example, the most basic money-in-the-utility-function model display these features.

of the variables is defined as the sum of a persistent component and a transitory component, this filter is a theory-free estimator of the persistent component given all the lags and leads of the proper variable. Moreover, he also notes that this filter needs β to be sufficiently close to the unity to be a good estimator of the persistent component when the transitory component is relatively more volatile than the persistent component.

Finally, Lucas makes inference about the slopes of b_1 and b_2 of the regressions

$$x_t^i(\beta) = b_i x_t^0(\beta) + \eta_t^i \quad \text{with} \quad E\left[x_t^0(\beta)\eta_t^i\right] = 0 \quad \text{for} \quad i = 1,2. \tag{2.2}$$

using simple eyeball analysis of the scatter plots of inflation on money growth and nominal interest rate on money growth and concludes a unitary slope in the two regressions held for the US data for the 1955-1975 period. I replicate this method for the 1990-2007 and the 1960-2007 period separately.

Figure 2.2 presents the scatter plot of the filtered series when $\beta = 0.95$ together with their respective fitted line. Left-panel shows the scatter plot when one uses data for the 1990-2007 period whereas right-panel shows the scatter plot when one uses data for 1960-2007 period. First row looks to the relation between inflation and money growth and second row looks to the relation between interest rate and money growth. It can be seen both relations are unitary in the 1960-2007 as points an almost perfect 45^o degree line in the 1960-2007 period. This is in line with the findings in Lucas (1980) for the 1955-1975 period as well as the Díaz-Giménez and Kirkby (2014) finding for the 1960-2009 for the unitary relation between inflation and money growth. However, when using the 1990-2007 period data one-to-one relations break down what is in line with the evidence found by Sargent and Surico (2011).

2.1.3 Evidence from a Time-Invariant VAR

In this section, I present the Sargent and Surico (2011) methodology. Suppose one runs a least squares regression of variable x_t^i for $i = 1,2$, on all lags and leads of money growth Δm_t

$$x_t^i = \sum_{j=-\infty}^{\infty} h_j^i \Delta m_{t-j} + \varepsilon_t \tag{2.3}$$

where $E\left[\Delta m_{t-j}\varepsilon_t\right] = 0$ for all j. Whiteman (1984) highlighted that making the sum of all the coefficients in this two sided least squares regression, $\sum_{j=-\infty}^{\infty} h_j^i$

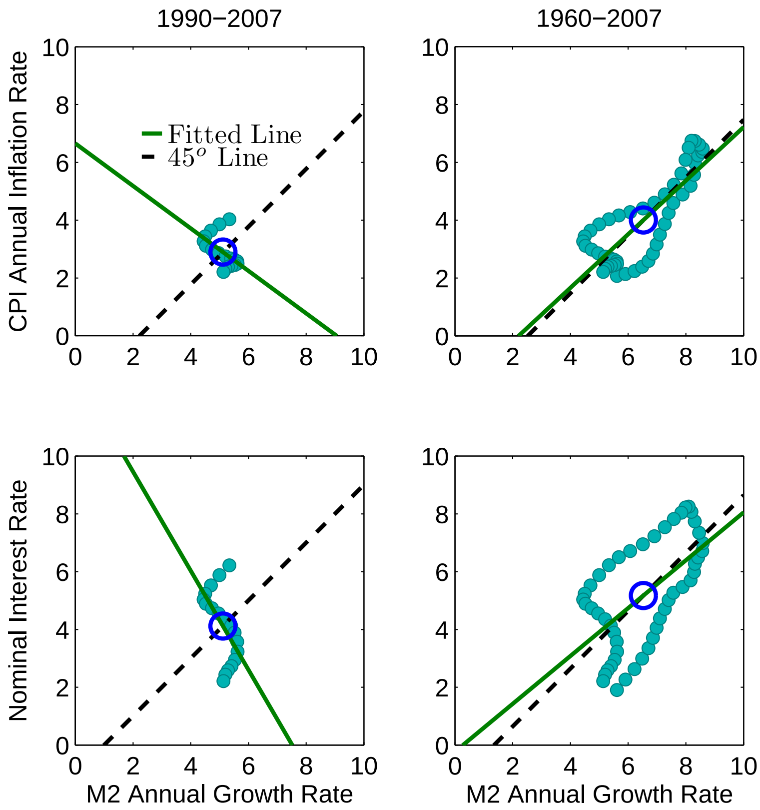

FIGURE 2.2. Scatter plot of trends of annual CPI annual inflation plus real GDP growth on Money growth. First quarters. $\beta = 0.95$. White circle represents grand mean of the series.

is approximates the Lucas' long-run slope regression whenever β is sufficiently close to 1.

Given that the sum of all lag coefficients is the Fourier transform of $\left\{h_j^i\right\}$ evaluated at zero frequency, Sargent and Surico (2011) used the following formulation to estimate its sum

$$\tilde{h}_i(0) = \frac{S_{x^i,\Delta m}(0)}{S_{\Delta m}(0)} \tag{2.4}$$

where $S_{x^i,\Delta m}(0)$ for $i = 1,2$ is the cross-spectral density of inflation and nominal interest rate with money growth, respectively, evaluated at zero frequency, and $S_{\Delta m}(0)$ is the spectral density of Δm_t evaluated at zero frequency.

Using this approach Sargent and Surico (2011) directly test for the one-to-one long-run relation between inflation and money growth. They use a VAR to estimate spectral densities and compute $\tilde{h}(0) = (h_1(0),\ h_2(0))$. Sargent and Surico (2011) make a Bayesian estimation of a time-variant VAR with stochastic volatility and estimate time-varying long-run relations. Differently, I make maximum likelihood estimation of a time-invariant version that can be seen as a constrained version of their model. Their approach is justified because their large sample includes different monetary policy regimes and data instability. The sample used here only accounts for two monetary regimes and I allow to estimate them separately.

The standard n-variate VAR(p) model that is used as the empirical model is then specified by

$$y_t = c + \sum_{j=1}^{p} B_j y_{t-j} + \varepsilon_t \equiv \Pi' X_t + \varepsilon_t \quad \forall \quad t = 1,2,\ldots T \tag{2.5}$$

where $y_t \equiv [y_{1,t},\ldots,y_{n,t}]'$ is a $n\times 1$ vector, $X_t \equiv [1, y_{t-1},\ldots,y_{t-p}]'$ is a $(np+1)\times 1$ vector, $\Pi' \equiv [c, B_1,\ldots,B_p]$ is the $n\times(np+1)$ matrix of lag coefficients that defines a stable process, and $\varepsilon_t \equiv [\varepsilon_{1,t},\ldots,\varepsilon_{n,t}]'$ is a $n\times 1$ *iid* Gaussian process with zero mean and variance S.

As in Sargent and Surico (2011) I take advantage of the state-space representation of this model to estimate the time-invariant spectral densities. The state-space representation in this environment is given by

$$\begin{aligned} Z_t &= AZ_{t-1} + BW_t \\ Y_t &= CZ_t + DW_t \end{aligned} \tag{2.6}$$

where X_t is the $np\times 1$ state vector, Y_t is the $n\times 1$ vector of observables, and W_t an $np\times 1$ vector that contains *iid* standard Gaussian innovations. The matrix of coefficients, A, B, C, and D are conformable matrices with A being the companion form matrix of lag coefficients defining a stable VAR, and B containing the Cholesky decomposition of S. Using this representation Cogley et al. (2011) find easily the spectral density of Y at zero frequency that is reduced to

$$S_Y(0) = C(I-A)^{-1}BB'(I-A')^{-1}C' \tag{2.7}$$

where it can be found the spectral densities to compute the sum of lags coefficients $\tilde{h}(0)_T(0)$ using equation 2.4. In the spirit of the Lucas' slope coefficients, they interpret that when the sum of lag coefficients equals unity they recover Lucas' results. They interestingly find that after the period analyzed by Lucas, it weakened and finally disappeared in the period comprised from the mid of the

90's decade to 2007. They interpret this is closely related to an important decline in the persistence of inflation that they link with the change in the monetary policy for the most recent period.

I use this methodology and estimate a VAR(1) with CPI annual inflation, M2 annual growth, real GDP annual growth and 3-month Treasury Bill rates and compute $\tilde{h}(0)$. For the 1990-2007 period, long-run relation coefficient of inflation and money growth is -0.29 with 84% confidence interval in $(-0.16, -0.57)$ constructed using bootstrapping. For the 1960-2007 period, long-run relation coefficient of inflation and money growth is 0.89 with 84% confidence interval in $(0.07, 1.70)$ constructed using bootstrapping. This evidence is consistent with the findings of Sargent and Surico (2011).

2.2 Low-Frequency Relation in the Rational Expectations and Learning Expectations Models

In this section I research if the model endogenous monetary policy models with rational expectations agents in section 1.3.2 and with learning expectations in section 1.3.4 are able to replicate the empirical observations discussed in the previous section. The objective is to compute the implicit long-run coefficients using the two methods in the previous section. Since the variables in the model economies are endowed without a drifting behavior, I feed the inflation target of the model with the inflation data trend observed. I make two exercises. First, for the 1990-2007 period and simulate the model economy for the 72 quarters in that period. In this simulations, I use the baseline calibration of section 1.3.2 that is the one obtained using the 1990-2007. Then, I make 20,000 simulations and compute average behavior of the long-run coefficients computed as in Lucas (1980) and as in Sargent and Surico (2011).

Second, I use the same procedure for the 1960-2007 sample. In this case, I use the same procedure as in the baseline calibration found in section 1.3.2 to calibrate the parameters affecting the money demand equation as to match data moments for that period. In particular, I calibrate η, and ρ_ω are statistically zero and σ_ω equal to 0.0025. Differently from the 1990-2007 period, in this period, there was no response of 3-month Treasury Bill to real money balances defined by M2 and CPI, and the first difference of the regression residual have no

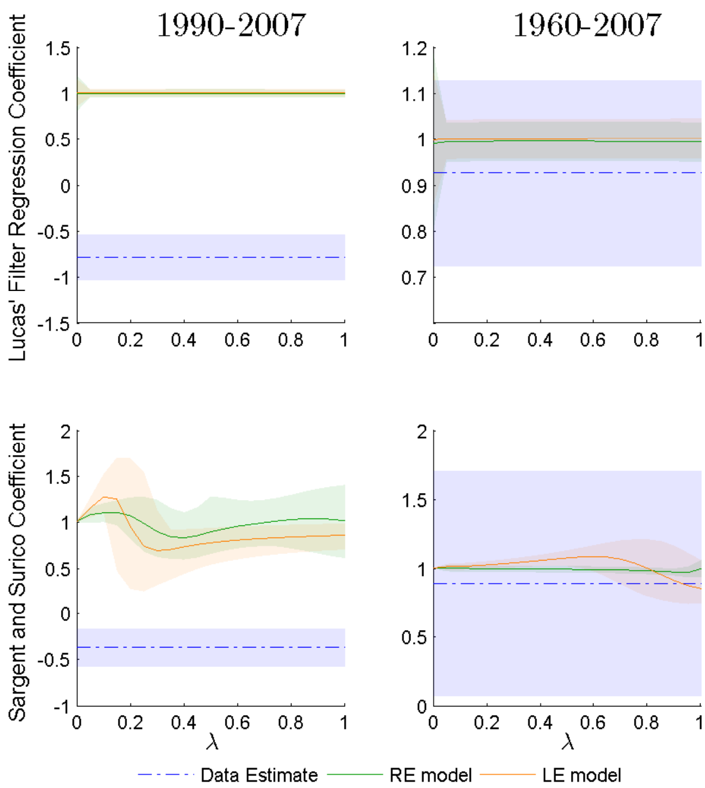

FIGURE 2.3. Low-Frequency slope coefficient of the regression of inflation on money growth. Shaded regions are the 95% confidence intervals for Lucas' method and 84% confidence interval after 10,000 iterations bootstrapping for Sargent and Surico method.

persistence. Hence, I make 20,000 simulations of 192 period length and compute the average across simulations of the long-run coefficients computed.

Figure 2.3 provides results of the first and second exercises in the left and right panels, respectively, paying attention to the long-run relation between inflation and money growth. First row shows the results when Lucas' method is used and second row includes results for Sargent and Surico's method. Several conclusions can be taken from both exercises. First, rational expectations and learning expectations models generate basically the same low-frequency relation between inflation and money growth for any of the two periods. Second, for the 1990-2007 period, low-frequency relation of the models is far from that in the data. Finally, both models generate a one-to-one relation once one uses the long sample from 1960-2007.

Figure 2.4 provides results paying attention to the long-run relation between nominal interest rate and money growth. Again, the models are able to generate

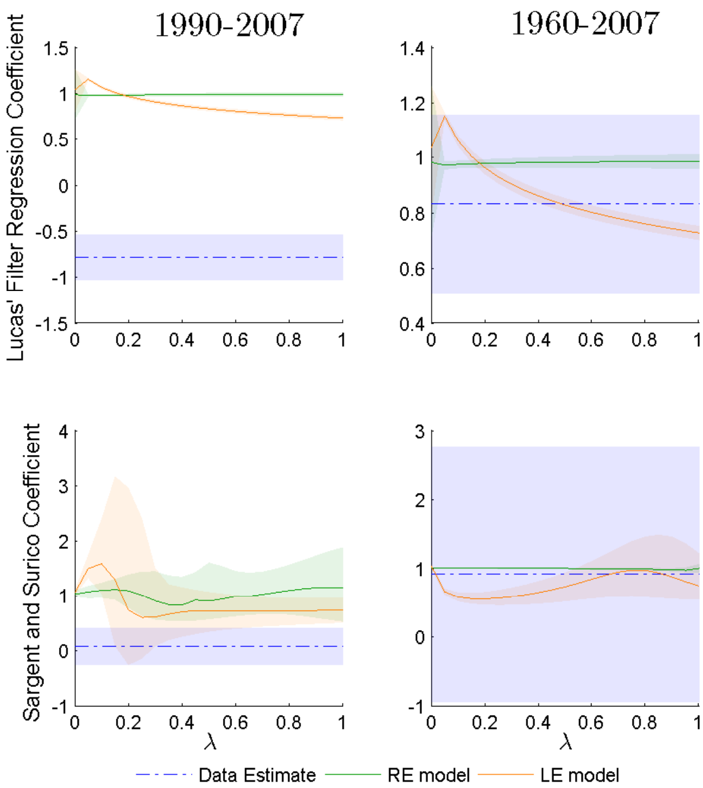

FIGURE 2.4. Low-Frequency slope coefficient of the regression of inflation on money growth. Shaded regions are the 95% confidence intervals for Lucas' method and 84% confidence interval after 10,000 iterations bootstrapping for Sargent and Surico method.

unitary coefficient when a long-run coefficient as in the data but fails when trying to match the short sample coefficient.

In summary, I conclude although the models presented in the previous chapter were successful to match high-frequency comovements there is some reason not included in the models that induces a low relation between trends of inflation and nominal interest rate with money growth trends during the 1990-2007 period.

2.3 Conclusion

This chapter evaluates if the endogenous monetary policy models introduced in the previous chapter are able to generate the empirical long-run relations between inflation, money growth, and nominal interest rates. To that aim, I use the methods in Lucas (1980) and Sargent and Surico (2011).

When estimating the empirical relation I use two different samples; the 1960-2007 period, and the 1990-2007 period. Whereas for the longer sample there exist a one-to-one relation between the trends of money growth, inflation, and nominal interest rate, in the shorter sample it does not. This is in line with the findings of Sargent and Surico (2011).

Then, I estimate the implicit long-run relation of the rational expectations and adaptive learning expectations model economies of the previous chapter when the endogenous monetary policy target inflation equals the one estimated from the data. I show that these models are not able to generate the low-frequency relations when calibrated for the 1990-2007 period.

I conclude although the models presented in the previous chapter were successful to match high-frequency comovements there is some reason not included in the models that induces a low relation between trends of inflation and nominal interest rate with money growth trends during the 1990-2007 period.

References

Ang, A., G. Bekaert, and M. Wei (2008). The term structure of real rates and expected inflation. *The Journal of Finance 63*(2), 797–849.

Balke, N. and R. J. Gordon (1986). Appendix b: historical data. In *The American business cycle: Continuity and change*, pp. 781–850. University of Chicago Press.

Calvo, G. A. (1983). Staggered prices in a utility-maximizing framework. *Journal of monetary Economics 12*(3), 383–398.

Campbell, J. Y., R. J. Shiller, and L. M. Viceira (2009). Understanding inflation-indexed bond markets. *Brookings Papers on Economic Activity 40*(1 (Spring), 79–138.

Chernov, M. and P. Mueller (2012). The term structure of inflation expectations. *Journal of financial economics 106*(2), 367–394.

Cogley, T., T. J. Sargent, and P. Surico (2011). The return of the gibson paradox. mimeo.

Cooray, A. (2002, September). The fisher effect: A review of the literature. Research Papers 0206, Macquarie University, Department of Economics.

D'Amico, S., D. H. Kim, and M. Wei (2010). Tips from tips: the informational content of treasury inflation-protected security prices. Finance and Economics Discussion Series 2010-19, Board of Governors of the Federal Reserve System (U.S.).

Díaz-Giménez, J. and R. Kirkby (2014). How to model money? racing monetary frameworks against the quantity theory of money. Working Paper.

Evans, C. L., C. T. Liu, and G. Pham-Kanter (2002). The 2001 recession and the chicago fed national index: identifying business cycle turning points. *Economic Perspectives* (Q III), 26–43.

Galí, J. (2008). *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework.* Princeton University Press.

Gürkaynak, R. S., B. Sack, and J. H. Wright (2008). The tips yield curve and inflation compensation. Finance and Economics Discussion Series 2008-05, Board of Governors of the Federal Reserve System (U.S.).

Haubrich, J. G., G. Pennacchi, and P. Ritchken (2011). Inflation expectations, real rates, and risk premia: evidence from inflation swaps. Working Paper 1107, Federal Reserve Bank of Cleveland.

Lang, D. and K. J. Lansing (2010). Forecasting growth over the next year with a business cycle index. *FRBSF Economic Letter 29*(Sep 27).

Lucas, Robert E, J. (1980). Two illustrations of the quantity theory of money. *American Economic Review 70*(5), 1005–14.

Pflueger, C. E. and L. M. Viceira (2011, March). An empirical decomposition of risk and liquidity in nominal and inflation-indexed government bonds. NBER Working Papers 16892, National Bureau of Economic Research, Inc.

Sargent, T. J. and P. Surico (2011). Two illustrations of the quantity theory of money: Breakdowns and revivals. *American Economic Review 101*(1), 109–28.

Stock, J. H. and M. W. Watson (1999, October). Forecasting inflation. *Journal of Monetary Economics 44*(2), 293–335.

Whiteman, C. H. (1984, September). Lucas on the quantity theory: Hypothesis testing without theory. *American Economic Review 74*(4), 742–49.

Chapter 3

Asset Pricing in an Heterogeneous Expectations Model

Consumption-based rational expectations modeling has been proved to fail when trying to replicate financial facts. Examples of this are the low volatility of stock returns as documented by Campbell (2003), or the high equity premium as showed by Mehra and Prescott (1985). A recent approach in the literature is to assume that agents learn about the processes leading the economy (henceforth, learners or learning agents). In this framework, Adam, Marcet, and Nicolini (2016) (hereafter, AMN) is able to match several asset pricing facts using a very simple version of the Lucas (1980) model completely populated by learning agents. However, their model would generate high stock returns persistence in contrast with empirical evidence, where a rational expectations model would generate low stock returns persistence.

In this chapter, I investigate whether an heterogeneous agents model, populated with learning agents and rational expectations agents, is able to generate facts in AMN together with a low stock returns persistence. I show that this model is able to generate low stock returns persistence but it looses capacity in reproducing some of the AMN facts. In particular, it does not reproduce empirical evidence on long-run excess return predictability, and price-dividend and stock returns volatility when it is able to generate low stock returns. The introduction of stock returns persistence originates a trade-off between itself and the volatility of stock returns and price dividends in the model where the proportion of learners plays a key role[1].

[1]Except the equity premium that is beyond the aims of this study.

I present a model populated by two types of agents; rational expectations agents and learning expectations agents. I try to model more closely the existing heterogeneity in stock markets. Empirically, there are different agents operating in financial markets. On the one extreme, there are small investors that could be model as learning agents that slowly recognize the actual law of motion of stock prices. Barber and Odean (2000) indicates that in 1996 about 47 percent of the total equity investments in the United States was explained by households. On the other extreme, hedge funds change their trading strategies rapidly given a superior set of information (Lasarte and Connor (2004)). They could be model as rational agents that recognize immediately the actual law of motion and act in consequence.

I compare its results in matching asset pricing facts with those of AMN. In order to do that, I compute the statistics in the same way and I make the estimation following the same procedure, the method of simulated moments (MSM). In addition to the facts in AMN (volatility of stock returns, volatility of price-dividend ratio, long-run excess returns predictability, equity premium, etc.) I try to match low predictability of excess returns in the short-run. There is a large literature accounting for the low predictability of excess returns in the short run. For instance, Shiller (1984) shows that just a small variability of the stock return can be explained in a VAR framework. Later, Campbell (1990) finds that stock returns are difficult to forecast and tries to understand the causes splitting unexpected returns into changes in expectations about future dividends and changes in expectations about future returns. I investigate the persistence of the stock return as a simple approximation. As can be seen in figure 3.1 the U.S. quarterly stock return evolution does not exhibit any persistence.

This chapter relates to learning expectations literature. One part of this literature define the equilibrium with one-step-ahead Euler equation, like Evans and Honkapohja (2003). For instance, the recent study of Benhabib and Dave (2011) use this method and conclude that the asymptotic distribution of the price dividends (PD) ratio follows a power law distribution under constant gain stochastic gradient algorithm. In a monetary economics environment, Preston (2005) demonstrates that agents decisions depend on forecasts of macroeconomic variables several periods in the future. This chapter is based on a very novel scheme is theoretically drawn by Adam and Marcet (2011). There, learning agents' decision problem is already engaged with the knowledge of the subjective belief and thereafter, they behave optimally. This framework is the one used by AMN.

This chapter also relates to heterogeneous agents in the literature. An example is Constantinides and Duffie (1996), where heterogeneity comes in the form of labor income shocks and obtain a set of Euler equations characterizing the equilibrium. In the heterogeneous expectations formation field, Honkapohja and Mitra (2006) apply heterogeneity in the sense that agents use different learning rules and conclude that different forms of heterogeneity change the way the adaptive learning expectations converge to the rational expectations equilibrium. Nunes (2009) includes heterogeneity in expectations to the New Keynesian model to replace the inclusion of lags of the data in the Phillips curve of the previous literature with a learning component. I use the way Nunes (2009) adds heterogeneity to the model, i.e. there is a representative agent whose expectations is a weighted average of the existing proportion of learners and rational expectations agents.

This chapter is organized as follows. Section 3.1 presents the facts. The model is shown in section 3.2. Section 3.3 presents the MSM estimation outcomes. Section 3.4 concludes.

3.1 The Facts: Stock Return Persistence

This section presents empirical facts in the US using a sample from Q2:1927 to Q4:2005. The purpose is to match all the facts in AMN. In addition, I include excess return persistence as an approximation to short-run excess return predictability. Figure 3.1 presents the evolution of stock returns. It can be seen there is no persistence in stock returns for the period studied. I include the first order quarterly autocorrelation of the stock returns, denoted by $\rho_{rs,-1}$, as the new fact and I report it in Table 3.1 to see if the model helps to understand any intuition behind this fact[2] and its relation with the rest of facts.

The rest of empirical facts highlighted are the following. The second row of Table 3.1 makes reference to the high volatility of the price-dividend ratio (PD) and it includes its mean, E_{PD}, and its standard deviation, σ_{PD}. Third row reports the high persistence of the PD ratio measured as the first order

[2] This is a simple approach. Another possibility is to include any VAR specification for short run excess return together with the specified for the long run and take the key statistics for estimation.

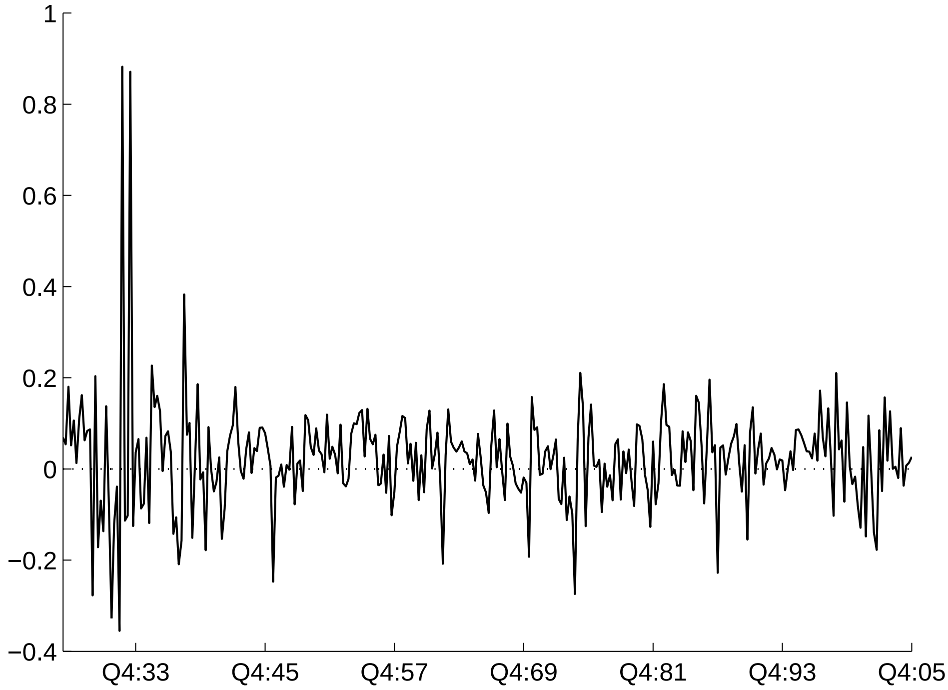

FIGURE 3.1. Quarterly Stock Returns in percentage terms from 1927:2 to 2005:4

autocorrelation, $\rho_{PD,-1}$. Fourth row presents stock returns volatility, σ_{r^s} which is much higher than the dividend growth volatility[3]. The dividend growth process is represented in the last row by the mean of the dividend growth $E_{\frac{\Delta D}{D}}$ and the standard deviation of the dividend growth $\sigma_{\frac{\Delta D}{D}}$. Long run excess stock returns predictability is showed in the form of an OLS regression of the excess returns in 5-year horizon on PD ratio. The results of this regression are summarized by the slope coefficient, c_2^5, and the R-square, R_5^2, displayed in the fifth row. Finally, the equity premium is represented in the sixth row by the sample mean of the stock returns, E_{r^s}, and the bond returns, E_{r^b}.

3.2 The Model

In this section I specify the model. I define a proportion of learners in the economy, $\mu \in (0,1)$, and of rational agents $(1-\mu)$. If $\mu = 1$ the model is exactly as the one of AMN where all agents are endowed with learning expectations. Therefore, deviations from that work comes from deviations of μ from 1, that is,

[3]This is related to the PD ratio volatility given that $1+r_t^s = \left(\frac{PD_t+1}{PD_{t-1}}\right)\frac{D_t}{D_{t-1}}$.

NEW FACT. STOCK RETURNS PERSISTENCE	$\rho_{rs,-1}$	-0.09
VOLATILITY OF PD RATIO	E_{PD}	113.20
	σ_{PD}	52.98
PERSISTENCE OF THE PD RATIO	$\rho_{PD,-1}$	0.92
STOCK RETURNS VOLATILITY	σ_{r^s}	11.65
EXCESS RETURNS	c_2^5	-0.0032
PREDICTABILITY	R_5^2	0.1968
EQUITY PREMIUM	E_{r^s}	2.41
	E_{r^b}	0.18
DIVIDEND GROWTH VOLATILITY	$\sigma_{\frac{\Delta D}{D}}$	2.98
	$E_{\frac{\Delta D}{D}}$	0.35

TABLE 3.1. Asset pricing facts, 1927:2-2005:4.
Growth rates and returns in percentage terms.

the introduction of a fraction of rational expectations agents and the extensions needed to accommodate it in that framework. Learning expectations agents know the true dividend process and the exogenous production process, but I kept them using a subjective probability measure $\mathscr{P}$ that explicitly suggests they ignore the actual stock price process. I assume that the rational agents are endowed with full knowledge of the economy, except that they are not aware that learners will change their expectation over time. In this sense, rational agents in this model economy are not fully rational as understood in the literature.

3.2.1 The Economy and the Competitive Equilibrium

In this section I set up the economy and define the competitive equilibrium. There is a mass of learning agents μ and a mass of rational agents $(1-\mu)$. They are infinitely-lived and there is a total endowment in the economy of one unit of stock attached with a dividend D_t that they trade in a competitive environment.

A representative agent i is assumed to solve the following optimization problem,

$$\max_{\{C_t^i, S_t^i, B_t^i\}_{t=0}^{\infty}} \tilde{E}_0 \sum_{t=0}^{\infty} \delta^t U(C_t^i) \tag{3.1}$$

where $U(.)$ denotes the utility function. The operator $\tilde{E}_t(.)$ represents the conditional expectation of the representative agent.

To solve the problem, agent i faces the following constraints,

$$\begin{gathered} C_t^i + B_t^i + P_t S_t^i \leq (P_t + D_t) S_{t-1}^i + (1 + r_{t-1}) B_t^i + Y_t, \ \text{ for } \ t \geq 0 \\ C_t^i \geq 0 \ \text{ for } \ t \geq 0 \\ B_{-1}^i \ \text{ and } \ S_{-1}^i \ \text{ given.} \end{gathered} \tag{3.2}$$

Notice that the endowment of Y_t units of consumption allows any correlation between dividends and exogenous consumption, so that weak correlation between dividends and consumption growth is feasible.

Hence, the definition of a competitive equilibrium for this economy is

Definition 1. A competitive equilibrium of this economy is a set of prices $\{P_t, r_{t-1}\}_{t=0}^{\infty}$ and a set of allocations $\left\{C_t^i, S_t^i, B_t^i\right\}_{t=0}^{\infty}$ such that:

1. Given prices $\{P_t, r_{t-1}\}_{t=0}^{\infty}$, consumers maximize (3.1) subject to (3.2) for all i.

2. Markets clear,

$$\begin{aligned} C_t^i &= C_t \quad \text{for} \quad t \geq 0 \\ S_t^i &= 1 \quad \text{for} \quad t \geq 0 \\ B_t^i &= 0 \quad \text{for} \quad t \geq 0. \end{aligned}$$

Consistent with the literature, I assume a constant relative risk aversion utility,

$$U(C_t^i) = \frac{(C_t^i)^{1-\gamma}}{1-\gamma}, \qquad \gamma \in (0, \infty) \tag{3.3}$$

where γ denotes the relative risk aversion parameter.

Dividends process and the exogenous consumption growth rate process laws of motion are assumed to be

$$\begin{aligned} \frac{D_t}{D_{t-1}} &= a\varepsilon_t^d, \quad \text{with} \quad a \geq 1 \quad \text{and} \quad log\varepsilon_t^d \sim ii\mathcal{N}(-\frac{s_d^2}{2}, s_d^2) \\ \frac{C_t}{C_{t-1}} &= a\varepsilon_t^c, \quad \text{with} \quad a \geq 1 \quad \text{and} \quad log\varepsilon_t^c \sim ii\mathcal{N}(-\frac{s_c^2}{2}, s_c^2), \end{aligned} \tag{3.4}$$

and the assumptions needed to ensure the no-Ponzi game as well as existence of a unique maximum for the consumer problem are satisfied by assuming the following conditions

$$\underline{B} < B_t^i < \overline{B}, \;\; \underline{S} < S_t^i < \overline{S} \tag{3.5}$$

where $\underline{B} < 0 < \overline{B}$ and $\underline{S} < 1 < \overline{S}$.

Finally, the initial endowments are assumed to be

$$B_{-1}^i = 0 \;\text{ and }\; S_{-1}^i = 1. \tag{3.6}$$

3.2.2 Characterization of the Equilibrium

This section characterizes the equilibrium given the setting developed in section 3.2.1. For that aim, I introduce how operator $\tilde{E}_t(.)$ is computed, I make explicit the subjective probability measure that learners are going to use, and the induced optimal updating rule for the expectation of learning agents.

Given the above discussion, the optimality conditions for the consumer problem are

$$1 = \delta(1+r_t)\tilde{E}_t\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}$$
$$P_t = \delta\tilde{E}_t\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}(P_{t+1}+D_{t+1})$$

Following AMN an easy way to have computability of the equilibrium is to have sufficiently large exogenous production such that the expected consumption derived from the asset trading decision is negligible and, then, the following approximations can be made

$$\begin{aligned} \tilde{E}_t\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma} &\simeq \tilde{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \\ \tilde{E}_t\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}(P_{t+1}+D_{t+1}) &\simeq \tilde{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(P_{t+1}+D_{t+1}). \end{aligned} \tag{3.7}$$

For simplicity, I use the way Nunes (2009) introduces heterogeneity in the expectation by decomposing it as a weighted sum of the expectations of the individuals in this economy in a New Keynesian model[4]. Under this assumption, the equilibrium can be characterized by the following two optimality conditions

$$1 = \delta(1+r_t)\left(\mu E_t^{\mathscr{P}}\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma} + (1-\mu)E_t\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}\right) \tag{3.8}$$

$$P_t = \delta\left(\mu E_t^{\mathscr{P}}\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}(P_{t+1}+D_{t+1}) + (1-\mu)E_t\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}(P_{t+1}+D_{t+1})\right) \tag{3.9}$$

where now operator $E_t^{\mathscr{P}}(.)$ denotes the expectation of learning agents and $E_t(.)$ denotes the expectation of the rational agents.

Assuming learners know the actual dividend process and the consumption process[5], but prevented to know the actual consumption process, and from the above discussion, the conditions (3.8) and (3.9) are finally redefined as

$$1 = \delta(1+r_t)a^{-\gamma}E_t\left(\varepsilon_{t+1}^c\right)^{-\gamma} \tag{3.10}$$

$$P_t = \left(\frac{1}{1-\delta\mu\beta_t}\right)\left(a^{1-\gamma}\delta D_t E_t\left(\varepsilon_{t+1}^c\right)^{-\gamma}\varepsilon_{t+1}^d + a^{-\gamma}(1-\mu)\delta E_t\left(\varepsilon_{t+1}^c\right)^{-\gamma}P_{t+1}\right) \tag{3.11}$$

where $E_t\left(\varepsilon_{t+1}^c\right)^{-\gamma} = e^{\gamma(1+\gamma)\frac{s_c^2}{2}}$ and $\beta_t \equiv E_t^{\mathscr{P}}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+1}}{P_t}\right)$.

I assume rational agents are not aware that learners' expectations will change in time. This assumption is also made in Nunes (2009). This is, computationally, a convenient assumption since it allows to get rid of some random variables that comes from cross-products. Notice that this assumption requires also bounded rationality for rational agents[6]. Therefore (3.11) can be simplified to

$$P_t = \frac{\delta a^{1-\gamma}\rho_\varepsilon}{1-\delta\mu\beta_t-\delta(1-\mu)a^{1-\gamma}\rho_\varepsilon}D_t \tag{3.12}$$

where $\rho_\varepsilon \equiv E_t\left(\varepsilon_{t+1}^c\right)^{-\gamma}\varepsilon_{t+1}^d = e^{\gamma(1+\gamma)\frac{s_c^2}{2}}e^{-\gamma s_c s_d}$. Notice that, when μ is 1, one recovers the model of learning expectations in AMN and also, if μ is 0, one gets the fully rational expectations case[7].

[4]Although this assumption largely simplifies the analysis. Further research requires to understand up to which point might affect results.

[5]This comes from the fact that they know the dividend process and the exogenous production process and that $Y_t = C_t + D_t$.

[6]Nunes (2009) names rational agents in this context as 'near-rationals'.

[7]The derivation of this equation is included in appendix 3.A.

Finally, to close the model it is necessary to characterize the probability measure $\mathscr{P}$ used by learners. I follow again AMN and assume that learners' perception about risk-adjusted stock price growth is given by

$$\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{P_t}{P_{t-1}} = b_t + \varepsilon_t$$
$$\text{with} \qquad b_t = b_{t-1} + \xi_t,$$

where $\varepsilon_t \sim iiN(0, \sigma_\varepsilon^2)$ and $\xi_t \sim iiN(0, \sigma_\xi^2)$. From there, one ends up having a updating rule for learners' expectations of the form

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha}\left(\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{P_t}{P_{t-1}} - \beta_{t-1}\right).$$

with $1/\alpha$ being the Kalman gain. Notice that this updating rule comes from the optimal filtering of the information observed by learning agents. Finally, assuming agents incorporate information about the risk-adjusted price growth with one lag, and a projection facility in the expectation error term then the optimal updating rule[8]

$$\beta_t = w\left(\beta_{t-1} + \frac{1}{\alpha}\left(\left(\frac{C_{t-1}}{C_{t-2}}\right)^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right)\right) \tag{3.13}$$

Hence, equations (3.10), (3.12) and (3.13) characterize the equilibrium in this model economy.

3.3 MSM estimation and Results

In this section I estimate the model parameters following the method of simulated moments (MSM) approach as in AMN and I compare their results with the produced by the model that is analyzed here. Differently AMN, there is a new

[8]These two assumptions are again taken from AMN. The first prevents simultaneity of price and forecast determination. The projection facility, $w(.)$ guarantees that the expectation β_t never exceeds the upper-bound β^U, and it takes the form

$$w(x) = \begin{cases} x & \text{for } x \leq \beta^L \\ \beta^L + \frac{x-\beta^L}{x+\beta^U-2\beta^L}\left(\beta^U - \beta^L\right) & \text{for } x < 0 \end{cases}$$

where $\beta^L \in \left(a^{1-\gamma}\rho_\varepsilon, \beta^U\right)$.

Statistic	US data (Std. Dev.)		Model Statistics: μ flexible. (t-statistic).		Model Statistics: $\mu = 1$ fixed. (t-statistic).	
$\rho_{rs,-1}$	-0.09	(0.01)	-0.05	(-0.55)	0.78	(-12.09)
E_{PD}	113.20	(15.15)	147.83	(-2.29)	104.68	(0.56)
σ_{PD}	52.98	(16.53)	0.82	(3.16)	59.79	(-0.41)
$\rho_{PD,-1}$	0.92	(0.02)	0.92	(-0.16)	0.94	(-1.10)
σ_{r^s}	11.65	(2.88)	1.88	(3.39)	10.15	(0.52)
c_2^5	-0.0037	(0.00)	-0.0015	(-1.09)	-0.0059	(1.11)
R_5^2	0.2647	(0.08)	0.0002	(3.23)	0.3573	(-1.13)
E_{r^s}	2.41	(0.45)	0.79	(3.61)	1.62	(1.76)
E_{r^b}	0.18	(0.23)	0.49	(-1.33)	0.66	(-2.05)
$\sigma_{\Delta D/D}$	2.98	(0.84)	1.87	(1.32)	3.32	(-0.41)
$E_{\Delta D/D}$	0.35	(0.19)	0.11	(1.28)	0.14	(1.10)
Parameter Estimates	$\hat{\mu}$		0.0302		1.0000	
Calibrated	δ		0.9972		0.9972	
	$1/\hat{\alpha}$		0.0073		0.0073	

TABLE 3.2. Estimation with $\gamma = 5$, and c_2^5 not included.

parameter that has to be estimated, the proportion of learners in the economy μ. For simplicity, I calibrate δ and α to be its estimated value in AMN for each of the estimation procedures. Likewise, I let $\gamma = 5$, the lowest value for the relative risk aversion found there[9]. In general, the parameter vector I estimate is $\theta = \left(E_{\Delta D/D}, \sigma_{\Delta D/D}, \mu\right)$.

The estimation is done by minimizing the overall distance between the sample statistics and the statistics computed in the simulations. That is, minimizing the objective function

$$\left[\hat{S}_N - \tilde{S}(\theta)\right]' \hat{\Sigma}_{S,N}^{-1} \left[\hat{S}_N - \tilde{S}(\theta)\right] \tag{3.14}$$

where $\hat{S}_N$ is the $v \times 1$ vector containing the v statistics computed from the data, $\tilde{S}(\theta)$ is the statistics vector function computed from the simulations that depend on the parameter vector θ, and $\hat{\Sigma}_{S,N}^{-1}$ is the weighting matrix (the inverse of an estimate of the variance-covariance matrix of the statistics $\hat{S}_N$) [10]. I take all the individual standard deviations of the statistics from this chapter, except for $\rho_{r^s,-1}$, and approximate $\hat{\Sigma}_{S,N}$ utilizing the individual variances of the statistics in the

[9] There is a long discussion in the literature about this issue. As it was firstly claimed in Mehra and Prescott (1985) Consumption-Based models are not able to reproduce equity premium for a microfounded value of this parameter. Attempts to find solution have been introduced, for example, by Campbell and Cochrane (1999).

[10] More on the technical discussion as well as a description of the method can be found in the appendix 7.6 of Adam, Marcet, and Nicolini (2016)

Statistic	US data (Std. Dev.)		Model Statistics: μ flexible. (t-statistic).		$\mu = 1$ fixed. (t-statistic).	
$\rho_{rs,-1}$	-0.09	(0.01)	0.82	(-12.67)	0.79	(-12.09)
E_{PD}	113.20	(15.15)	120.27	(-0.47)	110.04	(0.21)
σ_{PD}	52.98	(16.53)	61.69	(-0.53)	64.54	(-0.70)
$\rho_{PD,-1}$	0.92	(0.02)	0.95	(-1.35)	0.94	(-1.04)
σ_{r^s}	11.65	(2.88)	8.69	(1.03)	10.98	(0.23)
c_2^5	-0.0037	(0.00)	-0.0059	(1.07)	-0.0061	(1.17)
R_5^2	0.2647	(0.08)	0.4743	(-2.56)	0.3844	(-1.46)
E_{r^s}	2.41	(0.45)	1.32	(2.42)	1.58	(1.85)
E_{r^b}	0.18	(0.23)	0.49	(-1.33)	0.47	(-1.23)
$\sigma_{\Delta D/D}$	2.98	(0.84)	1.87	(1.32)	3.31	(-0.39)
$E_{\Delta D/D}$	0.35	(0.19)	0.10	(1.33)	0.10	(1.32)
Parameter Estimates	$\hat{\mu}$		0.9394		1.0000	
Calibrated	δ		0.9972		0.9972	
	$1/\hat{\alpha}$		0.0073		0.0073	

TABLE 3.3. Estimation with $\gamma = 5$, and c_2^5 and $\rho_{rs,-1}$ not included.

main diagonal, which are computed using the individual Newey-West estimator. In this approach, $\hat{\Sigma}_{S,N}^{-1}$ is no longer the optimal weighting matrix, however, it is enough to minimize the t-statistic value of the statistics estimates.

Table 3.2 presents the estimation results when all the statistics presented but c_2^5 are included in the objective function. Last three rows include the parameter estimate of the fraction of learners in the economy, $\hat{\mu}$, the calibrated δ, and calibrated Kalman gain, $1/\hat{\alpha}$. The second and third columns include the data statistics and their estimated standard deviations. Fourth and fifth columns presents the model statistics when c_2^5 is not incorporated in the objective function. Here the estimate for the fraction of learners is 0.0302 that is very close to the fully rational expectations model. One can see that the model is able to match stock returns persistence but at the same time fails to generate enough volatility to replicate the PD variance and stock return variance. Also, the equity premium is not matched as explained by AMN because of the low value of γ. Finally, long-run predictability of the excess return is not matched either.

In last two columns I estimate the model parameters excluding μ. In particular, I set $\mu = 1$ so that the model is as in AMN with all agents endowed with learning expectations. In this procedure, the model is able to reproduce very well all the statistics, except equity risk premium and stock returns persistence. Comparing these results with the ones where μ is a parameter estimate shows there is a trade-off between stock returns persistence, which increases with μ, and predictability of excess return in the long run, and volatilities of stock returns and PD ratio. When μ is flexible the model estimates points towards close the rational expectations model because of the weight of the stock return persistence. In that way, the model replicates low stock returns persistence but diminishes the volatility of stock return and PD ratio together with the long run predictability of excess return.

Table 3.3 investigates whether above results are consistent with the ones in AMN. For that aim, I exclude stock returns persistence from the objective function. Second and third column present results when μ is estimated. The exclusion of the stock return persistence statistic makes the estimation of the parameter $\hat{\mu}$ very close to one and hence, the model is back to a completely based learning environment. This is consistent with the estimates of AMN, when you do not want to match stock returns persistence the best model is one close to learning environment. In this case, statistics in the data are matched by those of the model except stock return persistence, the equity premium, and the predictive power of PD ratio over the long run excess return. Whereas the first is explained because it was not included in the objective function, and the second is explained by the low value of γ a discussed above, the third is different from the the findings of AMN and might come from the differences in μ.

Last two columns show results when μ is fixed to 1 . In this case, I replicate the results of AMN[11]. The model matches all the statistics, except for equity premium. Comparing with the results when μ is estimated, this shows the differences in predictability of long run stock return are due to the distance from the fully learning model.

In summary, I find a trade-off between long-run predictability of excess returns and the volatilities of PD ratio and stock returns with short-run predictability of excess returns. In that trade-off the proportion of learners μ is the key parameter. As μ approaches 1, that is, the model populated with only learning expectations agents, the model is able to get the data estimates on long-run predictability of excess returns and the volatilities of PD ratio and stock returns but fails to match short-run predictability of excess returns. On the contrary, when μ approaches 1, that is, the model populated with only rational expectations agents,

[11]Notice estimation is done through a grid for all the combination of parameters. This explains any quantitative difference with the results in AMN. The results presented here represent an upper bound the minimization of the distance of the model statistics with the empirical ones.

the model is able to match the data estimates on short-run predictability of excess returns but fails to get long-run predictability of excess returns and the volatilities of PD ratio and stock returns.

3.4 Conclusions

I deviate to a small extent from the learning agents model developed in Adam, Marcet, and Nicolini (2016) (AMN) to include learners and rational agents. Financial markets are operated by heterogeneous agents. That might be important to understand asset pricing behavior. This chapter tries to analyze whether a model economy with rational expectations agents and learning expectations agents is able to beat a model with only rational agents or learning agents. Learning agents optimally update their expectation for the risk-adjusted stock price growth given the observations available and their subjective belief and behaves rationally according to that.

I use the method of simulated moments to estimate the model as in AMN using the same statistics but including the stock returns persistence. The model does not reproduce the facts as well as it is done in that study. It suggests that the introduction of stock returns persistence creates a trade-off between itself and volatility in the model where the proportion of learners is key and its estimation is lower than the required to generate the volatility of stock return and price-dividend ratio. Also, it suggests the increase of proportion of learners in the economy generates more volatility but the needed to generate stock returns persistence and volatility could be lower than in an economy completely populated by learning agents.

Future research must be devoted to improve the way heterogeneous agents are introduced. The introduction of rational expectations agents and learning expectations separately, instead of assuming a representative agent whose expectations is the weighted sum of expectations of learning and rational expectations agents, could generate a more interesting dynamics and might change results.

This chapter has studied up to which point heterogeneous expectations is an avenue to improve the capacity of consumption-based models to achieve asset pricing facts. It is confirmed that the heterogeneity in financial markets matters in this framework.

References

Adam, K. and A. Marcet (2011). Internal rationality, imperfect market knowledge and asset prices. *Journal of Economic Theory 146*(3), 1224–1252.

Adam, K., A. Marcet, and J. P. Nicolini (2016). Stock market volatility and learning. *The Journal of Finance 71*(1), 33–82.

Barber, B. M. and T. Odean (2000). Trading is hazardous to your wealth: The common stock investment performance of individual investors. *The Journal of Finance 55*(2), 773–806.

Benhabib, J. and C. Dave (2011). Learning, large deviations and rare events. NBER Working Papers 16816, National Bureau of Economic Research, Inc.

Campbell, J. Y. (1990). Measuring the persistence of expected returns. *American Economic Review 80*(2), 43–47.

Campbell, J. Y. (2003). Consumption-based asset pricing. *Handbook of the Economics of Finance 1*, 803–887.

Campbell, J. Y. and J. Cochrane (1999). Force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy 107*(2), 205–251.

Constantinides, G. M. and D. Duffie (1996). Asset pricing with heterogeneous consumers. *Journal of Political Economy 104*(2), 219–40.

Evans, G. W. and S. Honkapohja (2003). Expectations and the stability problem for optimal monetary policies. *Review of Economic Studies 70*(4), 807–824.

Honkapohja, S. and K. Mitra (2006). Learning stability in economies with heterogeneous agents. *Review of Economic Dynamics 9*(2), 284–309.

Lasarte, T. and G. Connor (2004). An introduction to hedge fund strategies.

Lucas, Robert E, J. (1980). Two illustrations of the quantity theory of money. *American Economic Review 70*(5), 1005–14.

Mehra, R. and E. C. Prescott (1985). The equity premium: A puzzle. *Journal of Monetary Economics 15*(2), 145–161.

Nunes, R. (2009). Learning the inflation target. *Macroeconomic Dynamics 13*(02), 167–188.

Preston, B. (2005, September). Learning about monetary policy rules when long-horizon expectations matter. *International Journal of Central Banking 1*(2).

Shiller, R. J. (1984). Stock prices and social dynamics. Cowles Foundation Discussion Papers 719R, Cowles Foundation for Research in Economics, Yale University.

Appendix

Appendix 3.A Derivation of equation (3.12)

For simplicity, define $\Xi_t = \frac{\delta a^{-\gamma}}{1-\delta\mu\beta_t}$, then from (3.11)

$$\begin{aligned}
P_t &= \Xi_t \left(aE_t \left[\left(\varepsilon^c_{t+1}\right)^{-\gamma} \varepsilon^d_{t+1} \right] D_t + (1-\mu)E_t \left[\left(\varepsilon^c_{t+1}\right)^{-\gamma} P_{t+1} \right] \right) \\
&= \Xi_t a\rho_\varepsilon D_t + (1-\mu)\Xi_t \\
&\times E_t \left[\left(\varepsilon^c_{t+1}\right)^{-\gamma} \Xi_{t+1} \left(a\rho_\varepsilon E_{t+1} \left[a\varepsilon^d_{t+1} D_t \right] + (1-\mu)E_{t+1} \left[\left(\varepsilon^c_{t+2}\right)^{-\gamma} P_{t+2} \right] \right) \right] \\
&= \Xi_t a\rho_\varepsilon D_t + (1-\mu)\Xi_t \\
&\times E_t \left[\Xi_{t+1} a^2 \rho_\varepsilon \left(\varepsilon^c_{t+1}\right)^{-\gamma} \varepsilon^d_{t+1} D_t \right] + (1-\mu)E_t \left[\Xi_{t+1} E_{t+1} \left[\left(\varepsilon^c_{t+2}\right)^{-\gamma} P_{t+2} \right] \right] \\
&= \Xi_t a\rho_\varepsilon D_t (1 + (1-\mu)(\Xi_t a\rho_\varepsilon) \\
&+ ((1-\mu)\Xi_t)^2 E_t \left[\left(\varepsilon^c_{t+1}\right)^{-\gamma} E_{t+1} \left[\left(\varepsilon^c_{t+2}\right)^{-\gamma} P_{t+2} \right] \right]
\end{aligned}$$

where the second equality follows from the law of iterated expectations and the third equality follows from the fact that rational agents are not concerned about the expectation updating of the learning agents. By recursive substitutions,

$$P_t = \Xi_t a\rho_\varepsilon D_t \sum_{j=0}^{\infty} ((1-\mu)\Xi_t a\rho_\varepsilon)^j$$

where notice that here I have used the fact that $(1-\mu)\frac{\delta a^{-\gamma}}{1-\delta\mu\beta_t} < 1$. But notice that this is true given the upper-bound applied for β_t

$$\beta_t < \beta^+ = \frac{1-(1-\mu)\delta a^{1-\gamma}\rho_\varepsilon}{\delta\mu} \leq$$
$$\frac{1-(1-\mu)\delta a^{-\gamma}\rho_\varepsilon}{\delta\mu} \leq \frac{1-(1-\mu)\delta a^{-\gamma}}{\delta\mu}$$

provided that $\rho_\varepsilon \geq 1$ and $a \geq 1$. Moreover, since $\beta_t < \beta^+ = \frac{1-(1-\mu)\delta a^{1-\gamma}\rho_\varepsilon}{\delta\mu}$, it follows that $(1-\mu)\Xi_t a\rho_\varepsilon < 1$ and therefore, when substituting Ξ_t, the stock price equation can be expressed by

$$P_t = \frac{\delta a^{1-\gamma}\rho_\varepsilon}{1-\delta\mu\beta_t-\delta(1-\mu)a^{1-\gamma}\rho_\varepsilon} D_t$$

which is the equation (3.12).