Universitat Autònoma de Barcelona

Periodic orbits of differential systems via the averaging theory with special emphasis on Hamiltonian systems

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Periodic orbits of differential systems via the averaging theory with special emphasis on Hamiltonian systems

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Adviser. Dr. Jaume Llibre

To my family especially my parents,
To Dr. Jaume Llibre,
To all the people who helped me from near and far...
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Contents

Chapter 1

Introduction

1.1 Motivation

Over the last sixty years a significative advance in understanding dynamical systems governed by nonlinear equations has created a profound change in the scientific worldview. In order to predict the behavior of these dynamical systems, researchers adopted mathematical models coming from different fields of sciences. To solve the nonlinear equations created by these models, the scientists always look for new techniques either qualitative or quantitative. Every time new results are provided and a lot of techniques and theories are developed. As it is known, by numerical integration of the equations of motion we can observe a global dynamical properties. Along the last thirty years, other methods are extensively used to describe the local dynamical properties of Hamiltonian systems around their equilibrium points and periodic orbits. The stability around these objects determines the kind of motion in their neighborhood. For this reason my research is focused in finding analytically families of periodic orbits for three different Hamiltonian systems coming from both galactic and atomic dynamics. Until now the majority of these studies have been on cubic or quartic polynomial Hamiltonian systems of two degree of freedom.

In this work we extend these previous studies analyzing three fourth and sixth polynomial Hamiltonian systems of three degree of freedom, two of them depending on 6 parameters.

The principal aim of this research is to study analytically the existence of periodic orbits of these systems via the averaging theory of first order. In general, it is challenging to prove analytically the existence of periodic solutions of a system of differential equations. The tool used is the averaging theory of first order. This method will provide conditions on the parameters and on the Hamiltonian levels which guarantee the existence of the periodic solutions. Furthermore, we provide analytical estimations of the shape of these periodic orbits.

1.2 The three Hamiltonian systems studied

The Hamiltonian systems treated in this work are the Yang-Mills Hamiltonian, the Friedman- Robertson- Walker Hamiltonian, and the perturbed elliptic oscillator. All of them in dimension 6.

The organization of this dissertation is as follows, in the rest of this introduction, we present every Hamiltonian with his Hamiltonian system associated and we give a brief history about the studies related on them in the last years. At the end of this introduction we provide the statements of the main results obtained. After that we have three chapters where we provide in every one the proofs of the results on every Hamiltonian system studied. Finally we add an appendix writing about the tool used, the averaging theory of first order, remembering the important results of this technique that we need to use in the proofs of this work.

Throughout this dissertation, we denote by p_x, p_y, p_z the components of momentum per unit mass; ε is a small positive real parameter, in fact ε is the perturbation strength, and a,b,c,d,e and f are parameters. The dot denotes derivative with respect to the independent variable t, the time; h denotes the Hamiltonian level; α, β and θ are angular coordinates and r, R and ρ positive reals.

1.2.1 The generalized classical Yang-Mills Hamiltonian system in dimension 6

We study a generalized classical Yang–Mills Hamiltonian system in dimension 6. This Hamiltonian is formed by a harmonic oscillator plus the most general homogeneous potential of fourth degree with monomials having only even powers and with six parameters. .

$$\begin{aligned} H = \ & \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2) + \frac{1}{4}(ax^4 + 2bx^2y^2 \\ & +2cx^2z^2 + dy^4 + 2ey^2z^2 + fz^4). \end{aligned} \quad (1.1)$$

The principal aim is to study analytically the periodic solutions in the different energy levels $H = h$ of the Hamiltonian system associated to the Hamiltonian (1.1).

Until now the majority of articles related with the Hamiltonian (1.1) studied the planar classical Yang-Mills Hamiltonian system, i.e. the Hamiltonian (1.1) with $z = p_z = 0$. The periodic solutions of the planar system were studied in [26]. Contopoulos and co-workers during many years studied this planar Yang-Mills Hamiltonian system with $a = 0$. This Hamiltonian is now known as the Contopoulos Hamiltonian which describes the perturbed central part of an elliptical or barred galaxy without escapes. For more details see the references [12], [13] and [14]. Deprit and Elipe in [15] studied also several periodic orbits and bifurcations for this planar Hamiltonian system. When $d = 0$ and the quadratic part $(x^2 + y^2)/2$ is eliminated from the Hamiltonian we obtain the mechanical Yang-Mills Hamiltonian $H = (p_x^2 + p_y^2)/2 + bx^2y^2/2$. Many authors studied this quartic homogeneous potentials, for more details see the references [3], [6] and [19]. Moreover, if $b \neq 0$ the Hamiltonian of Yang-Mills is well known that it is non integrable and strongly chaotic. Other researches associated to this mechanical system of Yang-Mills with quartic potentials having three up to five terms were treated in [9], [17], [23], and [29]. Maciejewski et. al. [29] studied generalized Yang-Mills Hamiltonian systems having a quadratic potential plus a homogeneous potential of fourth degree with five parameters. They proved the existence of connected branches of non stationary periodic orbits starting at the origin. Caranicolas and Varvoglis [9] studied a Hamiltonian with a quartic potential of three parameters plus a quadratic harmonic potential with two frequencies ω_1 and ω_2 of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2 + \omega_1^2 x^2 + \omega_2^2 y^2) + \varepsilon(ax^4 + 2bx^2y^2 + cy^4).$$

They calculated numerically families of periodic orbits and its characteristic curves. In order to study the Hamiltonian system in a two-dimensional parameter space, the paper [26] treated the generalization of the Yang-Mills potentials $H = (p_x^2 + p_y^2 + x^2 + y^2)/2 + ax^4/4 + bx^2y^2/2$ with two real parameters a and b.

The Hamiltonian differential system associated to the Hamiltonian (1.1) is

$$\begin{aligned}
\dot{x} = & \ \frac{\partial H}{\partial p_x} = p_x, \\
\dot{y} = & \ \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} = & \ \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p}_x = & \ -\frac{\partial H}{\partial x} = -x - x(ax^2 + by^2 + cz^2), \\
\dot{p}_y = & \ -\frac{\partial H}{\partial y} = -y - y(bx^2 + dy^2 + ez^2), \\
\dot{p}_z = & \ -\frac{\partial H}{\partial z} = -z - z(cx^2 + ey^2 + fz^2).
\end{aligned} \tag{1.2}$$

1.2.2 The generalized classical Friedmann-Robertson-Walker Hamiltonian system in dimension 6

It is usual in galactic dynamics to consider potentials exhibiting a reflection symmetry with respect to the axes, here we make a change in the x-axis. We study a generalization of the Calzeta-Hasi's Hamiltonian as follows

$$\begin{aligned}
H = & \ \frac{1}{2}(p_y^2 + p_z^2 - p_x^2 + y^2 + z^2 - x^2) + \frac{1}{4}(ax^4 + 2bx^2y^2 \\
& +2cx^2z^2 + dy^4 + 2ey^2z^2 + fz^4),
\end{aligned} \tag{1.3}$$

In astrophysics, the study of the dynamic of galaxies progressed considerably thanks to the discovery of important theories coming from mathematical models, see for instance the articles [5, 30, 33, 37]. Thus, Calzeta and Hasi (1993) studied the simplified Friedmann-Robertson-Walker Hamiltonian system in dimension 4 given by the Hamiltonian $H = (p_y^2 - p_x^2 + y^2 - x^2)/2 + bx^2y^2/2$ which is a model for a universe. They proved analytically and numerically the existence of chaotic motion of the Hamiltonian system associated to the above Hamiltonian. Although this model is too simplified to be considered realistically, it is an interesting testing ground for the implications of chaos in cosmology, see for more details [7]. Hawking [21] and Page [31] considered similar models for understanding the relation between the thermodynamic and cosmological arrow of time, in the area of quantum cosmology. When $z = p_z = 0$ the previous Hamiltonian (1.3) contains the planar classical Friedmann-Robertson-Walker Hamiltonian system studied by Calzeta and Hasi. In [27] we find a study of its periodic solutions.

The Hamiltonian system associated to the Hamiltonian (1.3) is

$$
\begin{aligned}
\dot{x} = & \ \frac{\partial H}{\partial p_x} = -p_x, \\
\dot{y} = & \ \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} = & \ \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p}_x = & \ -\frac{\partial H}{\partial x} = x - x(ax^2 + by^2 + cz^2), \\
\dot{p}_y = & \ -\frac{\partial H}{\partial y} = -y - y(bx^2 + dy^2 + ez^2), \\
\dot{p}_z = & \ -\frac{\partial H}{\partial z} = -z - z(cx^2 + ey^2 + fz^2),
\end{aligned}
\tag{1.4}
$$

1.2.3 The Perturbed elliptic oscillators Hamiltonian system in dimension 6

The next Hamiltonian consists of a three coupled harmonic oscillators known as perturbed elliptic oscillators. It is formed by a harmonic oscillator plus a potential with a monomial of degree 6.

$$
H = \frac{1}{2}(x^2 + y^2 + z^2 + p_x^2 + p_y^2 + p_z^2) + \varepsilon(x^2y^2 + x^2z^2 + y^2z^2 - x^2y^2z^2). \tag{1.5}
$$

Perturbed elliptic oscillators appear very often in several fields of nonlinear mechanics, as in galactic dynamics, and in atomic physics. During the last three decades, in galactic dynamics, in order to describe the local dynamics properties of galaxies we consider Hamiltonian systems. Many studies have been made, see mainly [4, 8, 10, 11, 16, 18, 28]. In spite of the simple form of the perturbation (generally, they are cubic or quartic polynomials) they lead to chaotic phenomena as was shown in the work of Henon and Heiles [22]. In this work we analyze the periodic orbits of the Hamiltonian system associated to the Hamiltonian (1.5) studied by Caranicolas and Zotos in [11].

The Hamiltonian system associated to (1.5) is

$$
\begin{aligned}
\dot{x} =& \ \frac{\partial H}{\partial p_x} = p_x, \\
\dot{y} =& \ \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} =& \ \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p}_x =& -\frac{\partial H}{\partial x} = -x - \varepsilon(2xy^2 + 2xz^2 - 2xy^2z^2), \\
\dot{p}_y =& -\frac{\partial H}{\partial y} = -y - \varepsilon(2x^2y + 2yz^2 - 2x^2yz^2), \\
\dot{p}_z =& -\frac{\partial H}{\partial z} = -z - \varepsilon(2x^2z + 2y^2z - 2x^2y^2z),
\end{aligned} \tag{1.6}
$$

1.3 Statements of the main results

The principal goal of this research is to study analytically families of periodic orbits of the Hamiltonian systems (1.2), (1.4) and (1.6). In general, it is a hard task to find analytically these orbits. Reading publications and papers related with this kind of studies, we choose the averaging theory as the main tool in this research for providing the existence of periodic orbits in the mentioned Hamiltonian systems.

For applying this tool, it is necessary to transform the Hamiltonian systems into the canonical form of the averaging theory. For that, a change of variables is applied to obtain a perturbed 2π–periodic differential system depending on an angular coordinate instead of the time t. To find these changes of variables sometimes is a difficult task of this work. After, we must eliminate from the periodic differential system as many variables as first integrals the system has, otherwise the system will have in general a continuum of periodic orbits, and then the averaging method does not provide information on the periodic solutions of the system. The new differential system obtained is formed by the following four equations

$$
\begin{aligned}
r' &= \varepsilon F_{11}(\theta, r, \alpha, R, \beta) + O(\varepsilon^2), \\
\alpha' &= \varepsilon F_{12}(\theta, r, \alpha, R, \beta) + O(\varepsilon^2), \\
R' &= \varepsilon F_{13}(\theta, r, \alpha, R, \beta) + O(\varepsilon^2), \\
\beta' &= \varepsilon F_{14}(\theta, r, \alpha, R, \beta) + O(\varepsilon^2).
\end{aligned} \tag{1.7}
$$

where the prime denotes the derivative with respect to the angular coordinate

θ. Finally, we must calculate, f_{1i} for $i = 1, 2, 3, 4$ the integrals of F_{1j} for $j = 1, 2, 3, 4$ with respect to θ with $\theta \in [0, 2\pi]$. Thus, the problem of finding periodic solutions is reduced now to find the zeros of the averaged function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$. Sometimes this task is complicated to do.

When the Hamiltonian systems are written in the normal form of the averaging theory (1.7), we must compute the zeros of f_1. This calculation was not easy having 6 real parameters and also sometimes due to the complicated expression of the functions f_{1j} for $j = 1, 2, 3, 4$. Once the zeros $(r^*, \alpha^*, R^*, \beta^*)$ of f_1 are found, we must check that the Jacobian of f_1 on these zeros are nonzero. This condition guarantees the existence of the periodic orbits. Furthermore, going back to the original Hamiltonian differential system, we can get an estimation of the analytical shape of these periodic orbits. Note that in all this work the periodic orbits studied are isolated in every energy level.

The study of the periodic orbits of the three Hamiltonian systems leads to the following results and statements

Theorem 1. *At every positive energy level $H = h > 0$ the Yang–Mills Hamiltonian system* (1.2) *has at least*

(a) one periodic orbit if

$$\left|\frac{3a-2c}{c}\right| < 1, \left|\frac{3a-2b}{b}\right| < 1, \; bc(a-b)(3a-b)(a-c)(3a-c) \neq 0;$$

(b) two periodic orbits if one of the following six conditions hold:

(b.1) $\left|\dfrac{3d-2b}{b}\right| < 1, \left|\dfrac{2e-3d}{e}\right| < 1, \; eb(b-3d)(b-d)(d-e)(3d-e) \neq 0;$

(b.2) $\left|\dfrac{2e-3f}{e}\right| < 1, \left|\dfrac{2c-3f}{c}\right| < 1, \; ec(c-3f)(c-f)(e-3f)(e-f) \neq 0;$

(b.3) $\left|\dfrac{3c^2-2bc+2ae-2ce-3af+2bf}{bc-ae+ce-bf}\right| < 1, \; (f-c)(a-2c+f) > 0,$ $(a-c)(a-2c+f) > 0, \; c(bc-ae+af-bf-c^2+ce)(bc-ae+ce-bf)(bc-3c^2-ae+ce+3af-bf) \neq 0;$

(b.4) $\left|\dfrac{6ae-9af-2bc+6bf+c^2-2ce}{3ae+bc-3bf-ce}\right| < 1, \; (3f-c)(3a-2c+3f) > 0,$ $(3a-c)(3a-2c+3f) > 0, \; c(3bc-c^2-3ae+ce+9af-9bf)(bc-c^2-9ae+3ce+9af-3bf)(3ae+bc-3bf-ce) \neq 0;$

(b.5) $\left|\frac{3b^2-2bc-3ad+2cd+2ae-2be}{bc-cd-ae+be}\right|<1$, $b(3b^2-bc-3ad+cd+ae-be)(b^2-bc-ad+cd+ae-be)(bc-cd-ae+be)\neq 0$, $(d-b)(a-2b+d)>0, (a-b)(a-2b+d)>0$;

(b.6) $\left|\frac{b^2-2bc-9ad+6cd+6ae-2be}{bc-3cd+3ae-be}\right|<1$, $b(3cd-3ae-bc+be)(b^2-3bc-9ad+9cd+3ae-be)(b^2-bc-9ad+3cd+9ae-3be)\neq 0$, $(3d-b)(3a-2b+3d)>0, (3a-b)(3a-2b+3d)>0$;

(c) four periodic orbits if one of the following twenty six conditions hold:

(c.1) $\left|\frac{2ce-2cd+2be-3e^2-2bf+3df}{cd-be-ce+bf}\right|<1$, $e(cd-be-ce+3e^2+bf-3df)(cd-be-ce+e^2+bf-df)\neq 0$, $(f-e)(d-2e+f)>0, (d-e)(d-2e+f)>0$;

(c.2) $\left|\frac{2be+2ce-e^2-6cd-6bf+9df}{3cd+be-ce-3bf}\right|<1$, $(-9cd+be+3ce-e^2-3bf+9df)(3cd-3be-ce+e^2+9bf-9df)\neq 0$, $(3f-e)(3d-2e+3f)>0, (3d-e)(3d-2e+3f)>0$;

(c.3) $a+d=0$, $d(c+d-e)>0$, $(cd-ce+e^2-df)(c-a-e)(c+d-e)>0$, $(c^2-ac+ae-cd-ce+df)(c-a-e)(c+d-e)>0$, $ce(c+d-e)(c-a-e)(c^2-ad+2cd+2ae-2ce+e^2-af-df)\neq 0$;

(c.4) $c+d-e=0, d(a+d)>0, (cd+ae-ad)(a+d)(c+e-f)>0, (ac+ad-cd-af)(a+d)(c+e-f)>0$, $ce(c^2-ad+2cd+2ae-2ce+e^2-af-df)\neq 0$;

(c.5) $(cd-ce+e^2-df)(c^2+2cd-2ce+e^2-df-ad+2ae-af)>0, (cd+ae-ad)(c^2+2cd-2ce+e^2-df-ad+2ae-af)>0, (c^2-ce+ae-af)(c^2+2cd-2ce+e^2-df-ad+2ae-af)>0$, $ce(a+d)(c+d-e)\neq 0$;

(c.6) $a+d=0, d(3c+3d-e)>0, (-9cd+3ce-e^2+9df)(3c+3d-e)(3a-3c+e)>0, (3ac-ae-3c^2+3cd+ce-3df)(3c+3d-e)(3a-3c+e)>0,\ ce(3a-3c+e)(9c^2-9ad+18cd+6ae-6ce+e^2-9af-9df)\neq 0$;

(c.7) $3c+3d-e=0, d(a+d)>0, (3cd+ae-3ad)(a+d)(3c+e-3f)>0, (ac+ad-af-cd)(a+d)(3c+e-3f)>0,\ ce(-9c^2+9ad-18cd-6ae+6ce-e^2+9af+9df)\neq 0$;

(c.8) $(9cd-3ce+e^2-9df)(9c^2-9ad+18cd+6ae-6ce+e^2-9af-9df)>0, (-3ad+3cd+ae)(9c^2-9ad+18cd+6ae-6ce+e^2-9af-9df)>0, (3c^2-ce+ae-3af)(9c^2-9ad+18cd+6ae-6ce+e^2-9af-9df)>0, ce(a+d)(3c+3d-e)\neq 0$;

(c.9) $a+d=0, d(c+3d-e)>0, (3cd-ce+e^2-9df)(c-3a-e)(c+3d-e)>0, (c^2-3ac+3ae-3cd-ce+9df)(c-3a-e)(c+3d-e)>0, ce(c^2-9ad+6cd+6ae-2ce+e^2-9af-9df)\neq 0$;

(c.10) $c+3d-e=0, d(a+d)>0, (cd+ae-3ad)(a+d)(c+e-3f)>0, (ac+3ad-cd-3af)(a+d)(c+e-3f)>0,\ ce(c^2-9ad+6cd+6ae-2ce+e^2-9af-9df)\neq 0$;

(c.11) $(3cd-ce+e^2-9df)(c^2-9ad+6cd+6ae-2ce+e^2-9af-9df)>0, (cd-3ad+ae)(c^2-9ad+6cd+6ae-2ce+e^2-9af-9df)>0, (c^2-ce+3ae-9af)(c^2-9ad+6cd+6ae-2ce+e^2-9af-9df)>0, ce(a+d)(c+3d-e)\neq 0$;

(c.12) $a+d=0, d(c+3d-3e)>0, ce(c^2-9ad+6cd+18ae-6ce+9e^2-9af-9df)\neq 0, (ce-cd-3e^2+3df)(c+3d-3e)(3a-c+3e)>0, (3ac-c^2-9ae+3cd+3ce-9df)(c+3d-e)(3a-c+3e)>0$;

(c.13) $c+3d-3e=0, d(a+d)>0, ce(-c^2+9ad-6cd-18ae+6ce-9e^2+9af+9df)\neq 0, (cd+3ae-3ad)(a+d)(c+3e-3f)>0, (ac+3ad-3af-cd)(a+d)(c+3e-3f)>0$;

(c.14) $ce(a+d)(c+3d-3e)\neq 0, (cd-ce+3e^2-3df)(c^2-9ad+6cd+18ae-6ce+9e^2-9af-9df)>0, (-3ad+cd+3ae)(c^2-9ad+$

$6cd+18ae-6ce+9e^2-9af-9df)>0, (c^2-3ce+9ae-9af)(c^2-9ad+6cd+18ae-6ce+9e^2-9af-9df)>0$;

(c.15) $b-c-d+e=0, (d-b)(a-2b+d)>0, (-b^2-cd+be+bc-ae+ad)(a-2b+d)(b-c-e+f)>0, (cd+ab-ac-ad+af-bf)(a-2b+d)(b-c-e+f)>0, (b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c+b^2cd-bc^2d+ab^2e-abce+bc^2e-abde+acde-c^2de-ace^2+bce^2-b^2cf+bcdf+abef-b^2ef)\neq 0$;

(c.16) $a-2b+d=0, (b-d)(b-c-d+e)>0, (e^2+cd-ce-be+bf-df)(b-c-d+e)(a-b-c+e)>0, (cb-c^2-cd-ce-ac+ae-bf+df)(b-c-d+e)(a-b-c+e)>0, (2bc-b^2-c^2+ad-2cd-2ae+2be+2ce-e^2+af-2bf+df)(b^2ce-bc^2d+bc^2e+acde-c^2de-abe^2-bce^2+cde^2-b^2cf+bcdf+b^2ef-bdef)\neq 0$;

(c.17) $(b-c-d+e)(a-2b+d)(bc^2d-b^2ce-bc^2e-acde+c^2de+abe^2+ace^2-bce^2+b^2cf-bcdf-abef+b^2ef)\neq 0, (b^2-ad+cd+ae-bc-be)(b^2-2bc+c^2-ad-2cd+2ae-2be-2ce+e^2-af+2bf-df)>0, (cd-be-ce+e^2+bf-df)(b^2-2bc+c^2-ad-2cd+2ae-2be-2ce+e^2-af+2bf-df)>0, (c^2-ce+ae-af-bc+bf)(b^2-2bc+c^2-ad-2cd+2ae-2be-2ce+e^2-af+2bf-df)>0$;

(c.18) $3b-c-3d+e=0, (d-b)(a-2b+d)>0, (a-2b+d)(3b-c-e+3f)(-3b^2+3ad-cd-ae+bc+be)>0, (a-2b+d)(3b-c-e+3f)(cd-3bf+3ab-ac-3ad+3af)>0, (9b^2-6bc+c^2-9ad+6cd+6ae-6be-2ce+e^2-9af+18bf-9df)(b^2c^2-3b^3c+3b^2cd-bc^2d+3ab^2e-abce+4b^2ce-bc^2e-3abde-3acde+c^2de+ace^2-bce^2-3b^2cf+3bcdf+3abef-3b^2ef)\neq 0$;

(c.19) $a-2b+d=0, (b-d)(3b-c-3d+e)>0, (3a-3b-c+e)(3b-c-3d+e)(e^2+3cd-ce-3be+9bf-9df)>0, (3a-3b-c+e)(3b-c-3d+e)(3bc+c^2-3cd-ce-3ac+3ae-9bf+9df)>0, (6bc-9b^2-c^2+9ad-6cd-6ae+6be+2ce-e^2+9af-18bf+9df)(bc^2d-4abce+3b^2ce+bc^2e+3acde-4bcde-c^2de+abe^2-bce^2+cde^2+3b^2cf-3bcdf-3b^2ef+3bdef)\neq 0$;

(c.20) $(e^2+3cd-ce-3be+9bf-9df)(9b^2-6bc+c^2-9ad+6cd+$

$6ae - 6be - 2ce + e^2 - 9af + 18bf - 9df) > 0, (3b^2 - bc + 3ad + cd + ae - be)(9b^2 - 6bc + c^2 - 9ad + 6cd + 6ae - 6be - 2ce + e^2 - 9af + 18bf - 9df) > 0, (c^2 - 3bc + 3ae - ce - 9af + 9bf)(9b^2 - 6bc + c^2 - 9ad + 6cd + 6ae - 6be - 2ce + e^2 - 9af + 18bf - 9df) > 0, (3b - c - 3d + e)(a - 2b + d)(bc^2d - 5b^2ce + bc^2e + 3acde - c^2de + abe^2 - ace^2 + bce^2 + 3b^2cf - 3bcdf - 3abef + 3b^2ef) \neq 0;$

(c.21) $b - 3c - 3d + e = 0, (3d - b)(3a - 2b + 3d) > 0, (3cd + ab - 3ac - 3ad + 3af - bf)(3a - 2b + 3d)(b - 3c - e + 3f) > 0, (-9cd - b^2 + 3bc + be + 9ad - 3ae)(3a - 2b + 3d)(b - 3c - e + 3f) > 0, (b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df)(b^3c - 3b^2c^2 - 3b^2cd + 9bc^2d + 3ab^2e - 9abce - 3bc^2e - 9abde - 9acde + 9c^2de + 12bcde + 3ace^2 - bce^2 + 3b^2cf - 9bcdf + 9abef - 3b^2ef) \neq 0;$

(c.22) $3a - 2b + 3d = 0, (b - 3d)(b - 3c - 3d + e) > 0, (b - 3c - 3d + e)(3a - b - 3c + e)(e^2 + 9cd - be - 3ce + 3bf - 9df) > 0, (b - 3c - 3d + e)(3a - b - 3c + e)(-3ac + ae + bc + 3c^2 - 3cd - ce - bf + 3df) > 0, (b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df)(9bc^2d + 12abce - 5b^2ce - 15bc^2e - 9acde + 12bcde + 9c^2de - 3abe^2 + 5bce^2 - 3cde^2 + 3b^2cf - 9bcdf + 3b^2ef - 9bdef) \neq 0;$

(c.23) $(3a - 2b + 3d)(b - 3c - 3d + e)(3bc^2d + b^2ce - 5bc^2e - 3acde + 3c^2de - abe^2 + ace^2 + bce^2 + b^2cf - 3bcdf + 3abef - b^2ef) \neq 0, (9cd - be - 3ce + e^2 + 3bf - 9df)(b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df) > 0, (b^2 + 9cd - 3bc - be - 9ad + 3aeb)(b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df) > 0, (3c^2 - bc - ce + ae - 3af + bf)(b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df) > 0;$

(c.24) $3a - 2b + 3d = 0, (b - 3d)(b - c - 3d + 3e) > 0, (3a - b - c + 3e)(b - c - 3d + 3e)(cd - be - ce + 3e^2 + bf - 3df) > 0, (3a - b - c + 3e)(b - c - 3d + 3e)(9ae - 3ac + bc + c^2 - 3cd - 3ce - 3bf + 9df) > 0, (b^2 - 2bc + c^2 - 9ad + 6cd + 18ae - 6be - 6ce + 9e^2 - 9af + 6bf - 9df)(-bc^2d + b^2ce + bc^2e - 3acde + c^2de + 3abe^2 - 3bce^2 - 3cde^2 - b^2cf + 3bcdf - b^2ef + 3bdef) \neq 0;$

(c.25) $b - c - 3d + 3e = 0, (3d - b)(3a - 2b + 3d) > 0, (b - c - 3e + 3f)(3a - 2b + 3d)(-b^2 - 3cd + 9ad - 9ae + bc + 3be) > 0, (b - c - 3e + 3f)(3a -$

$2b+3d)(cd-bf+ab-ac-3ad+3af)>0, (-b^2+2bc-c^2+9ad-6cd-18ae+6be+6ce-9e^2+9af-6bf+9df)(b^3c-b^2c^2-3b^2cd+3bc^2d+3ab^2e-3abce-4b^2ce+bc^2e-9abde+9acde+12bcde-3c^2de-9ace^2+3bce^2+3b^2cf-9bcdf+9abef-3b^2ef)\neq 0$;

(c.26) $(b-c+3d-3e)(3a-2b+3d)(bc^2d-b^2ce-bc^2e+3acde-c^2de-3abe^2-3ace^2+5bce^2+b^2cf-3bcdf+3abef-b^2ef)\neq 0, (3e^2+cd--ce-be+bf-3df)(b^2+c^2+6cd-6ce+9e^2-2bc-6be+6bf-9df-9ad+18ae-9af)>0, (b^2-bc-9ad+3cd+9ae-3be)(b^2+c^2+6cd-6ce+9e^2-2bc-6be+6bf-9df-9ad+18ae-9af)>0, (c^2-bc+9ae-3ce-9af+3bf)(b^2+c^2+6cd-6ce+9e^2-2bc-6be+6bf-9df-9ad+18ae-9af)>0.$

Theorem 1 is proved in sections 2.2 and 2.3 of chapter 2.

In the next proposition we provide the shape of the periodic solutions studied in Theorem 1.

Proposition 2. *The family of periodic orbits* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), p_x(t,\varepsilon), p_y(t,\varepsilon), p_z(t,\varepsilon)\big)$ *of system* (1.2) *generated by the zeros* $(r^*,\alpha^*,R^*,\beta^*)$ *of the averaged function* $(f_{11}, f_{12}, f_{13}, f_{14})$ *given in* (2.7) *at every fixed energy level* $H=h>0$ *are given by*

$$\begin{aligned}
x(t,\varepsilon) &= \sqrt{\varepsilon}\quad r^*\cos t + O(\varepsilon^{3/2}),\\
y(t,\varepsilon) &= \sqrt{\varepsilon}\quad \sqrt{2h-r^{*2}-R^{*2}}\cos(\alpha^*+t) + O(\varepsilon^{3/2}),\\
z(t,\varepsilon) &= \sqrt{\varepsilon}\quad R^*\cos(\beta^*+t) + O(\varepsilon^{3/2}),\\
p_x(t,\varepsilon) &= \sqrt{\varepsilon}\quad r^*\sin t + O(\varepsilon^{3/2}),\\
p_y(t,\varepsilon) &= \sqrt{\varepsilon}\quad \sqrt{2h-r^{*2}-R^{*2}}\sin(\alpha^*+t) + O(\varepsilon^{3/2}),\\
p_z(t,\varepsilon) &= \sqrt{\varepsilon}\quad R^*\sin(\beta^*+t) + O(\varepsilon^{3/2}).
\end{aligned}$$

Proposition 2 is proved at the end of sections 2.3 of chapter 2.

Theorem 3. *At every fixed energy level* $H=h$ *with* $h\in\mathbb{R}$ *the Friedmann-Robertson-Walker Hamiltonian system* (1.4) *has at least*

(a) one periodic orbit if the following conditions hold

$h<0,\ bc\neq 0,\ (a+b)(3a+b)(a+c)(3a+c)\neq 0,\ \left|\dfrac{3a+2b}{b}\right|<1,$
$\left|\dfrac{3a+2c}{c}\right|<1$;

(b) two periodic orbits if one of the following six conditions hold

(b.1) $h > 0$, $eb \neq 0$, $(b+3d)(b+d) \neq 0$, $\left|\dfrac{3d+2b}{b}\right| < 1$, $\left|\dfrac{2e-3d}{e}\right| < 1$;

(b.2) $h > 0$, $ec \neq 0$, $(c+3f)(c+f) \neq 0$, $\left|\dfrac{2e-3f}{e}\right| < 1$, $\left|\dfrac{3f+2c}{c}\right| < 1$;

(b.3) $a+2c+f \neq 0$, $bc-ae-ce+bf \neq 0$, $h(a+2c+f)(c+f) < 0$, $h(a+2c+f)(a+c) > 0$, $(bc-c^2-ae-ce+af+bf)(bc-3c^2-ae-ce+3af+bf) \neq 0$, $\left|\dfrac{3c^2-2bc+2ae+2ce-3af-2bf}{bc-ae-ce+bf}\right| < 1$;

(b.4) $3a+2c+3f \neq 0$, $(bc+3ae+ce+3bf) \neq 0$, $h(3a+2c+3f)(c+3f) < 0$, $h(3a+2c+3f)(3a+c) > 0$, $c(c^2-3bc+3ae+ce-9af-9bf)(c^2-bc+9ae+3ce-9af-3bf) \neq 0$, $\left|\dfrac{-2bc+c^2+6ae+2ce-9af-6bf}{bc+3ae+ce+3bf}\right| < 1$;

(b.5) $a+2b+d \neq 0$, $b+d \neq 0$, $bc+cd-ae-be \neq 0$, $h(b+d)(a+2b+d) < 0$, $h(a+b)(a+2b+d) > 0$, $b(3b^2-bc-3ad-cd+ae+be)(b^2-bc-ad-cd+ae+be) \neq 0$, $\left|\dfrac{3b^2-2bc-3ad-2cd+2ae+2be}{bc+cd-ae-be}\right| < 1$;

(b.6) $3a+2b+3d \neq 0$, $3cd+3ae+bc+be \neq 0$, $h(b+3d)(3a+2b+3d) < 0$, $h(3a+b)(3a+2b+3d) > 0$, $b(b^2-3bc-9ad-9cd+3ae+be)(b^2-bc-9ad-3cd+9ae+3be) \neq 0$ $\left|\dfrac{b^2-2bc-9ad-6cd+6ae+2be}{3cd+3ae+bc+be}\right| < 1$.

(c) four periodic orbits if one of the following twenty eight conditions hold

(c.1) $cd-be-ce+bf \neq 0$, $d-2e+f \neq 0$, $h(d-2e+f)(f-e) > 0$, $h(d-2e+f)(d-e) > 0$, $e(-be+bf+cd-ce+3df-3e^2)(-be+bf+cd-ce+df-e^2) \neq 0$, $\left|\dfrac{-2cd+2be+2ce+3e^2-2bf-3df}{cd-be-ce+bf}\right| < 1$;

(c.2) $bd-cd-be+bf \neq 0$, $\left|\dfrac{3d+2b}{b}\right| < 1$, $h(bd-cd-be+bf)(cd-bf) > 0$, $hb(bd-cd-be+bf)(d-e) > 0$, $be(b+d)(b+3d)(d-2e+f)(ce-bf) \neq 0$;

(c.3) $3cd+be-ce-3bf \neq 0$, $3d-2e+3f \neq 0$, $h(3f-e)(3d-2e+3f) > 0$, $h(3d-e)(3d-2e+3f) > 0$, $(-9cd+be+3ce+e^2-3bf-9df)(3cd-3be-ce-e^2+9bf+9df) \neq 0$, $\left|\dfrac{2be+2ce+e^2-6cd-6bf-9df}{3cd+be-ce-3bf}\right| < 1$;

(c.4) $4bc+3bd+3cd-be+3bf \neq 0$, $hb(4bc+3bd+3cd-be+3bf)(3d-e) > 0$, $h(4bc+3bd+3cd-be+3bf)(4bc+3cd+3bf) > 0$, $eb(b+d)(b+3d)(3d-2e+3f)(4bc+ce+3bf) \neq 0$, $\left|\dfrac{3d+2b}{b}\right| < 1$;

(c.5) $a+c+e=0$, $e-c-f \neq 0$, $eh(e-c-f) > 0$, $h(cd+df-ce-e^2)(a+d)(e-c-f) > 0$, $ce\big(ad-c^2+2cd-2ae-2ce-e^2+af+df\big) \neq 0$, $h(ce+e^2+ac+af)(a+d)(e-c-f) < 0$;

(c.6) $a+c+e \neq 0$, $c-e+f=0$, $c-d+e \neq 0$, $eh(a+c+e) < 0$, $hc(c-d+e) > 0$, $h(ae-ad-cd)(a+c+e)(c-d+e) > 0$, $ce(c^2-ad-2cd+2ae+2ce+e^2-af-df) \neq 0$

(c.7) $\Sigma = c^2-ad-2cd+2ae+2ce+e^2-af-df \neq 0$, $a+c+e \neq 0$, $e-c-f \neq 0$, $h(cd+df-ce-e^2)\Sigma > 0$, $h(ae-ad-cd)\Sigma > 0$, $h(ce+c^2+ae-af)\Sigma > 0$, $ce \neq 0$;

(c.8) $3c-3d+e \neq 0$, $a+d=0$, $3a+3c+e \neq 0$, $dh(3c-3d+e) < 0$, $h(9cd-3ce-e^2+9df)(3c-3d+e)(3a+3c+e) > 0$, $h(3ac+ae+3c^2+3cd+ce+3df)(3c-3d+e)(3a+3c+e) > 0$, $cde(9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df) \neq 0$;

(c.9) $3c-3d+e=0$, $a+d \neq 0$, $3c-e+3f \neq 0$, $dh(a+d) < 0$, $h(3ad+3cd-ae)(a+d)(3c-e+3f) > 0$, $h(ac-ad+af-cd)(a+d)(3c-e+3f) > 0$, $cde(9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df) \neq 0$;

(c.10) $\Sigma_1 = 9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df \neq 0$, $a+d \neq 0$, $3c-3d+e \neq 0$, $h(9cd-3ce-e^2+9df)\Sigma_1 > 0$, $h(3c^2+ce+ae-3af)\Sigma_1 > 0$, $h(3ad+3cd-ae)\Sigma_1 < 0$, $ce \neq 0$;

(c.11) $3a+c+e=0$, $c-e+3f \neq 0$, $he(c-e+3f) < 0$, $h(a+d)(c-e+3f)(ce+e^2-9df-3cd) > 0$, $h(a+d)(c-e+3f)(ce+e^2+3ac+9af) > 0$, $a+d \neq 0$, $ce\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big) \neq 0$;

(c.12) $3a+c+e \neq 0$, $c-e+3f=0$, $3d-c-e \neq 0$, $he(3a+c+e)<0$, $hc(3d-c-e)>0$, $h(3ad+cd-ae)(3d-c-e)(3a+c+e)>0$, $ce\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big) \neq 0$;

(c.13) $\Sigma_2 = c^2-9ad-6cd+6ae+2ce+e^2-9af-9df \neq 0$, $h(ae-3ad-cd)\Sigma_2>0$, $3a+c+e \neq 0$, $c-e+3f \neq 0$, $h(3cd-ce-e^2+9df)\Sigma_2>0$, $h(c^2+ce+3ae-9af)\Sigma_2>0$, $ce \neq 0$;

(c.14) $a+d=0$, $c-3d+3e \neq 0$, $3a+c+3e \neq 0$, $dh(c-3d+3e)<0$, $h(cd-ce-3e^2+3df)(c-3d+3e)(3a+c+3e)>0$, $h(3ac+9ae+c^2+3cd+3ce+9df)(c-3d+3e)(3a+c+3e)>0$, $d(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-3abce-4bc^2e+9acde-3bcde+3c^2de-9abe^2-12bce^2+9cde^2+9bcdf-9bdef) \neq 0$;

(c.15) $a+d \neq 0$, $c-3d+3e=0$, $c-3e+3f \neq 0$, $dh(a+d)<0$, $h(3ad+cd-3ae)(a+d)(c-3e+3f)>0$, $h(ac+3af-3ad-cd)(a+d)(c-3e+3f)>0$, $(-b^2+2bc-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(bc^2d+abce-3abde+3acde-4bcde+c^2de-3ace^2+3bcdf+3abef) \neq 0$;

(c.16) $\Sigma_3 = c^2-9ad-6cd+18ae+6ce+9e^2-9af-9df \neq 0$, $a+d \neq 0$, $c-3d+3e \neq 0$, $h(3ae-3ad-cd)\Sigma_3>0$, $h(cd-ce-3e^2+3df)\Sigma_3>0$, $h(c^2+3ce+9ae-9af)\Sigma_3>0$, $(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-4bc^2e+9acde+3c^2de-9abe^2-9ace^2-12bce^2+9bcdf+9abef) \neq 0$;

(c.17) $b-c+d-e=0$, $a+2b+d \neq 0$, $b-c+e-f \neq 0$, $h(b+d)(a+2b+d)<0$, $h(b^2-cd+be-bc+ae-ad)(a+2b+d)(b-c+e-f)>0$, $h(cd+ab-ac+ad-af-bf)(a+2b+d)(b-c+e-f)>0$, $(b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c-b^2cd+bc^2d+ab^2e-abce-bc^2e+abde-acde-c^2de+ace^2+bce^2+b^2cf+bcdf-abef-b^2ef) \neq 0$;

(c.18) $b-c+d-e \neq 0$, $a+2b+d=0$, $a+b+c+e \neq 0$, $h(b+d)(b-c+d-e)>0$, $h(e^2-cd+ce+be-bf-df)(b-c+d-e)(a+b+c+e)>0$, $h(-ac-ae-cb-c^2-cd-ce-bf-df)(b-c+d-e)(a+b+c+e)>0$, $(2bc-b^2-c^2+ad+2cd-2ae-2be-2ce-e^2+af+2bf+df)(b^2ce-bc^2d+bc^2e+acde+c^2de-abe^2+bce^2+cde^2-b^2cf-bcdf-b^2ef-$

$bdef) \neq 0$;

(c.19) $\omega = b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df \neq 0$, $b-c+d-e \neq 0$, $a+2b+d \neq 0$, $h(cd-be-ce-e^2+bf+df)\omega > 0$, $h(b^2-ad-cd+ae-bc+be)\omega > 0$, $h(c^2+ce+ae-af-bc-bf)\omega > 0$, $(bc^2d-b^2ce-bc^2e-acde-c^2de+abe^2+ace^2+bce^2+b^2cf+bcdf-abef-b^2ef) \neq 0$;

(c.20) $3b-c+3d-e = 0$, $a+2b+d \neq 0$, $3b-c+e-3f \neq 0$, $h(b+d)(a+2b+d) < 0$, $h(a+2b+d)(3b-c+e-3f)(3b^2-cd-bc+be-3ad+ae) > 0$, $h(a+2b+d)(3b-c+e-3f)(3ab-ac+3ad+cd-3af-3bf) > 0$, $(9b^2-6bc+c^2-9ad-6cd+6ae+6be+2ce+e^2-9af-18bf-9df)(-3b^3c+b^2c^2-3b^2cd+bc^2d+3ab^2e-abce-4b^2ce+bc^2e+3abde+3acde+c^2de-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef) \neq 0$;

(c.21) $3b-c+3d-e \neq 0$, $a+2b+d = 0$, $3a+3b+c+e \neq 0$, $h(b+d)(3b-c+3d-e) > 0$, $h(3a+3b+c+e)(3b-c+3d-e)(3be+ce+e^2-3cd-9bf-9df) > 0$, $h(3a+3b+c+e)(3b-c+3d-e)(3ac+3ae+3bc+c^2+3cd+ce+9bf+9df) > 0$, $(6bc-9b^2-c^2+9ad+6cd-6ae-6be-2ce-e^2+9af+18bf+9df)(bc^2d+4abce+3b^2ce+bc^2e+3acde+4bcde+c^2de+abe^2+bce^2+cde^2+3b^2cf+3bcdf+3b^2ef+3bdef) \neq 0$;

(c.22) $\omega_1 = 9b^2-6bc+c^2-9ad-6cd+6ae+6be+2ce+e^2-9af-18bf-9df \neq 0$, $3b-c+3d-e \neq 0$, $a+2b+d \neq 0$, $h(3cd-3be-ce-e^2+9bf+9df)\omega_1 > 0$, $h(3b^2-3ad-cd+ae-bc+be)\omega_1 > 0$, $h(c^2-3bc+3ae+ce-9af-9bf)\omega_1 > 0$, $(bc^2d-5b^2ce+bc^2e+3acde+c^2de+abe^2-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef) \neq 0$;

(c.23) $b-3c+3d-e = 0$, $3a+2b+3d \neq 0$, $b-3c+e-3f \neq 0$, $h(b^2-9ad-9cd+3ae-3bc+be)(3a+2b+3d)(b-3c+e-3f) > 0$, $h(b+3d)(3a+2b+3d) < 0$, $h(3cd-bf+ab-3ac+3ad-3af)(3a+2b+3d)(b-3c+e-3f) > 0$, $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(b^3c-3b^2c^2+3b^2cd-9bc^2d+3ab^2e-9abce+3bc^2e+9abde+9acde+9c^2de+12bcde-3ace^2-bce^2-3b^2cf-9bcdf-9abef-3b^2ef) \neq 0$;

(c.24) $b-3c+3d-e \neq 0$, $3a+2b+3d = 0$, $3a+b+3c+e \neq 0$, $h(b+3d)(b-3c+3d-e) > 0$, $h(b-3c+3d-e)(3a+b+$

$3c+e)(e^2-9cd+be+3ce-3bf-9df)>0$, $h(b-3c+3d-e)(3a+b+3c+e)(3ac+ae+bc+3c^2+3cd+ce+bf+3df)<0$, $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(12abce-9bc^2d+5b^2ce+15bc^2e+9acde+12bcde+9c^2de+3abe^2+5bce^2+3cde^2-3b^2cf-9bcdf+3b^2ef+9bdef)\neq 0$;

(c.25) $b-3c+3d-e\neq 0$, $3a+2b+3d\neq 0$, $\omega_2=b^2+9c^2-9ad+6ae+e^2-18cd+6ce-6bc+2be-6bf-9af-9df\neq 0$, $h(9cd-be-3ce-e^2+3bf+9df)\omega_2>0$, $h(b^2-9ad-9cd+3aeb-3bc+be)\omega_2>0$, $h(3c^2+ce+ae-3af-bc-bf)\omega_2>0$, $(5bc^2e-3bc^2d-b^2ce+3acde+3c^2de+abe^2-ace^2+bce^2-b^2cf-3bcdf-3abef-b^2ef)\neq 0$;

(c.26) $c-b-3d+3e=0$, $3a+2b+3d\neq 0$, $b-c+3e-3f\neq 0$, $h(b+3d)(3a+2b+3d)<0$, $h(3a+2b+3d)(b-c+3e-3f)(b^2-bc-9ad-3cd+9ae+3be)>0$, $h(3a+2b+3d)(b-c+3e-3f)(cd+ab-ac+3ad-3af-bf)>0$, $(b^2-2bc+c^2-9ad-6cd+18ae+6be+6ce+9e^2-9af-6bf-9df)(b^3c-b^2c^2+3b^2cd-3bc^2d+3ab^2e-3abce+4b^2ce-bc^2e+9abde-9acde+12bcde-3c^2de+9ace^2+3bce^2-3b^2cf-9bcdf-9abef-3b^2ef)\neq 0$;

(c.27) $c-b-3d+3e\neq 0$, $3a+2b+3d=0$, $h(b+3d)(b-c+3d-3e)>0$, $h(b-c+3d-3e)(3a+b+c+3e)(be-bf+ce-cd+3e^2-3df)>0$, $h(b-c+3d-3e)(3a+b+c+3e)(3ac+bc+c^2+3cd+9ae+3ce+3bf+9df)<0$, $(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(b^2ce-bc^2d+bc^2e-3acde-c^2de+3abe^2+3bce^2-3cde^2-b^2cf-3bcdf+b^2ef+3bdef)\neq 0$;

(c.28) $c-b-3d+3e\neq 0$, $3a+2b+3d\neq 0$, $\omega_3=b^2-2bc+c^2-9ad-6cd+18ae+6be+6ce+9e^2-9af-6bf-9df\neq 0$ $h(cd-be-ce-3e^2+bf+3df)\omega_3>0$, $h(b^2-bc-9ad-3cd+9ae+3be)\omega_3>0$, $h(c^2-bc+9ae+3ce-9af-3bf)\omega_3>0$, $(bc^2d-b^2ce-bc^2e+3acde+c^2de-3abe^2-3ace^2-5bce^2+b^2cf+3bcdf+3abef+b^2ef)\neq 0$.

Theorem 3 is proved in sections 3.2 and 3.3 of chapter 3.

In the next proposition we provide the shape of the periodic solutions studied in Theorem 3.

Proposition 4. *The family of periodic orbits of the system* (1.4) *generated by the zeros* $(r^*,\alpha^*,R^*,\beta^*)$ *of the averaged function* $(f_{11},f_{12},f_{13},f_{14})$ *given* *in* (3.10) *at every fixed energy level* $H = h$ *with* $h \in \mathbb{R}$ *are given by*

$$\begin{aligned}
x(t,\varepsilon) &= \sqrt{\varepsilon} \quad r^* \cos t + O(\varepsilon^{3/2}),\\
y(t,\varepsilon) &= \sqrt{\varepsilon} \quad \sqrt{2h + r^{*2} - R^{*2}}\cos(\alpha^* - t) + O(\varepsilon^{3/2}),\\
z(t,\varepsilon) &= \sqrt{\varepsilon} \quad R^* \cos(\beta^* - t) + O(\varepsilon^{3/2}),\\
p_x(t,\varepsilon) &= \sqrt{\varepsilon} \quad r^* \sin t + O(\varepsilon^{3/2}),\\
p_y(t,\varepsilon) &= \sqrt{\varepsilon} \quad \sqrt{2h + r^{*2} - R^{*2}}\sin(\alpha^* - t) + O(\varepsilon^{3/2}),\\
p_z(t,\varepsilon) &= \sqrt{\varepsilon} \quad R^* \sin(\beta^* - t) + O(\varepsilon^{3/2}).
\end{aligned}$$

Proposition 4 is proved at the end of section 3.3 of chapter 3.

Theorem 5. *At every fixed energy level* $H = h$ *with* $h > 0$ *the perturbed elliptic oscillator Hamiltonian system* (1.6) *has at least*

(a) Ten periodic orbits if $h \in (0, 6/5)$;

(b) Twenty periodic orbits if $h \in \Big(6/5, 9/5\Big) \cup \Big(9/5, 27/10\Big) \cup \Big(27/10, 3]$;

(c) Twenty two periodic orbits if $h \in (3, 6)$;

(d) Thirty two periodic orbits if $h \in (6, 5 + 2\sqrt{6})$;

(e) Forty two periodic orbits if $h \in (5 + 2\sqrt{6}, +\infty)$.

Theorem 5 is proved in sections 4.2 and 4.3 of chapter 4.

In the next proposition we provide the shape of the periodic solutions studied in Theorem 5.

Proposition 6. *The family of periodic orbits of the system* (1.6) *generated by the zeros* $(r^*, \alpha^*, R^*, \beta^*)$ *of the averaged function* $(f_{11}, f_{12}, f_{13}, f_{14})$ *given in* (4.8) *at every fixed energy level* $H = h > 0$ *are given by*

$$\begin{aligned}
x(t,\varepsilon) &= r^* \cos t + O(\varepsilon),\\
y(t,\varepsilon) &= \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* + t) + O(\varepsilon),\\
z(t,\varepsilon) &= R^* \cos(\beta^* + t) + O(\varepsilon),\\
p_x(t,\varepsilon) &= r^* \sin t + O(\varepsilon),\\
p_y(t,\varepsilon) &= \sqrt{2h - r^{*2} - R^{*2}}\sin(\alpha^* + t) + O(\varepsilon),\\
p_z(t,\varepsilon) &= R^* \sin(\beta^* + t) + O(\varepsilon).
\end{aligned}$$

Proposition 6 is proved at the end of section 4.3 of chapter 4.

“Pure mathematics is, in its way, the poetry of logical ideas.” Albert Einstein.

Chapter 2

Yang-Mills Hamiltonian system in 6D

In this chapter we study analytically the periodic orbits of the Yang-Mills Hamiltonian system in dimension 6 using the averaging theory of first order.

2.1 Yang-Mills Hamiltonian system in 6D

We study a generalized classical Yang–Mills Hamiltonian system in dimension 6. It is formed by a harmonic oscillator plus the most general homogeneous potential of fourth degree with monomials having only even powers.

$$\begin{aligned} H = \; & \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2) + \frac{1}{4}(ax^4 + 2bx^2y^2 + 2cx^2z^2 \\ & + dy^4 + 2ey^2z^2 + fz^4). \end{aligned} \tag{1.1}$$

Note that the Hamiltonian (1.1) depends on six real parameters a,b,c,d,e and f.

The Hamiltonian differential system associated to the Hamiltonian (1.1) is

$$\begin{aligned}
\dot{x} = &\ \frac{\partial H}{\partial p_x} = p_x, \\
\dot{y} = &\ \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} = &\ \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p}_x = &\ -\frac{\partial H}{\partial x} = -x - x(ax^2 + by^2 + cz^2), \\
\dot{p}_y = &\ -\frac{\partial H}{\partial y} = -y - y(bx^2 + dy^2 + ez^2), \\
\dot{p}_z = &\ -\frac{\partial H}{\partial z} = -z - z(cx^2 + ey^2 + fz^2).
\end{aligned} \tag{1.2}$$

The dot denotes derivative with respect to the independent variable t, the time.

The principal objective is to compute analytically the periodic orbits of the Hamiltonian system (1.2), because after the equilibrium points the periodic orbits are the most simple non–trivial solutions of the system, and their stability determines the kind of motion in their neighborhood.

To do this study we use the *averaging theory of first order*, see Appendix for more details. This averaging method converts the problem of finding periodic solutions of a differential system in finding zeros of some convenient finite dimensional function depending on the six parameters a,b,c,d,e and f. The averaging method for studying periodic solutions of Hamiltonian systems has been used in other Hamiltonians different from the ones considered in this work, see for instance [2, 26, 25].

2.2 Applying averaging theory to Yang-Mills Hamiltonian system in 6D

To have a small parameter in system (1.2) we change their variables (x, y, z, p_x, p_y, p_z) by (X, Y, Z, p_X, p_Y, p_Z) where $x = \sqrt{\varepsilon}\,X$, $y = \sqrt{\varepsilon}\,Y$, $z = \sqrt{\varepsilon}\,Z$, $p_x = \sqrt{\varepsilon}\,p_X$, $p_y = \sqrt{\varepsilon}\,p_Y$ and $p_z = \sqrt{\varepsilon}\,p_Z$, with ε a small positive parameter. System (1.2) becomes

$$
\begin{aligned}
\dot{X} &= p_X, \\
\dot{Y} &= p_Y, \\
\dot{Z} &= p_Z, \\
\dot{p}_X &= -X - \varepsilon X(aX^2 + bY^2 + cZ^2), \\
\dot{p}_Y &= -Y - \varepsilon Y(bX^2 + dY^2 + eZ^2), \\
\dot{p}_Z &= -Z - \varepsilon Z(cX^2 + eY^2 + fZ^2).
\end{aligned}
\tag{2.1}
$$

This system again is Hamiltonian with Hamiltonian

$$
\begin{aligned}
H = \;& \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2 + X^2 + Y^2 + Z^2) + \\
& \varepsilon \frac{1}{4}\left(aX^4 + 2bX^2Y^2 + 2cX^2Z^2 + dY^4 + 2eY^2Z^2 + fZ^4\right).
\end{aligned}
$$

The original system (1.2) and the transformed system (2.1) have the same topological phase portrait because the change of variables is only a scale transformation for all $\varepsilon > 0$ fixed. Moreover, for ε sufficiently small the Hamiltonian system (2.1) is close to an integrable one.

The periodicity of the differential system is a basic requirement to apply the averaging theory. For that we need another change of variables to a kind of generalized polar coordinates $(r, \theta, \rho, \alpha, R, \beta)$ in $\mathbb{R}^6$ as follows

$$
\begin{aligned}
X &= r\cos\theta, & Y &= \rho\cos(\theta+\alpha), & Z &= R\cos(\theta+\beta), \\
p_X &= r\sin\theta, & p_Y &= \rho\sin(\theta+\alpha), & p_Z &= R\sin(\theta+\beta),
\end{aligned}
\tag{2.2}
$$

with $r \geq 0$, $\rho \geq 0$ and $R \geq 0$.

The first integral H becomes

$$
\begin{aligned}
H = \;& \frac{1}{2}(r^2 + \rho^2 + R^2) + \varepsilon\frac{1}{4}\Big[ar^4\cos^4\theta + 2br^2\rho^2 \\
& \cos^2\theta\cos^2(\theta+\alpha) + d\rho^4\cos^4(\theta+\alpha) + 2R^2\Big(cr^2\cos^2\theta \\
& + e\rho^2\cos^2(\theta+\alpha)\Big)\cos^2(\theta+\beta) + fR^4\cos^4(\theta+\beta)\Big],
\end{aligned}
\tag{2.3}
$$

The new equations of motion are

$$\begin{aligned}
\dot{r} &= -\varepsilon r\cos\theta\sin\theta\Big[a\,r^2\cos^2\theta + b\,\rho^2\cos^2(\theta+\alpha) \\
&\quad +c\,R^2\cos^2(\theta+\beta)\Big], \\
\dot{\theta} &= -1-\varepsilon\cos^2\theta\Big[a\,r^2\cos^2\theta + b\rho^2\cos^2(\theta+\alpha) \\
&\quad +cR^2\cos^2(\theta+\beta)\Big], \\
\dot{\rho} &= -\varepsilon\rho\cos(\theta+\alpha)\sin(\theta+\alpha)\Big[b\,r^2\cos^2\theta + d\rho^2\cos^2(\theta+\alpha) \\
&\quad +eR^2\cos^2(\theta+\beta)\Big], \\
\dot{\alpha} &= \varepsilon\Big[a\,r^2\cos^4\theta + \cos^2\theta\Big(b(-r^2+\rho^2)\cos^2(\theta+\alpha) \\
&\quad +cR^2\cos^2(\theta+\beta)\Big) \\
&\quad -\cos^2(\theta+\alpha)\Big(d\rho^2\cos^2(\theta+\alpha) \\
&\quad +eR^2\cos^2(\theta+\beta)\Big)\Big], \\
\dot{R} &= -R\varepsilon\cos(\theta+\beta)\sin(\theta+\beta)\Big[c\,r^2\cos^2\theta + e\rho^2\cos^2(\theta+\alpha) \\
&\quad +fR^2\cos^2(\theta+\beta)\Big], \\
\dot{\beta} &= \varepsilon\Big[a\,r^2\cos^4\theta + b\rho^2\cos^2\theta\cos^2(\theta+\alpha) - \Big(c(r^2-R^2) \\
&\quad \cos^2\theta + e\rho^2\cos^2(\theta+\alpha)\Big)\cos^2(\theta+\beta) - fR^2\cos^4(\theta+\beta)\Big].
\end{aligned} \tag{2.4}$$

We take from now on the variable θ as the new independent variable of the system instead of t and we get the differential system

$$
\begin{aligned}
r' &= \varepsilon\, r \sin\theta\cos\theta\Big[a\, r^2\cos^2\theta + b\rho^2\cos^2(\theta+\alpha) \\
&\quad +cR^2\cos^2(\theta+\beta)\Big] + O(\varepsilon^2), \\
\rho' &= \varepsilon\rho\cos(\theta+\alpha)\sin(\theta+\alpha)\Big[b\, r^2\cos^2\theta + d\rho^2\cos^2(\theta+\alpha) \\
&\quad +eR^2\cos^2(\theta+\beta)\Big] + O(\varepsilon^2), \\
\alpha' &= \varepsilon\Big[-\ ar^2\cos^4\theta + d\rho^2\cos^4(\theta+\alpha) + eR^2 \\
&\quad \cos^2(\theta+\alpha)\cos^2(\theta+\beta) + \cos^2\theta\Big(b(r^2-\rho^2)\cos^2(\theta+\alpha) \\
&\quad -cR^2\cos^2(\theta+\beta)\Big)\Big] + O(\varepsilon^2), \\
R' &= \varepsilon\, R\sin(\theta+\beta)\cos(\theta+\beta)\Big[c\, r^2\cos^2\theta + e\rho^2\cos^2(\theta+\alpha) \\
&\quad +fR^2\cos^2(\theta+\beta)\Big] + O(\varepsilon^2), \\
\beta' &= \varepsilon\Big[-\cos^2\theta\,\Big(ar^2\cos^2\theta + b\rho^2\cos^2(\theta+\alpha)\Big)+ \\
&\quad \Big(c(r^2-R^2)\cos^2\theta + e\rho^2\cos^2(\theta+\alpha))\cos^2(\theta+\beta)+ \\
&\quad fR^2\cos^4(\theta+\beta)\Big)\Big] + O(\varepsilon^2),
\end{aligned}
\tag{2.5}
$$

where the prime means the derivative with respect to the variable θ, and we have expanded the right-hand side of system (2.4) in Taylor series of powers of ε.

System (2.5) is now 2π-periodic respect to the variable θ. We shall restrict this system to every positive energy level $H = h$ with $h > 0$. Otherwise, when we should apply the averaging theory on it, the Jacobian (A.4) will be zero because the periodic orbits would be non-isolated living on a cylinder parameterized by the energy, see for more details [1].

Solving equation (2.3) $= h$ with respect to ρ we obtain two solutions, we choose the positive one, and expanding it in Taylor series in ε we have

$$\rho = \sqrt{2h - r^2 - R^2} + O(\varepsilon).$$

Substituting ρ in system (2.5) we obtain the differential system

$$
\begin{aligned}
r' &= \varepsilon\, r \sin\theta\cos\theta\Big[a\, r^2\cos^2\theta + b(2h - r^2 - R^2) \\
&\quad \cos^2(\theta+\alpha) + cR^2\cos^2(\theta+\beta)\Big] + O(\varepsilon^2), \\
\alpha' &= \varepsilon\Big[-\; ar^2\cos^4\theta + d(2h - r^2 - R^2)\cos^4(\theta+\alpha) \\
&\quad +eR^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta) \\
&\quad +\cos^2\theta\Big(b(-2h + 2r^2 + R^2)\cos^2(\theta+\alpha) \\
&\quad -cR^2\cos^2(\theta+\beta)\Big)\Big] + O(\varepsilon^2), \\
R' &= \varepsilon\, R\sin(\theta+\beta)\cos(\theta+\beta)\Big[c\, r^2\cos^2\theta + e(2h - r^2 \\
&\quad -R^2)\cos^2(\theta+\alpha) + fR^2\cos^2(\theta+\beta)\Big] + O(\varepsilon^2), \\
\beta' &= \varepsilon\Big[-\; ar^2\cos^4\theta + \cos^2(\theta+\beta)\Big(e(2h - r^2 - R^2)\cos^2(\theta+\alpha) \\
&\quad +fR^2\cos^2(\theta+\beta)\Big) + \cos^2\theta\Big(b(-2h + r^2 + R^2)\cos^2(\theta+\alpha) \\
&\quad +c(-R^2 + r^2)\cos^2(\theta+\beta)\Big)\Big] + O(\varepsilon^2).
\end{aligned}
\tag{2.6}
$$

The differential system (2.6) now is in the normal form (A.1) for applying the averaging theory. Moreover, it is exactly the Hamiltonian differential system (2.1) restricted to the energy level $H = h > 0$, i.e. to study the periodic solutions of system (2.6) is the same thing to study the periodic solutions of the Hamiltonian system (2.1) in the energy level $H = h > 0$.

For system (2.6) the function $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ which appear in system (A.1) is

$$
\begin{aligned}
F_{11} &= r\sin\theta\cos\theta\Big[a\,r^2\cos^2\theta + b(2h - r^2 - R^2)\cos^2(\theta+\alpha) \\
&\quad + cR^2\cos^2(\theta+\beta)\Big], \\
F_{12} &= -\,ar^2\cos^4\theta + d(2h - r^2 - R^2)\cos^4(\theta+\alpha) \\
&\quad + eR^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta) + \cos^2\theta\Big(b(-2h + 2r^2 + R^2) \\
&\quad \cos^2(\theta+\alpha) - cR^2\cos^2(\theta+\beta)\Big), \\
F_{13} &= R\sin(\theta+\beta)\cos(\theta+\beta)\Big[c\,r^2\cos^2\theta + e(2h - r^2 - R^2)\cos^2(\theta+\alpha) \\
&\quad + fR^2\cos^2(\theta+\beta)\Big], \\
F_{14} &= -\,ar^2\cos^4\theta + \cos^2(\theta+\beta)\Big(e(2h - r^2 - R^2)\cos^2(\theta+\alpha) \\
&\quad + fR^2\cos^2(\theta+\beta)\Big) + \cos^2\theta\Big(b(-2h + r^2 + R^2)\cos^2(\theta+\alpha) \\
&\quad + c(-R^2 + r^2)\cos^2(\theta+\beta)\Big).
\end{aligned}
$$

Computing the function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ using the expression (A.3) we obtain

$$
\begin{aligned}
f_{11}(r,\alpha,R,\beta) &= \frac{1}{8}r[b(-2h + r^2 + R^2)\sin 2\alpha - cR^2\sin 2\beta]\,, \\
f_{12}(r,\alpha,R,\beta) &= \frac{1}{8}[6dh - 3ar^2 - 3dr^2 - 3dR^2 + (-2bh + 2br^2 + bR^2) \\
&\quad (2 + \cos 2\alpha) + eR^2(2 + \cos 2(\alpha-\beta)) \\
&\quad - cR^2(2 + \cos 2\beta)]\,, \\
f_{13}(r,\alpha,R,\beta) &= \frac{1}{8}R[e(-2h + r^2 + R^2)\sin 2(\alpha-\beta) + cr^2\sin 2\beta]\,, \\
f_{14}(r,\alpha,R,\beta) &= \frac{1}{8}[4(-b+e)h - 3ar^2 + 2(b+c-e)r^2 + (2b - 2c - 2e \\
&\quad + 3f)R^2 + (-2h + r^2 + R^2)(b\cos 2\alpha - e\cos 2(\alpha-\beta)) \\
&\quad + c(r^2 - R^2)\cos 2\beta]\,.
\end{aligned}
$$

2.3 Periodic orbits of Yang-Mills Hamiltonian system in 6D

Our objective now, following Theorem 7 (see Appendix), is to find the zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ of

$$f_{1i}(r,\alpha,R,\beta) = 0 \quad for\ i = 1,2,3,4, \tag{2.7}$$

such that the Jacobian determinant (A.4) at these zeros be different from zero.

Solving $f_{11}(r,\alpha,R,\beta)=0$ we obtain three principal cases, case 1: $r=0$; case 2: $r=\sqrt{2h-R^2+\dfrac{cR^2\sin 2\beta}{b\sin 2\alpha}}$ if $b\sin 2\alpha\neq 0$; and case 3: $b\sin 2\alpha=0$.

Case 1: $r=0$. We have

$$\begin{aligned}
f_{12}(0,\alpha,R,\beta) = \; & \frac{1}{8}\Big[6dh-3dR^2+(-2bh+bR^2)(2+\cos 2\alpha)+eR^2 \\
& (2+\cos 2(\alpha-\beta))-cR^2(2+\cos 2\beta)\Big], \\
f_{13}(0,\alpha,R,\beta) = \; & \frac{1}{8}eR(-2h+R^2)\sin 2(\alpha-\beta), \\
f_{14}(0,\alpha,R,\beta) = \; & \frac{1}{8}\Big[4(-b+e)h+\Big(2b-2(c+e)+3f\Big)R^2+(-2h+R^2) \\
& \Big(b\cos 2\alpha-e\cos 2(\alpha-\beta)\Big)-cR^2\cos 2\beta\Big].
\end{aligned}$$

So $f_{13}=0$ implies four subcases, $e=0$, $R=0$, $R=\sqrt{2h}$ and $\alpha=\beta+\dfrac{k\pi}{2}$ with $k\in\mathbb{Z}$.

Subcase 1.1: $e=0$. Then $f_{13}=0$. The Jacobian vanishes and the averaging theory does not provide information.

Subcase 1.2: $R=0$. We obtain

$$\begin{aligned}
f_{12}(0,\alpha,R,\beta) = \; & \frac{1}{8}[6dh-2bh(2+\cos 2\alpha)], \\
f_{14}(0,\alpha,R,\beta) = \; & \frac{1}{4}h[-2b+2e-b\cos 2\alpha+e\cos 2(\alpha-\beta)].
\end{aligned}$$

Subcase 1.2.1: $b=0$. We get $f_{12}=3dh/4=$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 1.2.2: $b\neq 0$. Then we have

$$f_{12}(r,\alpha,R,\beta)=0\Rightarrow\left\{\begin{array}{l}\alpha_1=\dfrac{1}{2}\arccos\dfrac{3d-2b}{b}, \\[2ex] \alpha_2=-\dfrac{1}{2}\arccos\dfrac{3d-2b}{b}.\end{array}\right.$$

If we substitute α_1 in $f_{14}(0,\alpha,R,\beta)$ we get

$$f_{14}(0,\alpha,R,\beta) = -\frac{1}{4}h\left[e\cos\left(\arccos\frac{2b-3d}{b}+2\beta\right)+3d-2e\right].$$

Subcase 1.2.2.1: $e \neq 0$.

$$f_{14}(r,\alpha,R,\beta)=0 \Rightarrow \begin{cases} \beta_{11} = -\frac{1}{2}\left(\arccos\frac{2b-3d}{b}+\arccos\frac{2e-3d}{e}\right), \\ \beta_{12} = -\frac{1}{2}\left(\arccos\frac{2b-3d}{b}-\arccos\frac{2e-3d}{e}\right). \end{cases}$$

If we substitute α_2 in $f_{14}(0,\alpha,R,\beta)$ we get

$$f_{14}(0,\alpha,R,\beta) = \frac{1}{4}h\left[e\cos\left(\arccos\frac{3d-2b}{b}+2\beta\right)-3d+2e\right].$$

So

$$f_{14}(r,\alpha,R,\beta)=0 \Rightarrow \begin{cases} \beta_{21} = -\frac{1}{2}\left(\arccos\frac{3d-2b}{b}+\arccos\frac{3d-2e}{e}\right), \\ \beta_{22} = -\frac{1}{2}\left(\arccos\frac{3d-2b}{b}-\arccos\frac{3d-2e}{e}\right). \end{cases}$$

If

$$eb \neq 0, \left|\frac{3d-2b}{b}\right| < 1 \quad \text{and} \left|\frac{2e-3d}{e}\right| < 1, \tag{2.8}$$

hold, then system (2.7) has four zeros $S^* = (r^*,\alpha^*,R^*,\beta^*)$ with $\rho = \sqrt{2h}$ and $\beta^* = -\frac{1}{2}\Big[\arccos -\frac{3d-2b}{b} \pm \arccos\frac{2e-3d}{e}\Big]$ given by

$$\begin{aligned} &\left(0,\frac{1}{2}\arccos\frac{3d-2b}{b},0,\beta^*\right), \\ &\left(0,-\frac{1}{2}\arccos\frac{3d-2b}{b},0,\beta^*\right). \end{aligned} \tag{2.9}$$

which reduce to two solutions if either $\left|\frac{3d-2b}{b}\right| = 1$, $\left|\frac{2e-3d}{e}\right| < 1$ and $eb \neq 0$, or $|(3d-2b)/b| < 1$, $|(2e-3d)/e| = 1$ and $eb \neq 0$. We have one solution if $|(3d+2b)/b| = |(2e-3d)/e| = 1$, $h > 0$ and $eb \neq 0$.

Now we calculate the Jacobian of f_1 applied on these solutions. By definition the Jacobian is

$$J_{f_1(s*=(r^*,\alpha^*,R^*,\beta^*))} = \quad |D_{r\alpha R\beta} f_1(s*)| = \begin{vmatrix} \dfrac{\partial f_{11}}{\partial r} & \dfrac{\partial f_{11}}{\partial \alpha} & \dfrac{\partial f_{11}}{\partial R} & \dfrac{\partial f_{11}}{\partial \beta} \\ \dfrac{\partial f_{12}}{\partial r} & \dfrac{\partial f_{12}}{\partial \alpha} & \dfrac{\partial f_{12}}{\partial R} & \dfrac{\partial f_{12}}{\partial \beta} \\ \dfrac{\partial f_{13}}{\partial r} & \dfrac{\partial f_{13}}{\partial \alpha} & \dfrac{\partial f_{13}}{\partial R} & \dfrac{\partial f_{13}}{\partial \beta} \\ \dfrac{\partial f_{14}}{\partial r} & \dfrac{\partial f_{14}}{\partial \alpha} & \dfrac{\partial f_{14}}{\partial R} & \dfrac{\partial f_{14}}{\partial \beta} \end{vmatrix}.$$

The Jacobian is

$$J_{f_1(s*)} = -\frac{9}{64}h^4(b-3d)(b-d)(d-e)(3d-e).$$

When the conditions (2.8) hold and the Jacobian evaluated at these solutions is nonzero i.e. $(b-3d)(b-d)(d-e)(3d-e) \neq 0$, then the four solutions (2.9) of system (2.7) provide only two periodic solutions of differential system (2.6) because when $R = 0$ the four solutions of β provide the same initial conditions in (2.2). Note that the set of conditions on the parameters in this subcase is nonempty because of the value $b = 1, d = 2/3, e = 4/3$.

Subcase 1.2.2.2: $e = 0$. Then $f_{14} = -3dh/4$= constant. The Jacobian will be zero and the averaging theory does not give results.

In the rest of this chapter since when either the Jacobian evaluated at the solutions vanishes, or the set of conditions on the parameters that guarantee the existence of the periodic solutions is not satisfied, we know that the averaging theory does not provide information about the periodic solutions.

Subcase 1.3: $R = \sqrt{2h}$. Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = & \ \frac{1}{4}h\Big[e\Big(2+\cos 2(\alpha-\beta)\Big) - c\Big(2+\cos 2\beta\Big)\Big], \\ f_{14}(0,\alpha,R,\beta) = & \ -\frac{1}{4}h[2c - 3f + c\cos 2\beta]. \end{aligned}$$

Subcase 1.3.1: $c = 0$. Then $f_{14} = 3fh/4$= constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 1.3.2: $c \neq 0$.

$$f_{14}(0,\alpha,R,\beta)=0 \Rightarrow \begin{cases} \beta_3 = \dfrac{1}{2}\arccos\dfrac{3f-2c}{c}, \\ \beta_4 = -\dfrac{1}{2}\arccos\dfrac{3f-2c}{c}. \end{cases}$$

We substitute β_3 in f_{12} we have

$$f_{12}(0,\alpha,R,\beta) = \frac{h}{4}\left[2e-3f-e\cos\left(2\alpha+\arccos\frac{2c-3f}{c}\right)\right].$$

Subcase 1.3.2.1: $e \neq 0$. Then

$$f_{12}(0,\alpha,R,\beta)=0 \Rightarrow \begin{cases} \alpha_{31} &= \dfrac{1}{2}\left[\arccos\dfrac{2e-3f}{e}-\arccos\dfrac{2c-3f}{c}\right], \\ \alpha_{32} &= \dfrac{1}{2}\left[-\arccos\dfrac{2e-3f}{e}-\arccos\dfrac{2c-3f}{c}\right] \end{cases}$$

We substitute β_4 in f_{12} we have

$$f_{12}(0,\alpha,R,\beta) = \frac{h}{4}\left[2e-3f+e\cos\left(2\alpha+\arccos\frac{3f-2c}{c}\right)\right].$$

$$f_{12}(0,\alpha,R,\beta)=0 \Rightarrow \begin{cases} \alpha_{41} &= \dfrac{1}{2}\left[\arccos\dfrac{3f-2e}{e}-\arccos\dfrac{3f-2c}{c}\right], \\ \alpha_{42} &= \dfrac{1}{2}\left[-\arccos\dfrac{3f-2e}{e}-\arccos\dfrac{3f-2c}{c}\right], \end{cases}$$

Supposing that

$$ce \neq 0, \left|\frac{2e-3f}{e}\right| \leq 1, \quad \text{and} \left|\frac{2c-3f}{c}\right| < 1. \tag{2.10}$$

Then system (2.7) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\alpha_{1,\pm} = \dfrac{1}{2}\Big[\pm\arccos\dfrac{2e-3f}{e}-\arccos\dfrac{2c-3f}{c}\Big]$,
$\alpha_{2,\pm} = \dfrac{1}{2}\Big[\pm\arccos\dfrac{3f-2e}{e}-\arccos\dfrac{3f-2c}{c}\Big]$ and $\rho = 0$ given by

$$\begin{aligned} &\left(0,\alpha_{1,\pm},\sqrt{2h},\frac{1}{2}\arccos\frac{3f-2c}{c}\right), \\ &\left(0,\alpha_{2,\pm},\sqrt{2h},-\frac{1}{2}\arccos\frac{3f-2c}{c}\right), \end{aligned} \tag{2.11}$$

which reduce to two solutions if either $\left|\frac{2c-3f}{c}\right| = 1$, $\left|\frac{2e-3f}{e}\right| < 1$ and $ce \neq 0$, or $|(2c-3f)/c| < 1$, $|(2e-3f)/e| = 1$ and $ce \neq 0$. If $|(2c-3f)/c| = |(2e-3f)/e| = 1$ and $ce \neq 0$, we have only one solution. The Jacobian is

$$J_{f_1(s*)} = -\frac{9}{32}h^4(c-3f)(c-f)(e-3f)(e-f).$$

If (2.10) and $(c-3f)(c-f)(e-3f)(e-f) \neq 0$ hold. Then the four solutions (2.11) of system (2.7) provide only two periodic solutions of differential system (2.6) because when $\rho = 0$ the eight solutions of α provide the same initial conditions in (2.2). Note that the set of conditions on the parameters of this subcase is not empty because of the value $c = 1, e = 4/3, f = 2/3$.

Subcase 1.3.2.2: $e = 0$ then $f_{12} = -3fh/4=$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 1.4: $\alpha = \beta + \frac{k\pi}{2}$ with $k = 0, 1, 2, 3$.

Due to the periodicity of the cosinus we study the cases $k = 0$ and $k = 2$, and the cases $k = 1$ and $k = 3$ together.

Subcase 1.4.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. We have

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = \ & \frac{1}{8}\Big[-4bh + 6dh + (2b-2c-3d+3e)R^2 + \big(-2bh \\ & +(b-c)R^2\big)\cos 2\beta\Big], \\ f_{14}(0,\alpha,R,\beta) = \ & \frac{1}{8}\Big[-4bh + 6eh + (2b-2c-3e+3f)R^2 + \big(-2bh \\ & +(b-c)R^2\big)\cos 2\beta\Big]. \end{aligned}$$

Subcase 1.4.1.1 $2bh - bR^2 + cR^2 = 0$. The system $f_{12} = f_{14} = 0$ does not have solutions. Then the averaging theory cannot give information.

Subcase 1.4.1.2: $-2bh + bR^2 - cR^2 \neq 0$. Then solving $f_{12} = 0$ we have $\beta = \pm\frac{1}{2}\arccos\frac{-4bh+6dh+2bR^2-2cR^2-3dR^2+3eR^2}{2bh-bR^2+cR^2}$. Substituting β in f_{14} and solving $f_{14} = 0$ we get if $d - 2e + f \neq 0$, $R = \sqrt{\frac{2h(d-e)}{d-2e+f}}$. So $\beta = \pm\frac{1}{2}\arccos\frac{-3e^2-2cd+2ce+2be-2bf+3df}{cd-ce-be+bf}$ and $\rho = \sqrt{\frac{2h(f-e)}{d-2e+f}}$.

Assuming that

$$
\begin{array}{l}
d-2e+f \neq 0, \quad \left|\dfrac{-3e^2-2cd+2ce+2be-2bf+3df}{cd-ce-be+bf}\right| \leq 1, \\
(d-e)(d-2e+f)>0 \quad \text{and} \quad (f-e)(d-2e+f)>0.
\end{array} \tag{2.12}
$$

Therefore system (2.7) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\beta^* = \pm\dfrac{1}{2}\arccos\dfrac{-3e^2-2cd+2ce+2be-2bf+3df}{cd-ce-be+bf}$ and $\rho = \sqrt{\dfrac{2h(f-e)}{d-2e+f}}$ given by

$$
\begin{array}{l}
\left(0, \beta^*, \sqrt{\dfrac{2h(d-e)}{d-2e+f}}, \beta^*\right), \\
\left(0, \beta^*+\pi, \sqrt{\dfrac{2h(d-e)}{d-2e+f}}, \beta^*\right),
\end{array} \tag{2.13}
$$

which reduce to two zeros if $\left|\dfrac{-3e^2-2cd+2ce+2be-2bf+3df}{cd-ce-be+bf}\right| = 1, (d-e)(d-2e+f)>0, (f-e)(d-2e+f)>0$ and $d-2e+f \neq 0$.

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \; & \frac{9e(e-d)(e-f)h^4}{32(d-2e+f)^3}(cd-be-ce+3e^2+bf-3df) \\
& (cd-be-ce+e^2+bf-df).
\end{aligned}
$$

If (2.12) and $(cd-be-ce+3e^2+bf-3df)(cd-be-ce+e^2+bf-df) \neq 0$ hold. Then the four solutions (2.13) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters of this subcase is not empty because of the value $b=0, c=-2, d=-3/2, e=-1/2, f=-1$.

Subcase 1.4.2: Assume that either $k=1$ or $k=3$, i.e. either $\alpha = \beta + \dfrac{\pi}{2}$ or $\alpha = \beta + \dfrac{3\pi}{2}$. Then f_{12} and f_{14} become

$$
\begin{aligned}
f_{12}(0,\alpha,R,\beta) = \; & \frac{1}{8}\big[-4bh+6dh+\big(2b-2c-3d+e\big)R^2+\big(2bh \\
& -(b+c)R^2\big)\cos 2\beta)\big], \\
f_{14}(0,\alpha,R,\beta) = \; & \frac{1}{8}\big[-4bh+2eh+(2b-2c-e+3f)R^2+\big(2bh \\
& -(b+c)R^2\big)\cos 2\beta\big].
\end{aligned}
$$

Subcase 1.4.2.1: $2bh-bR^2-cR^2=0$. The averaging theory cannot give information in this subcase because the functions $f_{12}(0,\alpha,R,\beta)$ and

$f_{14}(0,\alpha,R,\beta)$ depend only of R and then we do not have solutions of α and β.

Subcase 1.4.2.2: $2bh - bR^2 - cR^2 \neq 0$. Then solving $f_{12} = 0$ we have $\beta = \pm\frac{1}{2}\arccos\frac{-4bh + 6dh + 2bR^2 - 2cR^2 - 3dR^2 + eR^2}{-2bh + bR^2 + cR^2}$. Substituting β in f_{14} and solving $f_{14} = 0$ we obtain if $3d-2e+3f \neq 0$, $R = \sqrt{\frac{2h(3d-e)}{3d-2e+3f}}$. So $\beta = \pm\frac{1}{2}\arccos\frac{2be + 2ce - e^2 - 6cd - 6bf + 9df}{3cd + be - ce - 3bf}$ and $\rho = \sqrt{\frac{2h(3f-e)}{3d-2e+3f}}$.

Considering that

$$\begin{array}{l} 3d - 2e + 3f \neq 0, \\ (3f-e)(3d-2e+3f) > 0, (3d-e)(3d-2e+3f) > 0 \quad \text{and} \\ \left|\dfrac{2be + 2ce - e^2 - 6cd - 6bf + 9df}{3cd + be - ce - 3bf}\right| < 1. \end{array} \tag{2.14}$$

Therefore system (2.7) for has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\beta^* = \pm\frac{1}{2}\arccos\frac{2be + 2ce - e^2 - 6cd - 6bf + 9df}{3cd + be - ce - 3bf}$ and $\rho = \sqrt{\frac{2h(3f-e)}{3d-2e+3f}}$ given by

$$\begin{array}{l} \left(0, \beta^* + \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3d-e)}{3d-2e+3f}}, \beta^*\right), \\ \left(0, \beta^* + \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3d-e)}{3d-2e+3f}}, \beta^*\right), \end{array} \tag{2.15}$$

which reduce to two zeros if we have $3d-2e+3f \neq 0, (3f-e)(3d-2e+3f) > 0, (3d-e)(3d-2e+3f) > 0$ and $\left|\frac{2be + 2ce - e^2 - 6cd - 6bf + 9df}{3cd + be - ce - 3bf}\right| = 1$.

$$\begin{aligned} J_{f_1(s*)} = \; & \frac{e(3d-e)(e-3f)h^4}{32(3d-2e+3f)^3}(9cd - be - 3ce + e^2 + 3bf - 9df) \\ & (3cd - 3be - ce + e^2 + 9bf - 9df). \end{aligned}$$

When (2.14) and the Jacobian ′s condition $(-9cd + be + 3ce - e^2 - 3bf + 9df)(3cd - 3be - ce + e^2 + 9bf - 9df) \neq 0$ hold, then the four solutions (2.15) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters of this subcase is not empty because the value $b = 0, c = -3, d = -1, e = -4, f = -1$ satisfy it.

Case 2: $r = \sqrt{2h - R^2 + \dfrac{cR^2 \sin 2\beta}{b \sin 2\alpha}}$ with the condition $b \sin 2\alpha \neq 0$. We have

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = {} & \frac{1}{8b}\Big[\frac{R^2 \sin 2\beta}{2\sin 2\alpha}(-6ac + 4bc\cos 2\alpha + 8bc - be\cos 4\alpha \\
& + be - 6cd) + b\big(R^2(3a - 2(b + c - e)) - 6ah \\
& + b\cos 2\alpha\big(2h - R^2\big) + 4bh\big) + bR^2\cos 2\beta(e\cos 2\alpha - c)\Big], \\
f_{13}(r,\alpha,R,\beta) = {} & \frac{cR\sin 2\beta}{8b\sin 2\alpha}\Big[2bh\sin 2\alpha - R^2(b\sin 2\alpha - c\sin(2\beta) \\
& - e\sin 2(\alpha - \beta))\Big], \\
f_{14}(r,\alpha,R,\beta) = {} & \frac{1}{16b}\Big[4b(-3a + 2c)h + \big(6ab - c(8b + e) + 6bf\big)R^2 \\
& + 4bc(h - R^2)\cos 2\beta + cR^2\Big[e\cos 4\beta + \frac{\sin 2\beta}{\sin 2\alpha}\big((-6a \\
& + 4(b + c - e) + 2b\cos 2\alpha) \\
& + (c - e\cos 2\alpha)\sin 4\beta)\Big]\Big].
\end{aligned}
$$

Solving $f_{13} = 0$ if $b\sin 2\alpha - c\sin 2\beta - e\sin 2(\alpha - \beta) = 0$ we have three possibilities, $c = 0, R = 0$ and $\beta = \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$ because $(b\sin 2\alpha \neq 0)$ in all of this case. In the case that $b\sin 2\alpha - c\sin 2\beta - e\sin 2(\alpha - \beta) \neq 0$ we get $R = \sqrt{\dfrac{2bh\sin 2\alpha}{b\sin 2\alpha - c\sin 2\beta - e\sin 2(\alpha - \beta)}}$.

Subcase 2.1: $c = 0$. Then $f_{13} = 0$ and the Jacobian is equal to zero so the averaging theory does not give information.

Subcase 2.2: $R = 0$. Therefore $r = \sqrt{2h}$ and $\rho = 0$. We get

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = {} & \frac{1}{4}h(-3a + b\cos(2\alpha) + 2b), \\
f_{14}(r,\alpha,R,\beta) = {} & \frac{1}{4}h(-3a + c\cos(2\beta) + 2c).
\end{aligned}
$$

Subcase 2.2.1: $b = 0$. We have $f_{12} = -3ah/4 =$ constant.

Subcase 2.2.2: $c = 0$ we obtain $f_{14} = -3ah/4 =$ constant.

Subcase 2.2.3: $b \neq 0$ and $c \neq 0$. Then solving $f_{12} = 0$ we get $\alpha = \pm\dfrac{1}{2}\arccos\dfrac{3a - 2b}{b}$ and solving $f_{14} = 0$ we obtain $\beta = \pm\dfrac{1}{2}\arccos\dfrac{3a - 2c}{c}$.

Assuming that

$$bc \neq 0, \left|\frac{3a-2c}{c}\right| < 1 \quad \text{and} \quad \left|\frac{3a-2b}{b}\right| < 1. \tag{2.16}$$

System (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ given by

$$\left(\sqrt{2h}, \pm\frac{1}{2}\arccos\frac{3a-2b}{b}, 0, \pm\frac{1}{2}\arccos\frac{3a-2c}{c}\right). \tag{2.17}$$

which reduce to two zeros if either $bc \neq 0, \left|\frac{3a-2c}{c}\right| < 1, \left|\frac{3a-2b}{b}\right| = 1$ or $bc \neq 0, \left|\frac{3a-2c}{c}\right| = 1, \left|\frac{3a-2b}{b}\right| < 1$, and to one zero if $bc \neq 0, \left|\frac{3a-2c}{c}\right| = 1, \left|\frac{3a-2b}{b}\right| = 1$.

The Jacobian is

$$J_{f_1(s*)} = -\frac{9}{32}h^4(a-b)(3a-b)(a-c)(3a-c).$$

We summarize the results of this subcase as follow, if (2.16) and the Jacobian 's condition $(a-b)(3a-b)(a-c)(3a-c) \neq 0$ hold, therefore the four solutions (2.17) of system (2.7) provide only one periodic solution of differential system (2.6) because when $\rho = 0$ and $R = 0$ the two solutions of α and β provide the same initial conditions in (2.2). The set of conditions on the parameters is not empty because the value $a = 1, b = 2, c = 2$ satisfy it.

Subcase 2.3: $R = \sqrt{\dfrac{2bh\sin 2\alpha}{b\sin 2\alpha - e\sin 2(\alpha-\beta) - c\sin 2\beta}}$.

Then $r = \sqrt{\dfrac{2eh\sin 2(\alpha-\beta)}{c\sin 2\beta - b\sin 2\alpha + e\sin 2(\alpha-\beta)}}$ and

$$\begin{aligned} f_{12}(r,\alpha,R,\beta) = \;& \frac{h}{D}\big[(3ae - bc - 2be) \\ & \sin 2(\alpha-\beta) + (2bc + be - 3cd)\sin 2\beta + 2b(e-c)\sin 2\alpha\big], \\ f_{14}(r,\alpha,R,\beta) = \;& -\frac{h}{D}\big[(bc - 3ae + 2ce) \\ & \sin 2(\alpha-\beta) + (2bc + ce - 3bf)\sin 2\alpha + 2c(e-b)\sin 2\beta\big]. \end{aligned}$$

Where $D = 4(b\sin 2\alpha - e\sin 2(\alpha-\beta) - c\sin 2\beta)$. To calculate the zeros of this two last functions we need their numerators, $h(\sin(2(\alpha-\beta))(3ae -$

$bc - 2be) + \sin(2\beta)(2bc + be - 3cd) + 2b\sin(2\alpha)(e - c))$ and $h\big(\sin 2(\alpha - \beta)(3ae - bc - 2ce) + 2c(b-e)\sin 2\beta + \sin(2\alpha)(-c(2b+e) + 3bf)\big)$. Expanding the trigonometrical terms of these numerators and using one notation of $\sin\alpha = s$; $\cos\alpha = \pm\sqrt{1-s^2}$; $\sin\beta = S$; $\cos\beta = \pm\sqrt{1-S^2}$ we obtain

$$\begin{aligned} P_{12}(s,S) = \;& 2hs\sqrt{1-s^2}\,(-6aeS^2 + 3ae + 2bcS^2 - 3bc + 4beS^2) \\ & -2hS\sqrt{1-S^2}\big(-6aes^2 + 3ae + 2bcs^2 - 3bc + 4bes^2 \\ & -3be + 3cd\big), \\ P_{14}(s,S) = \;& 2hs\sqrt{1-s^2}\big(-6aeS^2 + 3ae + 2bcS^2 - 3bc + 3bf + 4ceS^2 \\ & -3ce\big) - 2hS\sqrt{1-S^2}\,(-6aes^2 + 3ae + 2bcs^2 - 3bc + 4ces^2)\,. \end{aligned}$$

The other three notations provide the same previous expressions.
$P_{12}(s,S) = 0$ and $P_{14}(s,S) = 0$ implies that $Q_{12}(s,S) = 0$ and $Q_{14}(s,S) = 0$ where

$$\begin{aligned} Q_{12}(s,S) &= 4h^2S^2\,(1-S^2)\,\big(-6aes^2 + 3ae + 2bcs^2 \\ &\quad -3bc + 4bes^2 - 3be + 3cd\big)^2 - 4h^2s^2\,(1-s^2) \\ &\quad (-6aeS^2 + 3ae + 2bcS^2 - 3bc + 4beS^2)^2\,, \\ Q_{14}(s,S) &= 4h^2S^2\,(1-S^2)\,\big(-6aes^2 + 3ae + 2bcs^2 \\ &\quad -3bc + 4ces^2\big)^2 - 4h^2s^2\,(1-s^2)\,\big(-6aeS^2 + 3ae \\ &\quad +2bcS^2 - 3bc + 3bf + 4ceS^2 - 3ce\big)^2. \end{aligned} \tag{2.18}$$

We calculate the resultant of Q_{12} and Q_{14} with respect to s and S, we obtain

$$\begin{aligned} R_{12}(S) &= 47775744h^{16}(-1+S)^4S^8(1+S)^4K^2(S)L^2(S), \\ R_{14}(s) &= 47775744h^{16}(-1+s)^4s^8(1+s)^4M^2(s)N^2(s), \end{aligned} \tag{2.19}$$

with K(S) and M(s) two polynomials of the form $AS^2 + B$ and $Cs^2 + D$ respectively with A, B, C, D constants, and L(S) and N(s) two polynomials of the form $ES^4 + FS^2 + G$ and $Hs^4 + Is^2 + J$ respectively with E, F, G, H, I, J constants. So if we calculate s_0 and S_0 the zeros of R_{12} and R_{14} then (s_0, S_0) is a zero of (3.29).
Solving (3.30) we obtain 81 pairs of (s,S). Only 9 of this pairs are solutions of (3.29). When we calculate (α,β) corresponding to (s,S) solution we find the zeros $z^* = (r^*, \rho^*, \alpha^*, R^*, \beta^*)$ of (2.6) which are
$s^*_{1,\pm}(\sqrt{\frac{2eh}{e+c}}, \sqrt{\frac{2ch}{e+c}}, \pm\frac{\pi}{2}+2k\pi, 0, \pm\frac{\pi}{2}+2l\pi)$; $s^*_{2,\pm}(\sqrt{\frac{2eh}{e+c}}, \sqrt{\frac{2ch}{e+c}}, \pm\frac{\pi}{2}+2m\pi, 0, 0)$; $s^*_{3,\pm}(\sqrt{\frac{2eh}{e-c}}, \sqrt{\frac{2ch}{e-c}}, 0, 0, \pm\frac{\pi}{2}+2n\pi)$; $s^*_{4}(\sqrt{\frac{2eh}{e-c}}, \sqrt{\frac{2ch}{e-c}}, 0, 0, 0)$.
k, l, m, n $\in \mathbb{Z}$. The Jacobian applied in all this zeros is equal to zero. The averaging theory in this case cannot give information.

Subcase 2.4: $\beta = \dfrac{k\pi}{2}$. with $k = 0, 1, 2, 3$.

Due to the periodicity of the cosinus we study the cases $k = 0$ and $k = 2$, and the cases $k = 1$ and $k = 3$ together.

Subcase 2.4.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\beta = 0$ or $\beta = \pi$. Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(r, \alpha, R, \beta) = & \ \frac{1}{8}\big[-6ah + 4bh + (3a - 2b - 3c + 2e)R^2 + (2bh - bR^2 \\ & +eR^2)\cos 2\alpha\big], \\ f_{14}(r, \alpha, R, \beta) = & \ \frac{3}{8}\big[2(-a + c)h + (a - 2c + f)R^2\big]. \end{aligned}$$

Subcase 2.4.1.1: $a-2c+f = 0$. We have $f_{14} = -3(a-c)h/4=$ constant.

Subcase 2.4.1.2: $a - 2c + f \neq 0$. Then solving $f_{14} = 0$ we have $R = \sqrt{\dfrac{2h(a-c)}{a-2c+f}}$. Substituting R in f_{12} and solving $f_{12} = 0$ we get

Subcase 2.4.1.2.1: $bc - ae + ce - bf \neq 0$. Then we get $\alpha = \pm\dfrac{1}{2}\left|\dfrac{3c^2 - 2bc + 2ae - 2ce - 3af + 2bf}{bc - ae + ce - bf}\right|$. Then $r = \sqrt{\dfrac{2h(f-c)}{a-2c+f}}$ and $\rho = 0$.

Supposing that

$$\begin{aligned} & a - 2c + f \neq 0, \quad bc - ae + ce - bf \neq 0, \\ & (f - c)(a - 2c + f) > 0, (a - c)(a - 2c + f) > 0 \quad \text{and} \\ & \left|\frac{3c^2 - 2bc + 2ae - 2ce - 3af + 2bf}{bc - ae + ce - bf}\right| < 1. \end{aligned} \tag{2.20}$$

Then, system (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ and $\alpha^* = \pm\dfrac{1}{2}\left|\dfrac{3c^2 - 2bc + 2ae - 2ce - 3af + 2bf}{bc - ae + ce - bf}\right|$ given by

$$\begin{aligned} & \left(\sqrt{\frac{2h(f-c)}{a-2c+f}}, \alpha^*, \sqrt{\frac{2h(a-c)}{a-2c+f}}, 0\right), \\ & \left(\sqrt{\frac{2h(f-c)}{a-2c+f}}, \alpha^*, \sqrt{\frac{2h(a-c)}{a-2c+f}}, \pi\right). \end{aligned} \tag{2.21}$$

The Jacobian is

$$J_{f_1(s*)} = \frac{9ch^4(a-c)(f-c)(bc-c^2-ae+ce+af-bf)}{16(a-2c+f)^3(bc-ae+ce-bf)^2} (bc-3c^2-ae+ce+3af-bf)(-bc+ae-ce+bf)^2.$$

We conclude the results of this subcase as follows, when (2.20) and the Jacobian 's condition $c(bc-c^2-ae+ce+af-bf)(bc-3c^2-ae+ce+3af-bf) \neq 0$ hold then the four solutions (2.21) of system (2.7) provide only two periodic solutions of differential system (2.6) because when $\rho = 0$ the two solutions of α provide the same initial conditions in (2.2). Note that for $a = -3/2, b = -4, c = -1/2, d = 0, e = 0, f = -1$, the set of conditions on the parameters is not empty.

Subcase 2.4.1.2.2: $bc - ae + ce - bf = 0$. Then $f_{12} = h(3c^2 + 2ae - 2ce - 3af + 2b(-c+f))/(4(a-2c+f))=$ constant.

Subcase 2.4.2: Assume that either $k = 1$ or $k = 3$, i.e. either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$. Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(r,\alpha,R,\beta) = & \frac{1}{8}\big[-6ah+4bh+(3a-2b-c+2e)R^2+(2bh-bR^2 \\ & -eR^2)\cos 2\alpha\big], \\ f_{14}(r,\alpha,R,\beta) = & \frac{1}{8}\big[2(-3a+c)h+(3a-2c+3f)R^2\big]. \end{aligned}$$

Subcase 2.4.2.1: $3a - 2c + 3f = 0$. We have $f_{14} = 2h(-3a+c)h/4=$ constant.

Subcase 2.4.2.2: $3a - 2c + 3f \neq 0$. Then solving $f_{14} = 0$ we have $R = \sqrt{\dfrac{2h(3a-c)}{3a-2c+3f}}$. Substituting R in f_{12} and solving $f_{12} = 0$ we get two subcases.

Subcase 2.4.2.2.1: If $bc + 3ae - ce - 3bf \neq 0$. So $\alpha = \pm\dfrac{1}{2}\left|\dfrac{c^2-2bc+6ae-2ce-9af+6bf}{bc+3ae-ce-3bf}\right|$. Then $r = \sqrt{\dfrac{2h(3f-c)}{3a-2c+3f}}$ and $\rho = 0$.

Considering that

$$\begin{aligned} & 3a-2c+3f \neq 0, \quad bc+3ae-ce-3bf \neq 0, \\ & (3f-c)(3a-2c+3f) > 0, \quad (3a-c)(3a-2c+3f) > 0 \\ & \text{and} \quad \left|\frac{c^2-2bc+6ae-2ce-9af+6bf}{bc+3ae-ce-3bf}\right| < 1. \end{aligned} \tag{2.22}$$

Then, system (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ and $\alpha^* = \pm\frac{1}{2}\left|\frac{c^2 - 2bc + 6ae - 2ce - 9af + 6bf}{bc + 3ae - ce - 3bf}\right|$ given by

$$\begin{array}{c}\left(\sqrt{\frac{2h(3f-c)}{3a-2c+3f}}, \alpha^*, \sqrt{\frac{2h(3a-c)}{3a-2c+3f}}, \frac{\pi}{2}\right),\\ \left(\sqrt{\frac{2h(3f-c)}{3a-2c+3f}}, \alpha^*, \sqrt{\frac{2h(3a-c)}{3a-2c+3f}}, \frac{3\pi}{2}\right).\end{array} \tag{2.23}$$

The Jacobian is

$$\begin{aligned} J_{f_1(s*)} = & \frac{ch^4(3a-c)(3f-c)}{16(3a-2c+3f)^3}(3bc - c^2 - 3ae + ce + 9af - 9bf) \\ & (bc - c^2 - 9ae + 3ce + 9af - 3bf). \end{aligned}$$

If the Jacobian 's condition $c(3bc - c^2 - 3ae + ce + 9af - 9bf)(bc - c^2 - 9ae + 3ce + 9af - 3bf) \neq 0$ and (2.22) hold, then the four solutions (2.23) of system (2.7) provide only two periodic solutions of differential system (2.6) because when $\rho = 0$ the two solutions of α provide the same initial conditions in (2.2). Note that for $a = -1, b = -5/4, c = -4, e = -2, f = -1$ the set of conditions on the parameters is not empty.

Subcase 2.4.2.2.2: If $bc + 3ae - ce - 3bf = 0$, we get $f_{12} = h(c^2 - 2bc + 6ae - 2ce - 9af + 6bf)/(4(3a - 2c + 3f))$= constant.

Case 3: $b \sin 2\alpha = 0$. We have two main subcases: $b = 0$ and $\sin 2\alpha = 0$.

Subcase 3.1: $b = 0$. Then $f_{11} = -crR^2 \sin(2\beta)/8$. Solving $f_{11} = 0$ we obtain the four subcases: $c = 0$, $r = 0$ (studied in case 1), $R = 0$, $\beta = \frac{m\pi}{2}$ with $m \in \mathbb{Z}$.

Subcase 3.1.1: $c = 0$. No information as in subcase 1.1. Hence *in what follows in subcase* 3.1 *we suppose that* $c \neq 0$.

Subcase 3.1.2: $R = 0$. This subcase does not give results because f_{13} and f_{11} will be zero.

Subcase 3.1.3 $\beta = \frac{m\pi}{2}$ with $m \in \mathbb{Z}$. Due to the periodicity of the sinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 3.1.3.1: Assume that either $\beta = 0$ or $\beta = \pi$. So $f_{13} = \dfrac{eR}{8}(r^2 + R^2 - 2h)\sin 2\alpha$. Solving $f_{13} = 0$ we have four subcases, $e = 0$, $R = 0$, $R = \sqrt{2h - r^2}$, $\alpha = \dfrac{n\pi}{2}$ with $n \in \mathbb{Z}$.

Subcase 3.1.3.1.1: $e = 0$. No information as in subcase 1.1. Hence, *in what follows in subcase* 3.1.3.1 *we assume that* $e \neq 0$.

Subcase 3.1.3.1.2: $R = 0$. Then we have

$$\begin{aligned} f_{12} = & \ \frac{3}{8}[2dh - (a+d)r^2], \\ f_{14} = & \ \frac{1}{8}[(-3a + 3c - 2e - e\cos 2\alpha)r^2 + 2eh(2 + \cos 2\alpha)]. \end{aligned}$$

Subcase 3.1.3.1.2.1: $a + d = 0$. We have $f_{12} = 3dh/4$= constant.

Subcase 3.1.3.1.2.2: $a + d \neq 0$. We have $f_{12} = 0 \Rightarrow r = \sqrt{\dfrac{2dh}{a+d}}$. So $\rho = \sqrt{\dfrac{2ah}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ if

Subcase 3.1.3.1.2.2.1: $ae \neq 0$. We get $\alpha = \pm\dfrac{1}{2}\arccos\dfrac{3ad - 3cd - 2ae}{ae}$.

In the case that $\left|\dfrac{3ad - 3cd - 2ae}{ae}\right| < 1$, $ae(a+d) \neq 0$, $d(a+d) > 0$ and $a(a+d) > 0$, system (2.7) has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2ah}{a+d}}$ given by $\left(\sqrt{\dfrac{2dh}{a+d}}, \pm\dfrac{1}{2}\arccos\left(\dfrac{3ad - 3cd - 2ae}{ae}\right), 0, 0\right)$, and $\left(\sqrt{\dfrac{2dh}{a+d}}, \pm\dfrac{1}{2}\arccos\left(\dfrac{3ad - 3cd - 2ae}{ae}\right), 0, \pi\right)$,
which reduce to two zeros when $ae(a+d) \neq 0$, $d(a+d) > 0$, $a(a+d) > 0$ and $\left|\dfrac{3ad - 3cd - 2ae}{ae}\right| = 1$. The Jacobian evaluated on these solutions is zero, so the averaging theory does not give information in this subcase.

Subcase 3.1.3.1.2.2.2: $ae = 0$. We obtain $f_{14} = \dfrac{-3hd(a-c)}{4(a+d)}$= constant.

Subcase 3.1.3.1.3: $R = \sqrt{2h - r^2}$. Studied in the subcase 2.4.1.2.1.

Subcase 3.1.3.1.4: $\alpha = \frac{n\pi}{2}$. Due to the periodicity of the sinus we study the cases $n = 0$ and $n = 2$, and the cases $n = 1$ and $n = 3$ together.

Subcase 3.1.3.1.4.1: Assume that either $n = 0$ or $n = 2$, i.e. $\alpha = 0$ or $\alpha = \pi$.

$$\begin{aligned} f_{12} = & -\frac{3}{8}\left[-2dh + (a+d)r^2 + (c+d-e)R^2\right], \\ f_{14} = & \ \frac{3}{8}\left[2eh + (c-a-e)r^2 - (c+e-f)R^2\right]. \end{aligned}$$

To solve $f_{12} = 0$ we have four subcases to study.

Subcase 3.1.3.1.4.1.1: $a + d = 0$ and $c + d - e = 0$ then $f_{12} = \frac{3}{4}dh=$ constant.

Subcase 3.1.3.1.4.1.2: $a+d = 0$ and $c+d-e \neq 0$. Solving $f_{12} = 0$ with respect to R we obtain $R = \sqrt{\frac{2dh}{c+d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ we have

Subcase 3.1.3.1.4.1.2.1: If $c - a - e \neq 0$, $r = \sqrt{\frac{2h(cd - ce + e^2 - df)}{(c-a-e)(c+d-e)}}$.

Therefore $\rho = \sqrt{\frac{2h(c^2 - ac + ae - cd - ce + df)}{(c-a-e)(c+d-e)}}$.

Supposing that

$$\begin{aligned} & a + d = 0, \quad c + d - e \neq 0, \quad c - a - e \neq 0, \quad d(c + d - e) > 0, \\ & (cd - ce + e^2 - df)(c - a - e)(c + d - e) > 0, \quad \text{and} \\ & (c^2 - ac + ae - cd - ce + df)(c - a - e)(c + d - e) > 0. \end{aligned} \tag{2.24}$$

System (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with

$\rho = \sqrt{\dfrac{2h(c^2 - ac + ae - cd - ce + df)}{(c-a-e)(c+d-e)}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{(c-a-e)(c+d-e)}}, 0, \sqrt{\dfrac{2dh}{c+d-e}}, 0\right),\\ \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{(c-a-e)(c+d-e)}}, \pi, \sqrt{\dfrac{2dh}{c+d-e}}, 0\right),\\ \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{(c-a-e)(c+d-e)}}, 0, \sqrt{\dfrac{2dh}{c+d-e}}, \pi\right),\\ \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{(c-a-e)(c+d-e)}}, \pi, \sqrt{\dfrac{2dh}{c+d-e}}, \pi\right).\end{array} \tag{2.25}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} \; &= \; \frac{9cd^2eh^4}{16(c+d-e)^4(a-c+e)^2}(cd-ce+e^2-df)(c^2-ad+2cd+2ae \\ &\quad -2ce+e^2-af-df)(c^2-ac+ae-cd-ce+df). \end{aligned}$$

With the condition $ce(c^2 - ad + 2cd + 2ae - 2ce + e^2 - af - df) \neq 0$ and (2.24) we have $J_{f_1(S^*)} \neq 0$. The set of conditions on the parameters is not empty because the value $a = 1, c = -2, d = -1, e = -1, f = 0$ satisfy it. We deduce that the four zeros (2.25) of (2.7) provide four periodic solutions of (2.6).

Subcase 3.1.3.1.4.1.2.2: If $c - a - e = 0$, we have $f_{14} = 3h(ce - e^2 - dc + df)/(4(c + d - e))=$ constant.

Subcase 3.1.3.1.4.1.3: $a + d \neq 0$ and $c + d - e = 0$. we solve $f_{12} = 0$ with respect to r then $r = \sqrt{\dfrac{2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have,

Subcase 3.1.3.1.4.1.3.1: If $(c + e - f) \neq 0$, $R = \sqrt{\dfrac{2h(cd + ae - ad)}{(a+d)(c+e-f)}}$.

Therefore $\rho = \sqrt{\dfrac{2h(ac + ad - cd - af)}{(a+d)(c+e-f)}}$.

Supposing that

$$\begin{array}{l} d(a+d) > 0, \quad c+d-e = 0, \quad (cd+ae-ad)(a+d)(c+e-f) > 0 \\ \text{and} \quad (ac+ad-cd-af)(a+d)(c+e-f) > 0. \end{array} \tag{2.26}$$

System (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\dfrac{2h(ac+ad-cd-af)}{(a+d)(c+e-f)}}$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{2dh}{a+d}}, 0, \sqrt{\frac{2h(cd+ae-ad)}{(a+d)(c+e-f)}}, 0\right),\\
&\left(\sqrt{\frac{2dh}{a+d}}, \pi, \sqrt{\frac{2h(cd+ae-ad)}{(a+d)(c+e-f)}}, 0\right),\\
&\left(\sqrt{\frac{2dh}{a+d}}, 0, \sqrt{\frac{2h(cd+ae-ad)}{(a+d)(c+e-f)}}, \pi\right),\\
&\left(\sqrt{\frac{2dh}{a+d}}, \pi, \sqrt{\frac{2h(cd+ae-ad)}{(a+d)(c+e-f)}}, \pi\right).
\end{aligned} \tag{2.27}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} &= \frac{9cdeh^4}{16(a+d)^4(c+e-f)^3}(ad-cd-ae)^2(ac+ad-cd-af)\\
&\quad (c^2-ad+2cd+2ae-2ce+e^2-af-df).
\end{aligned}$$

In case that $ce(c^2-ad+2cd+2ae-2ce+e^2-af-df) \neq 0$ and (2.26) hold, we have $J_{f_1(S^*)} \neq 0$. The set of conditions on the parameters is not empty because the value $a = -3, c = 1, d = -2, e = -1, f = -2$ satisfy it. Moreover the four zeros (2.27) of (2.7) provide four periodic solutions of (2.6).

Subcase 3.1.3.1.4.1.3.2:If $(c+e-f) = 0$, $f_{14} = 3h(cd+ae-ad)/(4(a+d))=$ constant.

Subcase 3.1.3.1.4.1.4: $(a+d)(c+d-e) \neq 0$. Solving $f_{12} = 0$ with respect to r we obtain $r = \sqrt{\dfrac{2dh+R^2(e-c-d)}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have,

Subcase 3.1.3.1.4.1.4.1: If $\Sigma_1 = c^2+2cd-2ce+e^2-df-ad+2ae-af = 0$. Then $f_{14} = 3h(cd+ae-ad)/(4(a+d))=$ constant.

Subcase 3.1.3.1.4.1.4.2: If $\Sigma_1 = c^2+2cd-2ce+e^2-df-ad+2ae-af \neq 0$. Then $R = \sqrt{\dfrac{2h(cd+ae-ad)}{\Sigma_1}}$.

Therefore $r = \sqrt{\dfrac{2h(cd-ce+e^2-df)}{\Sigma_1}}$, and $\rho = \sqrt{\dfrac{2h(c^2-ce+ae-af)}{\Sigma_1}}$.

Assuming that

$$\begin{array}{ll} (a+d)(c+d-e)\neq 0, & (cd-ce+e^2-df)\Sigma_1>0,\\ (cd+ae-ad)\Sigma_1>0, & \text{and} \quad (c^2-ce+ae-af)\Sigma_1>0. \end{array} \tag{2.28}$$

System (2.7) for $n=0$ and $n=2$ has four solutions $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{2h(c^2-ce+ae-af)}{\Sigma_1}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{\Sigma_1}},0,\sqrt{\dfrac{2h(cd+ae-ad)}{\Sigma_1}},0\right),\\ \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{\Sigma_1}},\pi,\sqrt{\dfrac{2h(cd+ae-ad)}{\Sigma_1}},0\right),\\ \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{\Sigma_1}},0,\sqrt{\dfrac{2h(cd+ae-ad)}{\Sigma_1}},\pi\right),\\ \left(\sqrt{\dfrac{2h(cd-ce+e^2-df)}{\Sigma_1}},\pi,\sqrt{\dfrac{2h(cd+ae-ad)}{\Sigma_1}},\pi\right). \end{array} \tag{2.29}$$

The Jacobian is

$$\begin{array}{rcl} J_{f_1(S^*)} & = & \dfrac{9ceh^4}{16\Sigma_1^3}(cd+ae-ad)^2(c^2+ae-ce-af)\\ & & (cd-ce+e^2-df). \end{array}$$

If $ce\neq 0$ and (2.28) hold we have $J_{f_1(S^*)}\neq 0$. The set of conditions on the parameters is not empty because the value $a=-5, c=-2, d=-2, e=-1, f=-\dfrac{5}{2}$ satisfy it. Therefore the four zeros (2.29) of (2.7) provide four periodic solutions of (2.6).

Subcase 3.1.3.1.4.2: Assume that either $\alpha=\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. Then

$$\begin{array}{rl} f_{12}= & \dfrac{1}{8}\big[6dh-3(a+d)r^2-(3c+3d-e)R^2\big],\\ f_{14}= & \dfrac{1}{8}\big[2eh-(3a-3c+e)r^2-(3c+e-3f)R^2\big]. \end{array}$$

Subcase 3.1.3.1.4.2.1: $3c+3d-e=0$ and $a+d=0$. Then we get $f_{12}=3dh/4=$ constant.

Subcase 3.1.3.1.4.2.2: $3c+3d-e\neq 0$ and $a+d=0$. Solving $f_{12}=0$ we get

$R = \sqrt{\dfrac{6dh}{3c+3d-e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r we obtain two subcases.

Subcase 3.1.3.1.4.2.2.1: If $3a-3c+e \neq 0$. We have
$r = \sqrt{\dfrac{2h(-9cd+3ce-e^2+9df)}{(3c+3d-e)(3a-3c+e)}}$,
$\rho = \sqrt{\dfrac{6h(3ac-ae-3c^2+3cd+ce-3df)}{(3c+3d-e)(3a-3c+e)}}$.
Whenever

$$\begin{aligned}&a+d=0, \quad 3a-3c+e\neq 0, \quad d(3c+3d-e)>0,\\ &(-9cd+3ce-e^2+9df)(3c+3d-e)(3a-3c+e)>0 \quad \text{and}\\ &(3ac-ae-3c^2+3cd+ce-3df)(3c+3d-e)(3a-3c+e)>0,\end{aligned} \tag{2.30}$$

system (2.7) has four zeros $S^*=(r,\alpha,R,\beta)$ with
$\rho = \sqrt{\dfrac{6h(3ac-ae-3c^2+3cd+ce-3df)}{(3c+3d-e)(3a-3c+e)}}$ given by

$$\begin{aligned}&\left(\sqrt{\frac{2h(-9cd+3ce-e^2+9df)}{(3c+3d-e)(3a-3c+e)}}, \frac{\pi}{2}, \sqrt{\frac{6dh}{3c+3d-e}}, 0\right),\\ &\left(\sqrt{\frac{2h(-9cd+3ce-e^2+9df)}{(3c+3d-e)(3a-3c+e)}}, \frac{3\pi}{2}, \sqrt{\frac{6dh}{3c+3d-e}}, 0\right),\\ &\left(\sqrt{\frac{2h(-9cd+3ce-e^2+9df)}{(3c+3d-e)(3a-3c+e)}}, \frac{\pi}{2}, \sqrt{\frac{6dh}{3c+3d-e}}, \pi\right),\\ &\left(\sqrt{\frac{2h(-9cd+3ce-e^2+9df)}{(3c+3d-e)(3a-3c+e)}}, \frac{3\pi}{2}, \sqrt{\frac{6dh}{3c+3d-e}}, \pi\right).\end{aligned} \tag{2.31}$$

Its Jacobian is

$$\begin{aligned}J_{f_1(S^*)} = \; & \frac{27cd^2eh^4}{16(3a-3c+e)^2(3c+3d-e)^4}(9cd-3ce+e^2-9df)(3ac\\ & -ae-3c^2+3cd+ce-3df)(9c^2-9ad+18cd+6ae-6ce\\ & +e^2-9af-9df).\end{aligned}$$

In case that $ce(9c^2-9ad+18cd+6ae-6ce+e^2-9af-9df)\neq 0$ and (2.30) hold, Then $J_{f_1(S^*)}\neq 0$ and the four zeros (2.31) of (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a=1, c=-1, d=-1, f=0, e=-1$, satisfy it.

Subcase 3.1.3.1.4.2.2.2: If $3a-3c+e=0$. We have $f_{14}=h(9cd-3ce+e^2-9df)/(4(e-3c-3d))$= constant.

Subcase 3.1.3.1.4.2.3: $3c+3d-e=0$ and $a+d\neq 0$. Solving $f_{12}=0$ we obtain $r=\sqrt{\dfrac{2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R we get

Subcase 3.1.3.1.4.2.3.1: If $(3c+e-3f)\neq 0$. Then $R=\sqrt{\dfrac{2h(3cd+ae-3ad)}{(a+d)(3c+e-3f)}}$ and $\rho=\sqrt{\dfrac{6h(ac+ad-af-cd)}{(a+d)(3c+e-3f)}}$.
Assuming that

$$\begin{array}{l} 3c+3d-e=0, \quad d(a+d)>0, \quad (3cd+ae-3ad)(a+d) \\ (3c+e-3f)>0 \quad \text{and} \quad (ac+ad-af-cd) \\ (a+d)(3c+e-3f)>0. \end{array} \tag{2.32}$$

System (2.7) has four zeros $S^*=(r,\alpha,R,\beta)$ with
$\rho=\sqrt{\dfrac{6h(ac+ad-af-cd)}{(a+d)(3c+e-3f)}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2dh}{a+d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3cd+ae-3ad)}{(a+d)(3c+e-3f)}}, 0\right), \\ \left(\sqrt{\dfrac{2dh}{a+d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3cd+ae-3ad)}{(a+d)(3c+e-3f)}}, 0\right), \\ \left(\sqrt{\dfrac{2dh}{a+d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3cd+ae-3ad)}{(a+d)(3c+e-3f)}}, \pi\right), \\ \left(\sqrt{\dfrac{2dh}{a+d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3cd+ae-3ad)}{(a+d)(3c+e-3f)}}, \pi\right). \end{array} \tag{2.33}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{3cdeh^4}{16(3c+e-3f)^3(a+d)^4}(3ad-3cd-ae)^2(ac+ad-af \\ & -cd)(-9c^2+9ad-18cd-6ae+6ce-e^2+9af+9df). \end{aligned}$$

With the condition that $ce(-9c^2+9ad-18cd-6ae+6ce-e^2+9af+9df)\neq 0$ and (2.32), we have $J_{f_1(S^*)}\neq 0$ and the four zeros (2.33) of (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because it is satisfied for the value $a = -2, c = 2/3, d = -1, e = -1, f = -1$.

Subcase 3.1.3.1.4.2.3.2: $3c + e - 3f = 0$ we get $f_{14} = (3cd - 3ad + ae)h/(4(a + d))=$ constant.

Subcase 3.1.3.1.4.2.4: $(a + d)(3c + 3d - e) \neq 0$ we have $r = \sqrt{\dfrac{6dh - (3c + 3d - e)R^2}{3(a + d)}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.1.3.1.4.2.4.1: If

$$\Sigma_2 = 9c^2 - 9ad + 18cd + 6ae - 6ce + e^2 - 9af - 9df \neq 0,$$

then, $R = \sqrt{\dfrac{6h(-3ad + 3cd + ae)}{\Sigma_2}}$, $r = \sqrt{\dfrac{2h(9cd - 3ce + e^2 - 9df)}{\Sigma_2}}$ and $\rho = \sqrt{\dfrac{6h(3c^2 - ce + ae - 3af)}{\Sigma_2}}$.

Considering that

$$\begin{array}{c} (a + d)(3c + 3d - e) \neq 0, \quad (9cd - 3ce + e^2 - 9df)\Sigma_2 > 0, \\ (-3ad + 3cd + ae)\Sigma_2 > 0 \quad \text{and} \quad (3c^2 - ce + ae - 3af)\Sigma_2 > 0. \end{array} \tag{2.34}$$

System (2.7) has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{6h(3c^2 - ce + ae - 3af)}{\Sigma_2}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{2h(9cd - 3ce + e^2 - 9df)}{\Sigma_2}}, \dfrac{\pi}{2}, \sqrt{\dfrac{6h(-3ad + 3cd + ae)}{\Sigma_2}}, 0\right), \\ \left(\sqrt{\dfrac{2h(9cd - 3ce + e^2 - 9df)}{\Sigma_2}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{6h(-3ad + 3cd + ae)}{\Sigma_2}}, 0\right), \\ \left(\sqrt{\dfrac{2h(9cd - 3ce + e^2 - 9df)}{\Sigma_2}}, \dfrac{\pi}{2}, \sqrt{\dfrac{6h(-3ad + 3cd + ae)}{\Sigma_2}}, \pi\right), \\ \left(\sqrt{\dfrac{2h(9cd - 3ce + e^2 - 9df)}{\Sigma_2}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{6h(-3ad + 3cd + ae)}{\Sigma_2}}, \pi\right). \end{array} \tag{2.35}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{27ceh^4}{16{\Sigma_2}^3}(-3ad + 3cd + ae)^2(3c^2 + ae - ce - 3af)(9cd - 3ce \\ & + e^2 - 9df). \end{aligned}$$

Supposing $ce \neq 0$ and (2.34) hold. This assumption is not empty because the value $a = -2, c = -1, d = -1, e = -1, f = -1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (2.35) of system (2.7) provide four periodic solutions of differential system (2.6).

Subcase 3.1.3.1.4.2.4.2: If $\Sigma_2 = 0$, we get $f_{14} = h(-3ad + 3cd + ae)/(4(a+d))$= constant.

Subcase 3.1.3.2: Assume that either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$. So $f_{13} = -\dfrac{eR}{8}(r^2 + R^2 - 2h)\sin 2\alpha$. We get four subcases solving $f_{13} = 0$, $e = 0$, $R = 0$, $R = \sqrt{2h - r^2}$ and $\alpha = \dfrac{n\pi}{2}$, with $n \in \mathbb{Z}$.

Subcase 3.1.3.2.1: $e = 0$. No information as in subcase 1.1. Hence, *in what follows in subcase* 3.1.3.2.1 *we assume that* $e \neq 0$.

Subcase 3.1.3.2.2: $R = 0$. Then we have

$$\begin{aligned} f_{12} = &\ \frac{3}{8}[2dh - (a+d)r^2], \\ f_{14} = &\ \frac{1}{8}[(-3a + 3c - 2e + e\cos 2\alpha)r^2 + 2eh(2 - \cos 2\alpha)]. \end{aligned}$$

Subcase 3.1.3.2.2.1: $a + d = 0$. We have $f_{12} = 3dh/4$= constant.

Subcase 3.1.3.2.2.2: $a + d \neq 0$. Solving $f_{12} = 0$ we get $r = \sqrt{\dfrac{2dh}{a+d}}$. Then $\rho = \sqrt{\dfrac{2ah}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.1.3.2.2.2.1: If $ae \neq 0$ then $\alpha = \pm\dfrac{1}{2}\arccos\dfrac{-3ad + cd + 2ae}{ae}$.

In the case that $\left|\dfrac{-3ad + cd + 2ae}{ae}\right| < 1$, $ae(a+d) \neq 0$, $d(a+d) > 0$ and $a(a+d) > 0$, system (2.7) has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2ah}{a+d}}$ given by

$$\left(\sqrt{\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\left(\frac{-3ad + cd + 2ae}{ae}\right), 0, \frac{\pi}{2}\right),$$
$$\left(\sqrt{\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\left(\frac{-3ad + cd + 2ae}{ae}\right), 0, \frac{3\pi}{2}\right),$$

which reduce to two zeros when $ae(a+d) \neq 0$, $d(a+d) > 0$, $a(a+d) > 0$

and $\left|\dfrac{-3ad+cd+2ae}{ae}\right| = 1$. But when the Jacobian is evaluated on these solutions it becomes zero so the averaging theory does not give information in this subcase.

Subcase 3.1.3.2.2.2.1: If $ae = 0$, we get $f_{14} = \dfrac{-hd(3a-c)}{4(a+d)} =$ constant.

Subcase 3.1.3.2.3: $R = \sqrt{2h - r^2}$. Studied in the subcase 2.4.2.2.

Subcase 3.1.3.2.4: $\alpha = \dfrac{n\pi}{2}$. Due to the periodicity of the sinus we study the cases $n = 0$ and $n = 2$, and the cases $n = 1$ and $n = 3$ together.

Subcase 3.1.3.2.4.1: Assume that either $\alpha = 0$ or $\alpha = \pi$.

$$\begin{aligned} f_{12} &= \frac{1}{8}\left[6dh - 3(a+d)r^2 - (c+3d-e)R^2\right], \\ f_{14} &= \frac{1}{8}\left[2eh + (c-3a-e)r^2 - (c+e-3f)R^2\right]. \end{aligned}$$

Subcase 3.1.3.2.4.1.1: $a+d = 0$ and $c+3d-e = 0$. Then $f_{12} = \dfrac{3}{4}dh =$ constant.

Subcase 3.1.3.2.4.1.2: If $a+d = 0$ and $c+3d-e \neq 0$, then solving $f_{12} = 0$ we get $R = \sqrt{\dfrac{6dh}{c+3d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.1.3.2.4.1.2.1: If $c-3a-e \neq 0$, $r = \sqrt{\dfrac{2h(3cd-ce+e^2-9df)}{(c-3a-e)(c+3d-e)}}$.

Therefore ρ writes

$$\rho = \sqrt{\frac{2h(c^2-3ac+3ae-3cd-ce+9df)}{(c-3a-e)(c+3d-e)}}.$$

Supposing that

$$\begin{array}{l} a+d=0, \quad d(c+3d-e)>0, \\ (3cd-ce+e^2-9df)(c-3a-e)(c+3d-e)>0, \quad \text{and} \\ (c^2-3ac+3ae-3cd-ce+9df)(c-3a-e)(c+3d-e)>0. \end{array} \tag{2.36}$$

System (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with

$\rho = \sqrt{\dfrac{2h(c^2 - 3ac + 3ae - 3cd - ce + 9df)}{(c-3a-e)(c+3d-e)}}$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{(c-3a-e)(c+3d-e)}}, 0, \sqrt{\frac{6dh}{c+3d-e}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{(c-3a-e)(c+3d-e)}}, \pi, \sqrt{\frac{6dh}{c+3d-e}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{(c-3a-e)(c+3d-e)}}, 0, \sqrt{\frac{6dh}{c+3d-e}}, \frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{(c-3a-e)(c+3d-e)}}, \pi, \sqrt{\frac{6dh}{c+3d-e}}, \frac{3\pi}{2}\right).
\end{aligned} \tag{2.37}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} \quad = \quad & \frac{9cd^2eh^4}{16(c+3d-e)^4(3a-c+e)^2}(3cd - ce + e^2 - 9df)\\
& (c^2 - 9ad + 6cd + 6ae - 2ce + e^2 - 9af - 9df)\\
& (c^2 - 3ac + 3ae - 3cd - ce + 9df).
\end{aligned}$$

If $ce(c^2 - 9ad + 6cd + 6ae - 2ce + e^2 - 9af - 9df) \neq 0$ and (2.36) hold, we have $J_{f_1(S^*)} \neq 0$. Therefore the four zeros (2.37) of (2.7) provide four periodic solutions of (2.6). The set of conditions on the parameters is not empty because the value $a = 1, c = -2, d = -1, e = -1, f = -1/2$ satisfy it.

Subcase 3.1.3.2.4.1.2.2: If $c - 3a - e = 0$, $f_{14} = h(ce - e^2 - 3dc + 9df)/(4(c + 3d - e))$= constant.

Subcase 3.1.3.2.4.1.3: If $a + d \neq 0$ and $c + 3d - e = 0$ then solving $f_{12} = 0$ with respect to r we obtain $r = \sqrt{\dfrac{2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.1.3.2.4.1.3.1: If $c + e - 3f \neq 0$, $R = \sqrt{\dfrac{2h(cd + ae - 3ad)}{(a+d)(c+e-3f)}}$.

Therefore $\rho = \sqrt{\dfrac{2h(ac + 3ad - cd - 3af)}{(a+d)(c+e-3f)}}$.

Supposing that

$$\begin{aligned}
& d(a+d) > 0, \quad c + 3d - e = 0, \quad (cd + ae - 3ad)(a+d)\\
& (c + e - 3f) > 0, \quad \text{and}\\
& (ac + 3ad - cd - 3af)(a+d)(c+e-3f) > 0.
\end{aligned} \tag{2.38}$$

System (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(ac+3ad-cd-3af)}{(a+d)(c+e-3f)}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{2dh}{a+d}}, 0, \sqrt{\dfrac{2h(cd+ae-3ad)}{(a+d)(c+e-3f)}}, \dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{2dh}{a+d}}, \pi, \sqrt{\dfrac{2h(cd+ae-3ad)}{(a+d)(c+e-3f)}}, \dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{2dh}{a+d}}, 0, \sqrt{\dfrac{2h(cd+ae-3ad)}{(a+d)(c+e-3f)}}, \dfrac{3\pi}{2}\right),\\ \left(\sqrt{\dfrac{2dh}{a+d}}, \pi, \sqrt{\dfrac{2h(cd+ae-3ad)}{(a+d)(c+e-3f)}}, \dfrac{3\pi}{2}\right).\end{array} \tag{2.39}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} &= \frac{cdeh^4}{16(a+d)^4(c+e-3f)^3}(3ad-cd-ae)^2(ac+3ad-cd-3af) \\ &\quad (c^2-9ad+6cd+6ae-2ce+e^2-9af-9df). \end{aligned}$$

In the case that $ce(c^2-9ad+6cd+6ae-2ce+e^2-9af-9df) \neq 0$ and (2.38) hold, Then $J_{f_1(S^*)} \neq 0$. Therefore, the four zeros (2.39) of (2.7) provide four periodic solutions of (2.6). The set of conditions on the parameters is not empty because the value $a=-2, c=2, d=-1, e=-1, f=-1$ satisfy it.

Subcase 3.1.3.2.4.1.3.2: $(c+e-3f) = 0$. Then $f_{14} = h(cd+ae-3ad)/(4(a+d))$= constant.

Subcase 3.1.3.2.4.1.4: $(a+d)(c+3d-e) \neq 0$. Solving $f_{12} = 0$ with respect to R we have $R = \sqrt{\dfrac{3(2dh-ar^2+dr^2)}{c+3d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.1.3.2.4.1.4.1: $\Sigma_3 = c^2-9ad+6cd+6ae-2ce+e^2-9af-9df \neq 0$. Then we obtain $r = \sqrt{\dfrac{2h(3cd-ce+e^2-9df)}{\Sigma_3}}$.

Therefore, $R = \sqrt{\dfrac{6h(cd-3ad+ae)}{\Sigma_3}}$, and $\rho = \sqrt{\dfrac{2h(c^2-ce+3ae-9af)}{\Sigma_3}}$. Supposing that

$$\begin{array}{l}(a+d)(c+3d-e) \neq 0, \quad (3cd-ce+e^2-9df)\Sigma_3 > 0, \\ (cd-3ad+ae)\Sigma_3 > 0 \quad \text{and} \quad (c^2-ce+3ae-9af)\Sigma_3 > 0.\end{array} \tag{2.40}$$

Therefore system (2.7) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(c^2 - ce + 3ae - 9af)}{\Sigma_3}}$ given by

$$\begin{gathered}
\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{\Sigma_3}}, 0, \sqrt{\frac{6h(cd - 3ad + ae)}{\Sigma_3}}, \frac{\pi}{2}\right),\\
\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{\Sigma_3}}, \pi, \sqrt{\frac{6h(cd - 3ad + ae)}{\Sigma_3}}, \frac{\pi}{2}\right),\\
\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{\Sigma_3}}, 0, \sqrt{\frac{6h(cd - 3ad + ae)}{\Sigma_3}}, \frac{3\pi}{2}\right),\\
\left(\sqrt{\frac{2h(3cd - ce + e^2 - 9df)}{\Sigma_3}}, \pi, \sqrt{\frac{6h(cd - 3ad + ae)}{\Sigma_3}}, \frac{3\pi}{2}\right).
\end{gathered} \tag{2.41}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} \quad = \quad & \frac{9ceh^4}{16\Sigma_3^3}(3ad - cd - ae)^2(c^2 + 3ae - ce - 9af)\\
& (3cd - ce + e^2 - 9df).
\end{aligned}$$

Assuming that $ce \neq 0$ and (2.40) is satisfied. So we have $J_{f_1(S^*)} \neq 0$. The four zeros (2.41) of (2.7) provide four periodic solutions of (2.6). The set of conditions on the parameters is not empty because the value $a = -2, c = -2, d = -1, e = -1, f = -1$ satisfy it.

Subcase 3.1.3.2.4.1.4.2: $\Sigma_3 = 0$. Then $f_{14} = h(-3cd + ce - e^2 + 9df)/(4(c + 3d - e))$= constant.

Subcase 3.1.3.2.4.2: Assume that either $\alpha = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. Then

$$\begin{aligned}
f_{12} = \quad & \frac{1}{8}\big[6dh - 3(a + d)r^2 - (c + 3d - 3e)R^2\big],\\
f_{14} = \quad & \frac{1}{8}\big[6eh - (3a - c + 3e)r^2 - (c + 3e - 3f)R^2\big].
\end{aligned}$$

Subcase 3.1.3.2.4.2.1: $a + d = 0$ and $c + 3d - 3e = 0$. We get $f_{12} = 3dh/4$= constant.

Subcase 3.1.3.2.4.2.2: $a + d = 0$ and $c + 3d - 3e \neq 0$. Solving $f_{12} = 0$ we get $R = \sqrt{\dfrac{6dh}{c + 3d - 3e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to r we obtain two subcases

Subcase 3.1.3.2.4.2.2.1: If $3a - c + 3e \neq 0$. So, we have
$r = \sqrt{\dfrac{6h(ce - cd - 3e^2 + 3df)}{(c + 3d - 3e)(3a - c + 3e)}}$ and
$\rho = \sqrt{\dfrac{2h(3ac - c^2 - 9ae + 3cd + 3ce - 9df)}{(c + 3d - 3e)(3a - c + 3e)}}$.
Whenever

$$\begin{array}{l} a + d = 0, \quad d(c + 3d - 3e) > 0, \\ (ce - cd - 3e^2 + 3df)(c + 3d - 3e)(3a - c + 3e) > 0 \quad \text{and} \\ (3ac - c^2 - 9ae + 3cd + 3ce - 9df)(c + 3d - e)(3a - c + 3e) > 0, \end{array} \tag{2.42}$$

system (2.7) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\dfrac{2h(3ac - c^2 - 9ae + 3cd + 3ce - 9df)}{(c + 3d - 3e)(3a - c + 3e)}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{6h(ce - cd - 3e^2 + 3df)}{(c + 3d - 3e)(3a - c + 3e)}}, \dfrac{\pi}{2}, \sqrt{\dfrac{6dh}{c + 3d - 3e}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(ce - cd - 3e^2 + 3df)}{(c + 3d - 3e)(3a - c + 3e)}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{6dh}{c + 3d - 3e}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(ce - cd - 3e^2 + 3df)}{(c + 3d - 3e)(3a - c + 3e)}}, \dfrac{\pi}{2}, \sqrt{\dfrac{6dh}{c + 3d - 3e}}, \dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(ce - cd - 3e^2 + 3df)}{(c + 3d - 3e)(3a - c + 3e)}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{6dh}{c + 3d - 3e}}, \dfrac{3\pi}{2}\right). \end{array} \tag{2.43}$$

Its Jacobian

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{27cd^2eh^4}{16(3a - c + 3e)^2(c + 3d - 3e)^4}(cd - ce + 3e^2 - 3df)(3ac \\ & -9ae - c^2 + 3cd + 3ce - 9df)(c^2 - 9ad + 6cd + 18ae - 6ce \\ & +9e^2 - 9af - 9df). \end{aligned}$$

Assuming that $ce(c^2 - 9ad + 6cd + 18ae - 6ce + 9e^2 - 9af - 9df) \neq 0$ and (2.42) hold, therefore we have $J_{f_1(S^*)} \neq 0$ and the four zeros (2.43) of (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a = 1, c = -4, d = -1, e = -1, f = -\dfrac{3}{4}$, satisfy it.

Subcase 3.1.3.2.4.2.2.2: If $3a - c + 3e = 0$. We have $f_{14} = 3h(ce - cd - 3e^2 + 3df)/(4(c + 3d - 3e))$= constant.

Subcase 3.1.3.2.4.2.3: $a+d \neq 0$ and $c+3d-3e=0$. Solving $f_{12}=0$ we obtain $r=\sqrt{\dfrac{2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R if

Subcase 3.1.3.2.4.2.3.1: $c+3e-3f \neq 0$. We obtain
$R=\sqrt{\dfrac{2h(cd+3ae-3ad)}{(a+d)(c+3e-3f)}}$ and $\rho=\sqrt{\dfrac{2h(ac+3ad-3af-cd)}{(a+d)(c+3e-3f)}}$.
Supposing that

$$\begin{array}{l} c+3d-3e=0, \quad d(a+d)>0, \quad (cd+3ae-3ad)(a+d) \\ (c+3e-3f)>0 \quad \text{and} \quad (ac+3ad-3af-cd)(a+d) \\ (c+3e-3f)>0. \end{array} \tag{2.44}$$

System (2.7) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with
$\rho=\sqrt{\dfrac{2h(ac+3ad-3af-cd)}{(a+d)(c+3e-3f)}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2dh}{a+d}},\dfrac{\pi}{2},\sqrt{\dfrac{2h(cd+3ae-3ad)}{(a+d)(c+3e-3f)}},\dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2dh}{a+d}},\dfrac{3\pi}{2},\sqrt{\dfrac{2h(cd+3ae-3ad)}{(a+d)(c+3e-3f)}},\dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2dh}{a+d}},\dfrac{\pi}{2},\sqrt{\dfrac{2h(cd+3ae-3ad)}{(a+d)(c+3e-3f)}},\dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{2dh}{a+d}},\dfrac{3\pi}{2},\sqrt{\dfrac{2h(cd+3ae-3ad)}{(a+d)(c+3e-3f)}},\dfrac{3\pi}{2}\right). \end{array} \tag{2.45}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \quad & \frac{cdeh^4}{16(c+3e-3f)^3(a+d)^4}(3ad-cd-3ae)^2(ac+3ad-3af \\ & -cd)(-c^2+9ad-6cd-18ae+6ce-9e^2+9af+9df). \end{aligned}$$

Considering that $ce(-c^2+9ad-6cd-18ae+6ce-9e^2+9af+9df) \neq 0$ and (2.44) hold, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (2.45) of (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because it is satisfied for the value $a=-3, c=3, d=-2, e=-1, f=-2$.

Subcase 3.1.3.2.4.2.3.2: If $c+3e-3f=0$. We get $f_{14}=(cd-3ad+3ae)h/(4(a+d))=$ constant.

Subcase 3.1.3.2.4.2.4: $(a+d)(c+3d-3e) \neq 0$. We have $R = \sqrt{\dfrac{6dh-3(a+d)r^2}{c+3d-3e}}$. Substituting R in f_{14} and solving $f_{14}=0$ we obtain two subcases

Subcase 3.1.3.2.4.2.4.1: If

$$\Sigma_4 = c^2 - 9ad + 6cd + 18ae - 6ce + 9e^2 - 9af - 9df \neq 0,$$

we have $r = \sqrt{\dfrac{6h(cd-ce+3e^2-3df)}{\Sigma_4}}$, $R = \sqrt{\dfrac{6h(-3ad+cd+3ae)}{\Sigma_4}}$ and $\rho = \sqrt{\dfrac{2h(c^2-3ce+9ae-9af)}{\Sigma_4}}$.

Considering that

$$\begin{array}{ll} (a+d)(c+3d-3e) \neq 0, & (cd-ce+3e^2-3df)\Sigma_4 > 0, \\ (-3ad+cd+3ae)\Sigma_4 > 0 \quad \text{and} & (c^2-3ce+9ae-9af)\Sigma_4 > 0. \end{array} \tag{2.46}$$

System (2.7) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(c^2-3ce+9ae-9af)}{\Sigma_4}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{6h(cd-ce+3e^2-3df)}{\Sigma_4}}, \dfrac{\pi}{2}, \sqrt{\dfrac{6h(-3ad+cd+3ae)}{\Sigma_4}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(cd-ce+3e^2-3df)}{\Sigma_4}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{6h(-3ad+cd+3ae)}{\Sigma_4}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(cd-ce+3e^2-3df)}{\Sigma_4}}, \dfrac{\pi}{2}, \sqrt{\dfrac{6h(-3ad+cd+3ae)}{\Sigma_4}}, \dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(cd-ce+3e^2-3df)}{\Sigma_4}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{6h(-3ad+cd+3ae)}{\Sigma_4}}, \dfrac{3\pi}{2}\right). \end{array} \tag{2.47}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{27ceh^4}{16{\Sigma_4}^3}(3ad-cd-3ae)^2(c^2+9ae-3ce-9af)(cd-ce \\ & +3e^2-3df). \end{aligned}$$

Whenever $ce \neq 0$ and (2.46) hold, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (2.47) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a=-3, c=-4, d=-2, e=-1, f=-19/16$ satisfy it.

Subcase 3.1.3.2.4.2.4.2: If $\Sigma_4 = 0$. Then $f_{14} = 3h(ce - cd - 3e^2 + 3df)/(4(c + 3d - 3e))=$ constant.

Subcase 3.2: $\sin 2\alpha = 0$. Then we have four possibilities to study $\alpha = 0$, $\alpha = \pi$, $\alpha = \dfrac{\pi}{2}$ and $\alpha = \dfrac{3\pi}{2}$.

Subcase 3.2.1: Assume that either $\alpha = 0$ or $\alpha = \pi$. So if $f_{11} = -\dfrac{1}{8}crR^2 \sin 2\beta$. If $f_{11} = 0$ then consequently one of the following four subcases holds $c = 0$, $r = 0$ (studied in case 1), $R = 0$, $\beta = p\pi/2$ with $p \in \mathbb{Z}$.

Subcase 3.2.1.1: $c = 0$. No results as in subcase 1.1. So *in what follows in subcase* 3.2.1 *we assume that* $c \neq 0$.

Subcase 3.2.1.2: $R = 0$. We have

$$\begin{aligned} f_{12} = \;& -\frac{3}{8}\left[(a - 2b + d)r^2 + 2(b - d)h\right], \\ f_{14} = \;& -\frac{1}{8}\big[(3a - 3b - 2c + 2e - c\cos 2\beta + e\cos 2\beta)r^2 \\ & -2eh\cos 2\beta - 4eh + 6bh)\big]. \end{aligned}$$

Subcase 3.2.1.2.1: $a - 2b + d = 0$. Then $f_{12} = \dfrac{3h}{4}(d - b)=$ constant.

Subcase 3.2.1.2.2: $a - 2b + d \neq 0$, solving $f_{12} = 0$ we have $r = \sqrt{\dfrac{2h(d - b)}{a - 2b + d}}$ and $\rho = \sqrt{\dfrac{2h(a - b)}{a - 2b + d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.2.1.2.2.1: $bc - cd - ae + be \neq 0$. we get
$\beta = \pm\dfrac{1}{2}\arccos\left(\Delta_5 = \dfrac{3b^2 - 2bc - 3ad + 2cd + 2ae - 2be}{bc - cd - ae + be}\right)$.
Assuming that

$$\begin{aligned} a - 2b + d \neq 0, \quad bc - cd - ae + be \neq 0, \quad |\Delta_5| < 1, \\ (d - b)(a - 2b + d) > 0 \quad \text{and} \quad (a - b)(a - 2b + d) > 0, \end{aligned} \tag{2.48}$$

.

System (2.7) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(a - b)}{a - 2b + d}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{2h(d - b)}{a - 2b + d}}, 0, 0, \pm\frac{1}{2}\arccos\Delta_5\right), \\ &\left(\sqrt{\frac{2h(d - b)}{a - 2b + d}}, \pi, 0, \pm\frac{1}{2}\arccos\Delta_5\right), \end{aligned} \tag{2.49}$$

which reduce to two zeros if $a-2b+d \neq 0$, $bc-cd-ae+be \neq 0$, $(d-b)(a-2b+d) > 0$, $(a-b)(a-2b+d) > 0$ and $|\Delta_5| = 1$.

The Jacobian is

$$J_{f_1(S^*)} = -\frac{9bh^4}{32(a-2b+d)^3}(a-b)(d-b)(3b^2-bc-3ad+cd+ae -be)(b^2-bc-ad+cd+ae-be).$$

Supposing that $b(3b^2-bc-3ad+cd+ae-be)(b^2-bc-ad+cd+ae-be) \neq 0$ and (2.48) hold. So $J_{f_1(S^*)} \neq 0$ and the four zeros (2.49) of system (2.7) provide two periodic solutions of differential system (2.6) because when $R=0$ the two solutions of β provide the same initial conditions. The set of conditions on the parameters is not empty because it is satisfied for the value $a=0, b=-2, c=-7, d=-1, e=0$.

Subcase 3.2.1.2.2.2: $bc-cd-ae+be=0$. We have $f_{14} = h(3b^2-2bc-3ad+2cd+2ae-2be)/(4(a-2b+d)) =$ constant.

Subcase 3.2.1.3: $\beta = \dfrac{p\pi}{2}$ with $p \in \mathbb{Z}$. Due to the periodicity of the sinus we study the subcases $p=0$ and $p=2$, and the subcases $p=1$ and $p=3$ together.

Subcase 3.2.1.3.1: Assume that either $p=0$ or $p=2$, i.e. $\beta=0$ or $\beta=\pi$.

$$f_{12} = -\frac{1}{8}\left[6(b-d)h + 3(a-2b+d)r^2 - 3(b-c-d+e)R^2\right],$$
$$f_{14} = -\frac{3}{8}\left[2(b-e)h + (a-b-c+e)r^2 - (b-c-e+f)R^2\right].$$

Subcase 3.2.1.3.1.1: $b-c-d+e=0$ and $a-2b+d=0$, $f_{12} = -\dfrac{3h}{4}(b-d)=$ constant.

Subcase 3.2.1.3.1.2: $b-c-d+e=0$ and $a-2b+d \neq 0$. Solving $f_{12}=0$, we obtain $r = \sqrt{\dfrac{2h(d-b)}{a-2b+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R we have two subcases

Subcase 3.2.1.3.1.2.1: $b-c-e+f \neq 0$ we get $R = \sqrt{\dfrac{2hN_1}{\delta_1}}$ and $\rho = \sqrt{\dfrac{2hN_2}{\delta_1}}$ where $N_1 = -b^2-cd+be+bc-ae+ad$, $N_2 = cd+ab-ac-ad+af-bf$ and $\delta_1 = (a-2b+d)(b-c-e+f)$.

Assuming that

$$\begin{array}{l} b-c-d+e=0, \quad a-2b+d\neq 0, \quad b-c-e+f\neq 0, \\ (d-b)(a-2b+d)>0, \quad N_1\delta_1>0 \quad \text{and} \quad N_2\delta_1>0. \end{array} \tag{2.50}$$

System (2.7) for $\beta=0$ and $\beta=\pi$ has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{2hN_2}{\delta_1}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2h(d-b)}{a-2b+d}},0,\sqrt{\dfrac{2hN_1}{\delta_1}},0\right), \\ \left(\sqrt{\dfrac{2h(d-b)}{a-2b+d}},0,\sqrt{\dfrac{2hN_1}{\delta_1}},\pi\right), \\ \left(\sqrt{\dfrac{2h(d-b)}{a-2b+d}},\pi,\sqrt{\dfrac{2hN_1}{\delta_1}},0\right), \\ \left(\sqrt{\dfrac{2h(d-b)}{a-2b+d}},\pi,\sqrt{\dfrac{2hN_1}{\delta_1}},\pi\right). \end{array} \tag{2.51}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{9h^4(b-d)}{16(a-2b+d)^4(b-c-e+f)^3}N_1N_2(b^2-2bc+c^2-ad \\ & +2cd+2ae-2be-2ce+e^2-af+2bf-df)(b^2c^2-b^3c \\ & +b^2cd-bc^2d+ab^2e-abce+bc^2e-abde+acde-c^2de \\ & -ace^2+bce^2-b^2cf+bcdf+abef-b^2ef). \end{aligned}$$

Supposing that $(b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c+b^2cd-bc^2d+ab^2e-abce+bc^2e-abde+acde-c^2de-ace^2+bce^2-b^2cf+bcdf+abef-b^2ef)\neq 0$ and (2.50) hold, thus we have $J_{f_1(S^*)}\neq 0$ and consequently the four zeros (2.51) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because it is satisfied for the value $a=0, b=-3, c=-2, d=-2, e=-1, f=-2$.

Subcase 3.2.1.3.1.2.2: $b-c+e-f=0$ we have $f_{14}=3h(b^2-bc-ad+cd+ae-be)/(4(a-2b+d))=$ constant.

Subcase 3.2.1.3.1.3: $b-c-d+e\neq 0$ and $a-2b+d=0$. Solving $f_{12}=0$, we obtain $R=\sqrt{\dfrac{2h(b-d)}{b-c-d+e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r we have two subcases

Subcase 3.2.1.3.1.3.1: If $a-b-c+e\neq 0$. We get $r=\sqrt{\dfrac{2hN_3}{\delta_2}}$ and $\rho=\sqrt{\dfrac{2hN_4}{\delta_2}}$ where $N_3=e^2+cd-ce-be+bf-df$, $N_4=cb+c^2-cd-ce-ac+ae-bf+df$ and $\delta_2=(b-c-d+e)(a-b-c+e)$.

Assuming that

$$\begin{array}{l} b-c-d+e\neq 0, \quad a-2b+d=0, \quad a-b-c+e\neq 0, \\ (b-d)(b-c-d+e)>0, \quad N_3\delta_2>0 \quad \text{and} \quad N_4\delta_2>0. \end{array} \tag{2.52}$$

System (2.7) for $\beta=0$ and $\beta=\pi$ has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{2hN_4}{\delta_2}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2hN_3}{\delta_2}},0,\sqrt{\dfrac{2h(b-d)}{b-c-d+e}},0\right), \\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},\pi,\sqrt{\dfrac{2h(b-d)}{b-c-d+e}},0\right), \\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},0,\sqrt{\dfrac{2h(b-d)}{b-c-d+e}},\pi\right), \\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},\pi,\sqrt{\dfrac{2h(b-d)}{b-c-d+e}},\pi\right). \end{array} \tag{2.53}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & -\frac{9h^4(b-d)}{16(b-c-d+e)^4(a-b-c+e)^3}N_3N_4(2bc-b^2-c^2 \\ & +ad-2cd-2ae+2be+2ce-e^2+af-2bf+df)(b^2ce \\ & -bc^2d+bc^2e+acde-c^2de-abe^2-bce^2+cde^2-b^2cf \\ & +bcdf+b^2ef-bdef). \end{aligned}$$

Whenever $(2bc-b^2-c^2+ad-2cd-2ae+2be+2ce-e^2+af-2bf+df)(b^2ce-bc^2d+bc^2e+acde-c^2de-abe^2-bce^2+cde^2-b^2cf+bcdf+b^2ef-bdef)\neq 0$ and (2.52) hold, hence we have $J_{f_1(S^*)}\neq 0$. Therefore the four zeros (2.53) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because it is satisfied for the value $a=-2, b=-3, c=-2, d=-4, e=-1, f=-3$.

Subcase 3.2.1.3.1.3.1: If $a-b-c+e=0$. We have $f_{14}=3h(bf+cd-df-be-ce+e^2)/(4(b-c-d+e))$= constant.

Subcase 3.2.1.3.1.4: $(b-c-d+e)(a-2b+d) \neq 0$. Solving $f_{12}=0$ we get $r = \sqrt{\dfrac{2h(d-b)+(b-c-d+e)R^2}{a-2b+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R if

Subcase 3.2.1.3.1.4.1:

$$\Sigma_5 = b^2 - 2bc + c^2 - ad + 2cd + 2ae - 2be - 2ce + e^2 - af + 2bf - df \neq 0,$$

we get $r = \sqrt{\dfrac{2hN_5}{\Sigma_5}}$, $R = \sqrt{\dfrac{2hN_6}{\Sigma_5}}$ and $\rho = \sqrt{\dfrac{2hN_7}{\Sigma_5}}$, where $N_6 = b^2 - ad + cd + ae - bc - be$, $N_5 = cd - be - ce + e^2 + bf - df$, $N_7 = c^2 - ce + ae - af - bc + bf$.

In the case that

$$\begin{array}{l} (b-c-d+e)(a-2b+d) \neq 0, \quad N_5\Sigma_5 > 0, \\ N_6\Sigma_5 > 0 \quad \text{and} \quad N_7\Sigma_5 > 0, \end{array} \tag{2.54}$$

hold, then system (2.7) for $\beta = 0$ and $\beta = \pi$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_7}{\Sigma_5}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2hN_5}{\Sigma_5}}, 0, \sqrt{\dfrac{2hN_6}{\Sigma_5}}, 0\right), \\ \left(\sqrt{\dfrac{2hN_5}{\Sigma_5}}, 0, \sqrt{\dfrac{2hN_6}{\Sigma_5}}, \pi\right), \\ \left(\sqrt{\dfrac{2hN_5}{\Sigma_5}}, \pi, \sqrt{\dfrac{2hN_6}{\Sigma_5}}, 0\right), \\ \left(\sqrt{\dfrac{2hN_5}{\Sigma_5}}, \pi, \sqrt{\dfrac{2hN_6}{\Sigma_5}}, \pi\right). \end{array} \tag{2.55}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \;& \frac{9h^4}{16\Sigma_5^3} N_5 N_6 N_7 (bc^2d - b^2ce - bc^2e - acde + c^2de + abe^2 \\ & + ace^2 - bce^2 + b^2cf - bcdf - abef + b^2ef). \end{aligned}$$

Considering that $(bc^2d - b^2ce - bc^2e - acde + c^2de + abe^2 + ace^2 - bce^2 + b^2cf - bcdf - abef + b^2ef) \neq 0$ and (2.54) hold. We get $J_{f_1(S^*)} \neq 0$ and then the four zeros (2.55) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a=-13, b=-4, c=-3, d=-5, e=-1, f=-10$, satisfy it.

Subcase 3.2.1.3.1.4.2: $\Sigma_5 = 0$. we have $f_{14} = 3h(b^2 - bc - ad + cd + ae - be)/(4(a - 2b + d))$= constant.

Subcase 3.2.1.3.2: Assume that either $p = 1$ or $p = 3$, i.e. $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$.

$$\begin{aligned} f_{12} = &\ \frac{1}{8}\big[6(d-b)h - 3(a-2b+d)r^2 + (3b - c - 3d + e)R^2\big], \\ f_{14} = &\ \frac{1}{8}\big[2h(e-3b) - (3a - 3b - c + e)r^2 + (3b - c - e + 3f)R^2\big]. \end{aligned}$$

Subcase 3.2.1.3.2.1: $3b - c - 3d + e = 0$ and $a - 2b + d = 0$, $f_{12} = -\dfrac{3h}{4}(b - d)$= constant.

Subcase 3.2.1.3.2.2: If $3b - c - 3d + e = 0$ and $a - 2b + d \neq 0$, solving $f_{12} = 0$ we get $r = \sqrt{\dfrac{2h(d-b)}{a-2b+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.2.1.3.2.2.1: If $3b - c - e + 3f \neq 0$. Then we obtain $R = \sqrt{\dfrac{2hN_8}{\delta_3}}$ where $\delta_3 = (a - 2b + d)(3b - c - e + 3f)$, $N_8 = -3b^2 + 3ad - cd - ae + bc + be$ and $\rho = \sqrt{\dfrac{2hN_9}{\delta_3}}$ where $N_9 = (cd - 3bf + 3ab - ac - 3ad + 3af)$.

If we have

$$\begin{aligned} 3b - c - 3d + e = 0, \quad a - 2b + d \neq 0, \quad 3b - c - e + 3f \neq 0, \\ (d-b)(a-2b+d) > 0, \quad \delta_3 N_8 > 0 \quad \text{and} \quad \delta_3 N_9 > 0, \end{aligned} \tag{2.56}$$

then system (2.7) for $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_9}{\delta_3}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{2h(d-b)}{a-2b+d}}, 0, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{2h(d-b)}{a-2b+d}}, 0, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{3\pi}{2}\right), \\ &\left(\sqrt{\frac{2h(d-b)}{a-2b+d}}, \pi, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{2h(d-b)}{a-2b+d}}, \pi, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{3\pi}{2}\right). \end{aligned} \tag{2.57}$$

The Jacobian is

$$\begin{aligned}J_{f_1(S^*)} = \; & \frac{h^4(b-d)N_8N_9}{16(a-2b+d)^4(3b-c-e+3f)^3}(9b^2-6bc+c^2-9ad \\ & +6cd+6ae-6be-2ce+e^2-9af+18bf-9df)(b^2c^2 \\ & -3b^3c+3b^2cd-bc^2d+3ab^2e-abce+4b^2ce-bc^2e-3abde \\ & -3acde+c^2de+ace^2-bce^2-3b^2cf+3bcdf+3abef \\ & -3b^2ef).\end{aligned}$$

Supposing that $(9b^2-6bc+c^2-9ad+6cd+6ae-6be-2ce+e^2-9af+18bf-9df)(b^2c^2-3b^3c+3b^2cd-bc^2d+3ab^2e-abce+4b^2ce-bc^2e-3abde-3acde+c^2de+ace^2-bce^2-3b^2cf+3bcdf+3abef-3b^2ef) \neq 0$ and (2.56) hold. As a result of that, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (2.57) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because it is satisfied for the value $a=-1, b=-4/3, c=-2, d=-1, e=-1, f=-2$.

Subcase 3.2.1.3.2.2.2: If $3b-c-e+3f=0$. We have $f_{14} = (3b^2+cd-bc-be-3ad+ae)h/(4(a-2b+d))=$ constant.

Subcase 3.2.1.3.2.3: $3b-c-3d+e \neq 0$ and $a-2b+d=0$. Solving $f_{12}=0$ we get $R = \sqrt{\dfrac{6h(b-d)}{3b-c-3d+e}}$. Substituting R in f_{14} and solving $f_{14}=0$ we have two subcases

Subcase 3.2.1.3.2.3.1: If $3a-3b-c+e \neq 0$. Then we obtain $r = \sqrt{\dfrac{2hN_{10}}{\delta_4}}$ where $\delta_4 = (3a-3b-c+e)(3b-c-3d+e)$, $N_{10} = e^2+3cd-ce-3be+9bf-9df$ and $\rho = \sqrt{\dfrac{2hN_{11}}{\delta_4}}$ where $N_{11} = (3bc+c^2-3cd-ce-3ac+3ae-9bf+9df)$.

If we have

$$\begin{array}{l} 3b-c-3d+e \neq 0, \quad a-2b+d=0, \quad 3a-3b-c+e \neq 0, \\ (b-d)(3b-c-3d+e) > 0, \quad \delta_4 N_{10} > 0 \quad \text{and} \quad \delta_4 N_{11} > 0, \end{array} \tag{2.58}$$

then system (2.7) for $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$

with $\rho = \sqrt{\dfrac{2hN_{11}}{\delta_4}}$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, 0, \sqrt{\frac{6h(b-d)}{3b-c-3d+e}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, \pi, \sqrt{\frac{6h(b-d)}{3b-c-3d+e}}, \frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, 0, \sqrt{\frac{6h(b-d)}{3b-c-3d+e}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, \pi, \sqrt{\frac{6h(b-d)}{3b-c-3d+e}}, \frac{3\pi}{2}\right).
\end{aligned} \tag{2.59}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} = \; & \frac{9h^4(b-d)N_{10}N_{11}}{16(3b-c-3d+e)^4(3a-3b-c+e)^3}(6bc-9b^2-c^2\\
& +9ad-6cd-6ae+6be+2ce-e^2+9af-18bf+9df)\\
& (bc^2d-4abce+3b^2ce+bc^2e+3acde-4bcde-c^2de+abe^2\\
& -bce^2+cde^2+3b^2cf-3bcdf-3b^2ef+3bdef).
\end{aligned}$$

With the condition that $(6bc-9b^2-c^2+9ad-6cd-6ae+6be+2ce-e^2+9af-18bf+9df)(bc^2d-4abce+3b^2ce+bc^2e+3acde-4bcde-c^2de+abe^2-bce^2+cde^2+3b^2cf-3bcdf-3b^2ef+3bdef) \neq 0$ and (2.58) hold, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (2.59) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters of this subcase is not empty because it is satisfied for the value $a=-1/2, b=-1, c=-2, d=-3/2, e=-1, f=-1$.

Subcase 3.2.1.3.2.3.2: $3a-3b-c+e=0$. We get $f_{14} = (3cd-3be-ce+e^2+9bf-9df)h/4(3b-c-3d+e)=$ constant.

Subcase 3.2.1.3.2.4: $(3b-c-3d+e)(a-2b+d) \neq 0$ we have $r = \sqrt{\dfrac{6h(d-b)+(3b-c-3d+e)R^2}{3(a-2b+d)}}$. Substituting r in f_{14} and solving $f_{14}=0$ if

Subcase 3.2.1.3.2.4.1:

$\Sigma_6 = 9b^2-6bc+c^2-9ad+6cd+6ae-6be-2ce+e^2-9af+18bf-9df \neq 0$

so $r = \sqrt{\dfrac{2hN_{12}}{\Sigma_6}}$, $R = \sqrt{\dfrac{6hN_{13}}{\Sigma_6}}$ and $\rho = \sqrt{\dfrac{2hN_{14}}{\Sigma_6}}$ where $N_{12} = e^2 + 3cd - ce - 3be + 9bf - 9df$, $N_{13} = 3b^2 - bc - 3ad + cd + ae - be$, $N_{14} = c^2 - 3bc + 3ae - ce - 9af + 9bf$.

Supposing that

$$\begin{array}{l} (3b - c - 3d + e)(a - 2b + d) \neq 0, \\ N_{12}\Sigma_6 > 0, \quad N_{13}\Sigma_6 > 0 \quad \text{and} \quad N_{14}\Sigma_6 > 0. \end{array} \tag{2.60}$$

System (2.7) for $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_{14}}{\Sigma_6}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{2hN_{12}}{\Sigma_6}}, 0, \sqrt{\dfrac{6hN_{13}}{\Sigma_6}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{12}}{\Sigma_6}}, 0, \sqrt{\dfrac{6hN_{13}}{\Sigma_6}}, \dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{12}}{\Sigma_6}}, \pi, \sqrt{\dfrac{6hN_{13}}{\Sigma_6}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{12}}{\Sigma_6}}, \pi, \sqrt{\dfrac{6hN_{13}}{\Sigma_6}}, \dfrac{3\pi}{2}\right). \end{array} \tag{2.61}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{9h^4}{16\Sigma_6^3} N_{12}N_{13}N_{14}(bc^2d - 5b^2ce + bc^2e + 3acde - c^2de + abe^2 \\ & -ace^2 + bce^2 + 3b^2cf - 3bcdf - 3abef + 3b^2ef). \end{aligned}$$

Assuming that $(bc^2d - 5b^2ce + bc^2e + 3acde - c^2de + abe^2 - ace^2 + bce^2 + 3b^2cf - 3bcdf - 3abef + 3b^2ef) \neq 0$ and (2.60) hold. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (2.61) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a = -10, b = -4, c = -65/4, d = -25, e = -1, f = -8$ satisfy it.

Subcase 3.2.1.3.2.4.2: $\Sigma_6 = 0$ then $f_{14} = hN_{13}/(4(a - 2b + d))$= constant.

Subcase 3.2.2: If either $\alpha = \dfrac{\pi}{2}$ or $\alpha = \dfrac{3\pi}{2}$. So $f_{11} = -\dfrac{1}{8}crR^2 \sin 2\beta$. Then if $f_{11} = 0$ we have one of the following four subcases $c = 0$, $r = 0$ (studied in case 1), $R = 0$, $\beta = q\pi/2$ with $q \in \mathbb{Z}$.

Subcase 3.2.2.1: $c = 0$. No information about the periodic orbits as in subcase 1.1. So *in what follows in subcase* 3.2.2 *we assume that* $c \neq 0$.

Subcase 3.2.2.2: $R = 0$. Then

$$\begin{aligned} f_{12} = & -\frac{1}{8}\left[(3a-2b+3d)r^2 - 2(3d-b)h\right], \\ f_{14} = & -\frac{1}{8}\big[(3a-b-2c+2e-(c+e)\cos 2\beta)r^2 + 2h(b-2e+ \\ & e\cos 2\beta)\big]. \end{aligned}$$

Subcase 3.2.2.2.1: $3a-2b+3d = 0$, $f_{12} = \dfrac{h}{4}(3d-b) =$ constant.

Subcase 3.2.2.2.2: $3a-2b+3d \neq 0$. Solving $f_{12} = 0$ we get $r = \sqrt{\dfrac{2h(3d-b)}{3a-2b+3d}}$ and $\rho = \sqrt{\dfrac{2h(3a-b)}{3a-2b+3d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.2.2.2.2.1: If $3cd-3ae-bc+be \neq 0$. We have
$\beta = \pm\dfrac{1}{2}\arccos\left(\Delta_6 = \dfrac{b^2-2bc-9ad+6cd+6ae-2be}{bc-3cd+3ae-be}\right)$.
Whenever

$$\begin{aligned} & 3a-2b+3d \neq 0, \quad 3cd-3ae-bc+be \neq 0, \\ & (3d-b)(3a-2b+3d) > 0, \quad (3a-b)(3a-2b+3d) > 0 \\ & \quad \text{and} \quad |\Delta_6| < 1, \end{aligned} \tag{2.62}$$

system (2.7) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(3a-b)}{3a-2b+3d}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{2h(3d-b)}{3a-2b+3d}}, \frac{\pi}{2}, 0, \pm\frac{1}{2}\arccos\Delta_6\right), \\ &\left(\sqrt{\frac{2h(3d-b)}{3a-2b+3d}}, \frac{3\pi}{2}, 0, \pm\frac{1}{2}\arccos\Delta_6\right), \end{aligned} \tag{2.63}$$

which reduce to two zeros if $3a-2b+3d \neq 0$, $3cd-3ae-bc+be \neq 0$, $(3d-b)(3a-2b+3d) > 0$, $(3a-b)(3a-2b+3d) > 0$ and $|\Delta_6| = 1$.

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{bh^4}{32(3a-2b+3d)^3}(3a-b)(b-3d)(b^2-3bc-9ad+9cd \\ & +3ae-be)(b^2-bc-9ad+3cd+9ae-3be). \end{aligned}$$

Assuming that $b(b^2-3bc-9ad+9cd+3ae-be)(b^2-bc-9ad+3cd+9ae-3be) \neq 0$ and (2.62) hold. Consequently, we get $J_{f_1(S^*)} \neq 0$ and the four zeros (2.63) of system (2.7) provide only two periodic solutions of differential system (2.6) because when $R = 0$ the two solutions of β provide the same initial conditions in (2.2). The set of conditions on the parameters is not empty because the value $a = -1, b = -2, c = -3, d = -1, e = 0, f = 0$.

Subcase 3.2.2.2.2.2: If $3cd - 3ae - bc + be = 0$. Then we have
$f_{14} = \dfrac{h(b^2 - 9ad + 6cd + 6ae - 2bc - 2be)}{4(3a - 2b + 3d)} =$ constant.

Subcase 3.2.2.3: $\beta = \dfrac{q\pi}{2}$ with $q \in \mathbb{Z}$. Then due to the periodicity of the sinus we study the subcases $q = 0$ and $q = 2$, and the subcases $q = 1$ and $q = 3$ together.

Subcase 3.2.2.3.1: Assume that either $q = 0$ or $q = 2$, i.e. either $\beta = 0$ or $\beta = \pi$.

$$\begin{aligned} f_{12} = & -\frac{1}{8}\left[2(b-3d)h + (3a-2b+3d)r^2 - (b-3c-3d+e)R^2\right], \\ f_{14} = & -\frac{1}{8}\left[2(b-e)h + (3a-b-3c+e)r^2 - (b-3c-e+3f)R^2\right]. \end{aligned}$$

Subcase 3.2.2.3.1.1: $b - 3c - 3d + e = 0$ and $3a - 2b + 3d = 0$, $f_{12} = \dfrac{1}{4}(3d - b)h =$ constant.

Subcase 3.2.2.3.1.2: $b - 3c - 3d + e = 0$ and $3a - 2b + 3d \neq 0$. Solving $f_{12} = 0$, we obtain $r = \sqrt{\dfrac{2h(3d - b)}{3a - 2b + 3d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ with respect to R if

Subcase 3.2.2.3.1.2.1: $b - 3c - e + 3f \neq 0$ we get
$R = \sqrt{\dfrac{2h(-9cd - b^2 + 3bc + be + 9ad - 3ae)}{(3a - 2b + 3d)(b - 3c - e + 3f)}}$ and
$\rho = \sqrt{\dfrac{6h(3cd + ab - 3ac - 3ad + 3af - bf)}{(3a - 2b + 3d)(b - 3c - e + 3f)}}$.
Considering that

$$\begin{aligned} & b - 3c - 3d + e = 0, \quad 3a - 2b + 3d \neq 0, \quad b - 3c - e + 3f \neq 0, \\ & (-9cd - b^2 + 3bc + be + 9ad - 3ae)(3a - 2b + 3d) \\ & (b - 3c - e + 3f) > 0, \quad (3d - b)(3a - 2b + 3d) > 0 \quad \text{and} \\ & (3cd + ab - 3ac - 3ad + 3af - bf)(3a - 2b + 3d) \\ & (b - 3c - e + 3f) > 0. \end{aligned} \tag{2.64}$$

System (2.7) for $\beta = 0$ and $\beta = \pi$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\dfrac{6h(3cd + ab - 3ac - 3ad + 3af - bf)}{(3a - 2b + 3d)(b - 3c - e + 3f)}}$ and
$R = \sqrt{\dfrac{2h(-9cd - b^2 + 3bc + be + 9ad - 3ae)}{(3a - 2b + 3d)(b - 3c - e + 3f)}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{2h(3d-b)}{3a-2b+3d}}, \dfrac{\pi}{2}, R, 0\right),\\ \left(\sqrt{\dfrac{2h(3d-b)}{3a-2b+3d}}, \dfrac{\pi}{2}, R, \pi\right),\\ \left(\sqrt{\dfrac{2h(3d-b)}{3a-2b+3d}}, \dfrac{3\pi}{2}, R, 0\right),\\ \left(\sqrt{\dfrac{2h(3d-b)}{3a-2b+3d}}, \dfrac{3\pi}{2}, R, \pi\right).\end{array} \tag{2.65}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & -\frac{3h^4(3d-b)(b^2 - 3bc - 9ad + 9cd + 3ae - be)}{16(3a - 2b + 3d)^4(b - 3c - e + 3f)^3}(ab - 3ac \\ & -3ad + 3cd + 3af - bf)(b^2 - 6bc + 9c^2 - 9ad + 18cd \\ & +6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df)(b^3c - 3b^2c^2 \\ & -3b^2cd + 9bc^2d + 3ab^2e - 9abce - 3bc^2e - 9abde - 9acde \\ & +9c^2de + 12bcde + 3ace^2 - bce^2 + 3b^2cf - 9bcdf + 9abef \\ & -3b^2ef). \end{aligned}$$

In the case that $(b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df)(b^3c - 3b^2c^2 - 3b^2cd + 9bc^2d + 3ab^2e - 9abce - 3bc^2e - 9abde - 9acde + 9c^2de + 12bcde + 3ace^2 - bce^2 + 3b^2cf - 9bcdf + 9abef - 3b^2ef) \neq 0$ and (2.64) hold, then $J_{f_1(S^*)} \neq 0$ and the four zeros (2.65) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters in not empty because for the value $a = 2, b = -5, c = -1, d = -1, e = -1, f = -1/2$ it is satisfied.

Subcase 3.2.2.3.1.2.2: If $b - 3c - e + 3f = 0$. We have $f_{14} = (b^2 - 3bc - 9ad + 9cd + 3ae - be)/(4(3a - 2b + 3d))$

Subcase 3.2.2.3.1.3: $b - 3c - 3d + e \neq 0$ and $3a - 2b + 3d = 0$. Solving

$f_{12}=0$, we obtain $R=\sqrt{\dfrac{2h(b-3d)}{b-3c-3d+e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r we have two subcases

Subcase 3.2.2.3.1.3.1: If $3a-b-3c+e\neq 0$. Then we get $r=\sqrt{\dfrac{2hN_{15}}{\delta_5}}$ and $\rho=\sqrt{\dfrac{6hN_{16}}{\delta_5}}$ where $N_{15}=e^2+9cd-be-3ce+3bf-9df$, $N_{16}=-3ac+ae+bc+3c^2-3cd-ce-bf+3df$ and $\delta_5=(b-3c-3d+e)(3a-b-3c+e)$.

Supposing that

$$\begin{array}{c} b-3c-3d+e\neq 0,\quad 3a-2b+3d=0,\\ 3a-b-3c+e\neq 0,\quad (b-3d)(b-3c-3d+e)>0,\\ \delta_5 N_{15}>0 \quad\text{and}\quad \delta_5 N_{16}>0. \end{array} \tag{2.66}$$

System (2.7) for $\beta=0$ and $\beta=\pi$ has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{6hN_{16}}{\delta_5}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2hN_{15}}{\delta_5}},\dfrac{\pi}{2},\sqrt{\dfrac{2h(b-3d)}{b-3c-3d+e}},0\right),\\ \left(\sqrt{\dfrac{2hN_{15}}{\delta_5}},\dfrac{\pi}{2},\sqrt{\dfrac{2h(b-3d)}{b-3c-3d+e}},\pi\right),\\ \left(\sqrt{\dfrac{2hN_{15}}{\delta_5}},\dfrac{3\pi}{2},\sqrt{\dfrac{2h(b-3d)}{b-3c-3d+e}},0\right),\\ \left(\sqrt{\dfrac{2hN_{15}}{\delta_5}},\dfrac{3\pi}{2},\sqrt{\dfrac{2h(b-3d)}{b-3c-3d+e}},\pi\right). \end{array} \tag{2.67}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{3h^4(b-3d)N_{15}N_{16}}{16(b-3c-3d+e)^4(3a-b-3c+e)^3)}(b^2-6bc+9c^2\\ & -9ad+18cd+6ae-2be-6ce+e^2-9af+6bf-9df)\\ & (9bc^2d+12abce-5b^2ce-15bc^2e-9acde+12bcde+9c^2de\\ & -3abe^2+5bce^2-3cde^2+3b^2cf-9bcdf+3b^2ef-9bdef). \end{aligned}$$

Supposing that $(b^2-6bc+9c^2-9ad+18cd+6ae-2be-6ce+e^2-9af+6bf-9df)(9bc^2d+12abce-5b^2ce-15bc^2e-9acde+12bcde+9c^2de-3abe^2+5bce^2-3cde^2+3b^2cf-9bcdf+3b^2ef-9bdef)\neq 0$ and (2.66) hold. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (2.67) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a = 5/3, b = -2, c = -1, d = -3, e = -1, f = -1$ satisfy it.

Subcase 3.2.2.3.1.3.2: If $3a - b - 3c + e = 0$ then
$f_{14} = h(9cd - be - 3ce + e^2 - 3bf - 9df)/(4(b - 3c - 3d + e))=$ constant.

Subcase 3.2.2.3.1.4: $(b - 3c - 3d + e)(3a - 2b + 3d) \neq 0$. Solving $f_{12} = 0$ we get $R = \sqrt{\dfrac{2h(b - 3d) + (3a - 2b + 3d)r^2}{b - 3c - 3d + e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to r if

Subcase 3.2.2.3.1.4.1:

$\Sigma_7 = b^2 - 6bc + 9c^2 - 9ad + 18cd + 6ae - 2be - 6ce + e^2 - 9af + 6bf - 9df \neq 0,$

we get $r = \sqrt{\dfrac{2hN_{17}}{\Sigma_7}}$, $R = \sqrt{\dfrac{2hN_{18}}{\Sigma_7}}$ and $\rho = \sqrt{\dfrac{6hN_{19}}{\Sigma_7}}$ where $N_{17} = 9cd - be - 3ce + e^2 + 3bf - 9df$, $N_{18} = b^2 + 9cd - 3bc - be - 9ad + 3ae$ and $N_{19} = 3c^2 - bc - ce + ae - 3af + bf$.

Considering that

$$\begin{array}{ll} (b - 3c - 3d + e)(3a - 2b + 3d) \neq 0, & N_{17}\Sigma_7 > 0, \\ N_{18}\Sigma_7 > 0 \quad \text{and} \quad N_{19}\Sigma_7 > 0. & \end{array} \tag{2.68}$$

System (2.7) for $\beta = 0$ and $\beta = \pi$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{6hN_{19}}{\Sigma_7}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{2hN_{17}}{\Sigma_7}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2hN_{18}}{\Sigma_7}}, 0\right), \\ \left(\sqrt{\dfrac{2hN_{17}}{\Sigma_7}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2hN_{18}}{\Sigma_7}}, \pi\right), \\ \left(\sqrt{\dfrac{2hN_{17}}{\Sigma_7}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2hN_{18}}{\Sigma_7}}, 0\right), \\ \left(\sqrt{\dfrac{2hN_{17}}{\Sigma_7}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2hN_{18}}{\Sigma_7}}, \pi\right). \end{array} \tag{2.69}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{9h^4}{16\Sigma_7^3} N_{17}N_{18}N_{19}(3bc^2d + b^2ce - 5bc^2e - 3acde + 3c^2de \\ & -abe^2 + ace^2 + bce^2 + b^2cf - 3bcdf + 3abef - b^2ef). \end{aligned}$$

Wherever $(3bc^2d + b^2ce - 5bc^2e - 3acde + 3c^2de - abe^2 + ace^2 + bce^2 + b^2cf - 3bcdf + 3abef - b^2ef) \neq 0$ and (2.68) hold, $J_{f_1(S^*)} \neq 0$. Therefore the four zeros (2.69) of (2.7) provide four periodic solutions of (2.6). The set of conditions on the parameters is not empty because the value $a = -68, b = -259/16, c = -4, d = -21, e = -1, f = -92$ satisfy it.

Subcase 3.2.2.3.1.4.2: $\Sigma_7 = 0$ we have $f_{14} = hN_{17}/(4(b-3c-3d+e))=$ constant.

Subcase 3.2.2.3.2: Assume that either $q = 1$ or $q = 3$, i.e. either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$. Then

$$\begin{aligned} f_{12} = &\ -\frac{1}{8}\big[2h(b-3d) + (3a-2b+3d)r^2 - (b-c-3d+3e)R^2\big], \\ f_{14} = &\ -\frac{1}{8}\big[2h(b-3e) + (3a-b-c+3e)r^2 + (b-c-3e+3f)R^2\big]. \end{aligned}$$

Subcase 3.2.2.3.2.1: $3a - 2b + 3d = 0$ and $b - c - 3d + 3e = 0$. Then we get $f_{12} = \dfrac{1}{4}(3d-b)h=$ constant.

Subcase 3.2.2.3.2.2: $b - c - 3d + 3e \neq 0$ and $3a - 2b + 3d = 0$. Solving $f_{12} = 0$, we obtain $R = \sqrt{\dfrac{2h(b-3d)}{b-c-3d+3e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.2.2.3.2.2.1: If $3a - b - c + 3e \neq 0$. We obtain $r = \sqrt{\dfrac{6hN_{20}}{\delta_6}}$ and $\rho = \sqrt{\dfrac{2hN_{21}}{\delta_6}}$, where $N_{20} = cd - be - ce + 3e^2 + bf - 3df$, $N_{21} = 9ae-3ac+bc+c^2-3cd-3ce-3bf+9df$ and $\delta_6 = (3a-b-c+3e)(b-c-3d+3e)$.

Supposing that

$$\begin{aligned} &3a - 2b + 3d = 0, \quad (b-3d)(b-c-3d+3e) > 0, \\ &\delta_6 N_{20} > 0 \quad \text{and} \quad \delta_6 N_{21} > 0. \end{aligned} \tag{2.70}$$

System (2.7) for $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with

$\rho = \sqrt{\dfrac{2hN_{21}}{\delta_6}}$ given by

$$\begin{gathered}\left(\sqrt{\frac{6hN_{20}}{\delta_6}}, \frac{\pi}{2}, \sqrt{\frac{2h(b-3d)}{b-c-3d+3e}}, \frac{\pi}{2}\right), \\ \left(\sqrt{\frac{6hN_{20}}{\delta_6}}, \frac{\pi}{2}, \sqrt{\frac{2h(b-3d)}{b-c-3d+3e}}, \frac{3\pi}{2}\right), \\ \left(\sqrt{\frac{6hN_{20}}{\delta_6}}, \frac{3\pi}{2}, \sqrt{\frac{2h(b-3d)}{b-c-3d+3e}}, \frac{\pi}{2}\right), \\ \left(\sqrt{\frac{6hN_{20}}{\delta_6}}, \frac{3\pi}{2}, \sqrt{\frac{2h(b-3d)}{b-c-3d+3e}}, \frac{3\pi}{2}\right).\end{gathered} \tag{2.71}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{9h^4(3d-b)N_{20}N_{21}}{16(3a-b-c+3e)^3(b-c-3d+3e)^4}(b^2-2bc+c^2-9ad \\ & +6cd+18ae-6be-6ce+9e^2-9af+6bf-9df) \\ & (-bc^2d+b^2ce+bc^2e-3acde+c^2de+3abe^2-3bce^2-3cde^2 \\ & -b^2cf+3bcdf-b^2ef+3bdef). \end{aligned}$$

Assuming that $(b^2-2bc+c^2-9ad+6cd+18ae-6be-6ce+9e^2-9af+6bf-9df)(-bc^2d+b^2ce+bc^2e-3acde+c^2de+3abe^2-3bce^2-3cde^2-b^2cf+3bcdf-b^2ef+3bdef) \neq 0$ and (2.70) hold, thus $J_{f_1(S^*)} \neq 0$ and the four zeros (2.71) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because the value $a=-3/2, b=-3, c=-2, d=-1/2, e=-1, f=-9/8$ satisfy it.

Subcase 3.2.2.3.2.2.2: If $3a-b-c+3e=0$. We obtain $f_{14} = 3N_{20}/(4(b-c-3d+3e))$= constant.

Subcase 3.2.2.3.2.3: $b-c-3d+3e=0$ and $3a-2b+3d \neq 0$. We have $r = \sqrt{\dfrac{2h(3d-b)}{3a-2b+3d}}$. Substituting r in f_{14} and solving $f_{14}=0$ we have two subcases

Subcase 3.2.2.3.2.3.1: If $b-c-3e+3f \neq 0$. we get $R = \sqrt{\dfrac{2hN_{22}}{\delta_7}}$ and $\rho = \sqrt{\dfrac{6hN_{23}}{\delta_7}}$, where $N_{22} = -b^2-3cd+9ad-9ae+bc+3be$, $N_{23} = cd-bf+ab-ac-3ad+3af$ and $\delta_7 = (b-c-3e+3f)(3a-2b+3d)$.

Assuming that

$$
\begin{array}{l}
b-c-3d+3e=0,(3d-b)(3a-2b+3d)>0,\\
\delta_7 N_{22}>0 \quad \text{and} \quad \delta_7 N_{23}>0.
\end{array}
\tag{2.72}
$$

System (2.7) for $\beta=\frac{\pi}{2}$ and $\beta=\frac{3\pi}{2}$ has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\frac{6hN_{23}}{\delta_7}}$ given by

$$
\begin{array}{l}
\left(\sqrt{\frac{2h(3d-b)}{3a-2b+3d}},\frac{\pi}{2},\sqrt{\frac{2hN_{22}}{\delta_7}},\frac{\pi}{2}\right),\\
\left(\sqrt{\frac{2h(3d-b)}{3a-2b+3d}},\frac{\pi}{2},\sqrt{\frac{2hN_{22}}{\delta_7}},\frac{3\pi}{2}\right),\\
\left(\sqrt{\frac{2h(3d-b)}{3a-2b+3d}},\frac{3\pi}{2},\sqrt{\frac{2hN_{22}}{\delta_7}},\frac{\pi}{2}\right),\\
\left(\sqrt{\frac{2h(3d-b)}{3a-2b+3d}},\frac{3\pi}{2},\sqrt{\frac{2hN_{22}}{\delta_7}},\frac{3\pi}{2}\right).
\end{array}
\tag{2.73}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)}= \;& -\frac{3h^4(b-3d)N_{22}N_{23}}{16(b-c-3e+3f)^3(3a-2b+3d)^4}(-b^2+2bc-c^2+9ad\\
&-6cd-18ae+6be+6ce-9e^2+9af-6bf+9df)(b^3c-b^2c^2\\
&-3b^2cd+3bc^2d+3ab^2e-3abce-4b^2ce+bc^2e-9abde+9acde\\
&+12bcde-3c^2de-9ace^2+3bce^2+3b^2cf-9bcdf+9abef\\
&-3b^2ef).
\end{aligned}
$$

Assuming that $(-b^2+2bc-c^2+9ad-6cd-18ae+6be+6ce-9e^2+9af-6bf+9df)(b^3c-b^2c^2-3b^2cd+3bc^2d+3ab^2e-3abce-4b^2ce+bc^2e-9abde+9acde+12bcde-3c^2de-9ace^2+3bce^2+3b^2cf-9bcdf+9abef-3b^2ef)\neq 0$ and (2.72) hold. This assumption is satisfied for the value $a=2,b=-1,c=-4,d=0,e=-1,f=-3/4$. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (2.73) of system (2.7) provide four periodic solutions of differential system (2.6).

Subcase 3.2.2.3.2.3.2: If $b-c-3e+3f=0$. We obtain $f_{14}=-hN_{22}/(4(3a-2b+3d))=$ constant.

Subcase 3.2.2.3.2.4: $(b-c-3d+3e)(3a-2b+3d)\neq 0$. Solving $f_{12}=0$ we have

$r = \sqrt{\dfrac{6dh - 2bh + (b - c - 3d + 3e)R^2}{3a - 2b + 3d}}$, substituting r in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.2.2.3.2.4.1:

$\Sigma_8 = b^2 + c^2 + 6cd - 6ce + 9e^2 - 2bc - 6be + 6bf - 9df - 9ad + 18ae - 9af \neq 0,$

then $r = \sqrt{\dfrac{6hN_{24}}{\Sigma_8}}$, $R = \sqrt{\dfrac{2hN_{25}}{\Sigma_8}}$ and $\rho = \sqrt{\dfrac{2hN_{26}}{\Sigma_8}}$ where $N_{24} = 3e^2 + cd - ce - be + bf - 3df$, $N_{25} = b^2 - bc - 9ad + 3cd + 9ae - 3be$ and $N_{26} = c^2 - bc + 9ae - 3ce - 9af + 3bf$.

With the condition that

$$\begin{aligned} &(b - c - 3d + 3e)(3a - 2b + 3d) \neq 0, \quad N_{24}\Sigma_8 > 0, \\ &N_{25}\Sigma_8 > 0 \quad \text{and} \quad N_{26}\Sigma_8 > 0, \end{aligned} \tag{2.74}$$

system (2.7) for $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_{26}}{\Sigma_8}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{6hN_{24}}{\Sigma_8}}, \frac{\pi}{2}, \sqrt{\frac{2hN_{25}}{\Sigma_8}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{6hN_{24}}{\Sigma_8}}, \frac{\pi}{2}, \sqrt{\frac{2hN_{25}}{\Sigma_8}}, \frac{3\pi}{2}\right), \\ &\left(\sqrt{\frac{6hN_{24}}{\Sigma_8}}, \frac{3\pi}{2}, \sqrt{\frac{2hN_{25}}{\Sigma_8}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{6hN_{24}}{\Sigma_8}}, \frac{3\pi}{2}, \sqrt{\frac{2hN_{25}}{\Sigma_8}}, \frac{3\pi}{2}\right). \end{aligned} \tag{2.75}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \ & \frac{9h^4 N_{24} N_{25} N_{26}}{16\Sigma_8^3}(bc^2d - b^2ce - bc^2e + 3acde - c^2de - 3abe^2 \\ & -3ace^2 + 5bce^2 + b^2cf - 3bcdf + 3abef - b^2ef). \end{aligned}$$

Assuming that $(bc^2d - b^2ce - bc^2e + 3acde - c^2de - 3abe^2 - 3ace^2 + 5bce^2 + b^2cf - 3bcdf + 3abef - b^2ef) \neq 0$ and (2.74) hold. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (2.75) of system (2.7) provide four periodic solutions of differential system (2.6). The set of conditions on the parameters is not empty because it is satisfied for the value $a = -14802, b = -15, c = -225/16, d = -62, e = -1, f = -6$.

Subcase 3.2.2.3.2.4.2: $\Sigma_8 = 0$. Then we have $f_{14} = hN_{25}/(4(3a - 2b + 3d))$= constant.

Proof of Proposition 2. Following the averaging theory, $(r^*, \alpha^*, R^*, \beta^*)$ is a periodic solution of (2.6) means that

$$\begin{aligned} r(t,\varepsilon) = &\ r^* + O(\varepsilon), \\ \alpha(t,\varepsilon) = &\ \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = &\ R^* + O(\varepsilon), \\ \beta(0,\varepsilon) = &\ \beta^* + O(\varepsilon). \end{aligned} \tag{2.76}$$

Adding the fact that $\rho = \sqrt{2h - r^{*2} - R^{*2}}$ so system (2.76) becomes

$$\begin{aligned} r(t,\varepsilon) = &\ r^* + O(\varepsilon), \\ \rho(t,\varepsilon) = &\ \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) = &\ \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = &\ R^* + O(\varepsilon), \\ \beta(t,\varepsilon) = &\ \beta^* + O(\varepsilon). \end{aligned} \tag{2.77}$$

We reconsider the variable, t, the temps instead of θ. The (2.77) becomes

$$\begin{aligned} r(t,\varepsilon) = &\ r^* + O(\varepsilon), \\ \theta(t,\varepsilon) = &\ t + O(\varepsilon), \\ \rho(t,\varepsilon) = &\ \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) = &\ \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = &\ R^* + O(\varepsilon), \\ \beta(t,\varepsilon) = &\ \beta^* + O(\varepsilon). \end{aligned} \tag{2.78}$$

Using the change of variables (2.2): $X = r\cos\theta$, $Y = \rho\cos(\theta+\alpha)$, $Z = R\cos(\theta+\beta)$, $p_X = r\sin\theta$, $p_Y = \rho\sin(\theta+\alpha)$, $p_Z = R\sin(\theta+\beta)$, the system

(2.78) becomes

$$\begin{aligned}
X(t,\varepsilon) = &\quad r^*\cos t + O(\varepsilon^{3/2}),\\
Y(t,\varepsilon) = &\quad \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* + t) + O(\varepsilon^{3/2}),\\
Z(t,\varepsilon) = &\quad R^*\cos(\beta^* + t) + O(\varepsilon^{3/2}),\\
p_X(t,\varepsilon) = &\quad r^*\sin t + O(\varepsilon^{3/2}),\\
p_Y(t,\varepsilon) = &\quad \sqrt{2h - r^{*2} - R^{*2}}\sin(\alpha^* + t) + O(\varepsilon^{3/2}),\\
p_Z(t,\varepsilon) = &\quad R^*\sin(\beta^* + t) + O(\varepsilon^{3/2}).
\end{aligned} \tag{2.79}$$

Finally we reused the scaling $x = \sqrt{\varepsilon}\,X$, $y = \sqrt{\varepsilon}\,Y$, $z = \sqrt{\varepsilon}\,Z$, $p_x = \sqrt{\varepsilon}\,p_X$, $p_y = \sqrt{\varepsilon}\,p_Y$ and $p_z = \sqrt{\varepsilon}\,p_Z$ and (2.79) becomes

$$\begin{aligned}
x(t,\varepsilon) = &\quad \sqrt{\varepsilon}\quad r^*\cos t + O(\varepsilon^{3/2}),\\
y(t,\varepsilon) = &\quad \sqrt{\varepsilon}\quad \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* + t) + O(\varepsilon^{3/2}),\\
z(t,\varepsilon) = &\quad \sqrt{\varepsilon}\quad R^*\cos(\beta^* + t) + O(\varepsilon^{3/2}),\\
p_x(t,\varepsilon) = &\quad \sqrt{\varepsilon}\quad r^*\sin t + O(\varepsilon^{3/2}),\\
p_y(t,\varepsilon) = &\quad \sqrt{\varepsilon}\quad \sqrt{2h - r^{*2} - R^{*2}}\sin(\alpha^* + t) + O(\varepsilon^{3/2}),\\
p_z(t,\varepsilon) = &\quad \sqrt{\varepsilon}\quad R^*\sin(\beta^* + t) + O(\varepsilon^{3/2}).
\end{aligned}$$

□

The results of this chapter produced the following two articles:

F. Lembarki and J. Llibre, *Periodic orbits for the generalized Yang-Mills Hamiltonian system in dimension* 6, Nonlinear Dynam. **76** (2014), 1807–1819.

F. Lembarki and J. Llibre, *Periodic orbits for the generalized Yang-Mills Hamiltonian system in dimension* 6 *II*, Nonlinear Dynam. 2016 Submitted.

“The most incomprehensible about Mathematics is that it is comprehensible” Unknown author.

“Mathematics is like love, a simple idea, but it can get complicated.” Unknown author.

Chapter 3

Friedmann-Robertson-Walker Hamiltonian system in 6D

In this chapter we study analytically the periodic orbits of the Friedmann-Robertson-Walker Hamiltonian system in dimension 6 using the averaging theory of first order.

3.1 Friedmann-Robertson-Walker Hamiltonian system in 6D

We study the following generalized classical Friedmann-Robertson-Walker Hamiltonian system in dimension 6.

$$\begin{aligned} H = \; & \frac{1}{2}(p_y^2 + p_z^2 - p_x^2 + y^2 + z^2 - x^2) + \frac{1}{4}(ax^4 + 2bx^2y^2 + 2cx^2z^2 + dy^4 \\ & + 2ey^2z^2 + fz^4), \end{aligned} \tag{1.3}$$

Note that this Hamiltonian depends on six parameters a,b,c,d,e and f.

Our goal in this work is to study the periodic solutions in the different energy levels of the Hamiltonian system

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = -p_x, \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} &= \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = x - x(ax^2 + by^2 + cz^2), \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -y - y(bx^2 + dy^2 + ez^2), \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -z - z(cx^2 + ey^2 + fz^2),
\end{aligned} \tag{1.4}$$

associated to the Hamiltonian (1.3).

We study the existence of periodic orbits of system (1.4) and we compute them by using the *averaging theory*. Specifically we will provide through the averaging theory of first order sufficient conditions on the six parameters a,b,c,d,e and f for ensuring the existence of periodic orbits of Friedmann-Robertson-Walker system (1.4).

3.2 Applying averaging theory to Friedmann-Robertson-Walker Hamiltonian system in 6D

We scale using a small parameter $\varepsilon > 0$. In fact, In fact, in the Hamiltonian system (1.4), we change the variables (x, y, z, p_x, p_y, p_z) by (X, Y, Z, p_X, p_Y, p_Z) where $x = \sqrt{\varepsilon}\, X$, $y = \sqrt{\varepsilon}\, Y$, $z = \sqrt{\varepsilon}\, Z$, $p_x = \sqrt{\varepsilon}\, p_X$, $p_y = \sqrt{\varepsilon}\, p_Y$ and $p_z = \sqrt{\varepsilon}\, p_Z$. In the new variables, system (1.4) becomes

$$\begin{aligned}
\dot{X} &= -p_X, \\
\dot{Y} &= p_Y, \\
\dot{Z} &= p_Z, \\
\dot{p}_X &= X - \varepsilon X(aX^2 + bY^2 + cZ^2), \\
\dot{p}_Y &= -Y - \varepsilon Y(bX^2 + dY^2 + eZ^2), \\
\dot{p}_Z &= -Z - \varepsilon Z(cX^2 + eY^2 + fZ^2).
\end{aligned} \tag{3.1}$$

This differential system again is Hamiltonian with Hamiltonian

$$\begin{aligned} H = \; & \frac{1}{2}\Big(-p_X^2 + p_Y^2 + p_Z^2 - X^2 + Y^2 + Z^2 \Big) + \frac{\varepsilon}{4}\Big(aX^4 + 2bX^2Y^2 \\ & +2cX^2Z^2 + dY^4 + 2eY^2Z^2 + fZ^4 \Big). \end{aligned} \tag{3.2}$$

The original and the transformed systems (1.4) and (3.1) have the same topological phase portrait because the change of variables is only a scale transformation for all $\varepsilon > 0$, and also system (3.1) for ε sufficiently small is close to an integrable one.

The periodicity in the independent variable of the differential system is needed to apply the averaging theory, so we change the Hamiltonian (3.2) and its equations of motion (3.1) to a kind of generalized polar coordinates $(r, \theta, \rho, \alpha, R, \beta)$ in $\mathbb{R}^6$. Defined by

$$\begin{aligned} X = r\cos\theta, \quad & Y = \rho\cos(\alpha - \theta), \quad Z = R\cos(\beta - \theta), \\ p_X = r\sin\theta, \quad & p_Y = \rho\sin(\alpha - \theta), \quad p_Z = R\sin(\beta - \theta). \end{aligned} \tag{3.3}$$

Of course in this change of variables $r \geq 0$, $\rho \geq 0$ and $R \geq 0$.

The first integral H in the new variables becomes

$$\begin{aligned} H = \; & \frac{1}{2}(\rho^2 + R^2 - r^2) + \frac{\varepsilon}{4}\big[ar^4\cos^4\theta + 2br^2\rho^2\cos^2\theta \\ & \cos^2(\alpha - \theta) + d\rho^4\cos^4(\alpha - \theta) + 2R^2\big(cr^2\cos^2\theta \\ & +e\rho^2\cos^2(\alpha - \theta)\big)\cos^2(\beta - \theta) + fR^4\cos^4(\beta - \theta)\big], \end{aligned} \tag{3.4}$$

and the equations of motion (3.1) become

$$
\begin{aligned}
\dot{r} &= -\varepsilon r \cos\theta \sin\theta \big[a\, r^2 \cos^2\theta + b\, \rho^2 \cos^2(\alpha-\theta) \\
&\quad + c\, R^2 \cos^2(\beta-\theta)\big], \\
\dot{\theta} &= 1 - \varepsilon \cos^2\theta \big[a\, r^2 \cos^2\theta + b\rho^2 \cos^2(\alpha-\theta) \\
&\quad + cR^2 \cos^2(\beta-\theta)\big], \\
\dot{\rho} &= -\varepsilon \rho \cos(\alpha-\theta) \sin(\alpha-\theta) \big[b\, r^2 \cos^2\theta \\
&\quad + d\rho^2 \cos^2(\alpha-\theta) + eR^2 \cos^2(\beta-\theta)\big], \\
\dot{\alpha} &= \varepsilon \Big[- a\, r^2 \cos^4\theta - \cos^2\theta \Big(b(r^2+\rho^2) \cos^2(\alpha-\theta) \\
&\quad + cR^2 \cos^2(\beta-\theta) \Big) - \cos^2(\alpha-\theta) \Big(d\rho^2 \cos^2(\alpha-\theta) \\
&\quad + eR^2 \cos^2(\beta-\theta) \Big) \Big], \\
\dot{R} &= -\varepsilon R \cos(\beta-\theta) \sin(\beta-\theta) \big[c\, r^2 \cos^2\theta \\
&\quad + e\rho^2 \cos^2(\alpha-\theta) + fR^2 \cos^2(\beta-\theta)\big], \\
\dot{\beta} &= \varepsilon \Big[- a\, r^2 \cos^4\theta - b\rho^2 \cos^2\theta \cos^2(\alpha-\theta) \\
&\quad - \Big(c(r^2+R^2) \cos^2\theta - e\rho^2 \cos^2(\alpha-\theta) \Big) \\
&\quad \cos^2(\beta-\theta) - fR^2 \cos^4(\beta-\theta) \Big].
\end{aligned}
\tag{3.5}
$$

We note that in this system if we take the variable θ as the new independent variable instead of t, we obtain the necessary periodicity for writing the system in the normal form of the averaging theory. In what follows the independent variable will be θ. This means that the new differential system will have only five equations. We denote by a prime the derivative with respect to θ and we expand system (3.5) in Taylor series in ε. Thus, system (3.5) becomes

$$\begin{aligned}
r' &= -\varepsilon\, r\sin\theta\cos\theta\big[a\, r^2\cos^2\theta + b\rho^2\cos^2(\alpha-\theta) \\
&\quad +cR^2\cos^2(\beta-\theta)\big] + O(\varepsilon^2), \\
\rho' &= -\varepsilon\rho\cos(\alpha-\theta)\sin(\alpha-\theta)\big[b\, r^2\cos^2\theta + d\rho^2\cos^2(\alpha-\theta) \\
&\quad +eR^2\cos^2(\beta-\theta)\big] + O(\varepsilon^2), \\
\alpha' &= \varepsilon\big[-\ ar^2\cos^4\theta - d\rho^2\cos^4(\alpha-\theta) \\
&\quad -eR^2\cos^2(\alpha-\theta)\cos^2(\beta-\theta) - \cos^2\theta\big(b(r^2+\rho^2) \\
&\quad \cos^2(\alpha-\theta) + cR^2\cos^2(\beta-\theta)\big)\big] + O(\varepsilon^2), \\
R' &= -\varepsilon\, R\sin(\beta-\theta)\cos(\beta-\theta)\big[c\, r^2\cos^2\theta \\
&\quad +e\rho^2\cos^2(\alpha-\theta) + fR^2\cos^2(\beta-\theta)\big] + O(\varepsilon^2), \\
\beta' &= \varepsilon\big[-fR^2\cos^4(\beta-\theta) - \cos^2\theta\ \big(ar^2\cos^2\theta \\
&\quad +b\rho^2\cos^2(\alpha-\theta)\big) - \cos^2(\beta-\theta)\big(c(r^2+R^2)\cos^2\theta \\
&\quad +e\rho^2\cos^2(\alpha-\theta)\big)\big] + O(\varepsilon^2).
\end{aligned} \tag{3.6}$$

System (3.6) is 2π-periodic in respect to the variable θ, i.e. it is written as the normal form (A.1) but it is not ready for applying the averaging theory, we must fix the value of the first integral $H = h$ with $h \in \mathbb{R}$, otherwise the Jacobian (A.4) will be zero because the periodic orbits are non-isolated leaving on cylinders parameterized by the energy, see for more details [1].

Solving ρ from (3.4) $= h$, we get two positive solutions, but the unique with physical meaning expanded in Taylor series in ε is

$$\rho = \sqrt{2h + r^2 - R^2} + O(\varepsilon). \tag{3.7}$$

Since $\rho \geq 0$ we need that $2h + r^2 - R^2 \geq 0$.

Substituting ρ in system (3.6) we get the differential system

$$\begin{aligned}
r' &= -\varepsilon\, r\sin\theta\cos\theta\big[a\, r^2\cos^2\theta + b(2h+r^2-R^2)\cos^2(\alpha-\theta) \\
&\quad +cR^2\cos^2(\beta-\theta)\big] + O(\varepsilon^2), \\
\alpha' &= \varepsilon\Big[-\ ar^2\cos^4\theta + d(R^2-2h-r^2)\cos^4(\alpha-\theta) \\
&\quad -eR^2\cos^2(\alpha-\theta)\cos^2(\beta-\theta) + \cos^2\theta\Big(b(R^2-2h-2r^2) \\
&\quad \cos^2(\alpha-\theta) - cR^2\cos^2(\beta-\theta)\Big)\Big] + O(\varepsilon^2), \\
R' &= -\varepsilon\, R\sin(\beta-\theta)\cos(\beta-\theta)\big[c\, r^2\cos^2\theta + e(2h+r^2-R^2) \\
&\quad \cos^2(\alpha-\theta) + fR^2\cos^2(\beta-\theta)\big] + O(\varepsilon^2), \\
\beta' &= \varepsilon\Big[-\ ar^2\cos^4\theta + \cos^2(\beta-\theta)(-e(2h+r^2-R^2) \\
&\quad \cos^2(\alpha-\theta) - fR^2\cos^2(\beta-\theta)) - \cos^2\theta\Big(b(2h+r^2-R^2) \\
&\quad \cos^2(\alpha-\theta) + c(R^2+r^2)\cos^2(\beta-\theta)\Big)\Big] + O(\varepsilon^2).
\end{aligned} \tag{3.8}$$

Following the notation of the averaging theory given in section 2, the function $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ of (A.1) is

$$\begin{aligned}
F_{11} =& -r\sin\theta\cos\theta\big[a\,r^2\cos^2\theta + b(2h+r^2-R^2)\\
&\cos^2(\alpha-\theta) + cR^2\cos^2(\beta-\theta)\big],\\
F_{12} =& -ar^2\cos^4\theta - d(2h+r^2-R^2)\cos^4(\alpha-\theta)\\
&-eR^2\cos^2(\alpha-\theta)\cos^2(\beta-\theta) - \cos^2\theta\big(b(2h+2r^2-R^2)\\
&\cos^2(\alpha-\theta) + cR^2\cos^2(\beta-\theta)\big),\\
F_{13} =& -R\sin(\beta-\theta)\cos(\beta-\theta)\big[c\,r^2\cos^2\theta + e(2h+r^2-R^2)\\
&\cos^2(\alpha-\theta) + fR^2\cos^2(\beta-\theta)\big],\\
F_{14} =& -ar^2\cos^4\theta - e(2h+r^2-R^2)\cos^2(\alpha-\theta)\cos^2(\beta-\theta)\\
&-fR^2\cos^4(\beta-\theta) - \cos^2\theta\big(b(2h+r^2-R^2)\cos^2(\alpha-\theta)\\
&+c(R^2+r^2)\cos^2(\beta-\theta)\big),
\end{aligned} \tag{3.9}$$

where $F_{1j} = F_{1j}(\theta, r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

From (A.3) and (3.9) we compute the averaged function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ and we obtain

$$\begin{aligned}
f_{11} =& \ -\frac{1}{8}r\big[b(2h+r^2-R^2)\sin 2\alpha + cR^2\sin 2\beta\big],\\
f_{12} =& \ \frac{1}{8}\big[-6dh - 3ar^2 - 3dr^2 + 3dR^2 + (-2bh - 2br^2 + bR^2)\\
&(2+\cos 2\alpha) - eR^2(2+\cos 2(\alpha-\beta)) - cR^2(2+\cos 2\beta)\big],\\
f_{13} =& \ \frac{1}{8}R\big[e(2h+r^2-R^2)\sin 2(\alpha-\beta) - cr^2\sin 2\beta\big],\\
f_{14} =& \ \frac{1}{8}\Big[-3ar^2 - 3fR^2 - (2h+r^2-R^2)\Big(2(b+e) + b\cos 2\alpha\\
&+e\cos 2(\alpha-\beta)\Big) - c(r^2+R^2)(2+\cos 2\beta)\Big],
\end{aligned}$$

where $f_{1j} = f_{1j}(r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

3.3 Periodic orbits of Friedmann-Robertson-Walker Hamiltonian system in 6D

According to Theorem 7 (see Appendix), our aim is to find the zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ of

$$f_{1i}(r, \alpha, R, \beta) = 0 \quad \text{for } i = 1, 2, 3, 4, \tag{3.10}$$

and afterwards we must check that the Jacobian determinant (A.4) evaluated in these zeros are different from zero.

Then from $f_{11}(r,\alpha,R,\beta)=0$ we obtain either $r=0$, or $b\sin 2\alpha\neq 0$ and $r=\sqrt{R^2-2h-\dfrac{cR^2\sin 2\beta}{b\sin 2\alpha}}$, or $b\sin 2\alpha=0$.

Case 1: $r=0$. Substituting r in f_{1i}, for $i=2,3,4$, we get

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) =& \frac{1}{8}\big[-6dh+3dR^2+(-2bh+bR^2)(2+\cos 2\alpha)-eR^2 \\ & (2+\cos 2(\alpha-\beta))-cR^2(2+\cos 2\beta)\big], \\ f_{13}(0,\alpha,R,\beta) =& -\frac{1}{8}eR(-2h+R^2)\sin 2(\alpha-\beta), \\ f_{14}(0,\alpha,R,\beta) =& \frac{1}{8}\big[-4(b+e)h+(2b-2(c-e)-3f)R^2- \\ & (2h-R^2)\Big(b\cos 2\alpha+e\cos 2(\alpha-\beta)\Big)-cR^2\cos 2\beta\big]. \end{aligned}$$

From $f_{13}(0,\alpha,R,\beta)=0$ we have the following four subcases: $e=0$, $R=0$, $R=\sqrt{2h}$, $\alpha=\beta+\dfrac{k\pi}{2}$ with $k\in\mathbb{Z}$.

Subcase 1.1: $e=0$. The Jacobian is zero in this subcase because f_{13} vanish and the averaging theory cannot provide information about the periodic orbits. Hence, *in what follows in case* 1 *we assume that* $e\neq 0$.

Subcase 1.2: When $R=0$ then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0,\alpha,0,\beta) =& -\frac{1}{4}h\big[2b+3d+b\cos 2\alpha\big], \\ f_{14}(0,\alpha,0,\beta) =& -\frac{1}{4}h\big[2b+2e+b\cos 2\alpha+e\cos 2(\alpha-\beta)\big]. \end{aligned}$$

Subcase 1.2.1: $b\neq 0$. Then solving $f_{12}(0,\alpha,0,\beta)=0$ we obtain $\alpha_\pm=\pm\dfrac{1}{2}\arccos-\dfrac{3d+2b}{b}$. Substituting $\alpha_\pm$ respectively in $f_{14}(0,\alpha,R,\beta)$ and solving $f_{14}=0$ we get two subcases

Subcase 1.2.1.1: $e\neq 0$. Then we have respectively $\beta_{+\pm}=\dfrac{1}{2}\big[-\arccos\dfrac{3d+2b}{b}\pm\arccos\dfrac{2e-3d}{e}\big]$ and $\beta_{-\pm}=\dfrac{1}{2}\big[\arccos\dfrac{3d+2b}{b}\pm\arccos\dfrac{2e-3d}{e}\big]$.

Supposing that

$$h>0,\quad eb\neq 0,\quad \left|\frac{3d+2b}{b}\right|<1 \quad\text{and}\quad \left|\frac{2e-3d}{e}\right|<1. \tag{3.11}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2h}$ and $\beta^* = \frac{1}{2}\Big[-\arccos\frac{3d+2b}{b} \pm \arccos\frac{2e-3d}{e}\Big]$ given by

$$\begin{aligned}&\left(0, \frac{1}{2}\arccos -\frac{3d+2b}{b}, 0, \beta^*\right),\\ &\left(0, -\frac{1}{2}\arccos -\frac{3d+2b}{b}, 0, \beta^*\right),\end{aligned} \tag{3.12}$$

which reduces to two solutions if either $\left|\dfrac{3d+2b}{b}\right| = 1$, $\left|\dfrac{2e-3d}{e}\right| < 1$, $h > 0$ and $eb \neq 0$, or $|(3d+2b)/b| < 1$, $|(2e-3d)/e| = 1$, $h > 0$ and $eb \neq 0$. We have one solution if $|(3d+2b)/b| = |(2e-3d)/e| = 1$, $h > 0$ and $eb \neq 0$.

We should check if the Jacobian of f_1 evaluated in these solutions is different from zero. The Jacobian is

$J_{f_1(S^*)} = \dfrac{3}{64}h^4 e^2 (b+3d)(b+d)\left[1 - \left(\dfrac{2e-3d}{e}\right)^2\right].$

Assuming that $(b+d)(b+3d) \neq 0$ and (3.11) hold. This supposition is not empty because the value $b = 1, d = -2/3, e = -4/3, h = 1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (3.12) of system (3.10) provide only two periodic solutions of differential system (3.8) because when $R = 0$ the two solutions of β provide the same initial conditions in (3.3). In the remaining subcases throughout this chapter we have two possibilities and in both of them we do not have results. The first one when the $J_{f_1(S^*)} = 0$, therefore the averaging theory does not provide information about the periodic solution. The second one when the set of conditions on the parameters which guarantee the existence of the solutions is empty.

Subcase 1.2.1.2: If $e = 0$. Then $f_{14} = 3dh/4$= constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 1.2.2: $b = 0$. We obtain $f_{12} = -3dh/4$= constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 1.3: $R = \sqrt{2h}$ so $h > 0$. Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0, \alpha, \sqrt{2h}, \beta) &= -\frac{1}{4}h\big[2e + 2c + c\cos 2\beta + e\cos 2(\alpha - \beta)\big],\\ f_{14}(0, \alpha, \sqrt{2h}, \beta) &= -\frac{1}{4}h\big[2c + 3f + c\cos 2\beta\big]. \end{aligned}$$

Subcase 1.3.1: $c \neq 0$. Then solving $f_{14} = 0$ we get

$\beta_\pm = \pm\frac{1}{2}\arccos -\frac{3f+2c}{c}$. Substituting $\beta_\pm$ respectively in f_{12} and solving $f_{12}=0$ with respect to α we have two subcases

Subcase 1.3.1.1: $e \neq 0$. We obtain four solutions

$$\alpha_{\pm\mp} = \frac{1}{2}\left[\pm \arccos\frac{2e-3f}{e} \mp \arccos\frac{3f+2c}{c}\right].$$

Assuming

$$h > 0, \quad ec \neq 0, \quad \left|\frac{2e-3f}{e}\right| < 1 \quad \text{and} \quad \left|\frac{3f+2c}{c}\right| < 1. \tag{3.13}$$

System (3.10) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ and $\alpha^* = \pm\frac{1}{2}\arccos\frac{2e-3f}{e} - \frac{1}{2}\arccos\frac{3f+2c}{c}$ given by

$$\begin{aligned} &\left(0, \alpha^*, \sqrt{2h}, \frac{1}{2}\arccos -\frac{3f+2c}{c}\right), \\ &\left(0, \alpha^*, \sqrt{2h}, -\frac{1}{2}\arccos -\frac{3f+2c}{c}\right), \end{aligned} \tag{3.14}$$

which reduce to two solutions if either $h > 0$, $|(2e-3f)/e| < 1$, $ec \neq 0$ and $|(3f+2c)/c| = 1$ or $|(2e-3f)/e| = 1$ and $|(3f+2c)/c| < 1$. We have one solution if $h > 0$, $ec \neq 0$ and $|(2e-3f)/e| = |(3f+2c)/c| = 1$.

We evaluate the Jacobian on these solutions and we get

$$J_{f_1(S^*)} = \frac{3}{32}h^4e^2(c+3f)(c+f)\left[1 - \left(\frac{2e-3f}{e}\right)^2\right].$$

Supposing that $(c+3f)(c+f) \neq 0$ and (3.13) hold. This assumption is not empty because it is satisfied for the value $c = 1, f = -3/4, e = -7/4, h = 1$. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (3.14) of system (3.10) provide only two periodic solutions of differential system (3.8) because since $\rho = 0$ the two solutions of α provide the same initial conditions in (3.3).

Subcase 1.3.1.2: $e = 0$. Then $f_{12} = 3fh/4$= constant.

Subcase 1.3.2: $c = 0$. We get $f_{14} = -3fh/4$= constant.

Subcase 1.4: $\alpha = \beta + \frac{k\pi}{2}$. We consider four subcases $k = 0, 1, 2, 3$.

Due to the periodicity of the cosinus we study the cases $k = 0$ and $k = 2$, and the cases $k = 1$ and $k = 3$ together.

Subcase 1.4.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. Then

$$\begin{aligned} f_{12}(0, \alpha, R, \beta) = & -\frac{1}{8}\big[2h(2b + 3d + b\cos 2\beta) - \big(2b - 2c + 3d - 3e + \\ & (b - c)\cos 2\beta\big)R^2\big], \\ f_{14}(0, \alpha, R, \beta) = & -\frac{1}{8}\big[3fR^2 + cR^2(2 + \cos 2\beta) + (2h - R^2) \\ & (2b + 3e + b\cos 2\beta)\big]. \end{aligned}$$

Subcase 1.4.1.1: $D_1 = 2b - 2c + 3d - 3e + (b - c)\cos 2\beta \neq 0$. Solving $f_{12} = 0$ we get $R = \sqrt{2h\,(2b + 3d + b\cos 2\beta)\,/D_1}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to the variable β we get two subcases

Subcase 1.4.1.1.1: $cd - be - ce + bf \neq 0$. Then we have $\beta = \pm\frac{1}{2}\arccos[(-2cd+2be+2ce+3e^2-2bf-3df)/(cd-be-ce+bf)]$. Let $d-2e+f \neq 0$, $R = \sqrt{2h(d - e)/(d - 2e + f)}$ and $\rho = \sqrt{2h(f - e)/(d - 2e + f)}$.
With the condition that

$$\begin{array}{l} cd - be - ce + bf \neq 0, \quad (d - 2e + f) \neq 0, \quad \left|\Delta_1\right| < 1 \\ h(d - 2e + f)(f - e) > 0 \quad \text{and} \quad h(d - 2e + f)(d - e) > 0, \end{array} \tag{3.15}$$

where $\Delta_1 = (-2cd + 2be + 2ce + 3e^2 - 2bf - 3df)/(cd - be - ce + bf)$, system (3.10) for $k = 0$ and $k = 2$ has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(f - e)}{(d - 2e + f)}}$ given by

$$\begin{array}{l} \left(0, \beta, \sqrt{\dfrac{2h(d - e)}{(d - 2e + f)}}, \beta = \pm\dfrac{1}{2}\arccos\Delta_1\right), \\ \left(0, \beta + \pi, \sqrt{\dfrac{2h(d - e)}{(d - 2e + f)}}, \beta = \pm\dfrac{1}{2}\arccos\Delta_1\right), \end{array} \tag{3.16}$$

which reduce to two solutions if $cd - be - ce + bf \neq 0$, $(d - 2e + f) \neq 0$, $h(d - 2e + f)(f - e) > 0$, $h(d - 2e + f)(d - e) > 0$ and $\left|\Delta_1\right| = 1$.
Its Jacobian

$$\begin{aligned} J_{f_1(S^*)} = & \frac{9eh^4}{32(d - 2e + f)^3}(f - e)(d - e)(-be + bf + cd - ce + 3df - 3e^2) \\ & (-be + bf + cd - ce + df - e^2). \end{aligned}$$

Assuming that $e(-be+bf+cd-ce+3df-3e^2)(-be+bf+cd-ce+df-e^2) \neq 0$ and (3.15) hold. The set of the conditions on the parameters is not empty when $b = 5, c = 0, d = 0, f = -1/2, e = -1, h = 1$. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (3.16) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 1.4.1.1.2: $cd - be - ce + bf = 0$. We get $f_{14} = 3h(2be + 3e^2 - 2cd + 2ce - 2bf - 3df)/4D_1$. So either f_{14} never is zero, or f_{14} is identically zero. In both cases the averaging theory does not provide information.

Subcase 1.4.1.2: $D_1 = 0$. We have

$$
\begin{aligned}
f_{12}(0, \alpha, R, \beta) = & \ -\frac{1}{4}h\big[2b + 3d + b\cos 2\beta\big], \\
f_{14}(0, \alpha, R, \beta) = & \ -\frac{1}{4}h\big[2b + 3e + b\cos 2\beta + 2R^2\big(2b - 2c + 3e - 3f \\
& +(b - c)\cos 2\beta\big)\big].
\end{aligned}
$$

Subcase 1.4.1.2.1: $b \neq 0$. Solving $f_{12} = 0$ we have $\beta = \pm\frac{1}{2}\arccos -\frac{3d + 2b}{b}$. Substituting β in f_{14} and solving $f_{14} = 0$ we get two subcases

Subcase 1.4.1.2.1.1: $bd - cd - be + bf \neq 0$. Then we have $R = \sqrt{\frac{2hb(d-e)}{bd - cd - be + bf}}$ and $\rho = \sqrt{\frac{2h(cd - bf)}{bd - cd - be + bf}}$.

In the case that

$$
\begin{aligned}
& bd - cd - be + bf \neq 0, \quad b \neq 0, \quad \left|\frac{3d + 2b}{b}\right| < 1, \\
& \quad h(bd - cd - be + bf)(cd - bf) > 0 \quad \text{and} \\
& hb(bd - cd - be + bf)(d - e) > 0,
\end{aligned} \tag{3.17}
$$

system (3.10) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\frac{2h(cd - bf)}{(bd - cd - be + bf)}}$ given by

$$
\begin{aligned}
& \left(0, \beta, \sqrt{\frac{2hb(d-e)}{bd - cd - be + bf}}, \beta = \pm\frac{1}{2}\arccos -\frac{3d + 2b}{b}\right), \\
& \left(0, \beta + \pi, \sqrt{\frac{2hb(d-e)}{bd - cd - be + bf}}, \beta = \pm\frac{1}{2}\arccos -\frac{3d + 2b}{b}\right),
\end{aligned} \tag{3.18}
$$

which reduce to two solutions if $bd - cd - be + bf \neq 0$, $b \neq 0$, $\left|\frac{3d + 2b}{b}\right| = 1$, $h(bd - cd - be + bf)(cd - bf) > 0$ and $hb(bd - cd - be + bf)(d - e) > 0$.

$J_{f_1(S^*)} = \dfrac{9beh^4}{32(bd-cd-be+bf)^4}\big[(b+d)(b+3d)(d-e)(d-2e+f)(bf-cd)(ce-bf)^2\big]$.

Assuming that $e(b+d)(b+3d)(d-2e+f)(ce-bf) \neq 0$ and (3.17) hold. This supposition is not empty because the value $b=3, c=5, d=-2, f=-3, e=-1, h=1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (3.18) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 1.4.1.2.1.2: $bd-cd-be+bf=0$. We get $f_{14} = 3(d-e)h/4$= constant.

Subcase 1.4.1.2.2: $b=0$. we have $f_{12} = -3dh/4$= constant.

Subcase 1.4.2: Assume that either $k=1$ or $k=3$, i.e. either $\alpha = \beta + \dfrac{\pi}{2}$ or $\alpha = \beta + \dfrac{3\pi}{2}$ then

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = {} & -\frac{1}{8}\big[2h(2b+3d+b\cos 2\beta) + (2c-2b-3d+e+ \\ & (b+c)\cos 2\beta)R^2\big], \\ f_{14}(0,\alpha,R,\beta) = {} & \frac{1}{8}\big[-2(2b+e)h + (2b-2c+e-3f)R^2 + \big(2bh- \\ & (b+c)R^2\big)\cos 2\beta\big]. \end{aligned}$$

Subcase 1.4.2.1: $D_2 = 2c-2b-3d+e+b\cos 2\beta + c\cos 2\beta \neq 0$. So solving $f_{12}(0,\alpha,R,\beta)=0$ we get $R = \sqrt{2h\left(-2b-3d-b\cos 2\beta\right)/D_2}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to the variable β we obtain two subcases

Subcase 1.4.2.1.1: $3cd+be-ce-3bf \neq 0$. Then we have
$\beta = \pm\dfrac{1}{2}\arccos\Big(\Delta_2 = \dfrac{2be+2ce+e^2-6cd-6bf-9df}{3cd+be-ce-3bf}\Big)$.
Then $R = \sqrt{\dfrac{2h(3d-e)}{3d-2e+3f}}$ and $\rho = \sqrt{\dfrac{2h(3f-e)}{3d-2e+3f}}$.
Considering that

$$\begin{aligned} & 3cd+be-ce-3bf \neq 0, 3d-2e+3f \neq 0, \\ & h(3f-e)(3d-2e+3f) > 0, \quad h(3d-e)(3d-2e+3f) > 0 \\ & \quad \text{and} \quad |\Delta_2| < 1. \end{aligned} \tag{3.19}$$

System (3.10) for $k=1$ or $k=3$ has four solutions $S^* = (r^*,\alpha^*,R^*,\beta^*)$ with

$\rho = \sqrt{\dfrac{2h(3f-e)}{3d-2e+3f}}$ given by

$$\begin{aligned}&\left(0, \beta+\frac{\pi}{2}, \sqrt{\frac{2h(3d-e)}{3d-2e+3f}}, \beta = \pm\frac{1}{2}\arccos\Delta_2\right),\\ &\left(0, \beta+\frac{3\pi}{2}, \sqrt{\frac{2h(3d-e)}{3d-2e+3f}}, \beta = \pm\frac{1}{2}\arccos\Delta_2\right),\end{aligned} \tag{3.20}$$

which reduce to two solutions if $3cd+be-ce-3bf \neq 0$, $3d-2e+3f \neq 0$, $h(3f-e)(3d-2e+3f) > 0$, $h(3d-e)(3d-2e+3f) > 0$ and $|\Delta_2| = 1$.

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{eh^4(3f-e)(3d-e)}{32(3d-2e+3f)^3}(-9cd+be+3ce+e^2-3bf-9df) \\ & (3cd-3be-ce-e^2+9bf+9df). \end{aligned}$$

Assuming that $(-9cd+be+3ce+e^2-3bf-9df)(3cd-3be-ce-e^2+9bf+9df) \neq 0$ and (3.19) hold. This supposition is not empty because the value $b=5/8, c=1$,
$d=-1, f=-1, e=0, h=1$ satisfy it. Therefore $J_{f_1(S^*)} \neq 0$ and the four solutions (3.20) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 1.4.1.2.2: $3cd+be-ce-3bf=0$. We obtain $f_{14} = h(6cd-2be-2ce-e^2+6bf+9df)/(4D_2)$. So either f_{14} never is zero, or f_{14} is identically zero. In both cases the averaging theory does not provide information.

Subcase 1.4.2.2: $D_2 = 0$. Then $f_{12} = \dfrac{h}{4}(b\cos 2\beta - 2b - 3d)$. Solving $f_{12} = 0$ when $b \neq 0$ we get $\beta = \pm\dfrac{1}{2}\arccos(3d+2b)/b$. Substituting β in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 1.4.2.2.1: $4bc+3bd+3cd-be+3bf \neq 0$. Then we get $R = \sqrt{\dfrac{2hb(3d-e)}{4bc+3bd+3cd-be+3bf}}$ and $\rho = \sqrt{\dfrac{2h(4bc+3cd+3bf)}{4bc+3bd+3cd-be+3bf}}$.
Assuming that

$$\begin{aligned}&4bc+3bd+3cd-be+3bf \neq 0, \quad b \neq 0,\\ &hb(4bc+3bd+3cd-be+3bf)(3d-e) > 0,\\ &h(4bc+3bd+3cd-be+3bf)(4bc+3cd+3bf) > 0,\\ &\text{and} \quad \left|\frac{3d+2b}{b}\right| < 1.\end{aligned} \tag{3.21}$$

System (3.10) for $k = 1$ and $k = 3$ has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(4bc + 3cd + 3bf)}{4bc + 3bd + 3cd - be + 3bf}}$ and $\beta^* = \pm\dfrac{1}{2}\arccos(3d + 2b)/b$ given by

$$\begin{aligned} &\left(0, \beta^* + \frac{\pi}{2}, \sqrt{\frac{2hb(3d-e)}{4bc + 3bd + 3cd - be + 3bf}}, \beta^*\right), \\ &\left(0, \beta^* + \frac{3\pi}{2}, \sqrt{\frac{2hb(3d-e)}{4bc + 3bd + 3cd - be + 3bf}}, \beta^*\right), \end{aligned} \tag{3.22}$$

which reduce to two solutions when $4bc + 3bd + 3cd - be + 3bf \neq 0$, $b \neq 0$, $h(4bc + 3bd + 3cd - be + 3bf)(4bc + 3cd + 3bf) > 0$, $h(4bc + 3bd + 3cd - be + 3bf)b(3d - e) > 0$ and $\left|\dfrac{2b + 3d}{b}\right| = 1$.

We have

$$\begin{aligned} J_{f_1(S^*)} = \; & -\frac{3beh^4(b+d)(b+3d)}{32(4bc + 3bd + 3cd - be + 3bf)^4}(3d - e)(3d - 2e + 3f)(4bc \\ & +3cd + 3bf)(4bc + ce + 3bf)^2. \end{aligned}$$

In the case that $e(b + d)(b + 3d)(3d - 2e + 3f)(4bc + ce + 3bf) \neq 0$ and (3.21) hold, this supposition is true for the value $b = 2, c = 8, d = -1, f = -6, e = -9, h = 1$. Then we have $J_{f_1(S^*)} \neq 0$ and the four solutions (3.22) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 1.4.2.2.1.2: $4bc + 3bd + 3cd - be + 3bf = 0$. We get $f_{14} = h(3d - e)/(4b)$= constant.

Subcase 1.4.2.2.2: $b = 0$. Then $f_{12} = -3dh/4$= constant.

Case 2: $b\sin 2\alpha \neq 0$, $r = \sqrt{R^2 - 2h - \dfrac{cR^2 \sin 2\beta}{b \sin 2\alpha}}$. Then

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = \;& \frac{1}{32b}\big[4b(6ah + 4bh - (3a + 2(b + c + e))R^2 \\
& + b(2h - R^2)\cos 2\alpha - 4bR^2(c + e\cos 2\alpha)\cos 2\beta \\
& + \frac{R^2 \sin 2\beta}{\sin\alpha\cos\alpha}(6ac + 8bc + 6cd - be + 4bc\cos 2\alpha \\
& + be\cos 4\alpha)\big], \\
f_{13}(r,\alpha,R,\beta) = \;& \frac{cR\sin 2\beta}{4}\big[h - \frac{R^2}{2b\sin 2\alpha}[b\sin 2\alpha - c\sin 2\beta \\
& + e\sin 2(\alpha - \beta)]\big], \\
f_{14}(r,\alpha,R,\beta) = \;& \frac{1}{32b}\Bigg[8b(3a + 2c)h - 2R^2(6ab + 8bc - ce + 6bf) \\
& + 8bc(h - R^2)\cos 2\beta + cR^2\Big[-2e\cos 4\beta \\
& + \frac{1}{\sin\alpha\cos\alpha}\big[(c + e\cos 2\alpha)\sin 4\beta + \sin 2\beta \\
& + (2b\cos 2\alpha + 6a + 4b + 4c + 4e)\big]\Big]\Bigg].
\end{aligned}
$$

So if $D_3 = b\sin 2\alpha - c\sin 2\beta + e\sin 2(\alpha - \beta) = 0$,

$$
f_{13}(r,\alpha,R,\beta) = 0 \Rightarrow \begin{cases} c = 0, \\ R = 0, \\ \beta = \dfrac{k\pi}{2} \quad \text{with} \quad k \in \mathbb{Z}. \end{cases}
$$

and if $D_3 \neq 0$ we get

$$
f_{13}(r,\alpha,R,\beta) = 0 \Rightarrow R = \sqrt{\frac{2bh\sin 2\alpha}{D_3}}.
$$

Subcase 2.1: $D_3 = 0$ and $c = 0$. No information as in subcase 1.1. Hence, *in what follows in the rest of case 2 we assume that $c \neq 0$.*

Subcase 2.2: $D_3 = 0$ and $R = 0$. Then $r = \sqrt{-2h}$, $h < 0$ and $\rho = 0$.

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = \;& \frac{1}{4}\big[h(3a + 2b + b\cos 2\alpha)\big], \\
f_{14}(r,\alpha,R,\beta) = \;& \frac{1}{4}\big[h(3a + 2c + c\cos 2\beta)\big].
\end{aligned}
$$

Subcase 2.2.1: $b \neq 0$ and $c \neq 0$. Solving $f_{12} = f_{14} = 0$ we get
$\alpha = \pm\frac{1}{2}\arccos -\frac{3a+2b}{b}$ and $\beta = \pm\frac{1}{2}\arccos -\frac{3a+2c}{c}$.
With the condition that

$$h < 0, \quad bc \neq 0, \quad \left|\frac{3a+2b}{b}\right| < 1 \quad \text{and} \quad \left|\frac{3a+2c}{c}\right| < 1, \tag{3.23}$$

system (3.10) has four solutions $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ given by

$$(\sqrt{-2h}, \pm\frac{1}{2}\arccos -\frac{3a+2b}{b}, 0, \pm\frac{1}{2}\arccos -\frac{3a+2c}{c}), \tag{3.24}$$

which reduce to two solutions if either $h < 0$, $bc \neq 0$, $\left|\frac{3a+2b}{b}\right| < 1$ and $\left|\frac{3a+2c}{c}\right| = 1$ or $h < 0$, $bc \neq 0$, $\left|\frac{3a+2b}{b}\right| = 1$ and $\left|\frac{3a+2c}{c}\right| < 1$, and to one solution if $h < 0$, $bc \neq 0$, $\left|\frac{3a+2b}{b}\right| = \left|\frac{3a+2c}{c}\right| = 1$.

The Jacobian $J_{f_1(S^*)} = -\frac{9h^4}{32}(a+b)(3a+b)(a+c)(3a+c)$.

Assuming that $D_3 = 0$, $(a+b)(3a+b)(a+c)(3a+c) \neq 0$ and (3.23) hold. This supposition is true for the value $a = -(2/3), b = 1, c = 4/3, h = -1$. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (3.24) of system (3.10) provide only one periodic solution of differential system (3.8) because when $R = 0$ and $\rho = 0$ the two solutions of both α and β provide the same initial conditions in (3.3).

Subcase 2.2.1: $b = 0$. Then we have $f_{12} = 3ah/4=$ constant.

Subcase 2.2.1: $c = 0$. Then we obtain $f_{14} = 3ah/4=$ constant.

Subcase 2.3: $D_3 = 0$ and $\beta = \frac{k\pi}{2}$. Due to the periodicity of the cosinus we study the cases $k = 0$ and $k = 2$, and the cases $k = 1$ and $k = 3$, together.

Subcase 2.3.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\beta = 0$ or $\beta = \pi$. Then substituting β in r we obtain $r = \sqrt{R^2 - 2h}$. f_{12} and f_{14} become

$$\begin{aligned} f_{12}(r,\alpha,R,\beta) = {} & \frac{1}{8}\big[6ah + 4bh - (3a+2b+3c+2e)R^2 \\ & +(2bh - (b+e)R^2)\cos 2\alpha\big], \\ f_{14}(r,\alpha,R,\beta) = {} & \frac{3}{8}\big[2(a+c)h - (a+2c+f)R^2\big]. \end{aligned}$$

Subcase 2.3.1.1: $a+2c+f \neq 0$. Solving $f_{14}=0$ with respect to R we obtain
$R=\sqrt{2h(a+c)/a+2c+f}$. Substituting R in f_{12} and solving $f_{12}=0$ with respect to α we obtain two subcases

Subcase 2.3.1.1.1: $bc-ae-ce+bf \neq 0$. Then we obtain $\alpha = \pm\frac{1}{2}\arccos\Delta_3$, where $\Delta_3 = \dfrac{3c^2-2bc+2ae+2ce-3af-2bf}{bc-ae-ce+bf}$. Substituting R in r and in ρ we obtain $r=\sqrt{\dfrac{-2h(c+f)}{a+2c+f}}$ and $\rho=0$.

Supposing that

$$\begin{array}{l} a+2c+f\neq 0, \quad |\Delta_3|<1, \quad (bc-ae-ce+bf)\neq 0, \\ h(a+2c+f)(c+f)<0 \quad \text{and} \quad h(a+2c+f)(a+c)>0. \end{array} \tag{3.25}$$

System (3.10) for $k=0$ and $k=2$ has four zeros solutions $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=0$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{-2h(c+f)}{a+2c+f}}, \pm\dfrac{1}{2}\arccos\Delta_3, \sqrt{\dfrac{2h(a+c)}{a+2c+f}}, 0\right), \\ \left(\sqrt{\dfrac{-2h(c+f)}{(a+2c+f)}}, \pm\dfrac{1}{2}\arccos\Delta_3, \sqrt{\dfrac{2h(a+c)}{a+2c+f}}, \pi\right), \end{array} \tag{3.26}$$

which reduce to two solutions if $(bc-ae-ce+bf)\neq 0$, $a+2c+f\neq 0$, $|\Delta_3|=1$, $h(a+2c+f)(c+f)<0$ and $h(a+2c+f)(a+c)>0$.

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \ & \frac{-9ch^4}{16(a+2c+f)^3}(a+c)(c+f)\big(bc-c^2-ae-ce+af+bf\big) \\ & \big(bc-3c^2-ae-ce+3af+bf\big). \end{aligned}$$

In the case that $(bc-c^2-ae-ce+af+bf)\big(bc-3c^2-ae-ce+3af+bf\big)\neq 0$ and (3.25) hold, this supposition is not empty because the value $a=2, b=-3, c=-1, f=-1, e=0, h=-1$ satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (3.26) of system (3.10) provide only two periodic solutions of differential system (3.8) because since $\rho=0$ the two solutions of α provide the same initial conditions in (3.3).

Subcase 2.3.1.1.2: $bc+bf-ae-ce=0$. We get $f_{12}=h(3af+2b(c+f)-3c^2-2ae-2ce)/4(a+2c+f)=$ constant.

Subcase 2.3.1.2: $a+2c+f=0$. Then $f_{14}=3h(a+c)/4=$ constant.

Subcase 2.3.2: Assume that either $k=1$ or $k=3$, i.e. either $\beta=\dfrac{\pi}{2}$ or $\beta=\dfrac{3\pi}{2}$. Then substituting β in r we get $r=\sqrt{R^2-2h}$. f_{12} and f_{14} become

$$\begin{aligned}
f_{12}(r,\alpha,R,\beta) = & \ \frac{1}{8}\big[6ah+4bh-(3a+2b+c+2e)R^2 \\
& +(2bh-bR^2+eR^2)\cos 2\alpha\big], \\
f_{14}(r,\alpha,R,\beta) = & \ \frac{1}{8}\big[2h(3a+c)-(3a+2c+3f)R^2\big].
\end{aligned}$$

Subcase 2.3.2.1: $3a+2c+3f\neq 0$. Solving $f_{14}=0$ we get $R=\sqrt{\dfrac{2h(3a+c)}{3a+2c+3f}}$. Substituting R in f_{12} and solving $f_{12}=0$ we obtain two subcases

Subcase 2.3.2.1.1: $bc+3ae+ce+3bf\neq 0$. We have $\alpha=\pm\dfrac{1}{2}\arccos\Delta_4$ where $\Delta_4=(-2bc+c^2+6ae+2ce-9af-6bf)/bc+3ae+ce+3bf$. Substituting R in r and in ρ we have $r=\sqrt{\dfrac{-2h(c+3f)}{3a+2c+3f}}$ and $\rho=0$.

Assuming that

$$\begin{aligned}
&3a+2c+3f\neq 0, \quad bc+3ae+ce+3bf\neq 0, \\
&|\Delta_4|<1, \quad h(3a+2c+3f)(c+3f)<0, \\
&\text{and} \quad h(3a+2c+3f)(3a+c)>0.
\end{aligned} \tag{3.27}$$

System (3.10) for $k=1$ and $k=3$ has four solutions $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=0$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{-2h(c+3f)}{3a+2c+3f}}, \pm\frac{1}{2}\arccos\Delta_4, \sqrt{\frac{2h(3a+c)}{3a+2c+3f}}, \frac{\pi}{2}\right), \\
&\left(\sqrt{\frac{-2h(c+3f)}{3a+2c+3f}}, \pm\frac{1}{2}\arccos\Delta_4, \sqrt{\frac{2h(3a+c)}{3a+2c+3f}}, \frac{3\pi}{2}\right),
\end{aligned} \tag{3.28}$$

which reduce to two solutions if $3a+2c+3f\neq 0$, $bc+3ae+ce+3bf\neq 0$, $|\Delta_4|=1$, $h(3a+2c+3f)(c+3f)<0$ and $h(3a+2c+3f)(3a+c)>0$.

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} = & \ \frac{ch^4(3a+c)(c+3f)}{(163a+2c+3f)^3}\big(-3bc+c^2+3ae+ce-9af-9bf\big) \\
& (-bc+c^2+9ae+3ce-9af-3bf)\,.
\end{aligned}$$

Considering that $c\big(-3bc+c^2+3ae+ce-9af-9bf\big)\big(-bc+c^2+9ae+3ce-9af-3bf\big)\neq 0$ and (3.27) hold. This assumption is not empty because the value $a=1, b=-2, c=-1, f=-1, e=0, h=-1$ satisfy it. Therefore $J_{f_1(S^*)}\neq 0$ and the four zeros (3.28) of system (3.10) provide only two periodic solutions of differential system (3.8) because since $\rho=0$ the two solutions of α provide the same initial conditions in (3.3).

Subcase 2.3.2.1.2: $3a+2c+3f\neq 0$ and $bc+3ae+ce+3bf=0$. Then $f_{12}=h(9af+2b(c+3f)-c^2-6ae-2ce)/4(3a+2c+3f)$= constant.

Subcase 2.3.2.2: $3a+2c+3f=0$. Then we have $f_{14}=h(3a+c)/4$= constant.

Subcase 2.4: $D_3\neq 0$ and $R=\sqrt{\dfrac{2bh\sin 2\alpha}{D_3}}$. Then $r=\sqrt{\dfrac{2eh\sin 2(\beta-\alpha)}{D_3}}$. f_{12}, f_{14} become

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta)= \;& \frac{h}{4D_3}\big[(3ae-bc+2be)\sin 2(\alpha-\beta)+(2bc-be\\
&+3cd)\sin 2\beta-2b(c+e)\sin 2\alpha\big],\\
f_{14}(r,\alpha,R,\beta)= \;& \frac{h}{4D_3}\big[(-2bc+ce-3bf)\sin 2\alpha+(-bc+3ae+2ce)\\
&\sin 2(\alpha-\beta)+2c(b+e)\sin 2\beta\big].
\end{aligned}
$$

To calculate the zeros of these two last functions we need their numerators $h\big[(3ae-bc+2be)\sin 2(\alpha-\beta)+(2bc-be+3cd)\sin 2\beta-2b(c+e)\sin 2\alpha\big]$, $h\big[(-2bc+ce-3bf)\sin 2\alpha+(-bc+3ae+2ce)\sin 2(\alpha-\beta)+2c(b+e)\sin 2\beta\big]$. Expanding the trigonometrical terms of these numerators and using the notation $\sin\alpha=s$; $\cos\alpha=\pm\sqrt{1-s^2}$; $\sin\beta=S$; $\cos\beta=\pm\sqrt{1-S^2}$ we obtain using the sign $+$ for $\cos\alpha$ and $\cos\beta$

$$
\begin{aligned}
P^*_{12}(s,S)= \;& 2hs\sqrt{1-s^2}\,(-6aeS^2+3ae+2bcS^2-3bc-4beS^2)\\
&-2hS\sqrt{1-S^2}\big(-6aes^2+3ae+2bcs^2-3bc-4bes^2\\
&+3be-3cd\big),\\
P^*_{14}(s,S)= \;& 2hs\sqrt{1-s^2}\big(-6aeS^2+3ae+2bcS^2-3bc-3bf\\
&-4ceS^2+3ce\big)\\
&-2hS\sqrt{1-S^2}\,(-6aes^2+3ae+2bcs^2-3bc-4ces^2)\,.
\end{aligned}
$$

Note that the other three subcases provide, taking into account the different signs, the same zeros than the system $P^*_{12}(s,S)=P^*_{14}(s,S)=0$. This last system is equivalent to the system $Q^*_{12}(s,S) = Q^*_{14}(s,S) = 0$ where

$$
\begin{aligned}
Q^*_{12}(s,S) &= 4h^2S^2\,(1-S^2)\big(-6aes^2+3ae+2bcs^2 \\
&\quad -3bc-4bes^2+3be-3cd\big)^2-4h^2s^2\,(1-s^2) \\
&\quad \big(-6aeS^2+3ae+2bcS^2-3bc-4beS^2\big)^2, \\
Q^*_{14}(s,S) &= 4h^2S^2\,(1-S^2)\big(-6aes^2+3ae+2bcs^2 \\
&\quad -3bc-4ces^2\big)^2-4h^2s^2\,(1-s^2)\big(-6aeS^2 \\
&\quad +3ae+2bcS^2-3bc-3bf-4ceS^2+3ce\big)^2.
\end{aligned}
\tag{3.29}
$$

Calculating the resultant of Q^*_{12} and Q^*_{14} with respect to s and S, we obtain

$$
\begin{aligned}
R^*_{12}(S) &= 47775744h^{16}(-1+S)^4S^8(1+S)^4T^2(S)U^2(S), \\
R^*_{14}(s) &= 47775744h^{16}(-1+s)^4s^8(1+s)^4V^2(s)W^2(s),
\end{aligned}
\tag{3.30}
$$

with T(S) and V(s) two polynomials of the form AS^2+B and Cs^2+D respectively with A, B, C, D constants, and U(S) and W(s) two polynomials of the form ES^4+FS^2+G and Hs^4+Is^2+J respectively with E, F, G, H, I, J constants. Solving (3.30) we obtain 81 pairs (s,S). Only 9 of these pairs are solutions of (3.29). When we calculate (α,β) corresponding to an (s,S) solution we find the zeros $S^* = (r^*,\rho^*,\alpha^*,R^*,\beta^*)$ of (3.10) given by

$$S^*_{1,\pm,\pm} = \left(\sqrt{\frac{2eh}{c-e}},\sqrt{\frac{2ch}{c-e}},\pm\frac{\pi}{2},0,\pm\frac{\pi}{2}\right);$$

$$S^*_{2,\pm} = \left(\sqrt{\frac{2eh}{c-e}},\sqrt{\frac{2ch}{c-e}},\pm\frac{\pi}{2},0,0\right);$$

$$S^*_{3,\pm} = \left(\sqrt{\frac{-2eh}{c-e}},\sqrt{\frac{2h(c-2e)}{c-e}},0,0,\pm\frac{\pi}{2}\right);$$

$$S^*_{4} = \left(\sqrt{\frac{-2eh}{c-e}},\sqrt{\frac{2h(c-2e)}{c-e}},0,0,0\right).$$

The values of r and R are not well defined in these solutions. So the averaging theory in subcase 2.4 does not give information.

Case 3: $b\sin 2\alpha = 0$. Then $b=0$ or $\sin 2\alpha = 0$.

Subcase 3.1: $b=0$. Then $f_{11} = -crR^2\sin(2\beta)/8$. Solving $f_{11}=0$ we obtain the following four subcases: $c=0$, $r=0$ (studied in case 1), $R=0$, $\beta = \dfrac{m\pi}{2}$ with $m\in\mathbb{Z}$.

Subcase 3.1.1: $c=0$. No information as in subcase 1.1. Hence *in what follows in subcase* 3.1 *we assume that* $c\neq 0$.

Subcase 3.1.2: $R = 0$. This subcase does not give results because f_{13} and f_{11} will be zero.

Subcase 3.1.3 $\beta = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$. Due to the periodicity of the sinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 3.1.3.1: Assume that either $\beta = 0$ or $\beta = \pi$. So $f_{13} = \dfrac{eR}{8}(2h + r^2 - R^2)\sin 2\alpha$.

$$f_{13}(r, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} e = 0, \\ R = 0, \\ R = \sqrt{2h + r^2}, \\ \alpha = \dfrac{n\pi}{2} \quad \text{with} \quad n \in \mathbb{Z}. \end{cases}$$

Subcase 3.1.3.1.1: $e = 0$. No information as in subcase 1.1. Hence, *in what follows in subcase* 3.1.3 *we assume that* $e \neq 0$.

Subcase 3.1.3.1.2: $R = 0$. Then we have

$$\begin{aligned} f_{12} = & -\frac{3}{8}[2dh + (a+d)r^2], \\ f_{14} = & -\frac{1}{8}[(3a + 3c + 2e + e\cos 2\alpha)r^2 + 2eh(2 + \cos 2\alpha)]. \end{aligned}$$

Subcase 3.1.3.1.2.1: $a + d \neq 0$. Then solving $f_{12} = 0$ we have $r = \sqrt{\dfrac{-2dh}{a+d}}$. So $\rho = \sqrt{\dfrac{2ah}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.1.3.1.2.1.1: $ae \neq 0$. We get $\alpha = \pm\dfrac{1}{2}\arccos\dfrac{3ad + 3cd - 2ae}{ae}$.

In the case that $\left|\dfrac{3ad + 3cd - 2ae}{ae}\right| < 1$, $ae(a+d) \neq 0$, $hd(a+d) < 0$ and $ah(a+d) > 0$, system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2ah}{a+d}}$ given by

$$\left(\sqrt{-\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\left(\frac{3ad + 3cd - 2ae}{ae}\right), 0, 0\right),$$
$$\left(\sqrt{-\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\left(\frac{3ad + 3cd - 2ae}{ae}\right), 0, \pi\right),$$

which reduce to two zeros when $ae(a+d) \neq 0$, $hd(a+d) < 0$, $ah(a+d) > 0$

and $\left|\dfrac{3ad+3cd-2ae}{ae}\right| = 1$. But when the Jacobian is evaluated on these solutions it becomes zero so the averaging theory does not give information in this subcase.

Subcase 3.1.3.1.2.1.2: $ae = 0$. Then we have $f_{14} = \dfrac{3hd(a+c)}{4(a+d)}=$ constant.

Subcase 3.1.3.1.2.2: $a+d=0$. We have $f_{12} = -3dh/4=$ constant.

Subcase 3.1.3.1.3: $R=\sqrt{2h+r^2}$. Studied in the subcase 2.4.1.

Subcase 3.1.3.1.4: $\alpha = \dfrac{n\pi}{2}$. Due to the periodicity of the sinus we study the cases $n=0$ and $n=2$, and the cases $n=1$ and $n=3$ together.

Subcase 3.1.3.1.4.1: Assume that either $n=0$ or $n=2$, i.e. either $\alpha=0$ or $\alpha=\pi$.

$$\begin{aligned} f_{12} = & -\frac{1}{8}\left[6dh+3(a+d)r^2+3(c-d+e)R^2\right], \\ f_{14} = & -\frac{1}{8}\left[6eh+3(a+c+e)r^2-3(e-c-f)R^2\right]. \end{aligned}$$

Subcase 3.1.3.1.4.1.1: $a+c+e=0$ and $(e-c-f)\neq 0$. Solving $f_{14}=0$ we get $R=\sqrt{\dfrac{2eh}{e-c-f}}$. Substituting R in f_{12} and solving $f_{12}=0$ we obtain two subcases

Subcase 3.1.3.1.4.1.1.1: $a+d\neq 0$. Then we have $r=\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}}$ and $\rho=\sqrt{\dfrac{-2h(ce+e^2+ac+af))}{(a+d)(e-c-f)}}$. Supposing that

$$\begin{aligned} & a+c+e=0, \quad e-c-f\neq 0, \quad eh(e-c-f)>0 \\ & h(cd+df-ce-e^2)(a+d)(e-c-f)>0 \quad \text{and} \\ & h(ce+e^2+ac+af)(a+d)(e-c-f)<0. \end{aligned} \tag{3.31}$$

System (3.10) has the following four zeros $S^* = (r^*,\alpha^*,R^*,\beta^*)$ with $\rho =$

$\sqrt{\dfrac{-2h(ce+e^2+ac+af)}{(a+d)(e-c-f)}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},0,\sqrt{\dfrac{2eh}{e-c-f}},0\right),\\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},\pi,\sqrt{\dfrac{2eh}{e-c-f}},0\right),\\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},0,\sqrt{\dfrac{2eh}{e-c-f}},\pi\right),\\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},\pi,\sqrt{\dfrac{2eh}{e-c-f}},\pi\right).\end{array} \tag{3.32}$$

The Jacobian

$$\begin{aligned} J_{f_1(s*)} = \; & \frac{9ce^3h^4}{16(a+d)^2(e-c-f)^4}(ce+e^2+ac+af)(cd+df-ce \\ & -e^2)\big(ad-c^2+2cd-2ae-2ce-e^2+af+df\big). \end{aligned}$$

With the condition $ce\big(ad-c^2+2cd-2ae-2ce-e^2+af+df\big)\neq 0$ and (3.31) we have $J_{f_1(S^*)}\neq 0$. The condition is not empty because the value $a=2, c=-1, d=-3, f=-1, e=-1, h=-1$ satisfy it. Therefore the four zeros (3.32) of (3.10) provide four periodic solutions of (3.8).

Subcase 3.1.3.1.4.1.1.2: $a+d=0$. Then $f_{12}=3h(ce+e^2-dc-df)/(4(e-c-f))$= constant.

Subcase 3.1.3.1.4.1.2: $a+c+e=0$ and $(e-c-f)=0$. Then we have $f_{14}=-3eh/4$.

Subcase 3.1.3.1.4.1.3: $a+c+e\neq 0$ and $(e-c-f)=0$. Solving $f_{14}=0$ we get $r=\sqrt{\dfrac{-2eh}{a+c+e}}$. Substituting r in f_{12} and solving $f_{12}=0$ we obtain two subcases

Subcase 3.1.3.1.4.1.3.1 $c-d+e\neq 0$. Then we have $R=\sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}}$ and $\rho=\sqrt{\dfrac{2hc}{c-d+e}}$.

Assuming that

$$\begin{array}{l} a+c+e\neq 0, \quad c-e+f=0, \quad c-d+e\neq 0, \\ eh(a+c+e)<0, \quad hc(c-d+e)>0 \quad \text{and} \\ h(ae-ad-cd)(a+c+e)(c-d+e)>0. \end{array} \tag{3.33}$$

System (3.10) has the following four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hc}{c-d+e}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{-2eh}{a+c+e}}, 0, \sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}}, 0\right),\\ \left(\sqrt{\dfrac{-2eh}{a+c+e}}, \pi, \sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}}, 0\right),\\ \left(\sqrt{\dfrac{-2eh}{a+c+e}}, 0, \sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}}, \pi\right),\\ \left(\sqrt{\dfrac{-2eh}{a+c+e}}, \pi, \sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}}, \pi\right).\end{array} \tag{3.34}$$

The Jacobian

$$\begin{aligned} J_{f_1(S^*)} = \;& -\frac{9c^2e^2h^4}{16(a+c+e)^3(c-d+e)^3}(ad+cd-ae)^2(c^2-ad-2cd \\ & +2ae+2ce+e^2-af-df). \end{aligned}$$

With the condition $ce\big(c^2-ad-2cd+2ae+2ce+e^2-af-df\big) \neq 0$ and (3.33), we have $J_{f_1(S^*)} \neq 0$. The set of conditions is not empty because it is satisfied for the value $a = 9/4, c = -1, d = -4, e = -2, f = -1, h = -1$. Then the four zeros (3.34) of (3.10) provide four periodic solutions of (3.8).

Subcase 3.1.3.1.4.1.3.2: $c-d+e = 0$. Then we have $f_{12} = -3h(ad+cd-ae)/(4(a+c+e))=$ constant.

Subcase 3.1.3.1.4.1.4: $a+c+e \neq 0$ and $e-c-f \neq 0$. Then solving $f_{14} = 0$ we get $r = \sqrt{\dfrac{(e-c-f)R^2-2he}{a+c+e}}$. Substituting r in f_{12} and solving $f_{12} = 0$ we have two subcases

Subcase 3.1.3.1.4.1.4.1: $\Sigma = c^2-ad-2cd+2ae+2ce+e^2-af-df \neq 0$. We have

$R = \sqrt{\dfrac{2h(ae-ad-cd)}{\Sigma}}$ then $r = \sqrt{\dfrac{2h(cd+df-ce-e^2)}{\Sigma}}$ and

$\rho = \sqrt{\dfrac{2h(ce+c^2+ae-af)}{\Sigma}}$.

Wherever

$$\begin{array}{l}\Sigma = c^2-ad-2cd+2ae+2ce+e^2-af-df \neq 0,\\ a+c+e \neq 0, \quad e-c-f \neq 0, \quad h(cd+df-ce-e^2)\Sigma > 0,\\ h(ae-ad-cd)\Sigma > 0 \quad \text{and} \quad h(ce+c^2+ae-af)\Sigma > 0,\end{array} \tag{3.35}$$

system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\dfrac{2h(ce + c^2 + ae - af)}{\Sigma}}$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{2h(cd+df-ce-e^2)}{\Sigma}}, 0, \sqrt{\frac{2h(ae-ad-cd)}{\Sigma}}, 0\right),\\
&\left(\sqrt{\frac{2h(cd+df-ce-e^2)}{\Sigma}}, \pi, \sqrt{\frac{2h(ae-ad-cd)}{\Sigma}}, 0\right),\\
&\left(\sqrt{\frac{2h(cd+df-ce-e^2)}{\Sigma}}, 0, \sqrt{\frac{2h(ae-ad-cd)}{\Sigma}}, \pi\right),\\
&\left(\sqrt{\frac{2h(cd+df-ce-e^2)}{\Sigma}}, \pi, \sqrt{\frac{2h(ae-ad-cd)}{\Sigma}}, \pi\right).
\end{aligned} \tag{3.36}$$

The Jacobian is

$$J_{f_1(S^*)} = \frac{9ceh^4}{16\Sigma^3}(cd+df-ce-e^2)(ce+c^2+ae-af)(ae-ad-cd)^2.$$

Assuming that $ce \neq 0$ and (3.35) hold. This supposition is not empty because the value $a = 5, c = -2, d = -2, f = -2, e = -1, h = 1$, satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.36) of (3.10) provide four periodic solutions of (3.8).

Subcase 3.1.3.1.4.1.4.1: $\Sigma = 0$. We have $f_{12} = -3h(ad+cd-ae)/(4(a+c+e))$= constant.

Subcase 3.1.3.1.4.2: Assume that either $n = 1$ or $n = 3$, i.e. either $\alpha = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. Then

$$\begin{aligned}
f_{12} = & -\frac{1}{8}\big[6dh + 3(a+d)r^2 + (3c-3d+e)R^2\big],\\
f_{14} = & -\frac{1}{8}\big[2eh + (3a+3c+e)r^2 + (3c+3f-e)R^2\big].
\end{aligned}$$

Subcase 3.1.3.1.4.2.1: $3c - 3d + e \neq 0$ and $a + d = 0$. Solving $f_{12} = 0$ we get
$R = \sqrt{-\dfrac{6dh}{3c-3d+e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to R we have two subcases

Subcase 3.1.3.1.4.2.1.1: $3a + 3c + e \neq 0$. We obtain

$$r = \sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{(3c - 3d + e)(3a + 3c + e)}},$$
$$\rho = \sqrt{\frac{6h(3ac + ae + 3c^2 + 3cd + ce + 3df)}{(3c - 3d + e)(3a + 3c + e)}}.$$

Whenever

$$\begin{array}{l} 3c - 3d + e \neq 0, \quad a + d = 0, \quad 3a + 3c + e \neq 0, \\ dh(3c - 3d + e) < 0, \quad h(9cd - 3ce - e^2 + 9df)(3c - 3d + e) \\ (3a + 3c + e) > 0 \quad \text{and} \quad h(3ac + ae + 3c^2 + 3cd + ce + 3df) \\ (3c - 3d + e)(3a + 3c + e) > 0, \end{array} \tag{3.37}$$

system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\frac{6h(3ac + ae + 3c^2 + 3cd + ce + 3df)}{(3c - 3d + e)(3a + 3c + e)}}$ given by

$$\begin{array}{l} \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{(3c - 3d + e)(3a + 3c + e)}}, \frac{\pi}{2}, \sqrt{-\frac{6dh}{(3c - 3d + e)}}, 0\right), \\ \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{(3c - 3d + e)(3a + 3c + e)}}, \frac{3\pi}{2}, \sqrt{-\frac{6dh}{(3c - 3d + e)}}, 0\right), \\ \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{(3c - 3d + e)(3a + 3c + e)}}, \frac{\pi}{2}, \sqrt{-\frac{6dh}{(3c - 3d + e)}}, \pi\right), \\ \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{(3c - 3d + e)(3a + 3c + e)}}, \frac{3\pi}{2}, \sqrt{-\frac{6dh}{(3c - 3d + e)}}, \pi\right). \end{array} \tag{3.38}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & -\frac{27cd^2eh^4}{16(3a + 3c + e)^2(3c - 3d + e)^4}(9cd - 3ce - e^2 + 9df)(3ac \\ & + ae + 3c^2 + 3cd + ce + 3df)(9c^2 - 9ad - 18cd + 6ae + 6ce \\ & + e^2 - 9af - 9df). \end{aligned}$$

In the case that $cde(9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df) \neq 0$ and (3.37) hold, the set of these conditions is not empty because the value $a = 2, c = -1, d = -2, f = 0, e = -1, h = 1$, satisfy it. Therefore we have $J_{f_1(S^*)} \neq 0$ and the four zeros (3.38) of (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.1.3.1.4.2.1.2: $3a + 3c + e = 0$. We have $f_{14} = h(9cd - 3ce - e^2 + 9df)/(4(3c - 3d + e))$= constant.

Subcase 3.1.3.1.4.2.2: $3c - 3d + e = 0$ and $a + d = 0$. We get $f_{12} = -3dh/4$= constant.

Subcase 3.1.3.1.4.2.3: $3c - 3d + e = 0$ and $a + d \neq 0$. Then $f_{12} = 0 \Rightarrow r = \sqrt{\dfrac{-2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ with respect to R we have two subcases

Subcase 3.1.3.1.4.2.3.1: $3c - e + 3f \neq 0$. We obtain $R = \sqrt{\dfrac{2h(3ad + 3cd - ae)}{(a+d)(3c - e + 3f)}}$ and $\rho = \sqrt{\dfrac{6h(ac - ad + af - cd)}{(a+d)(3c - e + 3f)}}$.
Assuming that

$$\begin{array}{l} 3c - 3d + e = 0, \quad a + d \neq 0, \quad 3c - e + 3f \neq 0, \\ dh(a+d) < 0, \quad h(3ad + 3cd - ae)(a+d)(3c - e + 3f) > 0 \\ \text{and} \quad h(ac - ad + af - cd)(a+d)(3c - e + 3f) > 0. \end{array} \tag{3.39}$$

System (3.10) for $n = 1$ and $n = 3$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{6h(ac - ad + af - cd)}{(a+d)(3c - e + 3f)}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3ad + 3cd - ae)}{(a+d)(3c - e + 3f)}}, 0\right), \\ \left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3ad + 3cd - ae)}{(a+d)(3c - e + 3f)}}, 0\right), \\ \left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3ad + 3cd - ae)}{(a+d)(3c - e + 3f)}}, \pi\right), \\ \left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3ad + 3cd - ae)}{(a+d)(3c - e + 3f)}}, \pi\right). \end{array} \tag{3.40}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{3cdeh^4}{16(3c - e + 3f)^3(a+d)^4}(3ad + 3cd - ae)^2(ac - ad + af \\ & -cd)(9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df). \end{aligned}$$

With the condition that $cde(9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df) \neq 0$ and (3.39) hold, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (3.40) of (3.10) provide four periodic solutions of differential system (3.8). The set of conditions is not empty because it is satisfied for the value $a = 2, c = -2/3, d = -1, f = -1, e = -1, h = 1$.

Subcase 3.1.3.1.4.2.3.2: $3c - e + 3f = 0$. We get $f_{14} = (3ad + 3cd - ae)h/(4(a + d))$= constant.

Subcase 3.1.3.1.4.2.4: $a + d \neq 0$ and $3c - 3d + e \neq 0$. Then we have $r = \sqrt{-\dfrac{6dh + (3c - 3d + e)R^2}{3(a + d)}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.1.3.1.4.2.4.1: If

$$\Sigma_1 = 9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df \neq 0.$$

Then we have $R = \sqrt{-\dfrac{6h(3ad + 3cd - ae)}{\Sigma_1}}$, $r = \sqrt{\dfrac{2h(9cd - 3ce - e^2 + 9df)}{\Sigma_1}}$ and $\rho = \sqrt{\dfrac{6h(3c^2 + ce + ae - 3af)}{\Sigma_1}}$.

Considering that

$$\begin{gathered} \Sigma_1 = 9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df \neq 0, \\ a + d \neq 0, \quad 3c - 3d + e \neq 0, \quad h(9cd - 3ce - e^2 + 9df)\Sigma_1 > 0, \\ h(3c^2 + ce + ae - 3af)\Sigma_1 > 0 \quad \text{and} \quad h(3ad + 3cd - ae)\Sigma_1 < 0. \end{gathered} \tag{3.41}$$

System (3.10) for $n = 1$ and $n = 3$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{6h(3c^2 + ce + ae - 3af)/\Sigma_1}$ given by

$$\begin{gathered} \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{\Sigma_1}}, \frac{\pi}{2}, \sqrt{\frac{-6h(3ad + 3cd - ae)}{\Sigma_1}}, 0\right), \\ \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{\Sigma_1}}, \frac{3\pi}{2}, \sqrt{\frac{-6h(3ad + 3cd - ae)}{\Sigma_1}}, 0\right), \\ \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{\Sigma_1}}, \frac{\pi}{2}, \sqrt{\frac{-6h(3ad + 3cd - ae)}{\Sigma_1}}, \pi\right), \\ \left(\sqrt{\frac{2h(9cd - 3ce - e^2 + 9df)}{\Sigma_1}}, \frac{3\pi}{2}, \sqrt{\frac{-6h(3ad + 3cd - ae)}{\Sigma_1}}, \pi\right). \end{gathered} \tag{3.42}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \;& -\frac{27ceh^4}{16{\Sigma_1}^3}(3ad + 3cd - ae)^2(3c^2 + ae + ce - 3af)(9cd - 3ce \\ & -e^2 + 9df). \end{aligned}$$

Supposing $ce \neq 0$ and (3.41) hold. This assumption is not empty because the value $a = 1/2, c = -1, d = -1, f = 3, e = -1, h = -1$ satisfy it. Then

$J_{f_1(S^*)} \neq 0$ and the four zeros (3.42) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.1.3.1.4.2.4.2: $\Sigma_1 = 0$. Then we get $f_{14} = h(3ad + 3cd - ae)/(4(a + d))=$ constant.

Subcase 3.1.3.2: Assume that either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$. So if $f_{13} = \dfrac{eR}{8}(R^2 - 2h - r^2)\sin 2\alpha = 0$, then consequently one of the following four subcases holds: $e = 0$, $R = 0$, $R = \sqrt{2h + r^2}$ and $\alpha = l\pi/2$ with $l \in \mathbb{Z}$.

Subcase 3.1.3.2.1: $e = 0$. No information as in subcase 1.1. So *in what follows in this subcase we assume that* $e \neq 0$.

Subcase 3.1.3.2.2: $R = 0$. Then

$$\begin{aligned} f_{12} = &\ -\frac{3}{8}[2dh + (a + d)r^2], \\ f_{14} = &\ -\frac{1}{8}[4eh + (3a + c + 2e)r^2 - e(2h + r^2)\cos 2\alpha]. \end{aligned}$$

Subcase 3.1.3.2.2.1: $a + d \neq 0$. Solving $f_{12} = 0$, we obtain $r = \sqrt{-2dh/(a + d)}$ and $\rho = \sqrt{2ah/(a + d)}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.1.3.2.2.1.1: $ae \neq 0$. we get $\alpha = \pm\dfrac{1}{2}\arccos\dfrac{2ae - 3ad - cd}{ae}$. Therefore when $hd(a + d) < 0$, $ah(a + d) > 0$ and $\left|\dfrac{2ae - 3ad - cd}{ae}\right| < 1$, system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2ah/(a + d)}$ given by $\left(\sqrt{-\dfrac{2dh}{a + d}}, \pm\dfrac{1}{2}\arccos\dfrac{2ae - 3ad - cd}{ae}, 0, \dfrac{\pi}{2}\right)$, $\left(\sqrt{-\dfrac{2dh}{a + d}}, \pm\dfrac{1}{2}\arccos\dfrac{2ae - 3ad - cd}{ae}, 0, \dfrac{3\pi}{2}\right)$, which reduces to two zeros if $\left|\dfrac{2ae - 3ad - cd}{ae}\right| = 1$, $hd(a + d) < 0$ and $ah(a + d) > 0$. Evaluating the Jacobian on these zeros we get zero so the averaging theory does not give information in this subcase.

Subcase 3.1.3.2.2.1.2: $ae = 0$. We get $f_{14} = hd(3a + c)/(4(a + d))=$ constant.

Subcase 3.1.3.2.2.2: $a+d=0$. We have $f_{12} = -3dh/4=$ constant.

Subcase 3.1.3.2.3: $R = \sqrt{2h+r^2}$. Studied in the subcase 2.4.2.

Subcase 3.1.3.2.4: $\alpha = \dfrac{l\pi}{2}$ with $l \in \mathbb{Z}$. Due to the periodicity of the sinus we study the cases $l=0$ and $l=2$, and the cases $l=1$ and $l=3$ together.

Subcase 3.1.3.2.4.1: Assume that either $l=0$ or $l=2$, i.e. either $\alpha = 0$ or $\alpha = \pi$. Then

$$
\begin{aligned}
f_{12} = & -\frac{1}{8}\left[6dh + 3(a+d)r^2 + (c-3d+e)R^2\right],\\
f_{14} = & -\frac{1}{8}\left[2he + (3a+c+e)r^2 + (c-e+3f)R^2\right].
\end{aligned}
$$

Subcase 3.1.3.2.4.1.1: $3a+c+e=0$ and $c-e+3f \neq 0$. Solving $f_{14}=0$, we obtain $R = \sqrt{-\dfrac{2he}{c-e+3f}}$. Substituting R in f_{12} and solving $f_{12}=0$ with respect to r we have two subcases

Subcase 3.1.3.2.4.1.1.1: $a+d \neq 0$. Then we get
$r = \sqrt{\dfrac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}$ and $\rho = \sqrt{\dfrac{2h(ce+e^2+3ac+9af)}{3(a+d)(c-e+3f)}}$.
Supposing that

$$
\begin{aligned}
&3a+c+e=0, \quad c-e+3f \neq 0, \quad he(c-e+3f)<0,\\
&h(a+d)(c-e+3f)(ce+e^2-9df-3cd)>0, \quad h(a+d)\\
&(c-e+3f)(ce+e^2+3ac+9af)>0 \quad \text{and} \quad a+d\neq 0.
\end{aligned}
\tag{3.43}
$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\dfrac{2h(ce+e^2+3ac+9af)}{3(a+d)(c-e+3f)}}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, 0, \sqrt{\frac{-2he}{c-e+3f}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, \pi, \sqrt{\frac{-2he}{c-e+3f}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, 0, \sqrt{\frac{-2he}{c-e+3f}}, \frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, \pi, \sqrt{\frac{-2he}{c-e+3f}}, \frac{3\pi}{2}\right).
\end{aligned}
\tag{3.44}
$$

The Jacobian is

$$J_{f_1(S^*)} = \frac{ce^3h^4}{144(a+d)^2(c-e+3f)^4}(3ac+ce+e^2+9af)(3cd-ce -e^2+9df)\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big).$$

Considering that $ce\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big) \neq 0$ and (3.43) hold. This assumption is not empty because it is satisfied for the value $a=2/3, d=-1, c=-1, f=-1, e=-1, h=-1$. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.44) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.1.3.2.4.1.1.2: $a+d=0$. We get $f_{12} = h(2ec+2e^2-6dc-18df)/(8(c-e+3f))=$ constant.

Subcase 3.1.3.2.4.1.2: $3a+c+e=0$ and $c-e+3f=0$. We have $f_{14} = -eh/4=$ constant.

Subcase 3.1.3.2.4.1.3: $3a+c+e \neq 0$ and $c-e+3f=0$. Solving $f_{14}=0$ we obtain $r=\sqrt{-2he/(3a+c+e)}$. Substituting r in f_{12} and solving $f_{12}=0$ with respect to R we have two subcases

Subcase 3.1.3.2.4.1.3.1: $3d-c-e \neq 0$. We get

$$R = \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}} \text{ and } \rho = \sqrt{\frac{-2ch}{3d-c-e}}.$$

Assuming that

$$\begin{gathered} 3a+c+e \neq 0, \quad c-e+3f=0, \quad 3d-c-e \neq 0, \\ he(3a+c+e)<0, \quad hc(3d-c-e)<0 \quad \text{and} \\ h(3ad+cd-ae)(3d-c-e)(3a+c+e)>0. \end{gathered} \tag{3.45}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\frac{-2ch}{3d-c-e}}$ given by

$$\begin{gathered} \left(\sqrt{\frac{-2he}{3a+c+e}}, 0, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{\pi}{2}\right), \\ \left(\sqrt{\frac{-2he}{3a+c+e}}, \pi, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{\pi}{2}\right), \\ \left(\sqrt{\frac{-2he}{3a+c+e}}, 0, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{3\pi}{2}\right), \\ \left(\sqrt{\frac{-2he}{3a+c+e}}, \pi, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{3\pi}{2}\right). \end{gathered} \tag{3.46}$$

The Jacobian is

$$J_{f_1(S^*)} = \frac{9c^2e^2h^4(3ad+cd-ae)^2}{16(3d-c-e)^3(3a+c+e)^3}(c^2-9ad-6cd+6ae+2ce+e^2 -9af-9df).$$

Supposing that $ce\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big) \neq 0$ and (3.45) hold. This assumption is not empty because it is satisfied for the value $a=3/2, d=-2, c=-1, f=-1, e=-4, h=-1$. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.46) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.1.3.2.4.1.3.2: $3d-c-e=0$. We get $f_{12} = -3h(3ad+cd-ae)/(4(3a+c+e))$= constant.

Subcase 3.1.3.2.4.1.4: $3a+c+e \neq 0$ and $c-e+3f \neq 0$. Solving $f_{14}=0$ we have $R = \sqrt{-\dfrac{2he+(3a+c+e)r^2}{(c-e+3f)}}$. Substituting R in f_{12} and solving $f_{12}=0$ with respect to r if

Subcase 3.1.3.2.4.1.4.1: $\Sigma_2 = c^2-9ad-6cd+6ae+2ce+e^2-9af-9df \neq 0$. Then we get $r = \sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}}$, $R = \sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}}$ and $\rho = \sqrt{\dfrac{2h(c^2+ce+3ae-9af)}{\Sigma_2}}$.

Supposing that

$$\begin{array}{l} \Sigma_2 = c^2-9ad-6cd+6ae+2ce+e^2-9af-9df \neq 0, \\ h(ae-3ad-cd)\Sigma_2 > 0, \quad 3a+c+e \neq 0, \quad c-e+3f \neq 0, \\ h(3cd-ce-e^2+9df)\Sigma_2 > 0 \quad \text{and} \\ h(c^2+ce+3ae-9af)\Sigma_2 > 0. \end{array} \tag{3.47}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with

$\rho = \sqrt{\dfrac{2h(c^2+ce+3ae-9af)}{\Sigma_2}}$ given by

$$
\begin{array}{l}
\left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}}, 0, \sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}}, \dfrac{\pi}{2}\right),\\
\left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}}, \pi, \sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}}, \dfrac{\pi}{2}\right),\\
\left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}}, 0, \sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}}, \dfrac{3\pi}{2}\right),\\
\left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}}, \pi, \sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}}, \dfrac{3\pi}{2}\right).
\end{array} \tag{3.48}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = \; & \frac{9ceh^4}{16\Sigma_2^3}(ae-3ad-cd)^2(c^2+ce+3ae-9af)(3cd-ce-e^2\\
& +9df).
\end{aligned}
$$

Assuming that $ce \neq 0$ and (3.47) hold, this supposition is not empty because the value $a=-1, d=0, c=-1, f=-1, e=5/2, h=1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.48) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.1.3.2.4.1.4.2: $\Sigma_2 = 0$. We have $f_{12} = -3h(3ad+cd-ae)/(4(3a+c+e))=$ constant.

Subcase 3.1.3.2.4.2: Assume that either $l=1$ or $l=3$. Then

$$
\begin{aligned}
f_{12} = \; & -\frac{1}{8}\big[6dh+3(a+d)r^2+(c-3d+3e)R^2\big],\\
f_{14} = \; & -\frac{1}{8}\big[(3a+c+3e)r^2+(c-3e+3f)R^2+6eh\big].
\end{aligned}
$$

Subcase 3.1.3.2.4.2.1: $a+d=0$ and $c-3d+3e \neq 0$. Solving $f_{12}=0$ we get $R = \sqrt{\dfrac{-6dh}{c-3d+3e}}$. Substituting R in f_{14} and solving $f_{14}=0$ we obtain two subcases

Subcase 3.1.3.2.4.2.1.1: $3a+c+3e \neq 0$. Then we have
$r = \sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}}$ and

$\rho = \sqrt{\dfrac{2h(3ac+9ae+c^2+3cd+3ce+9df)}{(c-3d+3e)(3a+c+3e)}}$.

Considering that

$$\begin{array}{l} a+d=0, \quad c-3d+3e\neq 0, \quad 3a+c+3e\neq 0, \\ dh(c-3d+3e)<0, \quad h(cd-ce-3e^2+3df)(c-3d+3e) \\ (3a+c+3e)>0 \quad \text{and} \quad h(3ac+9ae+c^2+3cd+3ce+9df) \\ (c-3d+3e)(3a+c+3e)>0. \end{array} \tag{3.49}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(3ac+9ae+c^2+3cd+3ce+9df)}{(c-3d+3e)(3a+c+3e)}}$ given by

$$\begin{array}{l} \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}}, \dfrac{\pi}{2}, \sqrt{\dfrac{-6dh}{(c-3d+3e)}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{-6dh}{(c-3d+3e)}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}}, \dfrac{\pi}{2}, \sqrt{\dfrac{-6dh}{(c-3d+3e)}}, \dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{-6dh}{(c-3d+3e)}}, \dfrac{3\pi}{2}\right). \end{array} \tag{3.50}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \quad & \frac{9dh^4}{16(3a+c+3e)^3(c-3d+3e)^4}(cd-ce-3e^2+3df)(3ac \\ & +c^2+3cd+9ae+3ce+9df)(-b^2+2bc-c^2+9ad+6cd \\ & -18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-3abce \\ & -4bc^2e+9acde-3bcde+3c^2de-9abe^2-12bce^2+9cde^2 \\ & +9bcdf-9bdef). \end{aligned}$$

With the condition that $d(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-3abce-4bc^2e+9acde-3bcde+3c^2de-9abe^2-12bce^2+9cde^2+9bcdf-9bdef)\neq 0$ and (3.49) hold, we have $J_{f_1(S^*)}\neq 0$ and the four zeros (3.50) of system (3.10) provide four periodic solutions of differential system (3.8). The set of conditions is not empty because the value $a=4, b=-16, c=-169/32, d=-4, f=-1, e=-1, h=1$ satisfy it.

Subcase 3.1.3.2.4.2.1.2: $3a + c + 3e = 0$. Then we have $f_{14} = 3(cd - ce - 3e^2 + 3df)h/4(c - 3d + 3e)$= constant.

Subcase 3.1.3.2.4.2.2: $a + d = 0$ and $c - 3d + 3e = 0$. We get $f_{12} = -3dh/4$= constant.

Subcase 3.1.3.2.4.2.3: $a + d \neq 0$ and $c - 3d + 3e = 0$. Solving $f_{12} = 0$ we get $r = \sqrt{\frac{-2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.1.3.2.4.2.3.1: $c - 3e + 3f \neq 0$. Then we have
$R = \sqrt{\frac{2h(3ad + cd - 3ae)}{(a+d)(c-3e+3f)}}$ and $\rho = \sqrt{\frac{2h(ac + 3af - 3ad - cd)}{(a+d)(c-3e+3f)}}$.
Considering that

$$\begin{array}{l} a+d \neq 0, \quad c-3d+3e=0, \quad c-3e+3f \neq 0, \\ dh(a+d) < 0, \quad h(3ad+cd-3ae)(a+d) \\ (c-3e+3f) > 0 \quad \text{and} \quad h(ac+3af-3ad-cd) \\ (a+d)(c-3e+3f) > 0. \end{array} \tag{3.51}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\frac{2h(ac + 3af - 3ad - cd)}{(a+d)(c-3e+3f)}}$ given by

$$\begin{array}{l} \left(\sqrt{\frac{-2dh}{a+d}}, \frac{\pi}{2}, \sqrt{\frac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \frac{\pi}{2}\right), \\ \left(\sqrt{\frac{-2dh}{a+d}}, \frac{3\pi}{2}, \sqrt{\frac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \frac{\pi}{2}\right), \\ \left(\sqrt{\frac{-2dh}{a+d}}, \frac{\pi}{2}, \sqrt{\frac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \frac{3\pi}{2}\right), \\ \left(\sqrt{\frac{-2dh}{a+d}}, \frac{3\pi}{2}, \sqrt{\frac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \frac{3\pi}{2}\right). \end{array} \tag{3.52}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{dh^4}{16(a+d)^4(c-3e+3f)^3}(3ad + cd \\ & -3ae)(-ac + 3ad + cd - 3af)(-b^2 + 2bc - c^2 + 9ad + 6cd \\ & -18ae - 6be - 6ce - 9e^2 + 9af + 6bf + 9df)(bc^2d + abce \\ & -3abde + 3acde - 4bcde + c^2de - 3ace^2 + 3bcdf + 3abef). \end{aligned}$$

With the condition that $(-b^2+2bc-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(bc^2d+abce-3abde+3acde-4bcde+c^2de-3ace^2+3bcdf+3abef)\neq 0$ and (3.51) we have $J_{f_1(S^*)}\neq 0$ and the four zeros (3.52) of system (3.10) provide four periodic solutions of differential system (3.8). The set of conditions is not empty because the value $a=3, b=-68, c=-3, d=-2, f=-6, e=-1, h=1$ satisfy it.

Subcase 3.1.3.2.4.2.3.2: $c-3e+3f=0$. We have $f_{14}=(cd+3ad-3ae)h/(4(a+d))=$ constant.

Subcase 3.1.3.2.4.2.4: $a+d\neq 0$ and $c-3d+3e\neq 0$. Then we have $r=\sqrt{\dfrac{3dR^2-6dh-cR^2-3eR^2}{3(a+d)}}$. Substituting r in f_{14} and solving $f_{14}=0$ we obtain two subcases

Subcase 3.1.3.2.4.2.4.1:If
$\Sigma_3=c^2-9ad-6cd+18ae+6ce+9e^2-9af-9df\neq 0$

we obtain $R=\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}}$, $r=\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}}$

and $\rho=\sqrt{\dfrac{2h(c^2+3ce+9ae-9af)}{\Sigma_3}}$.
In the case that

$$\begin{array}{l}\Sigma_3=c^2-9ad-6cd+18ae+6ce+9e^2-9af-9df\neq 0,\\ a+d\neq 0,\quad c-3d+3e\neq 0,\quad h(3ae-3ad-cd)\Sigma_3>0,\\ h(cd-ce-3e^2+3df)\Sigma_3>0\quad\text{and}\\ h(c^2+3ce+9ae-9af)\Sigma_3>0,\end{array}\tag{3.53}$$

hold, system (3.10) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with
$\rho=\sqrt{\dfrac{2h(c^2+3ce+9ae-9af)}{\Sigma_3}}$ given by

$$\begin{array}{l}\left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{3\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{3\pi}{2}\right),\\ \left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{3\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{3\pi}{2}\right).\end{array}\tag{3.54}$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = \; & \frac{9h^4}{16\Sigma_3^4}(3ad+cd-3ae)(c^2+9ae+3ce-9af)(cd-ce \\
& -3e^2+3df)(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce \\
& -9e^2+9af+6bf+9df)(3bc^2d-4bc^2e+9acde+3c^2de \\
& -9abe^2-9ace^2-12bce^2+9bcdf+9abef).
\end{aligned}
$$

Whenever $(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-4bc^2e+9acde+3c^2de-9abe^2-9ace^2-12bce^2+9bcdf+9abef) \neq 0$ and (3.53) hold, then $J_{f_1(S^*)} \neq 0$ and the four zeros of system (3.10) provide four periodic solutions (3.54) of differential system (3.8). The set of conditions is not empty, the value $a=3, b=0, c=-4, d=-2, f=-2, e=-1, h=1$ satisfy it.

Subcase 3.1.3.2.4.2.4.2: $\Sigma_3 = 0$. We have $f_{14} = h(cd+3ad-3ae)/(4(a+d))$= constant.

Subcase 3.2: $\sin 2\alpha = 0$.

Subcase 3.2.1: Assume that either $\alpha = 0$ or $\alpha = \pi$. Then we get $f_{11} = -\frac{1}{8}crR^2 \sin 2\beta$. If $f_{11} = 0$ then consequently one of the following four subcases holds $c=0$, $r=0$ (studied in case 1), $R=0$, $\beta = p\pi/2$ with $p \in \mathbb{Z}$.

Subcase 3.2.1.1: $c=0$. No information as in subcase 1.1. So *in what follows in subcase* 3.2 *we assume that* $c \neq 0$.

Subcase 3.2.1.2: $R=0$. Then

$$
\begin{aligned}
f_{12} = \; & -\frac{3}{8}\left[(a+2b+d)r^2+2(b+d)h\right], \\
f_{14} = \; & -\frac{1}{8}\left[3ar^2+cr^2(2+\cos 2\beta)+(2h+r^2)(3b+2e+e\cos 2\beta)\right].
\end{aligned}
$$

Subcase 3.2.1.2.1: $a+2b+d \neq 0$. Then $f_{12} = 0 \Rightarrow r = \sqrt{\dfrac{-2h(b+d)}{a+2b+d}}$ and $\rho = \sqrt{\dfrac{2h(a+b)}{a+2b+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$

Subcase 3.2.1.2.1.1: $bc+cd-ae-be \neq 0$. We get
$\beta = \pm\frac{1}{2}\arccos\left(\Delta_5 = \dfrac{3b^2-2bc-3ad-2cd+2ae+2be}{bc+cd-ae-be}\right)$.

In the case that

$$\begin{array}{c} a+2b+d\neq 0, \quad bc+cd-ae-be\neq 0, \\ h(b+d)(a+2b+d)<0, \quad h(a+b)(a+2b+d)>0, \\ \text{and} \quad |\Delta_5|<1, \end{array} \tag{3.55}$$

system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(a+b)}{a+2b+d}}$ given by

$$\begin{array}{l} \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}}, 0, 0, \pm\dfrac{1}{2}\arccos\Delta_5\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}}, \pi, 0, \pm\dfrac{1}{2}\arccos\Delta_5\right), \end{array} \tag{3.56}$$

which reduce to two zeros if $a+2b+d\neq 0$, $bc+cd-ae-be\neq 0$, $h(b+d)(a+2b+d)<0$, $h(a+b)(a+2b+d)>0$ and $|\Delta_5|=1$.

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{9bh^4}{32(a+2b+d)^3}(a+b)(b+d)(3b^2-bc-3ad-cd+ae \\ & +be)(b^2-bc-ad-cd+ae+be). \end{aligned}$$

Supposing that $b(3b^2-bc-3ad-cd+ae+be)(b^2-bc-ad-cd+ae+be)\neq 0$ and (3.55) hold. This assumption is not empty because it is satisfied for the value $a=2, b=-1, c=-2, d=-1, e=0, h=-1$. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (3.56) of system (3.10) provide two periodic solutions of differential system (3.8) because when $R=0$, the two solutions of β provide the same initial conditions.

Subcase 3.2.1.2.1.2: $bc+cd-ae-be=0$. Then $f_{14} = -h(3b^2-2bc-3ad-2cd+2ae+2be)/(4(a+2b+d)) =$ constant.

Subcase 3.2.1.2.2: $a+2b+d=0$. Then we have $f_{12} = -\dfrac{3h}{4}(b+d)=$ constant.

Subcase 3.2.1.3: $\beta = \dfrac{p\pi}{2}$ with $p\in\mathbb{Z}$. Due to the periodicity of the sinus we study the subcases $p=0$ and $p=2$, and the subcases $p=1$ and $p=3$ together.

Subcase 3.2.1.3.1: Assume that either $p=0$ or $p=2$, i.e. $\beta=0$ or $\beta=\pi$.

$$\begin{aligned} f_{12} = & -\frac{1}{8}\left[6(b+d)h+3(a+2b+d)r^2-3(b-c+d-e)R^2\right], \\ f_{14} = & -\frac{1}{8}\left[6(b+e)h+3(a+b+c+e)r^2-3(b-c+e-f)R^2\right]. \end{aligned}$$

Subcase 3.2.1.3.1.1: $b-c+d-e=0$ and $a+2b+d\neq 0$. Solving $f_{12}=0$, we obtain $r=\sqrt{-\dfrac{2h(b+d)}{a+2b+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R we have two subcases

Subcase 3.2.1.3.1.1.1: $b-c+e-f\neq 0$ we get $R=\sqrt{\dfrac{2hN_1}{\delta_1}}$ and $\rho=\sqrt{\dfrac{2hN_2}{\delta_1}}$ where $N_1=b^2-cd+be-bc+ae-ad$, $N_2=cd+ab-ac+ad-af-bf$ and $\delta_1=(a+2b+d)(b-c+e-f)$.

Assuming that

$$\begin{array}{l} b-c+d-e=0, \quad a+2b+d\neq 0, \quad b-c+e-f\neq 0, \\ h(b+d)(a+2b+d)<0, \quad hN_1\delta_1>0 \quad \text{and} \quad hN_2\delta_1>0. \end{array} \tag{3.57}$$

System (3.10) for $p=0$ and $p=2$ has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{2hN_2}{\delta_1}}$ given by

$$\begin{array}{c} \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},0,\sqrt{\dfrac{2hN_1}{\delta_1}},0\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},0,\sqrt{\dfrac{2hN_1}{\delta_1}},\pi\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},\pi,\sqrt{\dfrac{2hN_1}{\delta_1}},0\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},\pi,\sqrt{\dfrac{2hN_1}{\delta_1}},\pi\right). \end{array} \tag{3.58}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = {} & -\frac{9h^4(b+d)}{16(a+2b+d)^4(b-c+e-f)^3}N_1N_2(b^2-2bc+c^2-ad \\ & -2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c \\ & -b^2cd+bc^2d+ab^2e-abce-bc^2e+abde-acde-c^2de \\ & +ace^2+bce^2+b^2cf+bcdf-abef-b^2ef). \end{aligned}$$

With the condition $(b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c-b^2cd+bc^2d+ab^2e-abce-bc^2e+abde-acde-c^2de+ace^2+bce^2+b^2cf+bcdf-abef-b^2ef)\neq 0$ and (3.57), we have $J_{f_1(S^*)}\neq 0$. The set of conditions is not empty because it is satisfied for the value $a=1, b=2, c=-1, d=-4, e=-1, f=-3, h=1$. Therefore the four zeros (3.58) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.1.3.1.1.2: $b-c+e-f=0$. We have $f_{14}=3h(-b^2+bc+ad+cd-ae-be)/(4(a+2b+d))=$ constant.

Subcase 3.2.1.3.1.2: $b-c+d-e=0$ and $a+2b+d=0$. Then $f_{12}=-\dfrac{3h}{4}(b+d)=$ constant.

Subcase 3.2.1.3.1.3: $b-c+d-e\neq 0$ and $a+2b+d=0$. Solving $f_{12}=0$, we obtain $R=\sqrt{\dfrac{2h(b+d)}{b-c+d-e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r we have two subcases

Subcase 3.2.1.3.1.3.1: $a+b+c+e\neq 0$. We get $r=\sqrt{\dfrac{2hN_3}{\delta_2}}$ and $\rho=\sqrt{\dfrac{2hN_4}{\delta_2}}$ where $N_3=e^2-cd+ce+be-bf-df$, $N_4=-ac-ae-cb-c^2-cd-ce-bf-df$ and $\delta_2=(b-c+d-e)(a+b+c+e)$.

Assuming that

$$\begin{array}{l} b-c+d-e\neq 0,\quad a+2b+d=0,\quad a+b+c+e\neq 0,\\ h(b+d)(b-c+d-e)>0,\quad hN_3\delta_2>0\quad \text{and}\quad hN_4\delta_2>0. \end{array} \tag{3.59}$$

System (3.10) for $p=0$ and $p=2$ has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{2hN_4}{\delta_2}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2hN_3}{\delta_2}},0,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},0\right),\\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},\pi,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},0\right),\\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},0,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},\pi\right),\\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},\pi,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},\pi\right). \end{array} \tag{3.60}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{9h^4(b+d)}{16(b-c+d-e)^4(a+b+c+e)^3}N_3N_4(2bc-b^2-c^2\\ & +ad+2cd-2ae-2be-2ce-e^2+af+2bf+df)(b^2ce\\ & -bc^2d+bc^2e+acde+c^2de-abe^2+bce^2+cde^2-b^2cf\\ & -bcdf-b^2ef-bdef). \end{aligned}$$

With the condition $(2bc - b^2 - c^2 + ad + 2cd - 2ae - 2be - 2ce - e^2 + af + 2bf + df)(b^2ce - bc^2d + bc^2e + acde + c^2de - abe^2 + bce^2 + cde^2 - b^2cf - bcdf - b^2ef - bdef) \neq 0$ and (3.59) we have $J_{f_1(S^*)} \neq 0$. The set of conditions is not empty because it is satisfied for the value $a = 7, b = -2, c = -1, d = -3, e = -1, f = -2, h = 1$. Therefore the four zeros (3.60) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.1.3.1.3.2: $a + b + c + e = 0$. Then we have
$f_{14} = -3h(bf + cd + df - be - ce - e^2)/(4(b - c + d - e))=$ constant.

Subcase 3.2.1.3.1.4: $b - c + d - e \neq 0$ and $a + 2b + d \neq 0$. Solving $f_{12} = 0$ we get
$R = \sqrt{\dfrac{2h(b + d) + (a + 2b + d)r^2}{(b - c + d - e)}}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to r we have two subcases

Subcase 3.2.1.3.1.4.1: If

$$\omega = b^2 - 2bc + c^2 - ad - 2cd + 2ae + 2be + 2ce + e^2 - af - 2bf - df \neq 0,$$

then we get $r = \sqrt{\dfrac{2hN_5}{\omega}}$, $R = \sqrt{\dfrac{2hN_6}{\omega}}$ and $\rho = \sqrt{\dfrac{2hN_7}{\omega}}$, where $N_5 = cd - be - ce - e^2 + bf + df$, $N_6 = b^2 - ad - cd + ae - bc + be$, $N_7 = c^2 + ce + ae - af - bc - bf$.

In the case that

$$\begin{aligned} &\omega = b^2 - 2bc + c^2 - ad - 2cd + 2ae + 2be + 2ce + e^2 - \\ &\qquad af - 2bf - df \neq 0, \\ &b - c + d - e \neq 0, \quad a + 2b + d \neq 0, \quad hN_5\omega > 0, \\ &hN_6\omega > 0 \quad \text{and} \quad hN_7\omega > 0, \end{aligned} \tag{3.61}$$

hold, then system (3.10) for $p = 0$ and $p = 2$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_7}{\omega}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{2hN_5}{\omega}}, 0, \sqrt{\frac{2hN_6}{\omega}}, 0\right), \\ &\left(\sqrt{\frac{2hN_5}{\omega}}, 0, \sqrt{\frac{2hN_6}{\omega}}, \pi\right), \\ &\left(\sqrt{\frac{2hN_5}{\omega}}, \pi, \sqrt{\frac{2hN_6}{\omega}}, 0\right), \\ &\left(\sqrt{\frac{2hN_5}{\omega}}, \pi, \sqrt{\frac{2hN_6}{\omega}}, \pi\right). \end{aligned} \tag{3.62}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{9h^4}{16\omega^3} N_5 N_6 N_7 (bc^2d - b^2ce - bc^2e - acde - c^2de + abe^2 \\ & + ace^2 + bce^2 + b^2cf + bcdf - abef - b^2ef). \end{aligned}$$

Considering that $(bc^2d - b^2ce - bc^2e - acde - c^2de + abe^2 + ace^2 + bce^2 + b^2cf + bcdf - abef - b^2ef) \neq 0$ and (3.61) hold. This assumption is not empty because the value $a = 4, b = -3, c = -2, d = -2, f = -5, e = -1, h = -1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.62) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.1.3.1.4.2: $\omega = 0$. Then we have $f_{14} = -3h(cd - be - ce - e^2 + bf + df)/(4(b - c + d - e))$= constant.

Subcase 3.2.1.3.2: Assume that either $p = 1$ or $p = 3$, i.e. $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$.

$$\begin{aligned} f_{12} = & -\frac{1}{8}\left[6(b+d)h + 3(a+2b+d)r^2 - (3b - c + 3d - e)R^2\right], \\ f_{14} = & -\frac{1}{8}\left[2h(3b+e) + (3a+3b+c+e)r^2 - (3b - c + e - 3f)R^2\right]. \end{aligned}$$

Subcase 3.2.1.3.2.1: $3b - c + 3d - e = 0$ and $a + 2b + d \neq 0$. Solving $f_{12} = 0$ we get $r = \sqrt{-2h(b+d)/(a+2b+d)}$. Substituting r in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.2.1.3.2.1.1: $3b - c + e - 3f \neq 0$. Then we obtain $R = \sqrt{\dfrac{2hN_8}{\delta_3}}$ where $\delta_3 = (a + 2b + d)(3b - c + e - 3f)$, $N_8 = 3b^2 - cd - bc + be - 3ad + ae$ and $\rho = \sqrt{\dfrac{2hN_9}{\delta_3}}$ where $N_9 = (3ab - ac + 3ad + cd - 3af - 3bf)$.

If we have

$$\begin{gathered} 3b - c + 3d - e = 0, \quad a + 2b + d \neq 0, \quad 3b - c + e - 3f \neq 0, \\ h(b+d)(a+2b+d) < 0, \quad h\delta_3 N_8 > 0 \quad \text{and} \quad h\delta_3 N_9 > 0, \end{gathered} \tag{3.63}$$

then system (3.10) for $p = 1$ and $p = 3$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$

with $\rho = \sqrt{\dfrac{2hN_9}{\delta_3}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{-2h(b+d)}{(a+2b+d)}}, 0, \sqrt{\dfrac{2hN_8}{\delta_3}}, \dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{-2h(b+d)}{(a+2b+d)}}, 0, \sqrt{\dfrac{2hN_8}{\delta_3}}, \dfrac{3\pi}{2}\right),\\ \left(\sqrt{\dfrac{-2h(b+d)}{(a+2b+d)}}, \pi, \sqrt{\dfrac{2hN_8}{\delta_3}}, \dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{-2h(b+d)}{(a+2b+d)}}, \pi, \sqrt{\dfrac{2hN_8}{\delta_3}}, \dfrac{3\pi}{2}\right).\end{array} \tag{3.64}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \;& -\frac{h^4(b+d)N_8N_9}{16(a+2b+d)^4(3b-c+e-3f)^3}(9b^2-6bc+c^2-9ad \\ & -6cd+6ae+6be+2ce+e^2-9af-18bf-9df)(b^2c^2 \\ & -3b^3c-3b^2cd+bc^2d+3ab^2e-abce-4b^2ce+bc^2e+3abde \\ & +3acde+c^2de-ace^2-bce^2+3b^2cf+3bcdf-3abef \\ & -3b^2ef). \end{aligned}$$

Supposing that $(9b^2-6bc+c^2-9ad-6cd+6ae+6be+2ce+e^2-9af-18bf-9df)(-3b^3c+b^2c^2-3b^2cd+bc^2d+3ab^2e-abce-4b^2ce+bc^2e+3abde+3acde+c^2de-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef) \neq 0$ and (3.63) hold. This assumption is not empty because it is satisfied for the value $a=0, b=13/3, c=-4, d=-6, f=-2, e=-1, h=1$. As a result of that, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (3.64) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.1.3.2.1.2: $3b-c+e-3f=0$. We have $f_{14} = (cd-3b^2+bc-be+3ad-ae)h/(4(a+2b+d))=$ constant.

Subcase 3.2.1.3.2.2: $3b-c+3d-e=0$ and $a+2b+d=0$. Then we have $f_{12} = -\dfrac{3h}{4}(b+d)=$ constant.

Subcase 3.2.1.3.2.3: $3b-c+3d-e \neq 0$ and $a+2b+d=0$. Solving $f_{12}=0$ we get $R = \sqrt{\dfrac{6h(b+d)}{3b-c+3d-e}}$. Substituting R in f_{14} and solving $f_{14}=0$ we obtain two subcases

Subcase 3.2.1.3.2.3.1: $3a + 3b + c + e \neq 0$. We obtain $r = \sqrt{\dfrac{2hN_{10}}{\delta_4}}$ where $\delta_4 = (3a+3b+c+e)(3b-c+3d-e)$, $N_{10} = 3be+ce+e^2-3cd-9bf-9df$ and $\rho = \sqrt{\dfrac{2hN_{11}}{\delta_4}}$ where $N_{11} = (3ac+3ae+3bc+c^2+3cd+ce+9bf+9df)$.

If we have

$$\begin{array}{l} 3b - c + 3d - e \neq 0, \quad a + 2b + d = 0, \\ 3a + 3b + c + e \neq 0, h(b + d)(3b - c + 3d - e) > 0, \\ \quad h\delta_4 N_{10} > 0 \quad \text{and} \quad h\delta_4 N_{11} > 0, \end{array} \tag{3.65}$$

then system (3.10) for $p = 1$ and $p = 3$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_{11}}{\delta_4}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2hN_{10}}{\delta_4}}, 0, \sqrt{\dfrac{6h(b+d)}{3b-c+3d-e}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{10}}{\delta_4}}, \pi, \sqrt{\dfrac{6h(b+d)}{3b-c+3d-e}}, \dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{10}}{\delta_4}}, 0, \sqrt{\dfrac{6h(b+d)}{3b-c+3d-e}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{10}}{\delta_4}}, \pi, \sqrt{\dfrac{6h(b+d)}{3b-c+3d-e}}, \dfrac{3\pi}{2}\right). \end{array} \tag{3.66}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{9h^4(b+d)N_{10}N_{11}}{16(3b-c+3d-e)^4(3a+3b+c+e)^3}(6bc - 9b^2 - c^2 \\ & +9ad + 6cd - 6ae - 6be - 2ce - e^2 + 9af + 18bf + 9df) \\ & (bc^2d + 4abce + 3b^2ce + bc^2e + 3acde + 4bcde + c^2de + abe^2 \\ & +bce^2 + cde^2 + 3b^2cf + 3bcdf + 3b^2ef + 3bdef). \end{aligned}$$

Supposing that $(6bc - 9b^2 - c^2 + 9ad + 6cd - 6ae - 6be - 2ce - e^2 + 9af + 18bf + 9df)(bc^2d + 4abce + 3b^2ce + bc^2e + 3acde + 4bcde + c^2de + abe^2 + bce^2 + cde^2 + 3b^2cf + 3bcdf + 3b^2ef + 3bdef) \neq 0$ and (3.65) hold. This assumption is not empty because it is satisfied for the value $a = 4, b = -2, c = -1, d = 0, f = -1/2, e = -1, h = 1$. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (3.66) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.1.3.2.3.2: $3a + 3b + c + e = 0$. We have $f_{14} = (-3cd + 3be + ce + e^2 - 9bf - 9df)h/4(3b - c + 3d - e)=$ constant.

Subcase 3.2.1.3.2.4: $3b - c + 3d - e \neq 0$ and $a + 2b + d \neq 0$. We have $R = \sqrt{\dfrac{6h(b+d) + 3(a+2b+d)r^2}{3b - c + 3d - e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ we have two subcases

Subcase 3.2.1.3.2.4.1: If $\omega_1 = 9b^2 - 6bc + c^2 - 9ad - 6cd + 6ae + 6be + 2ce + e^2 - 9af - 18bf - 9df \neq 0$ so $r = \sqrt{\dfrac{2hN_{12}}{\omega_1}}$, $R = \sqrt{\dfrac{6hN_{13}}{\omega_1}}$ and $\rho = \sqrt{\dfrac{2hN_{14}}{\omega_1}}$ where $N_{12} = 3cd - 3be - ce - e^2 + 9bf + 9df$, $N_{13} = 3b^2 - 3ad - cd + ae - bc + be$, $N_{14} = c^2 - 3bc + 3ae + ce - 9af - 9bf$.

Supposing that

$$\begin{array}{l}\omega_1 = 9b^2 - 6bc + c^2 - 9ad - 6cd + 6ae + 6be + 2ce + e^2 - 9af \\ -18bf - 9df \neq 0, \quad 3b - c + 3d - e \neq 0, \quad a + 2b + d \neq 0, \\ hN_{12}\omega_1 > 0, \quad hN_{13}\omega_1 > 0 \quad \text{and} \quad hN_{14}\omega_1 > 0.\end{array} \tag{3.67}$$

System (3.10) for $p = 1$ and $p = 3$ has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_{14}}{\omega_1}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{2hN_{12}}{\omega_1}}, 0, \sqrt{\dfrac{6hN_{13}}{\omega_1}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{12}}{\omega_1}}, 0, \sqrt{\dfrac{6hN_{13}}{\omega_1}}, \dfrac{3\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{12}}{\omega_1}}, \pi, \sqrt{\dfrac{6hN_{13}}{\omega_1}}, \dfrac{\pi}{2}\right), \\ \left(\sqrt{\dfrac{2hN_{12}}{\omega_1}}, \pi, \sqrt{\dfrac{6hN_{13}}{\omega_1}}, \dfrac{3\pi}{2}\right).\end{array} \tag{3.68}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{9h^4}{16\omega_1^3} N_{12}N_{13}N_{14}(bc^2d - 5b^2ce + bc^2e + 3acde + c^2de + abe^2 \\ & -ace^2 - bce^2 + 3b^2cf + 3bcdf - 3abef - 3b^2ef). \end{aligned}$$

Assuming that $(bc^2d - 5b^2ce + bc^2e + 3acde + c^2de + abe^2 - ace^2 - bce^2 + 3b^2cf + 3bcdf - 3abef - 3b^2ef) \neq 0$ and (3.67) hold. This set of conditions is not empty because the value $a = 7, b = -6, c = -24, d = -45, f = 17, e = -1, h = -1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.68) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.1.3.2.4.2: $\omega_1 = 0$. Then $f_{14} = -hN_{12}/(4(3b-c+3d-e))=$ constant.

Subcase 3.2.2: Assuming either $\alpha = \dfrac{\pi}{2}$ or $\alpha = \dfrac{3\pi}{2}$. Then we have $f_{11} = -\dfrac{1}{8}crR^2 \sin 2\beta$. If $f_{11} = 0$ then consequently one of the following four subcases holds $c = 0$, $r = 0$ (studied in case 1), $R = 0$, $\beta = q\pi/2$ with $q \in \mathbb{Z}$.

Subcase 3.2.2.1: $c = 0$. No information about the periodic orbits as in subcase 1.1. So *in what follows in subcase* 3.2.2 *we assume that* $c \neq 0$.

Subcase 3.2.2.2: $R = 0$. Then

$$
\begin{aligned}
f_{12} = & -\frac{1}{8}\left[(3a+2b+3d)r^2 + 2(b+3d)h\right],\\
f_{14} = & -\frac{1}{8}\big[(3a+b+2c+2e+(c-e)\cos 2\beta)r^2 + 2h(b+2e-\\
& e\cos 2\beta)\big].
\end{aligned}
$$

Subcase 3.2.2.2.1: $3a+2b+3d \neq 0$. Solving $f_{12} = 0$ we get $r = \sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}$ and $\rho = \sqrt{\dfrac{2h(3a+b)}{3a+2b+3d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.2.2.2.1.1: $3cd+3ae+bc+be \neq 0$. Then we have
$\beta = \pm\dfrac{1}{2}\arccos\left(\Delta_6 = \dfrac{b^2-2bc-9ad-6cd+6ae+2be}{3cd+3ae+bc+be}\right)$.
Whenever

$$
\begin{aligned}
&3a+2b+3d \neq 0, \quad 3cd+3ae+bc+be \neq 0,\\
&h(b+3d)(3a+2b+3d) < 0, \quad h(3a+b)(3a+2b+3d) > 0\\
&\quad \text{and} \quad |\Delta_6| < 1,
\end{aligned}
\tag{3.69}
$$

system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h(3a+b)}{3a+2b+3d}}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{-2h(b+3d)}{3a+2b+3d}}, \frac{\pi}{2}, 0, \pm\frac{1}{2}\arccos\Delta_6\right),\\
&\left(\sqrt{\frac{-2h(b+3d)}{3a+2b+3d}}, \frac{3\pi}{2}, 0, \pm\frac{1}{2}\arccos\Delta_6\right),
\end{aligned}
\tag{3.70}
$$

which reduce to two zeros if $3a + 2b + 3d \neq 0$, $3cd + 3ae + bc + be \neq 0$, $h(b + 3d)(3a + 2b + 3d) < 0$, $h(3a + b)(3a + 2b + 3d) > 0$ and $|\Delta_6| = 1$.

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = &\ \frac{bh^4}{32(3a+2b+3d)^3}(3a+b)(b+3d)(b^2-3bc-9ad-9cd \\ &+3ae+be)(b^2-bc-9ad-3cd+9ae+3be). \end{aligned}$$

Assuming that $b(b^2 - 3bc - 9ad - 9cd + 3ae + be)(b^2 - bc - 9ad - 3cd + 9ae + 3be) \neq 0$ and (3.69) hold. This supposition is not empty because the value $a = 1/2, b = -1, c = -3/8, d = 0, e = 0, h = -1$. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (3.70) of system (3.10) provide only two periodic solutions of differential system (3.8) because when $R = 0$ the two solutions of β provide the same initial conditions in (3.3).

Subcase 3.2.2.2.1.2: $3cd + 3ae + bc + be = 0$, we have
$f_{14} = -\dfrac{h(b^2 - 2bc - 9ad - 6cd + 6ae + 2be)}{4(3a + 2b + 3d)}$= constant.

Subcase 3.2.2.2.2: $3a + 2b + 3d = 0$. Then we get $f_{12} = -\dfrac{h}{4}(b + 3d)$= constant.

Subcase 3.2.2.3: $\beta = \dfrac{q\pi}{2}$ with $q \in \mathbb{Z}$. Then due to the periodicity of the sinus we study the subcases $q = 0$ and $q = 2$, and the subcases $q = 1$ and $q = 3$ together.

Subcase 3.2.2.3.1: Assume that either $q = 0$ or $q = 2$, i.e. either $\beta = 0$ or $\beta = \pi$.

$$\begin{aligned} f_{12} = &\ -\frac{1}{8}\left[2(b+3d)h + (3a+2b+3d)r^2 - (b-3c+3d-e)R^2\right], \\ f_{14} = &\ -\frac{1}{8}\left[2(b+e)h + (3a+b+3c+e)r^2 - (b-3c+e-3f)R^2\right]. \end{aligned}$$

Subcase 3.2.2.3.1.1: $b - 3c + 3d - e = 0$ and $3a + 2b + 3d \neq 0$. Solving $f_{12} = 0$, we obtain $r = \sqrt{-\dfrac{2h(b + 3d)}{3a + 2b + 3d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ with respect to R we have two subcases

Subcase 3.2.2.3.1.1.1: $b - 3c + e - 3f \neq 0$. Then we get $R = \sqrt{\dfrac{2h(b^2 - 9ad - 9cd + 3ae - 3bc + be)}{(3a + 2b + 3d)(b - 3c + e - 3f)}}$ and

$\rho = \sqrt{\dfrac{6h(3cd - bf + ab - 3ac + 3ad - 3af)}{(3a + 2b + 3d)(b - 3c + e - 3f)}}$.

Considering that

$$\begin{aligned} & b - 3c + 3d - e = 0, \quad h(b + 3d)(3a + 2b + 3d) < 0, \\ & h(b^2 - 9ad - 9cd + 3ae - 3bc + be)(3a + 2b + 3d) \\ & (b - 3c + e - 3f) > 0 \quad \text{and} \quad h(3cd - bf + ab \\ & -3ac + 3ad - 3af)(3a + 2b + 3d)(b - 3c + e - 3f) > 0. \end{aligned} \tag{3.71}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with
$\rho = \sqrt{\dfrac{6h(3cd - bf + ab - 3ac + 3ad - 3af)}{(3a + 2b + 3d)(b - 3c + e - 3f)}}$ and
$R^* = \sqrt{\dfrac{2h(b^2 - 9ad - 9cd + 3ae - 3bc + be)}{(3a + 2b + 3d)(b - 3c + e - 3f)}}$ given by

$$\begin{aligned} & \left(\sqrt{-\frac{2h(b + 3d)}{3a + 2b + 3d}}, \frac{\pi}{2}, R^*, 0\right), \\ & \left(\sqrt{-\frac{2h(b + 3d)}{3a + 2b + 3d}}, \frac{\pi}{2}, R^*, \pi\right), \\ & \left(\sqrt{-\frac{2h(b + 3d)}{3a + 2b + 3d}}, \frac{3\pi}{2}, R^*, 0\right), \\ & \left(\sqrt{-\frac{2h(b + 3d)}{3a + 2b + 3d}}, \frac{3\pi}{2}, R^*, \pi\right). \end{aligned} \tag{3.72}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & -\frac{3h^4(b + 3d)(b^2 - 3bc - 9ad - 9cd + 3ae + be)}{16(3a + 2b + 3d)^4(b - 3c + e - 3f)^3}(ab - 3ac \\ & +3ad + 3cd - 3af - bf)(b^2 - 6bc + 9c^2 - 9ad - 18cd \\ & +6ae + 2be + 6ce + e^2 - 9af - 6bf - 9df)(b^3c - 3b^2c^2 \\ & +3b^2cd - 9bc^2d + 3ab^2e - 9abce + 3bc^2e + 9abde + 9acde \\ & +9c^2de + 12bcde - 3ace^2 - bce^2 - 3b^2cf - 9bcdf - 9abef \\ & -3b^2ef). \end{aligned}$$

In the case that $(b^2 - 6bc + 9c^2 - 9ad - 18cd + 6ae + 2be + 6ce + e^2 - 9af - 6bf - 9df)(b^3c - 3b^2c^2 + 3b^2cd - 9bc^2d + 3ab^2e - 9abce + 3bc^2e + 9abde + 9acde + 9c^2de + 12bcde - 3ace^2 - bce^2 - 3b^2cf - 9bcdf - 9abef - 3b^2ef) \neq 0$ and (3.71) hold, then

$J_{f_1(S^*)} \neq 0$ and the four zeros (3.72) of system (3.10) provide four periodic solutions of differential system (3.8). The set of conditions in not empty because for the value $a = 1, b = 5, c = -2, d = -4, f = -1, e = -1, h = 1$ it is satisfied.

Subcase 3.2.2.3.1.1.2: $b-3c+e-3f = 0$. We have $f_{14} = (-b^2+3bc+9ad+9cd-3ae-be)/(4(3a+2b+3d))$.

Subcase 3.2.2.3.1.2: $b-3c+3d-e = 0$ and $3a+2b+3d = 0$. Then we get $f_{12} = -\dfrac{1}{4}(b+3d)h=$ constant.

Subcase 3.2.2.3.1.3: $b-3c+3d-e \neq 0$ and $3a+2b+3d = 0$. Solving $f_{12} = 0$, we obtain $R = \sqrt{\dfrac{2h(b+3d)}{b-3c+3d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to r we have two subcases

Subcase 3.2.2.3.1.3.1: $3a+b+3c+e \neq 0$, Then we get $r = \sqrt{\dfrac{2hN_{15}}{\delta_5}}$ and $\rho = \sqrt{\dfrac{-6hN_{16}}{\delta_5}}$ where $N_{15} = e^2-9cd+be+3ce-3bf-9df$, $N_{16} = 3ac+ae+bc+3c^2+3cd+ce+bf+3df$ and $\delta_5 = (b-3c+3d-e)(3a+b+3c+e)$.

Supposing that

$$\begin{gathered} b-3c+3d-e \neq 0, \quad 3a+2b+3d = 0, \\ 3a+b+3c+e \neq 0, \quad h(b+3d)(b-3c+3d-e) > 0, \\ h\delta_5 N_{15} > 0 \quad \text{and} \quad h\delta_5 N_{16} < 0. \end{gathered} \tag{3.73}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{-6hN_{16}}{\delta_5}}$ given by

$$\begin{gathered} \left(\sqrt{\frac{2hN_{15}}{\delta_5}}, \frac{\pi}{2}, \sqrt{\frac{2h(b+3d)}{b-3c+3d-e}}, 0\right), \\ \left(\sqrt{\frac{2hN_{15}}{\delta_5}}, \frac{\pi}{2}, \sqrt{\frac{2h(b+3d)}{b-3c+3d-e}}, \pi\right), \\ \left(\sqrt{\frac{2hN_{15}}{\delta_5}}, \frac{3\pi}{2}, \sqrt{\frac{2h(b+3d)}{b-3c+3d-e}}, 0\right), \\ \left(\sqrt{\frac{2hN_{15}}{\delta_5}}, \frac{3\pi}{2}, \sqrt{\frac{2h(b+3d)}{b-3c+3d-e}}, \pi\right). \end{gathered} \tag{3.74}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{3h^4(b+3d)N_{15}N_{16}}{16(b-3c+3d-e)^4(3a+b+3c+e)^3)}(b^2-6bc+9c^2 \\ & -9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df) \\ & (12abce-9bc^2d+5b^2ce+15bc^2e+9acde+12bcde+9c^2de \\ & +3abe^2+5bce^2+3cde^2-3b^2cf-9bcdf+3b^2ef+9bdef). \end{aligned}$$

Supposing that $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(12abce-9bc^2d+5b^2ce+15bc^2e+9acde+12bcde+9c^2de+3abe^2+5bce^2+3cde^2-3b^2cf-9bcdf+3b^2ef+9bdef) \neq 0$ and (3.73) hold. This assumption is not empty because the value $a = 11/3, b = -1, c = -2, d = -3, f = 0, e = -1, h = 1$ satisfy it. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (3.74) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.2.3.1.3.1: $3a+b+3c+e = 0$. Then $f_{14} = h(eb+e^2-9cd+3ce-3fb-9fd)/(4(b-3c+3d-e))$= constant.

Subcase 3.2.2.3.1.4: $b-3c+3d-e \neq 0$ and $3a+2b+3d \neq 0$. Solving $f_{12} = 0$ we get $R = \sqrt{\dfrac{2h(b+3d)+(3a+2b+3d)r^2}{b-3c+3d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to r if $\omega_2 = b^2+9c^2-9ad+6ae+e^2-18cd+6ce-6bc+2be-6bf-9af-9df \neq 0$ we get $r = \sqrt{\dfrac{2hN_{17}}{\omega_2}}$, $R = \sqrt{\dfrac{2hN_{18}}{\omega_2}}$ and $\rho = \sqrt{\dfrac{6hN_{19}}{\omega_2}}$ where $N_{17} = 9cd-be-3ce-e^2+3bf+9df$, $N_{18} = b^2-9ad-9cd+3aeb-3bc+be$ and $N_{19} = 3c^2+ce+ae-3af-bc-bf$.
Considering that

$$\begin{array}{lll} b-3c+3d-e \neq 0, & 3a+2b+3d \neq 0, & \omega_2 \neq 0, \\ hN_{17}\omega_2 > 0, & hN_{18}\omega_2 > 0 \quad \text{and} & hN_{19}\omega_2 > 0. \end{array} \tag{3.75}$$

System (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{6hN_{19}}{\omega_2}}$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{2hN_{17}}{\omega_2}},\frac{\pi}{2},\sqrt{\frac{2hN_{18}}{\omega_2}},0\right),\\
&\left(\sqrt{\frac{2hN_{17}}{\omega_2}},\frac{\pi}{2},\sqrt{\frac{2hN_{18}}{\omega_2}},\pi\right),\\
&\left(\sqrt{\frac{2hN_{17}}{\omega_2}},\frac{3\pi}{2},\sqrt{\frac{2hN_{18}}{\omega_2}},0\right),\\
&\left(\sqrt{\frac{2hN_{17}}{\omega_2}},\frac{3\pi}{2},\sqrt{\frac{2hN_{18}}{\omega_2}},\pi\right).
\end{aligned} \tag{3.76}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} = &-\frac{9h^4}{16\omega_2^3}N_{17}N_{18}N_{19}(5bc^2e-3bc^2d-b^2ce+3acde+3c^2de\\
&+abe^2-ace^2+bce^2-b^2cf-3bcdf-3abef-b^2ef).
\end{aligned}$$

Whenever $(5bc^2e-3bc^2d-b^2ce+3acde+3c^2de+abe^2-ace^2+bce^2-b^2cf-3bcdf-3abef-b^2ef)\neq 0$ and (3.75) hold, $J_{f_1(S^*)}\neq 0$. Therefore the four zeros (3.76) of (3.10) provide four periodic solutions of (3.8). The set of conditions is not empty because the value $a=24801$, $b=-\frac{1256502183358496925170226771841\,47}{738740718443550679046052380672}$, $c=-4232808, d=-23930, e=-72913, f=-92159, h=1$ satisfy it.

Subcase 3.2.2.3.1.4.2: $\omega_2=0$. Then $f_{14}=h(eb+e^2-9cd+3ce-3bf=$ constant.

Subcase 3.2.2.3.2: Assume that either $q=1$ or $q=3$, i.e. either $\beta=\frac{\pi}{2}$ or $\beta=\frac{3\pi}{2}$. Then

$$\begin{aligned}
f_{12} = &-\frac{1}{8}\big[2h(b+3d)+(3a+2b+3d)r^2+(c-b-3d+3e)R^2\big],\\
f_{14} = &-\frac{1}{8}\big[2h(b+3e)+(3a+b+c+3e)r^2+(3f-b-3e+c)R^2\big].
\end{aligned}$$

Subcase 3.2.2.3.2.1: $c-b-3d+3e=0$ and $3a+2b+3d\neq 0$. Solving $f_{12}=0$, we obtain $r=\sqrt{-\frac{2h(b+3d)}{3a+2b+3d}}$. Substituting r in f_{14} and solving $f_{14}=0$ we obtain two subcases

Subcase 3.2.2.3.2.1.1: $b-c+3e-3f\neq 0$. We obtain $R=\sqrt{\dfrac{2hN_{20}}{\delta_6}}$ and $\rho=\sqrt{\dfrac{6hN_{21}}{\delta_6}}$, where $N_{20}=b^2-bc-9ad-3cd+9ae+3be$, $N_{21}=cd+ab-ac+3ad-3af-bf$ and $\delta_6=(3a+2b+3d)(b-c+3e-3f)$.

Supposing that

$$\begin{array}{l} c-b-3d+3e=0, 3a+2b+3d\neq 0,\\ b-c+3e-3f\neq 0, \quad h(b+3d)(3a+2b+3d)<0,\\ h\delta_6 N_{20}>0 \quad \text{and} \quad h\delta_6 N_{21}>0. \end{array} \tag{3.77}$$

System (3.10) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{\dfrac{6hN_{21}}{\delta_6}}$ given by

$$\begin{array}{l} \left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}},\dfrac{\pi}{2},\sqrt{\dfrac{2hN_{20}}{\delta_6}},\dfrac{\pi}{2}\right),\\ \left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}},\dfrac{\pi}{2},\sqrt{\dfrac{2hN_{20}}{\delta_6}},\dfrac{3\pi}{2}\right),\\ \left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}},\dfrac{3\pi}{2},\sqrt{\dfrac{2hN_{20}}{\delta_6}},\dfrac{\pi}{2}\right),\\ \left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}},\dfrac{3\pi}{2},\sqrt{\dfrac{2hN_{20}}{\delta_6}},\dfrac{3\pi}{2}\right). \end{array} \tag{3.78}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = {} & \frac{3h^4(b+3d)N_{20}N_{21}}{16(3a+2b+3d)^4(b-c+3e-3f)^3}(b^2-2bc+c^2-9ad\\ & -6cd+18ae+6be+6ce+9e^2-9af-6bf-9df)(b^3c\\ & -b^2c^2+3b^2cd-3bc^2d+3ab^2e-3abce+4b^2ce-bc^2e\\ & +9abde-9acde+12bcde-3c^2de+9ace^2+3bce^2-3b^2cf\\ & -9bcdf-9abef-3b^2ef). \end{aligned}$$

Assuming that $(b^2-2bc+c^2-9ad-6cd+18ae+6be+6ce+9e^2-9af-6bf-9df)(b^3c-b^2c^2+3b^2cd-3bc^2d+3ab^2e-3abce+4b^2ce-bc^2e+9abde-9acde+12bcde-3c^2de+9ace^2+3bce^2-3b^2cf-9bcdf-9abef-3b^2ef)\neq 0$ and (3.77) hold, then $J_{f_1(S^*)}\neq 0$ and the four zeros (3.78) of system (3.10) provide four periodic solutions of differential system (3.8). The set of conditions is not empty because the value $a=2, b=1, c=-2, d=-2, f=-4, e=-1, h=1$.

Subcase 3.2.2.3.2.1.2: $b-c+3e-3f=0$. We have $f_{14}=-N_{20}/(4(3a+2b+3d))=constant$.

Subcase 3.2.2.3.2.2: $3a+2b+3d=0$ and $c-b-3d+3e=0$. Then we get $f_{12}=-\frac{1}{4}(b+3d)h=$constant.

Subcase 3.2.2.3.2.3: $c-b-3d+3e\neq 0$ and $3a+2b+3d=0$. We have $R=\sqrt{\frac{2h(b+3d)}{b-c+3d-3e}}$. Substituting R in f_{14} and solving $f_{14}=0$ we have two subcases

Subcase 3.2.2.3.2.3.1: $3a+b+c+3e\neq 0$. We get $r=\sqrt{\frac{6hN_{22}}{\delta_7}}$ and $\rho=\sqrt{-\frac{2hN_{23}}{\delta_7}}$, where $N_{22}=be-bf+ce-cd+3e^2-3df$, $N_{23}=3ac+bc+c^2+3cd+9ae+3ce+3bf+9df$ and $\delta_7=(b-c+3d-3e)(3a+b+c+3e)$.

Assuming that

$$\begin{array}{l} c-b-3d+3e\neq 0, \quad 3a+2b+3d=0, \quad 3a+b+c+3e\neq 0 \\ h(b+3d)(b-c+3d-3e)>0, \quad h\delta_7N_{22}>0 \quad \text{and} \quad h\delta_7N_{23}<0. \end{array} \tag{3.79}$$

System (3.10) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{-\frac{2hN_{23}}{\delta_7}}$ given by

$$\begin{array}{c} \left(\sqrt{\frac{6hN_{22}}{\delta_7}},\frac{\pi}{2},\sqrt{\frac{2h(b+3d)}{b-c+3d-3e}},\frac{\pi}{2}\right), \\ \left(\sqrt{\frac{6hN_{22}}{\delta_7}},\frac{\pi}{2},\sqrt{\frac{2h(b+3d)}{b-c+3d-3e}},\frac{3\pi}{2}\right), \\ \left(\sqrt{\frac{6hN_{22}}{\delta_7}},\frac{3\pi}{2},\sqrt{\frac{2h(b+3d)}{b-c+3d-3e}},\frac{\pi}{2}\right), \\ \left(\sqrt{\frac{6hN_{22}}{\delta_7}},\frac{3\pi}{2},\sqrt{\frac{2h(b+3d)}{b-c+3d-3e}},\frac{3\pi}{2}\right). \end{array} \tag{3.80}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)}= & \frac{9h^4(b+3d)N_{22}N_{23}}{16(b-c+3d-3e)^4(3a+b+c+3e)^3}(2bc-b^2-c^2+9ad \\ & +6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(b^2ce \\ & -bc^2d+bc^2e-3acde-c^2de+3abe^2+3bce^2-3cde^2-b^2cf \\ & -3bcdf+b^2ef+3bdef). \end{aligned}$$

Assuming that $(2bc - b^2 - c^2 + 9ad + 6cd - 18ae - 6be - 6ce - 9e^2 + 9af + 6bf + 9df)(b^2ce - bc^2d + bc^2e - 3acde - c^2de + 3abe^2 + 3bce^2 - 3cde^2 - b^2cf - 3bcdf + b^2ef + 3bdef) \neq 0$ and (3.79) hold. This assumption is satisfied for the value $a = 7, b = -3, c = -2, d = -5, f = -2, e = -1, h = 1$. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (3.80) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.2.3.2.3.2: $3a+b+c+3e = 0$. We obtain $f_{14} = 3hN_{22}/(4(b-c+3d-3e))=$ constant.

Subcase 3.2.2.3.2.4: $c - b - 3d + 3e \neq 0$ and $3a + 2b + 3d \neq 0$. Solving $f_{12} = 0$ we have from $f_{12} = 0$ that $R = \sqrt{\dfrac{2bh + 6dh + 3ar^2 + 2br^2 + 3dr^2}{b - c + 3d - 3e}}$, substituting R in f_{14} and solving $f_{14} = 0$ we obtain two subcases

Subcase 3.2.2.3.2.4.1: If

$$\omega_3 = b^2 - 2bc + c^2 - 9ad - 6cd + 18ae + 6be + 6ce + 9e^2 - 9af - 6bf - 9df \neq 0,$$

then $r = \sqrt{\dfrac{6hN_{24}}{\omega_3}}$, $R = \sqrt{\dfrac{2hN_{25}}{\omega_3}}$ and $\rho = \sqrt{\dfrac{2hN_{26}}{\omega_3}}$ where $N_{24} = cd - be - ce - 3e^2 + bf + 3df$, $N_{25} = b^2 - bc - 9ad - 3cd + 9ae + 3be$ and $N_{26} = c^2 - bc + 9ae + 3ce - 9af - 3bf$.

With the condition that

$$\begin{aligned} &\omega_3 = b^2 - 2bc + c^2 - 9ad - 6cd + 18ae + 6be + 6ce \\ &\qquad +9e^2 - 9af - 6bf - 9df \neq 0, \\ &c - b - 3d + 3e \neq 0, \quad 3a + 2b + 3d \neq 0, \\ &hN_{24}\omega_3 > 0, \quad hN_{25}\omega_3 > 0 \quad \text{and} \quad hN_{26}\omega_3 > 0, \end{aligned} \tag{3.81}$$

system (3.10) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2hN_{26}}{\omega_3}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{6hN_{24}}{\omega_3}}, \frac{\pi}{2}, \sqrt{\frac{2hN_{25}}{\omega_3}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{6hN_{24}}{\omega_3}}, \frac{\pi}{2}, \sqrt{\frac{2hN_{25}}{\omega_3}}, \frac{3\pi}{2}\right), \\ &\left(\sqrt{\frac{6hN_{24}}{\omega_3}}, \frac{3\pi}{2}, \sqrt{\frac{2hN_{25}}{\omega_3}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{6hN_{24}}{\omega_3}}, \frac{3\pi}{2}, \sqrt{\frac{2hN_{25}}{\omega_3}}, \frac{3\pi}{2}\right). \end{aligned} \tag{3.82}$$

The Jacobian is

$$J_{f_1(S^*)} = \frac{9h^4N_{24}N_{25}N_{26}}{16\omega_3^3}(bc^2d - b^2ce - bc^2e + 3acde + c^2de - 3abe^2 \\ -3ace^2 - 5bce^2 + b^2cf + 3bcdf + 3abef + b^2ef).$$

Assuming $(bc^2d - b^2ce - bc^2e + 3acde + c^2de - 3abe^2 - 3ace^2 - 5bce^2 + b^2cf + 3bcdf + 3abef + b^2ef) \neq 0$ and (3.81) hold. This supposition is not empty because the value $a = 5, b = -9, c = -8, d = -31, f = -2, e = -1, h = -1$ satisfy it. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (3.82) of system (3.10) provide four periodic solutions of differential system (3.8).

Subcase 3.2.2.3.2.4.2: $\omega_3 = 0$. We have $f_{14} = -3hN_{24}/(4(b - c + 3d - 3e))$ =constant.

Proof of Proposition 4. Following the averaging theory, $(r^*, \alpha^*, R^*, \beta^*)$ is a periodic solution of (3.8) means that

$$\begin{aligned} r(t,\varepsilon) &= r^* + O(\varepsilon), \\ \alpha(t,\varepsilon) &= \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) &= R^* + O(\varepsilon), \\ \beta(0,\varepsilon) &= \beta^* + O(\varepsilon). \end{aligned} \tag{3.83}$$

Adding the fact that $\rho = \sqrt{2h + r^{*2} - R^{*2}}$ so system (3.83) becomes

$$\begin{aligned} r(t,\varepsilon) &= r^* + O(\varepsilon), \\ \rho(t,\varepsilon) &= \sqrt{2h + r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) &= \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) &= R^* + O(\varepsilon), \\ \beta(t,\varepsilon) &= \beta^* + O(\varepsilon). \end{aligned} \tag{3.84}$$

We reconsider the variable,t, the temps instead of θ and the (3.84) becomes

$$\begin{aligned} r(t,\varepsilon) &= r^* + O(\varepsilon), \\ \theta(t,\varepsilon) &= t + O(\varepsilon), \\ \rho(t,\varepsilon) &= \sqrt{2h + r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) &= \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) &= R^* + O(\varepsilon), \\ \beta(t,\varepsilon) &= \beta^* + O(\varepsilon). \end{aligned} \tag{3.85}$$

Now using the change of variables (3.3): $X = r\cos\theta$, $Y = \rho\cos(\alpha-\theta)$, $Z = R\cos(\beta-\theta)$, $p_X = r\sin\theta$, $p_Y = \rho\sin(\alpha-\theta)$, $p_Z = R\sin(\beta-\theta)$, the system (3.85) becomes

$$\begin{aligned}
X(t,\varepsilon) &= r^*\cos t + O(\varepsilon),\\
Y(t,\varepsilon) &= \sqrt{2h + r^{*2} - R^{*2}}\cos(\alpha^* - t) + O(\varepsilon),\\
Z(t,\varepsilon) &= R^*\cos(\beta^* - t) + O(\varepsilon),\\
p_X(t,\varepsilon) &= r^*\sin t + O(\varepsilon),\\
p_Y(t,\varepsilon) &= \sqrt{2h + r^{*2} - R^{*2}}\sin(\alpha^* - t) + O(\varepsilon),\\
p_Z(t,\varepsilon) &= R^*\sin(\beta^* - t) + O(\varepsilon).
\end{aligned} \tag{3.86}$$

Finally, we reused the scaling $x = \sqrt{\varepsilon}\,X$, $y = \sqrt{\varepsilon}\,Y$, $z = \sqrt{\varepsilon}\,Z$, $p_x = \sqrt{\varepsilon}\,p_X$, $p_y = \sqrt{\varepsilon}\,p_Y$ and $p_z = \sqrt{\varepsilon}\,p_Z$ and (3.86) becomes

$$\begin{aligned}
x(t,\varepsilon) &= \sqrt{\varepsilon}\quad r^*\cos t + O(\varepsilon^{3/2}),\\
y(t,\varepsilon) &= \sqrt{\varepsilon}\quad \sqrt{2h + r^{*2} - R^{*2}}\cos(\alpha^* - t) + O(\varepsilon^{3/2}),\\
z(t,\varepsilon) &= \sqrt{\varepsilon}\quad R^*\cos(\beta^* - t) + O(\varepsilon^{3/2}),\\
p_x(t,\varepsilon) &= \sqrt{\varepsilon}\quad r^*\sin t + O(\varepsilon^{3/2}),\\
p_y(t,\varepsilon) &= \sqrt{\varepsilon}\quad \sqrt{2h + r^{*2} - R^{*2}}\sin(\alpha^* - t) + O(\varepsilon^{3/2}),\\
p_z(t,\varepsilon) &= \sqrt{\varepsilon}\quad R^*\sin(\beta^* - t) + O(\varepsilon^{3/2}).
\end{aligned}$$

□

The results of this study have been published, see

F. Lembarki and J. Llibre, *Periodic orbits for the generalized Friedmann-Robertson-Walker Hamiltonian in dim* 6, Discrete and Continuous Dynamical Systems, Series S Volume 8, Number 6, December 2015.

“Mathematics is a great motivator for all humans...because its carrer starts with ”zero” but it never end (infinity).” Unknown author.

Chapter 4

The Perturbed elliptic oscillators Hamiltonian system in 6D

In this chapter we study analytically the periodic orbits of the Perturbed Elliptic Oscillators Hamiltonian system in dimension 6 using the averaging theory of first order.

4.1 The Perturbed elliptic oscillators Hamiltonian system in 6D

The Hamiltonian studied in this chapter consists of a three coupled harmonic oscillators known as perturbed elliptic oscillators.

$$\begin{aligned} H = \; & \tfrac{1}{2}(x^2 + y^2 + z^2 + p_x^2 + p_y^2 + p_z^2) + \varepsilon(x^2y^2 + x^2z^2 \\ & +y^2z^2 - x^2y^2z^2). \end{aligned} \tag{1.5}$$

We present conditions on the energy level to guarantee the existence of periodic orbits which, we hope, will be useful information on the study of periodic motion not only in galactic dynamics but in the general field of nonlinear dynamics.

Our objective is to fix the families of the periodic solutions of the Hamiltonian system defined by the Hamiltonian (1.5)

The Hamiltonian system associated to (1.5)

$$
\begin{aligned}
\dot{x} = &\ \frac{\partial H}{\partial p_x} = p_x,\\
\dot{y} = &\ \frac{\partial H}{\partial p_y} = p_y,\\
\dot{z} = &\ \frac{\partial H}{\partial p_z} = p_z,\\
\dot{p}_x = &\ -\frac{\partial H}{\partial x} = -x - \varepsilon(2xy^2 + 2xz^2 - 2xy^2z^2),\\
\dot{p}_y = &\ -\frac{\partial H}{\partial y} = -y - \varepsilon(2x^2y + 2yz^2 - 2x^2yz^2),\\
\dot{p}_z = &\ -\frac{\partial H}{\partial z} = -z - \varepsilon(2x^2z + 2y^2z - 2x^2y^2z).
\end{aligned}
\tag{1.6}
$$

We study the existence of families of periodic orbits of the Hamiltonian system (1.6) and we compute them analytically using the averaging theory of first order.

4.2 Applying averaging theory to the perturbed elliptic oscillators Hamiltonian system in 6D

The periodicity of the independent variable of the differential system is needed to apply the averaging theory, so we change the Hamiltonian system (1.6) to a kind of generalized polar coordinates $(r, \theta, \rho, \alpha, R, \beta)$ in $\mathbb{R}^6$ defined by

$$
\begin{array}{lll}
x = r\cos\theta, & y = \rho\cos(\theta+\alpha), & z = R\cos(\theta+\beta),\\
p_x = r\sin\theta, & p_y = \rho\sin(\theta+\alpha), & p_z = R\sin(\theta+\beta),
\end{array}
\tag{4.1}
$$

where $r \geq 0$, $\rho \geq 0$ and $R \geq 0$.

The first integral H in the new variables is

$$
\begin{aligned}
H = &\ \frac{1}{2}(\rho^2 + r^2 + R^2) + \varepsilon\Big[r^2\cos^2\theta\Big(\rho^2\cos^2(\theta+\alpha)\\
&\ \big(1 - R^2\cos^2(\theta+\beta)\big) + R^2\cos^2(\theta+\beta)\Big)\\
&\ +\rho^2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)\Big],
\end{aligned}
\tag{4.2}
$$

and the equations of motion (1.6) become

$$\begin{aligned}
\dot{r} = & \ 2r\varepsilon \sin\theta\cos\theta\Big[\rho^2\cos^2(\theta+\alpha)\left(R^2\cos^2(\theta+\beta)-1\right) \\
& -R^2\cos^2(\theta+\beta)\Big], \\
\dot{\theta} = & \ -1+2\varepsilon\cos^2\theta\Big[\rho^2\cos^2(\theta+\alpha)\left(R^2\cos^2(\theta+\beta)-1\right) \\
& -R^2\cos^2(\theta+\beta)\Big], \\
\dot{\rho} = & \ 2\rho\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha)\Big[R^2\cos^2(\theta+\beta) \\
& \big(r^2\cos^2\theta-1\big)-r^2\cos^2\theta\Big], \\
\dot{\alpha} = & \ \varepsilon\Big[\cos^2\theta\Big((r^2-\rho^2)\cos^2(\theta+\alpha)\big(R^2\cos 2(\theta+\beta) \\
& +R^2-2\big)+2R^2\cos^2(\beta+\theta)\Big)-2R^2\cos^2(\theta+\alpha) \\
& \cos^2(\theta+\beta)\Big], \\
\dot{R} = & \ 2R\varepsilon\sin(\theta+\beta)\cos(\theta+\beta)\Big[\rho^2\cos^2(\theta+\alpha) \\
& \Big(r^2\cos^2(\theta)-1\Big)-r^2\cos^2(\theta)\Big], \\
\dot{\beta} = & \ \varepsilon\Big[-2\rho^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)+2\cos^2\theta \\
& \Big(\rho^2\cos^2(\theta+\alpha)+(r^2-R^2)\cos^2(\theta+\beta) \\
& \big(\rho^2\cos^2(\theta+\alpha)-1\big)\Big)\Big].
\end{aligned} \tag{4.3}$$

If we take the variable θ as the new independent variable instead of t in the system (4.3), we obtain the necessary periodicity to have the system of equations of motion in the normal form of the averaging theory. From now on the independent variable will be θ. Then the new differential system will have only five equations. Expanding system (4.3) in Taylor series in ε we have

$$
\begin{aligned}
r' = & \; -2r\varepsilon \sin\theta\cos\theta\Big[\rho^2\cos^2(\theta+\alpha)\Big(R^2\cos^2(\theta+\beta)-1\Big)\\
& -R^2\cos^2(\theta+\beta)\Big] + O(\varepsilon^2),\\
\rho' = & \; -2\rho\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha)\Big[R^2\cos^2(\theta+\beta)(r^2\cos^2\theta-1)\\
& -r^2\cos^2\theta\Big] + O(\varepsilon^2),\\
\alpha' = & \; \varepsilon\Big[2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta) - \cos^2\theta\Big((r^2-\rho^2)\cos^2(\theta+\alpha)\\
& \Big(R^2\cos 2(\theta+\beta)+R^2-2\Big) + 2R^2\cos^2(\theta+\beta)\Big)\Big] + O(\varepsilon^2),\\
R' = & \; -2R\varepsilon\sin(\theta+\beta)\cos(\theta+\beta)\Big[\rho^2(r^2\cos^2\theta-1)\\
& \cos^2(\theta+\alpha) - r^2\cos^2\theta\Big] + O(\varepsilon^2),\\
\beta' = & \; 2\varepsilon\Big[\rho^2\cos^2(\theta+\beta)\cos^2(\theta+\alpha) - \cos^2\theta\Big(\rho^2\cos^2(\theta+\alpha)\\
& +(r^2-R^2)(\rho^2\cos^2(\theta+\alpha)-1)\cos^2(\theta+\beta)\Big)\Big] + O(\varepsilon^2).
\end{aligned}
\tag{4.4}
$$

The prime denotes derivative with respect to the angle variable θ. System (4.4) is 2π-periodic respect to the variable θ, i.e. it is written in the normal form (A.1) of the averaging theory but we should fix the value of the first integral $H = h$ with $h \in \mathbb{R}^+$ to make it ready for applying the averaging theory. Otherwise, the Jacobian (A.4) will be zero because the periodic orbits are non-isolated leaving on cylinders parameterized by the energy, see for more details [1].

We fix the energy level in (4.2) and we solve $H = h$ with respect to ρ, we get two solutions, but we take only the one with physical meaning and we expanded it in Taylor series in ε

$$
\rho = \sqrt{2h - r^2 - R^2} + O(\varepsilon). \tag{4.5}
$$

Since $\rho \geq 0$ we need that $2h - r^2 - R^2 \geq 0$.

Substituting ρ in system (4.4) we get the differential system written in the normal form of the averaging theory

$$
\begin{aligned}
r' = \;& -2r\varepsilon \sin\theta\cos\theta\Big[\cos^2(\theta+\alpha)(2h-r^2-R^2)\\
& (R^2\cos^2(\theta+\beta)-1)-R^2\cos^2(\theta+\beta)\Big]+O(\varepsilon^2),\\
\alpha' = \;& \varepsilon\Big[\cos^2\theta\Big(\cos^2(\theta+\alpha)(2h-2r^2-R^2)\\
& (R^2\cos 2(\theta+\beta)+R^2-2)-2R^2\cos^2(\theta+\beta)\Big)\\
& +2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)\Big]+O(\varepsilon^2),\\
R' = \;& 2R\varepsilon\sin(\theta+\beta)\cos(\theta+\beta)\Big[r^2\cos^2\theta-\cos^2(\theta+\alpha)\\
& (2h-r^2-R^2)(r^2\cos\theta^2-1)\Big]+O(\varepsilon^2),\\
\beta' = \;& 2\varepsilon\Big[(R^2+r^2-2h)\cos^2\theta\cos^2(\theta+\alpha)-\cos^2(\theta+\beta)\\
& \Big((R^2-r^2)\cos^2\theta-(r^2+R^2-2h)((r^2-R^2)\cos^2\theta-1)\\
& \cos^2(\theta+\alpha)\Big)\Big]+O(\varepsilon^2).
\end{aligned}
\tag{4.6}
$$

Following the notation of the averaging theory given in section 2, the function $F_1=(F_{11},F_{12},F_{13},F_{14})$ of (A.1) is

$$
\begin{aligned}
F_{11} = \;& 2r\sin\theta\cos\theta\Big[(2h-r^2-R^2)\big(1-R^2\cos^2(\theta+\beta)\big)\\
& \cos^2(\theta+\alpha)+R^2\cos^2(\theta+\beta)\Big],\\
F_{12} = \;& 2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)-\cos^2\theta\Big[\cos^2(\theta+\alpha)\\
& (-2h+2r^2+R^2)\big(R^2\cos 2(\theta+\beta)+R^2-2\big)\\
& +2R^2\cos^2(\theta+\beta)\Big]\\
F_{13} = \;& 2R\sin(\theta+\beta)\cos(\theta+\beta)\Big[\cos^2(\theta+\alpha)(2h-r^2-R^2)\\
& (1-r^2\cos^2\theta)+r^2\cos^2\theta\Big],\\
F_{14} = \;& 2\cos^2(\theta+\beta)\Big[(2h-r^2-R^2)-2\cos^2(\theta+\alpha)\\
& (1-\cos^2\theta)-(r^2-R^2)\cos^2\theta\Big((2h-r^2-R^2)-1\Big)\\
& \cos^2(\theta+\alpha)\Big],
\end{aligned}
\tag{4.7}
$$

where $F_{1j}=F_{1j}(\theta,r,\alpha,R,\beta)$ for $j=1,2,3,4$.

From (A.3) and (4.7) we calculate the averaged function $f_1=(f_{11},f_{12},$

$f_{13}, f_{14})$ and we obtain

$$
\begin{aligned}
f_{11} = &\ \frac{1}{8} r\Big[(2-R^2)(R^2+r^2-2h)\sin 2\alpha - R^2(R^2+r^2-2h+2)\\
&\sin 2\beta\Big],\\
f_{12} = &\ \frac{1}{8}\Big[(2h-2r^2-R^2)\Big(2R^2\cos\alpha\cos(\alpha-2\beta)+(R^2-2)\cos 2\alpha\Big)\\
&+4h(R^2-2)-2\Big(2r^2(R^2-2)+R^4\Big)+2R^2(\cos 2(\alpha-\beta)+2)\\
&-2R^2\cos 2\beta\Big],\\
f_{13} = &\ \frac{1}{8} R\Big[(2-r^2)(-2h+r^2+R^2)\sin 2(\alpha-\beta)\\
&+r^2(-2h+r^2+R^2+2)\sin 2\beta\Big],\\
f_{14} = &\ \frac{1}{8}\Big[(r^2-R^2-2)\cos 2(\alpha-\beta)(-2h+r^2+R^2)+\\
&\cos 2\alpha(r^2-R^2+2)(-2h+r^2+R^2)+\\
&(r^2-R^2)(-2h+r^2+R^2+2)(\cos 2\beta+2)\Big],
\end{aligned}
$$

where $f_{1j} = f_{1j}(r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

4.3 Periodic orbits of the perturbed elliptic oscillators Hamiltonian system in 6D

According to Theorem 7 (see Appendix), our objective is to find the zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ of

$$
f_{1i}(r, \alpha, R, \beta) = 0 \quad \text{for } i = 1, 2, 3, 4, \tag{4.8}
$$

and after we must check that the Jacobian determinant (A.4) evaluated at these zeros are nonzero. Note that in all numerical calculations the numerical value h, the energy, is treated as a parameter.

Solving $f_{11}(r, \alpha, R, \beta) = 0$ we obtain three cases:

Case 1. $r = 0$;

Case 2. $r = \sqrt{\dfrac{(2h-R^2)(2-R^2)\sin 2\alpha + R^2(2-2h+R^2)\sin 2\beta}{2\sin 2\alpha - R^2(\sin 2\beta + \sin 2\alpha)}}$

if $\quad 2\sin 2\alpha - R^2(\sin 2\beta + \sin 2\alpha) \neq 0$;

Case 3. $2\sin 2\alpha - R^2(\sin 2\beta + \sin 2\alpha) = 0$.

Case 1: $r = 0$. Substituting r in f_{12}, f_{13} and f_{14} we have

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = {} & \frac{1}{8}\Big[(2h-R^2)\Big(2R^2\cos\alpha\cos(\alpha-2\beta)+(R^2-2)(2+ \\ & \cos 2\alpha)\Big) - 4R^2\sin\alpha\sin(\alpha-2\beta)\Big], \\ f_{13}(0,\alpha,R,\beta) = {} & \frac{1}{4}R(R^2-2h)\sin 2(\alpha-\beta), \\ f_{14}(0,\alpha,R,\beta) = {} & \frac{1}{8}\Big[(2h-R^2)(R^2-2)\cos 2\alpha + (2h-R^2)(2+R^2) \\ & \cos 2(\alpha-\beta) - R^2(2-2h+R^2)(2+\cos 2\beta)\Big]. \end{aligned}$$

Solving $f_{13}(0,\alpha,R,\beta) = 0$ we obtain three subcases: $R = 0$, $R = \sqrt{2h}$, $\alpha = \beta + \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$.

Subcase 1.1: $R = 0$, f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0,\alpha,0,\beta) = {} & -\frac{h}{2}(2+\cos 2\alpha), \\ f_{14}(0,\alpha,0,\beta) = {} & h\sin(2\alpha-\beta)\sin\beta. \end{aligned}$$

The system $f_{12} = f_{14} = 0$ has no solution in this subcase. The averaging theory does not give results.

Subcase 1.2: $R = \sqrt{2h}$, Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0,\alpha,\sqrt{2h},\beta) = {} & -h\sin(\alpha-2\beta)\sin\alpha, \\ f_{14}(0,\alpha,\sqrt{2h},\beta) = {} & -\frac{h}{2}(2+\cos 2\beta). \end{aligned}$$

As in the previous case, the averaging theory does not give information.

Subcase 1.3: $\alpha = \beta + \dfrac{k\pi}{2}$ with k $\in \mathbb{Z}$. Due to the periodicity of the cosinus and the sinus we study two subcases, either ($k = 0$ and $k = 2$) or ($k = 1$ and $k = 3$).

Subcase 1.3.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. Then

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = {} & \frac{1}{8}\big[6(h+1)R^2 - 2\cos 2\beta\big(R^4 - 2h(R^2-1)\big) - 8h - 3R^4\big], \\ f_{14}(0,\alpha,R,\beta) = {} & \frac{1}{8}\big[6(h-1)R^2 - 2\cos 2\beta\big(R^4 - 2h(R^2-1)\big) + 4h - 3R^4\big]. \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we obtain $R = \sqrt{h}$ and $\beta = \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}$. Substituting R in formula (4.5): $\rho = \sqrt{2h - r^2 - R^2}$, we get $\rho = \sqrt{h}$.

Supposing that

$$h > 0 \quad \text{and} \quad \left|\frac{2-3h}{2(h-2)}\right| < 1, \tag{4.9}$$

which is equivalent to $0 < h < \frac{6}{5}$.

System (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{h}$ given by

$$\begin{aligned} S^*_{1,2} = & \left(0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}\right), \\ S^*_{3,4} = & \left(0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)} + \pi\right), \end{aligned} \tag{4.10}$$

which reduce to two solutions if $h > 0$ and $\left|\frac{2-3h}{2(h-2)}\right| = 1$.

The Jacobian of f_1 evaluated at these solutions is

$J_{f_1(S^*)} = -\frac{3}{64}h^4(h+2)(5h-6)$.

If (4.9) hold, then $J_{f_1(S^*)} \neq 0$ and the four solutions (4.10) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back from the differential system (4.6) to the differential system (1.6) and using the Proposition (6) we obtain only two periodic orbits S^*_1 and S^*_3 since S^*_1 and S^*_2 are in the same family of periodic orbits as well as S^*_3 and S^*_4.

Subcase 1.3.2: Assume that either $k = 1$ or $k = 3$, i.e. either $\alpha = \beta + \frac{\pi}{2}$ or $\alpha = \beta + \frac{3\pi}{2}$. Then

$$\begin{aligned} f_{12}(0, \alpha, R, \beta) = & \ \frac{1}{8}\big[-8h + 2(1+h)R^2 - R^4 + 4(h - R^2)\cos 2\beta\big], \\ f_{14}(0, \alpha, R, \beta) = & \ \frac{1}{8}\big[-4h + 2(-1+h)R^2 - R^4 + 4(h - R^2)\cos 2\beta\big]. \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we do not have solutions. Therefore the averaging theory does not give information.

Case 2: $r = \sqrt{\frac{(2h - R^2)(2 - R^2)\sin 2\alpha + R^2(2 - 2h + R^2)\sin 2\beta}{(2 - R^2)\sin 2\alpha - R^2\sin 2\beta}}$ under the condition $D = 2\sin 2\alpha - R^2(\sin 2\alpha + \sin 2\beta) \neq 0$.

We cannot study this general case due to the difficulty of the calculations and to the huge expressions obtained when replacing r in f_{12}, f_{13}, f_{14}, but from the expression of D we can study the following two particular subcases, either $(2-R^2)\sin 2\alpha = 0$ and $R^2 \sin 2\beta \neq 0$ or $(2-R^2)\sin 2\alpha \neq 0$ and $R^2 \sin 2\beta = 0$.

Subcase 2.1: $(2-R^2)\sin 2\alpha = 0$ and $R^2 \sin 2\beta \neq 0$.

Subcase 2.1.1: $R = \sqrt{2}$ and $R^2 \sin 2\alpha \sin 2\beta \neq 0$. Replacing R in r we get $r = \sqrt{2(h-2)}$ then substituting R and r in formula (4.5): $\rho = \sqrt{2h - R^2 - r^2}$ we get $\rho = \sqrt{2}$. Substituting r in f_{13} we obtain $f_{13} = \dfrac{\sqrt{2}}{2}(h-3)\sin 2(\alpha - \beta)$. When $f_{13} = 0$, we have either $\alpha = \beta + \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$ or $h = 3$. Due to the periodicity of the cosinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 2.1.1.1: either $m = 0$ or $m = 2$ i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. Replacing α in f_{12} and f_{14} we get

$$
\begin{aligned}
f_{12} = & -\frac{1}{2}\big[(h-4) + (h-2)\cos 2\beta\big],\\
f_{14} = & -\frac{1}{2}\big[(h-4) + (h-2)\cos 2\beta\big].
\end{aligned}
$$

Solving $f_{12} = f_{14} = 0$ we get $\beta = \pm \arccos \dfrac{4-h}{h-2}$.

With the condition

$$
h > 0, \quad h-2 > 0 \quad \text{and} \quad \left|\frac{4-h}{h-2}\right| < 1, \tag{4.11}
$$

which is equivalent to $h > 3$, system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2}$ given by

$$
\begin{aligned}
S^*_{1,2} = & \left(\sqrt{2(h-2)}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}, \sqrt{2}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}\right),\\
S^*_{3,4} = & \left(\sqrt{2(h-2)}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}, \sqrt{2}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2} + \pi\right),
\end{aligned} \tag{4.12}
$$

If $h > 0$, $h-2 > 0$ and $\left|\dfrac{4-h}{h-2}\right| = 1$ i.e. $h = 3$ we have only two zeros. The Jacobian evaluated on these solutions is $J_{f_1(S^*)} = 8(h-3)^3$.

We conclude that if (4.11) hold we get $J_{f_1(S^*)} \neq 0$ and the four solutions (4.12) of system (4.8) provide four periodic solutions of differential system

(4.6). Nevertheless, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^* since S_1^* and S_2^* are in the same family of periodic orbits as well as S_3^* and S_4^*.

Subcase 2.1.1.2: Assume that either $m = 1$ and $m = 3$, i.e. either $\alpha = \beta + \dfrac{\pi}{2}$ or $\alpha = \beta + \dfrac{3\pi}{2}$. Then

$$\begin{aligned} f_{12} = &\ \frac{1}{2}\big[(h-4) - (h-2)\cos 2\beta\big], \\ f_{14} = &\ \frac{1}{2}\big[(h-4) + (h-2)\cos 2\beta\big]. \end{aligned}$$

The averaging theory does not give information because solving $f_{12} = f_{14} = 0$, we do not obtain any solutions.

Subcase 2.1.1.3: $h = 3$, $\sin 2(\alpha - \beta) \neq 0$ and $R^2 \sin 2\beta \neq 0$. We obtain $r = \sqrt{2}$ and $R = \sqrt{2}$ and f_{12} and f_{14} become

$$\begin{aligned} f_{12} = &\ -\sin\alpha \sin(\alpha - 2\beta), \\ f_{14} = &\ \sin\beta \sin(2\alpha - \beta). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ with respect to (α, β) we obtain eight solutions. System (4.8) has eight zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2}$ given by

$$\begin{aligned} S_1^* = &\ \left(\sqrt{2}, -\frac{2\pi}{3}, \sqrt{2}, -\frac{\pi}{3}\right), \\ S_2^* = &\ \left(\sqrt{2}, -\frac{2\pi}{3}, \sqrt{2}, \frac{2\pi}{3}\right), \\ S_3^* = &\ \left(\sqrt{2}, -\frac{\pi}{3}, \sqrt{2}, -\frac{2\pi}{3}\right), \\ S_4^* = &\ \left(\sqrt{2}, -\frac{\pi}{3}, \sqrt{2}, \frac{\pi}{3}\right), \\ S_5^* = &\ \left(\sqrt{2}, \frac{\pi}{3}, \sqrt{2}, -\frac{\pi}{3}\right), \\ S_6^* = &\ \left(\sqrt{2}, \frac{\pi}{3}, \sqrt{2}, \frac{2\pi}{3}\right), \\ S_7^* = &\ \left(\sqrt{2}, \frac{2\pi}{3}, \sqrt{2}, -\frac{2\pi}{3}\right), \\ S_8^* = &\ \left(\sqrt{2}, \frac{2\pi}{3}, \sqrt{2}, \frac{\pi}{3}\right), \end{aligned} \tag{4.13}$$

The Jacobian evaluated at these eight solutions $J_{f_1(S^*)} = -\dfrac{81}{16} \neq 0$.

Despite the eight solutions (4.13) of system (4.8) which provide eight periodic solutions of differential system (4.6), we have only four periodic orbits S_1^*, S_2^*, S_3^* and S_4^*. Because going back to the differential system (1.6) and using the Proposition 6, we get that S_1^* and S_8^* are in the same family of periodic orbits as well as S_2^* and S_7^*, S_3^* and S_6^*, S_4^* and S_5^*.

Subcase 2.1.2: $\alpha = \dfrac{l\pi}{2}$ with $l \in \mathbb{Z}$ and $(2 - R^2)R^2 \sin 2\beta \neq 0$. Due to the periodicity of the cosinus and sinus we study the subcases $l = 0$ and $l = 2$, and the subcases $l = 1$ and $l = 3$ together.

Subcase 2.1.2.1: either $l = 0$ or $l = 2$. i.e. either $\alpha = 0$ or $\alpha = \pi$. Substituting α in r we have $r = \sqrt{2h - 2 - R^2}$. Replacing r and α in f_{13} we get $f_{13} = \dfrac{1}{4}R(4 - 2h + R^2)\sin 2\beta$. Solving $f_{13} = 0$ we get $R = \sqrt{2h - 4}$ because by hypothesis $R^2 \sin 2\beta \neq 0$. Then $\rho = \sqrt{2}$. So replacing r and R in f_{12}, f_{14} we get

$$\begin{aligned} f_{12} = &\ 0, \\ f_{14} = &\ -\frac{1}{2}\big[h - 4 + (h - 2)\cos 2\beta\big]. \end{aligned}$$

Solving $f_{14} = 0$ we have $\beta = \pm\dfrac{1}{2}\arccos\dfrac{4 - h}{h - 2}$.

Assuming that

$$h > 0, \quad h - 2 > 0 \quad \text{and} \quad \left|\frac{4 - h}{h - 2}\right| < 1, \tag{4.14}$$

which is equivalent to $h > 3$.

Then system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2}$ given by

$$\begin{aligned} S_{1,2}^* = &\ \left(\sqrt{2}, 0, \sqrt{2h - 4}, \pm\frac{1}{2}\arccos\frac{4 - h}{h - 2}\right), \\ S_{3,4}^* = &\ \left(\sqrt{2}, \pi, \sqrt{2h - 4}, \pm\frac{1}{2}\arccos\frac{4 - h}{h - 2}\right), \end{aligned} \tag{4.15}$$

The Jacobian evaluated on these solutions $J_{f_1(S^*)} = 8(h - 3)^2$.

When (4.14) hold we have $J_{f_1(S^*)} \neq 0$ and the four solutions (4.15) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and by the Proposition 6, we get only two periodic orbits S_1^* and S_3^* since S_1^* and S_2^* are in the same family of periodic orbits likewise S_3^* and S_4^*.

Subcase 2.1.2.2: $l = 0$ and $l = 3$. i.e. $\alpha = \dfrac{\pi}{2}$ and $\alpha = \dfrac{3\pi}{2}$. Replacing α in r we get $r = \sqrt{2h - 2 - R^2}$. Substituting r and α in f_{13} we obtain $f_{13} = -\dfrac{1}{4}R(4 - 2h + R^2)\sin 2\beta$. Solving $f_{13} = 0$ we have $R = \sqrt{2h - 4}$ because by hypothesis $R \sin 2\beta \neq 0$. Then we get $r = \sqrt{2}$ and $\rho = \sqrt{2}$. f_{12} and f_{14} become

$$
\begin{aligned}
f_{12} =& \quad (2 - h)\cos 2\beta, \\
f_{14} =& \quad -\frac{1}{2}\big[h - 4 + (h - 2)\cos 2\beta\big].
\end{aligned}
$$

Solving $f_{12} = f_{14} = 0$ we do not have solutions for β. The averaging theory does not give information.

Subcase 2.1.3: $\sin 2\alpha = 0$, $(2 - R^2) = 0$ and $R^2 \sin 2\beta \neq 0$.

Subcase 2.1.3.1: Either $\alpha = 0$ or $\alpha = \pi$ and $R = \sqrt{2}$. Replacing α and $R = \sqrt{2}$ in r we get $r = \sqrt{2h - 4}$. Substituting r in f_{13} we obtain $f_{13} = -\dfrac{\sqrt{2}}{2}(h - 3)\sin 2\beta$. Solving $f_{13} = 0$ we have $h = 3$ because $\sin 2\beta \neq 0$. f_{12}, f_{14} become

$$
\begin{aligned}
f_{12} =& \quad (3 - h)\cos 2\beta, \\
f_{14} =& \quad -\frac{1}{2}\big[h - 2 + (h - 4)\cos 2\beta\big].
\end{aligned}
$$

System $f_{12} = f_{14} = 0$ does not give solutions for β.

Subcase 2.1.3.2: Either $\alpha = \dfrac{\pi}{2}$ or $\alpha = \dfrac{3\pi}{2}$ and $R = \sqrt{2}$. Substituting α and $R = \sqrt{2}$ in r we obtain $r = \sqrt{2h - 4}$. Replacing r in f_{13} we have $f_{13} = -\dfrac{\sqrt{2}}{2}(h - 3)\sin 2\beta$. Solving $f_{13} = 0$ we have $h = 3$ because $\sin 2\beta \neq 0$.

$$
\begin{aligned}
f_{12} =& \quad -\cos 2\beta, \\
f_{14} =& \quad \frac{1}{2}\big[h - 2 + (h - 4)\cos 2\beta\big].
\end{aligned}
$$

System $f_{12} = f_{14} = 0$ does not provide solutions for β.

Subcase 2.2: $(2 - R^2)\sin 2\alpha \neq 0$ and $R^2 \sin 2\beta = 0$.

Subcase 2.2.1: $R = 0$, $\sin 2\beta \neq 0$ and $(R^2-2)\sin 2\alpha \neq 0$. Then $r = \sqrt{2h}$ and

$$\begin{aligned} f_{12} = & \ \frac{h}{2}(2+\cos 2\alpha), \\ f_{14} = & \ \frac{h}{2}(2+\cos 2\beta). \end{aligned}$$

The system $f_{12} = f_{14} = 0$ does not have solutions. The averaging theory does give information.

Subcase 2.2.2: $R \neq 0$, $\sin 2\beta = 0$ and $(R^2-2)\sin 2\alpha \neq 0$. Because of the periodicity of the sinus and the cosinus we study $\beta = \dfrac{k\pi}{2}$ for $k = 0$ and $k = 2$ together and for $k = 1$ and $k = 3$ together.

Subcase 2.2.2.1: either $k = 0$ or $k = 2$, i.e. either $\beta = 0$ or $\beta = \pi$. So we have $r = \sqrt{2h - R^2}$ and

$$\begin{aligned} f_{12} = & \ \frac{1}{8}\Big[8h - 6(1+h)R^2 + 3R^4 + 2\big(R^4 - 2h(R^2-1)\big)\cos 2\alpha\Big], \\ f_{13} = & \ 0, \\ f_{14} = & \ \frac{3}{2}(h - R^2). \end{aligned}$$

Solving $f_{14} = 0$ we obtain $R = \sqrt{h}$ then $\rho = 0$. Replacing R in f_{12} and solving $f_{12} = 0$ we have $\alpha = \pm \arccos \dfrac{2-3h}{2(h-2)}$.

If

$$h > 0 \quad \text{and} \quad \left|\frac{2-3h}{2(h-2)}\right| < 1, \tag{4.16}$$

which is equivalent to $0 < h < \dfrac{6}{5}$.

System (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ given by

$$\begin{aligned} S^*_{1,2} = & \left(\sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, 0\right), \\ S^*_{3,4} = & \left(\sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, \pi\right), \end{aligned} \tag{4.17}$$

which reduce to two solutions if $h > 0$ and $\left|\dfrac{2-3h}{2(h-2)}\right| = 1$.

The Jacobian read $J_{f_1(S^*)} = -\dfrac{3}{64}h^4(h+2)(5h-6)$.

If (4.16) hold, then $J_{f_1(S^*)} \neq 0$ and the four solutions (4.17) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and by the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

Subcase 2.2.2.2: either $k = 1$ or $k = 3$. i.e. either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$. Then $r = \sqrt{2h - R^2}$ and

$$
\begin{aligned}
f_{12} =& \ \frac{1}{8}\Big[8h - 2(1+h)R^2 + R^4 + 4(h - R^2)\cos 2\alpha\Big],\\
f_{13} =& \ 0,\\
f_{14} =& \ \frac{1}{2}(h - R^2).
\end{aligned}
$$

Solving $f_{14} = 0$ we get $R = \sqrt{h}$. Replacing R in f_{12} we have $f_{12} = \dfrac{h}{8}(6-h) =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 2.2.3: $R = 0$, $\sin 2\beta = 0$ and $(R^2 - 2)\sin 2\alpha \neq 0$.

Subcase 2.2.3.1: $R = 0$, either $\beta = 0$ or $\beta = \pi$, and $(R^2 - 2)\sin 2\alpha \neq 0$. We have $r = \sqrt{2h}$ and

$$
\begin{aligned}
f_{12} =& \ \frac{h}{2}(2 + \cos 2\alpha),\\
f_{14} =& \ \frac{3h}{2}.
\end{aligned}
$$

We have $f_{14} =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 2.2.3.2: $R = 0$, either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$, and $(R^2 - 2)\sin 2\alpha \neq 0$. So $r = \sqrt{2h}$ and

$$
\begin{aligned}
f_{12} =& \ \frac{h}{2}(2 + \cos 2\alpha),\\
f_{14} =& \ \frac{h}{2}.
\end{aligned}
$$

As in the anterior subcase $f_{14} =$ constant. The Jacobian will be zero and the averaging theory does not give information.

Case 3: $D = 2\sin 2\alpha - R^2(\sin 2\alpha + \sin 2\beta) = 0$.

Subcase 3.1: $\sin 2\alpha + \sin 2\beta \neq 0$. Then solving $D = 0$ we have $R = \sqrt{\dfrac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}}$. The new averaged function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ becomes

$$
\begin{aligned}
f_{11} = &\; -\frac{r\sin 2\alpha \sin 2\beta}{4(\cos\alpha\sin\alpha + \cos\beta\sin\beta)},\\
f_{12} = &\; \frac{1}{4(\sin 2\alpha + \sin 2\beta)^2}\Bigg[\sin 2\beta\Big((h - r^2 + 1)\sin(4\alpha - 2\beta) + 2(h - r^2)\\
&\sin 2(\alpha - \beta) + \sin 2\alpha\big(-2\cos 2\beta - 3h + 3r^2 + 3\big)\Big)\\
&+ \sin 2\alpha\Big(-2\sin\alpha\cos(\alpha + 2\beta) + (h - r^2)\sin(4\alpha - 2\beta)\\
&+ (2h - 2r^2 - 3)\sin 2(\alpha - \beta)\Big)\\
&+ \big(-3h + 3r^2 + 1\big)\sin^2 2\beta\Bigg],\\
f_{13} = &\; \sqrt{\frac{\sin\alpha\cos\alpha}{16(\sin 2\alpha + \sin 2\beta)}}\Bigg[(2 - r^2)\sin 2(\alpha - \beta)\Big(r^2 - 2h +\\
&\frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\Big) + r^2\sin 2\beta\Big(2 - 2h + r^2 + \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\Big)\Bigg],\\
f_{14} = &\; \frac{1}{8}\Bigg[\cos 2(\alpha - \beta)\big(r^2 - 2 - \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\big)\big(r^2 - 2h\\
&+ \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\big) + \cos 2\alpha\big(2 + r^2 - \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\big)\big(r^2 - 2h\\
&+ \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\big) + (2 + \cos 2\beta)\big(r^2 - \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\big)\\
&\big(2 - 2h + r^2 + \frac{2\sin 2\alpha}{\sin 2\alpha + \sin 2\beta}\big)\Bigg].
\end{aligned}
$$

Solving $f_{11} = 0$ we obtain three main subcases: $r = 0$ (studied in case 1), $\alpha = \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$ and $\beta = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$.

Subcase 3.1.1: $\alpha = \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$. Due to the periodicity of the cosinus and sinus we study the subcases $k = 0$ and $k = 2$, and the subcases $k = 1$ and $k = 3$ together.

Subcase 3.1.1.1: either $k = 0$ or $k = 2$. i.e. either $\alpha = 0$ or $\alpha = \pi$. Substituting α in f_{12} we have $f_{12} = -\dfrac{3}{2}(h - r^2)$. Solving $f_{12} = 0$ we obtain $r =$

$\sqrt{h}$. Replacing r in f_{14} and solving $f_{14} = 0$ we get $\beta = \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}$. Then $R = 0$ and $\rho = \sqrt{h}$.

Supposing

$$h > 0 \quad \text{and} \quad \left|\frac{2-3h}{2(h-2)}\right| < 1. \tag{4.18}$$

which is equivalent to $0 < h < \frac{6}{5}$.

System (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{h}$ given by

$$\begin{aligned} S^*_{1,2} &= \left(\sqrt{h}, 0, 0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}\right), \\ S^*_{3,4} &= \left(\sqrt{h}, \pi, 0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}\right), \end{aligned} \tag{4.19}$$

which reduce to two solutions if $h > 0$ and $\left|\frac{2-3h}{2(h-2)}\right| = 1$.

The Jacobian evaluated on these solutions $J_{f_1(S^*)} = -\frac{3}{64}h^4(h+2)(5h-6)$.

When (4.18) hold, $J_{f_1(S^*)} \neq 0$ and the four solutions (4.19) of system (4.8) provide only two periodic solutions of differential system (4.6) because when $R = 0$ the two solutions of β provide the same initial conditions in (4.1).

Subcase 3.1.1.2: either $k = 1$ or $k = 3$. i.e. either $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$. Substituting α in f_{12} we obtain $f_{12} = \frac{1}{2}(r^2 - h)$. Solving $f_{12} = 0$ we get $r = \sqrt{h}$. Replacing r in f_{14} we have $f_{14} = -\frac{h}{8}(h-6) =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 3.1.2: $\beta = \frac{m\pi}{2}$ with $m \in \mathbb{Z}$. Due to the periodicity of the cosinus and sinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 3.1.2.1: either $m = 0$ or $m = 2$. i.e. either $\beta = 0$ or $\beta = \pi$. Substituting β in f_{13} we get $f_{13} = -\frac{\sqrt{2}}{8}(r^2-2)(r^2+2-2h)\sin 2\alpha$. Solving $f_{13} = 0$ under the hypothesis $\sin 2\alpha + \sin 2\beta \neq 0$ we get two subcases $r = \sqrt{2}$ and $r = \sqrt{2h-2}$.

Subcase 3.1.2.1.1: $r = \sqrt{2}$. Replacing r in f_{12} and solving $f_{12} = 0$ we obtain $\alpha = \pm\arccos\frac{4-h}{h-2}$. Then we get $R = \sqrt{2}$ and $\rho = \sqrt{2h-4}$.

With the condition

$$h > 0, \quad h-2>0 \quad \text{and} \quad \left|\frac{4-h}{h-2}\right| < 1. \tag{4.20}$$

which is equivalent to $h > 3$, system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2h-4}$ given by

$$\begin{aligned} S^*_{1,2} = & \left(\sqrt{2}, \pm\arccos\frac{4-h}{h-2}, \sqrt{2}, 0\right), \\ S^*_{3,4} = & \left(\sqrt{2}, \pm\arccos\frac{4-h}{h-2}, \sqrt{2}, \pi\right), \end{aligned} \tag{4.21}$$

which reduce to two solutions if $h > 0$, $h-2>0$ and $\left|\dfrac{4-h}{h-2}\right| = 1$.

The Jacobian evaluated on these solutions is $J_{f_1(S^*)} = 8(h-3)^3$.

If (4.20) hold, then $J_{f_1(S^*)} \neq 0$ and the four solutions (4.21) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we have only two periodic orbits S_1^* and S_3^*.

Subcase 3.1.2.1.2: $r = \sqrt{2h-2}$. Substituting r in f_{14} we have $f_{14} = \dfrac{3}{2}(h-2)=$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 3.1.2.2: either $m = 1$ or $m = 3$, i.e. $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$. Replacing β in f_{13} we have $f_{13} = \dfrac{\sqrt{2}}{8}(r^2-2)(r^2-2h+2)\sin 2\alpha$. Solving $f_{13} = 0$ we get either $r = \sqrt{2}$ or $r = \sqrt{2h-2}$.

Subcase 3.1.2.2.1: $r = \sqrt{2}$. Then

$$\begin{aligned} f_{12} = & \ \frac{1}{2}\Big(4-h-(h-2)\cos 2\alpha\Big), \\ f_{14} = & \ -(h-2)\cos 2\alpha. \end{aligned}$$

$f_{12} = f_{14} = 0$ does not have solution. So the averaging theory is not available.

Subcase 3.1.2.2.2: $r = \sqrt{2h-2}$. We have $f_{14} = \dfrac{1}{2}(h-2)=$ constant. The averaging theory does not give results.

Subcase 3.2: $\sin 2\alpha + \sin 2\beta = 0$. Solving $D = 2\sin 2\alpha - R^2(\sin 2\alpha + \sin 2\beta) = 0$ we obtain $\alpha = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$ and $\beta = \dfrac{n\pi}{2}$ with $n \in \mathbb{Z}$.

Subcase 3.2.1: For the values of $(\alpha, \beta) = (0,0), (\pi,\pi), (0,\pi), (\pi, 0)$ we have

$$\begin{aligned} f_{12} = &\ \frac{1}{8}(2h - 2r^2 - R^2)(5R^2 - 6), \\ f_{14} = &\ \frac{1}{8}(r^2 - R^2)(5R^2 + 5r^2 - 10h + 6). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we obtain four pairs of (r, R) :,

When $r_1 = \sqrt{\dfrac{6}{5}}$, $R_1 = \sqrt{\dfrac{2(5h-6)}{5}}$ then $\rho_1 = \sqrt{\dfrac{6}{5}}$.

Assuming that

$$h > 0 \quad \text{and} \quad 5h - 6 > 0, \tag{4.22}$$

which is equivalent to $h > \dfrac{6}{5}$, system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{6}{5}}$ given by

$$\begin{aligned} S_1^* = &\ \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{2(5h-6)}{5}}, 0\right), \\ S_2^* = &\ \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{2(5h-6)}{5}}, \pi\right), \\ S_3^* = &\ \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{2(5h-6)}{5}}, 0\right), \\ S_4^* = &\ \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{2(5h-6)}{5}}, \pi\right), \end{aligned} \tag{4.23}$$

The Jacobian evaluated at these solutions is

$$J_{f_1(S^*)} = \ -\frac{36}{15625}(5h-9)^2(5h-6)(10h-27).$$

Under the assumption (4.22), we get $J_{f_1(S^*)} \neq 0$. Therefore the four zeros (4.23) of system (4.8) provide four periodic solutions of differential system (4.6).

If $r_2 = \sqrt{\dfrac{2h}{3}}$, $R_2 = \sqrt{\dfrac{2h}{3}}$ we have $\rho_2 = \sqrt{\dfrac{2h}{3}}$.

If

$$h > 0, \tag{4.24}$$

system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2h}{3}}$ given by

$$\begin{aligned} S*_1 &= \left(\sqrt{\frac{2h}{3}}, 0, \sqrt{\frac{2h}{3}}, 0\right), \\ S^*_2 &= \left(\sqrt{\frac{2h}{3}}, 0, \sqrt{\frac{2h}{3}}, \pi\right), \\ S^*_3 &= \left(\sqrt{\frac{2h}{3}}, \pi, \sqrt{\frac{2h}{3}}, 0\right), \\ S^*_4 &= \left(\sqrt{\frac{2h}{3}}, \pi, \sqrt{\frac{2h}{3}}, \pi\right), \end{aligned} \tag{4.25}$$

The Jacobian evaluated at these solutions is

$$J_{f_1(S^*)} = -\frac{h^4}{729}(h-3)^2(5h-9)^2.$$

If $(5h-9)(h-3) \neq 0$ and (4.24) hold. The set of conditions on the energy level h is not empty because for $h = 1$ it satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (4.25) of system (4.8) provide four periodic solutions of differential system (4.6).

When $r_3 = \sqrt{\dfrac{6}{5}}$, $R_3 = \sqrt{\dfrac{6}{5}}$ we obtain $\rho_3 = \sqrt{\dfrac{2(5h-6)}{5}}$.

Assuming that

$$h > 0 \quad \text{and} \quad 5h - 6 > 0, \tag{4.26}$$

which is equivalent to $h > \dfrac{6}{5}$, then system (4.8) has four zeros

$S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\dfrac{2(5h-6)}{5}}$ given by

$$\begin{aligned} S^*_1 &= \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{6}{5}}, 0\right), \\ S^*_2 &= \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{6}{5}}, \pi\right), \\ S^*_3 &= \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{6}{5}}, 0\right), \\ S^*_4 &= \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{6}{5}}, \pi\right), \end{aligned} \tag{4.27}$$

The Jacobian evaluated on these solutions

$$J_{f_1(S^*)} = -\frac{36}{15625}(5h-9)^2(5h-6)(10h-27).$$

If $(5h-9)(10h-27) \neq 0$ and (4.26) hold. The set of conditions on the energy level h is not empty because for $h = 3/2$ it satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (4.27) of system (4.8) provide four periodic solutions of differential system (4.6).

Finally if $r_4 = \sqrt{\frac{2(5h-6)}{5}}$, $R_4 = \sqrt{\frac{6}{5}}$ we get $\rho_4 = \sqrt{\frac{6}{5}}$.

Assuming that

$$h > 0 \quad \text{and} \quad 5h-6 > 0, \tag{4.28}$$

which is equivalent to $h > \frac{6}{5}$, then system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{\frac{6}{5}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, 0, \sqrt{\frac{6}{5}}, 0\right), \\ S_2^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, 0, \sqrt{\frac{6}{5}}, \pi\right), \\ S_3^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, \pi, \sqrt{\frac{6}{5}}, 0\right), \\ S_4^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, \pi, \sqrt{\frac{6}{5}}, \pi\right), \end{aligned} \tag{4.29}$$

The Jacobian evaluated at these solutions is

$$J_{f_1(S^*)} = -\frac{36}{15625}(5h-9)^2(5h-6)(10h-27).$$

If $(5h-9)(10h-27) \neq 0$ and (4.28) hold. The set of conditions on the energy level h is not empty because for $h = 3/2$ it satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (4.29) of system (4.8) provide four periodic solutions of differential system (4.6).

Subcase 3.2.2: If $(\alpha, \beta) = (0, \frac{\pi}{2}), (0, \frac{3\pi}{2}), (\pi, \frac{\pi}{2}), (\pi, \frac{3\pi}{2})$ we get

$$\begin{aligned} f_{12} &= \frac{1}{8}(2h-2r^2-R^2)(R^2-6), \\ f_{14} &= \frac{1}{8}\Big(r^4+6r^2-2hr^2+2R^2(1+h)-R^4-8h\Big). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we obtain four pairs of (r, R) :,

When $r_1 = \sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$, $R_1 = \sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$ we get $\rho_1 = r_1$.

If

$$\begin{array}{l} h>0, \quad 1+h-\sqrt{h^2-10h+1}>0, \quad h^2-10h+1>0, \\ \text{and} \quad 2h-1+\sqrt{h^2-10h+1}>0, \end{array} \tag{4.30}$$

which is equivalent to $h > 5 + 2\sqrt{6}$, then system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_1 = \sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$ and $R^* = \sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, 0, R^*, \frac{\pi}{2}\right), \\ S_2^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, 0, R^*, \frac{3\pi}{2}\right), \\ S_3^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \pi, R^*, \frac{\pi}{2}\right), \\ S_4^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \pi, R^*, \frac{3\pi}{2}\right), \end{aligned} \tag{4.31}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{\sqrt{h^2-10h+1}}{34992}(1+h-\sqrt{h^2-10h+1})^2(5-h+\sqrt{h^2-10h+1}) \\ & (2h-10+\sqrt{h^2-10h+1})(2h-1+\sqrt{h^2-10h+1}) \\ & (h^2+17h-14+(14-h)\sqrt{h^2-10h+1}). \end{aligned}$$

The $J_{f_1(S^*)}$ does not vanish for $h \neq 5 \pm 2\sqrt{6}$. Therefore with the condition (4.30), the four solutions (4.31) of system (4.8) provide four periodic solutions of differential system (4.6). Although, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

If $r_2 = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$, $R_2 = \sqrt{\dfrac{2(2h-1-\sqrt{h^2-10h+1})}{3}}$ then we obtain $\rho_2 = r_2$.

With the condition

$$\begin{array}{l} h>0, \quad 1+h+\sqrt{h^2-10h+1}>0, \quad h^2-10h+1>0, \\ \text{and} \quad 2h-1-\sqrt{h^2-10h+1}>0, \end{array} \tag{4.32}$$

which is equivalent to $h > 5 + 2\sqrt{6}$, system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_2 = \sqrt{\dfrac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}$ and
$R^* = \sqrt{\dfrac{2(2h - 1 - \sqrt{h^2 - 10h + 1})}{3}}$ given by

$$\begin{aligned}
S_1^* &= \left(\sqrt{\frac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}, 0, R^*, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{\frac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}, 0, R^*, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{\frac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}, \pi, R^*, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{\frac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}, \pi, R^*, \frac{3\pi}{2}\right),
\end{aligned} \tag{4.33}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} = & -\frac{\sqrt{h^2 - 10h + 1}\left(1 + h + \sqrt{h^2 - 10h + 1}\right)^2}{34992}\Big(h - 5\\
& +\sqrt{h^2 - 10h + 1}\Big)\left(10 - 2h + \sqrt{h^2 - 10h + 1}\right)\Big(2h - 1\\
& -\sqrt{h^2 - 10h + 1}\Big)\left(h^2 + 17h - 14 + (h - 14)\sqrt{h^2 - 10h + 1}\right).
\end{aligned}$$

For $h \neq 5 \pm 2\sqrt{6}$, $J_{f_1(S^*)}$ does not vanish. Then with the condition (4.32), the four solutions (4.33) of system (4.8) provide four periodic solutions of differential system (4.6). Nevertheless, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

When $r_3 = \sqrt{h - 3 - \sqrt{h^2 - 10h + 33}}$ and $R_2 = \sqrt{6}$, then we get $\rho_3 = \sqrt{h - 3 + \sqrt{h^2 - 10h + 33}}$.
Assuming

$$\begin{aligned}
& h > 0, \quad h - 3 - \sqrt{h^2 - 10h + 33} > 0, \quad h^2 - 10h + 33 > 0,\\
& \text{and} \quad h - 3 - \sqrt{h^2 - 10h + 33} > 0,
\end{aligned} \tag{4.34}$$

which is equivalent to $h > 6$. Then system (4.8) provide four zeros $S^* =$

$(r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ given by

$$\begin{aligned}
S_1^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{3\pi}{2}\right),
\end{aligned} \tag{4.35}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(S^*)} = \; & \frac{3}{2}(h-6)(h^2-10h+33)(3-h+\sqrt{h^2-10h+33})\\
& (h-3+\sqrt{h^2-10h+33}).
\end{aligned}$$

For $h \neq 6$ the Jacobian does not vanish. So with the condition (4.34) we get $J_{f_1(S^*)} \neq 0$, therefore the four solutions (4.35) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

If $r_4 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ and $R_4 = \sqrt{6}$, we have $\rho_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$.

Supposing

$$\begin{aligned}
& h > 0, \quad h-3-\sqrt{h^2-10h+33} > 0, \quad h^2-10h+33 > 0,\\
& \text{and} \quad h-3+\sqrt{h^2-10h+33} > 0,
\end{aligned} \tag{4.36}$$

which is equivalent to $h > 6$. Then system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ given by

$$\begin{aligned}
S_1^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{3\pi}{2}\right),
\end{aligned} \tag{4.37}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \ & \frac{3}{2}(h-6)(h^2-10h+33)(3-h+\sqrt{h^2-10h+33}) \\ & (h-3+\sqrt{h^2-10h+33}). \end{aligned}$$

If $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. The condition (4.36) guarantees the fact that $J_{f_1(S^*)} \neq 0$. Moreover, the four solutions (4.37) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

Subcase 3.2.3: If $(\alpha, \beta) = (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0), (\frac{\pi}{2}, \pi), (\frac{3\pi}{2}, \pi)$ then f_{12}, f_{14} become

$$\begin{aligned} f_{12} = \ & -\frac{1}{8}\Big(R^4 - 2R^2(h-1-r^2) - 4r^2 + 4h\Big), \\ f_{14} = \ & \frac{1}{8}(r^2 - R^2)(R^2 + r^2 - 2h + 6). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we obtain four pairs of solutions (r, R),

$r_1 = R_1 = \sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$ and

$\rho_1 = \sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$.

Whenever

$$\begin{aligned} & h > 0, \quad 1+h-\sqrt{h^2-10h+1} > 0, \quad h^2-10h+1 > 0 \\ & \text{and} \quad 2h-1+\sqrt{h^2-10h+1} > 0, \end{aligned} \tag{4.38}$$

which is equivalent to $h > 5 + 2\sqrt{6}$, then system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_1 = \sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$ given by

$$\begin{aligned} S_1^* = \ & \left(r^* = \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}, r^*, 0\right), \\ S_2^* = \ & \left(r^* = \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}, r^*, 0\right), \\ S_3^* = \ & \left(r^* = \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}, r^*, \pi\right), \\ S_4^* = \ & \left(r^* = \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}, r^*, \pi\right), \end{aligned} \tag{4.39}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{(1+h-\sqrt{h^2-10h+1})^2}{17496}\Big(2h-10+\sqrt{h^2-10h+1}\Big) \\ & \Big(56-596h+413h^2-7h^3+7h^4-h^5-56\sqrt{h^2-10h+1} \\ & +316h\sqrt{h^2-10h+1}+9h^2\sqrt{h^2-10h+1}-2h^3\sqrt{h^2-10h+1} \\ & +h^4\sqrt{h^2-10h+1}\Big). \end{aligned}$$

The $J_{f_1(S^*)}$ does not vanish for $h \neq 5 \pm 2\sqrt{6}$. Therefore with the condition (4.38), the four zeros (4.39) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

When $r_2 = R_2 = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$ then we get $\rho_2 = \sqrt{\dfrac{2(2h-1-\sqrt{h^2-10h+1})}{3}}$.

With the condition

$$\begin{array}{l} h>0, \quad 1+h+\sqrt{h^2-10h+1}>0, \quad h^2-10h+1>0 \\ \text{and} \quad 2h-1-\sqrt{h^2-10h+1}>0, \end{array} \tag{4.40}$$

which is equivalent to $h > 5+2\sqrt{6}$, system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_2 = \sqrt{\dfrac{2(2h-1-\sqrt{h^2-10h+1})}{3}}$ given by

$$\begin{aligned} S_1^* = & \left(r^* = \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}, r^*, 0\right), \\ S_2^* = & \left(r^* = \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}, r^*, 0\right), \\ S_3^* = & \left(r^* = \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}, r^*, \pi\right), \\ S_4^* = & \left(r^* = \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}, r^*, \pi\right), \end{aligned} \tag{4.41}$$

The Jacobian evaluated at theses solutions is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{(1+h+\sqrt{h^2-10h+1})^2}{17496}\Big(2h-10-\sqrt{h^2-10h+1}\Big) \\ & \Big(56-596h+413h^2-7h^3+7h^4-h^5+56\sqrt{h^2-10h+1} \\ & -316h\sqrt{h^2-10h+1}-9h^2\sqrt{h^2-10h+1} \\ & +2h^3\sqrt{h^2-10h+1}-h^4\sqrt{h^2-10h+1}\Big). \end{aligned}$$

For $h \neq 5 \pm 2\sqrt{6}$, $J_{f_1(S^*)}$ does not vanish. Then with the condition (4.40), the four solutions (4.41) of system (4.8) provide four periodic solutions of differential system (4.6). Although, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

If $r_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ and $R_3 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ we have $\rho_3 = \sqrt{6}$.

Assuming that

$$\begin{array}{l} h > 0, \quad h-3+\sqrt{h^2-10h+33} > 0, \\ h-3-\sqrt{h^2-10h+33} > 0 \text{and} \quad h^2-10h+33 > 0, \end{array} \tag{4.42}$$

which is equivalent to $h > 6$. Therefore system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{6}$ and $R^* = \sqrt{h-3-\sqrt{h^2-10h+33}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{\pi}{2}, R^*, 0\right), \\ S_2^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{3\pi}{2}, R^*, 0\right), \\ S_3^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{\pi}{2}, R^*, \pi\right), \\ S_4^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{3\pi}{2}, R^*, \pi\right), \end{aligned} \tag{4.43}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{3}{2}(h-6)(h^2-10h+33)(h-3-\sqrt{h^2-10h+33}) \\ & (h-3+\sqrt{h^2-10h+33}). \end{aligned}$$

For $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. With the condition (4.42), the four solutions (4.43) of system (4.8) provide four periodic solutions of differential system (4.6). Nevertheless, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

If $r_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ and $R_4 = \sqrt{h-3+\sqrt{h^2-10h+33}}$. We get $\rho_4 = \sqrt{6}$.

Considering that

$$\begin{array}{l} h > 0, \quad h-3-\sqrt{h^2-10h+33} > 0, \\ h-3+\sqrt{h^2-10h+33} > 0 \quad \text{and} \quad h^2-10h+33 > 0, \end{array} \tag{4.44}$$

which is equivalent to $h > 6$ then system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_4 = \sqrt{6}$ and $R^* = \sqrt{h - 3 + \sqrt{h^2 - 10h + 33}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}, R^*, 0\right), \\ S_2^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}, R^*, 0\right), \\ S_3^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}, R^*, \pi\right), \\ S_4^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}, R^*, \pi\right), \end{aligned} \tag{4.45}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{3}{2}(h-6)(h^2-10h+33)(3-h+\sqrt{h^2-10h+33}) \\ & (h-3-\sqrt{h^2-10h+33}). \end{aligned}$$

If $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. Then with the condition (4.44), the four zeros (4.45) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we obtain only two periodic orbits S_1^* and S_3^*.

Subcase 3.2.4: If $(\alpha, \beta) = (\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{3\pi}{2})$ then f_{12}, f_{14} become

$$\begin{aligned} f_{12} = & -\frac{1}{8}\Big(R^4 - 2R^2(h+3-r^2) - 4r^2 + 4h\Big), \\ f_{14} = & -\frac{1}{8}\Big(R^4 - 2R^2(h-3) - r^4 + 2r^2(1+h) - 8h\Big). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we have four pairs of solutions (r, R),

If $r_1 = \sqrt{\frac{2}{3}(2h - 1 + \sqrt{h^2 - 10h + 1})}$ and $R_1 = \sqrt{\frac{1 + h - \sqrt{h^2 - 10h + 1}}{3}}$, we get $\rho_1 = \sqrt{\frac{1 + h - \sqrt{h^2 - 10h + 1}}{3}}$.

Whenever

$$\begin{aligned} & h > 0, \quad 2h - 1 + \sqrt{h^2 - 10h + 1} > 0, \quad h^2 - 10h + 1 > 0 \\ & \text{and} \quad 1 + h - \sqrt{h^2 - 10h + 1} > 0, \end{aligned} \tag{4.46}$$

which is equivalent to $h > 5 + 2\sqrt{6}$, then system (4.8) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_1 = \sqrt{\dfrac{1 + h - \sqrt{h^2 - 10h + 1}}{3}}$ and $R^* = \sqrt{\dfrac{1 + h - \sqrt{h^2 - 10h + 1}}{3}}$ given by

$$\begin{aligned}
S_1^* &= \left(\sqrt{\frac{2}{3}(2h - 1 + \sqrt{h^2 - 10h + 1})}, \frac{\pi}{2}, R^*, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{\frac{2}{3}(2h - 1 + \sqrt{h^2 - 10h + 1})}, \frac{\pi}{2}, R^*, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{\frac{2}{3}(2h - 1 + \sqrt{h^2 - 10h + 1})}, \frac{3\pi}{2}, R^*, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{\frac{2}{3}(2h - 1 + \sqrt{h^2 - 10h + 1})}, \frac{3\pi}{2}, R^*, \frac{3\pi}{2}\right),
\end{aligned} \tag{4.47}$$

The Jacobian evaluated on these zeros

$$\begin{aligned}
J_{f_1(S^*)} = {} & \frac{\sqrt{h^2 - 10h + 1}}{34992}\left(1 + h - \sqrt{h^2 - 10h + 1}\right)^2\Big(5 - h\\
& + \sqrt{h^2 - 10h + 1}\Big)\left(2h - 10 + \sqrt{h^2 - 10h + 1}\right)\Big(2h - 1\\
& + \sqrt{h^2 - 10h + 1}\Big)\left(h^2 + 17h - 14 + (14 - h)\sqrt{h^2 - 10h + 1}\right).
\end{aligned}$$

The $J_{f_1(S^*)}$ does not vanish for $h \neq 5 \pm 2\sqrt{6}$. Therefore with the condition (4.46), the four zeros (4.47) of system (4.8) provide four periodic solutions of differential system (4.6). Although, going back to the differential system (1.6) and using the Proposition 6, we obtain only one periodic orbits S_1^*.

If $r_2 = \sqrt{\dfrac{2}{3}(2h - 1 - \sqrt{h^2 - 10h + 1})}$ and $R_2 = \sqrt{\dfrac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}$, we get $\rho_2 = \sqrt{\dfrac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}$.

With the condition

$$\begin{aligned}
& h > 0, \quad 1 + h + \sqrt{h^2 - 10h + 1} > 0, \quad h^2 - 10h + 1 > 0\\
& \text{and} \quad 2h - 1 - \sqrt{h^2 - 10h + 1} > 0,
\end{aligned} \tag{4.48}$$

which is equivalent to $h > 5 + 2\sqrt{6}$, system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_2 = \sqrt{\dfrac{1 + h + \sqrt{h^2 - 10h + 1}}{3}}$ and

$R^* = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$ given by

$$\begin{aligned}
S_1^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{\pi}{2}, R^*, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{\pi}{2}, R^*, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{3\pi}{2}, R^*, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{3\pi}{2}, R^*, \frac{3\pi}{2}\right),
\end{aligned} \tag{4.49}$$

The Jacobian evaluated at theses solutions is

$$\begin{aligned}
J_{f_1(S^*)} = & -\frac{\sqrt{h^2-10h+1}}{34992}\left(2h-1-\sqrt{h^2-10h+1}\right)\\
& \left(h-5+\sqrt{h^2-10h+1}\right)\left(10-2h+\sqrt{h^2-10h+1}\right)\\
& \left(1+h+\sqrt{h^2-10h+1}\right)^2\left(h^2+17h-14+(h-14)\right.\\
& \left.\sqrt{h^2-10h+1}\right).
\end{aligned}$$

For $h \neq 5 \pm 2\sqrt{6}$, $J_{f_1(S^*)}$ does not vanish. Then with the condition (4.48), the four solutions (4.49) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we obtain only one periodic orbits S_1^*.

When $r_3 = \sqrt{6}$ and $R_3 = \sqrt{h-3-\sqrt{h^2-10h+33}}$, we have $\rho_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$.

Supposing that

$$\begin{aligned}
& h>0, \quad h-3+\sqrt{h^2-10h+33}>0,\\
& h-3-\sqrt{h^2-10h+33}>0 \quad \text{and} \quad h^2-10h+33>0,
\end{aligned} \tag{4.50}$$

which is equivalent to $h > 6$. Therefore system (4.8) provide four zeros

$S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{h - 3 + \sqrt{h^2 - 10h + 33}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right), \\ S_2^* &= \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right), \\ S_3^* &= \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right), \\ S_4^* &= \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right), \end{aligned} \tag{4.51}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{3}{2}\sqrt{h^2-10h+33}(h-6)(3-h+\sqrt{h^2-10h+33}) \\ & \left(h^2-10h+33+(h-3)\sqrt{h^2-10h+33}\right). \end{aligned}$$

For $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. Then with the condition (4.50), the four solutions (4.51) of system (4.8) provide four periodic solutions of differential system (4.6). However, going back to the differential system (1.6) and using the Proposition 6, we obtain only one periodic orbits S_1^*.

If $r_4 = \sqrt{6}$ and $R_4 = \sqrt{h - 3 + \sqrt{h^2 - 10h + 33}}$, we have $\rho_4 = \sqrt{h - 3 - \sqrt{h^2 - 10h + 33}}$.

Assuming that

$$\begin{aligned} & h > 0, \quad h - 3 + \sqrt{h^2-10h+33} > 0, \\ & h - 3 - \sqrt{h^2-10h+33} > 0 \quad \text{and} \quad h^2 - 10h + 33 > 0, \end{aligned} \tag{4.52}$$

which is equivalent to $h > 6$. Then system (4.8) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with

$\rho_4 = \sqrt{h - 3 - \sqrt{h^2 - 10h + 33}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right), \\ S_2^* &= \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right), \\ S_3^* &= \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right), \\ S_4^* &= \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right), \end{aligned} \tag{4.53}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{3}{2}\sqrt{h^2-10h+33}(h-6)(3-h-\sqrt{h^2-10h+33}) \\ & \left(h^2-10h+33+(h-3)\sqrt{h^2-10h+33}\right). \end{aligned}$$

If $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. With the condition (4.52), the four zeros (4.53) of system (4.8) provide four periodic solutions of differential system (4.6). Nevertheless, going back to the differential system (1.6) and using the Proposition 6, we obtain only one periodic orbits S_1^*.

Proof of Proposition 6. $(r^*, \alpha^*, R^*, \beta^*)$ is a periodic solution of (4.6) using the averaging theory means that

$$\begin{aligned} r(t,\varepsilon) = & \ r^* + O(\varepsilon), \\ \alpha(t,\varepsilon) = & \ \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = & \ R^* + O(\varepsilon), \\ \beta(0,\varepsilon) = & \ \beta^* + O(\varepsilon). \end{aligned} \tag{4.54}$$

Adding the expression of $\rho = \sqrt{2h - r^{*2} - R^{*2}}$ in system (4.54) we have

$$\begin{aligned} r(t,\varepsilon) = & \ r^* + O(\varepsilon), \\ \rho(t,\varepsilon) = & \ \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) = & \ \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = & \ R^* + O(\varepsilon), \\ \beta(t,\varepsilon) = & \ \beta^* + O(\varepsilon). \end{aligned} \tag{4.55}$$

Instead of θ we reconsider the variable, t, the temps, then (4.55) becomes

$$\begin{aligned} r(t,\varepsilon) = & \ r^* + O(\varepsilon), \\ \theta(t,\varepsilon) = & \ t + O(\varepsilon), \\ \rho(t,\varepsilon) = & \ \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) = & \ \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = & \ R^* + O(\varepsilon), \\ \beta(t,\varepsilon) = & \ \beta^* + O(\varepsilon). \end{aligned} \tag{4.56}$$

Finally, using the change of variables (4.1): $x = r\cos\theta$, $y = \rho\cos(\theta+\alpha)$, $z = R\cos(\theta+\beta)$, $p_x = r\sin\theta$, $p_y = \rho\sin(\theta+\alpha)$, $p_z = R\sin(\theta+\beta)$, the system

(4.56) becomes

$$
\begin{aligned}
x(t,\varepsilon) =&\ \ r^* \cos t + O(\varepsilon),\\
y(t,\varepsilon) =&\ \ \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* + t) + O(\varepsilon),\\
z(t,\varepsilon) =&\ \ R^* \cos(\beta^* + t) + O(\varepsilon),\\
p_x(t,\varepsilon) =&\ \ r^* \sin t + O(\varepsilon),\\
p_y(t,\varepsilon) =&\ \ \sqrt{2h - r^{*2} - R^{*2}}\sin(\alpha^* + t) + O(\varepsilon),\\
p_z(t,\varepsilon) =&\ \ R^* \sin(\beta^* + t) + O(\varepsilon).
\end{aligned}
$$

□

The results of this study have been submitted for publication.

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Appendix A

The averaging theory

We remember in this section the main results of averaging theory that we shall use for proving the results of this thesis work.

A.1 About the averaging theory

The idea of the averaging theory started at the eighteenth century when Laplace was studying Sun-Jupiter-Saturn system. After the averaging theory was considered by Lagrange (1788), in his study of three-body problem as a perturbation of the problem of the two bodies. During the 20^{th} century, this theory was developed by Fatou, Krylov and Bogoliubov and Bogoliubov and Metropolsky. A general introduction to the averaging theory and more references can be found in the books of Sanders, Verhulst and Murdock [35], and Verhulst [36].

We consider the following differential system

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{A.1}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Additionally, we assume that the functions $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. The averaged differential system in D is defined as follows

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{A.2}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt. \tag{A.3}$$

Later on, we see under convenient hypotheses that the equilibria solutions of the averaged system will provide $T-$periodic solutions of system (A.1).

A.2 The averaging theory of first order

Theorem 7. *Consider the two initial value problems* (A.1) *and* (A.2). *Suppose that*

(i) the functions F_1, $\partial F_1/\partial x$, $\partial^2 F_1/\partial x^2$, F_2 *and* $\partial F_2/\partial x$ *are defined, continuous and bounded by a constant independent of* ε *in* $[0,\infty)\times D$ *and* $\varepsilon \in (0, \varepsilon_0]$;

(ii) the functions F_1 *and* F_2 *are* $T-$*periodic in* t *(*T *independent of* ε*).*

Then the following statements hold.

(a) If p*, an equilibrium point of the averaged system* (A.2)*, satisfies*

$$\det\left(\frac{\partial f_1}{\partial \mathbf{y}}\right)\bigg|_{\mathbf{y}=p} \neq 0, \tag{A.4}$$

then there is a $T-$*periodic solution* $\varphi(t,\varepsilon)$ *of system* (A.1) *such that* $\varphi(0,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

(b) If p*, an equilibrium point of the averaged system* (A.2)*, is hyperbolic then it has the stability behavior of the Poincaré map associated to the periodic solution* $\varphi(t,\varepsilon)$.

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