Universitat Autònoma de Barcelona

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Contents

Chapter 1

Introduction

This thesis studies integrability and convergence properties of Fourier series/transforms of monotone (general monotone) functions or functions whose Fourier coefficients/transforms are monotone (general monotone).

1.1 Lorentz and weighted Lebesgue spaces

We start with the definitions of weighted Lebesgue and Lorentz spaces [9]. Let (Ω, μ) be a measure space.

Definition 1.1. *Let f be a μ-measurable function on Ω, then by f^* we denote the non-increasing rearrangement of f, i.e.,*

$$f^*(t) = \inf\{\sigma : \mu\{x \in \Omega : |f(x)| > \sigma\} \leq t\}.$$

Definition 1.2. *Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then the Lorentz spaces $L_{p,q}(\Omega)$ is the set of μ-measurable functions f for which, the functional*

$$\|f\|_{L_{p,q}(\Omega)} := \begin{cases} \left(\int\limits_0^{\mu(\Omega)} \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{for } 0 < p < \infty \text{ and } 0 < q < \infty, \\ \sup\limits_{0 \leq t \leq \mu(\Omega)} t^{\frac{1}{p}} f^*(t) & \text{for } 0 < p \leq \infty \text{ and } q = \infty, \end{cases}$$

is finite.

Definition 1.3. *Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then the weighted Lebesgue spaces $L^q_{w[p,q]}(\Omega)$ is the set of μ-measurable functions f for which, the functional*

$$\|f\|_{L^q_{w[p,q]}(\Omega)} := \begin{cases} \left(\int\limits_\Omega \left| t^{\frac{1}{p} - \frac{1}{q}} f(t) \right|^q dt \right)^{\frac{1}{q}} & \text{for } 0 < p < \infty \text{ and } 0 < q < \infty, \\ \operatorname{ess\,sup}\limits_{t \in \Omega} t^{\frac{1}{p}} |f(t)| & \text{for } 0 < p \leq \infty \text{ and } q = \infty, \end{cases}$$

is finite.

Here $w[p,q](t)$ stands for weight function $t^{\frac{1}{p}-\frac{1}{q}}$. We will denote by $l_{p,q}$ and $l^q_{w[p,q]}$ similarly defined Lorentz and weighted Lebesgue spaces of sequences, respectively.

Remark 1.1. *Note that* $\|f\|_{L_{p,p}(\Omega)} = \|f\|_{L^p_{w[p,p](\Omega)}} = \|f\|_{L_p(\Omega)}$. *Moreover, Hardy's rearrangement inequality implies*

$$\|f\|_{L_{p,q}(\Omega)} \geq \|f\|_{L^q_{w[p,q]}(\Omega)} \quad for\ q \leq p\,;$$
$$\|f\|_{L_{p,q}(\Omega)} \leq \|f\|_{L^q_{w[p,q]}(\Omega)} \quad for\ q \geq p\,.$$

Throughout this thesis, we denote by C a positive constant that may be different on different occasions. In addition, $T \lesssim S$ means that there exists $C > 0$ such that $T \leq CS$. Moreover, $T \asymp S$ means $T \lesssim S \lesssim T$. Throughout this thesis, p' denotes the conjugate index of p : $\frac{1}{p} + \frac{1}{p'} = 1$.

1.2 General monotonicity

Here we introduce the notion of general monotonicity, which is the key concept in this work. General monotone sequences (or functions) play an important role in many classical problems of harmonic analysis and approximation theory (see, for instance, [16], [17], [18], [34], [42], [58], [91], [95]). The definition of the $GM(\boldsymbol{\beta})$ sequences (see [59, 88, 91]) reads as follows.

Definition 1.4. *Let* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ *and* $\boldsymbol{\beta} = \{\beta_n\}_{n=1}^{\infty}$ *be two sequences of complex and non-negative numbers, respectively. The couple* $(\mathbf{a}, \boldsymbol{\beta})$ *determines a general monotone sequence* $\mathbf{a}$ *with majorant* $\boldsymbol{\beta}$, *written* $\mathbf{a} \in GM(\boldsymbol{\beta})$, *if there exists* $C > 0$ *such that for all* $n \in \mathbb{N}$,

$$\sum_{k=n}^{2n} |\Delta a_k| \leq C\beta_n. \tag{1.1}$$

It will be the key observation in our further study that $GM(\boldsymbol{\beta})$ sequences preserve some monotonicity properties. This is given by the following result.

Lemma 1.1 ([59, Lemma 3.1])**.** *Let* $\mathbf{a} \in GM(\boldsymbol{\beta})$, *then for all* $n \in \mathbb{N}$ *we have*

$$|a_k| \leq C\beta_n + |a_m| \quad for\ all \quad k, m = n, \ldots, 2n; \tag{1.2}$$

$$|a_k| \leq C\beta_n + \frac{1}{n}\sum_{j=n+1}^{2n} |a_j| \quad for\ all \quad k = n, \ldots, 2n; \tag{1.3}$$

$$|a_n| \leq \frac{C}{n}\left(\sum_{k=[\frac{n}{2}]}^{n-1} \beta_k + \sum_{j=n}^{2n-1} |a_j|\right). \tag{1.4}$$

Note that the widest class of general monotone sequences is when $\beta_n = \sum\limits_{k=n}^{2n} |a_k|$, since in this case any sequence belongs to this class with $C = 2$. Let us give some examples of majorants $\boldsymbol{\beta}$, which will be useful in study of trigonometric series.

1. $\beta_n^1 = |a_n|$;
2. $\beta_n^2 = \frac{1}{n} \sum\limits_{s=\frac{n}{\gamma}}^{\gamma n} |a_s|, \quad \gamma > 1$;
3. $\beta_n^3 = \frac{1}{n} \max\limits_{k \geq \frac{n}{\gamma}} \sum\limits_{s=k}^{2k} |a_s|, \quad \gamma > 1$.

It is known that $M \subsetneq QM \subsetneq GM(\boldsymbol{\beta}^1) \subsetneq GM(\boldsymbol{\beta}^2) \subsetneq GM(\boldsymbol{\beta}^3)$ (see [34, 91, 94]), where M is the class of non-increasing sequences, and QM is the class of quasi-monotone sequences. Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ is quasi-monotone if there exists $\tau > 0$ such that $\left\{\frac{a_n}{n^\tau}\right\}$ is non-increasing. More details about the various subclasses of general monotone sequences can be found in [59, 94].

Now we describe some classical problems that we study in this thesis.

1.3 Hardy-Littlewood's theorem: periodic case

We start with integrability properties of Fourier series. Integrability properties of Fourier series with monotone coefficients were first considered by Hardy and Littlewood ([43], [105, Ch. XII, §6], [14, §6]).

Theorem 1.1. *Let $1 < p < \infty$ and $f(x) \sim \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, where $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$ are non-increasing non-negative sequences vanishing at infinity. Then*

$$\|f\|_{L_p([0,2\pi])} \asymp \left(\sum_{n=1}^{\infty} n^{p-2} (a_n^p + b_n^p) \right)^{\frac{1}{p}}.$$

Generalizations of Theorem 1.1 for the Lorentz spaces $L_{p,q}([0, 2\pi])$ and weighted Lebesgue spaces $L^q_{w[p,q]}([0, 2\pi])$ were proved by Sagher [76, Theorems 1 and 2].

Theorem 1.2. *Let $f(x) \sim \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, where $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$ are non-increasing non-negative sequences vanishing at infinity. Then*

$$\|f\|_{L_{p,q}([0,2\pi])} \asymp \|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}} \quad 1 < p < \infty, \quad 0 < q \leq \infty, \tag{1.5}$$

$$\|f\|_{L^q_{w[p,q]}([0,2\pi])} \asymp \|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}} \quad 1 < p < \infty, \quad 1 \leq q \leq \infty. \tag{1.6}$$

Since for monotone sequence $\{a_n\}_{n=1}^{\infty}$, $a_n^* = a_n$, we obtain the following corollary.

Corollary 1.1. *Let* $f(x) \sim \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, *where* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$ *are non-increasing non-negative sequences vanishing at infinity. Then, for any* $1 < p < \infty$ *and* $1 \le q \le \infty$,

$$\|f\|_{L_{p,q}([0,2\pi])} \asymp \|f\|_{L^q_{w[p,q]}([0,2\pi])} \asymp \|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}}$$
$$\asymp \|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}}.$$

There are many generalizations of Theorem 1.1. In particular, for weighted Lebesgue spaces Theorem 1.1 was generalized in papers [4, 27, 32, 33, 34, 54, 76, 91, 103].

Analogues of Theorem 1.1 for the Lorentz spaces were proved in [16, 29, 31, 42, 68, 69, 76, 81].

1.4 Hardy-Littlewood's theorem and Boas' conjecture: non-periodic case

One of the first results on the integrability properties of the Fourier transforms of monotone functions is the well-known Hardy-Littlewood theorem [98, §4.12].

Theorem 1.3. *Let* $1 < p < 2$ *and* $f(x)$ *be a non-increasing non-negative on* $(0, +\infty)$ *function that vanishes at infinity, and* $\widehat{f}(t) = \int\limits_0^{+\infty} f(x) \cos tx \, dx$ *be a cosine transform of* f*. Then*

$$\|\widehat{f}\|_{L_p(0,\infty)} \le C \left(\int_0^{\infty} x^{p-2} f(x)^p \, dx \right)^{\frac{1}{p}}.$$

In the paper [15], Boas stated the following

Conjecture. *Let* $1 < p < \infty$ *and* f *be a non-increasing non-negative on* $(0, \infty)$ *function that vanishes at infinity. And put* $\widehat{f}$ *is cosine or sine transform of* f*. Then* $x^{-\gamma} \widehat{f}(x) \in L_p(0, \infty)$ *if and only if* $x^{1+\gamma-\frac{2}{p}} f(x) \in L_p(0, \infty)$ *provided* $-\frac{1}{p'} < \gamma < \frac{1}{p}$.

In [77], Sagher solved this problem in the setting of the weighted Lebesgue spaces and the Lorentz spaces.

By E we denote the set of non-negative, even on $\mathbb{R}$, non-increasing to 0 on $(0, +\infty)$ functions.

Theorem 1.4 ([77])**.** *Let* $f(x) \in E$ *and* $\widehat{f}(y) = \int\limits_{-\infty}^{\infty} f(x) e^{-ixy} dx$ *be the Fourier transform of* f*. Then*

$$\|f\|_{L_{p,q}(\mathbb{R})} \asymp \|\widehat{f}\|_{L_{p',q}(\mathbb{R})}, \quad 1 < p < \infty, \quad 0 < q \le \infty, \tag{1.7}$$

$$\|f\|_{L^q_{w[p,q]}(\mathbb{R})} \asymp \|\widehat{f}\|_{L^q_{w[p',q]}(\mathbb{R})}, \quad 1 < p < \infty, \quad 1 \leq q \leq \infty. \tag{1.8}$$

The equality $f^*(t) = f\left(\frac{t}{2}\right)$ for every $f \in E$ immediately implies the following corollary.

Corollary 1.2. *Let $f(x) \in E$, then*

$$\|\widehat{f}\|_{L^q_{w[p',q]}(\mathbb{R})} \asymp \|\widehat{f}\|_{L_{p',q}(\mathbb{R})}, \quad 1 < p < \infty, \quad 1 \leq q \leq \infty. \tag{1.9}$$

There are several results related with Boas' conjecture. In the paper [57] (see also [60]), Liflyand and Tikhonov proved Boas' conjecture for general monotone functions. Moreover, in the case of the sine transform the range of γ was enlarged. Later on Gorbachev, Liflyand, and Tikhonov [40] obtained the multidimensional version for radial functions. Also, Boas' conjecture was proved [28] for the wider class of general monotone functions than the one studied in [57].

Note that relation (1.7) of Theorem 1.4 was generalized in [67, 69] for weak monotone functions in the case when $1 < p < 2$. Later on, Kopezhanova, Nursultanov, and Persson [49] generalized (1.7) for weak monotone functions in the Lorentz spaces with general weights.

In [100, 101], Volosivets and Golubov proved the Boas' conjecture for multiplicative Fourier transforms of general monotone functions f, extending the results from [28] which deals with the trigonometric case.

1.5 Moduli of smoothness of Fourier series with monotone coefficients

Let f be an integrable 2π-periodic function. Denote by

$$\omega_l(f,\delta)_p := \sup_{|h|\leq\delta} \left\|\Delta_h^l f(\cdot)\right\|_p$$

the modulus of smoothness of function $f \in L_p$ of order $l \geq 1$, where

$$\Delta_h^l f(x) := \Delta_h(\Delta_h^{l-1} f(x)), \quad \Delta_h f(x) := f(x+h) - f(x).$$

The following relation between the modulus of the smoothness of the function f and its Fourier coefficients was proved by Aljančić [1] and Potapov-Berisha [72].

Theorem 1.5. *Let $1 < p < \infty$, $l \in \mathbb{N}$. Let also $f \in L_p([0, 2\pi])$,*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *be non-increasing sequences. Then*

$$\omega_l\left(f,\frac{1}{n}\right)_p \asymp \frac{1}{n^l}\left(\sum_{k=1}^{n}k^{(l+1)p-2}(a_k^p+b_k^p)\right)^{\frac{1}{p}}+\left(\sum_{k=n+1}^{\infty}k^{p-2}(a_k^p+b_k^p)\right)^{\frac{1}{p}}.$$

The result of Theorem 1.5 was generalized in many papers, in particular, [39, 41, 51, 53, 70, 72, 85, 98, 102]. Moreover, this result and its generalizations play important role to study characterizations and embedding theorems for smooth function classes.

1.6 Uniform convergence of sine and cosine series

Here we start with the well-known Chaundy-Jolliffee's result [20] on the sine series

$$\sum_{n=1}^{\infty}a_n\sin nx \tag{1.10}$$

with monotone coefficients $\{a_n\}_{n=1}^{\infty}$.

Theorem 1.6 **([20]).** *Let* $\{a_n\}_{n=1}^{\infty}$ *be a non-negative non-increasing sequence. Then series* (1.10) *converges uniformly on* $[0,2\pi]$ *if and only if* $na_n\to 0$ *as* $n\to\infty$.

For the cosine series

$$\sum_{n=1}^{\infty}a_n\cos nx, \tag{1.11}$$

we highlight the following obvious fact.

Theorem 1.7. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a non-negative sequence. Then series* (1.11) *converges uniformly on* $[0,2\pi]$ *if and only if* $\sum\limits_{n=1}^{\infty}a_n$ *converges.*

Very recently, several generalizations of these theorems have been proved where different extensions of monotonicity condition were considered (see, e.g., [33], [34], [53], [91], [94], [104] and the references therein). Many generalizations involve consideration of general monotone sequences. In particular, in the recent paper [37], the authors proved an analogue of Theorem 1.6 for $\{a_n\}_{n=1}^{\infty}\in GM(\boldsymbol{\beta}^2)$ without an assumption that $\{a_n\}_{n=1}^{\infty}$ is a non-negative sequence.

1.7 Structure of thesis and main results

In Chapter 2, we generalize the Hardy-Littlewood-type theorem (Theorem 1.2) for sequences from class $GM=GM(\boldsymbol{\beta^2})$. The main results of Chapter 2 are Theorems 2.3 and 2.4.

Theorem 2.3. *Let* $f(x) \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$*, and let sequences of real numbers* $\mathbf{a} = \{a_n\}_{n=0}^{\infty}, \mathbf{b} = \{b_n\}_{n=1}^{\infty} \in GM$*. Then*

$$\|f\|_{L_{p,q}([0,2\pi])} \asymp \|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}}, \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{1.12}$$

Theorem 2.4. *Let* $f(x) \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$*, and let sequences of real numbers* $\mathbf{a} = \{a_n\}_{n=0}^{\infty}, \mathbf{b} = \{b_n\}_{n=1}^{\infty} \in GM$*. Then*

$$\|f\|_{L^q_{w[p,q]}([0,2\pi])} \asymp \|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}}, \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{1.13}$$

The main novelty of Theorems 2.3 and 2.4 is that we do not assume that $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are of constant sign. This allows us to consider a very rich class of sequences.

Section 2.4 provide us some needed properties of $GM(\boldsymbol{\beta}^2)$ and WM sequences, where

$$WM = \left\{ \{a_n\}_{n=1}^{\infty} : |a_n| \le C \sum_{k=\frac{n}{\gamma}}^{\gamma n} \frac{|a_k|}{k} \quad C > 0, \ \gamma > 1 \right\}.$$

In Section 2.5, we prove the main results of this chapter.

In Chapter 3, we generalize Aljančić' and Potapov-Berisha's result for sequences from class $GM = GM(\boldsymbol{\beta}^2)$, see [1, 72, 41, 91]. Our main result reads as follows.

Theorem 3.5. *Let* $f(x) \in L_p([0, 2\pi])$*,* $1 < p < \infty$*,*

$$f(x) \sim \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$*. Then, for any* $l \in \mathbb{N}$*,*

$$\omega_l\left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^l}\left(\sum_{k=1}^{n} k^{lp+p-2}\left(|a_k|^p + |b_k|^p\right)\right)^{\frac{1}{p}} + \left(\sum_{k=n}^{\infty} k^{p-2}\left(|a_k|^p + |b_k|^p\right)\right)^{\frac{1}{p}}.$$

In Sections 3.1 and 3.2, we give some historical remarks on relations between the smoothness of a function and its Fourier coefficients. In Section 3.3, we formulate the main results. Section 3.4 provides us some needed properties of $GM(\boldsymbol{\beta}^2)$ sequences for this chapter. In Sections 3.5 and 3.6, we give upper estimates for the sums $\frac{1}{n^{lp}} \sum\limits_{k=1}^{n} |a_k|^p k^{(l+1)p-2}$ and $\sum\limits_{k=n}^{\infty} |a_k|^p k^{(l+1)p-2}$, respectively, which will be used in the proof of Theorem 3.5. In Section 3.7, we prove Theorem 3.5 and show that assertion of Theorem 3.5 is not true anymore for sequences from wider class $\overline{WM}$, where

$$\overline{WM} = \left\{ \{a_n\}_{n=1}^{\infty} : |a_n| \le C \sum_{k=\frac{n}{\gamma}}^{\infty} \frac{|a_k|}{k} \quad C > 0,\ \gamma > 1 \right\}.$$

In Section 3.8, we give some corollaries of Theorem 3.5, in particular, we give characterizations of norms of such functions in Besov spaces. As in Chapter 2 we do not assume non-negativity or non-positivity of the Fourier coefficients.

In Chapter 4, we prove a multidimensional analogue of Boas-Sagher's Theorem 1.4 for anisotropic Lorentz and weighted Lebesgue spaces. The main results of this chapter are the following theorems.

Theorem 4.2. *Let* $\mathbf{1} < \mathbf{p} < \infty$*,* $\mathbf{0} < \mathbf{q} \le \infty$ *and* $f \in E^n$*. Then*

$$\|f\|_{L_{\mathbf{p},\mathbf{q}}(\mathbb{R}^n)} \asymp \|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}(\mathbb{R}^n)}.$$

Theorem 4.3. *Let* $\mathbf{1} < \mathbf{p} < \infty$*,* $\mathbf{1} \le \mathbf{q} \le \infty$ *and* $f \in E^n$*. Then*

$$\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}(\mathbb{R}^n)} \asymp \|\widehat{f}\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p}',\mathbf{q}]}(\mathbb{R}^n)}.$$

Here, $\mathbf{p} = (p_1, p_2, \dots, p_n)$, $\mathbf{q} = (q_1, q_2, \dots, q_n)$ are n-dimensional vectors, E^n is a set of monotone in each variable functions on $\mathbb{R}^n$. By $L_{\mathbf{p},\mathbf{q}}(\mathbb{R}^n)$ and $L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}(\mathbb{R}^n)$ we denote the multidimensional anisotropic Lorentz and weighted Lebesgue spaces, respectively. The definitions of these spaces can be found in Section 4.2. In Section 4.3, we formulate our main results. Sections 4.4 and 4.5 are devoted to some auxiliary results. In Section 4.6, we prove Theorems 4.2 and 4.3. Note that the main results of this chapter were proved in [64].

In Chapter 5, we obtain generalizatons of Theorems 1.6 and 1.7. We consider a class of general monotone sequences $GM(\boldsymbol{\beta})$ with

$$\beta_n = \frac{1}{n} F_n(\widetilde{\mathbf{a}}),$$

where $\widetilde{\mathbf{a}} = \{\widetilde{a}_n\}_{n=1}^{\infty}$, $\widetilde{a}_n = \sum_{k=n}^{2n} |a_k|$, and $\{F_n\}_{n=1}^{\infty}$ is a sequence of admissible functionals defined on the set of sequences (see the definition in Section 5.1).

The main results of this chapter are the following theorems.

Theorem 5.2. *Let* $\{F_n\}_{n=1}^{\infty}$ *be admissible. Let also* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n} F_n(\widetilde{\mathbf{a}})$ *and* $\widetilde{\mathbf{a}}$ *is a bounded sequence. Then the following conditions are equivalent:*

(1) *the series* $\sum_{n=1}^{\infty} a_n \sin nx$ *converges uniformly on* $[0, 2\pi]$*;*

(2) $\lim\limits_{n\to\infty} na_n = 0$;

(3) $\lim\limits_{n\to\infty} \widetilde{a}_n = 0$.

Theorem 5.3. *Let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ *with admissible* $\{F_n\}_{n=1}^{\infty}$ *and bounded* $\widetilde{\mathbf{a}}$*. Then the series* $\sum\limits_{n=1}^{\infty} a_n \cos nx$ *converges uniformly on* $[0, 2\pi]$ *if and only if the series* $\sum\limits_{n=1}^{\infty} a_n$ *converges.*

Along with these problems, in Chapter 5, we study the rate of convergence of partial sums of series (1.10) and (1.11) in terms of growing properties of their coefficients. Denote by $g(x)$ and $f(x)$ the sums of (1.10) and (1.11) series, respectively.

Theorem 5.5. *Let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}\sum\limits_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$*. Then, for* $0 < \alpha \leq 1$*,*

$$\|f - S_n(f)\|_{C[0,2\pi]} = o\left(\frac{1}{n^{\alpha}}\right) \quad \Longleftrightarrow \quad a_n = o\left(\frac{1}{n^{\alpha+1}}\right).$$

$$\|g - S_n(g)\|_{C[0,2\pi]} = o\left(\frac{1}{n^{\alpha}}\right) \quad \Longleftrightarrow \quad a_n = o\left(\frac{1}{n^{\alpha+1}}\right).$$

Theorem 5.6. *Let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}\sum\limits_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$*. Then, for* $0 < \alpha \leq 1$*,*

$$\|f - S_n(f)\|_{C[0,2\pi]} = O\left(\frac{1}{n^{\alpha}}\right) \quad \Longleftrightarrow \quad a_n = O\left(\frac{1}{n^{\alpha+1}}\right).$$

$$\|g - S_n(g)\|_{C[0,2\pi]} = O\left(\frac{1}{n^{\alpha}}\right) \quad \Longleftrightarrow \quad a_n = O\left(\frac{1}{n^{\alpha+1}}\right).$$

As corollaries of Theorem 5.6 we obtain necessary and sufficient conditions for function to be in the Lipschitz space Lip α in terms of the rate of convergence of partial Fourier sums. In Sections 5.2 and 5.3, we formulate and prove the main results of Chapter 5, respectively. In Section 5.4, we give some examples of general monotone sequences. In particular, it is shown that Theorem 5.2 extends results from the paper [37]. In Section 5.5, we show that Theorem 5.2 does not hold without assumption on the boundedness of sequence $\{\widetilde{a}_n\}_{n=1}^{\infty}$. The main results of this chapter were proved in [30].

Chapter 2

Integrability theorems for the trigonometric series with general monotone coefficients

In this chapter, we consider the class of general monotone sequences $GM(\boldsymbol{\beta})$ with

$$\beta_n = \sum_{k=\frac{n}{\gamma}}^{\gamma n} \frac{|a_k|}{k}, \quad \gamma > 1.$$

For convenience, throughout this chapter, we denote this class by GM.

2.1 Historical remarks

Let f be an integrable 2π-periodic function with the Fourier series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Let us start with the following well-known Pitt's inequality (see [71, 84]) written in the setting of the weighted Lebesgue spaces $L^q_{w[p,q]}$ as follows

$$\|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}} \lesssim \|f\|_{L^q_{w[p,q]}([0,2\pi])},$$

where

$$1 < p \leq q \leq p'.$$

Note that the case $q = p'$ corresponds to the Hausdorff–Young inequality (see [105, Ch. XII, §2])

$$\left(\sum_{n=1}^{\infty}\left(|a_n|^{p'} + |b_n|^{p'}\right)\right)^{\frac{1}{p'}} \lesssim \|f\|_{L_p([0,2\pi])}, \quad 1 < p \le 2 \tag{2.1}$$

Moreover, the case $q = p$ corresponds to the Hardy–Littlewood inequality (see [105, Ch. XII, §3]).

$$\left(\sum_{n=1}^{\infty} n^{p-2}\left(|a_n|^{p} + |b_n|^{p}\right)\right)^{\frac{1}{p}} \lesssim \|f\|_{L_p([0,2\pi])}, \quad 1 < p \le 2. \tag{2.2}$$

Recall that for monotone sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ Hardy and Littlewood ([43], [105, Ch. XII, §6]) proved the equivalence

$$\left(\sum_{n=1}^{\infty} n^{p-2}\left(|a_n|^{p} + |b_n|^{p}\right)\right)^{\frac{1}{p}} \asymp \|f\|_{L_p([0,2\pi])}, \quad 1 < p < \infty. \tag{2.3}$$

Results of these type are of great importance in analysis since they provide a simple way to calculate the L_p norm of a function. We only mention the papers [41, 76, 82], where one can find applications of Hardy–Littlewood's equivalence in approximation theory, harmonic analysis, and functional analysis.

The equivalence (2.3) was generalized in [34] for general monotone sequences. In [34], Dyachenko and Tikhonov considered weighted Lebesgue spaces. We rewrite their results in terms of weighted Lebesgue spaces $L^q_{w[p,q]}$.

Theorem 2.1 ([34, Theorems 4.2 and 4.3]). *Let* $f(x) \sim \sum\limits_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, *where* $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$ *are non-negative sequences. Then*

$$\|f\|_{L^q_{w[p,q]}([0,2\pi])} \asymp \|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}}, \quad 1 < p < \infty, \quad 1 \le q < \infty.$$

Various generalizations of Hardy-Littlewood theorem in the setting of the weighted Lebesgue spaces can be found in [4, 27, 32, 33, 54, 89, 91, 103].

Now we consider some results related to the Lorentz spaces. Note that the estimates (2.1) and (2.2) in the case when $1 < p < 2$ follows from the general Hausdorff-Young inequality given by

$$\|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}} \lesssim \|f\|_{L_{p,q}([0,2\pi])}, \quad 1 < p < 2, \quad 0 < q \le \infty; \tag{2.4}$$

see, e.g., [75]. As was mentioned in Section 1.3, for monotone sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ Sagher [76] proved the following equivalence

$$\|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}} \asymp \|f\|_{L_{p,q}([0,2\pi])}, \quad 1 < p < \infty, \quad 0 < q \le \infty. \tag{2.5}$$

Recently, Grigoriev, Sagher, and Savage [42] generalized equivalence (2.5) for GM sequences. For given $0 \le \alpha < 2\pi,\ 0 \le \beta < \frac{\pi}{2}$, let us denote

$$S_{\alpha,\beta} = \{re^{i\varphi} : |\varphi - \alpha| \le \beta, r \ge 0\}. \tag{2.6}$$

Theorem 2.2 **([42]).** *Let* $h(x) \sim \sum\limits_{n=0}^{\infty} c_n e^{inx}$*, where sequence of complex numbers* $\mathbf{c} = \{c_n\}_{n=0}^{\infty} \in GM$ *such that, for any* $n \ge 0$, $c_n \in S_{\alpha,\beta}$ *for some* $0 \le \alpha < 2\pi, 0 \le \beta < \frac{\pi}{2}$. *Then*

$$\|h\|_{L_{p,q}([0,2\pi])} \asymp \|\mathbf{c}\|_{l_{p',q}}, \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{2.7}$$

Other generalizations of Hardy-Littlewood theorem in the setting of the Lorentz spaces obtained in [16, 29, 31, 42, 68, 69, 76, 81].

2.2 Main results

The main results of this chapter are Hardy-Littlewood theorems for functions with general monotone coefficients.

Theorem 2.3. *Let* $f(x) \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, *and let sequences of real numbers* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. *Then*

$$\|f\|_{L_{p,q}([0,2\pi])} \asymp \|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}}, \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{2.8}$$

Theorem 2.4. *Let* $f(x) \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, *and let sequences of real numbers* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. *Then*

$$\|f\|_{L^q_{w[p,q]}([0,2\pi])} \asymp \|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}}, \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{2.9}$$

The main novelty of Theorems 2.3 and 2.4 is that we do not assume additional conditions on $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ except general monotonicity as in the previous study. This allows us to consider a rich function class, for which the Hardy–Littlewood–Sagher type equivalences are valid.

Theorems 2.3 and 2.4 along with Lemma 2.3 below imply the following corollary.

Corollary 2.1. *Let* $f(x) \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, *and let sequences of real numbers* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. *Then, for any* $1 < p < \infty$ *and* $1 \le q \le \infty$,

$$\begin{aligned} \|f\|_{L_{p,q}([0,2\pi])} \asymp \|f\|_{L^q_{w[p,q]}([0,2\pi])} &\asymp \|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}} \\ &\asymp \|\mathbf{a}\|_{l_{p',q}} + \|\mathbf{b}\|_{l_{p',q}}. \end{aligned} \tag{2.10}$$

Remark 2.1. *Note that, for any* $0 \le \beta < \frac{\pi}{2}$*, the set* $S_{0,\beta}$ *(see Definition 2.6) contains a positive half-line* $\mathbb{R}_+$*, i.e., relation* (2.7) *holds for* $h(x) \sim \sum_{n=0}^{\infty} c_n e^{inx}$ *with* $\{c_n\}_{n=0}^{\infty} \in GM$*,* $c_n \ge 0$*. On the other hand, for any* $0 \le \alpha < 2\pi$ *and* $0 \le \beta < \frac{\pi}{2}$*, the set* $S_{\alpha,\beta}$ *does not cover a real line* $\mathbb{R}$*.*

Remark 2.2. *Theorem 2.4 was proved for non-negative sequences* $\{a_n\}_{n=0}^{\infty}$*,* $\{b_n\}_{n=1}^{\infty} \in GM$ *in [103] for* $1 < q < \infty$ *and in [34] for* $1 \le q < \infty$*.*

Throughout this chapter, we fix constants $C > 0$ and $\gamma > 1$ from the definition of GM sequences. All constants in this chapter may depend only on C, γ, p, and q.

2.3 Pitt-type inequalities involving averages of Fourier coefficients

Recall that Pitt's inequality for a function $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ in the setting of the weighted Lebesgue spaces $L^q_{w[p,q]}$ reads as follows

$$\|\mathbf{a}\|_{l^q_{w[p',q]}} + \|\mathbf{b}\|_{l^q_{w[p',q]}} \lesssim \|f\|_{L^q_{w[p,q]}},$$

where

$$1 < p \le q \le p';$$

see [71, 84].

The condition $1 < p \le q \le p'$ is sharp, see, e.g., [34]. In this section we will extend Pitt's inequality for the case $1 < p < \infty$ and $1 \le q \le \infty$ with the help of averages of Fourier coefficients. We start with the following known result.

Lemma 2.1 ([77, Theorem 2.4])**.** *Let* $1 < p < \infty$*,* $0 < q \le \infty$ *and let* $\{a_n\}_{n=1}^{\infty}$ *be the sequence of Fourier coefficients of an integrable function* f *with respect to* $\{\sin nx\}_{n=1}^{\infty}$*, or* $\{\cos nx\}_{n=1}^{\infty}$*. Then, for* $\sigma_n(\mathbf{a}) := \frac{1}{n} \sum_{k=1}^{n} a_k$*, we have*

$$\|m(\sigma_n(\mathbf{a}))\|_{l_{p',q}} \lesssim \|f\|_{L_{p,q}([0,2\pi])}, \tag{2.11}$$

where $m(\sigma_n) := \sup_{k \ge n} |\sigma_k|$*.*

Remark 2.3. *A stronger inequality than* (2.11) *was proved by Nursultanov in [68, Theorem 3] for* $p > 2$ *in setting of the net spaces [68, 69].*

An analogue of Lemma 2.1 for weighted Lebesgue spaces is written as follows.

Lemma 2.2. *Let $1 < p < \infty$, $1 \le q \le \infty$ and let $\{a_n\}_{n=1}^{\infty}$ be Fourier coefficients of an integrable function f with respect to $\{\sin nx\}_{n=1}^{\infty}$, or $\{\cos nx\}_{n=1}^{\infty}$. Then, for $\sigma_n(\mathbf{a}) := \frac{1}{n}\sum\limits_{k=1}^{n} a_k$, we have*

$$\|m(\sigma_n(\mathbf{a}))\|_{l^q_{w[p',q]}} \lesssim \|f\|_{L^q_{w[p,q]}([0,2\pi])}, \tag{2.12}$$

where $m(\sigma_n) := \sup\limits_{k\ge n}|\sigma_k|$.

We establish Lemma 2.2 following ideas from [77, Theorems 2.4 and 3.1] and using interpolation methods. Let us recall some notions. Let (A_0, A_1) be a compatible couple of quasi-normed spaces and

$$K(t,a) = K(t,a;A_0,A_1) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1$$

be the Peetre K-functional ([11]).

The space $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, consists of all elements $a \in A_0 + A_1$ for which the functional

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \begin{cases} \left(\int\limits_0^{\infty}(t^{-\theta}K(t,a))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup\limits_{0<t<\infty} t^{-\theta}K(t,a), & q = \infty, \end{cases}$$

is finite.

Recall that if (A_0, A_1) and (B_0, B_1) are compatible couples of quasi-normed spaces, and a quasi-linear operator $T : A_i \to B_i$, $i = 0, 1$ is bounded, then $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is bounded for any $0 < \theta < 1$ and $0 < q \le \infty$.

Proof of Lemma 2.2. We only present the proof for the sine series. For the cosine series and, consequently for the general trigonometric series, Lemma 2.2 follows from the boundedness of the Hilbert operator in the weighted Lebesgue spaces. We first establish the weak inequality

$$\|m(\sigma_n(\mathbf{a}))\|_{l_{s',\infty}} \lesssim \|f\|_{L^{\tau}_{w[s,\tau]}([0,2\pi])}, \quad 1 < s < \infty,\ 1 \le \tau \le \infty. \tag{2.13}$$

Let $1 < s < \infty$. Using simple calculations, we obtain

$$\begin{aligned} \sigma_n(\mathbf{a}) &= \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n}\int_0^{2\pi} f(x)\sin kx dx \\ &= \frac{2}{n}\sum_{k=1}^{n}\int_0^{\pi} f(x)\sin kx dx = \frac{2}{n}\int_0^{\pi} f(x)\left(\sum_{k=1}^{n}\sin kx\right)dx \\ &= \int_0^{\pi} f(x)\frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{n\sin\frac{x}{2}}dx \\ &= \int_0^{\pi} f(x)\left(\frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n}\right)dx. \end{aligned} \tag{2.14}$$

Assume first that $1 < \tau \le \infty$. Then Hölder's inequality yields

$$\begin{aligned} |\sigma_n(\mathbf{a})| &\le \int_0^\pi |f(x)| \left| \frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n} \right| dx \\ &= \int_0^\pi x^{\frac{1}{s}-\frac{1}{\tau}} |f(x)|\, x^{\frac{1}{s'}-\frac{1}{\tau'}} \left| \frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n} \right| dx \\ &\le \|f\|_{L^\tau_{w[s,\tau]}} \left(\int_0^\pi x^{\frac{\tau'}{s'}-1} \left| \frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n} \right|^{\tau'} dx \right)^{\frac{1}{\tau'}}. \end{aligned} \tag{2.15}$$

Now we estimate the integral on the right-hand side as follows

$$\begin{aligned} I &:= \int_0^\pi x^{\frac{\tau'}{s'}-1} \left| \frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n} \right|^{\tau'} dx \\ &\le 2^{\tau'-1} \int_0^\pi x^{\frac{\tau'}{s'}-1} \left(\left(\frac{1-\cos nx}{n\sin\frac{x}{2}} \right)^{\tau'} + \left(\frac{1}{n}\right)^{\tau'} \right) dx \\ &\le 2^{\tau'-1} \frac{s'}{\tau'} \pi^{\frac{\tau'}{s'}} \left(\frac{1}{n}\right)^{\tau'} + 2^{\tau'-1} \int_0^\pi x^{\frac{\tau'}{s'}-1} \left(\frac{1-\cos nx}{n\sin\frac{x}{2}} \right)^{\tau'} dx. \end{aligned}$$

Using the inequality $\sin\frac{x}{2} \ge \frac{x}{\pi}$, $x \in [0,\pi]$, and substituting nx on y in the last integral, we obtain

$$\begin{aligned} I &\lesssim \left(\frac{1}{n}\right)^{\tau'} + \int_0^\pi x^{\frac{\tau'}{s'}-1} \left(\frac{1-\cos nx}{n\sin\frac{x}{2}} \right)^{\tau'} dx \\ &\le \left(\frac{1}{n}\right)^{\tau'} + \pi^{\tau'} \int_0^\pi x^{\frac{\tau'}{s'}-1} \left(\frac{1-\cos nx}{nx} \right)^{\tau'} dx \\ &= \left(\frac{1}{n}\right)^{\tau'} + \pi^{\tau'} \int_0^{n\pi} \left(\frac{y}{n}\right)^{\frac{\tau'}{s'}-1} \left(\frac{1-\cos y}{y} \right)^{\tau'} \frac{dy}{n}. \end{aligned} \tag{2.16}$$

It follows from estimates (2.15) and (2.16) that

$$\begin{aligned} |\sigma_n(\mathbf{a})| &\lesssim \left(\left(\frac{1}{n}\right)^{\tau'} + \int_0^{n\pi} \left(\frac{y}{n}\right)^{\frac{\tau'}{s'}-1} \left(\frac{1-\cos y}{y} \right)^{\tau'} \frac{dy}{n} \right)^{\frac{1}{\tau'}} \|f\|_{L^\tau_{w[s,\tau]}} \\ &\le \left(\frac{1}{n} + \frac{1}{n^{\frac{1}{s'}}} \left(\int_0^\infty y^{\frac{\tau'}{s'}-1} \left(\frac{1-\cos y}{y} \right)^{\tau'} dy \right)^{\frac{1}{\tau'}} \right) \|f\|_{L^\tau_{w[s,\tau]}} \\ &\lesssim n^{-\frac{1}{s'}} \|f\|_{L^\tau_{w[s,\tau]}}. \end{aligned} \tag{2.17}$$

Let now $\tau = 1$. Taking into account (2.14), we have

$$\begin{aligned}|\sigma_n(\mathbf{a})| &\le \int_0^\pi |f(x)| \left|\frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n}\right| dx \\ &= \int_0^\pi x^{\frac{1}{s}-1}|f(x)|\, x^{\frac{1}{s'}} \left|\frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n}\right| dx \\ &\le \|f\|_{L^1_{w[s,1]}} \sup_{x\in[0,\pi]} x^{\frac{1}{s'}} \left|\frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}} + \frac{\sin nx}{n}\right|.\end{aligned}$$

Consider the function

$$H(x) := x^{\frac{1}{s'}} \frac{\cos\frac{x}{2}(1-\cos nx)}{n\sin\frac{x}{2}}.$$

If $0 < x \le \frac{1}{n}$, then

$$|H(x)| \le \frac{1}{n^{\frac{1}{s'}}}\left|\frac{2\sin^2\frac{nx}{2}}{n\sin\frac{x}{2}}\right| \le \frac{1}{n^{\frac{1}{s'}}}\frac{2\frac{nx}{2}}{n\frac{x}{\pi}} = \pi n^{-\frac{1}{s'}}.$$

If $\frac{1}{n} < x < \pi$, then

$$|H(x)| \le x^{\frac{1}{s'}}\frac{2}{n\sin\frac{x}{2}} \le x^{\frac{1}{s'}}\frac{1}{n\frac{x}{\pi}} = \frac{\pi}{n}\frac{1}{x^{1-\frac{1}{s'}}} < \frac{\pi}{n}n^{1-\frac{1}{s'}} = \pi n^{-\frac{1}{s'}}.$$

The last two estimates immediately imply that

$$\begin{aligned}|\sigma_n(\mathbf{a})| &\le \|f\|_{L^1_{w[s,1]}}\left(\sup_{x\in[0,\pi]}|H(x)| + \sup_{x\in[0,\pi]} x^{\frac{1}{s'}}\left|\frac{\sin nx}{n}\right|\right) \\ &\le \|f\|_{L^1_{w[s,1]}}\left(\pi n^{-\frac{1}{s'}} + \pi^{\frac{1}{s'}} n^{-1}\right) \le 2\pi n^{-\frac{1}{s'}}\|f\|_{L^1_{w[s,1]}}.\end{aligned} \tag{2.18}$$

Using (2.17) and (2.18) and the monotonicity of $\{m(\sigma_n(\mathbf{a}))\}_{n=1}^\infty$, we arrive at (2.13). In other words, the operator $Tf = m(\sigma_n)$ satisfies

$$T : L^\tau_{w[s,\tau]} \to l_{s',\infty}, \quad 1 < s < \infty,\ 1 \le \tau \le \infty.$$

Let now $1 < p < \infty$ and $1 \le q \le \infty$. Then there exist p_0, p_1 such that $p_0 < p < p_1$ and therefore, there exists $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Above arguments yield that

$$\begin{aligned}T &: L^q_{w[p_0,q]} \to l_{p_0',\infty}, \\ T &: L^q_{w[p_1,q]} \to l_{p_1',\infty}.\end{aligned}$$

Interpolating we obtain that

$$T : (L^q_{w[p_0,q]}, L^q_{w[p_1,q]})_{\theta,q} \to (l_{p_0',\infty}, l_{p_1',\infty})_{\theta,q}.$$

It remains to apply Stein-Weiss' and Marcinkiewicz' interpolation theorems(see [11, Ch. V]). □

2.4 Properties of general and weak monotone sequences

Let us start with the following definition of weak monotone sequences (see [61, 96]).

Definition 2.1. *A sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ is called weak monotone, written $\{a_n\}_{n=1}^{\infty} \in WM$, if there exist constants $C > 0$ and $\lambda > 1$ such that, for any $n \geq 1$,*

$$|a_n| \leq C \sum_{k=\frac{n}{\lambda}}^{\lambda n} \frac{|a_k|}{k}. \tag{2.19}$$

It is easy to see that the inequality (1.3) imply that $GM \subset WM$. We will use the following property of the weak monotone sequences.

Lemma 2.3 ([17, Theorem 3], [42, Theorem 3.11])**.** *Let $\{a_n\}_{n=1}^{\infty} \in WM$. Then, for all $1 < p < \infty$, $1 \leq q \leq \infty$ and for $p = q = \infty$, we have*

$$\|\mathbf{a}\|_{l_{p,q}} \asymp \|\mathbf{a}\|_{l^q_{w[p,q]}}. \tag{2.20}$$

Now we discuss several important notions related to GM sequences obtained in [35]. Without loss of generality, in the definition of general monotone sequences we may assume that $\gamma = 2^{\nu}$, where ν is a natural number. Let $\{a_n\}_{n=1}^{\infty} \in GM$. Denote for any $n > 2\nu$

$$A_n := \max_{2^n \leq k \leq 2^{n+1}} |a_k|$$

and

$$B_n := \max_{2^{n-2\nu} \leq k \leq 2^{n+2\nu}} |a_k|.$$

The following concept was introduced in [35].

Definition 2.2. *Let $\{a_n\}_{n=1}^{\infty} \in GM$. We say that a non-negative integer number n is* good, *if either $n \leq 2\nu$ or*

$$B_n \leq 2^{4\nu} A_n.$$

The rest of non-negative integer numbers we call bad.

Remark 2.4. *For given $C > 0$ and $\nu \in \mathbb{N}$, there exists sequence $\{a_n\}_{n=1}^{\infty} \in GM = GM(C, \nu)$ such that the set of good numbers of $\{a_n\}_{n=1}^{\infty}$ is finite. For example, for the sequence $\{2^{-n}\}_{n=1}^{\infty}$ the inequality $B_n \leq 2^{4\nu} A_n$ does not hold for any $n \geq 4\nu$.*

We set

$$M_n := \Big\{ k \in [2^{n-\nu}, 2^{n+\nu}] : |a_k| > \frac{A_n}{8C2^{2\nu}} \Big\},$$

$$M_n^+ := \{k \in M_n : a_k > 0\} \quad \text{and} \quad M_n^- := M_n \setminus M_n^+.$$

Lemma 2.4 ([35, Lemma 2.2]). *Let a vanishing sequence $\{a_n\}_{n=1}^{\infty} \in GM$. Denote $N_0 := [\log_2(C^3 2^{10\nu+8})] + 1$. Then, for any good n such that $n \geq N_0$, there exists an interval $[l_n, m_n] \subseteq [2^{n-\nu}, 2^{n+\nu}]$ such that at least one of the following condition holds:*
(i) *for any $k \in [l_n, m_n]$, we have $a_k \geq 0$ and*

$$|M_n^+ \cap [l_n, m_n]| \geq \frac{2^n}{C^3 2^{15\nu+8}};$$

(ii) *for any $k \in [l_n, m_n]$, we have $a_k \leq 0$ and*

$$|M_n^- \cap [l_n, m_n]| \geq \frac{2^n}{C^3 2^{15\nu+8}}.$$

Lemma 2.5. *Let a vanishing sequence $\{a_n\}_{n=1}^{\infty}$ be such that $\{a_n\}_{n=1}^{\infty} \in GM$. Then, for any bad number $r \in \mathbb{N}$, there exists either a set of integer numbers*

$$r = \xi_0 > \xi_1 > \xi_2 > \ldots > \xi_s =: \xi_{r,s} \tag{2.21}$$

or

$$r = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_s =: \xi_{r,s} \tag{2.22}$$

such that $\xi_1, \xi_2, \ldots, \xi_{s-1}$ are bad, $\xi_{r,s}$ is good and

$$A_r < 2^{-4\nu} A_{\xi_1} < 2^{-8\nu} A_{\xi_2} < \ldots < 2^{-4s\nu} A_{\xi_s},$$

$$|\xi_i - \xi_{i+1}| \leq 2\nu, \quad i = 0, \ldots, s-1.$$

The claim of Lemma 2.5 was proved in [35, Theorem 2.1]. For convenience, we sketch the proof.

Proof of Lemma 2.5. Let r be a bad number. Then $A_r < 2^{-4\nu} B_r$. Note that there exists an integer number ξ such that $B_r = A_\xi$ and $-2\nu \leq \xi - r \leq 2\nu - 1$. Set

$$\xi_1 := \min\{\xi : -2\nu \leq \xi - r \leq 2\nu - 1, B_r = A_\xi\}.$$

Assume first that $\xi_1 < r$. Then either ξ_1 is a good number or there exists an integer number ξ such that $-2\nu \leq \xi - \xi_1 < 2\nu - 1$ and

$$A_{\xi_1} < 2^{-4\nu} B_{\xi_1} = 2^{-4\nu} A_\xi.$$

Set

$$\xi_2 := \min\{\xi : -2\nu \leq \xi - \xi_1 < 2\nu - 1, B_{\xi_1} = A_\xi\}.$$

Since $\xi_1 < r$, it follows that

$$[2^{\xi_1}, 2^{\xi_1 + 2\nu}] \subset [2^{r-2\nu}, 2^{r+2\nu}].$$

Therefore, for any $k \in [2^{\xi_1}, 2^{\xi_1+2\nu}] \cap \mathbb{Z}$, we have $|a_k| \leq A_{\xi_1} = B_r$. If $\xi_2 > \xi_1$, then $[2^{\xi_2}, 2^{\xi_2+1}] \cap \mathbb{Z} \subset [2^{\xi_1}, 2^{\xi_1+2\nu}] \cap \mathbb{Z}$. On the other hand, we have $A_{\xi_2} > A_{\xi_1} = B_r$, arriving at contradiction. Hence, ξ_2 can not be greater than ξ_1, i.e., $\xi_2 < \xi_1$.

Continuing this procedure, we arrive at a finite sequence (since $\{\xi_j\}$ is the decreasing sequence)

$$r = \xi_0 > \xi_1 > \ldots > \xi_{s-1} > \xi_s,$$

where the numbers $\xi_0, \xi_1, \ldots, \xi_{s-1}$ are bad, and the number ξ_s is good. Moreover, $\xi_j - \xi_{j+1} \leq 2\nu$ and $A_{\xi_j} < 2^{-4\nu} A_{\xi_{j+1}}$ for any $0 \leq j \leq s-1$.

Let now $\xi_1 > r$. Then either ξ_1 is a good number, or there exists a number $\xi_2 > \xi_1$ such that $\xi_2 - \xi_1 \leq 2\nu - 1$ and $A_{\xi_1} < 2^{-4\nu} A_{\xi_2}$. Continuing this procedure and taking into account the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0, we obtain a finite sequence of numbers

$$r = \xi_0 < \xi_1 < \ldots < \xi_{s-1} < \xi_s,$$

where the numbers $\xi_0, \xi_1, \ldots, \xi_{s-1}$ are bad, and the number ξ_s is good. Moreover, for any $0 \leq j \leq s-1$ the inequalities $\xi_{j+1} - \xi_j \leq 2\nu$ hold and $A_{\xi_j} < 2^{-4\nu} A_{\xi_{j+1}}$. □

Remark 2.5. *From the proof of Lemma 2.5 it follows that, for any bad number r, we can construct a sequence of numbers like* (2.21) *or* (2.22) *uniquely. In particular, for any bad number r, the good number $\xi_{r,s}$ from* (2.21) *or* (2.22) *may be chosen uniquely. Therefore, for any good number n we can construct two sets of bad numbers Q_n^1 and Q_n^2 as follows.*

- *Q_n^1 is a set of bad numbers r such that* (2.21) *holds with $\xi_{r,s} = n$;*
- *Q_n^2 is a set of bad numbers r such that* (2.22) *holds with $\xi_{r,s} = n$.*

The number s will be called the length of the bad number r. Note that $Q_n^i \cap Q_m^j = \emptyset$ for any good numbers $n \neq m$ and for any $i, j = 1, 2$. Denote by G the set of good numbers of sequence $\{a_n\}_{n=1}^{\infty}$. Then we have

$$\mathbb{N}_0 = G \bigsqcup \left(\bigsqcup_{n \in G} Q_n^1 \right) \bigsqcup \left(\bigsqcup_{n \in G} Q_n^2 \right). \tag{2.23}$$

Remark 2.6. *Let us discuss the case when a sequence $\{a_n\}_{n=1}^{\infty}$ is such that the set of good numbers is finite. In this case, for any bad number r greater than the last good number, only the case* (2.21) *is possible. Moreover, the set Q_n^2 is finite for any good number n (in particular, Q_n^2 might be empty).*

Remark 2.7. *We note that*

1. *for any number $n \in G$, each of the sets Q_n^1 and Q_n^2 contain not more than 2ν bad numbers of length 1, not more than $(2\nu)^2$ bad numbers of length 2, etc;*

2. for any $r \in Q_n^1$ *of length* s*, the inequalities*

$$n < r \le n + 2s\nu \quad \text{and} \quad A_r < 2^{-4s\nu} A_n$$

hold;

3. for any $r \in Q_n^2$ *of length* s*, the inequalities*

$$n - 2s\nu \le r < n \quad \text{and} \quad A_r < 2^{-4s\nu} A_n$$

hold.

2.5 Proofs of main results

Proof of Theorem 2.3. We give the proof for functions

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

The general case follows from the boundedness of the Hilbert operator in the Lorentz spaces.

Part " $\gtrsim$ ". First, we consider the case of the sequence $\{a_n\}_{n=1}^{\infty}$ such that there exist good numbers $n \ge N_0$, where N_0 is given by Lemma 2.4. We divide the proof into two cases: $q < \infty$ and $q = \infty$.

A. Let $q < \infty$. By Lemma 2.3, it is sufficient to prove the inequality

$$\|f\|_{L_{p,q}} \gtrsim \|\mathbf{a}\|_{l^q_{w[p',q]}}.$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
\|f\|_{L_{p,q}}^q &\gtrsim \|m(\sigma_n(\mathbf{a}))\|_{l_{p',q}}^q = \sum_{n=1}^{\infty} n^{\frac{q}{p'}-1} \left(\sup_{k \ge n} \frac{1}{k} \left| \sum_{j=1}^{k} a_j \right| \right)^q \\
&\asymp \sum_{n=0}^{\infty} 2^{n\frac{q}{p'}} \left(\sup_{k \ge 2^n} \frac{1}{k} \left| \sum_{j=1}^{k} a_j \right| \right)^q = \sum_{n=N_0}^{\infty} 2^{\frac{(n-N_0)q}{p'}} \left(\sup_{k \ge 2^{n-N_0}} \frac{1}{k} \left| \sum_{j=1}^{k} a_j \right| \right)^q \\
&\asymp \sum_{n=N_0}^{\infty} 2^{\frac{nq}{p'}} \left(\sup_{k \ge 2^{n-N_0}} \frac{1}{k} \left| \sum_{j=1}^{k} a_j \right| \right)^q =: \sum_{n=N_0}^{\infty} P_n.
\end{aligned} \tag{2.24}$$

Let us denote

$$W_n := \sum_{k=2^n}^{2^{n+1}-1} k^{\frac{q}{p'}-1} |a_k|^q.$$

Taking into account (2.23), we have

$$\begin{aligned}\|\mathbf{a}\|^q_{l^q_{w[p',q]}} &= \sum_{n=1}^{\infty} n^{\frac{q}{p'}-1}|a_n|^q = \sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1} k^{\frac{q}{p'}-1}|a_k|^q = \sum_{n=0}^{\infty} W_n \\ &\le \sum_{n<N_0} W_n + \sum_{\substack{n\in G\\ n\ge N_0}} W_n + \sum_{n\in G}\sum_{v\in Q_n^1} W_v + \sum_{n\in G}\sum_{v\in Q_n^2} W_v \\ &=: J_1+J_2+J_3+J_4.\end{aligned} \tag{2.25}$$

Now we show the estimates $J_i \lesssim \|f\|^q_{L_{p,q}}$, $i=1,2,3,4$.

Step 1$_\mathbf{A}$. The estimate of J_2. Let n be a good number such that $n \ge N_0$. Without loss of generality, we can assume that condition (i) of Lemma 2.4 is valid. Since for integers l_n, m_n from Lemma 2.4 the inequalities $2^{n+\nu} \ge m_n > l_n - 1 \ge 2^{n-\nu} - 1 > 2^{n-N_0}$ hold, we derive

$$\begin{aligned}P_n &= 2^{n\frac{q}{p'}}\left(\sup_{k\ge 2^{n-N_0}} \frac{1}{k}\left|\sum_{j=1}^{k} a_j\right|\right)^q \\ &\ge 2^{n\frac{q}{p'}-1}\left(\frac{1}{l_n-1}\left|\sum_{j=1}^{l_n-1} a_j\right|\right)^q + 2^{n\frac{q}{p'}-1}\left(\frac{1}{m_n}\left|\sum_{j=1}^{m_n} a_j\right|\right)^q \\ &\ge 2^{n\frac{q}{p'}-1}\left[\left(\frac{1}{2^{n+\nu}}\left|\sum_{j=1}^{l_n-1} a_j\right|\right)^q + \left(\frac{1}{2^{n+\nu}}\left|\sum_{j=1}^{m_n} a_j\right|\right)^q\right] \\ &\ge 2^{-(\nu+1)q+n\frac{q}{p'}}\left(\frac{1}{2^n}\left|\sum_{j=l_n}^{m_n} a_j\right|\right)^q.\end{aligned}$$

Lemma 2.4 implies

$$\begin{aligned}P_n &\gtrsim 2^{n\frac{q}{p'}}\left(\frac{1}{2^n}\sum_{j=l_n}^{m_n} a_j\right)^q \ge 2^{n\frac{q}{p'}}\left(\frac{1}{2^n}\sum_{j\in[l_n,m_n]\cap M_n^+} a_j\right)^q \\ &\ge 2^{n\frac{q}{p'}}\left(\frac{1}{2^n}\frac{A_n}{8C2^{2\nu}}\frac{2^n}{C^3 2^{15\nu+8}}\right)^q \\ &\ge \frac{1}{C^{4q}2^{(17\nu+11)q}} 2^{n\frac{q}{p'}}\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}|a_k|^q \gtrsim \sum_{k=2^n}^{2^{n+1}-1} k^{\frac{q}{p'}-1}|a_k|^q = W_n.\end{aligned}$$

Hence, using (2.24), we obtain

$$J_2 = \sum_{\substack{n\in G\\ n\ge N_0}} W_n \lesssim \sum_{\substack{n\in G\\ n\ge N_0}} P_n \le \|f\|^q_{L_{p,q}}. \tag{2.26}$$

Step $2_{\mathbf{A}}$. The estimate of J_1. Using Hölder's inequality, we estimate

$$\begin{aligned} J_1 &= \sum_{n<N_0} \sum_{k=2^n}^{2^{n+1}-1} k^{\frac{q}{p'}-1}|a_k|^q = \sum_{n=1}^{2^{N_0}-1} n^{\frac{q}{p'}-1}|a_n|^q \\ &\lesssim \sum_{n=1}^{2^{N_0}-1} n^{\frac{q}{p'}-1}\left(\int_0^{2\pi}|f(x)|dx\right)^q \\ &\lesssim \sum_{n=1}^{2^{N_0}-1} n^{\frac{q}{p'}-1}\|f\|_{L_{p,q}}^q \lesssim \|f\|_{L_{p,q}}^q. \end{aligned} \tag{2.27}$$

Note that in the last inequality inequality depend only on p, q, C, and ν.

Step $3_{\mathbf{A}}$. The estimate of J_3. Let $n \in G$ and let $r \in Q_n^1$ be a bad number of length s. Then, by the definition of the set Q_n^1 (see Remark 2.7) we have $r \le n + 2s\nu$ and $A_r \le 2^{-4s\nu}A_n$. Therefore,

$$W_r = \sum_{k=2^r}^{2^{r+1}-1} k^{\frac{q}{p'}-1}|a_k|^q \lesssim A_r^q 2^{r\frac{q}{p'}} \le 2^{-4s\nu q}A_n^q 2^{(n+2s\nu)\frac{q}{p'}} \lesssim 2^{-2s\nu q}A_n^q 2^{n\frac{q}{p'}}. \tag{2.28}$$

Assume that $n \ge N_0$ and condition (i) of Lemma 2.4 is valid. Then Lemma 2.4 yields

$$\begin{aligned} W_r &\lesssim 2^{-2s\nu q}A_n^q 2^{n\frac{q}{p'}} \le 2^{-2s\nu q}2^{n\frac{q}{p'}}\left(\frac{8C2^{2\nu}}{|[l_n,m_n]\cap M_n^+|}\sum_{j\in[l_n,m_n]\cap M_n^+} a_j\right)^q \\ &\le 2^{-2s\nu q}2^{n\frac{q}{p'}}\left(\frac{C^4 2^{17\nu+11}}{2^n}\sum_{j=l_n}^{m_n} a_j\right)^q \\ &\lesssim 2^{-2s\nu q}2^{n\frac{q}{p'}}\left(\frac{1}{2^n}\left|\sum_{j=1}^{m_n} a_j\right| + \frac{1}{2^n}\left|\sum_{j=1}^{l_n-1} a_j\right|\right)^q. \end{aligned}$$

Taking into account that $2^{n+\nu} \ge m_n > l_n - 1 \ge 2^{n-\nu} - 1 > 2^{n-N_0}$, we derive

$$\begin{aligned} W_r &\lesssim 2^{-2s\nu q}2^{n\frac{q}{p'}}\left(\frac{1}{2^n}\left|\sum_{j=1}^{m_n} a_j\right| + \frac{1}{2^n}\left|\sum_{j=1}^{l_n-1} a_j\right|\right)^q \\ &\le 2^{(\nu+1)q-1}2^{-2s\nu q}2^{n\frac{q}{p'}}\left(\left(\frac{1}{m_n}\left|\sum_{j=1}^{m_n} a_j\right|\right)^q + \left(\frac{1}{l_n-1}\left|\sum_{j=1}^{l_n-1} a_j\right|\right)^q\right) \\ &\le 2^{(\nu+1)q}2^{-2s\nu q}2^{n\frac{q}{p'}}\left(\sup_{k\ge 2^{n-N_0}}\frac{1}{k}\left|\sum_{j=1}^{k} a_j\right|\right)^q = 2^{(\nu+1)q}2^{-2s\nu q}P_n. \end{aligned} \tag{2.29}$$

Let now $n < N_0$. Then from (2.28) by using Hölder's inequality, we establish

$$W_r \lesssim 2^{-2s\nu q}A_n^q 2^{n\frac{q}{p'}} \lesssim 2^{-2s\nu q}2^{n\frac{q}{p'}}\|f\|_{L_{p,q}}^q. \tag{2.30}$$

Recall that for any good number n the set Q_n^1 contains not more than 2ν bad numbers of length 1, not more than $(2\nu)^2$ bad numbers of length 2, etc. It follows that (2.29) and (2.30) imply

$$\begin{aligned} J_3 &= \sum_{n\in G}\sum_{r\in Q_n^1} W_r = \sum_{\substack{n\in G\\ n<N_0}}\sum_{r\in Q_n^1} W_r + \sum_{\substack{n\in G\\ n\geq N_0}}\sum_{r\in Q_n^1} W_r \\ &\lesssim \|f\|_{L_{p,q}}^q \sum_{\substack{n\in G\\ n<N_0}} 2^{n\frac{q}{p'}}\left(\sum_{r\in Q_n^1} 2^{-2s\nu q}\right) + \sum_{\substack{n\in G\\ n\geq N_0}} P_n\left(\sum_{r\in Q_n^1} 2^{-2s\nu q}\right) \\ &\leq \|f\|_{L_{p,q}}^q \sum_{\substack{n\in G\\ n<N_0}} 2^{n\frac{q}{p'}}\left(\sum_{s=1}^{\infty}(2\nu)^s 2^{-2s\nu q}\right) + \sum_{\substack{n\in G\\ n\geq N_0}} P_n\left(\sum_{s=1}^{\infty}(2\nu)^s 2^{-4s\nu q}\right). \end{aligned}$$

Hence, using (2.24), we obtain

$$J_3 \lesssim \|f\|_{L_{p,q}}^q \sum_{\substack{n\in G\\ n<N_0}} 2^{n\frac{q}{p'}} + \sum_{\substack{n\in G\\ n\geq N_0}} P_n \lesssim \left(2^{N_0\frac{q}{p'}}+1\right)\|f\|_{L_{p,q}}^q \lesssim \|f\|_{L_{p,q}}^q. \tag{2.31}$$

Step $4_{\mathbf{A}}$. The estimate of J_4. Using the same arguments as in the previous step, we estimate

$$J_4 \lesssim \|f\|_{L_{p,q}}^q. \tag{2.32}$$

Combining estimates (2.26), (2.27), (2.31) and (2.32), we finally obtain

$$\|\mathbf{a}\|_{l^q_{w[p',q]}}^q \lesssim J_1 + J_2 + J_3 + J_4 \lesssim \|f\|_{L_{p,q}}^q.$$

B. Let $q=\infty$. Recall that N_0 is given by Lemma 2.4. Lemma 2.1 implies

$$\begin{aligned} \|f\|_{L_{p,\infty}} &\gtrsim \sup_{n\geq 1} n^{\frac{1}{p'}} \sup_{k\geq n}\frac{1}{k}\left|\sum_{j=1}^k a_j\right| = \sup_{n\geq 0}\sup_{2^n\leq k\leq 2^{n+1}-1} k^{\frac{1}{p'}}\sup_{l\geq k}\frac{1}{l}\left|\sum_{j=1}^l a_j\right| \\ &\asymp \sup_{n\geq 0} 2^{n\frac{1}{p'}}\sup_{k\geq 2^n}\frac{1}{k}\left|\sum_{j=1}^k a_j\right| = \sup_{n\geq N_0} 2^{(n-N_0)\frac{1}{p'}}\sup_{k\geq 2^{n-N_0}}\frac{1}{k}\left|\sum_{j=1}^k a_j\right| \\ &\asymp \sup_{n\geq N_0} 2^{n\frac{1}{p'}}\sup_{k\geq 2^{n-N_0}}\frac{1}{k}\left|\sum_{j=1}^k a_j\right| =: \sup_{n\geq N_0}\widetilde{P}_n. \end{aligned}$$

Setting

$$\widetilde{W}_n := \max_{2^n\leq k\leq 2^{n+1}-1} k^{\frac{1}{p'}}|a_k|,$$

we obtain

$$\begin{aligned}\|\mathbf{a}\|_{l^{\infty}_{w(p',\infty)}} &= \sup_{n\geq 1} n^{\frac{1}{p'}}|a_n| = \sup_{n\geq 0}\max_{2^n\leq k\leq 2^{n+1}-1} k^{\frac{1}{p'}}|a_k| = \sup_{n\geq 0}\widetilde{W}_n \\ &= \sup\left\{\max_{n<N_0}\widetilde{W}_n,\ \sup_{\substack{n\in G\\ n\geq N_0}}\widetilde{W}_n, \sup_{n\in G}\sup_{r\in Q_n^1}\widetilde{W}_r, \sup_{n\in G}\sup_{r\in Q_n^2}\widetilde{W}_r\right\}.\end{aligned}$$

As in case **A** we divide the proof into four steps.

Step $1_{\mathbf{B}}$. The estimate of $\sup_{\substack{n\in G\\ n\geq N_0}}\widetilde{W}_n$. Let $n\geq N_0$ be a good number. Without loss of generality, we can assume that condition (i) of Lemma 2.4 is valid. Since $2^{n+\nu}\geq m_n > l_n - 1 \geq 2^{n-\nu}-1 > 2^{n-N_0}$, we have

$$\begin{aligned}\widetilde{P}_n &= 2^{n\frac{1}{p'}}\sup_{k\geq 2^{n-N_0}}\frac{1}{k}\left|\sum_{j=1}^{k} a_j\right| \\ &\geq 2^{n\frac{1}{p'}-1}\frac{1}{l_n-1}\left|\sum_{j=1}^{l_n-1} a_j\right| + 2^{n\frac{1}{p'}-1}\frac{1}{m_n}\left|\sum_{j=1}^{m_n} a_j\right| \\ &\geq 2^{n\frac{1}{p'}-1}\frac{1}{2^{n+\nu}}\left|\sum_{j=1}^{l_n-1} a_j\right| + 2^{n\frac{1}{p'}-1}\frac{1}{2^{n+\nu}}\left|\sum_{j=1}^{m_n} a_j\right| \gtrsim 2^{n\frac{1}{p'}}\frac{1}{2^n}\left|\sum_{j=l_n}^{m_n} a_j\right|.\end{aligned}$$

Using Lemma 2.4, we derive

$$\begin{aligned}\widetilde{P}_n &\gtrsim 2^{n\frac{1}{p'}}\frac{1}{2^n}\sum_{j=l_n}^{m_n} a_j \geq 2^{n\frac{1}{p'}}\frac{1}{2^n}\sum_{j\in[l_n,m_n]\cap M_n^+} a_j \geq 2^{n\frac{1}{p'}}\frac{1}{2^n}\frac{A_n}{8C2^{2\nu}}\frac{2^n}{C^3 2^{15\nu+8}} \\ &\gtrsim 2^{n\frac{1}{p'}}A_n \gtrsim \widetilde{W}_n.\end{aligned}$$

Therefore,

$$\sup_{\substack{n\in G\\ n\geq N_0}}\widetilde{W}_n \lesssim \sup_{\substack{n\in G\\ n\geq N_0}}\widetilde{P}_n \leq \sup_{n\in G}\widetilde{P}_n \lesssim \|f\|_{L_{p',\infty}}. \tag{2.33}$$

Step $2_{\mathbf{B}}$. The estimate of $\max_{n<N_0}\widetilde{W}_n$. It is easy to see that

$$\begin{aligned}\max_{n<N_0}\widetilde{W}_n &= \max_{n<N_0}\max_{2^n\leq k\leq 2^{n+1}-1} k^{\frac{1}{p'}}|a_k| \leq 2^{(N_0+1)\frac{1}{p'}}\int_0^{2\pi}|f(x)|dx \\ &= 2^{(N_0+1)\frac{1}{p'}}\int_0^{2\pi} x^{-\frac{1}{p'}}x^{\frac{1}{p'}}|f(x)|dx \\ &\leq 2^{(N_0+1)\frac{1}{p'}}\|f\|_{L_{p',\infty}}\int_0^{2\pi} x^{-\frac{1}{p'}}dx \lesssim \|f\|_{L_{p',\infty}}.\end{aligned} \tag{2.34}$$

Step $3_{\mathbf{B}}$. The estimate of $\sup\limits_{n\in G}\sup\limits_{r\in Q_n^1}\widetilde{W}_r$**.** Let $n\in G$ and let $r\in Q_n^1$ be a bad number of length s. By Remark 2.7, we have $r\le n+2s\nu$ and $A_r\le 2^{-4s\nu}A_n$, and therefore

$$\widetilde{W}_r=\max_{2^r\le k\le 2^{r+1}-1}k^{\frac{1}{p'}}|a_k|\le A_r 2^{(r+1)\frac{1}{p'}}\le 2^{-4s\nu}A_n 2^{(n+2\nu+1)\frac{1}{p'}}. \tag{2.35}$$

Suppose that $n\ge N_0$. The last inequality and Lemma 2.4 give

$$\begin{aligned}\widetilde{W}_r&\lesssim 2^{-4s\nu}A_n 2^{n\frac{1}{p'}}\lesssim 2^{-4s\nu}2^{n\frac{1}{p'}}\left(\frac{1}{|[l_n,m_n]|}\left|\sum_{j=l_n}^{m_n}a_j\right|\right)\\&\lesssim 2^{-4s\nu}2^{n\frac{1}{p'}}\left(\sup_{k\ge 2^{n-N_0}}\frac{1}{k}\left|\sum_{j=1}^{k}a_j\right|\right)=2^{-4s\nu}\widetilde{P}_n\le\|f\|_{L_{p,\infty}}.\end{aligned} \tag{2.36}$$

If $n<N_0$, then (2.35) yields

$$\widetilde{W}_r\lesssim 2^{-4s\nu}A_n 2^{n\frac{1}{p'}}\lesssim 2^{-4s\nu}2^{n\frac{1}{p'}}\|f\|_{L_{p,\infty}}\lesssim\|f\|_{L_{p,\infty}}. \tag{2.37}$$

Inequalities (2.36) and (2.37) imply

$$\sup_{n\in G}\sup_{r\in Q_n^1}\widetilde{W}_r\lesssim\|f\|_{L_{p,\infty}}. \tag{2.38}$$

Step $4_{\mathbf{B}}$. The estimate of $\sup\limits_{n\in G}\sup\limits_{r\in Q_n^2}\widetilde{W}_r$**.** Similarly to the argument in step $\mathbf{3_B}$ we estimate

$$\sup_{n\in G}\sup_{r\in Q_n^2}\widetilde{W}_r\lesssim\|f\|_{L_{p,\infty}}. \tag{2.39}$$

Combining inequalities (2.33), (2.34), (2.38), and (2.39), we derive that

$$\|\mathbf{a}\|_{l^\infty_{w(p',\infty)}}\lesssim\sup\left\{\max_{n<N_0}\widetilde{W}_n,\ \sup_{\substack{n\in G\\ n\ge N_0}}\widetilde{W}_n,\sup_{n\in G}\sup_{r\in Q_n^1}\widetilde{W}_r,\sup_{n\in G}\sup_{r\in Q_n^2}\widetilde{W}_r\right\}\lesssim\|f\|_{L_{p,\infty}}.$$

Thus, in both case $q<\infty$ and $q=\infty$ the inequality "$\gtrsim$" has been proved when the sequence $\{a_n\}_{n=1}^\infty$ contains good numbers $n\ge N_0$. In the case when all good numbers of $\{a_n\}_{n=1}^\infty$ are less than N_0 we can repeat the proof skipping Steps $\mathbf{1_A}$ and $\mathbf{1_B}$.

Part "$\lesssim$". It is known [37, Theorem 2.1] that for any $\{a_n\}_{n=1}^\infty\in GM$ there exists $B>0$ such that the sequences $\left\{b_n=\frac{B}{n}\sum\limits_{k=\frac{n}{\gamma}}^{\gamma n}|a_k|\right\}$ and $\{c_n=Bb_n-a_n\}$ are non-negative and belong to GM.

Let $\{a_n\}_{n=1}^{\infty} \in l_{p',q}$. Then by Hardy's inequality and by [42, Theorem 8.1] we obtain

$$\|\mathbf{a}\|_{l_{p',q}} \gtrsim \|\mathbf{a}\|_{l^q_{w[p',q]}} \gtrsim \|\mathbf{b}\|_{l^q_{w[p',q]}} + \|\mathbf{c}\|_{l^q_{w[p',q]}}$$
$$\gtrsim \|\mathbf{b}\|_{l_{p',q}} + \|\mathbf{c}\|_{l_{p',q}} \gtrsim \|g\|_{L_{p,q}} + \|h\|_{L_{p,q}} \gtrsim \|f\|_{L_{p,q}},$$

where $g(x) = \sum\limits_{n=1}^{\infty} b_n \cos nx$ and $h(x) = \sum\limits_{n=1}^{\infty} c_n \cos nx$. □

Proof of Theorem 2.4. The proof of the estimate " $\gtrsim$ " is similar to the proof of part "$\gtrsim$" of Theorem 2.3 using Lemma 2.2 in place of Lemma 2.1.

Regarding the estimate "$\lesssim$", if $q < \infty$, this part follows from [34]. Here we remark that in spite of the fact that Theorems 4.2 and 4.3 in [34] were proved for non-negative Fourier coefficients the proof of this part is also valid for general sequences. If $q = \infty$, the estimate "$\lesssim$" was proved in [35, Theorem 5.1].

□

Remark 2.8. *Under conditions of Theorem 2.3 we have*

$$\|f\|_{L_{p,q}([0,2\pi])} \asymp \|f\|_{L^q_{w[p,q]}([0,2\pi])} \asymp \left(\sum_{n=0}^{\infty} 2^{n\frac{q}{p'}} A_n^q \right)^{\frac{1}{q}}.$$

Chapter 3

Smoothness of functions and the Fourier coefficients

3.1 Behaviour of the Fourier coefficients of functions from L_p. The general case

Let f be an integrable 2π-periodic function with the Fourier series

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{3.1}$$

Recall that (see, e.g., [25])

$$\omega_l(f,\delta)_p := \sup_{|h|\le\delta}\left\|\Delta_h^l f(\cdot)\right\|_p$$

is the modulus of smoothness of the function $f \in L_p$ of order $l \ge 1$, where

$$\Delta_h^l f(x) := \Delta_h(\Delta_h^{l-1} f(x)), \quad \Delta_h f(x) := f(x+h) - f(x).$$

First we write simple estimates for the modulus of smoothness of the function $f \in L_p$, $1 \le p \le \infty$ in terms of its Fourier coefficients:

$$|a_n| + |b_n| \lesssim \omega_l\Big(f, \frac{1}{n}\Big)_p \lesssim \frac{1}{n^l}\sum_{k=1}^{n} k^l(|a_k| + |b_k|)$$
$$+ \sum_{k=n+1}^{\infty}(|a_k| + |b_k|).$$

The left-hand side inequality is the well-known Lebesgue estimate for $p = 1$ (see [5]), the right-hand side inequality follows from the Fourier representation of $\Delta_h^l f$ in the case of $p = \infty$ (see, for example, [90]).

By Lip (α, p) we denote the Lipschitz class:

$$\text{Lip}\,(\alpha, p) := \{f \in L_p([0, 2\pi]) : \omega(f, \delta)_p = O(\delta^\alpha)\},$$

where

$$\omega(f, \delta)_p := \omega_1(f, \delta)_p = \sup_{|h| \le \delta} \|\Delta_h f(\cdot)\|_p$$

is the L_p-modulus of continuity of f. In the middle of the last century, there were found necessary and sufficient conditions for belonging of a function f to the Lipschitz class. In particular, Lorentz [62] showed that for $2 \le p \le \infty$ and $0 < \alpha < 1$ the condition

$$\sum_{k=n}^{\infty} \left(|a_k|^{p'} + |b_k|^{p'}\right) = O\left(\frac{1}{n^{\alpha p'}}\right), \tag{3.2}$$

implies $f \in$ Lip (α, p). Note that condition (3.2) for any positive α is equivalent to the condition

$$\sum_{k=n}^{2n} \left(|a_k|^{p'} + |b_k|^{p'}\right) = O\left(\frac{1}{n^{\alpha p'}}\right).$$

For $1 < p \le 2$, condition (3.2) is necessary for $f \in$ Lip (α, p), see [97].

Gorbachev and Tikhonov [41, Theorem 2.1] obtained a more detailed relationship between the modulus of smoothness of a function $f \in L_p$ and its Fourier coefficients.

Theorem 3.1. *Let* (3.1) *be the Fourier series of a function* $f \in L_p([0, 2\pi])$*.*

(A) *Let* $1 < p \le 2$*. Then, for* $p \le q \le p'$*, we have*

$$\begin{aligned} &\frac{1}{n^l}\left(\sum_{k=1}^{n} k^{\left(l+1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_k|^q + |b_k|^q)\right)^{\frac{1}{q}} \\ &+ \left(\sum_{k=n+1}^{\infty} k^{\left(1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_k|^q + |b_k|^q)\right)^{\frac{1}{q}} \lesssim \omega_l\left(f, \frac{1}{n}\right)_p. \end{aligned} \tag{3.3}$$

(B) *Let* $2 \le p < \infty$ *and*

$$\left(\sum_{n=1}^{\infty} n^{\left(1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_n|^q + |b_n|^q)\right)^{\frac{1}{q}} < \infty,$$

where $p' \le q \le p$*. Then*

$$\begin{aligned} &\frac{1}{n^l}\left(\sum_{k=1}^{n} k^{\left(l+1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_k|^q + |b_k|^q)\right)^{\frac{1}{q}} \\ &+ \left(\sum_{k=n+1}^{\infty} k^{\left(1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_k|^q + |b_k|^q)\right)^{\frac{1}{q}} \gtrsim \omega_l\left(f, \frac{1}{n}\right)_p. \end{aligned} \tag{3.4}$$

The part (A) of this theorem recently was generalized in [24]. Note that Theorem 3.1 is sharp with respect to conditions on p and q. Moreover, for $0 < \alpha < l$, Theorem 3.1 implies the following fact. The condition $\omega_l(f,\delta)_p = O(\delta^\alpha)$ is sufficient (in case of $p \le 2$) and necessary (in case of $p \ge 2$), for condition (3.2). In other words, in this case the choice of the parameter $q = p'$ is the best possible. However, in some cases this is not true anymore. For instance, if

$$|a_n| = |b_n| \asymp n^{-l-1/p'} \ln^{-1/p} n,$$

then, for $q = p$, Theorem 3.1 gives

$$\omega_l\left(f, \frac{1}{n}\right)_p \lesssim \frac{(\ln\ln n)^{1/p}}{n^l}$$

in case of $p \ge 2$ and the inverse inequality in case of $p \le 2$. For other values of q the esimates are weaker.

3.2 Behaviour of the Fourier coefficients of functions from L_p with additional conditions

Assuming some additional conditions (monotonicity, general monotonicity) on the coefficients of series (3.1) it is possible to fully obtain interrelation between smoothness of a function and behaviour of its Fourier coefficients. In particular, Konyushkov [48], showed that for functions with monotone Fourier coefficients condition (3.2) is equivalent to the condition $f \in \mathrm{Lip}\,(\alpha, p)$.

Theorem 3.2. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let also (3.1) be the Fourier series of function $f \in L_p([0,2\pi])$, and $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ be non-increasing sequences. Then the following conditions are equivalent:*

(i) $f \in \mathrm{Lip}\,(\alpha, p)$;

(ii) $|a_n|, |b_n| = O\left(n^{\frac{1}{p}-\alpha-1}\right)$;

(iii) $\sum\limits_{k=n}^{\infty} (|a_k|^{p'} + |b_k|^{p'}) = O\left(n^{-\alpha p'}\right)$.

Later on it was shown that it is possible to characterize the behaviour of the modulus of smothness in terms of the Fourier coefficients of the function.

Theorem 3.3 ([1, 72])**.** *Under conditions of Theorem 3.2 the following equivalence*

$$\omega_l\left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^l}\left(\sum_{k=1}^{n} k^{(l+1)p-2}(a_k^p + b_k^p)\right)^{\frac{1}{p}} + \left(\sum_{k=n+1}^{\infty} k^{p-2}(a_k^p + b_k^p)\right)^{\frac{1}{p}}$$

holds.

Similar problems were considered in the papers [2, 3]. Such results are very important to characterize some smooth function spaces, see, e.g., [41, 73, 80, 86, 87, 91, 92, 93]. In particular, Askey [3] proved the following result.

Theorem 3.4. *Let $0 < \alpha < 2$, $1 < p < \infty$, $1 \le q \le \infty$. Let also $\sum_{n=1}^{\infty} a_n \cos nx$ be the Fourier series of a function $f \in L_p([0, 2\pi])$, and $\{a_n\}_{n=1}^{\infty}$ be a non-increasing sequence. Then*

$$\left(\sum_{n=1}^{\infty} a_n^q n^{q\left(\alpha+1-\frac{1}{p}\right)-1}\right)^{\frac{1}{q}} < \infty$$

if and only if

$$\left(\int_0^{\pi} \left[\int_0^{\pi} \left|\frac{f(x+t)-2f(x)+f(x-t)}{t^{\alpha}}\right|^p dx\right]^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} < \infty. \tag{3.5}$$

Note that the condition (3.5) is equivalent to the condition $f \in B^{\alpha}_{p,q}$, where the Besov space $B^{\alpha}_{p,q}$ is defined as follows.

Definition 3.1. *Let $1 \le p \le \infty$ and $\tau, r > 0$. The Besov space $B^r_{p,\tau}([0, 2\pi])$ is a set of functions $f \in L_p([0, 2\pi])$ such that*

$$\|f\|_{B^r_{p,\tau}} := \|f\|_{L_p} + \left(\int_0^1 \left(\frac{\omega_l(f,t)_p}{t^r}\right)^{\tau} \frac{dt}{t}\right)^{\frac{1}{\tau}} < \infty,$$

where $l > r$.

Note that Theorems 3.3 and 3.4 were generalized in various papers (see [39, 41, 51, 55, 70, 72, 85, 91, 102]), where the authors weaken the monotonicity condition on Fourier coefficients.

3.3 Main results

In this chapter we consider the class of general monotone sequences $GM(\boldsymbol{\beta})$ with

$$\beta_n = \sum_{k=\frac{n}{\gamma}}^{\gamma n} \frac{|a_k|}{k}, \quad \gamma > 1.$$

For convenience, throughout this chapter, we denote this class by GM. The main result of this chapter is the following theorem.

Theorem 3.5. *Let $f(x) \in L_p([0, 2\pi])$, $1 < p < \infty$,*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. Then, for any $l \in \mathbb{N}$,

$$\omega_l \left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^l} \left(\sum_{k=1}^{n} k^{lp+p-2} \left(|a_k|^p + |b_k|^p\right) \right)^{\frac{1}{p}} + \left(\sum_{k=n}^{\infty} k^{p-2} \left(|a_k|^p + |b_k|^p\right) \right)^{\frac{1}{p}}. \tag{3.6}$$

Remark 3.1. *It is sufficient to prove relation (3.6) for the functions*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

All auxiliary lemmas will be proved for the cosine Fourier series. The key estimates for the proof of Theorem 3.5 will be obtained in Lemmas 3.6 and 3.7. For non-negative general monotone sequences Theorem 3.5 follows from [41, 91].

Remark 3.2. *All constants in the proof of Theorem 3.5 depend on p, l, C, and γ.*

Remark 3.3. *Note that similar results in L_∞ were studied in [34, 92, 94].*

It is natural to ask if one can further extend the monotonicity condition in order that the results of Theorem 3.5 still hold. In this respect, we will show that if the sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ belong to the wider class of weak monotone sequences $\overline{WM}$, where

$$\overline{WM} = \left\{ \{a_n\}_{n=1}^{\infty} : |a_n| \le C \sum_{k=\frac{n}{\gamma}}^{\infty} \frac{|a_k|}{k} \quad C > 0,\ \gamma > 1 \right\},$$

then Theorem 3.5 is not true any more. Note that $GM \subset WM \subset \overline{WM}$. The following result, in particular, shows that relation (3.6) does not hold for weak monotone sequences, and the best possible estimates are given by Theorem 3.1.

Theorem 3.6. *Let $l \in \mathbb{N}$.*

(A) *Let $p > 2$, then there exists a continuous function*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \overline{WM}$, such that inequality (3.3) does not hold for any $q > 0$.

(B) *Let $1 < p < 2$, then there exists a continuous function*

$$f(x) \sim \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \overline{WM}$, such that inequality (3.4) does not hold for any $q > 0$.

Note that all results of the chapter are valid for moduli of smoothness of fractional order ($l > 0$). The importance of this remark follows, in particular, from consideration of sharp Ul'yanov inequalities [82].

3.4 Auxiliary results for general monotone sequences

In the proof of the main results of this chapter we will use the same technique and notation as in Chapter 2. Without loss of generality, we can assume in the definition of the class GM that $\gamma = 2^\nu$, where ν is a natural number. For a given sequence $\{a_n\}_{n=1}^{\infty}$, let us denote:

$$A_n := \max_{2^n \le k \le 2^{n+1}} |a_k|;$$

$$B_n := \max_{2^{n-2\nu} \le k \le 2^{n+2\nu}} |a_k|;$$

$$M_n := \left\{k \in [2^{n-\nu}, 2^{n+\nu}] : |a_k| > \frac{A_n}{8C2^{2\nu}}\right\};$$

$$M_n^+ := \{k \in M_n : a_k > 0\}, \quad \text{and} \quad M_n^- := M_n \setminus M_n^+.$$

Here we slightly modify the concepts of good and bad numbers (cf. Definition 2.2).

Definition 3.2. *Let $\{a_n\}_{n=1}^{\infty} \in GM$. We say that a non-negative integer number n is* good, *if either $n \le 2\nu$ or*

$$B_n \le 2^{4l\nu} A_n.$$

The rest of non-negative integer numbers we call bad.

The following result is a modified version of Lemma 2.4.

Lemma 3.1 ([35, Lemma 2.2])**.** *Let a vanishing sequence $\{a_n\}_{n=1}^{\infty}$ be such that $\{a_n\}_{n=1}^{\infty} \in GM$. Let $N_0 := [\log_2(C^3 2^{10\nu+8})] + 1$. Then, for any good $n \ge N_0$, there exists a segment of integer numbers $[l_n, m_n] \subseteq [2^{n-\nu}, 2^{n+\nu}]$ such that one of the following two conditions holds:*

(i) *for any $k \in [l_n, m_n]$, we have $a_k \ge 0$ and*

$$|M_n^+ \cap [l_n, m_n]| \ge \frac{2^n}{C^3 2^{(12l+6)\nu+8}};$$

(ii) *for any* $k \in [l_n, m_n]$, *we have* $a_k \le 0$ *and*

$$|M_n^- \cap [l_n, m_n]| \ge \frac{2^n}{C^3 2^{(12l+6)\nu+8}}.$$

Let us denote

$$I_s(x) := \sum_{k=2^s}^{2^{s+1}-1} a_k \cos kx.$$

Then

$$I_s^{(l)}(x) = \sum_{k=2^s}^{2^{s+1}-1} k^l a_k \cos\left(kx + \frac{l\pi}{2}\right).$$

Lemma 3.2. *Let* $\{a_n\}_{n=1}^{\infty} \in GM$ *and* $n \ge N_0$ *be a good number, where* N_0 *is given by Lemma 3.1. Then*

$$\left\| \sum_{s=n-\nu}^{n+\nu} I_s \right\|_p^p \gtrsim \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{p-2} |a_k|^p.$$

Proof. Let $n \ge N_0$ be a good number. Without loss of generality, we can assume that condition (i) of Lemma 3.1 holds, and we consider the sum

$$Q_n(x) := \sum_{k=l_n}^{m_n} a_k \cos kx.$$

Note that for any $0 \le x \le \frac{1}{2^{n+\nu}}$ all terms of $Q_n(x)$ are non-negative. Using the inequality $\cos t \ge \frac{3}{2\pi} t$ for any $t \in \left[0, \frac{\pi}{3}\right]$ and Lemma 3.1, we obtain, for any $0 \le x \le \frac{\pi}{3} \frac{1}{2^{n+\nu}}$,

$$\begin{aligned} Q_n(x) &= \sum_{k=l_n}^{m_n} a_k \cos kx \ge \frac{3}{2\pi} x \sum_{k=l_n}^{m_n} a_k k \\ &\ge \frac{3}{2\pi} x \sum_{k \in [l_n, m_n] \cap M_n^+} a_k k \\ &\ge \frac{3}{2\pi} x 2^{(n-\nu)} \frac{A_n}{8C2^{2\nu}} \frac{2^n}{C^3 2^{(12l+6)\nu+8}} \\ &\gtrsim 2^{2n} A_n x. \end{aligned}$$

Using the last inequality and the fact that $\|S_M(f,\cdot) - S_N(f,\cdot)\|_p \lesssim \|f\|_p$, we derive

$$\begin{aligned} \left\| \sum_{s=n-\nu}^{n+\nu} I_s \right\|_p^p &\gtrsim \|Q_n\|_p^p \ge \int_0^{\frac{\pi}{3}\frac{1}{2^{n+\nu}}} Q_n^p(x) dx \\ &\gtrsim 2^{2np} A_n^p \int_0^{\frac{\pi}{3}\frac{1}{2^{n+\nu}}} x^p dx \gtrsim 2^{(p-1)n} A_n^p \\ &\gtrsim \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{p-2} |a_k|^p. \end{aligned}$$

□

Using the fact that $\{a_k k^\delta\} \in GM$ whenever $\{a_k\} \in GM$, see [59], similarly to Lemma 3.2 one can obtain the following result.

Lemma 3.3. *Let $n \geq N_0$ be a good number, where N_0 is given by Lemma 3.1. Then*

$$\left\| \sum_{s=n-\nu}^{n+\nu} I_s^{(l)} \right\|_p^p \gtrsim \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{(l+1)p-2} |a_k|^p.$$

The following lemma is a slightly modified version of Lemma 2.5 from Chapter 2.

Lemma 3.4. *Let a vanishing sequence $\{a_n\}_{n=1}^\infty$ be such that $\{a_n\}_{n=1}^\infty \in GM$. Then, for any bad number $r \in \mathbb{N}$, there exists either a set of integer numbers*

$$r = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_s =: \xi_{r,s} \tag{3.7}$$

or

$$r = \xi_0 > \xi_1 > \xi_2 > \ldots > \xi_s =: \xi_{r,s} \tag{3.8}$$

such that $\xi_1, \xi_2, \ldots, \xi_{s-1}$ are bad, $\xi_{r,s}$ is good and

$$A_r < 2^{-4l\nu} A_{\xi_1} < 2^{-8l\nu} A_{\xi_2} < \ldots < 2^{-4ls\nu} A_{\xi_{r,s}},$$

$$|\xi_i - \xi_{i+1}| \leq 2\nu, \quad i = 0, \ldots, s-1.$$

Remark 3.4. *Recall that the sets* (3.7) *and* (3.8) *from Lemma 3.4 are constructed for any bad number r uniquely. In sequel, we say that sets* (3.7) *and* (3.8) *are increasing and decreasing chains, respectively, of the number r. Recall that the number s is called the length of the bad number r. Moreover, in this case, we will say that bad number r transforms into the good number $\xi_{r,s}$.*

We set

$$P_n := \sum_{k=2^n}^{2^{n+1}-1} k^{p-2} |a_k|^p, \quad P_{n,\nu} := \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{p-2} |a_k|^p,$$

$$\widetilde{P}_n := \sum_{k=2^n}^{2^{n+1}-1} k^{(l+1)p-2} |a_k|^p, \quad \widetilde{P}_{n,\nu} := \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{(l+1)p-2} |a_k|^p.$$

Lemma 3.5. *Let $n \geq N_0$ be a good number and R_n be a set of bad numbers transforming into n. Then*

$$\sum_{r \in R_n} P_r \lesssim P_{n,\nu}. \tag{3.9}$$

Moreover, if A is a subset of the set of good numbers such that $\min_{n\in A} n \geq N_0$, and B is a set of bad numbers transforming only into the numbers from A. Then

$$\sum_{r\in B} P_r \lesssim \sum_{m\in A} P_{m,\nu}. \tag{3.10}$$

Expressions P_r and $P_{m,\nu}$ in the estimates (3.9) *and* (3.10) *we can replace by $\widetilde{P}_r$ and $\widetilde{P}_{m,\nu}$, respectively.*

Proof. First we prove inequality (3.9). Divide the set R_n into two disjoint parts:

$$R_n = Q_n^1 \bigsqcup Q_n^2,$$

where Q_n^1 is the set of bad numbers which transform into n with decreasing chain, and Q_n^2 is the set of bad numbers which transform into n with increasing chain.

Consider the set Q_n^1. Let $r \in Q_n^1$ be a bad number of length s. Then, by Lemma 3.4, $r \leq n + 2s\nu$ and $A_r \leq 2^{-4ls\nu} A_n$. Therefore,

$$\begin{aligned} P_r = \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} &\lesssim A_r^p 2^{r(p-1)} \leq 2^{-4ls\nu p} A_n^p 2^{(n+2s\nu)(p-1)} \\ &\lesssim 2^{-2ps\nu} A_n^p 2^{n(p-1)}. \end{aligned} \tag{3.11}$$

On the other hand, by Lemma 3.1, we derive

$$\begin{aligned} P_{n,\nu} = \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p k^{p-2} &\geq \sum_{k=l_n}^{m_n} |a_k|^p k^{p-2} \\ &\geq \sum_{k\in[l_n,m_n]\cap M_n} |a_k|^p k^{p-2} \gtrsim A_n^p \sum_{k\in[l_n,m_n]\cap M_n} k^{p-2} \\ &\gtrsim A_n^p 2^{n(p-2)} \left|[l_n, m_n] \cap M_n\right| \gtrsim A_n^p 2^{n(p-1)}. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) for the bad number $r \in Q_n^1$ of length $s = s_r$, we get

$$P_r \lesssim 2^{-2ps_r\nu} P_{n,\nu}.$$

Note that the set Q_n^1 consists no more than 2ν bad numbers of length 1, no more than $(2\nu)^2$ bad numbers of length 2, etc. Hence,

$$\sum_{r\in Q_n^1} P_r \lesssim P_{n,\nu} \sum_{r\in Q_n^1} 2^{-2ps_r\nu} \leq P_{n,\nu} \sum_{s=1}^{\infty} (2\nu)^s 2^{-2ps\nu} \lesssim P_{n,\nu}, \tag{3.13}$$

since $\sum\limits_{s=1}^{\infty} (2\nu)^s 2^{-2ps\nu} < \infty$.

Similarly, we have that

$$\sum_{r\in Q_n^2} P_r \lesssim P_{n,\nu}. \tag{3.14}$$

From (3.13) and (3.14) we obtain

$$\sum_{r\in R_n} P_r = \sum_{r\in Q_n^1} P_r + \sum_{r\in Q_n^2} P_r \lesssim P_{n,\nu}.$$

Second, we show inequality (3.10). We enumerate the elements of the set A: $A = \{m_1, m_2, \dots, m_{|A|}\}$, where $|A|$ denotes the cardinality[1] of set A. We divide the set B into the following disjoint sets:

$$B = R_{m_1} \bigsqcup R_{m_2} \bigsqcup \dots \bigsqcup R_{m_{|A|}},$$

where R_{m_i} is a subset of bad numbers of the set B which transform into m_i, $i = 1, 2, \dots, |A|$. Then from inequality (3.9) it follows that

$$\sum_{r\in B} P_r = \sum_{i=1}^{|A|} \sum_{r\in R_{m_i}} P_r \lesssim \sum_{i=1}^{|A|} P_{m_i,\nu} = \sum_{m\in A} P_{m,\nu}.$$

□

3.5 Upper estimate for $n^{-lp} \sum\limits_{k=1}^{n} |a_k|^p k^{(l+1)p-2}$

Lemma 3.6. *Let $p > 2$, $f(x) \in L_p([0, 2\pi])$, $f(x) \sim \sum\limits_{n=1}^{\infty} a_n \cos nx$, and $\{a_n\}_{n=1}^{\infty} \in GM$. Then*

$$\frac{1}{n^{lp}} \sum_{k=1}^{n} |a_k|^p k^{(l+1)p-2} \lesssim \frac{1}{n^{lp}} \left\| S_{2^{\nu+2}n}^{(l)} \right\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p, \tag{3.15}$$

where $S_k(x) := \sum\limits_{s=1}^{k} a_s \cos sx$ is the k-th partial sum of Fourier series of the function f.

Proof. Choose N such that $2^{N-1} \le n < 2^N$. Then

$$\begin{aligned} \frac{1}{n^{lp}} \sum_{k=1}^{n} |a_k|^p k^{(l+1)p-2} &\le \frac{1}{2^{(N-1)lp}} \sum_{k=1}^{2^N-1} |a_k|^p k^{(l+1)p-2} \\ &\lesssim \frac{1}{2^{Nlp}} \sum_{r=0}^{N} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{(l+1)p-2} = \frac{1}{2^{Nlp}} \sum_{r=0}^{N} \widetilde{P}_r. \end{aligned} \tag{3.16}$$

[1]We admit cases when $|A| = \infty$.

A. Let first $N > N_0$. Let us denote by G the set of good numbers. Divide segment $[0, N] \cap \mathbb{Z}$ into six parts:

$$[0, N] \cap \mathbb{Z} = ([0, N_0) \cap \mathbb{Z}) \bigsqcup G_N \bigsqcup B_N^1 \bigsqcup B_N^2 \bigsqcup B_N^3 \bigsqcup B_N^4,$$

where

1. $G_N := G \cap [N_0, N]$ is a set of good numbers $r \in [N_0, N]$;
2. B_N^1 is a set of bad numbers $r \in [N_0, N]$ with increasing chain such that $\xi_{r,s} \leq N$ (see the definition of $\xi_{r,s}$ in Remark 3.4), and, hence, the following inequality holds:
$$N_0 \leq r < \xi_{r,s} \leq N;$$
3. B_N^2 is a set of bad numbers $r \in [N_0, N]$ with increasing chain such that $\xi_{r,s} > N$, and, hence, the following inequality holds:
$$N_0 \leq r \leq N < \xi_{r,s};$$
4. B_N^3 is a set of bad numbers $r \in [N_0, N]$ with decreasing chain such that $\xi_{r,s} \geq N_0$, and, hence, the following inequality holds:
$$N_0 \leq \xi_{r,s} < r \leq N;$$
5. B_N^4 is a set of bad numbers $r \in [N_0, N]$ with decreasing chain such that $\xi_{r,s} < N_0$, and, hence, the following inequality holds:
$$\xi_{r,s} < N_0 \leq r \leq N.$$

Therefore,

$$\begin{aligned} \frac{1}{2^{Nlp}} \sum_{r=0}^{N} \widetilde{P}_r = \frac{1}{2^{Nlp}} \sum_{r \in [0, N_0 - 1] \sqcup B_N^4} \widetilde{P}_r + \frac{1}{2^{Nlp}} \sum_{r \in G_N \sqcup B_N^1 \sqcup B_N^3} \widetilde{P}_r \\ + \frac{1}{2^{Nlp}} \sum_{r \in B_N^2} \widetilde{P}_r =: J_1 + J_2 + J_3. \end{aligned} \tag{3.17}$$

Step 1. The estimate of J_1.

Take $r \in B_N^4$. There exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$\xi_{r,s} \leq N_0 \leq r \leq \xi_{r,s} + 2s\nu.$$

Then

$$\begin{aligned}\widetilde{P}_r &= \sum_{k=2^r}^{2^{r+1}-1} k^{(l+1)p-2}|a_k|^p \lesssim A_r^p 2^{r((l+1)p-1)}\\ &\le 2^{-4ls\nu p} A^p_{\xi_{r,s}} 2^{(\xi_{r,s}+2s\nu)((l+1)p-1)} \le 2^{-2s\nu} A^p_{\xi_{r,s}} 2^{\xi_{r,s}((l+1)p-1)}\\ &< 2^{-2s\nu} A^p_{\xi_{r,s}} 2^{N_0((l+1)p-1)} \lesssim 2^{-2s\nu} A^p_{\xi_{r,s}}.\end{aligned}$$

Repeating arguments from the proof of Lemma 3.5, since the series $\sum\limits_{s=1}^{\infty}(2\nu)^s 2^{-2s\nu}$ converges, we get

$$\sum_{r\in B_N^4} \widetilde{P}_r \lesssim \sum_{\substack{\xi\in G\\ \xi\le N_0}} A_\xi^p \le N_0 \max_{1\le k\le 2^{N_0+1}} |a_k|^p \le N_0 \max_{1\le k\le 2^{N_0+1}} |k^l a_k|^p. \tag{3.18}$$

Consider now $r\in[0,N_0-1]$. It is easy to obtain that

$$\begin{aligned}\widetilde{P}_r &= \sum_{k=2^r}^{2^{r+1}-1} k^{(l+1)p-2}|a_k|^p\\ &\le \max_{1\le k\le 2^{N_0+1}} |k^l a_k|^p \sum_{k=1}^{2^{N_0}-1} k^{p-2} \lesssim \max_{1\le k\le 2^{N_0+1}} |k^l a_k|^p.\end{aligned}$$

Therefore,

$$\sum_{r=0}^{N_0-1} \widetilde{P}_r \lesssim \max_{1\le k\le 2^{N_0+1}} |k^l a_k|^p. \tag{3.19}$$

Note that, for any $k\le 2^{N_0+1}$, the expression $|k^l a_k|$ is an absolute value of the k-th Fourier coefficient of the function $S^{(l)}_{2^\nu+2n}(x)$. Using inequalities (3.18), (3.19), and Hölder inequality, we obtain

$$\begin{aligned}J_1 &= \frac{1}{2^{Nlp}} \sum_{r\in[0,N_0-1]\sqcup B_N^4} \widetilde{P}_r\\ &\lesssim \frac{1}{2^{Nlp}} \max_{1\le k\le 2^{N_0+1}} |k^l a_k|^p \lesssim \frac{1}{n^{lp}} \left\| S^{(l)}_{2^\nu+2n}\right\|_p^p.\end{aligned} \tag{3.20}$$

Step 2. The estimate of J_2. Since all bad numbers $r\in B_N^1\sqcup B_N^3$ transform only into good numbers $m\in[N_0,N]$, then, by Lemma 3.5, we get

$$J_2 = \frac{1}{2^{Nlp}} \sum_{r\in G_N} \widetilde{P}_r + \frac{1}{2^{Nlp}} \sum_{r\in B_N^1\sqcup B_N^3} \widetilde{P}_r \lesssim \frac{1}{2^{Nlp}} \sum_{m\in G_N} \widetilde{P}_{m,\nu}.$$

Using Lemma 3.3, we derive

$$
\begin{aligned}
J_2 &\lesssim \frac{1}{2^{Nlp}} \sum_{m\in G_N} \widetilde{P}_{m,\nu} \lesssim \frac{1}{2^{Nlp}} \sum_{m\in G_N} \left\| \sum_{s=m-\nu}^{m+\nu} I_s^{(l)} \right\|_p^p \\
&\le \frac{1}{2^{Nlp}} \sum_{m=\nu}^{N} \left\| \sum_{s=m-\nu}^{m+\nu} I_s^{(l)} \right\|_p^p .
\end{aligned}
\tag{3.21}
$$

Step 3. The estimate of J_3. Let $r \in B_N^2$, then there exists a good number $\xi_{r,s}$ such that

$$
A_r < 2^{-4ls\nu} A_{\xi_{r,s}} \tag{3.22}
$$

and

$$
r < \xi_{r,s} \le r + 2s\nu \le N + 2s\nu. \tag{3.23}
$$

Then (3.22) and $r < \xi_{r,s}$ imply

$$
\begin{aligned}
\frac{1}{2^{Nlp}} \widetilde{P}_r &= \frac{1}{2^{Nlp}} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{(l+1)p-2} \lesssim \frac{1}{2^{Nlp}} A_r^p 2^{r((l+1)p-1)} \\
&\le \frac{1}{2^{Nlp}} 2^{-4ls\nu p} A_{\xi_{r,s}}^p 2^{\xi_{r,s}((l+1)p-1)}.
\end{aligned}
\tag{3.24}
$$

To be definite, assume that condition (i) of Lemma 3.1 is valid. Using Lemma 3.1, from inequality (3.24) we get

$$
\begin{aligned}
&\frac{1}{2^{Nlp}} \widetilde{P}_r \le \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}((l+1)p-1)} A_{\xi_{r,s}}^p \\
&\le \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}((l+1)p-1)} \frac{(8C2^{2\nu})^p}{\left| [l_{\xi_{r,s}}, m_{\xi_{r,s}}] \cap M_{\xi_{r,s}}^+ \right|} \sum_{k\in[l_{\xi_{r,s}}, m_{\xi_{r,s}}]\cap M_{\xi_{r,s}}^+} |a_k|^p \\
&\lesssim \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}((l+1)p-1)} \frac{1}{2^{\xi_{r,s}}} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p \\
&= \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s} lp} 2^{\xi_{r,s}(p-2)} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p \\
&\lesssim \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s} lp} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2}.
\end{aligned}
\tag{3.25}
$$

Combining inequality (3.25) with the inequality $\xi_{r,s} \le N + 2s\nu$, we derive that

$$\begin{aligned}\frac{1}{2^{Nlp}}\widetilde{P}_r &\lesssim \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s} lp} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2} \\ &= 2^{(\xi_{r,s}-N-2s\nu)lp} 2^{-2ls\nu p} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2} \\ &\le 2^{-2ls\nu p} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2} = 2^{-2ls\nu p} P_{\xi_{r,s},\nu}.\end{aligned}$$

Thus, for any bad number $r \in B_N^2$, we have

$$\frac{1}{2^{Nlp}}\widetilde{P}_r \lesssim 2^{-2ls\nu p} P_{\xi_{r,s},\nu}.$$

Since $\sum_{s=1}^{\infty} (2\nu)^s 2^{-2ls\nu p} < \infty$, using similar arguments to those given in the proof of Lemma 3.5, we get

$$\sum_{r\in B_N^2} \frac{1}{2^{Nlp}}\widetilde{P}_r \lesssim \sum_{\substack{\xi\in G\\ \xi>N}} P_{\xi,\nu}.$$

Hence, by Lemma 3.2,

$$\begin{aligned} J_3 = \sum_{r\in B_N^2} \frac{1}{2^{Nlp}}\widetilde{P}_r &\lesssim \sum_{\substack{\xi\in G\\ \xi>N}} P_{\xi,\nu} \\ &\lesssim \sum_{\substack{\xi\in G\\ \xi>N}} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k \right\|_p^p \le \sum_{\xi=N+1}^{\infty} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k \right\|_p^p . \end{aligned} \tag{3.26}$$

Now we prove the estimate for the sum $\frac{1}{2^{Nlp}} \sum_{r=0}^{N} \widetilde{P}_r$. Applying (3.20), (3.21), and (3.26), we write

$$\begin{aligned} \frac{1}{2^{Nlp}} \sum_{r=0}^{N} \widetilde{P}_r &= J_1 + J_2 + J_3 \\ &\lesssim \frac{1}{2^{Nlp}} \sum_{m=\nu}^{N} \left\| \sum_{k=m-\nu}^{m+\nu} I_k^{(l)} \right\|_p^p \\ &+ \sum_{\xi=N+1}^{\infty} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k \right\|_p^p + \frac{1}{n^{lp}} \left\| S_{2^\nu n}^{(l)} \right\|_p^p . \end{aligned} \tag{3.27}$$

Using Jensen's inequality and the Littlewood-Paley theorem, we obtain that

$$
\begin{aligned}
\sum_{m=\nu}^{N}\left\|\sum_{k=m-\nu}^{m+\nu} I_k^{(l)}\right\|_p^p &= \int_0^{2\pi}\sum_{m=\nu}^{N}\left|\sum_{k=m-\nu}^{m+\nu} I_k^{(l)}(x)\right|^p dx \\
&\le \int_0^{2\pi}\left(\sum_{m=\nu}^{N}\left|\sum_{k=m-\nu}^{m+\nu} I_k^{(l)}(x)\right|^2\right)^{\frac{p}{2}} dx \\
&\le \int_0^{2\pi}\left((2\nu+1)^2\sum_{k=0}^{N+\nu}\left|I_k^{(l)}(x)\right|^2\right)^{\frac{p}{2}} dx \\
&\lesssim \left\|\left(\sum_{k=0}^{N+\nu}\left|I_k^{(l)}\right|^2\right)^{\frac{1}{2}}\right\|_p^p \lesssim \left\|S_{2^{\nu+2}n}^{(l)}\right\|_p^p.
\end{aligned}
\tag{3.28}
$$

In a similar way, we get

$$
\begin{aligned}
\sum_{\xi=N+1}^{\infty}\left\|\sum_{k=\xi-\nu}^{\xi+\nu} I_k\right\|_p^p &= \int_0^{2\pi}\sum_{\xi=N+1}^{\infty}\left|\sum_{k=\xi-\nu}^{\xi+\nu} I_k(x)\right|^p dx \\
&\le \int_0^{2\pi}\left(\sum_{\xi=N+1}^{\infty}\left|\sum_{k=\xi-\nu}^{\xi+\nu} I_k(x)\right|^2\right)^{\frac{p}{2}} dx \\
&\lesssim \left\|\left(\sum_{k=N+1-\nu}^{\infty}|I_k|^2\right)^{\frac{1}{2}}\right\|_p^p \lesssim \|f - S_{[2^{-\nu-2}n]}\|_p^p.
\end{aligned}
\tag{3.29}
$$

Combining inequalities (3.27)–(3.29), we complete the proof of Lemma 3.6 in the case when $N > N_0$.

B. Suppose now that $N \le N_0$. Then

$$
\begin{aligned}
\frac{1}{2^{Nlp}}\sum_{r=0}^{N}\widetilde{P}_r &= \frac{1}{2^{Nlp}}\sum_{r=0}^{N}\sum_{k=2^r}^{2^{r+1}-1} k^{(l+1)p-2}|a_k|^p \\
&\lesssim \frac{1}{2^{Nlp}}\max_{1\le k\le 2^{N+1}}|k^l a_k|^p.
\end{aligned}
$$

Since for any $1 \le k \le 2^{N+1}$ the expression $|k^l a_k|$ is an absolute value of the k-th Fourier coefficient of the function $S_{2^{\nu+2}n}^{(l)}(x)$, applying Hölder's inequality, we derive

$$
\frac{1}{2^{Nlp}}\sum_{r=0}^{N}\widetilde{P}_r \lesssim \frac{1}{2^{Nlp}}\max_{1\le k\le 2^{N+1}}|k^l a_k|^p \lesssim \frac{1}{n^{lp}}\left\|S_{2^{\nu+2}n}^{(l)}\right\|_p^p. \tag{3.30}
$$

□

3.6 Upper estimate for $\sum\limits_{k=n}^{\infty} |a_k|^p k^{p-2}$

Lemma 3.7. *Let $p > 2$, $f(x) \in L_p([0, 2\pi])$, $f(x) \sim \sum\limits_{n=1}^{\infty} a_n \cos nx$, and $\{a_n\}_{n=1}^{\infty} \in GM$. Then*

$$\sum_{k=n}^{\infty} |a_k|^p k^{p-2} \lesssim \frac{1}{n^{lp}} \left\| S^{(l)}_{2^{\nu+2}n} \right\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p. \tag{3.31}$$

Proof. Let $n \in \mathbb{N}$. Choose N such that $2^N \leq n < 2^{N+1}$. Then

$$\begin{aligned} \sum_{k=n}^{\infty} |a_k|^p k^{p-2} &\leq \sum_{k=2^N}^{\infty} |a_k|^p k^{p-2} \\ &= \sum_{r=N}^{\infty} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} = \sum_{r=N}^{\infty} P_r. \end{aligned} \tag{3.32}$$

A. Suppose that $N < N_0$. Divide the set $[N, \infty) \cap \mathbb{Z}$ into six sets as follows.

$$[N, \infty) \cap \mathbb{Z} = ([N, N_0) \cap \mathbb{Z}) \bigsqcup T_N \bigsqcup K_N^1 \bigsqcup K_N^2 \bigsqcup K_N^3 \bigsqcup K_N^4,$$

where

1. $T_N := G \cap [N_0, \infty)$ is a set of good numbers $r \in [N_0, \infty)$;
2. K_N^1 is a set of bad numbers $r \in [N_0, \infty)$ with increasing chain, and, hence, the following inequality holds:
$$N < N_0 \leq r < \xi_{r,s};$$
3. K_N^2 is a set of bad numbers $r \in [N_0, \infty)$ with decreasing chain such that $\xi_{r,s} \geq N_0$, and, hence, the following inequality holds:
$$N < N_0 \leq \xi_{r,s} < r;$$
4. K_N^3 is a set of bad numbers $r \in [N_0, \infty)$ with decreasing chain such that $N < \xi_{r,s} < N_0$, and, hence, the following inequality holds:
$$N < \xi_{r,s} < N_0 \leq r;$$
5. K_N^4 is a set of bad numbers $r \in [N_0, \infty)$ with decreasing chain such that $\xi_{r,s} \leq N$, and, hence, the following inequality holds:
$$\xi_{r,s} \leq N < N_0 \leq r.$$

Therefore,

$$\sum_{r=N}^{\infty} P_r = \sum_{r\in[N,N_0-1]\sqcup K_N^3} P_r + \sum_{r\in T_N\sqcup K_N^1\sqcup K_N^2} P_r + \sum_{r\in K_N^4} P_r$$
$$=: \Theta_1 + \Theta_2 + \Theta_3.$$

Step $1_{\mathbf{A}}$. The estimate of Θ_1. Let $r \in K_N^3$, then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$r \le \xi_{r,s} + 2s\nu.$$

The last inequalities yield

$$P_r = \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \le A_{\xi_{r,s}}^p 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)}$$
$$\le A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{N_0(p-1)} \lesssim A_{\xi_{r,s}}^p 2^{-2ls\nu p}.$$

Combining this with arguments from the proof of Lemma 3.5, we write

$$\sum_{r\in K_N^3} P_r \lesssim \sum_{\substack{\xi\in G\\ N\le\xi<N_0}} A_\xi^p \le N_0 \max_{2^N\le k\le 2^{N_0+1}} |a_k|^p \lesssim \max_{2^N\le k\le 2^{N_0+1}} |a_k|^p. \tag{3.33}$$

On the other hand, it is easy to get

$$\sum_{r=N}^{N_0-1} P_r = \sum_{r=N}^{N_0-1} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim \max_{2^N\le k\le 2^{N_0}} |a_k|^p. \tag{3.34}$$

From estimates (3.33) and (3.34) we have

$$\Theta_1 = \sum_{r=N}^{N_0-1} P_r + \sum_{r\in K_N^3} P_r \lesssim \max_{2^N\le k\le 2^{N_0+1}} |a_k|^p \lesssim \|f - S_{[2^{-\nu-2}n]}\|_p^p. \tag{3.35}$$

Step $2_{\mathbf{A}}$. The estimate of Θ_2. Since all bad numbers $r \in K_N^1 \sqcup K_N^2$ transform only into good $m \in [N_0, \infty)$, according to Lemma 3.5, we have

$$\Theta_2 = \sum_{r\in T_N} P_r + \sum_{r\in K_N^1\sqcup K_N^2} P_r \lesssim \sum_{m\in T_N} P_{m,\nu}.$$

Now, by Lemma 3.2,

$$\Theta_2 \lesssim \sum_{m\in T_N} P_{m,\nu} \lesssim \sum_{m\in T_N} \left\| \sum_{k=m-\nu}^{m+\nu} I_k \right\|_p^p \leq \sum_{m=N}^{\infty} \left\| \sum_{k=m-\nu}^{m+\nu} I_k \right\|_p^p . \tag{3.36}$$

Step $3_{\mathbf{A}}$. The estimate of Θ_3. Let $r \in K_N^4$, then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$r \leq \xi_{r,s} + 2s\nu.$$

Then, using the last inequalities, we get

$$\begin{aligned}
P_r &= \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \leq A_{\xi_{r,s}}^p 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \\
&\leq A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{N_0(p-1)} \lesssim A_{\xi_{r,s}}^p 2^{-2ls\nu p} \\
&\leq 2^{N_0 lp} \frac{1}{2^{Nlp}} A_{\xi_{r,s}}^p 2^{-2ls\nu p} \\
&\lesssim \frac{1}{2^{Nlp}} A_{\xi_{r,s}}^p 2^{-2ls\nu p} \leq 2^{-2ls\nu p} \frac{1}{2^{Nlp}} \max_{2^{\xi_{r,s}} \leq k \leq 2^{\xi_{r,s}+1}} |k^l a_k|^p.
\end{aligned}$$

Repeating the arguments from the proof of Lemma 3.5, we obtain

$$\begin{aligned}
\sum_{r\in K_N^4} P_r &\lesssim \sum_{\substack{\xi\in G\\ \xi<N}} \frac{1}{2^{Nlp}} \max_{2^{\xi}\leq k\leq 2^{\xi+1}} |k^l a_k|^p \\
&\leq \frac{N_0}{2^{Nlp}} \max_{1\leq k\leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{2^{Nlp}} \max_{1\leq k\leq 2^{N+1}} |k^l a_k|^p.
\end{aligned}$$

Since, for any $1 \leq k \leq 2^{N+1}$, the expression $|k^l a_k|$ is an absolute value of the k-th Fourier coefficient of the function $S_{2^\nu n}^{(l)}(x)$, we have

$$\Theta_3 = \sum_{r\in K_N^4} P_r \lesssim \frac{1}{2^{Nlp}} \max_{1\leq k\leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{n^{lp}} \left\| S_{2^{\nu+2}n}^{(l)} \right\|_p^p . \tag{3.37}$$

Now we estimate the sum $\sum\limits_{r=N}^{\infty} P_r$. From inequalities (3.35)–(3.37), we obtain that

$$\begin{aligned}
\sum_{r=N}^{\infty} P_r &= \Theta_1 + \Theta_2 + \Theta_3 \\
&\lesssim \sum_{m=N}^{\infty} \left\| \sum_{k=m-\nu}^{m+\nu} I_k \right\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p + \frac{1}{n^{lp}} \left\| S_{2^{\nu+2}n}^{(l)} \right\|_p^p .
\end{aligned}$$

In the same way as in the proof of inequality (3.28), we derive

$$\sum_{m=N}^{\infty} \left\| \sum_{k=m-\nu}^{m+\nu} I_k \right\|_p^p \lesssim \|f - S_{[2^{-\nu-2}n]}\|_p^p.$$

This completes the proof of Lemma 3.7 in this case.

B. Suppose now that $N \geq N_0$. Divide set $[N, \infty) \cap \mathbb{Z}$ into five sets:

$$[N, \infty) \cap \mathbb{Z} = T_N \bigsqcup K_N^1 \bigsqcup K_N^2 \bigsqcup K_N^3 \bigsqcup K_N^4,$$

where

1. $T_N := G \cap [N, \infty)$ is a set of good numbers $r \in [N, \infty)$;
2. K_N^1 is a set of bad numbers $r \in [N, \infty)$ with increasing chain, and, hence, the following inequality holds:
$$N_0 \leq N \leq r < \xi_{r,s};$$
3. K_N^2 is a set of bad numbers $r \in [N, \infty)$ with decreasing chain such that $\xi_{r,s} \geq N$, and, hence, the following inequality holds:
$$N_0 \leq N \leq \xi_{r,s} < r;$$
4. K_N^3 is a set of bad numbers $r \in [N, \infty)$ with decreasing chain such that $N_0 < \xi_{r,s} < N$, and, hence, the following inequality holds:
$$N_0 < \xi_{r,s} < N \leq r;$$
5. K_N^4 is a set of bad numbers $r \in [N, \infty)$ with decreasing chain such that $\xi_{r,s} \leq N_0$, and, hence, the following inequality holds:
$$\xi_{r,s} \leq N_0 \leq N \leq r.$$

Therefore,

$$\sum_{r=N}^{\infty} P_r = \sum_{r \in T_N \sqcup K_N^1 \sqcup K_n^2} P_r + \sum_{r \in K_N^3} P_r + \sum_{r \in K_N^4} P_r =: L_1 + L_2 + L_3.$$

Step $\mathbf{1_B}$. The estimate of L_1. Similarly to the estimate of Θ_2 from above (see Step $\mathbf{2_A}$) we get

$$L_1 = \sum_{r \in T_N} P_r + \sum_{r \in K_N^1 \sqcup K_N^2} P_r \lesssim \sum_{r=N}^{\infty} \left\| \sum_{k=r-\nu}^{r+\nu} I_k \right\|_p^p. \tag{3.38}$$

Step $2_{\mathbf{B}}$. The estimate of L_2. Let $r \in K_N^3$, then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}} \tag{3.39}$$

and

$$r \le \xi_{r,s} + 2s\nu \le N + 2s\nu. \tag{3.40}$$

Then (3.39) and the inequality $r \le \xi_{r,s} + 2s\nu$ yield

$$P_r = \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \le 2^{-4ls\nu p} A_{\xi_{r,s}}^p 2^{(\xi_{r,s}+2s\nu)(p-1)}. \tag{3.41}$$

To be definite, suppose that condition (i) of Lemma 3.1 is valid. By Lemma 3.1 and inequality (3.41), we obtain

$$\begin{aligned} P_r &\le 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} A_{\xi_{r,s}}^p \\ &\le 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \frac{(8C2^{2\nu})^p}{\left|[l_{\xi_{r,s}}, m_{\xi_{r,s}}] \cap M_{\xi_{r,s}}^+\right|} \sum_{k\in[l_{\xi_{r,s}}, m_{\xi_{r,s}}]\cap M_{\xi_{r,s}}^+} |a_k|^p \\ &\lesssim 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \frac{1}{2^{\xi_{r,s}}} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p \\ &\lesssim 2^{-2ls\nu p} 2^{-2s\nu} 2^{-\xi_{r,s} lp} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{(l+1)p-2}. \end{aligned}$$

Since $N \le \xi_{r,s} + 2s\nu$, it follows that

$$P_r \le 2^{-2s\nu} \frac{1}{2^{Nlp}} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{(l+1)p-2} = 2^{-2s\nu} \frac{1}{2^{Nlp}} \widetilde{P}_{\xi_{r,s},\nu}.$$

Therefore, for any bad number $r \in K_N^3$,

$$P_r \lesssim 2^{-2s\nu} \frac{1}{2^{Nlp}} \widetilde{P}_{\xi_{r,s},\nu}.$$

In similar manner as in the proof of Lemma 3.5, we derive

$$\sum_{r\in K_N^3} P_r \lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi\in G \\ \xi<N}} \widetilde{P}_{\xi,\nu}.$$

Hence, by Lemma 3.2,

$$\begin{aligned} L_2 &= \sum_{r\in K_N^3} P_r \lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi\in G\\ \xi<N}} \widetilde{P}_{\xi,\nu} \\ &\lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi\in G\\ \xi<N}} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k^{(l)} \right\|_p^p \le \frac{1}{2^{Nlp}} \sum_{\xi=\nu}^{N} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k^{(l)} \right\|_p^p . \end{aligned} \tag{3.42}$$

Step $3_{\mathbf{B}}$. The estimate of L_3. Let $r \in K_N^4$, then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$N \le r \le \xi_{r,s} + 2s\nu.$$

Last two inequalities imply

$$\begin{aligned} P_r &= \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \le A_{\xi_{r,s}}^p 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \\ &\le A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{-2s\nu} 2^{N_0(p-1)} \lesssim A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{-2s\nu} \\ &\lesssim 2^{-2ls\nu p} 2^{-2s\nu} \frac{1}{2^{\xi_{r,s} lp}} \max_{2^{\xi_{r,s}} \le k \le 2^{\xi_{r,s}+1}} |k^l a_k|^p \\ &\le 2^{-2s\nu} \frac{1}{2^{Nlp}} \max_{2^{\xi_{r,s}} \le k \le 2^{\xi_{r,s}+1}} |k^l a_k|^p . \end{aligned}$$

Using arguments from the proof of Lemma 3.5, we arrive at

$$\begin{aligned} L_3 &= \sum_{r\in K_N^4} P_r \lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi\in G\\ \xi<N_0}} \max_{2^{\xi} \le k \le 2^{\xi+1}} |k^l a_k|^p \\ &\le \frac{N_0}{2^{Nlp}} \max_{1\le k\le 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{2^{Nlp}} \max_{1\le k\le 2^{N+1}} |k^l a_k|^p \\ &\lesssim \frac{1}{n^{lp}} \left\| S_{2^{\nu+2}n}^{(l)} \right\|_p^p . \end{aligned} \tag{3.43}$$

From inequalities (3.38), (3.42), and (3.43) it follows that

$$\begin{aligned} \sum_{r=N}^{\infty} P_r &= L_1 + L_2 + L_3 \\ &\lesssim \sum_{r=N}^{\infty} \left\| \sum_{k=r-\nu}^{r+\nu} I_k \right\|_p^p + \frac{1}{2^{Nlp}} \sum_{\xi=\nu}^{N} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k^{(l)} \right\|_p^p + \frac{1}{n^{lp}} \left\| S_{2^{\nu+2}n}^{(l)} \right\|_p^p . \end{aligned}$$

It remains to apply the Jensen inequality and the Littlewood-Paley theorem to first two terms from the right-hand side of the last inequality. □

3.7 The proof of Theorems 3.5 and 3.6

Here we use the following realization theorem for the modulus of smoothness [26, 80]:

$$\omega_l^p\left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^{lp}} \left\| S_n^{(l)} \right\|_p^p + \|f - S_n\|_p^p, \qquad 1 < p < \infty.$$

Proof of Theorem 3.5. Upper estimate $\lesssim$. The upper estimate follows from [41, Theorem 6.1], where periodic functions with Fourier coefficients from a wider class $GM_1 \supset GM$, where

$$GM_1 = \left\{ \{a_n\}_{n=1}^{\infty} : \sum_{k=n}^{\infty} |a_k - a_{k+1}| \le C \sum_{k=\frac{n}{\gamma}}^{\infty} \frac{|a_k|}{k} \quad C > 0,\ \gamma > 1 \right\},$$

were considered. In spite of the fact that Theorem 6.1 from [41] is formulated for non-negative Fourier coefficients, the proof of this part of the theorem is also true for non-constant sign sequences.

Lower estimate $\gtrsim$. Let $p \le 2$, then from the realization theorem for the modulus of smoothness and the Hardy-Littlewood theorem on Fourier coefficients given by (2.2), we obtain

$$\begin{aligned} \omega_l^p\left(f, \frac{1}{n}\right)_p &\asymp \frac{1}{n^{lp}} \left\| S_n^{(l)} \right\|_p^p + \|f - S_n\|_p^p \\ &\gtrsim \frac{1}{n^{lp}} \sum_{k=1}^{n} k^{(l+1)p-2} |a_k|^p + \sum_{k=n}^{\infty} k^{p-2} |a_k|^p. \end{aligned}$$

Note that here we can also apply Theorem 3.1 in case of $q = p$.

Consider the case $p > 2$. By properties of the modulus of smoothness and Lemmas 3.6 and 3.7, we have

$$\begin{aligned} \omega_l^p\left(f, \frac{1}{n}\right)_p &\asymp \omega_l^p\left(f, \frac{1}{2^{\nu+2}n}\right)_p + \omega_l^p\left(f, \frac{1}{[2^{-\nu-2}n]}\right)_p \\ &\gtrsim \frac{1}{n^{lp}} \left\| S_{2^{\nu+2}n}^{(l)} \right\|_p^p + \left\| f - S_{[2^{-\nu-2}n]} \right\|_p^p \\ &\gtrsim \frac{1}{n^{lp}} \sum_{k=1}^{n} k^{(l+1)p-2} |a_k|^p + \sum_{k=n}^{\infty} k^{p-2} |a_k|^p. \end{aligned}$$

□

Remark 3.5. *In Theorem 3.5 we can assume that we deal with a trigonometric series* $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *such that* $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$*, and*

$$\left(\sum_{k=n}^{\infty} k^{p-2}\left(|a_k|^p + |b_k|^p\right)\right)^{\frac{1}{p}} < \infty.$$

In this case, the function f can be defined as corresponding an L_p-limit of the trigonometric polynomials.

Proof of Theorem 3.6. (A). We follow to proof of Theorem 4.2 from [35]. Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be Rudin-Shapiro's sequence, for which the inequality

$$\left|\sum_{n=0}^{N} \varepsilon_n e^{int}\right| < 5\sqrt{N+1}$$

holds for any $t \in [0, 2\pi]$ and $N = 0, 1, \cdots$.

In the paper [35] (see the proof of Theorem 4.2), it was proved that if an increasing on $(0, 1)$ function φ satisfies the condition

$$\int_0^u \varphi(t)\frac{dt}{t} = O(\varphi(u)) \qquad \text{as} \quad u \to 0,$$

then the function

$$f_\varphi(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n \varphi(1/n)}{n^{\frac{1}{2}}} \sin nx$$

satisfies the condition

$$\omega_l(f_\varphi, \delta)_C \lesssim \varphi(\delta) + \delta^l \sum_{k=1}^{[1/\delta]} k^{l-1}\varphi(1/k).$$

Let us consider $\varphi_0(x) = x^l$. Then the corresponding function

$$f_{\varphi_0}(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^{l+\frac{1}{2}}} \sin nx$$

is continuous and it satisfies

$$\omega_l\left(f_{\varphi_0}, \frac{1}{n}\right)_C \lesssim \frac{\ln n}{n^l}.$$

On the other hand, for $p > 2$, the Fourier coefficients of the function f_{φ_0} satisfy

$$\frac{1}{n^l}\left(\sum_{k=1}^{n} k^{\left(l+1-\frac{1}{p}-\frac{1}{q}\right)q}|b_k|^q\right)^{\frac{1}{q}} + \left(\sum_{k=n+1}^{\infty} k^{\left(1-\frac{1}{p}-\frac{1}{q}\right)q}|b_k|^q\right)^{\frac{1}{q}} \asymp \frac{n^{\frac{1}{2}-\frac{1}{p}}}{n^l}.$$

This shows that inequality (3.3) does not hold for any $q > 0$.

(B). Let us define a continuous function

$$g(x) = \sum_{n=1}^{\infty} \frac{\eta_n}{n^{l+\frac{1}{2}}}\left(\cos\left(nx - \frac{\pi l}{2}\right) + \sin\left(nx - \frac{\pi l}{2}\right)\right),$$

where the sequence $\eta_n = \pm 1$ is such that the series

$$\sum_{n=1}^{\infty} \frac{\eta_n}{n^{\frac{1}{2}}}\left(\cos nx + \sin nx\right) \tag{3.44}$$

is not the Fourier series (see [105, Ch. V, (8.14)]). Note that the Fourier coefficients of the function g, for $1 < p < 2$ and any $q > 0$, satisfy the condition

$$\begin{aligned}\frac{1}{n^l}&\left(\sum_{k=1}^{n} k^{\left(l+1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_k| + |b_k|)^q\right)^{\frac{1}{q}}\\ &+ \left(\sum_{k=n+1}^{\infty} k^{\left(1-\frac{1}{p}-\frac{1}{q}\right)q}(|a_k| + |b_k|)^q\right)^{\frac{1}{q}} \asymp \frac{1}{n^l}.\end{aligned}$$

Therefore, inequality (3.4) implies $\omega_l\left(f, \delta\right)_p = O(\delta^l)$. This relation is equivalent to $f^{(l)} \in L_p$. This contradicts the fact that series (3.44) is not the Fourier series. Hence, inequality (3.4) is not true for any $q > 0$. □

3.8 Applications

In approximation theory, the following direct and inverse estimates are well known (see [25, p. 210])

$$\begin{aligned}\frac{1}{n^l}\left(\sum_{\nu=0}^{n}(\nu+1)^{\tau l-1}E_\nu^\tau(f)_p\right)^{\frac{1}{\tau}} &\lesssim \omega_l\left(f, \frac{1}{n}\right)_p\\ &\lesssim \frac{1}{n^l}\left(\sum_{\nu=0}^{n}(\nu+1)^{ql-1}E_\nu^q(f)_p\right)^{\frac{1}{q}},\end{aligned} \tag{3.45}$$

where $f \in L_p([0,2\pi])$, $1 < p < \infty$, $l, n \in \mathbb{N}$, $q = \min(2,p)$, $\tau = \max(2,p)$, and $E_n(f)_p$ is the best approximation in L_p of function f by trigonometric polynomials of degree n. Note that inequalities (3.45) are equivalent (see [22]) to the relations

$$t^l \left(\int_t^1 u^{-\tau l - 1} \omega_{l+1}^{\tau}(f,u)_p \, du \right)^{\frac{1}{\tau}} \lesssim \omega_l(f,t)_p \lesssim t^l \left(\int_t^1 u^{-ql-1} \omega_{l+1}^{q}(f,u)_p \, du \right)^{\frac{1}{q}}.$$

The following theorem provides a sharp relation between moduli of smoothness $\omega_l(f,t)_p$ and $\omega_{l+1}(f,t)_p$, and modulus of smoothness $\omega_l(f,t)_p$ and the best approximations $E_k(f)_p$ for functions with general monotone Fourier coefficients.

Theorem 3.7. *Let $f \in L_p([0,2\pi])$, $1 < p < \infty$,*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. Then

$$\begin{aligned} \omega_l(f,t)_p &\asymp t^l \left(\int_t^1 u^{-lp} \omega_{l+1}^p(f,u)_p \frac{du}{u} \right)^{\frac{1}{p}} \\ &\asymp t^l \left(\sum_{k=0}^{[1/t]} (k+1)^{lp-1} E_k^p(f)_p \right)^{\frac{1}{p}}, \quad 0 < t < \frac{1}{2}. \end{aligned}$$

Remark 3.6. *The proof of Theorem 3.7 is similar to the proofs of Theorems 7.1 and 7.2 from [41] by using Theorem 3.5.*

From Theorem 3.5 it is possible to get the following description of Besov spaces $B_{p,\tau}^r([0,2\pi])$, cf. Definition 3.1.

Theorem 3.8. *Let $1 < \tau \le \infty$, $1 < p \le \tau$. Let also $f \in L_p([0,2\pi])$,*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. Then $f \in B_{p,\tau}^r([0,2\pi])$ if and only if

$$\sum_{n=1}^{\infty} n^{r\tau + \tau - \frac{\tau}{p} - 1} (|a_n|^\tau + |b_n|^\tau) < \infty, \quad \text{if} \quad 1 < \tau < \infty$$

and

$$\sup_n n^{r+1-\frac{1}{p}} (|a_n| + |b_n|) < \infty, \quad \text{if} \quad \tau = \infty, \ 1 < p < \infty.$$

Corollary 3.1. *Let $1 < p < q < \infty$. Let also $f \in L_p([0, 2\pi])$,*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in GM$. Then for $r = \frac{1}{p} - \frac{1}{q}$ we have

$$f \in B^r_{p,q}([0, 2\pi]) \quad \Longleftrightarrow \quad f \in L_q([0, 2\pi]).$$

Theorem 3.8 and Corollary 3.1 can be proved similarly to Theorem 7.3 and Corollary 7.2 from [41], respectively.

Remark 3.7. *1. If we consider a more restrictive condition in the definition of general monotone sequences (see [91])*

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \le C|a_n|,$$

then Theorem 3.8 holds for any $1 < p < \infty$ and $0 < \tau \le \infty$.

2. Theorem 3.8 for $l = 2$, $0 < r < 2$, and $1 < \tau \le \infty$ implies Theorem 3.4. In the case when $\tau = \infty$, $l = 1$, $0 < r < 1$, and $1 < p < \infty$, Theorem 3.8 implies Theorem 3.2.

3. Note that results similar to Theorem 3.8 for different function classes (under more restrictive monotonicity condition) were studied in [86, 87].

Chapter 4

Boas' conjecture in anisotropic spaces

4.1 Historical remarks

Let us start with some Fourier inequalities. For the Fourier transform given by

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) e^{-ixy} dx,$$

the Hausdorff-Young-Riesz inequality reads as follows (see [11, Ch. I], [75]):

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \lesssim \|f\|_{L_{p,q}(\mathbb{R})}, \quad 1 < p < 2, \quad 1 \le q \le \infty. \tag{4.1}$$

This inequality is a partial case of weighted Fourier inequalities, which has a long history (see [7], [19], [23], [44], [46], [47], [63]). In particular, for the weighted Lorentz spaces such problems were studied in [8, 19, 83]. In [69], Nursultanov and Tikhonov proved the following estimates:

$$\|Hf\|_{L_{p,q}(\mathbb{R})} \lesssim \|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \lesssim \|f\|_{L_{p,q}(\mathbb{R})} \quad 1 < p < 2, \quad 0 < q \le \infty, \tag{4.2}$$

where $Hf(x) = \frac{1}{|x|} \int\limits_{-|x|}^{|x|} f(t)dt$. Note that for a function f satisfying the condition

$$|f(x)| \le C|Hf(x)|,$$

inequality (4.2) implies the relation

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \asymp \|f\|_{L_{p,q}(\mathbb{R})}, \quad 1 < p < 2, \quad 0 < q \le \infty.$$

Recall that, for even monotone on $\mathbb{R}_+$ functions, Sagher obtained the following equivalences:

$$\|f\|_{L_{p,q}(\mathbb{R})} \asymp \|\widehat{f}\|_{L_{p',q}(\mathbb{R})}, \quad 1 < p < \infty, \quad 0 < q \le \infty, \tag{4.3}$$

$$\|f\|_{L^q_{w[p,q]}(\mathbb{R})} \asymp \|\widehat{f}\|_{L^q_{w[p',q]}(\mathbb{R})}, \quad 1 < p < \infty, \quad 1 \le q \le \infty, \tag{4.4}$$

$$\|\widehat{f}\|_{L^q_{w[p',q]}(\mathbb{R})} \asymp \|\widehat{f}\|_{L_{p',q}(\mathbb{R})}, \quad 1 < p < \infty, \ \ 1 \le q \le \infty, \tag{4.5}$$

see Theorem 1.4. Note that the equivalence (4.4) proves Boas' conjecture stated in [15].

In [57], Liflyand and Tikhonov proved Boas conjecture in the setting of the weighted Lebesgue spaces for general monotone functions. Let us denote

$$GMF = \left\{ h \in BV_{loc}(\mathbb{R}_+) : \int_x^{2x} |dh(x)| \le C \int_{x/\gamma}^{\gamma x} \frac{|h(u)|}{u}\, du \quad C > 0, \gamma > 1 \right\},$$

where $BV_{loc}(\mathbb{R}_+)$ is a set of locally of bounded variation on $\mathbb{R}_+$ functions vanishing at infinity.

Theorem 4.1 ([57, Corollary 1])**.** *Let f be a non-negative function on $\mathbb{R}_+$ such that $f \in GMF$. Then*

$$\|f\|_{L^q_{w[p,q]}(\mathbb{R}_+)} \asymp \|\widehat{f}\|_{L^q_{w[p',q]}(\mathbb{R}_+)}, \quad 1 < p < \infty, \quad 1 \le q < \infty. \tag{4.6}$$

Later on, this result was generalized in [28, 50]. Moreover, Gorbachev, Liflyand and Tikhonov [40] obtained the multidimensional version of (4.4) for radial functions.

4.2 Anisotropic weighted Lebesgue and Lorentz spaces

First we define multidimensional analogues of the Lorentz and weighted Lebesgue spaces. We will need the following notion. Let $f(x_1, \ldots, x_n)$ be a measurable function on $\mathbb{R}^n$. By $f^{*_1}(t_1, x_2, \ldots, x_n)$ we denote the rearrangement of $f(x_1, x_2, \ldots, x_n)$ with respect to x_1, i.e., $f^{*_1}(t_1, x_2, \ldots, x_n)$ is a non-increasing function on t_1 and the functions $f^{*_1}(t_1, \ldots, x_n)$ and $|f(x_1, \ldots, x_n)|$ are equimeasurable as functions of one variable for almost all $x_2, \ldots, x_n$. By rearranging $f^{*_1}(t_1, x_2, \ldots, x_n)$ with respect to other variables we obtain the function $f^{*_1 *_2 \cdots *_n}(t_1, t_2, \ldots, t_n)$ non-increasing in each variable and equimeasurable with f.

Throughout this chapter, by bold letters we denote vectors. And all operations on vectors are performed componentwisely. Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be n-dimensional vectors such that if $0 < q_i < \infty$, then $0 < p_i < \infty$, and if $q_i = \infty$, then

$0 < p_i \le \infty$. We define the functionals $\Phi_{\mathbf{p},\mathbf{q}}$ and $\Phi^+_{\mathbf{p},\mathbf{q}}$ by

$$\Phi_{\mathbf{p},\mathbf{q}}(\varphi) = \left(\int_{-\infty}^{\infty} \dots \left(\int_{-\infty}^{\infty} \left(|t_1|^{\frac{1}{p_1}} \dots |t_n|^{\frac{1}{p_n}} |\varphi(t_1, \dots, t_n)| \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}}$$

and

$$\Phi^+_{\mathbf{p},\mathbf{q}}(\varphi) = \left(\int_0^{\infty} \dots \left(\int_0^{\infty} \left| t_1^{\frac{1}{p_1}} \dots t_n^{\frac{1}{p_n}} \varphi(t_1, \dots, t_n) \right|^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}}.$$

In the case $q = \infty$, the expressions $\left(\int\limits_0^{\infty} (F(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$ and $\left(\int\limits_{-\infty}^{\infty} (F(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$ are considered as $\sup\limits_{t>0} F(t)$ and $\sup\limits_{t\in\mathbb{R}} F(t)$, respectively.

Definition 4.1. *The anisotropic Lorentz space ([6], [13], [65], [66])* $L_{\mathbf{p},\mathbf{q}}(\mathbb{R}^n)$ *is the set of measurable functions* f*, for which*

$$\|f\|_{L_{\mathbf{p},\mathbf{q}}} := \Phi^+_{\mathbf{p},\mathbf{q}}(f^{*_1 *_2 \dots *_n}) < \infty.$$

Definition 4.2. *The anisotropic weighted Lebesgue space* $L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}(\mathbb{R}^n)$ *is the set of measurable functions* f*, for which*

$$\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}} := \Phi_{\mathbf{p},\mathbf{q}}(f) < \infty.$$

Here, $\mathbf{w}[\mathbf{p},\mathbf{q}](t_1, \dots, t_n)$ stands for the weight function

$$\mathbf{w}[\mathbf{p},\mathbf{q}](t_1, \dots, t_n) = |t_1|^{\frac{1}{p_1} - \frac{1}{q_1}} \dots |t_n|^{\frac{1}{p_n} - \frac{1}{q_n}}.$$

Note that some interpolation properties of the spaces $L_{\mathbf{p},\mathbf{q}}(\mathbb{R}^n)$ were considered in [65].

4.3 The main results

Definition 4.3. *We say that a function* $f(x_1, x_2, \dots, x_n)$ *belongs to the class* E^n *if*

1. f *is non-negative on* $\mathbb{R}^n$*;*
2. $f(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n) = f(x_1, x_2, \dots, x_n)$ *for all* $\mathbf{x} = (x_1, x_2, \dots, x_n)$ *and* $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$*, where* $\varepsilon_i \in \{1, -1\}$*,* $i = 1, 2, \dots, n$*;*
3. $f(x_1, x_2, \dots, x_n)$ *is decreasing in each variable on* $\mathbb{R}_+$*, that is*

$$f(x_1, \dots, x_{i-1}, x_i^1, \dots, x_n) \le f(x_1, \dots, x_{i-1}, x_i^2, \dots, x_n)$$

for $0 \le x_i^2 \le x_i^1$, $1 \le i \le n$;

4. $f(x_1, x_2, \dots, x_n) \to 0$ *as* $|x_1| + |x_2| + \dots + |x_n| \to \infty$.

The main results of this chapter are the following Boas-Sagher-type theorems for the Fourier transform

$$\widehat{f}(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-\mathbf{ixy}} d\mathbf{x}, \quad n \ge 1.$$

Theorem 4.2. *Let* $\mathbf{1} < \mathbf{p} < \infty$, $\mathbf{0} < \mathbf{q} \le \infty$, *and* $f \in E^n$. *Then*

$$\|f\|_{L_{\mathbf{p},\mathbf{q}}} \asymp \|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}}.$$

Theorem 4.3. *Let* $\mathbf{1} < \mathbf{p} < \infty$, $\mathbf{1} \le \mathbf{q} \le \infty$, *and* $f \in E^n$. *Then*

$$\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}} \asymp \|\widehat{f}\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p}',\mathbf{q}]}}.$$

Corollary 4.1. *Let* $\mathbf{1} < \mathbf{p} < \infty$, $\mathbf{1} \le \mathbf{q} \le \infty$, *and* $f \in E^n$. *Then*

$$\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} \asymp \|\widehat{f}\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p}',\mathbf{q}]}}.$$

Remark 4.1. *For convenience, we prove Theorems 4.2 and 4.3 in the case* $n = 2$. *In the general case the arguments are similar. For functions* f *in* E^2, *we have*

$$\widehat{f}(y_1, y_2) = 4\widehat{f}_c(y_1, y_2) = 4\int_0^\infty \int_0^\infty f(x_1, x_2) \cos x_1 y_1 \cos x_2 y_2 \, dx_1 \, dx_2.$$

4.4 Auxiliary results

The following Hardy's inequality [9, p. 124] and Minkowski's inequality [56, p. 47] are often needed.

Lemma 4.1 (Hardy). *Let* ψ *be a non-negative measurable function on* $(0, \infty)$ *and suppose* $-\infty < \lambda < 1$, $1 \le q \le \infty$. *Then*

$$\left(\int_0^\infty \left[t^\lambda \frac{1}{t} \int_0^t \psi(s) ds \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left[t^\lambda \psi(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}. \tag{4.7}$$

Lemma 4.2 (Minkowski). *Let* (X, μ) *and* (Y, ν) *be measurable spaces. Let* $1 \le p \le \infty$, *and* $f(x, y)$ *be a measurable function on* $(X, \mu) \times (Y, \nu)$. *Then*

$$\left(\int_Y \left(\int_X |f(x, y)| d\mu(x) \right)^p d\nu(y) \right)^{\frac{1}{p}} \le \int_X \left(\int_Y |f(x, y)|^p d\nu(y) \right)^{\frac{1}{p}} d\mu(x). \tag{4.8}$$

Remark 4.2. *Note that for* $0 < q < 1$ *inequality* (4.7) *holds for monotone functions on* $\mathbb{R}_+$. *Moreover this inequality holds for quasi-monotone functions (see [10], [96]).*

We will need the following Hardy lemma for decreasing rearrangements [9, p. 44].

Lemma 4.3. *Let* μ *be Lebesgue measure on* $\mathbb{R}$, *and* f, g *be* μ*-measurable functions on* $\mathbb{R}$. *Then*

$$\int_{-\infty}^{\infty} |f(x)g(x)|\,dx \le \int_0^{\infty} f^*(u)g^*(u)\,du.$$

Also we will use the following [10, Lemma 2.1].

Lemma 4.4. *Let* f *be a non-negative non-increasing function on* $(0,\infty)$, *and let* $A > 0$, $0 < q < 1$. *Then the following inequality holds*

$$\left(\int_0^A f(x)dx\right)^q \le C\int_0^A (f(x))^q x^{q-1}dx. \tag{4.9}$$

Lemma 4.5. *Suppose* $f \in E^2$. *Then*

$$\widehat{f}(y_1,y_2) \le 36\int_0^{\frac{1}{|y_2|}}\int_0^{\frac{1}{|y_1|}} f(x_1,x_2)\,dx_1\,dx_2 \tag{4.10}$$

for all $(y_1,y_2) \in \mathbb{R}^2$.

Proof. From condition (2) of Definition 4.3 for E^2 it suffices to prove inequality (4.10) for $\mathbf{y} > \mathbf{0}$. Let $\mathbf{y} = (y_1,y_2)$, $y_i > 0$, $i = 1,2$. We have

$$\begin{aligned}
\widehat{f}(y_1,y_2) &= 4\int_0^{+\infty}\int_0^{+\infty} f(x_1,x_2)\cos x_1y_1\cos x_2y_2\,dx_1\,dx_2\\
&= 4\int_0^{\frac{1}{y_2}}\int_0^{\frac{1}{y_1}} f(x_1,x_2)\cos x_1y_1\cos x_2y_2\,dx_1\,dx_2\\
&+ 4\int_{\frac{1}{y_2}}^{+\infty}\int_0^{\frac{1}{y_1}} f(x_1,x_2)\cos x_1y_1\cos x_2y_2\,dx_1\,dx_2\\
&+ 4\int_0^{\frac{1}{y_2}}\int_{\frac{1}{y_1}}^{+\infty} f(x_1,x_2)\cos x_1y_1\cos x_2y_2\,dx_1\,dx_2\\
&+ 4\int_{\frac{1}{y_2}}^{+\infty}\int_{\frac{1}{y_1}}^{+\infty} f(x_1,x_2)\cos x_1y_1\cos x_2y_2\,dx_1\,dx_2\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Applying the second mean value theorem with respect to the second variable in I_2, we get

$$
\begin{aligned}
I_2 &= 4\int_0^{\frac{1}{y_1}} \left[\int_{\frac{1}{y_2}}^{+\infty} f(x_1,x_2)\cos x_2y_2\,dx_2\right]\cos x_1y_1\,dx_1\\
&= 4\int_0^{\frac{1}{y_1}} \left[f\left(x_1,\frac{1}{y_2}\right)\int_{\frac{1}{y_2}}^{\xi}\cos x_2y_2\,dx_2\right]\cos x_1y_1\,dx_1\\
&= 4\int_0^{\frac{1}{y_1}} \left[f\left(x_1,\frac{1}{y_2}\right)\frac{\sin y_2\xi-\sin 1}{y_2}\right]\cos x_1y_1\,dx_1.
\end{aligned}
$$

Hence,

$$
|I_2| \le \frac{8}{y_2}\int_0^{\frac{1}{y_1}} f\left(x_1,\frac{1}{y_2}\right)dx_1 \le 8\int_0^{\frac{1}{y_2}}\int_0^{\frac{1}{y_1}} f(x_1,x_2)\,dx_1\,dx_2.
$$

In the same way, we get

$$
|I_3| \le 8\int_0^{\frac{1}{y_2}}\int_0^{\frac{1}{y_1}} f(x_1,x_2)\,dx_1\,dx_2.
$$

Using twice the second value mean theorem for I_4, we have

$$
\begin{aligned}
I_4 &= 4\int_{\frac{1}{y_2}}^{+\infty} \left[f\left(\frac{1}{y_1},x_2\right)\int_{\frac{1}{y_1}}^{\zeta}\cos x_1y_1\,dx_1\right]\cos x_2y_2\,dx_2\\
&= 4\int_{\frac{1}{y_2}}^{+\infty} \left[f\left(\frac{1}{y_1},x_2\right)\frac{\sin y_1\zeta-\sin 1}{y_1}\right]\cos x_2y_2\,dx_2\\
&= 4f\left(\frac{1}{y_1},\frac{1}{y_2}\right)\frac{\sin y_1\zeta-\sin 1}{y_1}\,\frac{\sin y_2\alpha-\sin 1}{y_2}.
\end{aligned}
$$

Hence,

$$
|I_4| \le \frac{16}{y_1y_2}f\left(\frac{1}{y_1},\frac{1}{y_2}\right) \le 16\int_0^{\frac{1}{y_2}}\int_0^{\frac{1}{y_1}} f(x_1,x_2)\,dx_1\,dx_2.
$$

Therefore, we obtain

$$
\begin{aligned}
|\widehat{f}(y_1,y_2)| &\le |I_1|+|I_2|+|I_3|+|I_4|\\
&\le 36\int_0^{\frac{1}{y_2}}\int_0^{\frac{1}{y_1}} f(x_1,x_2)\,dx_1\,dx_2.
\end{aligned}
$$

□

Corollary 4.2. *Let $f \in E^2$. Then*

$$
\widehat{f}^{*_1,*_2}(t_1,t_2) \le C\int_0^{\frac{1}{t_2}}\int_0^{\frac{1}{t_1}} f(x_1,x_2)\,dx_1\,dx_2
$$

for all $(t_1,t_2) \in \mathbb{R}^2_+$.

4.5 Applications of interpolation results for Lorentz spaces to Fourier inequalities

Let $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$. According to [13], we say that a measurable function f belongs to the mixed norm Lorentz space $L_{p_2,q_2}[L_{p_1,q_1}](\mathbb{R}^2)$ if

$$\|f\|_{L_{p_2,q_2}[L_{p_1,q_1}]} := \left(\int_0^\infty t_2^{\frac{q_2}{p_2}-1} \left[\left(\int_0^\infty t_1^{\frac{q_1}{p_1}-1} \left(f^{*_1}(t_1,\cdot)\right)^{q_1} dt_1 \right)^{*_2} (t_2) \right]^{\frac{q_2}{q_1}} dt_2 \right)^{\frac{1}{q_2}} < \infty.$$

Now we need a special case of Corollary 2 in [65, p. 258]. For the reader's convenience, we state it in the following form.

Theorem 4.4. *Let $\mathbf{0} < \mathbf{p}^i, \mathbf{r}^i < \boldsymbol{\infty}$ be two-dimensional vectors, $i = 0, 1$. And let $p_j^0 \neq p_j^1$, $r_j^0 \neq r_j^1$, $j = 1, 2$. If T is a linear operator such that*

$$T : L_{p_2^0,1}[L_{p_1^0,1}] \to L_{(r_1^0,r_2^0)(\infty,\infty)},$$

$$T : L_{p_2^1,1}[L_{p_1^0,1}] \to L_{(r_1^0,r_2^1)(\infty,\infty)},$$

$$T : L_{p_2^0,1}[L_{p_1^1,1}] \to L_{(r_1^1,r_2^0)(\infty,\infty)},$$

$$T : L_{p_2^1,1}[L_{p_1^1,1}] \to L_{(r_1^1,r_2^1)(\infty,\infty)},$$

then

$$T : L_{\mathbf{p},\mathbf{q}} \to L_{\mathbf{r},\mathbf{q}},$$

where

$$\frac{\mathbf{1}}{\mathbf{p}} = \frac{\mathbf{1}-\boldsymbol{\theta}}{\mathbf{p}^0} + \frac{\boldsymbol{\theta}}{\mathbf{p}^1}, \quad \frac{\mathbf{1}}{\mathbf{r}} = \frac{\mathbf{1}-\boldsymbol{\theta}}{\mathbf{r}^0} + \frac{\boldsymbol{\theta}}{\mathbf{r}^1},$$

and $\mathbf{0} < \mathbf{q} \leq \boldsymbol{\infty}$, $\mathbf{0} < \boldsymbol{\theta} < \mathbf{1}$.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^2)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$.

Let $f \in L_{\mathbf{pq}}$ and $\varphi \in \mathcal{S}$. Define

$$T_\varphi(f)(\mathbf{t}) = (f, \varphi_{\mathbf{t}}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) \varphi_{\mathbf{t}}(x_1, x_2)\, dx_1\, dx_2,$$

where $\mathbf{t} = (t_1, t_2)$ and $\varphi_{\mathbf{t}}(u_1, u_2) = \frac{1}{t_1} \frac{1}{t_2} \varphi\left(\frac{u_1}{t_1}, \frac{u_2}{t_2}\right)$. Note that

$$\widehat{\varphi}_{\mathbf{t}}(u_1, u_2) = sign\, t_1\, sign\, t_2\, \widehat{\varphi}(t_1 u_1, t_2 u_2).$$

Put $\varphi(x_1, x_2) = e^{-\frac{x_1^2+x_2^2}{2}}$. Note that $\varphi = \widehat{\varphi}$, $\varphi \in \mathcal{S} \cap E^2$, and $\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$, where $\varphi_i(x_i) = e^{-\frac{x_i^2}{2}}$, $i = 1, 2$.

Theorem 4.5. *Let* $\mathbf{1} < \mathbf{p} < \boldsymbol{\infty}$, $\mathbf{0} < \mathbf{q} \leq \boldsymbol{\infty}$. *Let also* $\varphi(x_1, x_2) = e^{-\frac{x_1^2+x_2^2}{2}}$. *Then*

$$\|T_\varphi \widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} \leq C(p,\varphi)\|f\|_{L_{\mathbf{p},\mathbf{q}}}.$$

Proof. Let $\mathbf{1} < \mathbf{p} < \boldsymbol{\infty}$ and suppose $f \in L_{p_2,1}[L_{p_1,1}] \cap \mathcal{S}$. Using Lemma 4.3 twice, we obtain

$$\begin{aligned}
&|T_\varphi \widehat{f}(\mathbf{t})| = |(\widehat{f}, \varphi_{\mathbf{t}})| = |(f, \widehat{\varphi}_{\mathbf{t}})| \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_1,x_2)\widehat{\varphi}_{\mathbf{t}}(x_1,x_2)|\,dx_1\,dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_1,x_2)\widehat{\varphi}_{1t_1}(x_1)\widehat{\varphi}_{2t_2}(x_2)|\,dx_1\,dx_2 \\
&\leq \int_{-\infty}^{\infty}\int_{0}^{\infty} f^{*_1}(u_1,x_2)\widehat{\varphi}^{*}_{1t_1}(u_1)\widehat{\varphi}_{2t_2}(x_2)\,du_1\,dx_2 \\
&\leq \int_{0}^{\infty}\left(\int_{0}^{\infty} f^{*_1}(u_1,\cdot)\widehat{\varphi}^{*}_{1t_1}(u_1)\,du_1\right)^{*_2}(u_2)\widehat{\varphi}^{*}_{2t_2}(u_2)\,du_2 \\
&= \int_{0}^{\infty}\left(\int_{0}^{\infty} f^{*_1}(u_1,\cdot)\widehat{\varphi}_{1t_1}\left(\frac{u_1}{2}\right)du_1\right)^{*_2}(u_2)\widehat{\varphi}_{2t_2}\left(\frac{u_2}{2}\right)du_2 \\
&= \int_{0}^{\infty}\left(\int_{0}^{\infty} f^{*_1}(u_1,\cdot)u_1^{-\frac{1}{p_1'}}u_1^{\frac{1}{p_1'}}\widehat{\varphi}_1\left(\frac{t_1u_1}{2}\right)du_1\right)^{*_2}(u_2)\;u_2^{-\frac{1}{p_2'}}u_2^{\frac{1}{p_2'}}\widehat{\varphi}_2\left(\frac{t_2u_2}{2}\right)du_2 \\
&\leq \sup_{u_1\geq 0} u_1^{\frac{1}{p_1'}}\widehat{\varphi}_1\left(\frac{t_1u_1}{2}\right)\sup_{u_1\geq 0} u_2^{\frac{1}{p_2'}}\widehat{\varphi}_2\left(\frac{t_2u_2}{2}\right) \\
&\times \int_{0}^{\infty}\left(\int_{0}^{\infty} f^{*_1}(u_1,\cdot)u_1^{\frac{1}{p_1}-1}\,du_1\right)^{*_2}(u_2)u_2^{\frac{1}{p_2}-1}\,du_2 \\
&\leq Cu_1^{\frac{1}{p_1'}}u_2^{\frac{1}{p_2'}}(|t_1u_1|)^{-\frac{1}{p_1'}}(|t_2u_2|)^{-\frac{1}{p_2'}}\|f\|_{L_{p_2,1}[L_{p_1,1}]} \leq C\|f\|_{L_{p_2,1}[L_{p_1,1}]}|t_1|^{-\frac{1}{p_1'}}|t_2|^{-\frac{1}{p_2'}}.
\end{aligned}$$

Hence,

$$\|T_\varphi \widehat{f}\|_{L_{\mathbf{p}',\boldsymbol{\infty}}} \leq C\|f\|_{L_{p_2,1}[L_{p_1,1}]}$$

for all $f \in L_{p_2,1}[L_{p_1,1}] \cap \mathcal{S}$. By density of $\mathcal{S}$ in $L_{p_2,1}[L_{p_1,1}]$, we derive

$$\|T_\varphi \widehat{f}\|_{L_{\mathbf{p}',\boldsymbol{\infty}}} \leq C\|f\|_{L_{p_2,1}[L_{p_1,1}]}$$

for all $f \in L_{p_2,1}[L_{p_1,1}]$, that is, the operator $T_\varphi \circ \mathcal{F}$ is bounded from $L_{p_2,1}[L_{p_1,1}]$ to $L_{\mathbf{p}',\boldsymbol{\infty}}$.

Now let $\mathbf{1} < \mathbf{p} < \boldsymbol{\infty}$ and $\mathbf{0} < \mathbf{q} \leq \boldsymbol{\infty}$, then we can define vectors $\mathbf{1} < \mathbf{p}^0 < \mathbf{p} < \mathbf{p}^1 < \boldsymbol{\infty}$, $\mathbf{0} < \boldsymbol{\theta} < \mathbf{1}$ such that $\frac{\mathbf{1}}{\mathbf{p}} = \frac{\mathbf{1}-\boldsymbol{\theta}}{\mathbf{p^0}} + \frac{\boldsymbol{\theta}}{\mathbf{p^1}}$. From the above we get

$$T_\varphi \circ \mathcal{F} : L_{p_2^0,1}[L_{p_1^0,1}] \to L_{((p_1^0)',(p_2^0)')(\infty,\infty)},$$

$$T_\varphi \circ \mathcal{F} : L_{p_2^1,1}[L_{p_1^0,1}] \to L_{((p_1^0)',(p_2^1)')(\infty,\infty)},$$

$$T_\varphi \circ \mathcal{F} : L_{p_2^0,1}[L_{p_1^1,1}] \to L_{((p_1^1)',(p_2^0)')(\infty,\infty)},$$

$$T_\varphi \circ \mathcal{F} : L_{p_2^1,1}[L_{p_1^1,1}] \to L_{((p_1^1)',(p_2^1)')(\infty,\infty)}.$$

By Theorem 4.4, we get

$$T_\varphi \circ \mathcal{F} : L_{\mathbf{p},\mathbf{q}} \to L_{\mathbf{p}',\mathbf{q}}.$$

□

4.6 Proofs of main results

4.6.1 Proof of Theorem 4.2

We begin with the upper estimate of $\|f\|_{L_{\mathbf{p},\mathbf{q}}}$. Let $(t_1, t_2) \in \mathbb{R}^2$, and φ be as above. Then by non-negativity and monotonicity of f, we obtain

$$\begin{aligned}
sign\, t_1\, sign\, t_2 (f, \varphi_t) &= \frac{1}{|t_1|}\frac{1}{|t_2|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi\left(\frac{u_1}{t_1}, \frac{u_2}{t_2}\right) f(u_1, u_2)\, du_1\, du_2 \\
&\geq \frac{1}{|t_1|}\frac{1}{|t_2|}\int_{-t_2}^{t_2}\int_{-t_1}^{t_1} \varphi\left(\frac{u_1}{t_1}, \frac{u_2}{t_2}\right) f(u_1, u_2)\, du_1\, du_2 \\
&\geq f(t_1, t_2)\frac{1}{|t_1|}\frac{1}{|t_2|}\int_{-t_2}^{t_2}\int_{-t_1}^{t_1} \varphi\left(\frac{u_1}{t_1}, \frac{u_2}{t_2}\right) du_1\, du_2 \\
&= f(t_1, t_2)\|\varphi\|_{L_{\mathbf{1}}[-1,1]^2},
\end{aligned}$$

where

$$\|\varphi\|_{L_{\mathbf{1}}[-1,1]^2} = \int_{-1}^{1}\int_{-1}^{1} |\varphi(u_1, u_2)|\, du_1\, du_2$$

Therefore,

$$0 \leq f(t_1, t_2) \leq \frac{1}{\|\varphi\|_{L_{\mathbf{1}}}} T_\varphi f(t_1, t_2).$$

Hence, by Theorem 4.5 and the lattice property of anisotropic Lorentz spaces, we obtain

$$\|f\|_{L_{\mathbf{p},\mathbf{q}}} \leq \frac{1}{\|\varphi\|_{L_{\mathbf{1}}}} \|T_\varphi f\|_{L_{\mathbf{p},\mathbf{q}}} \leq C_{\mathbf{p},\varphi}\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}}.$$

Now we derive the lower estimate for f. Let $t_1, t_2 > 0$, then by Corollary 4.2 we have

$$\widehat{f}^{*_1,*_2}(t_1, t_2) \leq C\int_0^{\frac{1}{t_2}}\int_0^{\frac{1}{t_1}} f(x_1, x_2)\, dx_1\, dx_2.$$

Hence,

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\leq C\left(\int_0^\infty\left(\int_0^\infty\left[t_1^{\frac{1}{p_1'}} t_2^{\frac{1}{p_2'}}\int_0^{\frac{1}{t_2}}\int_0^{\frac{1}{t_1}} f(x_1, x_2)\, dx_1\, dx_2\right]^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&= C\left(\int_0^\infty\left(\int_0^\infty\left[t_1^{\frac{1}{p_1'}}\int_0^{\frac{1}{t_1}} \varphi(x_1, t_2)\, dx_1\right]^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}},
\end{aligned}$$

where

$$\varphi(x_1,t_2) = t_2^{\frac{1}{p_2'}} \int_0^{\frac{1}{t_2}} f(x_1,x_2)\,dx_2.$$

Substituting $\frac{1}{t_1}$ for z_1 and applying Hardy's inequality (4.7) to the inner integral we get

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\le C\left(\int_0^\infty\left(\int_0^\infty\left[t_1^{\frac{1}{p_1'}}\int_0^{\frac{1}{t_1}}\varphi(x_1,t_2)\,dx_1\right]^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}\\
&= C\left(\int_0^\infty\left(\int_0^\infty\left[z_1^{\frac{1}{p_1}-1}\int_0^{z_1}\varphi(x_1,t_2)\,dx_1\right]^{q_1}\frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}\\
&\le C\left(\int_0^\infty\left(\int_0^\infty\left[z_1^{\frac{1}{p_1}}\varphi(z_1,t_2)\right]^{q_1}\frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}\\
&= C\left(\int_0^\infty\left(\int_0^\infty\left[z_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2'}}\int_0^{\frac{1}{t_2}}f(z_1,x_2)\,dx_2\right]^{q_1}\frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}.
\end{aligned}$$

Let $q_1 \ge 1$. Then, by Minkowski's inequality (4.8) we have

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\le C\left(\int_0^\infty\left(\int_0^\infty\left[z_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2'}}\int_0^{\frac{1}{t_2}}f(z_1,x_2)\,dx_2\right]^{q_1}\frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}\\
&\le C\left(\int_0^\infty\left(t_2^{\frac{1}{p_2'}}\int_0^{\frac{1}{t_2}}\left[\int_0^\infty\left(z_1^{\frac{1}{p_1}-\frac{1}{q_1}}f(z_1,x_2)\right)^{q_1}dz_1\right]^{\frac{1}{q_1}}dx_2\right)^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}\\
&= C\left(\int_0^\infty\left(t_2^{\frac{1}{p_2'}}\int_0^{\frac{1}{t_2}}\psi(x_2)\,dx_2\right)^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}},
\end{aligned}$$

where

$$\psi(x_2) = \left[\int_0^\infty\left(z_1^{\frac{1}{p_1}-\frac{1}{q_1}}f(z_1,x_2)\right)^{q_1}dz_1\right]^{\frac{1}{q_1}}.$$

Again by changing variables $z_2 = \frac{1}{t_2}$ and by Hardy's inequality, we get

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\le C\left(\int_0^\infty\left(z_2^{\frac{1}{p_2}-1}\int_0^{z_2}\psi(x_2)dx_2\right)^{q_2}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}}\\
&\le C\left(\int_0^\infty\left(z_2^{\frac{1}{p_2}}\psi(z_2)\right)^{q_2}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}}\\
&= C\left(\int_0^\infty\left(z_2^{\frac{1}{p_2}}\left[\int_0^\infty\left(z_1^{\frac{1}{p_1}-\frac{1}{q_1}}f(z_1,z_2)\right)^{q_1}dz_1\right]^{\frac{1}{q_1}}\right)^{q_2}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}}\\
&= C\|f\|_{L_{\mathbf{p},\mathbf{q}}}.
\end{aligned}$$

Now, let $0 < q_1 < 1$. Then by inequality (4.9) we arrive at

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\leq C\left(\int_0^\infty \left(\int_0^\infty \left[z_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2'}} \int_0^{\frac{1}{t_2}} f(z_1,x_2)dx_2\right]^{q_1} \frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&\leq C\left(\int_0^\infty \left(t_2^{\frac{q_1}{p_2'}} \int_0^\infty z_1^{\frac{q_1}{p_1}-1}\left[\int_0^{\frac{1}{t_2}} (f(z_1,x_2))^{q_1} x_2^{q_1-1} dx_2\right] dz_1\right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&= C\left(\int_0^\infty \left(t_2^{\frac{q_1}{p_2'}} \int_0^{\frac{1}{t_2}} \int_0^\infty z_1^{\frac{q_1}{p_1}-1} (f(z_1,x_2))^{q_1} x_2^{q_1-1}\, dz_1\, dx_2\right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}.
\end{aligned}$$

We substitute $\frac{1}{t_2}$ for z_2

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\leq C\left(\int_0^\infty \left(z_2^{q_1\left(\frac{1}{p_2}-1\right)} \int_0^{z_2}\int_0^\infty z_1^{\frac{q_1}{p_1}-1} f^{q_1}(z_1,x_2) x_2^{q_1-1}\, dz_1\, dx_2\right)^{\frac{q_2}{q_1}} \frac{dz_2}{z_2}\right)^{\frac{1}{q_2}} \\
&= C\left(\int_0^\infty \left(z_2^{q_1\left(\frac{1}{p_2}-1\right)+1} \frac{1}{z_2}\int_0^{z_2} \xi(x_2)\, dx_2\right)^{\frac{q_2}{q_1}} \frac{dz_2}{z_2}\right)^{\frac{q_1}{q_2}\frac{1}{q_1}},
\end{aligned}$$

where

$$\xi(x_2) = \int_0^\infty z_1^{\frac{q_1}{p_1}-1} (f(z_1,x_2))^{q_1} x_2^{q_1-1}\, dz_1$$

is a quasi-monotone function, (since $\xi(x_2)x_2^{-(q_1-1)}$ is a non-increasing function). Applying Hardy's inequality given by Lemma 4.1, we obtain

$$\begin{aligned}
\|\widehat{f}\|_{L_{\mathbf{p}',\mathbf{q}}} &\leq \left(\int_0^\infty \left(z_2^{q_1\left(\frac{1}{p_2}-1\right)+1} \frac{1}{z_2}\int_0^{z_2} \xi(x_2)dx_2\right)^{\frac{q_2}{q_1}} \frac{dz_2}{z_2}\right)^{\frac{q_1}{q_2}\frac{1}{q_1}} \\
&\leq C\left(\int_0^\infty \left(z_2^{q_1\left(\frac{1}{p_2}-1\right)+1} \xi(z_2)\right)^{\frac{q_2}{q_1}} \frac{dz_2}{z_2}\right)^{\frac{q_1}{q_2}\frac{1}{q_1}} \\
&= C\left(\int_0^\infty \left(z_2^{q_1\left(\frac{1}{p_2}-1\right)+1} \int_0^\infty z_1^{\frac{q_1}{p_1}-1} (f(z_1,z_2))^{q_1} z_2^{q_1-1} dz_1\right)^{\frac{q_2}{q_1}} \frac{dz_2}{z_2}\right)^{\frac{q_1}{q_2}\frac{1}{q_1}} \\
&= C\|f\|_{L_{\mathbf{p},\mathbf{q}}}.
\end{aligned}$$

4.6.2 Proof of Theorem 4.3

Proof. The inequality

$$\|\widehat{f}\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p}',\mathbf{q}]}} \leq C\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}}$$

is proved in the same way as in the Theorem 4.2 by using Lemma 4.5. We will prove the reverse inequality. Let $\mathbf{z} = (z_1, z_2) > \mathbf{0}$, then

$$\begin{aligned}
H(\mathbf{z}) &:= \int_0^{z_2}\int_0^{z_1}\int_0^{u_2}\int_0^{u_1} \widehat{f}(x_1,x_2)\,dx_1\,dx_2\,du_1\,du_2 \\
&= \int_0^{z_2}\int_0^{z_1}\int_0^{u_2}\int_0^{u_1}\int_0^{\infty}\int_0^{\infty} f(t_1,t_2)\cos x_1 t_1 \cos x_2 t_2\,dt_1 dt_2\,dx_1 dx_2\,du_1 du_2 \\
&= \int_0^{z_2}\int_0^{z_1}\int_0^{\infty}\int_0^{\infty} f(t_1,t_2)\int_0^{u_2}\int_0^{u_1} \cos x_1 t_1 \cos x_2 t_2\,dx_1 dx_2\,dt_1 dt_2\,du_1 du_2 \\
&= \int_0^{z_2}\int_0^{z_1}\int_0^{\infty}\int_0^{\infty} \frac{f(t_1,t_2)}{t_1 t_2}\sin t_1 u_1 \sin t_2 u_2\,dt_1\,dt_2\,du_1\,du_2 \\
&= \int_0^{\infty}\int_0^{\infty} \frac{f(t_1,t_2)}{t_1 t_2}\int_0^{z_2}\int_0^{z_1}\sin t_1 u_1 \sin t_2 u_2\,du_1\,du_2\,dt_1\,dt_2 \\
&= 2\int_0^{\infty}\int_0^{\infty} \frac{f(t_1,t_2)}{t_1^2 t_2^2}\sin^2\frac{t_1 z_1}{2}\sin^2\frac{t_2 z_2}{2}\,dt_1\,dt_2 \geq 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
H\left(\frac{\pi}{2x_1},\frac{\pi}{2x_2}\right) &= \int_0^{\frac{\pi}{2x_2}}\int_0^{\frac{\pi}{2x_1}}\int_0^{u_2}\int_0^{u_1}\left|\widehat{f}(y_1,y_2)\right|dy_1\,dy_2\,du_1\,du_2 \\
&\geq \left|\int_0^{\frac{\pi}{2x_2}}\int_0^{\frac{\pi}{2x_1}}\int_0^{u_2}\int_0^{u_1}\widehat{f}(y_1,y_2)\,dy_1\,dy_2\,du_1\,du_2\right| \\
&= 2\int_0^{\infty}\int_0^{\infty}\frac{f(t_1,t_2)}{t_1^2 t_2^2}\sin^2\frac{\pi t_1}{4x_1}\sin^2\frac{\pi t_2}{4x_2}\,dt_1\,dt_2 \\
&\geq C\int_{\frac{x_2}{2}}^{2x_2}\int_{\frac{x_1}{2}}^{2x_1}\frac{f(t_1,t_2)}{t_1^2 t_2^2}\,dt_1\,dt_2 \geq C\frac{f(2x_1,2x_2)}{x_1 x_2}.
\end{aligned} \tag{4.11}$$

Denote $h(u_1,u_2) = \int_0^{u_2}\int_0^{u_1}\left|\widehat{f}(y_1,y_2)\right|dy_1\,dy_2$. Then (4.11) implies

$$\begin{aligned}
\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}} &= \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[|t_1|^{\frac{1}{p_1}}|t_2|^{\frac{1}{p_2}}f(t_1,t_2)\right]^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&= 2^{\frac{1}{q_1}+\frac{1}{q_2}}\left(\int_0^{\infty}\left(\int_0^{\infty}\left[t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}f(t_1,t_2)\right]^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&\leq C\left(\int_0^{\infty}\left(\int_0^{\infty}\left[t_1^{\frac{1}{p_1}+1}t_2^{\frac{1}{p_2}+1}\int_0^{\frac{\pi}{t_2}}\int_0^{\frac{\pi}{t_1}}h(u_1,u_2)\,du_1 du_2\right]^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}.
\end{aligned}$$

Further, we change variables $\mathbf{z} = \frac{\pi}{\mathbf{t}}$ and apply Hardy's inequality to get

$$
\begin{aligned}
&\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}} \\
&\leq C\left(\int_0^\infty\left(\int_0^\infty\left[z_1^{-\frac{1}{p_1}-1}z_2^{-\frac{1}{p_2}-1}\int_0^{z_2}\int_0^{z_1}h(u_1,u_2)du_1du_2\right]^{q_1}\frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}} \\
&\leq C\left(\int_0^\infty\left(\int_0^\infty\left[z_1^{-\frac{1}{p_1}}z_2^{-\frac{1}{p_2}-1}\int_0^{z_2}h(z_1,u_2)du_2\right]^{q_1}\frac{dz_1}{z_1}\right)^{\frac{q_2}{q_1}}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}} \\
&= C\left(\int_0^\infty z_2^{\left(-\frac{1}{p_2}-1\right)q_2}\left(\int_0^\infty\left[\int_0^{z_2}z_1^{-\frac{1}{p_1}-\frac{1}{q_1}}h(z_1,u_2)du_2\right]^{q_1}dz_1\right)^{\frac{q_2}{q_1}}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}}.
\end{aligned}
$$

Applying Minkowski's inequality, we estimate

$$
\begin{aligned}
&\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}} \\
&\leq C\left(\int_0^\infty z_2^{\left(-\frac{1}{p_2}-1\right)q_2}\left(\int_0^{z_2}\left[\int_0^\infty\left(z_1^{-\frac{1}{p_1}-\frac{1}{q_1}}h(z_1,u_2)\right)^{q_1}dz_1\right]^{\frac{1}{q_1}}du_2\right)^{q_2}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}} \\
&= C\left(\int_0^\infty\left(z_2^{-\frac{1}{p_2}-1}\int_0^{z_2}\left[\int_0^\infty\left(z_1^{-\frac{1}{p_1}-\frac{1}{q_1}}h(z_1,u_2)\right)^{q_1}dz_1\right]^{\frac{1}{q_1}}du_2\right)^{q_2}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}}.
\end{aligned}
$$

By Hardy's inequality, we get

$$
\begin{aligned}
\|f\|_{L^{\mathbf{q}}_{\mathbf{w}[\mathbf{p},\mathbf{q}]}} &\leq C\left(\int_0^\infty\left(z_2^{-\frac{1}{p_2}}\left[\int_0^\infty\left(z_1^{-\frac{1}{p_1}-\frac{1}{q_1}}h(z_1,z_2)\right)^{q_1}dz_1\right]^{\frac{1}{q_1}}\right)^{q_2}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}} \\
&= C\left(\int_0^\infty\left[\int_0^\infty\left(z_1^{-\frac{1}{p_1}}z_2^{-\frac{1}{p_2}}\int_0^{z_2}\int_0^{z_1}\left|\widehat{f}(y_1,y_2)\right|dy_1\,dy_2\right)^{q_1}\frac{dz_1}{z_1}\right]^{\frac{q_2}{q_1}}\frac{dz_2}{z_2}\right)^{\frac{1}{q_2}}.
\end{aligned}
$$

Using Hardy's inequality twice and Minkowski's inequality to the last expression, we arrive at the required estimate. □

Chapter 5

Uniform convergence of the trigonometric series with general monotone coefficients

We will study the trigonometric series

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{5.1}$$

$$\sum_{n=1}^{\infty} a_n \cos nx, \tag{5.2}$$

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{5.3}$$

with some conditions of their coefficients.

5.1 Several important classes of general monotone sequences

In this chapter, we consider the $GM(\boldsymbol{\beta})$ sequences with majorants $\boldsymbol{\beta}$ having the form described below. Let S be the set of numerical sequences. Denote by $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ any element of S.

We will say that a sequence of functionals on S, that is, $F_n : S \to \mathbb{R}_+$, $n \in \mathbb{N}$, is *admissible* if

(i) $F_n(\mathbf{x}) \to 0$ as $n \to \infty$ for any $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ vanishing at infinity,

(ii) $\{F_n(\mathbf{x})\}_{n=1}^{\infty}$ is bounded whenever $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ is bounded.

The examples of such $F = \{F_n\}_{n=1}^{\infty}$ are

(a) $F_n^1(\mathbf{x}) = |x_n|^{\alpha}$, $\alpha > 0$;

(b) $F_n^2(\mathbf{x}) = \sum\limits_{k=\frac{n}{\gamma}}^{\gamma n} \frac{|x_k|}{k}$, $\gamma > 1$;

(c) $F_n^3(\mathbf{x}) = \max\limits_{k \geq \frac{n}{\gamma}} |x_k|$, $\gamma > 1$;

(d) $F_n^4(\mathbf{x}) = \frac{1}{n} \sum\limits_{k=1}^{n} |x_k|$;

(e) $F_n^5(\mathbf{x}) = \sum\limits_{k=1}^{\infty} a_{nk}|x_k|$, where $\{a_{nk}\}_{n,k=1}^{\infty}$ is a regular matrix (see [105, Ch. III, §1]);

(f) The composition $F = G \circ H$, $F_n(\mathbf{x}) := G_n(H_k(\mathbf{x}))$, of admissible sequences $\{H_n\}_{n=1}^{\infty}$, $\{G_n\}_{n=1}^{\infty}$ is also admissible.

A typical example of non-admissible $\{F_n\}$ is

$$F_n(\mathbf{x}) = \sum_{k=n}^{n+\lambda_n} \frac{|x_k|}{k},$$

where a positive sequence $\{\lambda_n\}_{n=1}^{\infty}$ is such that $\lambda_n/n \to \infty$. Note also that conditions (i) and (ii) in the definition of admissible functionals are independent; take for example $F_n(\mathbf{x}) = |x_n|^{\alpha} + c$ with $\alpha, c > 0$ and $F_n(\mathbf{x}) = \sum\limits_{k=n}^{n^2} k^{x_k - 2}$.

For a given sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, denote by $\widetilde{\mathbf{a}}$ the following sequence:

$$\widetilde{a}_n := \sum_{k=n}^{2n} |a_k|.$$

Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ belongs to the class of general monotone sequences $GM(\boldsymbol{\beta})$ if there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\sum_{k=n}^{2n} |\Delta a_k| \leq C\beta_n.$$

Let us recall some important examples of majorants $\boldsymbol{\beta}$:

1. $\beta_n^1 = |a_n|$;
2. $\beta_n^2 = \frac{1}{n} \sum\limits_{s=\frac{n}{\gamma}}^{\gamma n} |a_s|, \quad \gamma > 1$;
3. $\beta_n^3 = \frac{1}{n} \max\limits_{k \geq \frac{n}{\gamma}} \sum\limits_{s=k}^{2k} |a_s|, \quad \gamma > 1$.

We study a class of general monotone sequences $GM(\boldsymbol{\beta})$ with

$$\beta_n = \frac{1}{n} F_n(\widetilde{\mathbf{a}}).$$

Note that the class $GM(\boldsymbol{\beta}^3)$ is the class $GM(\boldsymbol{\beta})$ with $\beta_n = \frac{1}{n}F_n^3(\widetilde{\mathbf{a}})$. Moreover, $GM(\boldsymbol{\beta}^2)$ coincides with $GM(\boldsymbol{\beta})$ with $\beta_n = \frac{1}{n}F_n^2(\widetilde{\mathbf{a}})$. Indeed, considering the sum $\sum\limits_{k=M}^{N} \frac{\widetilde{a}_k}{k}$, where $N > 2M$, we note that

$$\sum_{k=M}^{N} \frac{1}{k} \sum_{s=k}^{2k} |a_s| = \sum_{s=M}^{2M} |a_s| \sum_{k=M}^{s} \frac{1}{k} + \sum_{s=2M+1}^{N} |a_s| \sum_{k=\frac{s}{2}}^{s} \frac{1}{k} + \sum_{s=N+1}^{2N} |a_s| \sum_{k=\frac{s}{2}}^{N} \frac{1}{k}$$

and therefore,

$$C_1 \sum_{s=2M}^{N} |a_s| \le \sum_{k=M}^{N} \frac{\widetilde{a}_k}{k} \le C_2 \sum_{s=M}^{2N} |a_s|.$$

5.2 The main results

5.2.1 Historical remarks

The goal of this chapter is to study of uniform convergence of sine series. Recall that, according to Chaundy-Jolliffee, the series (5.1) with monotone coefficients $\{a_n\}_{n=1}^{\infty}$ converges uniformly if and only if $na_n \to 0$ as $n \to \infty$. This result was generalized, in particular, in [91] for non-negative sequences $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^1)$. In its turn, this was extended in [94, 104] for non-negative sequences $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^2)$ and in [34] for non-negative sequences $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^3)$. Various generalizations of Chaundy-Jolliffe's criterion can be found in the papers [34, 91, 104]. In the recent paper [37], the authors proved the following theorem.

Theorem 5.1. *Let $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^2)$. Then series* (5.1) *converges uniformly on* $[0, 2\pi]$ *if and only if $na_n \to 0$ as $n \to \infty$.*

Note that in Theorem 5.1 the authors do not assume non-negativity or non-positivity of sequence $\{a_n\}_{n=1}^{\infty}$.

5.2.2 Uniform convergence of the trigonometric series

Theorem 5.2. *Let $\{F_n\}_{n=1}^{\infty}$ be admissible. Let also $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ and $\widetilde{\mathbf{a}}$ is a bounded sequence. Then the following conditions are equivalent:*

(1) *the series* (5.1) *converges uniformly on* $[0, 2\pi]$;

(2) $\lim_{n\to\infty} na_n = 0$;

(3) $\lim_{n\to\infty} \widetilde{a}_n = 0$.

Remark 5.1. (i) *It is clear that the condition of boundedness of* $\widetilde{\mathbf{a}}$ *is needed only to show the implication* (1) $\Rightarrow$ (2).

(ii) *Generally speaking, the statement of Theorem 5.2 is not true without assuming that the sequence* $\{\widetilde{a}_n\}_{n=1}^{\infty}$ *is bounded. The corresponding counterexample is constructed in Theorem 5.9 below. More precisely, there exists the uniformly converging sine series with coefficients satisfying* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^3)$ *such that* $na_n \nrightarrow 0$ *and* $\widetilde{a}_n \nrightarrow 0$ *as* $n \to \infty$.

(iii) *It is easy to see that dealing with admissible* $\{F_n\}_{n=1}^{\infty}$ *allows us to expect that* $F_n(\widetilde{\mathbf{a}})$ *is bounded for a bounded sequence* $\widetilde{\mathbf{a}}$. *In light of the previous remark, this property is essential in the proof. In general, Theorem 5.2 is not valid for non-admissible sequnces. In particular, the corresponding example can be given using lacunary sequences. Take the non-admissible functional* $F_n(\mathbf{x}) = nx_n$ *and the lacunary sequence*

$$a_k = \begin{cases} m^{-2} & k = 2^m, \\ 0 & k \neq 2^m. \end{cases}$$

Then $\lim_{n\to\infty} \widetilde{a}_n = 0$, *the series* $\sum_{k=1}^{\infty} a_k \sin kx$ *converges uniformly, but* $\{ka_k\}_{k=1}^{\infty}$ *is not bounded.*

Another example can be given for non-admissible functional $F_n(\mathbf{x}) = \sum_{k=n}^{n+\lambda_n} \frac{|x_k|}{k}$ *with* $\lambda_n/n \to \infty$ *using Rudin-Shapiro construction; see Remark 5.7 (ii).*

(iv) *Regarding the fact that* $GM(\boldsymbol{\beta}^2) \subsetneq GM(\boldsymbol{\beta}^3)$, *we note that there exists a sequence* $\mathbf{a} \in GM(\boldsymbol{\beta}^3) \setminus GM(\boldsymbol{\beta}^2)$ *such that* $\widetilde{\mathbf{a}}$ *is bounded (see Section 5.4). This shows that Theorem 5.2 extends Theorem 5.1.*

A counterpart for the cosine series reads as follows.

Theorem 5.3. *Let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$, *where* $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ *with admissible* $\{F_n\}_{n=1}^{\infty}$ *and bounded* $\widetilde{\mathbf{a}}$. *Then series* (5.2) *converges uniformly on* $[0, 2\pi]$ *if and only if the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

Remark 5.2. *The condition of boundedness of* $\widetilde{\mathbf{a}}$ *in Theorem 5.3 is needed only to prove the "only if" part.*

The following can be seen as the main result of this section.

Corollary 5.1. *Let* $\{F_n\}_{n=1}^{\infty}$ *and* $\{G_n\}_{n=1}^{\infty}$ *be admissble. Let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$ *with* $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ *and* $\{b_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$ *with* $\beta_n = \frac{1}{n}G_n(\widetilde{\mathbf{b}})$.

Suppose that $\widetilde{\mathbf{a}}$ *and* $\widetilde{\mathbf{b}}$ *are bounded sequences. Then for the series* (5.3) *the following conditions are equivalent:*

(1) *the series* (5.3) *is the Fourier series of a continuous function;*

(2) *the series* (5.3) *converges uniformly on* $[0, 2\pi]$;

(3) $\sum_{n=1}^{\infty} a_n$ *converges and* $nb_n \to 0$ *as* $n \to \infty$.

5.2.3 Approximation by partial sums of Fourier series.

Here, we study the convergence rate of $\|h - S_n(h)\|_{C[0,2\pi]}$, where $S_n(h)$ is the n-th partial sum of the Fourier series of h. In [52] (see also [105, Ch. II, §10]), Lebesgue proved that for a function h from the Lipschitz space Lip α, given by

$$\mathrm{Lip}\ \alpha = \{f \in C[0, 2\pi] : \omega(f, \delta)_C = O(\delta^\alpha)\},$$

one has

$$\|h - S_n(h)\|_{C[0,2\pi]} = O\Big(\frac{\ln n}{n^\alpha}\Big). \tag{5.4}$$

Recall that $\omega(f, \delta)_C$ is the modulus of continuity of f, i.e.,

$$\omega(f, \delta)_C = \sup_{|h|\le\delta} \|\Delta_h f(\cdot)\|_C \quad \text{and} \quad \Delta_h f(x) = f(x + h) - f(x).$$

Salem and Zygmund [78] showed that the logarithm cannot be suppressed even if, in addition to the hypotesis $h \in \mathrm{Lip}\ \alpha$, we suppose that h is of bounded variation. However, they demonstrated that if a function $h \in \mathrm{Lip}\ \alpha$ is of monotonic type, then the logarithm can be omitted in (5.4).

Theorem 5.4 ([78, Theorem I]). *Let h be a continuous function of monotonic type; that is, there exists a real constant K such that the function $h(x) + Kx$ is either non-decreasing or non-increasing on $(-\infty, \infty)$. Let $h \in \mathrm{Lip}\ \alpha$, where $0 < \alpha < 1$. Then*

$$\|h - S_n(h)\|_{C[0,2\pi]} = O\Big(\frac{1}{n^\alpha}\Big). \tag{5.5}$$

We will show (see Corollaries 5.2–5.3 below) that estimate (5.5) holds for functions from Lip α having the Fourier series with coefficients from the $GM(\boldsymbol{\beta}^2)$ class. Denote by $g(x)$ and $f(x)$ the sums of (5.1) and (5.2) series, respectively. Here our main results read as follows.

Theorem 5.5. *Let $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n} \sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$. Then, for $0 < \alpha \le 1$,*

$$\|f - S_n(f)\|_{C[0,2\pi]} = o\left(\frac{1}{n^\alpha}\right) \iff a_n = o\left(\frac{1}{n^{\alpha+1}}\right). \tag{5.6}$$

$$\|g - S_n(g)\|_{C[0,2\pi]} = o\left(\frac{1}{n^\alpha}\right) \iff a_n = o\left(\frac{1}{n^{\alpha+1}}\right). \tag{5.7}$$

Theorem 5.6. *Let $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n}\sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$. Then, for $0 < \alpha \le 1$,*

$$\|f - S_n(f)\|_{C[0,2\pi]} = O\left(\frac{1}{n^\alpha}\right) \iff a_n = O\left(\frac{1}{n^{\alpha+1}}\right). \tag{5.8}$$

$$\|g - S_n(g)\|_{C[0,2\pi]} = O\left(\frac{1}{n^\alpha}\right) \iff a_n = O\left(\frac{1}{n^{\alpha+1}}\right). \tag{5.9}$$

Remark 5.3. (i) *Note that the condition $\|f - S_n(f)\|_{C[0,2\pi]} = O\left(\frac{1}{n^\alpha}\right)$ implies that the sum f is a continuous function and $\{a_n\}_{n=1}^\infty$ is the sequence of Fourier coefficients of f.*

(ii) *For $\alpha = 0$, Theorem 5.5 also holds in the case of the sine series, which gives an alternative proof of the main result in [37] (see Theorem 5.1).*

Moreover, Theorem 5.6 along with [35, Theorem 2.2 and Corollary 3.4] imply the following results.

Corollary 5.2. *Let $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n}\sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$. Also let* (5.2) *be the Fourier series of a continuous function f. Then, for $0 < \alpha \le 1$, the following conditions are equivalent:*

(i) $f \in \mathrm{Lip}\ \alpha$,

(ii) $\|f - S_n(f)\|_C = O\left(\frac{1}{n^\alpha}\right)$,

(iii) $E_n(f)_C = O\left(\frac{1}{n^\alpha}\right)$.

Here, $E_n(f)_C$ is the best approximation of a function f by trigonometric polynomials of degree n in $C[0, 2\pi]$.

Corollary 5.3. *Let $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{C}{n}\sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$. Let also* (5.1) *be the Fourier series of a continuous function g. Then, for $0 < \alpha < 1$, the following conditions are equivalent:*

(i) $g \in \mathrm{Lip}\ \alpha$,

(ii) $\|g - S_n(g)\|_C = O\left(\frac{1}{n^\alpha}\right)$,

(iii) $E_n(g)_C = O\left(\frac{1}{n^\alpha}\right)$.

Moreover, for $\alpha = 1$, conditions (ii), (iii), *and*

(iv) $a_n = O\left(\frac{1}{n^2}\right)$

are pairwise equivalent, but the condition $g \in \mathrm{Lip}\ 1$ is not equivalent to any of them.

Remark 5.4. *Regarding the case $\alpha = 1$ in Corollaries 5.2 and 5.3, we first note that the direct and inverse theorems of trigonometric approximation [25]; namely,*

$$E_n(\psi)_C \le C\omega\Big(\psi, \frac{1}{n}\Big)_C \le \frac{C}{n}\sum_{\nu=1}^{n+1} E_{\nu-1}(\psi)_C,$$

immediately imply that

$$\psi \in \text{Lip } \alpha \quad \textit{if and only if} \quad E_n(\psi)_C = O\left(\frac{1}{n^\alpha}\right) \quad \textit{for} \quad 0 < \alpha < 1.$$

We see that dealing with series with general monotone coefficients allows one to extend this result for the limiting case $\alpha = 1$ when $\psi = f$. A similar result does not hold for sine series ($\psi = g$), because of the following reason. For series with monotone coefficients, a necessary and sufficient condition for $g \in \text{Lip } 1$ is already given by $\sum_k ka_k < \infty$. This fact was first observed by Boas [14], and in turn is related to the behavior of the derivative of g at the origin. In particular, the function $g(x) = \sum_k \frac{\sin kx}{k^2}$ is such that $E_n(g)_C \le \|g - S_n(g)\|_C = O\Big(\frac{1}{n}\Big)$, but $g \notin \text{Lip } 1$. See [92, 93, 94] for the related results regarding series with non-negative GM coefficients.

5.3 Proofs of main results

Remark 5.5. *Without loss of generality, we may assume in Theorems 5.2 and 5.3 that the inequality*

$$\widetilde{a}_n \le F_n(\widetilde{\mathbf{a}}) \tag{5.10}$$

is valid for all sequences $\mathbf{a} = \{a_n\}_{n=1}^\infty$ and for all $n \in \mathbb{N}$. Indeed, if it is not the case, that we can consider the majorant:

$$G_n(\widetilde{\mathbf{a}}) = \max\{\widetilde{a}_n, F_n(\widetilde{\mathbf{a}})\}$$

which satisfies (5.10)*. It is clear that conditions* (i)–(ii) *hold for the sequence $\{G_n\}_{n=1}^\infty$. Moreover, instead of the class $GM(\boldsymbol{\beta})$ with $\beta_n = \frac{F_n(\widetilde{\mathbf{a}})}{n}$ we can consider the class $GM(\boldsymbol{\beta}^*) \supseteq GM(\boldsymbol{\beta})$, where $\beta_n^* = \frac{G_n(\widetilde{\mathbf{a}})}{n}$. Throughout this chapter, we will assume that $\{F_n\}_{n=1}^\infty$ satisfies* (5.10).

Lemma 5.1. *Let $\mathbf{a} \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{F_n(\widetilde{\mathbf{a}})}{n}$. Then, for all $n \in \mathbb{N}$,*

$$|a_k| \le C\frac{F_n(\widetilde{\mathbf{a}})}{n} \quad \textit{for all} \quad k = n, \dots, 2n. \tag{5.11}$$

Proof. The proof follows from (5.10) and inequality (1.3). □

5.3.1 Proof of Theorem 5.2

Here we need the following general result.

Theorem 5.7 ([33, Theorem 2.1, part (C)])**.** *Let $\boldsymbol{\beta} = \{\beta_n\}_{n=1}^{\infty}$ be a majorant such that $n\beta_n \to 0$ as $n \to \infty$. Then series (5.1) converges uniformly on $[0, 2\pi]$.*

Proof. We will prove Theorem 5.2 as follows: (2) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (2).

The implication (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let $\widetilde{a}_n \to 0$ as $n \to \infty$. Then from the property (i) of F_n, we get

$$F_n(\widetilde{\mathbf{a}}) \to 0 \quad \text{as} \quad n \to \infty.$$

Finally, we use Theorem 5.7 with $\beta_n = F_n(\widetilde{\mathbf{a}})/n$.

(1) $\Rightarrow$ (2). From Lemma 5.1 and property (ii) on $\{F_n\}_{n=1}^{\infty}$ it follows that it is sufficient to prove $\lim\limits_{n\to\infty} \widetilde{a}_n = 0$. Let $\varepsilon > 0$, then by Cauchy's criterion, we can choose $N \in \mathbb{N}$ such that for all $N \le k \le l$

$$\Big\| \sum_{j=k}^{l} a_j \sin jx \Big\|_{C[0,2\pi]} < \varepsilon. \tag{5.12}$$

Let $n > N$ and $\widetilde{a}_n \neq 0$. By (5.10), note that $F_n(\widetilde{\mathbf{a}}) \neq 0$. We put

$$A_n := \left\{ k : |a_k| \ge \frac{\widetilde{a}_n}{4n}, n \le k \le 2n \right\}. \tag{5.13}$$

Note that A_n is not empty set. Let us obtain a lower estimate for the cardinality of A_n denoted by $|A_n|$. By (5.11), we have $|a_k| \le \frac{C}{n} F_n(\widetilde{\mathbf{a}}), \quad n \le k \le 2n$, and therefore,

$$\begin{aligned}
\widetilde{a}_n = \sum_{s=n}^{2n} |a_s| &= \sum_{s\in[n,2n]\setminus A_n} |a_s| + \sum_{s\in A_n} |a_s| \\
&\le \sum_{s\in[n,2n]\setminus A_n} \frac{\widetilde{a}_n}{4n} + \sum_{s\in A_n} \frac{C}{n} F_n(\widetilde{\mathbf{a}}) \\
&\le \frac{2n\widetilde{a}_n}{4n} + |A_n| \frac{C}{n} F_n(\widetilde{\mathbf{a}}) = \frac{\widetilde{a}_n}{2} + |A_n| \frac{C}{n} F_n(\widetilde{\mathbf{a}}).
\end{aligned}$$

Hence,

$$|A_n| \ge \frac{n}{2C} \frac{\widetilde{a}_n}{F_n(\widetilde{\mathbf{a}})}. \tag{5.14}$$

Following [37], we construct disjoint subsets $S_1, \dots, S_{\kappa_n}$ of $[n, 2n]$. Put $m_1 = \min A_n$, and select ν_1 according to the following procedure:

(a) If there exists $j_0 \ge 1$ such that for $j = 0, 1, \dots, j_0$, $n \le m_1 + j \le 2n$ the numbers a_{m_1+j} have the same sign, and for $j = 0, 1, \dots, j_0 - 1$, $|a_{m_1+j}| \ge \frac{\widetilde{a}_n}{8n}$, and $|a_{m_1+j_0}| < \frac{\widetilde{a}_n}{8n}$, then we set $\nu_1 = j_0$.

(b) If such j_0 does not exist, then let $\nu_1 = l_0$ such that $m_1 + l_0 \in [n, 2n]$ and $a_{m_1+l_0}$ is the first element to become zero or of opposite sign than a_{m_1}.

(c) If neither (a) nor (b) happens, then simply $\nu_1 = l_0$, for which $m_1 + l_0$ is the first number greater than $2n$.

Define a set

$$S_1 = \{m_1, m_1 + 1, \dots, m_1 + \nu_1 - 1\}.$$

Next, if the set $A_n \setminus S_1$ is not empty, we put $m_2 = \min(A_n \setminus S_1)$. Using the same procedure as above, we select ν_2 and define

$$S_2 = \{m_2, m_2 + 1, \dots, m_2 + \nu_2 - 1\}.$$

We continue this procedure until we reach an S_{κ_n} for which

$$A_n \setminus (S_1 \cup \ldots \cup S_{\kappa_n}) = \emptyset.$$

Now we obtain the upper estimate for κ_n. If $\kappa_n > 1$, we note first that for all $1 \le j < \kappa_n$, we have

$$\sum_{k \in S_j} |\Delta a_k| \ge |a_{m_j} - a_{m_j+\nu_j}| \ge \frac{\widetilde{a}_n}{8n}.$$

From the definition of $GM(\boldsymbol{\beta})$, $\beta_n = \frac{F_n(\widetilde{\mathbf{a}})}{n}$, we get

$$\sum_{s=n}^{2n} |\Delta a_s| \le \frac{C}{n} F_n(\widetilde{\mathbf{a}}).$$

Hence,

$$\frac{C}{n} F_n(\widetilde{\mathbf{a}}) \ge \sum_{s=n}^{2n} |\Delta a_s| \ge \sum_{j=1}^{\kappa_n - 1} \sum_{k \in S_j} |\Delta a_k| \ge \sum_{j=1}^{\kappa_n - 1} \frac{\widetilde{a}_n}{8n} = (\kappa_n - 1) \frac{\widetilde{a}_n}{8n}.$$

Therefore,

$$\kappa_n \le \frac{8CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n} + 1 \le \frac{9CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n}. \tag{5.15}$$

If $\kappa_n = 1$, then (5.15) also holds. Let $x = \frac{\pi}{4n}$ and $n \le k \le 2n$. Then

$$\sin kx \ge \frac{2}{\pi} \frac{\pi k}{4n} \ge \frac{1}{2}.$$

Since all a_k, $k \in S_j$ have the same sign, we derive

$$\frac{1}{2} \sum_{k \in S_j} |a_k| \le \left| \sum_{k \in S_j} a_k \sin \frac{\pi k}{4n} \right| < \varepsilon \tag{5.16}$$

for all $n > N$. Hence,

$$\sum_{k\in A_n} |a_k| \le \sum_{j=1}^{\kappa_n} \sum_{k\in S_j} |a_k| < \varepsilon \frac{18CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n}.$$

From the definition (5.13) of the set A_n and estimate (5.14), we arrive at

$$\frac{1}{8C}\frac{\widetilde{a}_n^2}{F_n(\widetilde{\mathbf{a}})} \le \varepsilon \frac{18CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n}.$$

Hence,

$$\frac{\widetilde{a}_n^3}{F_n(\widetilde{\mathbf{a}})^2} \to 0 \qquad \text{as} \quad n \to \infty.$$

Since $\{F_n(\widetilde{\mathbf{a}})\}_{n=1}^{\infty}$ is bounded, we obtain that $\widetilde{a}_n \to 0$ as $n \to \infty$. Hence $F_n(\widetilde{\mathbf{a}}) \to 0$ as $n \to \infty$. □

5.3.2 Proof of Theorem 5.3

Here we need the following result.

Theorem 5.8 **(**[33, Theorem 2.1, part (B)]**).** *Let* $\mathbf{a} \in GM(\boldsymbol{\beta})$*. If* $n\beta_n = o(1)$ *as* $n \to \infty$*, then series* (5.2) *converges uniformly on* $[0, 2\pi]$ *if and only if the series* $\sum_n a_n$ *converges.*

Proof. The "only if" part is clear.

To show the "if" part, as in the proof of Theorem 5.2, it is enough to show that

$$\lim_{n\to\infty} \widetilde{a}_n = 0. \tag{5.17}$$

For $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that for all $l \ge k \ge N$

$$\Big|\sum_{j=k}^{l} a_j\Big| < \varepsilon.$$

Relation (5.17) is proved in the same way and with the same notation as in Theorem 5.2, using the inequality

$$\frac{1}{2}\sum_{k\in S_j} |a_k| = \frac{1}{2}\Big|\sum_{k\in S_j} a_k\Big| < \varepsilon, \quad S_j \subset [n, 2n], \quad n > N \tag{5.18}$$

instead of inequality (5.16). Then since $\{F_n\}_{n=1}^{\infty}$ is admissible, we obtain that $F_n(\widetilde{\mathbf{a}}) \to 0$ as $n \to \infty$. Thus, $\mathbf{a} \in GM(\boldsymbol{\beta})$ with $n\beta_n = o(1)$ and Theorem 5.8 concludes the proof. □

5.3.3 Proof of Corollary 5.1

We divide the proof into two parts: (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2). Let series (5.3) be the Fourier series of a continuous function $h(x)$. Note that for a sequence $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$ with $\boldsymbol{\beta}_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$, the boundedness of the sequence $\{na_n\}_{n=1}^{\infty}$ is equivalent to boundedness of the sequence $\{\widetilde{a}_n\}_{n=1}^{\infty}$. From boundedness of $\{na_n\}_{n=1}^{\infty}$ it follows that $a_n \geq -\frac{C}{n}$ for all $n \geq 1$ and some $C > 0$. The last inequality with the Paley–Fekete theorem (see [36, Theorem C]) implies the uniform convergence of series (5.3).

(2) $\Rightarrow$ (1). This part is clear.

(2) $\Rightarrow$ (3). Let series (5.3) converge uniformly. Denote by $h(x)$ the sum of series (5.3). Note that the series $\sum\limits_{n=1}^{\infty} a_n \cos nx$ and $\sum\limits_{n=1}^{\infty} b_n \sin nx$ are the Fourier series of the continuous functions

$$\frac{h(x)+h(-x)}{2} \quad \text{and} \quad \frac{h(x)-h(-x)}{2},$$

respectively. Since both series converge uniformly, Theorems 5.2 and 5.3 imply (3).

(3) $\Rightarrow$ (2). This part follows from Theorems 5.2 and 5.3.

5.3.4 Proof of Theorems 5.5 and 5.6

Here as in the proof of main results of Chapter 2 we follow the proof of [35]. In the definition of $GM(\boldsymbol{\beta})$ with $\beta_n = \frac{1}{n}\sum\limits_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$ we assume that $\gamma = 2^{\nu}$, where ν is an integer number. We use the notations introduced in Section 2.4

$$A_n = \max_{2^n \leq k \leq 2^{n+1}} |a_k|,$$

$$B_n = \max_{2^{n-2\nu} \leq k \leq 2^{n+2\nu}} |a_k|,$$

and

$$M_n = \left\{k \in [2^{n-\nu}, 2^{n+\nu}] : |a_k| > \frac{A_n}{8C2^{2\nu}}\right\},$$

$$M_n^+ := \{k \in M_n : a_k > 0\} \quad \text{and} \quad M_n^- := M_n \setminus M_n^+,$$

where C and ν are constants from the definition of $GM(\boldsymbol{\beta})$ class with $\beta_n = \frac{1}{n}\sum\limits_{k=\frac{n}{2^{\nu}}}^{2^{\nu}n} |a_k|$.

Recall that a natural number n is called *good* if either $n \leq 2\nu$ or $B_n \leq 2^{4\nu}A_n$. The rest of all natural numbers consists of *bad* numbers.

We need Lemma 2.4 mentioned in Section 2.4. For convenience we write this lemma here with the same number.

Lemma 2.4. *Let a vanishing sequence $\{a_n\}_{n=1}^{\infty} \in GM$. Denote $N_0 := [\log_2(C^3 2^{10\nu+8})]+1$. Then for any good n such that $n \geq N_0$ there exists an interval $[l_n, m_n] \subseteq [2^{n-\nu}, 2^{n+\nu}]$ such that at least one of the following condition holds:*

(i) *for any $k \in [l_n, m_n]$, we have $a_k \geq 0$ and*

$$|M_n^+ \cap [l_n, m_n]| \geq \frac{2^n}{C^3 2^{15\nu+8}};$$

(ii) *for any $k \in [l_n, m_n]$, we have $a_k \leq 0$ and*

$$|M_n^- \cap [l_n, m_n]| \geq \frac{2^n}{C^3 2^{15\nu+8}}.$$

Proof of Theorems 5.5 and 5.6. We will prove only the case of the sine series of Theorem 5.5. For the case of the cosine series in Theorem 5.5 and for both cases in Theorem 5.6, the proof is similar.

First, we prove the part "$\Longrightarrow$". Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\|g - S_n(g)\|_{C[0,2\pi]} \leq \frac{\varepsilon}{n^\alpha}.$$

Let n be a good number and $2^n > \max\{C^3 2^{15\nu+11}, 2^\nu N\}$. Assume Lemma 2.4(i) is valid and consider

$$Q_n(t) = \sum_{k=l_n+1}^{m_n} a_k \sin kt.$$

Then $|Q_n(t)| \leq \frac{2\varepsilon}{2^{(n-2\nu)\alpha}}$ for all $t \in [0, 2\pi]$. Setting $t = \frac{1}{2^{n+\nu}}$, we obtain

$$\begin{aligned}\frac{2\varepsilon}{2^{(n-2\nu)\alpha}} &\geq \sum_{k=l_n+1}^{m_n} a_k \sin \frac{k}{2^{n+\nu}} \geq \frac{2}{\pi} \frac{1}{2^{2\nu}} \frac{A_n}{8C 2^{2\nu}} \left(\frac{2^n}{C^3 2^{15\nu+8}} - 1 \right) \\ &\geq \frac{1}{2} \frac{1}{2^{2\nu}} \frac{A_n}{8C 2^{2\nu}} \frac{2^n}{C^3 2^{15\nu+9}} = \frac{2^n A_n}{C^4 2^{19\nu+13}}.\end{aligned}$$

Therefore,

$$A_n \leq \frac{L_1 \varepsilon}{2^{(\alpha+1)n}}.$$

Then

$$A_n \leq \frac{L_2 \varepsilon}{2^{(\alpha+1)n}}$$

holds for all good numbers, where $L_2 \geq L_1$ is another constant.

Let n be a bad number. Then $A_n < B_n 2^{-4\nu}$. Note that $B_n = A_{s_1}$, where $|s_1 - n| \leq 2\nu$.

Assume first that $s_1 < n$. Then either s_1 is a good number or there exists s_2 such that $|s_1 - s_2| \leq 2\nu$ and $A_{s_1} < A_{s_2} 2^{-4\nu}$. Also, we have

$$[2^{s_1}, 2^{s_1+2\nu}] \cup \mathbb{Z} \subset [2^{n-2\nu}, 2^{n+2\nu}] \cup \mathbb{Z}. \tag{5.19}$$

Then there is no a_k, $k \in [2^{s_1}, 2^{s_1+2\nu}] \cup \mathbb{Z}$, such that $|a_k| > A_{s_1}$. Hence, the case $s_2 > s_1$ is not possible.

Repeating the process, since s_j is a decreasing sequence, we arrive at a finite sequence $n = s_0 > s_1 > \ldots > s_{i-1} > s_i$, where numbers $s_0, s_1, \ldots, s_{i-1}$ are bad, and s_i is good. Moreover, $s_j - s_{j-1} \leq 2\nu$ and $A_{s_j} < A_{s_{j+1}} 2^{-4\nu}$ for any j. Since s_i is a good number, using the proof above we have $A_{s_i} \leq \frac{L_2\varepsilon}{2^{(\alpha+1)s_i}}$, which implies

$$A_n = A_{s_0} \leq \frac{A_{s_1}}{2^{4\nu}} \leq \ldots \leq \frac{L_2\varepsilon}{2^{4\nu i} 2^{(\alpha+1)s_i}}. \tag{5.20}$$

Now, since $n \leq s_i + 2i\nu$, we have

$$A_n \leq \frac{L_2\varepsilon}{2^{4\nu i} 2^{(\alpha+1)s_i}} = \frac{L_2\varepsilon}{2^{(\alpha+1)n}} \frac{2^{(\alpha+1)n}}{2^{(1+\alpha)(2\nu i + s_i)}} \frac{1}{2^{2\nu i(1-\alpha)}} \leq \frac{L_3\varepsilon}{2^{(\alpha+1)n}}. \tag{5.21}$$

Let now $s_1 > n$. Then either s_1 is a good number or there exists $s_2 > s_1$ such that $s_2 - s_1 \leq 2\nu$ and $A_{s_1} < A_{s_2} 2^{-4\nu}$. Continuing this process and taking into account that the sequence of the Fourier coefficients vanishes at infinity, we arrive at the finite sequence $n = s_0 < s_1 < \ldots < s_{i-1} < s_i$, where the numbers $s_0, s_1, \ldots, s_{i-1}$ are bad, and s_i is good. Then $A_{s_i} \leq \frac{L_2\varepsilon}{2^{(\alpha+1)s_i}}$ implies

$$A_n < A_{s_1} < A_{s_2} < \ldots A_{s_i} \leq \frac{L_2\varepsilon}{2^{(\alpha+1)s_i}} \leq \frac{L_2\varepsilon}{2^{(\alpha+1)n}}.$$

Then we have

$$A_n \leq \frac{L_3\varepsilon}{2^{(1+\alpha)n}}$$

for any n. Let $k \in \mathbb{N}$ such that $k \in [2^l, 2^{l+1}]$ and $2^l \geq N$. Then

$$|a_k| \leq A_l \leq \frac{L_2\varepsilon}{2^{(1+\alpha)l}} \leq \frac{L_2\varepsilon}{2^{(1+\alpha)l}} \leq \frac{L_3\varepsilon}{k^{1+\alpha}}.$$

We would like to remark that for certain sequences $\{a_k\}$ the number of good points is finite. In this case the proof of the "$\Longrightarrow$" part follows the same lines as above for all n being bad numbers. We repeat the procedure for $s_1 < n$; see (5.19)–(5.21), since the case $s_1 > n$ is impossible.

Now we prove the part "$\Longleftarrow$". Let $\varepsilon > 0$, then the inequality

$$\|g - S_n(g)\|_{C[0,2\pi]} \leq \sum_{k=n+1}^{\infty} |a_k| \leq \varepsilon \sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha+1}} \leq \varepsilon \frac{1}{n^{\alpha}}$$

holds for all $n \geq N$, where N is sufficiently large integer number depending on ε. □

Remark 5.6. *Regarding the Lebesgue and Salem-Zygmund estimates stated in Subsection 5.2.3, see* (5.4) *and* (5.5) *respectively, it is worth mentioning that if a function h*

belongs to the Lipschitz space Lip α*, then*

$$\|h(x)-\sigma_n(h,x)\|_{C[0,2\pi]} = O\Big(\frac{1}{n^\alpha}\Big), \quad \alpha<1, \tag{5.22}$$

$$\|h(x)-\sigma_n(h,x)\|_{C[0,2\pi]} = O\Big(\frac{\ln n}{n^\alpha}\Big), \quad \alpha=1, \tag{5.23}$$

where $\sigma_n(h,x)$ *is the first arithmetic mean of the Fourier series of* h*. These results were obtained by Bernstein [12]. Note that* (5.22) *implies that* $E_n(h)_C = O\Big(\frac{1}{n^\alpha}\Big)$*, which is equivalent to the condition* $h\in$ Lip α *for* $\alpha<1$*. Moreover, the function* $f(x)=\sum\limits_{n=1}^{\infty}\frac{\cos nx}{n^2}$ *belongs to* Lip 1*, but*

$$\|f(x)-\sigma_n(f,x)\|_{C[0,2\pi]} \ge \frac{\ln n}{n}.$$

It is important to note that there is a crucial difference between the results (5.4)–(5.5) *and* (5.22)–(5.23) *which becomes apparent only when we consider these relations for a particular value of* x*. Indeed, the relation* $h(x)-\sigma_n(h,x)=O\Big(\frac{1}{n^\alpha}\Big)$ *depends only on the behaviour of* x *in the neighborhood of the particular point* x *concerned but the relation* $h(x)-S_n(h,x)=O\Big(\frac{1}{n^\alpha}\Big)$ *depends on the behaviour of* x *in the entire interval* $[0,2\pi]$*; see the discussion in [38].*

5.4 Several examples of general monotone sequences

To compare Theorem 5.1 and Theorem 5.2, we construct several examples of sequences $\{a_k\}_{k=1}^\infty \in GM(\boldsymbol{\beta}^3)\setminus GM(\boldsymbol{\beta}^2)$. First, for convenience, we recall the definitions of $GM(\boldsymbol{\beta}^2)$ and $GM(\boldsymbol{\beta}^3)$ classes.

A sequence $\{a_k\}_{k=1}^\infty \in GM(\boldsymbol{\beta}^2)$, if there exist $C>0$, $\gamma>1$ such that for all $n\in\mathbb{N}$,

$$\sum_{k=n}^{2n}|\Delta a_k| \le \frac{C}{n}\sum_{s=\frac{n}{\gamma}}^{\gamma n}|a_s|. \tag{5.24}$$

A sequence $\{a_k\}_{k=1}^\infty \in GM(\boldsymbol{\beta}^3)$, if there exist $C>0$, $\gamma>1$ such that for all $n\in\mathbb{N}$,

$$\sum_{k=n}^{2n}|\Delta a_k| \le \frac{C}{n}\max_{k\ge\frac{n}{\gamma}}\sum_{s=k}^{2k}|a_s|. \tag{5.25}$$

We set,

$$N_1=1, \quad N_{j+1}=N_j+2M_j,$$

where $M_j > N_j$ and $\{M_j\}_{j=1}^{\infty}$ is an increasing sequence of integers. Consider the sequence

$$a_k = \begin{cases} \frac{(-1)^k}{C_j}, & N_j \le k < 2N_j; \\ \frac{1}{C_j}, & 2N_j \le k < 2N_j + M_j; \\ 0, & 2N_j + M_j \le k < N_{j+1}, \end{cases} \tag{5.26}$$

where $\{C_j\}_{j=1}^{\infty}$ is an increasing sequence.

1. We show that $\mathbf{a} \notin GM(\boldsymbol{\beta}^2)$. Let $k = N_j$, then

$$\sum_{s=k}^{2k} |\Delta a_s| = \sum_{s=N_j}^{2N_j} |\Delta a_s| \asymp \sum_{s=N_j}^{2N_j} \frac{2}{C_j} \asymp \frac{N_j}{C_j}.$$

On the other hand, we have

$$\frac{1}{k} \sum_{s=\frac{k}{\gamma}}^{\gamma k} |a_s| \le \frac{1}{N_j} \sum_{s=\frac{N_j}{\gamma}}^{\gamma N_j} \frac{1}{C_j} \asymp \frac{1}{N_j} N_j \frac{1}{C_j} = \frac{1}{C_j}.$$

Therefore, condition (5.24) does not hold.

2. Now we obtain sufficient conditions on $\{M_j\}_{j=1}^{\infty}$ for the sequence $\mathbf{a}$ to belong to the class $GM(\boldsymbol{\beta}^3)$. It is clear that it is enough to verify condition (5.25) for $k = N_j$. We have

$$\begin{aligned} \frac{1}{k} \max_{s \ge \frac{k}{\gamma}} \sum_{i=s}^{2s} |a_i| &= \frac{1}{N_j} \max_{s \ge \frac{N_j}{\gamma}} \sum_{i=s}^{2s} |a_i| \ge \frac{1}{N_j} \sum_{i=N_j+M_j/2}^{2N_j+M_j} |a_i| \\ &\asymp \frac{1}{N_j} \frac{1}{C_j} (N_j + M_j/2) = \frac{1 + \frac{M_j}{2N_j}}{C_j}. \end{aligned}$$

Comparing the expressions $\frac{1+\frac{M_j}{2N_j}}{C_j}$ and $\frac{N_j}{C_j}$, we obtain that if

$$N_j^2 = O(M_j) \quad \text{as} \quad j \to \infty,$$

then (5.25) holds, i.e., $\{a_k\}_{k=1}^{\infty} \in GM(\boldsymbol{\beta}^3)$.

3. Now we study the uniform boundedness of the sums $\sum\limits_{s=k}^{2k} |a_s|$. Let $2k = 2N_j + M_j$, then

$$\sum_{s=k}^{2k} |a_s| = \sum_{s=N_j+M_j/2}^{2N_j+M_j} |a_s| \asymp \frac{N_j + M_j}{C_j}.$$

Hence, the following hold:

(a) the condition

$$N_j + M_j = O(C_j) \quad \text{as} \quad j \to \infty.$$

implies the uniform boundedness of the sums $\sum_{s=k}^{2k} |a_s|$. In particular, the sequence $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ belongs to $GM(\boldsymbol{\beta}^3)$, where

$$a_k = \begin{cases} \frac{(-1)^k}{2^{N_j} N_j}, & N_j \le k < 2N_j; \\ \frac{1}{2^{N_j} N_j}, & 2N_j \le k < 2N_j + N_j 2^{N_j}; \\ 0, & 2N_j + N_j 2^{N_j} \le k < N_{j+1}, \end{cases}$$

and $\sum_{k=n}^{2n} |a_k| \le 2$, $n \ge 1$. But by Theorem 5.2, the series $\sum_{k=1}^{\infty} a_k \sin kx$ is not uniformly convergent, since $ka_k \nrightarrow 0$.

(b) the condition

$$N_j + M_j = o(C_j) \quad \text{as} \quad j \to \infty.$$

implies $\sum_{s=k}^{2k} |a_s| \to 0$ as $k \to \infty$. In particular, the series $\sum_{k=1}^{\infty} a_k \sin kx$ with coefficients $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$, where

$$a_k = \begin{cases} \frac{(-1)^k}{j^\alpha 2^{N_j} N_j}, & N_j \le k < 2N_j; \\ \frac{1}{j^\alpha 2^{N_j} N_j}, & 2N_j \le k < 2N_j + N_j 2^{N_j}; \\ 0, & 2N_j + N_j 2^{N_j} \le k < N_{j+1}, \end{cases}$$

and $\alpha > 0$, converges uniformly. Notice that $\{a_k\}_{k=1}^{\infty} \in GM(\boldsymbol{\beta}^3) \setminus GM(\boldsymbol{\beta}^2)$.

4. Note that if $\{C_j\}_{j=1}^{\infty}$ increases fast enough, then uniform convergence of $\sum_{k=1}^{\infty} a_k \sin kx$ simply follows from the absolute convergence of $\sum_{k=1}^{\infty} a_k$, since

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k| &= \sum_{j=1}^{\infty} \sum_{k=N_j}^{N_{j+1}-1} |a_k| = \sum_{j=1}^{\infty} \sum_{k=N_j}^{2N_j+M_j} |a_k| \\ &= \sum_{j=1}^{\infty} \frac{1}{C_j} \left(N_j + M_j + 1\right). \end{aligned}$$

In particular, the condition

$$j^\alpha (N_j + M_j) = O(C_j) \quad \text{as} \quad j \to \infty,$$

where $\alpha > 1$, implies convergence of the series $\sum_{k=1}^{\infty} a_k \sin kx$.

5.5 Counterexample to Theorem 5.2

Theorem 5.9. *There exists a uniformly convergent sine series* $\sum_{k=1}^{\infty} a_k \sin kx$ *such that*

(i) $\sum_{k=2^{n-1}}^{2^n} |a_k| \geq 2^{\frac{n}{2}-1} d_n, \quad n \geq 1,$

(ii) $k|a_k| \geq 2^{n-1}|a_{2^{n-1}}| = 2^{\frac{n}{2}-1} d_n, \quad 2^{n-1} \leq k < 2^n, \quad n \geq 1,$

where $\{d_n\}_{n=1}^{\infty}$ *is arbitrary positive sequence such that*

(a) $\sum_{n=1}^{\infty} d_n < \infty$;

(b) $2^{\frac{n}{2}} d_n \to \infty$ *as* $n \to \infty$.

Note that formally speaking, the constructed sequence $\{a_n\}_{n=1}^{\infty}$ is in $GM(\boldsymbol{\beta}^3)$. We will use the Rudin-Shapiro sequence, see [74, Theorem 1] and [79].

Lemma 5.2 (Rudin-Shapiro)**.** *There exists a sequence* $\{\varepsilon_k\}_{k=0}^{\infty}$, $\varepsilon_k = \pm 1$, $k \geq 0$ *such that*

$$\left| \sum_{k=0}^{N} \varepsilon_k e^{ikt} \right| < 5\sqrt{N+1} \tag{5.27}$$

for all $t \in [0, 2\pi]$ *and* $N = 0, 1, \dots$

Proof of Theorem 5.9. Let $\{d_n\}_{n=1}^{\infty}$ be a positive sequence satisfying conditions (a) and (b). Let also $\{\varepsilon_k\}_{k=0}^{\infty}$ be the Rudin-Shapiro sequence. Consider the series

$$\sum_{n=1}^{\infty} c_n \sum_{k=2^{n-1}}^{2^n-1} \varepsilon_k e^{ikt}, \tag{5.28}$$

with $c_n \in \mathbb{R}$ such that $|c_n| = 2^{-\frac{n}{2}} d_n$, $n \in \mathbb{N}$. By using the Rudin-Shapiro theorem, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left| c_n \sum_{k=2^{n-1}}^{2^n-1} \varepsilon_k e^{ikt} \right| &\leq \sum_{n=1}^{\infty} |c_n| \left(\left| \sum_{k=0}^{2^n-1} \varepsilon_k e^{ikt} \right| + \left| \sum_{k=0}^{2^{n-1}-1} \varepsilon_k e^{ikt} \right| \right) \\ &\leq C \sum_{n=1}^{\infty} |c_n| 2^{\frac{1}{2}n} \leq C \sum_{n=1}^{\infty} d_n. \end{aligned}$$

Hence, the convergence of the series $\sum_{n=1}^{\infty} d_n$ implies the uniform convergence of series (5.28). Then the series

$$\sum_{n=1}^{\infty} c_n \sum_{k=2^{n-1}}^{2^n-1} \varepsilon_k \sin kt \tag{5.29}$$

converges uniformly. Denote by f its sum and by $a_k(f)$ the Fourier coefficients of f. Then

$$\sum_{k=2^{n-1}}^{2^n} |a_k(f)| \geq \sum_{k=2^{n-1}}^{2^n-1} |c_n| = |c_n|2^{n-1} = 2^{\frac{n}{2}-1}d_n.$$

Condition (ii) is clear. □

Remark 5.7. (i) *As mentioned above in Section 1.2, the widest class of general monotone sequences satisfying condition (1.1) is when* $\beta_n = \sum_{k=n}^{2n} |a_k|$. *All sequences of the form*

$$a_k = c_n \varepsilon_k, \qquad 2^{n-1} \leq k < 2^n, \quad n \in \mathbb{N},$$

where $c_n \in \mathbb{R}$ *and* $\{\varepsilon_k\}_{k=1}^{\infty}$ *is the Rudin-Shapiro sequence (see the example in Theorem 5.9), belong to this extreme class. Moreover, such sequences do not belong to any smaller class since we always have*

$$\sum_{k=n}^{2n} |\Delta a_k| \asymp C \sum_{k=n}^{2n} |a_k|, \quad n \geq 6.$$

This follows from the fact that for any k, *the sequence* $\varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}, \varepsilon_{k+3}, \varepsilon_{k+4}$ *changes its sign at least once. Therefore, for any integer* $s \geq 6$ *such that* $2^{n-1} < s \leq 2^n$, $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=s}^{2s} |\Delta a_k| &\geq \Big(\sum_{k=s}^{2^n} + \sum_{k=2^n+1}^{2s}\Big)|\Delta a_k| \\ &\geq \frac{2^n - s + 1}{5}|c_n| + \frac{2s - 2^n}{5}|c_{n+1}| + |c_n \pm c_{n+1}| \\ &\geq C\Big(\sum_{k=s}^{2^n} + \sum_{k=2^n+1}^{2s+1}\Big)|a_k| = \sum_{k=s}^{2s+1} |a_k|. \end{aligned}$$

(ii) *Taking* $d_n = 2^{-\frac{n}{2}}$, $n \in \mathbb{N}$, *in Theorem 5.9 and following the construction, we see that* $|c_n| = |a_k| \asymp 2^{-n}$, $2^{n-1} \leq k < 2^n$. *In other words, there is a uniformly convergent series* $\sum_{k=1}^{\infty} a_k \sin kx$ *such that*

$$m|a_m| \asymp \sum_{k=m}^{2m} |a_k| \asymp 1.$$

Moreover, in view of part (i) *of this remark,* $\{a_k\}$ *satisfies the following condition*

$$\sum_{k=m}^{2m} |\Delta a_k| \asymp 1 \asymp \frac{C}{m} \sum_{k=m}^{m+\lambda_m} |a_k|, \quad \lambda_m = m2^m.$$

In other words, $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$, *where* $\beta_n = \frac{1}{n} F_n(\widetilde{\mathbf{a}})$ *with non-admissible functionals* $F_n(\mathbf{x}) = \sum_{k=m}^{m+\lambda_m} \frac{|x_k|}{k}$. *This shows that Theorem 5.2 does not hold for non-admissible* *functionals.*

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