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and
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Chapter 1

Introduction

In the traffic management field, mathematical models have been exploited to evaluate different engineering alternatives. The outcome of such models is crucial in order to take decisions on how links in the traffic network should be built or rebuilt, and furthermore, how a given city should be arranged with new industrial, residential, and commercial areas. Except for estimating the utility of constructing new roads, traffic models can be beneficial for measuring the effects of congestion, for instance, changing speed limit and capacity (number of lanes), locating road tolls, or re-designing the intersections (traffic signs, roundabouts, turning lanes, etc.). Another application of such models is to advance action plans for dealing with traffic interruptions that are caused by incidents. In this thesis, a fundamental issue is the mathematical modeling of route choices for vehicles in a congested traffic network, also known as *traffic assignment problem.*

A first main distinction between traffic models is drawn with respect to their level of details. A *macroscopic* model uses fluid variables, such as density and flow, and does not model individual vehicles. A *microscopic* model describes vehicles (and commonly even drivers) individually. A *mesoscopic* model is in between, and merge the ideas from macroscopic and microscopic models. In fact, mesoscopic model uses macroscopic flow-speed relations to represent the motions of individual vehicles.

Traditionally, macroscopic models have been implemented for planning in larger networks over a longer time period. They can be used to investigate how the city network should be developed and built in the coming years, with respect to some assumption of population growth in different subareas of the city. In contrast, microscopic models are deployed for smaller networks, and the result is used for more particular measurements, as for instance how to allocate the available space into lanes in the best way, to handle a difficult intersection or sequence of intersections. Also, microscopic models usually take data from a macroscopic model as input, for instance by stating an average situation. A mesoscopic model can be applied to capture the overall changes in the traffic, induced by some detailed changes of the infrastructure.

One can also classify traffic models as time-independent (static) or time-dependent (dynamic) models.

A dynamic model reproduces the reaction of the traffic to a current situation, which indeed describes how a traffic scenario is developed over time. A static model can be described as a steady state in a dynamic model, i.e. a situation, where reactions and contra-reactions are balanced. Static models display average descriptions of the traffic situations, though the input data must provide more details on the traffic situation and we must make assumptions on how the traffic flow propagates both in time and space. Dynamic models are mostly based on simulation. Traditionally, macroscopic models are static, whereas we need a smaller and more detailed network of microscopic type to analyze time-dependent effects. We work both on the static and dynamic models.

The first issue in this thesis concerns the estimation of OD matrices in a network. An Origin-Destination matrix, shown briefly by OD matrix, is a matrix with entries indicating number of trips (flow) between each origin and destination in a network. In general, one has

$$[\text{OD matrix}] = [\text{assignment matrix}] \times [\text{traffic counts}],$$

or symbolically

$$(g_i)_i = (p_{ia})_{ia} \times (v_a)_a.$$

Above, a traffic count v_a indicates the total flow through a link $a \in A$ of the network, and A is the set of links on the network. The assignment matrix has the mission of splitting each traffic count into OD pairs, that is, the entries of the OD matrix.

In real life, a historical OD matrix is acquired through a combination of real data and mathematical models (see [22], [55], and [58]). By historical, we mean an a priori matrix, which comes from historical data and information about OD flows during past time slots. The high costs of traffic measurements makes it impossible to know real-life OD matrices in full precision, and here is where the problem begins. Indeed, in reality, one can only measure the flow not on all of A, but on a small subset $\widehat{A} \subset A$ of the network, by installing count loop detectors. This gives the set of observed traffic counts $\{\widehat{v}_a\}_{a \in \widehat{A}}$. After doing this, the rest of the OD matrix estimation process is carried out by the means of some mathematical model.

Generally, since the historical OD matrix is often affected by substantial and unavoidable errors, fitting closely the estimated OD matrix to the historical OD matrix might not be an indication of a good OD estimation process (see [3]). The problem of finding an OD matrix which corresponds to some given traffic counts observations $\{\widehat{v}_a\}_{a \in \widehat{A}}$ is highly under-determined in most cases. It can be seen as the inverse of the traffic assignment problem, which is highly over-determined, and reversely distributes the flow of a given historical OD matrix throughout a network and assign traffic counts to every link. Concretely, we have

$$\begin{array}{ll} \underline{\text{OD adjustment (or estimation) problem}} & \underline{\text{Traffic assignment problem}} \\ (\widehat{v}_a) \quad \Longrightarrow \quad (g_i) & (g_i) \quad \Longrightarrow \quad (v_a) \\ \text{underdetermined} & \text{overdetermined} \end{array} \tag{1.1}$$

Still, if we were able to observe traffic counts on all the links, the data would most likely be neither error-free nor consistent. Even if correct and error-free traffic counts observations would be available for all links in the network, there are still many OD matrices, which may do the desired job, that is, when assigned onto the network would induce the same observed traffic counts. The latter motivates the usage of a historical OD matrix.

Another challenge regarding the OD matrices is that they are usually high dimensional objects, and more often not all the OD flows are important to understand the underlying phenomena of interest. Although certain computationally expensive methods can build highly accurate OD estimation models with high dimensional OD matrices (as for instance [67]), in real-time applications it is still a matter of interest to reduce the dimensionality of OD matrices prior to any modeling. Indeed, the specification and estimation of the OD matrices is both methodologically and computationally complex for real time applications, because the OD matrices are high dimensional multivariate data structures. Recently, to tackle the high dimensional OD estimation problem, researchers have presented decomposition of network into smaller sub-areas ([44], [52]). Nevertheless, the OD estimation problem stays computationally rigorous, indeed these methods have to handle not only the high dimensionality of the OD matrix, but also the complexity of the methods. In fact, there are three factors that increase the computation cost: the size of the state vector, the complexity of model components (e.g. adjustment matrix, gradient matrices, etc.), and the number of measurements to be processed. For instance, a standard method to estimate OD matrices is Kalman's algorithm ([4], [6]). The computational complexity of the Kalman algorithm is of order $\mathcal{O}(n^3)$, where n is the total number of the OD pairs in the network. This can stand as a potential computational obstruction. Obviously, reducing the dimensionality of the state vector is a way to improve computational efficiency without a significant loss of accuracy of the estimated OD matrix. In this direction, a good method is the Principal Component Analysis (PCA) ([49]), which has been widely used by researchers (see for instance [36]). In any case, there has been many research activity focused on the OD matrix estimation, with more sophisticated and less time consuming algorithms (for instance see [10], [51], and [67]).

Given the a priori OD matrix, there are many ways to assign traffic counts v_a on a set of links $a \in \widehat{A}$. For instance, Frank and Wolfe algorithm, which is an iterative first-order optimization algorithm for constrained convex optimization. This algorithm was originally proposed by Marguerite Frank and Philip Wolfe in 1956 ([43]), which in each iteration considers a linear approximation of the objective function, and moves towards a minimizer of this linear function (taken over the same domain). In this thesis, though, we focus on Wardrop assignment, whcih is a well-known equilibrium for trasportation experts. John Glen Wardrop [65] considered the situation where a large number of vehicles have to go from one origin to a destination, connected by a finite number of different routes. Each single vehicle has to choose a path along the route in order to minimize some transportation cost which depends not only on the chosen path, but also on the number of vehicles along it. According to Wardrop equilibrium, the cost of every actually used path should be equal or less than that which would be experienced by a single vehicle on any other routes. In particular, all actually used paths have the same cost. In other words, a Wardrop equilibrium is a situation where every traveler uses only shortest paths from his origin to his destination, given the overall congestion pattern resulting from the individual strategies of all the users of the network. This equilibrium was first introduced in finite-dimensional networks [65], and is well known among geographical economists. In the present thesis, we analyze this concept both for discrete and continuous networks.

The process of assigning traffic counts v_a for a set of links $a \in \widehat{A}$ using Wardrop's equilibrium is called *Wardrop assignment*. This particular assignment is carried out via a traffic simulator software. Without loss of generality, in this thesis we use the Aimsun Next traffic simulator [63], but one can use any other traffic simulator, including open-source software. Essentially, an important feature that any simulator has to exhibit is the numerical stability of its results, i.e. small OD flow perturbations should lead to small consistent variations in the simulated outputs, in order to have a fair comparison among the results using different algorithms. One of the Aimsun Next most outstanding features is its computation speed. The built-in microscopic simulator is the fastest on the market by far and the finest micro-simulation tool available in the world for large-scale projects. Furthermore, dynamic user equilibrium techniques and stochastic/discrete route choice models are both available in Aimsun Next in combination with either mesoscopic or microscopic modeling.

The main objective of traffic analysts is to estimate an OD matrix in such a way that the corresponding assigned traffic counts v_a are as close as possible to the observed traffic counts $\hat{v}_a$. In this direction, transportation experts have been using least-squared differences between the observed and the estimated traffic counts as the objective model,

$$Z(g) := \frac{1}{2} \sum_{a \in \widehat{A}} (v_a - \widehat{v}_a)^2, \tag{1.2}$$

where v_a are the estimated traffic counts, and $\widehat{v}_a$ are the observed traffic counts.

In this thesis, three different objective models rather than the one in (1.2) are studied. The so-called Ridge and Lasso shrinkage coefficients are used to estimate an OD demand. In one of the proposed methods, the objective function is structured in such a way that the OD estimation process does not only estimate an OD matrix, but also reduces the dimension of the estimated OD matrix, by converging some of the OD flows to zero. A new solution approach is applied based on the well-known gradient descent algorithm for the OD estimation process. The proposed methods are tested on a real-size network, with a high dimensional historical OD matrix.

One of the goals of this thesis is to improve the objective function given in (1.2), in such a way that minimizing it provides us not just the best fitted estimated traffic counts v_a to the observed ones $\widehat{v}_a$, but also an estimated OD matrix g on the network, whose distance to the historical OD matrix remains under control. Indeed, one could naturally think of adding a penalty term, involving the estimated OD matrix in the objective function (1.2), and then minimizing it. This is where shrinkage methods, known as regularization, come in play. These methods add a penalty term to the objective function used in the model, and minimizing the objective function maximizes the accuracy. Shrinking the coefficient estimates reduces their variance considerably. In fact, shrinkage performance makes the coefficient converge to zero, which leads to decrease the dimensionality of OD matrix. The latter is the second goal of this part of the thesis.

The most well-known shrinking methods are Ridge regression and Lasso regression (see [48] for more information). Ridge regression aims to minimize the residual sum of squares (RSS), but with a slight change, which is adding a squared difference penalty term in order to predict an estimated OD matrix g which is not so far from the historical OD matrix $\widehat{g}$. In this case, the objective function becomes

$$Z_{Ridge}(g) = \frac{1}{2}\sum_{a\in\widehat{A}}(v_a - \widehat{v}_a)^2 + \gamma\frac{1}{2}\sum_{i\in I}(g_i - \widehat{g}_i)^2, \tag{1.3}$$

where γ is a non-negative constant called tuning parameter (TP), showing how much we want to rely on the historical OD matrix $\widehat{g}$ to fit the estimated OD matrix g.

The necessity of shrinkage methods and the value of tuning parameter TP appear due to the issues of underfitting or overfitting the data. The bias-variance trade-off indicates the level of underfitting or overfitting of the data. A high-bias, low-variance means the model is underfitted and a low-bias, high-variance means that the model is overfitted. Minimizing mean squared error, needs optimized bias-variance trade-off. Small quantity of tuning parameter TP causes underfitted model with respect to the historical OD matrix, while large quantity for tuning parameter TP causes overerfitted model with respect to the historical OD matrix. A balance to trade-off between the bias and the variance is needed, in order to achieve the perfect combination for the minimum mean squared error. During this thesis we consider a range of values for tuning parameter TP, in order to check the sensitivity of the model to this parameter.

Ridge regression has some advantages. It capitalizes on the bias-variance trade-off. As tuning parameter γ increases, the coefficients shrink more towards zero. Another advantage of ridge regression is its differentiability which indeed facilitates calculating the derivative, since later on in the next section, the so-called gradient descent algorithm is being used to minimize the objective functions.

On the other hand, Ridge regression has a major disadvantage, that it includes all the OD pairs, in the estimated OD matrix, regardless of their value which can be challenging for a big network with large number of OD pairs. This disadvantage is overcome by Lasso regression, which performs OD pairs selection. Lasso regression [62] uses L^1 norm penalty as compared to L^2 norm penalty used in the Ridge. Depending on whether we want to rely on the historical OD matrix or not, the updated objective function for the Lasso regression can be stated as two different models as follows

$$Z_{Lasso_1}(g) = \frac{1}{2}\sum_{a\in\widehat{A}}(v_a - \widehat{v}_a)^2 + \gamma\sum_{i\in I}|g_i - \widehat{g}_i|, \tag{1.4}$$

$$Z_{Lasso_2}(g) = \frac{1}{2}\sum_{a\in\widehat{A}}(v_a - \widehat{v}_a)^2 + \gamma\sum_{i\in I}|g| \\ = \frac{1}{2}\sum_{a\in\widehat{A}}(v_a - \widehat{v}_a)^2 + \gamma\sum_{i\in I}g_i, \tag{1.5}$$

where the last equality comes from the fact that number of trips in a network are always non-negative, therefore $|g_i| = g_i$.

Ridge regression brings the value of OD pairs in $g_i - \widehat{g}_i$ close to zero, whereas Lasso regression forces some of the OD pairs in $g_i - \widehat{g}_i$ to be exactly equal to zero.

From now on, depending on the choice of objective function, we denote (1.3), (1.4), and (1.5) by ridge, lasso_1, and lasso_2 methods, respectively. In Chapter 3, a new solution approach based on gradient descent algorithm is implemented, in order to find an estimated OD matrix which minimizes the proposed objective functions in (1.3), (1.4), and (1.5).

While trying to extend the notion of Wardrop equilibrium to the continuous setting, Carlier, Jiménez and Santambrogio proposed in [19] a continuous model for congested traffic equilibrium. Since then, interest in the topic has grown, and connections have been found by researchers in partial differential equations, calculus of variations and traffic engineering. In this thesis, we also explore these connections from a mathematical point of view.

The construction of Wardrop equilibriums presented in [19] was improved later by Brasco, Carlier and Santambrogio [15]. There it is described a more systematic way of constructing Wardrop equilibriums, also in an autonomous setting, as a natural continuation of the model presented in [19]. In this model, the pointwise traffic cost depends only on the pointwise traffic intensity. However, it may happen that the traffic intensity does not determine the traffic cost. For instance, two different locations with the same traffic intensity may have very different road conditions. This explains the need for non-autonomous traffic optimizers, which have already been object of study (see for instance [16] or [18]). Here we study a non autonomous counterpart, thus allowing for non-constant same-traffic cost.

The model described in [15] requires to understand the regularity theory for minimizers of certain highly degenerate integral functionals. This is the second main issue in the present thesis. Following their ideas, here we focus on optimals for minimizing problems

$$\inf_u \left(\int \mathcal{F}(x, \nabla u)\, dx + \int f(x)\, u\, dx \right)$$

where $f \in L^s_{loc}(\Omega)$ for some $s > n$, and $\mathcal{F}$ is a somewhat special Carathéodory function. In the gradient variable, $\mathcal{F}$ is radial, and has growth $p \geq 2$ away from a ball. As an example,

$$\mathcal{F}(x, \xi) = \frac{(|\xi| - 1)_+^p}{p},$$

where $(\cdot)_+$ denotes the positive part. This means that $\mathcal{F}$ is highly degenerate, as it can even vanish on the set of points where the gradient has small norm. This is in contrast with other classical minimization problems, as for instance the classical p-energy

$$\inf_u \left(\int \frac{|\nabla u|^p}{p}\, dx + \int f(x)\, u\, dx \right)$$

in which the energy only vanishes when $\nabla u = 0$. In the spatial variable, we will assume for $\mathcal{F}$ some degree of Sobolev dependence $W^{1,s}_{loc}$. We will show that minimizers are locally Lipschitz, that is, $\nabla u \in L^\infty_{loc}$ (see Theorem 15). Moreover, in Theorem 22 we show that for smooth boundaries the local Lipschitz estimates for minimizers are indeed global. Boundedness of the gradient certainly requires that $s > n$. Indeed, the result is known to be false when $s \leq n$, even in the easiest setting of uniformly elliptic integral functionals $\mathcal{F}$ with growth $p = 2$.

The supercritical Sobolev smoothness $s > n$ of $\mathcal{F}$ in the spatial variable is also used to show that minimizers admit locally integrable distributional derivatives up to second order, away from the degeneracy set, see Theorem 24. Although this may seem surprising, it fits with other previous results in this direction. Among the most representative ones, let us mention a result by Colombo and Figalli [26], which roughly asserts that the composition $\varphi(\nabla u)$ is continuous whenever φ is continuous and vanishes on the degeneracy set of $\mathcal{F}$. Thus, on the set where $\mathcal{F}$ is uniformly elliptic, may things can be said for u. Away from this set, though, nothing is known. In any case, Theorems 15 and 24 extend, respectively, previous contributions to the L^∞ bounds (for instance [42, Theorem 3.1]) and the Sobolev bounds (for instance extending [16, Theorem 4.1]), in which Lipschitz regularity in the space variable was assumed for the energy.

The apriori estimates described above have their own independent interest. Indeed, they extend the well known theory for the p-laplace equation,

$$-\operatorname{div}(|\nabla u|^{p-2}\, \nabla u) = f$$

to the much more degenerate equation

$$-\operatorname{div}\left((|\nabla u| - 1)_+^{p-1} \frac{\nabla u}{|\nabla u|} \right) = f,$$

where $(\cdot)_+$ denotes the positive part. As we said before, in the p-laplace equation the degeneracy set reduces to a singleton (the zero set, with respect to the gradient variable), while in the second case, as it usually happens in traffic congestion problems, the equation degenerates even at non-vanishing gradient points. These highly degenerate equations have already been object of study by many authors, see for instance [14, 26, 57]. At this point, we refer again to Colombo and Figalli [26], where the following example is presented,

$$\operatorname{div} \mathcal{A}(\nabla u) = 0, \qquad \text{for} \qquad \mathcal{A}(w) = (|w| - 1)_+^{p-1} \frac{w}{|w|}.$$

For u, nothing better than $\nabla u \in \mathcal{L}^\infty$ can be said, as it is obvious that every 1-Lipschitz function u is a solution, and there are many 1-Lipschitz functions whose derivatives are just bounded functions.

However, one can conclude from the results at [26] that $\sigma = \mathcal{A}(\nabla u)$ is continuous. To understand this, just observe that on the set where $|\nabla u| \leqslant 1$ one has $\sigma = 0$, while on the set where $|\nabla u| > 1$ the equation becomes more elliptic.

The construction of Wardrop equilibriums from [15] is the third issue we pay attention to. Basically, each Wardrop equilibrium is represented by a measure Q on the metric space of curves $C([0,1];\mathbb{R}^n)$, supported only on the absolutely continuous ones, fitting certain boundary data, and minimizing some *traffic cost* function. This kind of measures appear in a natural way in the classical Monge-Kantorovich optimal transport problem,

$$\inf \int |x-y|\, d\gamma(x,y) \tag{1.6}$$

where the infimum is taken over all probability measures γ with prescribed projections μ_0, μ_1 onto the first and second components, respectively. It is well known that this problem always admits a solution γ. Yet, it may be not so well known that the expression

$$Q = \int \delta_{[x,y]}\, d\gamma(x,y) \tag{1.7}$$

defines a measure on $C([0,1];\mathbb{R}^n)$, whose push-forward under the time evaulations at times $t=0$ and $t=1$ fits with the given data μ_0 and μ_1, respectively. Moreover, among all possible probabilities Q with this property, Q is the one that minimizes the *total length* integral functional,

$$\int \ell(\alpha)\, dQ(\alpha)$$

where $\ell(\alpha) = \int_0^1 |\alpha'(t)|\, dt$. Morally, the construction of Wardrop equilibriums presented in [15] extends this construction to other integral functionals different than the total length. As a consequence, Wardrop equilibriums Q can be represented as product measures

$$Q = \gamma \otimes p^{x,y}$$

where γ solves a Monge-Kantorovich problem like (1.6) but with a different distance cost (now depending on the traffic cost), and $p^{x,y}$ is a probability measure on $C([0,1];\mathbb{R}^n)$, suported on rectifiable curves joinning x and y, and describing what are the trajectories used by travellers in an ideal equilibrium situation. For instance, in (1.7) one has

$$p^{x,y} = \delta_{[x,y]},$$

that is, $p^{x,y}$ is the Dirac mass on the line segment joinning x and y. In general, though, much more complicate situations can occur. In particular, each $p^{x,y}$ may be supported on more than one single curve, or, instead, on just one, depending on the situation. This connects to challenging problems on the uniqueness of solutions to ordinary differential equations, which will also appear in the discussion.

As in the autonomous case presented in [15], one first needs to reduce the congested traffic problem to a question on very degenerate elliptic equations. Such reduction is possible because of the non-autonomous duality result [18, Theorem 3.1]. After this reduction, as in [15], the construction of equilibriums involves the implementation of the so-called Dacorogna-Moser scheme, for which, in turn, DiPerna-Lions theory of flows for weakly differentiable vector fields needs to be used. This is, in fact, the reason why the above mentioned apriori L^∞ and Sobolev estimates are needed. In their celebrated paper [35], DiPerna and Lions established a systematic way to construct flows

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(t, X(t,x)), \\ X(0,x) = x, \end{cases} \tag{1.8}$$

of vector fields $b : [0,T] \times \Omega \to \mathbb{R}^n$. The classical Cauchy-Lipschitz theory determines existence, uniqueness and regularity of such flows when b is Lipschitz. In [35], DiPerna and Lions extended the theory to vector fields b such that

$$b \in L^1(0,T; W^{1,1}_{\mathrm{loc}}) \qquad \text{and} \qquad \mathrm{div}(b) \in L^1(0,T; L^\infty),$$

together with some growth condition on b. This theory relates the existence and uniqueness of a well-defined flow $X(t,\cdot) : \Omega \to \mathbb{R}^n$ of measurable maps solving the equation (1.8) to the solvability of linear transport equations

$$\begin{cases} \partial_t u + b \cdot \nabla u = cu, \\ u(0,\cdot) = u_0, \end{cases}$$

for which the renormalization property was shown to be the key notion. As a corollary, flows solving (1.8) are shown to exist and be unique, and moreover to be nice in a measure theory sense; indeed,

$$X(t,\cdot)_\# dx \le C(t) \cdot dx,$$

with a constant $C(t)$ that depends on $\|\mathrm{div}(b)\|_{L^1(0,t;L^\infty)}$. This theory is needed for traffic issues in the implementation of Dacorogna-Moser scheme. In particular, it allows us to provide a Wardrop equilibrium of this form

$$Q = X_\#(\mu_0)$$

where μ_0 is the given starting data, and $X : \mathbb{R}^n \to C([0,1]; \mathbb{R}^n)$ is the flow map.

Boundedness and Sobolev regularity of the gradient of minimizers guarantee the existence and uniqueness of a DiPerna-Lions flow. By means of Dacorogna-Moser scheme, one proves this flow supports an explicit traffic configuration that minimizes the traffic cost. Moreover, it turns out that this configuration is indeed a Wardrop equilibrium, whose support consists of the set of rectifiable trajectories of a reasonably nice velocity field in the sense of DiPerna and Lions. See Theorem 37 in Subsection 5.7 for the precise statement of our result.

The thesis is structured as follows: Chapter 2 provides requisite preliminaries. Chapter 3 deals with the OD matrix estimation process, and dimension reduction in the OD demand. Section 3.1, describes proposed solution approach, based on the so-called gradient descent algorithm, in order to solve the minimization problems presented in (1.3), (1.4), and (1.5). Section 3.2 presents experimental setup and design. In section 3.3, results are presented. In Chapter 4 and section 4.4 we prove respectively, the interior Lipschitz estimates and the interior second order Sobolev estimates for minimizers of certain very degenerate integral functionals. In Section 4.3 we show that the results in Chapter 4 and section 4.4 admit a global version, up to the boundary, under enough boundary regularity. This is needed in the proof of the results at Chapter 5. In Chapter 5 we recall the fundamentals of congested traffic theory, and prove our main results in this direction (we are using here the global estimates of Section 4.3). Finally Chapter 6 provides conclusions and future research directions.

Chapter 2

Preliminaries

2.1 Direct method in the calculus of variations

Let us start with some largely heuristic insights as to when the functional

$$\mathbb{F}[w] = \int_{\Omega} \mathcal{F}(x, Dw(x))\, dx,$$

defined for appropriate functions $u : \Omega \to \mathbb{R}$ satisfying

$$w = g \qquad \text{on } \partial\Omega,$$

should have a minimizer.

Coercivity. First of all, we note that even a smooth function f mapping $\mathbb{R}$ to $\mathbb{R}$ and bounded below need not attain its infimum. Consider, for instance $f = e^x$, or $(1 + x^2)^{-1}$. These examples suggest that in general, we will need some hypothesis controlling $\mathbb{F}[w]$ for "large" functions w. Certainly the most effective way to ensure this would be to hypothesize that $\mathbb{F}[w]$ "grows rapidly as $|w| \to \infty$". More specifically, let us assume

$$1 < q < \infty,$$

is fixed. We will then suppose there exists a constant $\alpha > 0, \beta \geqslant 0$, such that $\mathcal{F}(x, \xi) \geqslant \alpha|\xi|^q - \beta$ for all $\xi \in \mathbb{R}^n, x \in \Omega$.

Therefore

$$\mathbb{F}[w] \geqslant \alpha \left\| Dw \right\|_{L^q(\Omega)}^q - \gamma, \tag{2.1}$$

for $\gamma := \beta|\Omega|$. Thus $\mathbb{F}[w] \to \infty$ as $\left\| Dw \right\|_{L^q(\Omega)} \to \infty$. It is customary to call (2.1) a coercivity condition on $\mathbb{F}[\cdot]$.

Lower semi continuous function. On a metric space X, a function$f : X \to \mathbb{R} \cup \{+\infty\}$ is called lower-semi-continuous (l.s.c. in short), if for every sequence $x_n \to x$ we have $f(x) \leqslant \liminf_n f(x_n)$.

Weak convergence. Let U denote an open, bounded, smooth subset of $\mathbb{R}^n$ with $n \geqslant 2$. We assume $1 \leqslant p < \infty$ and let p' be the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. A sequence $\{u_n\}_{n\geqslant 1} \subset L^p(U)$ converges weakly to $u \in L^p(U)$, in which case we write

$$u_n \rightharpoonup u, \qquad \text{in } L^p(U),$$

if

$$\int_U u_n v \, dx \to \int_U uv \, dx, \qquad \forall v \in L^{p'}(U).$$

Theorem 1. ***(Existence of minimizer).*** *Assume that $\mathcal{F}$ is coercive and convex in the variable ξ. Suppose also the admissible set $\mathcal{A}$ is nonempty. Then there exists at least one function $u \in \mathcal{A}$ solving*

$$\min_{w\in\mathcal{A}} \mathbb{F}[w].$$

Theorem 2. ***(Uniqueness of minimizer).*** *Suppose the mapping $\xi \mapsto \mathcal{F}(x, \xi)$ is strictly convex for every x. Then the minimizer $u \in \mathcal{A}$ for $\mathbb{F}[\cdot]$ is unique.*

Proof of Theorems 1 and 2 can be found in [39].

Euler-Lagrange equations. Let $\mathcal{F} : \Omega \times \Omega \to \mathbb{R}$, such that $\mathcal{F} = \mathcal{F}(x, \xi)$, where We associate with $\mathcal{F}$ the functional

$$\mathbb{F}[w] := \int_\Omega \mathcal{F}(x, Dw(x)) \, dx, \tag{2.2}$$

defined for smooth functions $w : \Omega \to \mathbb{R}$, satisfying, let say, the boundary condition

$$w = g \text{ on } \partial\Omega, \tag{2.3}$$

where $g : \partial\Omega \to \mathbb{R}$. Now, let us additionally suppose some particular smooth function u, satisfying the boundary condition $w = g$ on $\partial\Omega$, to be a minimizer of $\mathbb{F}[\cdot]$ among all functions w satisfying (2.3). We will show that u is then automatically a solution of a certain nonlinear partial differential equation. To confirm this, first choose any smooth function $v \in C_c^\infty$ and consider the real-valued function

$$F(\tau) := \mathbb{F}[u + \tau v].$$

Since u is a minimizer of $\mathbb{F}[\cdot]$ and $u + \tau v = u = g$ on $\partial\Omega$, we observe that $F(\cdot)$ has a minimum at $\tau = 0$. Therefore

$$F'(0) = 0.$$

We explicitly compute this derivative (called the first variation) by writing out

$$F(\tau) = \int_\Omega \mathcal{F}(x, Du + \tau Dv) \, dx.$$

Thus

$$0 = F'(0) = \int_{\Omega} D_\xi \mathcal{F}(x, Du + \tau Dv) \cdot Dv\, dx.$$

Finally, since v has compact support, we can integrate by parts and obtain

$$0 = -\int_{\Omega} \Big[D_x D_\xi \mathcal{F}(x, Du + \tau Dv) \Big] v\, dx.$$

As this equality holds for all test functions v, we conclude u solves the nonlinear quasilinear PDE

$$-D_x D_\xi \mathcal{F}(x, Du + \tau Dv) = 0 \qquad \text{in } \Omega.$$

This is the *Euler-Lagrange equation* associated with the energy functional $\mathbb{F}[\cdot]$ defined in (2.2).

2.2 Optimal transport

Monge problem. Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$ and a cost function $c : \Omega \times \Omega \to [0, +\infty]$, Gaspard Monge proposed the classical optimal transport problem in [54] as follows

$$\text{(Monge Problem)} \qquad \inf \left\{ \int c\Big(x, T(x)\Big)\, d\mu_0; T_\# \mu_0 = \mu_1 \right\},$$

where the measure denoted by $T_\# \mu_0$ means the push-forward of the measure μ_0 through the map T, and it is defined by $T_\# \mu_0(A) := \mu_0(T^{-1}(A))$ for every set $A \subset \Omega$.

Kantorovich problem. The main idea by Kantorovich is that of looking at Monge's problem as connected to linear programming. Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$ and a cost function $c : \Omega \times \Omega \to [0, +\infty]$, Kantorovich in 1942 [50] proposed the following problem

$$\text{(Kantorovich Problem)} \qquad \inf \left\{ \int_{\Omega \times \Omega} c(x, y)\, d\gamma(x, y); \gamma \in \Pi(\mu_0, \mu_1) \right\},$$

where $\Pi(\mu_0, \mu_1)$ is the set of the so-called **transport plans**, i.e.

$$\Pi(\mu_0, \mu_1) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (e_0)_\# \gamma = \mu_0, (e_1)_\# \gamma = \mu_1\},$$

where e_0 and e_1 are projection maps. The minimizers for this problem are called **optimal transport plans** between μ_0 and μ_1.

Theorem 3. *Let Ω be a complete and separable metric spaces, $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$, and $c : \Omega \times \Omega \to [0, +\infty]$ be lower-semi-continuous. Then Kantorovich Problem admits a solution.*

The proof of above theorem can be found in [56], where the concept of tightness of product measure space $\Pi(\mu_0, \mu_1)$ and Prokhrov's theorem have been used. Here the direct method in calculus of variations works.

The consequence of all these continuity, semi-continuity, and compactness results is the existence, under very mild assumptions on the cost c, and the space Ω, of an optimal transport plan γ. Then, if one is interested in the Monge problem, the question may become: does this minimal γ come from a transport map T? Should the answer to this question be yes, then Monge problem would have a solution, which also solves a wider problem that of minimizing among transport plans, which is the idea of the so-called Dual problem. The Kantorovich problem is a linear optimization under convex constraints, given by linear equalities or inequalities. Hence, an important tool is duality theory, which is typically implemented for convex problems.

Dual problem. Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$ and a cost function $c : \Omega\times\Omega \to [0, +\infty]$, the dual formulation of Kantorovich problem is stated as follows

$$\text{(Dual Problem)} \qquad \max\left\{ \int_\Omega \varphi \mathrm{d}\mu_0 + \psi \mathrm{d}\mu_1 : \varphi, \psi \in C_b(\Omega) : \varphi \oplus \psi \leqslant c \right\}.$$

We notice that sup(Dual problem) $\leqslant$ min(Kantorovich problem). Yet, Dual problem does not admit a straightforward existence result, since the class of admissible functions lacks compactness.

c-transform. Given a function $\chi : X \to \mathbb{R}$ we define its c-transform (also called c-conjugate function) $\chi^c : Y \to \mathbb{R}$ by

$$\chi^c(y) = \inf_{x\in X} c(x, y) - \chi(x),$$

and furthermore, we also define the $\bar{c}$-transform of $\xi : Y \to \mathbb{R}$ by

$$\xi^{\bar{c}}(x) = \inf_{y\in Y} c(x, y) - \xi(y).$$

Moreover, we say that a function ψ defined on Y is $\bar{c}$-concave if there exists χ such that $\psi = \chi^c$ (and, analogously, a function φ on X is said to be c-concave if there exists $\xi : Y \to \mathbb{R}$ such that $\varphi = \xi^{\bar{c}}$) and we denote by $c\text{conc}(X)$ and $\bar{c}\text{conc}(Y)$ the sets of c- and $\bar{c}$-concave functions, respectively (when $X = Y$ and c is symmetric this distinction between c and $\bar{c}$ will play no more any role and will be dropped as soon as possible).

Theorem 4. *We have* min(*Kantrovich problem*) = max(*Dual problem*). *More preciesly*

$$\min(\textit{Kantrovich problem}) = \max_{\varphi\in c-conc(\Omega)} \int_\Omega \varphi \mathrm{d}\mu_0 + \varphi^c \mathrm{d}\mu_1,$$

which also shows that the minimum value of the Kantorovich problem is a convex function of (μ_0, μ_1), *as it is a supremum of linear functionals.*

A proof to the above Theorem can be found in [56].

2.3 Legendre-Fenchel transform

Legendre-Fenchel transform. For any given function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ its Legendre transform is defined as $f^*(y) := \sup_x x \cdot y - f(x)$.

Example 5. *The Legendre transform of* $x \mapsto \frac{1}{p}|x|^p$ *is* $x \mapsto \frac{1}{q}|x|^p$ *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Example 6. *As another example, if* $f(x) := ax + \dfrac{x^p}{p}$*, such that* a *is constant, then the Legendre transform of* f *is* $f^*(x) = \dfrac{(|x| - a)_+^q}{q}$*, where the plus subscript indicates the positive part, and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 7. *A function* $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *is convex and l.s.c. if and only if* $f^{**} = f$.

Proof. The proof of this proposition, which is an application of the Hahn-Banach theorem, can be found in [5]. □

2.4 DiPerna-Lions Theory

Given a vector field $b : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$, and a measure $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$, a measure valued map

$$\mu : [0, 1] \to \mathcal{M}(\mathbb{R}^n)$$
$$t \mapsto \mu_t$$

is said to be a *weak solution* of the Cauchy problem for the continuity equation

$$\begin{cases} \frac{d}{dt} u + \operatorname{div}(b\, u) = 0 \\ u(0, \cdot) = \mu_0 \end{cases} \tag{2.4}$$

if for any testing function $\psi \in C_c^\infty([0, 1) \times \mathbb{R}^n; \mathbb{R})$ one has that

$$\int \psi(0, x)\, d\mu_0(x) + \iint \frac{\partial \psi}{\partial t}\, d\mu_t\, dt + \iint b \cdot \nabla \psi\, d\mu_t\, dt = 0$$

Equivalently, for each $\varphi \in C_b^\infty(\mathbb{R}^n)$ the real-variable function $t \mapsto \int \varphi\, d\mu_t$ is absolutely continuous with derivative

$$\frac{d}{dt} \int \varphi\, d\mu_t = \int b \cdot \nabla \varphi\, d\mu_t \qquad \text{for a.e. } t \in [0, 1].$$

Under enough smoothness, the problem (2.4) has as its unique solution $u(t, \cdot)$ the image measure of the initial datum μ_0 throught the flow $X(t, \cdot)$at time t,

$$u(t, \cdot) = X(t, \cdot)_\#(\mu_0).$$

Above, $X(t, \cdot)$ is obtained by solving the following ODE

$$\begin{cases} \dot{X}(t, x) = b(t, X(t, x)), \\ X(0, x) = x. \end{cases} \tag{2.5}$$

It is well known, though, that such flow X need not be well defined in general. In their celebrated thesis, DiPerna and Lions [35] developed a systematic way to understand transport and continuity equations for non-smooth vector fields b. One of the main points in their theory was to establish a relation between the notion of well-defined flow and the solvability of initial value problems for scalar conservation laws and the notion of renormalized solution. As in [15], we recall that when $\mu_t \in L^1_{loc}$ then μ_t is said to be a renormalized solution of (2.4) if, for every $\beta \in C^1(\mathbb{R})$, the equation

$$\frac{d}{dt}\beta(\mu_t) + b \cdot \nabla(\beta(\mu_t)) + \operatorname{div}(b)\, \mu_t\, \beta'(\mu_t) = 0$$

is satisfied in $(0,1) \times \Omega$ in the sense of distributions. Clearly, renormalized solutions are weak solutions (simply take $\beta(u) = u$) while the converse is false in general. Since the work of DiPerna and Lions, it is known that if b is a Sobolev vector field then weak solutions to the continuity equation are renormalized. The following result summarizes the main ideas of DiPerna - Lions theory.

Theorem 8. *Let $b \in L^1([0,1]; W^{1,1}(\Omega))$ be such that $\operatorname{div}(b) \in L^1([0,1]; L^\infty(\Omega))$. Then there is a unique continuous map*

$$X : [0,1]^2 \to L^1(\Omega; \mathbb{R}^n)$$
$$(t,s) \mapsto X(t,s,\cdot)$$

that leaves $\overline{\Omega}$ invariant, and such that the following holds:

(a) $X(t,s,\cdot)_{\#}dx$ is absolutely continuous with respect to the Lebesgue measure dx, and

$$e^{-|A(t)-A(s)|} \le \frac{d}{dx}(X(t,s,\cdot)_{\#}dx) \le e^{|A(t)-A(s)|}$$

where $A(t) = \int_0^t \|\operatorname{div}(b)(s,\cdot)\|_{L^\infty(\Omega)}\, ds$.

(b) If $0 \le r < s < t \le 1$ then $X(t,r,x) = X(t,s,X(s,r,x))$ for almost every $x \in \Omega$.

(c) For almost every $x \in \Omega$, $X(t,s,\cdot)$ is an absolutely continuous solution of (2.5), that is,

$$X(t,s,x) = x + \int_s^t b(r, X(r,s,x))\, dr.$$

(d) If $\mu_0 \in L^p(\Omega)$ and $s \in [0,1)$, then $u(t,\cdot) = X(t,s,\cdot)_{\#}(\mu_0)$ is the unique renormalized solution $u \in C^0([s,1]; L^p(\Omega))$ of the Cauchy problem (2.4) with initial condition $u(s,\cdot) = \mu_0$.

2.5 Superposition principle

We say that μ_t is a superposition solution of (2.4) if there exists a measure Q on the metric space $C([0,1];\mathbb{R}^n)$, supported on the subset of absolutely continuous trajectories of b, that is,

$$\int_{C([0,1];\mathbb{R}^n)} \left| \alpha(t) - \alpha(0) - \int_0^t b(s,\alpha(s))\, ds \right| dQ(\alpha) = 0 \qquad \text{for every } t \in [0,1],$$

and such that $\mu_t = (\pi_t)_{\#}Q$ for each $t \in [0,1]$, that is,

$$\int \varphi(x)\, d\mu_t(x) = \int_{C([0,1];\mathbb{R}^n)} \varphi(\alpha(t))\, dQ(\alpha)$$

for each $t \in [0,1]$ and all $\varphi \in C_c(\mathbb{R}^n)$. It can be seen that superposition solutions to (2.4) are indeed weak solutions. Superposition solutions are nicer in the sense that they can be represented as time evaluations of a more intrinsic measure Q on the set of rectifiable trajectories of b. In particular, this obviously allows for vector fields for which uniqueness of solutions in (2.5) may not be true. The following result was proven in [2, Theorem 2].

Theorem 9. *If*

$$\int_0^1 \int_{\mathbb{R}^n} \frac{|b(t,x)|}{1+|x|} \, d\mu_t(x) \, dt < \infty \tag{2.6}$$

then non-negative measure valued weak solutions are superposition solutions.

As a corollary, the following result holds.

Corollary 10. *If* (2.6) *holds, then for each Borel set $A \subset \mathbb{R}^n$ the following statements are equivalent:*

(a) Solutions to (2.5) *are unique for every $x \in A$.*

(b) Non-negative measure valued weak solutions to (2.4) *are unique whenever μ_0 is supported on A.*

Proof. Proving that (b) implies (a) is easy: given two solutions to (2.5) $\alpha(t)$ and $\tilde{\alpha}(t)$, it suffices to see that $\delta_{\alpha(t)}$ and $\delta_{\tilde{\alpha}(t)}$ solve (2.4). For the converse, since each weak solution μ_t is a superposition solution, one can write $\mu_t = (\pi_t)_{\#}Q$ for some Q supported on absolutely continuous trajectories of b. Thus, from (a) we can denote by $X(t,x)$ the well defined solutions of (2.5). In this situation, Q formally desintegrates as $Q = \delta_{X(\cdot,x)} \otimes \mu_0$. It then follows that

$$\mu_t = (\pi_t)_{\#}Q = (\pi_t)_{\#}(\delta_{X(\cdot,x)} \otimes \mu_0) = X(t,\cdot)_{\#}(\mu_0),$$

or in other words

$$\int \varphi(x) \, d\mu_t(x) = \int \varphi(\pi_t(\alpha)) \, dQ(\alpha) = \int \left(\int \varphi(\alpha) \, d\delta_{X(t,x)}(\alpha) \right) d\mu_0(x) = \int \varphi(X(t,x)) \, d\mu_0(x).$$

In particular, the uniqueness of the trajectories $X(t,x)$ implies uniqueness of the desintegration measures $\delta_{X(\cdot,x)}$, which in turn gives uniqueness of μ_t. □

As a consequence, positive measure-valued weak solutions to the continuity equation (2.4) are superposition solutions only assuming (2.6). Let us mention that everything above can be reformulated on bounded domains $\Omega \subset \mathbb{R}^n$, by adding boundary conditions on b of Neumann type.

2.6 OD matrices, and OD estimation problem

Assume I is the set of all Origin-Destination pairs, or briefly OD pairs, in a network. Then an OD matrix, or OD demand $g = g_i, i \in I$, is a matrix with entries indicating the number of trips between each OD pair i. Without loss of generality, we use the term OD matrix for g, although, for a convenient calculation, g has been used in vector format.

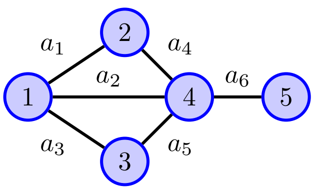


Figure 2.1: A simple network.

Concretely, Figure 2.1 shows a network with 5 centroids, $\{1, 2, 3, 4, 5\}$, and 6 links, $\{a_1, a_2, a_3, a_4, a_5, a_6\}$. For the OD pair $\{1, 5\}$, there are 3 paths $\{[a_1a_4a_6], [a_1a_2a_6], [a_3a_5a_6]\}$.

Let A be the set of all links in the network, and $\widehat{A} \subseteq A$ be the set of all links $a \in \widehat{A}$ with a loop detector on them. The traffic assignment of the OD matrix onto the links in the network induces an adjustment matrix $P = \{p_{ai}\}, i \in I, a \in A$, with p_{ai} being the proportion of the OD demand g_i that passes the link a. The traffic counts v and the OD matrix g are related to each other via the following formula

$$v_a := \sum_{i \in I} \sum_{k \in K_i} \sum_{a \in \widehat{A}} \delta_{ak} g_i,$$

where K_i is set of paths joining the OD pair i, and δ_{ak} is defined as follows

$$\delta_{ak} := \begin{cases} 1, & \text{if path } k \text{ includes the link } a, \\ 0, & \text{otherwise.} \end{cases}$$

We will use the notation $P = P(g)$ if needed, to emphasize that due to the congestion, these proportions depend on the the traffic volumes, i.e. on the OD matrix.

When assigned onto the network, the OD matrix induces a flow $v = \{v_a\}, a \in A$, on the links in the network. We assume that the set of observed traffic counts $\widehat{v} = \{\widehat{v}_a\}, a \in \widehat{A}$ is available, and that a historical matrix $\widehat{g}$ is also available.

In reality, we are often concerned about the reverse problem. More precisely, given a set of observed links $\widehat{A} \subseteq A$, the OD matrix estimation problem is about finding an OD matrix which, when it is assigned onto the network induces link flows $\{v_a\}_{a \in \widehat{A}}$ close to those which have been observed, i.e. $\{\widehat{v}_a\}_{a \in \widehat{A}}$. From now on, we denote $\{v_a\}_{a \in \widehat{A}}$ as the set of estimated traffic counts.

The OD matrix estimation problem is formulated as follows

$$\begin{aligned} & \min_{g,v} \ F(g, v), \\ & \qquad \text{subject to } \sum_{i \in I} p_{ai}(g) g_i = v_a, \quad \forall a \in \widehat{A}, \\ & \qquad g_i \geq 0, \quad \forall i \in I. \end{aligned} \tag{2.7}$$

In most of cases $F(g, v) = F_1(v, \widehat{v}) + \gamma F_2(g, \widehat{g})$, where $F_1(v, \widehat{v})$ and $F_2(g, \widehat{g})$ are generalized distance measures between the estimated traffic counts v and the observed traffic counts $\widehat{v}$, and between the estimated OD matrix g and the given historical OD matrix $\widehat{g}$, respectively. Also $\gamma > 0$ is called tuning parameter, and reflects the relative belief or uncertainty in the information provided by $\widehat{g}$. In one extreme case, using $\gamma = 0$, the historical OD matrix will have no influence.

Many types of distance measures for $F_1(v, \widehat{v})$ and $F_2(g, \widehat{g})$ have been considered by researchers during the last decades, such as maximum entropy [64], [8], and maximum likelihood [59] and [61]. Among all, the most common one, proposed in the in the models in the last decade is the least-square formulation. The least-square is a well-known deviation measure used in many types of estimation problems and is given by

$$F_1(v,\widehat{v}) := \frac{1}{2}\sum_{a\in\widehat{A}}(v_a - \widehat{v}_a)^2. \tag{2.8}$$

The objective function in Eq.(2.8) has a tractable solution, i.e. it is differentiable, and it has been traditionally used by many researchers ([60], [41], [53], and [37]).

Generally, there are many OD matrices which, when assigned onto the network, induce the same traffic counts. It is crucial thing to know, if the assignment matrix $P(g)$ is assumed to depend on g, i.e. if route choices are made with respect to congestion or not. In the latter case, $P(g) = P$ is a constant assignment matrix, and the first set of constraints in the generic description (2.7) is formulated as

$$\sum_{i\in I} p_{ai}(g) g_i = v_a, \quad \forall a \in \widehat{A}. \tag{2.9}$$

This equation, i.e. Eq.(2.9) is underdetermined as long as the number of OD pairs $|I|$ is greater than the number of observed traffic counts $|\widehat{A}|$. The problem (2.7) when the assignment matrix P is constant during the minimization process is in fact a linear problem. On the other way around, when $P(g)$ depends on OD matrix g, and changes during the minimization process, the problem (2.7) turns into a nonlinear problem. The latter is the proposed case in this thesis.

Considering constraint (2.9), equation (2.8) could be reformulated as follows

$$Z(g) := \frac{1}{2}\sum_{a\in\widehat{A}}(v_a - \widehat{v}_a)^2 = \frac{1}{2}\sum_{a\in\widehat{A}}\left(\sum_{i\in I} g_i P_{ia} - \widehat{v}_a\right)^2. \tag{2.10}$$

Many different terms as $F_2(g,\widehat{g})$ have been proposed by researchers. Among them, the first effort was made by Tao, et.al. in [66], where they have used Maximum Likelihood method for the term $F_2(g,\widehat{g})$. In Chapter 3, we will propose some distance measures for $F_2(g,\widehat{g})$, in order to improve the objective function in Eq.(2.10).

2.7 Gradient descent method

Gradient descent method is an iterative way to find a local minimum of a function. It starts with an initial guess of the solution. Then one takes the gradient of the function at that point. We step the solution in the negative direction of the gradient and repeat the process (shown in Figure 2.2). The algorithm will eventually converge where the gradient is zero (which correspond to a local minimum, under convenient convexity conditions). Its brother, the gradient ascent, finds the local maximum nearer the current solution by stepping it towards the positive direction of the gradient. They are both first-order algorithms because they take only the first derivative of the function.

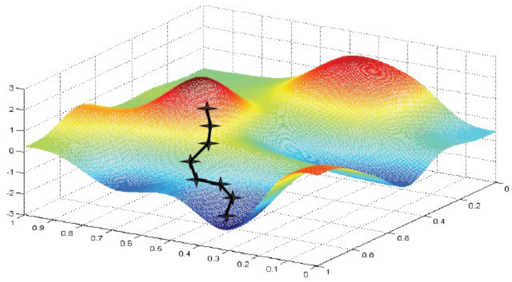


Figure 2.2: Gradient descent method.

Let's say we are trying to find the solution to the minimum of some function $f(x)$. Given some initial value x_0 for x, we can change its value in many directions (proportional to the dimension of x: with only one dimension, we can make it higher or lower). To figure out what is the best direction to minimize f, we take the gradient ∇f (the derivative along every dimension of x). Intuitively, the gradient will give the slope of the curve at that x and its direction will point to an increase in the function. So we change x in the opposite direction to lower the function value:

$$x_{k+1} = x_k - \gamma \nabla f(x_k).$$

The $\gamma > 0$ is a small number that forces the algorithm to make small jumps. This keeps the algorithm stable and its optimal value depends on the function. Given stable conditions (a certain choice of γ), it is guaranteed that $f(x_{k+1}) \leqslant f(x_k)$. A wrong step size γ may not reach convergence, so a careful selection of the step size is important; too large it will diverge, too small it will take a long time to converge. One option is to choose a fixed step size that will assure convergence wherever you start gradient descent. Another option is to choose a different step size at each iteration (adaptive step size). In this thesis, we use optimal step size by taking the gradient with respect to step size. Therefore, we choose the best step length in each step of algorithm. This choice has two advantageous at the same time: First, it converges, and second, the speed of convergence is fast.

2.8 Aimsun Next software

Aimsun Next is traffic modeling software that allows you to model anything from a single intersection to an entire region. With thousands of licensed users in government agencies, consultancies and universities all over the world, Aimsun Next stands out for the exceptionally high speed of its simulations and for fusing travel demand modeling, static and dynamic traffic assignment with mesoscopic, microscopic and hybrid simulation all within a single software application.

Without loss of generality, one can use any other traffic simulation software, such as open source ones. In this thesis we use Aimsun Next simulator because it allows users to carry out traffic operations assessments of any scale and complexity. Furthermore, Aimsun Next has more advantageous rather than other simulation software, which are the following

- Assessment and optimization of Transit Signal Priority (TSP) and Bus Rapid Transit (BRT) schemes.
- Priority (TSP) and Bus Rapid Transit (BRT) schemes.
- Feasibility studies for High Occupancy Vehicle (HOV) and High Occupancy Toll (HOT) lanes.
- Impact analysis of infrastructure design such as highway corridor improvement/construction.
- Environmental impact analysis.
- Evaluation of travel demand management (TDM) strategies.
- Evaluation of Variable Speed policies and other Intelligent Transportation Systems (ITS).

Aimsun Next software is available for download at: https://www.aimsun.com.

Chapter 3

Static OD matrix problem

In this chapter we propose a new solution approach base on the so-called gradient descent method, in order to minimize the proposed Ridge, Lasso_1, and Lasso_2 models in (1.3), (1.4), and (1.5), with respect to the OD matrix g.

3.1 New solution approach

In this section, the so-called gradient descent approach (steepest descent) is applied to solve OD estimation problems defined in equations (1.3), (1.4) and (1.5). Specifically, solution algorithm based on the gradient descent algorithm, proposed by Spiess (see [60] for more information), has been applied. The idea of the steepest descent method is to calculate the gradient of the objective functions given in equations (1.3), (1.4) and (1.5), with respect to g, and then move one step in the direction of negative gradient to determine new estimated OD matrix. The objective functions given in the equations (1.3), (1.4) and (1.5) are neither convex nor linear. Indeed, due to the change of OD matrix g, the assignment matrix $P(g)$ is not locally constant during the steepest descent algorithm, and is changing in each step of iteration. Therefore traffic counts v are changing in each step of iteration. Thus minimizing the objective functions given in the equations (1.3), (1.4) and (1.5), might not have a unique minimum.

Spiess in 1990 suggested a heuristic approach to solve the problem (2.10). It is an iterative procedure, in which $\lambda_1 = 0$, and $\widehat{g}$ is used as initial solution. In his approach, an approximate gradient of the objective function with respect to the OD matrix is computed, under the assumption that the assignment matrix $P(g)$ is locally constant.

According to his paper [60], the new estimated OD matrix in each iteration step s, in the gradient descent algorithm, has been defined as follows

$$g_{s+1} = g_s(1 - \lambda_s \nabla_{g_s} Z(g_s)). \tag{3.1}$$

However, to preserve zero OD flows in the estimated OD matrix, as given by starting point (i.e., historical OD matrix) we propose modification of the update step given in Eq.(3.1) in each iteration step as follows

$$g_{s+1} := g_s - \lambda_s \nabla_{g_s} Z(g_s). \tag{3.2}$$

The advantage of the update step modification is twofold. Firstly, computation process is speed up by omitting two matrix multiplication in each iteration step. Secondly, the zero flows in the historical OD matrix are preserved in the estimated OD matrix. Next, we demonstrate gradient derivation for OD estimation problems defined in Eq.(1.3), Eq.(1.4), and Eq.(1.5), based on the modified descent gradient solution.

3.1.1 Ridge OD objective function

At every external iteration, the modified gradient descent method is selected to minimize the objective function in Eq.(1.3), which uses the gradient of the objective function with respect to g as the search direction

$$\begin{aligned}\nabla Z_{Ridge}(g) &= \frac{\partial_t}{\partial_t g}\left[\frac{1}{2}\sum_{a\in\widehat{A}}(gp_a - \widehat{v}_a)^2 + \frac{1}{2}\gamma\sum_{i\in I}(g_i - \widehat{g}_i)^2\right] \\ &= \sum_{a\in\widehat{A}} P_a(v_a - \widehat{v}_a) + \gamma\sum_{i\in I}(g_i - \widehat{g}_i).\end{aligned} \tag{3.3}$$

The next step is to take the gradient of the Eq.(1.3), but this time with respect to λ and make it equal to zero, in order to find the best optimal step length λ in each step of iteration. The main advantage of choosing optimal step length over the choices like a fixed step length or line search is that, it guarantees the convergence of algorithm. Therefore taking the gradient of (1.3) with respect to λ, we get

$$\frac{\partial_t}{\partial_t \lambda}\left[\sum_{i\in I}\sum_{a\in\widehat{A}}\frac{1}{2}((g_i - \lambda\nabla Z_{Ridge})P_a - \widehat{v}_a)^2 + \gamma\frac{1}{2}(g_i - \widehat{g}_i)^2\right] = 0,$$

then calculating optimal step λ, within each step of iteration, using the gradient method, we obtain

$$\lambda_{Ridge}^{\text{optimal}} = \sum_{i\in I}\sum_{a\in\widehat{A}}\frac{P\nabla Z_{Ridge}(v_a - \widehat{v}_a) + \gamma\nabla Z_{Ridge}\cdot g_i - \gamma\nabla Z_{Ridge}\cdot\widehat{g}_i}{(P\nabla Z_{Ridge})^2 + \gamma(\nabla Z_{Ridge})^2}. \tag{3.4}$$

3.1.2 Lasso_2 OD objective function

Similarly, choosing (1.5) as the objective function and taking the gradients with respect to g and λ, we get

$$\nabla Z_{Lasso_2}(g) = \frac{\partial_t}{\partial_t g}\left[\frac{1}{2}\sum_{a\in\widehat{A}}(gp_a - \widehat{v}_a)^2 + \gamma\sum_{i\in I} g_i\right] = \sum_{a\in\widehat{A}} P_a(v_a - \widehat{v}_a) + \gamma\sum_{i\in I}\mathbf{Id}, \tag{3.5}$$

where **Id** indicates the identity matrix, and the step length λ will be obtained as follows

$$\lambda_{Lasso_2}^{\text{optimal}} = \sum_{i\in I}\sum_{a\in\widehat{A}}\frac{P\nabla Z_{Lasso_2}(v_a - \widehat{v}_a) + \gamma\nabla Z_{Lasso_2}}{(P\nabla Z_{Lasso_2})^2}. \tag{3.6}$$

3.1.3 Lasso_1 OD objective function

The last but not least is to apply proposed descent gradient solution algorithm on the objective function in Eq.(1.4). The difficulty here is to derivate the objective function, due to the non-differentiability of absolute value term $|g_i - \widehat{g}_i|$ at zero. This issue is addressed by using three different strategies.

Strategy 1.

The non-differentiability issue can be tackled, by using the idea of the smoothing the L^1 term. This is carried out by convolution with the so-called Gaussian kernel. The Gaussian kernel is defined as follows

$$\text{G\_kernel}(g) := \frac{\sigma}{\sqrt{2\pi}} \exp\left(\frac{-(g-\widehat{g})^2}{2\sigma^2}\right),$$

where σ is an arbitrary positive constant (called width of Gaussian curve). Now, convolving the L^1 term in Gaussian kernel, the corresponding objective function is updated as follows

$$Z_{Lasso_1}^{\text{strategy}_1}(g) = \sum_{a\in\widehat{A}} \frac{1}{2}(gp_a - \widehat{v}_a)^2 + \gamma \sum_{i\in I} |g - \widehat{g}| * \text{G\_kernel}, \tag{3.7}$$

then taking the gradient with respect to the OD matrix g, we get

$$\nabla Z_{Lasso_1}^{\text{strategy}_1}(g) = \frac{\partial_t}{\partial_t g}\left[\sum_{a\in\widehat{A}} \frac{1}{2}(gp_a - \widehat{v}_a)^2 + \gamma \sum_{i\in I} |g_i - \widehat{g}_i| * \text{G\_kernel}\right]. \tag{3.8}$$

By using the property of derivative of the convolution ($\frac{\partial_t}{dx}[f(x) * h(x)] = f(x) * \frac{\partial_t}{dx} h(x)$), equation (3.8) becomes

$$\nabla Z_{Lasso_1}^{\text{strategy}_1}(g) = \sum_{a\in\widehat{A}} P_a(v_a - \widehat{v}_a) + \gamma \sum_{i\in I}\left(|g_i - \widehat{g}_i| * \frac{-(g_i - \widehat{g}_i)}{\sigma^2} \cdot \text{G\_kernel}\right).$$

Respectively, taking the derivative of (3.8), but this time with respect to λ, results in

$$\begin{aligned}
&\frac{\partial_t}{\partial_t \lambda}\Bigg[\sum_{i\in I}\sum_{a\in\widehat{A}} \frac{1}{2}\left((g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1})P_a - \widehat{v}_a\right)^2 \\
&\quad + \gamma|g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1} - \widehat{g}_i| * \text{G_kernel}(g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1})\Bigg] \\
&\quad = \sum_{a\in\widehat{A}}\sum_{i\in I} -\nabla Z_{Lasso_1}^{\text{strategy}_1} \cdot P_a\left((g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1})P_a - \widehat{v}_a\right) \\
&\quad + \left(\gamma|g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1} - \widehat{g}_i|\right) * \Bigg(\frac{\nabla Z_{Lasso_1}^{\text{strategy}_1}(g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1} - \widehat{g}_i)}{\sigma^2} \\
&\quad \cdot \text{G_kernel}((g_i - \lambda\nabla Z_{Lasso_1}^{\text{strategy}_1})\Bigg),
\end{aligned}$$

which can not be solved analytically (due to the existence of exponential term), but it can be solved numerically by using methods such as Newton-Raphson interpolation (there are many other root-finding commands, such as Newton-Raphson interpolation, in most of language programs, including R language program).

Strategy 2.

The second way is to use the definition of absolute value, meaning that the objective function can be formulated as follows

$$Z_{Lasso_1}^{\text{strategy}_2}(g) = \begin{cases} \sum\limits_{i\in I}\sum\limits_{a\in\widehat{A}} \frac{1}{2}(gp_a - \widehat{v}_a)^2 + \gamma g_i - \widehat{g}_i, & \text{if } g_i \geqslant \widehat{g}_i, \\ \sum\limits_{i\in I}\sum\limits_{a\in\widehat{A}} \frac{1}{2}(g_i p_a - \widehat{v}_a)^2 - \gamma g_i + \widehat{g}_i, & \text{if } g_i < \widehat{g}_i, \end{cases} \tag{3.9}$$

hence, taking the derivative with respect to the OD matrix g, we get

$$[\nabla Z_{Lasso_1}^{\text{strategy}_2}(g)]_{i,j} = \begin{cases} \sum\limits_{a\in\widehat{A}} P_a(v_a - \widehat{v}_a) + \gamma \sum\limits_{i\in I} \mathbf{Id}, & \text{if } g_i \geqslant \widehat{g}_i, \\ \sum\limits_{a\in\widehat{A}} P_a(v_a - \widehat{v}_a) - \gamma \sum\limits_{i\in I} \mathbf{Id}, & \text{if } g_i < \widehat{g}_i, \end{cases} \tag{3.10}$$

and for the optimal step length λ, we have

$$[\lambda_{Lasso_1}^{\text{optimal}}]_{i,j} = \begin{cases} \sum\limits_{i\in I}\sum\limits_{a\in\widehat{A}} \frac{P\nabla Z_{Lasso_1}^{\text{strategy}_2}(v_a - \widehat{v}_a) + \gamma \nabla Z_{Lasso_1}^{\text{strategy}_2}}{(P\nabla Z_{Lasso_1}^{\text{strategy}_2})^2}, & \text{if } g_i \geqslant \widehat{g}_i, \\ \\ \sum\limits_{i\in I}\sum\limits_{a\in\widehat{A}} \frac{P\nabla Z_{Lasso_1}^{\text{strategy}_2}(v_a - \widehat{v}_a) - \gamma \nabla Z_{Lasso_1}^{\text{strategy}_2}}{(P\nabla Z_{Lasso_1}^{\text{strategy}_2})^2}, & \text{if } g_i < \widehat{g}_i. \end{cases} \tag{3.11}$$

Strategy 3.

The third way is to use the gradient formula achieved in *strategy 2*, and then by using the formula (3.4) as the optimal step length λ. Notice that due to the similarity of the objective functions Z_{Ridge} and Z_{Lasso_1} when plotting, one can think of using optimal step length for the objective function Z_{Ridge}, to be used as the one in Z_{Lasso_1} (at least, it is much more efficient than other non-optimized typically used step lengths seeker algorithms, such as line search or the fixed step length).

Naturally, all OD flows (number of trips), in an estimated OD matrix are considered to be positive, and if it is not the case, i.e. in each step of iteration, if there exists a negative OD flow in the estimated OD matrix, it is replaced by a small positive constant less than 1 (for instance 0.1 is used in this thesis). This technique is called projected gradient descent, which can be found in [9]. Substituting the negative OD flows with small positive constants prevents stopping the iteration in the algorithm.

Remark 11. *Notice that the optimal step length calculated in strategy 2 is not a constant, but a matrix.*

Remark 12. *Although, all 3 strategies above have been applied on the experimental samples, and it turned out they all provide the same result, we arbitrarily use the third strategy in the experimental section.*

3.2 Experimental setup

3.2.1 Experimental design

The benchmarking platform adopted in this thesis is based on a combination of Aimsun Next (2017) as traffic simulator, and R (2017) and Python (2017) as programming languages for running the OD matrix estimation algorithms, with the calls between these two programs are defined through Python. The algorithm is executed on a computer with Intel core-i7 quad-core processor, and 6 gigabyte of RAM. The static macroscopic network loading model and static user equilibrium (UE) route choice model in Aimsun Next have been used in the experiments.

First of all, lets see how the estimation OD matrix algorithm works using interaction between Aimsun Next and R:

- Aimsun Next receives an OD matrix generated by the estimation algorithm (or the historical OD matrix in the first iteration) as input;

- Inside the R code, it calls Aimsun Next for a new traffic simulation run with the new OD matrix and waits until the simulation ends. The outputs of this step are the vales of assigned traffic counts v, and the assignment matrix P. The actual communication of the instructions for the Aimsun Next call is done through Python;

- Next step is being carried out within the R language. The R code calculates gradient of the objective functions obtained in (3.3), (3.11) and (3.10), and afterward, it evaluates the optimal step lengths λ^{optimal} obtained in (3.4) and (3.6). Then it calculates the new estimated OD matrix using the gradient descent update rule (3.2) (recalling that, for the lasso_1 method we are following the idea presented in the third strategy).

3.2.2 Network layout

The algorithms have been tested out on two different networks:

- The first network, which we call toy network, includes 7 centriods, 49 OD pairs, and 7 count detectors. This small-sized network is used primarily for debugging and verification purposes.

- The second network, called Vitoria network (Vitoria city in Spain), shown in figure (3.1) is a medium-sized urban network, with a highway stretch north of the city centre. This network comprises 57 centroids, 3249 OD pairs, and a total road length of 631 km, corresponding to 5799 links. The network detection consists of 397 traffic loop detectors which are located both in the urban center as on the highway stretch. The Vitoria network resembles a reasonable sized real-life network, with representative congestion levels and route choice dimension, as found in many large urban areas.

Figure 3.1: The Vitoria network

3.2.3 Demand setup

In order to test the OD matrix estimation algorithms, a historical OD matrix should be provided as input. In general, the quality of such a historical OD matrix, in terms of both demand level (number of trips) and patterns, is a key element impacting the performance of the estimation algorithms, and it is even more relevant for congested networks. Consequently, selecting proper perturbations of an OD matrix coming from surveys, in order to derive the historical OD matrix as input for the OD estimation algorithm, is a crucial task. Accordingly, three different historical OD matrix scenarios have been chosen for the experiments in this thesis:

- ***Low-demand scenario (LD):*** This scenario deals with situations in which, for instance, historical OD matrix results from out-of-date surveys. More precisely, the low-demand scenario is defined as the 85% of the historical OD matrix coming from surveys, plus uniformly distributed random fluctuations over each OD pair in the range of $[-35\%, +35\%]$ of the historical OD matrix. In another way, the model under-fitted the historical OD matrix coming from survey. Therefore

$$[\widehat{g}^{\mathrm{LD}}]_{i,j} := [g_{\mathrm{survey}}]_{i,j} \cdot (0.5 + 0.7\alpha_{i,j}), \quad \alpha \in U(0,1).$$

- ***Medium-demand scenario (MD):*** This scenario is defined as the 95% of the historical OD matrix coming from surveys, plus uniformly distributed random fluctuations over each OD pair in the range of $[-15\%, +15\%]$ of the historical OD matrix. In another way, the model slightly under-fitted the historical OD matrix coming from survey. Therefore

$$[\widehat{g}^{\mathrm{MD}}]_{i,j} := [g_{\mathrm{survey}}]_{i,j} \cdot (0.8 + 0.3\alpha_{i,j}), \quad \alpha \in U(0,1).$$

- ***High-demand scenario (HD):*** This scenario deals with situations in which, for instance, the historical OD matrix reflects travel demand when congestion occurs on the network, in peak-hours. More precisely, the High-demand scenario is defined as the 115% of the historical OD matrix coming from surveys, plus uniformly distributed random fluctuations over each OD pair in the range of $[-35\%, +35\%]$ of the historical OD matrix. In another way, the model over-fitted the historical OD matrix coming from survey. Therefore

$$[\widehat{g}^{\text{HD}}]_{i,j} := [g_{\text{survey}}]_{i,j} \cdot (0.8 + 0.7\alpha_{i,j}), \quad \alpha \in U(0,1).$$

Remark 13. *We execute the required analysis on the toy network, under all these three scenarios, i.e. LD, MD, and HD. Without loss of generality, we just consider the LD scenario for the Vitoria network.*

Remark 14. *The criteria to stop the iteration in the OD estimation algorithm, described in subsection 3.2.3, are as follows:*

- *In the toy network experiment, the iteration is stopped once the difference between two consecutive objective function values is less than* 1.
- *In the Vitoria network experiment, the iteration is stopped once the difference between two consecutive objective function values is less than* 500.

3.2.4 Performance assessment

In order to assess the performance of all the three proposed OD matrix estimation methods, i.e., ridge, lasso_1 and lasso_2, to understand bounders of the proposed methods and identify which method performs the best, the following statistics have been used:

- ***Root mean squared error (RMSE):*** The RMSE represents the sample standard deviation of the differences between estimated and observed values. It is the square root of the average of squared differences between estimated and actual observation, and is defined as follows

  $$\text{RMSE} := \sqrt{\frac{1}{n}\sum_{j=1}^{n}(y_j - \widehat{y}_j)^2},$$

  where n is the number of samples, $\widehat{y}_j$'s are the observed values, and y_j's are the estimated values.

- ***GEH index:*** The GEH Statistic is another measure, used to compare two sets of traffic counts. It gets its name from Geoffrey E. Havers (UK Highways Agency, 1992). This index is commonly used as validation criteria in many applications following the recommendations suggested in many guidelines published by different road administrations, such as FHWA in USA, Highways Agency in UK, or ARRB in Australia (for more instances and the definition of GEH index, see [34], [21], and [33]). The GEH statistic of the estimated and observed traffic counts is formulated as follows

  $$\text{GEH} := \sqrt{\frac{2(v - \widehat{v})^2}{v + \widehat{v}}}.$$

- ***Trip length distribution (TLD):*** Trip length distribution, indicated by TLD is another measure of statistic, which has been commonly used in the traffic management field ([47], and [23]). TLD is defined as follows

  $$\text{TLD} := \frac{\text{flow of the OD pair AB}}{\text{distance of the shortest path between A and B}}$$

3.3 Results

In this section, the results of two case-studies, toy and Vitoria network, are presented for three developed OD matrix solution approaches, given the different scenarios that were described in the previous section.

3.3.1 Toy network

In order to examine sensitivity of the three proposed methods, using the toy network, four values as the tuning parameter TP have been selected; $TP \in \{0, 0.1, 1, 100\}$.

The performance of the proposed OD matrix estimation methods, in terms of reducing dimensionality, is presented in Figure (3.2). We analyze the sensitivity of the results under three conditions; the tuning parameters, demand scenarios, and the methods. From now on, in order to avoid eventual confusions, when using the ridge and lasso_1 methods, by the term converged OD flows, we mean the estimated OD flows which are converged to the corresponding historical OD flows. Furthermore, when using lasso_2 method, by the term converged OD flows, we mean the estimated OD flows which are converged to zero.

As the tuning parameter increases, number of OD flows converged to the corresponding historical OD flows in the ridge and lasso_1, and number of OD flows converged to zero in the lasso_2, increase too, using both demand scenarios. Indeed, when minimizing the objective functions (1.3), (1.4), and (1.5), increasing the tuning parameter in terms $\gamma(g_i - \widehat{g}_i)^2$, $\gamma|g_i - \widehat{g}_i|$, and γg_i leads the terms $(g_i - \widehat{g}_i)^2$, $|g_i - \widehat{g}_i|$, and g_i decrease more toward zero, and this results in converging higher number of OD flows.

Changing the demand scenario from the LD scenario to the HD scenario, leads in higher number of converged OD flows. This is due to the fact that when the HD scenario is chosen, the perturbed historical OD matrix is in fact overestimated. Consequently, the terms $(g_i - \widehat{g}_i)^2$, $|g_i - \widehat{g}_i|$, and g_i blow up, and when minimizing the objective functions (1.3), (1.4), and (1.5), it results in higher number of converged OD flows.

When minimizing the objective functions (1.3), (1.4), and (1.5), we have

$$\beta_{lasso\_1} := |g_i - \widehat{g}_i| \leq \beta_{ridge} := (g_i - \widehat{g}_i)^2 \leq \beta_{lasso\_2} := g_i, \qquad \forall g_i, \text{ s.t. } |g_i - \widehat{g}_i| \geq 1, \tag{3.12}$$

which indeed proves that switching between methods respectively, from the ridge to the lasso_1, and then to the lasso_2, the number of converged OD flows increases. This is also shown in the figure (3.2).

Comparing the ridge with the lasso_1, number of converged OD flows using the lasso_1, is higher than the number of converged OD flows using the ridge. Consequently $\beta_{lasso\_1} \leq \beta_{ridge}$, for coefficients β_{lasso_1} and β_{ridge} defined in (3.12). Therefore in the OD estimation process with the objective functions (1.3), (1.4), and (1.5), the dimensionality of the coefficient terms β_{lasso_1} and β_{ridge} reduces, and accordingly the dimensionality of their gradient in the OD estimation process reduces too. Therefore we expect to see that CPU running time using the lasso_1 method is less than CPU running time of the ridge method. The same discussion satisfies for the lasso_2, which is expected to be the fastest among the other methods, in terms of CPU running time of the OD estimation process.

The lasso_2 method reduces the dimensionality of the estimated OD matrix by removing the OD flows that converged to zero from the state vector. Practically, the lasso_2 method can be employed when the network is large, and it consists of long highways with large volumes, while there are also short-distance trips with small dispensable volumes, which can be discarded.

Speaking of reduction in dimensionality, the lasso_2 method performs the best. When the goal is to have higher number of converged OD flows, the optimal value of tuning parameter is $TP = 100$, for all the demand scenarios and all the methods.

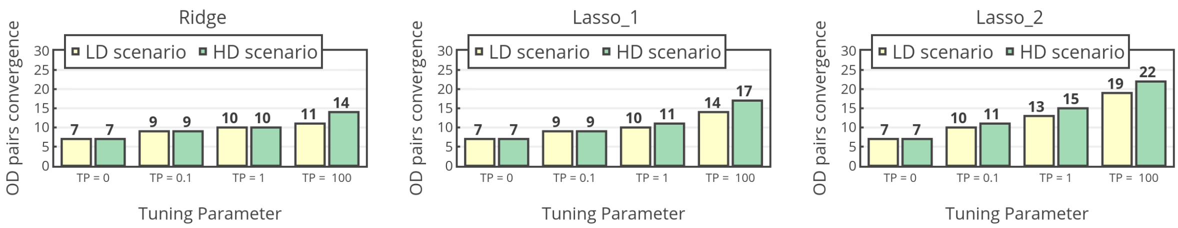


Figure 3.2: Number of non-zero estimated OD flows which converged exactly to the corresponding historical OD flows using the ridge and lasso_1 methods, and number of non-zero estimated OD flows which converged exactly to zero using the lasso_2 method (the toy network).

Figure (3.3) shows the RMSE between the estimated and the historical OD demand per scenario for tested OD matrix solution approaches. Increading the tuning parameter, lasso_2 method performs better than the ridge and lasso_1 methods, when the model is supposed to not have overfitting to the historical OD matrix. Furthermore, it is indicated that when the implemented methods are ridge and lasso_1, increasing the tuning parameter, changes the estimated OD matrix to fit better to the historical OD matrix.

Concerning demand scenarios, changing receptively from the LD scenario to the MD scenario and then to the HD scenario, the OD estimation process changes from an under-fitted historical matrix to a over-fitted historical matrix. Therefore indeed, coefficients β_{ridge} and β_{lasso_1} are decreasing, and consequently RMSE between the historical OD matrix and the estimated OD matrix is decreasing.

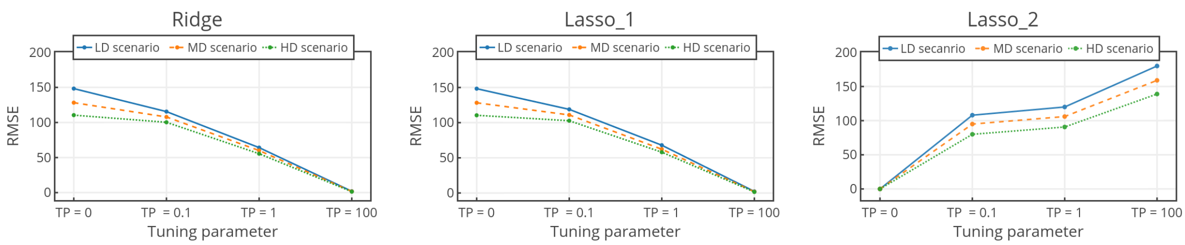


Figure 3.3: RMSE for the historical and the estimated OD matrix, for different methods, over all scenarios (the toy network).

Table (3.1)(a) shows the CPU computation time for the low demand (LD) scenario. We can observe in this table, significant CPU computation time reduction of the proposed lasso_2 method compared to the other two methods, since in each iteration step simplifies the estimated OD matrix by reducing its dimensionality, by omitting the OD flows equal to zero. Consequently, computation time decreases with the number of OD pairs to be estimated as iterations progress.

Recalling the definition of the trip length distribution (TLD) statistic, it is the proportion of the each OD flow to the distance of the shortest path in the corresponding OD pair. Thus, decreasing the OD flow will decrease the TLD. In another word, less congested OD pairs have less TLD, and using the lasso_2 method, the corresponding OD flows are expected to converge to zero. This reduces the dimensionality of the estimated OD matrix.

Table (3.1)(b) shows the percentage of the reduction in dimensionality of the estimated OD matrix, using the lasso_2 method, and for the low demand (LD) scenario. As the tuning parameter increases, the dimensionality of the estimated OD matrix reduces. Also, the table shows the maximum TLD over all the estimated OD pairs, for every specific tuning parameter. Therefore every estimated OD pair with TLD less than this maximum value, is omitted by converging its OD flow to zero. This measurement gives a characterization of the estimated OD flows which converged to zero.

Table 3.1: CPU computation time (sec), for the all three proposed methods and the LD scenario, and maximum TLD (veh/m) of the OD flows converged to zero, using the lasso_2 method and the LD scenario (the toy network).

(a) Execution time, for the LD scenario (the toy network).

	Ridge	**Lasso_1**	**Lasso_2**
TP = 0.1	234.11	125.88	117.5
TP = 1	296.78	67.2	60.56
TP = 100	33.81	29.22	25.61

(b) TLD, for the lasso_2 method, and by using the LD scenario (the toy network).

	OD dimension reduction	**TLD**
TP = 0.1	30%	8.678234e-05
TP = 1	39%	2.437603e-07
TP = 100	57%	2.440518e-07

Next, it is important to investigate how these estimated OD demand once assigned to network can produce traffic flows close to their real observations. Figure (3.4) provides a RMSE performance overview of solution approaches in terms of relationship between the estimated and observed link traffic counts. It appears clearly from Figure (3.4) that the lasso_2 method, in absence of the historical matrix within its objective function, performs better than the ridge and lasso_1 methods. In this regard, the ridge and lasso_1 have almost the same result. Optimal value of the tuning parameter when all the proposed methods are considered, is $TP = 0$, but when the implemented method is just lasso_2, the optimal tuning parameter can be any value between zero and one.

Concerning demand scenarios, changing receptively from the LD scenario to the MD scenario and then to the HD scenario, the OD estimation process changes from an under-fitted historical matrix to a over-fitted historical matrix. Therefore indeed, coefficients β_{ridge} and β_{lasso_1}, defined in (3.12), are decreasing, and consequently RMSE between the observed traffic counts and the estimated traffic counts is increasing.

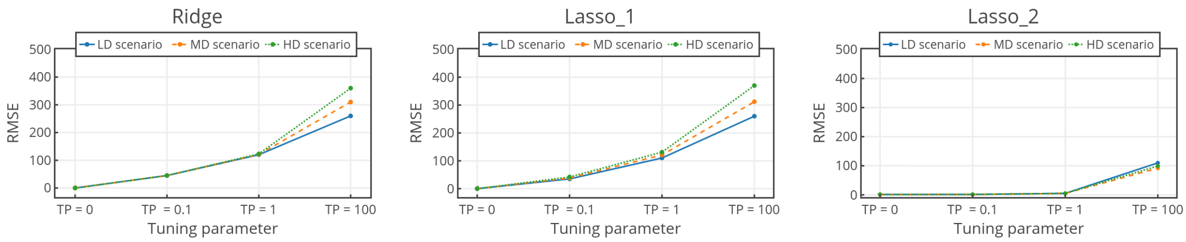


Figure 3.4: RMSE for the observed and estimated traffic counts, for the different methods and scenarios (the toy network).

GEH index is considered as additional performance indicator used by practitioners to evaluate performance of proposed methods. GEH index less than 5 in a measurement point is considered a good match between the estimated and the observed traffic counts. An indication for a good GEH statistic is to examine whether 85% of the traffic counts have a GEH value less than 5. Figure (3.5) shows that in the both ridge and lasso_1 methods, more than 85% of the traffic counts have the GEH indices less than 5, when tuning parameter (TP) is less or equal than 1. The performance of the lasso_2 method is perfect, having 100% of the traffic counts with the GEH indices less than 5. Indeed, the lasso_2 objective function is defined to fit the estimated and observed traffic counts. Similarly, the same argument discussed about changing the tuning parameters and the scenarios in Figure (3.4), can be used here too.

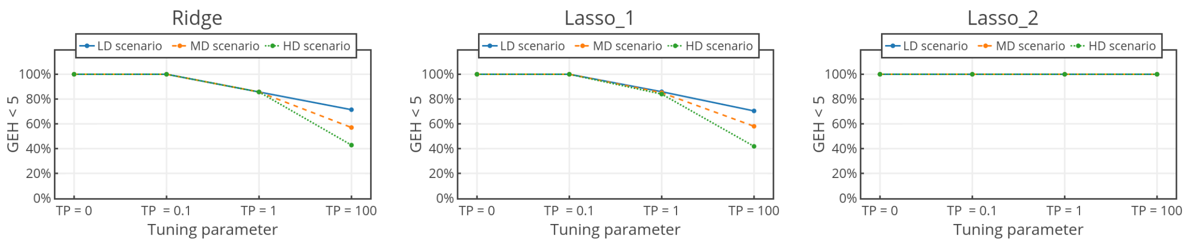


Figure 3.5: GEH statistic of the traffic counts, for the all scenarios and methods (the toy network).

3.3.2 Vitoria netwotk

In what follows, In order to examine sensitivity of the three proposed methods, but this time on the Vitoria network, four values as the tuning parameter TP have been selected; $TP \in \{0, 0.1, 1, 1000\}$. Without loss of generality, the low-demand (LD) scenario has been selected for the Vitoria network.

The performance of the proposed OD matrix estimation methods, in terms of reducing dimensionality, is presented in Figure (3.6). We analyze the sensitivity of the results under two conditions; the tuning parameters, and the methods.

As the tuning parameter increases, number of OD flows converged to the corresponding historical OD flows in the ridge and lasso_1, and number of OD flows converged to zero in the lasso_2, increase too. Indeed, when minimizing the objective functions (1.3), (1.4), and (1.5), increasing the tuning parameter in terms $\gamma(g_i - \widehat{g}_i)^2$, $\gamma|g_i - \widehat{g}_i|$, and γg_i leads the terms $(g_i - \widehat{g}_i)^2$, $|g_i - \widehat{g}_i|$, and g_i decrease more toward zero, and this results in converging higher number of OD flows.

Comparing the ridge with the lasso_1, number of converged OD flows using the lasso_1, is higher than the number of converged OD flows using the ridge. Consequently $\beta_{lasso\_1} \leq \beta_{ridge}$, for coefficients β_{lasso_1} and β_{ridge} defined in (3.12). Therefore in the OD estimation process with the objective functions (1.3), (1.4), and (1.5), the dimensionality of the coefficient terms β_{lasso_1} and β_{ridge} reduces, and accordingly the dimensionality of their gradient in the OD estimation process reduces too. Therefore we expect to see that CPU running using the lasso_1 method is less than the CPU running time of the ridge method. The same discussion satisfies for the lasso_2, which is expected to be the fastest among the other methods, in terms of CPU running time of the OD estimation process.

The lasso_2 method reduces the dimensionality of the estimated OD matrix by removing the OD flows that converged to zero from the state vector. Practically, the lasso_2 method can be employed when the network is large, and it consists of long highways with large volumes, while there are also short-distance trips with small dispensable volumes, which can be discarded.

Speaking of reduction in dimensionality, the lasso_2 method performs the best. When the goal is to have higher number of converged OD flows, the optimal value of tuning parameter is $TP = 1000$, for all the methods.

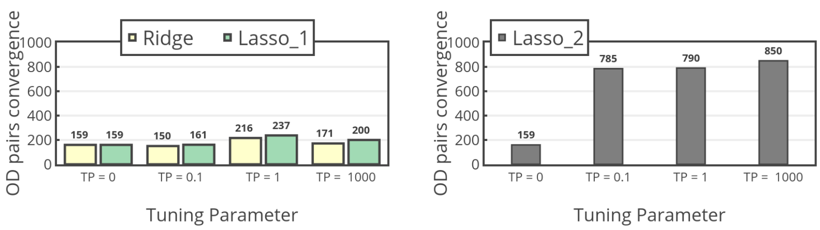


Figure 3.6: Number of non-zero estimated OD flows which converged exactly to the corresponding historical OD flows using the ridge and lasso_1 methods, and number of non-zero estimated OD flows which converged exactly to zero using the lasso_2 method (the Vitoria network).

Figure (3.7) shows the performance of objective functions defined in the equation (1.3), (1.4), and (1.5). There are some small fluctuations in the objective graph over the iteration process, but overall it decreases to its minimum. Since the minimum of objective functions for all the three methods are achieved up to the iteration step 100, the algorithm has been stopped after 100 iterations. Increasing the tuning parameter, the objective function converges faster to the minimum. Indeed, when minimizing the objective functions, choosing higher tuning parameter, the coefficient terms $\beta_{ridge}, \beta_{lasso\_1}$, and β_{lasso_2} have to minimize faster toward zero.

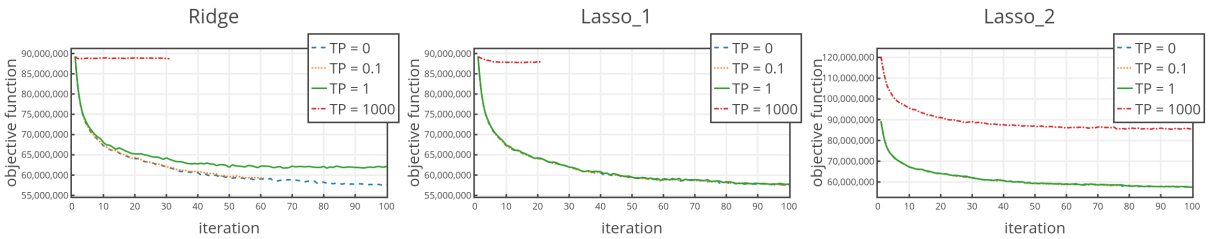


Figure 3.7: Objective functions for different methods, using the LD scenario (the Vitoria network).

Table (3.2)(a) shows the CPU computation time for the low demand (LD) scenario. We can observe in this table, significant CPU computation time reduction of the proposed lasso_2 method compared to the other two methods, since in each iteration step simplifies the estimated OD matrix by reducing its dimensionality, by omitting the OD flows equal to zero. Consequently, computation time decreases with the number of OD pairs to be estimated as iterations progress.

Table (3.2)(b) shows the percentage of the reduction in dimensionality of the estimated OD matrix, using the lasso_2 method, and for the low demand (LD) scenario. As the tuning parameter increases, the dimensionality of the estimated OD matrix reduces. Also, the table shows the maximum TLD over all the estimated OD pairs, for every specific tuning parameter. Therefore every estimated OD pair with TLD less than this maximum value, is omitted by converging its OD flow to zero. This measurement gives a characterization of the estimated OD flows which converged to zero.

Table 3.2: CPU computation time (sec), for the all three proposed methods and the LD scenario, and maximum TLD (veh/m) of the OD flows converged to zero, using the lasso_2 method and the LD scenario (the Vitoria network).

(a) Execution time, for the LD scenario (the Vitoria network).

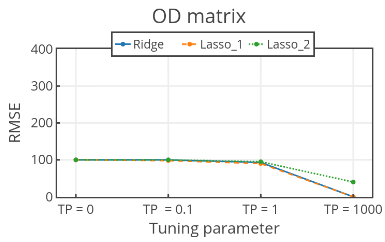
	Ridge	**Lasso_1**	**Lasso_2**
TP = 0.1	4720.43	3974.91	1560.54
TP = 1	2577.01	2090.39	1259.83
TP = 1000	1103.2	1004.19	930.22

(b) TLD, for the lasso_2 method, and by using the LD scenario (the Vitoria network).

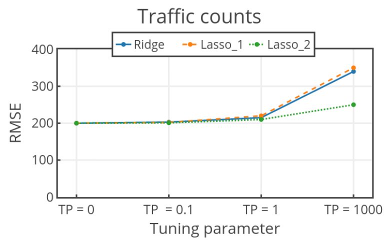
	OD dimension reduction	**TLD**
TP = 0.1	34%	1.678094e-05
TP = 1	34.5%	3.488646e-05
TP = 1000	37%	5.143583e-07

Figure (3.8)(a) shows the RMSE between the estimated and the historical OD demand for tested OD matrix solution approaches. Increading the tuning parameter, lasso_2 method performs better than the ridge and lasso_1 methods, when the model is supposed to not have overfitting to the historical OD matrix. Furthermore, it is indicated that when the implemented methods are ridge and lasso_1, increasing the tuning parameter, changes the estimated OD matrix to fit better to the historical OD matrix.

Next, it is important to investigate how these estimated OD demand once assigned to network can produce traffic flows close to their real observations. Figure (3.8)(b) provides a RMSE performance overview of solution approaches in terms of relationship between the estimated and observed link traffic counts. It appears clearly from Figure (3.8)(b) that the lasso_2 method, in absence of the historical matrix within its objective function, performs better than the ridge and lasso_1 methods. In this regard, the ridge and lasso_1 have almost the same result. Optimal value of the tuning parameter when all the proposed methods are considered, is $TP = 0$, but when the implemented method is just lasso_2, the optimal tuning parameter can be any value between zero and one.



(a) RMSE for the historical and the estimated OD matrix.



(b) RMSE for the observed and estimated traffic counts.

Figure 3.8: RMSE between the historical and the estimated OD matrix, and RMSE between the observed and the estimated traffic counts, for the different methods, and using the LD scenario (the Vitoria network).

Figure (3.9) shows the GEH statistic of method, for the different tuning parameters, and by using the low-demand (LD) scenario. The performance of the lasso_2 method is the best, among all the methods. Indeed, the lasso_2 objective function (1.5) is defined to fit the estimated and observed traffic counts. Similarly, the same argument discussed about changing the tuning parameters in Figure (3.8)(b), can be used here too. The optimal tuning parameter is $TP = 0$, with GEH index equal to 50%.

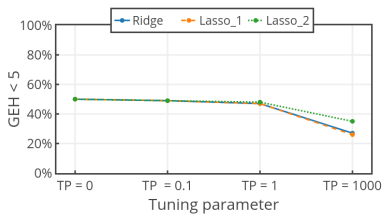


Figure 3.9: GEH statistic of the traffic counts, using the LD scenario, and all the methods (the Vitoria network).

Chapter 4

Regularity of minimizers for very degenerate integral functionals

We will follow the usual convention and denote by c or C a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{n\times m}$ we write $\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. By $B_r(x)$ we will denote the ball in $\mathbb{R}^n$ centered at x of radius r. The integral mean of a function u over a ball $B_r(x)$ will be denoted by $u_{x,r}$, that is

$$u_{x,r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy,$$

where $|B_r(x)|$ is the Lebesgue measure of the ball in $\mathbb{R}^n$. If no confusion may arise, we shall omit the dependence on the center.

In this chapter, we are given a bounded domain $\Omega \subset \mathbb{R}^n$, a real number $p \geq 2$, and an integer $N \geq 1$. We look at local minimizers $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of inhomogeneous functionals

$$\mathbb{F}(u, \Omega) = \int_\Omega \Big(\mathcal{F}(x, Du) + f(x) \cdot u(x) \Big)\, dx. \tag{4.1}$$

Roughly speaking, $\mathcal{F}$ is a Carathéodory function with growth p, which is assumed to be *only* assimptotically convex with respect to the gradient variable, and $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ is given. In this setting, and for $f \equiv 0$, Fonseca, Fusco and Marcellini proved [42] that local minimizers are Lipschitz continuous if

$$|D_{x,\xi}\mathcal{F}(x,\xi)| \leq k(x)\,(1 + |\xi|)^{p-1} \qquad (x,\xi) \in \Omega \times \mathbb{R}^{n\times N}$$

for some function $k \in L^\infty$. More recently, in [28] this result was extended to $k \in L^\sigma_{\mathrm{loc}}(\Omega)$ for some $\sigma > n$. Local boundedness for the gradient in the inhomogeneous case was proven in [15, Theorem 5.2] for $f \in C^\alpha$ and $\alpha > 0$, and in [14, Theorem 2.1] for $f \in L^\sigma$, $\sigma > n$. In both cases, though, the extra assumption that $\mathcal{F}(x,\xi)=\mathcal{F}(\xi)$ was needed.

To be more precise, we have a Carathéodory function

$$\mathcal{F}:\Omega\times\mathbb{R}^{n\times N}\to\mathbb{R}$$
$$(x,\xi)\to\mathcal{F}(x,\xi)$$

This means that, for every fixed $\xi\in\mathbb{R}^{n\times N}$, $x\mapsto\mathcal{F}(x,\xi)$ is measurable, and also that there is a null set $N\subset\Omega$ so that $\xi\mapsto\mathcal{F}(x,\xi)$ is continuous and convex in $\mathbb{R}^{n\times N}$, and C^2 on $\mathbb{R}^{n\times N}\setminus B_R(0)$. It will be assumed to satisfy the following properties:

F0 There exist positive constants ℓ, L such that for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^{n\times N}$

$$\ell|\xi|^p\le\mathcal{F}(x,\xi)\le L(|\xi|^p+1)$$

F1 $\mathcal{F}$ is radial at ∞: there is $F:\overline{\Omega}\times[R,\infty)\to\mathbb{R}$ such that

$$\mathcal{F}(x,\xi)=F(x,|\xi|)$$

for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^{n\times N}\setminus B_R(0)$

F2 Uniform p-convexity at ∞: there exists $\nu>0$ such that if $\lambda\in\mathbb{R}^{n\times N}$ then

$$\langle D_{\xi\xi}\mathcal{F}(x,\xi)\lambda,\lambda\rangle\geqslant\nu(1+|\xi|)^{p-2}|\lambda|^2,$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^{nN}\setminus B_R(0)$.

F3 There exists $L_1>0$, such that

$$|D_{\xi\xi}\mathcal{F}(x,\xi)|\leqslant L_1(1+|\xi|)^{p-2}$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^{nN}\setminus B_R(0)$.

F4 There exists $\sigma>0$ and a non-negative function $k\in L^{\sigma}_{\mathrm{loc}}(\Omega)$ such that

$$|D_{x\xi}\mathcal{F}(x,\xi)|\leqslant k(x)(1+|\xi|)^{p-1},$$

for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^{nN}\setminus B_R(0)$.

We can now state the main result of this section.

Theorem 15. *Let $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathbb{F}(u,\Omega)$ in (4.1), and assume that the energy density $\mathcal{F}(x,\xi)$ satisfies assumptions* **F0-F4**. *If $f\in L^{\sigma}{}_{\mathrm{loc}}(\Omega)$, then there exists a constant c, such that*

$$\sup_{B_\rho}|Du|\leqslant c(1+||k||_{L^\sigma(B_r)}+||f||_{L^\sigma(B_r)})^\tau\int_{B_r}(1+|Du|^p)\,dx,$$

for all balls $B_\rho\subset B_r\Subset\Omega$.

The proof of this Theorem is split into two parts: an apriori L^∞ estimate, and an approximation.

4.1 The apriori estimates

We introduce some auxiliary notation. For each $\gamma \in \mathbb{R}$, we will denote

$$\Phi(t) = \Phi_\gamma(t) = \frac{t^2}{(1+t)^{2-\gamma}}. \tag{4.2}$$

For such Φ, one easily sees that

$$t\Phi'(t) \leq 2(1+\gamma)\Phi(t). \tag{4.3}$$

We also introduce the following notation for the positive part of $|Du|-1$,

$$P = (|Du|-1)_+$$

so that

$$DP = \chi_{\{|Du>1|\}} \cdot \frac{Du}{|Du|} \cdot D^2 u \tag{4.4}$$

The following lemma is an important application in the so called hole-filling method. Its proof can be found for example in [46, Lemma 6.1] .

Lemma 16. *Let $h : [r, R_0] \to \mathbb{R}$ be a nonnegative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $r \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0 - r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

In this subsection, we prove an apriori estimate that will be used later in the approximation step (see Subsection 4.2) for proving Theorem 15. The precise statement is the following one. Recall that $\mathbb{F}$ is the one defined in (4.1).

Theorem 17. *Assume* **F0**–**F4** *hold, and let $f \in L^\sigma_{\mathrm{loc}}(\Omega)$, where σ is the exponent appearing in assumption* **F4**. *Fix a ball $B_r(x_0) \Subset \Omega$ and two functions $u, \bar{u} \in W^{1,p}(B_r(x_0); \mathbb{R}^N)$, and define*

$$\widetilde{\mathbb{F}}(v; B_r(x_0)) := \mathbb{F}(v; B_r(x_0)) + \int_{B_r(x_0)} \arctan(|v - \bar{u}(x)|^2)\, dx.$$

Let $v \in u + W^{1,p}_0(B_r(x_0); \mathbb{R}^N)$ be a minimizer of $\widetilde{\mathbb{F}}$, satisfying

$$v \in W^{2,2}_{\mathrm{loc}}(B_r(x_0); \mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(B_r(x_0); \mathbb{R}^N) \quad \text{and} \quad (1+|Dv|^2)^{\frac{p-2}{2}} |D^2 v|^2 \in L^1_{\mathrm{loc}}(B_r(x_0)).$$

Then, for every $B_{\bar{r}}(\bar{x}) \Subset B_r(x_0)$, every $0 < \rho < r' \leq \bar{r}$

$$\sup_{B_\rho} |Du| \leqslant C(1 + ||k||_{L^\sigma(B_{r'})} + ||f||_{L^\sigma(B_{r'})})^\tau \left[\int_{B_{r'}(x_0)} (1+|Du|)^p \mathrm{d}x \right]^{\frac{1}{p}},$$

for some constant $C = C(n, N, p, L_1, \nu, \operatorname{diam}\Omega)$. Moreover, one has

$$\int_{B_\rho(x_0)} \frac{((|Du|-1)_+)^2}{(1+(|Du|-1)_+)^2} |Du|^{p-2} |D^2 u|^2 \, dx \leq C(1+||k||_{L^\sigma(B_{r'})} + ||f||_{L^\sigma(B_{r'})})^\tau \int_{B_{r'}(x_0)} (1+|Du|^p))\, dx, \tag{4.5}$$

for some $C = C(n, N, p, L, L_1, \nu, \rho, r'$.

In the above result, we are assuming without loss of generality that $R = 1$ in **F0**–**F4**. For the proof of the above result, the integral $\int_{B_r(x_0)} \arctan(|v-\bar{u}|^2)\,dx$ is a perturbation of $\mathcal{F}(v; B_r(x_0))$ that provides no difficulties. Indeed, denoting $g(x,v) := \arctan(|v-\bar{u}(x)|^2)$, we have that g and its derivatives g_{v^α}, $\alpha = 1,\dots,m$, are bounded. Thus, for the sake of clarity, we prefer to drop this perturbation term, and to state, and prove an a priori estimate for local minimizers of $\mathbb{F}(\cdot;\Omega)$ only, see Theorem 18 below.

Theorem 18. *Let $\mathcal{F}(x,\xi)$ satisfy conditions* **F0**–**F4**, *and let $f \in L^\sigma_{\text{loc}}(\Omega)$, where σ is the exponent appearing in assumption* **F4**. *Assume that $u \in W^{2,2}_{\text{loc}}(\Omega,\mathbb{R}^N) \cap W^{1,\infty}_{\text{loc}}(\Omega,\mathbb{R}^N)$ is a local minimizer of the functional $\mathbb{F}(\cdot;\Omega)$, and that $(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2 \in L^1_{\text{loc}}(\Omega)$. Then the estimate*

$$\sup_{B_\rho} |Du| \leqslant C(1 + ||k||_{L^\sigma(B_{2R})} + ||f||_{L^\sigma(B_{2R})})^\tau \left[\int_{B_R} (1+|Du|)^p \mathrm{d}x\right]^{\frac{1}{p}}, \tag{4.6}$$

holds for every concentric balls $B_\rho \subset B_R \Subset \Omega$. Moreover, the following Caccioppoli inequality holds,

$$\int_{B_\rho} \frac{(|Du|-1)_+^2}{(1+(|Du|-1)_+)^2} |Du|^{p-2}|D^2u|^2\,dx \le C(1 + ||k||_{L^\sigma(B_{2R})} + ||f||_{L^\sigma(B_{2R})})^\tau \int_{B_{2R}} (1+|Du|^p)\;dx, \tag{4.7}$$

for some $C = C(n,N,p,L,L_1,\nu,\sigma,\rho,R,\operatorname{diam}\Omega)$, an exponent $\tau = \tau(n,\sigma) > 0$ and for every concentric balls $B_\rho \subset B_R \subset B_{2R} \Subset \Omega$.

Proof. We will prove the theorem in 3 steps.

Step 1. The first step is to prove that if $\gamma \geqslant 0$ and if $\eta \in C_0^\infty(\Omega)$ is a non-negative cutoff function, then one has

$$\begin{aligned}\int_\Omega \eta^2\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx &\le C(\gamma+1)^2 \int_\Omega \eta^2 k^2 (1+P)^{\gamma+p}\,dx \\ &+ C\int_\Omega |D\eta|^2(1+P)^{\gamma+p}\,dx + C\int_\Omega \eta^2|f|^2(1+P)^{\gamma-p+2}\,dx\end{aligned} \tag{4.8}$$

with $C = C(n,N,\nu,L,L_1)$. Since u is a local minimizer of $\mathbb{F}(\cdot;\Omega)$, it satisfies the following integral identity

$$\int_\Omega \langle D_\xi\mathcal{F}(x,Du), D\psi\rangle = \int_\Omega f\cdot\psi \qquad \forall\psi \in C_0^\infty(\Omega,\mathbb{R}^N).$$

By our assumption on u and a standard approximation argument, we can choose

$$\psi \equiv \sum_s D_{x_s}\Big(\eta^2\cdot\Phi(P)\cdot D_{x_s}u\Big),$$

where $\eta \in C_0^\infty(\Omega)$. Such a choice, together with an integration by parts in the left hand side of previous identity, yields

$$-\sum_s \int_\Omega \big\langle D_{x_s\xi}\mathcal{F}(x,Du) + D_{\xi\xi}\mathcal{F}(x,Du)\cdot D_{x_s}Du, D\big(\eta^2\cdot\Phi(P)\cdot D_{x_s}u\big)\big\rangle = \int_\Omega f\cdot\psi \tag{4.9}$$

We now use the product rule to calculate the derivatives of $\eta^2\cdot\Phi(P)\cdot D_{x_s}u$. This converts (4.9) into

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 = 0,$$

where

$$I_1 = 2\sum_s \int_\Omega \langle D_{\xi\xi}\mathcal{F}(x,Du)\cdot D_{x_s}Du, D\eta\cdot D_{x_s}u\rangle\,\eta\,\Phi(P)\,dx,$$

$$I_2 = \sum_s \int_\Omega \langle D_{\xi\xi}\mathcal{F}(x,Du)\cdot D_{x_s}Du, D_{x_s}Du\rangle\, \eta^2\, \Phi(P)\, dx,$$

$$I_3 = \sum_s \int_\Omega \langle D_{\xi\xi}\mathcal{F}(x,Du)\cdot D_{x_s}Du, \Phi'(P)\, DP\cdot D_{x_s}u\rangle\, \eta^2\, dx,$$

$$I_4 = 2\sum_s \int_\Omega \langle D_{x_s\xi}\mathcal{F}(x,Du), D\eta\cdot D_{x_s}u\rangle\, \eta\, \Phi(P)\, dx,$$

$$I_5 = \sum_s \int_\Omega \langle D_{x_s\xi}\mathcal{F}(x,Du), D_{x_s}Du\rangle\, \eta^2\, \Phi(P)\, dx,$$

$$I_6 = \sum_s \int_\Omega \langle D_{x_s\xi}\mathcal{F}(x,Du), \Phi'(P)\, DP\cdot D_{x_s}u\rangle\, \eta^2\, dx,$$

$$I_7 = 2\sum_s \int_\Omega f\, \eta\, \Phi(P)\, D_{x_s}\eta\cdot D_{x_s}u\, dx$$

$$I_8 = \sum_s \int_\Omega f\, \eta^2\, \Phi(P)\, D_{x_sx_s}u\, dx$$

$$I_9 = \sum_s \int_\Omega f\, \eta^2\, \Phi'(P)\, D_{x_s}u\cdot D_{x_s}P\, dx$$

We will estimate each term separately. It is worth pointing out that the integrals I_i, with $i = 1,\dots,6$ will be estimate as done in [38] ,[27]. We will report here for the sake of completeness.

For the estimate of I_1 we use assumption **F3** and Young's inequality as follows

$$\begin{aligned} |I_1| \leqslant & 2L_1 \sum_s \int_\Omega \eta(1+|Du|)^{p-2}\, |D_{x_s}Du|\, |D\eta|\, |D_{x_s}u|\, \Phi(P)\ dx \\ \leqslant & \epsilon \sum_s \int_\Omega \eta^2(1+|Du|)^{p-2}|D(D_{x_s}u)|^2\Phi(P)\, dx + C(\epsilon, L_1)\int_\Omega |D\eta|^2(1+|Du|)^p\Phi(P)\, dx. \end{aligned} \tag{4.10}$$

In order to estimate I_4,we use assumption **F4** as follows,

$$\begin{aligned} |I_4| &\leqslant \int_\Omega \eta\, |D\eta|\, k\, (1+|Du|)^p\, \Phi(P)\, dx \\ &\leqslant C\int_\Omega \eta^2\, k^2\, (1+|Du|)^p\, \Phi(P)\, dx + C\int_\Omega |D\eta|^2\, (1+|Du|)^p\, \Phi(P)\, dx. \end{aligned} \tag{4.11}$$

We estimate I_5 arguing as we did before, again using **F4**. Indeed

$$\begin{aligned} |I_5| &\leqslant \int_\Omega \eta^2 k\, (1+|Du|)^{p-1}\, |D_{x_s}Du|\, \Phi(P)\, dx \\ &\leqslant \epsilon\int_\Omega \eta^2\, (1+|Du|)^{p-2}\, |D^2u|^2\, \Phi(P)\ dx + C(\epsilon)\int_\Omega \eta^2\, k^2\, (1+|Du|)^p\, \Phi(P)\, dx. \end{aligned} \tag{4.12}$$

For the estimate of I_6, again by virtue of assumption **F4**, we have

$$\begin{aligned} |I_6| &\leqslant \sum_s \int_\Omega \eta^2\, k\, (1+|Du|)^{p-1}\, |D_{x_s}u|\, \Phi'(P)\, |DP|\, dx \\ &\leqslant C\int_\Omega \eta^2\, k\, (1+|Du|)^p\, \Phi'(P)\, |DP|\, dx \\ &\leqslant C\int_\Omega \eta^2\, k\, (1+|Du|)^p\, \Phi'(P)|D^2u|, \end{aligned} \tag{4.13}$$

where we used the second equality in (4.4). After setting $C_\gamma = 2(1+\gamma) > 0$, we multiply and divide the last integrand in (4.13) by $\left(\frac{\delta+P}{C_\gamma}\right)^{1/2}$ with $0 < \delta < 1$, and use Young's inequality, and obtain

$$\begin{aligned}
|I_6| &\leqslant C \int_\Omega \eta^2\, \Phi'(P) \left\{ \frac{\delta+P}{C_\gamma} (1+|Du|)^{p-2}\, |D^2u|^2 \right\}^{\frac{1}{2}} \times \left\{ \frac{C_\gamma}{\delta+P} k^2\, (1+|Du|)^{p+2} \right\}^{\frac{1}{2}} dx \\
&\leqslant \epsilon \int_\Omega \eta^2\, \Phi'(P)\, \frac{\delta+P}{C_\gamma} (1+|Du|)^{p-2}\, |D^2u|^2\, dx + C(\epsilon) \int_\Omega \eta^2\, \Phi'(P)\, \frac{C_\gamma}{\delta+P} k^2\, (1+|Du|)^{p+2}\, dx
\end{aligned} \tag{4.14}$$

Using that

$$\delta + P \leqslant 2P \qquad \text{on the set} \quad \{|Du| > 2\}$$

and that, by the definition of C_γ and by (4.3),

$$t\Phi'(t) \leq C_\gamma \Phi(t),$$

we can estimate the first integral in the left hand side of (4.14) as follows,

$$\begin{aligned}
&\frac{\epsilon}{C_\gamma} \int_\Omega \eta^2 \cdot \Phi'(P) \cdot (\delta+P) (1+|Du|)^{p-2} |D^2u|^2\, dx \\
&= \frac{\epsilon}{C_\gamma} \int_{\{|Du|>2\}} \eta^2\, \Phi'(P)\, (\delta+P)\, (1+|Du|)^{p-2}\, |D^2u|^2 dx \\
&\quad + \frac{\epsilon}{C_\gamma} \int_{\{1 \leqslant |Du| \leqslant 2\}} \eta^2\, \Phi'(P)\, (\delta+P)\, (1+|Du|)^{p-2}\, |D^2u|^2\, dx \\
&\leqslant \frac{2\epsilon}{C_\gamma} \int_{\Omega \cap \{|Du|>2\}} \eta^2\, \Phi'(P) P\, (1+|Du|)^{p-2}\, |D^2u|^2\, dx \\
&\quad + \frac{\epsilon}{C_\gamma} \int_{\Omega \cap \{1 \leqslant |Du| \leqslant 2\}} \eta^2\, \Phi'(P)\, P\, (1+|Du|)^{p-2}\, |D^2u|^2\, dx \\
&\quad + \frac{\epsilon\delta}{C_\gamma} \int_{\Omega \cap \{1 \leqslant |Du| \leqslant 2\}} \eta^2\, \Phi'(P)\, (1+|Du|)^{p-2}\, |D^2u|^2\, dx \\
&\leqslant 2\epsilon \int_{\Omega \cap \{|Du|>2\}} \eta^2\, \Phi(P)\, (1+|Du|)^{p-2}\, |D^2u|^2\, dx \\
&\quad + \epsilon \int_{\Omega \cap \{1 \leqslant |Du| \leqslant 2\}} \eta^2 \Phi(P) (1+|Du|)^{p-2} |D^2u|^2\, dx \\
&\quad + \frac{\epsilon\delta}{C_\gamma} \int_{\Omega \cap \{1 \leqslant |Du| \leqslant 2\}} \eta^2 \cdot \Phi'(1) (1+|Du|)^{p-2} |D^2u|^2\, dx \\
&\leqslant 3\epsilon \int_\Omega \eta^2 \Phi(P) (1+|Du|)^{p-2} |D^2u|^2\, dx \\
&\quad + \frac{\epsilon\delta}{2} \int_{\{1 \leq |Du| \leq 2\}} \eta^2 \cdot \Phi'(1) (1+|Du|)^{p-2} |D^2u|^2\, dx,
\end{aligned} \tag{4.15}$$

where, in the last line we used that $\Phi'(t)$ is increasing and that $P \leq 1$ on the set $\{1 \leqslant |Du| \leqslant 2\}$.

Plugging (4.15) into (4.14), we get

$$
\begin{aligned}
|I_6| &\leqslant 3\epsilon \int_\Omega \eta^2 \Phi(P)(1+|Du|)^{p-2}|D^2u|^2\,dx + \frac{\epsilon\delta}{2}\int_{\{1\le|Du|\le 2\}} \eta^2\Phi'(1)(1+|Du|)^{p-2}|D^2u|^2\,dx \\
&+ C(\epsilon,\gamma)\int_\Omega \eta^2\Phi'(P)k^2(x)(\delta+P)^{-1}(1+|Du|)^{p+2}\,dx \\
&\leqslant 3\epsilon\int_\Omega \eta^2\Phi(P)(1+|Du|)^{p-2}|D^2u|^2\,dx + C_p\epsilon\delta\,\Phi'(1)\int_{\{1\le|Du|\le 2\}}\eta^2|D^2u|^2\,dx \\
&+ C(\epsilon)C_\gamma\int_\Omega \eta^2\Phi'(P)k^2(x)(\delta+P)^{-1}(1+|Du|)^{p+2}\,dx.
\end{aligned}
$$

Using the definition of Φ, and the fact that $\dfrac{P}{\delta+P}\leqslant 1$, we have

$$
\begin{aligned}
(\delta+P)^{-1}\Phi'(P) &= \Phi'(P)\cdot\frac{P}{\delta+P}\cdot P^{-1} \\
&\leqslant C_\gamma\Phi(P)\cdot P^{-2} = C_\gamma\,(1+P)^{\gamma-2}
\end{aligned}
\tag{4.16}
$$

Therefore inserting (4.16) into the last integral on the right hand side of previous estimate, and using the definition of $\Phi(P)$

$$
\begin{aligned}
|I_6| &\leqslant 3\epsilon\int_\Omega \eta^2\Phi(P)(1+|Du|)^{p-2}|D^2u|^2\,dx + C_p\epsilon\delta\,\Phi'(1)\int_{\{1\le|Du|\le 2\}}\eta^2|D^2u|^2\,dx \\
&+ C(\epsilon)C_\gamma^2\int_\Omega \eta^2k^2(x)(1+|Du|)^{p+\gamma}\,dx.
\end{aligned}
$$

By virtue of the assumption $|D^2u|\in L^2_{\rm loc}(\Omega)$, we can let $\delta\to 0$ in previous estimate thus getting

$$
\begin{aligned}
|I_6| &\le 3\epsilon\int_\Omega \eta^2\Phi(P)(1+|Du|)^{p-2}|D^2u|^2\,dx \\
&+ C(\epsilon)(\gamma+1)^2\int_\Omega \eta^2k^2(x)(1+|Du|)^{p+\gamma}\,dx,
\end{aligned}
\tag{4.17}
$$

where we used that $C_\gamma\sim(\gamma+1)$. For I_7, using Young's inequality, we get

$$
\begin{aligned}
|I_7| &\le 2\int_\Omega \eta|f||D\eta||Du|\Phi(P) = 2\int_\Omega \eta|f||D\eta||Du|(1+|Du|)^{\frac{p-2}{2}}(1+|Du|)^{\frac{2-p}{2}}\Phi(P) \\
&\le C\int_\Omega \eta^2|f|^2(1+|Du|)^{2-p}\Phi(P) + C\int_\Omega |D\eta|^2(1+|Du|)^p\Phi(P).
\end{aligned}
\tag{4.18}
$$

Concerning I_8 and I_9, we have

$$
|I_8|+|I_9| \le \int_\Omega \eta^2|f||D^2u|\Phi(P) + \int_\Omega \eta^2|f||Du||D^2u|\Phi'(P).
\tag{4.19}
$$

The first integral in the right hand side of (4.19) can be estimated by Young's inequality as follows

$$
\begin{aligned}
\int_\Omega \eta^2\,|f|\,|D^2u|\,\Phi(P) &= \int_\Omega \eta^2|f|\,\frac{(1+|Du|)^{\frac{p-2}{2}}}{(1+|Du|)^{\frac{p-2}{2}}}|D^2u|\Phi(P) \\
&\le \epsilon\int_\Omega \eta^2(1+|Du|)^{p-2}|D^2u|^2\Phi(P) + C(\epsilon)\int_\Omega \eta^2(1+|Du|)^{2-p}|f|^2\Phi(P).
\end{aligned}
\tag{4.20}
$$

To estimate the second integral in (4.19) we argue as we did for I_6 multiplying and dividing for $C_\gamma^{\frac{1}{2}}(\delta+P)^{\frac{1}{2}}$ with $0<\delta<1$ and we use Young's inequality, thus getting

$$\begin{aligned}
\int_\Omega \eta^2|f||Du||D^2u|\Phi'(P) &= \int_\Omega \eta^2|f||Du||D^2u|\frac{C_\gamma^{\frac{1}{2}}(\delta+P)^{\frac{1}{2}}(1+|Du|)^{\frac{p-2}{2}}}{C_\gamma^{\frac{1}{2}}(\delta+P)^{\frac{1}{2}}(1+|Du|)^{\frac{p-2}{2}}}\Phi'(P)\\
&\le \frac{\epsilon}{C_\gamma}\int_\Omega \eta^2(1+|Du|)^{p-2}(\delta+P)|D^2u|^2\Phi'(P)\\
&+C(\epsilon)C_\gamma\int_\Omega \eta^2\frac{|Du|^2|f|^2}{(\delta+P)(1+|Du|)^{p-2}}\Phi'(P)\\
&=\frac{\epsilon}{C_\gamma}\int_{\Omega\cap\{|Du|>2\}}\eta^2(1+|Du|)^{p-2}(\delta+P)|D^2u|^2\Phi'(P)\\
&+\frac{\epsilon}{C_\gamma}\int_{\Omega\cap\{1<|Du|\le 2\}}\eta^2(1+|Du|)^{p-2}(\delta+P)|D^2u|^2\Phi'(P)\\
&+C(\epsilon)C_\gamma\int_\Omega \eta^2\frac{|Du|^2|f|^2}{(\delta+P)(1+|Du|)^{p-2}}\Phi'(P)\\
&\le 2\frac{\epsilon}{C_\gamma}\int_{\Omega\cap\{|Du|>2\}}\eta^2(1+|Du|)^{p-2}P|D^2u|^2\Phi'(P)\\
&+\frac{\epsilon}{C_\gamma}\int_{\Omega\cap\{1<|Du|\le 2\}}\eta^2(1+|Du|)^{p-2}P|D^2u|^2\Phi'(P)\\
&+\frac{\epsilon}{C_\gamma}\delta\int_{\Omega\cap\{1<|Du|\le 2\}}\eta^2(1+|Du|)^{p-2}|D^2u|^2\Phi'(1)\\
&+C(\epsilon)C_\gamma\int_\Omega \eta^2\frac{|Du|^2|f|^2}{(\delta+P)(1+|Du|)^{p-2}}\Phi'(P)\\
&\le 3\frac{\epsilon}{C_\gamma}\int_\Omega \eta^2(1+|Du|)^{p-2}P|D^2u|^2\Phi'(P)\\
&+\frac{\epsilon}{C_\gamma}\delta\int_{\Omega\cap\{1<|Du|\le 2\}}\eta^2(1+|Du|)^{p-2}|D^2u|^2\Phi'(1)\\
&+C(\epsilon)C_\gamma\int_\Omega \eta^2\frac{|Du|^2|f|^2}{(\delta+P)(1+|Du|)^{p-2}}\Phi'(P),
\end{aligned}$$

where we used that $\delta+P\le 2P$ in the set $\{|Du|>2\}$ and that $\Phi'(t)$ is increasing and that $P\le 1$ in the set $\{1<|Du|\le 2\}$. Using the inequalities $t\Phi'(t)\le C_\gamma\Phi(t)$ and (4.16) in previous estimate, we obtain

$$\begin{aligned}
\int_\Omega \eta^2|f||Du||D^2u|\Phi'(P) &\le 3\epsilon\int_\Omega \eta^2(1+|Du|)^{p-2}|D^2u|^2\Phi(P)\\
&+C\frac{\epsilon}{C_\gamma}\delta\int_{\Omega\cap\{1<|Du|\le 2\}}\eta^2|D^2u|^2\Phi'(1)\\
&+C(\epsilon)C_\gamma\int_\Omega \eta^2(1+|Du|)^{2-p+\gamma}|f|^2. \qquad (4.21)
\end{aligned}$$

Combining (4.21) and (4.20) and letting $\delta\to 0$, we get

$$|I_8|+|I_9|\le 4\epsilon\int_\Omega \eta^2(1+|Du|)^{p-2}|D^2u|^2\Phi(P)$$

$$+ C(\epsilon)C_\gamma \int_\Omega \eta^2(1+|Du|)^{2-p+\gamma}|f|^2. \tag{4.22}$$

We remind that

$$I_2 + I_3 = -I_1 - I_4 - I_5 - I_6 - I_7 - I_8 - I_9. \tag{4.23}$$

We now ellaborate on the precise form of $D_{\xi\xi}\mathcal{F}(x,\xi)$ to estimate I_3. To do this, we abbuse of notation and for every scalar t we denote $F'(x,t) = \partial_t F(x,t)$ and $F''(x,t) = \partial_{tt}F(x,t)$. By **F1**, for every $\xi \in \mathbb{R}^{n\times N} \setminus \{0\}$ one has

$$D_{\xi\xi}\mathcal{F}(x,\xi) = \left(\frac{F''(x,|\xi|)}{|\xi|^2} - \frac{F'(x,|\xi|)}{|\xi|^3}\right)\xi\otimes\xi + \frac{F'(x,|\xi|)}{|\xi|}\,\mathbf{Id}_{\mathbb{R}^{n\times N}}$$

Componentwise,

$$\begin{aligned} D_{\xi_j^\beta,\xi_i^\alpha}\mathcal{F}(x,\xi) &= D_{\xi_j^\beta}\left(F'(x,|\xi|)\,\frac{\xi_i^\alpha}{|\xi|}\right) \\ &= \left(\frac{F''(x,|\xi|)}{|\xi|^2} - \frac{F'(x,|\xi|)}{|\xi|^3}\right)\xi_i^\alpha\xi_j^\beta + \frac{F'(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha\xi_j^\beta} \end{aligned}$$

Recalling the second equality in (4.4), for a.e. $x \in \{|Du| \geq 1\}$, we have

$$\begin{aligned} &\sum_s \langle D_{\xi\xi}\mathcal{F}(x,Du)\cdot D_{x_s}Du, DP\cdot D_{x_s}u\rangle = \sum_{s,i,j,\alpha,\beta} D_{\xi_j^\beta,\xi_i^\alpha}\mathcal{F}(x,Du)u_{x_s}^\alpha u_{x_s x_j}^\beta P_{x_i} = \\ &= \left(\frac{F''(x,|Du|)}{|Du|^2} - \frac{F'(x,|Du|)}{|Du|^3}\right)\sum_{s,i,j,\alpha,\beta} u_{x_s}^\alpha u_{x_s x_j}^\beta u_{x_i}^\alpha u_{x_j}^\beta P_{x_i} + \frac{F'(x,|Du|)}{|Du|}\sum_{s,i,\alpha} u_{x_s}^\alpha u_{x_s x_i}^\alpha P_{x_i} \\ &= \left(\frac{F''(x,|Du|)}{|Du|} - \frac{F'(x,|Du|)}{|Du|^2}\right)\sum_{s,i,\alpha} u_{x_s}^\alpha(|Du|)_{x_s}u_{x_i}^\alpha(|Du|)_{x_i} + \frac{F'(x,|Du|)}{|Du|}\sum_{s,i,\alpha} u_{x_s}^\alpha u_{x_s x_i}^\alpha(|Du|)_{x_i} \\ &= \left(\frac{F''(x,|Du|)}{|Du|} - \frac{F'(x,|Du|)}{|Du|^2}\right)\sum_\alpha\left(\sum_i u_{x_i}^\alpha(|Du|)_{x_i}\right)^2 + F'(x,|Du|)|D(|Du|)|^2. \end{aligned} \tag{4.24}$$

Thus,

$$\begin{aligned} I_3 &= \int_\Omega \eta^2\Phi'(P)\frac{F''(x,|Du|)}{|Du|}\sum_\alpha\left(\sum_i u_{x_i}^\alpha(|Du|)_{x_i}\right)^2 dx \\ &+ \int_\Omega \eta^2\Phi'(P)\,F'(x,|Du|)\left(|D(|Du|)|^2 - \frac{\sum_\alpha\left(\sum_i u_{x_i}^\alpha(|Du|)_{x_i}\right)^2}{|Du|^2}\right)dx. \end{aligned}$$

Now, if we use the Cauchy-Schwartz inequality, we have

$$\sum_\alpha\left(\sum_i u_{x_i}^\alpha(|Du|)_{x_i}\right)^2 \leq |Du|^2|D(|Du|)|^2.$$

Since

$$\Phi'(t) = (1+t)^{\gamma-3}t(\gamma t+2) \tag{4.25}$$

is nonnegative for every $t \geq 0$ and, by **F2**, $F'(x,|Du|) \geq 0$, then we conclude that

$$I_3 \geq \int_\Omega \eta^2\Phi'(P)\,\frac{F''(x,|Du|)}{|Du|}\sum_\alpha\left(\sum_i u_{x_i}^\alpha(|Du|)_{x_i}\right)^2 dx \geq 0. \tag{4.26}$$

Therefore, using $I_3 \geqslant 0$ together with (4.23) we have

$$I_2 \leqslant |I_1| + |I_4| + |I_5| + |I_6| + |I_7| + |I_8| + |I_9|. \tag{4.27}$$

On the other hand, the ellipticity assumption **F2** gives us that

$$I_2 \geqslant \nu \int_\Omega \eta^2 \cdot \Phi(P) \cdot (1+|Du|)^{p-2} \cdot |D^2u|^2 \, dx. \tag{4.28}$$

Inserting estimates (4.10), (4.11), (4.12), (4.17), (4.18), (4.20), (4.28) into (4.27), we obtain

$$\begin{aligned}
&\nu \int_\Omega \eta^2 \cdot \Phi(P)\,(1+|Du|)^{p-2}\,|D^2u|^2\,dx \\
&\leq 6\,\epsilon \int_\Omega \eta^2 \Phi(P)\,(1+|Du|)^{p-2}\,|D^2u|^2\,dx + C(\epsilon, L_1) \int_\Omega |D\eta|^2|\,(1+|Du|)^p \Phi(P)\,dx \\
&+ C \int_\Omega \eta^2 \Phi(P) k^2(x)\,(1+|Du|)^p\,dx + C(\epsilon)(\gamma+1)^2 \int_\Omega \eta^2 k^2(x)\,(1+|Du|)^{p+\gamma}\,dx \\
&+ C(\gamma+1)^2 \int_\Omega \eta^2 |f|^2 \Phi(P)|(1+|Du|)^{2-p}\,dx.
\end{aligned} \tag{4.29}$$

We now choose $\epsilon = \frac{\nu}{12}$, and reabsorb the first integral in the right hand side by the left hand side. We obtain

$$\begin{aligned}
&\int_\Omega \eta^2 \cdot \Phi(P)\,(1+|Du|)^{p-2}\,|D^2u|^2\,dx \\
&\leq C \int_\Omega |D\eta|^2|\,(1+|Du|)^p\,\Phi(P)\,dx + C \int_\Omega \eta^2 \Phi(P) k^2(x)\,(1+|Du|)^p\,dx \\
&+ C(\gamma+1)^2 \int_\Omega \eta^2 k^2(x)\,(1+|Du|)^{p+\gamma}\,dx + C(\gamma+1)^2 \int_\Omega \eta^2 |f|^2 \Phi(P)|(1+|Du|)^{2-p}\,dx
\end{aligned}$$

Inequality (4.8) follows taking into account the definition of $\Phi(P)$, and recalling $P = (|Du|-1)_+$.

Step 2. Fix a ball $B_R(x_0) \Subset \Omega$ and radii $0 < \rho < s < t < R$. Due to the local nature of our results, without loss of generality we may suppose $R < 1$. Let $\eta \in C_0^\infty(B_t)$ be a cut off function such that $\eta \equiv 1$ on B_s and $|D\eta| \leqslant \dfrac{C}{t-s}$. Inequality (4.8) can be written as follows

$$\int_\Omega \eta^2 \Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx \leq J_1 + J_2 + J_3, \tag{4.30}$$

where we used the notations

$$\begin{aligned}
J_1 &:= C \int |D\eta|^2 (1+P)^{\gamma+p}\,dx, \\
J_2 &:= C(\gamma+1)^2 \int \eta^2 k^2\,(1+P)^{\gamma+p}\,dx, \\
J_3 &:= C(\gamma+1)^2 \int \eta^2 |f|^2 (1+P)^{\gamma-p+2}\,dx.
\end{aligned}$$

We estimate J_i, $i = 1,2,3$ using the assumptions $k,\ f \in L^\sigma_{\mathrm{loc}}(\Omega)$, Hölder's inequality and the properties of η. Setting $b = \frac{\sigma}{\sigma-2}$, we get

$$J_1 \leqslant C \frac{|B_R|^{2/\sigma}}{(t-s)^2} \left(\int_{B_t} (1+P)^{b(\gamma+p)}\,dx \right)^{\frac{1}{b}} \tag{4.31}$$

and

$$J_2 \leqslant C(\gamma+1)^2\left(\int_{B_t} k^\sigma\,dx\right)^{\frac{2}{\sigma}}\cdot\left(\int_{B_t}(1+P)^{b(\gamma+p)}\,dx\right)^{\frac{1}{b}}, \tag{4.32}$$

and

$$\begin{aligned} J_3 &\leq (\gamma+1)^2\int_{B_t}|f|^2(1+P)^{-2p+2}(1+P)^{\gamma+p}\,dx \\ &\leqslant (\gamma+1)^2\left(\int_{B_t}|f|^\sigma\,dx\right)^{\frac{2}{\sigma}}\cdot\left(\int_{B_t}(1+P)^{b(\gamma+p)}\,dx\right)^{\frac{1}{b}}, \end{aligned} \tag{4.33}$$

where we used that $(1+P)^{2-2p}\leqslant 1$. Let us now define the constant E_R as follows

$$E_R^2 := \|k\|^2_{L^\sigma(B_R)} + \|f\|^2_{L^\sigma(B_R)}\,. \tag{4.34}$$

Inserting (4.31), (4.32) and (4.33) into (4.30), and using (4.34), we get

$$\int_\Omega \eta^2\,\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx \leqslant C(\gamma+1)^2\frac{1+E_R^2}{(t-s)^2}\left(\int_{B_R}(1+P)^{b(\gamma+p)}\,dx\right)^{\frac{1}{b}}, \tag{4.35}$$

Following [?], we now introduce the auxilliary function

$$G(t) = 1+\int_0^t (1+s)^{\frac{\gamma+p-4}{2}}\,s\,ds.$$

It is easy to see that

$$\frac{1}{2(\gamma+p)}(1+t)^{\frac{\gamma+p}{2}} \leq G(t) \leq 2(1+t)^{\frac{\gamma+p}{2}}, \qquad G'(t) = t(1+t)^{\frac{\gamma+p-4}{2}}\,. \tag{4.36}$$

We now observe that $2\leq n<\sigma<\infty$ implies $1<b<\frac{n}{n-2}\leq\infty$ and so there exists m such that

$$b<m\leq\frac{n}{n-2}.$$

For such a choice of m, one has by the Sobolev embedding that

$$W^{1,2}_{loc}\hookrightarrow L^{2^*}_{loc}\hookrightarrow L^{2m}_{loc}.$$

Indeed, for $n>2$ this is clear, while for $n=2$ one simply abbuses of notation and replace L^{2^*} by BMO. Thus

$$\left(\int_\Omega \left|\eta\,G(P)\right|^{2m}dx\right)^{\frac{1}{m}} \leq C\int_\Omega\left|D(\eta\,G(P))\right|^2dx \leq C\int_\Omega |D\eta|^2G(P)^2\,dx + C\int_\Omega \eta^2G'(P)^2|DP|^2\,dx.$$

Using the properties of $G(t)$ at (4.36) in the previous inequality, we obtain

$$\begin{aligned} &\frac{1}{(\gamma+p)^2}\left(\int_\Omega \eta^{2m}\,(1+P)^{m(\gamma+p)}\,dx\right)^{\frac{1}{m}} \\ &\leq c\int_\Omega |D\eta|^2\,(1+P)^{\gamma+p}\,dx + c\int_\Omega \eta^2\,(1+P)^{\gamma+p-4}\,P^2\,|DP|^2\,dx \\ &\leq \frac{c}{(t-s)^2}\int_{B_t}(1+P)^{\gamma+p}\,dx + c\int_\Omega \eta^2\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx \\ &\leq \frac{c}{(t-s)^2}\left(\int_{B_t}(1+P)^{b(\gamma+p)}\,dx\right)^{\frac{1}{b}} + c\int_\Omega \eta^2\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx\,, \end{aligned} \tag{4.37}$$

where we also used the properties of the function η and Hölder's inequality. Combining estimates (4.42) and (4.37), using that $\eta \equiv 1$ on B_s and recalling that $P = (|Du| - 1)_+$, we get

$$\left(\int_{B_s} (1+P)^{m(\gamma+p)}\,dx\right)^{\frac{1}{m}} \leqslant C(\gamma+p)^4 \frac{1+E_R^2}{(t-s)^2}\left(\int_{B_t} (1+P)^{b(\gamma+p)}\,dx\right)^{\frac{1}{b}} \tag{4.38}$$

Using now that $1 < b < m$, and setting $\theta = \frac{1-1/b}{1-1/m}$, one has $0 < \theta < 1$ and moreover

$$\frac{1}{b(\gamma+p)} = \frac{1-\theta}{\gamma+p} + \frac{\theta}{m(\gamma+p)}$$

and so, the use of the interpolation inequality in the right hand side (4.38) yields

$$\begin{aligned}&\left(\int_{B_s} (1+P)^{m(\gamma+p)}\,dx\right)^{\frac{1}{m}} \\ &\leq C(\gamma+p)^4 \frac{1+E_R^2}{(t-s)^2}\left(\int_{B_t} (1+P)^{\gamma+p}\,dx\right)^{1-\theta}\left(\int_{B_t} (1+P)^{m(\gamma+p)}\,dx\right)^{\frac{\theta}{m}}\end{aligned} \tag{4.39}$$

We now use Young's inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$ in the right hand side of (4.39) thus getting

$$\begin{aligned}\left(\int_{B_s} (1+P)^{m(\gamma+p)}\,dx\right)^{\frac{1}{m}} &\leqslant \frac{1}{2}\left(\int_{B_t} (1+P)^{m(\gamma+p)}\,dx\right)^{\frac{1}{m}} \\ &\quad + C\left((\gamma+p)^4 \frac{1+E_R^2}{(t-s)^2}\right)^{\frac{1}{1-\theta}} \int_{B_R} (1+P)^{\gamma+p}\,dx\end{aligned} \tag{4.40}$$

The iteration Lemma 16 now gives that

$$\left(\int_{B_\rho} (1+P)^{m(\gamma+p)}\,dx\right)^{\frac{1}{m}} \leqslant C\left((\gamma+p)^4 \frac{1+E_R^2}{(R-\rho)^2}\right)^{\frac{1}{1-\theta}} \int_{B_R} (1+P)^{\gamma+p}\,dx. \tag{4.41}$$

It is important to notice that here $\gamma \geq 0$ can take any value.

Step 3. Let us define the decreasing sequence of radii ρ_j, $j \in \mathbb{N}$, by setting

$$\rho_j := \rho + \frac{R-\rho}{2^j},$$

and the increasing sequence of exponents

$$p_j = (\gamma+p)\, m^j.$$

Estimate (4.41) can be written on every ball B_{ρ_j} as follows

$$\left(\int_{B_{\rho_{j+1}}} (1+P)^{p_{j+1}}\,dx\right)^{\frac{1}{p_{j+1}}} \leqslant \left[C\frac{p_j^2(1+E_R)}{(\rho_j-\rho_{j+1})}\right]^{\frac{2m}{p_j}\frac{1}{1-\theta}} \left(\int_{B_{\rho_j}} (1+P)^{p_j}\,dx\right)^{\frac{1}{p_j}}. \tag{4.42}$$

Iterating estimate (4.42) we obtain

$$\begin{aligned}\left(\int_{B_{\rho_{j+1}}} (1+P)^{p_{j+1}}\,dx\right)^{\frac{1}{p_{j+1}}} &\leqslant \prod_{j=0}^{+\infty}\left[C\frac{p_j^2(1+E_R)}{(\rho_j-\rho_{j+1})}\right]^{\frac{2m}{p_j}\frac{1}{1-\theta}} \left(\int_{B_{\rho_0}} (1+P)^{p_0}\,dx\right)^{\frac{1}{p_0}} \\ &= \prod_{j=0}^{+\infty}\left[C\frac{2^{j+1}p_j^2\,(1+E_R)}{(R-\rho)}\right]^{\frac{2m}{p_j}\frac{1}{1-\theta}} \left(\int_{B_{\rho_0}} (1+P)^{p_0}\,dx\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used the definition of ρ_j. It is easy to prove that

$$\prod_{j=0}^{+\infty}\left[C\,\frac{2^{j+1}p_j^2(1+E_R)}{(R-\rho)}\right]^{\frac{2m}{p_j}\frac{1}{1-\theta}}\le \widehat{C}\left(\frac{(1+E_R)}{(R-\rho)}\right)^{\tau}$$

for $\tau=\frac{4m^3}{\sigma(m-1)^2}$, and a constant $\widehat{C}=\widehat{C}(n,N,p,\sigma,\nu,L,L_1,m)$. Therefore

$$\left(\int_{B_{\rho_{j+1}}}(1+P)^{p_{j+1}}\,dx\right)^{\frac{1}{p_{j+1}}}\leqslant \widehat{C}\frac{(1+E_R)^{\tau}}{(R-\rho)^{\tau}}\left(\int_{B_t}(1+P)^{\gamma+p}\,dx\right)^{\frac{1}{\gamma+p}},$$

for every $j\in\mathbb{N}$. Now, letting $j\to\infty$, and recalling that $\rho_j\ge\rho$, for every $j\in\mathbb{N}$, we end up with

$$\begin{aligned}\sup_{B_\rho}|Du|&\le\sup_{B_\rho}(1+P)\\&=\lim_{j\to\infty}\left(\int_{B_\rho}(1+P)^{p_j}\right)^{\frac{1}{p_j}}\le\widehat{C}\frac{(1+E_R)^{\tau}}{(R-\rho)^{\tau}}\left(\int_{B_R}(1+|Du|)^{(\gamma+p)}\right)^{\frac{1}{\gamma+p}},\end{aligned}$$

that, choosing $\gamma=0$, gives (4.6).

Step 4. In this Step we are going to establish estimate (4.7). To this aim we write (4.8) for $\gamma=0$,

$$\begin{aligned}\int_\Omega\eta^2\Phi(P)\,|Du|^{p-2}\,|D^2u|^2\,dx&\le C\int_\Omega\eta^2k^2\,(1+P)^p\,dx\\&\quad+C\int_\Omega|D\eta|^2\,(1+P)^p\,dx+C\int_\Omega\eta^2|f|^2\,(1+P)^{-p+2}\,dx,\end{aligned}\tag{4.43}$$

for every $\eta\in C_0^\infty(\Omega)$. Choosing radii $0<\rho<R$, η a cut off function between B_ρ and B_R and using estimate (4.6), we obtain

$$\begin{aligned}\int_{B_\rho}\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx&\le C\sup_{B_R}|Du|^p\int_{B_R}k^2\,dx+\frac{C|B_R|}{(R-\rho)^2}\sup_{B_R}|Du|^p+C\int_{B_R}|f|^2\,dx\\&\le C(1+E_{2R})^{\tau'}\int_{B_{2R}}(1+|Du|^p)\,dx,\end{aligned}\tag{4.44}$$

for a constant $C=C(n,N,p,L,L_1,\nu)$ and an exponent $\tau'=\tau'(\sigma,p,n)$. The proof is finished. □

4.2 Proof of Theorem 15

In this section, we use the apriori estimate of Theorem 17 to prove Theorem 15. In particular, we show that local minimizers $u\in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ of the functional in (4.1),

$$\mathbb{F}(u,\Omega)=\int_\Omega\mathcal{F}(x,Du)+f(x)\cdot u\,dx$$

are locally Lipschitz.

To this end, we state first the following approximation result, which we take from [27]. It shows that one can find a sequence of uniformly elliptic integrands $\mathcal{F}_m$ that approximate the given $\mathcal{F}$. The approximants can be chosen to be Lipschitz in the x variable, and also to have ellipticity bounds on all the domain of the ξ variable, although these bounds may depend on m (see conditions $\tilde{\mathbf{F}}_{\mathbf{m}}\mathbf{2}$-$\tilde{\mathbf{F}}_{\mathbf{m}}\mathbf{4}$ below). Furthermore, these ellipticity conditions may be assumed uniform in m away from a ball of the ξ variable (see conditions $\tilde{\mathbf{F}}\mathbf{0}$-$\tilde{\mathbf{F}}\mathbf{4}$ below). We recall that, without loss of generality, we assumed that the radius R appearing in the assumptions **F0**–**F4** is equal to 1.

Proposition 19. *Let $\mathcal{F} : \Omega \times \mathbb{R}^{n\times N} \to [0, +\infty)$ be a Carathéodory function, convex and C^2 with respect to the second variable, and satisfying assumptions* **F0**–**F4**. *Fixed an open set $\Omega' \Subset \Omega$, there exists a sequence $\mathcal{F}_m : \Omega' \times \mathbb{R}^{n\times N} \to [0, +\infty)$ of Carathéodory functions, C^2 and convex in the second variable, such that $\mathcal{F}_m$ converges to $\mathcal{F}$ pointwise a.e. on Ω' and everywhere in $\mathbb{R}^{n\times N}$. Moreover, each $\mathcal{F}_m$ can be chosen so that the following properties are satisfied:*

$\tilde{\mathbf{F}}\mathbf{0}$ *there exist constants $\tilde{L}, \tilde{c}_1, \tilde{c}_2 > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{n\times N}$*

$$\tilde{c}_1|\xi|^p - \tilde{c}_2 \le \mathcal{F}_m(x, \xi) \le \tilde{L}(1 + |\xi|)^p,$$

$\tilde{\mathbf{F}}\mathbf{1}$ *for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$ one has $\mathcal{F}_m(x, \xi) = F_m(x, |\xi|)$,*

$\tilde{\mathbf{F}}\mathbf{2}$ *there exists $\tilde{\nu} = \tilde{\nu}(\nu, p)$ such that for every $x \in \Omega'$, $\xi \in \mathbb{R}^{n\times N} \setminus B_2(0)$ and $\lambda \in \mathbb{R}^{n\times N}$*

$$\tilde{\nu}(1 + |\xi|)^{p-2}|\lambda|^2 \le \langle D_{\xi\xi}\mathcal{F}_m(x, \xi)\lambda, \lambda\rangle,$$

$\tilde{\mathbf{F}}\mathbf{3}$ *there exists $\tilde{L}_1 > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN} \setminus B_2(0)$*

$$|D_{\xi\xi}\mathcal{F}_m(x, \xi)| \le \tilde{L}_1(1 + |\xi|)^{p-2},$$

$\tilde{\mathbf{F}}\mathbf{4}$ *for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$,*

$$|D_{\xi x}\mathcal{F}_m(x, \xi)| \le 2^{p-1}k_m(x)(1 + |\xi|)^{p-1},$$

where $k_m \in C^\infty(\Omega')$ is a non-negative function, such that $k_m \to k$ in $L^\sigma(\Omega')$,

Moreover, the above properties can be extended to all $\xi \in \mathbb{R}^{n\times N}$ in the following way:

$\tilde{\mathbf{F}}_{\mathbf{m}}\mathbf{2}$ *There is $\mu_m > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{n\times N}$ and for all $\lambda \in \mathbb{R}^{n\times N}$*

$$\mu_m(1 + |\xi|)^{p-2}|\lambda|^2 \le \langle D_{\xi\xi}\mathcal{F}_m(x, \xi)\lambda, \lambda\rangle.$$

$\tilde{\mathbf{F}}_{\mathbf{m}}\mathbf{3}$ *there exists $\sigma_m > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$*

$$|D_{\xi\xi}\mathcal{F}_m(x, \xi)| \le \sigma_m(1 + |\xi|)^{p-2}.$$

$\tilde{\mathbf{F}}_{\mathbf{m}}\mathbf{4}$ *There is $\Lambda_m > 0$ such that for every $x \in \Omega'$ and $\xi \in \overline{B_2(0)}$*

$$|D_{\xi x}\mathcal{F}_m(x, \xi)| \le \Lambda_m(1 + |\xi|)^{p-1}.$$

We now recall a regularity result for minimizers of functionals of the form

$$\inf_w \int_\Omega \mathcal{F}(x, Dw) + \arctan(|w-\bar{u}|^2)\, dx$$

where $\mathcal{F}$ has standard growth conditions and smooth dependence on the x-variable, and $\bar{u}$ is fixed. In absence of the perturbation term $\arctan(|w-\bar{u}|^2)$, this regularity result is well known. We refer to [45] for the higher differentiability result, and to [29, Theorem 1.1] as far as the Lipschitz continuity of the local minimizers is concerned. In presence of the perturbation term, the proofs can be easily adapted because of the boundedness of its integrand and of its derivative with respect to the variable w. More precisely, we have the following.

Theorem 20. *Let $\mathcal{F} : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$, $\mathcal{F} \in C^2(\Omega \times \mathbb{R}^{nN})$, and define the functional*

$$\int_\Omega \mathcal{F}(x, Dw) + \arctan(|w-\bar{u}|^2)\, dx \tag{4.45}$$

with $\bar{u} \in C^2(\Omega; \mathbb{R}^N)$. Assume that there exists $p \geq 2$ such that for every $x \in \Omega$ and every $\xi, \lambda \in \mathbb{R}^{nN}$,

$$c_1|\xi|^p - c_2 \leq g(x, \xi) \leq L(1 + |\xi|)^p,$$

$$\nu\,(1 + |\xi|)^{p-2}|\lambda|^2 \leq \langle D_{\xi\xi} g(x, \xi)\lambda, \lambda\rangle,$$

$$|D_{\xi\xi} g(x, \xi)| \leq L_1\,(1 + |\xi|)^{p-2},$$

$$|D_{\xi x} g(x, \xi)| \leq K(1 + |\xi|)^{p-1},$$

with positive constants c_1, c_2, L, L_1, ν, K. Then any local minimizer v of (4.45) is in $W^{2,2}_{loc}(\Omega; \mathbb{R}^N)$ and

$$(1 + |Dv|^2)^{\frac{p-2}{2}} |D^2 v|^2 \in L^1_{loc}(\Omega).$$

Moreover, if there exists $F : \Omega \times [0, +\infty) \to [0, +\infty)$ such that $\mathcal{F}(x, \xi) = F(x, |\xi|)$, then one also has $v \in W^{1,\infty}_{loc}(\Omega; \mathbb{R}^N)$.

We are now ready for proving Theorem 15.

Proof of Theorem 15. Let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a local minimizer of the functional $\mathbb{F}(u, \Omega)$, and let $B_r \subset \Omega$ be a fixed ball. We consider the sequence of energy densities $\mathcal{F}_m(x, \xi)$ obtained after applying Proposition 19 to the integrand $\mathcal{F}$. For a standard sequence of mollifiers ρ_ϵ, we set $u_\epsilon = u * \rho_\epsilon$, $f_\epsilon = f * \rho_\epsilon$. We define

$$\mathbb{F}_{\epsilon,m}(w, B_r) := \int_{B_r} \mathcal{F}_m(x, Dw) + f_\epsilon(x) w + \arctan|w - u_\epsilon|^2\ dx.$$

The lower semi-continuity and strict convexity of $\mathbb{F}_{\epsilon,m}$ ensures that the minimization problem

$$\min\left\{\mathbb{F}_{\epsilon,m}(w; B_r) : w \in u + W^{1,p}_0(B_r, \mathbb{R}^N)\right\} \tag{4.46}$$

has a unique solution $v_{\epsilon,m} \in u + W_0^{1,p}(B_r, \mathbb{R}^N)$. By the growth of $\mathcal{F}_m$ (Proposition 19 , condition ($\mathcal{A}0$)) and the minimality of $v_{\epsilon,m}$, there exists a constant C independent of m and such that

$$\begin{aligned}
\int_{B_r} |Dv_{\epsilon,m}|^p\,dx &\le C\int_{B_r} 1 + \mathcal{F}_m(x, Dv_{\epsilon,m})\,dx \\
&\le C\,\mathbb{F}_{\epsilon,m}(v_{\epsilon,m}, B_r) - \int_{B_r} f_\epsilon(x) v_{\epsilon,m}\,dx \\
&\le C\,\mathbb{F}_{\epsilon,m}(u, B_r) - \int_{B_r} f_\epsilon(x) v_{\epsilon,m}\,dx \\
&= C\int_{B_r} \left(1 + \mathcal{F}_m(x, Du) + \arctan|u-u_\epsilon|^2\right)dx - \int_{B_r} f_\epsilon(x)(v_{\epsilon,m}-u)\,dx \\
&\le C\int_{B_r} \left(1 + |Du|^p + \left(\frac{\pi}{2}\right)^2\right)dx + \int_{B_r} |f_\epsilon(x)||v_{\epsilon,m}-u|\,dx.
\end{aligned}$$

We now use Young's inequality, and Poincaré inequality, and the previous estimate yields

$$\begin{aligned}
\int_{B_r} |Dv_{\epsilon,m}|^p\,dx &\le C\int_{B_r} 1 + |Du|^p\,dx + c_\alpha \int_{B_r} |f_\epsilon(x)|^{p'}\,dx + \alpha\int_{B_r} |v_{\epsilon,m}-u|^p\,dx \\
&\le C\int_{B_r} 1 + |Du|^p\,dx + c_\alpha \int_{B_r} |f_\epsilon(x)|^{p'}\,dx + c_{n,p,r}\alpha\int_{B_r} |Dv_{\epsilon,m}-Du|^p\,dx, \\
&\le C\int_{B_r} 1 + |Du|^p\,dx + c_\alpha \int_{B_r} |f_\epsilon(x)|^{p'}\,dx + c_{n,p,r}\alpha\int_{B_r} |Du|^p\,dx + c_{n,p,r}\alpha\int_{B_r} |Dv_{\epsilon,m}|^p\,dx,
\end{aligned}$$

We now choose $\alpha = \frac{1}{2c_{n,p,r}}$, and the above inequality becomes

$$\int_{B_r} |Dv_{\epsilon,m}|^p\,dx \le C\int_{B_r} 1 + |Du|^p\,dx + c_{n,p,r}\int_{B_r} |f_\epsilon(x)|^{p'}\,dx$$

for a constant C independent of ϵ and m. By weak compactness, we deduce that $\{v_{\epsilon,m}\}_m$ weakly converges to a function $v_\epsilon \in u + W_0^{1,p}(B_r; \mathbb{R}^N)$ as $m \to +\infty$, up to a subsequence $m_j = m_j(\epsilon)$. Set now

$$\mathbb{F}_\epsilon(w, B_r) := \int_{B_r} \mathcal{F}(x, Dw) + f_\epsilon(x)w + \arctan|w-u_\epsilon|^2\,dx.$$

For every fixed $\epsilon > 0$, one can see that $\mathbb{F}_{\epsilon,m}$ Γ-converges to $\mathbb{F}_\epsilon$ as $m \to \infty$ (see Theorem 5.14 and Corollary 7.20 in [30]). As a consequence, v_ϵ is a minimizer of $\mathbb{F}_\epsilon$. Now, the lower semicontinuity of the L^p norm implies that

$$\int_{B_r} |Dv_\epsilon|^p\,dx \le \liminf_{m\to\infty}\int_{B_r} |Dv_{\epsilon,m}|^p\,dx \le C\int_{B_r} 1 + |Du|^p\,dx + c_{n,p,r}\int_{B_r} |f_\epsilon(x)|^{p'}\,dx$$

From $p' < 2 < \sigma$, it turns out that $\int_{B_r} |f_\epsilon(x)|^{p'}\,dx \leqslant C$, with C is independent of ϵ and so

$$\int_{B_r} |Dv_\epsilon|^p\,dx \le \liminf_{m\to\infty}\int_{B_r} |Dv_{\epsilon,m}|^p\,dx \leqslant C. \tag{4.47}$$

As before, by compactness there exists $v \in u + W_0^{1,p}(B_r; \mathbb{R}^N)$ such that v_ϵ converges to v in the weak $W_0^{1,p}(B_r, \mathbb{R}^N)$ topology. Also, again by the lower semicontinuity of the norm,

$$\int_{B_r} |Dv|^p\,dx \le \lim_{\epsilon\to 0}\int_{B_r} |Dv_\epsilon|^p\,dx \leqslant C. \tag{4.48}$$

Observe that, as $\epsilon \to 0$, the functionals $\mathbb{F}_\epsilon$ Γ-converge to

$$\mathbb{F}_0(w, B_r) := \int_{B_r} \mathcal{F}(x, Dw) + f(x)\cdot w + \arctan|w-u|^2\, dx,$$

whence v is a minimizer of $\mathbb{F}_0$, and therefore $\mathbb{F}_0(v, B_r) \leq \mathbb{F}_0(u, B_r)$. This, together with the minimality of u, implies that

$$\begin{aligned}\mathbb{F}(u, B_r) &\leq \mathbb{F}(v, B_r)\\ &\leq \mathbb{F}_0(v, B_r) \leq \mathbb{F}_0(u, B_r) = \mathbb{F}(u, B_r)\end{aligned}$$

Hence the above inequalities are an equality, and as a consequence

$$\int_{B_r} \arctan|u-v|^2\, dx = 0, \qquad \Rightarrow \qquad u = v \qquad \text{a.e. in } B_r,$$

The functionals $G_{\epsilon,m}$, for every ϵ and m, satisfy the assumptions of Theorem 20. Therefore their minimizers $v_{\epsilon,m}$ belong to $W^{2,2}_{\text{loc}} \cap W^{1,\infty}$, and so we are legitimate to use the a priori estimate (4.6) of Theorem 17, thus getting

$$\sup_{B_\rho} |Dv_{\epsilon,m}| \leqslant C(1 + ||k_\epsilon||_{L^\sigma(B_r)} + ||f_\epsilon||_{L^\sigma(B_r)})^\tau \left(\int_{B_{\rho'}} (1 + |Dv_{\epsilon,m}|^p\, dx\right)^{\frac{1}{p}},$$

for every $B_\rho \subset B_{\rho'} \Subset B_r$ and for a constant C independent of ϵ and m. By virtue of (4.47) and (4.48), and since $k_\epsilon \to k$ and $f_\epsilon \to f$ strongly in L^σ, passing to the limit, first as $m \to +\infty$, and then as $\epsilon \to 0$, we conclude that

$$\sup_{B_\rho} |Du| \leqslant C(1 + ||k||_{L^\sigma(B_r)} + ||f||_{L^\sigma(B_r)})^\tau \left(\int_{B_{\rho'}} (1 + |Du|^p + |f|^\sigma)\, dx\right)^{\gamma}.$$

This finishes the proof. □

4.3 Boundary Lipschitz estimates

In this section we show that a typical reflection method extends the previous interior estimates up to the boundary. To this end, we will proceed as in [15]. We will start with the following auxilliary standard result.

Lemma 21. *Let $B \subset \mathbb{R}^n$ be a ball centered at the origin. Set $B^+ = B \cap \{x_n > 0\}$ and $B^- = B \cap \{x_n < 0\}$, and let $f : B \longrightarrow \mathbb{R}$ be such that $f \in W^{1,p}(B^+)$ and $f \in W^{1,p}(B^-)$. If f is continuous on B, then $f \in W^{1,p}(B)$.*

This result is classical, and we omit its proof. Our first main result states that Theorem 15 gives global bounds when Ω has nice boundary.

Theorem 22. *Suppose Ω is a bounded domain with $\partial\Omega \in C^{3,1}$. Let $\mathcal{F}$ be as in Theorem 15, and let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of following Neumann boundary problem*

$$\begin{cases} -\operatorname{div}(D_\xi \mathcal{F}(x, \nabla u(x))) = f, & \text{in } \Omega,\\ D_\xi \mathcal{F}(x, \nabla u(x)).\nu = 0, & \text{on } \partial\Omega,\end{cases} \tag{4.49}$$

for some $f \in L^\sigma(\Omega)$, and $\sigma > n$. Then $Du \in L^\infty(\Omega)$.

Proof. Let $x_0 \in \partial\Omega$, and let V be a neighborhood of x_0 in $\mathbb{R}^n$. We write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. We denote $B = \{|x| < 1\}$, $B^+ = \{x \in B; x_n > 0\}$. Since Ω is $C^{3,1}$, there exist a diffeomorphism

$$\psi : B^+ \to V \cap \Omega$$

which is onto, and which extends to ∂B^+ in a C^1-smooth way, with $D\psi(y) \neq 0$ for $y \in \partial B^+$. In particular, we can assume that $\psi(\partial B^+) = \partial\Omega \cap V$. Set $\mathcal{R}$ to be reflection in $\mathbb{R}^n$ with respect to the hyperplane $\{x_n = 0\}$, and $B^- = \mathcal{R}B^+$, $B = B^+ \cup B^-$. We also set $L = \partial B^+ \cap \partial B^-$. Define

$$\Psi(y) =: \begin{cases} \psi(y), & \text{if } y \in B^+, \\ \psi \circ \mathcal{R}(y), & \text{if } y \in B^-, \end{cases}$$

By construction, Ψ is continuous on B, and smooth on $B \setminus L$, with

$$D\Psi(y) =: \begin{cases} D\psi(y), & \text{if } y \in B^+, \\ D\psi(\mathcal{R}(y)) \cdot \mathcal{R}, & \text{if } y \in B^-, \end{cases}$$

In particular, the jacobian $J(y, \Psi) = \det D\Psi(y)$ is well defined on $B \setminus L$. Let us set

$$\widehat{u}(y) = u(\Psi(y)), \qquad \widehat{f}(y) = f(\Psi(y)) \cdot |J(y, \Psi)|. \tag{4.50}$$

Let φ be a smooth testing function with $\mathrm{supp}(\varphi) \subset (\overline{\Omega} \cap \overline{V})$. Without loss of generality, we can assume that φ vanishes on $\partial(V \cap \Omega) \setminus \partial\Omega$. Using that u solves the problem (4.49),

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla\varphi \rangle \, dx = \int_\Omega f \, \varphi \, dx,$$

where $\mathcal{A}(x, \xi) = D_\xi \mathcal{F}(x, \xi)$. After the change of variables $x = \psi(y)$, this reads

$$\int_{B^+} \langle \mathcal{A}(\psi, \nabla u(\psi)), \nabla\varphi(\psi) \rangle \cdot |J\psi| \, dx = \int_{B^+} f(\psi)\varphi(\psi) \cdot |J\psi| \, dx, \tag{4.51}$$

and similarly, with the change of variable $x = \psi(\mathcal{R}y)$,

$$\int_{B^-} \langle \mathcal{A}(\psi(\mathcal{R}), \nabla u(\psi(\mathcal{R}))), \nabla\varphi(\psi(\mathcal{R})) \rangle \cdot |J\psi(\mathcal{R})| \, dx = \int_{B^-} f(\psi(\mathcal{R}))\varphi(\psi(\mathcal{R})) \cdot |J\psi(\mathcal{R})| \, dx. \tag{4.52}$$

Using the chain rule, (4.51) becomes

$$\int_{B^+} \langle \mathcal{A}\left(\Psi, \nabla(u \circ \Psi) \cdot (D\Psi)^{-1}\right), \nabla(\varphi \circ \Psi) \cdot (D\Psi)^{-1} \rangle \cdot |J\Psi| \, dx = \int_{B^+} f(\Psi)\varphi(\Psi) \cdot |J\Psi| \, dx, \tag{4.53}$$

while for (4.52) we get

$$\int_{B^-} \langle \mathcal{A}\left(\Psi, \nabla(u \circ \Psi) \cdot (D\Psi)^{-1}\right), \nabla(\varphi \circ \Psi) \cdot (D\Psi)^{-1} \rangle \cdot |J\Psi| \, dx = \int_{B^-} f(\Psi)\varphi(\Psi) \cdot |J\Psi| \, dx. \tag{4.54}$$

Putting things together, we get

$$\int_{B^+ \cup B^-} \langle \mathcal{B}(y, \nabla\widehat{u}), \nabla\widehat{\varphi} \rangle \, dx = \int_{B^+ \cup B^-} \widehat{f} \cdot \widehat{\varphi} \, dx, \tag{4.55}$$

where we have defined

$$\mathcal{B}(y,\xi) := \mathcal{A}\Big(\Psi(y), \xi \cdot \mathcal{M}(y)\Big) \cdot \mathcal{M}^t(y) \cdot \mathcal{J}(y), \qquad \forall x \in B, \tag{4.56}$$

as well as $\widehat{\varphi(y)} := \varphi(\Psi(y))$, $\mathcal{M}(y) = (D\Psi(y))^{-1}$ and $\mathcal{J}(x) = |J(y,\Psi)|$. It is clear that (4.55) holds not only for functions of the form $\widehat{\varphi} = \varphi(\Psi)$, but also for any function $\widehat{\varphi} \in W_0^{1,p}(B)$.

Arguing as in [15], we now claim that texist a matrix function $x \mapsto \mathcal{O}(x)$, such that

$$\mathcal{O} \in C^{1,1}(B), \qquad \mathcal{O}^t \cdot \mathcal{O} = I, \tag{4.57}$$

and moreover

$$[D\psi(x')]^{-1} = \mathcal{R}[D\psi(x')]^{-1}\mathcal{O}(x'), \qquad \forall x' \in \partial B^+ \cap \partial B^-. \tag{4.58}$$

Since $D\psi$ has positive determinant, we can write $D\psi = OU$, with O orthogonal and U symmetric and positive definite. To see this, note that $O = D\psi\, U^{-1}$ and $[D\psi]^t D\psi = U^2$, therefore it suffices to choose $O = D\psi([D\psi]^t D\psi)^{-\frac{1}{2}}$. Such a choice certainly gives an O wich is orthogonal,

$$\Big(D\psi([D\psi]^t D\psi)^{-\frac{1}{2}}\Big)^t = \Big(([D\psi]^t D\psi)^{-\frac{1}{2}}\Big)^t (D\psi)^t = \Big(([D\psi]^t D\psi)^{-\frac{1}{2}}\Big)(D\psi)^t,$$

so that O is indeed orthogonal. Let us then set $\mathcal{O}$ to be the matrix O in the decompositon $D\psi = OU$. Proving that (4.58) holds deserves some more effort, as it depends on the precise choice of the parametrization ψ. Let us choose ψ having the following form,

$$\psi(x', x_n) = (x', g(x')) - x_n\,(\nabla g(x'), -1) \tag{4.59}$$

where $g \in C^{3,1}(\mathbb{R}^{n-1})$ is restricted to L. With this choice, we obtain

$$D\psi(x) = \begin{pmatrix} \mathbf{Id}_{n-1} - x_n \cdot D^2 g(x') & -\nabla g(x')^t \\ \nabla g(x') & 1 \end{pmatrix}$$

Especially, $\det D\psi(x', 0) = 1 - |\nabla g(x')|^2$. As a consequence, we have the equality

$$[D\psi]^t D\psi = ((D\psi)\mathcal{R})^2 \qquad \text{at points } x = (x', 0) \in L. \tag{4.60}$$

Therefore, using our choice for matrix $\mathcal{O}$, together with (4.60), we get

$$\mathcal{R}(D\psi)^{-1}\mathcal{O} = \mathcal{R}(D\psi)^{-1} \cdot D\psi \cdot ((D\psi)^t D\psi)^{-\frac{1}{2}} = \mathcal{R}(((D\psi)\mathcal{R})^2)^{-\frac{1}{2}} = (D\psi)^{-1}$$

as claimed.

We now observe that

$$\begin{aligned} \mathcal{B}(x,\xi) &= D_\xi \mathcal{F}(\Psi(x), \xi \cdot \mathcal{M}(x))\, \mathcal{M}(x)^t\, \mathcal{J}(x) \\ &= \nabla_\xi \left(\mathcal{F}(\Psi(x), \mathcal{M}(x)^t\, \xi)\, \mathcal{J}(x)\right) \\ &= \nabla_\xi \mathcal{G}(x,\xi) \end{aligned}$$

where $\mathcal{G}(x,\xi) = \mathcal{F}(\Psi(x), \mathcal{M}(x)^t\,\xi)\,\mathcal{J}(x) = F(\Psi(x), |\mathcal{M}(x)^t\xi|)\,\mathcal{J}(x)$. We now justify that $\mathcal{G}$ satisfies the assumptions **F0**–**F4** of Theorem 15. Condition **F0** is clear, since

$$\begin{aligned} \frac{1}{c} \leq & \mathcal{J}(x) \leq c \\ \frac{1}{c}\,|\xi| \leq |M(x)^t\xi| & \leq c\,|\xi| \end{aligned} \tag{4.61}$$

where $c = \|M\|_\infty = \max\{\|(D\psi)^{-1}\|_\infty, \|J(\cdot,\psi)\|_\infty\}$. Concerning **F1**, even though one cannot say that $\mathcal{G}$ is genuinely radial, we can certainly say that $\mathcal{G}(x,\xi) = G(x, |\mathcal{M}(x)^t\xi|)$. Having in mind that

$$D_\xi(|\mathcal{M}(x)^t\xi|) = \frac{\mathcal{M}(x)\cdot\mathcal{M}(x)^t\xi}{|\mathcal{M}(x)^t\xi|}$$

and the bounds (4.61), one can show that Theorem 15 still holds (with a similar proof) if one replaces **F1** by $\mathcal{G}(x,\xi) = G(x, |\mathcal{M}(x)^t\xi|)$. With respect to **F2** and **F3**, one only needs to recall (4.61). Finally, we prove **F4**. First, at points $x \notin L$, we can clearly differentiate $D_\xi\mathcal{G}$ in x and use the chain rule to obtain

$$|D_{x\xi}\mathcal{G}(x,\xi)| \leq k(\Psi(x))\,|\mathcal{M}(x)^t\xi|^{p-1}|\mathcal{M}(x)^t|\mathcal{J}(x) + |D_\xi\mathcal{F}(\Psi(x), \mathcal{M}(x)^t\xi)\,D_x(\mathcal{M}(x)\,\mathcal{J}(x))$$

Thus, arguing separately for $x \in B^+$ and $x \in B^-$, on which Ψ is separately bilipschitz, one sees that

$$|D_{x,\xi}\mathcal{G}(x,\xi)| \leq \tilde{k}(x)|\xi|^{p-1}$$

for a function $\tilde{k} \in L^\sigma$. Concerning the points $x_0 \in L$, we recall that $D\psi$ is continuous in B^+ up to ∂B^+. Having in mind also that $D_\xi\mathcal{F}(x,\xi\mathcal{O}) = \mathcal{A}(x,\xi)\cdot\mathcal{O}$, we get at points $x_0 = (x_0', 0) \in L$ the following equality

$$\begin{aligned}
\lim_{x\to x_0, x\in B^+} D_\xi\mathcal{G}(x,\xi) &= \lim_{x\to x_0, x\in B^+} D_\xi\mathcal{F}\left(\psi(x), \xi\cdot\mathcal{M}(x)\right)\cdot\mathcal{M}(x)^t\cdot|\mathcal{J}(x)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\right)\cdot\mathcal{M}(x_0)^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\right)\cdot(\mathcal{R}\,\mathcal{M}(x_0)\,\mathcal{O}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\right)\cdot\mathcal{O}(x_0)^t\,\mathcal{R}\,\mathcal{M}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\cdot\mathcal{O}(x_0)^t\right)\,\mathcal{R}\,\mathcal{M}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{R}\,\mathcal{M}(x_0)\right)\,\mathcal{R}\,\mathcal{M}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(\mathcal{R}x_0), \xi\cdot\mathcal{R}\,\mathcal{M}(\mathcal{R}x_0)\right)\,\mathcal{R}\,\mathcal{M}(\mathcal{R}x_0))^t\cdot|\mathcal{J}(\mathcal{R}x_0)| \\
&= \lim_{y\to x_0, x\in B^-} D_\xi\mathcal{G}(y,\xi),
\end{aligned}$$

and so $y \mapsto D_\xi\mathcal{G}(y,\xi)$ is continuous on L. It then follows from Lemma 21 that $x \mapsto D_\xi\mathcal{G}(x,\xi)$ is $W^{1,\sigma}$ on B. Thus **F4** follows.

We have just shown that $\mathcal{G}$ is in the assumptions of Theorem 15. Now, since $f \in L^\sigma(\Omega)$ implies $\widehat{f} \in L^\sigma$ we can deduce from Theorem 15 that $|D\widehat{u}| \in L^\infty(\frac{1}{2}B)$. But using again the bilipschitz caracter of ψ we obtain that $|Du| \in L^\infty(\psi(\frac{1}{2}B^+))$. In particular, Du is bounded up to $\partial\Omega$. The claim follows. $\square$

4.4 The regularity of D^2u

In this section, we establish the integrability of second order distributional derivatives of the local minimizers of the functional $\mathbb{F}(\cdot,\Omega)$. For this, we will need the following Proposition, in which we denote

$$\mathcal{G}(t) := \int_0^t s(s+1)^{\frac{p-4}{2}}\, ds \tag{4.62}$$

This proposition is inspired by a result in [27], but the result we present here goes a bit further, as it states convergence on the set $\{|Df|>1\}$, and not only on $\{|Df|>\lambda\}$ for each $\lambda>1$. This has also consequences in Theorem 24. The precise statement is as follows.

Proposition 23. *Let $p\geq 2$, $N\geq 1$, and let $f_k, f\in W^{1,p}(\Omega;\mathbb{R}^N)$ be given, and denote $P_k=(|Df_k|-1)_+$. Assume that:*

(a) $f_k \rightharpoonup f$ in $W^{1,p}(\Omega;\mathbb{R}^N)$,

(b) $P_k\in L^\infty$ and

$$\|P_k\|_{L^\infty(\Omega)} \leq M \tag{4.63}$$

for some M independent of k.

(c) Assume that

$$\int_\Omega \frac{P_k^p}{(1+P_k)^2}\,|DP_k|^2\,dx \leq N \tag{4.64}$$

for some N independent of k.

Then one has $(|Df|-1)_+^{\frac{p}{2}+1}\in W^{1,2}(\Omega)$. Moreover, there exists a not relabeled subsequence f_k such that

$$\lim_{k\to\infty} |Df_k| = |Df| \quad \text{strongly in } L^{p+2}(\Omega\cap\{|Df|>1\})\ ,$$

and

$$\lim_{k\to\infty} |Df_k| = |Df| \quad \text{a.e. in } \Omega\cap\{|Df|>1\}.$$

Proof. First, it is immediate to see that

$$\begin{aligned}\int |D(P_k^{\frac{p+2}{2}})|^2 &= c(p)\int P_k^p\,|DP_k|^2\\ &\leq c(p,M)\int \frac{P_k^p}{(1+P_k)^2}\,|DP_k|^2 \leq c(p,M,N)\end{aligned}$$

By compactness, there exist $\varphi\in W^{1,2}$ such that $P_k^{\frac{p+2}{2}}\rightharpoonup\varphi$ in $W^{1,2}$. In particular, the convergence is strong in L^2. As a consequence, $\varphi\geq 0$ almost everywhere. Using also that $r\mapsto r^{\frac{2}{2+p}}$ is $\frac{2}{2+p}$-Hölder continuous on $[0,\infty)$, we can deduce that

$$\int |P_k-\varphi^{\frac{2}{p+2}}|^{p+2} \leq c(p)\int |P_k^{\frac{p+2}{2}}-\varphi|^2$$

and therefore $P_k\to\varphi^{\frac{2}{p+2}}$ strongly in L^{p+2}. From now on, let us denote $P=\varphi^{\frac{2}{p+2}}$. We now denote, for each $t\geq 0$,

$$A(t)=\{P>t\}$$

The integrability of P guarantees that $t \mapsto |A(t)|$ is a right-continuous functions on $[0, \infty)$. Moreover, since $\|P_k - P\|_{p+2} \to 0$ as $k \to \infty$, the convengence also occurs in measure. Therefore, recalling the definition of P_k

$$\lim_{k\to+\infty} \int |P_k - P|^{p+2}$$
$$= \lim_{k\to+\infty} \int_{\{|Df_k|\le 1\}} |P|^{p+2} + \lim_{k\to+\infty} \int_{\{|Df_k|>1\}} ||Df_k| - 1 - P|^{p+2} = 0$$

that, in particular, implies

$$\lim_{k\to+\infty} \int_{\{|Df_k|>1\}} ||Df_k| - 1 - P|^{p+2} = 0. \tag{4.65}$$

Moreover by the convergence in measure of P_k to P, for every $t > 0$, we have

$$0 = \lim_{k\to+\infty} |\{x \in \Omega : \ \big|\big(|Df_k| - 1\big)_+ - P| \ge t\}\big|$$
$$= \lim_{k\to+\infty} |\{x \in \Omega : \ |Df_k| \le 1 \text{ and } \big|P| \ge t\}\big|$$
$$+ \lim_{k\to+\infty} |\{x \in \Omega : \ |Df_k| > 1 \text{ and } \big||Df_k| - 1 - P| \ge t\}\big|$$

that, in particular, yields

$$0 = \lim_{k\to+\infty} |\{x \in \Omega : \ |Df_k| \le 1 \text{ and } \big|P| \ge t\}\big| \tag{4.66}$$

For $t > 0$, we write

$$\int_{\{P\ge t\}} \big||Df_k| - 1 - P\big|^{p+2}$$
$$= \int_{\{|Df_k|\le 1 \text{ and } P\ge t\}} \big||Df_k| - 1 - P\big|^{p+2} + \int_{\{|Df_k|>1 \text{ and } P\ge t\}} \big||Df_k| - 1 - P\big|^{p+2}$$
$$= I_{1,k} + I_{2,k} \tag{4.67}$$

By (4.80) we have that

$$\lim_{k\to+\infty} I_{2,k} \le \lim_{k\to+\infty} \int_{\{|Df_k|>1\}} ||Df_k| - 1 - P|^{p+2} = 0 \tag{4.68}$$

Since by virtue of the assumption (b), we have $||P||_\infty \le c(M)$, we get

$$\lim_{k\to+\infty} I_{1,k} = \lim_{k\to+\infty} \int_{\{|Df_k|\le 1 \text{ and } P\ge t\}} ||Df_k| - 1 - P|^{p+2}$$
$$\le \lim_{k\to+\infty} \int_{\{|Df_k|\le 1 \text{ and } P\ge t\}} (2 + |P|)^{p+2}$$
$$\le c(M, p) \lim_{k\to+\infty} |\{|Df_k| \le 1 \text{ and } P \ge t\}| = 0, \tag{4.69}$$

by the equality in (4.66). Inserting (4.69) and (4.68) in (4.67), we conclude

$$\lim_{k\to+\infty}\int_{\{P\geq t\}}\big||Df_k|-1-P\big|^{p+2}=0 \tag{4.70}$$

for every $t>0$. Note now that

$$B(0):=\{x\in\Omega\,:\,P>0\}=\bigcup_{n\in\mathbb{N}}\left\{x\in\Omega\,:\,P>\frac{1}{n}\right\}=:\bigcup_{n\in\mathbb{N}}B\left(\frac{1}{n}\right)$$

and

$$|\{x\in\Omega\,:\,P>0\}|=\lim_{n\to+\infty}\left|\left\{x\in\Omega\,:\,P>\frac{1}{n}\right\}\right|.$$

Since $B\left(\frac{1}{n}\right)\subset B(0)$, for every $n\in\mathbb{N}$,

$$\|\chi_{B(0)}-\chi_{B(1/n)}\|_{L^1(\Omega)}=\|\chi_{B(0)\setminus B(1/n)}\|_{L^1(\Omega)}=|B(0)-B(1/n)|\to 0.$$

Therefore

$$\begin{aligned}
&\int_{\{P>0\}}\big||Df_k|-1-P\big|^{p+2}\\
&=\int_{\{P>0\}}\big||Df_k|-1-P\big|^{p+2}-\int_{\{P>1/n\}}\big||Df_k|-1-P\big|^{p+2}\\
&+\int_{\{P>1/n\}}\big||Df_k|-1-P\big|^{p+2}\\
&+\int_{\{P>1/n\}}\big||Df_k|-1-P\big|^{p+2}\\
&=\int\big||Df_k|-1-P\big|^{p+2}(\chi_{B(0)}-\chi_{B(1/n)})\\
&+\int_{\{P>1/n\}}\big||Df_k|-1-P\big|^{p+2}\\
&\leq C(M)\|\chi_{B(0)}-\chi_{B(1/n)}\|_{L^1(\Omega)}+\int_{\{P>1/n\}}\big||Df_k|-1-P\big|^{p+2}
\end{aligned} \tag{4.71}$$

Passing to the limit first as $k\to\infty$ and using (4.70) with $t=1/n$, and then as $n\to\infty$ in (4.71), we conclude that

$$\lim_{k\to\infty}\int_{\{P>0\}}\big||Df_k|-1-P\big|^{p+2}=0 \tag{4.72}$$

From previous equality we deduce that

$$|Df_k|\to P+1 \qquad \text{strongly in} \quad L^{p+2}(\{P>0\}) \tag{4.73}$$

and, of course, modulo subsequences, also weakly and almost everywhere. By assumption, (f_k) is weakly convergent in $W^{1,p}(\Omega;\mathbb{R}^N)$ to f, so, by the essential uniqueness of the weak limit we conclude that

$$|Df|=P+1 \qquad\qquad \text{a.e. in } \{P>0\}. \tag{4.74}$$

Therefore

$$\{P > 0\} = \{x \in \Omega \,:\, |Df| > 1\}.$$

By (4.73) and (4.74) and the equality above, there exists a subsequence (not relabeled) of f_k such that

$$|Df_k| \to_{k\to\infty} |Df| \qquad \text{strongly in } L^{p+2}(\{x \in \Omega \,:\, |Df| > 1\}). \tag{4.75}$$

The other assertions follows easily. □

We can now proceed with the main result in this section.

Theorem 24. *Let $u \in W^{1,p}(\Omega; \mathbb{R}^{n\times N}$ be a local minimizer of the functional $\mathbb{F}(\cdot, \Omega)$ in (4.1), and so that assumptions* **(F0)-(F4)** *are satisfied with $R = 1$. If $f \in L^s_{\text{loc}}(\Omega)$, then*

$$\mathcal{G}((|Du| - 1)_+) \in W^{1,2}_{\text{loc}}(\Omega) \tag{4.76}$$

and the following Caccioppoli type inequality holds,

$$\int_{B_\rho(x_0)} |D(\mathcal{G}((|Du| - 1)_+))|^2 \le C(1 + \|k\|_{L^s} + \|f\|_{L^s})^\tau \int_{B_r(x_0)} (1 + |Du|^p)\, dx, \tag{4.77}$$

for every ball $B_r(x_0) \Subset \Omega$, every $0 < \rho < r$, for some $C = C(n, N, p, s, c_1, c_2, L, L_1, \nu, \rho, r)$ and for some $\tau = \tau(s, n) > 0$.

Let us mention that in (4.77) the term on the left hand side is equivalent to

$$\int_{B_\rho(x_0)} |D(\mathcal{G}((|Du| - 1)_+))|^2 = \int_{B_\rho(x_0)} (|Du| - 1)^2_+ \,|Du|^{p-4}\,|D^2u|^2\, dx$$

so that the above result is, in fact, a weighted bound for D^2u with the weight $(|Du| - 1)^2_+ \,|Du|^{p-4}$.

Proof. Let $v_{\varepsilon,m}$ be the solution of the problem (4.46). The functionals $\mathbb{F}_{\varepsilon,m}$, for every ϵ and m, satisfy the assumptions of Theorem 20. Therefore their minimizers $v_{\varepsilon,m}$ belong to $W^{2,2}_{\text{loc}} \bigcap W^{1,\infty}$ and are such that $|Dv_{\epsilon,m}|^{p-2}|D^2v_{\varepsilon,m}|^2 \in L^1$. As a consequence, we are legitimate to use the a priori estimate (4.7) of Theorem 17, thus getting

$$\int_{B_\rho(x_0)} \frac{P^2_{\varepsilon,m}}{(1 + P_{\varepsilon,m})^2}\, |Dv_{\varepsilon,m}|^{p-2}|D^2v_{\varepsilon,m}|^2\, dx \le C(1 + ||k_\varepsilon||_{L^s} + ||f_\varepsilon||_{L^s})^\tau \int_{B_{2R}(x_0)} (1 + |Dv_{\varepsilon,m}|^p)\; dx,$$

with a constant C independent of m and of ε. As usually, $P_{\varepsilon,m} = (|Dv_{\varepsilon,m}| - 1)_+$. Now using estimate (4.47) and the fact that $||k_\varepsilon||_{L^s} + ||f_\varepsilon||_{L^s} \to ||k||_{L^s} + ||f||_{L^s}$, we obtain

$$\int_{B_\rho(x_0)} |D(\mathcal{G}(P_{\varepsilon,m}))|^2\, dx = \int_{B_\rho(x_0)} \frac{P^2_{\varepsilon,m}}{(1 + P_{\varepsilon,m})^2}\, |Dv_{\varepsilon,m}|^{p-2}|D^2v_{\varepsilon,m}|^2\, \mathrm{d}x \le C \tag{4.78}$$

with a constant C independent of m and of ε. Furthermore, we have that

$$\|P_{\varepsilon,m}\|_{L^\infty(B_\rho)} \le \sup_{B_\rho} |Dv_{\varepsilon,m}| \le C,$$

with a constant C independent of m and ε. By Proposition 23,

$$|Dv_{\varepsilon,m}| \to |Dv_\varepsilon| \text{ strongly in } \; L^p(B_\rho \cap \{|Dv_\varepsilon| > 1\}), \tag{4.79}$$

and

$$|Dv_{\varepsilon,m}| \to |Dv_\varepsilon| \text{ a.e. in } B_\rho \cap \{|Dv_\varepsilon| > 1\}. \tag{4.80}$$

If we now set $w_{\varepsilon,m} = \mathcal{G}(P_{\varepsilon,m})$, then from (4.78) we deduce that (up to a subsequence) one has $w_{\varepsilon,m} \to w_\varepsilon$ as $m \to \infty$ for some $w_\varepsilon \in W^{1,2}(B_\rho)$, with weak convergence in $W^{1,2}(B_\rho)$, strong convergence in $L^2(B_\rho)$, and a.e. convergence in B_ρ. The latter, together with (4.80), implies that $w_\varepsilon = \mathcal{G}((|Dv_\varepsilon| - 1)_+)$ almost everywhere on $B_\rho \cap \{|Dv_\varepsilon| > 1\}$. But then the lower semicontinuity of the norm implies that

$$\int_{B_\rho(x_0)} |D(\mathcal{G}((|Dv_\varepsilon| - 1)_+))|^2 \leq C, \tag{4.81}$$

with a constant C independent of ε. We now argue for the sequence v_ε as we did for $v_{\varepsilon,m}$, and the theorem follows . □

Chapter 5

Congested traffic minimization problems

We first summarize the models and results in [15, 19] for the reader's convenience.

5.1 Traffic strategies

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and μ_0 and μ_1 be two probability measures, representing both sources and destinations of travellers on a urban network. We denote by $\Gamma(\overline{\Omega}) = C([0,1];\overline{\Omega})$ the metric space of all paths in Ω, together with the uniform norm, and let

$$\pi_t : \Gamma(\overline{\Omega}) \to \overline{\Omega},$$
$$\alpha \mapsto \pi_t(\alpha) = \alpha(t),$$

denote the evaluation at time t. One calls *traffic strategy* to any measure Q on the set of paths $\Gamma(\overline{\Omega})$, concentrated on absolutely continuous curves, and such that

$$(\pi_t)_\# Q = \mu_t, \qquad t \in \{0,1\}.$$

The set of all such traffic strategies is denoted by $\mathcal{Q}(\mu_0, \mu_1)$.

Example 25. *Let us assume that $\phi : [0,1] \times \Omega \to \Omega$ is a flow of diffeomorphisms $\phi(t,x) = \phi_t(x)$ of a reasonably smooth vector field $b : [0,1] \times \Omega \to \mathbb{R}^n$, and such that at time $t = 1$ one has $(\phi_1)_\# \mu_0 = \mu_1$. Also, set $\Phi : \Omega \to \Gamma(\Omega)$ to be the flow map, so that $\Phi(x) = \phi(\cdot, x)$ is the trajectory that passes through the point x at time 0. One then sets $Q = Q_\phi$ to be the image measure of μ_0 under Φ,*

$$Q = \Phi_\#(\mu_0).$$

Equivalently, on each set $G \subset \Gamma(\Omega)$,

$$Q(G) = \mu_0\{x \in \Omega : \phi(.,x) \in G\}.$$

By construction,

$$(\pi_0)_\# Q(E) = Q(\pi_0^{-1}(E)) = \mu_0(\Phi^{-1}(\pi_0^{-1}(E))) = \mu_0((\pi_0 \circ \Phi)^{-1}(E)) = \mu_0(E),$$
$$(\pi_1)_\# Q(E) = Q(\pi_1^{-1}(E)) = \mu_0(\Phi^{-1}(\pi_1^{-1}(E))) = \mu_0((\pi_1 \circ \Phi)^{-1}(E)) = \mu_0(\phi_1^{-1}(E)) = \mu_1(E).$$

This shows that Q is a traffic strategy between μ_0 and μ_1, and moreover Q is supported on the trajectories of ϕ.

Example 26. *There is a trivial way to construct traffic strategies from classical optimal transport. More precisely, if $\gamma \in \mathcal{P}(\Omega \times \Omega)$ is a transport plan between μ_0 and μ_1, then one can construct a traffic strategy $Q = Q_\gamma$ associated to γ as follows,*

$$Q_\gamma(G) = \gamma\{(x, y) \in \Omega \times \Omega : [x, y] \in G\} \qquad \forall G \subset \Gamma(\overline{\Omega}).$$

Roughly, it corresponds to choose as $p^{x,y}$ a Dirac delta on the line segment $[x, y]$. It is an exercise to see that Q_γ is certainly a traffic strategy.

In general, the above example can be generalized. By means of a disintegration result, one can always decompose any traffic strategy Q as

$$dQ = d\gamma \otimes dp^{x,y}$$

where $\gamma \in \Pi(\mu_0, \mu_1)$ is a transport plan between μ_0 and μ_1, and $(p^{x,y})_{x,y\in\Omega}$ is a set of probability measures on $\Gamma(\overline{\Omega})$ whose support is contained on the set of curves joining x to y. In Example 26 we obviously had $p^{x,y} = \delta_{[x,y]}$, the Dirac delta on the line segment. In general, though, the situation is much more complicated and $p^{x,y}$ may even not reduce to a single curve.

To each traffic strategy Q, one can associate a scalar measure $i = i_Q \in \mathcal{M}(\overline{\Omega})$, called the *traffic intensity*, and defined as follows,

$$\langle i_Q, \varphi \rangle := \int_{\Gamma(\overline{\Omega})} \left(\int_\alpha \varphi \right) dQ(\alpha).$$

Above,

$$\int_\alpha \varphi = \int_0^1 \varphi(\alpha(t))\, |\alpha'(t)|\, dt.$$

As equivalent definition, one may say that on nice sets $A \subset \Omega$ the traffic intensity is the amount of flow (counted in terms of Q) that passes through the set,

$$i_Q(A) = \int_{\Gamma(\overline{\Omega})} \mathcal{H}^1(A \cap \alpha)\, dQ(\alpha),$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. This quantity makes sense, since all traffic strategies concentrate (by definition) on absolutely continuous curves α.

Example 27. *If Q is the strategy defined by the flow Φ of a smooth vector field b, as in example 25,* *then*

$$\begin{aligned}
\langle i_Q, \varphi \rangle &= \int_{\Gamma(\Omega)} \left(\int_\alpha \varphi \right) dQ(\alpha) \\
&= \int_\Omega \left(\int_{\Phi(x)} \varphi \right) d\mu_0(x) \\
&= \int_\Omega \left(\int_0^1 \varphi(\phi(t,x)) \, |b(t,\phi(t,x))| \, dt \right) d\mu_0(x) \\
&= \int_0^1 \int_\Omega \varphi(\phi(t,x)) \, |b(t,\phi(t,x))| \, d\mu_0(x) \, dt \\
&= \int_0^1 \int_\Omega \varphi(y) \, |b(t,y)| \, d((\phi_t)_\# \mu_0)(y) \, dt
\end{aligned}$$

whence

$$i_Q = \int_0^1 |b(t,y)| \, d((\phi_t)_\# \mu_0) \, dt$$

in other words, i_Q *is the measure obtained after superposing (continuously in time) the measures* $|b(t,\cdot)| \, d\mu_t$*, where* $\mu_t = (\phi_t)_\# \mu_0$.

Example 28. *If* Q *is as in Example 26, then it is easy to see that the corresponding traffic intensity takes the form*

$$i_Q(A) = \int_{\Omega \times \Omega} \mathcal{H}^1(A \cap [x,y]) \, d\gamma(x,y).$$

It turns out that this quantity had already been introduced by Feldmann-McCann [40] and Ambrosio [1] in the optimal transport setting, and named *transport density.* This fact will be essential later on when we see later on in the next sectin, some equivalent forms of transport equation, with usage of concept of duality.

5.2 Optimal transport and congested traffic problems

Traffic strategies appear in a natural way in the classical optimal transport problem. To see this, we now recall four equivalent formulations of the same question. Let us remember that the classical Monge-Kantorovich problem

$$\inf \left\{ \int_{\Omega \times \Omega} |x-y| \, d\gamma(x,y); (\pi_i)_\# \gamma = \mu_i \right\} \tag{5.1}$$

always does admit at least one solution γ. Above, $\pi_i : \Omega \times \Omega$ is the componentwise projection. In this setting, the expression

$$Q = \int \delta_{[x,y]} \, d\gamma(x,y)$$

defines a measure on the set of curves $C([0,1],\Omega)$, and whose support is precisely the set of transport rays of the optimal transport plan γ. Moreover, the measure Q defined above is, indeed, a traffic strategy. To see this, just note that Q is clearly supported on absolutely continuous curves, and moreover

$$(\pi_i)_\# Q = \mu_i, \qquad i = 0, 1$$

where (abusing of notation) now $\pi_t : C([0,1;\Omega) \to \Omega$ is the evaluation at time t. In fact, it can be shown that Q is optimal for the following *total length* minimization problem,

$$\inf\left\{\int_{C([0,1];\Omega)} \ell(\alpha)\, dQ(\alpha); Q \in \mathcal{Q}(\mu_0,\mu_1)\right\} \tag{5.2}$$

where $\ell(\alpha) = \int_0^1 |\alpha'(t)|\, dt$ for any absolutely continuous $\alpha \in C([0,1];\Omega)$. One further equivalent formulation is given by the classical Kantorovich problem,

$$\sup\left\{\int \varphi\, d(\mu_0 - \mu_1); \|\nabla\varphi\|_\infty \le 1\right\} \tag{5.3}$$

whose equivalence with (5.1) is the classical Kantorovich duality. It was first shown in [?] that (5.3) is also equivalent to

$$\inf\left\{\int_\Omega d|V|; -\operatorname{div} V = \mu_0 - \mu_1, V \cdot \vec{n}_{|\partial\Omega} = 0\right\} \tag{5.4}$$

where V are vector valued Radon measures solving a divergence equation with Neumann boundary values.

The last problem pertains to a wide family of problems, introduced by Beckmann in [7], in which instead of the total variation $\int_\Omega d|V|$ one minimizes an integral functional with the same divergence constraint,

$$\inf\left\{\int_\Omega \mathcal{H}(V); -\operatorname{div} V = \mu_0 - \mu_1, V \cdot \vec{n}_{|\partial\Omega} = 0\right\} \tag{5.5}$$

for a given convex function $\mathcal{H}$. Finding equivalent formulations for (5.5) with the spirit of (5.1), (5.2) or (5.3) was the main issue in [18]. It turns out that these equivalent formulations can be easily proven between (5.5) and a sort of $\mathcal{H}$-counterpart for (5.3) (see Lemma 34 below), even under greater generality. However, the situation changes when the equivalence between (5.5) and an $\mathcal{H}$-counterpart for (5.2) is investigated. This was already noticed in [15], and was used there as a starting point for developping a congested traffic model which we recall below.

5.3 The scalar problem

In traffic analysis, one usually models the total traffic cost of a given urban traffic configuration as an integral functional of the traffic intensity,

$$\int H(i_Q(x))\, dx$$

That is, each traffic strategy Q induces a traffic intensity i_Q which fully determines the total traffic cost, by means of a given cost function $H = H(i)$ that depends on the traffic intensity. This principle was used in [19] to establish the traffic model that has inspired this part of this thesis. If two traffic strategies Q_1 and Q_2 have the same traffic intensity $i_{Q_1} = i_{Q_2}$, then the induced traffic cost does not change. This assumption has the restriction that points x with the same traffic intensity generate the same cost, which may well not be the case in real life. Indeed, the same traffic intensity at two points with different road conditions may provide significantly different costs. This was already noticed in

[18], and is one of the motivations to include in H the dependence on the x variable.

In general, we will be given a function

$$H : \overline{\Omega} \times [0, \infty) \to [0, \infty)$$
$$(x, i) \to H(x, i)$$

such that:

$(H0)$ $H(x, 0) = 0$.

$(H1)$ For every fixed $x \in \overline{\Omega}$, $i \mapsto H(x, i)$ is non-decreasing and convex.

$(H2)$ If $(x, i) \in \overline{\Omega} \times [0, \infty)$ then

$$a\, i^q \leqslant H(x, i) \leqslant b\,(i^q + 1) \tag{5.6}$$

$(H3)$ If $(x, i) \in \overline{\Omega} \times [0, \infty)$ and $g(x, i) = \partial_i H(x, i)$ then

$$0 \leqslant g(x, i) \leqslant c(i^{q-1} + 1). \tag{5.7}$$

Then, we will call *optimal traffic strategy* to any traffic strategy Q that solves the following variational problem,

$$\inf_{Q \in \mathcal{Q}^q(\mu_0, \mu_1)} \int_\Omega H(x, i_Q(x))\, dx. \tag{5.8}$$

Above, $\mathcal{Q}^q(\mu_0, \mu_1)$ denotes the set of all possible measures Q such that

$$(\pi_0)_\# Q = \mu_0,$$
$$(\pi_1)_\# Q = \mu_1,$$

and, at the same time, induce a traffic intensity $i_Q \in L^q(\Omega)$. The following result concerns the non-emptiness of $\mathcal{Q}^q(\mu_0, \mu_1)$ under rather general assumptions.

Theorem 29. *If $\mu_0, \mu_1 \in L^q(\Omega)$, then $\mathcal{Q}^q(\mu_0, \mu_1)$ is non-empty.*

The above Theorem is a consequence of works by De Pascale, Evans and Pratelli. For a proof, one only needs to construct for the given datum μ_0 and μ_1 the trivial traffic strategy

$$Q = \gamma \otimes \delta_{[x,y]}$$

where $d\gamma(x, y)$ is an optimal transport plan between μ_0 and μ_1 for the Monge problem, and $\delta_{[x,y]}$ is the Dirac delta on the line segment $[x, y]$, understood as a measure on $C([0, 1]; \Omega)$. For this particular Q, one easily sees that i_Q is precisely the transport density associated to γ (see Example 28 for the definition of transport density). Now, one just needs to remind that De Pascale, Evans and Pratelli proved that the transport density inherits the Lebesgue integrability of μ_0, μ_1. In particular, if $\mu_0, \mu_1 \in L^q$ then the transport density also belongs to L^q (see [32] for $1 < q < 2$, and [31] for $2 \leq q \leq \infty$), as claimed.

At this point, it is worth mentioning that Brasco and Petrache proved that $\mathcal{Q}^q(\mu_0, \mu_1)$ is non-empty if and only if $\mu_0 - \mu_1 \in W^{-1,q}$. See [18, Proposition 4.4] for details. As an example, one can look at the case of two Dirac masses, $\mu_i = \delta_{x_i}$, $i = 0, 1$. In this particular case, one has $\mu_0 - \mu_1 \in W^{-1,q}$ if and only if $1 \leq q < \frac{n}{n-1}$.

The non-emptiness of $\mathcal{Q}^q(\mu_0, \mu_1)$ guarantees the existence of minimizing sequences. It turns out that much more is true.

Theorem 30. *The minimization problem* (5.8) *admits a solution if* $\mathcal{Q}^q(\mu_0, \mu_1) \neq \emptyset$.

The above result was proven in [19, Theorem 2.10] when $n = 2$ and $1 < q < \infty$, in the autonomous setting, that is, with the extra assumption that $H(x, i) = H(i)$ (see [20, Theorem 2] for a non-autonomous counterpart). The proof comes together with a precise description of the relation between the minimizers and Wardrop's equilibrium, see Subsection 5.5 for details. Moreover, if $H(x, \cdot)$ is strictly convex and Q_1, Q_2 are equilibriums one can see that $i_{Q_1} = i_{Q_2}$, althoug it is false in general that $Q_1 = Q_2$, see [19]. Concerning the non-emptiness of $\mathcal{Q}^q(\mu_0, \mu_1)$, we refer to [18, Theorem 4.4] as well as to the following result (see also [19, Remarks 2.5 and 2.6]).

It is because of Theorem 29 that, from now on, we will assume that μ_0, μ_1 are absolutely continuous with respect to the Lebesgue measure. Abusing of notation, we will also denote by μ_0, μ_1 their Radon-Nikodym derivatives, so that $d\mu_j(x) = \mu_j(x)\, dx$.

5.4 The vector problem

There is though a different way to prove existence of minimizers at (5.8), with the advantage of being more explicit. The basic idea (see [15] for details) is to compare (5.8) with the following variational problem,

$$\inf_{\sigma \in \Sigma^q(\mu_0,\mu_1)} \int_\Omega \mathcal{H}(x, \sigma(x))\, dx \tag{5.9}$$

where $\mathcal{H}(x, \xi) = H(x, |\xi|)$ for each $\xi \in \mathbb{R}^n$, and H satisfies $(H0)$, $(H1)$, $(H2)$ and $(H3)$. Also, $\Sigma^q(\mu_0, \mu_1)$ denotes the set of all $L^q(\Omega)$ weak solutions of the following Neumann problem

$$\begin{cases} \operatorname{div} \sigma = \mu_0 - \mu_1 & \Omega, \\ \sigma \cdot \vec{n} = 0 & \partial\Omega. \end{cases} \tag{5.10}$$

Lemma 31. *We have that* (5.9) $\leqslant$ (5.8).

Proof. Let $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$. Define a vector measure σ_Q as follows,

$$\langle \sigma_Q, F \rangle = \int_{C([0,1];\overline{\Omega})} \left(\int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle\, dt \right) dQ(\alpha), \qquad \forall F \in C(\overline{\Omega}, \mathbb{R}^n), \tag{5.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. It is clear that σ_Q defines a vector measure, with total variation $|\sigma_Q| \le i_Q \in L^q(\Omega)$. Also, for any testing function $\varphi \in C^1(\Omega, \mathbb{R})$

$$\begin{aligned}\langle \sigma_Q, \nabla\varphi\rangle &= \int_{C([0,1];\overline{\Omega})} \left(\int_0^1 \nabla\varphi(\alpha(t)) \cdot \alpha'(t)\, dt\right) dQ(\alpha) \\ &= \int_{C([0,1];\overline{\Omega})} \varphi(\alpha(1))\, dQ(\alpha) - \int_{C([0,1];\overline{\Omega})} \varphi(\alpha(0))\, dQ(\alpha) \\ &= \int_{C([0,1];\overline{\Omega})} \varphi(\pi_1(\alpha))\, dQ(\alpha) - \int_{C([0,1];\overline{\Omega})} \varphi(\pi_0(\alpha))\, dQ(\alpha) \\ &= \int_\Omega \varphi(x)\, d\mu_1(x) - \int_\Omega \varphi(x)\, d\mu_0(x) = \langle \varphi, \mu_1 - \mu_0\rangle\end{aligned}$$

It then follows that $\sigma_Q \in \Sigma^q(\mu_0, \mu_1)$. Moreover, since by definition $\mathcal{H}(x, \sigma) = H(x, |\sigma|)$, we have

$$\int \mathcal{H}(x, \sigma_Q(x))\, dx = \int H(x, |\sigma_Q(x)|)\, dx \le \int H(x, i_Q(x))\, dx,$$

from which one infers that (5.9) $\leqslant$ (5.8). □

According to the above proof, to each traffic strategy Q one can associate a sort of *vector valued* traffic intensity $\sigma = \sigma_Q \in \Sigma^q(\mu_0, \mu_1)$ whose total variation $|\sigma_Q|$ satisfies that

$$|\sigma_Q| \le i_Q.$$

It is a question of interest to determine to which extent this vector strategy σ_Q can be constructed in such a way that it concentrates the overall traffic intensity, that is, $\sigma_Q = i_Q$. The appropiate tool to do this is the so-called Dacorogna-Moser scheme. In this context, the following result was proven in [15, Theorem 3.2] in the autonomous setting (see also [20, Section 5] for a non-autonomous counterpart). We include below a proof for the reader's convenience.

Theorem 32. *If $\mu_0, \mu_1 \in L^q$, and $\frac{1}{\mu_0}, \frac{1}{\mu_1} \in L^\infty$, then the infimums at* (5.8) *and* (5.9) *are equal.*

Proof. The easy inequality follows from Lemma 31 and holds under greater generality. To prove that (5.8) $\leqslant$ (5.9), we implement Dacorogna-Moser scheme as follows. Given $\sigma \in \Sigma^q(\mu_0, \mu_1)$, we know that

$$\int \sigma(x) \cdot \nabla\psi(x)\, dx = \int \psi(x)\, d(\mu_1 - \mu_0)(x) \tag{5.12}$$

for any $\psi = \psi(x) \in C_c^\infty(\mathbb{R}^n)$. Setting $\mu_t = (1-t)\mu_0 + t\mu_1$ and using that $\frac{1}{\mu_j} \in L^\infty$, the vector field

$$\hat\sigma(t, x) = \frac{\sigma(x)}{(1-t)\,\mu_0(x) + t\,\mu_1(x)}, \quad (t, x) \in [0,1] \times \Omega,$$

is well defined, and equation (5.12) is equivalent to

$$\int \hat\sigma(t, x) \cdot \nabla\psi(x)\, d\mu_t(x) = \int \psi(x)\, d(\mu_1 - \mu_0)(x).$$

We now take any $\varphi = \varphi(t)$ of class $C_c^\infty([0, \infty))$. Then, we multiply both sides in (5.12) by φ and integrate in time on $[0, \infty)$, and obtain

$$\iint \hat\sigma \cdot \nabla(\varphi\psi)\, d\mu_t(x)\, dt = \iint \varphi\psi\, d(\mu_1 - \mu_0)(x)\, dt \tag{5.13}$$

Now, we use Fubini's theorem, and integrate by parts in time on the right hand side, and get that

$$\begin{aligned}
\iint \varphi(t)\psi(x)\, d(\mu_1-\mu_0)(x)\, dt &= \int \psi(x)\left(\int \varphi(t)\,(\mu_1-\mu_0)(x)\, dt\right)\, dx \\
&= \int \psi(x)\left(\int \varphi(t)\,\frac{d\mu_t(x)}{dt}\, dt\right)\, dx \\
&= \int \psi(x)\left(-\varphi(0)\,\mu_0(x) - \int \frac{d\varphi(t)}{dt}\,\mu_t(x)\, dt\right)\, dx \\
&= -\int \psi(x)\varphi(0)\,\mu_0(x)\, dx - \iint \psi(x)\frac{d\varphi(t)}{dt}\,\mu_t(x)\, dt\, dx.
\end{aligned}$$

We end up getting from (5.13) that

$$\iint \hat{\sigma}\cdot\nabla(\varphi\psi)\, d\mu_t(x)\, dt + \int \psi\varphi(0)\,\mu_0\, dx + \iint \psi\frac{d\varphi}{dt}\,\mu_t\, dt\, dx = 0$$

It then follows that μ_t is a weak solution of the following continuity equation

$$\partial_t\mu_t + \operatorname{div}(\widehat{\sigma}\mu_t) = 0,$$

with initial datum μ_0. Moreover, using that $|\Omega| < \infty$, we see that

$$\int_0^1\int_\Omega |\widehat{\sigma}(t,x)|\, d\mu_t(x)\, dt = \int_0^1\int_\Omega |\sigma(x)|\, dx\, dt \leq \|\sigma\|_{L^q(\Omega)}\, |\Omega|^{1-\frac{1}{q}} < +\infty.$$

Thus we are legitimate to use the Superposition Principle (i.e. Theorem 9) and deduce that μ_t is indeed a superposition solution. In particular, there exists a probability measure $Q \in \mathcal{P}(C([0,1];\overline{\Omega}))$ with the property that $\mu_t = (\pi_t)_\# Q$, that is

$$\int_{\overline{\Omega}} \varphi(x)\, d\mu_t(x) = \int_{C([0,1];\overline{\Omega})} \varphi(\alpha(t))\, dQ(\alpha), \quad \forall\varphi \in C(\overline{\Omega}).$$

It turns out that $Q \in \mathcal{Q}^q(\mu_0,\mu_1)$. Indeed, by construction $\mu_t = (\pi_t)_\# Q$ at times $t = 0$ and $t = 1$. On the other hand, concerning its traffic intensity i_Q we have

$$\begin{aligned}
\langle i_Q, \varphi\rangle &= \int_{C([0,1];\Omega)}\int_0^1 \varphi(\alpha(t))|\alpha'(t)|\, dt\, dQ(\alpha) \\
&= \int_0^1\int_{C([0,1];\Omega)} \varphi(\alpha(t))|\hat{\sigma}(t,\alpha(t))|\, dQ(\alpha)\, dt \\
&= \int_0^1\int_\Omega \varphi(x)|\hat{\sigma}(t,x)|\, d\mu_t(x)\, dt \\
&= \int_0^1\int_\Omega \varphi(x)|\sigma(x)|\, dx\, dt = \int_\Omega \varphi(x)|\sigma(x)|\, dx,
\end{aligned}$$

thus i_Q is absolutely continuous with density $|\sigma| \in L^q$. Therefore $Q \in \mathcal{Q}^q(\mu_0,\mu_1)$ as claimed. Now it just remains to note that

$$\int H(x, i_Q(x))\, dx = \int H(x, |\sigma(x)|)\, dx = \int \mathcal{H}(x,\sigma(x))\, dx,$$

so we get (5.8) $\leqslant$ (5.9). $\square$

It is worth mentioning here that if $H(x,\cdot)$ is strictly convex then (5.9) has a unique minimizer. Indeed, it consists of minimizing a strictly convex and coercive functional on L^q, subject to a closed constraint. Thus, applying the above proof to the optimizer σ_0 one obtains an traffic strategy Q_0 which optimizes (5.8). It may be, though, that (5.8) has other optimizers Q, but they all must have the same traffic intensity i_Q, see [19, Section 2.3].

5.5 Wardrop equilibriums

As mentioned in [19] (see also [20]), the congestion effects in a given traffic strategy $Q \in \mathcal{Q}^q(\mu_0,\mu_q)$ are captured by the traffic intensity i_Q through the function $g(x,i) = \partial_i H(x,i)$. Namely, one thinks the function $g(x, i_Q(x))$ as the density of a Riemannian metric

$$d_Q(x,y) = \inf_{\alpha\in C^{x,y}} \int_\alpha g(z, i_Q(z))\,|dz| = \inf_{\alpha\in C^{x,y}} \int_0^1 g(\alpha(t), i_Q(\alpha(t)))\,|\alpha'(t)|\,dt$$

where $C^{x,y}$ denotes the subset of $C([0,1];\overline{\Omega})$ of rectifiable curves such that $\alpha(0) = x$ and $\alpha(1) = y$. In this context, a curve $\alpha_0 \in C^{x,y}$ is called a Q-geodesic if

$$d_Q(x,y) = \int_{\alpha_0} g(z, i_Q(z))\,|dz| \tag{5.14}$$

With this terminology, a *Wardrop equilibrium* is a traffic strategy $Q \in \mathcal{Q}(\mu_0,\mu_1)$ that satisfies the following two properties:

(a) Q is supported on Q-geodesics, that is,

$$Q\left(\left\{\alpha \in C([0,1];\overline{\Omega}) : d_Q(\alpha(0),\alpha(1)) = \int_\alpha g(z, i_Q(z))\,|dz|\right\}\right) = 1$$

(b) The transport plan $\gamma_Q = (\pi_0,\pi_1)_\# Q$ solves the following Monge problem

$$\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\Omega\times\Omega} d_Q(x,y)\,d\gamma(x,y)$$

The following result was given in [19, Theorem 4.2] in the autonomous case (see [20, Theorem 2] for the non-autonomous counterpart).

Theorem 33. *A traffic strategy $Q \in \mathcal{Q}(\mu_0,\mu_1)$ is a minimizer of* (5.8) *if and only if Q is a Wardrop equilibrium.*

This result was proven in [19] in the autonomous case. A sketch for a non-autonomous counterpart was given in [20]. In boths cases, the proof is quite involved, as regularity issues need to be handled in detail. Indeed, already at the very beginning, the definition d_Q may have problems, since $i_Q \in L^q$ only implies $g(x, i_Q(x)) \in L^{\frac{q}{q-1}}$ and this may not be enough to define line integrals.

5.6 Duality and Regularity Theory

The transition from congested traffic problems to the classical regularity theory for elliptic PDE relies on the following duality result. It was proven by Brasco and Petrache in [18, Theorem 3.1], but we remind it here for the reader's convenience. By $W^{1,r}_*(\Omega)$ we denote the class of functions $\varphi \in W^{1,r}(\Omega)$ such that $\int_\Omega \varphi = 0$.

Lemma 34. *Let $T \in W^{-1,q}(\Omega)$, $1 < q < \infty$, and $p = \frac{q}{q-1}$. Let $\mathcal{H} : \Omega \times \mathbb{R}^n \to [0, \infty)$ be a Carathéodory function, such that*

$$a\,|\xi|^q \leq \mathcal{H}(x, \xi) \leq b(|\xi|^q + 1), (x, \xi) \in \Omega \times \mathbb{R}^n.$$

Assume also that $\xi \mapsto \mathcal{H}(x, \xi)$ is convex, for each fixed x. Then

$$\inf_{\sigma \in \Sigma^q(T)} \int_\Omega \mathcal{H}(x, \sigma(x))\, dx = \sup_{\varphi \in W^{1,p}_*(\Omega)} \left(\langle T, \varphi \rangle - \int_\Omega \mathcal{H}^*(x, \nabla\varphi(x))\, dx \right) \tag{5.15}$$

where $\mathcal{H}^*(x, \cdot)$ is the Legendre transform of $\mathcal{H}(x, \cdot)$. Moreover, one has

$$\sigma(x) \in \partial\mathcal{H}^*(x, \nabla\varphi(x)) \qquad \text{for a.e.} x \in \Omega,$$

where $\partial\mathcal{H}^*(x, \xi)$ denotes the subdifferential $\partial\mathcal{H}^*(x, \xi) = \{z \in \mathbb{R}^n; \mathcal{H}^*(x, \xi) + \mathcal{H}(x, z) = \xi \cdot z\}$.

For the traffic applications of the present thesis, one is left to see if the distribution $T = \mu_0 - \mu_1$ belongs to $W^{-1,q}$. In this direction, if both μ_0 and μ_1 are L^q functions, one can choose for any $\varphi \in W^{1,q'}_*$ an appropiate constant φ_Ω so that the Poincaré inequality applies, and one gets

$$\begin{aligned} \left| \int \varphi\, d(\mu_0 - \mu_1) \right| &= \left| \int (\varphi - \varphi_\Omega)\, d(\mu_0 - \mu_1) \right| \\ &\leq \|\varphi - \varphi_\Omega\|_{L^{q'}} \|\mu_0 - \mu_1\|_{L^q} \\ &\leq C(\Omega, q) \|D\varphi\|_{L^{q'}} \|\mu_0 - \mu_1\|_{L^q}, \end{aligned}$$

and therefore $T \in W^{-1,q}$. Another interesting example is that of the point masses, that is, $\mu_j = \delta_{x_j}$ for $j = 0, 1$. In this case, $\mu_0 - \mu_1 \in W^{-1,q}$ if and only if $1 \leq q < \frac{n}{n-1}$, see [18, Example 2.4].

As explained in [18] (see also [15, Theorem 2.1]), when $\xi \mapsto \mathcal{H}(x, \xi)$ is strictly convex the Legendre transform $\xi \mapsto \mathcal{H}^*(x, \xi)$ becomes C^1-smooth, and therefore the subdifferential $\partial\mathcal{H}^*(x, \xi)$ reduces to a singleton

$$\partial\mathcal{H}^*(x, \xi) = \{\nabla_\xi \mathcal{H}^*(x, \xi)\},$$

whence *the* optimals σ and φ must necessarily be related by

$$\sigma(x) = \nabla_\xi \mathcal{H}^*(x, \nabla\varphi(x)).$$

In particular, in that strictly convex setting the optimizer $\sigma \in \Sigma^p(T)$ is unique. Moreover, φ can be found by solving the following Neumann problem

$$\begin{cases} -\operatorname{div} \nabla_\xi \mathcal{H}^*(x, \nabla\varphi) = T & \Omega \\ \nabla_\xi \mathcal{H}^*(x, \nabla\varphi) \cdot \vec{n} = 0 & \partial\Omega \end{cases}$$

The following examples explain a bit better the situation.

Example 35. *If* $H(x,i) = \frac{i^q}{q}$, *then* $\mathcal{H}^*(x,\xi) = \frac{|\xi|^p}{p}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *In particular,* $\mathcal{H}^*$ *is strictly convex, and* $\nabla\mathcal{H}^*(x,\xi) = |\xi|^{p-2}\xi$ *degenerates at just one point. The relation between optimals* $\sigma = |\varphi|^{p-2}\varphi$ *also says that* φ *is a solution of an inhomogeneous Neumann problem for the p-Laplace operator,*

$$\begin{cases} \Delta_p \varphi = \mu_0 - \mu_1 \\ \nabla\varphi \cdot \vec{n} = 0 \end{cases}$$

On the other hand, since $g(x,0) = \partial_i H(x,0) = 0$, *according to* (5.14) *and Theorem 33, this example gives zero cost to zero traffic. This is not completely realistic, as a positive cost must be expected even in absence of traffic.*

Example 36. *If* $H(x,i) = \frac{i^q}{q} + \lambda i$ *one gets* $g(x,0) = \partial_i H(x,0) = \lambda$, *and so for* $\lambda > 0$ *this model is more realistic. At the same time, though,* $\mathcal{H}^*(x,\xi) = \frac{(|\xi|-\lambda)_+^p}{p}$ *is not strictly convex, and therefore*

$$\nabla\mathcal{H}^*(x,\xi) = (|\xi| - \lambda)^{p-1}\frac{\xi}{|\xi|}$$

degenerates not only when $|\xi| = 0$ *but on the whole ball* $\{|\xi| \le \lambda\}$ *of the gradient variable. Still,* $\mathcal{H}^*$ *is* C^1 *in the second variable, and so one gets a Neumann boundary value problem for the following Euler-Lagrange equation*

$$-\operatorname{div}\left((|\nabla\varphi| - \lambda)^{p-1}\frac{\nabla\varphi}{|\nabla\varphi|}\right) = \mu_0 - \mu_1$$

although it signifficantly differs from the p-Laplace equation.

5.7 Traffic applications of the Main Theorem

One of the fundamental results of [15] consists of proving that at least one of the optimal strategies Q_0 in 5.8 is supported on the flow of a reasonably nice vector field. In the present thesis, we want extend this result from its original autonomous formulation to certain non-autonomous situations. To this end, some regularity needs to be assumed in the spatial x variable. We have in mind the following model,

$$H(x,i) = a(x)\,\frac{i^q}{q} + b\,i \tag{5.16}$$

where in principle one can have $1 < q < \infty$, $b > 0$, and $a > 0$ is a measurable function, with $a, \frac{1}{a} \in L^\infty(\Omega)$, and more importantly $a \in W^{1,s}(\Omega)$ for some $s > n$. Of course, other models may fit into the discussion. Here Ω is a domain in $\mathbb{R}^n$ with nice boundary. This is our main result.

Theorem 37. *Let* $1 < q \le 3/2$. *If* $x \mapsto H(x,i)$ *is as in* (5.16), *and* μ_0, μ_1 *are probability densities such that* $\frac{1}{\mu_0}, \frac{1}{\mu_1} \in W^{1,\infty}$, *then* (5.8) *admits an equilibrium strategy* $Q \in \mathcal{Q}^q(\mu_0,\mu_1)$ *of the form*

$$Q = X_\#(\mu_0)$$

for some DiPerna - Lions flow map X.

The proof of this theorem repeats the structure of [15]. This means that optimal traffic strategies will be constructed by means of the classical Dacorogna-Moser scheme. Such structures will be proven to have support on the trajectories of a DiPerna-Lions velocity field. This obviously will require the uses of DiPerna-Lions theory (see Subsection 2.4 for the definition of DiPerna - Lions flow), for which L^∞ and Sobolev estimates are essential, and here is where the results from Sections 4, 4.4 and 4.3 will enter the game.

Let us start by observing that if H is as in (5.16) then

$$\mathcal{H}^*(x,\xi) = \frac{a(x)}{p}\left(\frac{|\xi|-b}{a(x)}\right)_+^p$$

so that

$$D_\xi \mathcal{H}^*(x,\xi) = \left(\frac{|\xi|-b}{a(x)}\right)_+^{p-1}\frac{\xi}{|\xi|}$$

$$D_{\xi\xi}\mathcal{H}^*(x,\xi) = \frac{p-1}{a(x)}\left(\frac{|\xi|-b}{a(x)}\right)_+^{p-2}\frac{\xi\otimes\xi}{|\xi|^2} + \left(\frac{|\xi|-b}{a(x)}\right)_+^{p-1}\left(\frac{1}{|\xi|}\mathbf{Id} - \frac{\xi\otimes\xi}{|\xi|^3}\right)$$

This immediately gives the following bounds

$$\begin{aligned} \langle D_{\xi\xi}\mathcal{H}^*(x,\xi)\lambda,\lambda\rangle &\geq c\,(|\xi|-b)_+^{p-1}\,|\xi|^{-1}\,|\lambda|^2 \\ |D_{\xi\xi}\mathcal{H}^*(x,\xi)| &\leq C\,(|\xi|-b)_+^{p-2} \\ |D_{x\xi}\mathcal{H}^*(x,\xi)| &\leq k(x)\,(|\xi|-b)_+^{p-1} \end{aligned} \tag{5.17}$$

for all $\xi \in \mathbb{R}^n$, $x \in \Omega$ and $\lambda \in \mathbb{R}^n$, and some function $k \in L^s(\Omega)$. In particular, for H as in (5.16), equations (5.17) guarantee that $\mathcal{F} = \mathcal{H}^*$ satisfies the assumptions **(F0)**–**(F4)** from Section 4.

Towards the proof of Theorem 37, we start with the following regularity result. It is a direct consequence of Theorem 15 and its global version from Theorem 22.

Proposition 38. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with C^3-smooth boundary. Suppose also that $\mu_0, \mu_1 \in L^s$ for some $s > n$. If H is as in* (5.16)*, $q \leq 2$, and σ is a minimizer in* (5.9)*, then $\sigma \in L^\infty$.*

Proof. Since H is strictly convex in the second variable, the remarks after Lemma 34 apply. So if σ is a minimizer then

$$\sigma = \nabla_\xi \mathcal{H}^*(x,\nabla u) \tag{5.18}$$

where u is necessarily a minimizer of

$$\inf_{u\in W^{1,p}(\Omega;\mathbb{R})}\int \mathcal{H}^*(x,\nabla u) + (\mu_0-\mu_1)\,u\,dx.$$

Since $\mathcal{F} = \mathcal{H}^*$ satisfies conditions **(F0)**–**(F4)** of Section 4, we are legitimate to use Theorem 15. As a consequence, and since $\mu_0, \mu_1 \in L^s$ for some $s > n$, we deduce that $u \in W^{1,\infty}_{loc}$. The global boundedness of ∇u is a consequence of the boundary reflection result, see Theorem 22. So the claim follows. □

It is worth mentioning that the essential condition on H here is that $\mathcal{F} = \mathcal{H}^*$ satisfies **(F0)-(F4)**. Now, we continue with the Sobolev estimates for σ.

Proposition 39. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with C^3-smooth boundary. Suppose also that $\mu_0, \mu_1 \in L^s$ for some $s > n$. If σ is a minimizer in* (5.9)*, H is as in* (5.16)*, and $q \leq 3/2$, then $\sigma \in W^{1,1}_{loc}(\Omega)$.*

Proof. We start as in the proof of Proposition 38. So we have

$$\sigma = \nabla_\xi \mathcal{H}^*(x, \nabla u) \tag{5.19}$$

where u is necessarily a minimizer of

$$\inf_{u \in W^{1,p}(\Omega;\mathbb{R})} \int \mathcal{H}^*(x, \nabla u) + (\mu_0 - \mu_1)\, u\, dx.$$

Now, we have for $\mathcal{F}(x,\xi) = \mathcal{H}^*(x,\xi)$ the bounds in (5.17). In particular, this means that

$$\begin{aligned} |D\sigma| &\leq |D_{x,\xi}\mathcal{H}^*(x, Du)| + |D_{\xi,\xi}\mathcal{H}^*(x, Du)\, D^2 u| \\ &\leq k(x)\,(|Du| - b)_+^{p-1} + C\,(|Du| - b)_+^{p-2}\,|D^2 u| \end{aligned}$$

Using now Theorem 15 together with the fact that $k \in L^s$, we see that the first term on the right hand side above belongs to L^s, and so in particular it is locally integrable. Concerning the second, we use Theorem 24 to deduce that if $1 < q \leq \frac{3}{2}$ then $p \geq 3$ and so

$$\int (|Du| - b)_+^{p-2}\,|D^2 u| \leq \left(\int (|Du| - b)_+^2\,|D^2 u|^2\right)^{\frac{1}{2}} \left(\int (|Du| - b)_+^{2(p-3)}\right)^{\frac{1}{2}}, \tag{5.20}$$

which is easily seen to be finite as a consequence of Theorems 15 and 24. The claim follows. □

We can now prove Theorem 37.

Proof of Theorem 37. For the reader's convenience, we sketch the proof of existence of optimal traffic strategies Q (see [20, Theorem 2] or [19, Theorem 2.10] for the precise proofs). Let us take $\sigma \in \Sigma^q(\mu_0, \mu_1)$ to be the unique optimizer at (5.9), and call

$$\hat{\sigma}(t,x) = \frac{\sigma(x)}{(1-t)\,\mu_0(x) + t\,\mu_1(x)}.$$

Then, $u(t,\cdot) = (1-t)\,\mu_0 + t\,\mu_1$ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t u + \operatorname{div}(\hat{\sigma}\, u) = 0, \\ u(0,\cdot) = \mu_0. \end{cases} \tag{5.21}$$

Moreover, using thaty $\sigma \in L^q(\Omega)$ and Ω is bounded, one actually deduces from the superposition principle (see Subsection 2.5) that $(1-t)\,\mu_0 + t\,\mu_1$ is indeed a superpositon solution. Thus, there exists a measure Q on $C([0,1];\overline{\Omega})$ such that

$$(1-t)\,\mu_0 + t\,\mu_1 = (\pi_t)_\# Q \tag{5.22}$$

and in particular $Q \in \mathcal{Q}(\mu_0, \mu_1)$ can be chosen so that $i_Q = |\sigma|$, whence $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$. As a consequence, one also has

$$\int \mathcal{H}(x, \sigma(x))\, dx = \int H(x, i_Q(x))\, dx.$$

The optimality of σ in (5.9) forces Q to be an optimal for (5.8). It just remains to prove that $Q = X_\#(\mu_0)$ for some DiPerna-Lions flow X. To this end, we need to see that $\hat{\sigma}$ admits such a flow. But this is a consequence of Theorem 8. Indeed,

- $\hat{\sigma} \in L^\infty(\Omega)$, because $\sigma \in L^\infty$ (by Proposition 38) and $\frac{1}{\mu_j} \in L^\infty$ (by assumption).
- $\hat{\sigma} \in W^{1,1}(\Omega)$, because $\sigma \in W^{1,1}$ (by Proposition 39), $\sigma \in L^\infty$ (by Proposition 38) and $\frac{1}{\mu_j} \in L^\infty \cap W^{1,1}$ (by assumption).
- $\operatorname{div} \hat{\sigma} \in L^\infty$, because by assumption $D(\mu_0 - \mu_1) \in L^\infty$.

By Theorem 8, $\hat{\sigma}$ induces a well-defined DiPerna-Lions flow X, and $u(t, \cdot) = X(t, \cdot)_\#(\mu_0)$ must necessarily be a weak solution of (5.21). However, the fact that $\hat{\sigma} \in L^1([0,1]; L^1(\Omega))$ allows us to use the Superposition Principle (Theorem 9), and hence all non-negative measure valued weak solutions to (5.21) are superposition solutions. Moreover, Corollary 10 also applies to (5.21), and so there is only one superposition solution. As a consequence,

$$(1-t)\,\mu_0 + t\,\mu_1 = (X_t)_\#(\mu_0)$$

for each $t \in [0,1]$. As a consequence, one gets that $(\pi_t)_\# Q = (X_t)_\#(\mu_0)$ for each time $t \in [0,1]$. Thus $Q = X_\#(\mu_0)$, and the theorem is proved. □

Although there is little room for improvement in DiPerna - Lions theory, recent investigations allow to relax the assumption $D(1/\mu_j) \in L^\infty$ in Theorem 37 by the weaker Orlicz-type one

$$D(1/\mu_j) \in L^P, \qquad P(t) = \exp\left(\frac{t}{\log(t)\,\log\log(t)\,\ldots}\right),$$

allowing in particular for non-Lipschitz data. The interested reader is adressed to see the references [25, 24] for details.

Chapter 6

Conclusions and future lines

6.1 Conclusions

The OD matrix estimation problem is one of the major concerns in modern traffic controlling systems. OD matrices are very volatile, being affected by a number of intentional and unintentional parameters, such as weather conditions. In addition, OD matrices are not observable. In this thesis, regarding the static traffic problem, we explore new OD demand problem formulation that allows the modeller to define structural similarity between the historical and estimated OD matrix while ensuring computationally fast and tractable solution. Furthermore, an OD estimation model based on Lasso shrinkage method has been introduced to reduce dimensionality of the OD demand vector.

The experiment setup is defined in such a way that sensitivity of the proposed OD estimation methods is evaluated for a range of user-pre-defined tuning parameters. A new solution approach is applied based on the well-known gradient descent algorithm for the OD estimation process, both to estimate the OD demand and step length in the gradient method, in each step of algorithm. In this respect, non-differentiability issues in the Lasso_1 method with respect to the OD demand are addressed by using three different strategies, including smoothening the non-differentiable term by mollification with the Gaussian kernel.

In order to test the applicability of the proposed methods, in terms of running-time of the estimation process, they are tested on Vitoria city in Spain, with high dimensional historical OD demand. Increasing the tuning parameter value, increases the number of OD flows which are converging to zero, and therefore reduces the dimensionality of the estimated OD demand vector. The Lasso_2 and Lasso_1 methods, increase the iterpretability of OD demand, and reduce the computational time of the OD estimation process significantly, in comparison to the Ridge method. This is done indeed, by preserving the structure and sparsity of the historical OD matrix, and reduction in dimensionality of OD demand.

In chapter 4 we investigated on L^{∞} bounds for minimizers of a certain type of functional with a large degeneracy set. In this respect, weighted Sobolev bounds are a delicate issue. Giaquinta and Modica [45] have shown for $\Delta_p u = f$ that we have $|\nabla u|^{p-2}\nabla^2 u \in L^1_{\rm loc}$, provided that $f \in L^s_{\rm loc}$, $s > n$. Along with this result, Brasco, Carlier and Santambrogio have shown that if $\mathrm{div}(|\nabla u - 1|_+^{p-1} \frac{\nabla u}{|\nabla u|}) = f$ the we get $(|\nabla u| - 1)_+^{p-2} D^2 u \in L^1_{\rm loc}$. At the first glance, we would expect the same result, but, instead, in the non-autonomous setting, we got a different regularity result as follows

$$(|\nabla u| - 1)_+ D^2 u \in L^2_{\rm loc},$$

and the reason is that a different method has been used.

Indeed, as discussed in Chapter 4, we show that $\mathcal{G}(P) \in W^{1,2}$, for $\mathcal{G}'(t) = t(t+1)^{\frac{p-4}{2}}$ and $P = (|Du| - 1)_+$. As a consequence

$$D\Big(\mathcal{G}(P)\Big) = \mathcal{G}'(P) \cdot DP = P \cdot (P+1)^{\frac{p-4}{2}} \cdot DP = (|\nabla u| - 1)_+ \cdot D^2 u \cdot |Du|^{\frac{p-4}{2}} \in L^2. \tag{6.1}$$

This has been proven by testing the PDE against an appropriately chosen testing function φ. This contrasts with Brasco, Carlier and Santambrogio [15], who used difference quotient method. Unfortunately, this method does not work for the non-autonomous case, because an extra term appears which is impossible to be controlled without global ellipticity.

Concerning the dynamic congested traffic problem, one of the fundamental results of [15] consists of proving that at least one of the optimal strategies Q_0 in (5.8) is supported on the flow of a reasonably nice vector field. In the present thesis, we found a solution consisting of an optimal traffic strategy with support on the trajectories of a DiPerna-Lions flow, for a non-autonomous congested traffic problem. The original autonomous scheme was introduced in [15]. We presented a non-autonomous counterpart, in which the spatial variable is supercritically Sobolev dependent. As in [15], it requires both Lipschitz and Sobolev $W^{2,1}$ apriori bounds for minimizers of integral functionals whose ellipticity bounds are satisfied only away from a ball of the gradient variable.

We showed that the construction of Wardrop equilibriums in [15] can be extended to non-autonomous, Sobolev-dependent counterparts with growth $1 < q \leq 3/2$. The reason behind this growth limitation is that our Sobolev estimates (6.1) do not provide enough integrability. As a consequence, when applying Dacorogna-Moser scheme, the vector field $\sigma = \nabla\mathcal{H}^*(x, \nabla u)$ can only be proven to meet DiPerna-Lions requirements when $p \geqslant 3$, as

$$\begin{aligned} |D\sigma| &\leqslant |D_x D_\xi \mathcal{H}^*(x, Du)| + |D_{\xi\xi}\mathcal{H}^*(x, Du)| \cdot |D^2 u| \\ &\leqslant L^\infty + (|Du| - 1)_+^{p-2} \cdot |D^2 u|, \end{aligned}$$

which is $L^1_{\rm loc}$ for $p \geqslant 3$. Further research is due to fill the missing range $3/2 < q \leq 2$ at the traffic side.

6.2 Future lines

A question arises here, whether if there is any chance to modify testing function φ in order to get the desired $(p-2)$ exponent? This is an open problem which we are trying to solve.

By the moment, it is known that $\nabla u \in L^\infty$ may fail for $s = n = 2$, even when $f \equiv 0$ in the linear, uniform elliptic setting, and so other arguments are needed if one wants to find such nice strategies, as the lack of boundedness forbids us to implement Dacorogna-Moser scheme. In practice, specifically for the sake of application, It would be desirable to understand what is the situation when the x dependence is not continuous, and if there is any chance to find such nice strategies in that case.
As another interesting idea, one can think of a different definition for the function $H(x, i)$, such that

$$H(x, i) = a(x)\frac{i^q}{q} + b(x) \cdot i.$$

Thus, the model PDE becomes

$$\operatorname{div}\left((|\nabla u| - b(x))_+^{p-2} \frac{\nabla u}{|\nabla u|}\right) = f,$$

which is very difficult to carry out, as b is variable, which seems to make the argument significantly tougher.
It was shown that under sufficient assumptions we get $\widehat{\sigma} \in L^\infty$, now one can ask if there is any chance to get the same regularity without having $\sigma \in L^\infty$, and $\frac{1}{\mu_t} \in L^\infty$. For instance, recent developments [24, 25] show that there is some room for DiPerna-Lions theory with unbounded $\operatorname{div}(b)$. We would like to see what consequences does this have in traffic analysis.

Recently, researchers have been investigating on the regularity of minimizers of different functionals with a wide degree of degeneracy and orthotropic structure (see [11], [12], [13], and [17]). It is desirable to extend our results to a non-autonomous orthotropic setting.

The static OD estimation problem has been considered typically as an initial step towards online real-time OD estimation. Thus, as a potential research idea, one can think of employing the presented methodology in this thesis in the online real-time OD estimation settings.

Appendix A

R and Python codes

This part provides the explicit codes required for the OD estimation process in Chapter 3. We have used two language programs: R (2017), and Python (2017).

A.1 Implemented R code in the Vitoria network

```
# Main file
#source('Lasso.R')
source('Gradient.R')
source('Step_length.R')
source('Update_matrix.R')
source('Results.R')
source('Sparsity.R')
# Inputs
s <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/Results/Current_matrix_and_flows/states_Lasso_without_original_matrix_1000.csv", sep = ",", header = FALSE)[,1]
penalization_coefficient_seq <- c(1)
for(penalization_coefficient in penalization_coefficient_seq){
  #penalization_coefficient <- 100
  number_of_iteration <- 31
  flow_floor <- 0.0001

  Aimsun_install_location <- "C:\\apps\\Aimsun"
  # aconsole -project path to your ANG file -cmd execute -target scriptId
  project_path_to_ang_file <- 'aconsole -project "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/vitoria2.ang" -cmd execute -target 15058686'
  #project_path_to_ang_file <- 'aconsole -project "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/vitoria2.ang" -cmd execute -target 15058688'
  volumes <- "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/volumes.txt"

  length(which(historical_matrix==0))
  length(historical_matrix)
  # Matrices
  centroidsN <- sqrt(length(read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/states.txt", sep = ",", header = FALSE)[ ,1]))
  real_matrix <- matrix(data = read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/states.txt", sep = ",", header = FALSE)[ ,1],nrow = centroidsN, ncol = centroidsN, byrow = TRUE)
  historical_matrix <- as.matrix(read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/historical_matrix.csv", sep =",", header = FALSE))
```

```
original_matrix <- historical_matrix
current_matrix <- original_matrix

# Flows
volumes_with_the_same_separator <- read.table(textConnection(gsub(":", ",", readLines("E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/volumes.txt"))))
write.table(volumes_with_the_same_separator, 'E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/volumes_with_the_same_separator.txt', col.names = FALSE, row.names = FALSE, quote = FALSE)
detectors_id <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/volumes_with_the_same_separator.txt", sep = ",", header = FALSE)[ ,1]
observed_flows <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/volumes_with_the_same_separator.txt", sep = ",", header = FALSE)[ ,2]
observed_flows <- observed_flows[order(detectors_id)]
detectorsN <- length(observed_flows)

# Probability matrix
assigned_prob_col_pos <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/matrix.txt", sep = ";", header = FALSE)[,1]+1
assigned_prob_row_pos <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/matrix.txt", sep = ";", header = FALSE)[,2]+1
assigned_prob_entry <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/matrix.txt", sep = ";", header = FALSE)[,3]
assigned_probabilities <- matrix(data = 0, nrow = (centroidsN * centroidsN), ncol = detectorsN)
for (i in 1:length(assigned_prob_entry)) {
  assigned_probabilities[assigned_prob_row_pos[i] , assigned_prob_col_pos[i]] <- assigned_prob_entry[i]
}
row_name <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/origins.txt", sep = "\n", header = FALSE, stringsAsFactors=FALSE)
row_name <- as.vector(row_name)
col_name <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/original/destinations.txt", sep = "\n", header = FALSE, stringsAsFactors=FALSE)
col_name <- as.vector(col_name)
col_name <- c('id',t(col_name))

# Required objects for Lasso
width_gaussian <- 0.01
height_gaussian <- width_gaussian / sqrt(2 * pi)
center_gaussian <- original_matrix
# Choose which method you want to utilize to calculate assignment matrix;
# "Modified_Spiess"                          : sum(assigned_flows - observed_flows)^2,

# "Ridge"                            : sum(assigned_flows - observed_flows)^2,
#                                      + (penalization_coefficient * sum(current_matrix - original_matrix)^2),

# "Lasso_without_original_matrix" : sum(assigned_flows - observed_flows)^2
#                                      + (penalization_coefficient * sum(current_matrix)),

# "Lasso"                            : sum(assigned_flows - observed_flows)^2
#                                      + (penalization_coefficient * sum(|current_matrix - original_matrix|).
cat("\014")
#method <- readline("Choose a method ( Modified_Spiess, Ridge, Lasso, Lasso_without_original_matrix, Lasso ) : ")
method <- "Ridge"
assignment_loop <- 0
cost <- as.vector(0)
step_length <- as.numeric(0)
```

```
ptm <- proc.time()

#for(iter in c(1:number_of_iteration)){
while(assignment_loop >= 0) {
  if((assignment_loop > 1) && (abs(cost[assignment_loop] - cost[assignment_loop - 1]) < 500)){
    break
  }

  if(assignment_loop != 0){
    gradient <- calculate_gradient()
    step_length <- calculate_step_length()
    last_step_matrix <- current_matrix
    current_matrix <- update_matrix()
  }

  #if(assignment_loop != 0 && method == "Lasso"){
  #  current_matrix <- calculate_lasso()
  #}

  wrt_current_matrix <- rbind(t(row_name), current_matrix)
  wrt_current_matrix <- cbind(col_name, wrt_current_matrix)
  wrt_current_matrix <- wrt_current_matrix[rowSums(wrt_current_matrix[,-1] != 0) != 0, ]
  write.table(wrt_current_matrix, 'E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/current_matrix.txt', sep ='\t', col.names = FALSE, row.names = FALSE, quote = FALSE)
  Sys.setenv(PATH = paste(Sys.getenv("PATH"), Aimsun_install_location, sep = ";"))
  shell(project_path_to_ang_file, wait = TRUE)

  assigned_flows <- read.table(file = volumes, sep = ",", header = FALSE)[ ,2]
  assigned_flows <- assigned_flows[order(detectors_id)]
  if(assignment_loop == 0){
    assigned_flows_historical <- assigned_flows
  }

  assigned_prob_col_pos <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/matrix.txt", sep = ";", header = FALSE)[,1]+1
  assigned_prob_row_pos <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/matrix.txt", sep = ";", header = FALSE)[,2]+1
  assigned_prob_entry <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/matrix.txt", sep = ";", header = FALSE)[,3]
  assigned_probabilities <- matrix(data = 0, nrow = (centroidsN * centroidsN), ncol = detectorsN)
  for (i in 1:length(assigned_prob_entry)) {
    assigned_probabilities[assigned_prob_row_pos[i] , assigned_prob_col_pos[i]] <- assigned_prob_entry[i]
  }

  assignment_loop <- assignment_loop + 1
  cat("\014")
  cat("cost =", cost)
  cat(sep = "\n")
  cat(sep = "\n")
  if(assignment_loop > 9){
    cat("Latest 10 costs =", cost[(assignment_loop-9):assignment_loop])
    cat(sep = "\n")
  }
  # Calculating cost (objective) function
  if (method == "Modified_Spiess") {
    cost[assignment_loop] <- 1/2 * sum((assigned_flows - observed_flows) ^ 2)

  }else if(method == "Ridge"){
```

```
      cost[assignment_loop] <- (1/2 * sum((assigned_flows - observed_flows) ^ 2)) + (penalization_coefficient * (1/2 * sum((current_matrix - original_matrix) ^ 2)))

    }else if (method == "Lasso_without_original_matrix"){
      cost[assignment_loop] <- (1/2 * sum((assigned_flows - observed_flows) ^ 2)) + (penalization_coefficient * (sum(current_matrix)))

    }else if (method == "Lasso"){
      cost[assignment_loop] <- (1/2 * sum((assigned_flows - observed_flows) ^ 2)) + (penalization_coefficient * sum(abs(current_matrix - original_matrix)))  }
  }
  # Results
  run_time <- proc.time()-ptm
  print(penalization_coefficient)
  print(current_matrix)
  degree_of_sparsity <- calculate_sparsity(matrix = if(method == "Lasso" || method == "Ridge"){abs(current_matrix - original_matrix)}else{current_matrix})
  sum_of_cells_of_current_matrix <- sum(current_matrix)
  results <- calculate_results()
}
```

Listing A.1: R code for OD estimation process in the Vitoria network; Main_scrip.R

```
# calculating gradient

calculate_gradient <- function(){
  identity_matrix <- matrix (data = 1, nrow = centroidsN*centroidsN, ncol = 1)
   if (method == "Modified_Spiess") {
    gradient <- gradient <- matrix (data = assigned_probabilities %*% (assigned_flows - observed_flows),
                                    nrow = centroidsN, ncol = centroidsN, byrow = TRUE)
      }else if(method == "Ridge"){
    gradient <- (matrix (data = assigned_probabilities %*% (assigned_flows - observed_flows), nrow = centroidsN, ncol = centroidsN, byrow = TRUE)) + (penalization_coefficient * (current_matrix - original_matrix))
     }else if (method == "Lasso_without_original_matrix"){
    gradient <- (matrix (data = assigned_probabilities %*% (assigned_flows - observed_flows), nrow = centroidsN, ncol = centroidsN, byrow = TRUE)) + penalization_coefficient
    }else if (method == "Lasso"){
    for( rowN in 1:nrow(original_matrix) ){
      for(colN in 1:ncol(original_matrix) ){

        if(current_matrix[rowN, colN] - original_matrix[rowN, colN] >= 0 ){
          gradient[rowN, colN] <- ((matrix (data = assigned_probabilities %*% (assigned_flows - observed_flows), nrow = centroidsN, ncol = centroidsN, byrow = TRUE)) + penalization_coefficient)[rowN, colN]
        }
        if(current_matrix[rowN, colN] - original_matrix[rowN, colN] < 0 ){
          gradient[rowN, colN] <- ((matrix (data = assigned_probabilities %*% (assigned_flows - observed_flows), nrow = centroidsN, ncol = centroidsN, byrow = TRUE)) - penalization_coefficient)[rowN, colN]
        }
      }
    }

    #gradient <- (matrix (data = assigned_probabilities %*% (assigned_flows - observed_flows), nrow = centroidsN, ncol = centroidsN, byrow = TRUE)) + convolve((penalization_coefficient * (current_matrix - original_matrix)), ((-(height_gaussian * (current_matrix - center_gaussian)) / (width_gaussian ^ 2)) %*% (exp(-((current_matrix - center_gaussian) ^ 2) / (2 * (width_gaussian ^ 2))))))
```

```
  }
  return(gradient)
}
```

Listing A.2: R code for OD estimation process in the Vitoria network; Gradient.R

```
# calculating step length "lambda"
calculate_step_length <- function() {
  flows_derivative <- as.vector(t(assigned_probabilities) %*% as.vector(t(gradient)))
      if (method == "Modified_Spiess") {
    step_length <- sum(flows_derivative * (assigned_flows - observed_flows)) / sum(flows_derivative ^ 2)
      }else if(method == "Ridge"){
    step_length_nominator <- sum(flows_derivative * (assigned_flows - observed_flows)) + sum(penalization_coefficient * c(gradient) * c(current_matrix)) - sum(penalization_coefficient * c(gradient) * c(original_matrix))

    step_length_denominator <- sum(flows_derivative ^ 2) + sum(penalization_coefficient * (c(gradient) ^ 2))

    step_length <- (step_length_nominator / step_length_denominator)
      }else if (method == "Lasso_without_original_matrix"){

    step_length_nominator <- sum(flows_derivative * (assigned_flows - observed_flows))  + sum(penalization_coefficient * c(gradient))

    step_length_denominator <- sum(flows_derivative ^ 2)

    step_length <- step_length_nominator / step_length_denominator

      }else if (method == "Lasso"){

    step_length_nominator <- sum(flows_derivative * (assigned_flows - observed_flows)) + sum(penalization_coefficient * c(gradient) * c(current_matrix)) - sum(penalization_coefficient * c(gradient) * c(original_matrix))

    step_length_denominator <- sum(flows_derivative ^ 2) + sum(penalization_coefficient * (c(gradient) ^ 2))

    step_length <- (step_length_nominator / step_length_denominator)

    #positive_indices_in_gradient <- which(gradient >= 0)

    #b1 <- penalization_coefficient * (c(gradient)[positive_indices_in_gradient] ^ 0.0001)
    #b2 <- penalization_coefficient * -(abs(c(gradient)[-positive_indices_in_gradient]) ^ 0.0001)
    #b <- sum(b1 + b2)

    #step_length_nominator <- sum(flows_derivative * (assigned_flows - observed_flows))
    #+ sum(penalization_coefficient * c(gradient) * c((current_matrix ^ 0.0001))) + b
    #+ sum(penalization_coefficient * c((original_matrix ^ 0.0001)))

    #step_length_denominator <- sum(flows_derivative ^ 2)

    #step_length <- step_length_nominator / step_length_denominator

    #step_length_nominator <- sum(flows_derivative %*% (observed_flows - assigned_flows)) + sum(penalization_coefficient * (c(current_matrix) ^ 2) * c(gradient)) - sum(penalization_coefficient * c(current_matrix) * c(original_matrix) * c(gradient))

    #step_length_denominator <- sum(flows_derivative %*% t(flows_derivative)) + sum(penalization_
```

```
        coefficient * c(current_matrix) * (c(gradient) ^ 2))

    #step_length <- (step_length_nominator / step_length_denominator)

    #step_length <- matrix(data = 0, nrow = centroidsN, ncol = centroidsN)
    #for(rowN in 1:nrow(original_matrix) ){
    #  for(colN in 1:ncol(original_matrix) ){
    #    if(current_matrix[rowN, colN] - original_matrix[rowN, colN] >= 0 ){
    #      step_length_nominator <- sum(flows_derivative * (assigned_flows - observed_flows)) + sum(
        penalization_coefficient * c(gradient))
    #      step_length_denominator <- sum(flows_derivative ^ 2)
    #      step_length[rowN, colN] <- step_length_nominator / step_length_denominator
    #    }
    #    if(current_matrix[rowN, colN] - original_matrix[rowN, colN] < 0 ){
    #      step_length_nominator <- sum(flows_derivative %*% (observed_flows - assigned_flows)) -
        sum(penalization_coefficient * c(gradient))
    #      step_length_denominator <- sum(flows_derivative %*% t(flows_derivative))
    #      step_length[rowN, colN] <- -step_length_nominator / step_length_denominator
    #    }
    #  }
    #}

  }
    #if(method == "Lasso_without_original_matrix"){
  # pos<-1
  #for(step in c(step_length)){
  # positive_indices_in_current_matrix <- c(current_matrix > 0)
  #positive_indices_in_gradient <- c(gradient > 0)
  #identity_matrix <- matrix (data = 1, ncol = centroidsN * centroidsN, byrow = TRUE )

  #  is_step_length_Correct <- all(step * c(gradient)[positive_indices_in_current_matrix] <= c(
      current_matrix)[positive_indices_in_current_matrix])

  #  if (!is_step_length_Correct) {
  #   step_length[pos] <- min(c(current_matrix)[positive_indices_in_current_matrix & positive_
      indices_in_gradient] / c(gradient)[positive_indices_in_current_matrix & positive_indices_in_
      gradient])

  #for(rowN in 1:nrow(original_matrix) ){
  # for(colN in 1:ncol(original_matrix) ){
  #    if(is.na(step_length[rowN, colN]) || is.infinite(step_length[rowN, colN])){
  #     step_length[rowN, colN] <- 0
  #    }
  #  }
  #}

  #}
  #pos<-pos+1
  #  }
  #}
  print(step_length)
  return(step_length)
}
```

Listing A.3: R code for OD estimation process in the Vitoria network; Step_length.R

```
calculate_results <- function(){

  RMSE_current_historical <- sqrt((sum(((current_matrix - historical_matrix) ^ 2))) / (centroidsN ^
      2))
```

```
U_current_historical <- (RMSE_current_historical)/((sqrt((sum((current_matrix ^ 2))) / (centroidsN ^ 2)))+(sqrt((sum((historical_matrix ^ 2))) / (centroidsN ^ 2))))
MAE_current_historical <- (1/(centroidsN ^ 2))*sum(abs(current_matrix - historical_matrix))
SE_current_historical <- sum(((current_matrix - historical_matrix)^2))
ME_current_historical <- (1/(centroidsN ^ 2))*sum((current_matrix - historical_matrix))
R2_current_historical <- sum(((historical_matrix - current_matrix)^2))/sum(((historical_matrix - (((1/centroidsN^2)*sum(historical_matrix))))^2))
GEH_current_historical <- sqrt((2*((current_matrix - historical_matrix)^2))/(current_matrix + historical_matrix))
GEH_percent_current_historical <- (100*length(GEH_current_historical[GEH_current_historical<5]))/(centroidsN ^ 2)

RMSE_flow_current_real <- sqrt((sum(((assigned_flows - observed_flows) ^ 2))) / (length(assigned_flows)))
RMSE_flow_historical_real <- sqrt((sum(((assigned_flows_historical - observed_flows) ^ 2))) / (length(assigned_flows_historical)))
RMSE_flow_current_historical <- sqrt((sum(((assigned_flows - assigned_flows_historical) ^ 2))) / (length(assigned_flows)))

U_flow_current_real <- (RMSE_flow_current_real)/((sqrt((sum((assigned_flows ^ 2))) / (length(assigned_flows))))+(sqrt((sum((observed_flows ^ 2))) / (length(assigned_flows)))))
MAE_flow_current_real <- (1/(length(assigned_flows)))*sum(abs(assigned_flows - observed_flows))
SE_flow_current_real <- sum(((assigned_flows - observed_flows)^2))
ME_flow_current_real <- (1/(length(assigned_flows)))*sum((assigned_flows - observed_flows))
R2_flow_current_real <- sum(((observed_flows - assigned_flows)^2))/sum(((observed_flows - ((((length(assigned_flows))^2)*sum(observed_flows))))^2))
GEH_flow_current_real <- sqrt((2*((assigned_flows - observed_flows)^2))/(assigned_flows + observed_flows))
GEH_percent_flow_current_real <- (100*length(GEH_flow_current_real[GEH_flow_current_real<5]))/length(observed_flows)

 results_header <- c('Lambda',
                    'RMSE (estimated and historical OD matrices)',
                    'U (estimated and historical OD matrices)',
                    'MAE (estimated and historical OD matrices)',
                    'SE (estimated and historical OD matrices)',
                    'ME (estimated and historical OD matrices)',
                    'R^2 (estimated and historical OD matrices)',
                    '%GEH < 5 (estimated and historical OD matrices)',
                    'RMSE (estimated and real traffic counts)',
                    'RMSE (historical and real traffic counts)',
                    'RMSE (estimated and historical traffic counts)',
                    'U (estimated and real traffic counts)',
                    'MAE (estimated and real traffic counts)',
                    'SE (estimated and real traffic counts)',
                    'ME (estimated and real traffic counts)',
                    'R^2 (estimated and real traffic counts)',
                    '%GEH < 5 (estimated and real traffic counts)',
                    'Number of OD pairs which converge to zero',
                    'Total demand',
                    'Computational time')

results_data <- c(penalization_coefficient,
                  RMSE_current_historical,
                  U_current_historical,
                  MAE_current_historical,
                  SE_current_historical,
                  ME_current_historical,
                  R2_current_historical,
                  GEH_percent_current_historical,
```

```
                    RMSE_flow_current_real,
                    RMSE_flow_historical_real,
                    RMSE_flow_current_historical,
                    U_flow_current_real,
                    MAE_flow_current_real,
                    SE_flow_current_real,
                    ME_flow_current_real,
                    R2_flow_current_real,
                    GEH_percent_flow_current_real,
                    degree_of_sparsity,
                    sum_of_cells_of_current_matrix,
                    as.numeric(run_time[3]))

results_data_frame <- data.frame(results_header, results_data)
existence_of_results <- file.exists(file = file.path('E:/Google Drive/TSS/MatrixAdjustment/
    Codevitoria/Data/Results', paste0('Results_',method,'.csv')))

if(existence_of_results == FALSE){
  write.table(results_data_frame, file = file.path('E:/Google Drive/TSS/MatrixAdjustment/
      Codevitoria/Data/Results', paste0('Results_',method,'.csv')), sep =",", col.names = FALSE,
      row.names = FALSE, quote = FALSE)
}

new_data_frame <- read.table(file = file.path('E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/
    Data/Results', paste0('Results_',method,'.csv')), sep =",", header = FALSE)
new_data_frame <- new_data_frame[,-1]

if(is.vector(new_data_frame)){
  test_existence_lambda <- new_data_frame[1]
}else{
  test_existence_lambda <- new_data_frame[1,]
}

if(!any(test_existence_lambda == penalization_coefficient)){
  new_data_frame <- cbind(new_data_frame, results_data)
  new_data_frame_sorted <- new_data_frame[,order(new_data_frame[1,])]
  new_data_frame_sorted <- cbind(results_header, new_data_frame_sorted)
  write.table(new_data_frame_sorted, file= file.path('E:/Google Drive/TSS/MatrixAdjustment/
      Codevitoria/Data/Results', paste0('Results_',method,'.csv')), sep =",", col.names = FALSE,
      row.names = FALSE, quote = FALSE)
}
# saving current_matrix and current_flows
#file.copy("E:/Google Drive/TSS/MatrixAdjustment/Code/Data/states.txt", "E:/Google Drive/TSS/
    MatrixAdjustment/Code/Data/Results/Current_matrix_and_flows")
# rename the file
new_states <- read.table(file = "E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/states.txt"
    , sep = ",", header = FALSE)
new_states <- cbind(as.vector(t(historical_matrix)),new_states)
new_states <- rbind(c("historical OD matrix","estimated OD matrix"),new_states)
write.table(new_states, file= file.path('E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/
    Results/Current_matrix_and_flows', paste0('states_',method,'_',penalization_coefficient,'.csv
    ')), sep =",", col.names = FALSE, row.names = FALSE, quote = FALSE)
#file.rename(from = file.path('E:/Google Drive/TSS/MatrixAdjustment/Code/Data/Results/Current_
    matrix_and_flows', 'states.txt'), to = file.path('E:/Google Drive/TSS/MatrixAdjustment/Code/
    Data/Results/Current_matrix_and_flows', paste0('states_',method,'_',penalization_coefficient
    ,'.txt')))
# saving current flows
write.table(assigned_flows, file = file.path('E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/
    Data/Results/Current_matrix_and_flows', paste0('flows_',method,'_',penalization_coefficient,'
    .txt')), sep ="\t", col.names = FALSE, row.names = FALSE, quote = FALSE)
```

```
  # saving cost function
  new_cost <- cbind(c(1:length(cost)), cost)
  new_cost <- rbind(c("iteration","objective function"), new_cost)
  write.table(new_cost, file= file.path('E:/Google Drive/TSS/MatrixAdjustment/Codevitoria/Data/
      Results/Current_matrix_and_flows', paste0('objective_function_',method,'_',penalization_
      coefficient,'.csv')), sep =",", col.names = FALSE, row.names = FALSE, quote = FALSE)

  cat("RMSE (estimated and historical OD matrix) = ", RMSE_current_historical)
  cat(sep = "\n")
  cat("RMSE (estimated and real traffic counts) = ", RMSE_flow_current_real)
  cat(sep = "\n")
  cat("RMSE (historical and real traffic counts) = ", RMSE_flow_historical_real)
  cat(sep = "\n")
  cat("RMSE (estimated and historical traffic counts) = ", RMSE_flow_current_historical)
}
```

Listing A.4: R code for OD estimation process in the Vitoria network; Results.R

```
calculate_sparsity <- function(matrix){
  degree_of_sparsity <- 0
  positive_indices_of_original_matrix <- which(c(original_matrix) > 0)
  for(i in c(positive_indices_of_original_matrix)){
    if(c(matrix)[i]<=0.1){
      degree_of_sparsity <- degree_of_sparsity + 1
    }
  }
  cat("\n")
  if(method == "Lasso"){
    cat("Number of additional zero cells (in (current_matrix - original_matrix)) =", degree_of_
        sparsity)
  }else{
    cat("Number of additional zero cells (in current_matrix) =", degree_of_sparsity)
  }
  return(degree_of_sparsity)
}
```

Listing A.5: R code for OD estimation process in the Vitoria network; Sparsity.R

A.2 Python script to call the R code from Aimsun Next

In what follows, the Python code, in which Aimsun Next simulator software is called to run the assignment, is represented.

```
experiment = model.getCatalog().find( 15058688 )
path = str( QFileInfo( model.getDocumentFileName() ).absolutePath() )
def observations():
  res = []
  for detector in GK.GetObjectsOfType( model.getType( "GKDetector" ) ):
    section = detector.getSection()
    res.append( section.getId() )
  return res
def getAssignedVolume( id ):
  section = model.getCatalog().find( id )
  column = model.getColumn( "MACRO::" + str( experiment.getId() ) + "_GKSection_
      macroAssignedVolume_15058284" )
  ts = section.getDataValueTS( column )
  if ts != None:
```

```
    return ts.getValue( GKTimeSerieIndex( 0 ) )[0]
  return 0.0
def getDetector( sectionId ):
  section = model.getCatalog().find( sectionId )
  for topObject in section.getTopObjects():
    if topObject.getType() == model.getType( "GKDetector" ):
      return topObject
def getObservedVolume( sectionId ):
  detector = getDetector( sectionId )
  column = model.getColumn( "GKRealData::RD_GKDetector_count_" +str( experiment.getScenario().
      getRealDataSet().getId() )+ "_0" )
  ts = detector.getDataValueTS( column )
  if ts != None:
    return ts.getValue( GKTimeSerieIndex( 0 ) )[0]
  return 0.0
def getDemand( originId, destinationId ):
  origin = model.getCatalog().find( originId )
  destination = model.getCatalog().find( destinationId )
  return experiment.getScenario().getDemand().getTrips( origin, destination, None, None )
def states():
  res = []
  centroids = experiment.getScenario().getDemand().getCentroidConfigurations()[0].getCentroids()
  for origin in centroids:
    for destination in centroids:
      res.append( ( origin.getId(), destination.getId() ) )
  return res
def main():
  rds = experiment.getScenario().getRealDataSet()
  rds.restoreData()
  GKSystem.getSystem().executeAction( "execute", experiment, [], "" )
  forest = experiment.getForest()
  matrix = forest.getAssignmentMatrix( model, observations(), states() )
  matrix.writeToFile( path + "/matrix.txt" )
  f = open( path + "/meta_info.txt", 'w' )
  f.write( "states: [" + ",".join( [ "({},{})".format(origin, dest) for origin, dest in states() ] )
       + "]\n" )
  f.write( "observations: [" + ",".join(["{}".format(i) for i in observations()]) + "]\n" )
  f.close()
  f = open( path + "/volumes.txt", 'w' )
  for section in observations():
    f.write( str( section ) + ": " + str( getObservedVolume(section) ) + "," + str(
        getAssignedVolume(section) )  + "\n" )
  f.close()
  f = open( path + "/states.txt", 'w' )
  for state in states():
    f.write( str( getDemand( state[0], state[1] ) ) + "\n" )
  f.close()
main()
```

Listing A.6: Python code to call Aimsun Next to run the assignment, in the Vitoria network; Python_script.py

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