Universitat Autònoma de Barcelona

Advanced Stock Price Models, Concave Distortion Functions and Liquidity Risk in Finance

Gero Quintus Rudolf Junike

Advanced Stock Price Models, Concave Distortion Functions and Liquidity Risk in Finance

Doctoral Thesis

Author: Gero Quintus Rudolf Junike
Director: Prof. Dr. Alejandra Cabaña

January 2, 2019

Doctorat en Matemàtiques
Departament de Matemàtiques
Universitat Autònoma de Barcelona

Für Amalia.

Für Malte, Nils, Arne, Sönke, Viktor, Merle und Maria.

All models are approximations.
Essentially, all models are wrong,
but some are useful.
George Edward Pelham Box (1919-2013).

Abstract

This thesis consists of three essays. In the first essay, we test empirically the pricing performance of several advanced financial models. We calibrate six advanced stock price models to a time series of real market data of European options on the DAX, a German blue chip index. Via a Monte Carlo simulation, we price barrier down-and-out call options for all models and compare the modelled prices to given real market data of the barrier options. The Bates model reproduces barrier option prices well. The BNS model overvalues and Lévy models with stochastic time-change and leverage undervalue the exotic options. A heuristic analysis suggests that the different degree of fluctuation of the random paths of the models are responsible of producing different prices for the barrier options.

The second essay of this thesis discusses the relationship between coherent risk measures and concave distortion functions. A family of concave distortion functions is a set of concave and increasing functions, mapping the unity interval onto itself. Distortion functions play an important role defining coherent risk measures. We prove that any family of distortion functions which fulfils a certain translation equation, can be represented by a distribution function. An application can be found in actuarial science: moment based premium principles are easy to understand but in general are not monotone and cannot be used to compare the riskiness of different insurance contracts with each other. Our representation theorem makes it possible to compare two insurance risks with each other consistent with a moment based premium principle by defining an appropriate coherent risk measure.

In the last essay of this thesis, we investigate financial markets with frictions, where bid and ask prices of financial intruments are described by sublinear pricing functionals. Such functionals can be defined recursively using coherent risk measures. We prove the convergence of bid and ask prices for various European and American possible path-dependent options, in particular plain vanilla, Asian, lookback and barrier options in a binomial model in the presence of transaction costs. We perform several numerical experiments to confirm the theoretical findings. We apply the results to real market data of European and American plain vanilla options and compute an implied liquidity to describe the bid-ask spread. This method describes liquidity over time very well, compared to the classical approach of describing the bid-ask spread by quoting bid and ask implied volatilities.

Resumen

Esta tesis consta de tres ensayos. En el primer ensayo, probamos empíricamente el desempeño de los precios de varios modelos financieros avanzados para opciones exóticas. Calibramos seis modelos avanzados para precios de acciones a una serie de datos de mercado reales de opciones europeas en el DAX, el índice de referencia de la Bolsa alemana. A través de una simulación de Monte Carlo, calculamos precios de opciones de barrera para todos los modelos y comparamos los precios modelados con los precios del mercado de las opciones de barrera. El modelo Bates reproduce bien los precios de las opciones de barrera. El modelo BNS sobrevalora y los modelos Lévy con cambio temporal estocástico y con efecto de palanca subestiman los precios de las opciones exóticas. Un análisis heurístico sugiere que el diferente grado de fluctuación de las trayectorias aleatorias de los modelos es el responsable de producir diferentes precios para las opciones de barrera.

En el segundo ensayo de esta tesis se examinan medidas de riesgo coherentes y funciones de distorsión cóncavas. Una familia de funciones cóncavas de distorsión es un conjunto de funciones cóncavas y crecientes, con dominio e imagen igual al intervalo unitario. Se usan las funciones de distorsión para definir medidas de riesgo coherentes. Demostramos que cualquier familia de funciones de distorsión que cumpla una ecuación de traslación, puede ser representada por una función de distribución. Una aplicación se puede encontrar en la ciencia actuarial: los principios de primas basados en los momentos son fáciles de entender, pero en general no son monótonos y no se pueden utilizar para comparar los riesgos de diferentes contratos de seguros entre sí. Nuestro teorema de representación permite comparar dos riesgos de seguros entre sí de acuerdo con un principio de primas basado en un momento, definiendo adecuadamente una medida de riesgo coherente.

En el último ensayo de esta tesis, investigamos los mercados financieros con fricciones, donde los precios de compra y venta de instrumentos financieros se describen mediante funciones de precios sublineares. Estas funciones pueden definirse recursivamente utilizando medidas de riesgo coherentes. En un modelo binomial y en la presencia de costes de transacción, demostramos la convergencia de los precios de compra y venta para varias opciones europeas y americanas, en particular opciones *plain vanillas*, asiáticas, *lookback* y de barrera. Realizamos varios experimentos numéricos para confirmar los hallazgos teóricos. Aplicamos los resultados a los datos de mercado reales de las opciones plain vanilla europeas y americanas y calculamos una liquidez implícita para describir la diferencia de precios de compra y venta. Este método describe muy bien la liquidez en comparación con el enfoque clásico de describir la diferencia entre los precios de compra y venta con las volatilidades implícitas de dichos precios.

Contents

1 Introduction

Derivative pricing is an important branch of finance requiring advanced quantitative techniques. Derivatives *derive* their value from a market-based reference like a stock, an index or a foreign exchange rate. It is often criticized that derivatives are too complex to understand[1]. The present thesis attempts to bring some light into the discussion. In this thesis, several existing advanced stock price models are presented and analysed empirically. A good model might help to find a "fair" price for a derivative, which satisfies all contracting parties.

An investor holding an asset with an uncertain future cash flow like a derivative takes several risks. The *market risk* can be describe by coherent risk measures. Many coherent risk measures like the expected shortfall are represented by concave distortion functions, which are functions mapping the unity interval onto itself.

The financial crisis in 2008 led to fundamental questions about *liquidity risk*. During a crisis the market is less willing to trade, it is less liquid, the bid-ask spread of many products widens. Using coherent risk measures, we extend classical frictionless financial market models to markets with frictions (for example caused by transaction costs), where prices of assets depend on the direction of the trade. There is an ask price to buy the asset from the market and a usually lower bid price to sell the asset to the market. Such models allow to model (il)liquidity of financial markets very well as we show by several empirical studies.

The structure of this thesis is as follows: Chapter 2 introduce simple frictionless markets. We define many technical and financial terms which are used throughout the thesis. Chapter 3 analyses empirically advanced frictionless stock price market models. Chapter 4 introduces coherent risk measures and discusses representation of concave distortion functions and applications to insurance science. In Chapter 5 we define markets with frictions based on results from Chapter 4. The structure of the thesis is visualized in Figure 1.1.

In Chapter 3 we analyse advanced stock price models empirically. There is an endless list of advanced stock price models generalizing the Black-Scholes model and being able to capture many stylized facts typically observed in financial time series like fat-tail behaviour of log-returns, volatility clustering and negative correlation between volatility and stock price movements known as the *leverage effect*. There are several studies in the literature showing that many advanced stock price models can be calibrated very well to plain vanilla option data, in the sense that different models lead to almost identical

[1] See Somanathan and Nageswaran (2015) for an introduction to derivatives containing a critical view on the subject, in particular a discussion of the role of derivatives in the global financial crisis in 2008.

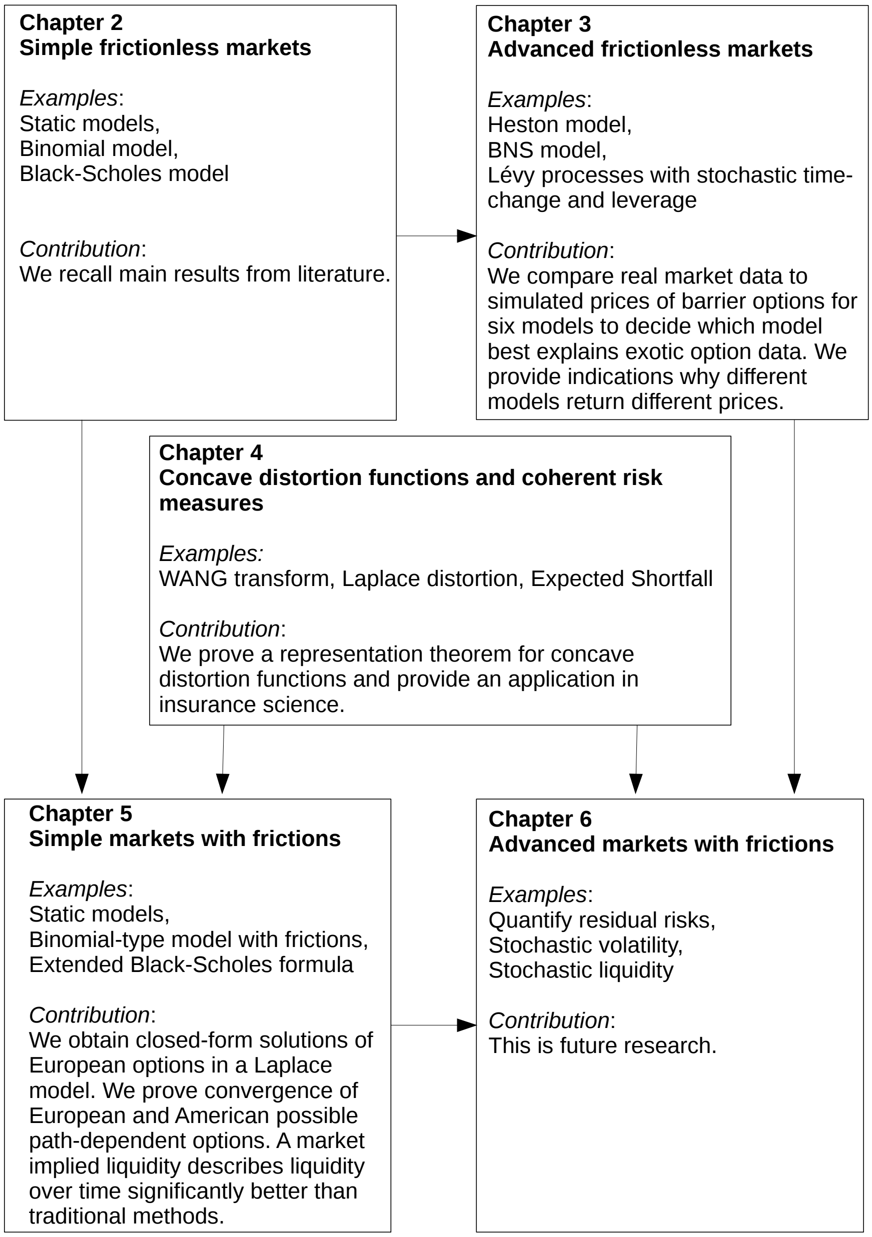

Figure 1.1: Key themes

plain vanilla prices. However, the calibrated advanced stock price models may lead to very different prices for exotic options. This is known as *model risk*. In Chapter 3 we investigate the question which model best explains exotic option data. We calibrate six advanced stock price models to a time series of real market data of European options on the DAX, a German blue chip index. Via a Monte Carlo simulation, we price barrier down-and-out call options for all models and compare the modelled prices to given real market data of the barrier options. There are three contributions in this work: a) this is the first study which focus only on models describing the volatility by a stochastic process and incorporating a leverage effect. To the best of our knowledge, this is the first time exotic option prices are simulated under Lévy processes with stochastic time-change and leverage. b) in contrast to former studies, we also compare the simulated prices to real market data of equity barrier options. Hence for our particular data set, we are able to decide which model reproduces barrier option data best. c) we provide some analysis *why* some models overvalue and other models undervalue barrier options.

We devote Chapter 4 to introduce coherent risk measures induced by concave distortion functions. Fundamental research motivates us to study the connection between concave distortion functions and distribution functions. We prove a representation theorem and show that a family of concave distortion functions satisfying a certain translation equation can be represented by a distribution function. The representation theorem helps to interpret concave distortion functions. An application of this theorem can also be found in insurance science. Premium principles in actuarial science are used to determine the initial payment, known as the *premium*, an insured has to pay to the insurance company in return for an insurance contract. For example the premium can be defined by the expected loss of the insured object plus a multiple of the standard deviation of the loss. Moment based premium principles are easy to understand but in general are not monotone and cannot be used to compare the riskiness of different insurance contracts with each other. Our representation theorem makes it possible to compare two insurance risks with each other consistent with a moment based premium principle by defining an appropriate coherent risk measure. Coherent risk measures are also a useful tool to define markets with frictions, as we will see in Chapter 5.

In Chapter 5 we turn our attention to markets with frictions. Stocks are usually traded on stock exchanges which demand some fee from the investor to trade the stock. It is therefore reasonable to assume that the prices for buying and selling the stock differ. The investor has to pay more money for purchasing the stock than she receives when selling it. We introduce markets with frictions in *discrete time*, i.e. there are only a finite number N of timepoints when the risky asset can be traded. Bid and ask prices are defined using concave distortions as discussed in Chapter 4. We look at two special cases: in the static case, $N = 1$, we obtain closed-form solutions for bid and ask prices of European options, if the log-returns are normal or Laplace distributed. We also look at the asymptotic case $N \to \infty$ and prove convergence of bid and ask prices for many American and Exotic options in a binomial-type model. We obtain closed-form solutions for bid and ask prices of European plain vanilla and barrier options. We are motivated to study the asymptotic behaviour of the model to obtain closed-form solutions for efficient numerical applications. The static case is interesting because the Laplace distribution might be better suited to model log-returns than the normal distribution, which appears asymptomatically in the binomial model. Working with real market data, we show that a market implied liquidity can be computed from a European or an American plain vanilla option surface. This implied liquidity measure describes liquidity over time significantly better than traditional methods.

Each Chapter 3, 4 and 5 is surrounded by a suitable introduction and conclusion. Chapter 6 of the thesis suggests ideas for future research. This thesis is based on journal articles which are either published or are submitted during the doctoral studies.

2 Introduction to Classical Financial Markets

This Chapter introduces some basic concepts of mathematical finance. The main notation and financial terms used in the thesis are defined and presented. Readers familiar with financial terms and concepts may just glance over this Chapter. We will mention and explain the following terms, which frequently appear in the whole thesis:

- Economic terms: underlying, stocks, equity market, stock market index, exchange traded fund, bank account, riskless and risky asset, dividend yield, interest rates, portfolio, diversification, discounting, future random cash flow, log-returns.
- Financial market classifications: frictionless markets, markets with frictions, complete and incomplete markets, static, discrete and continuous time trading.
- Arbitrage: arbitrage opportunity, arbitrage-free price, bid, ask and risk-neutral price, equivalent martingale measure, martingale, self-financing trading strategy, zero initial investment.
- Derivatives: contingent claim, call or put plain vanilla option, strike, maturity, put-call parity, moneyness, barrier, down-and-out, down-and-in, knock-out, option premium, holder, owner or buyer and writer or seller of an option, Black–Scholes formula, implied volatility, implied volatility index.
- Mathematical terms: random path, filtration, usual conditions, stochastic process, adapted and predictable process, objective function, calibration.

In Section 2.2 we formally introduce discrete time frictionless financial markets, which are technical much easier to handle than continuous time financial markets. In Section 2.3 we comment on continuous time financial markets as well. We try to answer the following questions:

i How can a financial (stock) market be modelled over time?

ii How can we (mathematically) guarantee that the market model makes economically sense? The market should work efficiently, in particular it should not be possible to make money out of nothing. The analogue of a *perpetual motion machine* in physics is called *arbitrage* in economics. We will provide some conditions to guarantee that the market model is *arbitrage-free*.

iii From a mathematical point of view not only the stock itself but, *contracts* on the stock, which change their value depending where the stock is moving, are very interesting. How do we price such contracts without introducing arbitrage opportunities in the market? If there are more than one arbitrage-free price for the contract, which one do we choose?

2.1 Assets

Financial markets usually consist of at least one *risky asset* and a bank-account. The risky assets might be stocks, commodities like oil or sugar or foreign exchange rates. The risky assets are also often called *underlyings.* The future prices of the risky assets are unpredictable and modelled by a stochastic process. The bank-account on the other hand has a totally deterministic behaviour, and is therefore also called *riskless asset.* The bank account is characterized by some fixed continuously compounded *interest rate* $r > -1$: an investor can deposit any amount of money and one currency unit will grow to e^{rt} after t years. t is measured in fractions of a year, if the investor deposits money for one month we have $t = \frac{1}{12}$. She can also borrow money from the bank and for each currency united borrowed, she has to return e^{rT} to the bank. Any model has to abstract from reality and assuming that the interest rates and borrowing rates are equal and independent of the time-horizon is such a simplifying abstraction.

In this thesis, there is usually only one risky asset, which models a *stock* (Stocks represent a partial ownership of a company like Apple Inc., CaixaBank S.A. or Siemens AG) or an index on stocks. Markets where stock can be traded are called *equity markets.* A *stock market index* represents a basket of stocks, also called *portfolio,* and is computed as an weighted average from the prices of selected stocks. For example the S&P 500 Index is based on the market capitalizations of 500 large American companies. The DAX 30 on the other hand consists of 30 mayor German companies. For many indices there are funds, called *exchange traded funds* (ETF), replicating the index. For example the SPDR S&P 500 ETF Trust replicates the S&P 500 index. It contains the same stocks in the same ratio as the S&P 500 index and can be traded as a usual stock. Funds replicating the S&P 500 are examples of *diversified* portfolios because they do not focus on a single asset or single line of business but rather invest in a variety of different stocks and reduce thereby the exposure to any particular stock.

Companies may pay dividends to its shareholders. This may happen if the company earns a profit and does not re-invest the profit. Simplifying, we assume that the stock pays a continuous annualized *dividend yield* at rate q, i.e. a stock paying dividends is modelled by

$$S_t = e^{-qt}\bar{S}_t,$$

where $\left(\bar{S}_t\right)$ describes the stock price process, if the dividend would be reinvested in the company, see Schoutens (2003, Section 2.6) for details.

2.2 Discrete Time Market Model

In this Section we introduce financial markets in discrete time. We follow closely Föllmer and Schied (2011, Section 5) and Schoutens (2003, Section 1 and 2).

Let $T > 0$ be some time-horizon. Let $N \in \mathbb{N}$ be the number of possible trading periods in the interval $[0, T]$. Trading can take place at the timepoints

$$\left\{0, \frac{T}{N}, ..., \frac{T(N-1)}{N}, T\right\},$$

i.e. the interval $[0, T]$ is divided into N equidistant time steps. There are no trading restrictions: stocks can be bought and sold for the same price in any quantities. The time-horizon T is usually measured in fractions of a year and one trading period might for example correspond to a day, an hour or just a second. The special case $N = 1$ is called a *static market model*, if $N > 1$ we speak of a *multiperiod* or *discrete market model*. In a static model, trading can take place only twice: today and at the end of the time-horizon.

We assume there are $d + 1$ assets (one bank account and d risky assets). We model the k^{th} asset by a discrete time stochastic process, which is a collection of random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over the index set $\{0, .., N\}$ and denoted by $S^k = \left(S_i^k\right)_{i=0,..,N}$. Every $\omega \in \Omega$ corresponds to a particular scenario of market evolution and $S_i^k(\omega)$ is the price of the k^{th} asset after i trading periods if scenario ω occurs.

For a fixed trading period i, the map $S_i^k : \Omega \to [0, \infty)$ is a random variable and for a fixed $\omega \in \Omega$, the map

$$\begin{array}{rcl} \{0, .., N\} & \to & [0, \infty) \\ i & \mapsto & S_i^k(\omega) \end{array}$$

is called a *realization of a random path* of asset S^k. Assets $S^1, ..., S^d$ represent risky assets, for example different stocks. We assume the asset S^0 is a riskless bank account and can be modelled by the deterministic process

$$S_i^0 = e^{r\frac{iT}{N}}, \;\; i = 0, ..., N,$$

where r are some interest rates. Today the prices of all assets are known and deterministic, we denote them by the tuple $\left(S_0^0, ..., S_0^k\right) \in \mathbb{R}_+^{d+1}$. The prices of the risky assets at trading period $i \in \{1, .., N\}$ are modelled by nonnegative random variables S_i^k, $k = 1, .., d$. The random variable S_i^k is assumed to be measurable with respect to a $\sigma-$Algebra $\mathcal{F}_i \subset \mathcal{F}$. One should think of $\mathcal{F}_i$ as information available after i trading periods. It is therefore assumed that

$$\mathcal{F}_i \subset \mathcal{F}_j \subset \mathcal{F} \text{ for } i < j, \;\; i, j \in \{0, .., N\} \text{ and } \mathcal{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{F}_N = \mathcal{F}.$$

The family $(\mathcal{F}_i)_{i=0,...,N}$ is called *filtration* and models the knowledge of an investor over time. At period i the investors only knows $\mathcal{F}_i$. Let $I \subset \mathbb{R}$ be some interval. After i trading periods, the investor knows sets like

$$A := \{\omega \in \Omega : S_i^k(\omega) \in I\}$$

because the fact that S_i^k is $\mathcal{F}_i-$measurable means $A \in \mathcal{F}_i$, i.e. the investor is able to decide if the price of the k^{th} asset lays within the interval I. Sets like

$$\{\omega \in \Omega : S_{i+1}^k(\omega) \in I\}$$

on the other hand are elements of $\mathcal{F}_{i+1}$ and are not yet known to the investor. This means after i trading periods, the history of the stock prices and the present stock prices, i.e. the stock prices at period $0,..,i$ are known to the investors but future stock prices at periods $i+1,...,N$ are unknown.

Definition 2.2.1. A stochastic process $(Y_i)_{i=0,..,N}$ is called *adapted* with respect to the filtration $(\mathcal{F}_i)_{i=0,..,N}$ if each Y_i is $\mathcal{F}_i-$measurable.

The risky assets are modelled by adapted processes. We denote the collection of the riskless and the risky asset by the $d+1$ dimensional stochastic process

$$\overline{S} = \left(\overline{S}_i\right)_{i=0,...,N} = \left(S_i^0, ..., S_i^d\right)_{i=0,..,N}.$$

S_N^k describes the price of the risky asset S^k at time T. If $d=1$ and ambiguity can be excluded we also may write S_T instead of S_N^1 in particular in static time financial markets.

In order to compare different assets over time, investors often compare their relative price changes, which leads to the following Definition:

Definition 2.2.2. The *return* of asset S^k between the periods $i<j$ is defined by

$$\frac{S_j^k - S_i^k}{S_i^k}.$$

The *log-return* is defined by

$$\log\left(\frac{S_j^k}{S_i^k}\right).$$

Often it is more convenient to work with log-returns. Many stock price models are defined by the exponential of some stochastic process. Using log-returns is then a natural choice. Furthermore log-returns can be added up over several periods. The log-return between period i and $i+2$ is the same as the sum of the log-returns between periods i and $i+1$ and $i+1$ and $i+2$.

2.2.1 Trading Strategies

Next we define a trading strategy, i.e. a way to invest in the different risky assets. At the beginning of a trading period, an investor has to decide which stock to sell and which to buy. Hence the investors knows at the beginning of each period which stock she will hold in which amount at the end of the period. Trading strategies are plannable or predictable. Before we define a trading strategy, let us define what is meant by a predictable process.

Definition 2.2.3. A process $(Z_i)_{i=1,..,N}$ is called *predictable* with respect to the filtration $(\mathcal{F}_i)_{i=0,..,N-1}$, if Z_i is $\mathcal{F}_{i-1}-$measurable for $i = 1, .., N$.

At time $t = 0$ the investor may choose a *portfolio*, i.e. numbers

$$\overline{\xi}_1 = (\xi_1^0, ..., \xi_1^d) \in \mathbb{R}^{d+1},$$

which correspond to quantity of shares invested in the various assets. If ξ_1^0 is negative, she borrows $|\xi_1^0|$ currency units from the bank otherwise she makes a deposit. Similarly if ξ_1^k is positive, she buys asset S^k, otherwise she sells it. If $\xi^k = 0$ she does not trade asset S^k. The amount of money invested in asset S^k today is $\xi_1^k S_0^k$. After one period, i.e. at time $t = \frac{T}{N}$, her portfolio has the (random) value

$$\overline{\xi}_1 \cdot \overline{S}_1 = \sum_{k=0}^{d} \xi_1^k S_1^k.$$

After the first trading period she may rearrange the portfolio, which leads to the definition of a trading strategy:

Definition 2.2.4. A *trading strategy* is a predictable $\mathbb{R}^{d+1}$ valued process

$$\overline{\xi} = (\xi_i^0, ..., \xi_i^d)_{i=1,...,N}.$$

The values

$$\overline{\xi}_i = \left(\xi_i^0, ..., \xi_i^d\right)$$

correspond to the quantity of shares held during period i, i.e. between the timepoints $\frac{(i-1)T}{N}$ and $\frac{iT}{N}$.

Definition 2.2.5. A trading strategy is called *self-financing* if

$$\overline{\xi}_i \cdot \overline{S}_i = \overline{\xi}_{i+1} \cdot \overline{S}_i, \ \ i = 1, ..., N-1.$$

At period $i-1$ we invested some amount of money which is worth $\overline{\xi}_i \cdot \overline{S}_i$ at period i. At period i, we then rearrange the portfolio in such a way that the value of the portfolio does not change. A self-financing trading strategy is a way to invest in the market without exogenous infusion or withdrawal of money except for an initial investment; the purchase of new assets must be financed by the sale of old ones.

Example 2.2.6. Assume $d = 1$, i.e. there is just one risky asset. If the investor has no own capital, she can still trade. For example, she could simply set

$$\xi_1^0 = -S_0^1 \text{ and } \xi_1^1 = 1,$$

i.e. finance the purchase of the stock completely by debts. At time T she owns

$$-S_0^1 e^{rT} + S_N^1$$

currency units, if she does not change her portfolio any further, i.e

$$\xi_i^0 = S_0^1 \text{ and } \xi_i^1 = 1, i = 0, .., N.$$

Such a strategy is a self-financing strategy with *zero initial investment.*

2.2.2 Discounting

Definition 2.2.7. Let Z be a $\mathcal{F}-$measurable random variable. A contract which promises to pay $Z(\omega)$ at time T, if the event ω occurs is called a *future random cash flow.*

Mathematically, a future random cash flow is just a random variable. But it has some economic meaning, it describes the movement of money from one counterparty of the contract to another. The amount of money

$$Z = -S_0^1 e^{rT} + S_N^1$$

an investor receives by following the strategy described in Example 2.2.6 is a future random cash flow. It can be negative, in which case the investor makes a loss.

Remark 2.2.8. One currency unit today grows to e^{rt} currency units at timepoint t. An investor is therefore indifferent to receive $x > 0$ currency units at time t or xe^{-rt} currency units today. If she received xe^{-rt} currency units today, she could deposit the money at the bank account and obtain x currency units at time t. If Z is a random variable modelling some future random cash flow promised to be paid at time t, the value $e^{-rt}Z$ is called the *discounted value* of Z. Discounting is necessary in order to be able to compare today two future random cash flows, which are promised to be paid at different future timepoints.

The *discounted price process of the risky asset*s are defined by

$$X_i^k := \frac{S_i^k}{S_i^0}, ...i = 0, ..., N, \quad k = 0, ..., d.$$

The discounted price process describing the bank account is equal to

$$\frac{S_i^0}{S_i^0} = 1$$

all times. We summarize the discounted price processes in a vector

$$\overline{X} = (X^0, ..., X^d).$$

Definition 2.2.9. The *discounted value process* $V = (V_i)_{i=0,..,N}$ associated with a trading strategy $\overline{\xi}$ is given by

$$V_0 := \overline{\xi}_1 \cdot \overline{S}_0 \text{ and } V_i := \overline{\xi}_i \cdot \overline{X}_i, \ \ i = 1, ..., N.$$

The *discounted gains process* associated with $\overline{\xi}$ is defined as

$$G_0 := 0 \text{ and } G_i := \sum_{j=1}^{i} \overline{\xi}_j \cdot \left(\overline{X}_j - \overline{X}_{j-1} \right), \ \ i = 1, ..., N. \tag{2.1}$$

The value V_i can be interpreted as the portfolio value after i trading periods. The gain process represents the net gains which have accumulated through the trading strategy $\overline{\xi}$ after i trading periods. The sum in Equation (2.1) could be seen as a discrete version of a stochastic integral, compare with Equation 2.5. It can be shown that a strategy is self-financing, if and only if

$$V_i = V_0 + G_i, \ \ i = 0, ..., N, \tag{2.2}$$

see Föllmer and Schied (2011, Proposition 5.7).

2.2.3 Arbitrage Opportunities

In classical finance the absence of arbitrage is a standard assumption. Arbitrage opportunities are trading strategies with zero initial investment, without the possibility of losing money but with some chance to make a positive return and are precisely defined in the next Definition:

Definition 2.2.10. An *arbitrage opportunity* is a self-financing trading strategy whose value process satisfies

$$V_0 \leq 0, \ \ V_N \geq 0 \ \ \mathbb{P} - a.s., \ \text{ and } \ \mathbb{P}(V_N > 0) > 0.$$

The trading strategy mentioned in Example 2.2.6 is *not* an arbitrage opportunity, if

$$\mathbb{P}\left(S_N^1 < S_0^1 e^{rT} \right) > 0,$$

which is usually the case. The very essence of a stock and its risky structure is the fact that it might perform worse than a bank account. The existence of an arbitrage opportunity in a financial market can be regarded as a market inefficiency in the sense that the risky asset is not priced in a reasonable way. We say the market is *arbitrage-free* if there do not exist any arbitrage opportunities. From an economic point of view it is therefore necessary to prove that the market model we are working with is arbitrage free. We have the following result, which is known in literature as *the first fundamental theorem of asset pricing*. Before we state it, we introduce the notation of an equivalent martingale measure.

Definition 2.2.11. A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called an *equivalent martingale measure* if $\mathbb{Q}$ is equivalent to $\mathbb{P}$, i.e.

$$\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0, \;\; A \in \mathcal{F},$$

and the discounted price process $\overline{X}$ is a $\mathbb{Q}-$*martingale*, i.e.

$$E_{\mathbb{Q}}[\left|X_i^k\right|] < \infty \;\text{ and }\; X_i^k = E_{\mathbb{Q}}[X_j^k \middle| \mathcal{F}_i], \;\; 0 \le i \le j \le N, \;\; k = 0, .., d. \tag{2.3}$$

The term $E_{\mathbb{Q}}[X_j^k \big| \mathcal{F}_i]$ denotes the conditional expectation of X_j^k under the measure $\mathbb{Q}$ conditioned on the $\sigma-$Algebra $\mathcal{F}_i$. In particular it holds

$$S_0^k = e^{-rT} E_{\mathbb{Q}}[S_N^k], \;\; k = 0, ..., d,$$

or vectorized

$$\overline{S}_0 = e^{-rT} E_{\mathbb{Q}}[\overline{S}_N] = E_{\mathbb{Q}}[\overline{X}_N], \tag{2.4}$$

i.e. the prices of the risky assets can be seen as the expected values of (discounted) future stock prices. This sounds like a fair game. Indeed an equivalent martingale measure can be seen as an artificial measure under which the discounted stock price process behaves like a fair game. Under the *real world measure* $\mathbb{P}$ the stock might behave quite differently: $\mathbb{P}$ and $\mathbb{Q}$ only have the same Null-sets. An equivalent martingale measure is also called a *pricing measure* or an *equivalent risk-neutral measure*, because this measure can be used for pricing and does not take any risk preferences of investors into consideration. The usefulness becomes clear in Theorem 2.2.12 and Theorem 2.2.17. Let $\mathcal{Q}$ denote the set of all equivalent martingale measures.

Theorem 2.2.12. *The market is arbitrage-free if the set of equivalent martingale measures $\mathcal{Q}$ is not empty.*

Due to the fundamental importance of this theorem, we highlight the main idea of the proof:

Proof. We only show the static case $N = 1$. Assume $\mathcal{Q} \neq \emptyset$ and let $\mathbb{Q} \in \mathcal{Q}$ be an equivalent martingale measure. For a trading strategy $\overline{\xi}$ let $V_N = \overline{\xi}_1 \cdot \overline{X}_N$ and assume $V_N \geq 0$ $\mathbb{P}-$a.s. and $\mathbb{P}(V_N > 0) > 0$. Both properties remain valid if we replace $\mathbb{P}$ by $\mathbb{Q}$ because the measures are equivalent to each other. Then it follows

$$V_0 = \overline{\xi}_1 \cdot \overline{S}_0 = \overline{\xi}_1 \cdot E_{\mathbb{Q}}\left[\overline{X}_N\right] = E_{\mathbb{Q}}\left[V_N\right] > 0,$$

see Equation (2.4), i.e. $\overline{\xi}$ is not an arbitrage opportunity. □

The converse also holds true but is much harder to prove, i.e. the absence of arbitrage guarantees the existence of at least one equivalent martingale measure. The complete proof can be found in most textbooks about financial mathematics, see e.g. Föllmer and Schied (2011, Theorem 5.16).

2.2.4 European Contingent Claims

Definition 2.2.13. A nonnegative future random cash flow, i.e. a nonnegative random variable C on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *European contingent claim.*

A contingent claim C is a contract. The *holder, owner* or *buyer* of the contract receives the nonnegative amount $C(\omega)$ from the *writer* or *seller* of the claim at timepoint T if the event ω occurs. Today, the holder has to pay a non-stochastic *premium* $\pi \geq 0$ to the writer to receive the contract C in return. Many contingent claims contain some optionality and are therefore also called *options.*

Example 2.2.14. The holder of a *European plain vanilla call option with maturity T on asset S^k* has the right, but not the obligation, to buy asset S^k at time T for a fixed price K, called the *strike price.* This contract can be described by the random variable

$$C_{\text{Call}} = \max(S_N^k - K, 0) =: (S_N^k - K)^+.$$

A contract called *European plain vanilla put option with strike K and maturity T on asset S^k* can be described by the random variable

$$C_{\text{Put}} := (K - S_N^k)^+.$$

The ratio between the stock price today and the strike is called *moneyness.* For call options it is defined as $\frac{S_0^k}{K}$ and for put options it is defined as $\frac{K}{S_0^k}$. We say an European plain vanilla option is *in-, at-,* and *out-of-the-money* if the moneyness is greater than 1, equal to 1 and smaller than 1, respectively. The holder of an in-the-money option would receive a positive amount from the writer of the option if the maturity of the option was today.

A *European down-and-out barrier call option with strike K and barrier B* is similar to a plain vanilla call option, but it becomes worthless, if the risky asset drops below a certain barrier B. This contract can be described by

$$C_{\text{DOCall}} = \begin{cases} (S_N^k - K)^+ & , \min_{i \in \{0,\dots,N\}} S_i > B \\ 0, & \text{otherwise.} \end{cases}$$

While a barrier option depends on the whole path of the asset S^k, a plain vanilla option only depends on the value of the risky asset at maturity, i.e. only depends on S_N^k. If the barrier is not hit, i.e. if the stock remains above B at the timepoints $0, .., N$, the holder of an ordinary plain vanilla option receives the same amount of money as the holder of a barrier option. When the barrier is hit, it becomes worthless and disappears from the market. In this case we say the barrier option is *knocked-out.*

Another example of a barrier option is a *European down-and-in barrier call option with strike K and barrier B*, which can be described by

$$C_{\text{DICall}} = \begin{cases} (S_N^k - K)^+ & , \min_{i \in \{0,\dots,N\}} S_i < B \\ 0, & \text{otherwise.} \end{cases}$$

This barrier option only returns a positive cash flow at maturity under the condition that the barrier is at least once hit by the stock price process.

Remark 2.2.15. How are options used? For example for hedging purposes: an airline sells flight tickets today for flights taking place in a year. We therefore set $T = 1$. The airline of course does not know the price S_T of jet fuel, which need to be bought from some oil company in a year. The airline will probably get into trouble if S_T is much higher then today's jet fuel prices S_0. Therefore the airline might be interested to buy today an *insurance protecting against high fuel prices*, i.e. for example a simple at-the-money European plain vanilla call option on a certain amount of jet fuel with maturity T and strike $K := S_0$. The airline then has the right at time T to buy jet fuel for the price K instead of S_T and will execute this right if $S_T > K$. The airline has to pay some premium π to the option seller. In this Section we answer the question, *how to compute the premium of European contingent claims.* Of course the airline is interested in paying as little as possible for the insurance against high jet fuel prices. This is one reason for the existence of barrier options: they are cheaper than plain vanilla options (but also offer less protection).

For an broad introduction to contingent claims and its use in finance and economics, see Hull (2017). It should be mentioned that contingent claims, which are also called derivatives, can be used for purely speculative (gambling) purposes, too. Assume an investor thinks a stock, which costs $S_0 = 100$ currency units today, will rise by 5% after one year. Assume she is correct, then she could make a return of 5% by investing directly in the stock.

A plain vanilla European call option on the stock with strike $K = 100$ could cost about 10 currency units today. Buying ten options for 100 currency units instead of the stock, she makes a return of

$$\frac{(105 - 100) \times 10}{100} = 50\%,$$

if her prediction about the future stock price is correct. But if the stock costs, say 99 currency units in a year, she loses 1% of her initial investment when buying the stock directly, but 100% when betting on the stock using options. There are certainly many situations for companies to use options to protect against unpredictable events, but using options for speculative purposes is very risky.

Usually we know the contract C, see Example 2.2.14, and have to find a price π for C. There is a minimum requirement on π: the *extended* market model containing the assets $S^0, ..., S^d$ and C still should be arbitrage-free. This leads to the following definition.

Definition 2.2.16. A real number $\pi \geq 0$ is called an *arbitrage-free price* of discounted contingent claim

$$H := e^{-rT}C,$$

if there is a nonnegative adapted process X^{d+1} such that

$$X_0^{d+1} = \pi \text{ and } X_N^{d+1} = H$$

and the extended market model $(X^0, ..., X^d, X^{d+1})$ is arbitrage-free.

The following theorem shows how to find an arbitrage-free price of C.

Theorem 2.2.17. *Let $\mathbb{Q}$ be an equivalent martingale measure for the original market consisting of the assets $(X^0, ..., X^d)$. Let C be a contingent claim. Assume*

$$\pi := e^{-rT} E_{\mathbb{Q}}[C] < \infty.$$

The price π is an arbitrage-free price for C.

Proof. Define

$$X_i^{d+1} := e^{-rT} E_{\mathbb{Q}}[C|\,\mathcal{F}_i], \;\; i = 0, .., N.$$

Then it follows

$$\pi = X_0^{d+1} \;\text{ and }\; e^{-rT} C = X_N^{d+1}$$

As $\mathbb{Q}$ an equivalent martingale measure for the original market and X^{d+1} is by definition a $\mathbb{Q}-$martingale, $\mathbb{Q}$ is also an equivalent martingale measure for the extended market model, which is hence by Theorem 2.2.12 arbitrage-free. The proof is taken from Föllmer and Schied (2011, Theorem 5.29). □

2.2.5 Market Incompleteness

Most financial markets are *incomplete*, i.e. $\mathcal{Q}$ contains more then one element, otherwise the market is called *complete*. The binomial model, see Cox, Ross and Rubinstein (1979) in discrete time and the Black-Scholes model in continuous time, see Black and Scholes (1973), are prominent exceptions of complete markets. In the incomplete case two different equivalent martingale measures $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}$ might lead to different prices for a contingent claim C, i.e

$$\pi_1 := e^{-rT} E_{\mathbb{Q}_1}[C] \neq \pi_2 := e^{-rT} E_{\mathbb{Q}_2}[C].$$

By Föllmer and Schied (Theorem, 5.29), lower and upper bounds of possible arbitrage-free prices for C are given by

$$\pi_{\text{inf}} := \inf_{\mathbb{Q}\in\mathcal{Q}} e^{-rT} E_{\mathbb{Q}}[C] \;\text{ and }\; \pi_{\text{sup}} := \sup_{\mathbb{Q}\in\mathcal{Q}} e^{-rT} E_{\mathbb{Q}}[C].$$

The interval $(\pi_{\text{inf}}, \pi_{\text{sup}})$ might be very large and it is unclear which price

$$\pi \in (\pi_{\text{inf}}, \pi_{\text{sup}})$$

is an adequate price for C.

We cannot simultaneously price contingent claim C_A using equivalent martingale measure $\mathbb{Q}_A$ and price another contingent claim C_B using a different equivalent martingale measure $\mathbb{Q}_B$ without running into the danger of introducing arbitrage into the market, i.e. we need to price all contingent claims consistently by the same equivalent martingale measure.

Example 2.2.18. For any real numbers x, y, it holds

$$(x-y)^+ - (y-x)^+ = x - y.$$

Hence European put and call options with the same strike K and maturity T are related with each other by

$$C_{\text{Call}} - C_{\text{Put}} = S_N^k - K.$$

Discounting and taking expectations under any $\mathbb{Q} \in \mathcal{Q}$ it follows

$$\pi_{\text{Call}} - \pi_{\text{Put}} = S_0^k - e^{-rT} K,$$

which is known as the *put-call-parity.* The arbitrage-free price of the put option is totally determined by the price of the call option and vice versa, independently of the chosen equivalent martingale measure.

To identify this measure, we can use other information available in financial markets: on many stocks, in particular on stock indices, there exist an option chain, e.g. a set of plain vanilla put and call options with different strikes and maturities on the stock index with *known prices.* Those prices are determined by supply and demand similar to the prices of ordinary stocks. Generally speaking, both academics and practitioners use such option chains to choose a particular equivalent martingale measure $\mathbb{Q}$. In Chapter 5 parametric stock prices models are calibrated to given market prices of plain vanilla options by minimizing the mean-square error between model and market prices. This procedure is for example described by Bakshi et al. (1997). Let us assume there is a set of parameters $\Theta \subset \mathbb{R}^n$ and a bijective function

$$f : \Theta \to \mathcal{Q},$$

i.e. each equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}$ can be identified with a parameter $\theta \in \Theta$. Let us further assume there are $M > n$ plain vanilla call and put options $C_1, ..., C_M$ with known market prices $\pi_{C^1}, ..., \pi_{C^M}$ and maturities $T_1, ..., T_M$. We minimize the mean-square error between market and model prices, i.e. we minimize the *objective function*

$$\theta \mapsto \sqrt{\sum_{m=1}^{M} \frac{\left(\pi_{C_m} - e^{-rT_m} E_{f(\theta)}[C_m]\right)^2}{M}}.$$

Let us denote the minimum by $\hat{\theta}$. The existence and uniqueness of a global minimum is a delicate numerical issue and further discussed in Chapter 3. The thereby uniquely identified equivalent martingale measure $\hat{\mathbb{Q}} := f(\hat{\theta})$ can then be chosen to price other plain vanilla options with unknown prices or complex contracts, such as barrier options. This means we compute the price of a contract (with unknown price) consistently with current market prices of a certain set of plain vanilla options. No historic data is used, only present market data. We do this by taking expectations and expectations are linear, therefore the operator

$$\begin{aligned} p_{\mathbb{Q}} : L^1(\Omega, \mathcal{F}, \mathbb{Q}) &\to [0, \infty) \\ C &\mapsto e^{-rT} E_{\mathbb{Q}}[C] \end{aligned}$$

assigning a price to a contingent claim C is also called *linear pricing rule* or *risk-neutral price operator with respect to the equivalent martingale measure* $\mathbb{Q}$. We call the value $e^{-rT}E_{\mathbb{Q}}[C]$ the *risk-neutral price of* C *with respect to the measure* $\mathbb{Q}$.

In Chapter 5 we look at markets with frictions and introduce *bid and ask prices* for C. The ask price is the price an investor has to pay to purchase C and the investor only receives the usually lower bid price when selling C. The risk-neutral price of C lays between the bid and the ask price.

2.3 Continuous Time Finance

The notation and many of the results of stochastic finance in discrete time can be passed to finance in continuous time. See Björk (2009), Bingham and Kiesel (2013), Musiela and Rutkowski (1997) and Karatzas and Shreve (1998) for an introduction to stochastic finance in continuous time. See Protter (2001) for a compact introduction to mathematical finance using semi-martingales. In this section, we very briefly provide the main notation as a continuous-time extension of Section 2.2.

In continuous-time finance, we work with a filtered probability

$$\left(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$$

that satisfies the usual conditions, i.e.

1. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $s \leq t$.
2. $\mathcal{F}_0$ contains all the $\mathbb{P}-$null sets of $\mathcal{F}$.
3. The filtration is right-continuous: $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$ for all $0 \leq t < 0$.

As in discrete time, there are $d+1$ assets with price processes

$$\overline{S} = \left(\overline{S}_t\right)_{t\in[0,T]} = \left(S_t^0, ..., S_t^d\right)_{t\in[0,T]},$$

where

$$S_t^0 := e^{rt}, \ \ t \in [0,T],$$

models a riskless bank account in continuous time and $S^1, ..., S^d$ are semimartingales with càdlàg (right-continuous with left limits) paths. The $d+1$ random variable $\overline{S}_t$ is $\mathcal{F}_t-$measurable, i.e. the process $\overline{S}$ is adapted. For example Lèvy-processes are semi-martingales, see Protter (2004, Section 3, Theorem 9). The process

$$\overline{X} = \left(\overline{X}\right)_{t\in[0,T]} := \left(\frac{S_t^0}{S_t^0}, ..., \frac{S_t^d}{S_t^0}\right),$$

is called the discounted price process.

A trading strategy

$$\overline{\xi} = \left(\overline{\xi}_t\right)_{t\in[0,T]} = \left(\xi_t^0, ..., \xi_t^d\right)_{t\in[0,T]}$$

is an adapted càglàd (left-continuous with right limits) process. ξ_t^k represents the quantities of shares in asset S^k at time t. The value-process $(V_t)_{t\in[0,T]}$ associated with a trading strategy $\overline{\xi}$ is exactly defined as in the discrete case, i.e.

$$V_t := \overline{\xi}_t \cdot \overline{X}_t, \;\; t \in [0,1].$$

The gain-process associated with a trading strategy $\overline{\xi}$ is defined using the notation of a *stochastic integral*, see Protter (2004) for a straightforward introduction to stochastic integration and differentiation: for an adapted, càglàd process H and a semimartingale S, Protter (2001, 2004) defines a stochastic integral by

$$\int_0^t H_u dS_u := \lim_{n\to\infty} \sum_{t_i \in \pi^n[0,t]} H_{t_i} \Delta_i S, \tag{2.5}$$

where $\pi^n[0,t]$ is a sequence of partitions of $[0,t]$ with mesh tending to 0 as $n \to \infty$. Protter defines $\Delta_i S := S_{t_{i+1}} - S_{t_i}$ and the convergence is in u.c.p (uniform in time on compacts and converging in probability). Protter defines semimartingales as the set of processes, for which the limit of such sums exists. Using this notation, we define the gain process in continuous time:

$$G_t := \int\limits_0^t \overline{\xi}_u dS_u = \sum_{k=0}^{d} \int_0^t \xi_u^i dS_u^i,$$

which can be seen as a continuous version of Equation (2.1). A trading strategy is then called self-financing, if

$$V_t = V_0 + G_t, t \in [0,T],$$

compare with Equation (2.2).

An arbitrage opportunity is defined as in discrete time. It is a self-financing trading strategy $\overline{\xi}$ such that $V_0 = 0$ and

$$V_T \geq 0 \, \mathbb{P} - \text{a.s. and } \mathbb{P}\,(V_T > 0) > 0.$$

A measure $\mathbb{Q}$ is called an equivalent martingale measure if it is equivalent to $\mathbb{P}$ and $\overline{X}$ is a local martingale (martingales are also local martingales) under $\mathbb{Q}$. See for example Protter (2004, Section 6) for a definition of a local martingale.

Theorem 2.2.12 can be passed from discrete to continuous time: the market is arbitrage-free, if there exists an equivalent martingale measure. In continuous time, the converse is not true. Only the stronger assumption of *no free lunch with vanishing risk*, which can be seen as “approximately” arbitrage opportunities, guarantees the existence of at least one equivalent martingale measure, see Delbaen and Schachermayer (1994).

As in discrete time, a contingent claim C described by a $\mathcal{F}-$measurable random variable can be introduced in the original market. The question arises how to find a price π for C, which does not introduce arbitrage. Fortunately, also Theorem 2.2.17 can be passed to continuous time: if

$$\pi := e^{-rT} E_{\mathbb{Q}}[C] < \infty,$$

then π is an arbitrage-free price of C.

Example 2.3.1. The Black-Scholes model. Black and Scholes (1973) modelled the risky asset in continuous time via a geometric Brownian motion. We define

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}, \ \ t \geq 0, \tag{2.6}$$

where $(W_t)_{t\geq 0}$ is an adapted standard Brownian motion, r describes interest rate of the risk-free bank account and $\sigma > 0$ is equal to the standard deviation of the log-returns of the stock price at $t = 1$. σ is called *annual volatility* or just *volatility.* The process $(S_t)_{t\geq 0}$ is a martingale, we present in Equation (2.6) the Black-Scholes model as described under the equivalent martingale measure $\mathbb{Q}$. As a nice feature of this model, there exist closed-form solutions of European plain vanilla put and call options. For example let C_{Call} be a call option with maturity T and strike K on the risky asset $(S_t)_{t\geq 0}$, i.e. $C_{\text{Call}} = (S_T - K)^+$. An arbitrage-free price π_{BS} of C_{Call} is

$$\begin{aligned} \pi_{\text{BS}}(S_0, K, r, T, \sigma) &= e^{-rT} \int_{\Omega} (S_T - K)^+ d\mathbb{Q} \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\mathbb{R}} (S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}y} - K)^+ e^{-\frac{y^2}{2}} dy & (2.7) \\ &= S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2), & (2.8) \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$. The step from Equation (2.7) to (2.8) is simple calculus, see e.g. Föllmer and Schied (2011, Example 5.56). The closed-form solution in Equation (2.8) is called the *Black–Scholes formula for the price of a call option.* There is a similar formula for put options.

The Black-Scholes formula is usually not directly used for pricing but as a transformation between volatilities and option prices: for a European plain vanilla option with known market price π_{market}, one can invert the Black-Scholes formula and compute numerically a σ_{implied} such that

$$\pi_{\text{market}} = \pi_{\text{BS}}(S_0, K, r, T, \sigma_{\text{implied}}).$$

σ_{implied} is called *implied Black-Scholes volatility* or just *implied volatility* of the option. The values r and S_0 can be observed in the market and the values K and T are defined by the contract of the option. Given many different call and put options with different strikes and maturities, one can construct an *implied volatility surface* as a three-dimensional plot on the grid spanned by the strikes and maturities.

Remark 2.3.2. In industry, traders usually prefer quoting the implied volatility instead of the option prices, because the implied volatility is comparable across strikes, maturities and underlying assets. Historically at-the-money index options, i.e. options on a stock market index, have been used to compute an *implied volatility index* via the Black-Scholes formula. An implied volatility index is also called *fear index* because a high implied volatility expresses the expectation of market participation of possible large up or down moves of the stock market index. Nowadays model-free methods have been developed. For example the CBOE Volatility Index (VIX) is implied by S&P 500 index options by aggregating the weighted prices of puts and calls over a certain range of strike prices, see CBOE (2018).

3 Performance of Advanced Stock Price Models when it becomes Exotic: an Empirical Study

3.1 Introduction

In this Chapter, we analyse six advanced stock price models generalizing the famous Black-Scholes (BS) model as introduced by Black and Scholes (1973). The BS model describes a stock price process by a geometric Brownian motion with constant volatility.

Unfortunately, log-returns of stock prices are modelled very poorly by the normal distribution. Usually log-returns exhibit some fat tail behaviour and are skewed. Furthermore there are periods with high log-return variance and periods with low log-return variance. Hence volatility of log returns are not constant over time but there is some volatility clustering, which motivates to model volatility by a stochastic process. Also there exist some negative correlation between log-returns and volatility, known as the *leverage effect*: if volatility and uncertainty in a financial market increases, stock prices tend to decrease, see Cont (2001) for a more extended list of stylized facts of financial times series.

There is an endless list of models generalizing the Black-Scholes model and incorporating some or all of the stylized facts we just mentioned. The constant elasticity of variance model has been introduced by Cox (1975). The variance is not modelled by a stochastic process but the model captures the leverage effect. Local volatility models replace the constant volatility of the BS model by a deterministic function of both time and current underlying level, see Dupire (1994). The Heston model, see Heston (1993), replaces the volatility of the BS model by a mean-reverting stochastic process, the square root process of Cox, Ingersoll, and Ross (1985), which is allowed to be correlated with the uncertainty driving the log-returns. The Heston model is hence able to model both stochastic volatility and the leverage effect. Bates (1996) generalized the Heston model incorporating the possibility of jumps of the stock price. Barndorff-Nielsen and Shephard (2001) developed the BNS model replacing the constant volatility of the BS model by an Ornstein-Uhlenbeck process. It is possible to integrate a leverage effect into the BNS model.

One could as well replace the Brownian motion of the BS model by a more flexible Lévy process. A prominent example is the variance gamma (VG) model developed by Madan et al. (1998). Pure Lévy models are quite flexible and are able to describe skewness and fat tail behaviour of log-returns but are stationary over time. Carr et al. (2003) integrated stochastic volatility by replacing the time by an independent stochastic process. Lévy models with stochastic time-change have been described in detail by Schoutens (2003) including algorithms to simulate such processes. As in the BNS model, it is possible to allow the stochastic time change to directly effect the log-returns. We call such models *Lévy models with stochastic time-change and leverage*.

Often, a set of prices of European plain vanilla options are known and financial models are calibrated to market data of plain vanilla options. The calibrated models are then used to price exotic options or to construct trading strategies to replicate plain vanilla or exotic options as good as possible. Most of the presented models can be calibrated very well to plain vanilla option data in the sense that different models lead to almost identical plain vanilla option prices. The question is whether prices under different models of an exotic option are also approximately equal. The clear answer is *no* as the following three studies show.

i) Hirsa et al. (2003) calibrated the VG model, a local volatility model, the constant elasticity of variance model and the VG model with stochastic time-change (but without leverage) to plain vanilla options on the S&P 500 index and priced barrier options under the different models. They concluded: "regardless of the closeness of the vanilla fits to different models, prices of up-and-out call options (a simple case of exotic options) differ noticeably when different stochastic processes are used to calibrate the vanilla options surface".

ii) Schoutens et al. (2003) calibrated the Heston model (with and without jumps), the BNS model and various Lévy processes with stochastic time change (but without leverage) to plain vanilla option data on the Eurostoxx 50 index and used the calibrated models to price various exotic options among them different types of barrier options. They concluded that all those models can be calibrated almost perfectly to plain vanilla option data but the resulting exotic option price can differ significantly.

iii) Jessen and Poulsen (2013) calibrated the Black- Scholes model, the constant elasticity of variance model, the Heston model (with and without jumps), the Merton jump-diffusion model, the VG model and the VG model with stochastic time-change (but without leverage) to plain vanilla options on the USD/EUR exchange rate and priced different types of foreign exchange barrier options. They concluded: "Models may produce very similar prices of plain vanilla options yet differ markedly for exotic options."

In contrast to the two former studies i) and ii) which only worked with real data of plain vanilla options, Jessen and Poulsen (2013) compared the modelled prices of barrier options to given market data of the barrier options. For their particular data, the constant elasticity of variance model best explained the market data of barrier options, leading to an average relative error of just 0.13%. The Heston model undervalued the barrier options by 3.48% on average. The Heston model with jumps priced barrier options significantly worse than the Heston model with an average absolute error of 24.7%.

What can we add to the studies i)−iii)? To the best of our knowledge only Jessen and Poulsen (2013) compared model prices of exotic options to real market foreign-exchange data. We repeat that study for equity data. In contrast to former studies, we focus on models incorporating both stochastic volatility and the leverage effect. We think this is the first study which applies Lévy models with stochastic time-change and leverage to simulate exotic options and which compares model prices and market prices of barrier options on a stock market index. The leverage effect extends Lévy models significantly, see Section 3.2.6.

For a time-series of about 102 timepoints we are given prices of plain vanilla options: put and call options with maturities ranging from $0-3$ years and moneyness ranging from $0.5-1.0$ issues by some large international banks. At each timepoint, we calibrate six different models to the plain vanilla prices: the Heston model (HES), the Bates model (HESJ), the BNS model and three Lévy models with stochastic time-change and leverage. In this thesis we take the Normal Inverse Gaussian (NIG) process, see Barndorff-Nielsen (1997a), the VG process and the Merton jump-diffusion model (MJD), see Merton (1976), and subordinate them by a random time-change modelled by the integrated square root process of Cox, Ingersoll, and Ross (1985), abbreviated by CIR. Other choices are possible, see Carr et al. (2003). The three resulting stock price models are abbreviated by NIG_CIRL, VG_CIRL and MJD_CIRL, where 'L' stands for leverage. We chose these six models, because all those models are able to capture stochastic volatility and the leverage effect and are flexible enough to model plain vanilla options reasonably well, see Schoutens et al. (2003). They are further mathematically tractable, can be calibrated relatively fast to real market data and it is straightforward to implement the models in order to perform a Monte Carlo simulation.

The practise of recalibrating the models at each timepoint is difficult to justify economically but it is an industrial standard to ensure that a model prices liquid plain vanilla options as good as possible, see e.g. Jessen and Poulsen (2013). Over time a financial market changes, new information arrives etc. which leads to the need of recalibrating the models.

Additionally we have for each timepoint prices of exotic barrier options issued by the same banks. After calibrating the six models to the plain vanilla market data, we simulate the prices of the exotic options via Monte Carlo for each model and each timepoint and compare the real market prices of the exotic options with the model prices.

Both the plain vanilla and the barrier options are issued by financial institutions which might default. In this study we do not model the credit risk of the issuers. We argue that we can neglect the default risk of the issuers because the issuers have a very high creditworthiness. By this argumentation, we follow Chen and Kensinger (1990), Chen and Sears (1990), Wasserfallen and Schenk (1996), Burth et al. (2001) and Henderson and Pearson (2011) among others. Furthermore the price of an option emitted by some issuer is directly influenced by the issuer's default risk. Therefore the calibrated models implicitly contain the credit risk already. For a direct approach to model the credit risk of the issuer of derivatives, see Hull and White (1995).

Figure 3.1 summarizes the main result of this Chapter quite well. It shows a time-series of market and model prices of one single exemplary exotic option. We see that HESJ reproduces market prices extraordinarily well. The BNS model overvalues quite a lot and HES undervalues the barrier option slightly. The Lévy models MJD_CIRL, NIG_CIRL and VG_CIRL behave similarly though undervaluing the option. The model MJD_CIRL seems to be less robust, being sometimes close and sometimes far away from the real market price.

This Chapter is structured as follows. In Section 3.2 we present the models HES, HESJ, BNS, NIG_CIRL, VG_CIRL and MJD_CIRL and recall the characteristic functions of the log stock price of the six models from the literature. In Section 3.3 we describe in detail the data set used in this empirical study. Sections 3.4 comments on the calibration and pricing procedure via Monte Carlo. Section 3.5 compares the six models, when applied to exotic option data and offers some explanation *why* some models overvalue and other models undervalue barrier options. Section 3.6 concludes.

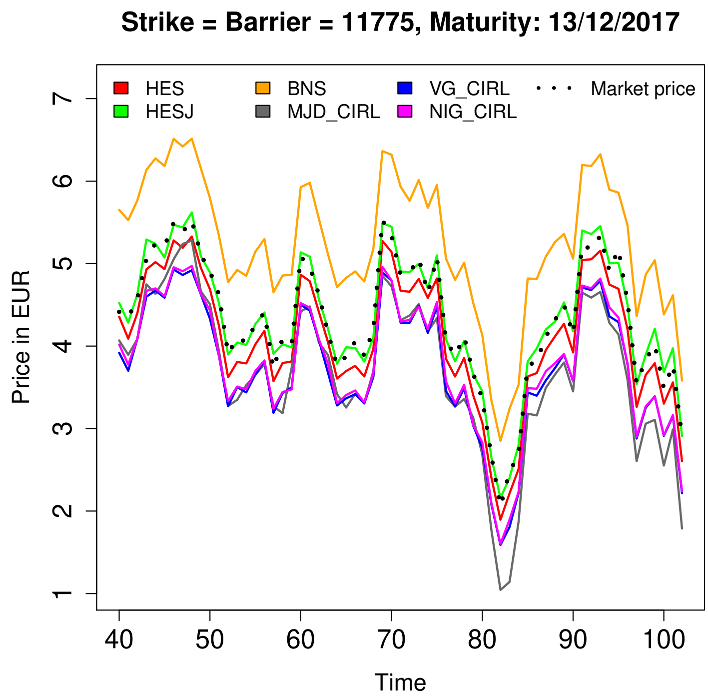

Figure 3.1: Market and model prices of an exemplary exotic option over time from 24/07/2017 till 21/08/2017. Each day consists of three timepoints. All options have a ratio of 1:100, i.e. they are entitled to receive $\frac{1}{100}$ of the difference between underlying and strike in cash at maturity if the difference is positive and the barrier is not hit.

3.2 The Models

3.2.1 Overview and Calibration

In continuous time finance the risky-asset is modelled by some stochastic process

$$S = (S_t)_{t \geq 0}$$

on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. The market is arbitrage-free, if there exists an equivalent martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. $\mathbb{P}$ is called *the real world*, *the subjective, the historical, the physical* or *the statistical measure*. We are not interested in estimating $\mathbb{P}$. The equivalent martingale measure $\mathbb{Q}$ on the other hand is an artificial measure and is used in finance to price financial contract such as barrier options: the price is equal to the discounted expectation of the contract under the measure $\mathbb{Q}$. We discuss the parametric models BNS, HES, HESJ, MJD_CIRL, NIG_CIRL and VG_CIRL. Model n depends on some parameter set

$$\Theta_n \subset \mathbb{R}^{D_n}, \;\; D_n \in \mathbb{N},$$

and for each parameter $\theta_n \in \Theta_n$ the price process

$$S^{\theta_n} = \left(S_t^{\theta_n}\right)_{t \geq 0}$$

is a martingale under some equivalent martingale measure $\mathbb{Q}^{\theta_n}$. The characteristic function

$$\varphi_{\log\left(S_t^{\theta_n}\right)}(u) = E\left[\exp\left(iu \log\left(S_t^{\theta_n}\right)\right)\right], \;\; t \geq 0,$$

is known analytically for the six models. Here i denotes the imaginary unit. By Carr and Madan (1999), the prices of European plain vanilla call options under the stock price dynamics S^{θ_n} can be computed very efficiently using Fourier techniques. Prices of put options can be obtained by the put-call parity.

We are given for a time series of $i = 0, ..., N$ timepoints M_i plain vanilla call and put options

$$C_i^1, ..., C_i^{M_i}$$

at each timepoint with different strikes and maturities and with known market prices

$$\pi_{C_i^1}^{\text{market}}, ..., \pi_{C_i^{M_i}}^{\text{market}}.$$

At each timepoint, we minimize the mean-square error between market and model prices for all models, i.e. we minimize for model n the *objective function*

$$\theta_i^n \mapsto \sqrt{\sum_{m=1}^{M_i} \frac{\left(\pi_{C_i^m}^{\text{market}} - \pi_{C_i^m}^{\text{model}(n)}\left(\theta_i^n\right)\right)^2}{M_i}}, \;\; \theta_i^n \in \Theta_n.$$

The price of the plain vanilla option C_i^m at timepoint i under model n for the parameter set θ_i^n is denoted by $\pi_{C_i^m}^{\text{model}(n)}\left(\theta_i^n\right)$ and can be computed very fast by the Carr-Madan formula, which is explained in the Appendix. Let us denote the minimum by $\hat{\theta}_i^n$. For most models the existences and uniqueness of a global minimum is simply not known. Theoretical and numerical problems arising when minimizing the objective function to plain vanilla market option data are discussed in Gilli and Schumann (2012). Difficulties of the optimization procedure for particular models like the Heston model are mentioned in Cui et al. (2017). The *calibration risk*, i.e. the risk of mispricing exotic options such as barrier options, is discussed in Guillaume and Schoutens (2012).

Nevertheless stochastic optimisation methods like differential evolution, see Storn and Price (1997), can be used to minimize the objective function heuristically and return satisfactory results, see Gilli and Schumann (2012).

3.2.2 Relation of Advanced Stock Price Models to the Black-Scholes Model

In the following Subsections, we introduce the models discussed in this thesis. In Section 3.4 we calibrate the models to real market data of plain vanilla options on the DAX, a German stock market index. The DAX is a performance index, all dividends are already included in its calculation. Therefore the models do not include dividends.

All models are somewhat related to the Black-Scholes model. The stock price process under the Black-Scholes model, see Black and Scholes (1973) and Merton (1973), satisfies the stochastic differential equation (SDE)

$$dS_t = S_t \left(r dt + \sigma dW_t \right), \;\; S_0 > 0, \;\; t \geq 0, \tag{3.1}$$

where $r > 0$, $\sigma > 0$ and (W_t) is a standard Brownian motion. The SDE can equivalently be written for the log stock price process $Z_t := \log(S_t)$,

$$dZ_t = d\,\log(S_t) = \left(r - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t, \;\; Z_0 := \log(S_0), \;\; t \geq 0. \tag{3.2}$$

The solution of Equation (3.1) is well known, see for example Björk (2009):

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}, \;\; t \geq 0. \tag{3.3}$$

The HES model generalizes Equation (3.1) replacing the constant volatility parameter σ by a CIR process. A leverage effect is introduced by allowing the Brownian motion driving the CIR process be correlated with (W_t). The HES model can generalised by adding an independent jump process to the stock price process. We abbreviates the resulting process by HESJ.

The deviation of the BNS model starts from Equation (3.2) and replaces the constant volatility σ by an Ornstein-Uhlenbeck (OU) process. A leverage effect can be incorporated by adding the randomness driving the OU-process to the log stock price.

One can as well start directly from Equation (3.3) and make the *time* stochastic by subordination of the Brownian motion, which also introduce stochastic volatility. This idea goes back to Clark (1973) and can be generalized further by replacing the Brownian motion by a more flexible Lévy process X. Like in the BNS model, a leverage effect can be integrated. The resulting model is called *Lévy model with stochastic time-change and leverage.* A general treatment of this approach can be found in Carr et al. (2003).

Carr and Wu (2004) instead directly paired the Lévy-process with the random time-change to introduce a leverage effect. They show that the problem of finding the characteristic function of a paired Lévy-process with correlated random time-change reduces to the problem of finding the Laplace transform of the random time under a complex-valued measure, evaluated at the characteristic exponent of the Lévy process. In this thesis, we do not follow this approach.

In order to calibrate a parametric model to real market data of plain vanilla options, usually the mean-square error between market and model prices is minimized. This usually means an optimizer calls the objective function very often, it is therefore essential to price plain vanilla options numerically very fast. By the Carr-Madan formula, see Carr and Madan (1999), that is possible, if we have an analytic expression of the characteristic function of the logarithm of the underlying. In the following Subsections, we present the models HES, HESJ, BNS, NIG_CIRL, VG_CIRL and MJD_CIRL in more detail and recall their characteristic function from literature.

3.2.3 The Heston (HES) and Bates (HESJ) Models

The HESJ model with parameters $\kappa > 0$, $\eta > 0$, $\lambda > 0$, $\rho \in [-1, 1]$, $\sigma_0^2 > 0$, $\theta \geq 0$, $\mu_J > -1$ and $\sigma_J > 0$ is described by the following system of differential equations

$$\begin{aligned} \frac{dS_t}{S_t} &= (r - \theta\mu_J)dt + \sigma_t dW_t + J_t dN_t, \;\; S_0 \geq 0 \\ d\sigma_t^2 &= \kappa(\eta - \sigma_t^2)dt + \lambda\sigma_t d\tilde{W}_t, \;\; \sigma_0^2 > 0. \end{aligned} \tag{3.4}$$

(W_t) and $\left(\tilde{W}_t\right)$ are correlated Brownian motions such that $\mathrm{cov}\left[dW_t d\tilde{W}_t\right] = \rho dt$. If $\theta > 0$, (N_t) is an Poisson process with intensity θ, i.e. $E[N_t] = \theta t$, modelling the dates at which the stock jumps. If $\theta = 0$, (N_t) is set equal to zero for all $t \geq 0$. (J_t) is the percentage jump size (conditional on a jump occurring), such that $J_t + 1$ is log-normal distributed:

$$\log(1 + J_t) \sim N\left(\log(1 + \mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2\right), \;\; t \geq 0.$$

The Poisson process and the percentage jumps size are assumed to be independent and are independent of the two Brownian motions.

The CIR process, described by Equation (3.4), stays positive if $2\kappa\eta \geq \lambda^2$, which is known as the *Feller condition*, see Andersen et al. (2010). During the calibration procedure we ensure that the Feller condition is satisfied. The characteristic function of the log stock price is given by, see Bakshi (1997),

$$
\begin{aligned}
\varphi_{\log(S_t)}(u) &= E\left[\exp(iu\log\left(S_t\right)\right] \\
&= \exp\left(iu\left(\log S_0 + rt\right)\right) \\
&\quad\times \exp(\eta\kappa\lambda^{-2}\left(\left(\kappa-\rho\lambda ui-d\right)t - 2\log\left(\frac{1-ge^{-dt}}{1-g}\right)\right) \\
&\quad\times \exp\left(\sigma_0^2\lambda^{-2}\frac{\left(\kappa-\rho\lambda iu-d\right)\left(1-e^{-dt}\right)}{1-ge^{-dt}}\right) \\
&\quad\times \exp\left(-\theta\mu_J iut + \theta t\left(\left(1+\mu_J\right)^{iu}\exp\left(\frac{1}{2}\sigma_J^2 iu\left(iu-1\right)\right)-1\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
d &= \left(\left(\rho\lambda ui-\kappa\right)^2 - \lambda^2\left(-iu-u^2\right)\right)^{\frac{1}{2}} \\
g &= \frac{\kappa-\rho\lambda ui-d}{\kappa-\rho\lambda ui+d}
\end{aligned}
$$

The HES model, introduced by Heston (1993), is a special case of the HESJ model, setting $\theta = 0$, i.e. removing possible jumps of the stock price.

3.2.4 The BNS Model

The Barndorff-Nielsen and Shephard model (BNS) was introduced by Barndorff-Nielsen and Shephard (2001, 2002) and Nicolato and Venardos (2003). In the risk-neutral setting, the log stock-price follows the dynamics

$$
\begin{aligned}
dZ_t &= d\,\log\left(S_t\right) = \left(r-\lambda k(-\rho)-\frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t + \rho dz_{\lambda t}, \;\; Z_0 = \log(S_0), \\
d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dz_{\lambda t}, \;\; \sigma_0^2 > 0,
\end{aligned}
$$

where (z_t) is called a *Background-driving Lévy process* and follows in the classical BNS model a Gamma-Ornstein-Uhlenbeck process, i.e.

$$
z_t = \sum_{n=1}^{N_t} x_n.
$$

(W_t) is a standard Brownian motion, (N_t) is a Poisson process, with intensity parameter $a >$, i.e. $E[N_t] = at$ and (x_n) is a sequence of exponential distributed random variables with mean $\frac{1}{2} > 0$. The function k is defined by $k(u) = \frac{-au}{b+u}$. The three sources of randomness, (W_t), (N_t) and (x_n) are mutually independent.

The Poisson process jumps in any compact interval a finite number of times. Each time it jumps, the variance process (σ_t^2) jumps up and the jump-size is exponentially distributed. Between two jumps, the variance decreases exponentially, where $\lambda > 0$

indicates the speed of the down-move of the variance process. At time $t = 0$, the variance process starts in σ_0^2. The constant $\rho \leq 0$ introduces a leverage effect: each time the variance jumps up, the log stock price jumps down.

The characteristic function of the log stock price is taken from Schoutens et al. (2003) and can be expressed as

$$\begin{aligned}\varphi_{\log(S_t)}(u) &= E\left[\exp(iu \log\left(S_t\right)\right] \\ &= \exp\left(iu\left(\log S_0 + \left(r - a\lambda\rho(b-\rho)^{-1}\right)t\right)\right) \\ &\quad \times \exp\left(-\lambda^{-1}\left(u^2 + iu\right)\left(1 - \exp\left(-\lambda t\right)\right)\frac{\sigma_0^2}{2}\right) \\ &\quad \times \exp\left(a(b-f_2)^{-1}\left(b \log\left(\frac{b-f_1}{b-iu\rho}\right) + f_2\lambda t\right)\right),\end{aligned}$$

where

$$\begin{aligned}f_1 &= iu\rho - \frac{1}{2}\lambda^{-1}(u^2 + iu)(1 - \exp(-\lambda t)), \\ f_2 &= iu\rho - \frac{1}{2}\lambda^{-1}(u^2 + iu).\end{aligned}$$

3.2.5 Lévy Models with Stochastic Time-Change and Leverage

Exponential Lévy processes with stochastic time-change and leverage have been introduced by Carr et al. (2003). The uncertainty of the stock model is modelled by a homogeneous Lévy processes X and stochastic volatility is introduced by subordinating the Lévy process by a mean-reverting positive process, the so-called square root process of Cox, Ingersoll, and Ross (1985),

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW, \ \ y_0 > 0, \tag{3.5}$$

where W_t is a standard Brownian motion and $\kappa > 0$, $\eta > 0$ and $\lambda > 0$. The mean reversion in this process induces volatility clustering. As for the HES and HESJ models, we ensure that the CIR process satisfies the Feller condition during the calibration procedure. We define the integrated stochastic volatility process by

$$Y_t = \int_0^t y_s ds, \ \ t \geq 0. \tag{3.6}$$

The expected value of the integrated CIR process is given by, see e.g. Schoutens (2003, Section 7.2.1):

$$E\left[Y_t\right] = \eta t + \kappa^{-1}(y_0 - \eta)(1 - \exp(-kt)), \ \ t \geq 0. \tag{3.7}$$

The stock price is modelled by

$$S_t = S_0 e^{rt + \omega(t) + Z_t}, \ \ t \geq 0,$$

where

$$Z_t = X_{Y_t} + \rho y_t. \tag{3.8}$$

(X_t) is a homogeneous Lévy process with Lévy exponent ψ_X, i.e.

$$E\left[e^{iuX_t}\right] = e^{t\psi_X(u)},$$

and Y_t and y_t are defined as in Equations (3.5) and (3.6), see Carr et al. (2003). The processes $(X_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ are stochastically independent. $\omega(t)$ is a deterministic mean-correting function, which makes the underlying a martingale and puts us in a risk-neutral setting. The mean-correting function is defined below. As in the BNS model, a leverage effect is introduced by the term ρy_t. If $\rho < 0$, a rise of volatility leads to a fall of the stock price. The characteristic function of $\log(S_t)$ is given by

$$\begin{aligned}
\varphi_{\log(S_t)}(u) &= E[\exp(iu\log(S_t))] \\
&= \exp\left(iu\left(\log\left(S_0\right) + rt + \omega(t)\right)\right) \\
&\quad \times \Phi_t(-i\psi_X(u), \rho u),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_t(a,b) = \Phi_t(a,b,\kappa,\eta,\lambda,y_0) &= A(t,a,b)\exp\left(\frac{\kappa^2\eta t}{\lambda^2} + B(t,a,b)y_0\right), \\
A(t,a,b) &= \left(c + \frac{\bar{\kappa}}{\gamma}s\right)^{-\frac{2\kappa\eta}{\lambda^2}}, \\
B(t,a,b) &= \frac{ib\left(\gamma c - \kappa s\right) + 2ias}{\gamma c + \bar{\kappa}s}, \\
\gamma &= \sqrt{\kappa^2 - 2\lambda^2 ia}, \\
c &= \cosh\left(\frac{\gamma t}{2}\right), \\
s &= \sinh\left(\frac{\gamma t}{2}\right), \\
\bar{\kappa} &= \kappa - ib\lambda^2, \\
\omega(t) &= -\log\left(\Phi_t(-i\psi_X(-i), -i\rho)\right).
\end{aligned} \tag{3.9}$$

In the following subsections, we describe briefly three famous homogeneous Lévy processes and provide analytic formulas of the corresponding Lévy exponents. A general introduction to Lévy processes can be found in Sato (1999), Schoutens (2003) and Applebaum (2009).

Normal-Inverse Gaussian Process (NIG)

The normal inverse Gaussian (NIG) model has been introduced by Barndorff-Nielsen (1995, 1997a,b). The NIG process with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$, and $\delta > 0$

can be represented as a subordination of a Brownian motion with drift by the Inverse Gaussian (IG) process:

$$\mathrm{NIG}_t := \beta\delta^2 I_t + \delta W_{I_t}, \ \ t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion and $(I_t)_{t\geq 0}$ is a stochastically independent IG process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$. The density of an inverse Gaussian distribution $\mathrm{IG}(a, b)$ with parameters $a, b > 0$ is given by

$$f_{\mathrm{IG}}(x) = \frac{a}{\sqrt{2\pi}} \exp(ab) x^{-\frac{3}{2}} \exp(-\frac{1}{2}(a^2 x^{-1} + b^2 x)), \ \ x > 0.$$

A IG process with parameters a and $b > 0$ is a non-decreasing Lévy process defined by

$$I_t := \inf\left\{ s > 0; \tilde{B}_s + bs = at \right\}, \ \ t \geq 0,$$

it denotes the first time the standard Brownian motion $\tilde{B}$ with drift b hits the barrier at. The increments

$$I_{t+s} - I_s, \ \ t > s \geq 0,$$

have an Inverse Gaussian distribution $\mathrm{IG}(at, b)$, see Applebaum, (2009, Example 1.3.21). The Lévy exponent of a NIG process is given by,

$$\psi_{\mathrm{NIG}}(u) = \psi_{\mathrm{NIG}}(u, \alpha, \beta, \delta) = -\delta \left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2} \right).$$

Alternatively, NIG can be parametrized by $\sigma > 0$, $\nu > 0$, $\theta \in \mathbb{R}$ such that

$$\psi_{\mathrm{NIG}}(u) = \psi_{\mathrm{NIG}}(u, \sigma, \nu, \theta) = -\sigma \left(\sqrt{\frac{\nu^2}{\sigma^2} + \frac{\theta^2}{\sigma^4} - \left(\frac{\theta}{\sigma^2} + iu \right)^2} - \frac{\nu}{\sigma} \right),$$

where we corrected a typing error in Carr et al. (2003, p. 349) and used the transformation $\beta = \frac{\theta}{\sigma^2}$, $\alpha^2 = \frac{\nu^2}{\sigma^2} + \frac{\theta^2}{\sigma^4}$ and $\delta = \sigma$.

The symmetric NIG process is a subclass of the NIG process, setting $\beta = 0$. In this thesis, we only work with the symmetric NIG process, because calibration of the process NIG_CIRL to real market data is more robust in the symmetric case.

Variance Gamma Process (VG)

The Variance-Gamma (VG) process as been introduced by Madan and Seneta (1987, 1990), Madan and Milne (1991) and Madan et al. (1998). The VG process with parameters $\sigma > 0$, $\nu > 0$ and $\theta \in \mathbb{R}$, can be represented as a subordination of a Brownian motion with drift by a gamma process:

$$\mathrm{VG}_t := \theta\Gamma_t + \sigma W_{\Gamma_t}, \ \ t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion and $(\Gamma_t)_{t\geq 0}$ is an independent gamma process with mean rate 1 and variance rate ν. The increments

$$\Gamma_{t+s} - \Gamma_s, \ \ t > s \geq 0,$$

of the gamma process have a gamma density with mean t and variance νt. The Lévy exponent of the VG process is

$$\psi_{\text{VG}}(u) = \psi_{\text{VG}}(u, \sigma, \nu, \theta) = -\frac{1}{\nu} \log \left(1 - iu\theta\nu + \left(\frac{\sigma^2 \nu}{2} \right) u^2 \right).$$

The symmetric VG process is a subclass of the VG process, setting $\theta = 0$. In this thesis, we only work with the symmetric VG processes, because calibration of the process VG_CIRL to real market data is more robust in the symmetric case.

Merton jump-diffusion (MJD)

The uncertainty of the Merton jump-diffusion (MJD) model, Merton (1976), is modelled by

$$\text{MJD}_t := \sigma W_t + \sum_{k=1}^{N_t} z_k, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, $\sigma > 0$, N_t is a Poisson process with intensity parameter $\pi > 0$, i.e. $E(N_t) = \pi t$, and independent of $(W_t)_{t\geq 0}$. The jump-sizes are independent and identically distributed and are modelled by

$$z_k \sim \text{Normal}(\gamma, \sigma_J^2), \quad k = 1, 2..., \gamma \in \mathbb{R}, \quad \sigma_J > 0.$$

The MDJ process has Lévy exponent

$$\psi_{\text{MJD}}(u) = \psi_{\text{MJD}}(u, \sigma, \pi, \gamma, \sigma_J) = -\frac{\sigma^2 u^2}{2} + \pi \left(\exp \left(i\gamma u - \frac{\sigma_J^2 u^2}{2} \right) - 1 \right).$$

The MJD_CIRL model has the same ingredients as the HESJ model. For the HESJ model, a change of the CIR process describing the volatility does not change the possibility of a (big) jump due to the independence assumption of the jump comonent. From an economic point of view, if uncertainty in financial markets increase, it might be desirable that the chance of large stock price movements, i.e. the chance of a crises, also increase. This effect can be modelled by the MJD_CIRL model, because the CIR process directly influences the rate of big jumps for the MJD_CIRL model.

Fix some Parameters of the Various Models

For any $\xi > 0$ it holds

$$\Phi_t(-i\xi\psi_X(u), \rho u, \kappa, \eta, \lambda, y_0) = \Phi_t \left(-i\psi_X(u), \frac{\rho u}{\xi}, \kappa, \xi\eta, \sqrt{\xi}\lambda, \xi y_0 \right).$$

Without loss of generality, we therefore may fix some of the parameters of the various models. For NIG_CIRL, we set $\delta = 1$, because

$$\psi_{\text{NIG}}(u, \alpha, \beta, \delta) \quad = \quad \delta\psi_{\text{NIG}}(u, \alpha, \beta, 1).$$

Fixing $\delta = 1$ is equivalent of rescaling ρ, η, λ and y_0. With the same argument, we set $\nu = 1$ in the VG_CIRL model, because

$$\psi_{\mathrm{VG}}(u,\sigma,\nu,\theta) \quad = \quad \frac{1}{\nu}\psi_{\mathrm{VG}}(u,\sigma\sqrt{\nu},1,\theta\nu).$$

And for MJD_CIRL, we set $\pi = 1$, because

$$\psi_{\mathrm{MJD}}(u,\sigma,\pi,\gamma,\sigma_J) \quad = \quad \pi\psi_{\mathrm{MJD}}(u,\frac{\sigma}{\sqrt{\pi}},1,\gamma,\sigma_J).$$

For all three models one could as well set $y_0 = 1$. This has been done by Schoutens et al. (2003) and Carr et al. (2003). We find our choices numerically more stable.

3.2.6 The Leverage makes the Difference

Lévy models including only stochastic volatility ($\rho = 0$) seem to have a lower path fluctuation compared to models incorporating also a leverage effect ($\rho < 0$) and therefore a lower probability of hitting the barrier.

Indeed Schoutens et al. (2003) reported that Lévy models with stochastic time-change (but without leverage) tend to overvalue some down-and-out barrier call options relative to the Heston model by roughly 70% but undervalue down-and-in barrier options by roughly 35%. A down-and-out (down-and-in) option is less (more) worth, the higher the probability hitting the barrier. We also calibrate the VG model with stochastic time-change and leverage to the plain vanilla option data used by Schoutens et al. (2003) and show that prices of down-and-out and down-and-in barrier call options of the Lévy model with leverage are quite close to the prices prognosticated by the Heston model, see Table 3.1. This simply highlights the fact that one cannot ignore the leverage effect in a model when pricing exotic options.

Model	RMSE	Down-and-out call	Down-and-in call
HES	3.0	173.03	336.35
VG_CIR	2.4	293.28	218.51
VG_CIRL	1.8	191.72	319.22

Table 3.1: This table reports the root mean-square error (RMSE) between model and market prices of plain vanilla options and simulated prices of two exotic options for three models: the Heston model, the variance gamma model with stochastic time-change (VG_CIR) and the VG_CIR model incorporating a leverage effect (VG_CIRL). The first two rows are taken from Schoutens et al. (2003). The underlying is equal to $S_0 = 2461.44$. The strikes of the exotic options are equal to S_0 and the barriers are equal to $0.95S_0$. The maturities are three years and the risk-free interest rate is set to 0.03, the dividends are assumed to be zero. The VG_CIRL process is described in Section 3.2.5, its calibrated parameters are: $\sigma = 0.205396$, $\nu = 0.009892$, $\theta = -0.115094$, $\kappa = 0.582229$, $\eta = 0.98048$, $\lambda = 1.347724$, $y_0 = 1.0$, $\rho = -0.114496$.

3.3 Market Data

3.3.1 Plain Vanilla Options

We look at a time series from 05/07/2017 till 21/08/2017, which contain 34 trading days. At each trading day, there are three timepoints, namely “morning” (10am-10:30am), “midday” (1pm-1:30pm) and “afternoon” (4pm-4:30pm) on which prices of in total about 471.000 European plain vanilla put and call options with maturities ranging from $0-3$ years and moneyness ranging from $0.5-1.0$ are available on the DAX, a blue chip stock market index consisting of the 30 major German companies.

We follow the methodology applied for the volatility index (VIX) by the Chicago board of exchange, see CBOE (2018), and only use out-of-the-money options for calibration, see also Carr and Wu (2009).

The DAX is a performance index, all dividends are included in its calculation. In total there are thus 102 timepoints. The DAX and the VDAX-New, the corresponding volatility index, are plotted in Figure 3.2 to provide an indication of the market situation.

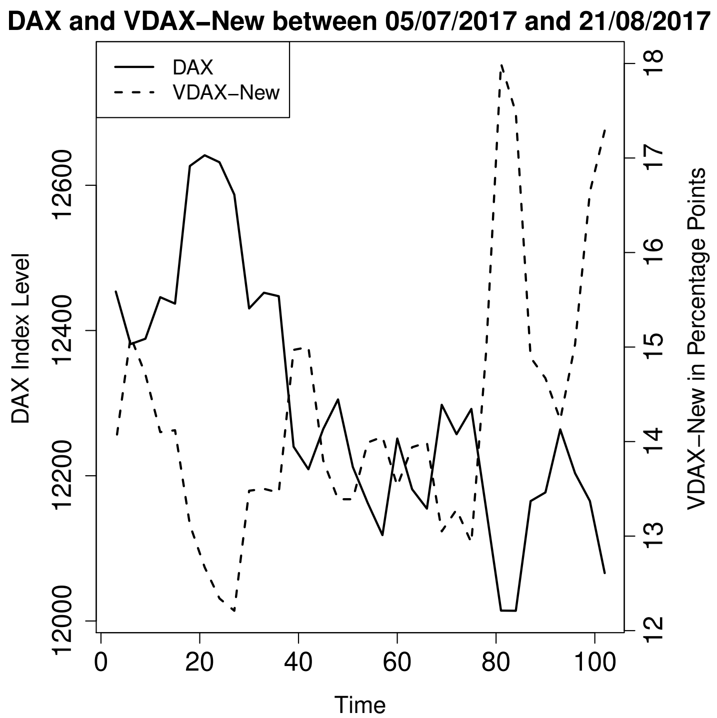

Figure 3.2: End of day values of DAX and and its volatility index, the VDAX-New. The VDAX-New reached its hightest value of 18.0% at 10/08/2017 which corresponds to timepoint 81. In mid-Agust, the North Korea crisis escalated rhetorically, which explains the rise of the volatility index economically. The VDAX-New was lowest on 17/07/2017 (timepoint 27) at a level of 12.2%.

Those options are issued by the financial institutions Commerzbank, UBS and UniCredit, which usually also act as a market maker. The options are listed on different stock exchanges in Germany, mainly on Frankfurt stock exchange and Stuttgart stock exchange. They can also be traded over the counter, directly with the issuer. Due to some missing data, not all option prices are available at each timepoint. But each option with a short maturity (less than three month) and being deep out of the money (moneyness is smaller than 0.85) can be found at least on 65% of all observations. Prices of all other options are available at least on 85% of all observations[1].

Figures 3.7 and 3.8 show a relative stable distribution over time for different maturity and moneyness buckets for one exemplary issuer, the option data of the other issuers look similar.

We use the following abbreviations: "dOTM" means "deep out-ouf-the money" and refers to a moneyness smaller than 0.85. "OTM" stands for "out of the money", which is a moneyness between 0.85 and 1.00. The maturity is measured in months. For all issuers, there are roughly the same number of OTM call and OTM put options for all maturities. But there are usually much more dOTM put options than dOTM call options.

The absolute average spread (ask − bid) over all products of all banks is 0.03 EUR. The average relative spread $\left(\frac{\text{ask}-\text{bid}}{\frac{1}{2}(\text{ask}+\text{bid})}\right)$ is 5%. In Table 3.4 several quantiles of the absolute and the relative spread are shown. For most options, the spread is small.

3.3.2 Exotic Options

We obtained in total 303.000 bid and ask quotes of down-and-out barrier (DOB) call-options for the same time series consisting of 102 timepoints in the period from 05/07/2017 till 21/08/2017 and issued by the same issuers as described in the previous section. The payoff of such option with strike K and maturity T is

$$\text{DOB}(K,T) = \begin{cases} \max(S_T - K, 0) & , \inf\limits_{0\leq t\leq T} S_t > K \\ 0 & , \text{otherwise.} \end{cases}$$

The strike and the barrier are equal. The process $(S_t)_{t\geq 0}$ describes the stock price under an equivalent martingale measure. A call option with the same maturity T and strike K is called *the corresponding plain vanilla option.* If the barrier is hit before maturity, the DOB option become worthless, otherwise it has the same payoff as the corresponding plain vanilla option. It is clear that the corresponding plain vanilla option is always in-the-money otherwise the barrier would be knocked-out. For each exotic option, we compute the price of the corresponding plain vanilla option using an implied volatility surface which we obtain from the plain vanilla data set, see Section 3.3.1. We focus on all exotic options, which are "exotic" enough, i.e. whose exotic prices are smaller than 0.75 times the corresponding plain vanilla prices. This essentially means removing all exotic options whose corresponding plain vanilla options are deep in-the-money. We are

[1] In the case the option expires before the 21/08/2017, the percentages relates to the number of observations till maturity of the option.

then left with about twenty thousands exotic options with maturities ranging from a few days to half a year. The moneyness of the corresponding plain vanilla options lay between 1.01 and 1.07.

Due to our selection-procedure we focus on exotic options whose barriers lay slightly below the underlying. We therefore face a different set of exotic options at each timepoint: as the underlying is changing over time, some barrier option might be knock-out, if the underlying decreases, and hence disappear from the data set. Or, if the underlying increases, some barrier options might move too far in-the-money, and are also removed from the data set because they are not exotic enough any more and are filtered by our selection procedure. There are some exotic options with a timeline with missing data as well. Therefore the set of identification numbers of the exotic options and the strikes differ over time. But the maturities and the moneyness of the corresponding plain vanilla options are more or less stable over time. 99.9% of the absolute bid-ask spreads of all exotic options are smaller or equal than 0.02 EUR.

3.4 Methodology

3.4.1 Calibration

For each issuer and at each timepoint, we calibrate the models BNS, HES, HESJ, MJD_CIRL, NIG_CIRL and VG_CIRL to prices of plain vanilla options by minimizing the mean-square error between market prices and model prices. We obtain for each model a time series of parameters. Table 3.5 shows the average root mean-square error (RMSE) for the various models over all timepoints and the average estimated parameters. We conclude that all models can be fitted very well to real market data. This is in line with the results of Schoutens et al. (2003).

3.4.2 Pricing via Monte-Carlo

For each issuer, each timepoint and each model, we take the parameters obtained by calibrating the various models to plain vanilla option market prices, see Section 3.4.1, and price all available exotic barrier options via Monte-Carlo simulations. For each options we use $M = 200,000$ simulations and a timestep of $\delta = 4.0 \cdot 10^{-5}$ business years, which corresponds to a grid-size of about five minutes. Such a narrow grid-size is necessary, to keep the discretion error small. The price of an option whose barrier is very close to the underlying reacts quite sensitive to the number of time-steps chosen to discretize the underlying. We use variance reduction techniques by control-variates. The simulation of jump-diffusion models are standard, see for example Glasserman (2013). Simulation techniques for Lévy processes with stochastic time-change and the BNS model can be found e.g. in Schoutens (2003).

3.5 Pricing Ability of The Models

We present the simulation results in two steps. First we give an aggregated overview of all simulated exotic options, then we turn to about 38 particular options and analyse the corresponding time series of the residuals: the relative difference between model and market prices.

In Figure 3.3 and Figure 3.4, we see the relative differences between market and model prices of all exotic options for the various models. On average HES undervalues exotic options with short maturities (less than three months) by about 7% and options with long maturities (between three and six months) by about 5%. HESJ explains exotic option market data best, but still undervalues options with short and long maturities by about 4% and 1% respectively.

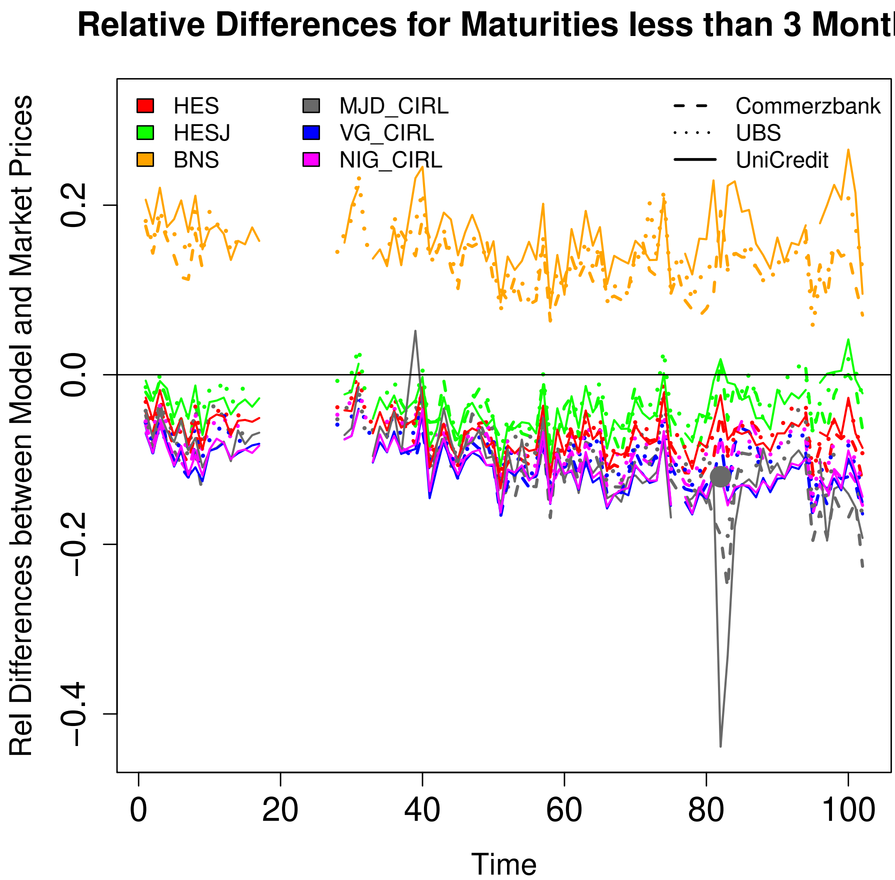

Figure 3.3: Relative Differences between market and model prices, maturity less than three months. The point at timepoint 82 corresponds to another parameter set for the MJD_CIRL model.

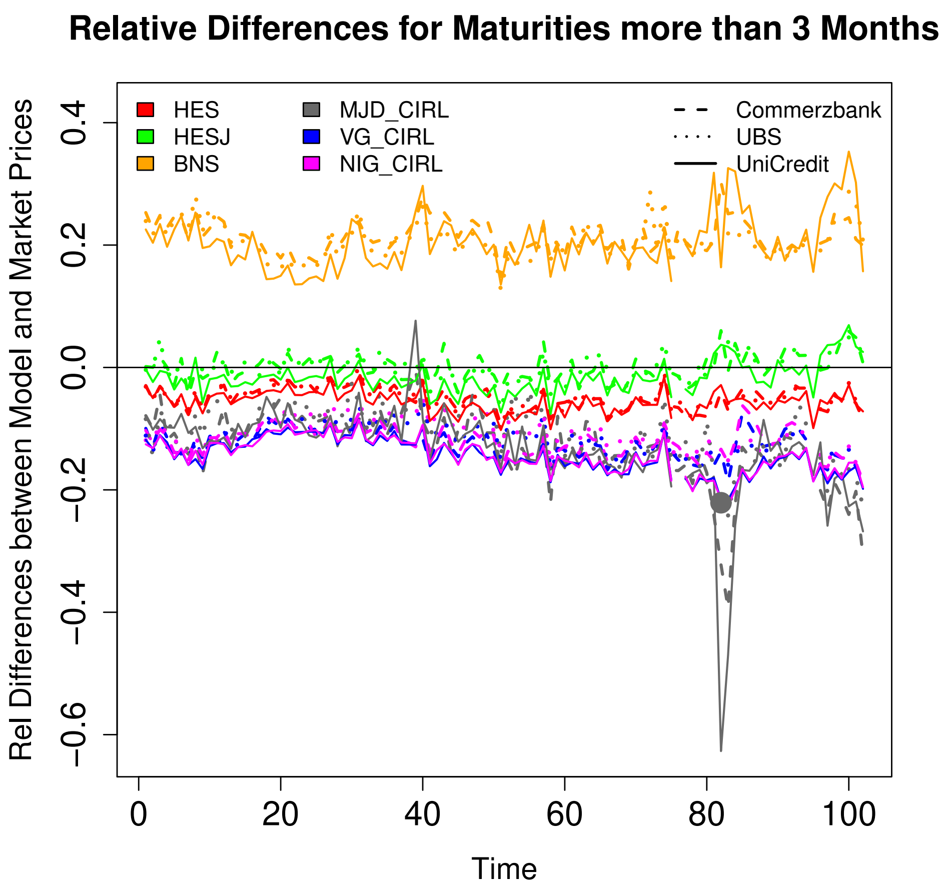

Figure 3.4: Relative differences between market and model prices, maturity more than three months. The point at timepoint 82 corresponds to another parameter set for the MJD_CIRL model.

BNS has the greatest bias overvaluing the options by about 19% on average. The Lévy models MJD_CIRL, NIG_CIRL and VG_CIRL undervalue the barrier options by about 13% on average.

The patterns are independent of the issuer of the exotic options and are similar also under different market environments: while the first half of the analysed timepoints correspond to a rather calm market environment (the volatility index get as low as 12.2%, see Figure 3.2, in the second half, uncertainty measured by the volatility index raises at its top to 18%, which is a difference of 50% to its lowest level.

MJD_CIRL is less stable than the other models. On August, 11th in the morning, there is a big peak at timepoint 82 in Figures 3.3 and 3.4 demonstrating that MJD_CIRL is undervaluing the barrier options by up to 60%. The model is calibrated using differential evolution, which is a stochastic optimizer independent of any starting point.

	DE	NM
RMSE	0.0526	0.0599
σ	0.18	0.26
π	1.00	1.00
γ	-0.24	-0.30
σ_J	0.17	0.14
κ	10.71	3.41
η	0.32	0.21
λ	1.98	1.00
y_0	0.12	0.13
ρ	-0.26	-0.45

Table 3.2: Calibration details for the model MJD_CIRL on August, 11th in the morning, using two different optimizer: the differential evolution (DE) and the Nelder-Mead (NM) algorithms.

The estimated parameters and the RMSE between model and market prices are shown in Table 3.2. Note the particular high values of the parameters describing the speed of mean reversion, κ and the "vol of var", λ.

Choosing the optimal parameter set of the previous timepoint 81 as starting point and the Nelder-Mead algorithm, see Nelder and Mead (1965), instead to search for an optimum, leads to a slightly less satisfactory parameter set. But pricing the barrier options using these new parameters, leads to a much better result: the relative difference between model and market prices shrinks to about -12% and -22% as shown by a point in Figures 3.3 and 3.4, respectively. Calibrating MJD_CIRL is less robust than calibrating the other models.

As described in Section 3.3.2, at each timepoint we are aggregating a different set of options because over time not all options are available. We therefore choose 38 options, their security identifier code can be found in Table 3.7, for which exotic option prices are available at all timepoints in the interval [50,102], except for seven options, which knock-out at timepoint 81 and 82. For those options, prices at about 30 timepoints are available.

All exotic options of our sample are numbers from 1,2,...,38, ordered by strike, see Table 3.7. The maturities lay between 0.3 and 0.4 years throughout the whole time series. The mean and standard deviation of the time series of the relative residuals between market and model prices of each option can be found in Figure 3.5. We see that the standard deviation of the residuals for the models HES, HESJ, NIG_CIR and VG_CIR are similar for all 38 options and lower than 0.05. The standard deviation of the residuals of the MJD_CIRL model vary between the options a lot and are higher than 0.1 for many options. This once more underlines that MJD_CIRL is less robust than the other models.

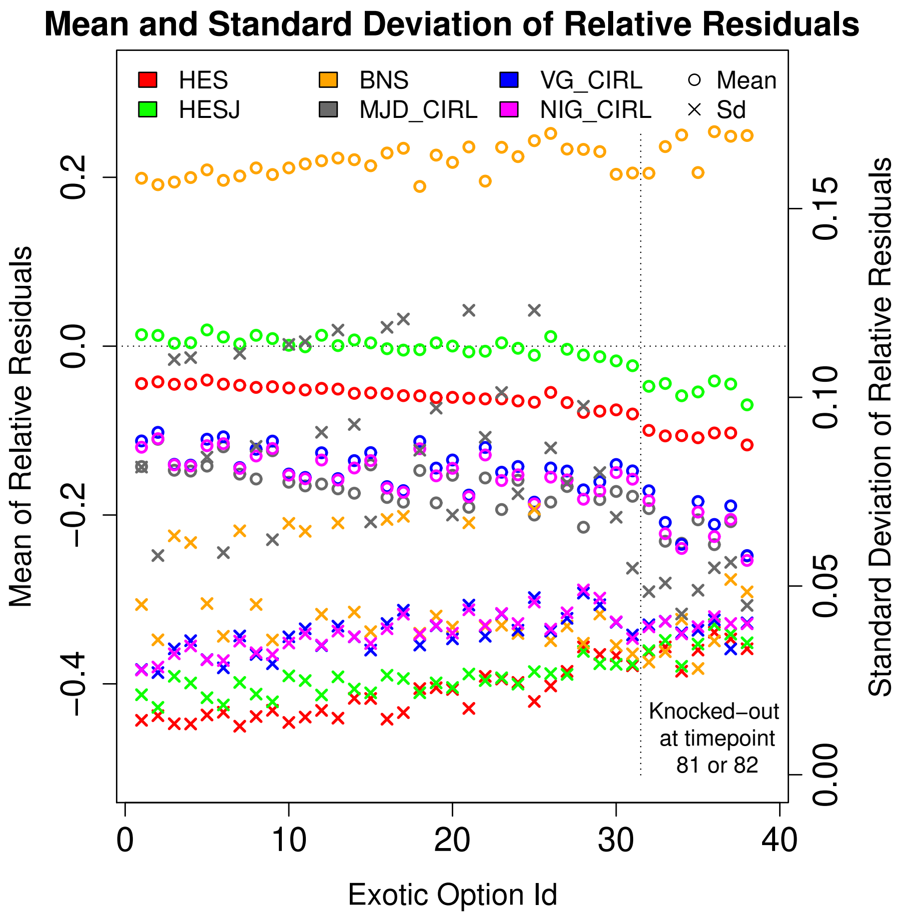

Figure 3.5: Mean and standard deviation of the time series of the relative residuals between market and model prices for various models for the options of Table 3.7.

3.5.1 Path Characteristics

In this section, we attempt to provide some characteristics of the path-behaviour of the various models under the risk-neutral measure. That helps to understand the models and might (partially) answer the question why some models overvalue and other models undervalue the barrier option data set from Section 3.3.2.

For a stochastic process $S_{t\in[0,1]}$ describing a stock price we define some characteristics as follows. All characteristics refer to the interval $[0,1]$, i.e. to the period of one year.

$$\begin{aligned} h &= \text{Probability the process hits } 0.95S_0 \text{ at least once,} \\ \sigma &= \sqrt{\mathrm{Var}(S_1)}, \\ J_{\log} &= \text{Expected number of times the absolute value} \\ &\quad \text{of the log-returns jump by more than 5\%,} \\ \mathrm{ARV} &= \text{Average realized variance (defined below),} \\ U &= \text{Expected number of times the process crosses } [0.98S_0, 1.02S_0]. \end{aligned}$$

We estimate the values by a Monte Carlo simulation for the six models based on the average parameters of Table 3.5. Those parameters do not belong to any particular set of plain vanilla options, nevertheless the computed characteristics shown in Table 3.3 are similar when choosing the parameter sets of a particular timepoint and issuer.

Model	h	σ	$J_{\log}$	ARV	U
BNS	0.66	0.26	1.32	0.063	1.22
HESJ	0.72	0.27	0.03	0.073	1.47
HES	0.74	0.19	0.00	0.032	1.59
MJD_CIRL	0.83	0.22	0.15	0.068	2.35
NIG_CIRL	0.84	0.24	0.91	0.079	2.47
VG_CIRL	0.86	0.26	0.69	0.096	2.82

Table 3.3: Some characteristics of the path-behaviour of the various models. The table is ordered by column h. Each model is simulated $M = 200,000$ times on the time interval $[0,1]$ using a step-size of $\delta = 4.0 \cdot 10^{-5}$ business years or equivalently using $N = 25,000$ time-steps. We assume $S_0 = 100$ and $r = 0$ for all models.

Some of the values can be estimated analytically as well. Define by

$$p^{\pm 0.05}_{(\mu,\sigma)}$$

the probability of a normal random variable with mean μ and standard deviation σ to be outside the interval $(-0.05, 0.05)$.

The expected number of times the absolute value of the log-returns jump by more than 5% in the interval $[0,1]$ can be estimated for the BNS model by

$$a\lambda \exp\left(-b\frac{0.05}{|\rho|}\right),$$

for the HESJ model by

$$\theta p^{\pm 0.05}_{\left(\log(1+\mu_J)-\frac{1}{2}\sigma_J^2, \sigma_J^2\right)},$$

and for the MJD_CIRL model by

$$\left(\eta + \kappa^{-1}(y_0 - \eta)(1 - \exp(-\kappa))\right) p^{\pm 0.05}_{(\gamma,\sigma_J)},$$

where we use Equation (3.7). Those estimates only capture the jump-part of the models ignoring the diffusion part.

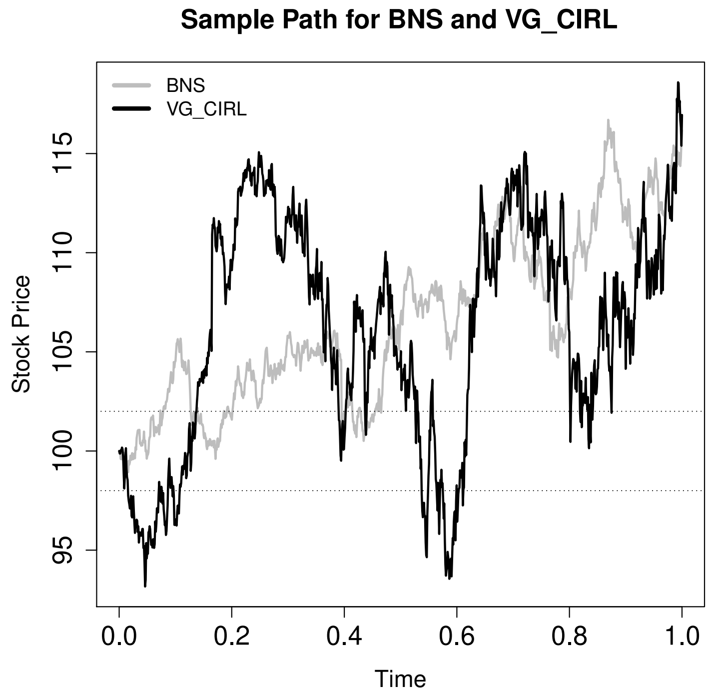

Figure 3.6: Sample path for the BNS and the VG_CIRL model.

Clearly, the probability of hitting the barrier directly influences the price of a barrier option. The value h is comparatively low for the BNS model and high for the Lévy models. Indeed in Section 3.5, we show that the BNS model overvalues and the Lévy models undervalue the barrier options. But the probability of hitting the barrier can neither be explained very well by the standard deviation nor by the expected number of (big) jumps of the processes: the standard deviations are approximately in the same range for all models. For instance the BNS model and the VG_CIRL model have the same standard deviation at time $t = 1$ but significantly different probabilities of hitting the barrier. The number of expected jumps are particularly high for the BNS *and* the Lévy models.

We also compute for the six models the average realized variance ARV: we discretize the interval $[0, 1]$ by $N = 25,000$ time-steps and for a simulated path we add up the squared log-returns between two successive time-points. We repeat this $M = 200,000$ times and take the average. See for example Barndorff-Nielsen and Shephard (2002) for a precise definition of ARV and its relation to quadratic variation of semimartingales. (We also computed the ARV based on daily returns, but the result does no differ much). It turns out that the ARV of the Lévy models NIG_CIRL and VG_CIRL are significantly higher than the ARV of the BNS model, which indicates that the Lévy models have a higher path-fluctuation compared to the BNS model. But the ARV of the BNS model is about twice as large than the ARV of the HES model. The low ARV of the HES model might be explained by the fact that HES is a continuous model without any jumps. However, the probability of hitting the barrier is considerable higher for the HES than for the BNS model and we conclude that the ARV does not really help to understand the different pricing behaviour of the various models.

The characteristic U seems to provide some explanation: U measures the expected number of up-crossings of some interval around the starting point of the stock price process. U indicates how often a stock price process changes its direction. The higher U, the higher the fluctuation of the sample random paths under the risk-neutral measure.

Even though the BNS model jumps quite often (and each jump is directed downwards increasing the probability of hitting the barrier), we think between the jumps the BNS behaves too calmly to explain real market data of barrier options very well. This is probably due to the structure of the Ornstein-Uhlenbeck process process modelling the stochastic volatility of the BNS model.

As indicated by U, the random paths generated by the Lévy models have quite a high fluctuations leading to a (too) high probability of hitting the barrier and therefore underestimating the prices of the barrier options. Lévy models with stochastic volatility (but without leverage) tend to overvalue down-and-out barrier options compared to the HES model, see Schoutens et al. (2003).

We think the high fluctuations of the models VG_CIRL, NIG_CIRL and MJD_CIRL are due to the direct linkage of the CIR process and the log stock price, i.e. the way a leverage is incorporated into the Lévy models is responsible of the high fluctuations of the random sample path and the relative low prices of barrier down-and-out options. Figure 3.6 show a sample path for both the BNS and the VG_CIRL model, illustrating typical high fluctuations of the VG_CIRL model.

The HES model is a continuous model, it does not contain any jumps but the random sample paths generated by the HES model have moderately higher fluctuations compared to the HESJ model and measured by U, and the HES model undervalues the barrier options relative to the HESJ model slightly.

3.6 Conclusion

We test the performance of six advanced stock price models on a given set of time series of market prices of European plain vanilla put and call options and barrier down-and-out call options for the period between 05/07/2017 and 21/08/2017 issued by different banks. At each timepoint and for each issuer we calibrate six models by minimizing the mean-square error between model and market prices of the plain vanilla options. The models are: the Heston model, see Heston (1993) and its generalization, see Bates (1996), the BNS model, see Barndorff-Nielsen and Shephard (2001) and three Lévy models with stochastic time-change and leverage, see Carr et al. (2003). We apply the Lévy models MJD_CIRL, NIG_CIRL and VG_CIRL, see Section 3.2.5. MJD_CIRL for example corresponds to a Merton jump-diffusion process, subordinated by the square root process of Cox, Ingersoll, and Ross (1985). As in the BNS model, a leverage effect is incorporated in the Lévy models. We chose those models, because they are able to capture both stochastic volatility and the leverage effect and are flexible enough to model plain vanilla options reasonably well. They are further mathematically tractable, can be calibrated relatively fast to real market data and it is straightforward to implement the models in order to perform a Monte Carlo simulation.

All six models can be fitted almost equally well to market data of plain vanilla options. But computing the prices of the barrier options via a Monte Carlo simulation using the various calibrated models, leads to significantly different prices for the exotic options. In particular, the BNS model overvalues barrier options by about 19% on average, the Heston model undervalues those options slightly and the Bates model reproduces barrier option prices very well.

Jessen and Poulsen (2013) worked with real market data of foreign-exchange barrier options and similarly concluded that the Heston model slightly undervalues barrier options, but the Bates model performs significantly worse than the Heston model, which stands in contrast to our results. Future research need to be done to explain why adding jumps to the Heston model increase the valuation ability of the model when applied to equity data and decrease the model performance for foreign-exchange barrier options.

Lévy models with stochastic time-change and leverage undervalue the exotic options by about 13% on average. There is barely any difference between the models NIG_CIRL and VG_CIRL. Compared to the other models, the model MJD_CIRL is the least robust one. Calibrating the MJD_CIRL model sometimes lead to unreasonably parameter sets. The results are similar for all issuers.

The findings that advanced stock price models can be fitted very well to plain vanilla market data and the fact that those calibrated models predict very different prices for exotic options are in line with other studies in literature. In contrast to other studies, we are able to compare the simulated prices of exotic options to real market data and conclude that for the particular data set and period we looked at, the Heston model with jumps best explains barrier option market prices.

A heuristic analysis suggests that the different degree of fluctuation of the random paths of the models under the risk-neutral measure are responsible of producing different prices for the barrier options. The fluctuations are measured by the expected number of up-crossings of the stock price process of some interval. Further research need to be done in this direction.

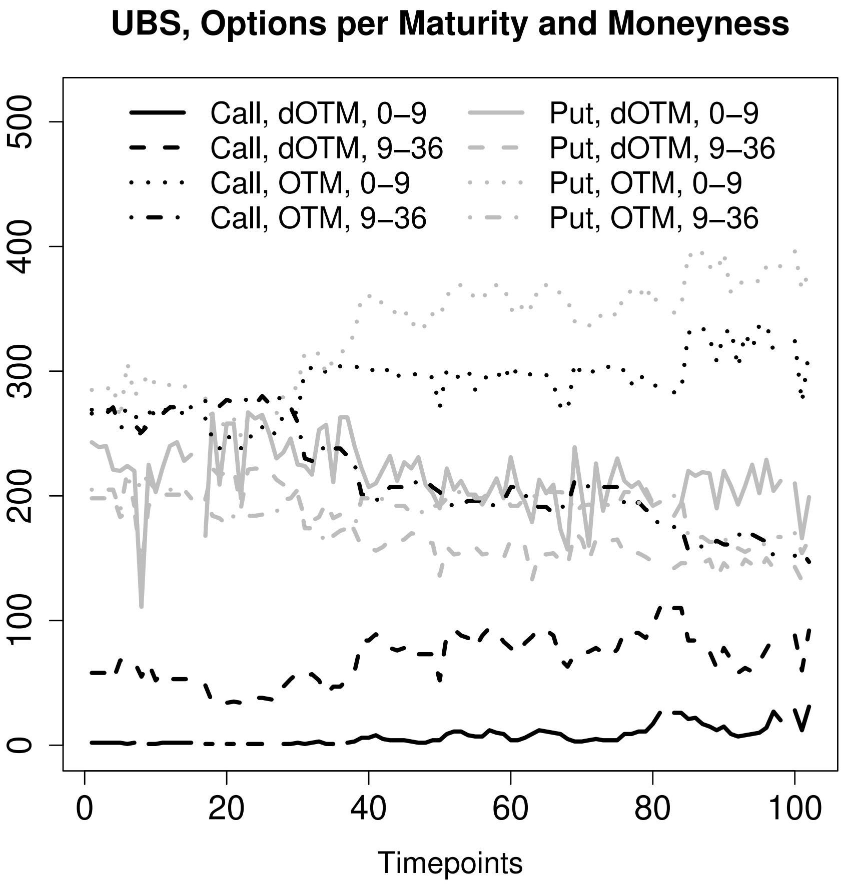

Figure 3.7: UBS, distribution of maturity and moneyness buckets and over time

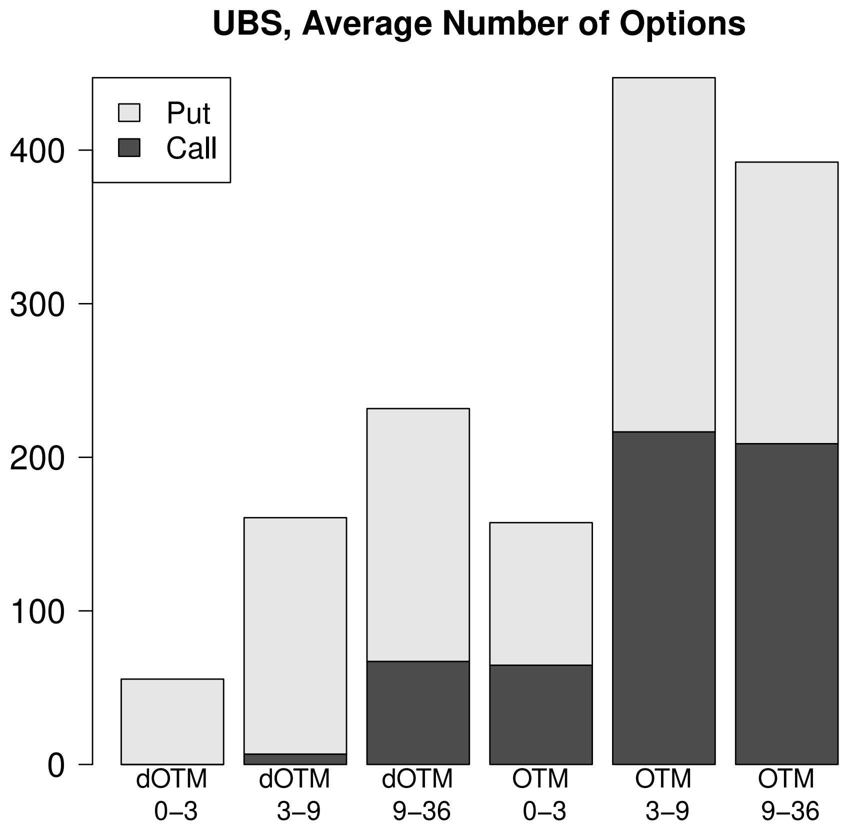

Figure 3.8: UBS, average number of options for different maturity and moneyness buckets. The average is taken over all timepoints.

Quantile	5%	50%	75%	95%	99%	Max
Absolute spread in EUR	0.01	0.01	0.02	0.1	0.2	0.3
Relative spread	0%	1%	3%	12%	107%	192%

Table 3.4: Relative and absolute spread for plain vanilla options of all banks.

	BNS	HES	HESJ	MJD_CIRL	NIG_CIRL	VG_CIRL
RMSE	0.08 (0.01)	0.09 (0.02)	0.05 (0.01)	0.06 (0.01)	0.09 (0.02)	0.10 (0.02)
κ		2.35 (0.82)	4.85 (3.4)	1.53 (0.93)	0.6 (0.13)	0.61 (0.13)
η		0.04 (0.01)	0.03 (0.01)	0.26 (0.1)	0.37 (0.25)	1.58 (1.06)
λ		0.45 (0.05)	0.45 (0.13)	0.61 (0.21)	0.58 (0.19)	1.12 (0.4)
y_0 or σ_0^2	0.01 (0.002)	0.02 (0.004)	0.01 (0.005)	0.07 (0.03)	0.04 (0.03)	0.19 (0.13)
ρ	-4.04 (2.94)	-0.78 (0.04)	-0.76 (0.09)	-0.8 (0.36)	-1.24 (0.58)	-0.35 (0.25)
v_1	0.79 (1.00)		0.03 (0.02)	1 (0)	9.32 (5.24)	0.16 (0.05)
v_2	2.95 (6.41)		-0.45 (0.18)	-0.24 (0.06)	0 (0)	1 (0)
v_3	45.7 (20.5)		0.83 (0.68)	0.22 (0.04)	1 (0)	0 (0)
v_4				0.27 (0.07)		

Table 3.5: Root mean square error (RMSE) between market and model prices of European plain vanilla options and average values and standard deviation (in brackets) of the estimated parameters of the various models. The variables $v_1, ..., v_4$ are defined in Table 3.6. Some parameters are fixed, which explains the standard deviation of zero.

	BNS	HESJ	MJD_CIRL	NIG_CIRL	VG_CIRL
v_1	λ	θ	π	α	σ
v_2	a	μ_J	γ	β	ν
v_3	b	σ_J	σ_J	δ	θ
v_4			σ		

Table 3.6: Definition of parameters $v_1, ..., v_4$.

Id	WKN	Issuer	Strike	Maturity	Knocked-out
1	CE9Q7M	Commerzbank	11675	13/12/2017	No
2	UW9JMM	UBS	11675	22/12/2017	No
3	HW2FWL	UniCredit	11675	12/12/2017	No
4	HU9B3X	UniCredit	11690	12/12/2017	No
5	CE9Q7N	Commerzbank	11700	13/12/2017	No
6	UW9HBE	UBS	11700	22/12/2017	No
7	HW2FWM	UniCredit	11700	12/12/2017	No
8	CE9Q7P	Commerzbank	11725	13/12/2017	No
9	UW9HTW	UBS	11725	22/12/2017	No
10	HW2FWN	UniCredit	11725	12/12/2017	No
11	HU9B3Z	UniCredit	11740	12/12/2017	No
12	CE9Q7Q	Commerzbank	11750	13/12/2017	No
13	HW2FWP	UniCredit	11750	12/12/2017	No
14	CE9Q7R	Commerzbank	11775	13/12/2017	No
15	UW9LU0	UBS	11775	22/12/2017	No
16	HW2FWQ	UniCredit	11775	12/12/2017	No
17	HU9B4B	UniCredit	11790	12/12/2017	No
18	CV1MU7	Commerzbank	11800	15/11/2017	No
19	CE9Q7S	Commerzbank	11800	13/12/2017	No
20	UW9M60	UBS	11800	22/12/2017	No
21	HW2FWR	UniCredit	11800	12/12/2017	No
22	CV1MU8	Commerzbank	11825	15/11/2017	No
23	CE9Q7T	Commerzbank	11825	13/12/2017	No
24	UW9F8A	UBS	11825	22/12/2017	No
25	HW2FWS	UniCredit	11825	12/12/2017	No
26	CE9Q7U	Commerzbank	11850	13/12/2017	No
27	UW9L4T	UBS	11850	22/12/2017	No
28	CE9Q7V	Commerzbank	11875	13/12/2017	No
29	UW9EWA	UBS	11875	22/12/2017	No
30	CV1MUB	Commerzbank	11900	15/11/2017	No
31	CV1MUC	Commerzbank	11925	15/11/2017	No
32	CV1MUE	Commerzbank	11975	15/11/2017	11/08/2017
33	CE9SRT	Commerzbank	11975	13/12/2017	11/08/2017
34	HW2FWY	UniCredit	11975	12/12/2017	11/08/2017
35	CV1MGZ	Commerzbank	12000	15/11/2017	10/08/2017
36	CE9SRU	Commerzbank	12000	13/12/2017	10/08/2017
37	UW9EWG	UBS	12000	22/12/2017	10/08/2017
38	HW2FWZ	UniCredit	12000	12/12/2017	10/08/2017

Table 3.7: Barrier down-and-out options. Barriers and strikes are equal. “WKN” is a German securities identification code.

4 Concave Distortion Functions

4.1 Introduction

Concave distortion functions play a very important role in insurance and financial mathematics. They are used to define coherent risk measures, as introduced axiomatically by Artzner et al. (1999). A famous coherent risk measures is the expected shortfall. Risk measures are for example applied by insurances to compute the premium of an insurance contract or may describe a potential loss from a capital investment.

In this Chapter, we introduce concave distortion functions. We provide a list of desirable properties for families of concave distortion functions and look at many examples. Some of those example are new to literature.

We also prove a novel representation theorem and show that a family of concave distortion functions satisfying a certain translation equation can be represented by a distribution function. A famous example is the Wang transform, which is defined via the normal distribution and its inverse. It is well known that a coherent risk measure induced by the Wang transform reduces to the standard deviation premium principle for normal distributed random variables. Our representation theorem helps to interpret general concave distortion functions in a similar spirit.

An application of this theorem can be found in insurance science. Premium principles in actuarial science are used to determine the premium an insured has to pay to the insurance company in return for an insurance contract. For example the premium can be calculated by the expected loss of the insured object plus a multiple of the standard deviation of the loss. Moment based premium principles are easy to understand but in general are not monotone and cannot be used to compare the riskiness of different insurance contracts with each other. Our representation theorem makes it possible to compare two insurance risks with each other consistent with a moment based premium principle by defining an appropriate coherent risk measure.

In particular, we answer the following question: if an insurance company insures risk X for a certain premium and the premium is computed using a classical moment based premium principle, what would be an adequate premium for another risk Z consistent with the premium of X? We are able to answer this question even if Z as infinite second moments. Consistency between the premium for X and for Z is measured using performance measures as axiomatically introduced by Cherny and Madan (2008).

In Section 4.2 we introduce coherent risk measures. In Section 4.3 acceptability indices (performance measures) are presented. Section 4.4 defines concave distortion functions and points out the relation to coherent risk measures. In Section 4.5 we prove our main result of this Chapter: a representation theorem for a family of concave distortion functions. The proof of the theorem is devoted to Section 4.5.1. In Section 4.5.2 we apply the theorem to insurance science. Section 4.6 provides conclusions of this Chapter.

Concave distortion functions are also applied in Chapter 5 to construct financial markets with frictions.

4.2 Coherent Risk Measures

Before defining concave distortion functions, we recall the definition of coherent risk measures and acceptability indices from literature. Throughout this Chapter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define by

$$L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$$

the set of bounded random variables and by

$$L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$$

the set of integrable random variables on this probability space. In general for $p \in [1, \infty)$ we denote by

$$L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$$

the space of all measurable random variables whose absolute value raised to the $p-$th power has a finite integral.

Coherent risk measure are widely used in finance. For example portfolio managers may use a risk measure to get an idea of the potential loss from an investment. Let a random variable $X \in L^\infty$ describe the future random cash flow an investor will face at some future date. For example assume some stock on a certain company costs S_0 currency units today and can be modelled by the nonnegative random variable S_T at some future timepoint T. The time-horizon $0 < T < \infty$ is usually measured in fractions of a business year and might be equal to a year, a week, a second or any other positive time-interval depending on the time-horizon of the investor.

Today, the investor might lend exactly S_0 currency units from a bank, promising the bank to return the money plus interest rates at timepoint T, and buy the stock. At timepoint T, she sells the stock and receives S_T currency units. She has to return $S_0 e^{rT}$ to the bank, where r are some interest rates. At time T, she hence faces the future random cash flow $S_T - S_0 e^{rT}$. We discount that cash flow and define

$$X := e^{-rT} S_T - S_0.$$

If things go badly for the investor and the stock loses significantly on value, she makes considerable loses. Hence the need of a good risk management. There is a famous real-world example, where poor risk management led to a disastrous bankruptcy: the hedge fond "Long-Term Capital Management" (LTCM) founded by John Meriwether in 1994 financed its investments to a great extent by debts. LTCM went bankruptcy and was closed in 2000, see Jorion (2000).

In the following, we present the formal definition of risk measures. We assume all future random cash flows are already discounted. The random variables X and Y represent the value of a financial position, not a loss. We provide several applications to insurance science in which case $Y \geq 0$ describes some claim costs or insurance risk and $-Y$ models the loss of the insurance company. We find this perspective more convenient, as we will later analyse the performance of the residual cash flow $\pi - Y$ of an insurance company insuring risk $Y \geq 0$ in return for gaining premium π.

Definition 4.2.1. (Coherent risk measure). A map $\rho : L^\infty \to \mathbb{R}$ is called a *coherent risk measure* if it satisfies the following properties for all $X, Y \in L^\infty$:

R1: *Cash invariance:* $\rho(X + c) = \rho(X) - c$ for any $c \in \mathbb{R}$.

R2: *Monotonicity:* $X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$.

R4: *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $0 \leq \lambda \leq 1$.

R5: *Positive homogeneity:* $\rho(\lambda X) = \lambda\rho(X)$ where $0 \leq \lambda$.

A coherent risk measure assesses the riskiness of a future random cash flow X. The higher $\rho(X)$, the more risky is an investment in X. We say some future random cash flow X is *acceptable* if $\rho(X) \leq 0$. By some supervising agency, an investor might only be allowed to invest in acceptable investments. The value $\rho(X)$ can be interpreted as the amount of money which need to be added to the position X to make it acceptable: by R1 it holds

$$\rho(X + \rho(X)) = 0.$$

The value $\rho(X)$ could be seen as the minimal amount of capital an investor must own and deposit in order to be allowed by the supervising agency to enter into the trade X.

The cash invariance axiom means the risk measures fulfils a translation property. Cash can be added or subtracted to a position and the risk of the position will change exactly by that amount. Monotonicity means that a higher cash flow is less risky than a smaller one. The convexity axiom encourages diversification, i.e. a portfolio is less risky then the sum of its components. By the positive homogeneity property, the risk of a position changes linearly with its size.

For a detailed survey and economic interpretation of static coherent risk measures, see Artzner et al. (1999), Delbaen (2002) and Föllmer and Schied (2011, Section 4). The domain of some coherent risk measures can meaningfully be defined on L^p, $p \geq 1$, as well. See Remark 4.4.6 for important examples of coherent risk measures which can be defined on L^1 and L^2.

Example 4.2.2. Define the *worst-case* risk measure by

$$\rho_{\mathrm{W}}(X) := \inf\{x \in \mathbb{R},\ \ X + x \geq 0\ \ \mathbb{P} - \text{a.s.}\},\ \ X \in L^\infty.$$

This is the most conservative risk measure. If Ω is finite, it is minus the minimal value of X. Under ρ_{W} a position is only acceptable if it is always nonnegative. On the other side the map

$$\rho_{\mathrm{E}}(X) := -E_{\mathbb{P}}[X], \quad X \in L^{\infty},$$

also defines a coherent risk measure, which could be seen as the less conservative risk measure. A position is acceptable, if its expectation is nonnegative.

Let us assume a stock cost $S_0 = 100$ currency units today and can be modelled by a log-normal random variable after one year, i.e. $T = 1$ and

$$S_T = S_0 e^{r - \frac{\sigma^2}{2} + \sigma Z},$$

where Z is a standard normal variable and $\sigma > 0$. Say an investor is only allowed to invest in *acceptable* future random cash flows. She would like to know how much money m she is allowed to borrow from a bank in order to finance the purchase of the stock. Let

$$X = e^{-rT} S_T - m$$

be the discounted future random cash flow of the investor. It holds

$$\rho_{\mathrm{W}}(X) = m,$$

which is less or equal to zero if and only if $m \leq 0$. Under the worst-case risk measure, the investor is not allowed to borrow a cent from the bank and has to finance the purchase of the stock completely by her own capital. It holds

$$\rho_{\mathrm{E}}(X) = -S_0 + m.$$

Under the risk measure ρ_{E} the investor can borrow up to S_0 currency units from the bank and could finance the purchase of the stock completely by debts.

The risk measure the investor has to use depends on the conservativeness of the supervising agency. In this thesis, all risk measures lay somewhere between the worst-case risk measure and the risk measure based on the expectation operator.

Another application of risk measures appears in insurance science:

Example 4.2.3. By Artzner et al. (1999), a coherent risk measure could be seen as a *premium principle*, i.e. can be used to determine the price of an insurance contract. Let us assume an insurance company sells contracts to people exposed to risk from some natural disaster, e.g. an earthquake. Let us further assume, the possible financial loss due to the disaster can be modelled by a nonnegative random variable X and the insurance receives a total premium of π currency units for selling the contracts. Using a coherent risk measure ρ, the contracts are acceptable from the perspective of the insurance, if

$$\rho(-X + \pi) \leq 0,$$

i.e. if the insurance receives at least a premium greater or equal to $\rho(-X)$.

Coherent risk measures can be represented by the supremum over a set of probability measures, as the following theorem shows.

Theorem 4.2.4. *Let $\rho : L^\infty \to \mathbb{R}$ be a coherent risk measure. Then the following conditions are equivalent:*

i) *ρ is continuous from above: if $X_n \searrow X$ $\mathbb{P}-a.s.$ then $\rho(X_n) \nearrow \rho(X)$.*

ii) *ρ satisfies the Fatou property: for any bounded sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ which converges $\mathbb{P}-a.s.$ to $X \in L^\infty$, the following holds:*

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

iii) *There is a set of probability measures $\mathcal{P}$ such that any $P \in \mathcal{P}$ is absolutely continuous with respect to $\mathbb{P}$ and*

$$\rho(X) = \sup_{P\in\mathcal{P}} \left\{E_P[-X]\right\}.$$

For the proof see Delbaen (2002) or Föllmer and Schied (2011, Corollary 4.37).

4.3 Acceptability Index

Similar to the axiomatic approach of describing coherent risk measure, see Section 4.2, Cherny and Madan (2008) introduced an axiomatic approach to characterise maps measuring the *performance* or the *attractiveness* of a future random cash flow and defined an *acceptability index* as map from the set of bounded random variables L^∞ to the extended half line $[0, \infty]$ fulfilling the following axioms. We use an acceptability index in Section 4.5.2 in the context of insurance science and in Section 5.3.1 where the *conic finance* theory is presented.

Definition 4.3.1. A map $\alpha : L^\infty \to [0, \infty]$ is called an *acceptability index* if it satisfies the following properties for $X, Y \in L^\infty$:

A1: *Quasi-concavity:* If $\alpha(X) \geq \gamma$ and $\alpha(Y) \geq \gamma$, then $\alpha(\lambda X + (1-\lambda)Y) \geq \gamma$ for $\gamma \geq 0$ and $0 \leq \lambda \leq 1$.

A2: *Monotonicity:* If $X \leq Y$ then $\alpha(X) \leq \alpha(Y)$.

A3: *Fatou property:* If $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ is a bounded sequence of random variables such that $\alpha(X_n) \geq \gamma$ for all $n \in \mathbb{N}$ and (X_n) converges to X $\mathbb{P}-$a.s., then $\alpha(X) \geq \gamma$.

A4: *Scale invariance:* It holds $\alpha(\lambda X) = \alpha(X)$ where $\lambda > 0$.

Quasi-concavity states that a diversified portfolio has a performance level at least as high as its components. Monotonicity is a basic property: generally a greater cash flows is preferred over a smaller one. The Fatou property is a technical axiom, it is a weak form of continuity and needed in order to prove a certain representation theorem. The scale invariance of performance measures states that the performance of a cash flow does not depend on its size.

We refer to Cherny and Madan (2008) for a more detailed economic interpretation of the axioms presented here and a discussion of additional properties an acceptability index can possibly have, e.g. law invariance.

Example 4.3.2. For a given coherent risk measure ρ satisfying the Fatou property, the *coherent Risk-Adjusted Return on Capital* (RAROC) is defined by

$$\alpha_{\text{RAROC}}(X) := \begin{cases} \frac{E[X]}{\rho(X)} & , E[X] > 0,\ \rho(X) > 0 \\ \infty & , \rho(X) \leq 0 \\ 0 & , E[X] \leq 0. \end{cases}$$

For an investor, an investment is more attractive, the higher the expected return, measured by $E[X]$ and the lower the risk related to X and measured by $\rho(X)$. Indeed, the greater the expected value of the future random cash flow X and the smaller its risk, the greater the performance of X measured by α_{RAROC}.

Cherny and Madan showed that any unbounded acceptability index can be represented by a family of coherent risk measures and proved the following theorem:

Theorem 4.3.3. *Let $\mathcal{P}$ be the set of probability measures absolutely continuous with respect to $\mathbb{P}$. A map $\alpha : L^\infty \to [0, \infty]$ unbounded above is an acceptability index if and only if there exists a family of subsets $(\mathcal{D}_\gamma)_{\gamma \geq 0}$ of $\mathcal{P}$ such that $\mathcal{D}_{\gamma_1} \subset \mathcal{D}_{\gamma_2}$ for $0 \leq \gamma_1 \leq \gamma_2$ and*

$$\alpha(X) = sup\left\{\gamma \geq 0 \,:\, \rho^\gamma(X) \leq 0\right\}, \tag{4.1}$$

where $\rho^\gamma(X) := \sup_{P \in \mathcal{D}_\gamma} E_P[-X]$, $\gamma \geq 0$ and $\sup \emptyset = 0$.

Axiom A3 plays an important part to prove this theorem. $(\rho^\gamma)_{\gamma \geq 0}$ is a set of coherent risk measures, see Föllmer and Schied (2011, Proposition 2.84 and Chapter 4). The family of coherent risk measures is increasing, i.e. for $0 \leq \gamma_1 \leq \gamma_2$ it follows $\rho^{\gamma_1}(X) \leq \rho^{\gamma_2}(X)$. The economic interpretation of the map α defined via Equation (4.1) is the following: the performance of X is the greatest level γ, such that the risk of X under ρ^γ is still acceptable.

The domain of an index of acceptability can be defined as L^p if the acceptability index is induced via Equation (4.1) by a family of coherent risk measures, which have domain L^p.

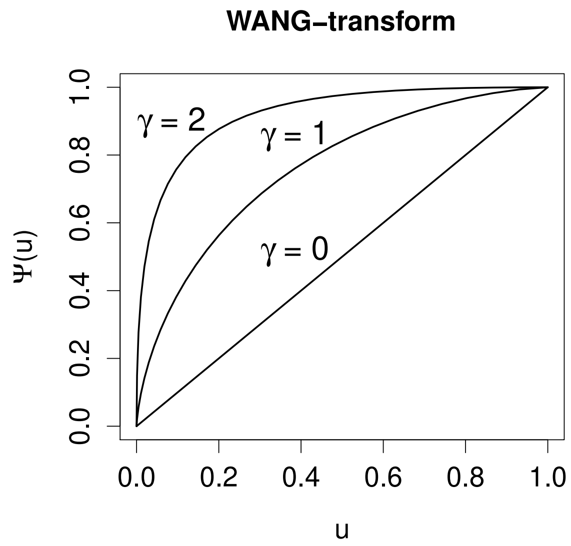

Figure 4.1: The Wang transform.

4.4 Concave Distortions and their Connections to Risk Measures

In this Section, we define concave distortion functions and provide a connection to coherent risk measures.

Definition 4.4.1. A *concave distortion function* $\Psi : [0,1] \to [0,1]$ is monotonically increasing and concave and it holds $\Psi(0) = 0$ and $\Psi(1) = 1$.

Example 4.4.2. The *Wang transform* is defined by

$$\Psi(u) = \Psi^{\gamma}_{\text{WANG}}(u) = \Phi(\Phi^{-1}(u) + \gamma), \quad u \in [0,1], \quad \gamma \geq 0, \tag{4.2}$$

which involves the standard cumulative normal distribution Φ and its inverse, see Wang (2000), and is widely used in actuarial science. The WANG transform is drawn for different values of γ in Figure 4.1.

Example 4.4.3. The *ess sup-expectation convex combinations* distortion function jumps at zero and is defined for $\lambda \in [0,1]$ by

$$\Psi(u) = \begin{cases} 0 & , u = 0 \\ \lambda + (1-\lambda)u, & u \in (0,1], \end{cases}$$

see Bannör and Scherer (2014). The risk measure induced by Ψ involves a convex combination of the essential supremum and the ordinary expectation. Bannör and Scherer (2014) applied this distortion function to calibrate a non-linear pricing model to quoted bid and ask prices.

For a concave distortion function Ψ one can define a *Choquet integral* for a random variable $Y \in L^\infty$ by

$$H(Y) = \int_{-\infty}^{0} \left(\Psi\left(\mathbb{P}\left[Y > y\right]\right) - 1\right) dy + \int_{0}^{\infty} \Psi\left(\mathbb{P}\left[Y > y\right]\right) dy. \tag{4.3}$$

According to Föllmer and Schied (2011, Theorem 4.70) the map

$$\rho_\Psi(Y) := H(-Y) \tag{4.4}$$

$$= \int_{0}^{\infty} \left(\Psi\left(\mathbb{P}\left[Y < y\right]\right) - 1\right) dy + \int_{-\infty}^{0} \Psi\left(\mathbb{P}\left[Y < y\right]\right) dy \tag{4.5}$$

is a coherent risk measure with domain L^∞, see also Kusuoka (2001). We say the risk measure ρ_Ψ is *induced* by the distortion function Ψ. Let

$$F_Y(x) := \mathbb{P}(Y \leq x)$$

be the distribution function of Y. For continuous Ψ it holds

$$\rho_\Psi(Y) = -\int_{\mathbb{R}} xd\left(\Psi(F_Y(x))\right) \tag{4.6}$$

$$= -\int_{\mathbb{R}} x\Psi'\left(F_Y(x)\right) f_Y(x)dx, \tag{4.7}$$

see Föllmer and Schied (2004, Theorem 4.64). The function $\Psi(F_Y(.))$ is called the *distorted distribution function of F_Y with respect to the concave distortion* Ψ: smaller values of Y get higher probabilities and the probabilities of greater values of Y are reduced. We *distort* the distribution function of Y. The value $-\rho_\Psi(Y)$ can be seen as the expectation of a random variable with distribution function $\Psi(F_Y(x))$. For Equation (4.7) we assume that the distribution function F_Y of Y is differentiable with density f_Y and the first derivative of Ψ exists.

Remark 4.4.4. Several authors interpret Y as a loss and defined a coherent risk measure directly via Equation (4.3), see Wang (2000, eq. (2)) and Tsanakas (2004, eq. (3)). If Ψ is continuous it holds

$$H(Y) = \int_{0}^{1} F_Y^{-1}(y)d\hat{\Psi}(y), \tag{4.8}$$

where $\hat{\Psi}(u) = 1 - \Psi(1-u)$ is the *dual* distortion of Ψ, see Föllmer and Schied (2011, Theorem 4.70). Acerbi (2002) and Tsukahara (2009, eq. (1.1)) among others work with the convex dual distortion to define coherent risk measures via Equation (4.8).

Example 4.4.5. The risk-measure based on the expectation operator ρ_{E} and the worst-case risk measure ρ_{W}, see Example 4.2.2, can be represented by concave distortion functions. It holds

$$\rho_\Psi(.) = \begin{cases} \rho_{\mathrm{E}}(.) & \text{, if } \Psi(u) = u \ \ \forall u \in [0,1] \\ \rho_{\mathrm{W}}(.) & \text{, if } \Psi(u) = 1_{(0,1]}(u) \ \ \forall u \in [0,1], \end{cases}$$

see also Föllmer and Schied (2011, Remark 4.50).

Remark 4.4.6. If the distortion function Ψ is continuous and differentiable, let

$$\sigma(u) := \Psi'(1-u).$$

For $q \in [1,\infty]$, let $p \in [1,\infty]$ such that $\frac{1}{q} + \frac{1}{p} = 1$. If $\sigma \in L^q$ it follows that the coherent risk measure ρ_Ψ is finite on the domain L^p. This follows using Hölder's inequality, see Pichler (2013). In particular if the derivative Ψ' is bounded on $[0,1]$, the functional ρ_Ψ is well defined on L^1. A little calculus shows that

$$\int_0^1 \left(\frac{\partial}{\partial u} \Psi^\gamma_{\mathrm{WANG}}(u) \right)^2 du = \exp(\gamma^2).$$

Hence the coherent risk measure induced by the Wang transform, see Equation (4.2), is well defined on L^2.

4.4.1 Parametric Families of Concave Distortion Functions

Often, one would like work with a parametric family of risk measures $(\rho_\gamma)_{\gamma \geq 0}$, where γ models the view of the risk manager: the greater γ, the more conservative the risk measure ρ_γ. For example Wang (1995) and Wang (2000) proposed the proportional hazard transform and the Wang transform as distortion functions for insurance premium calculation of an insurance risk X. The premium is computed according to Equation (4.4). Both distortions depend on a single parameter γ: the premium of a risk is thus a function of γ and varies continuously between the smallest and greatest reasonably premium: the expected value and maximal value of X. The insurance company may choose γ depending on many external circumstances and the risk-attitude of the company. Wang (2000) proposed that possible changes in court rulings or in the interest rate yield curve, moral hazards by insurance buyers and competition with other insurance companies, should be taken into consideration when choosing the parameter γ.

Another use of a family of risk measures is discussed in Cherny and Madan (2008), who proved that an acceptability index, which measures the performance of a future random cash flow, can be represented by an increasing family of coherent risk measures, compare with Section 4.3.

If the parametric family of risk measures is induced by distortion functions, we need to work with a family of concave distortion functions, which is defined as follows:

Definition 4.4.7. A *family of concave distortion functions* (FCDF) $(\Psi^\gamma)_{\gamma \geq 0}$ is a set of functions

$$\Psi^\gamma : [0,1] \to [0,1], \ \ \gamma \geq 0,$$

that are monotonically increasing and concave for all $\gamma \geq 0$ and for which

$$\Psi^\gamma(0) = 0 \text{ and } \Psi^\gamma(1) = 1.$$

Moreover the family is monotonically increasing and continuous at γ, i.e. it holds for all $u \in [0,1]$ that

$$\Psi^{\gamma_1}(u) \leq \Psi^{\gamma_2}(u), \quad \gamma_1 \leq \gamma_2,$$

and the map $\gamma \mapsto \Psi^\gamma(u)$ is continuous for all $u \in [0,1]$.

We note that the map $u \mapsto \Psi^\gamma(u)$ is continuous on $(0,1]$ for all $\gamma \geq 0$ but might jump at zero. Let us additionally assume the following conditions:

[E] It holds $\Psi^0(u) = u$, for $u \in [0,1]$.

[W] It holds $\lim_{\gamma\to\infty} \Psi^\gamma(u) = 1$, for $u \in (0,1]$.

[T] It holds $\Psi^{\gamma_2}(\Psi^{\gamma_1}(u)) = \Psi^{\gamma_1+\gamma_2}(u)$, for $\gamma_1, \gamma_2 \geq 0$ and $u \in [0,1]$.

Define $\Psi'_\gamma(u) = \frac{\partial}{\partial u}\Psi^\gamma(u)$, if the partial derivative exists.

[L] It holds $\lim_{u\searrow 0} \Psi'_\gamma(u) = \infty, \quad \gamma > 0$.

[G] It holds $\lim_{u\nearrow 1} \Psi'_\gamma(u) = 0, \quad \gamma > 0$.

[A] It holds $\Psi^{\xi_\gamma}\left(\frac{1}{2} + \xi_p\right) = \frac{1}{2} + \xi_p + \frac{1}{2}\xi_\gamma + o\left(|\xi_p| + |\xi_\gamma|\right), \quad \xi_p \in (0,1), \quad \xi_\gamma > 0.$

The interpretation of Definition 4.4.7 is the following: the greater γ, the greater the distortion and the more conservative the risk measure induced by Ψ^γ. Conditions [E] and [W] are quite natural: it is usually desirable that for $\gamma = 0$ no distortion occurs, the risk measure induced by Ψ^0 should be equal to the negative expectation operator.

For $\gamma \to \infty$ the risk measure induced by Ψ^γ should converge to the worst-case risk measure, i.e. $\Psi^\gamma(u)$ should converge to 1 for $u > 0$, which is expressed in condition [W].

Condition [T] means distorting the probability u first at level γ_1 and then at level γ_2 is the same as distorting the probability once at level $\gamma_1 + \gamma_2$. This condition is called *translation equation* in functional equation theory, see Aczél (1966, Section 6.1.1.). Its use becomes clear in Section 4.5, where we prove a representation theorem for FCDF.

Assumption [L] and [G] have a purely financial background. Assumption [L] ensures loss aversion. Large losses should be overstated up to infinity. An example best explains this point.

Example 4.4.8. Let $L < -1$ and define a random variable X taking the large loss L with probability

$$p_L := -\frac{1}{L}$$

and the value $\frac{L}{L+1}$ with probability $1-p_L$. The expectation of X is zero. For a distortion function Ψ, it holds

$$\begin{aligned}\rho_\Psi(X) &= -L\Psi(p_L) - \frac{L}{L+1}\left(1-\Psi(p_L)\right)\\ &= -Lp_L\frac{\Psi(p_L)-\Psi(0)}{p_L} - \frac{L}{L+1}\left(1-\Psi(p_L)\right)\\ &\leq \Psi'(0).\end{aligned}$$

If $\Psi'(0)$ was finite, there would exist an upper bound of the risk of X independent of the maximum loss L.

Similarly assumptions [G] ensures being enticed by large gains, see Wang (1996), Cherny and Madan (2008) and Balbás et al. (2009) for details. From a purley financial point of view, there are some other properties a FCDF should satisfy, e.g. the map $u \mapsto \Psi^\gamma(u)$ should be strictly increasing, such distortion functions are called *complete* by Balbás et al. (2009).

Assumption [A] describes a linear approximation of the FCDF around $(u,\gamma) = \left(\frac{1}{2}, 0\right)$ by the total differential and holds if the function $(u,\gamma) \mapsto \Psi^\gamma(u)$ is partial differentiable and all partial derivatives at $\left(\frac{1}{2}, 0\right)$ are continuous with

$$\left.\frac{\partial}{\partial u}\Psi^\gamma(u)\right|_{(u,\gamma)=(\frac{1}{2},0)} = 1 \text{ and } \left.\frac{\partial}{\partial \gamma}\Psi^\gamma(u)\right|_{(u,\gamma)=(\frac{1}{2},0)} = \frac{1}{2}. \tag{4.9}$$

This approximation is used in Section 5.4 in a discrete time model to prove convergence of various plain vanilla and exotic options if the number of trading periods approaches infinity.

Let us provide some FCDF often used in literature.

Example 4.4.9. The FCDF corresponding to the expected shortfall at level $e^{-\gamma} \in (0,1]$ can be defined by

$$\Psi^\gamma_{\text{ExpShortfall}}(u) = \min(ue^\gamma, 1), \quad u \in (0,1), \quad \gamma \geq 0,$$

see e.g. Föllmer and Schied (2011, Example 4.71) and is drawn for $\gamma = 1$ as Ψ_1 in Figure 4.3. For u close enough to $\frac{1}{2}$ and γ close enough to zero, it holds

$$\Psi^\gamma_{\text{ExpShortfall}}(u) = ue^\gamma.$$

This distortion function obviously satisfies assumptions [E, W, T, G, A] but does not satisfy assumption [L].

Let Φ be the cumulative standard normal distribution function and φ the normal density. The Wang transform

$$\Psi^\gamma_{\text{WANG}}(u) = \Phi\left(\Phi^{-1}(u) + \frac{1}{2\varphi(0)}\gamma\right), \quad u \in (0,1), \gamma \geq 0,$$

was introduced by Wang (2000) and fulfils [E-A]. The Wang transform is widely used in actuarial science and was originally defined without the scaling factor $\frac{1}{2\varphi(0)}$. We need to rescale γ of the original Wang transform, to satisfy assumption [A].

Cherny and Madan (2008) introduced four FCDF: the MAXVAR and MINVAR distortions, which are reparametrizations of the power distortion and its dual, the proportional hazards distortion, see Wang (1995, 1996), and the MINMAXVAR and MAXMINVAR, which are compositions of the former two.

Example 4.4.10. The first two FCDF are defined by

$$\Psi^{\gamma}_{\mathrm{MINVAR}}(u) = 1 - (1-u)^{1+\gamma}, \;\; u \in (0,1), \gamma \geq 0,$$

and

$$\Psi^{\gamma}_{\mathrm{MAXVAR}}(u) = u^{\frac{1}{1+\gamma}}, \;\; u \in (0,1), \gamma \geq 0,$$

and satisfy assumption [A] after rescaling, i.e. replacing γ by $-\frac{\gamma}{\log\left(\frac{1}{2}\right)}$.

Example 4.4.11. The FCDF MINMAXVAR and MAXMINVAR are defined respectively by

$$\Psi^{\gamma}_{\mathrm{MINMAXVAR}}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}, \;\; u \in (0,1), \gamma \geq 0,$$

and

$$\Psi^{\gamma}_{\mathrm{MAXMINVAR}}(u) = \left(1 - (1-u)^{1+\gamma}\right)^{\frac{1}{1+\gamma}}, \;\; u \in (0,1), \gamma \geq 0.$$

Both FCDF satisfy assumption [A], replacing γ by $-\frac{\gamma}{2\log\left(\frac{1}{2}\right)}$ which again corresponds to a simple rescaling of the FCDF.

Table 4.1 summarizes the properties of the presented FCDF, compare with a similar table in Madan and Schoutens (2016a, Table 4.1). For most FCDF it is not so obvious to see whether Assumption [T] is satisfied or not. That point is discussed in more detail in Section 4.5. In particular Examples 4.5.2 - 4.5.5 provide some more FCDF with a special focus on Assumption [T].

	[E]	[W]	[T]	[L]	[G]	[A]
Expected Shortfall	y	y	y	n	y	y
Wang transform	y	y	y	y	y	y
MINVAR	y	y	n*	n	y	y
MAXVAR	y	y	n*	y	n	y
MINMAXVAR	y	y	n	y	y	y
MAXMINVAR	y	y	n	y	y	y
Laplace	y	y	y	n	n	n

Table 4.1: Properties of several FCDF.
*) Examples 4.5.7 and 4.5.8 show that there exist a reparametrization of the FCDF satisfying assumption [T].

FCDF induced by Distribution Functions

In this thesis, generalizations of the Wang transform play a special role in Section 4.5. The Wang transform, see Equation (4.2) has been modified by Wang (2002) to a two factor model replacing the normal distribution in Equation (4.2) by Student's t-distribution but leaving the inverse normal distribution inside the brackets untouched. Kijima and Muromachi (2006) introduced a new transform involving a non-central t-distribution and the inverse of a standard t-distribution. The classical Wang transform has been extended to the multidimensional case by Kijima (2006). Kijima and Muromachi (2008) generalized the Wang transform and constructed a transformation using the normal distribution and the inverse of the cumulative distribution function of the quotient of a normal distributed random variable and some independent positive random variable Y. For $Y = 1$ the classical Wang transform is obtained. Tsukahara (2009) generalized the Wang transform by replacing the normal distribution function Φ in Equation (4.2) by a general distribution function F and its inverse by F^{-1}. Tsukahara called such a distribution function a *one-parameter distortion group*. We say a FCDF (Ψ^γ) is *induced by the distribution function* F if

$$\Psi^\gamma(u) = F\left(F^{-1}(u) + \gamma\right), \quad u \in [0,1], \quad \gamma \geq 0, \tag{4.10}$$

where F^{-1} is understood to be the generalized inverse of F and we define $F(-\infty) = 0$ and $F(\infty) = 1$.

Remark 4.4.12. The function Ψ^γ defined by Equation (4.10) is continuous in the variable u if F and F^{-1} are continuous. It is easy to see that the map $u \mapsto \Psi^\gamma(u)$ is concave for all $\gamma \geq 0$, if the corresponding density $f = F'$ has support on a possibly unbounded interval U, i.e. $f(x) > 0$ for $x \in U$, and is log-concave, compare with Tsukahara (2009, p. 697). The function f is called *log-concave* if $\log(f)$ is concave, see Bagnoli and Bergstrom (2005).

A FCDF induced by the Laplace distribution has been introduced by Guillaume, Junike, Leoni and Schoutens (2018):

Example 4.4.13. A random variable with mean zero and variance one which is Laplace distributed, has the following distribution function

$$F(x) = \begin{cases} \frac{1}{2}e^{\sqrt{2}x} & , x < 0 \\ 1 - \frac{1}{2}e^{-\sqrt{2}x} & , x \geq 0. \end{cases}$$

It induces via Equation (4.10) the *Laplace distortion*

$$\Psi^{\gamma}_{\text{Laplace}}(u) = \begin{cases} ue^{\sqrt{2}\gamma} & , u \in \left[0, \frac{1}{2}e^{-\sqrt{2}\gamma}\right) \\ 1 - \frac{e^{-\sqrt{2}\gamma}}{4u} & , u \in \left[\frac{1}{2}e^{-\sqrt{2}\gamma}, \frac{1}{2}\right) \\ ue^{-\sqrt{2}\gamma} + 1 - e^{-\sqrt{2}\gamma} & , u \in \left[\frac{1}{2}, 1\right], \end{cases}$$

which is linear for $u < \frac{1}{2}e^{-\sqrt{2}\gamma}$ and for $u \geq \frac{1}{2}$. In between it behaves like a reciprocal function and it is clearly concave. Applying the Laplace distortion to a uniform distribution function, which appears e.g. via a Monte Carlo simulation, leads to a new interesting interpretation of the parameter γ. In Figure 4.2 the density of a Laplace distorted uniform distribution is drawn. It is high and constant at the beginning, then it drops sharply and is quite low and constant at the right hand side of the median. So if we only wish to distort the $q-$quantile of a uniform distribution strongly, we can simply choose $\gamma = -\frac{1}{\sqrt{2}}\log(2q)$. On the other hand, if real data is given and we calculate implicitly the parameter γ, the value $\frac{1}{2}e^{-\sqrt{2}\gamma}$ can be interpreted as the quantile that is the most strongly distorted.

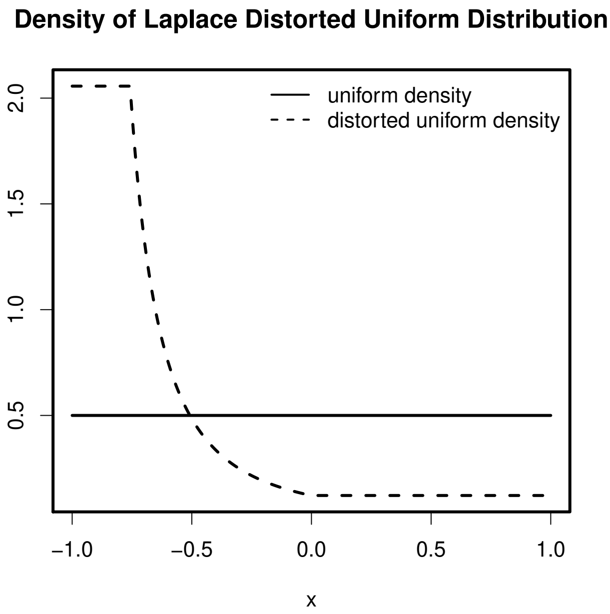

Figure 4.2: Density of Laplace distorted uniform distribution on $[-1, 1]$ at distortion level $\gamma = 1$.

4.5 A Representation Theorem

We are able to prove that for any FCDF (Ψ^γ), which satisfies conditions [E], [W] and [T] of Definition 4.4.7, there exists a distribution function G such that

$$\Psi^\gamma(u) = G(G^{-1}(u) + \gamma), \quad u \in (0, 1), \quad \gamma \geq 0. \tag{4.11}$$

Conversely, if a FCDF is represented as in Equation (4.11), it satisfies conditions [E], [W] and [T].

Based on results from functional equation theory, see Aczél (1966, Section 6.1.), Tsukahara (2009) obtained a similar result, under the additional assumptions that the FCDF is continuous in the variable u and strictly increasing in the variable γ and that G is strictly increasing. While we interpret a random variable X as a net worth, Tsukahara interprets X as a loss, see also Remark 4.4.4. Therefore Tsukahara works with the *convex dual*

$$u \mapsto 1 - \Psi(1 - u)$$

of the concave distortion function Ψ.

Examples 4.5.2 - 4.5.5 provides various FCDF, which are not continuous at $u = 0$ or are not strictly increasing in γ but can be represented by a distribution function. Some of those FCDF are applied in Section 4.5.2 to actuarial science and we develop a new FCDF using the gamma distribution, which includes the expected shortfall and the Wang transform as special cases. In the following theorem, we present our main result of this Section: a relationship between distribution functions and FCDF. The proof is devoted to Section 4.5.1. An application to insurance science can be found in Section 4.5.2.

Theorem 4.5.1. *Let (Ψ^γ) be a FCDF. Let $u_0 \in (0,1)$. The following two statements are equivalent.*

i) *The* FCDF (Ψ^γ) *satisfies conditions [E], [W] and [T].*

ii) *There exists a unique distribution function G, such that $G(0) = u_0$ and*

$$\Psi^\gamma(u) = G(G^{-1}(u) + \gamma), \quad \gamma \geq 0, \quad u \in (0,1). \tag{4.12}$$

Necessary conditions for a function $u \mapsto \Psi^\gamma(u)$, defined via Equation (4.12), to be concave are given in Remark 4.4.12. The constant u_0 mentioned in the theorem can be chosen arbitrarily: if G induces Ψ^γ then also the shifted distribution $\tilde{G}(x) := G(x+\mu)$ for any $\mu \in \mathbb{R}$ induces Ψ^γ. Hence we could reformulate Theorem 4.5.1 and say that G is unique up to location translation. The distribution function G can be identified by

$$G(x) = \begin{cases} \Psi^x(u_0) & , x \geq 0 \\ \overline{\Psi}^{-x}(u_0) & , x < 0, \end{cases} \tag{4.13}$$

where $\overline{\Psi}^\gamma$ is the generalized inverse of the function $u \mapsto \Psi^\gamma(u)$, in particular for $\gamma \geq 0$

$$\begin{aligned} \overline{\Psi}^\gamma : [0,1] &\to [0,1] \\ p &\mapsto \inf\{u \in [0,1] : \Psi^\gamma(u) \geq p\}. \end{aligned}$$

We provide four examples of FCDF satisfying conditions [E], [W] and [T]. The four distortions are also shown in Figure 4.3.

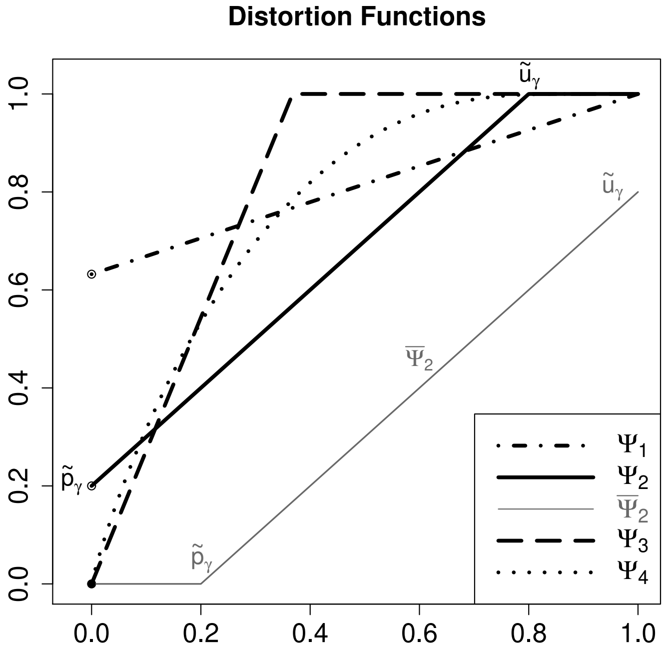

Figure 4.3: Distortion Functions from Examples 4.5.2 - 4.5.5. We set $\gamma = 1$. $\overline{\Psi}_2$ denotes the generalized inverse of Ψ_2. The jump-size of Ψ_2 at zero is defined by $\tilde{p}_\gamma$ and the point, where Ψ_2 first reaches one, is defined by $\tilde{u}_\gamma$.

Example 4.5.2. The following FCDF is not continuous at $u = 0$. Let

$$\Psi_1^\gamma(u) := \begin{cases} 0 & , u = 0 \\ 1 - (1-u)e^{-\gamma} & , u > 0, \end{cases}$$

The FCDF (Ψ_1^γ) is called "ess sup-*expectation convex combination*" by Bannör and Scherer (2014) because the Choquet integral induced by (Ψ_1^γ) involves a convex combination of the essential supremum and the ordinary expectation. Bannör and Scherer (2014) applied this FCDF to calibrate a non-linear pricing model to quoted bid-ask prices. $(\Psi_1^\gamma)_{\gamma \geq 0}$ is induced by the exponential distribution function

$$G_1(x) = \begin{cases} 1 - e^{-x} & , x > 0 \\ 0 & , \text{otherwise}. \end{cases}$$

Example 4.5.3. Let

$$\Psi_2^\gamma(u) := \begin{cases} 0 & , u = 0 \\ \min\left(u + \frac{\gamma}{\lambda}, 1\right) & , u > 0. \end{cases}$$

This FCDF is induced by the uniform distribution function on $\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$ for any $\lambda > 0$.

Example 4.5.4. The FCDF corresponding to the expected shortfall at level $e^{-\gamma} \in (0,1]$, see e.g. Föllmer and Schied (2011, Example 4.71), can be defined by

$$\Psi_3^\gamma(u) := \min(ue^\gamma, 1).$$

This FCDF is induced by the distribution $G_3(x) = \min(e^x, 1)$, $x \in \mathbb{R}$ and is increasing in the variable γ but not strictly increasing.

Let X be exponential distributed. It holds

$$\rho_{\Psi_3^\gamma}(-X) = E[X]e^\gamma,$$

i.e. the expected shortfall reduces to the expected value premium principle when applied to exponential risks.

The next example is applied in Section 4.5.2 to insurance science.

Example 4.5.5. Let

$$\Psi_4^\gamma(u) := \tilde{G}(\tilde{G}^{-1}(u) + \gamma),$$

The FCDF (Ψ_4^γ) is similar to the Wang transform but replacing the normal distribution function by the function

$$\tilde{G}(x) = 1 - \Gamma_{\alpha,\beta}\left(-\frac{\sqrt{\alpha}}{\beta}x\right), \;\; x < 0,$$

where $\Gamma_{\alpha,\beta}$ is the gamma distribution with shape α and rate β. (Ψ_4^γ) generalizes the expected shortfall: for $\alpha = 1$ and $\beta = 1$, (Ψ_3^γ) and (Ψ_4^γ) are identical. Setting $\beta := \sqrt{\alpha}$, (Ψ_4^γ) converges to the Wang transform for large α. We will see in Example 4.5.13, that the coherent risk measure induced by (Ψ_4^γ), reduces to the *standard deviation premium principle* when applied to gamma distributed random variables.

Cherny and Madan (2008) proposed some FCDF, called MINVAR, MAXVAR, MINMAXVAR and MAXMINVAR, see Example 4.4.9, which do not satisfy condition [T]. But as we shall see, sometimes it is possible find a reparametrization, by rescaling the parameter γ, such that the reparameterized FCDF does satisfy condition [T] and hence can be represented by a distribution function. In the following definition we state more precisely what we mean by a reparametrization.

Definition 4.5.6. We say that the FCDF $\left(\tilde{\Psi}^\gamma\right)_{\gamma\geq 0}$ is a *reparametrization* of the FCDF $(\Psi^\gamma)_{\gamma\geq 0}$ if there exist bijective function

$$t : [0,\infty) \to [0,\infty)$$

such that $t(0) = 0$ and

$$\Psi^{t(\gamma)}(u) = \tilde{\Psi}^\gamma(u), \;\; u \in [0,1], \;\; \gamma \geq 0.$$

Example 4.5.7. The MAXVAR FCDF is defined by $\Psi^{\gamma}_{\text{MAXVAR}}(u) = u^{\frac{1}{1+\gamma}}$ and there is a slight modification which indeed satisfies condition [T], in particular let

$$\tilde{\Psi}^{\gamma}_{\text{MAXVAR}}(u) := u^{\exp(-\gamma)},$$

which is a reparametrization of $\Psi^{\gamma}_{\text{MAXVAR}}$. By Theorem 4.5.1, the FCDF $\left(\tilde{\Psi}^{\gamma}_{\text{MAXVAR}}\right)$ is induced by the distribution function

$$F_{\text{MAXVAR}}(x) \quad = \quad e^{-\exp(-x)}, \;\; x \in \mathbb{R},$$

which is the Gumbel distribution with location zero and scale one. Let X be a Gumbel distributed random variable with location $\mu \in \mathbb{R}$ and scale $\sigma > 0$. X has distribution function

$$F_X(x) = e^{-\exp\left(-\frac{x-\mu}{\sigma}\right)}, \;\; x \in \mathbb{R}.$$

It holds

$$\rho_{\tilde{\Psi}^{\gamma}_{\text{MAXVAR}}}(X) = -E[X] + \sigma\gamma$$

i.e. the coherent risk measures induced by the MAXVAR FCDF and applied to a Gumbel distributed random variable X can be expressed by a linear mapping of the expectation of X.

Example 4.5.8. The MINVAR FCDF is defined by $\Psi^{\gamma}_{\text{MINVAR}}(u) = 1 - (1-u)^{\gamma+1}$ and can be represented after a reparametrization by $1 - G(-x)$, where G is the Gumbel distribution function with location zero and scale one.

We have seen in Example 4.5.7 that the MAXVAR and MINVAR FCDF defined by Cherny and Madan (2008) do not satisfy the condition [T] but there exist a reparametrization satisfying condition [T]. The following proposition is useful to check whether a FCDF can be reparameterized into a FCDF satisfying condition [T].

Proposition 4.5.9. *Let (Ψ^{γ}) be a FCDF. If there exist a reparametrization $\left(\tilde{\Psi}^{\gamma}\right)$ which satisfies condition [T], then it holds*

$$\Psi^{\gamma_1}\left(\Psi^{\gamma_2}(u)\right) = \Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u)\right), \;\; \gamma_1, \gamma_2 \geq 0, \;\; u \in [0,1], \tag{4.14}$$

i.e. the original FCDF is permutable.

Proof. Let $\gamma_1, \gamma_2 \geq 0$ and $u_0 \in [0,1]$. Then it follows

$$\Psi^{\gamma_1}\left(\Psi^{\gamma_2}(u_0)\right) = \tilde{\Psi}^{t(\gamma_1)}\left(\tilde{\Psi}^{t(\gamma_2)}(u_0)\right) = \tilde{\Psi}^{t(\gamma_1)+t(\gamma_2)}(u_0) = \Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u_0)\right),$$

for a suitable function t. □

Simple numerical examples and Proposition 4.5.9 show that the following FCDF

$$\Psi^{\gamma}_{\text{MINMAXVAR}}(u) \quad = \quad 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma},$$

$$\Psi^{\gamma}_{\text{MAXMINVAR}}(u) \quad = \quad \left(1 - (1-u)^{\gamma+1}\right)^{\frac{1}{\gamma+1}},$$

cannot be reparameterized into a FCDF satisfying condition [T], i.e. cannot be represented by a distribution function.

4.5.1 Proof of Theorem 4.5.1.

The following lemma shows that a FCDF can only be represented by a distribution function G with a certain structure, e.g. G is continuous on the whole real line and strictly increasing on its support until it hits its upper limit 1.

Lemma 4.5.10. *Let $u_0 \in (0,1)$. Let $G : \mathbb{R} \to [0,1]$ be a distribution function such that $G(0) = u_0$. Define $G(-\infty) = 0$. Let G^{-1} be the generalized inverse of G, for instance*

$$G^{-1}(u) := \inf \{x \in \mathbb{R} \, : \, G(x) \geq u\}.$$

Define

$$x_0 := \inf \{x \in \mathbb{R}, \;\; G(x) > 0\}$$

and

$$x_1 := G^{-1}(1).$$

It then holds $x_0 < x_1$. Let (Ψ^γ) be a FCDF. If

$$\Psi^\gamma(u) = G(G^{-1}(u) + \gamma), \;\; u \in (0,1), \;\; \gamma \geq 0,$$

then it holds $G(x_0) = 0$ and G is continuous on $\mathbb{R}$ and strictly increasing on (x_0, x_1). We further have

$$G^{-1}(G(x)) = x, \;\; x \in (x_0, x_1) \tag{4.15}$$

and

$$G(G^{-1}(u)) = u, \;\; u \in (0,1). \tag{4.16}$$

Proof. We trivially have $x_0 \leq 0 < x_1$. Assume $0 < p_0 := G(x_0)$. Then $p_0 \leq u_0 < 1$ and $G^{-1}(p) = x_0$ for $p \in (0, p_0]$. Hence the map $u \mapsto G(G^{-1}(u))$ is constant and equal to p_0 on $(0, p_0)$, which is a contraction as the map $u \mapsto \Psi^0(u)$ is concave and increasing and $\Psi^0(1) = 1$. Thus it holds $G(x_0) = 0$.

As G is a distribution function, G is right-continuous and increasing, i.e. for all $x \in \mathbb{R}$ it holds

$$G(x+) := \lim_{\varepsilon \downarrow 0} G(x + \varepsilon) = G(x).$$

Assume there is a $\bar{x} \in (x_0, x_1]$ such that

$$\bar{u} := G(\bar{x}-) := \lim_{\varepsilon \uparrow 0} G(\bar{x} + \varepsilon) < G(\bar{x})$$

i.e. G jumps at $\bar{x}$. Then $G(G^{-1}(\bar{u}-)) < G(\bar{x}) \leq G(G^{-1}(\bar{u}+))$, which is a contradiction because the map $u \mapsto \Psi^0(u)$ is continuous on $(0,1]$. We conclude that G is continuous on $\mathbb{R}$.

Now we show, that G is strictly increasing on (x_0, x_1). Assume there are $x_0 < \tilde{x}_1, \tilde{x}_2 < x_1$ such that $\tilde{x}_1 < \tilde{x}_2$ and $G(\tilde{x}_1) = G(\tilde{x}_2) =: \tilde{u}$. Then it follows $0 < \tilde{u} < 1$ and there exists $\gamma > 0$ such that

$$G(G^{-1}(\tilde{u}-) + \gamma) \leq G(\tilde{x}_1 + \gamma) < G(\tilde{x}_2 + \gamma) \leq G(G^{-1}(\tilde{u}+) + \gamma),$$

which is again a contraction. The second assertion, expressed by Equations (4.15) and (4.16), follows immediately, because $\tilde{G} : (x_0, x_1) \to (0,1)$, $x \mapsto G(x)$, is bijective. □

Prove of Theorem 4.5.1. We show the direction i)⇒ ii). Let $u_0 \in (0,1)$ and define $G : \mathbb{R} \to [0,1]$ by Equation (4.13).

First step: Show that $p \mapsto \overline{\Psi}^{\gamma}(p)$ is continuous.

By Definition 4.4.7, for a fixed $\gamma \geq 0$, the function $u \mapsto \Psi^{\gamma}(u)$ is monotonically increasing and concave and it holds $\Psi^{\gamma}(0) = 0$ and $\Psi^{\gamma}(1) = 1$. This implies a strong structure on Ψ^{γ}: There exists a constant $\tilde{u}_{\gamma} \in [0,1]$, namely

$$\tilde{u}_{\gamma} = \inf \{u \,:\, \Psi^{\gamma}(u) = 1\}, \tag{4.17}$$

such that $u \mapsto \Psi^{\gamma}(u)$ is strictly increasing and continuous on $(0, \tilde{u}_{\gamma}]$ and constant on $(\tilde{u}_{\gamma}, 1]$. At zero, $u \mapsto \Psi^{\gamma}(u)$ might jump. Let $\tilde{p}_{\gamma} := \lim_{\varepsilon \downarrow 0} \Psi^{\gamma}(\varepsilon)$ be the jump-size at $u = 0$. For a particular distortion function, $\tilde{u}_{\gamma}$ and $\tilde{p}_{\gamma}$ are visualized in Figure 4.3. By definition of $p \mapsto \overline{\Psi}^{\gamma}(p)$, it holds for $0 \leq p \leq \tilde{p}_{\gamma}$

$$\overline{\Psi}^{\gamma}(p) = \inf \{u \in [0,1] \,:\, \Psi^{\gamma}(u) \geq p\} = \inf \{u \in (0,1]\} = 0. \tag{4.18}$$

Continuity of $p \mapsto \overline{\Psi}^{\gamma}(p)$ follows immediately: define

$$\Theta^{\gamma}(u) := \begin{cases} \tilde{p}_{\gamma} & , u = 0 \\ \Psi^{\gamma}(u) & , u > 0. \end{cases}$$

Then $u \mapsto \Theta^{\gamma}(u)$ is continuous and bijective as a function from $[0, \tilde{u}_{\gamma}]$ to $[\tilde{p}_{\gamma}, 1]$ and hence its inverse $\overline{\Theta}^{\gamma}$ is also continuous. We further have $\overline{\Psi}^{\gamma}(p) = \overline{\Theta}^{\gamma}(p)$ for $p \in [\tilde{p}_{\gamma}, 1]$, which shows continuity of $p \mapsto \overline{\Psi}^{\gamma}(p)$.

Second step: show that $\gamma \mapsto \overline{\Psi}^{\gamma}(u_0)$ is decreasing and continuous, hence G is a distribution function.

While $\gamma \mapsto \Psi^{\gamma}(u_0)$ is increasing and continuous in the variable γ by definition, it is easy to see that its generalized inverse is decreasing in the variable γ. The function $\gamma \mapsto \overline{\Psi}^{\gamma}(u_0)$ is continuous, which can be seen by the following auxiliary result:

If $\gamma_2 \geq \gamma_1 \geq 0$ and $\Psi^{\gamma_2 - \gamma_1}(u_0) < 1$ and $u_0 > \tilde{p}_{\gamma_1}$, it follows

$$\Psi^{\gamma_2 - \gamma_1}(u_0) = \Psi^{\gamma_2 - \gamma_1}\left(\Psi^{\gamma_1}\left(\overline{\Psi}^{\gamma_1}(u_0)\right)\right) = \Psi^{\gamma_2}\left(\overline{\Psi}^{\gamma_1}(u_0)\right).$$

Applying $\overline{\Psi}^{\gamma_2}$ on both sides, yields

$$\overline{\Psi}^{\gamma_1}(u_0) = \overline{\Psi}^{\gamma_2}\left(\Psi^{\gamma_2 - \gamma_1}(u_0)\right). \tag{4.19}$$

Let $\gamma_0 := \inf \{\gamma \geq 0 : \tilde{p}_{\gamma} \geq u_0\}$, where $\inf \emptyset = \infty$. γ_0 is the smallest number, such that the jump-size of Ψ^{γ_0} at zero is greater or equal to u_0. The map $\gamma \mapsto \overline{\Psi}^{\gamma}(u_0)$ is identical to zero on $[\gamma_0, \infty)$, compare with Equation (4.18). It remains to show continuity from below at $\gamma \in (0, \gamma_0]$ and continuity from above at $\gamma \in (0, \gamma_0)$. Let $0 < \gamma \leq \gamma_0$ and $(\gamma_n)_{n \in \mathbb{N}}$ be a positive sequence converging from below to γ. Without loss of generality, we assume $\gamma_n < \gamma$ for all n. For n large enough, it holds $\Psi^{\gamma - \gamma_n}(u_0) < 1$ because Ψ^{γ} is continuous at γ and $\Psi^{0}(u_0) = u_0 < 1$. We have $u_0 > \tilde{p}_{\gamma_n}$ because $\gamma_n < \gamma_0$ and by Equation (4.19), it holds

$$\overline{\Psi}^{\gamma_n}(u_0) = \overline{\Psi}^{\gamma}\left(\Psi^{\gamma - \gamma_n}(u_0)\right) \to \overline{\Psi}^{\gamma}(u_0), \;\; n \to \infty,$$

where we used that $p \mapsto \overline{\Psi}^{\gamma}(p)$ is continuous on $[0,1]$. If $\gamma < \gamma_0$ and (γ_n) is a sequence converging from above to γ, let $\varepsilon > 0$ such that $\Psi^{\varepsilon\gamma}(u_0) < 1$ and choose n large enough so that $(1+\varepsilon)\gamma - \gamma_n \geq 0$ and $\gamma_n < \gamma_0$. It follows $\Psi^{(1+\varepsilon)\gamma-\gamma_n}(u_0) < 1$ and using Equation (4.19) twice and continuity of $p \mapsto \overline{\Psi}^{\gamma}(p)$, shows continuity from above.

Thus G is monotonically increasing and continuous. Continuity at zero can be shown using condition [E]: it holds $G(0) = \Psi^0(u_0) = u_0$. By condition [W] it follows $\lim_{x\to\infty} G(x) = 1$ and $\lim_{x\to-\infty} G(x) = 0$. G is thereby a distribution function.

Third step: show that Equation (4.12) holds.

We distinguish three cases and use that $(\Psi^{\gamma})_{\gamma\geq 0}$ satisfies condition [T]. Let $\gamma \geq 0$ and $u \in (0,1)$. As G is continuous, it is a surjective function from $\mathbb{R}$ to $(0,1)$ and there exists $x \in \mathbb{R}$ such that $G(x) = u$ and $G^{-1}(u) = x$. If $x \geq 0$, it follows

$$\begin{aligned} G(x+\gamma) &= \Psi^{x+\gamma}(u_0) \\ &= \Psi^{\gamma}(\Psi^{x}(u_0)) \\ &= \Psi^{\gamma}(G(x)). \end{aligned}$$

If $x < 0$, it holds

$$\overline{\Psi}^{-x}(u_0) = G(x) > 0$$

and therefore $u_0 > \tilde{p}_{-x}$. If $x < 0$ and $x + \gamma \geq 0$, it follows

$$\begin{aligned} G(x+\gamma) &= \Psi^{x+\gamma}(u_0) \\ &= \Psi^{x+\gamma}\left(\Psi^{-x}\left(\overline{\Psi}^{-x}(u_0)\right)\right) \\ &= \Psi^{\gamma}(G(x)). \end{aligned}$$

If $x < 0$ and $x + \gamma < 0$ we have

$$1 > u_0 = \Psi^{-x}\left(\overline{\Psi}^{-x}(u_0)\right) = \Psi^{-\gamma-x}\left(\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right)\right)$$

and thereby

$$\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right) < \tilde{u}_{-\gamma-x},$$

compare with Equation (4.17). We further have

$$\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right) > 0$$

as

$$\overline{\Psi}^{-x}(u_0) = G(x) = u > 0.$$

Because the function

$$\Psi^{-\gamma-x} : (0, \tilde{u}_{\gamma}] \to (\tilde{p}_{\gamma}, 1]$$

is bijective, it follows

$$\begin{aligned} G(x+\gamma) &= \overline{\Psi}^{-x-\gamma}(u_0) \\ &= \overline{\Psi}^{-x-\gamma}\left(\Psi^{-\gamma-x}\left(\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right)\right)\right) \\ &= \Psi^{\gamma}(G(x)). \end{aligned}$$

Fourth step: Show the uniqueness of G.

Let assume there is another distribution function F such that $F(0) = u_0$ and

$$F(F^{-1}(u) + \gamma) = \Psi^{\gamma}(u), \;\; u \in (0,1), \;\; \gamma \geq 0.$$

For $x \geq 0$ it follows by Lemma 4.5.10,

$$F(x) = F(F^{-1}(u_0) + x) = \Psi^{x}(u_0) = G(x).$$

Let $x_0 := \inf\{x, \, F(x) > 0\}$. For $x_0 < x < 0$, it follows $0 < F(x) < 1$ and it holds

$$\Psi^{-x}\left(F(x)\right) = F(F^{-1}(F(x)) - x) = F(0) = u_0$$

and hence

$$F(x) = \overline{\Psi}^{-x}(u_0) = G(x).$$

If $-\infty < x_0$, we further have

$$\tilde{p}_{-x_0} = \lim_{\varepsilon \downarrow 0} F(F^{-1}(\varepsilon) - x_0) = F(0) = u_0$$

and therefore $G(x_0) = \overline{\Psi}^{-x_0}(u_0) = 0 = F(x_0)$. Hence it holds $G(x) = F(x)$ for all $x \in \mathbb{R}$.

Now let us show the other direction ii)$\Rightarrow$ i). We use lemma 4.5.10. Let $u_0 \in (0,1)$. If there is a distribution function G such that $G(0) = u_0$ and Equation (4.12) holds, it follows for any $u \in (0,1]$

$$\lim_{\gamma \to \infty} \Psi^{\gamma}(u) = \lim_{\gamma \to \infty} G(G^{-1}(u) + \gamma) = 1,$$

i.e. (Ψ^{γ}) satisfies condition [W]. We further have

$$\Psi^{0}(u) = G(G^{-1}(u)) = u, \;\; u \in (0,1),$$

which shows that the FCDF satisfies condition [E]. Now let $\gamma_1, \gamma_2 \geq 0$ and $u \in (0,1)$. Assume $\Psi^{\gamma_1}(u) < 1$, then it holds

$$\begin{aligned} \Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u)\right) &= G(G^{-1}\left[G(G^{-1}(u) + \gamma_1)\right] + \gamma_2) \\ &= G(G^{-1}(u) + \gamma_1 + \gamma_2) \\ &= \Psi^{\gamma_1 + \gamma_2}(u). \end{aligned}$$

The case $\Psi^{\gamma_1}(u) = 1$ is trivial. Thus (Ψ^{γ}) satisfies condition [T]. $\square$

4.5.2 Application to Moment Based Premium Principles

In Section 4.2 we introduced a coherent risk measure ρ as a map from set of bounded random variables to the real numbers describing the riskiness of future random cash flows. In insurance science we are usually dealing with nonnegative random variables describing for example the possible financial loss due to a natural disaster. In an insurance context, we call a nonnegative random variable X *insurance risk* or just *risk* and the value $\rho(-X)$ a *premium*, see also Example 4.2.3.

It is possible to apply our representation result Theorem 4.5.1 to compare different insurance risks with each other. Let us assume an insurance company is insuring a risk, which can be described by a nonnegative random variable X. The amount of money charged by the insurer to the insured for the coverage of the loss due to the risk X, is called the *risk-adjusted premium*, excluding acquisition or internal expenses. There are several method for assigning a risk-adjusted premium π to the risk X. The premium π could be defined via a coherent risk measure ρ by $\pi(X) = \rho(-X)$. But many premium principles used in practice are equal to the expected value of the risk plus some security loading, so called *moment based premium principles*:

$$\begin{aligned} \text{the } \textit{Expected Value Premium} \text{ is defind by} \quad & E[X] + \gamma E[X], \\ \text{the } \textit{Standard Deviation Premium} \text{ is defind by} \quad & E[X] + \gamma\sqrt{\mathrm{Var}(X)}, \\ \text{and the } \textit{Variance Premium} \text{ is defind by} \quad & E[X] + \gamma \mathrm{Var}(X), \end{aligned}$$

where $\gamma \geq 0$, see Straub (1988), Daykin et al. (1994) and Rolski et al. (2009). The moment based premium principles are *not* coherent, the standard deviation premium principle for example is not monotone, i.e. two different risks cannot really be compared with each other[1]. But for a particular random variable X, it is possible to construct a FCDF (Ψ_X^γ), such that for a fixed $\xi > 0$ it holds

$$\rho_{\Psi_X^\gamma}(-X) = E[X] + \gamma\xi, \;\; \gamma \geq 0.$$

The value $\rho_{\Psi_X^\gamma}(-X)$ is equal to a particular moment based premium of X for all $\gamma \geq 0$ if

$$\xi \in \{E[X], \sqrt{\mathrm{Var}(X)}, \mathrm{Var}(X)\}.$$

What are the benefits? An insurance which mainly insures a risk X and uses a moment based premium principle to assign a premium to X, might wish to compare risk X to another risk Z, which can be archived by comparing the values $\rho_{\Psi_X^\gamma}(-X)$ and $\rho_{\Psi_X^\gamma}(-Z)$ with each other.

On the one hand, the moment based premium principles are not coherent, they are arguably not very well suited to compare different risks with each other. They may even be infinite, e.g. if the second moments of Z do not exist.

[1]For example let X take the values 10 or 90, each with probability $\frac{1}{2}$. Clearly, X is less risky than the constant $Z = 100$. But the mean plus standard deviation of X is about 106. The standard deviation premium of Z is just 100.

On the other hand, moment based premium principles are easy to understand and explain to policyholders. That is why the insurance may use a moment based premium principle in the first place, to compute the premium of the risk X.

Note that already Wang (2000) observed, that the Wang transform leads to the standard deviation premium principle, if X is normal distributed. Our representation result for FCDF makes a straightforward computation of Ψ_X^γ possible, in particular for non-negative and skewed random variables X.

Construction of a Coherent Risk Measure Reproducing a Moment-Based Premium Principle

In this section we construct a coherent risk measure, based on a concave distortion function and depending on a risk X, such that the premium principle of this risk measure reduces to the expected value, the standard deviation or the variance premium principle for risk X. Let an integrable, nonnegative random variable X on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. We make the following assumptions on the risk X:

Assumption 1. *The density f_X of X is continuous with support on $(0, \infty)$.*

Assumption 2. *The density $f_X = F_X^{'}$ is log-concave*

Assumption 3. *For the density it holds: $\lim_{x\to\infty} \frac{f_X(x-\gamma)}{f_X(x)} < \infty$ for all $\gamma > 0$.*

Those assumptions are made to keep the notation simple and could be relaxed. See Remark 4.4.12 for a definition of log-concavity. For example the densities of the normal distribution and the gamma, the beta and the Weibull distribution, respectively with shape parameter $\alpha \geq 1$, are log-concave, see Bagnoli and Bergstrom (2005). Assumption 3 is used to show that a coherent risk measure induced by the distribution function of X is well defined on the whole space of integrable random variables L^1. In particular the gamma and the Weibull distributions satisfy assumptions $1-3$, both distributions are frequently used in insurance science to model insurance risks.

Proposition 4.5.11. *Let X satisfy Assumptions $1-3$. Let $\xi > 0$. Let*

$$G(x) := 1 - F_X(-x\xi), \;\; x \in \mathbb{R}.$$

The set of functions

$$\Psi_X^\gamma(u) := G(G^{-1}(u) + \gamma), \;\; \gamma \geq 0, \;\; u \in (0,1), \tag{4.20}$$

define a FCDF and it holds

$$\rho_{\Psi_X^\gamma}(-X) = E[X] + \gamma\xi, \;\; \gamma \geq 0, \tag{4.21}$$

where $\rho_{\Psi_X^\gamma}$ is a coherent risk measure with domain L^1 induced by the concave distortion Ψ_X^γ, see Equation (4.4).

Remark 4.5.12. The value $\rho_{\Psi_X^\gamma}(-X)$ is then equal to the expected value premium, the standard deviation premium or the variance premium of X if

$$\xi \in \left\{E[X], \sqrt{\mathrm{Var}(X)}, \mathrm{Var}(X)\right\},$$

The inverse of G can easily be computed:

$$G^{-1}(p) = -\frac{1}{\xi} F_X^{-1}(1-p).$$

Proof. For $\gamma \geq 0$, we define Ψ_X^γ pointwise: $\Psi_X^\gamma(0) := 0$, $\Psi_X^\gamma(1) := 1$. Let $u \in (0,1)$ and let $x > 0$ such that

$$u = H(x) := 1 - F_X(x).$$

H is the decumulative distribution function of X. By Assumption 1, F_X is a bijective function from $(0,\infty)$ to $(0,1)$. It holds $x = H^{-1}(u)$ and we define

$$\Psi_X^\gamma(u) := H(H^{-1}(u) - \gamma\xi).$$

It follows

$$\Psi_X^\gamma(H(x)) = H(x - \gamma\xi), \;\; x > 0, \;\; \gamma \geq 0. \tag{4.22}$$

It is straightforward to see that $\gamma \mapsto \Psi_X^\gamma(u)$ is continuous and increasing and that $u \mapsto \Psi_X^\gamma(u)$ is increasing and concave, because the density corresponding to F_X is log-concave. Hence the family $(\Psi_X^\gamma)_{\gamma\geq 0}$ is a FCDF. It additionally satisfies conditions [E], [W] and [T], hence by Theorem 4.5.1, there exist a unique distribution function $\hat{G}$ such that $\hat{G}(0) = \frac{1}{2}$ and

$$\Psi_X^\gamma(u) = \hat{G}(\hat{G}^{-1}(u) + \gamma).$$

By Equation (4.13), $\hat{G}$ can be identified by

$$\hat{G}(x) = 1 - F_X\left(F_X^{-1}\left(\frac{1}{2}\right) - x\xi\right), \;\; x \in \mathbb{R}, \;\; \xi>0.$$

We shift $\hat{G}$ and define

$$G(x) = 1 - F_X(-x\xi)$$

and we have

$$\Psi_X^\gamma(u) = G(G^{-1}(u) + \gamma). \tag{4.23}$$

Let $g(x) := \xi f_X(-x\xi)$. It follows for $\gamma > 0$ by Assumption 3:

$$\begin{aligned}\lim_{u \searrow 0} \frac{\partial}{\partial u} \Psi_X^\gamma(u) &= \lim_{u \searrow 0} \frac{g\left(G^{-1}(u) + \gamma\right)}{g\left(G^{-1}(u)\right)} \\ &= \lim_{x \to \infty} \frac{f_X(x - \xi\gamma)}{f_X(x)} < \infty.\end{aligned}$$

Hence because Ψ_X^γ is concave for all $\gamma \geq 0$ its partial derivative is bounded on the unit uniterval and the coherent risk measures induced by the family (Ψ_X^γ) are well defined on L^1. It follows by Equation (4.22) for all $\gamma \geq 0$

$$\begin{aligned} E[X] + \gamma\xi &= \int_0^\infty 1 - F_X(x - \gamma\xi)dx \\ &= \int_0^\infty \Psi_X^\gamma(1 - F_X(x))dx \\ &= \rho_{\Psi_X^\gamma}(-X). \end{aligned}$$

□

Example 4.5.13. Let $X \sim \Gamma(\alpha, \beta)$ be a gamma distributed random variable with mean $\frac{\alpha}{\beta}$ and variance $\frac{\alpha}{\beta^2}$ modelling a risk or an aggregated risk insured by the insurance company. The gamma distribution satisfies Assumption $1 - 3$, if $\alpha \geq 1$. We apply the standard deviation premium principle and choose

$$\xi = \sqrt{\mathrm{Var}(X)} = \frac{\sqrt{\alpha}}{\beta}.$$

Additionally, assume that the insurance faces another risk Z and wishes to compare both risks using a coherent risk measure, which reproduces the standard deviation premium for X and is induced by the FCDF $\left(\rho_{\Psi_X^\gamma}\right)$, defined via Equation (4.20). Table 4.2 compares the standard deviation premium of X, to the premium of various other risks computed using $\rho_{\Psi_X^\gamma}$. The premium of a nonnegative risk $Z \in L^1$ under $\rho_{\Psi_X^\gamma}$ is equal to

$$\rho_{\Psi_X^\gamma}(-Z) = \int_0^\infty \Psi_X^\gamma(1 - F_Z(s))ds. \tag{4.24}$$

The integral appearing in Equation (4.24) can be computed using standard numeric methods.

We compare risk X to an exponential, a Gaussian, a Bernoulli and a Pareto risk. If $Z \sim \mathrm{Pareto}(x_m, a)$ is Pareto distributed with scale $x_m > 0$ and shape $a > 0$ and if $a \in (1, 2]$, then Z has finite first and infinite second moments. In particular, the standard deviation premium principle cannot be applied to Z. The expected value of Z is $\frac{ax_m}{1-a}$ for $a > 1$. We further compare risk X to a risk W defined by the loss occurring in a layer with deductible $D \geq 0$ and cover $C > D$ of a Pareto distributed loss Z, i.e.

$$W := (Z - D)^+ - (Z - C - D)^+.$$

Let the distribution of W be denoted by

$$F_W^{\alpha, x_m, D, C}(x) := \begin{cases} 1 - \left(\frac{x_m}{x+D}\right)^\alpha & , (x_m - D, 0)^+ \leq x < C \\ 1 & , x \geq C. \end{cases}$$

It turns out that for $\gamma = 1$, the Standard Deviation Premia of the exponential and the Gaussian risk are very similar to the corresponding premia computed using $\rho_{\Psi^1_X}$. The differences between both premia for Bernoulli or Pareto risks are very large.

	X	Z_{exp}	Z_{Gauss}	Z_{B}	Z_∞	Z_{250}	Z_{10}
Expected Value	1	1	1	1	1	1	1
SD premium	1.47	2	1.20	10.95	∞	8.1	2.92
Premium under $\rho_{\Psi^1_X}$	1.47	1.99	1.19	4.25	4.31	3.46	2.64

Table 4.2: Compare the standard deviation (SD) premium principle to the premium principle using the coherent risk measure $\rho_{\Psi^1_X}$ applied to various risks: $X \sim \Gamma\left(\frac{9}{2}, \frac{9}{2}\right)$, $Z_{\text{exp}} \sim \exp(1)$, $Z_{\text{Gauss}} \sim N(1, \frac{2}{10})$, Z_B is Bernoulli distributed taking the value 100 with probability $\frac{1}{100}$. $Z_\infty \sim \text{Pareto}(\frac{1}{10}, \frac{10}{9})$, $Z_{250} \sim F_W^{\frac{10}{9},0.2,0.2,250}$ and $Z_{10} \sim F_W^{\frac{10}{9},0.36,0.36,10}$. The concave distortion function Ψ^1_X is drawn in Figure 4.3 as Ψ_4.

Interpretation of the Coherent Risk Measure $\rho_{\Psi^\gamma_X}$

As above let X describe some insurance risk and let π_X be the premium of X obtained by a moment based premium principle. Let the FCDF (Ψ^γ_X) be defined such that

$$\pi_X = \rho_{\Psi^\gamma_X}(-X).$$

The following proposition offers an interpretation of the premium principle based on the coherent risk measure $\rho_{\Psi^\gamma_X}$. There is an acceptability index α such that the performance of the future random cash flow

$$\rho_{\Psi^\gamma_X}(-Z) - Z$$

for any risk $Z \in L^1$ is at least as high as the performance of the cash flow $\pi_X - X$. Using only the acceptability index α as a criterion, the insurance is indifferent insuring risk X and obtaining premium π_X or insuring another risk Z in return for premium $\rho_{\Psi^\gamma_X}(-Z)$.

Proposition 4.5.14. *Let X satisfy Assumptions $1-3$. For some $\xi > 0$, let the FCDF $(\Psi^\gamma_X)_{\gamma\geq 0}$ be defined as in Equation (4.20). Let $\gamma_0 \geq 0$ and*

$$\pi_X := E[X] + \gamma_0 \xi.$$

There exist an acceptability index $\alpha : L^1 \to [0, \infty]$ such that

$$\alpha\left(\pi_X - X\right) = \gamma_0 \leq \alpha\left(\rho_{\Psi^{\gamma_0}_X}(-Z) - Z\right), \tag{4.25}$$

for all $Z \in L^1$ with $Z \geq 0$.

By the Fatou property, the performance of the null-position is infinite. Therefore the right-hand side of Equation (4.25) can be equal to infinity, for example if $Z = 0$.

Proof. The family of coherent risk measures $\left(\rho_{\Psi_X^\gamma}\right)_{\gamma\geq 0}$ has domain L^1 and defines an acceptability index α by

$$\begin{aligned}\alpha: L^1 &\to [0,\infty]\\ Y &\mapsto \sup\left\{\gamma\geq 0:\ \rho_{\Psi_X^\gamma}(Y)\leq 0\right\},\end{aligned}$$

see Section 4.3. Let $Z\in L^1$ such that $Z\geq 0$. It holds using the translation property for coherent risk measures

$$\begin{aligned}\alpha\left(\rho_{\Psi_X^{\gamma_0}}(-Z)-Z\right) &= \sup\left\{\gamma\geq 0:\ \rho_{\Psi_X^\gamma}\left(\rho_{\Psi_X^{\gamma_0}}(-Z)-Z\right)\leq 0\right\}\\ &= \sup\left\{\gamma\geq 0:\ \rho_{\Psi_X^\gamma}(-Z)\leq \rho_{\Psi_X^{\gamma_0}}(-Z)\right\}\\ &\geq \gamma_0\end{aligned}$$

and similarly

$$\alpha\left(\pi_X-X\right)=\sup\left\{\gamma\geq 0:\ \rho_{\Psi_X^\gamma}(-X)\leq E[X]+\gamma_0\xi\right\}=\gamma_0.$$

□

4.6 Conclusion

In this Chapter we point out the relation between a family of concave distortion function (FCDF) and coherent risk measures. A concave distortion function is a concave function mapping the unity interval onto itself. A coherent risk measures can be defined by distorting the original distribution function of a random variable: losses are given more weight and gains are given less weight. We have shown that a FCDF satisfying a certain translation equation, can be represented by a distribution function. Our representation theorem is novel, it generalizes a comparable result obtained by Tsukahara (2009).

In contrast to Tsukahara (2009), our representation results also covers FCDF which are not strictly increasing in the distortion level like the FCDF related to the expected shortfall and FCDF which jump like the "ess sup-*expectation convex combination*" distortion function defined and applied to finance by Bannör and Scherer (2014).

On the other hand, Tsukahara's result does not require the family of distortion functions to be concave. But concavity is a natural requirement when dealing with coherent risk measures. A risk measure should encourage diversification, i.e. the risk of a portfolio must not exceed the sum of the risk of its components. A risk measures induced by a distortion function which is not concave, is in general not sub-additive and does not encourage diversification.

An application of the representation result can be found in actuarial science: assume there is an insurance company selling mainly contracts to insure a risk X. The risk X may describe a loss due to some natural disaster like fire. The insurance company computes the premium of the insurance contract using a moment based premium principle, e.g. the premium is calculated as the expected value of X plus a multiple of the standard deviation of X. Such a premium principle is easy to understand and to explain to policyholders but it is not monotone, i.e. different insurance risks cannot be compared with each other and cannot be priced in a consistent way.

Our representation theorem makes it possible to construct a coherent risk measure ρ_X, induced by a concave distortion function and depending on the distribution function of X, such that the premium principle of that risk measure reduces to a moment based premium principle when applied to risk X. The price of another insurance risk Z may then be compared to the standard deviation premium of X, even if the variance of Z does not exist, by applying ρ_X both to X and to Z.

The premium principle based on ρ_X is consistent with a moment based premium principle like the standard deviation premium principle. The residual cash flow of the insurance company insuring risk X in return for the (standard deviation premium) is the difference of the premium and the insurance risk X. We show that there exists an acceptability index (performance measure) such that the performance of the residual cash flow insuring risk X is equal to the performance of the residual cash flow insuring any other risk Z, if the premium of Z is computed based on ρ_X.

Using only this acceptability index as a criterion, the insurance company is indifferent insuring risk X and obtaining a standard deviation premium or insuring another risk Z in return for the premium $\rho_X(-Z)$.

5 Financial Markets with Frictions

5.1 Introduction

In this Chapter, we obtain closed-form solutions of bid and ask prices of European plain vanilla and barrier options in markets with frictions. The construction of bid and ask prices is heavily based on concave distortion functions as introduced in Chapter 4. Markets with frictions are markets with transaction costs. While in frictionless markets the risky underling, e.g. a stock, can be bought and sold for the same price S_t at time t, in market with frictions, there are two prices: an investor can purchase the stock for the *ask price* S_t^a and sell the stock for the usually lower *bid price* S_t^b. Simple proportional transaction costs models assume the underlying "fair" price of the stock can be modelled by a stochastic process (S_t) and define for numbers $\mu \in (0,1)$ and $\lambda > 1$ the bid and ask prices of the stock by

$$S_t^b := \mu S_t \text{ and } S_t^a := \lambda S_t.$$

The spread $S_t^a - S_t^b$ measures the fee an investor has to pay to the exchange for trading the stock.

In contrast to complete financial markets without any imperfections where prices are obtained by a *linear* pricing rule, prices in markets with frictions can be described by *sublinear* pricing functionals, see Jouini (2000). Such pricing functionals may also describe prices in markets with additional or different kind of frictions than (proportional) transaction costs, like short sales costs or constrains, borrowing costs, taxes and other market imperfections, see Jouini and Kallal (2001), Koehl and Pham (2000), Bion-Nadal (2009) and references therein.

Jouini and Kallal (1995, 2001) and Jouini (2000) introduced an axiomatic approach to describe financial markets with frictions. They considered a finite time-horizon $T > 0$ and a multiperiod economy, where investors can trade a riskless and a risky asset. Let N be the number of trading periods in $[0, T]$. Jouini (2000) modelled the bid and ask price processes of the risky asset by adapted processes

$$0 < S_i^b \le S_i^a, \ \ i = 0, .., N.$$

They postulate the existence of a pricing functional p and define the ask price of a contingent claim C by $p(C)$, and the bid price by $-p(-C)$, hence buying the contingent claim C is the same as selling $-C$.

Furthermore, p is assumed to satisfy the following axioms: (i) p is monotone, i.e. no agent is willing to pay more for less, (ii) p is sub-additive, i.e. it is less expensive buying the portfolio $C + C'$ than buying C and C' separately. Indeed an agent might save transaction costs hedging a portfolio instead of hedging the components of the portfolio separately: due to possible diversification effects, some orders in the risky asset may cancel out. (iii) p is positively homogeneous, i.e. the ask price of a position scales linearly with its size:

$$p(\lambda C) = \lambda p(C), \ \ \lambda \geq 0.$$

There is some criticism about this axiom. For instance Föllmer and Schied (2002) argued that '[...] an additional liquidity risk may arise if a position is multiplied by a large factor. This suggests to relax the conditions of positive homogeneity [...].' But positive homogeneity is a standard assumption in classical financial markets and holds approximately for reasonable values of λ. Cetin et al. (2004) and Bion-Nadal (2009) extended Jouini (2000), allowing prices to depend on the size of the position. (iv) p does not introduce arbitrage. This is a natural requirement of any financial market. (v) p is lower-semi-continuous, which is a rather technical axiom. (vi) For a future random cash flow C, $p(C)$ is less than or equal to the price of the smallest self-financing trading strategy dominating[1] C, i.e. it is not possible to obtain a better payoff than C for lower cost by directly investing in the underlying. See Jouini and Kallal (1995, 2001) and Jouini (2000) for a more detailed discussion and economic interpretation of the axioms (i)-(vi).

Jouini and Kallal (1995) showed that the market is arbitrage-free, if and only if there exist a measure $\mathbb{Q}$, equivalent to the physical measure $\mathbb{P}$, and a process $Z^{\mathbb{Q}}$, which is a martingale under $\mathbb{Q}$, such that $S^b \leq Z^{\mathbb{Q}} \leq S^a$. This leads to an easy construction of arbitrage-free financial markets with frictions: we take a frictionless market, where the risky underlying is described by some martingale $(S_i)_{i=0,..,N}$ under the risk-neutral measure. Introducing a sequence of dynamic coherent risk measures $(\rho_i)_{i=0,..,N}$ and defining a pricing functional by $p_i(.) := \rho_i(-.)$, we introduce frictions into the market by defining the ask price process of the underlying by $(p_i(S_N))$ and the bid price process by $(-p_i(-S_N))$. The sequence (p_i) induced by a sequence of coherent risk measures fulfils axioms (i)-(vi).

Contribution

Bid and ask prices are recursively defined in a discrete time model with N trading periods. We look at two special cases: the static case $N = 1$ and the asymptotic case $N \to \infty$. In both cases we obtain closed-form solutions of bid and ask prices of European options by introducing a new parameter γ, which enters into the dividend yield.

In the static case we obtain closed-form solutions for bid and ask prices of European options, if the log-returns are normal or Laplace distributed. Existing closed-form solutions of the risk-neutral price of European options are extended with a new parameter $\gamma_{\text{static}} \geq 0$, which adjusts the dividend yield. The greater γ_{static}, the greater the bid-ask

[1] A self-financing trading strategy is a way to invest in the market, i.e. going long and short in the risky asset without exogenous infusion or withdrawal of money except for an initial investment; the purchase of new assets must be financed by the sale of old ones. For a given contingent claim C, *the price of the smallest self-financing trading strategy dominating* C, is the smallest investment in a self-financing trading strategy, which is always greater or equal to C.

spread. The static case is certainly of interest because the Laplace distribution has fatter tails than the normal distribution, which appears in the Black-Scholes model, and might therefore be better suited to model stock price log-returns.

We also look at the asymptotic case $N \to \infty$ and prove convergence of bid and ask prices for many American and Exotic options in a binomial-type model. We are interested in the asymptotic behaviour of the model to obtain closed-form solutions for efficient numerical applications.

In a binomial-type model with frictions, we develop closed-form solutions for European plain vanilla and some barrier options and obtain in the limit an extended Black-Scholes formula with a new parameter $\gamma_{\text{continuous}} \geq 0$. The limit bid or ask price of a possible path-dependent option is given by the Black-Scholes price of the option but on a stock with an adjusted dividend yield. Hence existing numerical methods, developed to price options in a Black-Scholes setting in classical finance, can also be used to compute bid and ask prices of such options. No new software need to be written to apply our formulas in financial institutions.

Practical Relevance

We think the main application area is the possibility of computing *implicitly* a parameter γ, such that given bid and ask market prices of a European or an American plain vanilla option are exactly matched by our two-price formula.

This idea is comparable to the concept of implied volatility. In principle volatilities could be constant across strikes, maturities, and underlying assets, hence the preference by practitioners for quoting implied volatilities instead of (mid-)prices. Similarly the parameter γ could be constant across all three dimensions, even though there are many non-linearities between the bid-ask spread and strikes, maturities, and underlying assets. It should therefore be beneficial to quote an implicitly computed γ instead of the absolute bid-ask spread of a plain vanilla option. Indeed Corcuera et al. (2012) used a setting similar to our model, but in static time, and showed empirically that the liquidity dry up during the period 2007-2009 is described very well by the parameter γ. Our discrete time model makes it possible to analyse also path-dependent and American options.

Up to now, traders quote the difference between implied bid and ask volatilities to describe the current market liquidity of plain vanilla options. Both this heuristic method and our proposal of computing implicitly the parameter γ have the advantage of using only present market data and of being extremely fast in terms of computational time, in both cases one has to invert the Black-Scholes formula. However, we show in two empirical studies that both the static and the continuous time-model describe (il)liquidity of European and American plain vanilla options very well over time compared to the heuristic method of quoting implied bid and ask volatilities.

Limitations

In the static model, we obtain closed-form for European plain vanilla options if the pricing functional is defined in terms of the distribution of the log-returns of the underlying. In particular, the choice of the pricing functional depends on the model of the log-returns.

In the discrete binomial-type model, convergence is only proven for monotone payoffs, for example European or American plain vanilla, lookback, Asian and some barrier options (up-and-out put, down-and-out call, down-and-in put and up-and-in call). The underlying is essentially modelled by a binomial model in discrete time and by geometric Brownian motion in continuous time. Hence the volatility is assumed to be constant over time and log-returns are assumed to be (approximately) normal distributed. Future research need to be done to treat contingent claims which are not monotone with respect to the underlying, e.g. a barrier up-and-out call option and to generalize the market model replacing for example the constant volatility by a mean-reverting stochastic process.

Literature Review

In general, liquidity is effected by many factors like the ability of trading large quantities, by the speed, the cost and the price impact of the trade. Several measures have been developed in literature to capture some or all of these factors. Amihud (2002) defines the liquidity of a stock by the average of the ratio of absolute daily returns to volume, where the average is taken over a month. Acharya and Pedersen (2005) developed a liquidity adjusted capital asset pricing model and measured liquidity using a normalized version of Amihud's liquidity measure. Liu (2006) analyse the relation between liquidity risk and asset pricing using a liquidity measure based on historic data. Goyenko et al. (2009) compared several well known liquidity measures using stock data from 1993 to 2005.

In contrast to the above studies, which define (il)liquidity mainly using a historic time series of the stock, both our static model and the extended Black-Scholes formula are well suited to be applied to an option surface and needs only present market data to compute the market implied liquidity parameter γ.

Recently, Madan and Cherny (2010) developed the conic finance theory. Out discrete time market model with frictions is connected to conic finance by the common approach of using recursively defined sublinear functionals to describe bid and ask prices. Indeed our discrete market model is closely related to discrete time conic finance models, where bid and ask prices are defined recursively using nonlinear expectations, see Leippold and Schärer (2017), Madan (2010), Madan et al. (2013, 2017a) and Madan and Schoutens (2012b). Time-consistent nonlinear expectations are connected with solutions to backward stochastic difference equations, see Cohen and Elliott (2010). See Bielecki et al. (2013, 2015) for a framework incorporating transaction costs in discrete time conic finance models.

Our work is related to Madan et al. (2017b) who showed in a general context that, under some technical conditions, an iterated spectral risk measure, which is a risk measure in a multiperiod setting based on distortion functions, converges to some g-expectation. A g-expectation is a non-linear expectation proposed by Peng (2004).

Relative to these papers our contribution is to proof convergence of bid and ask prices in a binomial-type model with frictions when the number of trading periods approaches infinity and to obtain closed-form solutions for bid and ask prices for plain vanilla and barrier options in the limit.

Contents

The remainder of this Chapter is organized as follows. In Section 5.2, we introduce a discrete time-model for a market with frictions. A special case of the discrete model, the static case, is analysed in Section 5.3. In the static case we are able to derive closed-form solutions for bid and ask prices of European options if the log-returns are normal or Laplace distributed. Those formulas are applied to real market data of European options in Section 5.3.5.

In Section 5.4.2, we present the classical binomial model. In Section 5.4.3 we prove convergence of bid and ask prices for European and American possibly path-dependent options. In Section 5.4.7, we apply the results to real market data of American options. Section 5.5 concludes.

5.2 The Formal Setup

We make the following economic assumptions: we assume all investors have a finite time-horizon and trading can take place only finitely many times. There is a very liquid bank account and a risky-asset whose bid and ask prices can be described by binomial trees. There exists a pricing functional and bid and ask prices of a contingent claim can be computed via the pricing functional. At the end of the time-horizon, the bid-ask spread of all products is assumed to be zero.

Formally, we assume the following framework: Let $T > 0$ be some time-horizon and $N \in \mathbb{N}$ be the number of trading periods, each trading period has length $\frac{T}{N}$. We introduce a frictionless market and extend it to a market with frictions using a pricing functional. Let the risky-asset

$$(S_i)_{i=0,1,..,N}$$

be described by a nonnegative adapted stochastic process on a given filtered probability space

$$\left(\Omega, (\mathcal{F}_i)_{i=0,\ldots,N}, \mathcal{F}, \mathbb{P}\right)$$

satisfying the usual conditions. By

$$B_i = (1+r)^i, \ \ i = 0, .., N,$$

we denote a risk-free bank account. We assume the market is arbitrage-free and denote by $\mathbb{Q}$ a risk-neutral measure, such that the discounted price process of the risky-asset is a $\mathbb{Q}-$martingale. The process (S_i) describes the *risky asset of the underlying frictionless market*. In this section we assume that the interest rates are equal to zero, i.e. we work with discounted cash flows, and that the stock is not paying any dividends. Those assumptions are only made to keep the notation simple and are relaxed in Section 5.4.2 and 5.4.3. Let

$$L^\infty := L^\infty(\Omega, \mathbb{Q}, \mathcal{F})$$

be the set of $\mathcal{F}-$measurable bounded random variables with respect to the probability measure $\mathbb{Q}$ and $(\rho_i)_{i=0,\dots,N}$ be a set of dynamic, time-consistent coherent risk measures being continuous from above

$$\rho_i : L^\infty \to L_i^\infty := L^\infty(\Omega, \mathbb{Q}, \mathcal{F}_i).$$

The following definition of dynamic coherent risk measures are a direct extension of Definition 4.2.1 and is taken from Föllmer and Schied (2011, Definition 11.1)

Definition 5.2.1. (Coherent conditional risk measure). A map $\rho_i : L^\infty \to L_i^\infty$ is called a *coherent conditional risk measure* if it satisfies the following properties for all $X, Y \in L^\infty$:

R1: *Conditional cash invariance:* $\rho_i(X + X_i) = \rho_i(X) - X_i$ for any $X_i \in L_i^\infty$.

R2: *Monotonicity:* $X \leq Y \Rightarrow \rho_i(X) \geq \rho_i(Y)$.

R3: *Conditional convexity:* $\rho_i(\lambda X + (1-\lambda)Y) \leq \lambda\rho_i(X) + (1-\lambda)\rho_i(Y)$ for $\lambda \in L_i^\infty$ and $0 \leq \lambda \leq 1$.

R4: *Conditional positive homogeneity:* $\rho_i(\lambda X) = \lambda\rho_i(X)$ where $\lambda \in L_i^\infty$ and $0 \leq \lambda$.

If X is some future random cash flow, the random value $\rho_i(X)$ can be interpreted as the risk of X as if measured at the (future) trading period i. The interpretation of axioms R1-R4 can directly be adopted from the static case. For example the conditional cash invariance axiom means we can add certain amount with respect to the information available at time i to the position X and the risk will thereby reduce exactly about that amount.

The dynamic risk measure is called *continuous from above*, if it holds

$$X_n \searrow X \Rightarrow \rho_i(X_n) \nearrow \rho_i(X) \text{ for any sequence } (X_n) \subset L^\infty \text{ and } X \in L^\infty,$$

The dynamic risk measure is called *time-consistent*, if

$$\rho_{i+1}(X) \leq \rho_{i+1}(Y) \Rightarrow \rho_i(X) \leq \rho_i(Y), \;\; X, Y \in L^\infty, i = 0, .., N-1$$

or equivalently

$$\rho_i(X) = \rho_i(\rho_{i+1}(X)), \;\; X \in L^\infty, i = 0, ..., N-1,$$

see Föllmer and Schied (2011, Definition 11.10 and Lemma 11.11).

Note that by R4, axiom R3 is equivalent to the axiom

R3' *Conditional sub-additivity*: $\rho_i(X + Y) \leq \rho_i(X) + \rho_i(Y)$.

We call the operator

$$p_i(.) := \rho_i(-.), \;\; i = 0, ..., N,$$

a *price functional.* Then (p_i) fulfils the following properties, see Föllmer and Schied (2011, Definition 11.1, Theorem 11.2. and Lemma 11.11). Let $X, Y \in L^\infty$. It holds for $i = 0, ..., N$,

P1: *Cash invariance:* $p_i(X + X_i) = p_i(X) + X_i$ for any $X_i \in L_i^\infty$.

P2: *Monotonicity:* $X \leq Y \Rightarrow p_i(X) \leq p_i(Y)$.

P3: *Sub-additivity:* $p_i(X + Y) \leq p_i(X) + p_i(Y)$.

P4: *Positive homogeneity:* $p_i(\lambda X) = \lambda p_i(X)$, where $\lambda \in L_i^\infty$ and $0 \leq \lambda$.

P5: *Continuity from below*: It holds $X_n \nearrow X \Rightarrow p_i(X_n) \nearrow p_i(X)$ for any sequence $(X_n) \subset L^\infty$.

P6: *Time-consistency:* $p_i(X) = p_i(p_{i+1}(X)), \;\; i = 0, ..., N-1$.

We additionally assume

P7: *No-arbitrage:* $E_{\mathbb{Q}}[X|\mathcal{F}_i] \leq p_i(X)$.

Jouini (2000) modelled the risk-free bank account as perfectly liquid. Property P1 states the same: investors may insert or withdraw any amount of cash to or from the risk-free bank account without transaction costs. Properties P2-P5 have been proposed in similar form and are discussed by Jouini and Kallal (1995, 2001) and Jouini (2000). Time-consistency has been introduced by Peng (2004) for nonlinear expectations. It means that prices behave consistently over time: prices can be computed either directly or using an intermediate instant of time, see Bion-Nadal (2009). Property P7 guarantees that the bid-ask spread is always greater or equal to zero and that the market is arbitrage-free, see Proposition 5.2.2. Our model of bid and ask prices can be seen as a discrete version of the continuous time model via dynamic convex risk-measures developed by Bion-Nadal (2009).

Bid and ask prices of a contingent claim $C \in L^\infty$ at trading period i are defined by

$$\text{bid}_i(C) := -p_i(-C) \;\text{ and }\; \text{ask}_i(C) := p_i(C), \;\; i = 0, ..., N,$$

i.e. as in Jouini and Kallal (1995), Staum (2004) and Bion-Nadal (2009), we consider that selling C is the same as buying $-C$. By property R1, we assume that at the end of the time-horizon the bid-ask spread of C is zero. We therefore do not have to distinguish between contingent claims with asset delivery and cash settlement. Bid and ask prices of the risky asset are then defined by the processes

$$S_i^b := -p_i(-S_N) \;\text{ and }\; S_i^a := p_i(S_N), \;\; i = 0, ..., N.$$

American contingent claims can be described by adapted stochastic processes, bid and ask prices of such claims are defined in Section 5.4.1.

We call the tuple $\left((B_i), \left(S_i^b\right), (S_i^a), (p_i)\right)$ a *security price model.* We show that our security price model does not admit arbitrage. Furthermore, it is not possible to construct a self-financing portfolio, which super-replicates C but can be bought for less than $p_0(C)$.

Proposition 5.2.2. *The security price model $\left((B_i), \left(S_i^b\right), (S_i^a), (p_i)\right)$ admits no arbitrage and the ask price $p_0(C)$ of a contingent claim $C \in L^\infty$ is less or equal to the price of the smallest self-financing trading strategy dominating C.*

Proof. We trivially have

$$0 \leq S_i^b \leq E_{\mathbb{Q}}\left[S_N | \mathcal{F}_i\right] = S_i \leq S_i^a, \;\; i = 0, ..., N,$$

hence by Jouini and Kallal (1995, Theorem 3.2), the security price model admits no multiperiod free lunch and is hence arbitrage-free. Let

$$\mathcal{A}_0 := \left\{X \in L^\infty, \rho_0(X) \leq 0\right\}.$$

For a probability measure Q equivalent to $\mathbb{Q}$, define

$$\alpha_0^{\min}(Q) := \sup_{X \in \mathcal{A}_0} E_Q[X].$$

It holds $\alpha_0^{\min}(\mathbb{Q}) \leq 0$, hence by Föllmer and Penner (2006, Corollary 4.12.), there exist a set of probability measures $(\mathcal{Q}^e)$, such that each element of $\mathcal{Q}^e$ is equivalent to $\mathbb{Q}$ and

$$p_i(X) = \sup_{Q \in \mathcal{Q}^e} E_Q[X | \mathcal{F}_i], \;\; i = 0, .., N.$$

Let $\mathcal{P}$ be a set of probability measures containing all probability measures P which are equivalent to $\mathbb{Q}$ and for which exist a $P-$martingale $\left(Z_i^P\right)$ with

$$S_i^b \leq Z_i^P \leq S_i^a, \;\; i = 0, .., N.$$

It follows $\mathcal{Q}^e \subseteq \mathcal{P}$: for each $Q^e \in \mathcal{Q}^e$ there is a Q^e-martingale $\left(Z_i^{Q^e}\right)$, namely

$$Z_i^{Q^e} := E_{Q^e}[S_N | \mathcal{F}_i], \;\; i = 0, .., N,$$

such that

$$S_i^b \leq Z_i^{Q^e} \leq S_i^a, \;\; i = 0, .., N.$$

Let $C \in L^\infty$. By Jouini (2000, Theorem 1.), the value $p^*(C) := \sup_{P \in \mathcal{P}} E_P[C]$ is less or equal to the price of the smallest self-financing trading strategy dominating C. As $p_0(C) \leq p^*(C)$, we conclude. □

5.2.1 Parametrization of the Pricing Functional

In this Section, we introduce a parametric model for the pricing functional (p_i). We allow the pricing functional to depend on a parameter $\gamma \geq 0$ with the following interpretation: the greater γ, the greater the bid-ask spread; for $\gamma = 0$, the spread is equal to zero. To obtain such parametrization, we let the coherent risk measures, defining the pricing functional, be induced by a family of concave distortion functions (FCDF), as defined by Definition 4.4.7. We assume that the FCDF satisfies assumption [E] and [A].

Assumption [A] is used to prove convergence of bid and ask prices. By Assumption [E] the FCDF (Ψ^γ) fulfils

$$\Psi^0\left(\frac{1}{2}\right) = \frac{1}{2}.$$

All FCDF satisfying Assumption [A] are also (approximately) equal in a small neighbourhood around the point $(u, \gamma) = \left(\frac{1}{2}, 0\right)$. Therefore we will see that the particular choice of the FCDF to model the pricing functional in the discrete time model does not matter when the number of trading periods tends to infinity.

As in Madan et al. (2013, 2017a) and Leippold and Schärer (2017), we generalize the static coherent risk measure defined in Equation (4.4) to the dynamic case. For $X \in L^\infty$ and $i = 0, ..., N$ let

$$\tilde{\rho}_i^\gamma(X) := \int_0^\infty \left(\Psi^\gamma\left(\mathbb{Q}_i\left[X < y\right]\right) - 1\right) dy + \int_{-\infty}^0 \Psi^\gamma\left(\mathbb{Q}_i\left[X < y\right]\right) dy, \quad \gamma \geq 0,$$

where

$$\mathbb{Q}_i[A] := E_{\mathbb{Q}}\left[1_A \,|\mathcal{F}_i\right], \quad i = 0, ..., N, \quad A \in \mathcal{F},$$

is a conditional probability. Define

$$\rho_N^\gamma := \tilde{\rho}_N^\gamma$$

and recursively

$$\rho_i^\gamma := \tilde{\rho}_i^\gamma(-\rho_{i+1}^\gamma), \quad i = 0, ..., N-1, \quad \gamma \geq 0.$$

The pricing functional used in this article is then defined by

$$p_i^\gamma(.) := \rho_i^\gamma(-.), \quad i = 0, ..., N, \quad \gamma \geq 0. \tag{5.1}$$

The recursive definition makes the pricing functional time-consistent. By assumption [E] it holds

$$E_{\mathbb{Q}}[X|\,\mathcal{F}_i] = p_i^0(X).$$

The parameter $\gamma \geq 0$ describes the liquidity of the market: the greater γ, the greater the bid-ask spread. For $\gamma = 0$, bid and ask prices coincide and are identical to the risk neutral price operator.

Due to the time-consistency, for a fixed $\gamma_N \geq 0$, bid and ask prices of a future random cash flow $C^E \in L^\infty$ can be obtained by recursions:

$$\begin{aligned} \text{bid}_N\left(C^E\right) &= \text{ask}_N\left(C^E\right) = C^E, \\ \text{bid}_i\left(C^E\right) &= -p_i^{\gamma_N}\left(-\text{bid}_{i+1}\left(C^E\right)\right), \;\; i = 0, .., N-1, \\ \text{and} \quad \text{ask}_i\left(C^E\right) &= p_i^{\gamma_N}\left(\text{ask}_{i+1}\left(C^E\right)\right), \;\; i = 0, .., N-1. \end{aligned} \tag{5.2}$$

We explicitly allow the parameter γ_N, which describes the bid-ask spread in the N^{th} model, to depend on N, in order to obtain convergence results for $N \to \infty$.

5.3 Static Time: Implied Liquidity in Option Markets

The economic model presented in Section 5.2 contains a static model as a special case by setting the number of trading periods N equal to one. Let us look at a nonnegative future random cash flow X, e.g. a plain vanilla European put or call option, which only depends on the value S_N of the stock. By Section 5.2.1, the bid price of X in static time with respect to some continuous FCDF (Ψ^γ) is defined as follows:

$$\begin{aligned} \text{bid}_0^\gamma(X) &= -p_0(-e^{-rT}X) \\ &= -\rho_0(e^{-rT}X) \\ &= e^{-rT}\int_{-\infty}^{\infty} x d\Psi^\gamma\left(F_X(x)\right) & (5.3) \\ &= e^{-rT}\int_{-\infty}^{\infty} x \Psi_\gamma'\left(F_X(x)\right) f_X(x) dx, \;\; \gamma \geq 0, & (5.4) \end{aligned}$$

compare also with Equation (4.6). F_X is the distribution function of X with respect to the equivalent martingale measure $\mathbb{Q}$. The factor e^{-rT} discounts the future random cash flow X. For Equation (5.4), we assume that the distribution function F_X of X is differentiable with density f_X and that the partial derivative

$$\Psi_\gamma'(u) = \frac{\partial}{\partial u}\Psi^\gamma(u)$$

exists. Similarly, the ask price is defined by

$$\text{ask}_0^\gamma(X) = -e^{-rT}\int_{-\infty}^{\infty} x d\Psi_\gamma\left(F_{-X}(x)\right). \tag{5.5}$$

Note that the functionals $\text{bid}_0^\gamma(.)$ and $\text{ask}_0^\gamma(.)$ are well defined on L^1 if the FCDF is induced by the Laplace distribution, see Example 5.3.3 and Remark 4.4.6. They are well defined on L^2 if the FCDF corresponds to the Wang transform, see Equation (4.2). Both the Laplace and the normal distribution play an important part in this Section.

Interestingly, the recently developed conic finance theory, see Madan and Cherny (2010), provides identically formulas for bid and ask prices as stated in Equations (5.3) and (5.5). We therefore discuss conic finance in Section 5.3.1 briefly.

Let us assume that F_X is log-concave. If we use the family of distortion functions that is induced by F_X, see Equation (4.10), we are able to derive explicit formulas for the bid and ask prices of the future random cash flow X. In particular the bid and ask prices of European vanilla options are equal to the risk-neutral price of an option on the underlying with an adjusted dividend yield. It is then possible to derive closed-form solutions provided that the log-returns are normal or Laplace distributed.

Section 5.3 is structured as follows: In Section 5.3.1 we introduce conic finance. In Section 5.3.3 we derive closed-form solutions for bid and ask prices of European options. Two important examples, the Black-Scholes and the Laplace-model are discussed. In Section 5.3.4 the concept of implied liquidity is defined and applied to real data in Section 5.3.5.

5.3.1 Introduction to Conic Finance

Madan and Cherny (2010) developed the *conic finance* theory extending classical financial models. For applications of the conic finance theory, see Corcuera et al. (2012), Dhaene et al. (2012), Guillaume (2015), Guillaume and Schoutens (2015), Guillaume et al. (2018), Madan (2012a,b, 2014, 2016b, 2018), Madan et al. (2016) and Madan and Schoutens (2011, 2012a, 2016a,b).

Madan and Cherny (2010) modelled the market as a passive counterparty that demands a minimal performance $\gamma > 0$ to be willing to enter into a contract with an investor. The performance is measured by an acceptability index α, see Section 4.3. Conic finance replaces the classical one-price market model by a two-price market model, where an investor has to pay an ask price to buy an asset from the market and receives a usually smaller bid price for selling the same asset to the market. Prices are defined from the perspective of the market. Motivated by competition, the ask price of a future random cash flow X is determined as the minimal price a such that the residual future random cash flow $a - X$ has at least the performance γ, i.e.

$$\begin{aligned}
\operatorname{ask}(X) &= \inf\{a \in \mathbb{R} : \alpha(a - X) \geq \gamma\} \\
&= \inf\{a \in \mathbb{R} : \rho^{\gamma}(a - X) \leq 0\} \\
&= \inf\{a \in \mathbb{R} : \rho^{\gamma}(-X) \leq a\} \\
&= \rho^{\gamma}(-X)
\end{aligned}$$

By this argumentation, they derived the following formulas for bid and ask prices at level $\gamma \geq 0$ in a static world.

$$\operatorname{bid}(X) = -\rho^{\gamma}(X) \text{ and } \operatorname{ask}(X) = \rho^{\gamma}(-X), \;\; \gamma \geq 0. \tag{5.6}$$

The parameter γ describes the liquidity of the market. The greater γ, the greater the bid-ask spread. Madan and Cherny (2010) also discussed the existence of a set of hedging cash flows $\mathcal{H}$ with zero initial cost, which are assumed to be perfectly liquid. The ask price is then defined by

$$\operatorname{ask}^{\text{hedging}}(X) = \inf\{a : \text{there exisits } H \in \mathcal{H} \text{ such that } \alpha(a + H - X) \geq \gamma\}.$$

The bid price using hedging opportunities is defined similarly. We work in a market with frictions, which means even the stock cannot be bought and sold for the same price and hence we can reasonably assume that there are no perfectly liquid hedging cash flows, i.e. $\mathcal{H} = \emptyset$.

We see that the "conic bid and ask prices" coincide with the static version of the bid and ask prices defined in Section 5.2. Both approaches model bid and ask prices using coherent risk measures. While the axiomatic approach by Jouini (2000) and Bion-Nadal (2009) *postulate* the definitions for bid and ask prices, conic finance is able to explain this definition economically using the theory of performance measures.

5.3.2 An Exponential Stock Price Model

In this Section, we provide a concrete static model of the stock price. Inspired by Corcuera et al. (2009), let us model the stock at the date T by a random variable S_T, which is defined in the following way: let Z be a random variable with mean zero and variance equal to 1. Its distribution function is denoted by F_Z, its density by f_Z. The random variable $\sqrt{T}Z$ has then variance T and the underlying S_T at time T is defined by

$$S_T = S_0 e^{(r-q+\omega)T+\sigma\sqrt{T}Z} \tag{5.7}$$

where $\sigma > 0$, r is the risk-free rate, q the dividend yield and $\omega \in \mathbb{R}$ is a mean correcting term, i.e. ω is chosen such that

$$e^{-(r-q)T} E\left(S_T\right) = S_0, \tag{5.8}$$

where the expectation is taken under an equivalent martingale measure $\mathbb{Q}$. In the following, we assume that F_Z is symmetric about zero, i.e.

$$F_Z(-x) = 1 - F_Z(x), \;\; x \in \mathbb{R}.$$

Remark 5.3.1. Note that Equation (5.7) describes the stock price at maturity, where we assume a bid-ask spread of zero. At time zero, the stock may have a positive bid-ask spread. The "process" $\{S_0, S_T\}$ describes the risky asset of the underlying frictionless market.

5.3.3 Bid and Ask prices of European Options

We now introduce a call option with strike K and maturity T on the underlying S_T. It is easy to see that the distribution of the call option $C = (S_T - K)^+$ is

$$\begin{aligned} F_C(x) &= F_{S_T}(x+K), \;\; x \geq 0 \\ &= F_Z\left(\frac{\log(x+K) - \log(S_0) - (r-q+\omega)T}{\sigma\sqrt{T}}\right), \;\; x \geq 0, \end{aligned}$$

see for example Madan and Schoutens (2016a, p. 110). Let us assume that F_Z induces a family of distortion functions by

$$\Psi_Z^\gamma(u) = F_Z\left(F_Z^{-1}(u) + \gamma\right), \;\; \gamma \geq 0, \tag{5.9}$$

compare with Remark 4.4.12. In particular, we assume that f_Z belongs to the family of log-concave densities. This leads to particularly simple formulas for the bid and ask prices because the distorted distribution function can then explicitly be calculated via Equations (5.3) and (5.5). For a call option it holds

$$\Psi_Z^{\gamma}\left(F_C(x)\right) = F_Z\left(\frac{\log\left(x+K\right)-\log\left(S_0\right)-(r-q+\omega)T}{\sigma\sqrt{T}}+\gamma\right), \quad x \geq 0.$$

We calculate the bid price of a call option by

$$\begin{aligned}
\operatorname{bid}^{\gamma}(C) &= e^{-rT}\int_{-\infty}^{\infty} x d\Psi_Z^{\gamma}\left(F_C(x)\right) \\
&= e^{-rT}\int_{0}^{\infty} x\frac{f_Z\left(\frac{\log(x+K)-\log(S_0)-(r-q+\omega)T}{\sigma\sqrt{T}}+\gamma\right)}{\sigma\sqrt{T}(x+K)}dx \\
&= e^{-rT}\int_{-d+\gamma}^{\infty}\left(S_0 e^{\sigma\sqrt{T}y+(r-q+\omega)T-\sigma\sqrt{T}\gamma}-K\right)f_Z\left(y\right)dy, \qquad (5.10)
\end{aligned}$$

where

$$d = \frac{\log\left(\frac{S_0}{K}\right)+(r-q+\omega)T}{\sigma\sqrt{T}}. \qquad (5.11)$$

From Equation (5.10) we see that the bid price of an option C on a stock with dividend yield q at level $\gamma \geq 0$ equals the risk neutral price of an option on a stock with a different dividend yield

$$\tilde{q} = q + \frac{\gamma\sigma}{\sqrt{T}}.$$

Similarly, the ask price can be obtained by evaluating F_{-C}, it holds

$$\begin{aligned}
\operatorname{ask}^{\gamma}(C) &= e^{-rT}\int_{-d-\gamma}^{\infty}\left(S_0 e^{\sigma\sqrt{T}y+(r-q+\omega)T+\sigma\sqrt{T}\gamma}-K\right)f_Z\left(y\right)dy \qquad (5.12) \\
&= \operatorname{bid}^{-\gamma}(C).
\end{aligned}$$

Hence, if we have an analytic formula for the bid price, we just need to substitute γ by $-\gamma$ to get an analytic formula for the ask price.

Analogically, it holds for the bid price of an European Put option $P = (K - S_T)^+$

$$\operatorname{bid}^{\gamma}(P) = e^{-rT}\int_{d+\gamma}^{\infty}\left(K - S_0 e^{-\sigma\sqrt{T}y+(r-q+\omega)T+\sigma\sqrt{T}\gamma}\right)f_Z\left(y\right)dy \qquad (5.13)$$

and the ask price of a put option can be expressed by

$$\begin{aligned}
\operatorname{ask}^{\gamma}(P) &= e^{-rT}\int_{d-\gamma}^{\infty}\left(K - S_0 e^{-\sigma\sqrt{T}y+(r-q+\omega)T-\sigma\sqrt{T}\gamma}\right)f_Z\left(y\right)dy \qquad (5.14) \\
&= \operatorname{bid}^{-\gamma}(P).
\end{aligned}$$

Summarizing, the bid price of a call option and the ask price of a put option are equal to the risk-neutral prices of a call and a put option respectively, replacing the dividend yield q of the stock by $q + \frac{\sigma\gamma}{\sqrt{T}}$. The ask price of a call option and the bid price of a put option are equal to risk-neutral price of a call and a put option respectively, replacing the dividend yield by $q - \frac{\sigma\gamma}{\sqrt{T}}$.

We provide two examples where Equation (5.10) can be calculated explicitly.

Example 5.3.2. As already mentioned by Madan and Schoutens (2016a, Example 5.5), assuming a Black-Scholes setting, i.e. Z is standard normal distributed with distribution function Φ and $\omega = -\frac{1}{2}\sigma^2$ and using the Wang transform, leads to the following formulas for the bid price of a call option and a put option

$$\begin{aligned} \mathrm{bid}_{\mathrm{WANG}}(C) &= S_0 e^{-\left(q+\frac{\gamma\sigma}{\sqrt{T}}\right)T} \Phi\left(d_1 - \gamma\right) - e^{-rT} K \Phi\left(d_2 - \gamma\right) && (5.15) \\ \mathrm{bid}_{\mathrm{WANG}}(P) &= e^{-rT} K \Phi\left(-d_2 - \gamma\right) - S_0 e^{-\left(q-\frac{\gamma\sigma}{\sqrt{T}}\right)T} \Phi\left(-d_1 - \gamma\right), && (5.16) \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$ are defined as in the classical Black-Scholes model. The ask prices are equal to the bid prices, replacing γ by $-\gamma$. For $\gamma = 0$, we obtain the classical Black-Scholes formula.

The Laplace distribution is particularly interesting because mathematically it is even easier to handle than the normal distribution and it has fatter tails. While the logarithm of the density of the normal distribution decays quadratically, the logarithm of the Laplace density decreases linearly. Thus using the Laplace distribution instead of the normal distribution can overcome some of the criticism of the Black-Scholes model.

Example 5.3.3. Let $T > 0$ and let Z be Laplace distributed with mean zero and variance 1. In particular, Z has density

$$f_Z(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|}.$$

Let us use the Laplace distortion as defined in Example 4.4.13 and assume

$$\sigma^2 T < 2$$

and let

$$\omega = \frac{1}{T} \log\left(1 - \frac{1}{2}\sigma^2 T\right),$$

which makes the discounted underlying in Equation (5.7) a martingale. See also Madan (2016a) for the use of the Laplace distribution in pricing European options. Note that the integral in Equation (5.10) is infinite if $\sigma^2 T \geq 2$, independently of the choice of ω.

We should not worry too much about this: from a practical point of view, the maturity T and volatility σ usually do not exceed the limit, i.e. $\sigma^2 T < 2$. E.g. if we look at a time-horizon of less then eight years and a yearly volatility of 50% or less, we are well below the limit. From a mathematical point of view, we know by Equation (5.8) that the expectation of S_T under the equivalent martingale measure must be finite, in particular it holds

$$E(S_T) < \infty.$$

This is equivalent to $E\left(e^{\sigma\sqrt{T}Z}\right) < \infty$. On the other hand it holds

$$E\left(e^{\sigma\sqrt{T}Z}\right) = \int_{\mathbb{R}} e^{\sigma x}\frac{1}{\sqrt{2T}}e^{-\frac{\sqrt{2}}{\sqrt{T}}|x|}dx = \begin{cases}\infty & ,\sigma\sqrt{T} \geq \sqrt{2} \\ \frac{2}{2-\sigma^2 T} & ,\sigma\sqrt{T} < \sqrt{2}.\end{cases}$$

Therefore $\sigma^2 T < 2$ *must* hold but as

$$\frac{2}{2-\sigma^2 T} \to \infty \text{ for } \sigma^2 T \nearrow 2,$$

the integral may be arbitrary large. Closed-form solutions for bid and ask prices of European options can be obtained by taking the corresponding closed-form solutions in Madan (2016a, Section 2.1) and replacing q by $q + \frac{\gamma\sigma}{\sqrt{T}}$, respectively by $q - \frac{\gamma\sigma}{\sqrt{T}}$.

5.3.4 Implied Liquidity (IL)

The concept of implied liquidity has been introduced by Corcuera et al. (2012), Dhaene et al. (2012) and Albrecher et al. (2013) and Guillaume et al. (2018). It is similar to the idea of implied volatility and computes implicitly two parameters γ_b and γ_a such that modelled bid and ask prices match real market prices.

Given some real market data of bid and ask prices of a cash-flow X, we assume that the equivalent martingale measure $\mathbb{Q}$ is chosen, such that the mid price is equal to the discounted expectation of X under $\mathbb{Q}$. For example if X is an European option, and the underlying is described by the Black-Scholes model, one would compute an implied volatility such that the Black-Scholes price matches the given mid price of the option. Thus we assume the distribution F_X is known and call a non-negative number γ_b such that $\text{bid}^{\gamma_b}(X)$, defined in Equation (5.3), exactly matches the given market bid-price as *the implied liquidity at the bid-side.* We similarly define $\gamma_a \geq 0$ such that $\text{ask}^{\gamma_a}(X)$ is equal to the given ask price as *the implied liquidity at the ask-side.* The pair (γ_b, γ_a) is simply called the *implied liquidity* (IL).

5.3.5 Application to real Market Data

We apply both the Black-Scholes model and the Laplace-model from Example 5.3.2 and 5.3.3 to bid and ask prices of real option data and compute the IL. For a time-series of 21 days, between August, 5th and September, 2nd 2015, we look at 1820 end-of-day bid and ask prices of European plain vanilla call and put options on the S&P 500 with maturities ranging from about 0.42 to 2.36 years and moneyness between 0.83 and 1.09. The options are obtained from the Chicago Board Options Exchange.

As shown in Figure 5.1, the uncertainty of Standard & Poor's 500 stock market index rose sharply during that period. On August, 24th, which was termed "Black Monday" by China's media due to the China's stock market crash, the CBOE Volatility Index (VIX) reached 53.29 points during the day and closed at 40.74 points. Only a week before, on August, 17th, the VIX closed at 13.02 points. It is well known that liquidity of stock markets usually drops, when uncertainty rises. Indeed, while at the beginning of the time-series, the relative bid-ask spread is less than 1% for at-the-money call options with maturity of about half a year, it rises to more than 5.6% on August, 24th for the same type of options.

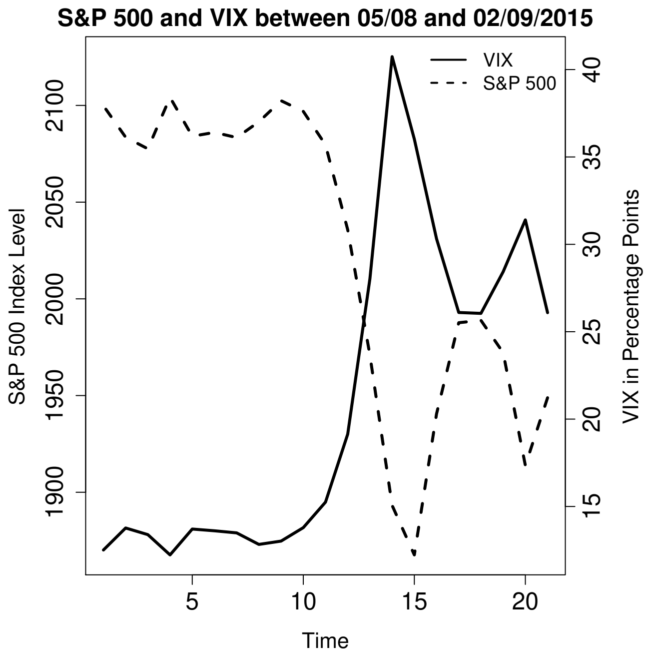

Figure 5.1: S&P 500 and VIX between August, 5th and September, 2nd 2015.

In the following, we are going to compare the relative bid-ask spread, the IL and the difference between implied bid and ask volatilities. We compute for each option at each timepoint the Black-Scholes implied volatility $\sigma^{\mathrm{BS}}_{\mathrm{Mid}}$ and Laplacian implied volatility $\sigma^{\mathrm{L}}_{\mathrm{Mid}}$ matching exactly the mid-price using the classical Black-Scholes formula and the formulas derived by Madan (2016a, Section 2.1) for the Laplace-model, i.e. the formulas in Equations (5.12) and (5.14) setting $\gamma = 0$. We get a typical volatility smile for both models, even though the Laplacian implied volatility surface is slightly flatter than the Black-Scholes implied volatility surface.

For each option, at each timepoint and for both models, we use the mid-price implied volatility as estimate of the volatility in Equation (5.7) and compute the IL, $(\gamma_b, \gamma_a) \in$

$\mathbb{R}_+$, such that the model bid and ask prices match exactly the quoted market bid and ask prices, see Section 5.3.4. For example, the Black-Scholes bid-price in Equation (5.15), is equal to the quoted market bid price of a call option, when using the implied γ_b and $\sigma_{\text{Mid}}^{\text{BS}}$ as input parameters. Note that for most options γ_b and γ_a are almost identical, only for very deep out-of-the money options the difference between both values is more pronounced.

In industry, traders usually prefer quoting the implied volatility instead of the mid-price, because the implied volatility is comparable across strikes, maturities and underlying assets. With the same argument, it seems more appropriate to quote the bid-ask spread in terms of the IL because spreads behave in a non-linear way across strikes, maturities and underlyings while the IL improves comparability across all three dimensions.

So far traders quote implied bid and ask volatilities and describe the bid-ask spread implicitly by the difference of the implied bid and ask volatilities. This procedure needs to be compared to the approach to describe the bid-ask spread by the IL. Note that while for some options it is not possible to compute the implied bid volatility, because the bid-price is below the arbitrage-free price, for all options there exists an implied γ_b matching the bid-price exactly. We removed all options from the data set where it is not possible to compute an implied bid volatility.

In Figure 5.2, the time-series of mean values $\frac{\gamma_a+\gamma_b}{2}$ for the Black-Scholes model and the Laplace-model are shown for at-the-money call options with maturity of about half a year and are compared to the relative bid-ask spread and the implied bid-ask volatilities over time. While the relative bid-ask spread rose from timepoint 12 (August, 20th) to timepoint 14 (August 24th) from 1.2% to 5.6%, hence by the factor 4.87, the IL make a similar move and rose by the factor 4.81. But the difference between bid and ask implied volatilities changed by the factor 6.85. Hence describing the bid-ask spread by quoting implied bid and ask volatilities, overestimates the change in liquidity by about 35%. Looking at put options instead or analysing options with different maturities or moneyness levels, gives a similar picture.

Figure 5.3 illustrates the relative difference of four liquidity measures, respectively between two successive timepoints. The relative bid-ask spread, the Black-Scholes and the Laplacian IL and the difference between bid and ask Black-Scholes implied volatilities are compared for at-the-money and out-of-the-money call and put options with maturities of about half a year. It is not unusual that quoting the bid-ask spread using implied volatilities overestimates an up or down move in liquidity by 40% and more compared to the relative bid-ask spread. Only for in-the-money options, all four liquidity measures behave similarly. The correlation between the relative bid-ask spread and γ_b or γ_a for the different maturities and option types (call and put), lay between 0.91 and 0.99 for the Black-Scholes and the Laplace-model. That makes the IL a more intuitive measure for liquidity than quoting the spread implicitly by stating implied volatilities for both bid and ask prices.

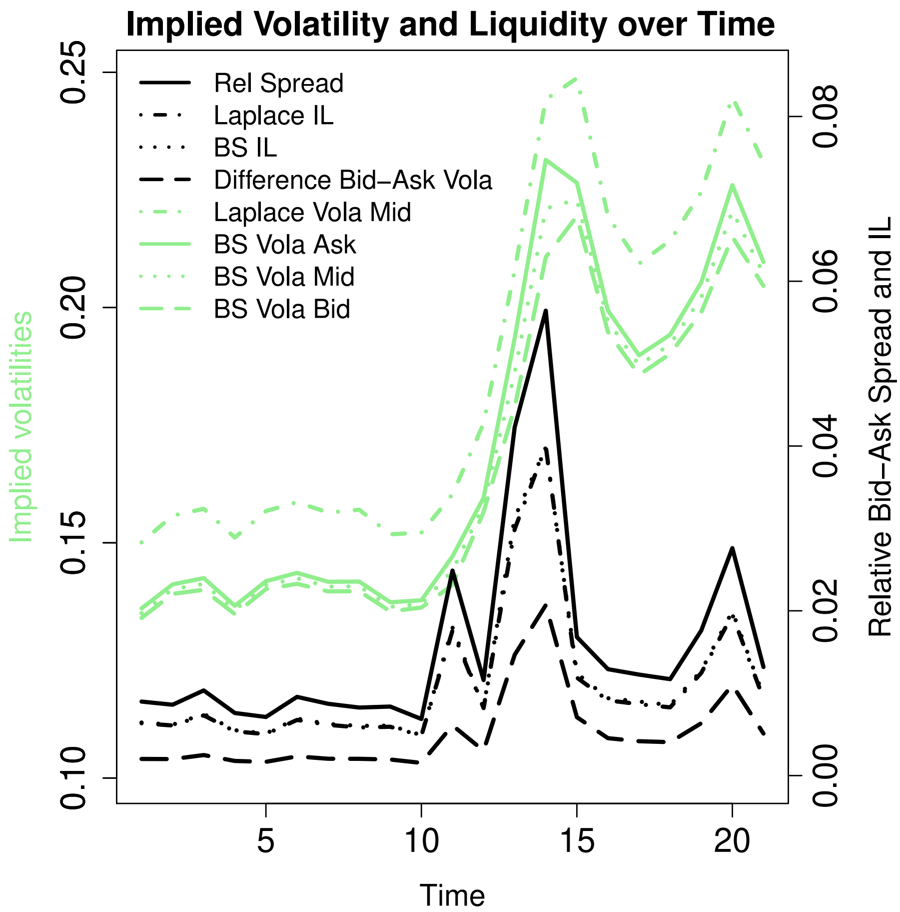

Figure 5.2: Average implied volatility and liquidity over time of ATM-Call options with maturity varying between 0.42 and 0.55 years and moneyness ranging between 0.98 and 1.02 between August, 5th and September, 2nd 2015

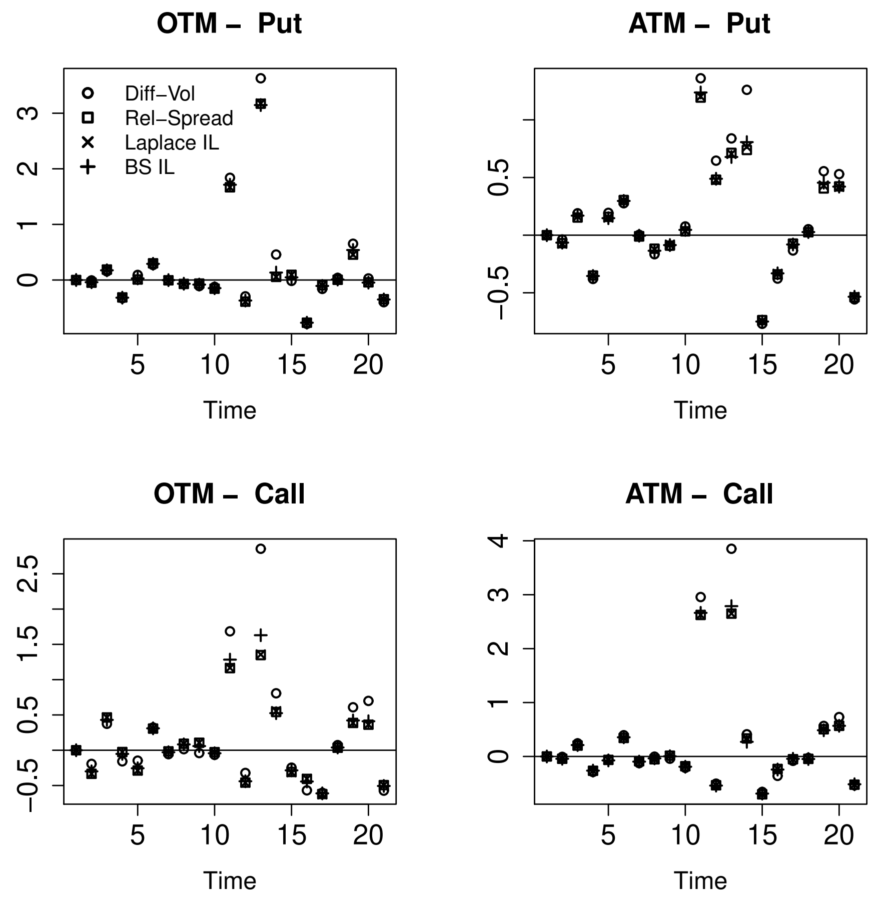

Figure 5.3: The Figure shows for call and put options with maturities between 0.42 and 0.55 years and moneyness within the two ranges 0.9-0.95 (OTM) and 0.98-1.02 (ATM), the relative difference of four liquidity measures, respectively between two successive timepoints. The liquidity measures are: the relative bid-ask spread (Rel-Spread), the Black-Scholes and the Laplacian IL (BS and Laplace IL) and the difference between bid and ask Black-Scholes implied volatilities (Diff-Vol).

5.4 American and Exotic Options in a Market with Frictions

The main goal of this Section is to prove convergence of bid and ask prices of different European and American contingent claims in a binomial-type model with frictions, when the number of trading periods approaches infinity. We focus on contingent claims which are monotonically increasing or decreasing with respect to the underlying, this has a technical reason, see Remark 5.4.1. In Section 5.4.1 we precisely define increasing and decreasing European and American contingent claims and provide a selection of examples.

Remark 5.4.1. In a binomial-type model, the bid and ask prices are recursively defined and can be computed going backwards through a tree. Figure 5.4 shows a binomial tree with $N = 2$ time-steps. The ask prices at the final nodes are equal to the value of the option at expiration. The ask price at the first node (today) can be computed going iteratively through the tree using the recursions (5.2). For example the ask price a_{11} can be computed using the two successive nodes a_{21} and a_{22}.

$$a_{11} = \begin{cases} \Psi^\gamma(1-p)a_{22} + (1-\Psi^\gamma(1-p))a_{21} & , a_{21} \le a_{22} \\ (1-\Psi^\gamma(p))a_{22} + \Psi^\gamma(p)a_{21} & , a_{21} > a_{22}, \end{cases}$$

where p denotes the up-move probability in a classical binomial model and (Ψ^γ) is a FCDF. The formula is deduced from the definition of pricing functional, see Equation (5.1). In contrast to the iterative computation of the risk-neutral price in the classical binomial model, the bid and ask prices depend on the *sorting* of the successive nodes. Therefore in this thesis we only prove convergence for monotone payoffs, which are precisely defined in Definition 5.4.2. Bid and ask prices of general payoffs can be computed in the discrete time model going backwards through the tree and checking at each node the sorting of the two successive nodes.

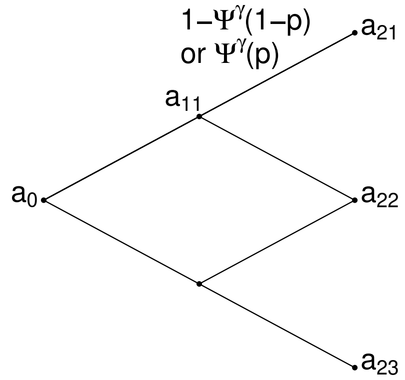

Figure 5.4: Binomial tree to compute the ask price. The up-move probability depends on the sorting of the successive nodes.

5.4.1 Payoffs

In this Section, we define increasing and decreasing European and American contingent claims.

Definition 5.4.2. A *European contingent claim* C^E is a bounded random variable on $(\Omega, \mathcal{F})$, such that there is a measurable function h, with

$$C^E = h(S_0, ..., S_N).$$

The claim is called *increasing* if

$$h(x_0, ..., x_N) \geq h(y_0, ..., y_N), \;\; x_i \geq y_i, \;\; i = 0, ..., N$$

and *decreasing* if

$$h(x_0, ..., x_N) \leq h(y_0, ..., y_N), \;\; x_i \geq y_i, \;\; i = 0, ..., N.$$

An *American contingent claim* C^A is a bounded adapted process

$$C^A = \left(C_i^A\right)_{i=0,..,N},$$

such that for each i there is a measurable function h_i, with

$$C_i^A = h_i(S_0, ..., S_i).$$

The claim is called *increasing* if

$$h_i(x_0, ..., x_i) \geq h_i(y_0, ..., y_i), \;\; x_k \geq y_k, \;\; i = 0, ..., N, \;\; , k = 0, ..., i$$

and *decreasing* if

$$h_i(x_0, ..., x_i) \leq h_i(y_0, ..., y_i), \;\; x_k \geq y_k, \;\; i = 0, ..., N, \;\; , k = 0, ..., i.$$

A European claim C^E can be interpreted as a random payoff at maturity T. For each i, the random variable C_i^A is interpreted as the payoff of the American contingent claim if the claim is exercised after i trading periods. We assume the American option is cash-settled, and the reference price is the process (S_i). If the holder of an American option exercises the option early after i trading periods, she will receive the amount $h_i(S_0, ..., S_i)$, which is independent of the current bid-ask spread or the processes $\left(S_i^b\right)$ and (S_i^a). This may in particular hold for cash-settled index options and it holds approximately for options with physically delivery if the transaction costs of trading the stock are small. Similar to European contingent claims, see Equation (5.2), bid and ask prices of an American contingent claim C^A can be defined recursively, incorporating the possibility of an early exercise:

$$\begin{aligned} \mathrm{bid}_N(C^A) &= \mathrm{ask}_N(C^A) = C_N^A, \\ \mathrm{bid}_i(C^A) &= C_i^A \vee -p_i^{\gamma_N}(-\mathrm{bid}_{i+1}(C^A)), \;\; i = 0, .., N-1, \\ \text{and} \quad \mathrm{ask}_i(C^A) &= C_i^A \vee p_i^{\gamma_N}(\mathrm{ask}_{i+1}(C^A)), \;\; i = 0, .., N-1. \end{aligned} \tag{5.17}$$

Let Ξ be the set all American contingent claims. To simplify notation, the operators assigning bid and ask prices to American contingent claims

$$\mathrm{bid}_i : \Xi \to L_i^\infty \text{ and } \mathrm{ask}_i : \Xi \to L_i^\infty,$$

have the same names as the operators describing prices of European contingent claims, which can be seen as functionals from L^∞ to L_i^∞.

We provide some examples of European and American contingent claims. Let $K \geq 0$ be a strike price and $B \geq 0$ be a barrier. By $\mathcal{N}$ we denote the time, the option is exercised. If $\mathcal{N} \in \{0, ..., N\}$ can be chosen by the holder of the option, we speak of an American contingent claim, exercised at time $\mathcal{N}$. If only $\mathcal{N} = N$ is allowed, i.e. the option can only be exercised at maturity, we speak of a European contingent claim.

Example 5.4.3. The following derivatives are increasing contingent claims.

- Call option: $C_{\text{Call}} = (S_{\mathcal{N}} - K)^+$
- Lookback call option: $C_{\text{LbCall}} = \left(\max\limits_{i \in \{0,...,\mathcal{N}\}} S_i - K \right)^+$
- Asian call option: $C_{\text{AsianCall}} = \left(\frac{1}{\mathcal{N}} \sum_{i=0}^{\mathcal{N}} S_i - K \right)^+$
- Barrier up-and-in call option:

$$C_{\text{UICall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \max\limits_{i \in \{0,...,\mathcal{N}\}} S_i \geq B \\ 0, & \text{otherwise} \end{cases}$$

- Barrier down-and-out call option:

$$C_{\text{DOCall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \min\limits_{i \in \{0,...,\mathcal{N}\}} S_i > B \\ 0, & \text{otherwise} \end{cases}$$

Example 5.4.4. Decreasing payoffs are for example:

- Put option: $C_{\text{Call}} = (K - S_{\mathcal{N}})^+$
- Lookback put option: $C_{\text{LbPut}} = \left(K - \max\limits_{i \in \{0,...,\mathcal{N}\}} S_i \right)^+$
- Asian put option: $C_{\text{AsianPut}} = \left(K - \frac{1}{\mathcal{N}} \sum_{i=0}^{\mathcal{N}} S_i \right)^+$
- Barrier up-and-out put option:

$$C_{\text{UOPut}} = \begin{cases} (K - S_{\mathcal{N}})^+ & , \max\limits_{i \in \{0,...,\mathcal{N}\}} S_i < B \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.4.5. The following two derivatives are neither increasing nor decreasing payoffs.

- Barrier up-and-out call option

$$C_{\text{UOCall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \max_{i \in \{0,\dots,\mathcal{N}\}} S_i < B \\ 0, & \text{otherwise} \end{cases}$$

- Barrier down-and-in call option

$$C_{\text{DICall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \min_{i \in \{0,\dots,\mathcal{N}\}} S_i \leq B \\ 0, & \text{otherwise.} \end{cases}$$

5.4.2 Classical Binomial Model

In this Section, we recall the classical binomial model. Let $T > 0$ be some time-horizon and assume there are $N \in \mathbb{N}$ trading periods between $[0, T]$, each trading period is of length $\frac{T}{N}$. There is a riskless bond

$$B_i^{(N)} = (1 + r_N)^i, \;\; i = 0, 1, ..., N,$$

paying interest

$$r_N = \frac{rT}{N} > -1$$

in each trading period and just one risky asset, paying dividends

$$q_N = \frac{qT}{N}$$

in each period, and whose price process takes the form

$$S_i^{(N)} = S_0 \prod_{k=1}^{i} (1 + R_k^{(N)}), \;\; i = 1, 2, ..., N,$$

where $S_0 > 0$, and the returns

$$R_i^{(N)} = \frac{S_i^{(N)} - S_{i-1}^{(N)}}{S_{i-1}^{(N)}}, \;\; i = 1, 2, ..., N$$

are random variables with values in $\{a_N, b_N\} \subset \mathbb{R}$, such that

$$d_N := 1 + a_N = e^{-\sigma\sqrt{\frac{T}{N}}} \text{ and } u_N := 1 + b_N = e^{\sigma\sqrt{\frac{T}{N}}}. \tag{5.18}$$

The market is arbitrage-free and complete if

$$-1 < a_N < r_N - q_N < b_N,$$

which holds for N large enough. In this case, the returns $R_1^{(N)}, ..., R_N^{(N)}$ are independent, their distributions are characterized by

$$P_N^*(R_i^{(N)} = b_N) := p_N^* := p_N^q + \varphi(N), \;\; i = 1, 2, ..., N, \tag{5.19}$$

where P_N^* is the unique risk neutral measure, $\varphi(N) \in o\left(\frac{1}{\sqrt{N}}\right)$ and

$$p_N^q := \frac{1}{2} + \frac{r-q}{2\sigma}\sqrt{\frac{T}{N}}. \tag{5.20}$$

We say the classical binomial model is *characterized by the tuple* $(S_0, r, q, \sigma, T, N)$.

Let $C_E^{(N)}$ be a possibly path-dependent European contingent claim. The discounted claim

$$H_E^{(N)} = \frac{C_E^{(N)}}{B_N^{(N)}}$$

can be written as

$$H_E^{(N)} = h\left(S_0^{(N)}, ..., S_N^{(N)}\right) \tag{5.21}$$

for a suitable function h. The value process in the N^{th} model,

$$V_i^{(N)} = E_{P_N^*}\left[H_E^{(N)} | \mathcal{F}_i\right], \;\; i = 1, 2, ..., N,$$

of a replicating strategy for $H_E^{(N)}$ at time $t = \frac{iT}{N}$ is of the form

$$V_i^{(N)}(\omega) = v_i^{(N)}(S_0, S_1(\omega), .., S_i(\omega)),$$

where the function $v_i^{(N)}$ is given by recursion

$$\begin{aligned} v_N^{(N)}(x_0, ..., x_N) &= h(x_0, ..., x_N) \\ v_i^{(N)}(x_0, ..., x_i) &= (1 - p_N^*)\, v_{i+1}^{(N)}(x_0, ..., x_i, x_i d_N) \\ &\quad + p_N^* v_{i+1}^{(N)}(x_0, ..., x_i, x_i u_N), \;\; i = 0, 1, ..., N-1, \end{aligned}$$

see e.g. Föllmer and Schied (2011, Proposition 5.41).

On the other hand, dealing with an American contingent claim

$$C_A^{(N)} = \left(C_{A,i}^{(N)}\right)_{i=0,..,N},$$

and the corresponding discounted claim

$$H_{A,i}^{(N)} = \frac{C_{A,i}^{(N)}}{B_i^{(N)}}, \;\; i = 0, .., N,$$

for each $i = 0, 1, .., N$ there is a suitable function h_i such that

$$H_{A,i}^{(N)} = h_i(S_0, ..., S_i).$$

By no-arbitrage-arguments, the value process $(V_i)_{i=0,...,N}$ of a replicating strategy for $H_A^{(N)}$ can be found by recursion, compare with Föllmer and Schied (2011, Chapter 6):

$$V_N := H_{A,N}^{(N)}, \;\; V_i := H_{A,i}^{(N)} \vee E_{P_N^*}\left[V_{i+1} \middle| \mathcal{F}_i\right], \;\; i = 0, .., N-1.$$

Hence, there are functions $v_i^{(N)}$ such that

$$V_i = v_i^{(N)}(S_0, ..., S_i), \;\; i = 0, ..., N,$$

namely

$$\begin{aligned} v_N^{(N)}(x_0, ..., x_N) &= h_N(x_0, ..., x_N) \\ v_i^{(N)}(x_0, ..., x_i) &= h_i(x_0, ..., x_i) \vee \Big\{(1-p_N^*)\, v_{i+1}^{(N)}(x_0, ..., x_i, x_i d_N) \\ &\qquad + p_N^* v_{i+1}^{(N)}(x_0, ..., x_i, x_i u_N)\Big\}, \;\; i = 0, 1, ..., N-1. \end{aligned}$$

It is well known that in a classical binomial-tree model, which is characterized by the tuple $(S_0, r, q, \sigma, T, N)$, the risk-neutral price

$$\pi^{(N)}(C^{(N)}, S_0, r, q, \sigma, T) := v_0^{(N)}(S_0) \tag{5.22}$$

of a European or American contingent claim $C^{(N)} = C_E^{(N)}$ or $C^{(N)} = C_A^{(N)}$, converge for many products as $N \to \infty$. If the limit exists, we define

$$\pi(C, S_0, r, q, \sigma, T) := \lim_{N \to \infty} \pi^{(N)}(C^{(N)}, S_0, r, q, \sigma, T).$$

Convergence of plain vanilla European options to the Black-Scholes price are discussed in Cox et al. (1979). For plain vanilla American options we refer to Amin and Khanna (1994), for European and American Asian options and lookback options and some other path-dependent options, see Jiang and Dai (2004). For a proof of convergence for European barrier option, see Carbone (2004) and Lin and Palmer (2013) and references therein. Those convergence results can directly be applied to prove convergence of bid and ask prices as Theorem 5.4.6 shows.

5.4.3 Convergence of Bid and Ask Prices

In this Section we prove our main result and show that bid and ask prices of European or American contingent claims converge, if the risk-neutral price of the claim converges in the classical binomial model. The theorem has an important practical implication: is states that bid and ask prices of monotone payoffs, in particular plain vanilla European and American options, can be computed using the classical Black-Scholes model with an adjusted drift. Bid and ask prices of such options can therefore be computed very fast.

Theorem 5.4.6. *Let $(\Psi^\gamma)_{\gamma\geq 0}$ be a FCDF fulfilling Assumptions [E] and [A] of Definition 4.4.7. Let a classical binomial-tree model be given, which is characterized by the tuple*

$$(S_0, r, q, \sigma, T, N).$$

Let $C^{(N)}$ be an increasing (decreasing) European or American contingent claim. Let $\gamma \geq 0$ and

$$\gamma_N := \gamma\sqrt{\frac{T}{N}}. \tag{5.23}$$

Define bid and ask prices of a European claim by recursions (5.2) and of an American claim by recursions (5.17). If risk-neutral price, defined via Equation (5.22),

$$\pi^{(N)}(C^{(N)}, S_0, r, \tilde{q}, \sigma, T)$$

converges in the classical binomial model for all dividends

$$\tilde{q} \in [q - \sigma\gamma, q + \sigma\gamma]$$

to some non-negative number $\pi(C, S_0, r, \tilde{q}, \sigma)$, then the ask (bid) of the contingent claim converges to

$$\lim_{N\to\infty} \pi^{(N)}(C^{(N)}, S_0, r, q - \sigma\gamma, \sigma)$$

and the bid (ask) price converges to

$$\lim_{N\to\infty} \pi^{(N)}(C^{(N)}, S_0, r, q + \sigma\gamma, \sigma).$$

Proof. We first assume $C^{(N)}$ models a European contingent claim and can be described by a function h as in Equation (5.21). Let u_N, d_N and p_N^q be defined as in Section 5.4.2. Then it holds for the processes describing the ask price $(A_i)_{i=0,1,..,N}$ and the bid price $(B_i)_{i=0,1,..,N}$ of $C^{(N)}$,

$$A_i(\omega) = a_i((S_0, S_1(\omega), .., S_i(\omega)),$$

and

$$B_i(\omega) = b_i((S_0, S_1(\omega), .., S_i(\omega)),$$

where the functions a_i and b_i are recursively defined:

$$b_N(x_0, ..., x_N) = a_N(x_0, ..., x_N) = h(x_0, ..., x_N)$$

and for $i = 0, 1, .., N-1$, if the European contingent claim is increasing

$$\begin{aligned} a_i(x_0, ..., x_i) &= (1 - \Psi^{\gamma_N}(p_N^q + \varphi(N)))\, a_{i+1}(x_0, ..., x_i, x_i d_N) \\ &\quad + \Psi^{\gamma_N}(p_N^q + \varphi(N))\, a_{i+1}(x_0, ..., x_i, x_i u_N) \end{aligned}$$

and

$$\begin{aligned} b_i(x_0, ..., x_i) &= \Psi^{\gamma_N}(1 - (p_N^q + \varphi(N)))\, b_{i+1}(x_0, ..., x_i, x_i d_N) \\ &\quad + (1 - \Psi^{\gamma_N}(1 - (p_N^q + \varphi(N))))\, b_{i+1}(x_0, ..., x_i, x_i u_N). \end{aligned}$$

If h defines a decreasing European contingent claim, it follows for $i = 0, .., N-1$,

$$\begin{aligned} a_i(x_0, ..., x_i) &= \Psi^{\gamma_N}\left(1 - \left(p_N^q + \varphi(N)\right)\right) a_{i+1}\left(x_0, ..., x_i, x_i d_N\right) \\ &\quad + \left(1 - \Psi^{\gamma_N}\left(1 - \left(p_N^q + \varphi(N)\right)\right)\right) a_{i+1}\left(x_0, ..., x_i, x_i u_N\right) \end{aligned}$$

and

$$\begin{aligned} b_i(x_0, ..., x_i) &= \left(1 - \Psi^{\gamma_N}\left(p_N^q + \varphi(N)\right)\right) b_{i+1}\left(x_0, ..., x_i, x_i d_N\right) \\ &\quad + \Psi^{\gamma_N}\left(p_N^q + \varphi(N)\right) b_{i+1}\left(x_0, ..., x_i, x_i u_N\right). \end{aligned}$$

The ask (bid) price of an increasing European payoff and the bid (ask) price of a decreasing European payoff at level $\gamma_N \geq 0$ are exactly defined as the risk-neutral price in the classical binomial model, when replacing the up-move probability in Equation (5.19), i.e,

$$p_N^q + \varphi(N),$$

by

$$\Psi^{\gamma_N}\left(p_N^q + \varphi(N)\right),$$

respectively by

$$\left(1 - \Psi^{\gamma_N}\left(1 - \left(p_N^q + \varphi(N)\right)\right)\right).$$

This observation can directly be carried forward to American contingent claims and is explained by the structure of the binomial model and the recursive definition of bid and ask prices.

By Assumption [A], there is a sequence $\tilde{\varphi}(N) \in o\left(\frac{1}{\sqrt{N}}\right)$ such that

$$\begin{aligned} \Psi^{\gamma_N}\left(p_N^q + \varphi(N)\right) &= \frac{1}{2} + \frac{r-q}{2\sigma}\sqrt{\frac{T}{N}} + \frac{\gamma}{2}\sqrt{\frac{T}{N}} + \tilde{\varphi}(N) \\ &= \frac{1}{2} + \frac{r-(q-\sigma\gamma)}{2\sigma}\sqrt{\frac{T}{N}} + \tilde{\varphi}(N) \\ &= p_N^{q-\sigma\gamma} + \tilde{\varphi}(N). \end{aligned}$$

Similarly it holds for a suitable $\hat{\varphi}(N) \in o\left(\frac{1}{\sqrt{N}}\right)$,

$$1 - \Psi^{\gamma_N}\left(1 - \left(p_N^q + \varphi(N)\right)\right) = p_N^{q+\sigma\gamma} + \hat{\varphi}(N).$$

Hence the up-move probability of the distorted binomial model describing bid and ask prices can be expressed as in the classical binomial model with an adjusted dividend yield:

$$\frac{1}{2} + \frac{r-(q \pm \sigma\gamma)}{2\sigma}\sqrt{\frac{T}{N}} + o\left(\frac{1}{\sqrt{N}}\right). \tag{5.24}$$

As the up and down moves u_N and d_N remain unchanged compared to the classical binomial model, we conclude. $\square$

Remark 5.4.7. A look at the proof of Theorem 5.4.6 shows that one could define γ_N in Equation (5.23) arbitrarily, as long as it converges to zero as fast as $\frac{1}{\sqrt{N}}$. Under our particular choice, γ can be interpreted as a drift-adjustment via the dividend yield in the continuous time limit scaled by the volatility. The drift adjustment is

$$\hat{q}T = (q \pm \sigma\gamma)\, T.$$

In a static setting we obtained similar formulas for bid and ask prices if the log-returns are normal distributed, see Example 5.3.2. In a static setting the term $q \pm \frac{\sigma\gamma}{\sqrt{T}}$ appears. The different scaling is mainly convention. We could match the formulas in continuous and in static time by either rescaling γ_N, see Equation (5.23) or by replacing in the static setting Equation (5.9) by the FCDF

$$\Psi_Z^{\gamma}(u) = F_Z\left(F_Z^{-1}(u) + \gamma\sqrt{T}\right).$$

Remark 5.4.8. In the binomial model, the underlying is modelled by a bounded stochastic process. Therefore there is no restriction of the definition of bid and ask prices via recursions (5.2) and (5.17) requiring the contingent claims to be bounded. In particular call options are bounded in discrete time. Bid and ask prices of a contingent claim form a two-dimensional sequence with the natural numbers $1, 2, 3, ..$ as index set. The index N corresponds to the N^{th} binomial model. In Theorem 5.4.6, we prove convergence of such a sequence. Hence Theorem 5.4.6 says that bid and ask prices of a possibly unbounded contingent claim like a European call option in continuous time can be approximated arbitrary closely by the bid and ask prices of a bounded contingent claim in discrete time.

5.4.4 European Plain Vanilla Options

In the classical Black-Scholes world, there exist closed-form solutions for the risk-neutral price of European plain vanilla and barrier options. By Theorem 5.4.6, we obtain closed-form solutions for bid and ask prices of European plain vanilla and barrier options, which are stated in the next corollaries, by taking the corresponding closed-form solutions for the risk-neutral price and adjusting the dividend yield.

The Black-Scholes prices of plain vanilla European call and put options with strike K and maturity T are given in closed-form and denoted by

$$\text{BS}_{\text{Call}}(S_0, T, K, r, q, \sigma) = S_0 e^{-qT}\Phi\left(d_1\right) - e^{-rT}K\Phi\left(d_2\right)$$

and

$$\text{BS}_{\text{Put}}(S_0, T, K, r, q, \sigma) = e^{-rT}K\Phi\left(-d_2\right) - S_0 e^{-qT}\Phi\left(-d_1\right),$$

where

$$d_1 = \frac{\log\frac{S_0}{K} + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$ and Φ denotes the distribution function of the standard normal distribution, see Black and Scholes (1973).

Corollary 5.4.9. *Under the notation of Theorem 5.4.6, let $C^{(N)}$ be a European plain vanilla option with strike $K > 0$ and maturity T. Bid and ask prices of a put option converge to*

$$\text{bid}^{\gamma}_{\text{Put}} = \text{BS}_{\text{Put}}(S_0, T, K, r, q - \sigma\gamma, \sigma)$$

and

$$\text{ask}^{\gamma}_{\text{Put}} = \text{BS}_{\text{Put}}(S_0, T, K, r, q + \sigma\gamma, \sigma).$$

Bid and ask prices of a call option converge to

$$\text{bid}^{\gamma}_{\text{Call}} = \text{BS}_{\text{Call}}(S_0, T, K, r, q + \sigma\gamma, \sigma)$$

and

$$\text{ask}^{\gamma}_{\text{Call}} = \text{BS}_{\text{Call}}(S_0, T, K, r, q - \sigma\gamma, \sigma).$$

Figure 5.5 shows the relative bid-ask spread surface of European call options over strikes and maturities. Long term options and options being deep out-of-the money are less liquid, the relative bid-ask spread is greater.

Remark 5.4.10. Similarly to the existence of an implied volatility smile, there exist an implied liquidity smile. Computing γ implicitly from given bid and ask prices of options, Corcuera et al. (2012) show that there is a non-linear dependence of γ, with respect to the term structure and the moneyness of the option surface. In particular, we cannot expect to predict the bid-ask spread of one option from given bid and ask prices of another option, if the corresponding strikes and maturities are too distant from each other.

5.4.5 Path-dependent and American Options

In a classical Black-Scholes framework, there exist closed-form solution for many barrier options, see Rubinstein and Reiner (1991) and Cheng (2003). For example the arbitrage-free price of an up-and-in barrier call option with maturity T, strike K and barrier $B > K$ is

$$\begin{aligned}\text{BS}_{\text{UICall}}(S_0, T, K, B, r, q, \sigma) \quad = \quad & S_0 e^{-qT}\Phi(x_1) - Ke^{-rT}\Phi(x_1 - \sigma\sqrt{T}) \\ & - S_0 e^{-qT}\left(\frac{B}{S_0}\right)^{2m}(\Phi(-y) - \Phi(-y_1)) \\ & + Ke^{-rT}\left(\frac{B}{S_0}\right)^{2m-2}(\Phi(-y + \sigma\sqrt{T}) \\ & - \Phi(-y_1 + \sigma\sqrt{T})),\end{aligned}$$

where

$$m = \frac{r - q + \frac{1}{2}\sigma^2}{\sigma^2}, \qquad y = \frac{\log\left(\frac{B^2}{S_0 K}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T},$$

$$x_1 = \frac{\log\left(\frac{S_0}{B}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}, \qquad y_1 = \frac{\log\left(\frac{B}{S_0}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}.$$

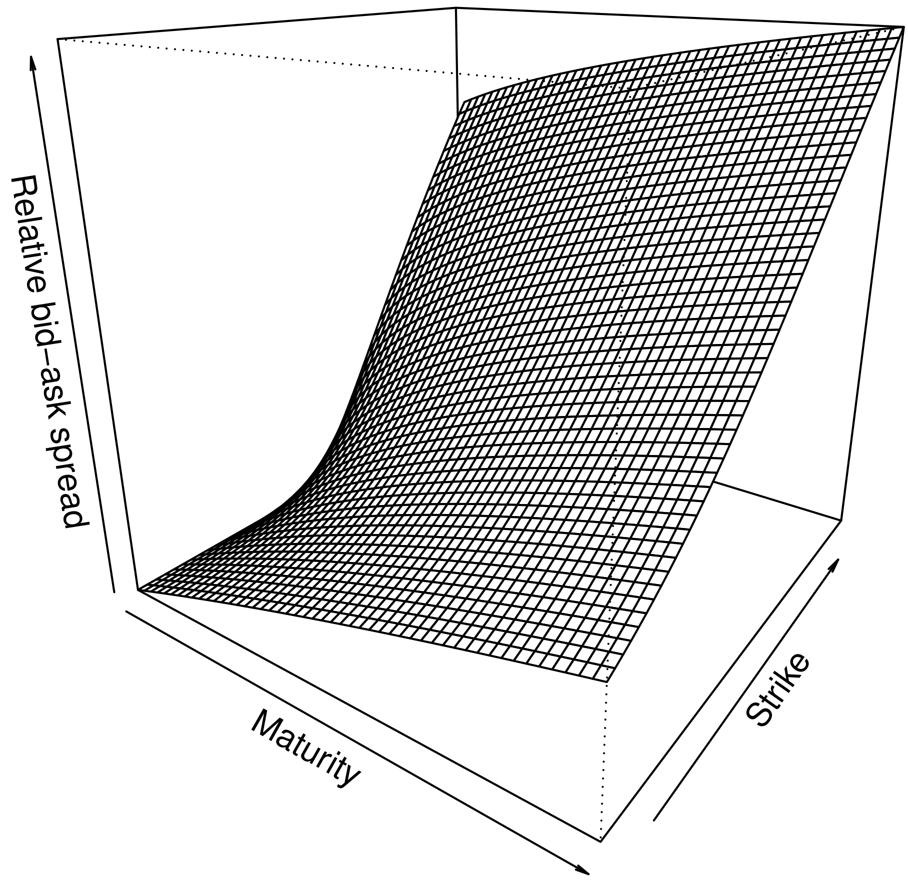

Figure 5.5: Relative bid-ask spread surface for European plain vanilla call options. We use the following parameters: the underlying is equal to 100, the strikes vary between 50 and 150, annual interest rates are set to 0.01, the dividend yield is assumed to be 0.03, the time left to maturity lays in the interval $[0, 2]$, the annual volatility is 0.2 and the annual γ is set to 0.05.

Corollary 5.4.11. *Under the notation of Theorem 5.4.6, let $C^{(N)}$ be an up-and-in barrier call option with maturity T, strike $K > 0$, and barrier $B > K$. The bid price converges to*

$$\mathrm{bid}^{\gamma}_{\mathrm{UICall}} = BS_{\mathrm{UICall}}(S_0, T, K, B, r, q + \sigma\gamma, \sigma)$$

and the ask converges to

$$\mathrm{ask}^{\gamma}_{\mathrm{UICall}} = BS_{\mathrm{UICall}}(S_0, T, K, B, r, q - \sigma\gamma, \sigma).$$

In the following, we treat bid prices of American put options but the findings can be transferred directly to ask prices of American put options and American call options as well. Let the risk-neutral Black-Scholes prices of a plain vanilla American put option with strike K and maturity T be denoted by

$$\mathrm{BS}^{\mathrm{A}}_{\mathrm{Put}}(S_0, T, K, r, q, \sigma).$$

There are no closed-form solutions for American plain vanilla options in a classical Black-Scholes framework, but there exist efficient numerical methods to approximate $\mathrm{BS}^{\mathrm{A}}_{\mathrm{Put}}$, see for example Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993). We denote the numerical approximation by

$$\widetilde{\mathrm{BS}}^{\mathrm{A}}_{\mathrm{Put}}(S_0, T, K, r, q, \sigma)$$

and the error by

$$\varepsilon^{q}_{\mathrm{Put}} := \left| \mathrm{BS}^{\mathrm{A}}_{\mathrm{Put}}(S_0, T, K, r, q, \sigma) - \widetilde{\mathrm{BS}}^{\mathrm{A}}_{\mathrm{Put}}(S_0, T, K, r, q, \sigma) \right|.$$

The next corollary follows immediately:

Corollary 5.4.12. *Under the notation of Theorem 5.4.6, let $C^{(N)}$ be an American plain vanilla put option with strike $K > 0$ and maturity T. The bid price converge to*

$$\mathrm{bid}^{\gamma}_{\mathrm{Put}} = \mathrm{BS}^{\mathrm{A}}_{\mathrm{Put}}(S_0, T, K, r, q - \sigma\gamma, \sigma).$$

The error approximating the bid price using $\widetilde{BS}^{A}_{Put}$ as an estimate for BS^{A}_{Put} is less or equal to $\varepsilon^{q-\sigma\gamma}_{Put}$.

The corollary states the following: the bid price of an American put option on a stock with dividend yield q is equal to the risk-neutral price of an American put option but on a stock with dividend yield $q - \sigma\gamma$. The bid price directly inherits the numerical error from the approximation of the the risk-neutral price of the American option by some numeric algorithm. A similar corollary could easily be stated for other options, like Asian options, which do not have closed-form solutions in the classical Black-Scholes model and can only be approximated for example with Monte Carlo methods. Bid and ask prices can then also be computed using Monte Carlo methods and the absolute error does not increase compared to classical risk-neutral pricing.

5.4.6 Numeric Simulations

In this Section, we try to investigate how fast the recursively defined bid and ask prices converge. We make two approximations: we approximate the concave distortion function by a linear function, see Assumption [A] of Definition 4.4.7 and we approximate the Black-Scholes model by a binomial tree. The error of the first approximation approaches zero faster than $\frac{1}{\sqrt{N}}$, see Equation (5.24).

The convergence rate of the classical binomial model is well studied in literature for many products: Heston and Zhou (2000) show that the risk-neutral price of a plain vanilla European call option converges at least as fast as $\frac{1}{\sqrt{N}}$. Lamberton (1998) prove that the risk-neutral price of an American put option converges from below and from above at least as fast as $N^{-\frac{2}{3}}$ and $N^{-\frac{3}{4}}$ respectively. Leisen and Reimer (1996) and Leisen (1998) analysed three different approaches to build a binomial tree, in particular the definitions for the returns of one trading period differ. They show that European plain vanilla options converge at least as fast as $\frac{1}{N}$ but American put options may only converges from below as fast as $\frac{1}{\sqrt{N}}$ depending on the exact tree definition. Lin and Palmer (2013) treat barrier options.

In our setting the up-move probability to obtain bid and ask prices has only asymptotically the martingale property, which makes it difficult to directly apply convergence results for classical binomial trees to our framework.

We therefore rely on simulations and compute bid and ask prices of a European call option, an American put option and a European up-and-in call option using recursions (5.2) and (5.17) for time-steps ranging from $N \in \{5, .., 2000\}$. We compare the tree-prices to their continuous counterpart, which can be obtained via the Corollaries 5.4.9, 5.4.11 and 5.4.12. Slightly abusing notation, we denote by e_N the absolute difference (error) between the bid or ask price of a contingent claim $C^{(N)}$ in the N^{th} binomial model and the limit of the bid and ask price. We say the sequence of errors *converges with order* $\rho > 0$, if there is a constant $\kappa > 0$ such that

$$\forall N \in \mathbb{N} : e_N \leq \frac{\kappa}{N^\rho}.$$

The order of convergence can be indicated straightforwardly by a simulation, see Leisen and Reimer (1996). As

$$\log\left(\frac{\kappa}{N^\rho}\right) = \log(\kappa) - \rho \log(N),$$

the negative slope of a straight line obtained from a log-log plot of the errors e_N against the refinement N can be used as an indicator for ρ. Figure 5.6 indicates an order of convergence between 1 and $\frac{1}{2}$ of the recursions (5.2) and (5.17) for different European and American options.

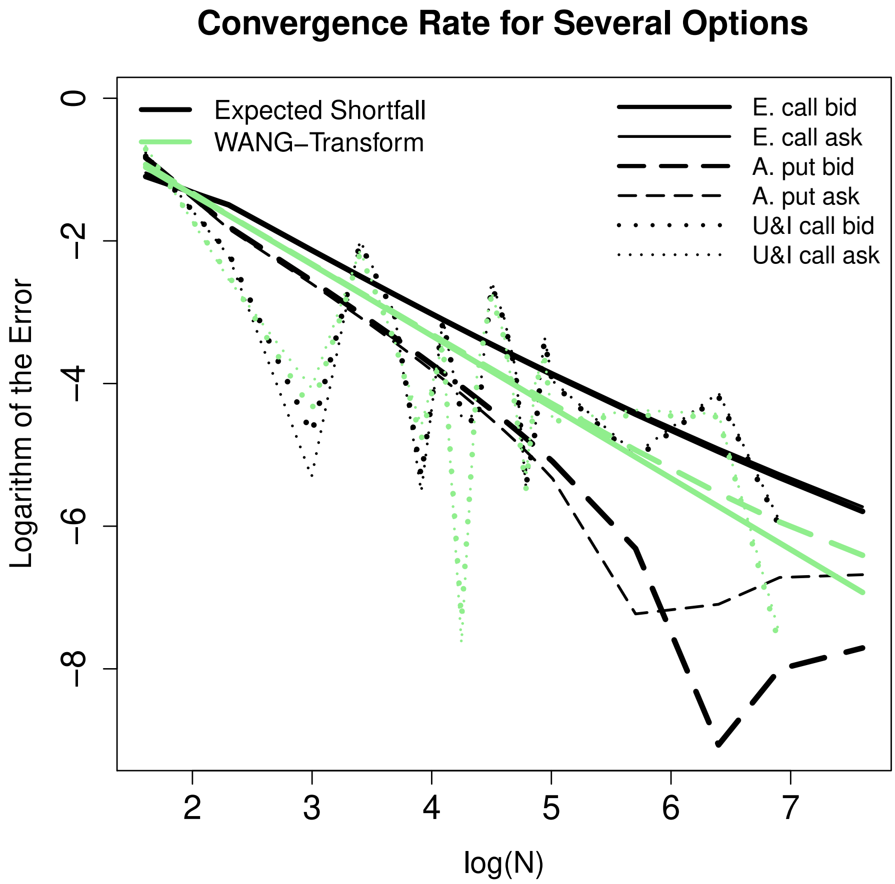

Figure 5.6: Log-log plot of the binomial tree approximation error for a European call option, an American put option and an up-and-in barrier option with barrier $B = 110$. All option have the strike $K = 100$ and the maturity is set to one year. The stock starts in $S_0 = 100$, annual interest rates are set to 0.01, the dividend yield is assumed to be 0.03, the annual volatility is 0.2 and the annual γ is set to 0.05. N goes in non-equidistant steps from 5 to 2000. The up-and-in barrier option is only simulated up to $N = 1000$.

5.4.7 Application to real Market Data

In Section 5.3.4 the concept of *implied liquidity* (IL) is defined. It is similar to the idea of implied volatility and returns two implicitly computed parameters γ_b and γ_a such that modelled bid and ask prices match real market prices. The benefits of quoting the IL instead of bid-ask spreads are comparable to the benefits of quoting implied volatilities instead of mid-prices: in principle the IL can be constant across strikes, maturities and underlyings and hence makes it possible to compare bid-ask spreads across all three dimensions.

For a time-series of two days, February, 2nd and February, 5th, 2018, we obtained end of day bid and ask prices of 80 plain vanilla, at-the-money American put and call options on the S&P500, or rather the SPDR S&P 500 ETF Trust, an exchange traded fund replicating the S&P500, with maturities ranging from about 3 to 8 month. The option prices were obtained from the Chicago Board Options Exchange and can be found in Table 7.1 in the appendix.

The CBOE Volatility Index (VIX), tracking short-term market volatility, jumped from 17.31 points on February, 2nd to 37.32 points on February, 5th, thus by 116%, which is the highest daily relative change recorded so far. The S&P500 lost about 4% between the two dates. It is well-known that liquidity dries up, when uncertainty in financial market rises. Therefore the chosen dates are well suited to analyse how different measures for the bid-ask spread behave, when liquidity changes.

For each American option, on both dates, we first compute an implied Black-Scholes volatility σ_{Mid} matching exactly the mid-price. Then we use the mid-price implied volatility and compute the IL, $(\gamma_b, \gamma_a) \in \mathbb{R}_+^2$, such that the modelled bid and ask prices match exactly the quoted market bid and ask prices. In particular for an American call option C, we solve numerically

$$\text{bid}_{\text{quoted market price}}(C) = \text{BS}^{\text{A}}_{\text{Call}}(S_0, T, K, r, q + \sigma_{\text{Mid}}\gamma_b, \sigma_{\text{Mid}}),$$

for γ_b. The function $\text{BS}^{\text{A}}_{\text{Call}}(S_0, T, K, r, q, \sigma)$ describes the risk-neutral price of an American call option in a Black-Scholes setting with strike K and maturity T on a stock with initial value S_0, volatility σ and paying a continuously dividend yield q. The risk-free interest rate is denoted by r. The parameter γ_a for the call option and the IL $(\tilde{\gamma}_b, \tilde{\gamma}_a)$ of put options can be found analogously, see Theorem 5.4.6. For most options γ_b and γ_a are almost identical.

The average relative bid-ask spread of the American option data set is 1.6% on February, 2nd and 6.6% on February, 5th. The relative bid-ask spread changed by the factor 4.1. The average IL $\frac{\gamma_a+\gamma_b}{2}$, changed from an average value of 0.011 for all options on February, 2nd to 0.043 on February, 5th, which corresponds to a change by the factor 3.9. The average difference of implied bid and ask volatilities on the other hand, rose by the factor 6.1, hence about 49% more than the relative bid-ask spread. In Figure 5.7, we show the multiplicative factor describing the change of the relative bid-ask spread, the IL and the difference of implied bid and ask volatilities from February, 2nd to February 5th separately for put and call options and different maturities.

The overall picture is the following: a change in liquidity of American options, due to a rise in uncertainty in the market and measured by the change of the relative bid-ask spread, is described by the IL very well. On the other hand, an overestimation of the change of liquidity by 50% and more are no exceptions, when describing the bid-ask spread by the classical way of quoting implied bid and ask volatilities. Our findings for American options are in line with a similar empirical study for European options done by Guillaume et al. (2018), see Section 5.3.5.

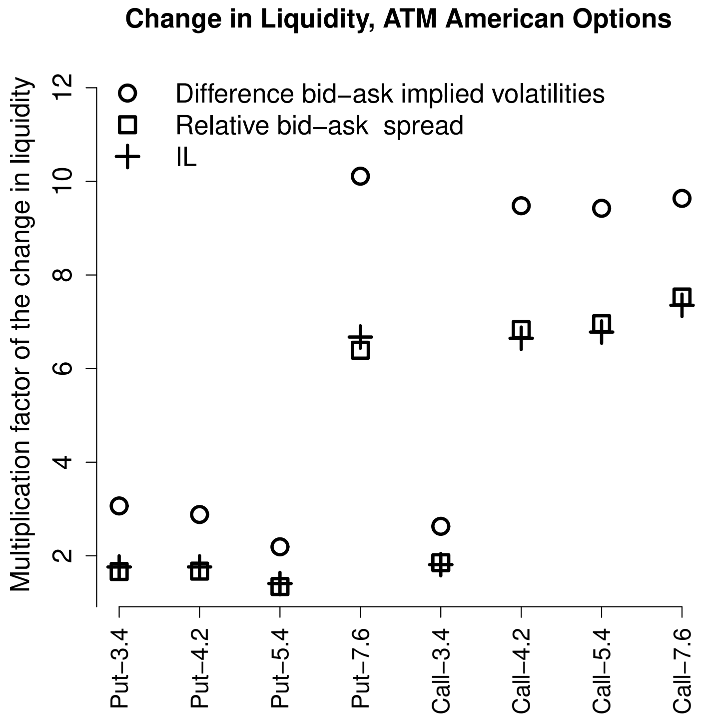

Figure 5.7: This Figure describes the change of the relative bid-ask spread, the IL and the difference of implied bid and ask volatilities from February, 2nd to February, 5th of various American at-the-money put and call options by a multiplicative factor. Maturities are measured in months.

5.5 Conclusion

We model a financial market with frictions in discrete time using a pricing functional, which is defined recursively via coherent risk measures. The risk measures are defined via a family of concave distortion functions (FCDF). Economically, the discrete time market model with frictions is justified in Jouini and Kallal (1995, 2001) and Jouini (2000).

The discrete time model contains two special cases: a) the static case where we model the log-returns by a normal or a Laplace distribution, and b) a binomial-type model where it is possible to prove convergence for many European and American path-dependent options when the number of trading periods tend to infinity. We call the limit of the binomial-type model *extended Black-Scholes model*.

In all three settings (normal, Laplace or extended Black-Scholes), we obtain closed-form solutions for bid and ask prices of European plain vanilla options. Bid and ask prices of European options can be calculated as the risk neutral price of the same option but on an underlying with an adjusted dividend yield. The bid-ask spread depends on an additional parameter $\gamma \geq 0$. The greater γ, the greater the bid-ask spread. For $\gamma = 0$, the spread is equal to zero and bid and ask prices coincide with the risk-neutral price of the option.

The static case is interesting because it allows a more flexible distribution for the log-returns. In particular, the Laplace distribution has fatter tails than the normal distribution, which appears in the asymptotic case of the binomial-type model, and might therefore be better suited to model log-returns. But the flexibility of the static model is reduced because the FCDF, modelling the pricing functional, cannot be chosen arbitrary, we assumed that the FCDF is induced by the distribution function of the log-returns. In Section 4.4.1 we mentioned some desirable properties that distortion functions should have, in particular the first derivative of the distortion function should approach infinity at zero and should be equal to zero at one. If log-returns are assumed to be normal distributed, we would use the Wang transform which has all desirable properties. However, those properties are not satisfied by the family of distortion functions induced by the Laplace distribution.

On the other hand, we are motivated to study convergence of bid and ask price in the binomial-type model for American or European options to find fast numerical methods to compute those prices. We have shown that bid and ask prices of monotone payoffs, for example European or American plain vanilla, Asian, lookback and some barrier options, can be computed as fast as the risk-neutral price of such an option in a classical Black-Scholes framework. In contrast to the static model, the particular choice of the FCDF to model bid and ask prices in discrete time, does not matter in the limit.

The three new models (normal, Laplace or extended Black-Scholes) may find a similar application in practise as the classical Black-Scholes model. Trader usually prefer to quote implied volatilities instead of prices, because there are many nonlinearities in prices making comparisons across strikes, maturities and underlying assets difficult and understand. In principle volatilities could be constant across all three dimensions, hence the preference for quoting implied volatilities.

With the same argument it might be more convenient to quote an implied liquidity instead of the bid-ask spread. It is then possible to compare bid-ask spreads across different strikes, maturities and underlyings. To demonstrate this idea we conducted two empirical studies: we computed implicitly two parameters γ_b and γ_a such that modelled bid and ask prices match real market prices. We did this for a set of European options using the static normal and Laplace-model and for a set of American options using the extended Black-Scholes model. In principle the tuple (γ_b, γ_a) could be constant across strikes, maturities and underlyings and hence makes it possible to compare bid-ask spreads across all three dimensions.

Up to now, traders usually describe bid-ask spreads by quoting both bid and ask implied volatilities. There are several advantages using the IL instead: it is not always possible to compute an implied bid volatility, because bid prices of options sometimes lay below the theoretical arbitrage-free price, i.e. are lower than the risk-neutral price of an option on a stock with volatility zero. The concept of IL overcomes this inconsistency. When uncertainty in financial market rises and liquidity dries up, looking only at the difference of Black-Scholes implied bid and ask volatilities often overestimates a change in liquidity by 40% and more, because the difference of the implied volatilities changes by a higher factor than the relative bid-ask spread. However, the correlation between the relative bid-ask spread and the IL is strong, which makes the IL an intuitive measure for liquidity.

There are almost no differences between the normal and Laplace static models with respect to the concept of IL. We therefore recommend to us the extended Black-Scholes model to compute the IL in practise, mainly because it is a continuous time model and the Black-Scholes model is well known and understood in industry.

6 Outlook and Future Research

In Chapter 3 we calibrate six advanced stock price model to a time series of European plain vanilla options, simulate barrier option prices via a Monte Carlo simulation and compare the simulated prices to real market data of barrier options. For our particular data set, the three Lévy models we looked at, do not reproduce barrier option data very well. It would therefore be interesting to look at other Lévy models. So far we model the time-change of Lévy processes by a CIR process. The time-change may as well be modelled by a linear combination of Ornstein–Uhlenbeck processes or by an integrated Inverse Gaussian process.

Future research need to be done to include different and longer time series and to analyse other exotic options than barrier options.

In Chapter 3 we investigate empirical which advanced stock price model returns the smallest *pricing error*. An interesting research question is which advanced stock price model has the smallest *hedging error* for exotic options.

In Chapter 4 we analyse concave distortion functions, which play an important role in Chapter 5 to define bid and ask prices in a market with frictions. A concave distortion function is a concave function mapping the unity interval onto itself and is used to distort distribution functions. We prove that a family of concave distortions (FCDF) satisfying a certain translation equation can be represented by a distribution function.

There are FCDF which only satisfy the translation equation after a certain reparametrization. In Proposition 4.5.9 we prove that if there exist a reparametrization of a FCDF satisfying the translation equation, then the original FCDF is permutable. From a mathematical point of view, it would be interesting to see whether the reverse also holds true, i.e. if a permutable FCDF can be reparameterized into a FCDF satisfying a translation equation and hence can be represented by a distribution function.

A natural question for a future work is: which properties of a family of coherent risk measures, induced by a FCDF, are implied by the translation equation? Beside the application in Section 4.5.2, what is the precise economic and actuarial meaning of the translation equation?

In Chapter 5 we construct a binomial-type with frictions, which may model a market with transaction costs. Bid and ask prices are recursively defined by a pricing functional, which is induced by a sequence of time-consistent coherent risk measures. We are able to prove that bid and ask prices converge for many European or American possible path-dependent options. The limit of the bid and ask prices of European plain vanilla options can be expressed by the Black-Scholes formula with an adjusted dividend yield. Both the volatility and the liquidity parameter are assumed to be constant over time. It would be interesting to model both parameters by mean-reverting stochastic processes. We leave these extensions for future research.

By Soner et al. (1995), the least expensive superreplication strategy dominating a European call in a Black-Scholes model with proportional transaction costs is the trivial strategy of buying one share and holding it till maturity. The ask price of a call option in our market model with frictions presented in Chapter 5 lays significantly below the price of this trivial superreplication strategy. However, our market model with frictions is not a simple proportional transaction costs model because at maturity the bid-ask spread of the underlying is zero. Nevertheless, the results by Soner et al. (1995) indicate that an investor who agrees to buy the option for the ask price is ready to take some residual risks. In a future research we would like quantify that risk using a coherent risk-measure ρ for instance. Results from Xu (2006) and Föllmer and Schied (2011, Section 8) might help to answer this question. But both authors considered a frictionless market, while our market model contains frictions.

7 Appendix

Curriculum Vitæ

2016-2019: PhD Student, Department of Mathematics, Universitat Autònoma de Barcelona, Spain.
Visiting Scholar at the Department of Mathematics, KU Leuven, Belgium with Wim Schoutens[1] from Jan. 2017 till June 2017 and from Oct. 2018 till Feb. 2019.

2013-2015: Manager at the HSBC, Risk Department, Dusseldorf, Germany.

2011-2013: Master in Mathematics (with honors), TU Braunschweig, Germany. Major field of study: stochastic.
Erasmus study at Queen Mary University of London, UK, from Sept. 2011 till Dec. 2011.

2008-2011: Bachelor in Mathematics, TU Braunschweig, Germany. Major field of study: stochastic.
Erasmus study at Universidad de la Rioja, Spain, from Sept. 2010 till Mar. 2011.

[1] Wim Schoutens a Professor in Financial Engineering at the KU Leuven. He has extensive practical experience of model implementation and validation and is well known for his consulting work to the banking industry. He is an independent expert advisor to the European Commission, has worked for the IMF and is the author of several books on quantitative finance.

Publications and Conferences

Some part of this thesis has been published or is about to be published in international journals or conferences. Due to copyright transfer statements, we explicitly state which part of this thesis has been published in which journal.

- Chapter 3 is a version of the pre-print
  - Junike, G. and Schoutens, W. (2018). *Performance of Advanced Stock Price Models when it becomes Exotic: an Empirical Study*, submitted.
- Chapter 4 is a version of the pre-print
  - Junike, G. (2018). *Representation of Distortion Risk Measures and Applications*, submitted.
  - The pre-print has been accepted for a poster presentation at the *Actuarial and Financial Mathematics Conference* taking place February, 7-8th 2019 in Brussels, Belgium.
- Section 5.3 is a post-peer-review, pre-copyedit version of an article published in *Annals of Finance*. The final authenticated version is available online at:
  - Guillaume, F., Junike, G., Leoni, P., and Schoutens, W. (2018). *Implied liquidity risk premia in option markets*. Annals of Finance, https://doi.org/10.1007/s10436-018-0339-y.
- Section 5.4 is a is a post-peer-review version of an article submitted to *The European Journal of Finance*
  - Junike, G., Arratia, A., Cabaña, A. and Schoutens, W. (2018). *American and Exotic Options in a Market with Frictions* submitted to The European Journal of Finance. The pre-print has been resubmitted to the journal.
  - Some of the result of the pre-print have also been presented in a talk at the conference *Mathematical and Statistical Methods for Actuarial Sciences and Finance* (MAF 2018) in Madrid, Spain on April, 4-6th, 2018.

Acknowledgement

First and foremost I would like to thank my supervisor Alejandra Cabaña for guiding and supporting me during the three years of my doctorate. In particular, I highly appreciated the freedom Alejandra has granted me in the direction of my research and I would like to thank Alejandra for carefully proofreading the entire thesis and for her very valuable contribution to improve this work. Her willingness to give her time so generously has been very much appreciated.

I would also like to express my very great appreciation to Wim Schoutens for his brilliant ideas and constant support. Wim has become a second supervisor during my two stays at KU Leuven.

My special thanks go to Argimiro Arratia for many insightful discussions and his valuable and constructive suggestions during the planning and development of this research work.

Collaboration provided by my other co-authors Florence Guillaume and Peter Leonie, was greatly appreciated.

Assistance and help with respect to my teaching duties by Rosa Camps, Mercè Farré, Angel Calsina and Ana Vazquez were thankfully accepted.

Finally, I wish to thank Carolina for her love, support and encouragement throughout my study.

American Option Data

Today	Maturity	Strike	Bid-call	Ask-call	Bid-put	Ask-put
02/02/2018	18/05/2018	273	10.1	10.25	6.99	7.14
02/02/2018	15/06/2018	273	11.23	11.39	8.15	8.27
02/02/2018	20/07/2018	273	12.32	12.51	9.29	9.48
02/02/2018	21/09/2018	273	14.85	15.08	11.32	11.51
02/02/2018	18/05/2018	274	9.42	9.57	7.32	7.46
02/02/2018	15/06/2018	274	10.55	10.7	8.48	8.61
02/02/2018	20/07/2018	274	11.66	11.84	9.64	9.83
02/02/2018	21/09/2018	274	14.21	14.41	11.66	11.86
02/02/2018	18/05/2018	275	8.77	8.9	7.67	7.82
02/02/2018	15/06/2018	275	9.9	10.03	8.84	8.97
02/02/2018	20/07/2018	275	11.01	11.2	9.99	10.18
02/02/2018	21/09/2018	275	13.56	13.76	12.02	12.2
02/02/2018	18/05/2018	276	8.13	8.26	8.03	8.2
02/02/2018	15/06/2018	276	9.25	9.37	9.2	9.33
02/02/2018	20/07/2018	276	10.38	10.55	10.35	10.55
02/02/2018	21/09/2018	276	12.93	13.11	12.38	12.57
02/02/2018	18/05/2018	277	7.51	7.64	8.42	8.58
02/02/2018	15/06/2018	277	8.62	8.74	9.59	9.72
02/02/2018	20/07/2018	277	9.75	9.92	10.74	10.94
02/02/2018	21/09/2018	277	12.29	12.48	12.75	12.95
05/02/2018	18/05/2018	262	13.07	13.44	12.69	13.15
05/02/2018	15/06/2018	262	13.83	15.2	13.76	14.11
05/02/2018	20/07/2018	262	14.79	16.48	14.8	15.2
05/02/2018	21/09/2018	262	17.07	18.98	16.62	18.4
05/02/2018	18/05/2018	263	12.42	12.77	13.04	13.49
05/02/2018	15/06/2018	263	13.16	14.52	14.1	14.46
05/02/2018	20/07/2018	263	14.15	15.81	15.14	15.55
05/02/2018	21/09/2018	263	16.44	18.32	16.98	18.79
05/02/2018	18/05/2018	264	11.76	12.11	13.4	13.84
05/02/2018	15/06/2018	264	12.54	13.84	14.46	14.81
05/02/2018	20/07/2018	264	13.52	15.15	15.51	15.91
05/02/2018	21/09/2018	264	15.81	17.65	17.35	19.2
05/02/2018	18/05/2018	265	11.12	11.47	13.76	14.2
05/02/2018	15/06/2018	265	11.92	13.19	14.83	15.18
05/02/2018	20/07/2018	265	12.91	14.5	15.88	16.29
05/02/2018	21/09/2018	265	15.2	17.01	17.73	19.63
05/02/2018	18/05/2018	266	10.52	10.83	14.13	14.59
05/02/2018	15/06/2018	266	11.3	12.15	15.2	15.55
05/02/2018	20/07/2018	266	12.32	13.85	16.26	16.67
05/02/2018	21/09/2018	266	14.6	16.37	18.11	20.04

Table 7.1: End-of day prices.

Pricing through the Characteristic Function

In Example 2.3.1, we expressed the price of a European plain vanilla call option in simple terms. This is possible, because the density of the log-returns in the Black-Scholes model are known at all times. For most advanced stock price models, see Chapter 3, the density of the log-returns are unknown but the characteristic function of the log-returns are usually given in analytic form. The Fourier transform can then by used to compute the call price of an option with strike K and maturity T. Let $k := \log(K)$, by Carr and Madan (1999), the call price is given by

$$C(k,T) \quad = \quad e^{-\alpha k}\frac{1}{\pi}\Re\left(\int_0^{\infty} e^{-ivk}\rho_T(v)dv\right), \tag{7.1}$$

where $\alpha > 0$ and φ_T is the characteristic function of the logarithm of the risky asset at time $T > 0$, i.e.

$$\varphi_T(u) = E_{\mathbb{Q}}\left[\exp\left(iu\log\left(S_T\right)\right)\right]$$

and $\rho_T(.)$ is defined by

$$\rho_T(v) = \frac{e^{-rT}\varphi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}.$$

Carr and Madan introduced the scaling term α to make the call pricing function square integrable, which is necessary to apply the Fourier transform. The integral appearing in Equation (7.1) can be computed using the fast Fourier transform.

The fast Fourier transform (FFT) is used to compute the discrete Fourier transform very fast. Let $x_1,..,x_N$ be a sequence of complex numbers. The discrete Fourier transform of the sequence $x_1,..,x_N$ is defined by the sequence

$$\hat{x}_n = \sum_{j=1}^{N}\exp\left(-\frac{2\pi i(j-1)(n-1)}{N}\right)x_j, \;\; n = 1,...,N.$$

Evaluating the discrete Fourier transform directly requires $O(N^2)$ operations. The FFT calculates the discrete Fourier transform using only $O(N\log(N))$ operations. N should be of power of two, because most FFT algorithms split the problem at each recursion into two parts, see e.g Van Loan (1992) for numerical aspects of the FFT.

Simpson's rule is applied to approximate the integral appearing in Equation (7.1), Let

$$v_j = \eta(j-1), \;\; j = 1,..,N.$$

It holds

$$C(k,T) \approx e^{-\alpha k}\frac{1}{\pi}\Re\left(\sum_{j=1}^{N}e^{-iv_jk}\rho_T(v_j)\eta\left(\frac{3+(-1)^j-\delta_{j-1}}{3}\right)\right),$$

where $\eta > 0$ is the step-size and $N \in \mathbb{N}$ being power of two and

$$\delta_j = \begin{cases} 1 & ,j = 0 \\ 0 & ,j > 0. \end{cases}$$

The integral is approximated by zero in the interval $(\eta N, \infty)$. Let $\lambda := \frac{2\pi}{N\eta}$ and define

$$k_n = -\frac{1}{2}N\lambda + \lambda(n-1), \;\; n = 1, ..., N.$$

We will see that λ is exactly chosen such that the FFT can be applied. We will obtain simultaneously prices for the strikes

$$\exp(k_n), \;\; n = 1, .., N.$$

The positive constant η has to be chosen small enough such that $k \in [k_1, k_N]$. Let

$$x_j := e^{iv_j \frac{\pi}{\eta}} \rho_T(v_j)\eta \left(\frac{3 + (-1)^j - \delta_{j-1}}{3} \right), \;\; j = 1, ..., N.$$

As it holds

$$-iv_j k_n = -i\eta(j-1)(-\frac{\pi}{\eta} + \frac{2\pi}{N\eta}(n-1)) = iv_j \frac{\pi}{\eta} - \frac{2\pi i}{N}(j-1)(n-1),$$

it follows

$$C(k_n, T) \approx e^{-\alpha k_n} \frac{1}{\pi} \Re \left(\sum_{j=1}^{N} e^{-\frac{2\pi i}{N}(j-1)(n-1)} x_j \right), ...n = 1, ..., N.$$

the sum corresponds to a discrete Fourier transform of the sequence $x_1, ..., x_N$ and can be evaluated very efficiently using FFT. Carr and Madan suggest to use

$$\begin{aligned} \alpha &= 1.5 \\ N &= 4096 \\ \eta &= 0.25. \end{aligned}$$

We obtain simultaneously prices for the strikes $\exp(k_n)$, $n = 1, .., N$. Because the interval $[k_1, k_N]$ is large, in most practical cases, $\log(K)$ is inside. By (linear) interpolation, we get the price for strike K.

Bibliography

[1] Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance*, 26(7), 1505-1518.

[2] Acharya, V. V., and Pedersen, L. H. (2005). Asset pricing with liquidity risk. *Journal of financial Economics*, 77(2), 375-410.

[3] Aczél, J. (1966). *Lectures on Functional Equations and Their Applications*, New York–London: Academic Press.

[4] Albrecher, H., Guillaume, F., and Schoutens, W. (2013). Implied liquidity: Model sensitivity. *Journal of Empirical Finance*, 23, 48-67.

[5] Amihud, Y. (2002). Illiquidity and stock returns: cross-section and time-series effects. *Journal of financial markets*, 5(1), 31-56.

[6] Amin, K., Khanna, A. (1994). Convergence of American option values from discrete to continuous-time financial models. *Mathematical Finance* 4.4, 289-304

[7] Andersen, L. B., Jäckel, P., and Kahl, C. (2010). Simulation of square-root processes. *Encyclopedia of Quantitative Finance*, 1642-1649.

[8] Applebaum, D. (2009). Lévy processes and stochastic calculus. *Cambridge university press.*

[9] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, 9(3), 203-228.

[10] Bagnoli, M., and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic theory*, 26(2), 445-469.

[11] Bakshi, G., Cao, C., and Chen, Z. (1997). Empirical performance of alternative option pricing models. *The Journal of finance*, 52(5), 2003-2049.

[12] Balbás, A., Garrido, J., and Mayoral, S. (2009). Properties of distortion risk measures. *Methodology and Computing in Applied Probability*, 11(3), 385.

[13] Bannör, K. F., and Scherer, M. (2014). On the calibration of distortion risk measures to bid-ask prices. *Quantitative Finance*, 14(7), 1217-1228.

[14] Barndorff-Nielsen, O. E. (1995). Normal inverse Gaussian distributions and the modeling of stock returns *Research report* (Vol. 300).

[15] Barndorff-Nielsen, O. E. (1997a). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of statistics*, 24(1), 1-13.

[16] Barndorff-Nielsen, O. E. (1997b). Processes of normal inverse Gaussian type. *Finance and stochastics*, 2(1), 41-68.

[17] Barndorff-Nielsen, O. E., and Shephard, N. (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society*: Series B, 63(2), 167-241.

[18] Barndorff-Nielsen, O. E., and Shephard, N. (2002). Estimating quadratic variation using realized variance. *Journal of Applied econometrics*, 17(5), 457-477.

[19] Barone-Adesi, G., and Whaley, R. E. (1987). Efficient analytic approximation of American option values. *The Journal of Finance*, 42(2), 301-320.

[20] Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies*, 9(1), 69-107.

[21] Bernardo, A. E., and Ledoit, O. (2000). Gain, loss, and asset pricing. *Journal of political economy*, 108(1), 144-172.

[22] Bielecki, T. R., Cialenco, I., and Chen, T. (2015). Dynamic conic finance via backward stochastic difference equations. *SIAM Journal on Financial Mathematics*, 6(1), 1068-1122.

[23] Bielecki, T. R., Cialenco, I., Iyigunler, I., and Rodriguez, R. (2013). Dynamic conic finance: Pricing and hedging in market models with transaction costs via dynamic coherent acceptability indices. *International Journal of Theoretical and Applied Finance*, 16(01), 1350002.

[24] Bingham, N. H., and Kiesel, R. (2013). *Risk-neutral valuation: Pricing and hedging of financial derivatives.* Springer Science & Business Media.

[25] Bion-Nadal, J. (2009). Bid-ask dynamic pricing in financial markets with transaction costs and liquidity risk. *Journal of Mathematical Economics*, 45(11), 738-750.

[26] Bjerksund, P., and Stensland, G. (1993). Closed-form approximation of American options. *Scandinavian Journal of Management*, 9, S87-S99.

[27] Björk, T. (2009). *Arbitrage theory in continuous time.* Oxford university press.

[28] Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of political economy*, 81(3), 637-654.

[29] Burth, S., Kraus, T., and Wohlwend, H. (2001). The pricing of structured products in the Swiss market. *The Journal of Derivatives*, 9(2), 30-40.

[30] Carbone, R. (2004). Binomial approximation of Brownian motion and its maximum. *Statistics & probability letters*, 69(3), 271-285.

[31] Carr, P., and Madan, D. M. (1999). Option valuation using the fast Fourier transform. *Journal of computational finance*, 2(4), 61-73.

[32] Carr, P., and Wu, L. (2004). Time-changed Lévy processes and option pricing. *Journal of Financial economics*, 71(1), 113-141.

[33] Carr, P., and Wu, L. (2009). Stock options and credit default swaps: A joint framework for valuation and estimation. *Journal of Financial Econometrics*, 8(4), 409-449.

[34] Carr, P., Geman, H., Madan, D. B., and Yor, M. (2003). Stochastic volatility for Lévy processes. *Mathematical finance*, 13(3), 345-382.

[35] CBOE (2018). White Paper Cboe Volatility Index, http://www.cboe.com/micro/vix/vixwhite.pdf, accessed on September 1st, 2018.

[36] Cetin, U., Jarrow, R. A., and Protter, P. (2004). Liquidity risk and arbitrage pricing theory. *Finance and stochastics*, 8(3), 311-341.

[37] Chen, A. H., and Kensinger, J. W. (1990). An analysis of market-index certificates of deposit. *Journal of Financial Services Research*, 4(2), 93-110.

[38] Chen, K. C., and Sears, R. S. (1990). Pricing the SPIN. *Financial Management*, 36-47.

[39] Cheng, K. (2003). An overview of barrier options. *Global Derivatives*, working paper.

[40] Cherny, A., and Madan, D. (2008). New measures for performance evaluation. *The Review of Financial Studies*, 22(7), 2571-2606.

[41] Clark, P. K. (1973). A subordinated stochastic process model with finite variance for speculative prices. *Econometrica: journal of the Econometric Society*, 135-155.

[42] Cohen, S. N., and Elliott, R. J. (2010). A general theory of finite state backward stochastic difference equations. *Stochastic Processes and their Applications*, 120(4), 442-466.

[43] Cont, R (2001). Empirical properties of asset returns: stylized facts and statistical issues, *Quantitative Finance* 1, 223-236.

[44] Corcuera, J. M., Guillaume, F., Leoni, P., and Schoutens, W. (2009). Implied Lévy volatility. *Quantitative Finance*, 9(4), 383-393.

[45] Corcuera, J. M., Guillaume, F., Madan, D. B., and Schoutens, W. (2012). Implied liquidity: towards stochastic liquidity modelling and liquidity trading. *International Journal of Portfolio Analysis and Management* 1, no. 1: 80-91.

[46] Cox, J. (1975). Notes on option pricing I: Constant elasticity of variance diffusions. *Unpublished note*, Stanford University, Graduate School of Business.

[47] Cox, J.C., J.E. Ingersoll and S.A. Ross (1985). A Theory of the Term Structure of Interest Rates. *Econometrica.* 53: 385–407.

[48] Cox, J. C., Ross, S. A., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of financial Economics*, 7(3), 229-263.

[49] Cui, Y., del Baño Rollin, S., and Germano, G. (2017). Full and fast calibration of the Heston stochastic volatility model. *European Journal of Operational Research*, 263(2), 625-638.

[50] Daykin, C. D., Pentikainen, T., and Pesonen, M. (1993). *Practical risk theory for actuaries.* Chapman and Hall, London.

[51] Delbaen, F. (2002). Coherent risk measures on general probability spaces. *In Advances in finance and stochastics* (pp. 1-37). Springer, Berlin, Heidelberg.

[52] Delbaen, F., and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Mathematische annalen*, 300(1), 463-520.

[53] Dhaene, J., Dony, J., Forys, M. B., Linders, D., and Schoutens, W. (2012). FIX: The Fear Index−Measuring Market Fear. *In Topics in Numerical Methods for Finance* (pp. 37-55). Springer, Boston, MA.

[54] Dupire, B. (1994). Pricing with a smile. *Risk*, 7(1), 18-20.

[55] Föllmer, H., and Penner, I. (2006). Convex risk measures and the dynamics of their penalty functions. *Statistics & Decisions*, 24(1/2006), 61-96.

[56] Föllmer, H., and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and stochastics*, 6(4), 429-447.

[57] Föllmer, H. and Schied, A. (2004). *Stochastic finance: an introduction in discrete time.* 2nd Edition, Walter de Gruyter.

[58] Föllmer, H., and Schied, A. (2011). *Stochastic finance: an introduction in discrete time.* 3rd Edition, Walter de Gruyter.

[59] Gilli, M., and Schumann, E. (2012). Heuristic optimisation in financial modelling. *Annals of operations research*, 193(1), 129-158.

[60] Glasserman, P. (2013). *Monte Carlo methods in financial engineering* (Vol. 53). Springer Science & Business Media.

[61] Goyenko, R. Y., Holden, C. W., and Trzcinka, C. A. (2009). Do liquidity measures measure liquidity?. *Journal of financial Economics*, 92(2), 153-181.

[62] Guillaume, F. (2015). The LIX: A model-independent liquidity index. *Journal of Banking & Finance*, 58, 214-231.

[63] Guillaume, F., and Schoutens, W. (2012). Calibration risk: Illustrating the impact of calibration risk under the Heston model. *Review of Derivatives Research*, 15(1), 57-79.

[64] Guillaume, F., and Schoutens, W. (2015). Bid-Ask Spread for Exotic Options under Conic Finance. *In Innovations in Quantitative Risk Management* (pp. 59-74). Springer, Cham.

[65] Guillaume, F., Junike, G., Leoni, P., and Schoutens, W. (2018). Implied liquidity risk premia in option markets. *Annals of Finance*, https://doi.org/10.1007/s10436-018-0339-y.

[66] Henderson, B. J., and Pearson, N. D. (2011). The dark side of financial innovation: A case study of the pricing of a retail financial product. *Journal of Financial Economics*, 100(2), 227-247.

[67] Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2), 327-343.

[68] Heston, S., and Zhou, G. (2000). On the rate of convergence of discrete-time contingent claims. *Mathematical Finance*, 10(1), 53-75.

[69] Hirsa, A., Courtadon, G., and Madan, D. B. (2003). The effect of model risk on the valuation of barrier options. *The Journal of Risk Finance*, 4(2), 47-55.

[70] Hull, J. (2017). Options, futures, and other derivatives. 10th edition. Pearson, University of Toronto.

[71] Hull, J., and White, A. (1995). The impact of default risk on the prices of options and other derivative securities. *Journal of Banking & Finance*, 19(2), 299-322.

[72] Jessen, C., and Poulsen, R. (2013). Empirical performance of models for barrier option valuation. *Quantitative Finance*, 13(1), 1-11.

[73] Jiang, L., and Dai, M. (2004). Convergence of binomial tree methods for European/American path-dependent options. *SIAM Journal on Numerical Analysis*, 42(3), 1094-1109.

[74] Jorion, P. (2000). Risk management lessons from long-term capital management. *European financial management*, 6(3), 277-300.

[75] Jouini, E. (2000). Price functionals with bid–ask spreads: an axiomatic approach. *Journal of Mathematical Economics*, 34(4), 547-558.

[76] Jouini, E., and Kallal, H. (1995). Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66(1), 178-197.

[77] Jouini, E., and Kallal, H. (2001). Efficient trading strategies in the presence of market frictions. *The Review of Financial Studies*, 14(2), 343-369.

[78] Karatzas, I., Shreve, S. E., Karatzas, I., and Shreve, S. E. (1998). *Methods of mathematical finance*. Springer, New York.

[79] Kijima, M. (2006). A multivariate extension of equilibrium pricing transforms: The multivariate Esscher and Wang transforms for pricing financial and insurance risks. *ASTIN Bulletin: The Journal of the IAA*, 36(1), 269-283.

[80] Kijima, M., and Muromachi, Y. (2006). On the Wang transform with fat-tail distributions. *In Proceedings of Stanford– Tsukuba Workshop.*

[81] Kijima, M., and Muromachi, Y. (2008). An extension of the Wang transform derived from Bühlmann's economic premium principle for insurance risk. Insurance: *Mathematics and Economics*, 42(3), 887-896.

[82] Koehl, P. F., and Pham, H. (2000). Sublinear price functionals under portfolio constraints. *Journal of Mathematical Economics*, 33(3), 339-351.

[83] Kusuoka, S. (2001). On law invariant coherent risk measures. *In Advances in mathematical economics* (pp. 83-95). Springer, Tokyo.

[84] Lamberton, D. (1998). Error estimates for the binomial approximation of American put options. *The Annals of Applied Probability*, 8(1), 206-233.

[85] Leippold, M., and Schärer, S. (2017). Discrete-time option pricing with stochastic liquidity. *Journal of Banking & Finance*, 75, 1-16.

[86] Leisen, D. P. (1998). Pricing the American put option: A detailed convergence analysis for binomial models. *Journal of Economic Dynamics and Control*, 22(8-9), 1419-1444.

[87] Leisen, D. P., and Reimer, M. (1996). Binomial models for option valuation-examining and improving convergence. *Applied Mathematical Finance*, 3(4), 319-346.

[88] Lin, J., and Palmer, K. (2013). Convergence of barrier option prices in the binomial model. *Mathematical Finance* 23(2), 318-338.

[89] Liu, W. (2006). A liquidity-augmented capital asset pricing model. *Journal of financial Economics*, 82(3), 631-671.

[90] Madan, D. B. (2010). Conserving capital by adjusting deltas for gamma in the presence of skewness. *Journal of Risk and Financial Management*, 3(1), 1-25.

[91] Madan, D. B. (2012a). Execution costs and efficient execution frontiers. *Annals of Financial Economics*, 7(01), 1250002.

[92] Madan, D. B. (2012b). On pricing contingent capital notes. *In Recent Advances In Financial Engineering* 2011 (pp. 21-42).

[93] Madan, D. B. (2014). Modeling and monitoring risk acceptability in markets: The case of the credit default swap market. *Journal of Banking & Finance*, 47, 63-73.

[94] Madan, D. B. (2016a). Adapted hedging. *Annals of Finance*, 12(3-4), 305-334.

[95] Madan, D. B. (2016b). Conic portfolio theory. *International Journal of Theoretical and Applied Finance*, 19(03), 1650019.

[96] Madan, D. B. (2018). Nonlinear equity valuation using conic finance and its regulatory implications. *Mathematics and Financial Economics*, 1-35.

[97] Madan, D. B., Carr, P. P., and Chang, E. C. (1998). The variance gamma process and option pricing. *Review of Finance*, 2(1), 79-105.

[98] Madan, D. B., and Cherny, A. (2010). Markets as a counterparty: an introduction to conic finance. International *Journal of Theoretical and Applied Finance*, 13(08), 1149-1177.

[99] Madan, D. B., and Milne, F. (1991). Option Pricing With VG Martingale Components 1. *Mathematical finance*, 1(4), 39-55.

[100] Madan, D. B., Pistorius, M., and Schoutens, W. (2013). The valuation of structured products using Markov chain models. *Quantitative Finance*, 13(1), 125-136.

[101] Madan, D. B., Pistorius, M., and Schoutens, W. (2016). Dynamic conic hedging for competitiveness. *Mathematics and Financial Economics*, 10(4), 405-439.

[102] Madan, D. B., Pistorius, M., and Schoutens, W. (2017a). Conic trading in a markovian steady state. *International Journal of Theoretical and Applied Finance*, 20(02), 1750010.

[103] Madan, D., Pistorius, M., and Stadje, M. (2017b). On dynamic spectral risk measures, a limit theorem and optimal portfolio allocation. *Finance and Stochastics*, 21(4), 1073-1102.

[104] Madan, D. B., and Schoutens, W. (2011). Conic coconuts: the pricing of contingent capital notes using conic finance. *Mathematics and Financial Economics*, 4(2), 87-106.

[105] Madan, D. B., and Schoutens, W. (2012a). Structured products equilibria in conic two price markets. *Mathematics and Financial Economics*, 6(1), 37-57.

[106] Madan, D. B., and Schoutens, W. (2012b). Tenor specific pricing. *International Journal of Theoretical and Applied Finance*, 15(06), 1250043.

[107] Madan, D. B, and Schoutens, W. (2016a). *Applied Conic Finance.* Cambridge University Press.

[108] Madan, D. B., and Schoutens, W. (2016b). Conic CVA and DVA. *Wilmott*, 2016(82), 45-50.

[109] Madan, D. B., and Seneta, E. (1987). Chebyshev polynomial approximations and characteristic function estimation. *Journal of the Royal Statistical Society*: Series B, 163-169.

[110] Madan, D. B., and Seneta, E. (1990). The variance gamma (VG) model for share market returns. *Journal of Business*, 511-524.

[111] Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of economics and management science*, 141-183.

[112] Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2), 125-144.

[113] Musiela, M. and Rutkowski, M. (1997), *Martingale Methods in Financial Modelling*, Springer, Berlin.

[114] Nelder, J. A., and Mead, R. (1965). A simplex method for function minimization. *The computer journal*, 7(4), 308-313.

[115] Nicolato, E., and Venardos, E. (2003). Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Mathematical Finance*: 13(4), 445-466.

[116] Peng, S. (2004). Nonlinear expectations, nonlinear evaluations and risk measures. *In Stochastic methods in finance* (pp. 165-253). Springer, Berlin, Heidelberg.

[117] Pichler, A. (2013). The natural Banach space for version independent risk measures. *Insurance: Mathematics and Economics*, 53(2), 405-415.

[118] Protter, P. (2001). A partial introduction to financial asset pricing theory. *Stochastic processes and their applications*, 91(2), 169-203.

[119] Protter, P. E. (2004). *Stochastic differential equations.* In Stochastic integration and differential equations. Springer, Berlin, Heidelberg.

[120] Rockafellar, R. T. (1970): *Convex analysis*, Princeton: University Press.

[121] Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. L. (2009). *Stochastic processes for insurance and finance* (Vol. 505). John Wiley & Sons.

[122] Rubinstein, M. and Reiner, E. (1991). Breaking down the barriers, *Risk Magazine*, 48, 28-35.

[123] Sato, K. (1999). *Lévy processes and infinitely divisible distributions*. Cambridge university press.

[124] Schoutens, W. (2003). *Lévy Processes in Finance: Pricing Financial Derivatives*. Wiley.

[125] Schoutens, W., Simons, E., and Tistaert, J. (2003). A perfect calibration! Now what?. *The best of Wilmott*, 281.

[126] Somanathan, T. V., and Nageswaran, V. A. (2015). *The Economics of Derivatives*. Cambridge University Press.

[127] Soner, H. M., Shreve, S. E., and Cvitanic, J. (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability*, 5(2), 327-355.

[128] Staum, J. (2004). Fundamental theorems of asset pricing for good deal bounds. *Mathematical Finance*: 14(2), 141-161.

[129] Storn, R., and Price, K. (1997). Differential evolution−a simple and efficient heuristic for global optimization over continuous spaces. *Journal of global optimization*, 11(4), 341-359.

[130] Straub, E. (1988). *Non-life insurance mathematics*. Springer, Berlin.

[131] Tsanakas, A. (2004). Dynamic capital allocation with distortion risk measures. *Insurance: Mathematics and Economics*, 35(2), 223-243.

[132] Tsukahara, H. (2009). One-parameter families of distortion risk measures, *Mathematical Finance*, 19(4), 691-705.

[133] Van Loan, C. (1992). Computational frameworks for the fast Fourier transform (Vol. 10). Siam.

[134] Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance Mathematics & Economics*, 17(1), 43-54.

[135] Wang, S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin: The Journal of the IAA*, 26(1), 71-92.

[136] Wang, S. S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of risk and insurance*, 15-36.

[137] Wang, S. S. (2002). A universal framework for pricing financial and insurance risks. *ASTIN Bulletin: The Journal of the IAA*, 32(2), 213-234.

[138] Wasserfallen, W., and Schenk, C. (1996). Portfolio Insurance for the Small Investor in Switzerland. *Journal of Derivatives*, 3, Spring, 37-

[139] Xu, M. (2006). Risk measure pricing and hedging in incomplete markets. *Annals of Finance*, 2(1), 51-71.