Universitat Autònoma de Barcelona

Combinatorics of plethysm via Segal groupoids and operads

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Abstract

In the present thesis we study the combinatorics of plethysm from the perspective of incidence bialgebras and objective combinatorics. The objective algebra is carried out at the level of Segal groupoids, by using homotopy slices and homotopy pullbacks of groupoids and simplicial methods.

The first main contribution is to exhibit plethystic substitution as a convolution tensor product obtained from an explicit simplicial groupoid, $\mathsf{T}\mathbf{S}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd}$, by the standard general constructions of incidence coalgebras and homotopy cardinality, in analogy with how ordinary substitution is obtained from the fat nerve $\mathsf{N}\mathbf{S}$ of the category of finite sets and surjections $\mathbf{S}$. The simplicial groupoid $\mathsf{T}\mathbf{S}$ arises from $\mathbf{S}$ as its T-*construction*, a new categorical construction which is reminiscent of Quillen's Q and Waldhausen's S-constructions. Furthermore, it is closely related to the Nava–Rota theory of partitionals, and in fact it gives a new interpretation of transversals, a key concept in this theory.

The simplicial groupoid $\mathsf{N}\mathbf{S}$ is equivalent to the two-sided bar construction of the operad Sym. We observe that $\mathsf{T}\mathbf{S}$ too is equivalent to the two-sided bar construction of a certain operad, and that the way to obtain this operad from Sym can be generalized to any (nice enough) operad.

This leads to the second main contribution: a functorial construction on generalized operads, called the $\mathcal{T}$-construction, which establishes a passage from ordinary substitutions to plethystic substitutions. This construction is staged in the setting of P-operads for P a strong cartesian monad on a cartesian category. This generality allows for treating simultaneously a variety of notions of plethysm (and in fact leads to new notions of plethysm), such as ordinary plethysm, plethysm in several variables, plethysm of series with coefficients in a noncommutative ring, plethysm of series with non-commuting variables and Y-plethysm for Y a monoid. In the last case the construction agrees with the Giraudo T-construction, which produces an operad from a monoid. For all these notions of plethysm, a combinatorial model is exhibited in the form of a monoidal Segal groupoid.

Keywords: plethysm, simplicial groupoids, homotopy combinatorics, incidence bialgebras, generalized operads, bar construction.

MSC classes: 05A18, 05A19, 18B40, 18N50, 18M60, 18M65, 18M80, 16T10, 13F25, 18C15, 18D25.

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Introduction

The art of counting is a fundamental aspect of mathematics, and is the primary concern of enumerative combinatorics [1, 66, 67].

The usage of power series and their operations has long been an essential tool to solve enumeration problems. It goes back to the 18th century, with the study of Euler on partitions of integers by means of generating functions [27]. This was one starting point for algebraic combinatorics, where algebraic structures are exploited to establish combinatorial identities, and vice-versa. It has long been appreciated, though, that bijective proofs give a deeper understanding than algebraic manipulation. This is the aim of the so-called objective method, pioneered by Lawvere [45], Joyal [43], and Baez and Dolan [2].

With Joyal's theory of species [43], the intuition behind manipulation of generating functions found a proper combinatorial foundation. A species is a functor from the category of finite sets and bijections to itself (see Section 1.1). Operations between species can be explicitly defined, and they correspond in a natural way to operations on their generating functions. Paramount among these is the substitution of species, which corresponds to the functional substitution of generating functions, hence providing an objective counterpart to substitution of one variable power series.

Plethysm is a substitution of power series in infinitely many variables (see (2) below) that was introduced by Pólya [62] in the context of unlabelled enumeration theory, as a tool to count equivalence classes of a set under a symmetry action.

Independently, it was introduced by Littlewood [48], who actually coined the term, in the framework of symmetric functions and representation theory of the general linear groups [52] (see Section 1.5). This notion has later been used in algebraic topology, in connection with λ-rings [10] and power operations in cohomology [4].

The first combinatorial model for plethysm was given by Nava and Rota [61]. They developed the notion of partitional (Section 1.4), a functor from the groupoid of partitions to the category of finite sets, and showed that a suitable notion of composition of partitionals yields plethystic substitution of their generating functions, in analogy with the theory of species. A variation of this combinatorial interpretation was given shortly after by Bergeron [7], who instead of partitionals considered permutationals, functors from the groupoid of permutations to the category of finite sets. This approach is nicely related to the theory of species through an adjunction. Later on, Nava [60] studied both partitionals and permutationals from the point of view of incidence algebras, and added a third class of functors called linear partitionals.

Incidence algebras and coalgebras are another essential tool in algebraic combinatorics [39]. Coalgebras arise from the ability to decompose structures into smaller ones. Rota [64] showed that many of these coalgebras admit an interpretation in terms of *incidence coalgebras of posets*: from any locally finite poset, form the free vector space on its intervals, and endow this with a coalgebra structure by defining the comultiplication as

$$\Delta([x,y]) = \sum_{x\leqslant m\leqslant y} [x,m]\otimes[m,y].$$

The same construction works for locally finite monoids [14]. Observe that a poset can be viewed as a category where there is at most one arrow between any two objects, and a monoid is a category with only one object. The notion of incidence coalgebra was generalized to the context of categories by Leroux [20, 47], and goes as follows: a category is *locally finite* if every arrow admits only a finite number of 2-step factorizations. The *incidence coalgebra* of a locally finite category is the free vector space spanned by its arrows, with comultiplication

$$\Delta(f) = \sum_{b\circ a=f} a\otimes b$$

and counit $\epsilon(\mathrm{id}_x) = 1$ and $\epsilon(f) = 0$ else. The coassociativity of Δ comes from the associativity of composition of arrows.

In the classical theory of posets, often it is not the raw incidence coalgebra that is most interesting, but rather a *reduced* incidence coalgebra, where two intervals are identified if they are equivalent in some specific sense (e.g. isomorphic as abstract posets).

In the 21st century, an objective approach to Leroux theory was taken up by Lawvere and Menni [45], by using linear algebra over sets. Let **Set** be the category of sets. The objective counterpart of the vector space spanned by a set S is the slice category $\mathbf{Set}_{/S}$ (cf. [31]). An object in this category is a morphism $A \xrightarrow{f} S$ of sets, and it corresponds to the vector whose s-entry (for $s \in S$) is $|f^{-1}(s)|$. Linear functors $\mathbf{Set}_{/S} \to \mathbf{Set}_{/R}$ (those that preserve sums) are given by spans $S \leftarrow M \to R$, and obtained by taking pullback and postcomposition, as in Equation (4) below. A coalgebra in $\mathbf{Set}_{/S}$ is thus given by a comultiplication span $S \leftarrow M \to S\times S$ and a counit span $S \leftarrow N \to 1$.

However, combinatorial structures have symmetries, and to deal with them it is useful to upgrade this objective method to groupoids and homotopy linear algebra over groupoids [3, 30, 31]. On the other hand, the importance of factorizations of arrows in incidence coalgebras suggests a simplicial viewpoint, via the nerve construction [23]. These two facts lead to the recent generalization of the theory of Leroux to ∞-categories by Gálvez, Kock and Tonks [32, 33, 35]. They introduced the notion of decomposition space, a general homotopical framework for incidence coalgebras and Möbius inversion. These are the same as 2-Segal spaces, introduced by Dyckerhoff and Kapranov [24] in the context of homological algebra and representation theory [29].

Decomposition spaces are simplicial groupoids that satisfy an exactness condition weaker than the Segal condition: while the Segal condition essentially characterizes the ability to compose, the decomposition-space axioms express the ability to decompose. Many incidence coalgebras in combinatorics do not arise from posets or categories but do arise from decomposition spaces. Examples of this include the Butcher–Connes–Kreimer bialgebra of rooted trees [13, 19, 32], various Hall algebras, or some coalgebras coming from the Waldhausen S-construction [24, 32]. Bialgebras, rather than coalgebras, are obtained from monoidal decomposition spaces [32]. In examples from combinatorics, the monoidal structure is often disjoint union. In the present work we use the machinery of decomposition spaces, but all our examples will be Segal groupoids (Section 2.2).

To recover the algebraic incidence coalgebra from the categorified incidence coalgebra (Section 2.3) one takes homotopy cardinality, a cardinality functor defined from groupoids to the rationals [31] (Section 2.4).

Example (see Section 1.3 below)**.** Let $\mathbb{Q}[\![x]\!]$ be the ring of formal power series in x without constant term, and let $F, G \in \mathbb{Q}[\![x]\!]$. The *Faà di Bruno bialgebra* $\mathcal{F}$ is the free algebra $\mathbb{Q}[A_1, A_2, \ldots]$, where $A_n \in \mathbb{Q}[\![x]\!]^*$ is the linear map defined by

$$A_n(F) = \frac{d^n F}{dx^n}.$$

Its comultiplication is defined to be dual to substitution of power series. That is

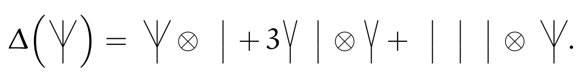
$$\Delta(A_n)(F, G) = A_n(G \circ F).$$

The Faà di Bruno bialgebra is based on the original computations made by Faà di Bruno in [28] on the derivatives of the composition of two functions. The combinatorial interpretation of $\mathcal{F}$ in terms of incidence algebras was first found by Doubilet [21]. It is the reduced incidence bialgebra of the lattice of partitions reduced modulo type equivalence. Joyal showed that this bialgebra can be objectified by using the category of finite sets and surjections $\mathbf{S}$ [43, §7.4]. In the context of Segal spaces and incidence bialgebras the result reads as follows: *the Faà di Bruno bialgebra $\mathcal{F}$ is equivalent to the homotopy cardinality of the incidence bialgebra of the fat nerve* $N\mathbf{S}$ *of the category* $\mathbf{S}$ (Remark 3.3.3). The comultiplication here is given by summing over factorizations of surjections:

$$\Delta(n \twoheadrightarrow l) = \sum_{n \twoheadrightarrow k \twoheadrightarrow l} (n \twoheadrightarrow k) \otimes (k \twoheadrightarrow l). \tag{1}$$

The algebra structure is given by disjoint union of sets. This sum runs over isomorphism classes of factorizations $n \twoheadrightarrow k \twoheadrightarrow l$, meaning up to isomorphism $k \xrightarrow{\sim} k'$ making the diagram commute. The advantage of working with groupoids is precisely that one does not have to worry about these isomorphism classes: this information is encoded in the simplicial groupoid $N\mathbf{S}$. Of course, the surjection $n \twoheadrightarrow 1$ corresponds to A_n.

The Faà di Bruno bialgebra can also be interpreted through operads: it is well-known that $N\mathbf{S}$ is equivalent to the two-sided bar construction of Sym, the terminal reduced symmetric operad [42] (see also Theorem 7.0.1). This equivalence takes the surjection $n \twoheadrightarrow 1$ to the unique operation of arity n, and the comultiplication of an operation runs through all possible 2-step factorizations. For example



We arrive thus at the last ingredient of this thesis. The theory of operads has long been a standard tool in topology and algebra [49, 53], and in category theory [46], and it is getting increasingly important also in combinatorics [37, 55].

It was shown in [42] that the two-sided bar construction [54, 74] of an operad is a Segal groupoid, and classical constructions of bialgebras from operads factor through this construction (see [18, 68, 69] for related constructions).

Contributions of the present thesis

In this thesis we study the combinatorics of plethysm within the framework of incidence bialgebras and objective combinatorics. The objective algebra is carried out at the level of Segal groupoids, by using homotopy slices and homotopy pullbacks of groupoids and simplicial methods. The construction of these Segal groupoids is itself a subject of study, and leads to many other (old and new) notions of plethysm.

The first main contribution of the present work is to exhibit plethystic substitution as a convolution tensor product obtained from an explicit simplicial groupoid,

$$\mathsf{T}\mathbf{S}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd},$$

by the standard general constructions of incidence coalgebras and homotopy cardinality, in pleasant close analogy with the example above with ordinary substitution and $\mathsf{N}\mathbf{S}$. This simplicial groupoid arises from $\mathbf{S}$ as its T-*construction*, a categorical construction that we introduce in Section 3.1, and which is reminiscent of Quillen's Q [63] and Waldhausen's S-constructions [72]. The simplicial groupoid $\mathsf{T}\mathbf{S}$ is closely related to partitionals, and in fact it gives a new interpretation of transversals, a key concept in this theory.

We have mentioned several different approaches to the combinatorics of plethysm in the beginning: partitionals, permutationals and linear partitionals. Besides these different approaches, distinct variations of plethysm have emerged with time. Most prominently, plethysm of power series with variables indexed by a (locally finite) monoid, introduced by Méndez and Nava [56] in the course of generalizing Joyal's theory of colored species to an arbitrary set of colors, and plethysm in several variables used in multisort species.

The second main contribution of this thesis is a construction on operads, called the $\mathcal{T}$-construction, which establishes a passage from ordinary substitutions to plethystic substitutions. In particular, this construction produces Segal groupoids for the partitional and the linear partitional (also called exponential) cases, for the other aforementioned variations of plethysm, for plethysm of series with coefficients in a noncommutative ring, and for plethysm of series with noncommuting variables. The last two have never appeared before, but it is appropriate to mention that the noncommutative version of the Faà di Bruno bialgebra comes up in algebraic topology [58], combinatorics [11, 25], numerical analysis [50, 59] and number theory [38].

The starting point for the $\mathcal{T}$-construction is the observation that, in the same way as $\mathsf{N}\mathbf{S}$, the simplicial groupoid $\mathsf{T}\mathbf{S}$ is equivalent to the two-sided bar construction of an operad, and that the way to obtain this operad from Sym can be generalized to any (nice enough) operad.

In the present work, for maximal flexibility, we work with operads in the form of P-operads [12, 46], where is P a cartesian monad in a cartesian category $\mathcal{E}$. This is a technical but powerful machinery, which allows us to cover simultaneously notions such as monoids, categories, nonsymmetric and symmetric operads, colored and noncolored operads, and to work over the category $\mathcal{E} = \mathbf{Grpd}$ of groupoids. The two-sided bar construction is worked out in this context for the first time. An essential condition for the $\mathcal{T}$-construction to be well-defined is that P is strong [41, 57, 71]. The present work represents the first manifestation of strong monads in combinatorics.

The main results of this thesis have already been released in paper form. The content of Chapter 3 constitutes the paper *A simplicial groupoid for plethysm* [15], to appear in

Algebraic and Geometric Topology. This material was exposed at the international conferences Category Theory 2018 and CSASC 2018. Moreover, an expository survey titled *Combinatorics and simplicial groupoids* [16], containing the ideas of [15] and its relation to the theory of species and partitionals, will appear in TEMat.

The content of Chapters 4 to 8 constitutes the paper *Plethysms and operads* [17], submitted for publication.

Summary

The thesis is divided into eight chapters.

Chapter 1: Plethysm

This chapter is a summary on plethysm and the combinatorics surrounding it. Let us begin by saying what this substitution is about.

Definition ([62]). Let $\mathbb{Q}[\![\mathbf{x}]\!]$ be the ring of power series in the variables $\mathbf{x} = (x_1, x_2, \dots)$ without constant term. Given two power series, $F(x_1, x_2, \dots)$ and $G(x_1, x_2, \dots)$ in $\mathbb{Q}[\![\mathbf{x}]\!]$, their *plethystic substitution* is defined as

$$(G \circledast F)(x_1, x_2, \dots) = G(F_1, F_2, \dots), \tag{2}$$

with $F_k = F(x_k, x_{2k}, \dots)$.

As mentioned before, this notion originated within unlabelled enumeration theory. In Section 1.1 we broadly explain how plethysm is used in this setting, and we treat it from the perspective of the theory of species.

We continue in Section 1.2 with the theory of colored species. On the one hand, this gives rise to plethysm in several variables. For two variables, it is defined as follows:

$$\big(H \circledast (G, F)\big)(x_1, x_2, \dots; y_1, y_2, \dots) = H(G_1, G_2, \dots; F_1, F_2, \dots),$$

with $G_k = G(x_k, x_{2k}, \dots; y_k, y_{2k}, \dots)$ and $F_k = F(x_k, x_{2k}, \dots; y_k, y_{2k}, \dots)$. On the other hand, if the species are colored over a monoid Y, one can define their Y-plethysm [56], which at the level of power series is

$$(G \circledast F)(x_i; i \in Y) = G\big(F_i(x_j; j \in Y); i \in Y\big),$$

where $F_i(x_j; j \in Y) = F(x_{ij}; j \in Y)$. Note that now the variables are indexed by Y. For this to be well-defined, Y has to be locally finite.

In Section 1.3 we expand a bit on the Faà di Bruno bialgebra and some of its variations, including the ordinary Faà di Bruno bialgebra, the multivariate case, and the noncommutative case.

In the same way the theory of species categorifies substitution of one variable power series, the theory of partitionals of Nava–Rota [61] categorifies plethysm. We explain this theory in Section 1.4. In this approach the power series $F \in \mathbb{Q}[\![\mathbf{x}]\!]$ are written as

$$F(\mathbf{x}) = \sum_{\lambda} F_\lambda \frac{\mathbf{x}^\lambda}{\mathrm{aut}(\lambda)}, \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$ is an infinite vector of natural numbers with a finite number of nonzero entries, and

$$\operatorname{aut}(\lambda) = 1!^{\lambda_1}\lambda_1! \cdot 2!^{\lambda_2}\lambda_2! \cdots, \qquad \mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots.$$

As we shall see, $\operatorname{aut}(\lambda)$ is the number of automorphisms of the partition or surjection represented by λ. The key concept for defining the substitution of partitionals is that of a transversal (Definition 1.4.2, Example 1.4.3), a complex notion of partition of a partition. We also explain linear partitions and linear transversals [60], which are concerned with exponential power series, in which $\operatorname{aut}(\lambda) = \lambda! = \lambda_1!\lambda_2! \cdots$.

Finally, in Section 1.5, for completeness, we describe Littlewood's plethysm of symmetric functions and its relation to Pólya plethysm.

Chapter 2: Segal groupoids and incidence coalgebras

In Section 2.1 we review the homotopy theory of the 2-category of groupoids, including in particular the extensively used notions of homotopy pullback, weak slice and fibration. In the rest of the chapter we review the theory of Segal groupoids and their incidence coalgebras. Let us present a synopsis of these. A Segal groupoid is a simplicial groupoid

$$X \colon \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$$

satisfying certain conditions encoding the ability to compose (2.2.1). The spans

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2, d_0)} X_1 \times X_1, \qquad\qquad X_1 \xleftarrow{s_0} X_0 \xrightarrow{t} 1,$$

define two functors

$$\begin{array}{rcl} \Delta \colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd}_{/X_1 \times X_1} \\ S \xrightarrow{s} X_1 & \longmapsto & (d_2, d_0)_! \circ d_1^*(s), \end{array} \qquad \begin{array}{rcl} \epsilon \colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd} \\ S \xrightarrow{s} X_1 & \longmapsto & t_! \circ s_0^*(s), \end{array} \tag{4}$$

where upperstar is homotopy pullback, lowershriek is postcomposition and $\mathbf{Grpd}_/$ denotes the weak slice category. If X is a Segal groupoid then Δ is coassociative and ϵ counital (in a homotopy sense, see Theorem 2.3.1) [32].

If X satisfies some finiteness conditions [33], which we explain in Section 2.4, then we can take homotopy cardinality $|\cdot|$ of Δ and ϵ to get a coalgebra structure

$$\begin{array}{rcl} \Delta \colon \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ \left|S \xrightarrow{s} X_1\right| & \longmapsto & \left|(d_2, d_0)_! \circ d_1^*(s)\right| \end{array} \qquad \begin{array}{rcl} \epsilon \colon \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q}, \\ \left|S \xrightarrow{s} X_1\right| & \longrightarrow & \left|t_! \circ s_0^*(s)\right|, \end{array}$$

in the vector space $\mathbb{Q}_{\pi_0 X_1}$ spanned by the connected components of X_1. Moreover, if X is monoidal then $\mathbb{Q}_{\pi_0 X_1}$ acquires a bialgebra structure.

Chapter 3: A Segal groupoid for plethysm

This chapter contains the first main contribution of the thesis, published in [15]. We introduce, in Section 3.2, the *plethystic bialgebra* $\mathcal{P}$ as the free algebra $\mathbb{Q}\left[\{A_\lambda\}_\lambda\right]$ on the linear functionals A_λ which return the F_λ coefficient (3) of a power series $F \in \mathbb{Q}[\![\mathbf{x}]\!]$, and with comultiplication dual to plethystic substitution

$$\Delta(A_\lambda)(F, G) = A_\lambda(G \circledast F).$$

We study this bialgebra to obtain a formula for extracting the comultiplication of A_λ (Proposition 3.2.3) which can be viewed as a generalization of the Bell polynomial expression for extracting the coefficients of ordinary composition of one-variable power series.

In Section 3.3 we show (Theorem 3.3.1) that this bialgebra is in fact the homotopy cardinality of the incidence bialgebra of a certain new Segal groupoid, called $T\mathbf{S}$. This Segal groupoid arises from the category of finite sets and surjections $\mathbf{S}$ from the T-construction, a main contribution of this thesis, that we previously introduce in Section 3.1. Let us see what $T\mathbf{S}$ looks like. The objects of its 1 and 2-simplices can be pictured as

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{11} \end{tikzcd} \in T_1\mathbf{S} \qquad \begin{tikzcd} & & t_{02} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, near start, "\lrcorner"] & & \\ & t_{01} \arrow[dl, two heads, "\mu"'] \arrow[dr, two heads] & & t_{12} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{11} \arrow[rr, two heads] & & t_{22} \end{tikzcd} \in T_2\mathbf{S}.$$

where the t_{ij} are finite sets and the arrows surjections. Morphisms of such shapes are levelwise bijections $t_{ij} \xrightarrow{\sim} t'_{ij}$ compatible with the diagram. In general $T_n\mathbf{S}$ is an analogous pyramid, with t_{0n} in the peak, all of whose squares are pullbacks of sets. The face maps d_i remove all the sets containing an i index, and the degeneracy maps s_i repeat the ith diagonals, as will be detailed in Section 3.1.

Diagrams whose last set is singleton are called *connected*. The main point of $T\mathbf{S}$ is that the connected objects in $T_1\mathbf{S}$ parametrize precisely the summation of the series, and, rather strikingly, the connected objects in $T_2\mathbf{S}$ encode all the combinatorics of plethystic substitution, as we shall see. Including also the non-connected objects is essential for having a simplicial object.

The comultiplication is given by

$$\Delta\left(\begin{tikzcd} & t_{02} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{22} \end{tikzcd}\right) := \sum_{\begin{tikzcd}[sep=small] & & t_{02} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, near start, "\lrcorner"] & & \\ & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] & & t_{12} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{11} \arrow[rr, two heads] & & t_{22} \end{tikzcd}} \begin{tikzcd} & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{11} \end{tikzcd} \otimes \begin{tikzcd} & t_{12} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{11} \arrow[rr, two heads] & & t_{22} \end{tikzcd}.$$

As in (1), this sum is over isomorphism classes of such diagrams, meaning up to isomorphisms $t_{01} \xrightarrow{\sim} t'_{01}$ and $t_{12} \xrightarrow{\sim} t'_{12}$ making the diagram commute.

In Section 3.4 we see how the notion of transversal of a partition, which is the key concept in the theory of partitionals [61], is encoded in $T_2\mathbf{S}$. To make the comparison we use the fact that the groupoid of partitions is equivalent to the groupoid of surjections.

Finally, in Section 3.5 we take the opportunity to describe a plethystic analogue of the Faà di Bruno formula for the connected Green function associated to the incidence bialgebra of $T\mathbf{S}$.

The fact that $T\mathbf{S}$ is based on surjections just like $N\mathbf{S}$, leads to an important question which is the second part of the thesis: what is the general mechanism by which plethystic substitution arises from ordinary substitution?

The key observation is that, in the same way the fat nerve of the category of surjections $N\mathbf{S}$ is equivalent to the bar construction of Sym, the simplicial groupoid $T\mathbf{S}$ is also equivalent to the bar construction of an operad, $\mathcal{T}_{S^r}\mathrm{Sym}$. The comparison between $T\mathbf{S}$ and $\mathcal{T}_{S^r}\mathrm{Sym}$ is treated in Chapter 8.

The mechanism by which $\mathcal{T}_{S^r}\mathrm{Sym}$ arises from Sym can be generalized to include all the other plethysms and beyond, and we call it the $\mathcal{T}$-constructions.

Chapter 4: Monads and operads

It turns out that that the operad $\mathcal{T}_{S^r}\mathrm{Sym}$ and many of the other operads related to plethysms, are not operads over **Set** but over **Grpd**. Moreover, we encounter symmetric and nonsymmetric operads, as well as colored and noncolored operads. This makes the framework of P-operads (generalized multicategories in the terminology of Leinster [46]) the ideal level of generality for the $\mathcal{T}$-construction.

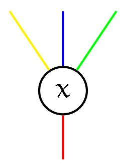
We explain the basics of this theory in Section 4.2. Let P be a cartesian monad on a cartesian category $\mathcal{E}$. A P-operad is represented by a span and two arrows

$$\begin{tikzcd} & Q_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ PQ_0 & & Q_0 \end{tikzcd} \qquad \begin{array}{c} PQ_1 \times_{PQ_0} Q_1 \xrightarrow{m} Q_1 \\[2em] Q_0 \xrightarrow{e} Q_1, \end{array}$$

where Q_0 is thought of as the object of colors, Q_1 is thought of as the object of operations, s returns the P-configuration of input colors, t returns the output color, e is the unit and m is composition. All these arrows have to satisfy associativity and unit axioms. These axioms have been relegated to Appendix A, since they are routinary.

For instance, if Id is the identity monad, then an Id-operad is a category internal to $\mathcal{E}$. The $\mathcal{T}$-construction is in fact a composition of two constructions, one from P-operads to (internal) categories and one from categories to P-operads.

We will mainly be interested in $\mathcal{E} = \mathbf{Grpd}$. In particular, nonsymmetric operads will be considered as M^r-operads, where M^r is the free semimonoidal category monad in **Grpd**, and symmetric operads as S^r-operads, where S^r is the free symmetric semimonoidal category monad in **Grpd**. As it is usual, the (objects of the) operations of M^r or S^r-operads Q will be depicted as



where the colors symbolize objects of Q_0.

Note that unlike nonsymmetric operads, symmetric operads cannot be portrayed as P-operads in **Set**, because the free commutative monoid monad is not cartesian. Furthermore, working in **Grpd** adapts better with the theory of decomposition spaces and incidence coalgebras. This theory uses weak notions of simplicial groupoids, slice categories, and pullbacks, but by keeping track of fibrancy we can stay within strict notions and strict monads in the style of [73].

In Section 4.4 we explain the notion of strong monad (Definition 4.4.1). It is essential for the $\mathcal{T}$-construction that P is strong. This notion goes back to the work of A. Kock [41] in enriched category theory, but it has turned out to be fundamental for the role monads play in functional programming [57, 71]. Strong monads have recently found their way to algebraic topology [6]. The present work represents their first manifestation in combinatorics.

Finally, in Section 4.5 we introduce the two-sided bar construction for P-operads. This is a standard construction [54], but has never appeared in the setting of P-operads before. We denote by $\mathcal{B}Q$ the two-sided bar construction of the P-operad Q. As a heuristic description we can say that the n-simplices of $\mathcal{B}Q$ are forests of n-level trees of operations of Q.

Chapter 5: The $\mathcal{T}$-construction

The $\mathcal{T}$-construction is named after the T-construction of Chapter 3. By coincidence Giraudo [36] had used the same letter T for a functor from monoids to nonsymmetric operads. The $\mathcal{T}$-construction of the present work encompasses both these constructions, and the letter T has been maintained, but now in a fancier font.

The $\mathcal{T}$-construction is composed in fact by two constructions. The first one, introduced in Section 5.1, produces a P-operad $\mathcal{T}_P C$ from a category C internal to $\mathcal{E}$. The entire section is devoted to the definition $\mathcal{T}_P C$ (5.1.1), together with its composition (Definition 5.1.1) and unit (Definition 5.1.4) and to show they satisfy associativity and unit axioms (Propositions 5.1.3 and 5.1.6).

The second construction, introduced in Section 5.2, produces a category $\mathcal{T}^P Q$ from a P-operad Q. The entire section is devoted to the definition $\mathcal{T}^P Q$ (5.2.1), together with its composition (Definition 5.2.1) and unit (Definition 5.2.4) and to show they satisfy associativity and unit axioms (Propositions 5.2.3 and 5.2.6).

Some of the technical proofs of these two sections have been shifted to Appendix B, for the sake of readability.

We will extensively use the composite of these two, which produces a P-operad $\mathcal{T}_P Q$ from a P'-operad Q. The only purpose of Section 5.3 is to emphasize this fact.

In Section 5.4 we explore the finiteness conditions that the agents of the $\mathcal{T}$-construction have to satisfy, in the case $\mathcal{E} = \mathbf{Grpd}$, so that later on we can take homotopy cardinality. These include finiteness conditions on P, C and Q (Definition 5.4.1), and verification that the conditions are preserved by the $\mathcal{T}$-construction (Lemmas 5.4.3 and 5.4.4).

This chapter is the most technical, and there does not seem to be much of a point in going into these details in the present summary. Instead, we will see some examples in the subsequent subsection.

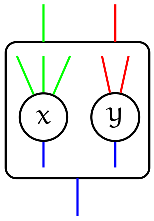
Chapter 6: $\mathcal{T}$-construction for M^r and S^r-operads

In Section 6.1 we begin by unraveling the $\mathcal{T}$-construction from categories in **Set** to M^r-operads (nonsymmetric operads). The idea is that an operation in $\mathcal{T}_{\mathsf{M}^r} C$ is given by a sequence of arrows of C with the same output. For instance, an operation with input c_1, c_2, c_3 and output c can be pictured as

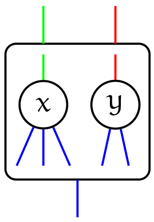
$$\begin{tikzcd}
c_1 \arrow[dr] & c_2 \arrow[d] & c_3 \arrow[dl] \\
& c &
\end{tikzcd}$$

Composition goes as expected (6.1.4). Clearly this can be made a symmetric operad too. As shown in Example 6.1.3, when C is a monoid this construction coincides with the Giraudo T-construction, which was introduced by Giraudo [36] as a generic method to build combinatorial operads from monoids.

In Section 6.2 we unravel the $\mathcal{T}$-construction from nonsymmetric to symmetric operads and some variations, such as from symmetric operads to S^r-operads. Recall that this means producing first a category $\mathcal{T}^{M^r}Q$ and then a S^r-operad $\mathcal{T}^{S^r}Q$. The general idea is that an operation in $\mathcal{T}^{S^r}Q$ is a sequence of operations $(x_1, \dots, x_n)$ of Q satisfying that for each i all the input objects of x_i coincide, and that all the n output objects coincide. The following picture shows an operation of input (•, •) and output • consisting of the sequence (x, y):



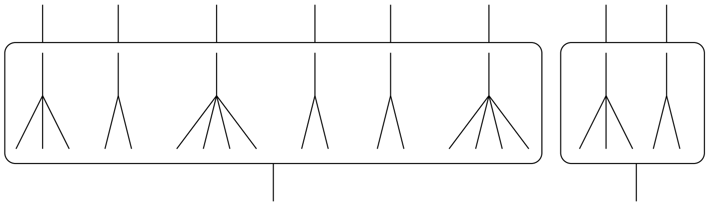
Composition is obtained by repetition of operations and composition in Q (6.2.1). Nevertheless, from a combinatorial point of view, when dealing with plethysm it is more natural to apply the $\mathcal{T}$-construction to the opposite category $(\mathcal{T}^{M^r}Q)^{op}$. It is the purpose of Section 6.3 to develop this point of view. The general idea is that an operation in $\mathcal{T}_{S^r}Q^{op}$ is a sequence of operations $(x_1, \dots, x_n)$ of Q satisfying that all the input objects coincide. But then these inputs become the outputs, as shown in the following picture:



Composition is again obtained by repetition of operations and composition in Q (6.3.1). From a purely formal perspective, in all our cases of interest it makes no difference to take this opposite convention.

Chapter 7: Plethysms and operads

We finally explain in this chapter the relation between the $\mathcal{T}$-construction and plethysm. We begin with the classical plethystic bialgebra $\mathcal{P}$: it is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{S^r}\mathsf{Sym}$ (Theorem 7.0.2). For instance, under this isomorphism, the following 1-simplex of $\mathcal{B}\mathcal{T}_{S^r}\mathsf{Sym}$

$s =$  (5)

corresponds to $A_{(0,3,1,2)}A_{(0,1,1)}$ in $\mathcal{P}$. In this statement we have of course incurred in a slight abuse of language, since the real correspondence is

$$\left|1 \xrightarrow{\ulcorner s \urcorner} \mathcal{B}_1\mathcal{T}_{S^r}\mathsf{Sym}\right| \longmapsto A_{(0,3,1,2)}A_{(0,1,1)}.$$

In Section 7.1 we give a summary of all the variations of the plethystic bialgebra that can be obtained from the operads Sym, Ass, Sym_2 and Ass_2 and the monads M^r and S^r. Looking at s in Equation (5) one can already see several possible noncolored variations: commutativity of the two operations of s; commutativity of the operations inside s, and whether they belong to Sym or to Ass.

At the level of power series, the variations include: exponential plethysm of Nava [60] (Theorem 7.3.3); noncommutative variables (Theorems 7.2.1 and 7.3.4); noncommutative variables and coefficients (Theorems 7.2.2 and 7.3.6), and power series in two sets of infinitely many variables (Theorems 7.2.4 and 7.3.9).

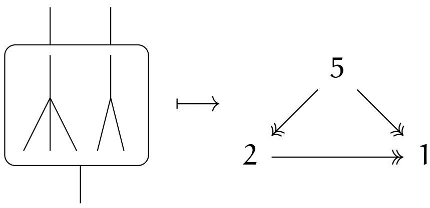
Section 7.2 is devoted to the bialgebras arising from Sym and Sym_2, and Section 7.3 is devoted to the bialgebras arising from Ass and Ass_2. The difference between the first ones and the second ones relies on the number of automorphisms. Finally, in Section 7.4 we deal with the Y-plethysm of Méndez–Nava [56] (Theorem 7.4.2).

Along the way we encounter some variations of the Faà di Bruno bialgebra, such as: the ordinary Faà di Bruno bialgebra (Theorem 7.3.1); noncommutative [11, 25, 50] (Theorem 7.3.2); two-variables (Theorem 7.2.3); two noncommuting variables (Theorem 7.3.7), and noncommutative Faà di Bruno bialgebra in two noncommuting variables (Theorem 7.3.8).

*Chapter 8: T**S** revisited*

As mentioned above, it is the operadic nature of T**S** that motivates the $\mathcal{T}$-construction. We begin, in Section 8.1, by showing that indeed $\mathcal{B}\mathcal{T}_{S^r}\mathsf{Sym}$ is equivalent to T**S**. This in particular will prove Theorem 7.0.2.

Here is an example of a correspondence under this equivalence at the level of 1-simplices:



In Example 8.1.6 we establish a correspondence of 2-simplices.

Finally, in Section 8.2 we characterize some of the two-sided bar constructions of Chapter 7 as simplicial groupoids similar to $\mathsf{T}\mathbf{S}$. In particular, in Example 8.2.1 we recover the linear transversals of Nava [60].

List of notations

$\circledast$	plethystic substitution (1.0.1)
Grpd	category of groupoids and groupoid morphisms
Cat	category of small categories
$\mathcal{C}$	generic category
Set	category of sets and set maps
$\mathbf{S}$	category of finite sets and surjections
$\mathbb{B}$	category of finite sets and bijections
$\mathbb{P}$	category of partitions (page 24)
$\mathbb{S}$	category of surjections (page 47)
Tw^{+}	twisted arrow category with additional arrows between the identities (page 36)
$\mathbb{\Delta}$	simplex category (page 30)
$\mathsf{N}\mathbf{S}$	fat nerve (page 31) of $\mathbf{S}$
$\mathsf{T}\mathbf{S}$	simplicial groupoid obtained from the T-construction of $\mathbf{S}$ (page 39)
Λ	set of infinite vectors $\lambda = (\lambda_1, \lambda_2, \dots)$ of natural numbers with $\lambda_i = 0$ for all i large enough (page 39)
V^k	kth Verschiebung operator (page 39)
W	set of finite words $\omega = \omega_1 \dots \omega_n$ of positive natural numbers (page 88)
$\mathcal{F}$	Faà di Bruno bialgebra (page 21)
$\mathcal{P}$	plethystic bialgebra (page 40)
$\mathcal{E}$	generic ambient cartesian category, mainly **Set** or **Grpd** (page 54)
(P, μ, η)	generic strong cartesian monad (4.2.1 and 4.4.1)
Id	identity monad (4.4.3)
M	free monoid monad (page 54)
M^{r}	free semigroup monad (4.4.5)
S	free symmetric monoidal category monad (4.4.7)
S^{r}	free symmetric semimonoidal category monad (4.4.8)
L	monad $A \mapsto \mathsf{P}1 \times A$ (page 65)
Y	generic (locally finite) monoid (4.4.6)
Y	monad given by $A \mapsto Y \times A$, for Y a monoid (4.4.6)
C	category internal to $\mathcal{E}$ (page 53)

Q P-operad internal to $\mathcal{E}$ (page 54)

Q_0, Q_1 objects and operations of Q (page 54)

(Q, μ^Q, η^Q) monad on $\mathcal{E}/Q_0$ defined by the P-operad Q (page 60)

$D_{A,B}$ strength natural transformation (4.4.1)

D_B strength for $A = 1$ (page 65)

R_A projection $P1 \times A \mapsto A$ (page 65)

$\mathcal{B}$ two-sided bar construction (page 61)

$\mathcal{B}^P$ two-sided bar construction relative to a monad P (page 63)

$\mathcal{B}_n$ n-simplices of the two-sided bar construction $\mathcal{B}$ (page 94)

$\mathcal{B}_n^{\circ}$ subgroupoid of connected objects of $\mathcal{B}_n$ (page 94)

Sym the reduced symmetric operad (4.4.8)

Ass the reduced associative operad (4.4.5)

$\mathcal{T}_P C$ $\mathcal{T}$-construction from C to a P-operad (page 65)

$\mathcal{T}^P Q$ $\mathcal{T}$-construction from a P-operad to a category C (page 71)

$\mathcal{T}_P Q$ $\mathcal{T}$-construction from a P'-operad to a P-operad (page 77)

$\mathcal{F}_{\mathrm{ord}}$ ordinary Faà di Bruno bialgebra (page 22)

$\mathcal{F}^{\mathrm{nc}}$ noncommutative Faà di Bruno bialgebra (page 21)

$\mathcal{F}^2$ Faà di Bruno bialgebra in two variables (page 91)

$\mathcal{F}^2_{\mathrm{ord}}$ ordinary Faà di Bruno bialgebra in two variables

$\mathcal{F}^{\langle 2 \rangle}$ Faà di Bruno bialgebra with two noncommuting variables (page 95)

$\mathcal{F}^{\langle 2 \rangle, \mathrm{nc}}$ noncommutative Faà di Bruno with two noncommuting variables (page 95)

A_n, a_n generators for the several Faà di Bruno bialgebras

$\mathcal{P}_{\mathrm{exp}}$ exponential plethystic bialgebra (page 93)

$\mathcal{P}^{\diamondsuit}$ plethystic bialgebra with noncommuting variables (page 90)

$\mathcal{P}^{\diamondsuit}_{\mathrm{lin}}$ linear plethystic bialgebra with noncommuting variables (page 94)

$\mathcal{P}^{\diamondsuit, \mathrm{nc}}$ noncommutative plethystic bialgebra with noncommuting variables (page 91)

$\mathcal{P}^{\diamondsuit, \mathrm{nc}}_{\mathrm{lin}}$ noncommutative linear plethystic bialgebra in noncommuting variables (page 94)

$\mathcal{P}^2$ plethystic bialgebra in two variables (page 92)

$\mathcal{P}^2_{\mathrm{exp}}$ exponential plethystic bialgebra in two variables (page 95)

$\mathcal{P}^Y$ plethystic bialgebra relative to a monoid Y (page 96)

A_λ, a_λ
A_ω, a_ω generators for the several plethystic bialgebras

1 Plethysm

This chapter is a summary on plethysm and the combinatorics surrounding it. We start by explaining its relevance in unlabelled enumeration theory. Although this goes back to Pólya [62], we treat it from the more modern point of view of combinatorial species. We also take the opportunity to introduce the Faà di Bruno bialgebra, since it is closely related both to species and to the plethystic bialgebra. In Section 1.4 we introduce the theory of partitionals [61]. Finally in Section 1.5 we give a glimpse of plethysm of symmetric functions. But before that, let us define again the notion of plethystic substitution, since it is the thread of this chapter.

Definition 1.0.1 ([62]). Given two power series, $F(x_1, x_2, \dots)$ and $G(x_1, x_2, \dots)$ in $\mathbb{Q}[\![\mathbf{x}]\!]$, their *plethystic substitution* is defined as

$$(G \circledast F)(x_1, x_2, \dots) = G(F_1, F_2, \dots),$$

with $F_k = F(x_k, x_{2k}, \dots)$.

Example 1.0.2. For instance,

$$\begin{aligned} x_n \circledast x_m &= x_{n \cdot m} \\ x_n \circledast (x_{m_1} + x_{m_2}) &= x_{n \cdot m_1} + x_{n \cdot m_2} \\ (x_2^3 + x_5^7) \circledast (x_4^6 + x_1^2) &= (x_8^6 + x_2^2)^3 + (x_{20}^6 + x_5^2)^7. \end{aligned}$$

Observe that a one-variable power series can be regarded as a power series in $\mathbb{Q}[\![\mathbf{x}]\!]$ where only the monomials x_1^n may have nonzero coefficient. Under this correspondence plethysm agrees with ordinary substitution of one-variable power series.

1.1 Species

The theory of species, introduced by Joyal [43], is one of the starting points for objective combinatorics. Through the notion of species, Joyal showed that manipulations with power series and generating functions can be carried out directly on the combinatorial structures themselves. A *species* is a functor

$$F \colon \mathbb{B} \longrightarrow \mathbb{B}$$

from the category $\mathbb{B}$ of finite sets and bijections to itself. To each finite set S the species F associates another finite set $F[S]$, whose elements are called F-*structures* on the set S. Each bijection $S \to R$ gives a bijection $F[S] \to F[R]$. Two F-structures $\sigma, \sigma' \in F[S]$ are *isomorphic* if there exists a bijection $f \colon S \xrightarrow{\sim} S$ such that $F[f](\sigma) = \sigma'$.

Example 1.1.1. The species of partitions Π sends a set E to $\Pi(E)$, the set of all its partitions, and a bijection $E \xrightarrow{f} E'$ to the obvious bijection $\Pi(E) \to \Pi(E')$ given by f. For instance

$$\Pi([3]) = \left\{ \boxed{1\ \ 2\ \ 3}\ \boxed{1\ \ 2 \mid 3}\ \boxed{1 \mid 2\ \ 3}\ \boxed{1\ \ 3 \mid 2}\ \boxed{1 \mid 2 \mid 3} \right\}$$

Other examples include structures of graphs, trees, linear orders, etc.

We may attach different kinds of power series to a species F in order to enumerate the F-structures or the isomorphism classes of F-structures. The first ones are often called *labelled* structures, while the second ones are called *unlabelled* structures. The *exponential generating function* associated to F is

$$F(x) = \sum_{n \geqslant 0} |F[n]| \frac{x^n}{n!},$$

where $|F[n]|$ is the number of F-structures on a set of n elements. This function is used for labelled enumeration. The *type generating function* associated to F is

$$\widetilde{F}(x) = \sum_{n \geqslant 0} |\widetilde{F}[n]| \, x^n,$$

where $|\widetilde{F}[n]|$ is the number of unlabelled structures of F.

Example 1.1.2. For instance, in view of the example above we have that $|\Pi[3]| = 5$, while $|\widetilde{\Pi}[3]| = 3$, because the partitions $\boxed{1\ \ 2 \mid 3}$, $\boxed{1 \mid 2\ \ 3}$ and $\boxed{1\ \ 3 \mid 2}$ are isomorphic. The number of partitions of an n-element set $|\Pi[n]|$ is generally called the Bell number B_n [66].

Several operations of generating functions can be lifted to the level of species [43]. For instance, given two species F and G we define their *sum* and *product* by

$$(F+G)[S] = F[S] + G[S] \quad \text{and} \quad (F \cdot G)[S] = \sum_{\substack{S_1 + S_2 = S \\ S_1 \cap S_2 = \varnothing}} F[S_1] \times G[S_2]$$

respectively. Both operations are compatible with addition and multiplication of generating functions, so that $(F+G)(x) = F(x) + G(x)$, $(F \cdot G)(x) = F(x) \cdot G(x)$ and similarly for the type generating functions. Nevertheless, the operation that interests us most is *substitution* [43, §2.2]. Suppose that $G[\emptyset] = \emptyset$. Then

$$(F \circ G)[S] = \sum_{\pi \in \Pi[S]} F[\pi] \times \prod_{B \in \pi} G[B], \tag{1.1.1}$$

where $F[\pi]$ interprets π as a set. In other words, an $(F \circ G)$-structure on S is a partition π of S together with an F-structure of π and G-structures for each block of π. Notice that this is not the composition of F and G as functors. Substitution of species is compatible with the exponential generating function,

$$(F \circ G)(x) = F(x) \circ G(x), \tag{1.1.2}$$

but not with the type generating function.

Example 1.1.3. Let E be the species of sets, defined by $E[S] = \{S\}$, so that for each finite set S there is a unique E-structure, the set S itself. Also, denote by E^+ the species of nonempty sets, defined by $E_+[S] = \{S\}$ if S is nonempty and $E_+[S] = \emptyset$ if $S = \emptyset$. It is clear then that the exponential and type generating functions for these species are given by

$$E(x) = \sum_{n=0} \frac{x^n}{n!} = e^x, \qquad \widetilde{E}(x) = \sum_{n=0} x^n = \frac{1}{1-x},$$
$$E_+(x) = \sum_{n=1} \frac{x^n}{n!} = e^x - 1, \qquad \widetilde{E}_+(x) = \sum_{n=1} x^n = \frac{1}{1-x} - 1.$$

Observe that the species of partitions Π is precisely the substitution of these two:

$$\Pi = E \circ E_+,$$

because, in view of Equation (1.1.1),

$$\Pi[S] = \sum_{\pi \in \Pi(S)} \{\pi\} \times \prod_{B \in \pi} \{B\} = \sum_{\pi \in \Pi(S)} E[\pi] \times \prod_{B \in \pi} E_+[B].$$

By using Equation (1.1.2) we obtain the exponential generating function of Π,

$$\Pi(x) = e^{e^x - 1},$$

from which the Bell numbers can be computed. Notice that we had to use E_+ instead of E as the second argument of substitution because a partition does not allow empty blocks.

We can easily see that (1.1.2) does not hold for the type generating functions. Indeed,

$$\widetilde{E}(x) \circ \widetilde{E}_+(x) = 1 + x + 2x^2 + 4x^3 + \cdots,$$

which already fails at x^3, since we had seen that $|\widetilde{\Pi}[3]| = 3$, not 4.

To obtain a power series for unlabelled enumeration compatible with substitution, a third kind of generating function is required. The *cycle index series* of a species F [43, §3] is the formal power series (in infinitely many variables $x_1, x_2, \ldots$)

$$Z_F(x_1, x_2, \ldots) = \sum_{n \geqslant 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} |\mathrm{Fix}(F[\sigma])|\, x_1^{\sigma_1} x_2^{\sigma_2} \cdots \right),$$

where $\mathfrak{S}_n$ denotes the group of permutations of $[n]$, σ_k is the number of cycles of size k of σ and $\mathrm{Fix}(F[\sigma])$ is the set of F-structures fixed by $F[\sigma]$.

Example 1.1.4. Consider the species E. For every n and $\sigma \in \mathfrak{S}_n$ it is obvious that $E[\sigma]$ fixes $E[n]$, since it has only one element. This implies that its cycle index series is given by [9]

$$Z_E(x_1, x_2, \ldots) = \sum_{n \geqslant 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} x_1^{\sigma_1} x_2^{\sigma_2} \cdots \right) = e^{x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots}.$$

Let us now see that the term $n = 3$ of Z_Π is

$$\sum_{\sigma \in \mathfrak{S}_3} |\mathrm{Fix}(\Pi[\sigma])|\, x_1^{\sigma_1} x_2^{\sigma_2} \cdots = 5x_1^3 + 9x_1x_2 + 4x_3. \tag{1.1.3}$$

We have six permutations in $\mathfrak{S}_3$. The only permutation with three cycles of order 1 is the identity, and it fixes the five partitions of [3]. This gives the first term, $5x_1^3$. There are two cyclic permutations, and each of them fixes the partitions $\boxed{1 \,|\, 2 \,|\, 3}$ and $\boxed{1 \;\; 2 \;\; 3}$. This yields the last term, $4x_3$. Finally, there are three permutations with a cycle of order two, and each of them fixes the partitions $\boxed{1 \,|\, 2 \,|\, 3}$ and $\boxed{1 \;\; 2 \;\; 3}$ plus the partition they induce. This gives the middle term, $9x_1x_2$.

Theorem 1.1.5 (Joyal [43], see also [9])**.** *Given two species* F *and* G*, their cycle index series satisfy the following properties:*

(i) $Z_{F+G} = Z_F + Z_G$,

(ii) $Z_{F \cdot G} = Z_F \cdot Z_G$,

(iii) $Z_{F \circ G} = Z_F \circledast Z_G$,

(iv) $Z_F(x, 0, 0, \dots) = F(x)$,

(v) $Z_F(x, x^2, x^3, \dots) = \widetilde{F}(x)$,

where $\circledast$ *denotes plethystic substitution.*

In particular, we can compute the type generating function of the composite species $F \circ G$ from the plethystic substitution of the cycle index series of F and G. From the viewpoint of the theory of species, this is the motivation for the cycle index series and plethystic substitution, often referred to as Pólya theory.

Example 1.1.6. The cycle index series of Π can be computed as

$$\begin{aligned} Z_\Pi = Z_E \circledast Z_{E_+} &= Z_E\big(Z_{E_+}(x_1, x_2, \dots), Z_{E_+}(x_2, x_4, \dots), \dots\big) \\ &= e^{Z_{E_+}(x_1, x_2, \dots) + \frac{Z_{E_+}(x_2, x_4, \dots)}{2} + \cdots}. \end{aligned}$$

Expanding the exponentials and collecting the terms by degree we obtain

$$\begin{aligned} Z_\Pi &= 1 + x_1 + \frac{1}{2!}(2x_1^2 + 2x_2) + \frac{1}{3!}(5x_1^3 + 9x_1x_2 + 4x_3) \\ &\quad + \frac{1}{4!}(15x_1^4 + 42x_1^2x_2 + 21x_2^2 + 24x_1x_3 + 18x_4) + \cdots \end{aligned}$$

Notice that the term $n = 3$ coincides, as expected, with the one computed above (1.1.3). Also, the type generating function is given by

$$\widetilde{\Pi}(x) = Z_\Pi(x, x^2, \dots) = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots .$$

1.2 *Colored species*

The theory of species has a natural generalization to several variables [8,43]. Consider the set $\underline{k} = \{1, \dots, k\}$. A k*-sort species* is a functor

$$F \colon \mathbb{B}/\underline{k} \longrightarrow \mathbb{B},$$

where $\mathbb{B}/\underline{k}$ is the slice category of finite sets over $\underline{k}$. To each map $\chi\colon S \to \underline{k}$ the species F associates a finite set $F[\chi]$, whose elements are called F-*structures* on χ. Isomorphisms of F-structures are defined in the same way as before. Each bijection

$$\begin{tikzcd} S \arrow[rr, "\sim"] \arrow[dr, "\chi"'] & & R \arrow[dl, "\rho"] \\ & \underline{k} & \end{tikzcd}$$

in $\mathbb{B}/\underline{k}$ gives a bijection $F[\chi] \to F[\rho]$.

Example 1.2.1. For example, there is a 2-sort species Π^2 that sends $\chi\colon E \to \underline{2}$ to $\Pi(E)$, the set of partitions of E, and a bijection $\chi \xrightarrow{f} \chi'$ to the obvious bijection $\Pi^2(E) \to \Pi^2(E')$ given by f. For instance, let $\chi\colon \underline{3} \to \underline{2}$ given by $\chi(1) = \chi(2) = 1$ and $\chi(3) = 2$. Then,

$$\Pi^2(\chi) = \left\{ \boxed{\textcolor{red}{1}\ \textcolor{red}{2}\ \textcolor{blue}{3}}\ \boxed{\textcolor{red}{1}\ \textcolor{red}{2} \mid \textcolor{blue}{3}}\ \boxed{\textcolor{red}{1} \mid \textcolor{red}{2}\ \textcolor{blue}{3}}\ \boxed{\textcolor{red}{1}\ \textcolor{blue}{3} \mid \textcolor{red}{2}}\ \boxed{\textcolor{red}{1} \mid \textcolor{red}{2} \mid \textcolor{blue}{3}} \right\}$$

We have colored the elements of $\underline{3}$ according to χ. It may seem this example does not add much to Example 1.1.1, but notice that the bijections between structures are different, since they have to preserve the colors. In general we can get other colored species by coloring the vertices of graphs, trees, etc.

As before, we can associate generating functions to a k-sort species. For simplicity we define them for $k = 2$, but the generalization to arbitrary k is obvious:

$$F(x, y) = \sum_{n,k \geqslant 0} |F[n,k]| \frac{x^n}{n!} \frac{y^k}{k!},$$

$$\widetilde{F}(x, y) = \sum_{n,k \geqslant 0} |\widetilde{F}[n,k]|\, x^n y^k,$$

$$Z_F(x_1, x_2, \ldots; y_1, y_2, \ldots) = \sum_{n,k \geqslant 0} \frac{1}{n!k!} \left(\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \tau \in \mathfrak{S}_k}} |\mathrm{Fix}(F[\sigma,\tau])|\, x_1^{\sigma_1} x_2^{\sigma_2} \cdots y_1^{\tau_1} y_2^{\tau_2} \cdots \right),$$

where $F[n,k]$ is the set of F-structures of a map $\underline{n} + \underline{k} \to \underline{2}$, $\widetilde{F}[n,k]$ is the set of its unlabelled structures, and $\mathrm{Fix}(F[\sigma,\tau])$ is the set of F-structures fixed by $F[\sigma,\tau]$.

Example 1.2.2. For instance, in view of Example 1.2.1, we have that $|\Pi^2(2,1)| = 5$, while $|\Pi^2(2,1)| = 4$, because the two partitions $\boxed{\textcolor{red}{1} \mid \textcolor{red}{2}\ \textcolor{blue}{3}}$ and $\boxed{\textcolor{red}{1}\ \textcolor{blue}{3} \mid \textcolor{red}{2}}$ are isomorphic. Note that in this case the partition $\boxed{\textcolor{red}{1}\ \textcolor{red}{2} \mid \textcolor{blue}{3}}$ is not isomorphic to these two. It is easy to see that the $(2,1)$ term of Z_{Π^2} is

$$\sum_{\substack{\sigma \in \mathfrak{S}_2 \\ \tau \in \mathfrak{S}_1}} |\mathrm{Fix}(F[\sigma,\tau])|\, x_1^{\sigma_1} x_2^{\sigma_2} \cdots y_1^{\tau_1} y_2^{\tau_2} \cdots = 5x_1^2 y_1 + 3x_2 y_1.$$

Indeed, τ can only be the identity and σ can be either the identity or the transposition of 1 and 2. In the first case all five partitions are fixed, while in the second case the partitions $\boxed{\textcolor{red}{1} \mid \textcolor{red}{2} \mid \textcolor{blue}{3}}$, $\boxed{\textcolor{red}{1}\ \textcolor{red}{2}\ \textcolor{blue}{3}}$ and $\boxed{\textcolor{red}{1}\ \textcolor{red}{2} \mid \textcolor{blue}{3}}$ are fixed.

Notions of sum, product and substitution of k-sort species can be defined analogously as before, and they are compatible with these series in the same way as before. We define only substitution of species. Given k-sort species $F, G_1, \dots, G_k$ we define

$$F(G_1,\dots,G_k)[S \xrightarrow{\chi} \underline{k}] = \sum_{\substack{\pi\in\Pi(S)\\ \rho:\,\pi\to\underline{k}}} F[\rho] \times \prod_{B\in\pi} G_{\rho(B)}[B]. \tag{1.2.1}$$

The theory of k-sort species was further generalized by Méndez and Nava [56] to colored species over any set I: an I-*species* is a functor

$$M\colon \mathbb{B}/I \longrightarrow \mathbb{B}.$$

The *generating function* of an I-species is given by

$$M(x_i; i \in I) = \sum_{\mathbf{n}} |M[\mathbf{n}]| \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!},$$

where $\mathbf{n} = (n_i; i \in I)$ with $n_i = 0$ for almost all i, $\mathbf{n}! = \prod_{i\in I} n_i!$ and $\mathbf{x}^{\mathbf{n}} = \prod_{i\in I} x_i^{n_i}$. Most of the notions defined for species and k-sort species can be defined in this case. We are, however, mostly interested in one of its differences, that comes into play when I is a monoid. We now replace I by Y for the sake of notation coherence. Suppose Y is a locally finite monoid satisfying the left cancellation property and indivisibility of the identity. This means that Y has a well-defined divisibility relation $|$. For an object $\chi\colon S \to Y$, we write $i|\chi$ if $i|\chi(s)$ for all $s \in S$, and χ/i the map $\chi/i(s) := \chi(s)/i$. Given a Y-species M, we define a new Y-species $F_i M$ for each $i \in Y$ as follows:

$$F_i M[\chi] = \begin{cases} M[\chi/i] & \text{if } i|\chi, \\ \emptyset & \text{otherwise} \end{cases}$$

Given another Y-species, their Y-*plethysm* is defined as

$$(N \circledast M)[S \xrightarrow{\chi} Y] = \sum_{\substack{\pi\in\Pi(S)\\ \rho:\,\pi\to Y}} N[\rho] \times \prod_{B\in\pi} F_{\rho(B)} M[B]. \tag{1.2.2}$$

Notice the similarity between this definition and Equation (1.2.1). This substitution is compatible with the Y-plethysm of their generating functions:

$$(N \circledast M)(\mathbf{x}) = N\big(M(x_{ij}; j \in Y); i \in Y\big).$$

We treat this substitution in Section 7.4, but we only need Y to be locally finite. Notice that if $Y = (\mathbb{N}^+, \times)$ then Y-plethysm reduces to classical plethysm.

1.3 *The Faà di Bruno bialgebra*

Back to the definition of substitution of species (1.1.1), observe that the relevant information comes from a decomposition of S, for S a finite set. This decomposition is in fact the comultiplication of the isomorphism class of the interval $[\hat{0}, \hat{1}]$ of partitions of S in the incidence bialgebra of the poset of partitions. Here $\hat{0}$ denotes the finest partition (bottom element in the poset), and $\hat{1}$ denotes the coarsest partition (top element in the poset). Indeed,

$$\Delta\big([\hat{0},\hat{1}]\big) = \sum_{\hat{0}\leqslant\pi\leqslant\hat{1}} [\hat{0},\pi]\otimes[\pi,\hat{1}], \tag{1.3.1}$$

which becomes the same as (1.1.1) after expressing partitions as the disjoint union of their blocks. Disjoint union gives this coalgebra a structure of bialgebra, known as the Faà di Bruno bialgebra.

As already explained in the introduction, the *Faà di Bruno bialgebra* $\mathcal{F}$ is the free commutative algebra $\mathbb{Q}[A_1, A_2, \dots]$, where A_n is the dual map $A_n \in \mathbb{Q}[\![x]\!]^*$ defined by

$$A_n(F) = \frac{d^nF}{dx^n}.$$

Its comultiplication is defined to be dual to substitution of power series. That is

$$\Delta(A_n)(F,G) = A_n(G\circ F).$$

The comultiplication of A_n corresponds to the comultiplication of $[n]$ in the incidence coalgebra of partitions, and can be expressed with the *Bell polynomials* $B_{n,k}(A_1, A_2, \dots)$, which count the number of partitions of a set with n elements into k blocks [66]:

$$\Delta(A_n) = \sum_{k=1}^{n} B_{n,k}(A_1, A_2, \dots)\otimes A_k.$$

For example,

$$\Delta(A_4) = A_4\otimes A_1 + (4A_1A_3 + 3A_2^2)\otimes A_2 + 6A_1^2A_2\otimes A_3 + A_1^4\otimes A_4.$$

The category of partitions is equivalent to the category of surjections, so that $\mathcal{F}$ can be expressed from surjections too [43, §7.4], and in fact it looks simpler. The interval $[\hat{0},\hat{1}]$ corresponds to the surjection $S\to 1$, and its partitions correspond to 2-step factorizations (see (ii) of Lemma 3.4.1), so that Equation (1.3.1) corresponds to

$$\Delta(S\twoheadrightarrow 1) = \sum_{S\twoheadrightarrow R\twoheadrightarrow 1} (S\twoheadrightarrow R)\otimes(R\twoheadrightarrow 1).$$

The algebra structure is again given by disjoint union of sets. This sum is over isomorphism classes of factorizations $S\twoheadrightarrow R\twoheadrightarrow 1$, meaning up to isomorphism $R\xrightarrow{\sim}R'$ making the diagram commute. The precise statement that this comultiplication on surjections (or partitions) gives in fact the Faà di Bruno bialgebra fits very well into the theory of Segal spaces, where all the issues with isomorphism classes take care of themselves (see Remark 3.3.3).

If instead we took the monoidal category of finite ordered sets and monotone surjections we would obtain the noncommutative Faà di Bruno bialgebra [11, 25, 50] (see Section 7.3). It is defined as the free associative unital algebra $\mathbb{Q}\langle a_1, a_2, \dots\rangle$ on the linear maps defined by

$$a_n(F) = \frac{1}{n!}\frac{d^nF}{dx^n}.$$

In this case the coproduct runs over all possible monotone 2-step factorizations. For example,

$$\Delta(a_4) = a_4\otimes a_1 + (a_1a_3 + a_3a_1 + a_2^2)\otimes a_2 + (a_1^2a_2 + a_1a_2a_1 + a_2a_1^2)\otimes a_3 + a_1^4\otimes a_4.$$

Moreover, if we symmetrize the above monoidal category we obtain the ordinary Faà di Bruno bialgebra (see Section 7.3). This is defined as the free commutative algebra $\mathbb{Q}[a_1, a_2, \ldots]$. For example,

$$\Delta(a_4) = a_4 \otimes a_1 + (2a_1a_3 + a_2^2) \otimes a_2 + 3a_1^2a_2 \otimes a_3 + a_1^4 \otimes a_4.$$

There is also a k-variate Faà di Bruno bialgebra, in connection to substitution of k-sort species. Find its definition in Section 7.2. For now we content ourselves to express the decomposition of k-colored sets inherent to Equation (1.2.1) as a comultiplication of colored surjections:

$$\Delta\left(\begin{array}{ccc} S & \twoheadrightarrow & 1 \\ {\scriptstyle\chi}\downarrow & & \\ \underline{k} & & \end{array}\right) = \sum_{\substack{S \xrightarrow{\pi} R \twoheadrightarrow 1 \\ \chi\downarrow \quad \downarrow\rho \\ \underline{k} \quad \underline{k}}} \left(\begin{array}{ccc} S & \overset{\pi}{\twoheadrightarrow} & R \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\rho} \\ \underline{k} & & \underline{k} \end{array}\right) \otimes \left(\begin{array}{ccc} R & \twoheadrightarrow & 1 \\ {\scriptstyle\rho}\downarrow & & \\ \underline{k}. & & \end{array}\right).$$

As before, this sum is over isomorphism classes of factorizations. The analogy between this equation and (1.2.1) is not difficult to see: S, χ, π and ρ have the same meaning, and the blocks B are the elements of R.

The monomials of ordinary power series are indexed by natural numbers, which coincide with isomorphism classes of sets. However, the monomials of power series in infinitely many variables are indexed by isomorphism classes of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$. This is why Nava and Rota [61] developed the notion of partitional as a generalization of species, in order to give an interpretation of plethystic substitution analogous to the species interpretation of ordinary substitution. A sequence like λ could also represent the isomorphism class of a permutation, and in fact Bergeron [7] gave a similar interpretation but in terms of permutationals, rather than partitionals.

1.4 *Partitionals*

The content on partitions featuring in this section is due to Nava and Rota [61]. A *partition* π of a finite set E is a family of subsets of E, called *blocks*, such that every block of π is nonempty, the blocks are pairwise disjoint, and every element of E is contained in some block. Given two partitions π, σ of E we say that π is finer than σ (or σ is coarser than π), and write $\pi \leqslant \sigma$ if every block of π is a subset of some block of σ. In this case we define the *induced partition* $\sigma|\pi$ to be the partition on the set of blocks of π given by the blocks of σ. Also, the *restriction* of a partition π to a subset $B \subseteq E$ is the partition of B given by the intersections of B with the blocks of π; it is denoted by π_B.

Notice that the relation $\leqslant$ defines a partial order on the set $\Pi(E)$ of all partitions of E, and this order has a minimum, given by the partition with singleton blocks and denoted by $\hat{0}$, and a maximum, given by the partition with one block and denoted by $\hat{1}$. In particular, for every $\pi, \sigma \in \Pi(E)$ the supremum $\pi \vee \sigma$ and the infimum $\pi \wedge \sigma$ of π and σ exist in $(\Pi(E), \leqslant)$, and they are respectively called the *join* and the *meet*. For example if $E = \{1,2,3,4,5,6\}$, $\pi = \boxed{1\ 2 \mid 3\ 4\ 5 \mid 6}$ and $\sigma = \boxed{1\ 2\ 6 \mid 3\ 4 \mid 5}$, then

$$\pi \wedge \sigma = \boxed{1\ 2 \mid 3\ 4 \mid 5 \mid 6}, \quad \pi \vee \sigma = \boxed{1\ 2\ 6 \mid 3\ 4\ 5},$$

$$\sigma|(\pi\vee\sigma) = \boxed{\boxed{1\ \ 2\ \ 6} \ \Big|\ \boxed{3\ \ 4 \mid 5}}, \quad \text{and} \quad \pi_{\{1,3,4,6\}} = \boxed{1 \mid 3\ \ 4 \mid 6}.$$

It is easy to see that in fact

$$\pi \wedge \sigma = \{B \cap C \mid B \in \pi, C \in \sigma, B \cap C \neq \emptyset\}.$$

The join is, roughly speaking, the union of all blocks with common elements. However to give a precise and simple definition of the join it is preferable to view partitions as equivalence relations. This will also help us to introduce other notions later. It is clear that a partition π on a set E defines an equivalence relation $\sim_\pi$ on E, where two elements are related if they belong to the same block of π. In this setting the meet of two partitions π, σ is given by (for $p, q \in E$) $p \sim_{\pi\wedge\sigma} q$ if $p \sim_\pi q$ and $p \sim_\sigma q$, and their join is given by $p \sim_{\pi\vee\sigma} q$ if there exists a finite sequence $r_0, \ldots, r_n$ such that

$$p = r_0 \sim_1 r_1 \sim_2 \cdots \sim_n r_n = q, \tag{1.4.1}$$

where each relation $\sim_k$ is either $\sim_\pi$ or $\sim_\sigma$. We say that two partitions π, σ are *independent* if every block of π meets every block of σ. We say that they *commute* if, for every $p, q \in E$, we have that $p \sim_\pi r \sim_\sigma q$ for some $r \in E$ if and only if $p \sim_\sigma s \sim_\pi q$ for some $s \in E$. It is straightforward to see that two partitions are independent if and only if they commute and their join is $\hat{1}$. The following result says that commuting is the same as being blockwise independent.

Proposition 1.4.1 ([22])**.** *Let $\pi, \sigma \in \Pi(E)$. Then π and σ commute if and only if or every $B \in \pi \vee \sigma$ the restrictions π_B and σ_B are independent partitions of B.*

We are finally ready to define the most intricate and important definition regarding partitions in this section.

Definition 1.4.2 ([61])**.** Let σ be a partition of E. A pair (π, τ) of partitions of E is called a *transversal* of σ when

(i) $\pi \leqslant \sigma$,

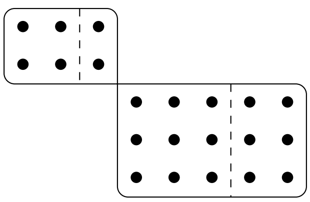
(ii) $\pi \wedge \tau = \hat{0}$,

(iii) π and τ commute, and

(iv) $\sigma \vee \tau = \pi \vee \tau$.

One can visualize this concept as follows: if (π, τ) is a transversal of (E, σ), then the elements of E can be arranged in a collection of matrices whose rows are the blocks of τ and whose columns are the blocks of π (because of (ii), (iii) and Proposition 1.4.1), such that the blocks of σ are unions of columns (by (i)) of the same matrix (by (iv)). The matrices are the blocks of $\sigma \vee \tau = \pi \vee \tau$. Let us see an example of a transversal:

Example 1.4.3. Consider a set E with 21 elements, and σ a partition with blocks of size $2, 4, 6$ and 9. Then the following arrangement represents a transversal of (E, σ):



(1.4.2)

The rows are the blocks of τ, the columns are the blocks of π, the dashed lines delimit the blocks of σ, and the two matrices are the blocks of $\sigma \vee \tau = \pi \vee \tau$. Observe that the matrix arrangement is not unique, since we could permute rows and some columns, but any arrangement compatible with σ defines a unique transversal, and any pair (π, τ) admitting such a representation is a transversal.

There is a category $\mathbb{P}$ whose objects are pairs (E, π), where E is a set and $\pi \in \Pi(E)$, and whose morphisms are defined as follows: if (F, σ) is another object of $\mathbb{P}$ a morphism

$$f\colon (E, \pi) \longrightarrow (F, \sigma)$$

is a bijection $f\colon E \to F$ which maps blocks of π to blocks of σ. Notice that $\mathbb{P}$ is in fact a groupoid, since all the morphisms are invertible. In this groupoid, the isomorphism class of a partition (E, π) can be described by the sequence of natural numbers

$$\lambda = (\lambda_1, \lambda_2, \dots), \quad \text{where } \lambda_k = \text{number of blocks of size } k \text{ of } (E, \pi).$$

Observe that $|E| = 1 \cdot \lambda_1 + 2 \cdot \lambda_2 + 3 \cdot \lambda_3 + \cdots$ and that the number of blocks of π is $|\pi| = \lambda_1 + \lambda_2 + \cdots$. Also, notice that the number of automorphisms of (E, π) is

$$\mathrm{aut}(\lambda) = 1!^{\lambda_1} \lambda_1! \cdot 2!^{\lambda_2} \lambda_2! \cdot 3!^{\lambda_3} \lambda_3! \cdots,$$

because an automorphism of π permutes the elements inside each block and permutes the blocks of the same size.

A *partitional* [61] is a functor $M\colon \mathbb{P} \to \mathbb{B}$ from the category of partitions $\mathbb{P}$ to the category of sets and bijections $\mathbb{B}$. The image $M[E, \pi]$ of (E, π) under M is the set of M-*structures*. By functoriality, the cardinality $|M[E, \pi]|$ depends only on the isomorphism class λ of the partition (E, π), and is denoted by $M[\lambda]$. Therefore we can define the *generating function* of M as

$$M(x_1, x_2, \dots) = \sum_{\lambda} \frac{M[\lambda]}{\mathrm{aut}\,(\lambda)} x_1^{\lambda_1} x_2^{\lambda_2} \cdots. \tag{1.4.3}$$

As in the case of species, several operations of generating functions can be lifted to the level of partitionals. The sum and the product are defined in a similar way as for species, and we will not do it here. The substitution, however, is more complex and involves the notion of transversal. Let M and R be two partitionals, then their *substitution* [61, §6] is defined as

$$(M \circ R)[E, \sigma] := \sum_{\substack{(\pi,\tau) \\ \text{transversal of } \sigma}} M\big[\tau, (\sigma \vee \tau)|\tau\big] \times \prod_{B \in \sigma \vee \tau} R[\pi_B, \sigma_B|\pi_B].$$

Substitution of partitionals is compatible with plethystic substitution of generating functions. That is, $(M \circ R)(x_1, x_2, \dots) = M(x_1, x_2, \dots) \circledast R(x_1, x_2, \dots)$ [61, §6]. Notice that, as before, this definition is also based on a decomposition of (E, σ),

$$\Delta(E, \sigma) := \sum_{\substack{(\pi,\tau) \\ \text{transversal of } \sigma}} \big(\tau, (\sigma \vee \tau)|\tau\big) \times \prod_{B \in \sigma \vee \tau} (\pi_B, \sigma_B|\pi_B). \tag{1.4.4}$$

which under isomorphism classes gives rise to a comultiplication in a bialgebra, the *plethystic bialgebra* [15, §3], introduced in this thesis (see section 3.2), and closely related to the incidence algebra of Nava [60].

In [60] Nava gave a variation of partitionals, called linear partitionals, as a set-theoretic counterpart to power series given in *exponential* form, that is, with denominator $\lambda! = \lambda_1!\lambda_2!\cdots$,

$$F(x_1, x_2, \dots) = \sum_{\lambda} \frac{f_\lambda}{\lambda!} x_1^{\lambda_1} x_2^{\lambda_2} \cdots . \tag{1.4.5}$$

Partitions and transversals are replaced by linear partitions and linear transversals: a *linear partition* E is pair $(\sigma, \leqslant)$ where σ is a partition of E and $\leqslant$ is a partial order on E consisting of linear orders for each block of σ. Observe that if σ is a linear partition of type λ, the number of automorphisms of σ that preserve the order is precisely $\lambda!$. A *linear transversal* of σ is a transversal (π, τ) such that every block of π is a segment of $\leqslant$.

As mentioned in the introduction, a similar model was given by Bergeron [7] in terms of permutationals, rather than partitionals. These are functors from the category of permutations to the category of sets and bijections, and their substitution relies on a notion of transversal compatible with a permutation. This was also later studied by Nava in [60], alongside partitionals and linear partitionals. In this approach $\mathrm{aut}(\lambda) = 1^{\lambda_1}\lambda_1 \cdot 2^{\lambda_2}\lambda_2! \cdots$, which is the number of automorphisms a permutation with λ_k cycles of length k.

1.5 *Littlewood plethysm*

Although we are interested in Pólya's notion of plethysm, it is appropriate, in order to show the scope of this subject, to give a brief review of Littlewood's plethysm [48] and their relation.

Denote by Λ the ring of symmetric functions [67]. That is, the subring of $\mathbb{Q}[\![x_1, x_2, \dots]\!]$ whose elements are invariant under permutations of the variables and whose monomials have bounded degree. It can also be described as the direct limit of Λ_n, the subring of symmetric polynomials of $\mathbb{Q}[x_1, x_2, \dots]$, with the inclusion $\phi_n \colon \Lambda_n \longrightarrow \Lambda_{n+1}$ that amounts to adding all monomials containing the new variable obtained by symmetry.

The power sum symmetric functions,

$$p_k = \sum_{i \geqslant 1} x_i^k,$$

form an algebra basis of Λ. This means that

$$\Lambda \simeq \mathbb{Q}[p_1, p_2, \dots].$$

The following result characterizes Littlewood's plethysm of symmetric functions.

Proposition 1.5.1 (cf. [52, 67]). *There is a unique binary operation* $\bullet : \Lambda \times \Lambda \to \Lambda$*, called* plethysm*, satisfying the following properties:*

(i) For all $n, m \geqslant 1$, $p_n \bullet p_m = p_{nm}$.

(ii) For all $n \geqslant 1$*, the map* $p_n \bullet \_ : \Lambda \to \Lambda$ *is a* $\mathbb{Q}$*-algebra homomorphism.*

(iii) For all $f \in \Lambda$*, the map* $\_ \bullet f : \Lambda \to \Lambda$ *is a* $\mathbb{Q}$*-algebra homomorphism.*

For example, it is not difficult to see that for all $f \in \Lambda$

$$p_n \bullet f = f \bullet p_n = f(x_1^n, x_2^n, \dots).$$

Notice that the plethysm of symmetric functions is precisely the same as the Pólya plethysm in the ring $\mathbb{Q}[p_1, p_2, \dots]$. But of course, we defined Pólya plethysm in the ring of power series, rather than the polynomial ring.

Littlewood's plethysm was first defined in connection with the representation theory of the general linear groups. More concretely, the characters of polynomial representations of the general linear groups of finite vector spaces can be identified with symmetric functions (cf. [67]). Consider finite dimensional vector spaces V, W and Y, and polynomial representations $\phi\colon GL(V) \to GL(W)$ and $\psi\colon GL(W) \to GL(Y)$. It turns out that their characters satisfy

$$\mathrm{char}(\psi \circ \phi) = \mathrm{char}(\psi) \bullet \mathrm{char}(\phi).$$

Plethysm may be also naturally understood from the point of view of λ-rings [52]. In fact, the ring Λ is the free λ-ring on one generator, e_1, where the operations λ^n are defined by plethysm with the elementary symmetric functions e_n:

$$\lambda^n(f) = e_n \bullet f.$$

Also, composition of operations $\lambda^n \lambda^m$ is given by plethysm $e_n \bullet e_m$. As a λ-ring, the plethysm with the power sum symmetric function p_n is precisely the nth Adams operation.

2 Segal groupoids and incidence coalgebras

The theory of incidence coalgebras for Segal groupoids and, more generally, for decomposition spaces, was developed by Gálvez, Kock and Tonks [31–33] in the context of ∞-groupoids. We stick however to 1-groupoids, which is enough for our combinatorial applications.

2.1 Groupoids

A *groupoid* is a small category whose arrows are all isomorphisms. For instance, a set is a groupoid with only identity arrows, and a group is a groupoid with only one object. We denote by $\mathrm{Iso}(x,y)$ the set of arrows between two elements x,y of a groupoid X. A *map* of groupoids is just a functor, and a *homotopy* between two maps is a natural transformation. Since every arrow in a groupoid is invertible, every homotopy is in fact a natural isomorphism. We denote by **Grpd** the 2-category of groupoids.

A map $f : X \to Y$ is a *homotopy equivalence* if it is an equivalence of categories, that is, there exists a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. As for categories, a map is a homotopy equivalence if it is essentially surjective and fully faithful. Two groupoids X and Y are called *homotopy equivalent* if there exists an equivalence between them. This equivalence relation is denoted $\simeq$, and it is the appropriate notion of sameness for groupoids. All the notions involved in this section are invariant under homotopy equivalence.

We may henceforth say just equivalence, instead of homotopy equivalence, as well as pullback, fiber, etc., for the homotopy notions.

A groupoid X is called *discrete* if it is homotopy equivalent to a set. In other words, if $\mathrm{Aut}(x) := \mathrm{Hom}_X(x,x)$ is trivial for every $x \in X$. It is *connected* if $\mathrm{Hom}_X(x,y)$ is nonempty for all $x,y \in X$. A maximal connected subgroupoid of X is termed a *component* of X, and the set of components is denoted $\pi_0(X)$. Also, the group $\mathrm{Aut}(x)$ is often denoted $\pi_1(X,x)$ or just $\pi_1(x)$. Finally, if X is both discrete and connected then it is called *contractible*. This means homotopy equivalent to the terminal groupoid 1.

A *homotopy pullback* is a homotopy limit of a functor $F : \{\bullet \to \bullet \leftarrow \bullet\} \to \mathbf{Grpd}$. More concretely, given groupoid maps $X \xrightarrow{f} B \xleftarrow{g} Y$, their homotopy pullback is a square

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner\ \nearrow\psi"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & B \end{tikzcd} \tag{2.1.1}$$

which is universal in a homotopy sense. Usually we will not draw the natural transformation, but all squares are understood to commute up to homotopy.

By the homotopy universal property, the homotopy pullback is only defined up to homotopy equivalence. We shall exploit two specific models for it. The first is the homotopy fiber product, which we proceed to define; the second is the strict pullback in certain fibrant situations (see Lemma 2.1.5 below).

The *homotopy fiber product* of $X \xrightarrow{f} B \xleftarrow{g} Y$ is the groupoid $X \times_B Y$ defined as follows: its objects are triples (x, y, ϕ) with $x \in X$, $y \in Y$ and $\phi\colon f(x) \to g(y)$ an arrow of B. Its arrows are pairs $(\alpha, \beta)\colon (x, y, \phi) \to (x', y', \phi')$ consisting of two arrows $\alpha\colon x \to x'$ and $\beta\colon y \to y'$ satisfying $g(\beta) \circ \phi = \phi' \circ f(\alpha)\colon f(x) \to g(y')$. The maps q and p (2.1.1) are given by the projections of x and y, while the natural isomorphism ψ is given by the third components ϕ. It is a standard fact that this homotopy fibre product is a model for the homotopy pullback, i.e. satisfies the 2-categorical universal property.

Given a map of groupoids $X \xrightarrow{f} B$ and an object $b \in B$, the *homotopy fiber* of b along f, or fiber of f over b, is the groupoid X_b obtained by taking the homotopy pullback of the diagram $X \xrightarrow{f} B \xleftarrow{\ulcorner b \urcorner} 1$. In the rest of the chapter all the pullbacks and fibers are homotopy.

Example 2.1.1. Consider a groupoid X and two objects $x, y \in X$. There is a pullback square

$$\begin{tikzcd} \mathrm{Iso}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\mathrm{diag}"] \\ 1 \arrow[r, "\ulcorner (x,y) \urcorner"'] & X \times X. \end{tikzcd}$$

Let us see this. The objects of this pullback can be given by triples (z, ϕ_x, ϕ_y), where $z \in X$ and $z \xrightarrow{\phi_x} x$ and $z \xrightarrow{\phi_y} y$ are arrows in X. An arrow $(z, \phi_x, \phi_y) \to (z', \phi'_x, \phi'_y)$ is given by an arrow $z \xrightarrow{f} z'$ of X such that the following diagram commutes:

$$\begin{tikzcd} z \arrow[r, "\phi_x"] \arrow[d, "\phi_y"'] \arrow[dr, "f"] & x \\ y & z'. \arrow[l, "\phi'_y"] \arrow[u, "\phi'_x"'] \end{tikzcd}$$

Clearly there is at most one arrow between any two objects. One of these squares is given by

$$\begin{tikzcd} z \arrow[rr, "\phi_x"] \arrow[d, "\phi_y"'] \arrow[drr, "\phi_x"] & & x \\ y & z \arrow[l, "\phi_y"] & x, \arrow[l, "\phi_x^{-1}"] \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

which shows that the connected components are given by the arrows $x \to y$, as we wanted to see.

Lemma 2.1.2. *Consider a diagram of groupoids*

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d] \\ Y_2 \arrow[r] & Y_1 \arrow[r] & Y_0 \end{tikzcd}$$

such that the right square is a homotopy pullback. Then the left square is a homotopy pullback if and only if the outer square is a homotopy pullback.

A map $p \colon E \to B$ of groupoids is a *fibration* if for any object $e \in E$ and arrow $\phi : b \to p(e)$ there exists an arrow $\psi : e' \to e$ such that $p(\psi) = \phi$. If the arrow ψ is unique then p is called a *discrete fibration*. For example, if B is discrete and p is surjective on objects then p is a fibration.

Remark 2.1.3. A functor $p \colon E \to B$ between categories is an *isofibration* if for any object $e \in E$ and isomorphism $\phi : b \to p(e)$ there exists an isomorphism $\psi : e' \to e$ such that $p(\psi) = \phi$. Hence a fibration of groupoids is also an isofibration. We will use this in Section 3.1.

It is important to note that these notions of fibrations are not homotopy invariant notions: a map of groupoids equivalent to a fibration is not in general a fibration. Nevertheless, naturally defined maps of groupoids are surprisingly often fibrations. In particular, the projections from the homotopy fibre product $X \leftarrow X \times_B Y \to Y$ are always fibrations.

The great advantage of fibrations is that they allow to work with strict pullbacks as a model for the homotopy pullback, as expressed in the following result.

Lemma 2.1.4. *The strict pullback along a fibration is equivalent to the homotopy pullback.*

To ensure a good supply of fibrations, the following result is useful.

Lemma 2.1.5. *Fibrations and discrete fibrations are preserved by strict pullbacks.*

Whenever a group G acts on a groupoid X, the *homotopy quotient* $X /\!\!/ G$ [2] is defined as the groupoid whose objects are the same those of X, and whose arrows $x \to y$ are pairs (g, ϕ) such that $g \in G$ and $\phi : gx \to y$ an arrow in X. Intuitively, it is the smallest groupoid containing the arrows of X and an arrow $x \to gx$ for every $x \in X$ and $g \in G$.

Lemma 2.1.6. *Given a map of groupoids $X \xrightarrow{f} B$, there is a canonical equivalence between X and the* homotopy sum *of its fibers over f,*

$$X \simeq \int^{b} X_b := \sum_{\pi_0 B} X_b /\!\!/ \operatorname{Aut}(b).$$

Let I be a groupoid. The *weak slice* $\mathbf{Grpd}_{/I}$ is the category whose objects are maps of groupoids $X \to I$ and whose arrows are triangles

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Rightarrow"] & Y \arrow[dl] \\ & I & \end{tikzcd}$$

If 1 is the terminal groupoid, then $\mathbf{Grpd}_{/1} \simeq \mathbf{Grpd}$. Be aware that in Chapter 4 we also use the notion of strict slice category $\mathbf{Grpd}/I$, whose objects are again maps $X \to I$ but whose arrows are strictly commutative triangles.

A map of groupoids $f : X \to Y$ defines a functor

$$f^* : \mathbf{Grpd}_{/Y} \longrightarrow \mathbf{Grpd}_{/X}$$

by taking pullback along f. It also defines a functor

$$f_! : \mathbf{Grpd}_{/X} \longrightarrow \mathbf{Grpd}_{/Y}$$

given by post-composition with f.

A *span* is a pair of groupoid maps with common domain $I \xleftarrow{f} X \xrightarrow{g} J$. A span thus induces a map between the slices given by pullback and post-composition,

$$\mathbf{Grpd}_{/I} \xrightarrow{f^*} \mathbf{Grpd}_{/X} \xrightarrow{g_!} \mathbf{Grpd}_{/J}.$$

A functor is *linear* if it preserves homotopy sums. The linear functors are precisely those that are given by spans [31]. The following Beck–Chevalley rule holds: for any pullback square

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & B \end{tikzcd}$$

the functors $p_! q^*, g^* f_! : \mathbf{Grpd}_{/X} \longrightarrow \mathbf{Grpd}_{/Y}$ are homotopy equivalent [31]. By the Beck–Chevalley rule composition of linear functors is linear. We denote by **LIN** the monoidal 2-category of all slice categories $\mathbf{Grpd}_{/X}$ and linear functors between them, with the tensor product induced by the cartesian product,

$$\mathbf{Grpd}_{/X} \otimes \mathbf{Grpd}_{/Y} := \mathbf{Grpd}_{X \times Y}.$$

The neutral object is $\mathbf{Grpd} \simeq \mathbf{Grpd}_{/1}$. In homotopy linear algebra [31], the category $\mathbf{Grpd}_{/X}$ plays the role of the vector space with basis X, and the linear functors play the role of linear maps. Moreover, for a linear functor induced by the span $I \leftarrow X \rightarrow J$, the groupoid X plays the role of the matrix defining the linear map.

2.2 *Segal groupoids*

We denote by $\mathbb{\Delta}$ the *simplex category*, whose objects are finite nonempty standard ordinals

$$[n] = \{0 < 1 < \cdots < n\}$$

and whose morphisms are order preserving maps between them. These maps are generated by the *coface* maps $\partial^i \colon [n-1] \to [n]$, which skip i, and the *codegeneracy* maps $\sigma^i \colon [n+1] \to [n]$, which repeats i. The obvious relations between this maps, such as $\partial^i \partial^j = \partial^{j-1} \partial^i$ for $i < j$, are called *cosimplicial identities*.

A simplicial groupoid is a pseudo-functor $X \colon \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$. The image of $[n]$ is denoted by X_n and called the groupoid of n-simplices. The images of ∂^i and σ^i are denoted d_i and s_i and called face and degeneracy maps respectively. Explicitly, a simplicial groupoid is a sequence of groupoids $(X_n)_{n \geqslant 0}$ together with morphisms $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$ for $0 \leqslant i \leqslant n$, satisfying the *simplicial identities*, induced by the cosimplicial identities:

$$d_i s_i \simeq d_{i+1} s_i = 1, \quad d_i d_j \simeq d_{j-1} d_i, \quad d_{j+1} s_i \simeq s_i d_j,$$
$$d_i s_j \simeq s_{j-1} d_i, \quad s_j s_i \simeq s_i s_{j-1}, \qquad (i < j).$$

The simplicial identities are not equalities but coherent isomorphisms.

A simplicial groupoid $X\colon \mathbb{\Delta}^{op} \longrightarrow \mathbf{Grpd}$ is a *Segal space* [32, §2.9, Lemma 2.10] if the following square is a pullback for all $n > 0$:

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_0"] \arrow[d, "d_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_n"] \\ X_n \arrow[r, "d_0"'] & X_{n-1}. \end{tikzcd} \tag{2.2.1}$$

Segal spaces arise prominently through the fat nerve construction: the *fat nerve* of a category $\mathcal{C}$ is the simplicial groupoid $X = N\mathcal{C}$ with $X_n = \mathrm{Fun}([n], \mathcal{C})^{\simeq}$, the groupoid of functors $[n] \to \mathcal{C}$. For instance, the objects of the groupoid X_3 are chains of three composable arrows of $\mathcal{C}$,

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot,$$

and the arrows of X_3 are diagrams

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\wr"] & \cdot \arrow[r] \arrow[d, "\wr"] & \cdot \arrow[r] \arrow[d, "\wr"] & \cdot \arrow[d, "\wr"] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot, \end{tikzcd}$$

with all the vertical arrows invertible.

In this case the outer face maps are fibrations, so that by Lemma 2.1.5 the pullbacks (2.2.1) are strict. This implies that all the simplices are strictly determined by X_0 and X_1, respectively the objects and arrows of $\mathcal{C}$, and the inner face maps are given by composition of arrows in $\mathcal{C}$. In particular, the groupoid X_2 is equivalent to $X_1 \times_{X_0} X_1$, the groupoid of composable pairs arrows of $\mathcal{C}$, and $d_1 : X_2 \to X_1$ is composition in $\mathcal{C}$. The map $d_0 : X_2 \to X_1$ assigns to a composable pair the second arrow, and $d_2 : X_2 \to X_1$ assigns to a composable pair the first arrow.

In the general case, X_n is determined from X_0 and X_1 only up to equivalence, but one may still think of it as a "category" object whose composition is defined only up to equivalence.

Remark 2.2.1. Despite the Segal conditions (2.2.1) require the squares to be homotopy pullbacks, if the top or bottom face maps are fibrations, the ordinary pullbacks are also homotopy pullbacks, by Lemma 2.1.5. In the present work, homotopy pullbacks mostly arise n this way.

2.3 *Incidence coalgebras*

Let X be a simplicial groupoid. The spans

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1, \qquad\qquad X_1 \xleftarrow{s_0} X_0 \xrightarrow{t} 1,$$

define two functors

$$\begin{array}{rcl} \Delta\colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd}_{/X_1 \times X_1} \\ S \xrightarrow{s} X_1 & \longmapsto & (d_2, d_0)_! \circ d_1^*(s), \end{array} \qquad \begin{array}{rcl} \epsilon\colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd} \\ S \xrightarrow{s} X_1 & \longmapsto & t_! \circ s_0^*(s)\ . \end{array} \tag{2.3.1}$$

Recall that upperstar is homotopy pullback and lowershriek is postcomposition.

Theorem 2.3.1 ([32]). *If X is a Segal groupoid then $\mathbf{Grpd}_{/X_1}$ has the structure of strong homotopy comonoid in the symmetric monoidal category* **LIN**, *with the comultiplication and counit defined by the spans above.*

This comonoid is called the *incidence coalgebra* of X. The original statement of the theorem involves decomposition spaces, which are a special kind of simplicial ∞-groupoids encoding the ability to decompose. Segal groupoids are a particular case of decomposition spaces [32, Proposition 3.7] and the only one we shall need. The coassociativity square looks like this:

$$\begin{tikzcd}
\mathbf{Grpd}_{/X_1} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathbf{Grpd}_{/X_1 \times X_1} \arrow[d, "\Delta \otimes \mathrm{id}"] \\
\mathbf{Grpd}_{/X_1 \times X_1} \arrow[r, "\mathrm{id} \otimes \Delta"'] & \mathbf{Grpd}_{/X_1 \times X_1 \times X_1}.
\end{tikzcd}$$

It commutes up to equivalence, and it is a consequence of Beck–Chevalley and some of the Segal conditions.

The morphisms of Segal spaces that induce coalgebra homomorphisms are the so-called CULF functors [32, §4], standing for conservative and unique-lifting-of-factorisations (ULF). A functor $F : X \to Y$ of simplicial groupoids is *conservative* if it is cartesian with respect to codegeneracy maps,

$$\begin{tikzcd}
X_n \arrow[r, "s_i"] \arrow[d, "F"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+1} \arrow[d, "F"] \\
Y_n \arrow[r, "s_i"'] & Y_{n+1},
\end{tikzcd}$$

and it is *ULF* if it is cartesian with respect to inner coface maps

$$\begin{tikzcd}
X_{n+1} \arrow[d, "F"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+2} \arrow[l, "d_i"'] \arrow[d, "F"] \\
Y_{n+1} & Y_{n+2}, \arrow[l, "d_i"]
\end{tikzcd} \qquad (0 \leqslant i \leqslant n).$$

We say that F is *CULF* if it is both conservative and ULF. This means that it is cartesian with respect to any generic (i.e. induced by an endpoint-preserving map of $\mathbb{\Delta}$). Observe that F induces a linear functor

$$F_{1!} : \mathbf{Grpd}_{/X_1} \longrightarrow \mathbf{Grpd}_{/Y_1}.$$

Lemma 2.3.2. *If F is CULF, then $F_{1!}$ is a coalgebra homomorphism, meaning that it preserves the comultiplication and the counit up to coherent homotopy:*

$$(F_{1!} \otimes F_{1!})\Delta_X \simeq \Delta_Y F_{1!}, \quad \epsilon_X \simeq \epsilon_Y F_{1!}.$$

A Segal space X is *monoidal* if it comes equipped with an associative unital monoid structure given by CULF functors $m : X \times X \to X$ and $e : 1 \to X$.

Proposition 2.3.3 ([32]). *If* X *is a monoidal Segal space then* $\mathbf{Grpd}_{/X_1}$ *is naturally a bialgebra, termed its* incidence bialgebra*. Monoidal CULF functors induce bialgebra homomorphisms.*

For example the fat nerve of a monoidal extensive category is a monoidal Segal space. Recall that a category $\mathcal{C}$ is monoidal extensive if it is monoidal $(\mathcal{C}, +, 0)$ and the natural functors $\mathcal{C}_{/A} \times \mathcal{C}_{/B} \to \mathcal{C}_{/A+B}$ and $\mathcal{C}_{/0} \to 1$ are equivalences.

If X is monoidal the multiplication of its incidence bialgebra is given by

$$\begin{array}{rcccc} \odot\colon \mathbf{Grpd}_{/X_1} \otimes \mathbf{Grpd}_{/X_1} & \xrightarrow{\sim} & \mathbf{Grpd}_{/X_1 \times X_1} & \xrightarrow{+_!} & \mathbf{Grpd}_{/X_1} \\ (G \to X_1) \otimes (H \to X_1) & \longmapsto & G \times H \to X_1 \times X_1 & \longmapsto & G \times H \to X_1. \end{array}$$

2.4 *Homotopy cardinality*

A groupoid X is *finite* if $\pi_0(X)$ is a finite set and $\pi_1(x) = \mathrm{Aut}(x)$ is a finite group for every point x. If only the latter is satisfied then it is called *locally finite*. A map of groupoids is called finite when all its fibers are finite. The *homotopy cardinality* [2], [31, §3] of a finite groupoid X is defined as

$$|X| := \sum_{x \in \pi_0 X} \frac{1}{|\,\mathrm{Aut}(x)|} \in \mathbb{Q}.$$

A span $I \xleftarrow{f} X \to J$, and the corresponding linear functor $\mathbf{Grpd}_{/I} \to \mathbf{Grpd}_{/J}$, are termed *finite* when f is finite. We denote by $\mathbf{grpd}$ the 2-category of finite groupoids.

Proposition 2.4.1 ([31]). *Let* I, J *and* X *be locally finite groupoids and* $\mathbf{Grpd}_{/I} \to \mathbf{Grpd}_{/J}$ *a finite span. Then the induced linear functor* $\mathbf{Grpd}_{/I} \to \mathbf{Grpd}_{/J}$ *restricts to a functor*

$$\mathbf{grpd}_{/I} \to \mathbf{grpd}_{/J}. \tag{2.4.1}$$

To a slice category $\mathbf{grpd}_{/X}$, with X locally finite, we associate the free vector space $\mathbb{Q}_{\pi_0 X}$ spanned by the connected components of X, with canonical basis $\{\delta_x\}_{x \in \pi_0 X}$. A finite map $p : Y \to X$ is associated to the vector given by its *homotopy cardinality*:

$$|p| := \sum_{x \in \pi_0 X} \frac{|Y_x|}{|\,\mathrm{Aut}(x)|} \delta_x,$$

In this sum, Y_x. A simple computation shows that $|1 \xrightarrow{\ulcorner x \urcorner} X| = \delta_x$. A finite linear functor as in 2.4.1 defines a linear map

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0 X} & \longrightarrow & \mathbb{Q}_{\pi_0 Y} \\ |Z \xrightarrow{z} X| & \longmapsto & |g_! f^*(z)|, \\ \delta_x & \longmapsto & \displaystyle\sum_{y \in \pi_0 Y} \frac{|M_{x,y}|}{\pi_1(y)}, \end{array}$$

where $M_{x,y}$ is the homotopy fiber along the map $M \xrightarrow{(f,g)} X \times Y$.

A Segal space X is *locally finite* [33, §7] if X_1 is a locally finite groupoid and both $s_0\colon X_0 \to X_1$ and $d_1\colon X_2 \to X_1$ are finite maps. In this case one can take homotopy cardinality of the incidence coalgebra (2.3.1) to get the *numerical incidence coalgebra* (cf. [33, §7]):

$$\begin{array}{rclcrcl} \Delta\colon \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} & \qquad & \epsilon\colon \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q} \\ \left|S \xrightarrow{s} X_1\right| & \longmapsto & \left|(d_2, d_0)_! \circ d_1^*(s)\right| & & \left|S \xrightarrow{s} X_1\right| & \longrightarrow & \left|t_! \circ s_0^*(s)\right|. \end{array}$$

Moreover, if X is monoidal then $\mathbb{Q}_{\pi_0 X_1}$ acquires a bialgebra structure with the product $\cdot = |\odot|$. In particular, if we denote by $+$ the monoidal product in X, then $\delta_a \cdot \delta_b = \delta_{a+b}$ for any $|1 \xrightarrow{\ulcorner a \urcorner} X_1|$ and $|1 \xrightarrow{\ulcorner b \urcorner} X_1|$. Furthermore, if the monoidal structure of X is symmetric, then the bialgebra $\mathbb{Q}_{\pi_0 X_1}$ is commutative. Note that in the special case where X is the nerve of a poset, this construction becomes the classical incidence coalgebra [64, 65].

The following result gives a closed formula for the computation of the comultiplication when X is a Segal space.

Lemma 2.4.2 ([15]). *Let X be a Segal space. Then for f in X_1 we have*

$$\Delta(\delta_f) = \sum_{b \in \pi_0 X_1} \sum_{a \in \pi_0 X_1} \frac{|\mathrm{Iso}(d_0 a, d_1 b)_f|}{|\mathrm{Aut}(b)||\mathrm{Aut}(a)|} \delta_a \otimes \delta_b, \tag{2.4.2}$$

where $\mathrm{Iso}(d_0 a, d_1 b)$ *is the set of morphisms from* $d_0 a$ *to* $d_1 b$ *and* $\mathrm{Iso}(d_0 a, d_1 b)_f$ *is its homotopy fiber along* d_1.

Proof. It is enough to see that the following square is a pullback:

$$\begin{tikzcd} \mathrm{Iso}(d_0 a, d_1 b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_2 \arrow[d, "(d_2, d_0)"] \\ 1 \arrow[r, "\ulcorner a, b \urcorner"'] & X_1 \times X_1 \end{tikzcd}$$

Consider the following diagram,

$$\begin{tikzcd} \mathrm{Iso}(d_0 a, d_1 b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_2 \arrow[r, "d_0 \circ d_2"] \arrow[d, "(d_2, d_0)"'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "\mathrm{diag}"] \\ 1 \arrow[r, "\ulcorner (a, b) \urcorner"'] & X_1 \times X_1 \arrow[r, "d_0 \times d_1"'] & X_0 \times X_0. \end{tikzcd}$$

That the right square is a pullback follows from the Segal square (2.2.1) for $n = 1$ by general properties of the pullback. The outer square is precisely the square of Example 2.1.1. As a consequence, by Lemma 2.1.2, the left square is a pullback too. Equation (2.4.2) now follows from the definitions of Δ and homotopy cardinality. Notice that the fiber of f is taken along the map

$$\mathrm{Iso}(d_0 a, d_1 b) \longrightarrow X_2 \xrightarrow{d_1} X_1.$$

Moreover, since $\mathrm{Iso}(d_0 a, d_1 b)$ is discrete we have that

$$\mathrm{Iso}(d_0 a, d_1 b)_f \simeq \mathrm{Aut}(f) \times \{\phi \in \mathrm{Iso}(d_0 a, d_1 b) \mid d_1(\phi)\}.$$

□

Remark 2.4.3. A wrong version of the result was originally published in [33]. The mistake was found and corrected as a result of [15], and the correct statement and proof appeared in [34].

3

A Segal groupoid for classical plethysm

In this chapter we give a combinatorial interpretation for classical plethysm. This interpretation is modeled on Joyal's construction of the Faà di Bruno bialgebra from the category of surjections [43], as explained in the introduction. Instead of the fat nerve, we use the T-construction, a formal construction that we introduce next. In Section 3.2 we develop the plethystic bialgebra $\mathcal{P}$ to derive a formula for extracting the comultiplication of the elements of its basis (Proposition 3.2.3). In Ssection 3.3 we finally apply the T-construction to the category $\mathbf{S}$ of finite sets and surjections to obtain a Segal groupoid whose incidence bialgebra is isomorphic to $\mathcal{P}$ (Theorem 3.3.1). In Section 3.4 we show that the notion of transversal (see Section 1.4) is encoded in the groupoid of 2-simplices of $\mathrm{T}\mathbf{S}$. We end the chapter by deriving a Faà di Bruno formula for the "connected Green function" in the incidence bialgebra of $\mathrm{T}\mathbf{S}$.

3.1 *The* T*-construction*

The present construction is inspired by Lurie's account [51, §1.2.2] of the Waldhausen S-construction [72]. The use of transversal complexes (which we introduce here) instead of "gap complexes", on the other hand, is reminiscent of Quillen's Q-construction [63], which is the "twisted arrow category" (edgewise subdivision) of the S-construction.

The T-construction can be applied to any category which possesses a class of *distinguished squares* satisfying the following axioms inspired by the properties of pullbacks:

(i) The identity squares are distinguished and the class of distinguished squares is closed under isomorphisms. Equivalently, the squares

$$\begin{tikzcd} \cdot \arrow[r, "\sim"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "\sim"'] & \cdot \end{tikzcd}$$

are distinguished.

(ii) Given

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd},$$

if the right and the left squares are distinguished then the outer rectangle is distinguished.

(iii) For any two maps $b \xrightarrow{p} a, c \xrightarrow{q} a$ there are maps $d \xrightarrow{p'} c, d \xrightarrow{q'} b$ making a distinguished square. Moreover, for any other two maps $d' \xrightarrow{p''} c, d' \xrightarrow{q''} b$ making a distinguished square there is a unique isomorphism $d' \xrightarrow{\phi} d$ making the diagram commute,

$$\begin{tikzcd} d' \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "p''"] \arrow[ddr, bend right, "q''"'] & & \\ & d \arrow[r, "p'"] \arrow[d, "q'"'] & c \arrow[d, "q"] \\ & b \arrow[r, "p"'] & a. \end{tikzcd}$$

For instance in any category with pullbacks, these form a class of distinguished squares. Moreover, in any subcategory of a category with pullbacks whose arrows are stable under pullbacks these form again a class of distinguished squares. This is our motivating example, since surjections are stable under pullbacks in the category of sets, although the category of finite sets and surjections does not have pullbacks. Distinguished squares are indicated with the same symbol as pullbacks.

Let I be a linearly ordered set. Consider the category $\mathrm{Tw}^+(I)$, whose objects are pairs $i \leqslant j$ in I and whose morphisms are relations $(i,j) \leqslant (i',j')$ whenever $i' \leqslant i$ and $j \leqslant j'$ or whenever $i = j \leqslant i' = j'$. This construction can be viewed as the twisted arrow category of I together with arrows between the identities of I.

For example, for $I = [n]$ the objects and arrows of $\mathrm{Tw}^+([n])$ can be pictured as (picturing $n = 3$)

$$\begin{tikzcd}[column sep=small] & & & 03 \arrow[dl] \arrow[dr] & & & \\ & & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & \\ & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & 23 \arrow[dl] \arrow[dr] & \\ 00 \arrow[rr] & & 11 \arrow[rr] & & 22 \arrow[rr] & & 33. \end{tikzcd}$$

The set of categories $\mathrm{Tw}^+([n])$ for all n form a cosimplicial object $\Delta \to \mathbf{Cat}$ given by $[n] \mapsto \mathrm{Tw}^+([n])$. The face map $d_k \colon \mathrm{Tw}^+([n-1]) \to \mathrm{Tw}^+([n])$ is the obvious induced map $d_k(i,j) = (d_k(i), d_k(j))$, and similarly for the degeneracy maps.

Let $\mathcal{C}$ be a category with distinguished squares. A functor $F\colon \mathrm{Tw}^+(I) \to \mathcal{C}$ is called *transversal complex* if for every $i \leqslant j \leqslant k \leqslant l$ the associated diagram

$$\begin{tikzcd} F(i,l) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(j,l) \arrow[d] \\ F(i,k) \arrow[r] & F(j,k) \end{tikzcd} \tag{3.1.1}$$

is a distinguished square, as indicated in the picture. The word transversal comes of course from the Nava–Rota notion (see section 1.4). Let $\mathrm{Trans}(I, \mathcal{C})$ be the full subgroupoid of $\mathrm{Fun}(\mathrm{Tw}^+(I), \mathcal{C})^{\simeq}$ containing only the transversal complexes. Then the assignment

$$[n] \longmapsto \mathrm{Trans}([n], \mathcal{C})$$

defines a simplicial groupoid $T\mathcal{C}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd}$. The groupoid $T_n\mathcal{C} = \mathrm{Trans}([n], \mathcal{C})$ has as objects diagrams in $\mathcal{C}$ (picturing $n = 4$)

$$\begin{tikzcd}[column sep=small]
 & & & & t_{04} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & & & \\
 & & & t_{03} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & t_{14} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & & \\
 & & t_{02} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & t_{13} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & t_{24} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
 & t_{01} \arrow[dl] \arrow[dr] & & t_{12} \arrow[dl] \arrow[dr] & & t_{23} \arrow[dl] \arrow[dr] & & t_{34} \arrow[dl] \arrow[dr] & \\
t_{00} \arrow[rr] & & t_{11} \arrow[rr] & & t_{22} \arrow[rr] & & t_{33} \arrow[rr] & & t_{44}.
\end{tikzcd}$$

The morphisms of such diagrams are levelwise isomorphisms $t_{ij} \xrightarrow{\sim} t'_{ij}$ making the diagram commute. In particular $T_0\mathcal{C} = \mathcal{C}^{\simeq}$. The face map d_i removes all the objects containing an i index. The degeneracy map s_i repeats the ith diagonals. For example

$$s_1\left(\begin{tikzcd}[column sep=small]
 & & t_{02} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
 & t_{01} \arrow[dl] \arrow[dr] & & t_{12} \arrow[dl] \arrow[dr] & \\
t_{00} \arrow[rr] & & t_{11} \arrow[rr] & & t_{22},
\end{tikzcd}\right) = \begin{tikzcd}[column sep=small]
 & & & t_{02} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & & \\
 & & t_{01} \arrow[dl, equal] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & t_{12} \arrow[dl] \arrow[dr, equal] \arrow[dd, phantom, "\vee" very near start] & & \\
 & t_{01} \arrow[dl] \arrow[dr] & & t_{11} \arrow[dl, equal] \arrow[dr, equal] & & t_{12} \arrow[dl] \arrow[dr] & \\
t_{00} \arrow[rr] & & t_{11} \arrow[rr, equal] & & t_{11} \arrow[rr] & & t_{22}.
\end{tikzcd}$$

Remark 3.1.1. The Quillen Q-construction of an abelian category $\mathcal{A}$, denoted $Q\mathcal{A}$, can be described in a similar way [63]. It is the simplicial groupoid such that $Q_n\mathcal{A}$ is the full subgroupoid of $\mathrm{Fun}(\mathrm{Tw}([n]), \mathcal{A})^{\simeq}$ consisting of functors F satisfying the same pullback condition as the transversal complexes (8.1.1) and the additional conditions that for all $i \leqslant j \leqslant k$ the map $F(i,k) \twoheadrightarrow F(i,j)$ is an epimorphism and the map $F(i,k) \rightarrowtail F(j,k)$ is a monomorphism. Thus, an object of $Q_2\mathcal{A}$ is essentially a diagram

$$\begin{tikzcd}[column sep=small]
 & & t_{02} \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, phantom, "\vee" very near start] & & \\
 & t_{01} \arrow[dl, two heads] \arrow[dr, tail] & & t_{12} \arrow[dl, two heads] \arrow[dr, tail] & \\
t_{00} & & t_{11} & & t_{22}.
\end{tikzcd}$$

The main difference between T and Q are the horizontal arrows $a_{ii} \to a_{jj}$ which appear in T but not in Q, coming from the additional arrows $(i,i) \to (j,j)$ of $\mathrm{Tw}^+([n])$ compared to $\mathrm{Tw}([n])$. For the significance of these extra arrows see Section 3.4.

We proceed to show that $T\mathcal{C}$ is a Segal space. We will make use of the following standard result.

Lemma 3.1.2. *Let $F \colon \mathcal{A} \to \mathcal{B}$ be a functor between two categories and let $\mathcal{C}$ be another category. Then if F is injective on objects the induced functor $\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is an isofibration.*

Proposition 3.1.3. *Let $\mathcal{C}$ be a category with distinguished squares. Then the simplicial groupoid $T\mathcal{C}$ is a Segal space.*

Proof. By the previous lemma, since the face maps $d_i \colon \mathrm{Tw}^+([n-1]) \to \mathrm{Tw}^+([n])$ are injective on objects, the face maps

$$d_i \colon \mathrm{Fun}\left(\mathrm{Tw}^+([n]), \mathcal{C}\right) \to \mathrm{Fun}\left(\mathrm{Tw}^+([n-1]), \mathcal{C}\right)$$

are isofibrations. But because $T_n\mathcal{C}$ is a full subgroupoid of $\mathrm{Fun}\left(\mathrm{Tw}^+([n]), \mathcal{C}\right)^{\simeq}$ and is closed under isomorphisms it follows that the face maps of $T\mathcal{C}$ are isofibrations. As a consequence it is sufficient to see (see Diagram (2.2.1)) that $T_{n+1}\mathcal{C}$ is equivalent to the strict pullback

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T_n\mathcal{C} \arrow[d, "d_n"] \\ T_n\mathcal{C} \arrow[r, "d_0"'] & T_{n-1}\mathcal{C}. \end{tikzcd} \tag{3.1.2}$$

But this is indeed the case: the objects of this pullback are pairs in $T_n\mathcal{C} \times T_n\mathcal{C}$ coinciding at the last and first face respectively. That is, pairs

$$\left(\begin{tikzcd}[column sep=small] & & * \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\ & * \arrow[dl] \arrow[dr] & {} & * \arrow[dl, dashed] \arrow[dr, dashed] & \\ * \arrow[rr] & & * \arrow[rr, dashed] & & * \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] & & * \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\ & * \arrow[dl, dashed] \arrow[dr, dashed] & {} & * \arrow[dl] \arrow[dr] & \\ * \arrow[rr, dashed] & & * \arrow[rr] & & * \end{tikzcd} \right)$$

such that the dashed regions are equal. This gives a groupoid whose objects are

$$\begin{tikzcd}[column sep=small] & & * \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & * \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\ & * \arrow[dl] \arrow[dr] & {} & * \arrow[dl] \arrow[dr] & {} & * \arrow[dl] \arrow[dr] & \\ * \arrow[rr] & & * \arrow[rr] & & * \arrow[rr] & & * \end{tikzcd}$$

and morphisms are levelwise isomorphisms making the diagram commute. But this is equivalent to $T_{n+1}\mathcal{C}$, since the missing apex can be uniquely filled with a distinguished square. $\square$

The following result characterizes the categories whose T-construction is CULF monoidal.

Proposition 3.1.4. *Let $(\mathcal{C}, +, 0)$ be a monoidal category with distinguished squares. Then $T\mathcal{C}$ is a CULF monoidal Segal space if and only if the following conditions hold.*

(i) *$(\mathcal{C}, +, 0)$ is monoidal extensive,*

(ii) *Given Ω, Γ, Λ commutative squares in $\mathcal{C}$ such that $\Omega = \Gamma + \Lambda$, then*

$$\Omega \text{ is distinguished} \iff \Gamma \text{ and } \Lambda \text{ are distinguished}.$$

Proof. Condition (i) is the necessary and sufficient condition to be able to sum arrows and commutative diagrams in $\mathcal{C}$, as we know from the nerve of a monoidal extensive category. However we need sums of distinguished squares to be distinguished squares in order for $T_n\mathcal{C} \times T_n\mathcal{C} \to T_n\mathcal{C}$ to be well-defined. This gives the left implication of (ii). Finally, we have to impose right implication of (ii) to ensure that the monoidal structure is given by CULF functors. $\square$

As explained in the introduction, the central object of this chapter is $T\mathbf{S}$, the T-construction of the category $\mathbf{S}$ of finite sets and surjections. Let us now see that $T\mathbf{S}$ meets all the requirements for the main theorem to be stated.

Lemma 3.1.5. *The category* $\mathbf{S}$

(i) has a class of distinguished squares,

(ii) is monoidal extensive with disjoint union $(+)$ *and empty set as monoidal structure,*

(iii) satisfies (ii) of Proposition 3.1.4,

Proof. We declare the distinguished squares to be the commutative squares in $\mathbf{S}$ that are pullbacks in the category of sets (note that $\mathbf{S}$ itself does not have pullbacks). For (ii) observe that taking disjoint union clearly gives an equivalence $\mathbf{S}_{/A} \times \mathbf{S}_{/B} \simeq \mathbf{S}_{/A+B}$. It is the restriction to surjections of the monoidal structure of finite sets and their coproduct. The fact the pullback in sets is the disjoint union of the product of the fibers implies (iii). □

In view of this lemma and Propositions 3.1.3 and 3.1.4 we obtain the following result.

Proposition 3.1.6. $T\mathbf{S}$ *is a CULF monoidal Segal space.*

By Theorem 2.3.1 and Proposition 2.3.3 this implies that $T\mathbf{S}$ has an associated incidence bialgebra. Observe that $\mathbf{S}^{\simeq}$ is locally finite, so that $T_1\mathbf{S}$ is locally finite and $s_0\colon T_0\mathbf{S} \to T_1\mathbf{S}$ is finite. Moreover, every arrow of $\mathbf{S}$ admits, up to isomorphism, a finite number of 2-step factorizations, therefore $d_1\colon T_2\mathbf{S} \to T_1\mathbf{S}$ is also finite. This means that $T\mathbf{S}$ is locally finite in the sense of [33]. As a consequence we can take homotopy cardinality of the incidence bialgebra of $T\mathbf{S}$.

Remark 3.1.7. Notice that $N\mathbf{S}$, the fat nerve of the category of finite sets and surjections, is the full subsimplicial groupoid of $T\mathbf{S}$ containing the simplices whose left-down arrows (i.e. $F(i,j) \to F(i,k), j < k$) are identities.

3.2 *Plethystic bialgebra*

The following notation is used:

- $\mathbf{x} = (x_1, x_2, \dots)$,
- Λ: set of infinite vectors of natural numbers with $\lambda_i = 0$ for all i large enough,
- $\Lambda \ni \lambda = (\lambda_1, \lambda_2, \dots)$ and $(\lambda_1, \dots, \lambda_n) := (\lambda_1, \dots, \lambda_n, 0, \dots)$,
- $|\lambda| = \sum_k \lambda_k$,
- $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$,
- $\mathrm{aut}(\lambda) = 1!^{\lambda_1}\lambda_1! \cdot 2!^{\lambda_2}\lambda_2! \cdots$,
- $\lambda + \mu$ is coordinate-wise sum.

First of all, we define the n-*th Verschiebung operator* V^n as

$$(V^n\lambda)_i = \begin{cases} \lambda_{i/n} & \text{if } n \mid i \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } \lambda = (\lambda_1, \lambda_2, \dots).$$

For example $V^2(5,9,2,0,\dots) = (0,5,0,9,0,2,0,\dots)$. It is clear that V^n preserves sums. Note that the F_k introduced in the definition of plethystic substitution (Definition 1.0.1) can be expressed as

$$F_k(x_1,x_2,\dots) = F(x_k,x_{2k},\dots) = \sum_\lambda F_\lambda \frac{\mathbf{x}^{V^k\lambda}}{\mathrm{aut}(\lambda)}.$$

Remark 3.2.1. Note that λ can be viewed as the isomorphism class of a surjection $X \twoheadrightarrow B$ with λ_k fibers of size k. With this identification, $\mathrm{aut}(\lambda)$ is precisely the cardinal of $\mathrm{Aut}(X \twoheadrightarrow B)$ in the groupoid of surjections $\mathcal{S}$, whose objects are surjections and whose arrows are pairs of compatible bijections, one for the source and one for the target. Moreover, the Verschiebung operators can also be defined at the objective level of surjections,

$$V^S(X \twoheadrightarrow B) := X \times S \twoheadrightarrow X \twoheadrightarrow B,$$

which is nothing but the scalar multiplication of $X \twoheadrightarrow B$ and S in $\mathbf{Set}_{/B}$ [31]. It is clear that V^S corresponds numerically to $V^{|S|}$.

For each λ define the functional $A_\lambda \in \mathbb{Q}[\![\mathbf{x}]\!]^*$ by $A_\lambda(F) = F_\lambda$. We define the *plethystic bialgebra* to be the free polynomial algebra $\mathcal{P} = \mathbb{Q}\big[\{A_\lambda\}_\lambda\big]$ along with the comultiplication dual to plethystic substitution. That is, for each λ and $F, G \in \mathbb{Q}[\![\mathbf{x}]\!]$,

$$\Delta(A_\lambda)(F,G) = A_\lambda(G \circledast F).$$

The counit is given by $\epsilon(A_\lambda) = A_\lambda(x_1)$.

Now, consider a list $\boldsymbol{\mu} \in \Lambda^n$ of n infinite vectors, regarded as a representative element of a multiset $\overline{\boldsymbol{\mu}} \in \Lambda^n/\mathfrak{S}_n$. We denote by $R(\boldsymbol{\mu}) \subseteq \mathfrak{S}_n$ the set of automorphisms that maps the list $\boldsymbol{\mu}$ to itself. For example if $\boldsymbol{\mu} = \{\alpha,\alpha,\beta,\gamma,\gamma,\gamma\}$ then $R(\boldsymbol{\mu})$ has $2! \cdot 1! \cdot 3!$ elements. Notice that if $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $R(\boldsymbol{\mu}) \cong R(\boldsymbol{\mu}')$. We may thus refer to $R(\boldsymbol{\mu})$ for a multiset $\overline{\boldsymbol{\mu}} \in \Lambda^n/\mathfrak{S}_n$ by taking a representative, since we are only interested in its cardinality.

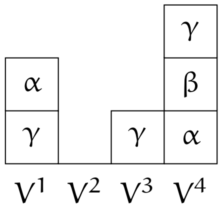
Remark 3.2.2. Observe that $\sum_n \Lambda^n/\mathfrak{S}_n \simeq \pi_0 T_1 \mathbf{S}$. Furthermore, the number of automorphisms of a representative element in $T_1\mathbf{S}$ of the image of $\overline{\boldsymbol{\mu}}$ under this bijection is precisely

$$\mathrm{aut}(\boldsymbol{\mu}) = |R(\boldsymbol{\mu})| \cdot \prod_{\mu \in \boldsymbol{\mu}} \mathrm{aut}(\mu).$$

Fix two infinite vectors, $\sigma, \lambda \in \Lambda$, and a list of infinite vectors $\boldsymbol{\mu} \in \Lambda^n$, with $n = |\lambda|$. We define the set of $(\lambda,\boldsymbol{\mu})$*–decompositions* of σ as

$$T^{\boldsymbol{\mu}}_{\sigma,\lambda} := \left\{ p \colon \boldsymbol{\mu} \xrightarrow{\sim} \sum_k \{1,\dots,\lambda_k\} \mid \sigma = \sum_{\mu\in\boldsymbol{\mu}} V^{q(\mu)}\mu \right\},$$

where p is a bijection of n-element sets and q returns the index of $p(\mu)$ in the sum. A useful way to visualize an element of this set is as a placement of the elements of $\boldsymbol{\mu}$ over a grid with λ_k cells in the kth column such that if we apply V^k to the kth column and sum the cells the result is σ. For example, if $\lambda = (2,0,1,3)$ and $\boldsymbol{\mu} = \{\alpha,\alpha,\beta,\gamma,\gamma,\gamma\}$ the placement



belongs to $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ if $\sigma = V^1(\gamma+\alpha)+V^3(\gamma)+V^4(\alpha+\beta+\gamma)$, where the sum is a pointwise vector sum in Λ. Note that each such placement appears $|R(\boldsymbol{\mu})|$ times in $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$. Observe also that if $\boldsymbol{\mu},\boldsymbol{\mu}' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $T^{\boldsymbol{\mu}}_{\sigma,\lambda} \cong T^{\boldsymbol{\mu}'}_{\sigma,\lambda}$. We may thus refer to $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ for a class $\overline{\boldsymbol{\mu}} \in \Lambda^{|\lambda|}/\mathfrak{S}_{|\lambda|}$ by taking a representative, since we are only interested in its cardinality.

Proposition 3.2.3. *Let $\sigma \in \Lambda$ be an infinite vector. Then the comultiplication of A_σ in $\mathcal{P}$ is given by*

$$\Delta(A_\sigma) = \sum_{\lambda}\sum_{\overline{\boldsymbol{\mu}}} \frac{\mathrm{aut}(\sigma)\cdot|T^{\boldsymbol{\mu}}_{\sigma,\lambda}|}{\mathrm{aut}(\lambda)\cdot\mathrm{aut}(\boldsymbol{\mu})}\left(\prod_{\mu\in\boldsymbol{\mu}} A_\mu\right)\otimes A_\lambda, \tag{3.2.1}$$

where λ runs through Λ and $\overline{\boldsymbol{\mu}}$ runs through $\Lambda^{|\lambda|}/\mathfrak{S}_{|\lambda|}$.

Remark 3.2.4. Not surprisingly, if $\sigma = (n,0,0,\dots)$ this expression gives the comultiplication of A_n for ordinary composition of one-variable power series. Extending this analogy between classical and plethystic we define the polynomials $P_{\sigma,\lambda}(\{A_\mu\}_\mu)$,

$$\Delta(A_\sigma) =: \sum_\lambda P_{\sigma,\lambda}(\{A_\mu\}_\mu)\otimes A_\lambda,$$

which are the generalization of the Bell polynomials to the plethystic case. Hence in particular $P_{(n,0,\dots),(k,0,\dots)}(\{A_\mu\}_\mu) = B_{n,k}\left(\{A_{(i,0,\dots)}\}_i\right)$.

Example 3.2.5. Let us see that

$$\begin{aligned} &P_{(0,0,0,1,0,2),(1,2)}(\{A_\mu\}_\mu) = \\ &\quad = \frac{6!^2 2!4!\cdot 2!}{2!^2 2!\cdot 4!3!^2 2!}A_{(0,0,0,1)}A^2_{(0,0,1)} + \frac{6!^2 2!4!\cdot 2!}{2!^2 2!\cdot 6!3!2!}A_{(0,0,0,0,0,1)}A_{(0,0,1)}A_{(0,1)}. \end{aligned}$$

Indeed, there are only two elements $\overline{\boldsymbol{\mu}} \in \Lambda^3/\mathfrak{S}_3$ (where $3 = |(1,2)|$) such that $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ is not empty. Namely

$$\begin{aligned} \boldsymbol{\mu_1} &= \{(0,0,0,1),(0,0,1),(0,0,1)\} \\ \boldsymbol{\mu_2} &= \{(0,0,0,0,0,1),(0,0,1),(0,1)\}. \end{aligned}$$

The vector $(0,0,0,1,0,2)$ is obtained respectively as

$$\begin{aligned} (0,0,0,1,0,2) &= V^1(0,0,0,1)+V^2(0,0,1)+V^2(0,0,1), \\ (0,0,0,1,0,2) &= V^1(0,0,0,0,0,1)+V^2(0,0,1)+V^2(0,1). \end{aligned}$$

The colors will be used to connect this example to Example 3.3.2. In both cases it is straightforward to check that $|T^{\boldsymbol{\mu_1}}_{\sigma,\lambda}| = |T^{\boldsymbol{\mu_2}}_{\sigma,\lambda}| = 2$. Notice however that $|R(\boldsymbol{\mu_1})| = 2$ while $|R(\boldsymbol{\mu_2})| = 1$. The rest comes from the automorphisms of the vectors involved.

This can be visualized at the level of power series as follows. The vector σ corresponds to (the linear map returning the coefficient of) $x_4x_6^2$. In the same way, the vector λ corresponds to $x_1x_2^2$. The monomials associated to $\boldsymbol{\mu_1}$ are x_4 and x_3 and the monomials associated to $\boldsymbol{\mu_2}$ are x_6, x_3 and x_2. If we take power series with only these monomials and substitute them we obtain

$$\begin{aligned}(x_1x_2) \circledast (x_4+x_3) &= (x_4+x_3)(x_8+x_6)^2 = x_4x_6^2 + \cdots, \\ (x_1x_2) \circledast (x_6+x_3+x_2) &= (x_6+x_3+x_2)(x_{12}+x_6+x_4)^2 = 2x_4x_6^2 + \cdots.\end{aligned}$$

Before proving Proposition 3.2.3 we shall need the following two lemmas. For the sake of notation we work from now on with another basis of $\mathcal{P}$, $\{a_\lambda\}_\lambda$, defined as $a_\lambda := \frac{A_\lambda}{\mathrm{aut}(\lambda)}$. Let $\mathbf{z} = z_1, z_2, \ldots$ be a set of infinitely many formal variables. Consider, in the style of [11, Remark 2.3], the map $\Delta\colon \mathcal{P}[\![\mathbf{z}]\!] \to (\mathcal{P}\otimes\mathcal{P})[\![\mathbf{z}]\!]$ given by linearly extending the comultiplication defined above for $\mathcal{P}$. Given power series $F, G \in \mathbb{Q}[\![\mathbf{x}]\!]$ and elements $\phi \in \mathcal{P}[\![\mathbf{z}]\!]$ and $\psi \in (\mathcal{P}\otimes\mathcal{P})[\![\mathbf{z}]\!]$, we introduce the notation

$$\langle \phi, F\rangle := \phi(F) \in \mathbb{Q}[\![\mathbf{z}]\!] \quad \text{and} \quad \langle \psi, (F,G)\rangle := \psi(F,G) \in \mathbb{Q}[\![\mathbf{z}]\!],$$

for the sake of readability. For each positive natural number i, define the power series

$$A_i(\mathbf{z}) = \sum_\lambda a_\lambda \mathbf{z}^{V^i\lambda} \in \mathcal{P}[\![\mathbf{z}]\!].$$

Observe that by definition $\Delta(A_i(\mathbf{z})) = \sum_\lambda \Delta(a_\lambda)\mathbf{z}^{V^i\lambda}$. The following result is straightforward.

Lemma 3.2.6. *For any* $i, j \geqslant 0$ *and* $F, G \in \mathbb{Q}[\![\mathbf{x}]\!]$

(i) $\langle A_i(\mathbf{z}), F\rangle = F_i(\mathbf{z})$,

(ii) $\langle \Delta(A_i(\mathbf{z})), (F,G)\rangle = (G \circledast F)_i(\mathbf{z})$,

(iii) $\langle A_i(\mathbf{z}), F\cdot G\rangle = \langle A_i(\mathbf{z}), F\rangle \cdot \langle A_i(\mathbf{z}), G\rangle$,

(iv) $\langle A_i(\mathbf{z})\cdot A_j(\mathbf{z}), F\rangle = \langle A_i(\mathbf{z}), F\rangle \cdot \langle A_j(\mathbf{z}), F\rangle$.

Lemma 3.2.7.

$$\Delta(A_1(\mathbf{z})) = \sum_\lambda \left(\prod_i A_i^{\lambda_i}(\mathbf{z})\right) \otimes a_\lambda.$$

Proof. Let $F, G \in \mathbb{Q}[\![\mathbf{x}]\!]$. By (ii) of Lemma 3.2.6 we have

$$\langle \Delta(A_1(\mathbf{z})), (F,G)\rangle = (G \circledast F)(\mathbf{z}).$$

Now, by definition of plethystic substitution

$$(G \circledast F)(\mathbf{z}) = \sum_\lambda \left(\prod_i F_i^{\lambda_i}(\mathbf{z})\right) \cdot a_\lambda(G(\mathbf{z})),$$

but (iv) and (i) of Lemma 3.2.6 tell us respectively that

$$\left\langle \sum_\lambda \left(\prod_i A_i^{\lambda_i}(\mathbf{z})\right), F \right\rangle = \sum_\lambda \left(\prod_i \langle A_i(\mathbf{z}), F\rangle^{\lambda_i}\right) = \sum_\lambda \left(\prod_i F_i^{\lambda_i}(\mathbf{z})\right).$$

Therefore

$$(G \circledast F)(\mathbf{z}) = \left\langle \sum_{\lambda} \left(\prod_{i} A_i^{\lambda_i}(\mathbf{z}) \right) \otimes a_\lambda, (F, G) \right\rangle,$$

as we wanted to see. □

Proof of Proposition 3.2.3. Define sets

$$T_\lambda := \left\{ \{\mu_{i,j}\}_{i \geqslant 1, j \in \{1,\ldots,\lambda_i\}} \right\},$$

$$T_{\sigma,\lambda} := \left\{ \{\mu_{i,j}\}_{i \geqslant 1, j \in \{1,\ldots,\lambda_i\}} \mid \sigma = \sum_i \sum_{j=1}^{\lambda_i} V^i \mu_{i,j} \right\}.$$

We now compute

$$\begin{aligned}
\Delta(A_1(\mathbf{z})) = \sum_{\lambda} \left(\prod_{i} A_i^{\lambda_i}(\mathbf{z}) \right) \otimes a_\lambda &= \sum_{\lambda} \left(\sum_{\{\mu_{i,j}\}_{i,j} \in T_\lambda} \prod_i \prod_{j=1}^{\lambda_i} a_{\mu_{i,j}} \mathbf{z}^{V^i \mu_{i,j}} \right) \otimes a_\lambda = \\
&= \sum_{\lambda} \left(\sum_{\sigma} \left(\sum_{\{\mu_{i,j}\}_{i,j} \in T_{\sigma,\lambda}} \prod_i \prod_{j=1}^{\lambda_i} a_{\mu_{i,j}} \right) \mathbf{z}^{\sigma} \right) \otimes a_\lambda = \\
&= \sum_{\sigma} \sum_{\lambda} \left(\sum_{\{\mu_{i,j}\}_{i,j} \in T_{\sigma,\lambda}} \prod_i \prod_{j=1}^{\lambda_i} a_{\mu_{i,j}} \right) \mathbf{z}^{\sigma} \otimes a_\lambda.
\end{aligned}$$

But on the other hand $\Delta(A_1(\mathbf{z})) = \sum_\sigma \Delta(a_\sigma)\mathbf{z}^\sigma$. Hence by Lemma 3.2.7

$$\Delta(a_\sigma) = \sum_{\lambda} \left(\sum_{\{\mu_{i,j}\}_{i,j} \in T_{\sigma,\lambda}} \prod_i \prod_{j=1}^{\lambda_i} a_{\mu_{i,j}} \right) \otimes a_\lambda. \tag{3.2.2}$$

Notice that lists $\{\mu_{i,j}\}_{i,j}$ with the same elements ordered in different ways may be λ-decompositions of distinct σ. Therefore the comultiplications of different generators may have terms in common. Finally, observe that the multiset $\bar{\boldsymbol{\mu}}$ represented by any list $\boldsymbol{\mu} = \{\mu_{i,j}\}_{i,j}$ appears precisely $\dfrac{|T^{\boldsymbol{\mu}}_{\sigma,\lambda}|}{|R(\boldsymbol{\mu})|}$ times in $T_{\sigma,\lambda}$. This implies that expression (3.2.2) is equivalent to

$$\Delta(a_\sigma) = \sum_{\lambda} \left(\sum_{\bar{\boldsymbol{\mu}}} \frac{|T^{\boldsymbol{\mu}}_{\sigma,\lambda}|}{|R(\boldsymbol{\mu})|} \prod_{\mu \in \boldsymbol{\mu}} a_\mu \right) \otimes a_\lambda.$$

Changing again the basis from $\{a_\lambda\}_\lambda$ to $\{A_\lambda\}_\lambda$ we obtain Equation (3.2.1). □

3.3 T-*construction for surjections*

We are now ready to state and prove the main result of this chapter. The proof is essentially a question of unpacking the abstract constructions. A pleasant feature is the way in which the subtle symmetry factors come out naturally from the groupoid formalism.

Theorem 3.3.1. *The homotopy cardinality of the incidence bialgebra of* $T\mathbf{S}$ *is isomorphic to* $\mathcal{P}$.

Proof. Recall from Chapter 2 that the homotopy cardinality of the incidence bialgebra of $T\mathbf{S}$ is denoted by $\mathbb{Q}_{\pi_0 T_1 \mathbf{S}}$. We split the proof into three parts. First we define an isomorphism $\mathbb{Q}_{\pi_0 T_1 \mathbf{S}} \overset{\theta}{\simeq} \mathcal{P}$ of algebras, next we explore the relation between the Verschiebung operator and $T_2\mathbf{S}$, and finally we show that θ preserves the comultiplication.

The isomorphism. We call *connected* the elements of $T_n\mathbf{S}$ with a singleton at the nn position. Notice that as a vector space $\mathbb{Q}_{\pi_0 T_1 \mathbf{S}}$ is spanned by $\pi_0 T_1 \mathbf{S}$, and as a free algebra it is generated by the classes of the connected elements of $T_1\mathbf{S}$, since every diagram is a sum of connected ones. The isomorphism class δ_λ of a connected element

$$\begin{tikzcd} & t_{01} \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\lambda"] & \\ t_{00} \arrow[rr, two heads] & {} & 1 \end{tikzcd} \quad \in T_1\mathbf{S}$$

is given by the infinite vector $\lambda = (\lambda_1, \lambda_2, \dots)$ representing the class of $t_{01} \twoheadrightarrow t_{00}$. Be aware that the same notation is used for either the connected elements of $T_1\mathbf{S}$ and the infinite vectors representing their isomorphism class. This being said, the assignment

$$\begin{aligned} \mathbb{Q}_{\pi_0 T_1 \mathbf{S}} &\longrightarrow \mathcal{P} \\ \delta_\lambda &\longmapsto A_\lambda \\ \delta_{\lambda+\mu} = \delta_\lambda \delta_\mu &\longmapsto A_\lambda A_\mu, \end{aligned}$$

for λ and μ connected, defines an isomorphism of algebras. Notice that $\lambda + \mu$ is the monoidal sum in $T_1\mathbf{S}$, which does not correspond to the pointwise sum of their corresponding infinite vectors, since it has two connected components.

The Verschiebung operator. Pick a connected element t,

$$\begin{tikzcd} & & t_{02} \arrow[dl] \arrow[dr, two heads] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & t_{01} \arrow[dl, "\mu"'] \arrow[dr, two heads] & & t_{12} \arrow[dl] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{11} \arrow[rr, two heads] & & 1 \end{tikzcd} \quad \in T_2\mathbf{S}.$$

For each $r \in t_{11}$, consider the map on the fibers $\mu_r \colon (t_{01})_r \twoheadrightarrow (t_{00})_r$. Since the square is a pullback of sets we have that the surjection $(t_{02})_r \longrightarrow (t_{00})_r$ is isomorphic to the composite

$$(t_{01})_r \times (t_{12})_r \xrightarrow{p_1} (t_{01})_r \xrightarrow{\mu_r} (t_{00})_r,$$

which is precisely $V^{(t_{12})_r}\mu_r$ (see Remark 3.2.1). Therefore the surjection $(t_{02})_r \longrightarrow (t_{00})_r$ belongs to the isomorphism class of $V^{|(t_{12})_r|}\mu_r$. But recall that

$$d_1(t) = \begin{tikzcd} & t_{02} \arrow[dl] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & 1, \end{tikzcd}$$

therefore the isomorphism class of $d_1(t)$ is precisely

$$\sum_{r \in t_{11}} V^{|(t_{12})_r|} \mu_r.$$

In fact, continuing with the scalar multiplication interpretation of Remark 3.2.1, we could just say that $t_{02} \twoheadrightarrow t_{00}$ is a linear combination in $\bigoplus_r \mathbf{Set}_{/(t_{00})_r}$, namely

$$t_{02} \longrightarrow t_{00} = \sum_r ((t_{02})_r \longrightarrow (t_{00})_r) \cdot (t_{12})_r = \sum_r V^{(t_{12})_r}((t_{02})_r \longrightarrow (t_{00})_r).$$

The comultiplication. We have to show that $\Delta(\delta_\sigma)$ for σ connected yields Equation (3.2.1). By Lemma 2.4.2 we have that

$$\Delta(\delta_\sigma) = \sum_{\lambda \in \pi_0 T_1 \mathbf{S}} \sum_{\tau \in \pi_0 T_1 \mathbf{S}} \frac{|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma|}{|\operatorname{Aut}(\lambda)||\operatorname{Aut}(\tau)|} \delta_\tau \otimes \delta_\lambda. \tag{3.3.1}$$

Since σ is connected, also λ must be connected,

$$\begin{tikzcd} & t_{12} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lambda"] & \\ t_{11} \arrow[rr, two heads] & & 1. \end{tikzcd}$$

It is clear that $\operatorname{Iso}(d_0\tau, d_1\lambda)$ is nonempty if and only if $|d_0\tau| = |d_1\lambda|$. Without loss of generality we can assume that $d_0(\tau) = d_1(\lambda) = t_{11}$ and write

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\tau"] & \\ t_{00} \arrow[rr, two heads] & & t_{11}. \end{tikzcd}$$

Note that in particular $|t_{11}| = |\lambda|$. If, as above, we denote by $\mu_r \colon (t_{01})_r \twoheadrightarrow (t_{00})_r$ the fiber surjection of $r \in t_{11}$ and $\boldsymbol{\mu} = \{\mu_r\}_r$, we have that δ_τ corresponds to $\prod_{\mu \in \boldsymbol{\mu}} A_\mu$ and that

$$|\operatorname{Aut}(\tau)| = \operatorname{aut}(\boldsymbol{\mu}).$$

Hence it only remains to show that $|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma| = \operatorname{aut}(\sigma) \cdot |T^{\boldsymbol{\mu}}_{\sigma,\lambda}|$. Since $\operatorname{Iso}(d_0\tau, d_1\lambda)$ is a discrete groupoid, i.e. just a set, it makes sense to consider the subset

$$\{\phi \in \operatorname{Iso}(d_0\tau, d_1\lambda) \mid d_1(\phi) \simeq \sigma\}$$

consisting of those ϕ that give an object isomorphic to σ after composition by ϕ, pullback and d_1, as shown in the picture,

$$d_1 \left(\begin{tikzcd} & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\tau"] \arrow[rrr, dashed, two heads] & & & t_{12} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lambda"] & \\ t_{00} \arrow[rr, two heads] & & t_{11} \arrow[r, "\phi"'] & t_{11} \arrow[rr, two heads] & & 1 \end{tikzcd} \right) \simeq \sigma.$$

Note that in this picture the pullback is taken along the dashed arrow. Observe that this condition on ϕ can be written as

$$\sigma = \sum_{r \in t_{11}} V^{|(t_{12})_{\phi(r)}|} \mu_r,$$

where now σ and μ_r represent the corresponding infinite vectors. This is in bijection with morphisms $\boldsymbol{\mu} \xrightarrow{\sim} \sum\{1,\dots,\lambda_k\}$ satisfying

$$\sigma = \sum_{\mu \in \boldsymbol{\mu}} V^{q(\mu)}\mu,$$

since summing over $r \in t_{11}$ is equivalent to summing over $\mu \in \boldsymbol{\mu}$ and the same Verschiebung operators appear in both sums. Hence there is a bijection

$$\{\phi \in \mathrm{Iso}(d_0\tau, d_1\lambda) \mid d_1(\phi) \simeq \sigma\} \simeq T^{\boldsymbol{\mu}}_{\sigma,\lambda} = \left\{ p \colon \boldsymbol{\mu} \xrightarrow{\sim} \sum_k \{1,\dots,\lambda_k\} \mid \sigma = \sum_{\mu \in \boldsymbol{\mu}} V^{q(\mu)}\mu \right\}.$$

But the homotopy fiber of $\mathrm{Iso}(d_0\tau, d_1\lambda)$ over σ is precisely this subset described above times the set of automorphisms of σ in $T_1\mathbf{S}$, that is

$$\mathrm{Iso}(d_0\tau, d_1\lambda)_\sigma \simeq \mathrm{Iso}(\sigma,\sigma) \times \{\phi \in \mathrm{Iso}(d_0\tau, d_1\lambda) \mid d_1(\phi) \simeq \sigma\}.$$

Therefore $|\mathrm{Iso}(d_0\tau, d_1\lambda)_\sigma| = \mathrm{aut}(\sigma) \cdot |T^{\boldsymbol{\mu}}_{\sigma,\lambda}|$ and Equation (3.3.1) corresponds to Equation (3.2.1), as we wanted to see. $\square$

Example 3.3.2. Let us see the interpretation of $P_{(0,0,0,1,0,2),(1,2)}(\{A_\mu\}_\mu)$ (see Example 3.2.5) in the simplicial groupoid $T\mathbf{S}$. The vector $\sigma = (0,0,0,1,0,2)$ corresponds to

$$\begin{tikzcd} & 16 \arrow[dl] \arrow[dr, two heads] & \\ 3 \arrow[rr, two heads] & & 1, \end{tikzcd} \qquad \text{or, specifying the size of each fiber,} \qquad \begin{tikzcd} & (6,6,4) \arrow[dl] \arrow[dr, two heads] & \\ 3 \arrow[rr, two heads] & & 1. \end{tikzcd}$$

In the same way, the vector $\lambda = (1,2)$ corresponds to

$$\begin{tikzcd} & (1,2,2) \arrow[dl] \arrow[dr, two heads] & \\ 3 \arrow[rr, two heads] & & 1. \end{tikzcd}$$

Now, what we want to compute is, roughly speaking, all the 2-simplices of $T\mathbf{S}$ that give σ under d_1 and λ under d_0. There are essentially two such simplices,

$$\begin{tikzcd}[column sep=small] & & (4,6,6) \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\vee"] & & \\ & (4,3,3) \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\boldsymbol{\mu}_1"] & & (1,2,2) \arrow[dl] \arrow[dr, two heads] & \\ 3 \arrow[rr, two heads] & & 3 \arrow[rr, two heads] & & 1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & & (6,6,4) \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\vee"] & & \\ & (6,3,2) \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\boldsymbol{\mu}_2"] & & (1,2,2) \arrow[dl] \arrow[dr, two heads] & \\ 3 \arrow[rr, two heads] & & 3 \arrow[rr, two heads] & & 1, \end{tikzcd}$$

and it is clear that the two left-down diagrams, given by the face map d_2, correspond to $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ of Example 3.2.5.

Remark 3.3.3. We end this section by deriving the analogous result for one-variable power series. As mentioned in the introduction, the statement reads: the Faà di Bruno bialgebra $\mathcal{F}$ is equivalent to $\mathbb{Q}_{\pi_0 S}$, the homotopy cardinality of the incidence bialgebra of the fat nerve $\mathbf{N}\mathbb{S} \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd}$ of the category of surjections. First of all, observe that $\mathcal{F}$ is generated by the functionals A_n returning the nth coefficient of a power series. The connected elements of $\mathbb{S}$ are the surjections with singleton target. Hence, $\delta_n = |1 \xrightarrow{\ulcorner n \twoheadrightarrow 1 \urcorner} \mathbb{S}|$ corresponds to A_n. Using Lemma 2.4.2 we get

$$\Delta(\delta_n) = \sum_{b:k \twoheadrightarrow 1} \sum_{a:n \twoheadrightarrow k} \frac{|\operatorname{Iso}(k,k)_{n \twoheadrightarrow 1}|}{|\operatorname{Aut}(b)||\operatorname{Aut}(a)|} \delta_a \otimes \delta_k.$$

It is clear that $|\operatorname{Aut}(b)| = k!$, and that any element of $\operatorname{Iso}(k,k)$ gives $n \twoheadrightarrow 1$, so that $|\operatorname{Iso}(k,k)_{n \twoheadrightarrow 1}| = n! \cdot k!$. Moreover, $\delta_a = \delta_{n_1} \dots \delta_{n_k}$, where n_i are the fibers of $a \colon n \twoheadrightarrow k$. Altogether we obtain

$$\Delta(\delta_n) = \sum_{k \twoheadrightarrow 1} \sum_{n \twoheadrightarrow k} \frac{n!}{|\operatorname{Aut}(n \twoheadrightarrow k)|} \left(\prod_{i=1}^{k} \delta_{n_i} \right) \otimes \delta_k,$$

which is easily checked to correspond to the comultiplication of A_n,

$$\Delta(A_n) = \sum_{k} \sum_{n_1 + \dots + n_k = n} \binom{n}{n_1, \dots, n_k} \left(\prod_{i=1}^{k} A_{n_i} \right) \otimes A_k.$$

In view that $N\mathbf{S}$ is contained in $T\mathbf{S}$ (see Remark 3.1.7), this result is a particular case of Theorem 3.3.1, as also pointed out in Remark 3.2.4.

3.4 *Transversals from surjections*

The category of partitions $\mathbb{P}$ is equivalent to the category of surjections $\mathbb{S}$, so that as in the case of Faà di Bruno we can use surjections to describe transversals. Among the advantages of surjections over partitions there is the fact that partitions of partitions are pairs of composable surjections. In what follows we introduce a new approach to transversals by using surjections.

First of all, let us translate all the notions regarding partitions to surjections. The category of surjections has as objects surjections $E \twoheadrightarrow S$ between finite sets and as morphisms commutative squares

$$\begin{array}{ccc} E & \xrightarrow{\sim} & F \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ S & \xrightarrow{\sim} & R, \end{array}$$

where the horizontal arrows are bijections. It is clear that $f \colon E \twoheadrightarrow S$ corresponds to the partition π of E given by $p \sim_\pi q$ if $f(p) = f(q)$ or, what is the same, the partition whose blocks are the fibers of f. Hence, the isomorphism class of a surjection is given by a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ where λ_k is the number of fibers of size k, and the number of automorphisms of f is also $\operatorname{aut}(\lambda)$: in this case $\lambda_1! \cdot \lambda_2! \cdots$ is the number of bijections $S \xrightarrow{\sim} S$ permuting elements with a fiber of the same size, and $1!^{\lambda_1} \cdot 2!^{\lambda_2} \cdots$ is the number of fiberwise bijections $E \xrightarrow{\sim} E$.

Consider $\pi, \tau \in \Pi(E)$ and let $\pi\colon E \twoheadrightarrow S$ and $\tau\colon E \twoheadrightarrow X$ be their corresponding surjections. Construct the diagram of sets

$$\begin{tikzcd}
E \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "\tau"] \arrow[ddr, bend right, "\pi"'] & & \\
& S\times_I X \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads] \\
& S \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner", very near end] & I
\end{tikzcd}$$

by taking pushout along π and τ and pullback of the pushout diagram. Note that all the arrows are surjections except perhaps ϕ. Note also that any pullback of surjections is also a pushout square.

Lemma 3.4.1. *Let π and τ be two partitions of E presented as surjections as above, and $\sigma\colon E \twoheadrightarrow B$ another partition. Let also $A \subseteq E$.*

(i) $\pi \leqslant \sigma$ *if and only if σ factors through π: $E \twoheadrightarrow S \twoheadrightarrow B$. Moreover the surjection $S \twoheadrightarrow B$ corresponds to $\sigma|\pi$.*

(ii) π_A *corresponds to the unique surjection $A \twoheadrightarrow R$ (up to isomorphism) that factors the morphism $A \hookrightarrow E \twoheadrightarrow S$ as a surjection followed by an injection $A \twoheadrightarrow R \hookrightarrow S$.*

(iii) $\hat{0}$ *is $E \twoheadrightarrow E$ and $\hat{1}$ is $E \twoheadrightarrow 1$.*

(iv) The join $\pi \vee \tau$ corresponds to the pushout surjection $E \twoheadrightarrow I$.

(v) The meet $\pi \wedge \tau$ corresponds to the surjection $\phi\colon E \twoheadrightarrow \mathrm{Im}(\phi)$. Hence $\pi \wedge \tau = \hat{0}$ if and only if ϕ is injective.

(vi) π and τ commute if and only if ϕ is surjective.

(vii) π and τ are independent if and only if ϕ is surjective and $I = 1$.

Proof. (i), (ii) and (iii) are clear. (iv) follows from the fact that the pushout is precisely $X \sqcup_{/\sim} S$ where $\sim$ is the equivalence relation generated by the relation of belonging to the same fiber along π and τ. This is precisely the same relation defined in (1.4.1).

For (v), recall that for every $p, q \in E$ we have that $p \sim_{\pi\wedge\tau} q$ if and only if $p \sim_\pi q$ and $p \sim_\tau q$, but this is the same as $\pi(p) = \pi(q)$ and $\tau(p) = \tau(q)$, which is the same as $\phi(p) = \phi(q)$. But this is equivalent to $p \sim_\phi q$, considering ϕ as a surjection to its image.

Finally, if π and τ are independent then $\pi \wedge \tau = \hat{1}$, so that $I = 1$, and every fiber along π has nonempty intersection with every fiber along τ, which means that ϕ is surjective. The converse is similar. This, together with Proposition 1.4.1 shows (vi), since the set $\pi \vee \tau$ is precisely I. □

In view of this lemma, a transversal of the surjection $\sigma\colon E \twoheadrightarrow B$ is a diagram

$$\begin{tikzcd}
& & E \arrow[dl, two heads, "\pi"'] \arrow[dr, two heads, "\tau"] \arrow[dll, bend right, two heads, "\sigma"'] \arrow[d, phantom, "\vee", very near start] & & \\
& S \arrow[dl, two heads] \arrow[dr, two heads] & & X \arrow[dl, two heads] \arrow[dr, two heads] & \\
B \arrow[rr, two heads] & & I \arrow[rr, two heads] & & 1
\end{tikzcd} \tag{3.4.1}$$

where the square is obtained as the pushout of π and τ. The fact that $\pi \wedge \tau = \hat{0}$ and that π and τ commute implies that this square is also a pullback. Furthermore the condition that the pushouts $\pi \vee \tau$ and $\sigma \vee \tau$ coincide gives a map $B \twoheadrightarrow I$. Conversely, any commutative square of the form

$$\begin{tikzcd}
 & S \arrow[dl, two heads] \arrow[dr, two heads] & \\
B \arrow[dr, two heads] & & I \arrow[dl, equal] \\
 & I &
\end{tikzcd}$$

is a pushout in the category of surjections. Therefore the map $B \twoheadrightarrow I$ says that $\pi \vee \tau$ coincides with $\sigma \vee \tau$.

Hence, a transversal is just an object of $T_2\mathbf{S}$.

Example 3.4.2. The diagram corresponding to the transversal of Example 1.4.3 is

$$\begin{tikzcd}
 & & 21 \arrow[dl, two heads, "\pi"'] \arrow[dr, two heads, "\tau"] \arrow[dd, phantom, "\vee" near start] \arrow[ddll, two heads, bend right, "\sigma"'] & & \\
 & 8 \arrow[dl, two heads] \arrow[dr, two heads] & & 5 \arrow[dl, two heads] \arrow[dr, two heads] & \\
4 \arrow[rr, two heads] & & 2 \arrow[rr, two heads] & & 1
\end{tikzcd}$$

where each number corresponds to the number of blocks of each partition, and 21 is the cardinality of E.

3.5 *Faà di Bruno formula for the Green function*

In this section we take the opportunity to describe a plethystic analogue of the Faà di Bruno formula for the connected Green function, originating in quantum field theory. This formula is a compact and elegant way of encapsulating the comultiplication of the connected elements of coalgebras of graphs, trees, operads, etc. (cf. [30, 42]). The original connected Green function was defined in the Connes–Kreimer Hopf algebra of Feynman graphs [19] as the sum of all connected graphs divided by their symmetry factors. For its role in quantum field theory, see also Bellon–Schaposnik [5], Ebrahimi-Fard–Patras [26], and van Suijlekom [70].

In the case of the Faà di Bruno bialgebra, the formula was first noticed by Brouder, Frabetti and Krattenthaler [11]. The role of the Green function is played by the infinite series

$$A = \sum_{k=1}^{\infty} \frac{A_k}{k!} \in \mathbb{Q}[\![A_1, A_2, \ldots]\!].$$

They showed that the comultiplication of $\mathcal{F}$ extends to a comultiplication in $\mathbb{Q}[\![A_1, A_2, \ldots]\!]$ and that

$$\Delta(A) = \sum_{k=1}^{\infty} A^k \otimes \frac{A_k}{k!},$$

thus synthesizing the comultiplication of the individual coefficients in a single formula.

Let us obtain this formula for the plethystic bialgebra. In the completion of $\mathcal{P}$ define the *Green function* to be the series

$$A := \sum_{\lambda} a_{\lambda} \quad \left(\text{recall that } a_{\lambda} = \frac{A_{\lambda}}{\mathrm{aut}(\lambda)}\right).$$

Proposition 3.5.1. *Let* $a_k := \sum_{|\lambda|=k} a_{\lambda}$. *Then*

$$\Delta(A) = \sum_k A^k \otimes a_k.$$

Proof. This could be proved directly in $\mathcal{P}$, but we will prove it more elegantly by deriving an equivalence of groupoids related to $\mathrm{T}\mathbf{S}$ and taking homotopy cardinality. First of all, let G be the inclusion $\mathbb{S} \xhookrightarrow{G} \mathrm{T}_1\mathbf{S}$ taking $a \twoheadrightarrow b$ to

$$\begin{tikzcd} & a \arrow[dl, two heads] \arrow[dr, two heads] & \\ b \arrow[rr, two heads] & & 1, \end{tikzcd}$$

It is clear that

$$|G| = \sum_{\lambda} \frac{A_{\lambda}}{|\mathrm{Aut}(\lambda)|} = \sum_{\lambda} a_{\lambda} = A.$$

Denote by $\mathbb{T}$ the subgroupoid of connected objects of $\mathrm{T}_2\mathbf{S}$. Observe that $d_1^*(G)$ is precisely the inclusion $\mathbb{T} \hookrightarrow \mathrm{T}_2\mathbf{S}$. Therefore $\Delta(G)$ is the map $\mathbb{T} \xrightarrow{(d_2,d_0)_! d_1^*(G)} \mathrm{T}_1\mathbf{S} \times \mathrm{T}_1\mathbf{S}$. Now, since $\mathrm{T}\mathbf{S}$ is a Segal space we have that $\mathrm{T}_2\mathbf{S} \simeq \mathrm{T}_1\mathbf{S} \times_{\mathbb{B}} \mathrm{T}_1\mathbf{S}$, where $\mathbb{B} = \mathrm{T}_0\mathbf{S}$ is the groupoid of finite sets and bijections. As a consequence $\mathbb{T} \simeq \mathrm{T}_1\mathbf{S} \times_{\mathbb{B}} \mathbb{S}$. In pictures this equivalence looks like

$$\mathbb{T} \simeq \left\{ \begin{tikzcd}[ampersand replacement=\&] \& * \arrow[dl, two heads] \arrow[dr, two heads] \& \\ * \arrow[rr, two heads] \& \& t_{11}, \end{tikzcd} \quad \begin{tikzcd}[ampersand replacement=\&] \& * \arrow[dl, two heads] \arrow[dr, two heads] \& \\ t_{11} \arrow[rr, two heads] \& \& 1, \end{tikzcd} \right\}$$

We can decompose this as the homotopy sum of its fibers

$$\mathbb{T} \simeq \int^{b \in \mathbb{B}} (\mathrm{T}_1\mathbf{S})_b \times {}_b\mathbb{S},$$

where now the right b subscript means d_0-fiber over b and the left b subscript means d_1-fiber over b. A fancier way to express this equation is

$$\mathbb{T} \simeq \int^{b \in \mathbb{B}} \mathbb{S}^b \times \mathbb{B}^b,$$

in view of the equivalences $(\mathrm{T}_1\mathbf{S})_b \simeq \mathbb{S}^b$ and ${}_b\mathbb{S} \simeq \mathbb{B}^b$.

Let us take homotopy cardinality. To compute $\Delta(A)$ we only need to know $|\mathbb{S}^b \to \mathrm{T}_1\mathbf{S}|$ and $|\mathbb{B}^b \to \mathrm{T}_1\mathbf{S}|$. The latter, if we call $|b| = k$, is clearly

$$|\mathbb{B}^b \to \mathrm{T}_1\mathbf{S}| = \sum_{|\lambda|=k} a_{\lambda} =: a_k.$$

For the former, just notice that $\mathbb{S}^b \to \mathsf{T}_1\mathbf{S} \simeq (\mathbb{S} \to \mathsf{T}_1\mathbf{S})^{\odot k}$, since $(\mathsf{T}_1\mathbf{S})_b \simeq (\mathsf{T}_1\mathbf{S})_1^{+k}$. Here $\odot$ is the monoidal product in $\mathbf{Grpd}_{/\mathsf{T}_1\mathbf{S}}$. This implies that

$$|\mathbb{S}^b \to \mathsf{T}_1\mathbf{S}| = |(\mathbb{S} \to \mathsf{T}_1\mathbf{S})^{\odot k}| = |\mathbb{S} \to \mathsf{T}_1\mathbf{S}|^k = A^k,$$

and therefore

$$\Delta(A) = \sum_k A^k \otimes a_k,$$

as asserted. $\square$

Remark 3.5.2. We shall see later that the plethystic bialgebra is actually the incidence bialgebra of a certain locally finite operad (see Theorem 7.0.2 and Proposition 8.1.5). Once this is established, the above plethystic Faà di Bruno formula can actually be seen as a special case of the abstract Faà di Bruno formula for operads of Kock–Weber [42].

4 Monads and operads

As mentioned in the introduction, the $\mathcal{T}$-construction fits neatly within the context of generalized operads and strong monads. The following discussion of generalized operads is taken from [46]. Let us start by expressing the notions of category and of nonsymmetric operad in this setting.

A small category C can be described by sets and functions

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd} \qquad \begin{array}{c} C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \\ \\ C_0 \xrightarrow{e} C_1 \end{array}$$

where the pullback is taken along $C_1 \xrightarrow{s} C_0 \xleftarrow{t} C_1$, satisfying associativity and identity axioms, which can be expressed with commutative diagrams in **Set** (see Appendix A.1). The set C_0 is the set of objects and C_1 is the set of arrows of C. The map s returns the source of an arrow and t returns its target. The maps m and e represent composition and identities.

4.1 Classical operads

Recall that a nonsymmetric operad in **Set** is a collection

$$O := \coprod_{n \geqslant 0} O(n)$$

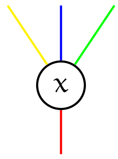
of sets together with composition maps

$$\circ : O(k) \times \big(O(n_1) \times \cdots \times O(n_k)\big) \longrightarrow O(n_1 + \cdots + n_k)$$

and a distinguished element $1 \in O(1)$, called the identity. These maps have to satisfy the following identity and associativity axioms:

$$\begin{aligned} x \circ (1, \ldots, 1) = 1 \circ x = x, \\ \big(x \circ (x_1, \ldots, x_n)\big) \circ \big(x_1^1, \ldots, x_1^{m_1}, \ldots, x_n^1, \ldots, x_n^{m_n}\big) \\ = x \circ \big(x_1 \circ (x_1^1, \ldots, x_1^{m_1}), \ldots, x_n \circ (x_n^1, \ldots, x_n^{m_n})\big). \end{aligned}$$

A symmetric operad consists on the same data as above together with an action of the symmetric group $\mathfrak{S}_n$ on $O(n)$ compatible with $\circ$. A more compact way to define an operad is as a monoid in a monoidal category of symmetric sequences [40], which are equivalent to species [44]. Different notions of species give different flavors of operads, such as nonsymmetric, symmetric, and colored operads. Recall that operations of classical operads are pictured as



Let us define nonsymmetric operads in a similar way as the category C above. First we take the opportunity to recall what a monad is. Let $\mathcal{E}$ be a category. A *monad* is a triple (P, μ, η) where $P\colon \mathcal{E} \to \mathcal{E}$ is a functor, and $\mu\colon P^2 \to P$ and $\eta\colon 1_{\mathcal{E}} \to P$ are natural transformations, called *multiplication* and *unit*, satisfying the following commutative squares:

$$\begin{tikzcd} P^3 \arrow[r, "P(\mu)"] \arrow[d, "\mu_P"'] & P^2 \arrow[d, "\mu"] \\ P^2 \arrow[r, "\mu"'] & P, \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "P(\eta)"] \arrow[d, "\eta_P"'] \arrow[dr, equal] & P^2 \arrow[d, "\mu"] \\ P^2 \arrow[r, "\mu"'] & P. \end{tikzcd}$$

Let $M\colon \mathbf{Set} \to \mathbf{Set}$ be the free monoid monad: it sends a set A to $\coprod_{n\in\mathbb{N}} A^n$ (see Example 4.4.4 below). Then a nonsymmetric operad can be described as consisting of sets and functions

$$\begin{tikzcd} & Q_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ MQ_0 & & Q_0 \end{tikzcd} \qquad \begin{array}{c} MQ_1 \times_{MQ_0} Q_1 \xrightarrow{m} Q_1 \\ \\ Q_0 \xrightarrow{e} Q_1 \end{array} \tag{4.1.1}$$

satisfying associativity and identity axioms, which can be expressed with commutative diagrams in **Set** (see Appendix A.2) and involve the monad structure on M. The set Q_0 is the set of objects and Q_1 is the set of operations of Q. The map s assigns to an operation the sequence of objects constituting its source, and t returns its target. The maps m and e represent composition and identities.

4.2 P-*operads*

The above characterization of nonsymmetric **Set** operads can be generalized to any ambient category and any monad P as long as they are cartesian. The classical case is **Set**; we shall be concerned also with **Grpd**.

Definition 4.2.1. A category is *cartesian* if it has all pullbacks. A functor is *cartesian* if it preserves pullbacks. A natural transformation is *cartesian* if all its naturality squares are pullbacks. A monad (P, μ, η) is *cartesian* if P is cartesian as a functor and μ and η are cartesian natural transformations.

Given a cartesian category $\mathcal{E}$ and a cartesian monad (P, μ, η), we define $\mathcal{E}_{(P)}$ as the bicategory whose 0-cells are the objects E of $\mathcal{E}$, whose 1-cells $E \to E'$ are spans $PE \leftarrow M \to E'$, and 2-cells are the usual morphisms $M \to N$ between spans:

$$
\begin{tikzcd}
 & M \arrow[dl, "d"'] \arrow[dr, "c"] \arrow[dd] & \\
PE & & E' \\
 & N. \arrow[ul, "q"] \arrow[ur, "p"'] &
\end{tikzcd}
$$

Given two 1-cells

$$
\begin{tikzcd}
 & M \arrow[dl, "d"'] \arrow[dr, "c"] & & & M' \arrow[dl, "d'"'] \arrow[dr, "c'"] & \\
PE & & E' & PE' & & E''
\end{tikzcd}
$$

the composite is given by taking a pullback and using the multiplication μ of P, and the 1-cell identity is given by η and id. They are shown in the following diagram:

$$
\begin{tikzcd}
 & & & N \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & & & & E \arrow[dl, "\eta_E"'] \arrow[dr, "id"] & \\
 & & PM \arrow[dl, "Pd"'] \arrow[dr, "Pc"] & & M' \arrow[dl, "d'"'] \arrow[dr, "c'"] & & & PE & & E \\
 & P^2E \arrow[dl, "\mu_E"'] & & PE' & & E'' & & & & \\
PE & & & & & & & & &
\end{tikzcd}
$$

Composition and identity of 2-cells are obvious. Since composition assumes a global choice of pullbacks, and since the pasting of two chosen pullbacks is not generally a chosen pullback, composition is associative up to coherent isomorphism. The coherence 2-cells are defined using the universal property of the pullback.

Definition 4.2.2 (Burroni [12])**.** Let P be a cartesian monad in a cartesian category $\mathcal{E}$. A P-*operad* is a monad in the bicategory $\mathcal{E}_{(P)}$.

This means that a P-operad Q consists precisely of objects Q_0 and Q_1 of $\mathcal{E}$ together with maps s, t, composition m and identities e as in Diagram (4.1.1) satisfying associativity and identity axioms (Appendix A.2). A *morphism* $Q \to Q'$ of P-operads is defined as a pair of arrows $Q_0 \xrightarrow{f_0} Q_0'$, $Q_1 \xrightarrow{f_1} Q_1'$, satisfying the following diagrams,

$$
\begin{tikzcd}
 & Q_1 \arrow[dl] \arrow[dr] \arrow[dd, "f_1"] & \\
PQ_0 \arrow[dd, "Pf_0"'] & & Q_0 \arrow[dd, "f_0"] \\
 & Q_1' \arrow[dl] \arrow[dr] & \\
PQ_0' & & Q_0',
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_0 \arrow[r, "e"] \arrow[d, "f_0"'] & Q_1 \arrow[d, "f_1"] \\
Q_0' \arrow[r, "e"'] & Q_1',
\end{tikzcd}
\qquad
\begin{tikzcd}
PQ_1 \times_{PQ_0} Q_1 \arrow[r, "m"] \arrow[d, "Pf_1 \times_{Pf_0} f_1"'] & Q_1 \arrow[d, "f_1"] \\
PQ_1' \times_{PQ_0'} Q_1' \arrow[r, "m"'] & Q_1',
\end{tikzcd}
\tag{4.2.1}
$$

regarding compatibility with the spans, identities and composition maps. Notice that this is not an arrow in $\mathcal{E}_{(\mathsf{P})}$. The category of P-operads is denoted P-**Operad**.

4.3 *Morphisms of spans*

In Chapter 5 we will deal with morphisms between long horizontal composites of spans. It is thus worth to set up a framework for such morphisms: consider the following diagrams, named blocks, made of maps in $\mathcal{E}$,

$$\begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot & \cdot \arrow[l] \arrow[r] & \cdot \end{tikzcd} \tag{4.3.1}$$

$$\begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] & \cdot & \cdot \arrow[l] \arrow[r] & \cdot \arrow[d] \\ \cdot & & \cdot \arrow[ll] \arrow[rr] \arrow[ul] \arrow[ur] \arrow[u, phantom, "\wedge"] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[d] & & \cdot \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \cdot \arrow[d] \\ \cdot & \cdot \arrow[l] \arrow[r] & \cdot & \cdot \arrow[l] \arrow[r] & \cdot \end{tikzcd} \tag{4.3.2}$$

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \arrow[r] & \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] \\ & & \cdot & \cdot \arrow[l] \arrow[r] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] & \cdot \\ \cdot & \cdot \arrow[l] \arrow[r] & \cdot & & \end{tikzcd} \tag{4.3.3}$$

Notice that (4.3.2) induce isomorphisms of spans if the vertical maps are isomorphisms, since in this case they represent horizontal composition of spans. Diagram (4.3.1) is an isomorphism when all the vertical arrows are isomorphisms, and (4.3.3) are isomorphisms when all the vertical arrows and the span projected away are isomorphisms. Besides, the blocks can be horizontally and vertically attached in the obvious way to get morphisms of longer spans, with the only restriction that the diagrams (4.3.3) can be attached to the right and to the left respectively.

Lemma 4.3.1. *Any pasting of blocks defines a morphism between the limit of the top row and the limit of the bottom row. Moreover, such a morphism is an isomorphism if it can be constructed from blocks that are isomorphisms.*

The morphisms between long spans are pictured with diagrams

$$\begin{tikzcd} \boldsymbol{\cdot} \arrow[d] & \cdot \arrow[d, phantom, "\vdots"] & \cdot \arrow[l] & \cdots & \cdot \arrow[r] & \cdot \arrow[d, phantom, "\vdots"] \\ \boldsymbol{\cdot} & \cdot & \cdot \arrow[l] & \cdots & \cdot \arrow[r] & \cdot \end{tikzcd}$$

where the left bold part is the limit of the diagram: the upper dot is the limit of the upper row, and same for the bottom row. Observe that the decomposition of a morphism into blocks is not unique, and there may be decompositions of isomorphisms whose blocks are not necessarily isomorphisms. Here is an example that will be used later on.

Example 4.3.2. The following diagram represents an isomorphism of composites of spans:

$$
\begin{tikzcd}
\cdot \arrow[d, equal] & \cdot \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "g"] & \cdot \arrow[l, "c"'] \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] \\
\cdot & \cdot \arrow[l, "a'"] \arrow[r, "b'"'] & \cdot & \cdot \arrow[l, "c'"] \arrow[r] & \cdot
\end{tikzcd}
\tag{4.3.4}
$$

Indeed, it can be expressed by pasting isomorphism blocks:

$$
\begin{tikzcd}
\cdot \arrow[d, equal] & & \cdot \arrow[ll, "a"'] \arrow[rr, "b"] \arrow[dl, "f"'] \arrow[dr, "b"] \arrow[d, phantom, "\vee", very near start] & & \cdot \arrow[d, equal] & \cdot \arrow[l, "c"'] \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] \\
\cdot \arrow[d, equal] & \cdot \arrow[l, "a'"'] \arrow[r, "b'"] \arrow[d, equal] & \cdot \arrow[r, leftarrow, "g"] \arrow[d, equal] & \cdot \arrow[r, equal] & \cdot & \cdot \arrow[l, "c"'] \arrow[r] & \cdot \arrow[d, equal] \\
\cdot & \cdot \arrow[l, "a'"] \arrow[r, "b'"'] & \cdot & & \cdot \arrow[ll, "c'"] \arrow[rr] \arrow[ul, "c"] \arrow[ur, equal] \arrow[u, phantom, "\wedge", very near start] & & \cdot
\end{tikzcd}
\tag{4.3.5}
$$

4.4 *Strong monads*

We now recall the notion of strong monad [41], which is central in the $\mathcal{T}$-construction. From now on the ambient category $\mathcal{E}$ is required to have a terminal object, hence all finite limits.

Definition 4.4.1. Let (P, μ, η) be a monad on $\mathcal{E}$. A *strength* for P is a natural transformation with components $D_{A,B} \colon A{\times}PB \to P(A{\times}B)$, satisfying the following two axioms concerning tensoring with 1 and consecutive applications of D,

$$
\begin{tikzcd}
1{\times}PA \arrow[r, "D_{1,A}"] \arrow[dr, "p_2"'] & P(1{\times}A) \arrow[d, "Pp_2"] \\
& PA
\end{tikzcd}
\tag{4.4.1a}
$$

$$
\begin{tikzcd}
(A{\times}B){\times}PC \arrow[r] \arrow[d, "D_{A\times B,C}"'] & A{\times}(B{\times}PC) \arrow[r, "A\times D_{B,C}"] & A{\times}P(B{\times}C) \arrow[d, "D_{A,B\times C}"] \\
P((A{\times}B){\times}C) \arrow[rr] & & P(A{\times}(B{\times}C))
\end{tikzcd}
\tag{4.4.1b}
$$

and two axioms concerning compatibility with monad unit and multiplication,

$$
\begin{tikzcd}
A{\times}B \arrow[r, "A\times\eta_B"] \arrow[dr, "\eta_{A\times B}"'] & A{\times}PB \arrow[d, "D_{A,B}"] \\
& P(A{\times}B)
\end{tikzcd}
\tag{4.4.2a}
$$

$$
\begin{tikzcd}
A{\times}P^2B \arrow[r, "D_{A,PB}"] \arrow[d, "A\times\mu_B"'] & P(A{\times}PB) \arrow[r, "PD_{A,B}"] & P^2(A{\times}B) \arrow[d, "\mu_{A\times B}"] \\
A{\times}PB \arrow[rr, "D_{A,B}"'] & & P(A{\times}B)
\end{tikzcd}
\tag{4.4.2b}
$$

Before seeing some examples of P-operads and strong monads, we prove the following lemma, which will be useful in Chapter 5.

Lemma 4.4.2. *Let* u *be the unique morphism* $u\colon P1 \to 1$*. Then the square*

$$\begin{tikzcd} A\times P^2 1 \arrow[r, "A\times Pu"] \arrow[d, "D_{A,P1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A\times P1 \arrow[d, "D_{A,1}"] \\ P(A\times P1) \arrow[r, "P(A\times u)"'] & P(A\times 1) \end{tikzcd} \tag{4.4.3}$$

is a pullback.

Proof. Observe that if we project the bottom rows of this square to the second component,

$$\begin{tikzcd} A\times P^2 1 \arrow[r, "A\times Pu"] \arrow[d, "D_{A,P1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A\times P1 \arrow[d, "D_{A,1}"] \\ P(A\times P1) \arrow[r, "P(A\times u)"] \arrow[d, "Pp_2"'] & P(A\times 1) \arrow[d, "Pp_2"] \\ P^2 1 \arrow[r, "Pu"'] & P1, \end{tikzcd}$$

then the lower square is a pullback because P is cartesian, and the outer square is a pullback because it is a projection, by (4.4.1a). Therefore the upper square is a pullback too. □

Let us see some examples of strong monads.

Example 4.4.3. Obviously the identity monad is strong. If we take the identity monad Id on any cartesian cartesian category $\mathcal{E}$ then a Id-operad is the same as a category internal to $\mathcal{E}$, and a noncolored Id-operad is a monoid in $\mathcal{E}$. In particular if $\mathcal{E} = \mathbf{Set}$ they are small categories and monoids, respectively.

Example 4.4.4. Let (M, μ, η) be the free monoid monad on the category $\mathcal{E}{=}\mathbf{Set}$. As mentioned above, a M-operad is the same thing as a nonsymmetric operad. Here is the full explicit description of M. Let A be a set and $a_0, \dots, a_n {\in} A$, then

$$\begin{aligned} MA &= \bigsqcup_{n\in N} A^n, \\ \eta_A(a_0) &= (a_0), \\ \mu_A\big((a_1,\dots,a_i),\dots,(a_j,\dots,a_n)\big) &= (a_1,\dots,a_n). \end{aligned} \tag{4.4.4}$$

The free monoid monad is strong with the following strength:

$$\begin{aligned} D_{A,B}\colon A\times MB &\longrightarrow M(A\times B) \\ (a,(b_1,\dots,b_n)) &\longrightarrow \big((a,b_1),\dots,(a,b_n)\big). \end{aligned}$$

It is straightforward to check that the diagrams (4.4.2b) and (4.4.2a) are satisfied and clear that $D_{A,B}$ is injective. This last feature is relevant because to define the $\mathcal{T}$-construction, in Chapter 5, it is be necessary that D_{1,C_0} is a monomorphism.

Example 4.4.5. The free semigroup monad M^r on **Set** is defined in the same way as the free monoid monad, except that in this case $M^r A = \bigsqcup_{n \geqslant 1} A^n$. This means that a M^r-operad is a nonsymmetric operad without nullary operations. The terminal M^r-operad be denoted Ass, which is of course the reduced associative operad. Notice that M^r is also a strong cartesian monad on **Grpd**. In this sense the operad Ass can also be considered as an M^r-operad in **Grpd**, with discrete groupoid of objects and discrete groupoid of operations. The context will suffice to distinguish between **Set** and **Grpd**, but in the main applications (Chapter 7) we work over **Grpd**.

Example 4.4.6. Let Y be a monoid. Denote by Y the monad on **Set** given by $YA = Y \times A$ with unit and multiplication given by those of Y. Then Y is strong with strength given by the associator of the cartesian product. Therefore in this case the strength is an isomorphism. The same holds if Y is a monoid in **Grpd** and Y is then a monad on **Grpd**.

Example 4.4.7. Let (S, μ, η) be the free symmetric monoidal category monad on **Grpd**. An S-operad is an operad internal to groupoids, so that it has a groupoid of colors and a groupoid of operations. Let A be a groupoid and $\mathfrak{S}_n$ the symmetric group on n elements. The monad S acts on A by

$$SA = \bigsqcup_{n \in \mathbb{N}} A^n /\!\!/ \mathfrak{S}_n,$$

where $/\!\!/$ means homotopy quotient [2, 30]. Hence it is analogous to M, but we add an arrow

$$(a_1, \ldots, a_n) \xrightarrow{\sigma} (a_{\sigma 1}, \ldots, a_{\sigma n})$$

for every element $\sigma \in \mathfrak{S}_n$. The multiplication and unit natural transformations are defined as in (4.4.4) for both objects and operations. Notice that any symmetric operad Q is in particular an S-operad, where the groupoid of objects Q_0 is discrete and the groupoid Q_1 has only the arrows coming from the permutations of its source sequence. In other words, a symmetric operad is an S-operad

$$SQ_0 \xleftarrow{s} Q_1 \xrightarrow{t} Q_0$$

such that Q_0 is discrete and s is a discrete fibration. The strength for S is defined the same way as for M,

$$\begin{aligned} D_{A,B} \colon A \times SB &\longrightarrow S(A \times B) \\ (a, (b_1, \ldots, b_n)) &\longrightarrow ((a, b_1), \ldots, (a, b_n)), \end{aligned}$$

and it is again a monomorphism, since it is injective both on objects and morphisms.

Observe that symmetric operads cannot be expressed as P-operads in **Set**, since the actions of the symmetric groups have to be encoded necessarily as morphisms in Q_1. Also, the only monad P one could attempt to use to define them is the free commutative monoid monad, but it is not cartesian.

Example 4.4.8. As for M and M^r, we can remove the empty sequence from S to get a monad S^r on **Grpd** whose operads do not have nullary operations. We denote by Sym the terminal S^r-operad, which is the reduced commutative operad.

4.5 *The two-sided bar construction for* P*-operads*

The two-sided bar construction for operads is standard [54]. In this section we introduce the construction in the more general setting of P-operads by using induced monads. Any P-operad Q defines a monad (Q, μ^Q, η^Q) on the slice category of $\mathcal{E}$ over Q_0

$$Q : \mathcal{E}/Q_0 \longrightarrow \mathcal{E}/Q_0,$$

given by pullback and composition, as shown in the following diagram for an element $X \xrightarrow{f} Q_0$ of $\mathcal{E}/Q_0$

$$\begin{tikzcd}
 & QX \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
PX \arrow[dr, "Pf"'] & & Q_1 \arrow[dl, "s"] \arrow[dr, "t"] & \\
 & PQ_0 & & Q_0.
\end{tikzcd} \tag{4.5.1}$$

The image of f is thus the red composite. The multiplication μ^Q and the unit η^Q are defined by the following morphisms

$$\begin{tikzcd}
Q^2X \arrow[dd, "\mu^Q_X"'] & P^2X \arrow[r, "P^2f"] \arrow[d, equal] & P^2Q_0 \arrow[d, equal] & PQ_1 \arrow[l, "Ps"'] \arrow[r, "Pt"] & PQ_0 & Q_1 \arrow[l, "s"'] \arrow[r, "t"] & Q_0 \arrow[d, equal] \\
 & P^2X \arrow[r, "P^2f"] \arrow[d, "\mu_X"] & P^2Q_0 \arrow[d, "\mu_{Q_0}"] & & PQ_1 \times_{PQ_0} Q_1 \arrow[ll] \arrow[rr] \arrow[ul, "s"] \arrow[ur, "t"'] \arrow[u, phantom, "\wedge"] \arrow[d, "m"] & & Q_0 \arrow[d, equal] \\
QX & PX \arrow[r, "Pf"] & PQ_0 & & Q_1 \arrow[ll, "s"] \arrow[rr, "t"'] & & Q_0,
\end{tikzcd} \tag{4.5.2}$$

$$\begin{tikzcd}
X \arrow[d, "\eta^Q_X"'] & X \arrow[r, "f"] \arrow[d, "\eta_X"] & Q_0 \arrow[r, equal] \arrow[d, "\eta_{Q_0}"] & Q_0 \arrow[r, equal] \arrow[d, "e"] & Q_0 \arrow[d, equal] \\
P_QX & X \arrow[r, "Pf"'] & PQ_0 & Q_1 \arrow[l, "s"] \arrow[r, "t"'] & Q_0.
\end{tikzcd} \tag{4.5.3}$$

Definition 4.5.1. An *algebra* over the P-operad Q is an algebra over the monad Q.

Notice that the category $\mathcal{E}/Q_0$ has a terminal object, $Q_0 \xrightarrow{1} Q_0$, so that there is an algebra over Q given by the unique arrow $q : Q1 \to 1$. Moreover, since $\mathcal{E}$ has a terminal object, the P-operad $P : \mathcal{E}/1 \to \mathcal{E}/1$ itself can be represented by the span

$$P1 \longleftarrow P1 \longrightarrow 1,$$

and is the terminal P-operad. Now, the terminal arrow $u : Q_0 \to 1$ induces, by postcomposition, a functor $u_! : \mathcal{E}/Q_0 \to \mathcal{E}/1$. The diagram

$$\begin{tikzcd}
PQ_0 \arrow[d, "Pu"'] & Q_1 \arrow[l] \arrow[r] \arrow[d] & Q_0 \arrow[d, "u"] \\
P1 & P1 \arrow[l] \arrow[r] & 1
\end{tikzcd} \tag{4.5.4}$$

represents a natural transformation $u_!Q \overset{\phi}{\Rightarrow} Pu_!$ which is compatible with the comultiplications and units of Q and P, meaning that

$$u_!Q^2 \xRightarrow{\phi^2} P^2u_! \xRightarrow{\mu u_!} Pu_! = u_!Q^2 \xRightarrow{u_!\mu^Q} u_!Q \overset{\phi}{\Rightarrow} Pu_! \qquad \text{and}$$
$$u_! \xRightarrow{u_!\eta^Q} u_!Q \overset{\phi}{\Rightarrow} Pu_! = u_! \xRightarrow{\eta u_!} Pu_!$$

or, equivalently,

$$\begin{tikzcd}
PQ_0 \arrow[d, "Pu"'] & P^2Q_0 \arrow[l, "\mu_{Q_0}"'] \arrow[d, "P^2u"] & PQ_1 \arrow[l] \arrow[r] \arrow[d] & PQ_0 \arrow[d, "Pu"'] & Q_1 \arrow[l] \arrow[r] \arrow[d] & Q_0 \arrow[d, "u"] \\
P1 \arrow[dd, equal] & P^21 \arrow[l, "\mu_1"] & P^21 \arrow[l] \arrow[r, "Pu"] & P1 \arrow[d, phantom, "\wedge"] & P1 \arrow[l] \arrow[r] & 1 \arrow[dd, equal] \\
 & & & P^21 \arrow[ul] \arrow[ur, "Pu"'] \arrow[d, "\mu_1"] & & \\
P1 & & & P1 \arrow[lll] \arrow[rr] & & 1
\end{tikzcd}
=
\begin{tikzcd}
PQ_0 \arrow[dd, equal] & P^2Q_0 \arrow[l, "\mu_{Q_0}"'] & PQ_1 \arrow[l] \arrow[r] & PQ_0 \arrow[d, phantom, "\wedge"] & Q_1 \arrow[l] \arrow[r] & Q_0 \arrow[dd, equal] \\
 & & & Q_2 \arrow[ul] \arrow[ur] \arrow[d] & & \\
PQ_0 \arrow[d, "Pu"'] & & & Q_1 \arrow[lll] \arrow[rr] \arrow[d] & & Q_0 \arrow[d, "u"] \\
P1 & & & P1 \arrow[lll] \arrow[rr] & & 1,
\end{tikzcd} \tag{4.5.5}$$

$$\begin{tikzcd}
Q_0 \arrow[r, equal] \arrow[d, "\eta_{Q_0}"'] & Q_0 \arrow[r, equal] \arrow[d, "e"] & Q_0 \arrow[d, equal] \\
PQ_0 \arrow[d, "Pu"'] & Q_1 \arrow[l] \arrow[r] \arrow[d] & Q_0 \arrow[d, "u"] \\
P1 & P1 \arrow[l] \arrow[r] & 1.
\end{tikzcd}
=
\begin{tikzcd}
Q_0 \arrow[r, equal] \arrow[d, "u"'] & Q_0 \arrow[r, equal] \arrow[d, "u"] & Q_0 \arrow[d, "u"] \\
1 \arrow[d, "\eta_1"'] & 1 \arrow[l] \arrow[r] \arrow[d, "\eta_1"] & 1 \arrow[d, equal] \\
P1 & P1 \arrow[l] \arrow[r] & 1.
\end{tikzcd} \tag{4.5.6}$$

Lemma 4.5.2. *The natural transformation ϕ is cartesian.*

Proof. Let us describe the naturality squares of ϕ. Let H be a map in $\mathcal{E}/Q_0$, that is, a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & Q_0. &
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
PX \underset{PQ_0}{\times} Q_1 \arrow[r, "u_!QH"] \arrow[d, "\phi_X"'] & PY \underset{PQ_0}{\times} Q_1 \arrow[r] \arrow[d, "\phi_Y"'] \arrow[dr, phantom, "\lrcorner"] & Q_1 \arrow[d, "s"] \\
PX \arrow[r, "Pu_!H"'] & PY \arrow[r, "Pg"'] & PQ_0.
\end{tikzcd}$$

From (4.5.1) it is clear that the pullback square on the right is precisely the definition of $u!Qg$. From (4.5.1) and (4.5.4) we have that the square on the left is the naturality square for ϕ at H, and moreover that ϕ_X and ϕ_Y are projections. But $Pu_!H = Ph$ and $Pg \circ Ph = Pf$, so that the composite square is precisely the definition of $u!Qf$, which is a pullback. As a consequence, the naturality square is a pullback too. □

Given a P-operad Q, we define its *two-sided bar construction* [42, 54, 74]

$$\mathcal{B}Q : \mathbb{\Delta}^{op} \longrightarrow \mathcal{E}$$

as the two-sided bar construction of Q, ϕ and the terminal algebra 1. This means that the space of n-simplices $\mathcal{B}_n Q$ is given by

$$Pu_! Q^n 1,$$

the inner face maps are given by the monad multiplication μ^Q, the bottom face map is given by $c : Q1 \to 1$ and the top face maps are given by ϕ and μ. Similarly, the degeneracy maps are given by η^Q. Diagrams (4.5.5) and (4.5.6) and the monad axioms for P and Q guarantee that the simplicial identities are satisfied.

In practice, the bar construction of Q is simply

$$PQ_0 \leftleftarrows PQ_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; PQ_2 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; PQ_3 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots, \tag{4.5.7}$$

where

(i) $Q_2 := PQ_1 \underset{PQ_0}{\times} Q_1$ and $Q_3 := P^2 Q_1 \underset{P^2 Q_0}{\times} PQ_1 \underset{PQ_0}{\times} Q_1$, etc.;

(ii) the bottom face maps d_0 are induced by t;

(iii) the top face maps d_n are induced by s and μ;

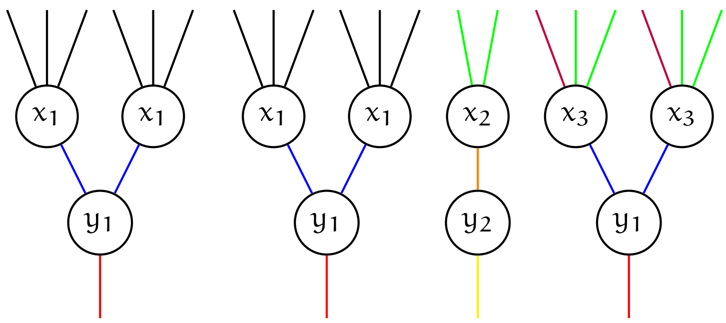
(iv) the inner face maps are induced by m and μ, and

(v) the degeneracy maps are induced by e and η.

Henceforth we will indiscriminately use this simplicial notation. Let us see some examples.

Example 4.5.3. Let C be a small category. Hence C is a Id-operad in **Set**. Then $\mathcal{B}C$ is the nerve of C. Moreover, we can consider C as a category internal to **Grpd** whose groupoid of objects has as morphisms the isomorphisms of C, and whose groupoid of arrows has as morphisms the isomorphisms of the arrow category of C. In this case $\mathcal{B}C$ is the fat nerve of C, whose groupoid of n-simplices is the groupoid $\mathrm{Map}(\Delta[n], C)$. In the theory of incidence coalgebras, this is often more interesting than the ordinary nerve, cf. [15, 32, 33].

Example 4.5.4. If Q is a symmetric operad, as in Example 4.4.7, then $\mathcal{B}Q$ is the usual operadic two-sided bar construction. Its n-simplices have as objects forests of n-level Q-trees, and as morphisms permutations at each level. For example, the following picture



is an object of PQ_2 with $(2! \cdot 2!^2 \cdot 3!^4) \cdot (2!) \cdot (2! \cdot 2!^2)$ automorphisms.

The following result is a reformulation of [74, Proposition 4.4.1] and [42, Proposition 3.3] in the context of P-operads.

Proposition 4.5.5. *The simplicial object $\mathcal{B}Q$ is a strict category object.*

Proof. We have to check that the squares

$$\begin{tikzcd} \mathcal{B}_{n+2}Q \arrow[r, "d_0"] \arrow[d, "d_{n+2}"'] & \mathcal{B}_{n+1}Q \arrow[d, "d_{n+1}"] \\ \mathcal{B}_{n+1}Q \arrow[r, "d_0"'] & \mathcal{B}_n Q \end{tikzcd} \tag{4.5.8}$$

are pullbacks for $n \geqslant 0$. We show the case $n = 0$, the rest are similar. The square is given by

$$\begin{tikzcd} Pu_!QQ1 \arrow[r, "Pu_!Qc"] \arrow[d, "P(\phi_{Q1})"'] & Pu_!Q1 \arrow[d, "P(\phi_1)"] \\ PPu_!Q1 \arrow[r, "PPu_!c"] \arrow[d, "\mu^P_{u_!Q1}"'] & PPu_!1 \arrow[d, "\mu^P_{u_!1}"] \\ Pu_!Q1 \arrow[r, "Pu_!c"'] & Pu_!1. \end{tikzcd} \tag{4.5.9}$$

The bottom square is cartesian because it is a naturality square for μ^P, and P is a cartesian monad. The top square is P applied to a naturality square of ϕ, which is cartesian, by Lemma 4.5.2. Since P preserves pullbacks, the square is cartesian. □

This allows to obtain the following result, in the special case where $\mathcal{E} = \mathbf{Grpd}$.

Proposition 4.5.6. *Let $P : \mathbf{Grpd} \to \mathbf{Grpd}$ be a cartesian monad that preserves fibrations. Let Q be a P-operad such that Q_0 is a discrete groupoid. Then the simplicial groupoid $\mathcal{B}Q$ is a Segal groupoid.*

Proof. It is enough to see that the strict pullbacks (4.5.8) are also homotopy pullbacks. For $n = 0$, notice that $Pu_!Q1 \xrightarrow{Pu_!c} Pu_!1$ is precisely the map $PQ_1 \xrightarrow{Pm} PQ_0$. But since Q_0 is discrete, m is a fibration, which means that Pm is a fibration, because P preserves fibrations. This implies that the square is also a homotopy pullback. Moreover, since pullbacks preserve fibrations, the map $Pu_!QQ1 \xrightarrow{Pu_!Qc} Pu_!Q1$ is again a fibration. The same argument then implies that the square for $n = 1$ is also a homotopy pullback, and so on. □

Suppose now that $R : \mathcal{E} \to \mathcal{E}$ is another cartesian monad and that there is a cartesian monad map $P \overset{\psi}{\Rightarrow} R$. Then we can take the bar construction over R

$$\mathcal{B}^R Q : \Delta^{op} \longrightarrow \mathcal{E}$$

whose n-simplices are given by

$$Ru_!Q^n 1 \quad (\text{or } \; RQ_n).$$

In this case all the face maps coincide with the previous ones except the top face map, which is given by

$$Ru_!Q^{n+1}1 \xrightarrow{R(\varphi_{Q1})} RPu_!Q^n1 \xrightarrow{R(\psi_{u_!Q^n1})} RRu_!Q^n1 \xrightarrow{\mu^R_{u_!Q^n1}} Ru_!Q^n1.$$

Since ψ is cartesian, the simplicial object $\mathcal{B}^R$ is also a strict category object. Moreover, if R preserves fibrations, it is a Segal groupoid, for the same reason as $\mathcal{B}Q$ in Proposition 4.5.6. The main examples of this bar construction that we use come from the natural transformations $M^r \Rightarrow S$, $S^r \Rightarrow S$ and $M^r \Rightarrow M$, as in [42].

5

The $\mathcal{T}$-construction

Throughout this chapter (P, μ, η) is a cartesian strong monad on a cartesian category $\mathcal{E}$, and category means a category internal to $\mathcal{E}$. As mentioned in the introduction, the $\mathcal{T}$-construction consists of two constructions, one from internal categories to P-operads and another from P-operads to categories. With the purpose of reducing the diagrams and fiber products, we use the following notation for the endofunctors and natural transformations featuring in this chapter:

$$\begin{aligned} L : \mathcal{E} &\longrightarrow \mathcal{E} & F : \mathrm{Id} &\Longrightarrow L \\ A &\longmapsto A \times P1, & F_A : A \times 1 &\xrightarrow{\mathrm{id} \times \eta_1} LA, \\ D : L &\Longrightarrow P & R : L &\Longrightarrow \mathrm{Id} \\ D_A : A \times P1 &\longrightarrow PA, & R_A : A \times P1 &\xrightarrow{p_1} A. \\ D^2_A &:= D_{PA} \circ D_{A,P1}, & R^2_A &:= R_A \circ R_{LA}. \end{aligned}$$

Observe that L is cartesian as a functor. Also, notice that R and F are cartesian natural transformations. Finally, by monomorphism we refer to the 1-categorical notion. In the case of most interest where $\mathcal{E}$ is **Set** or **Grpd**, this means injective on objects and injective on arrows.

The material of this chapter is highly technical. The casual or application-oriented reader might wish to regard it as a black box and take on faith the well-definedness of the constructions, and still be able to appreciate the examples worked out in Chapters 6 and 7.

5.1 *From categories to* P*-operads*

Let C be a category such that $D_{C_0} : P1 \times C_0 \to PC_0$ is a monomorphism. It is convenient in this section to adopt a simplicial nomenclature. Hence C is represented by the span

$$\begin{tikzcd} & C_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\ C_0 & & C_0 \end{tikzcd} \qquad \begin{aligned} C_1 \times_{C_0} C_1 =: C_2 &\xrightarrow{d_1} C_1 \\ C_0 &\xrightarrow{e} C_1, \end{aligned}$$

with the only inconvenience that some of the face maps share their names. Notice that we still denote by e the degeneracy map s_0. We now construct a P-operad $\mathcal{T}_P C$ from the category C. To keep notation short, the simplicial nomenclature for $\mathcal{T}_P C$ is $\widetilde{C}_i$ for the simplices and $\widetilde{d}_i$ for the face maps. The span defining the objects and operations of $\mathcal{T}_P C$ is given by the pullback

$$
\begin{tikzcd}
& & \widetilde{C}_1 \arrow[dl, tail, "i_1"'] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] \arrow[ddll, bend right, "\widetilde{d}_1"'] \arrow[ddrr, bend left, "\widetilde{d}_0"] & & \\
& PC_1 \arrow[dl, "Pd_1"] \arrow[dr, "Pd_0"'] & & LC_0 \arrow[dl, tail, "D_{C_0}"'] \arrow[dr, "R_{C_0}"] & \\
PC_0 & & PC_0 & & C_0 .
\end{tikzcd}
\tag{5.1.1}
$$

Observe that $\widetilde{C}_0 = C_0$, so that $\mathcal{T}_P C$ has the same objects as C. Besides, the morphism i_1 is a monomorphism, since monomorphisms are preserved by pullbacks and D_{C_0} is a monomorphism.

To define composition we need to specify a map $\widetilde{C}_2 \xrightarrow{\widetilde{d}_1} \widetilde{C}_1$, where $\widetilde{C}_2 := P\widetilde{C}_1 \underset{PC_0}{\times} \widetilde{C}_1$, satisfying the axioms of Appendix A.1. However, to describe it we have to express $\widetilde{C}_2$ in a way we can naturally use composition in the original category $C_2 \xrightarrow{d_1} C_1$. The following diagram represents an isomorphism

$$
\widetilde{C}_2 \cong P^2C_1 \underset{P^2C_0}{\times} PLC_1 \underset{PLC_0}{\times} (C_0 \times P^2 1) =: \widetilde{C}_2',
$$

$$
\begin{tikzcd}
P^2C_0 \arrow[d, equal] & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "P^2d_0"] \arrow[d, equal] & P^2C_0 \arrow[d, equal] & & PLC_1 \arrow[ll] \arrow[rr, "PLd_0"] \arrow[dl, "PLd_1"'] \arrow[dr, "PR_{C_1}"] \arrow[d, phantom, "\vee\ (A)"] & & PLC_0 \arrow[d, "PR_{C_0}"'] \arrow[dr, phantom, "(B)"] & C_0 \times P^2 1 \arrow[l, "D_{C_0,P1}"'] \arrow[r, "p_1"] \arrow[d, "\mathrm{id} \times Pu"] \arrow[dl, phantom, very near start, "\llcorner"] & C_0 . \arrow[d, equal] \\
P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"] \arrow[r, "P^2d_0"'] & P^2C_0 & PLC_0 \arrow[l, "PD_{C_0}"] \arrow[r, "PR_{C_0}"'] & PC_0 & PC_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] & PC_0 & LC_0 \arrow[l, "D_{C_0}"] \arrow[r, "R_{C_0}"'] & C_0
\end{tikzcd}
\tag{5.1.2}
$$

It is clear that all the squares in (5.1.2) commute. Moreover, the square (A) is cartesian because R and P are cartesian, and the square (B) is the same as (4.4.3) of Lemma 4.4.2.

Definition 5.1.1. The composition of $\mathcal{T}_P C$ is given by the following arrow $\widetilde{C}_2' \xrightarrow{\widetilde{d}_1'} \widetilde{C}_1$,

$$
\begin{tikzcd}
\widetilde{C}_2' \arrow[dddd, "\widetilde{d}_1'"] & P^2C_0 \arrow[d, equal] & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "P^2d_0"] \arrow[d, equal] & P^2C_0 \arrow[d, equal] & PLC_1 \arrow[l] \arrow[r, "PLd_0"] \arrow[d, "PD_{C_1}"] \arrow[dr, phantom, "(A)"] & PLC_0 \arrow[d, "PD_{C_0}"] \arrow[dr, phantom, "(B)"] & C_0 \times P^2 1 \arrow[l, "D_{P1,C_0}"'] \arrow[r, "p_1"] \arrow[d, equal] & C_0 \arrow[d, equal] \\
& P^2C_0 \arrow[dd, equal] & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "P^2d_0"] \arrow[ddl, phantom, "(C)"] & P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "P^2d_0"'] \arrow[ddr, phantom, "(D)"] & P^2C_0 \arrow[dd, equal] \arrow[ddr, phantom, "(G)"] & C_0 \times P^2 1 \arrow[l, "D^2_{C_0}"] \arrow[r, "p_1"] \arrow[ddd, "\mathrm{id} \times \mu_1"] & C_0 \arrow[ddd, equal] \\
& & & P^2C_2 \arrow[ul, "P^2d_2"] \arrow[ur, "P^2d_0"'] \arrow[d, "P^2d_1"] \arrow[u, phantom, "\wedge"] & & & & \\
& P^2C_0 \arrow[d, "\mu_{C_0}"] \arrow[dr, phantom, "(E)"] & & P^2C_1 \arrow[ll, "P^2d_1"'] \arrow[rr, "P^2d_0"] \arrow[d, "\mu_{C_1}"] \arrow[dr, phantom, "(F)"] & & P^2C_0 \arrow[d, "\mu_{C_0}"] & & \\
\widetilde{C}_1 & PC_0 & & PC_1 \arrow[ll, "Pd_1"] \arrow[rr, "d_0"'] & & PC_0 & LC_0 \arrow[l, "D_{C_0}"] \arrow[r, "R_{C_0}"'] & C_0 .
\end{tikzcd}
\tag{5.1.3}
$$

It is clear that the diagram commutes: (A) is P applied to a naturality square of D; (B) is the definition of D^2; (C) and (D) are P^2 applied to axioms (A.1.1a) and (A.1.1b) for composition in C; (E) and (F) are naturality squares of μ, and (G) is again axiom (4.4.2b) for strong monads. The remaining squares are trivial.

Notice that from this definition it is clear that $\widetilde{d}_1$ satisfies axioms (A.2.1a) and (A.2.1b). Furthermore, there is a map

$$\widetilde{C}'_2 \xrightarrow{i'_2} P^2C_2,$$

given by the diagram

$$\begin{tikzcd}
P^2C_0 \arrow[d, equal] & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "Pd_0"] \arrow[d, equal] & P^2C_0 \arrow[d, equal] & PLC_1 \arrow[l] \arrow[r, "PLd_0"] \arrow[d, "PD_{C_1}"] & PLC_0 \arrow[d, "PD_{C_0}"] & C_0\times P^2 1 \arrow[l, "D_{C_0,P1}"'] \arrow[r] & C_0 \\
P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "P^2d_0"'] & P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"] \arrow[r, "P^2d_0"'] & P^2C_0, & &
\end{tikzcd} \tag{5.1.4}$$

which clearly makes the square

$$\begin{tikzcd}
\widetilde{C}'_2 \arrow[r, "i'_2"] \arrow[dd, "\widetilde{d}'_1"'] & P^2C_2 \arrow[d, "P^2d_1"] \\
& P^2C_1 \arrow[d, "\mu_{C_1}"] \\
\widetilde{C}_1 \arrow[r, tail, "i_1"'] & PC_1,
\end{tikzcd}
\quad \text{and therefore also the square} \quad
\begin{tikzcd}
\widetilde{C}_2 \arrow[r, "i_2"] \arrow[dd, "\widetilde{d}_1"'] & P^2C_2 \arrow[d, "P^2d_1"] \\
& P^2C_1 \arrow[d, "\mu_{C_1}"] \\
\widetilde{C}_1 \arrow[r, tail, "i_1"'] & PC_1,
\end{tikzcd} \tag{5.1.5}$$

commute, for the corresponding arrow i_2. This says, roughly speaking, that composition in $\mathcal{T}_P C$ is "the same" as composition in P^2C, as it is also clear in most of the examples.

We have to check that composition is associative (A.2.3). Let us first state the following lemma.

Lemma 5.1.2. *There is a map $\widetilde{C}_3 \xrightarrow{i_3} P^3C_3$ such that the following diagrams commute*

$$\begin{tikzcd}
\widetilde{C}_3 \arrow[r, "i_3"] \arrow[dd, "\widetilde{d}_1"'] & P^3C_3 \arrow[d, "P^3d_1"] \\
& P^3C_2 \arrow[d, "P\mu_{C_2}"] \\
\widetilde{C}_2 \arrow[r, "i_2"'] & P^2C_2,
\end{tikzcd} \tag{5.1.6a}$$

$$\begin{tikzcd}
\widetilde{C}_3 \arrow[r, "i_3"] \arrow[dd, "\widetilde{d}_2"'] & P^3C_3 \arrow[d, "P^3d_2"] \\
& P^3C_2 \arrow[d, "\mu_{PC_2}"] \\
\widetilde{C}_2 \arrow[r, "i_2"'] & P^2C_2.
\end{tikzcd} \tag{5.1.6b}$$

Proof. See Appendix B. □

Proposition 5.1.3. *Composition is associative.*

Proof. In view of Lemma 5.1.2 there is a diagram

$$\begin{tikzcd}
\widetilde{C}_3 \arrow[rrrr, "\widetilde{d}_1"] \arrow[dr, "i_3"] \arrow[dddd, "\widetilde{d}_2"'] & & & & \widetilde{C}_2 \arrow[dl, "i_2"] \arrow[dddd, "\widetilde{d}_1"] \\
& P^3C_3 \arrow[r, "P^3d_1"] \arrow[d, "P^3d_2"'] \arrow[dr, phantom, "(A)"] & P^3C_2 \arrow[r, "P\mu_{C_2}"] \arrow[d, "P^3d_1"] \arrow[dr, phantom, "(B)"] & P^2C_2 \arrow[d, "P^2d_1"] & \\
& P^3C_2 \arrow[r, "P^3d_1"] \arrow[d, "\mu_{PC_2}"'] \arrow[dr, phantom, "(C)"] & P^3C_1 \arrow[r, "P\mu_{C_1}"] \arrow[d, "\mu_{PC_1}"] \arrow[dr, phantom, "(D)"] & P^2C_1 \arrow[d, "\mu_{C_1}"] & \\
& P^2C_2 \arrow[r, "P^2d_1"] & P^2C_1 \arrow[r, "\mu_{C_1}"] & PC_1 & \\
\widetilde{C}_2 \arrow[rrrr, "\widetilde{d}_1"] \arrow[ur, "i_2"] & & & & \widetilde{C}_1, \arrow[ul, tail, "i_1"']
\end{tikzcd}$$

where the four trapeziums are diagrams (5.1.5) and (5.1.6) of Lemma 5.1.2. The inner squares are the following: (A) is P^3 applied to associativity of C; (B) is P applied to naturality of μ at d_1; (C) is naturality of μ at Pd_1 and (D) is the associativity law of μ. Since i_1 is a monomorphism (5.1.1) and all the inner diagrams commute, so does the outer square, as we wanted to see. □

The unit morphism of $\mathcal{T}_PC$ is easier to obtain than composition. Recall that the unit is a morphism $\widetilde{e}\colon C_0 \to \widetilde{C}_1$ such that the following diagram (A.2.2) commutes,

$$\begin{tikzcd}
& C_0 \arrow[dl, "\eta_{C_0}"'] \arrow[d, "\widetilde{e}"] \arrow[dr, "id"] & \\
PC_0 & \widetilde{C}_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & C_0.
\end{tikzcd}$$

Definition 5.1.4. The unit of $\mathcal{T}_PC$ is given by the following arrow:

$$\begin{tikzcd}
\mathbf{C_0} \arrow[dd, "\widetilde{e}"] & PC_0 \arrow[dd, equal] \arrow[ddr, phantom, "(A)"] & C_0 \arrow[l, "\eta_{C_0}"'] \arrow[d, "\eta_{C_0}"] \arrow[r, equal] \arrow[ddr, phantom, "(B)"] & C_0 \arrow[dd, "\eta_{C_0}"] \arrow[r, equal] \arrow[ddr, phantom, "(C)"] & C_0 \arrow[dd, "F_{C_0}"] \arrow[r, equal] \arrow[ddr, phantom, "(D)"] & C_0 \arrow[dd, equal] \\
& & PC_0 \arrow[d, "Pe"'] & & & \\
\widetilde{\mathbf{C}}_1 & PC_0 & PC_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] & PC_0 & LC_0 \arrow[l, "D_{C_0}"] \arrow[r, "R_{C_0}"'] & C_0.
\end{tikzcd} \tag{5.1.7}$$

It is clear that all the diagrams commute: (A) and (B) come from P applied to (A.1.2a) and (A.1.2b), this is $d_1 \circ e = id = d_0 \circ e$; (C) is the compatibility between D and η (4.4.2a), and (D) is obvious from the definitions of R and F.

We have to verify that composition with the unit morphism is the identity (A.2.4). To prove it we follow the same strategy as for associativity. That is, we project the diagrams into diagrams in the original category C containing the corresponding unit axioms. Recall first that

$$C_2 := C_1 \times_{C_0} C_1 \quad \text{and} \quad \widetilde{C}_2 := P\widetilde{C}_1 \underset{PC_0}{\times} \widetilde{C}_1.$$

Lemma 5.1.5. *We have commutative squares*

$$
\begin{tikzcd}
PC_0 \underset{PC_0}{\times} \widetilde{C}_1 \arrow[r, "i_1^l"] \arrow[dd, "P\widetilde{e} \times_{id} id"] & PC_0 \underset{PC_0}{\times} PC_1 \arrow[d, "Pe \times_{id} id"'] \\
 & PC_2 \arrow[d, "P\eta_{C_2}"'] \\
\widetilde{C}_2 \arrow[r, "i_2"'] & P^2C_2,
\end{tikzcd}
\tag{5.1.8a}
$$

$$
\begin{tikzcd}
\widetilde{C}_1 \underset{C_0}{\times} C_0 \arrow[r, "i_1^r"] \arrow[dd, "\eta_{\widetilde{C}_1} \times_{\eta_{C_0}} \widetilde{e}"] & PC_1 \underset{PC_0}{\times} PC_0 \arrow[d, "id \times_{id} Pe"'] \\
 & PC_2 \arrow[d, "\eta_{PC_2}"'] \\
\widetilde{C}_2 \arrow[r, "i_2"'] & P^2C_2,
\end{tikzcd}
\tag{5.1.8b}
$$

where i_1^l and i_1^r are the morphisms corresponding to i_1.

Proof. See Appendix B. □

Proposition 5.1.6. *The unit morphism $\widetilde{e}$ of $\mathcal{T}_P C$ satisfies the left and right composition axioms (A.2.4).*

Proof. For the left composition (A.2.4a), the required commutative triangle is the outline of the diagram

$$
\begin{tikzcd}
PC_0 \underset{PC_0}{\times} \widetilde{C}_1 \arrow[rrrr, "P\widetilde{e} \times_{id} id"] \arrow[dr, "i_1^r"] \arrow[ddddrr, bend right] & & & & \widetilde{C}_2 \arrow[dl, "i_2"'] \arrow[ddddll, bend left, "\widetilde{d}_1"] \\
 & PC_0 \underset{PC_0}{\times} PC_1 \arrow[r, "Pe \times_{id} id"] \arrow[ddr, "p_2"] \arrow[u, phantom, "(B)"] & PC_2 \arrow[r, "P\eta_{C_2}"] \arrow[d, "id"'] & P^2C_2 \arrow[dl, "\mu_{C_2}"] \arrow[d, "P^2d_1"] & \\
 & (A) & PC_2 \arrow[d, "Pd_1"] & P^2C_1 \arrow[dl, "\mu_{C_1}"] & \\
 & & PC_1 & & \\
 & & \widetilde{C}_1 \arrow[u, hook, "i_1"] & &
\end{tikzcd}
\tag{5.1.9}
$$

We have that Diagram (A) commutes by definition of i_1^l; (B) is precisely (5.1.8a) of Lemma 5.1.5; (C) is P applied to the left composition with the unit axiom in the category C (A.1.4a); (D) is naturality of μ at d_1; (E) is P of the unit axiom of P applied to C_2, and (F) is the same as (5.1.5). Since i_1 is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see.

For the right composition (A.2.4b), the required commutative triangle is the outline of the diagram

$$\begin{tikzcd}
\tilde{C}_1 \underset{C_0}{\times} C_0 \arrow[rrrr, "\eta_{\tilde{C}_1} \times_{\eta_{C_0}} \tilde{e}"] \arrow[dr, "i_1^l"] \arrow[ddddrr, bend right] \arrow[ddr, phantom, "(A)"] & & & & \tilde{C}_2 \arrow[dl, "i_2"'] \arrow[ddddll, bend left, "\tilde{d}_1"] \\
& PC_1 \underset{PC_0}{\times} PC_0 \arrow[r, "\mathrm{id} \times_{\mathrm{id}} Pe"] \arrow[ddr] \arrow[dr, phantom, "(C)"] & PC_2 \arrow[r, "\eta_{PC_2}"] \arrow[d, "\mathrm{id}"] \arrow[u, phantom, "(B)"] \arrow[dr, phantom, "(E)"] & P^2C_2 \arrow[d, "P^2 d_1"] \arrow[dl, "\mu_{C_2}"] & \\
& & PC_2 \arrow[d, "Pd_1"] \arrow[r, phantom, "(D)"] & P^2C_1 \arrow[dl, "\mu_{C_1}"] \arrow[d, phantom, "(F)"] & \\
& & PC_1 & & \\
& & \tilde{C}_1 \arrow[u, "i_1"] & &
\end{tikzcd} \tag{5.1.10}$$

We have that Diagram (A) commutes by definition of i_1^r, (B) is precisely (5.1.8a) of Lemma 5.1.5, (C) is P applied to the right composition with the unit axiom in the category C (A.1.4b); (D) is again naturality of μ at d_1, (E) is the unit axiom of P applied to PC_2 and (F) is the same as (5.1.5), as before. Since i_1 is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see. $\square$

The last thing to check is that the construction is functorial. First of all we have to specify how the construction acts on morphisms. Let C and C' be two categories and $C \xrightarrow{f} B$ a functor, that is a diagram

$$\begin{tikzcd}
C_0 \arrow[d, "f_0"'] & C_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\
B_0 & B_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] & B_0
\end{tikzcd}$$

satisfying the commutative squares of (4.2.1). Then $\mathcal{T}_P f$ is the morphism given by

$$\begin{tikzcd}
\tilde{\mathbf{C}}_\mathbf{1} \arrow[d, "\tilde{f}_1"] & PC_0 \arrow[d, "Pf_0"] & PC_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"] \arrow[d, "Pf_1"] & PC_0 \arrow[d, "Pf_0"] & LC_0 \arrow[l, "D_{C_0}"'] \arrow[r, "R_{C_0}"] \arrow[d, "Lf_0"] & C_0 \arrow[d, "f_0"] \\
\tilde{\mathbf{B}}_\mathbf{1} & PB_0 & PB_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] & PB_0 & LB_0 \arrow[l, "D_{B_0}"] \arrow[r, "R_{B_0}"'] & B_0.
\end{tikzcd}$$

Proposition 5.1.7. *The morphism $\mathcal{T}_P f$ satisfies again the commutative squares of Equation* (4.2.1).

Proof. The first of the squares of (4.2.1) is clear from the definition of $\mathcal{T}_P f$. For the compatibility with the unit, let us write the square in expanded form:

$$\begin{tikzcd}
\widetilde{B}_1 & PB_0 & PB_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"'] & PB_0 & LB_0 \arrow[l, "D_{B_0}"'] \arrow[r, "R_{B_0}"'] & B_0 \\
 & & PB_0 \arrow[u, "Pe"'] & & & \\
B_0 \arrow[uu, "\widetilde{e}"] & PB_0 \arrow[uu, equal] & B_0 \arrow[l, "\eta_{B_0}"'] \arrow[u, "\eta_{B_0}"] \arrow[r, equal] & B_0 \arrow[uu, "\eta_{B_0}"] \arrow[r, equal] & B_0 \arrow[uu, "F_{B_0}"] \arrow[r, equal] & B_0 \arrow[uu, equal] \\
C_0 \arrow[u, "f_0"] \arrow[dd, "\widetilde{e}"] & PC_0 \arrow[u, "Pf_0"] \arrow[dd, equal] & C_0 \arrow[l, "\eta_{C_0}"'] \arrow[u, "f_0"'] \arrow[d, "\eta_{C_0}"'] \arrow[r, equal] & C_0 \arrow[u, "f_0"'] \arrow[dd, "\eta_{C_0}"'] \arrow[r, equal] & C_0 \arrow[u, "f_0"'] \arrow[dd, "F_{C_0}"'] \arrow[r, equal] & C_0 \arrow[u, "f_0"'] \arrow[dd, equal] \\
 & & PC_0 \arrow[d, "Pe"] & & & \\
\widetilde{C}_1 \arrow[d, "\widetilde{f}_1"] & PC_0 \arrow[d, "Pf_0"] & PC_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"'] \arrow[d, "Pf_1"] & PC_0 \arrow[d, "Pf_0"] & LC_0 \arrow[l, "D_{C_0}"'] \arrow[r, "R_{C_0}"'] \arrow[d, "Lf_0"] & C_0 \arrow[d, "f_0"] \\
\widetilde{B}_1 & PB_0 & PB_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"'] & PB_0 & LB_0 \arrow[l, "D_{B_0}"'] \arrow[r, "R_{B_0}"'] & B_0
\end{tikzcd}$$

Observe that the five columns of the expanded diagram represent five squares, since the top row and the bottom row are the same. We refer to this squares as vertical squares. The red and green diagrams are precisely the definition of the unit morphism $\widetilde{e}$ for $\mathcal{T}_P B$ and $\mathcal{T}_P C$ respectively (Definition 5.1.4). The rest is the definition of $\mathcal{T}_P f$, given above. We have to check that the vertical squares commute. The first and the last one are trivial; the second one is a combination of naturality of η at f_0 and the middle square of (4.2.1) for $C \xrightarrow{f} B$, and the other two are naturality of η at f_0 and naturality of F at f_0 respectively.

The compatibility of $\mathcal{T}_P f$ with composition follows from an analogous argument. Moreover, given another morphism $B \xrightarrow{g} A$ it is clear that $\mathcal{T}_P(g \circ f) = \mathcal{T}_P g \circ \mathcal{T}_P f$, just because of the functoriality of P and L. $\square$

Since the construction is functorial, if the strength D_A is a monomorphism for every object $A \in \mathcal{E}$ then $\mathcal{T}_P$ is in fact a functor from categories internal to $\mathcal{E}$ to P-operads.

5.2 *From* P*-operads to categories*

This construction has a similar structure as the construction above, so we follow the same steps. Let Q be a P-operad,

$$\begin{tikzcd}
 & Q_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\
PQ_0 & & Q_0
\end{tikzcd}
\qquad
\begin{array}{c}
PQ_1 \underset{PQ_0}{\times} Q_1 := Q_2 \xrightarrow{\ d_1\ } Q_1 \\[2em]
Q_0 \xrightarrow{\ e\ } Q_1,
\end{array}$$

and assume that $D_{Q_0} : P1 \times Q \to PQ$ is a monomorphism. We construct a category $\mathcal{T}^P Q$ from the P-operad Q. In this case, the simplicial nomenclature for $\mathcal{T}^P Q$ is $\overline{Q}_i$ for the simplices and $\overline{d}_i$ for the face maps. The following pullback defines the objects and arrows of $\mathcal{T}^P Q$:

$$\begin{tikzcd}
 & & \overline{Q}_1 \arrow[dl] \arrow[dr, tail, "j_1"] \arrow[d, phantom, "\vee"] \arrow[ddll, bend right, "\overline{d}_1"'] \arrow[ddrr, bend left, "\overline{d}_0"] & & \\
 & LQ_0 \arrow[dl, "R_{Q_0}"'] \arrow[dr, tail, "D_{Q_0}"] & {} & Q_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\
Q_0 & & PQ_0 & & Q_0.
\end{tikzcd} \tag{5.2.1}$$

Observe that $\overline{Q}_0 = Q_0$, so that again $\mathcal{T}^P Q$ has the same objects as Q. Besides, the morphism j_1 is a monomorphism, since monomorphisms are preserved by pullbacks and D_{Q_0} is a monomorphism.

To define composition we need to define a map $\overline{Q}_2 \xrightarrow{\overline{d}_1} \overline{Q}_1$, where $\overline{Q}_2 := \overline{Q}_1 \times_{Q_0} \overline{Q}_1$, satisfying the axioms of Appendix A.2. However, to specify this map we need to express it in a way we can naturally use composition in the original P-operad $Q_2 \xrightarrow{d_1} Q_1$. The following diagram represents an isomorphism

$$\overline{Q}_2 \cong L^2 Q_0 \underset{PLQ_0}{\times} LQ_1 \underset{PQ_0}{\times} Q_1 =: \overline{Q}'_2,$$

$$\begin{tikzcd}
Q_0 \arrow[d, equal] & L^2Q_0 \arrow[l] \arrow[r, "LD_{Q_0}"] \arrow[d, "R_{LQ_0}"] \arrow[dr, phantom, "\lrcorner\ (A)"] & LPQ_0 \arrow[d, "R_{PQ_0}"] & & LQ_1 \arrow[ll, "Ld_1"'] \arrow[rr] \arrow[dl, "R_{Q_1}"'] \arrow[dr, "Ld_0"] \arrow[d, phantom, "\vee\ (B)"] & & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, equal] & Q_0 \arrow[d, equal] \\
Q_0 & LQ_0 \arrow[l, "R_{Q_0}"] \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & Q_0 & LQ_0 \arrow[l, "R_{Q_0}"] \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & Q_0
\end{tikzcd} \tag{5.2.2}$$

It is clear that all the squares in (5.2.2) commute. Moreover, the squares (A) and (B) are cartesian because so is R.

Definition 5.2.1. The composition of $\overline{Q}$ is given by the following arrow $\overline{Q}'_2 \xrightarrow{\overline{d}'_1} \overline{Q}_1$,

$$\begin{tikzcd}
\overline{Q}'_2 \arrow[ddd] & Q_0 \arrow[ddd, equal] & L^2Q_0 \arrow[l] \arrow[r, "LD_{Q_0}"] \arrow[d, "\mathrm{id}\times D_{P1}"'] \arrow[dr, phantom, "(A)"] & LPQ_0 \arrow[d, "D_{PQ_0}"] \arrow[dr, phantom, "(B)"] & LQ_1 \arrow[l, "Ld_1"'] \arrow[r] \arrow[d, "D_{Q_1}"] & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, equal] & Q_0 \arrow[d, equal] \\
 & & Q_0 \times P^2 1 \arrow[r, "D^2_{Q_0}"'] \arrow[dd, "\mathrm{id}\times\mu_1"'] \arrow[ddr, phantom, "(C)"] & P^2Q_0 \arrow[dd, "\mu_{Q_0}"] & PQ_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"] \arrow[d, phantom, "(D)"] & PQ_0 & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"'] \arrow[d, phantom, "(E)"] & Q_0 \arrow[dd, equal] \\
 & & & & {} & Q_2 \arrow[ul, "d_2"] \arrow[ur, "d_0"'] \arrow[u, phantom, "\wedge"] \arrow[d, "d_1"] & {} & \\
\overline{Q}_1 & Q_0 & PQ_0 \arrow[l, "R_{Q_0}"] \arrow[r, "D_{Q_0}"'] & PQ_0 & & Q_1 \arrow[ll, "d_1"] \arrow[rr, "d_0"'] & & PQ_0.
\end{tikzcd} \tag{5.2.3}$$

Let us see that all the diagrams commute: (A) is a combination of naturality of D applied to D_{Q_0} and axiom (4.4.1b) concerning consecutive applications of the strength, (B) is naturality of D at d_1, (C) is axiom (4.4.2b) for strong monads and (D) and (E) are respectively axioms (A.2.1a) and (A.2.1b) for composition in Q. The remaining diagrams are clear.

Notice that from this definition it is clear that $\overline{d}_1$ satisfies axioms (A.1.1a) and (A.1.1b). Furthermore, there is a morphism

$$\overline{Q}_2' \xrightarrow{j_2'} Q_2,$$

given by the diagram

$$\begin{tikzcd}
Q_0 & L^2Q_0 \arrow[l] \arrow[r, "LD_{Q_0}"] & LPQ_0 \arrow[d, "D_{PQ_0}"'] & LQ_1 \arrow[l, "Ld_1"'] \arrow[r] \arrow[d, "D_{Q_1}"] & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, equal] & Q_0 \arrow[d, equal] \\
 & & P^2Q_0 & PQ_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] & PQ_0 & Q_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & Q_0,
\end{tikzcd} \tag{5.2.4}$$

which clearly makes the square

$$\begin{tikzcd}
\overline{Q}_2' \arrow[r, "j_2'"] \arrow[d, "\overline{d}_1'"'] & Q_2 \arrow[d, "d_1"] \\
\overline{Q}_1 \arrow[r, tail, "j_1"'] & Q_1,
\end{tikzcd}
\quad \text{and therefore also the square} \quad
\begin{tikzcd}
\overline{Q}_2 \arrow[r, "j_2"] \arrow[d, "\overline{d}_1"'] & Q_2 \arrow[d, "d_1"] \\
\overline{Q}_1 \arrow[r, tail, "j_1"'] & Q_1,
\end{tikzcd} \tag{5.2.5}$$

commute, for the corresponding j_2. This says, roughly speaking, that composition in $\overline{Q}$ is "the same" as composition in Q, as is also clear in most of the examples.

We have to check that composition is associative (A.1.3).

Lemma 5.2.2. *There is a morphism $\overline{Q}_3 \xrightarrow{j_3} Q_3$ such that the following diagrams commute*

$$\begin{tikzcd}
\overline{Q}_3 \arrow[r, "j_3"] \arrow[d, "\overline{d}_1"'] & Q_3 \arrow[d, "d_1"] \\
\overline{Q}_2 \arrow[r, "j_2"'] & Q_2,
\end{tikzcd} \tag{5.2.6a}$$

$$\begin{tikzcd}
\overline{Q}_3 \arrow[r, "j_3"] \arrow[d, "\overline{d}_2"'] & Q_3 \arrow[d, "d_2"] \\
\overline{Q}_2 \arrow[r, "j_2"'] & Q_2.
\end{tikzcd} \tag{5.2.6b}$$

Proof. See Appendix B. □

Proposition 5.2.3. *Composition is associative.*

Proof. In view of Lemma 5.2.2 there is a diagram

$$\begin{tikzcd}
\overline{C}_3 \arrow[rrr, "\overline{d}_2"] \arrow[dr, "j_3"] \arrow[ddd, "\overline{d}_1"'] & & & \overline{C}_2 \arrow[dl, "j_2"] \arrow[ddd, "\overline{d}_1"] \\
 & C_3 \arrow[r, "d_2"] \arrow[d, "d_1"'] & C_2 \arrow[d, "d_1"] & \\
 & C_2 \arrow[r, "d_1"'] & C_1 & \\
\overline{C}_2 \arrow[ur, "j_2"] \arrow[rrr, "\overline{d}_1"'] & & & \overline{C}_1. \arrow[ul, tail, "j_1"']
\end{tikzcd}$$

The four trapeziums are the commutative diagrams (5.2.5) and (5.2.6) of Lemma 5.2.2 respectively, and the inner square is associativity of composition in C (A.2.3). Since j_1 is a monomorphism and all the inner diagrams commute, so does the outer square, as we wanted to see. □

The unit morphism of the new category is easier to obtain than composition. Recall that the unit is a morphism $e\colon Q_0 \to \overline{Q}_1$ such that the following diagram (A.1.2) commutes

$$\begin{tikzcd} & Q_0 \arrow[dl, "id"'] \arrow[d, "e"] \arrow[dr, "id"] & \\ Q_0 & \overline{Q}_1 \arrow[l, "\overline{d}_1"] \arrow[r, "\overline{d}_0"'] & Q_0. \end{tikzcd}$$

Definition 5.2.4. The unit of $\overline{Q}$ is given by the following arrow:

$$\begin{tikzcd} \mathbf{Q_0} \arrow[d, "\overline{e}"'] & Q_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "(A)"] & Q_0 \arrow[r, equal] \arrow[d, "F_{Q_0}"] \arrow[dr, phantom, "(B)"] & Q_0 \arrow[r, equal] \arrow[d, "\eta_{Q_0}"] \arrow[dr, phantom, "(C)"] & Q_0 \arrow[r, equal] \arrow[d, "e"] \arrow[dr, phantom, "(D)"] & Q_0 \arrow[d, equal] \\ \overline{\mathbf{Q}}_\mathbf{1} & Q_0 & LQ_0 \arrow[l, "R_{Q_0}"] \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & Q_0. \end{tikzcd} \tag{5.2.7}$$

It is clear that all the diagrams commute: (A) is obvious from the definitions of R and F, (B) is axiom (4.4.2a) for strong monads, and (C) and (D) are respectively the unit axioms (A.2.2b) and (A.2.2a) of Q.

We have to check that composition with the unit morphism is the identity (A.1.4). To prove it we follow the same strategy as for associativity. That is, we project the diagrams into diagrams in the original P-operad Q containing the corresponding unit axioms. Recall first that

$$Q_2 := PQ_1 \underset{PQ_0}{\times} Q_1 \quad \text{and} \quad \overline{Q}_2 := \overline{Q}_1 \times_{Q_0} \overline{Q}_1.$$

Lemma 5.2.5. *We have commutative squares*

$$\begin{tikzcd} Q_0 \underset{Q_0}{\times} \overline{Q}_1 \arrow[r, "j_1^l"] \arrow[d, "\overline{e} \times_{id} id"'] & PQ_0 \underset{PQ_0}{\times} Q_1 \arrow[d, "Pe \times_{id} id"] \\ \overline{Q}_2 \arrow[r, "j_2"'] & Q_2, \end{tikzcd} \tag{5.2.8a}$$

$$\begin{tikzcd} \overline{Q}_1 \underset{Q_0}{\times} Q_0 \arrow[r, "j_1^r"] \arrow[d, "id \times_{id} \overline{e}"'] & Q_1 \underset{Q_0}{\times} Q_0 \arrow[d, "\eta_{Q_1} \times_{\eta_{Q_0}} e"] \\ \overline{Q}_2 \arrow[r, "j_2"'] & Q_2, \end{tikzcd} \tag{5.2.8b}$$

where j_1^l and j_1^r are the morphisms corresponding to j_1.

Proof. See Appendix B. □

Proposition 5.2.6. *The unit morphism $\overline{e}$ of $\overline{Q}$ satisfies the left and right composition axioms (A.1.4).*

Proof. For the left composition (A.1.4a), the required commutative triangle is the outline of the diagram

$$\begin{tikzcd}
Q_0 \underset{Q_0}{\times} \overline{Q}_1 \arrow[rrrr, "\overline{e}\times_{\mathrm{id}}\mathrm{id}"] \arrow[dr, "j_1^l"] \arrow[dddrr, bend right] \arrow[rrrr, phantom, shift right=6ex, "(B)"] & & & & \overline{Q}_2 \arrow[dl, "j_2"'] \arrow[dddll, bend left, "\overline{d}_1"] \\
& PQ_0 \underset{PQ_0}{\times} Q_1 \arrow[rr, "Pe\times_{\mathrm{id}}\mathrm{id}"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "(C)"] \arrow[ddr, phantom, shift right=6ex, "(A)"] & & PQ_1 \underset{PQ_0}{\times} Q_1 \arrow[dl, "d_1"'] \arrow[ddl, phantom, shift left=6ex, "(D)"] & \\
& & Q_1 & & \\
& & \overline{Q}_1, \arrow[u, hook, "j_1"] & &
\end{tikzcd} \tag{5.2.9}$$

We have that Diagram (A) commutes by definition of j_1^l, (B) is precisely (5.2.8a) of Lemma 5.2.5, (C) is the left composition with unit axiom in the P-operad C (A.2.4a) and (D) is the same as (5.2.5). Since j_1 is a monomorphism and all the inner diagrams commute, so does the outer triangle, as we wanted to see.

For the right composition (A.1.4b), the required commutative triangle is the outline of the diagram

$$\begin{tikzcd}
\overline{Q}_1 \underset{Q_0}{\times} Q_0 \arrow[rrrr, "\mathrm{id}\times_{\mathrm{id}}\overline{e}"] \arrow[dr, "j_1^r"] \arrow[dddrr, bend right] \arrow[rrrr, phantom, shift right=6ex, "(B)"] & & & & \overline{Q}_2 \arrow[dl, "j_2"'] \arrow[dddll, bend left, "\overline{d}_1"] \\
& Q_1 \underset{Q_0}{\times} Q_0 \arrow[rr, "\eta_{Q_1}\times_{\eta_{Q_0}} e"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "(C)"] \arrow[ddr, phantom, shift right=6ex, "(A)"] & & PQ_1 \underset{PQ_0}{\times} Q_1 \arrow[dl, "d_1"'] \arrow[ddl, phantom, shift left=6ex, "(D)"] & \\
& & Q_1 & & \\
& & \overline{Q}_1, \arrow[u, hook, "j_1"] & &
\end{tikzcd} \tag{5.2.10}$$

We have that Diagram (A) commutes by definition of j_1^r, (B) is precisely (5.2.8b) of Lemma 5.2.5, (C) is the right composition with unit axiom in the P-operad Q (A.2.4b), and (D) is the same as (5.2.5), as before. Since j_1 is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see. □

The last thing to check is that the construction is functorial. First of all we have to specify how the construction acts on morphisms. Let Q and Q' be two P-operads and $Q \xrightarrow{f} B$ a morphism, that is a diagram

$$\begin{tikzcd}
PQ_0 \arrow[d, "Pf_0"'] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, "f_1"] & Q_0 \arrow[d, "f_0"] \\
PB_0 & B_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] & B_0
\end{tikzcd}$$

satisfying the commutative squares of (4.2.1). Then $\mathcal{T}^P f$ is the functor given by

$$
\begin{tikzcd}
\overline{\mathbf{Q}}_1 \arrow[d, "\overline{f}_1"'] & Q_0 \arrow[d, "f_0"'] & LQ_0 \arrow[l, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"] \arrow[d, "Lf_0"] & PQ_0 \arrow[d, "Pf_0"] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, "Pf_1"] & Q_0 \arrow[d, "f_0"] \\
\overline{\mathbf{B}}_1 & B_0 & LB_0 \arrow[l, "R_{B_0}"] \arrow[r, "D_{B_0}"'] & PB_0 & B_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & B_0.
\end{tikzcd}
$$

Proposition 5.2.7. *The morphism $\mathcal{T}^P f$ satisfies again the commutative squares of Equation* (4.2.1).

Proof. The first of the squares of (4.2.1) is clear from the definition of $\mathcal{T}^P f$. For the compatibility with the unit, let us write the square in expanded form:

$$
\begin{tikzcd}
\overline{\mathbf{B}}_1 & B_0 & LB_0 \arrow[l, "R_{B_0}"'] \arrow[r, "D_{B_0}"] & PB_0 & B_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] & B_0 \\
\mathbf{B}_0 \arrow[u, "\overline{e}"'] & B_0 \arrow[u, equal] \arrow[r, equal] & B_0 \arrow[u, "F_{B_0}"] \arrow[r, equal] & B_0 \arrow[u, "\eta_{B_0}"] \arrow[r, equal] & B_0 \arrow[u, "e"] \arrow[r, equal] & B_0 \arrow[u, equal] \\
\mathbf{Q}_0 \arrow[u, "f_0"] \arrow[d, "\overline{e}"'] & Q_0 \arrow[u, "f_0"] \arrow[r, equal] \arrow[d, equal] & Q_0 \arrow[u, "f_0"] \arrow[r, equal] \arrow[d, "F_{Q_0}"] & Q_0 \arrow[u, "f_0"] \arrow[r, equal] \arrow[d, "\eta_{Q_0}"] & Q_0 \arrow[u, "f_0"] \arrow[r, equal] \arrow[d, "e"] & Q_0 \arrow[u, "f_0"'] \arrow[d, equal] \\
\overline{\mathbf{Q}}_1 \arrow[d, "\overline{f}_1"'] & Q_0 \arrow[d, "f_0"'] & LQ_0 \arrow[l, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"] \arrow[d, "Lf_0"] & PQ_0 \arrow[d, "Pf_0"] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, "Pf_1"] & Q_0 \arrow[d, "f_0"] \\
\overline{\mathbf{B}}_1 & B_0 & LB_0 \arrow[l, "R_{B_0}"] \arrow[r, "D_{B_0}"'] & PB_0 & B_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & B_0.
\end{tikzcd}
$$

The red and green diagrams correspond to the definition of the unit morphisms of $\mathcal{T}^P Q$ and $\mathcal{T}^P B$ respectively (Definition 5.2.4). The rest is the definition of $\mathcal{T}^P f$, given above. We have to see that the vertical squares commute. The first and the last one are trivial; the second is naturality of F at f_0; the third one is naturality of η at f_0, and the fourth one is the middle square of (4.2.1) for $Q \xrightarrow{f} B$.

The compatibility of $\mathcal{T}^P f$ with composition follows from an analogous argument. Moreover, given another morphism $B \xrightarrow{g} A$ it is clear that $\mathcal{T}^P(g \circ f) = \mathcal{T}^P g \circ \mathcal{T}^P f$, just because of the functoriality of P and L. $\square$

Since the construction is functorial, if the strength D_A is a monomorphism for every object $A \in \mathcal{E}$ then $\mathcal{T}^P$ is in fact a functor from P-operads to categories internal to $\mathcal{E}$.

5.3 *The composite construction*

Since we have defined a construction from P-operads to categories and a construction from categories to P-operads, we obtain a composite construction from P'-operads to P-operads, for P' and P not necessarily the same monad. In particular, since a category is the same as an Id-operad, the composite construction for $P' = \mathrm{Id}$ is the same as the functor from categories to P-operads. From now on we call $\mathcal{T}$-construction any of the three constructions; the context will suffice to distinguish, but we are mainly interested in landing on a P-operad, rather than a category. To keep notation short, we denote by

$$\mathcal{T}_P Q := \mathcal{T}_P \mathcal{T}^{P'} Q$$

the composite construction that produces a P-operad from the P'-operad Q. The monad P' will always be clear from the context.

5.4 *Finiteness conditions*

In Chapter 7 we will be interested in computing the incidence bialgebra of the bar construction of several P-operads in $\mathcal{E} = \mathbf{Grpd}$. Recall that to be able to take the homotopy cardinality, the bar construction has to be locally finite as a simplicial groupoid (in the sense of [33]). We now define the notion of locally finite operad (in the sense of [42]) in the setting of P-operads, which is the sufficient condition for its bar construction to be locally finite, and we give sufficient conditions on the $\mathcal{T}$-construction to preserve locally finiteness.

Definition 5.4.1. A natural transformation is *finite* if all its components are finite. A monad (P, μ, η) on **Grpd** is *locally finite* if μ and η are finite natural transformations. A P-operad Q is locally finite if Q_1 is locally finite, and the maps d_1 and e are finite.

In the special case of $P = \mathrm{Id}$, P-operads are just categories, and the notion of locally finite agrees with the standard notion [47]. Notice that Q can be locally finite even if P is not. The condition of P being locally finite appears in the $\mathcal{T}$-construction.

Example 5.4.2. For a classical symmetric or nonsymmetric operad, the locally finiteness condition amounts to saying that every operation can be expressed as a composition of operations in a finite number of ways. For instance, the operads Ass and Sym are locally finite. For this it is important that nullary operations are excluded. The nonreduced versions, where there is a nullary operation, are *not* locally finite.

The bar construction of Q is locally finite if Q is locally finite and P preserves locally finite groupoids and finite maps (see Chapter 4). Also, given another locally finite monad R on **Grpd** that preserves locally finite groupoids and finite maps, if there is a cartesian monad map $P \overset{\psi}{\Rightarrow} R$ with ψ finite, then the bar construction $\mathcal{B}^R$ is also locally finite. Let us see that the $\mathcal{T}$-construction interacts well with finiteness, as long as some simple conditions are satisfied.

Lemma 5.4.3. *Let $P\colon \mathbf{Grpd} \to \mathbf{Grpd}$ be a locally finite strong monad that preserves locally finite groupoids, finite maps and fibrations. Assume moreover that the strength D is finite. Consider a locally finite category C in **Grpd** such that C_0 is discrete and D_{C_0} is a monomorphism. Then the P-operad $\mathcal{T}_P C$ is locally finite.*

Proof. Recall from Diagram (5.1.1) that $\widetilde{C}_1$ is defined as the pullback

$$\begin{tikzcd} \widetilde{C}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LC_0 \arrow[d, tail, "D_{C_0}"] \\ PC_1 \arrow[r, "Pd_0"'] & PC_0. \end{tikzcd}$$

Notice that the pullback and the monomorphism refer to the 1-categorical notions, while the finite map condition is a homotopy notion.

Let us see first that $\widetilde{C}_1$ is locally finite. Since C_1 is locally finite and P preserves locally finite groupoids, PC_1 is locally finite. Now, an automorphism in $\widetilde{C}_1$ is a pair of automorphisms $(f, g) \in PC_1 \times LC_0$ coinciding at PC_0, but there is only a finite number of f's, since PC_1 is locally finite, and for each f at most one g, since D_{C_0} is a monomorphism.

We have to prove also that $\widetilde{d}_1$ and $\widetilde{e}$ are finite maps. This follows from their definitions, 5.1.1 and 5.1.4: since C_0 is discrete, we have that d_0 is a fibration, and because P (and also L) preserves fibrations, all the right arrows in diagrams (5.1.3) and (5.1.7) are fibrations. As a consequence their limit is equivalent to their homotopy limit. Finally, notice that all the vertical maps involved in these two diagrams are finite. This implies that their homotopy limit, and hence their limit, is also finite. □

Lemma 5.4.4. *Let $P\colon \mathbf{Grpd} \to \mathbf{Grpd}$ be a locally finite strong monad that preserves locally finite groupoids, finite maps and fibrations. Assume moreover that the strength D is finite. Consider a locally finite P-operad Q such that Q_0 is discrete and D_{Q_0} is a monomorphism. Then the P-operad $\mathcal{T}_P C$ is locally finite.*

Proof. The proof is analogous to the proof of Lemma 5.4.3. □

In particular these results imply of course that if P and P' are monads satisfying the conditions of Lemmas 5.4.3 and 5.4.4 and Q is a locally finite P'-operad then $\mathcal{T}_P Q$ is locally finite.

Remark 5.4.5. In the sequel, we deal with the free semigroup monad M^r and the free symmetric semimonoidal category monad S^r, which preserve locally finite groupoids, finite maps and fibrations, as required by Lemmas 5.4.3 and 5.4.4. Moreover, their strength is finite, as can be easily seen from its definition (see Examples 4.4.4 and 4.4.7). Also, we use the reduced operads Ass and Sym, as well as their colored versions. They are all locally finite and have discrete groupoid of colors.

6 $\mathcal{T}$-construction for M^r and S^r-operads

In this chapter we unravel the $\mathcal{T}$-construction with some of the main examples. We begin discussing the construction from categories to M^r-operads and S^r-operads. When the category is just a monoid we get the Giraudo T-construction [36], which we recall next. Lastly we treat symmetric and nonsymmetric operads.

The choice of working with the reduced version of the operads (excluding nullary operations), is irrelevant for the sake of the $\mathcal{T}$-construction itself, which is abstract enough to work with any operad. The reason for preferring the reduced version is to stay within the realm of locally finite operads, as mentioned in Example 5.4.2 and Remark 5.4.5. Moreover, it is also easy to see that the cartesian monad maps $M^r \Rightarrow S$, $S^r \Rightarrow S$ and $M^r \Rightarrow M$ are finite.

6.1 The $\mathcal{T}$-construction for categories

Let C be a category internal to **Set**, represented by the span $C_0 \leftarrow C_1 \rightarrow C_0$, and take the free semigroup monad M^r. The set of objects of $\mathcal{T}_{M^r}C$ is again C_0, while $\widetilde{C}_1$ is given by

$$\begin{tikzcd}
 & & \widetilde{C}_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "\widetilde{d}_1"'] \arrow[ddrr, bend left, "\widetilde{d}_0"] & & \\
 & M^r C_1 \arrow[dl, "Pd_1"] \arrow[dr, "Pd_0"] & & LC_0 \arrow[dl, "D_{C_0}"'] \arrow[dr, "R_{C_0}"] & \\
M^r C_0 & & M^r C_0 & & C_0 .
\end{tikzcd} \tag{6.1.1}$$

Recall from Example 4.4.7 that the strength is given by

$$\begin{aligned} D_{C_0} \colon LC_0 &\longrightarrow M^r C_0 \\ (c, (1, \ldots, 1)) &\longrightarrow ((c,1), \ldots, (c,1)). \end{aligned} \tag{6.1.2}$$

Therefore, the pullback condition means that the elements in $\widetilde{C}_1$ that have input $c_1, \ldots, c_n$ and output c are the sequences of n arrows in C whose sources are $c_1, \ldots, c_n$ and whose targets are all c. Hence

$$\widetilde{C}_1 = \sum_{(c_1, \ldots, c_n; c)} \prod_{i=1}^{n} \mathrm{Hom}(c_i, c).$$

Substitution in $\mathcal{T}_{M^r}C$,

$$\circ \colon \prod_{i=1}^{k} \mathrm{Hom}(c_i, c) \times \prod_{i=1}^{k} \prod_{j=1}^{n_i} \mathrm{Hom}(d^i_j, c_i) \longrightarrow \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n_i}} \mathrm{Hom}(d^i_j, c),$$

goes as follows: for an operation $x \in \prod_{i=1}^{k} \mathrm{Hom}(c_i, c)$ and a sequence of k operations $y^i \in \prod_{j=1}^{n_i} \mathrm{Hom}(d^i_j, c_i)$, with $i = 1, \dots, k$,

$$x \circ (y^1, \dots, y^k) = (x_1 \circ y^1_1, \dots, x_1 \circ y^1_{n_1}, \dots, x_k \circ y^k_1, \dots, x_k \circ y^k_{n_k}) \in \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n_i}} \mathrm{Hom}(d^i_j, c).$$

Note that now the composition inside the parenthesis is composition of morphisms of C, while the composition on the left-hand side of the equation is composition in $\mathsf{M}^{\mathrm{r}}C$. It is not difficult to see that the composition we get from Definition 5.1.1 agrees with the one defined above: both use the fact that $\widetilde{C}_2$ is a subset of $(\mathsf{M}^{\mathrm{r}})^2 C$ together with $(\mathsf{M}^{\mathrm{r}})^2 \circ$ and the monad multiplication. The identity elements of this operad are given by the identity morphisms of C. If the category C has coproducts $(+)$ then

$$\prod_{i=1}^{n} \mathrm{Hom}(c_i, c) = \mathrm{Hom}(c_1 + \cdots + c_n, c),$$

so that the operations of $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$ are in fact arrows of C.

Since C can be considered as a category internal to **Grpd**, we can also compute $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$ to get a symmetric operad. It is clear that $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C_1 = \mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C_1 /\!\!/ \mathfrak{S}$, where the action of the symmetric group $\mathfrak{S}_n$ is given by permutation of tuples, that is

$$\begin{array}{rcl} \mathfrak{S}_n \times \prod_{i=1}^{n} \mathrm{Hom}(c_i, c) & \longrightarrow & \prod_{i=1}^{n} \mathrm{Hom}(c_{\sigma(i)}; c) \\ \big(\sigma, (x_1, \dots, x_n)\big) & \longmapsto & (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{array}$$

It is useful to picture elements $(c_1, \dots, c_n; c)$ as (picturing $n = 3$)

$$s = \begin{tikzcd}[column sep=small] c_3 \arrow[dr] & c_2 \arrow[d] & c_1 \arrow[dl] \\ & c & \end{tikzcd} \tag{6.1.3}$$

Under this representation, composition in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$ (or $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$) looks like

$$\begin{tikzcd}[column sep=small] c^2_3 \arrow[d] & c^1_3 \arrow[dl] & c^1_2 \arrow[d] & c^3_1 \arrow[dr] & c^2_1 \arrow[d] & c^1_1 \arrow[dl] \\ c_3 \arrow[drr] & & c_2 \arrow[d] & & c_1 \arrow[dll] & \\ & & c & & & \end{tikzcd} = \begin{tikzcd}[column sep=small] c^2_3 \arrow[drr] & c^1_3 \arrow[dr] & c^1_2 \arrow[d] & c^3_1 \arrow[dl] & c^2_1 \arrow[dll] & c^1_1 \arrow[dlll] \\ & & c & & & \end{tikzcd} \tag{6.1.4}$$

Example 6.1.1. Take $C = \{0 \rightleftarrows 1\}$. For any pair of objects of C there is exactly one morphism between them. Hence $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$ has one operation for each given sequence of inputs and output, so that it is the 2-colored associative operad Ass_2. In the same way $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$ is the 2-colored symmetric operad Sym_2. In fact it is straightforward to see that the $\mathcal{T}$-constructions of the discrete connected groupoid of n elements are Ass_n and Sym_n.

Example 6.1.2. Consider the category $C = \{0 \longrightarrow 1\}$. Note that in this case there is either one or no morphism between two objects of C. Thus clearly

$$\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C(c_1, \dots, c_n; c) = \begin{cases} (c \to c_1, \dots, c \to c_n) & \text{if } c = 0 \ \text{ or } c = c_1 = \cdots = c_n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Of course this operad is a suboperad of the previous one, since this category is a subcategory of the previous one. In particular, composition is obvious.

Example 6.1.3. We now specialize to the case of categories with only one object, that is monoids, recovering the T-construction of Giraudo. This construction was introduced by Giraudo [36] as a generic method to build combinatorial operads from monoids.

Since a monoid is just a category with one object, it is represented by the span $1 \leftarrow Y \rightarrow 1$, and because the morphism $L1 \xrightarrow{D_1} M^r 1$ is an isomorphism, we have that $\mathcal{T}_{M^r}Y$ is given by

$$\begin{tikzcd}
& & M^rY \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee" very near start] & & \\
& M^rY \arrow[dl, "M^r s"'] \arrow[dr, "M^r t"] & {} & L1 \arrow[dl, "D_1"'] \arrow[dr] & \\
M^r1 & & M^r1 & & 1.
\end{tikzcd}$$

It is easy to see that this gives the same operad TY defined in [36], since TY is precisely M^rY, and both compositions are defined by using composition in $(M^r)^2Y$ and the monad multiplication.

Example 6.1.4. If Y_1 is the singleton monoid, then $\mathcal{T}_{M^r}Y_1 = \mathsf{Ass}$, the associative operad, and $\mathcal{T}_{S^r}Y_1 = \mathsf{Sym}$, the commutative operad.

It is interesting to have a heuristic look at the incidence coalgebra that arises from the operads $\mathcal{T}_{S^r}C$. Recall from Section 4.5 that the Segal groupoid $\mathcal{B}\mathcal{T}_{S^r}C$ has as 1-simplices families of operations. The 0-simplices are families of objects of $\mathcal{C}$, and n-simplices are families of n-level trees. For example, the left-hand side of (6.1.4) is a 2-simplex (consisting of a family of one operation).

The face and degeneracy maps are clear. For instance, the degeneracy maps s_0 and s_1 of the operation s of (6.1.3) are respectively

$$\begin{tikzcd}
c_1 \arrow[d] & c_2 \arrow[d] & c_3 \arrow[d] \\
c_1 \arrow[dr] & c_2 \arrow[d] & c_3 \arrow[dl] \\
& c &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c_1 \arrow[dr] & c_2 \arrow[d] & c_3 \arrow[dl] \\
& c \arrow[d] & \\
& c &
\end{tikzcd}$$

where the morphisms between equal objects are identities. Recall that the comultiplication is given by all the possible two-step factorizations:

$$\Delta(s) = \sum_{\substack{t \in \mathcal{B}_2\mathcal{T}^{S^r} C \\ d_1(t) \simeq r}} d_2(t) \otimes d_0(t).$$

A decomposition of the s has thus one of the following forms,

$$\begin{tikzcd}[column sep=small]
c_1 \arrow[d] & c_2 \arrow[d] & c_3 \arrow[d] \\
d_1 \arrow[dr] & d_2 \arrow[d] & d_3 \arrow[dl] \\
& c &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
c_1 \arrow[dr] & c_2 \arrow[d] & c_3 \arrow[dl] \\
& d \arrow[d] & \\
& c &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
c_1 \arrow[d] & & c_2 \arrow[dr] & & c_3 \arrow[dl] \\
d_1 \arrow[drr] & & & d_2 \arrow[dl] & \\
& & c & &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
c_1 \arrow[dr] & & c_2 \arrow[dl] & & c_3 \arrow[d] \\
& d_1 \arrow[dr] & & & d_2 \arrow[dll] \\
& & c & &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
c_1 \arrow[dr] & & c_2 \arrow[dl] & & c_2 \arrow[d] \\
& d_1 \arrow[dr] & & & d_2 \arrow[dll] \\
& & c. & &
\end{tikzcd}
\tag{6.1.5}$$

Note the change of order of the outputs in the last factorization. This happens because $(c \to c_1, c \to c_2, c \to c_3) \simeq (c \to c_1, c \to c_3, c \to c_2)$ since we are dealing with symmetric operads. In the nonsymmetric situation of $\mathcal{T}_{M^r}C$, the comultiplication would have to respect the order, so that in (6.1.5) the last decomposition would not appear. Hence, the comultiplication of an operation involves all the possible simultaneous and non-simultaneous factorizations of the morphisms constituting it. In particular, the comultiplication of unary operation $c \to d$ coincides with its comultiplication as a morphism of C. Of course the simultaneous factorizations also appear as non-simultaneous factorizations, for example the second factorization in (6.1.5) also appears as

$$\begin{tikzcd}
c_1 \arrow[d] & c_2 \arrow[d] & c_3 \arrow[d] \\
d \arrow[dr] & d \arrow[d] & d \arrow[dl] \\
& c, &
\end{tikzcd}$$

but they are not isomorphic in $\mathcal{T}_{S^r}C$, since the first one is a unary operation composed with a 3-ary operation, while this one is a 3-ary operation composed with three unary operations.

Example 6.1.5. Consider the divisibility poset $(\mathbb{N}, |)$, where $d \to n$ means $d|n$. The comultiplication of $d \to n$ in the incidence coalgebra of this poset is, as it is well known

$$\Delta(d \to n) = \sum_{d|k|n} (k \to n) \otimes (d \to k).$$

Let us see what the comultiplication in the incidence coalgebra of $\mathcal{B}\mathcal{T}_{S^r}(\mathbb{N}, |)$ looks like. Consider the 3-ary operation

$$\begin{tikzcd}
6 \arrow[dr] & 4 \arrow[d] & 8 \arrow[dl] \\
& 24. &
\end{tikzcd}$$

It is clear that $4 \to 24$ and $8 \to 24$, as well as $4 \to 24$ and $6 \to 24$, admit a nontrivial simultaneous decomposition, given respectively by $8 \to 24$ and $12 \to 24$, but $8 \to 24$ and $6 \to 24$ do not admit any non trivial simultaneous decompositions. Hence the decompositions of this operation are

$$\begin{tikzcd}
6 \arrow[d] & 4 \arrow[d] & 8 \arrow[d] \\
6,12,24 \arrow[dr] & 4,8,12,24 \arrow[d] & 8,24 \arrow[dl] \\
& 24 &
\end{tikzcd}
\qquad
\begin{tikzcd}
6 \arrow[dr] & 4 \arrow[d] & 8 \arrow[dl] \\
& 24 \arrow[d] & \\
& 24 &
\end{tikzcd}
\qquad
\begin{tikzcd}
6 \arrow[dr] & & 4 \arrow[dl] & 8 \arrow[d] \\
& 12,24 \arrow[dr] & & 8,24 \arrow[dl] \\
& & 24 &
\end{tikzcd}$$

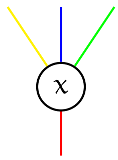
$$\begin{tikzcd}
6 \arrow[d] & & 4 \arrow[dr] & & 8 \arrow[dl] \\
6,12,24 \arrow[drr] & & & 8,24 \arrow[dl] & \\
& & 24 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
4 \arrow[d] & & 6 \arrow[dr] & & 8 \arrow[dl] \\
4,8,12,24 \arrow[drr] & & & 24, \arrow[dl] & \\
& & 24 & &
\end{tikzcd}$$

which yield $2 \cdot 4 \cdot 3 + 1 + 2 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 = 39$ terms.

6.2 *The $\mathcal{T}$-construction for operads*

We now unravel the full $\mathcal{T}$-construction from nonsymmetric operads to S^r-operads. As we already know, the first ones are the same as M^r-operads in **Set**, but we view them as M^r-operads in **Grpd** with discrete groupoids of objects and arrows. At the end we comment on other variations similar to this case, such as from symmetric operads to S^r-operads.

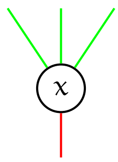
Let Q be an M^r-operad represented by the span $M^r Q_0 \leftarrow Q_1 \rightarrow Q_0$. Recall that elements of Q_1 are depicted as



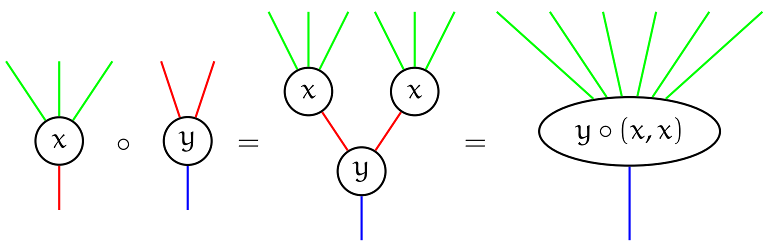
We apply first the $\mathcal{T}$-construction to get a category $\mathcal{T}^{M^r} Q$:

$$\begin{tikzcd}
 & & \overline{Q}_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & LQ_0 \arrow[dl, "R_{Q_0}"'] \arrow[dr, "D_{Q_0}"] & & Q_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\
Q_0 & & SQ_0 & & Q_0 .
\end{tikzcd}$$

The strength morphism is the same as in (6.1.2). Therefore the elements of $\overline{Q}_1$ are the elements of Q_1 such that all the input objects coincide,

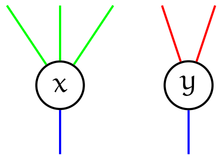
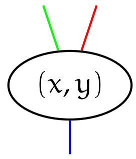


so that x is an arrow $c \xrightarrow{x} d$ in $\mathcal{T}^{M^r} Q$. Notice that $\overline{Q}_2$ is a subset of Q_2. Therefore composition in $\mathcal{T}^{M^r} Q$ is the same as composition in Q. For example



(6.2.1)

where $y \circ (x, x)$ is composition in Q. Hence the recipe is to repeat x for each input of y and use composition in Q. Now we have to apply again the $\mathcal{T}$-construction to get a S^r-operad from the category $\mathcal{T}^{M^r} Q$. This step was made above for any category: the objects of $\widetilde{Q}_1$ are sequences $(x_1, \ldots, x_n)$ of elements $x_i \in \overline{Q}_1$. For instance the pair

 is an operation 

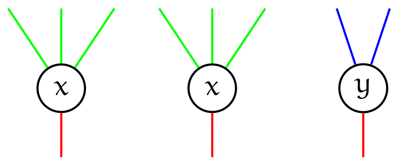
in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}Q$. Clearly $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}Q$ is a symmetric operad, since the groupoid of objects is discrete and the morphisms in the groupoid $\widetilde{Q}_1$ are given by permutation of tuples.

Example 6.2.1. If the starting M^{r}-operad is Ass, which is a noncolored operad, then it is easy to see that the monoid $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ is isomorphic to $(\mathbb{N}^+, \times)$. Therefore the operations of $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}$ are sequences of natural numbers and composition is given by multiplication. For example

$$\big((2,3),(4,7)\big) \circ (5,9) = (5\cdot 2, 5\cdot 3, 9\cdot 4, 9\cdot 7) = (10,15,36,63).$$

If the starting M^{r}-operad is Ass_2 the 2-colored associative operad, then the category $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}_2$ has two objects and a morphism $\xrightarrow{n}$ for every pair of objects and positive natural number n. Composition is given by multiplication. The operations of $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}_2$ are thus sequences of such arrows with the same output.

Suppose we start instead from a symmetric operad Q. Recall from Example 4.4.7 that a symmetric operad is an S-operad in **Grpd** such that Q_0 is discrete and $\mathsf{S}^{\mathrm{r}}Q_0 \xleftarrow{s} Q_1$ is discrete fibration. The $\mathcal{T}$-construction to get another S^{r}-operad is completely analogous to the previous case, but in this case the groupoid $\widetilde{Q}_1$ inherits morphisms from Q, so that for instance the element



has $2!\cdot 3!^2 \cdot 2!$ automorphisms, corresponding to $2!$ invariant permutations of (x,x,y) and permutations of the inputs. The latter contribution did not appear in the previous case, since Q was a planar operad. Notice that this means that $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}Q$ is not a symmetric operad, but just an S^{r}-operad in **Grpd**.

Example 6.2.2. If the starting S^{r}-operad is Sym, which is a noncolored symmetric operad, then it is easy to see that the monoid $\mathcal{T}^{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$ is isomorphic to the monoid $(\mathbb{N}^+, \times)$ internal to groupoids where $\mathrm{Aut}(n) \cong \mathfrak{S}_n$. The objects of $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$ are the same as the objects in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}$, and the morphisms are given by permutation of tuples (as in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}$) plus the ones given by $\mathrm{Aut}(n)$ for each n. The colored case is analogous.

6.3 *The opposite convention*

It is not difficult to see that the categories $\mathcal{T}^{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}$ and $\mathcal{T}^{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$ (as well as $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ and $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}}\mathsf{Sym}$) are self dual. In the case of Ass this means that the monoid $(\mathbb{N}^+, \times)$ is commutative. An obvious consequence of this self duality is that we get equivalent operads by applying the $\mathcal{T}$-construction to their opposite categories. Nevertheless, when dealing with plethysm it is in fact more natural, from a combinatorial point of view, to apply the $\mathcal{T}$-construction to the opposite categories. This is particularly apparent when we interpret the simplicial groupoid $T\mathbf{S}$ as an operad (see Examples 8.1.6 and 8.2.6).

We end this chapter by developing this variant in a general context, since we believe it is interesting in its own right. From a formal perspective there is not much to say, since in the context of internal categories if C is represented by

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$$

then C^{op} is represented by

$$\begin{tikzcd} & C_1 \arrow[dl, "t"'] \arrow[dr, "s"] & \\ C_0 & & C_0 \end{tikzcd}$$

and thus the $\mathcal{T}$-construction can be applied the same way. Let us see what $\mathcal{T}_{S^r} C^{op}$ looks like. We have that

$$\widetilde{C_1^{op}} = \sum_{(c_1,\ldots,c_n;c)} \prod_{i=1}^{n} \mathrm{Hom}_{C^{op}}(c_i, c) = \sum_{(c_1,\ldots,c_n;c)} \prod_{i=1}^{n} \mathrm{Hom}_C(c, c_i),$$

for each tuple $(c_1, \ldots, c_n; c)$ of elements of C_0. In this case elements $(c_1, \ldots, c_n; c)$ can be pictured as (picturing $n = 3$)

$$\begin{tikzcd} & c \arrow[dl] \arrow[d] \arrow[dr] & \\ c_1 & c_2 & c_3 \end{tikzcd}$$

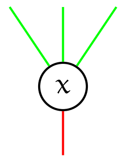
and under this representation, composition in $\mathcal{T}_{S^r} C^{op}$ looks like

$$\begin{tikzcd}[column sep=small] & & & c \arrow[dll] \arrow[d] \arrow[drr] & & \\ & c_1 \arrow[dl] \arrow[d] \arrow[dr] & & c_2 \arrow[d] & & c_3 \arrow[dl] \arrow[d] \\ c_1^1 & c_1^2 & c_1^3 & c_2^1 & c_3^1 & c_3^2. \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=small] & & c \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\ c_1^1 & c_1^2 & c_1^3 & c_2^1 & c_3^1 & c_3^2 \end{tikzcd}$$

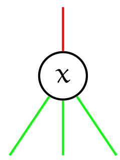
Furthermore, if C has products then

$$\prod_{i=1}^{n} \mathrm{Hom}_C(c, c_i) = \mathrm{Hom}_C(c, c_1 \times \cdots \times c_n).$$

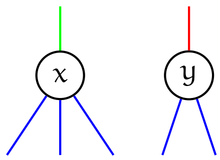
Suppose now that we start from an M^r-operad Q. The first step is the same as before: we obtain a category $\mathcal{T}^{M^r} Q$ whose arrows are operations of Q



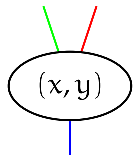
all of whose inputs coincide. Next, we take the opposite category $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} Q^{\mathrm{op}}$, and depict its arrows as



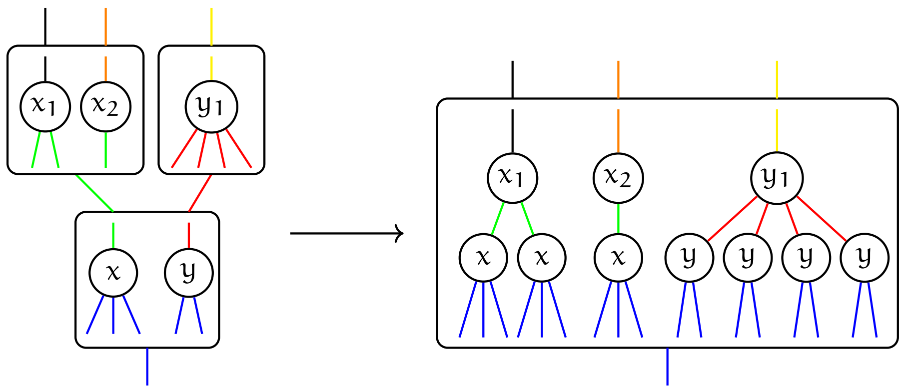
The $\mathcal{T}$-construction $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Q^{\mathrm{op}}$ has as operations sequences $(x_1, \dots, x_n)$ of arrows $x_i \in \overline{Q_1^{\mathrm{op}}}$ with the same output. For instance the pair



is an operation



in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Q^{\mathrm{op}}$. We show once and for all an example of composition in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Q^{\mathrm{op}}$:



(6.3.1)

and then of course we compose in Q, as in (6.2.1).

7

Plethysms and operads

Let us present the relation between the several plethystic bialgebras, operads and the $\mathcal{T}$-construction. Some proofs are omitted, since most of them are similar. The operads involved are the reduced symmetric operad Sym, the reduced associative operad Ass and their 2-colored versions. Also, playing the same role as these operads, we have a locally finite monoid Υ. On the other hand, the $\mathcal{T}$-constructions are taken with respect to the monads S^{r} and M^{r}, as in Chapter 6, and everything is internal to $\mathcal{E} = \mathbf{Grpd}$.

Let us stress again that by Proposition 4.5.6, Lemmas 5.4.3 and 5.4.4 and the discussion of Chapter 6 all the bar constructions featuring in the present chapter are locally finite Segal groupoids, so that we can take cardinality to arrive at their incidence bialgebra in the classical sense of vector spaces.

It is appropriate to begin with the classical bialgebras, which are the main cases:

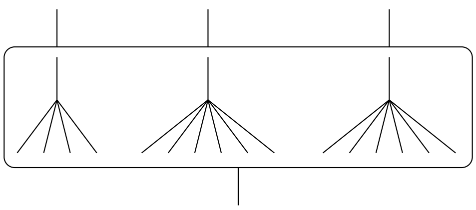
Theorem 7.0.1 (Joyal, cf. modern reformulation in [30])**.** *The Faà di Bruno bialgebra $\mathcal{F}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}}\mathsf{Sym}$.*

Recall that $\mathcal{B}^{\mathsf{S}}\mathsf{Sym} \simeq \mathrm{N}\mathbf{S}$, so that Remark 3.3.3 already proves this result. Notice that Sym is of course the same as $\mathcal{T}_{\mathsf{Id}}\mathsf{Sym}$ and, as explained in Chapter 6, it is also $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}}$ of the trivial monoid. This connects the Faà di Bruno bialgebra to the $\mathcal{T}$-construction in an analogous way as the plethystic bialgebras.

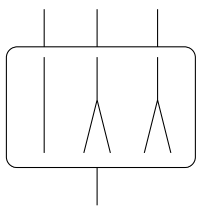
Theorem 7.0.2. *The plethystic bialgebra $\mathcal{P}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^{r}}\mathsf{Sym}$.*

Proof. The comparison between these two incidence bialgebras has been explained in Chapter 3, where the simplicial interpretation of plethysm was established. In Chapter 8 we will see that indeed $\mathcal{B}\mathcal{T}_{\mathsf{S}^{\mathsf{r}}}\mathsf{Sym}$ is equivalent to $\mathrm{T}\mathbf{S}$. □

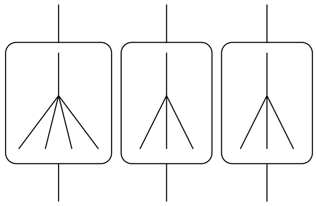
Example 7.0.3. Nevertheless, let us see again the interpretation of $\mathrm{P}_{(0,0,0,1,0,2),(1,2)}(\{A_\mu\}_\mu)$ (see Examples 3.2.5 and 3.3.2) but now from the point of view of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^{\mathsf{r}}}\mathsf{Sym}$. The vectors $\sigma = (0,0,0,1,0,2)$ and $\lambda = (1,2)$ are represented by



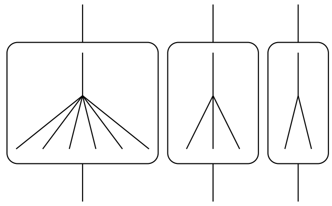
and



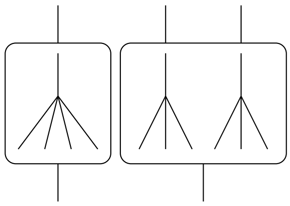
respectively. We have used the opposite convention (Section 6.3), which in this case does not affect the result. What we want to count is, roughly speaking, the number of ways we can obtain σ as a composition of λ with three operations. It is straightforward to see that there are essentially two choices:



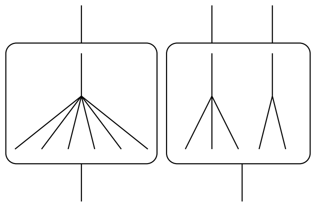
and



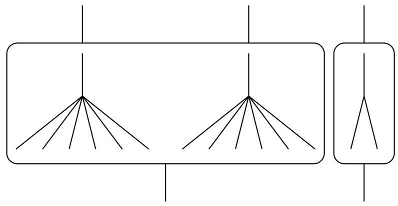
which clearly coincide with the ones of Examples 3.2.5 and 3.3.2. This example could be misleading, in the sense that each operation above contains only one operation of Sym. This happens of course because $|\sigma| = |\lambda|$. For instance, if we take instead $\lambda = (1,1)$ we obtain three possible choices:



,



and



Remark 7.0.4. We mentioned in Remark 3.1.7 that NS is contained in T**S**. From the present operadic perspective, it is clear that Sym is the suboperad of $\mathcal{T}_{S^r}$Sym consisting of sequences of the unique unary operation.

Before showing the rest of results, we give a brief summary of all the different plethystic bialgebras we study and how they relate to the various operads. The following standard notation is used:

- $\mathbf{x} = (x_1, x_2, \dots)$,
- Λ: set of infinite vectors of natural numbers with $\lambda_i = 0$ for all i large enough,
- $\Lambda \ni \lambda = (\lambda_1, \lambda_2, \dots)$ and $(\lambda_1, \dots, \lambda_n) := (\lambda_1, \dots, \lambda_n, 0, \dots)$,
- $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$,
- $\mathrm{aut}(\lambda) = 1!^{\lambda_1}\lambda_1! \cdot 2!^{\lambda_2}\lambda_2! \cdots$,
- $\lambda! = \lambda_1! \cdot \lambda_2! \cdots$,
- W: set of finite words of positive natural numbers,
- $W \ni \omega = \omega_1 \dots \omega_n$,
- $\mathbf{x}_\omega = x_{\omega_1} \cdots x_{\omega_n}$,
- $\omega! = \omega_1! \cdots \omega_n!$.

7.1 *Overview of variations*

We proceed to introduce the variations of the plethystic bialgebra we explore. For the set of variables $(x_1, x_2, \dots)$, there are three sources of variations. At the level of power series they are the following:

(i) Commuting or noncommuting variables: of course in the classical case the variables commute. When the variables do not commute we will index them by $\omega \in W$, rather than $\lambda \in \Lambda$.

(ii) Commuting or noncommuting coefficients.

(iii) Two types of automorphisms: $\operatorname{aut}(\lambda)$ or $\lambda!$ for commuting variables, and $\omega!$ or 1 for noncommuting variables.

These variations are not independent: if the variables commute then the coefficients commute. Analogous variations can be obtained of the Faà di Bruno bialgebra, except that in this case there is only one variable.

At the objective level, these three variations correspond (respectively) to the following choices:

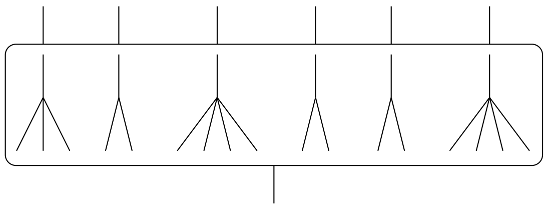
(i) $\mathcal{T}$-construction over S^{r} or over M^{r}.

(ii) Bar construction over S or over M.

(iii) Taking Sym or Ass as input operads.

The reason why they are not independent is clear here: there is a cartesian natural transformation $\mathsf{M}^{\mathrm{r}} \Rightarrow \mathsf{S}^{\mathrm{r}}$ that allows taking $\mathcal{B}^{\mathsf{S}^{\mathrm{r}}}$ of an M^{r}-operad (see Chapter 4), but no natural transformation in the opposite direction.

Let us give a brief justification of these correspondences. Consider the following operation:

$a =$ 

This could be an operation in any one of the following operads:

(i) $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$: in this case each operation has automorphisms, coming from the action of the symmetric group on Sym, and since the $\mathcal{T}$-construction is over S^{r} we can permute the operations inside a. This means that the isomorphism class of a is given by $\lambda = (0, 3, 1, 2)$, since the order of the operations does not matter, and it has $\operatorname{aut}(\lambda) = 2!^3 3! \cdot 3!^1 1! \cdot 4!^2 2!$ automorphisms. The corresponding bialgebra is thus $\mathcal{P}$, the classical plethystic bialgebra, and this particular operation corresponds to $A_{(0,3,1,2)}$, the linear map returning the coefficient of $x_2^3 x_3 x_4^2 / \operatorname{aut}(\lambda)$.

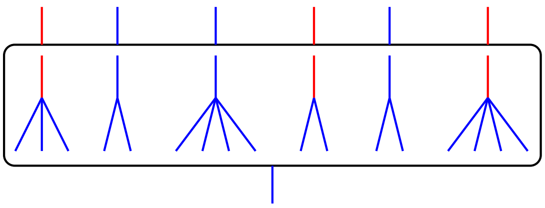
(ii) $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Sym}$: in this case the operations have automorphisms again, but since the $\mathcal{T}$-construction is over M^{r} we cannot permute them. This means that the isomorphism class of a is given by $\omega = (3, 2, 4, 2, 2, 4)$, so that it corresponds to noncommuting variables. Clearly it has $3!2!4!2!2!4!$ automorphisms. Now, depending on the bar construction it corresponds to commuting or noncommuting coefficients. This particular operation corresponds to $A_{(3,2,4,2,2,4)}$, the linear map returning the coefficient of $x_3 x_2 x_4 x_2 x_2 x_4 / \omega!$.

(iii) $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$: in this case the operations do not have automorphisms, and since the $\mathcal{T}$-construction is over M^{r} we cannot permute them. This means that the isomorphism class of a is given by $\omega = (3, 2, 4, 2, 2, 4)$, so that it corresponds to noncommuting variables, and it has no automorphisms. Now, depending on the bar construction it corresponds to commuting or noncommuting coefficients, as in the previous case. This particular operation corresponds to $a_{(3,2,4,2,2,4)}$, the linear map returning the coefficient of $x_3x_2x_4x_2x_2x_4$.

(iv) $\mathcal{T}_{S^r}\mathsf{Ass}$: in this case the operations do not have automorphisms, and since the $\mathcal{T}$-construction is over S^r we can permute them. This means that the isomorphism class of a is given by $\lambda = (0,3,1,2)$, and it has $\lambda! = 3! \cdot 1! \cdot 2!$ automorphisms. Therefore it corresponds to commuting variables and coefficients. This particular operation corresponds to $a_{(0,3,1,2)}$, the linear map returning the coefficient of $x_2^3x_3x_4^2/\lambda!$.

The cases of Sym and Ass are developed in Sections 7.2 and 7.3 respectively. In Section 7.4 we generalize $\mathcal{P}$ to power series in the set of variables $(x_m \,|\, m \in Y)$ indexed over a locally finite monoid.

In Sections 7.2 and 7.3 we also study the Faà di Bruno bialgebra in two variables and the plethystic bialgebra in the two sets of variables $(x_1, x_2, \dots), (y_1, y_2, \dots)$. For the plethystic case we consider only commuting variables and coefficients. Let us give a similar digression as above for the plethystic cases. Consider the following 2-colored operation:

$a =$ 

The isomorphism class of this operation is given by $(\lambda^1, \lambda^2) = \big((0,2,0,2),(0,1,1,1)\big)$ (since everything commutes now), and it can be an operation in either $\mathcal{T}_{S^r}\mathsf{Sym}_2$ or $\mathcal{T}_{S^r}\mathsf{Ass}_2$. It thus corresponds to $A_{\left((0,2,0,2),(0,1,1,1)\right)}$ the linear map returning the coefficients of $x_2^2x_4^2y_2y_3y_4/\operatorname{aut}(\lambda^x)\operatorname{aut}(\lambda^y)$, or to $a_{\left((0,2,0,2),(0,1,1,1)\right)}$, the linear map returning the coefficients of $x_2^2x_4^2y_2y_3y_4/\lambda^x!\lambda^y!$.

Remark 7.1.1. There is no Segal groupoid whose incidence bialgebra corresponds to permutational power series, where $\operatorname{aut}(\lambda) = 1^{\lambda_1}1! \cdot 2^{\lambda_2}2! \cdots$, because, roughly speaking, circular orders cannot be composed, and hence cannot be assembled into a Segal groupoid nor an operad. They can, however, be encoded into a decomposition space in the sense of [32]. Nevertheless, at this point, no decomposition space has been found to give a simplicial-groupoid interpretation of Bergeron permutationals.

7.2 *Bialgebras from* Sym *and* Sym_2

Replace $\mathbb{Q}[\![\mathbf{x}]\!]$ by $\mathbb{Q}\langle\!\langle \mathbf{x} \rangle\!\rangle$, that is, noncommuting variables. Elements of $\mathbb{Q}\langle\!\langle \mathbf{x} \rangle\!\rangle$ are written

$$F(\mathbf{x}) = \sum_{\omega \in W} \frac{F_\omega}{\omega!}\mathbf{x}^\omega,$$

Substitution of power series in $\mathbb{Q}\langle\!\langle \mathbf{x} \rangle\!\rangle$ is defined in the same way as before (Definition 1.0.1). The *plethystic bialgebra with noncommuting variables* $\mathcal{P}^\diamondsuit$ is defined as the free polynomial algebra $\mathbb{Q}\big[\{A_\omega\}_\omega\big]$ on the set maps A_ω and comultiplication and counit as usual.

Theorem 7.2.1. *The plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamondsuit}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{M}^r}\mathsf{Sym}$.*

If we now take $R\langle\langle \mathbf{x} \rangle\rangle$ with R a noncommutative unital ring, then we get the *noncommutative plethystic bialgebra with noncommuting variables* $\mathcal{P}^{\diamondsuit,\mathrm{nc}}$, which is the free associative unital algebra $\mathbb{Q}\langle\{A_\omega\}_\omega\rangle$ together with the usual comultiplication and counit. In this case, substitution of power series is defined in the same way but it is not associative. However the comultiplication is still coassociative. A proof of this can be found in [11] for the one variable case, which is obtained below.

Theorem 7.2.2. *The noncommutative plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamondsuit,\mathrm{nc}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{M}}\mathcal{T}_{\mathsf{M}^r}\mathsf{Sym}$.*

Let us move forward to power series in two variables. All the results are also valid for any number of variables, but for simplicity and notation we have chosen to show the two variables case. Also, for the bivariate plethystic bialgebras, we do not enter into noncommutativity of the variables or of the coefficients.

Let $\mathbb{Q}[\![x,y]\!]$ be the ring of formal power series in the variables x and y with coefficients in $\mathbb{Q}$ without constant term. Elements of $\mathbb{Q}[\![x,y]\!]$ are written

$$F(x,y) = \sum_{n+m\geqslant 1} \frac{F_{n,m}}{n!m!} x^n y^m.$$

The set $\mathbb{Q}[\![x,y]\!]\times\mathbb{Q}[\![x,y]\!]$ forms a (noncommutative) monoid with substitution of power series:

$$\begin{aligned}\big(\mathbb{Q}[\![x,y]\!]\times\mathbb{Q}[\![x,y]\!]\big)\times\big(\mathbb{Q}[\![x,y]\!]\times\mathbb{Q}[\![x,y]\!]\big) &\xrightarrow{\ \circ\ } \mathbb{Q}[\![x,y]\!]\times\mathbb{Q}[\![x,y]\!] \\ \big((F^1,F^2),(G^1,G^2)\big) &\longrightarrow \big(G^1(F^1,F^2),G^2(F^1,F^2)\big).\end{aligned}$$

We define the *Faà di Bruno bialgebra in two variables* $\mathcal{F}^2$ as the free polynomial algebra $\mathbb{Q}\big[\{A^i_{n,m}\}^{i=1,2}_{n+m\geqslant 1}\big]$ generated by the set maps

$$\begin{aligned} A^i_{n,m} : \mathbb{Q}[\![x,y]\!]\times\mathbb{Q}[\![x,y]\!] &\longrightarrow \mathbb{Q} \\ (F^1,F^2) &\longrightarrow F^i_{n,m}\end{aligned}$$

together with the comultiplication induced by substitution, meaning that

$$\Delta(A^i_{n,m})\big((F^1,F^2),(G^1,G^2)\big) = A^i_{n,m}\big((G^1,G^2)\circ(F^1,F^2)\big),$$

and counit given by $\epsilon(A^i_{n,m}) = A^i_{n,m}(x,y)$.

Theorem 7.2.3. *The Faà di Bruno bialgebra in two variables $\mathcal{F}^2$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathsf{Sym}_2$. The same holds for n variables and Sym_n.*

Notice that Sym_2 is the same as $\mathcal{T}_{\mathsf{Id}}\mathsf{Sym}$ and the same as $\mathcal{T}_{\mathsf{S}^r}C$, where $C=\{0 \rightleftarrows 1\}$ (Example 6.1.1). This connects the Faà di Bruno bialgebra in two variables to the $\mathcal{T}$-construction in an analogous way as the plethystic bialgebras.

We can do the same with the power series ring in two sets of infinitely many variables $\mathbb{Q}[\![\mathbf{x},\mathbf{y}]\!]$ with coefficients in $\mathbb{Q}$. We write

$$\mathbf{X} = (\mathbf{x},\mathbf{y}), \quad \lambda = (\lambda^1,\lambda^2) \in \Lambda^2, \quad \mathrm{aut}(\lambda) = \mathrm{aut}(\lambda^1)\,\mathrm{aut}(\lambda^2) \text{ and } \mathbf{X}^\lambda = \mathbf{x}^{\lambda^1}\mathbf{y}^{\lambda^2},$$

so that elements of $\mathbb{Q}[\![\mathbf{X}]\!]$ are written

$$F(\mathbf{X}) = \sum_\lambda \frac{F_\lambda}{\mathrm{aut}(\lambda)}\mathbf{X}^\lambda.$$

The set $\mathbb{Q}[\![\mathbf{X}]\!]\times\mathbb{Q}[\![\mathbf{X}]\!]$ forms a (noncommutative) monoid with plethystic substitution of power series:

$$\begin{aligned} (\mathbb{Q}[\![\mathbf{X}]\!]\times\mathbb{Q}[\![\mathbf{X}]\!])\times(\mathbb{Q}[\![\mathbf{X}]\!]\times\mathbb{Q}[\![\mathbf{X}]\!]) &\xrightarrow{\quad\circledast\quad} \mathbb{Q}[\![\mathbf{X}]\!]\times\mathbb{Q}[\![\mathbf{X}]\!] \\ \big((F^1,F^2),(G^1,G^2)\big) &\longmapsto \big(G^1(F^1,F^2),G^2(F^1,F^2)\big) \end{aligned}$$

The *plethystic bialgebra in two variables* $\mathcal{P}^2$ is defined as the free polynomial algebra $\mathbb{Q}\big[\{A^i_\lambda\}^{i=1,2}\big]$ generated by the set maps

$$\begin{aligned} A^i_\lambda : \mathbb{Q}[\![\mathbf{X}]\!]\times\mathbb{Q}[\![\mathbf{X}]\!] &\longrightarrow \mathbb{Q} \\ (F^1,F^2) &\longmapsto F^i_\lambda \end{aligned}$$

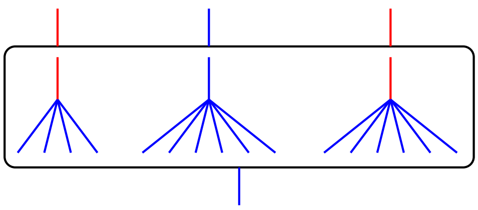
together with the comultiplication induced by substitution, meaning that

$$\Delta(A^i_\lambda)\big((F^1,F^2),(G^1,G^2)\big) = A^i_\lambda\big((G^1,G^2)\circ(F^1,F^2)\big),$$

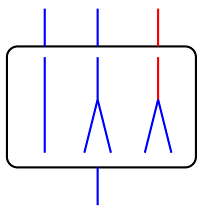
and counit given by $\epsilon(A^i_\lambda) = A^i_\lambda(x,y)$.

Theorem 7.2.4. *The plethystic bialgebra in two variables $\mathcal{P}^2$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^S\mathcal{T}_{S^r}\mathsf{Sym}_2$.*

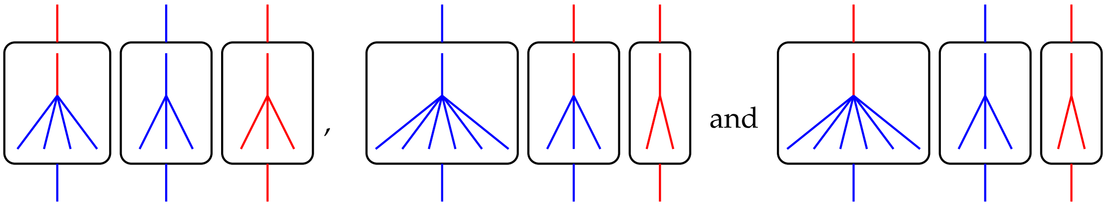
Example 7.2.5. We could define polynomials $P^2_{\sigma,\lambda}\big(\{A_\mu\}^i_\mu\big)$ to express the comultiplication of A^i_σ, in analogy to the univariate case. Let us see again the interpretation of $P^2_{\sigma,\lambda}\big(\{A_\mu\}^i_\mu\big)$, for $\sigma = \big((0,0,0,0,0,1),(0,0,0,1,0,1)\big)$ and $\lambda = \big((1,1),(0,1)\big)$, from the point of view of $\mathcal{B}^S\mathcal{T}_{S^r}\mathsf{Sym}_2$. These two vectors are represented by



and



respectively. The output color depends on whether we are computing the comultiplication of A^1_σ or A^2_σ. We assume the former, without loss of generality. It is easy to see that there are essentially three options, which are the only possible colorings of the solutions for the analogous case of Example 7.2.5:



At the level of power series, in the style of Example 3.2.5, we have the following substitutions

$$
\begin{aligned}
(x_1x_2y_2) \circledast (y_4+x_3, y_3) &= (y_4+x_3)(y_8+x_6)y_6 = y_4x_6y_6 + \cdots \\
(x_1x_2y_2) \circledast (x_6+y_3, y_2) &= (x_6+y_3)(x_{12}+y_6)y_4 = x_6y_6y_4 + \cdots \\
(x_1x_2y_2) \circledast (y_6+x_3, y_2) &= (y_6+x_3)(y_{12}+x_6)y_4 = y_6x_6y_4 + \cdots .
\end{aligned}
$$

7.3 *Bialgebras from* Ass *and* Ass_2

Take again $\mathbb{Q}[\![x]\!]$, but now write elements of $\mathbb{Q}[\![x]\!]$ as

$$F(x) = \sum_{n \geqslant 1} f_n x^n.$$

The *ordinary Faà di Bruno bialgebra* $\mathcal{F}_{\mathrm{ord}}$ is the free polynomial algebra $\mathbb{Q}[a_1, a_2, \dots]$ generated by the linear maps $a_i(F) = f_i$ together with the comultiplication induced by substitution and counit given by $\epsilon(a_n) = a_n(x)$, as before.

Theorem 7.3.1. *The ordinary Faà di Bruno bialgebra $\mathcal{F}_{\mathrm{ord}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}^r}\mathsf{Ass}$.*

It is clear that $\mathcal{F}$ and $\mathcal{F}_{\mathrm{ord}}$ are isomorphic bialgebras, since we have only changed the basis. However their combinatorial meaning is slightly different, and indeed $\mathcal{B}^{\mathsf{S}^r}\mathsf{Sym}$ and $\mathcal{B}^{\mathsf{S}^r}\mathsf{Ass}$ are not equivalent. Note that Ass is of course the same as $\mathcal{T}_{\mathsf{Id}}\mathsf{Ass}$ and, as explained in Chapter 6, it is also $\mathcal{T}_{\mathsf{M}^r}$ of the trivial monoid. This connects the ordinary Faà di Bruno bialgebra to the $\mathcal{T}$-construction.

If above we replace $\mathbb{Q}$ by R (a noncommutative unital ring), we obtain the *noncommutative Faà di Bruno bialgebra* $\mathcal{F}^{\mathrm{nc}}$ [11,25,50], the free associative unital algebra $\mathbb{Q}\langle a_1, a_2, \dots\rangle$ generated by the set maps $a_i(F) = f_i$, together with the comultiplication induced by substitution and counit $\epsilon(a_n) = a_n(x)$, as before. In this case, substitution of power series is not associative, but the comultiplication is still coassociative [11]. It is clear that $\mathcal{F}$ and $\mathcal{F}_{\mathrm{ord}}$ are the abelianization of $\mathcal{F}^{\mathrm{nc}}$ [11].

Theorem 7.3.2. *The noncommutative Faà di Bruno bialgebra $\mathcal{F}^{\mathrm{nc}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{M}^r}\mathsf{Ass}$.*

We now move to the plethystic bialgebras. The *exponential plethystic bialgebra* $\mathcal{P}_{\mathrm{exp}}$ is the same bialgebra as $\mathcal{P}$, but in this case $\mathrm{aut}(\lambda) = \lambda! = \lambda_1!\lambda_2!\cdots$ [60]. The generators of this bialgebra are denoted by a_λ.

Theorem 7.3.3. *The exponential plethystic bialgebra $\mathcal{P}_{\mathrm{exp}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^r}\mathsf{Ass}$.*

The *linear plethystic bialgebra with noncommuting variables* $\mathcal{P}^{\diamondsuit}_{\mathrm{lin}}$ is the same bialgebra as $\mathcal{P}^{\diamondsuit}$ but without automorphisms of ω. The generators for this bialgebra are denoted a_ω.

Theorem 7.3.4. *The linear plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamondsuit}_{\mathrm{lin}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$.*

The *noncommutative linear plethystic bialgebra with noncommuting variables* $\mathcal{P}^{\diamondsuit,\mathrm{nc}}_{\mathrm{lin}}$ is the same as $\mathcal{P}^{\diamondsuit,\mathrm{nc}}$ but without automorphisms on ω. We write a_ω for its generators. Contrary to what it may seem, the noncommutativity simplifies the explicit formula for the comultiplication of the generators. Denote by $|w|$ the length of a word. Let also W^W_n be the set of length n words of words of W. Finally, for $k \in \mathbb{N}$ and $\omega = \omega_1 \dots \omega_n \in W$, define the kth Verschiebung operator as

$$k\omega = (k\omega_1)\dots(k\omega_n).$$

Proposition 7.3.5. *The comultiplication of $\mathcal{P}^{\diamondsuit,\mathrm{nc}}_{\mathrm{lin}}$ is given by*

$$\Delta(a_\nu) = \sum_{\omega \in W} \sum_{\kappa \in W^W_{|\omega|}} T^{\kappa}_{\nu,\omega} \left(\prod_{i=1}^{|\omega|} a_{\kappa_i} \right) \otimes a_\omega,$$

where

$$T^{\kappa}_{\nu,\omega} = \begin{cases} 1 & \text{if } \nu = \sum_{i=1}^{n} \omega_i \kappa_i \\ 0 & \text{otherwise.} \end{cases}$$

This proposition is analogous to Proposition 3.2.3.

Theorem 7.3.6. *The noncommutative linear plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamondsuit,\mathrm{nc}}_{\mathrm{lin}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{M}}\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$.*

Proof. Notice that $\mathcal{B}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$ is discrete. Its elements are given by sequences of tuples

$$(m^1_1, \dots, m^1_{n_1}), \dots, (m^k_1, \dots, m^k_{n_k})$$

of elements of positive natural numbers (see Example 6.2.1), but there is only the identity morphisms between them. Thus juxtaposition of sequences gives $\mathcal{B}\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$ a (nonsymmetric) monoidal structure. Sequences containing one tuple are called connected, and form an algebra basis of the incidence bialgebra. The subgroupoid of connected sequences is denoted $\mathcal{B}^{\circ}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$. It is clear that $\pi_0\mathcal{B}^{\circ}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass} = \mathcal{B}^{\circ}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$ is isomorphic to W, and that $\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass} = \mathcal{B}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$ is isomorphic to W^W. Although $\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass} = \mathcal{B}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$ we keep using the notation δ_ω for the isomorphism class of $\omega \in \mathcal{B}_1\mathcal{T}_{\mathsf{M}^r}\mathsf{Ass}$. It only remains to compute the comultiplication:

$$\Delta(\delta_\nu) = \sum_{\omega} \sum_{\kappa} |\operatorname{Iso}(d_0\kappa, d_1\omega)_\nu| \delta_\kappa \otimes \delta_\omega.$$

By the discussion above we only have to check that

$$|\operatorname{Iso}(d_0\kappa, d_1\omega)_\nu| = T^{\kappa}_{\nu,\omega},$$

but this is clear because there is only one morphism between $d_0\kappa$ and $d_1\omega$ and fibering over ν means taking the subset of those morphisms that give ν after composing, hence $|\operatorname{Iso}(d_0\kappa, d_1\omega)_\nu| = 1$ if $d_1(\kappa, \omega) = \nu$ and 0 otherwise, exactly as $T^{\kappa}_{\nu,\omega}$. $\square$

Let us move forward to power series in two variables. Again, all the results are also valid for any number of variables, but for simplicity and notation we have chosen to show the two variables case.

Let $\mathbb{Q}\langle\langle x, y\rangle\rangle$ be the ring of formal power series in the noncommutative variables x and y with coefficients in $\mathbb{Q}$ without constant term. Elements of $\mathbb{Q}\langle\langle x, y\rangle\rangle$ are written

$$F(x, y) = \sum_{\omega} f_{\omega}\omega,$$

where ω is a nonempty word in x and y. The set $\mathbb{Q}\langle\langle x, y\rangle\rangle$ forms a noncommutative monoid with substitution of power series.

We define the *Faà di Bruno bialgebra in two noncommuting variables* $\mathcal{F}^{\langle 2\rangle}$ as the free polynomial algebra $\mathbb{Q}\big[\{a^i_{\omega}\}\big]$ generated by the set maps

$$\begin{aligned} a^i_{\omega} : \mathbb{Q}\langle\langle x, y\rangle\rangle \times \mathbb{Q}\langle\langle x, y\rangle\rangle &\longrightarrow \mathbb{Q} \\ (F^1, F^2) &\longmapsto f^i_{\omega} \end{aligned}$$

together with the counit given by $\epsilon(a^i_{\omega}) = a^i_{\omega}(x, y)$ and the comultiplication induced by substitution.

Theorem 7.3.7. *The Faà di Bruno bialgebra in two noncommuting variables $\mathcal{F}^{\langle 2\rangle}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{S^r}\mathsf{Ass}_2$.*

We obtain the *noncommutative Faà di Bruno bialgebra in two noncommuting variables* $\mathcal{F}^{\langle 2\rangle,nc}$ by taking above power series with coefficients in R.

Theorem 7.3.8. *The noncommutative Faà di Bruno bialgebra in two noncommuting variables $\mathcal{F}^{\langle 2\rangle,nc}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathsf{Ass}_2$.*

Finally, the *exponential plethystic bialgebra in two variables* $\mathcal{P}^2_{\exp}$ is the same as $\mathcal{P}^2$ but with exponential automorphisms $\mathrm{aut}(\lambda) = \lambda_1!\lambda_2!\cdots$. The generators of this bialgebra are denoted a^i_{λ}.

Theorem 7.3.9. *The exponential plethystic bialgebra in two variables $\mathcal{P}^2_{\exp}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^S\mathcal{T}_{S^r}\mathsf{Ass}_2$.*

7.4 Y-plethysm and bialgebras from Y

In Section 7.3 we could have taken the locally finite monoid $(\mathbb{N}, \times)$ instead of Ass, since $\mathcal{T}^{M^r}\mathsf{Ass} = (\mathbb{N}, \times)$ (Example 6.2.1). In fact, we have indirectly done so in the proof of Theorem 7.3.6. It is the case that the three plethystic bialgebras of Section 7.3 can be generalized to any locally finite monoid. In this section we explain the generalization of $\mathcal{P}_{\exp}$, which arises from Y-plethysm, introduced by Méndez and Nava [56] in the context of colored species.

Let Y be a locally finite monoid; this means that any $m \in Y$ there has a finite number of two-step factorizations $m = nk$. This is the same as the finite decomposition property of Cartier–Foata [14]. Consider the ring of formal power series $\mathbb{Q}[\![x_m | m \in Y]\!]$ without constant term. Following the same conventions as above, the set of variables $\{x_m\}_{m\in Y}$ is denoted $\mathbf{x}$. Elements of $\mathbb{Q}[\![\mathbf{x}]\!]$ are written

$$F(\mathbf{x}) = \sum_{\lambda\in\Lambda} \frac{f_\lambda}{\lambda!}\mathbf{x}^\lambda,$$

where now the sum is indexed by the subset $\Lambda \subseteq \mathrm{Hom}_{\mathbf{Set}}(Y,\mathbb{N})$ of maps with finite support, and $\mathbf{x}^\lambda$ is the obvious monomial, for $\lambda \in \Lambda$. In this case $\lambda! = \prod \lambda_m!$.

The monoid structure of Y defines an operation $x_m \circledast x_n = x_{mn}$, which extends to a binary operation on $\mathbb{Q}[\![\mathbf{x}]\!]$ as

$$\begin{aligned}(G \circledast F)(x_m|m \in Y) &:= G(F_m|m \in Y), \qquad \text{where}\\ F_m(x_n|n \in Y) &:= F(x_{mn}|n \in Y).\end{aligned}$$

This substitution operation was introduced in [56] in the context of species colored over a monoid, although their conditions on the monoid are more restrictive (see Section 1.2). The main example comes from the monoid $(\mathbb{N}^+,\times)$, which gives ordinary plethysm. Another relevant example is $(\mathbb{N},+)$, which gives $F_k(\mathbf{x}) = F(x_k, x_{k+1},\dots)$, which appears in [55]. The power series F_m can be described by using the Verschiebung operators: for each $m \in Y$ we define the mth Verschiebung operator V^m on $\mathrm{Hom}_{\mathbf{Set}}(Y,\mathbb{N})$ as follows: for each $\lambda \in \mathrm{Hom}_{\mathbf{Set}}(Y,\mathbb{N})$ and $n \in Y$,

$$V^m\lambda(n) = \sum_{mk=n} \lambda_k.$$

Clearly if $Y = (\mathbb{N}^+,\times)$ this gives the usual Verschiebung operators [15,60,61]. The power series F_m can be expressed as

$$F_m(\mathbf{x}) = \sum_\lambda \frac{f_\lambda}{\mathrm{aut}(\lambda)}\mathbf{x}^{V^m\lambda}.$$

As usual, we define the Y-*plethystic bialgebra* $\mathcal{P}^Y$ as the polynomial algebra $\mathbb{Q}\big[\{a_\lambda\}_\lambda\big]$ on the set maps $a_\lambda : \mathbb{Q}[\![\mathbf{x}]\!] \to \mathbb{Q}$ defined by $a_\lambda(F) = f_\lambda$, with comultiplication dual to plethystic substitution, that is

$$\Delta(a_\lambda)(F,G) = a_\lambda(G \circledast F),$$

and counit given by $\epsilon(a_\lambda) = a_\lambda(x_1)$.

What follows is devoted to expressing the comultiplication of $\mathcal{P}^Y$. We develop it in an analogous way as $\mathcal{P}$ in Section 3.2. Consider a list $\boldsymbol{\mu} \in \Lambda^n$ of n infinite vectors, regarded as a representative element of a multiset $\overline{\boldsymbol{\mu}} \in \Lambda^n/\mathfrak{S}_n$. We denote by $R(\boldsymbol{\mu}) \subseteq \mathfrak{S}_n$ the set of automorphisms that maps the list $\boldsymbol{\mu}$ to itself. For example if $\boldsymbol{\mu} = \{\alpha,\alpha,\beta,\gamma,\gamma,\gamma\}$ then $R(\boldsymbol{\mu})$ has $2!\cdot 1!\cdot 3!$ elements. Notice that if $\boldsymbol{\mu},\boldsymbol{\mu}' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $R(\boldsymbol{\mu}) \cong R(\boldsymbol{\mu}')$. We may thus refer to $R(\boldsymbol{\mu})$ for a multiset $\overline{\boldsymbol{\mu}} \in \Lambda^n/\mathfrak{S}_n$ by taking a representative, since we are only interested in its cardinality.

Fix two infinite vectors, $\sigma,\lambda \in \Lambda$, and a list of infinite vectors $\boldsymbol{\mu} \in \Lambda^n$, with $n = |\lambda|$. We define the set of $(\lambda,\boldsymbol{\mu})$–*decompositions* of σ as

$$T^{\boldsymbol{\mu}}_{\sigma,\lambda} := \left\{ p\colon \boldsymbol{\mu} \xrightarrow{\sim} \sum_{m\in Y}\{1,\dots,\lambda_m\} \mid \sigma = \sum_{\mu\in\boldsymbol{\mu}} V^{q(\mu)}\mu \right\},$$

where p is a bijection of n-element sets and q returns the index of $p(\mu)$ in the sum. A useful way to visualize an element of this set is as a placement of the elements of $\boldsymbol{\mu}$ over a grid with λ_m cells in the mth column such that if we apply V^m to the mth column and sum the cells the result is σ. For example, if $\lambda = (\lambda_{m_1}, \lambda_{m_2}, \lambda_{m_3}) = (2, 1, 3)$ and $\boldsymbol{\mu} = \{\alpha, \alpha, \beta, \gamma, \gamma, \gamma\}$ the placement

$$\begin{array}{ccc} & & \boxed{\gamma} \\ \boxed{\alpha} & & \boxed{\beta} \\ \boxed{\gamma} & \boxed{\gamma} & \boxed{\alpha} \\ V^{m_1} & V^{m_2} & V^{m_3} \end{array}$$

belongs to $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ if $\sigma = V^{m_1}(\gamma + \alpha) + V^{m_2}(\gamma) + V^{m_3}(\alpha + \beta + \gamma)$, where the sum is a pointwise vector sum in Λ. Note that each such placement appears $|R(\boldsymbol{\mu})|$ times in $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$. Observe also that if $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $T^{\boldsymbol{\mu}}_{\sigma,\lambda} \cong T^{\boldsymbol{\mu}'}_{\sigma,\lambda}$. We may thus refer to $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ for a class $\overline{\boldsymbol{\mu}} \in \Lambda^{|\lambda|}/\mathfrak{S}_{|\lambda|}$ by taking a representative, since we are only interested in its cardinality.

Proposition 7.4.1. *The comultiplication of $\mathcal{P}^Y$ is given by*

$$\Delta(\sigma) = \sum_{\lambda} \sum_{\overline{\boldsymbol{\mu}}} \frac{\mathrm{aut}(\sigma) \cdot |T^{\boldsymbol{\mu}}_{\sigma,\lambda}|}{\mathrm{aut}(\lambda) \cdot \mathrm{aut}(\boldsymbol{\mu})} \prod_{\mu \in \boldsymbol{\mu}} a_{\mu}. \tag{7.4.1}$$

This proposition is analogous to Proposition 3.2.3.

Theorem 7.4.2. *The Y-plethystic bialgebra $\mathcal{P}^Y$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^S\mathcal{T}_{S^r}Y$.*

Proof of 7.4.2. Let us compute the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{S^r}Y$. First of all, notice that the elements of $\mathcal{B}_1\mathcal{T}_{S^r}Y = S^r\mathcal{T}_{S^r}Y$ are sequences of tuples

$$(m^1_1, \ldots, m^1_{n_1}), \ldots, (m^k_1, \ldots, m^k_{n_k})$$

of elements of Y. Juxtaposition of sequences gives $\mathcal{B}\mathcal{T}_{S^r}Y$ a symmetric monoidal structure. Sequences containing only one tuple are called connected, and form an algebra basis of the incidence bialgebra. Since the morphisms between tuples are given by permutations, it is clear that the set of isomorphism classes of connected elements $\pi_0\mathcal{B}^{\circ}_1\mathcal{T}_{S^r}Y$ is isomorphic to Λ, the subset of $\mathrm{Hom}_{\mathbf{Set}}(Y, \mathbb{N})$ consisting of maps with finite support. The isomorphism class δ_λ of a connected element λ is given by the map $Y \xrightarrow{\lambda} \mathbb{N}$ such that λ_m is the number of times m appears in λ. Be aware that the same notation is used for either the connected elements of $\mathcal{B}_1\mathcal{T}_{S^r}Y$ and the maps representing their isomorphism class. Moreover,

$$\pi_0\mathcal{B}_1\mathcal{T}_{S^r}Y \cong \sum_n \Lambda^n /\!\!/ \mathfrak{S}_n,$$

so that an element $\tau \in \pi_0\mathcal{B}_1\mathcal{T}_{S^r}Y$ may be identified with a multiset $\overline{\boldsymbol{\mu}}$ of maps. With these identifications we clearly have

$$|\mathrm{Aut}(\lambda)| = \lambda! \quad \text{and} \quad |\mathrm{Aut}(\tau)| = \mathrm{aut}(\boldsymbol{\mu}),$$

for λ connected and τ not necessarily connected. The left-hand sides refer to the automorphism groups in $\mathcal{B}_1\mathcal{T}_{S^r}Y$, while the right-hand sides were introduced above.

The assignment

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0\mathcal{B}_1\mathcal{T}_{S^r}Y} & \longrightarrow & \mathcal{P}_{\exp} \\ \delta_\lambda & \longmapsto & a_\lambda \\ \delta_{\lambda+\mu} = \delta_\lambda\delta_\mu & \longmapsto & a_\lambda a_\mu, \end{array}$$

for λ and μ connected, defines an isomorphism of algebras. Notice that $\lambda+\mu$ is the monoidal sum in $\mathcal{B}_1\mathcal{T}_{S^r}Y$, which does not correspond to the pointwise sum of their corresponding infinite vectors, since it has two connected components.

We now have to compute the coproduct in $\mathbb{Q}_{\pi_0\mathcal{B}_1\mathcal{T}_{S^r}Y}$. It is enough to compute it for connected elements. From Lemma 2.4.2 we have, for σ connected,

$$\Delta(\delta_\sigma) = \sum_\lambda\sum_\tau \frac{|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma|}{|\operatorname{Aut}(\lambda)||\operatorname{Aut}(\tau)|}\delta_\tau\otimes\delta_\lambda. \tag{7.4.2}$$

In view of the discussion above, it only remains to show that

$$|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma| = \operatorname{aut}(\sigma)\cdot|T^{\boldsymbol{\mu}}_{\sigma,\lambda}|.$$

Consider representatives for τ and λ,

$$\begin{aligned} \tau &= \big((m^1_1,\dots,m^1_{n_1}),\dots,(m^k_1,\dots,m^k_{n_k})\big) \\ \lambda &= (m_1,\dots,m_k), \end{aligned}$$

then $d_0\tau = d_1\lambda = (1,\dots,1)$, k times. This means that

$$\operatorname{Iso}(d_0\tau, d_1\lambda) = \operatorname{Aut}(1,\dots,1)\cong\mathfrak{S}_k.$$

Any element $\phi\in\operatorname{Iso}(d_0\tau, d_1\lambda)$ induces a map between sequences

$$\big((m^1_1,\dots,m^1_{n_1}),\dots,(m^k_1,\dots,m^k_{n_k})\big)\xrightarrow{\phi}(m_1,\dots,m_k).$$

We express it as a permutation on τ and write

$$\phi(\tau) = \big((m^{\phi(1)}_1,\dots,m^{\phi(1)}_{n_{\phi(1)}}),\dots,(m^{\phi(k)}_1,\dots,m^{\phi(k)}_{n_{\phi(k)}})\big).$$

Now, consider the subset

$$\Big\{\phi\in\operatorname{Iso}(d_0\tau, d_1\lambda)\mid d_1\big((\phi(\tau),\lambda)\big)\simeq\sigma\Big\}.$$

It is straightforward to see that this subset is isomorphic to

$$T^{\boldsymbol{\mu}}_{\sigma,\lambda} := \left\{p\colon\boldsymbol{\mu}\xrightarrow{\sim}\sum_{m\in M}\{1,\dots,\lambda_m\}\mid\sigma=\sum_{\mu\in\boldsymbol{\mu}}V^{q(\mu)}\mu\right\},$$

under the identifications $\tau\to\boldsymbol{\mu}$ and $\phi\to p$. The summation of the Verschiebung operators is precisely composition of $\phi(\tau)$ and λ. Finally, since $\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma$ is a homotopy fiber we have that

$$\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma\cong\operatorname{Aut}(\sigma)\times\Big\{\phi\in\operatorname{Iso}(d_0\tau, d_1\lambda)\mid d_1\big((\phi(\tau),\lambda)\big)\simeq\sigma\Big\}\cong\operatorname{Aut}(\sigma)\times T^{\boldsymbol{\mu}}_{\sigma,\lambda}$$

and therefore

$$|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma| = \operatorname{aut}(\sigma)\cdot|T^{\boldsymbol{\mu}}_{\sigma,\lambda}|,$$

as we wanted to see. □

This proves also Theorem 7.3.3 by taking the monoid $(\mathbb{N}^+,\times)$.

8

$\mathsf{T}\mathbf{S}$ *revisited*

We end this work by exploring the relations between the $\mathcal{T}$-construction and the simplicial groupoid $\mathsf{T}\mathbf{S}$. We first prove that $\mathsf{T}\mathbf{S}$ and $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ are equivalent simplicial groupoids. This proves in particular Theorem 7.0.2. Then we show that the operads of Chapter 7 arising from Ass or Sym are also equivalent to similar simplicial groupoids.

8.1 *Equivalence between* $\mathsf{T}\mathbf{S}$ *and* $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$

Consider the category of finite ordinals $[n] = \{1, \ldots, n\}$ and set maps. We say that a square

$$\begin{tikzcd}
{[m]} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[n]} \arrow[d, "f"] \\
{[l]} \arrow[r, "g"'] & {[k]}
\end{tikzcd} \tag{8.1.1}$$

is *monotone* if it is a pullback of sets, p is monotone and q is monotone at each fiber over p, that is, $q_{|p^{-1}(i)|}$ is monotone for all $i \in [l]$.

Lemma 8.1.1. *Consider the category of finite ordinals and set maps.*

(i) The class of monotone pullback squares is closed under composition of squares.

(ii) Given a diagram $[l] \xrightarrow{g} [k] \xleftarrow{f} [n]$, there is a unique monotone square as (8.1.1).

Proof. (i) is clear, and (ii) follows from the fact that we can totally order the pullback,

$$P = \sum_{i \in [k]} [l]_i \times [n]_i,$$

by using the orders of $[l]$ and $[n]$. That is, given $a, b \in P$, then $a < b$ if $p(a) < p(b)$ or $p(a) = p(b)$ and $q(a) < q(b)$. □

Consider the full subsimplicial groupoid $\mathcal{V} \subseteq \mathsf{T}\mathbf{S}$ containing only the simplices whose entries are the finite ordinals $[k]$, whose left-down arrows and right arrows are monotone surjections and whose left-down arrows are fiber-monotone in the sense of (8.1.1), and whose pullback squares are monotone. Note that Lemma 8.1.1 ensures that $\mathcal{V}$ is well defined, meaning that the inclusion $\mathcal{V} \hookrightarrow \mathsf{T}\mathbf{S}$ is a morphism of simplicial groupoids.

Lemma 8.1.2. $\mathcal{V} \hookrightarrow \mathsf{T}\mathbf{S}$ *is an equivalence of simplicial groupoids.*

Proof. Given an element of $T_n\mathbf{S}$,

$$\begin{tikzcd}[column sep=small]
 & {} \arrow[dl, dashed] & & & {} \arrow[dr, dashed] & \\
 & t_{01} \arrow[dl] \arrow[dr] & & & & t_{n-1,n} \arrow[dl] \arrow[dr] & \\
t_{00} \arrow[rr] & & t_{11} \arrow[r, dashed] & & t_{n-1,n-1} \arrow[rr] & & t_{nn},
\end{tikzcd}$$

it is clear we can choose an ordering of the t_{ii} and the $t_{i,i+1}$ such that all the arrows between them are monotone. Then by Lemma 8.1.1 there exists a unique ordering on the rest of the t_{ij}'s making the pullback squares monotone. Hence the inclusion is essentially surjective. Since we have taken the full inclusion, the automorphism group of any element of $\mathcal{V}_n$ is equal to to its automorphism group as an element of $T_n\mathbf{S}$. Hence the inclusion is an equivalence. □

Note that in $\mathcal{V}$ the uniqueness of the monotone squares implies that the Segal maps are in fact isomorphisms,

$$\mathcal{V}_n \cong \mathcal{V}_1 \times_{\mathcal{V}_0} \cdots \times_{\mathcal{V}_0} \mathcal{V}_1.$$

In other words, there is a well-defined composition $d_1 \colon \mathcal{V}_1 \times_{\mathcal{V}_0} \mathcal{V}_1 \to \mathcal{V}_1$. In view of this we may drop the elements t_{ij} with $j \geqslant i+2$ from the diagrams.

Lemma 8.1.3. *Let* V *be the operad whose* n*-ary operations are diagrams*

$$\begin{tikzcd}
 & {[m]} \arrow[dl] \arrow[dr] & \\
{[n]} \arrow[rr] & & 1
\end{tikzcd}$$

where $[m] \twoheadrightarrow [n]$ *is monotone, whose morphisms are entrywise bijections, and whose composition is given by monotone pullback squares. Then* $\mathcal{V} \cong \mathcal{B}\mathsf{V}$.

Proof. The isomorphism is given by

$$\begin{tikzcd}[column sep=small]
 & {[m_1]} \arrow[dl] \arrow[dr] & & & {[m_k]} \arrow[dl] \arrow[dr] & & & & {[m_1+\cdots+m_k]} \arrow[dl] \arrow[dr] & \\
{[n_1]} \arrow[rr] & & 1, \cdots, & {[n_2]} \arrow[rr] & & 1, & \mapsto & {[n_1+\cdots+n_k]} \arrow[rr] & & {[k]}
\end{tikzcd}$$

at the level of 1-simplices and similarly in general. □

Lemma 8.1.4. V *is isomorphic to* $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$.

Proof. An operation of $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ is a family of operations of Sym, which is equivalent to a monotone surjection $[m] \twoheadrightarrow [n]$. It is also clear that morphims between operations of $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ are the same as morphisms in V. Thus we only need to see that the compositions coincide. Let us denote by x the unique x-ary operation of Sym. Thus a general element of $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ is a tuple $(x_1, \ldots, x_n)$. By definition of the $\mathcal{T}$-construction

$$\begin{aligned}
(x_1, \ldots, x_n) \circledast \big((y_1^1, \ldots, y_{k_1}^1), \ldots, (y_1^n, \ldots, y_{k_n}^n)\big) &= \\
&= \big(y_1^1 \cdot x_1, \ldots, y_{k_1}^1 \cdot x_1, \ldots, y_1^n \cdot x_n, \ldots, y_{k_n}^n \cdot x_n\big),
\end{aligned}$$

which is nothing but the pullback

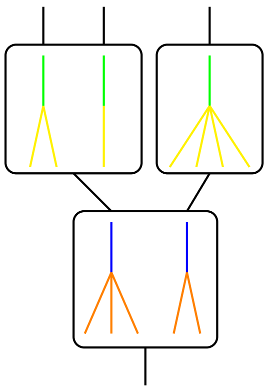
$$\begin{tikzcd}
 & & \left[\sum_{i,j} x_i y^i_j\right] \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & \left[\sum_{i,j} y^i_j\right] \arrow[dl, two heads] \arrow[dr, two heads] & & \left[\sum_i x_i\right] \arrow[dl, two heads] \arrow[dr, two heads] & \\
\left[\sum_i k^i_i\right] \arrow[rr, two heads] & & [n] \arrow[rr, two heads] & & 1,
\end{tikzcd}$$

the composition of their corresponding operations in V. □

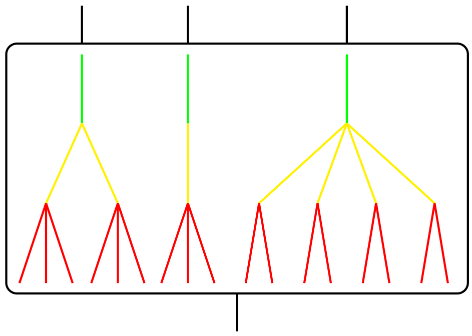
Proposition 8.1.5. *The simplicial groupoids* T**S** *and* $\mathcal{B}^{S}\mathcal{T}_{S'}$Sym *are equivalent.*

Proof. It is direct from Lemmas 8.1.1, 8.1.2 and 8.1.3. □

Example 8.1.6. Consider the following 2-simplex of $\mathcal{B}^{S}\mathcal{T}_{S'}$Sym:



, or



(8.1.2)

We use colors here only to make the comparison more pleasant, but of course this is not a colored operad. This 2-simplex corresponds, in T**S**, to

$$\begin{tikzcd}
 & & 17 \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & 7 \arrow[dl, two heads] \arrow[dr, two heads] & & 5 \arrow[dl, two heads] \arrow[dr, two heads] & \\
3 \arrow[rr, two heads] & & 2 \arrow[rr, two heads] & & 1.
\end{tikzcd} \tag{8.1.3}$$

It is opportune in this example to show that indeed the opposite convention comes out more naturally in order to interpret T**S** as an operad. Recall from Section 3.3 that using the Verschiebung operators as a scalar multiplication (see Remark 3.2.1) we can write the information on (8.1.3) as

$$17 \twoheadrightarrow 3 = (9 \twoheadrightarrow 2) + (8 \twoheadrightarrow 1) = V^3(3 \twoheadrightarrow 2) + V^2(4 \twoheadrightarrow 1) = (3 \times 3 \twoheadrightarrow 2) + (2 \times 4 \twoheadrightarrow 1), \tag{8.1.4}$$

This was used in the proof of Theorem 3.3.1, and we can clearly see it in (8.1.2). On the contrary, it is not difficult to check that without the opposite convention Equation (8.1.4) would rather appear as

$$17 \twoheadrightarrow 3 = (9 \twoheadrightarrow 2) + (8 \twoheadrightarrow 1) = \big((2 \times 3 \twoheadrightarrow 1) + (1 \times 3 \twoheadrightarrow 1)\big) + (4 \times 2 \twoheadrightarrow 1).$$

8.2 *Other* TS*-like simplicial groupoids*

We now present other equivalences between variations of TS and some of the bar constructions treated before. First of all we introduce some notation: monotone surjections between ordered sets are denoted $a \overset{\bullet}{\twoheadrightarrow} b$. We call *linear surjection* $a \overset{\circ}{\twoheadrightarrow} b$ a surjection between finite sets $f : a \twoheadrightarrow b$ with an order on $f^{-1}(r)$ for each $r \in b$, as in Section 3.4.

Notice that the composite of two monotone surjections is again a monotone surjection, and the composite of two linear surjections is again a linear surjection, with the obvious order. Moreover, given pullback squares

$$\begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d, two heads, "p\,\bullet"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, two heads, "\bullet\, f"] \\ \cdot \arrow[r, two heads] & \cdot \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d, two heads, "q\,\circ"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, two heads, "\circ\, g"] \\ \cdot \arrow[r, two heads] & \cdot \end{tikzcd},$$

we say that p and f are *compatible* if the order of p is induced by the order of f, in the sense of Lemma 8.1.1. Similarly, we say that q and g are compatible if the order of q is induced by the order of g.

The proofs of all the following results are similar to the one of Proposition 8.1.5. To avoid repetitiveness we give only intuitive explanations.

Example 8.2.1. The simplicial groupoid $\mathcal{B}^{S}\mathcal{T}_{S^r}\mathsf{Ass}$ is equivalent to the simplicial groupoid constructed as TS but with the additional structure that all the left-down surjections are linear and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads, "\circ"'] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads] & & t_{11}. \end{tikzcd}$$

The isomorphism classes of connected diagrams are again infinite vectors $\lambda = (\lambda_1, \lambda_2, \dots)$ as in TS, and the number of automorphisms of a connected element of class λ is precisely $\lambda_1! \cdot \lambda_2! \cdots$, since t_{01} is fixed.

Recall from Section 1.4 the definition of linear transversal. In the language of surjections, a linear partition corresponds to a surjection $\sigma \colon E \twoheadrightarrow B$ with a partial order on E consisting of linear orders on each fiber of σ. We shall name these surjections *linear*, and denote them $\sigma \colon E \overset{\circ}{\twoheadrightarrow} B$. A linear transversal of σ corresponds to a transversal

$$\begin{tikzcd} & & E \arrow[dll, two heads, bend right, "\sigma\,\circ"'] \arrow[dl, two heads, "\pi\,\circ"'] \arrow[dr, two heads, "\tau"] \arrow[dd, phantom, "\vee", very near start] & & \\ & S \arrow[dl, two heads, "\circ\,\gamma"] \arrow[dr, two heads] & & X \arrow[dl, two heads, "\circ\,\theta"] \arrow[dr, two heads] & \\ B \arrow[rr, two heads] & & I \arrow[rr, two heads] & & 1 \end{tikzcd} \tag{8.2.1}$$

such that all the left-down surjections are linear and whose linear orders are compatible. By compatible we mean that the order of σ is the one induced by the orders of π and γ and the order of π is induced by the one of θ along the pullback.

We can clearly recognize this as an object in the groupoid of 2-simplices of $\mathcal{B}^{S}\mathcal{T}_{S^r}\mathsf{Ass}$.

Example 8.2.2. The simplicial groupoid $\mathcal{B}^S\mathcal{T}_{M^r}\mathsf{Ass}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the left-down surjections and the right surjections are linear and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads, "\circ" marking] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads, "\circ" marking] & & t_{11}. \end{tikzcd}$$

Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by words $\omega = \omega_1\omega_2\ldots\omega_n$ where ω_i is the size of the ith fiber. It does not have any automorphisms, since t_{01} and t_{00} are fixed.

Example 8.2.3. The simplicial groupoid $\mathcal{B}^S\mathcal{T}_{M^r}\mathsf{Sym}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the right surjections are linear and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads, "\circ" marking] & & t_{11}. \end{tikzcd}$$

Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by finite words $\omega = \omega_1\omega_2\ldots\omega_n$ where $\omega_i > 0$ is the size of the ith fiber. It has $\omega! := \omega_1!\omega_2!\cdots\omega_n!$ automorphisms, since t_{00} is fixed.

Example 8.2.4. The simplicial groupoid $\mathcal{B}^M\mathcal{T}_{M^r}\mathsf{Sym}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the right surjections are monotone. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads, "\bullet" marking] & & t_{11}. \end{tikzcd}$$

Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by finite words $\omega = \omega_1\omega_2\ldots\omega_n$ where $\omega_i > 0$ is the size of the ith fiber. It has $\omega! := \omega_1!\omega_2!\cdots\omega_n!$ automorphisms, since t_{00} is fixed. The difference between this simplicial groupoid and the one of Example 8.2.3 is that in this case t_{11} is also ordered. As a consequence the monoidal structure is not symmetric, so that the resulting incidence bialgebra is not commutative.

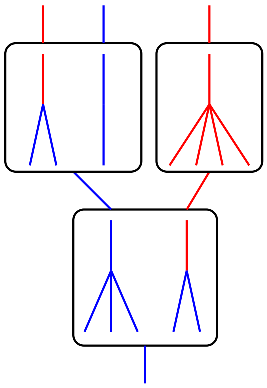
Example 8.2.5. The simplicial groupoid $\mathcal{B}^M\mathcal{T}_{M^r}\mathsf{Ass}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the left-down surjections and the right surjections are monotone and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{tikzcd} & t_{01} \arrow[dl, two heads, "\bullet" marking] \arrow[dr, two heads] & \\ t_{00} \arrow[rr, two heads, "\bullet" marking] & & t_{11}. \end{tikzcd}$$

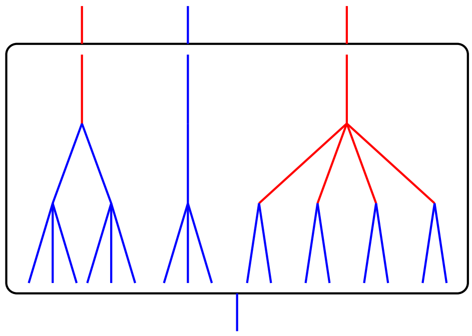
Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by words $\omega = \omega_1\omega_2\dots\omega_n$ where ω_i is the size of the ith fiber. It does not have any automorphisms, since t_{01} and t_{00} are fixed. Again, the difference between this simplicial groupoid and the one of Example 8.2.2 is that in this case t_{11} is ordered.

Example 8.2.6. Finally, the simplicial groupoid $\mathcal{B}^S\mathcal{T}_{S^r}\mathsf{Sym}_2$ is equivalent to the simplicial groupoid constructed as $\mathbf{TS}$ but with the additional structure that the objects are 2-colored and the right-down surjections are color preserving. Morphisms are color-preserving levelwise bijections.

For instance, the following 2-simplex of $\mathcal{B}^S\mathcal{T}_{S^r}\mathsf{Sym}_2$,



, or



(8.2.2)

where now the colors do refer to the input and output colors, corresponds to the following 2-simplex:

$$\begin{tikzcd}[column sep=small]
 & & 17 \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\vee" very near start] & & \\
 & 3+4 \arrow[dl] \arrow[dr, two heads] & {} & 5 \arrow[dl, two heads] \arrow[dr, two heads] & \\
1+2 \arrow[rr, two heads] & & 1+1 \arrow[rr, two heads] & & 1.
\end{tikzcd} \qquad (8.2.3)$$

Observe that indeed the right-down surjections are color-preserving. Notice also that if we had not used here the opposite convention the colors of (8.2.2) would not match the colors of (8.2.3) in such a direct way.

A
Some axioms

A.1 Axioms for internal category

Let $\mathcal{E}$ be a cartesian category. A category C internal to $\mathcal{E}$ can be described by objects and arrows of $\mathcal{E}$

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd} \qquad \begin{array}{c} C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \\ \\ C_0 \xrightarrow{e} C_1 \end{array}$$

where the pullback is taken along $C_1 \xrightarrow{s} C_0 \xleftarrow{t} C_1$, satisfying the following commutative diagrams:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "p_1"'] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "s"'] & C_0 \end{tikzcd} \tag{A.1.1a}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "p_2"'] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd} \tag{A.1.1b}$$

$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "id"'] & C_1 \arrow[d, "s"] \\ & C_0 \end{tikzcd} \tag{A.1.2a}$$

$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "id"'] & C_1 \arrow[d, "t"] \\ & C_0 \end{tikzcd} \tag{A.1.2b}$$

$$\begin{tikzcd} (C_1 \times_{C_0} C_1) \times_{C_0} C_1 \arrow[r, "m \times_{C_0} C_1"] \arrow[d] & C_1 \times_{C_0} C_1 \arrow[dd, "m"] \\ C_1 \times_{C_0} (C_1 \times_{C_0} C_1) \arrow[d, "C_1 \times_{C_0} m"'] & \\ C_1 \times_{C_0} C_1 \arrow[r, "m"'] & C_1 \end{tikzcd} \tag{A.1.3}$$

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[rr, "e \times_{C_0} C_1"] \arrow[dr, "p_2"'] & & C_1 \times_{C_0} C_1 \arrow[dl, "m"] \\ & C_1 & \end{tikzcd} \tag{A.1.4a}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_0 \arrow[rr, "C_1 \times_{C_0} e"] \arrow[dr, "p_1"'] & & C_1 \times_{C_0} C_1 \arrow[dl, "m"] \\ & C_1 & \end{tikzcd} \tag{A.1.4b}$$

A.2 *Axioms for* P*-operad*

Let $\mathcal{E}$ be a cartesian category and (P, μ, η) a cartesian monad. A P-multicategory Q can be described by objects and arrows of $\mathcal{E}$

$$\begin{tikzcd} & Q_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ PQ_0 & & Q_0 \end{tikzcd} \qquad \begin{array}{c} PQ_1 \times_{PQ_0} Q_1 \xrightarrow{m} Q_1 \\[2ex] Q_0 \xrightarrow{e} Q_1 \end{array}$$

where the pullback is taken along $PQ_1 \xrightarrow{t} PQ_0 \xleftarrow{s} Q_1$, satisfying the following commutative diagrams:

$$\begin{tikzcd} PQ_1 \times_{PQ_0} Q_1 \arrow[r, "p_1"] \arrow[dd, "m"'] & PQ_1 \arrow[d, "Ps"] \\ & P^2 Q_0 \arrow[d, "\mu_{Q_0}"] \\ Q_1 \arrow[r, "s"'] & PQ_0 \end{tikzcd} \tag{A.2.1a}$$

$$\begin{tikzcd} PQ_1 \times_{Q_0} Q_1 \arrow[r, "m"] \arrow[d, "p_2"'] & Q_1 \arrow[d, "t"] \\ Q_1 \arrow[r, "t"'] & Q_0 \end{tikzcd} \tag{A.2.1b}$$

$$\begin{tikzcd} Q_0 \arrow[r, "e"] \arrow[dr, "\eta_{Q_0}"'] & Q_1 \arrow[d, "s"] \\ & PQ_0 \end{tikzcd} \tag{A.2.2a}$$

$$\begin{tikzcd} Q_0 \arrow[r, "e"] \arrow[dr, "id"'] & Q_1 \arrow[d, "t"] \\ & Q_0 \end{tikzcd} \tag{A.2.2b}$$

$$\begin{tikzcd}
(P^2Q_1 \times_{P^2Q_0} PQ_1) \times_{PQ_0} Q_1 \arrow[rr, "Pm \times_{Q_0} Q_1"] \arrow[d] & & PQ_1 \times_{PQ_0} Q_1 \arrow[dd, "m"] \\
P^2Q_1 \times_{P^2Q_0} (PQ_1 \times_{PQ_0} Q_1) \arrow[d, "\mu_{Q_1} \times_{\mu Q_0} m"'] & & \\
PQ_1 \times_{PQ_0} Q_1 \arrow[rr, "m"'] & & Q_1
\end{tikzcd} \tag{A.2.3}$$

$$\begin{tikzcd}
PQ_0 \times_{PQ_0} Q_1 \arrow[rr, "Pe \times_{Q_0} Q_1"] \arrow[dr, "p_2"'] & & PQ_1 \times_{PQ_0} Q_1 \arrow[dl, "m"] \\
& Q_1 &
\end{tikzcd} \tag{A.2.4a}$$

$$\begin{tikzcd}
Q_1 \times_{Q_0} Q_0 \arrow[rr, "Q_1 \times_{Q_0} e"] \arrow[dr, "p_1"'] & & PQ_1 \times_{PQ_0} Q_1 \arrow[dl, "m"] \\
& Q_1 &
\end{tikzcd} \tag{A.2.4b}$$

B

Some proofs

B.1 Associativity of $\mathcal{T}_P C$

Proof of Lemma 5.1.2. Let us begin by proving (5.1.6a). We have from (5.1.2) that

$$\widetilde{C}_3 \cong P\widetilde{C}'_2 \times_{PC_0} \widetilde{C}_1,$$

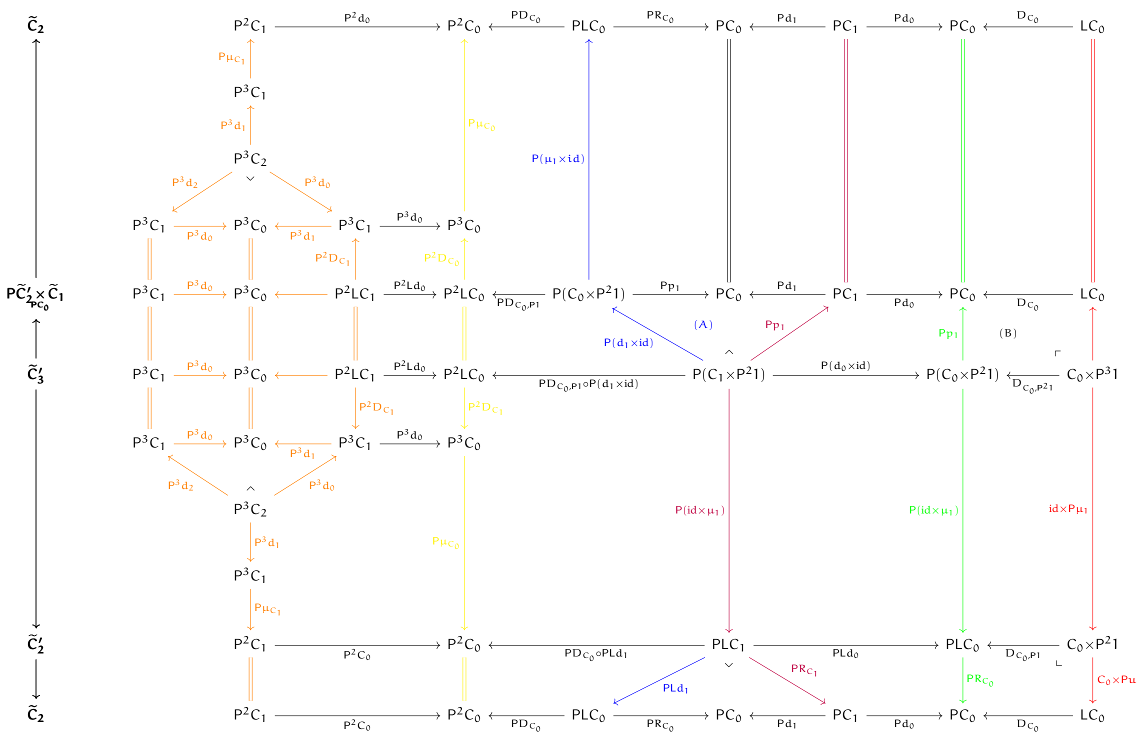
so that the inner face map $\widetilde{d}_1$ is isomorphic to

$$P\widetilde{C}'_2 \times_{PC_0} \widetilde{C}_1 \xrightarrow{P\widetilde{d}'_1 \times_{\mathrm{id}} \mathrm{id}} \widetilde{C}_2.$$

However, in order to see (5.1.2) it is convenient to express $\widetilde{C}_3$ and $\widetilde{d}_1$ in another form. Diagram (B.1.2) represents a commutative square

$$\begin{tikzcd} \widetilde{C}'_3 \arrow[r] \arrow[d, "\widetilde{d}'_1"'] & P\widetilde{C}'_2 \times_{PC_0} \widetilde{C}_1 \arrow[d, "P\widetilde{d}'_1 \times_{\mathrm{id}} \mathrm{id}"] \\ \widetilde{C}'_2 \arrow[r] & \widetilde{C}_2 \end{tikzcd} \tag{B.1.1}$$

introducing both $\widetilde{C}'_3 \cong \widetilde{C}_3$ and its corresponding map $\widetilde{d}'_1$. The square above is pictured in bold letters at the left of (B.1.2), indicating that each element is the total fiber product of its row, and similarly for the arrows:



(B.1.2)

Note that the square (A) is clearly a pullback, and (B) is a consequence of Lemma 4.4.2. The rest of diagrams inside (B.1.2) are the same as the ones in (5.1.2) and (5.1.3), with perhaps some extra P or L. This, together with the fact that all the arrows to the left of (A) are identities, ensures that the morphism $\widetilde{C}'_3 \to P\widetilde{C}'_2 \times_{PC_0} \widetilde{C}_1$ is indeed an isomorphism.

Hence, we only need to see that the vertical diagrams, given by the coloring, commute, but they are all either trivial or projections. Now, the following diagram defines a morphism

$$\widetilde{C}'_3 \xrightarrow{i'_3} P^3C_3,$$

$$\begin{tikzcd}
P^3C_1 \arrow[r, "P^3d_0"] \arrow[d, equal] & P^3C_0 \arrow[d, equal] & P^2LC_1 \arrow[l] \arrow[r, "P^2Ld_0"] \arrow[d, "P^2D_{C_1}"] & P^2LC_0 \arrow[d, "P^2D_{C_1}"] & P(C_1\times P^21) \arrow[l] \arrow[r, "P(d_0\times id)"] \arrow[d, "PD^2_{C_1}"] & P(C_0\times P^21) \arrow[d, "PD^2_{C_0}"] & C_0\times P^31 \arrow[l, "D_{C_0,P^21}"'] \\
P^3C_1 \arrow[r, "P^3d_0"'] & P^3C_0 & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"'] & P^3C_0 & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"'] & P^3C_0,
\end{tikzcd}$$

(B.1.3)

which is given by applying the strength D to every factor. Thus, the last thing to check is that the square

$$
\begin{tikzcd}
\widetilde{C}'_3 \arrow[r, "i'_3"] \arrow[dd, "\widetilde{d}'_1"'] & P^3C_3 \arrow[d, "P^3d_1"] \\
 & P^3C_2 \arrow[d, "P\mu_{C_2}"] \\
\widetilde{C}'_2 \arrow[r, "i'_2"'] & P^2C_2,
\end{tikzcd}
\tag{B.1.4}
$$

From the definitions of i'_3, $\widetilde{C}_2 \xrightarrow{\widetilde{d}'_1} \widetilde{C}_1$ and i'_2 (5.1.3) and $\widetilde{C}'_3 \xrightarrow{\widetilde{d}'_1} \widetilde{C}'_2$ (B.1.2) this square is given by the following diagram,

$$
\begin{tikzcd}[column sep=small]
P^2C_2 & & P^2C_1 \arrow[rr, "P^2d_0"] & & P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"'] \arrow[r, "P^2d_0"] & P^2C_0 & \\
P^3C_2 \arrow[u] & & P^3C_1 \arrow[rr, "P^3d_0"] \arrow[u, "P\mu_{C_1}"] & & P^3C_0 \arrow[u, "P\mu_{C_0}"] & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"] \arrow[u, "P\mu_{C_1}"] & P^3C_0 \arrow[u, "P\mu_{C_0}"] & \\
 & & P^3C_2 \arrow[u, "P^3d_1"] \arrow[dl, "P^3d_2"'] \arrow[dr, "P^3d_0"] & & & & & \\
P^3C_3 \arrow[uu] & P^3C_1 \arrow[r, "P^3d_0"] & P^3C_0 & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"] & P^3C_0 \arrow[uu, equal] & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"] \arrow[uu, equal] & P^3C_0 \arrow[uu, equal] & \\
\widetilde{C}'_3 \arrow[u] \arrow[dddd] & P^3C_1 \arrow[r, "P^3d_0"] \arrow[u, equal] & P^3C_0 \arrow[u, equal] & P^2LC_1 \arrow[l] \arrow[r, "P^2Ld_0"] \arrow[u, "P^2D_{C_1}"] \arrow[d, "P^2D_{C_1}"] & P^2LC_0 \arrow[u, "P^2D_{C_0}"] \arrow[d, "P^2D_{C_1}"] & P(C_1 \times P^21) \arrow[l] \arrow[r, "P(d_0\times \mathrm{id})"] \arrow[u, "PD^2_{C_1}"] \arrow[dddd, "P(\mathrm{id}\times\mu_1)"'] & P(C_0 \times P^21) \arrow[u, "PD^2_{C_0}"] \arrow[dddd, "P(\mathrm{id}\times\mu_1)"'] & C_0 \times P^31 \arrow[l, "D_{C_0P^21}"] \arrow[dddd, "\mathrm{id}\times P\mu_1"'] \\
 & P^3C_1 \arrow[r, "P^3d_0"] \arrow[u, equal] & P^3C_0 \arrow[u, equal] & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"] & P^3C_0 \arrow[ddd, "P\mu_{C_0}"'] & & & \\
 & & P^3C_2 \arrow[ul, "P^3d_2"] \arrow[ur, "P^3d_0"'] \arrow[d, "P^3d_1"] & & & & & \\
 & & P^3C_1 \arrow[d, "P\mu_{C_1}"] & & & & & \\
\widetilde{C}'_2 \arrow[d] & & P^2C_1 \arrow[rr, "P^2C_0"'] \arrow[d, equal] & & P^2C_0 \arrow[d, equal] & PLC_1 \arrow[l] \arrow[r, "PLd_0"'] \arrow[d, "PD_{C_1}"] & PLC_0 \arrow[d, "PR_{C_0}"] & C_0 \times P^21 \arrow[l, "D_{C_0,P1}"] \\
P^2C_2 & & P^2C_1 \arrow[rr, "P^2C_0"'] & & P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"] \arrow[r, "P^2d_0"'] & P^2C_0 &
\end{tikzcd}
\tag{B.1.5}
$$

which clearly commutes. Therefore (5.1.6a) also commutes, as we wanted to see.

Let us do the same for (5.1.6b). We have from (5.1.2) that

$$
\widetilde{C}_3 \cong P^2\widetilde{C}_1 \underset{P^2C_0}{\times} \widetilde{C}'_2,
$$

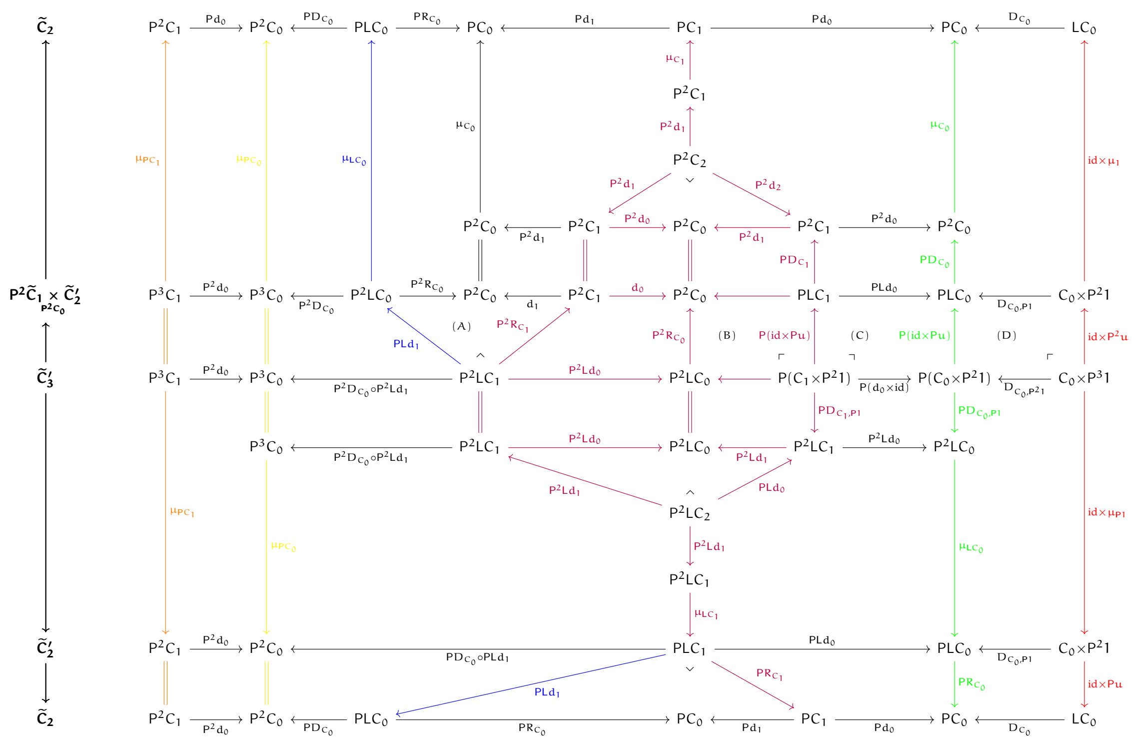
so that the inner face map $\widetilde{d}_2$ is isomorphic to

$$
P^2\widetilde{C}_1 \underset{P^2C_0}{\times} \widetilde{C}'_2 \xrightarrow{\mu_{\widetilde{C}_1} \times_{\mu_{C_0}} \widetilde{d}'_1} \widetilde{C}_2.
$$

However, in order to see (5.2.2) it is convenient to work again with the corresponding face map $\widetilde{C}_3 \xrightarrow{\widetilde{d}'_2} \widetilde{C}'_2$. Diagram (B.1.7) represents a commutative square

$$
\begin{tikzcd}
\widetilde{C}'_3 \arrow[r] \arrow[d, "\widetilde{d}'_2"'] & P^2\widetilde{C}_1 \underset{P^2C_0}{\times} \widetilde{C}'_2 \arrow[d, "\mu_{\widetilde{C}_1} \times_{\mu_{C_0}} \widetilde{d}'_1"] \\
\widetilde{C}'_2 \arrow[r] & \widetilde{C}_2
\end{tikzcd}
\tag{B.1.6}
$$

defining $\widetilde{d}'_2$. The square above is pictured in bold letters at the left of (B.1.7), indicating that each element is the total fiber product of its row, as before:



(B.1.7)

Note that the squares (A) and (C) are clearly a cartesian, and (B) and (D) are a consequence of Lemma 4.4.2. The rest of diagrams inside (B.1.7) are the same as the ones in (5.1.2) and (5.1.3), with perhaps some extra P or L. This, together with the fact that all the arrows to the left of (A) are identities, ensures that the morphism $\widetilde{C}'_3 \to P^2\widetilde{C}_1 \underset{P^2C_0}{\times} \widetilde{C}'_2$ is indeed an isomorphism. Hence, we only need to see that the vertical diagrams, given by the coloring, commute: the orange, yellow and blue diagrams are trivial, the red diagram is naturality of μ at u, the green one is naturality of μ at R_{C_0} plus naturality of D at u and the purple diagram is naturality of μ at R_{C_1} plus naturality of R at d_1.

The last thing to check is that the square

$$\begin{tikzcd}
\widetilde{C}'_3 \arrow[r, "i'_3"] \arrow[dd, "\widetilde{d}'_2"'] & P^3C_3 \arrow[d, "P^3d_2"] \\
& P^3C_2 \arrow[d, "\mu_{PC_2}"] \\
\widetilde{C}'_2 \arrow[r, "i'_2"'] & P^2C_2,
\end{tikzcd} \tag{B.1.8}$$

commutes. From the definitions of i'_3 (B.1.3), $\widetilde{C}_2 \xrightarrow{\widetilde{d}'_1} \widetilde{C}_1$ and i'_2 (5.1.3) and $\widetilde{C}'_3 \xrightarrow{\widetilde{d}'_2} \widetilde{C}'_2$ (B.1.7) we have that this square is given by the diagram

$$\begin{tikzcd}
P^2C_2 & P^2C_1 \arrow[r, "P^2d_0"] & P^2C_0 & & P^2C_1 \arrow[ll, "P^2d_1"'] \arrow[rr, "P^2d_0"] & & P^2C_0 & \\
P^3C_2 \arrow[u] & P^3C_1 \arrow[u, "\mu_{PC_1}"] \arrow[r, "P^3d_0"] \arrow[dd, equal] & P^3C_0 \arrow[u, "\mu_{PC_1}"] \arrow[dd, equal] & & P^3C_1 \arrow[u, "\mu_{PC_0}"] \arrow[ll, "P^3d_1"'] \arrow[rr, "P^3d_0"] & & P^3C_0 \arrow[u, "\mu_{PC_0}"] \arrow[dd, equal] & \\
& & & & P^3C_2 \arrow[u, "P^3d_1"] \arrow[dl, "P^3d_2"'] \arrow[dr, "P^3d_0"] \arrow[d, phantom, "\vee"] & & & \\
P^3C_3 \arrow[uu] & P^3C_1 \arrow[r, "P^3d_0"] \arrow[d, equal] & P^3C_0 \arrow[d, equal] & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"] & P^3C_0 & P^3C_1 \arrow[l, "P^3d_1"] \arrow[r, "P^3d_0"] & P^3C_0 & \\
\widetilde{C}'_3 \arrow[u] \arrow[dddd] & P^3C_1 \arrow[r, "P^3d_0"] \arrow[dddd, "\mu_{PC_1}"] & P^3C_0 \arrow[d, equal] & P^2LC_1 \arrow[u, "P^2D_{C_1}"] \arrow[l] \arrow[r, "P^2Ld_0"] \arrow[d, equal] & P^2LC_0 \arrow[u, "P^2D_{C_0}"] \arrow[d, equal] & P(C_1 \times P^21) \arrow[u, "PD^2_{C_1}"] \arrow[l] \arrow[r, "P(d_0 \times id)"] \arrow[d, "PD_{C_1,P1}"] & P(C_0 \times P^21) \arrow[u, "PD_{\mathcal{C}_0,P1}"] \arrow[d, "PD^2_{C_0}"] & C_0 \times P^31 \arrow[l, "D_{C_0,P^21}"] \arrow[dddd, "id \times \mu_{P1}"] \\
& & P^3C_0 \arrow[ddd, "\mu_{PC_0}"] & P^2LC_1 \arrow[l] \arrow[r, "P^2Ld_0"] & P^2LC_0 & P^2LC_1 \arrow[l] \arrow[r, "P(id \times d_0)"] & P^2LC_0 \arrow[ddd, "\mu_{LC_0}"] & \\
& & & & P^2LC_2 \arrow[ul, "P^2Ld_2"] \arrow[ur, "P^2Ld_0"'] \arrow[d, "P^2Ld_1"] \arrow[u, phantom, "\wedge"] & & & \\
& & & & P^2LC_1 \arrow[d, "\mu_{LC_1}"] & & & \\
\widetilde{C}'_2 \arrow[d] & P^2C_1 \arrow[r, "P^2d_0"] \arrow[d, equal] & P^2C_0 \arrow[d, equal] & & PLC_1 \arrow[ll] \arrow[rr, "PLd_0"] \arrow[d, "PD_{C_1}"] & & PLC_0 \arrow[d, "PD_{C_0}"] & C_0 \times P^21 \arrow[l, "D_{C_0,P1}"] \\
P^2C_2 & P^2C_1 \arrow[r, "P^2d_0"'] & P^2C_0 & & P^2C_1 \arrow[ll, "P^2d_1"] \arrow[rr, "P^2d_0"'] & & P^2C_0 &
\end{tikzcd} \tag{B.1.9}$$

which commutes by naturality of D. Therefore (5.1.6b) also commutes, as we wanted to see. □

B.2 *Left and right composition with the unit in* $\mathcal{T}_P C$

Proof of Lemma 5.1.5. Let us begin by showing (5.1.8a). The following diagram is an expanded version of (5.1.8a) in the way shown by the bold diagram in the left, so that the fiber product of each row is the bold element at its left, and similarly for the arrows.

$$
\begin{tikzcd}
\mathbf{P^2C_2} & & & P^2C_1 \arrow[r, "P^2d_0"] & P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"'] & & \\
\mathbf{PC_2} \arrow[u] & & & PC_1 \arrow[u, "P\eta_{C_1}"] \arrow[r, "Pd_0"'] & PC_0 \arrow[u, "P\eta_{C_0}"] & PC_1 \arrow[u, "P\eta_{C_1}"] \arrow[l, "Pd_1"] & & \\
\mathbf{PC_0 \times_{PC_0} PC_1} \arrow[u] & & & PC_0 \arrow[u, "Pe"] \arrow[r] & PC_0 \arrow[u] & PC_1 \arrow[u] \arrow[l, "Pd_1"] & & \\
\mathbf{PC_0 \times_{PC_0} \widetilde{C}_1} \arrow[u] \arrow[dd] & PC_0 \arrow[r, equal] \arrow[d, "P\eta_{C_0}"] & PC_0 \arrow[r, equal] \arrow[dd, "P\eta_{C_0}"] & PC_0 \arrow[u] \arrow[r, equal] \arrow[dd, "PF_{C_0}"] & PC_0 \arrow[u] \arrow[dd] & PC_1 \arrow[u] \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] \arrow[dd] & PC_0 \arrow[dd] & LC_0 \arrow[l, "D_{C_0}"] \arrow[dd] \\
 & P^2C_0 \arrow[d, "P^2e"] & & & & & & \\
\mathbf{\widetilde{C}_2} \arrow[dd] & P^2C_1 \arrow[r, "P^2d_0"'] \arrow[dd] & P^2C_0 \arrow[dd] & PLC_0 \arrow[l, "PD_{C_0}"] \arrow[r, "PR_{C_0}"] & PC_0 & PC_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] & PC_0 & LC_0 \arrow[l, "D_{C_0}"] \\
 & & & & PLC_1 \arrow[ul, "PLd_1"] \arrow[ur, "PR_{C_1}"'] \arrow[d, "PD_{C_1}"] & & & \\
\mathbf{P^2C_2} & P^2C_1 \arrow[r, "P^2d_0"'] & P^2C_0 & & P^2C_1 \arrow[ll, "P^2d_1"] & & &
\end{tikzcd}
\tag{B.2.1}
$$

The arrows without a label are identities. All the diagrams in (B.2.1) have appeared before, in the definitions of $\widetilde{C}_1$ (5.1.1), i'_2 (5.1.4) or $\widetilde{e}$ (5.1.7). Hence we only have to see that the three diagrams given by the colors commute. The red square is just P applied to naturality of η at e, and the green one is trivial. For the blue diagram notice that the pullback induces a morphism $PC_1 \to PLC_1$ which is easily seen to be PF_{C_1}, but by axiom (4.4.2a) we have that $PD_{C_1} \circ PF_{C_1} = P\eta_{C_1}$, which coincides with the upper blue diagram, as we wanted to see.

Let us now see (5.1.8b). We expand the diagram in the same way:

$$
\begin{tikzcd}
P^2C_2 & P^2C_1 \arrow[r, "P^2d_0"] & P^2C_0 & P^2C_1 \arrow[l, "P^2d_1"'] & & & & \\
PC_2 \arrow[u] & PC_1 \arrow[r, "Pd_0"] \arrow[u, red, "\eta_{PC_1}"] & PC_0 \arrow[u, green, "\eta_{PC_0}"'] & PC_1 \arrow[l, "Pd_1"] \arrow[u, blue, "\eta_{PC_1}"'] & & & & \\
PC_1 \underset{PC_0}{\times} PC_0 \arrow[u] & PC_1 \arrow[r, "Pd_0"] \arrow[u, red] & PC_0 \arrow[u, green] & PC_0 \arrow[l] \arrow[u, blue, "Pe"'] & & & & \\
\widetilde{C}_1 \underset{C_0}{\times} C_0 \arrow[u] \arrow[dd] & PC_1 \arrow[r, "Pd_0"] \arrow[u, red] \arrow[dd, red, "\eta_{PC_1}"] & PC_0 \arrow[u, green] \arrow[dd, green, "\eta_{PC_0}"] & LC_0 \arrow[l, "D_{C_0}"] \arrow[u, blue, "D_{C_0}"] \arrow[dd, blue, "\eta_{LC_0}"] \arrow[r, blue, "R_{C_0}"] & C_0 \arrow[r, equal] \arrow[dd, "\eta_{C_0}"] & C_0 \arrow[r, equal] \arrow[d, blue, "\eta_{C_0}"] & C_0 \arrow[r, equal] \arrow[dd, "\eta_{C_0}"] & C_0 \arrow[dd, "F_{C_0}"] \\
 & & & & & PC_0 \arrow[d, blue, "Pe"] & & \\
\widetilde{C}_2 \arrow[dd] & P^2C_1 \arrow[r, "P^2d_0"'] \arrow[dd, red] & P^2C_0 \arrow[dd, green] & PLC_0 \arrow[l, "PD_{C_0}"] \arrow[r, blue, "PR_{C_0}"] & PC_0 & PC_1 \arrow[l, blue, "Pd_1"'] \arrow[r, "Pd_0"'] & PC_0 & LC_0 \arrow[l, "D_{C_0}"] \\
 & & & & PLC_1 \arrow[ul, blue, "PLd_1"] \arrow[ur, blue, "PR_{C_0}"'] \arrow[u, phantom, "\wedge"] \arrow[d, blue, "PD_{C_1}"] & & & \\
P^2C_2 & P^2C_1 \arrow[r, "P^2d_0"'] & P^2C_0 & & P^2C_1 \arrow[ll, "P^2d_1"] & & &
\end{tikzcd}
\tag{B.2.2}
$$

As before, all the diagrams in (B.2.1) have appeared before, in the definitions of $\widetilde{C}_1$ (5.1.1), i'_2 (5.1.4) or $\widetilde{e}$ (5.1.7). Hence we only have to see that the three diagrams given by the colors commute. In this case both the red and the green diagrams are trivial. For the blue one, notice that the pullback induces a morphism $LC_0 \to PLC_1$ which is easily seen to be $PLe \circ \eta_{LC_0}$, and as a consequence the lower blue diagram coincides with the upper blue diagram, by naturality of D and η. □

B.3 *Associativity in* $\mathcal{T}^P Q$

Proof of Lemma 5.2.2. Let us begin by proving (5.2.6a). We have from (5.2.2) that

$$\overline{Q}_3 \cong \overline{Q}'_2 \underset{Q_0}{\times} \overline{Q}_1,$$

so that the inner face map $\overline{d}_1$ is isomorphic to

$$\overline{Q}'_2 \underset{Q_0}{\times} \overline{Q}_1 \xrightarrow{\overline{d}'_1 \times_{\mathrm{id}} \mathrm{id}} \overline{Q}_2.$$

However, in order to see (5.2.2) it is convenient to express $\overline{Q}_3$ and $\overline{d}_1$ in another form. Diagram (B.3.2) represents a commutative square

$$
\begin{tikzcd}
\overline{Q}'_3 \arrow[r] \arrow[d, "\overline{d}'_1"'] & \overline{Q}'_2 \underset{Q_0}{\times} \overline{Q}_1 \arrow[d, "\overline{d}'_1 \times_{\mathrm{id}} \mathrm{id}"] \\
\overline{Q}'_2 \arrow[r] & \overline{Q}_2
\end{tikzcd}
\tag{B.3.1}
$$

introducing both $\overline{Q}'_3 \cong \overline{Q}_3$ and its corresponding map $\overline{d}'_1$. The square above is pictured in bold letters at the left of (B.3.2), indicating that each element is the total fiber product of its row, and similarly for the arrows:

$$\begin{tikzcd}
\overline{\mathbf{Q}}_2 & LQ_0 \arrow[r, "D_{Q_0}"] & PQ_0 & & Q_1 \arrow[ll, "d_1"'] \arrow[rr, "d_0"] & & Q_0 \arrow[dd, equal] & LQ_0 \arrow[l, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"] \arrow[ddd, equal] & PQ_0 \arrow[ddd, equal] & Q_1 \arrow[l, "d_1"'] \arrow[ddd, equal] \\
& & & & Q_2 \arrow[u, "d_1"] \arrow[dl, "d_2"'] \arrow[dr, "d_0"] & & & & & \\
& Q_0\times P^2 1 \arrow[uu, "\mathrm{id}\times\mu_1"] \arrow[r, "D^2_{Q_0}"] & P^2Q_0 \arrow[uu, "\mu_{Q_0}"] & PQ_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"] & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d, equal] & Q_0 \arrow[d, equal] & & & \\
\overline{\mathbf{Q}}'_2\times_{\mathbf{Q}_0}\overline{\mathbf{Q}}_1 \arrow[uuu] & L^2Q_0 \arrow[u, "\mathrm{id}\times D_{P1}"] \arrow[r, "LD_{Q_0}"] & LPQ_0 \arrow[u, "D_{PQ_0}"] & LQ_1 \arrow[l, "Ld_1"'] \arrow[u, "D_{Q_0}"] \arrow[r] & PQ_0 & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] & Q_0 & LQ_0 \arrow[l, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"] & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[d, equal] \\
\overline{\mathbf{Q}}'_3 \arrow[u] \arrow[dddd] & L^3Q_0 \arrow[u, "R_{L^2Q_0}"] \arrow[r, "L^2D_{Q_0}"] \arrow[d, "L(\mathrm{id}\times D_{P1})"'] & L^2PQ_0 \arrow[u, "R_{LPQ_0}"] \arrow[d, "LD_{PQ_0}"] & L^2Q_1 \arrow[l, "L^2d_1"'] \arrow[u, "R_{LQ_0}"] \arrow[r] \arrow[d, "LD_{Q_1}"] & LPQ_0 \arrow[u, "R_{PQ_0}"] \arrow[d, equal] & & LQ_1 \arrow[ll, "Ld_1"'] \arrow[ul, "R_{Q_1}"] \arrow[ur, "Ld_0"'] \arrow[rr, "D_{Q_0}\circ Ld_0"] \arrow[d, equal] & & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[ddd, equal] \\
& L(Q_0\times P^2 1) \arrow[r, "LD^2_{Q_0}"] \arrow[dd, "L(\mathrm{id}\times\mu_1)"'] & LP^2Q_0 \arrow[dd, "L\mu_{Q_0}"] & LPQ_1 \arrow[l, "LPd_1"'] \arrow[r] & LPQ_0 & & LQ_1 \arrow[ll, "Ld_1"'] \arrow[rr] & & PQ_0 \arrow[dd, equal] & \\
& & & & LQ_2 \arrow[ul, "Ld_2"'] \arrow[urr, "Ld_0"] \arrow[d, "Ld_1"] & & & & & \\
\overline{\mathbf{Q}}'_2 \arrow[d] & L^2Q_0 \arrow[r, "LD_{Q_0}"] \arrow[d, "R_{LQ_0}"'] & LPQ_0 \arrow[d, "R_{PQ_0}"] & & LQ_1 \arrow[ll, "Ld_1"'] \arrow[rrrr, "D_{Q_0}\circ Ld_0"] \arrow[dl, "R_{Q_1}"'] \arrow[drrr, "Ld_0"] & & & & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[d, equal] \\
\overline{\mathbf{Q}}_2 & LQ_0 \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"'] & Q_0 & & & LQ_0 \arrow[lll, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"']
\end{tikzcd} \tag{B.3.2}$$

All the pullback squares are naturality squares of R, which is cartesian. All the diagrams inside (B.3.2) are the same as the ones in (5.2.2) and (5.2.3), with perhaps some extra P or L. Thus, it is clear that the morphism $\overline{Q}'_3 \to \overline{Q}'_2 \times_{Q_0} \overline{Q}_1$ is indeed an isomorphism. Hence, we only need to see that the vertical diagrams, given by the coloring, commute: the red, green and purple diagrams are trivial, and the rest are naturality squares for R.

Now, the following diagram represents a morphism,

$$\overline{Q}'_3 \xrightarrow{j'_3} Q_3,$$

$$\begin{tikzcd}
L^3Q_0 \arrow[r, "L^2R_{Q_0}"] & L^2PQ_0 & L^2C_1 \arrow[l, "L^2d_1"'] \arrow[r, "LR_{Q_0}"] \arrow[d, "D^2_{Q_1}"] & LPQ_0 \arrow[d, "D_{PQ_0}"] & LQ_1 \arrow[l] \arrow[r, "D_{Q_0}"] \arrow[d, "D_{Q_1}"] & PQ_0 \arrow[d] & Q_1 \arrow[l, "d_1"'] \arrow[d] \\
& & P^2Q_1 \arrow[r, "Pd_0"'] & P^2Q_0 & PQ_1 \arrow[l, "Pd_1"] \arrow[r, "Pd_0"'] & PQ_0 & Q_1 \arrow[l, "d_1"]
\end{tikzcd} \tag{B.3.3}$$

which is given by applying the strength D to every factor. The last thing we have to check is that the square

$$\begin{tikzcd}
\overline{Q}'_3 \arrow[r, "j'_3"] \arrow[d, "\overline{d}'_1"'] & Q_3 \arrow[d, "d_1"] \\
\overline{Q}'_2 \arrow[r, "j'_2"'] & Q_2,
\end{tikzcd} \tag{B.3.4}$$

commutes. From the definitions of j_3', $\overline{Q}_2 \xrightarrow{\overline{d}_1'} \overline{Q}_1$ and j_2' (5.2.3) and $\overline{Q}_3' \xrightarrow{\overline{d}_1'} \overline{Q}_2'$ (B.3.2) we have that this square is given by the following diagram

$$\begin{tikzcd}
\overline{Q}_2 & & P^2Q_0 & & PQ_1 \arrow[ll, "Pd_1"'] \arrow[rr, "Pd_0"] & & PQ_0 & Q_1 \arrow[l, "d_1"] \\
& & & & PQ_2 \arrow[u, "Pd_1"] \arrow[dl, "Pd_2"'] \arrow[dr, "Pd_0"] \arrow[d] & & & \\
\overline{Q}_3 \arrow[uu] & & P^3Q_0 \arrow[uu, "P\mu_{Q_0}"] & P^2Q_1 \arrow[l, "P^2d_1"'] \arrow[r] & P^2Q_0 & PQ_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"] & PQ_0 \arrow[uu, equal] & Q_1 \arrow[l, "d_1"] \arrow[uu, equal] \\
\overline{Q}_3' \arrow[u] \arrow[ddd] & L^3Q_0 \arrow[r, "L^2D_{Q_0}"] \arrow[d, "L(\mathrm{id}\times D_{P1})"'] & L^2PQ_0 \arrow[u] \arrow[d, "LD_{PQ_0}"] & L^2Q_1 \arrow[l, "L^2d_1"] \arrow[u] \arrow[r] \arrow[d, "LD_{Q_1}"] & LPQ_0 \arrow[u] \arrow[d, equal] & LQ_1 \arrow[l, "Ld_1"] \arrow[u, "D_{Q_1}"'] \arrow[r, "D_{Q_0}\circ Ld_0"] \arrow[d, equal] & PQ_0 \arrow[u, equal] \arrow[d, equal] & Q_1 \arrow[l, "d_1"] \arrow[u, equal] \arrow[ddd, equal] \\
& L(Q_0\times P^2 1) \arrow[r, "LD^2_{Q_0}"] \arrow[dd, "L(\mathrm{id}\times \mu_1)"'] & LP^2Q_0 \arrow[dd, "L\mu_{Q_0}"] & LPQ_1 \arrow[l, "LPd_1"] \arrow[r] & LPQ_0 & LQ_1 \arrow[l, "Ld_1"] \arrow[r] & PQ_0 \arrow[dd, equal] & \\
& & & & LQ_2 \arrow[ul, "Ld_2"] \arrow[ur, "Ld_0"'] \arrow[u] \arrow[d, "Ld_1"] & & & \\
\overline{Q}_2' \arrow[d] & L^2Q_0 \arrow[r, "LD_{Q_0}"'] & LPQ_0 \arrow[d, "D_{PQ_0}"] & & LQ_1 \arrow[ll, "Ld_1"] \arrow[rr, "D_{Q_0}\circ Ld_0"] \arrow[d, "D_{Q_1}"] & & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"] \arrow[d, equal] \\
\overline{Q}_2 & & P^2Q_0 & & PQ_1 \arrow[ll, "Pd_1"'] \arrow[rr, "Pd_0"'] & & PQ_0 & Q_1 \arrow[l, "d_1"']
\end{tikzcd} \tag{B.3.5}$$

which commutes by naturality of D. Therefore (5.2.6a) also commutes, as we wanted to see.

Let us do the same for (5.2.6b). We have from (5.2.2) that

$$\overline{Q}_3 \cong \overline{Q}_1 \underset{Q_0}{\times} \overline{Q}_2',$$

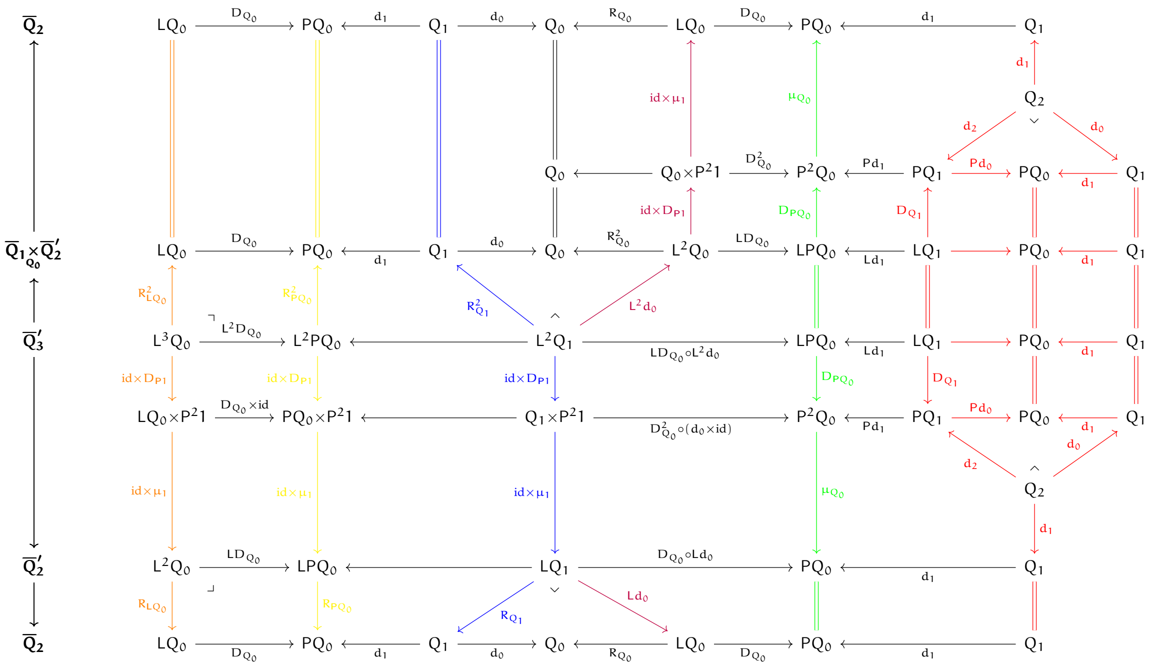
so that the inner face map $\overline{d}_2$ is isomorphic to

$$\overline{Q}_1 \underset{Q_0}{\times} \overline{Q}_2' \xrightarrow{\mathrm{id}\times_{\mathrm{id}}\overline{d}_1'} \overline{Q}_2.$$

However, in order to see (5.2.2b) it is convenient to work again with $\overline{Q}_3'$. Diagram (B.3.7) represents a commutative square

$$\begin{tikzcd}
\overline{Q}_3' \arrow[r] \arrow[d, "\overline{d}_2'"'] & \overline{Q}_1 \underset{Q_0}{\times} \overline{Q}_2' \arrow[d, "\overline{d}_1'\times_{\mathrm{id}}\mathrm{id}"] \\
\overline{Q}_2' \arrow[r] & \overline{Q}_2
\end{tikzcd} \tag{B.3.6}$$

defining the corresponding map $\overline{d}'_2$. The square above is pictured in bold letters at the left of (B.3.7), indicating that each element is the total fiber product of its row, and similarly for the arrows:



(B.3.7)

All the pullback squares are naturality squares of R, which is cartesian. All the diagrams inside (B.3.7) are the same as the ones in (5.2.2) and (5.2.3), with perhaps some extra P or L. Thus, it is clear that the morphism $\overline{Q}'_3 \to \overline{Q}_1 \times_{Q_0} \overline{Q}'_2$ is indeed an isomorphism. Hence, we only need to see that the vertical diagrams, given by the coloring, commute: the red, green and purple diagrams are trivial, and the rest are just projections.

The last thing to check is that the square

$$\begin{tikzcd} \overline{Q}'_3 \arrow[r, "j'_3"] \arrow[d, "\overline{d}'_2"'] & Q_3 \arrow[d, "d_2"] \\ \overline{Q}'_2 \arrow[r, "j'_2"'] & Q_2, \end{tikzcd} \tag{B.3.8}$$

commutes. From the definitions of j_3' (B.3.3), $\overline{Q}_2' \xrightarrow{\overline{d}_1'} \overline{Q}_1$ and j_2' (5.2.3) and $\overline{Q}_3' \xrightarrow{\overline{d}_2'} \overline{Q}_2'$ (B.3.7) it is not difficult to see that this square is represented by the diagram

$$
\begin{tikzcd}
\mathbf{Q_2} & & P^2Q_0 & PQ_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"] & PQ_0 & & Q_1 \arrow[ll, "d_1"'] & \\
& & & & & & Q_2 \arrow[u, "d_1"] \arrow[dl, "d_2"'] \arrow[dr, "d_0"] & \\
\mathbf{Q_3} \arrow[uu] & & P^3Q_0 \arrow[uu, "P\mu_{Q_0}"] & P^2Q_1 \arrow[l, "P^2d_1"'] \arrow[r] \arrow[uu, "\mu_{Q_1}"] & P^2Q_0 \arrow[uu, "\mu_{Q_0}"] & PQ_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"] & PQ_0 & Q_1 \arrow[l, "d_1"'] \\
\overline{\mathbf{Q}}_3' \arrow[u] \arrow[ddd] & L^3Q_0 \arrow[r, "L^2D_{Q_0}"] \arrow[d, "L(\mathrm{id}\times D_{P1})"'] & L^2PQ_0 \arrow[u] \arrow[d, "LD_{PQ_0}"] & L^2Q_1 \arrow[l, "L^2d_1"'] \arrow[r] \arrow[u] \arrow[d, "LD_{Q_1}"] & LPQ_0 \arrow[u] \arrow[d, equal] & LQ_1 \arrow[l, "Ld_1"'] \arrow[r, "D_{Q_0}\circ Ld_0"] \arrow[u, "D_{Q_1}"] \arrow[d, equal] & PQ_0 \arrow[u, equal] \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[u, equal] \arrow[d, equal] \\
& LQ_0\times P^21 \arrow[r, "D_{Q_0}\times\mathrm{id}"] \arrow[dd, "\mathrm{id}\times\mu_1"'] & PQ_0\times P^21 \arrow[dd, "\mathrm{id}\times\mu_1"] & Q_1\times P^21 \arrow[l, "d_1\times\mathrm{id}"'] \arrow[r] \arrow[dd, "\mathrm{id}\times\mu_1"] & P^2Q_0 \arrow[dd, "\mu_{Q_0}"] & PQ_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"] & PQ_0 & Q_1 \arrow[l, "d_1"'] \\
& & & & & & Q_2 \arrow[ul, "d_2"] \arrow[ur, "d_0"'] \arrow[d, "d_1"] & \\
\overline{\mathbf{Q}}_2' \arrow[d] & L^2Q_0 \arrow[r, "LD_{Q_0}"'] & LPQ_0 \arrow[d, "D_{PQ_0}"] & LQ_1 \arrow[l, "Ld_1"'] \arrow[r] \arrow[d, "D_{Q_1}"] & PQ_0 \arrow[d, equal] & & Q_1 \arrow[ll, "d_1"'] \arrow[d, equal] & \\
\mathbf{Q_2} & & P^2Q_0 & PQ_1 \arrow[l, "Pd_1"'] \arrow[r, "Pd_0"'] & PQ_0 & & Q_1 \arrow[ll, "d_1"'] &
\end{tikzcd}
\tag{B.3.9}
$$

which clearly commutes. Therefore (5.2.6b) also commutes, as we wanted to see. □

B.4 *Left and right composition with the unit in* $\mathcal{T}^P Q$

Proof of Lemma 5.2.5. Let us begin by showing (5.2.8a). The following diagram is an expanded version of (5.2.8a) in the way shown by the bold diagram in the left, so that the fiber product of each row is the bold element at its left, and similarly for the arrows.

$$
\begin{tikzcd}
\mathbf{Q_2} & & & & & PQ_1 \arrow[r, "Pd_0"] & PQ_0 & Q_1 \arrow[l, "d_1"'] \\
\mathbf{PQ_0 \times_{PQ_0} Q_1} \arrow[u] & & & & & PQ_0 \arrow[u, "Pe"] \arrow[r] & PQ_0 \arrow[u, equal] & Q_1 \arrow[l, "d_1"'] \arrow[u, equal] \\
\mathbf{Q_0 \times_{Q_0} \overline{Q}_1} \arrow[u] \arrow[d] & Q_0 \arrow[r, equal] \arrow[d, "F_{Q_0}"'] & Q_0 \arrow[r, equal] \arrow[d, "\eta_{Q_0}"'] & Q_0 \arrow[r, equal] \arrow[d, "e"'] & Q_0 \arrow[d, equal] & LQ_0 \arrow[l, "R_{Q_0}"] \arrow[r, "D_{Q_0}"] \arrow[u, "D_{Q_0}"] \arrow[d, equal] & PQ_0 \arrow[u, equal] \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[u, equal] \arrow[d, equal] \\
\overline{\mathbf{Q}}_2 \arrow[dd] & LQ_0 \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] & Q_0 & LQ_0 \arrow[l, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"] & PQ_0 \arrow[dd, equal] & Q_1 \arrow[l, "d_1"'] \arrow[dd, equal] \\
& & & & LQ_1 \arrow[ul, "R_{Q_0}"] \arrow[ur, "Ld_0"'] \arrow[d, "D_{Q_1}"] & & & \\
\mathbf{Q_2} & & & & PQ_1 \arrow[rr, "Pd_0"'] & & PQ_0 & Q_1 \arrow[l, "d_1"']
\end{tikzcd}
\tag{B.4.1}
$$

The arrows without a label are identities. All the diagrams in (B.4.1) have appeared before, in the definitions of $\overline{Q}_1$ (5.2.1), j_2' (5.2.4) or $\overline{e}$ (5.2.7). Hence we only have to see that the three diagrams given by the colors commute. The green and blue diagrams are trivial. For the red one, notice that the pullback induces a morphism $LQ_0 \to LQ_1$ which is easily seen to be Le, so that the red square is just naturality of D at e.

Let us now see (5.2.8b). We expand the diagram in the same way:

$$\begin{tikzcd}
\mathbf{Q_2} & & & PQ_1 \arrow[r, "Pd_0"] & PQ_0 & Q_1 \arrow[l, "d_1"'] & & \\
\mathbf{Q_1 \times_{Q_0} Q_0} \arrow[u] & & & Q_1 \arrow[u, red, "\eta_{Q_1}"] \arrow[r, "d_0"] & Q_0 \arrow[u, green, "\eta_{Q_0}"] \arrow[r, equal] & Q_0 \arrow[u, blue, "e"'] & & \\
\mathbf{\overline{Q}_1 \times_{Q_0} Q_0} \arrow[u] \arrow[d] & LQ_0 \arrow[r, "D_{Q_0}"] \arrow[d, equal] & PQ_0 \arrow[d, equal] & Q_1 \arrow[l, "d_1"'] \arrow[u, red, equal] \arrow[r, red, "d_0"] \arrow[d, red, equal] & Q_0 \arrow[u, green, equal] \arrow[r, red, equal] \arrow[d, red, equal] & Q_0 \arrow[u, blue, equal] \arrow[r, equal] \arrow[d, red, "F_{Q_0}"] & Q_0 \arrow[r, equal] \arrow[d, green, "\eta_{Q_0}"] & Q_0 \arrow[d, blue, "e"] \\
\mathbf{\overline{Q}_2} \arrow[dd] & LQ_0 \arrow[r, "D_{Q_0}"'] & PQ_0 & Q_1 \arrow[l, "d_1"] \arrow[r, red, "d_0"] & Q_0 & LQ_0 \arrow[l, red, "R_{Q_0}"'] \arrow[r, "D_{Q_0}"] & PQ_0 \arrow[dd, green, equal] & Q_1 \arrow[l, "d_1"] \arrow[dd, blue, equal] \\
 & & & & LQ_1 \arrow[ul, red, "Ld_0"] \arrow[ur, red, "R_{Q_1}"'] \arrow[d, red, "D_{Q_1}"] \arrow[u, phantom, "\wedge" near start] & & & \\
\mathbf{Q_2} & & & & PQ_1 \arrow[rr, "Pd_0"'] & & PQ_0 & Q_1 \arrow[l, "d_1"]
\end{tikzcd} \tag{B.4.2}$$

All the diagrams in (B.4.2) have appeared before, in the definitions of $\overline{Q}_1$ (5.2.1), j_2' (5.2.4) or $\overline{e}$ (5.2.7). Hence we only have to see that the three diagrams given by the colors commute. The green and blue diagrams are trivial. For the red one notice that the pullback induces a morphism $Q_1 \to LQ_1$ which is easily seen to be F_{Q_1}, but by axiom (4.4.2a) we have that $D_{C_1} \circ F_{Q_1} = \eta_{Q_1}$, so that the red square commutes. $\square$

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