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Essays in Fair Allocation Rules

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A Thesis submitted to Departament d'Economia i d'Història Econòmica of The Universitat Autònoma de Barcelona in fulfillment of the requirements for the degree of Doctor of Philosophy by the International Doctorate Program of Economic Analysis (IDEA) in May, 2021

To Chenghong, Lili, Jianzhong, and Lingling

Acknowledgment

First and foremost, I am grateful to David Pérez-Castrillo for guidance and support. Among many good things about him, I appreciate David for taking my ideas at preliminary stage seriously and treating me as an equal collaborator. Besides, I owe an immense intellectual debt through David's joint work with another David, David Wettstein, who is a great researcher and wonderful person. To be fair, without their work on multi-bidding mechanisms, the production of this thesis would be unthinkable.

During the course of pursuing this Ph.D., I benefit from discussions with my teachers, colleagues and friends. I would like to thank Ramon Caminal, Alfredo Contreras, Ezra Einy, Chunyang Fu, Itzhak Gilboa, Roweno Heijmans, Ori Haimanko, Flip Klijn, Chenghong Luo, Inés Macho-Stadler, Juan Enrique Martínez-Legaz, Andreu Mas-Colell, Jordi Massó, Pau Milán, Abraham Neyman, David Wettstein, Eyal Winter, Yue Zhu.

I would also like to thank the administrative staff at IDEA, especially Madam Àngels López-Garcia.

I am blessed with unbounded love from my family. In particular, I thank my wife, Chenghong Luo, my mom & dad, Lili Chen & Jianzhong Sun, and my aunt, Lingling Chen for everything.

I gratefully acknowledge the financial support the research received from the Spanish Ministry of Science and Innovation and Feder (PGC2018-094348-B-I00) and the Generalitat de Catalunya (FI fellowship).

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Chapter 0

General Introduction

It has long been recognized that human actions are influenced by fairness considerations with regard to the sharing of resulting cost and benefit. For example, more than 2000 years ago, Confucius stressed that a governor should concern more about the allocation of wealth than economic growth. Almost everyone agrees that fairness is essential, but there are one thousand theories of fairness in one thousand eyes. Furthermore, how to apply these theories to different economic environments and how to address the associated incentive problems are also subject to discussion. This thesis is an exploration of the above important yet challenging problems.

Chapter 1 is based on a joint work with David Pérez-Castrillo. We introduce the *value-free (v-f) reductions*, which are operators that map a coalitional game played by a set of players to another "similar" game played by a subset of those players. We propose properties of v-f reductions, some of which have an appeal of fairness, such as permanent null player, null player out, and 1-addition invariance. Moreover, we characterize several v-f reductions (among which the value-free version of the reduced games proposed by Hart and Mas-Colell, 1989, and Oishi et al., 2016). Unlike reduced games, introduced to characterize values in terms of consistency, v-f reductions are not defined in reference to values. However, a v-f reduction induces a value. To put it into perspective, we may see v-f reductions as schemes leading to fair allocations. In particular, we characterize v-f reductions that induce the Shapley value, the stand-alone value, and the Banzhaf value.

Our new approach is not only interesting for its own, but also has the potential for making contributions to other existing fields. First, our approach may contribute to enrich to literature of characterizing values in terms of consistency. We find new reduced games that are useful to characterize the Banzhaf and the stand-alone values in terms of consistency. Second, our approach may be connected to implementation theory. We consider the v-f reduction of the Pérez-Castrillo and Wettstein's (2001) bidding mechanism. Dualizing this v-f reduction prefigures a new mechanism that implements the Shapley value, which serves as a starting point of Chapter 2.

In contrast to Chapter 1, which relies on the axiomatic approach, Chapter 2 pursues the strategic approach to implement solution concepts for transferable utility games (TU games). We introduce two mechanisms that implement the Shapley value and the equal surplus value, respectively. The main feature of both mechanisms is that multiple proposers put forth allocation plans simultaneously, which is reminiscent of the Nash demand game (Nash, 1953). The implementation of a plan requires both consensus among proposers and acceptance of respondents. In case of a disagreement among proposers, we resort to the Pérez-Castrillo and Wettstein's bidding procedure, which facilitates a buyout of one proposer in each round. The difference between the two values now comes down to the difference in the protocols of two mechanisms. In our case, we find that the difference between the two values defined on 0-monotonic games lies in how proposers negotiate with respondents. This finding provides another example demonstrating that the strategic approach is complementary to the axiomatic approach in studying allocation rules.

Chapters 1 and 2 focus on TU games, which are highly idealized, and rely on the often unrealistic assumption that every participant's utility function is linear with respect to a numeraire. Chapter 3, which is based on a joint work with David Pérez-Castrillo, turns to pure exchange economies. We define the *proportional ordinal Shapley* (the *POSh*) solution, an ordinal allocation rule for pure exchange economies in the spirit of the Shapley value. Our construction is inspired by Hart and Mas-Colell's (1989) classic characterization of the Shapley value with the aid of a potential function. We establish the existence and uniqueness of the *POSh* and show that it is essentially single-valued for a fairly general class of economies. It satisfies individual rationality, anonymity, and counterpart properties of the null-player and null-player out properties in TU games. It is worth mentioning that all these properties, except for individual rationality, are satisfied by the Shapley value defined on the unrestricted domain. Individual rationality is satisfied by the Shapley value for 0-monotonic games. Moreover, the *POSh* is immune to agents' manipulation of their initial endowments: It is not manipulable via disposal of one's own endowment and does not suffer from the transfer paradox. Finally, we construct a bidding mechanism à la Pérez-Castrillo and Wettstein (2006) that implements the *POSh* in every subgame perfect Nash equilibrium for economies where agents have homothetic preferences and positive endowments.

Chapter 1

Value-Free Reductions

1.1 Introduction

We consider environments where a set of participants can collaborate to obtain and share surplus, that is, we study coalitional games with transferable utility (TU games). In such environments, we look at the consequences of removing some players from the game. In the new game faced by the remaining participants, the worth of each coalition of players is a function of the strategic possibilities of all the players in the initial game.

This problem is relevant in many economic contexts. For instance, when a group of shareholders leave a company, the remaining shareholders reorganize the ownership among themselves. The process through which the outstanding shareholders acquire the shares of the leaving shareholders will determine the strategic environment where they will interact from then on, that is, the worth of each possible coalition in the new environment.

Thus, in this paper, we look at TU games from a new perspective. We study "operators" that map a TU game played by a set N of players to another, similar but "reduced" game, played by a subset of N. We propose properties that such functions may satisfy, and we use these properties to characterize several operators. Our research question is different but related to the search for consistency properties of values for TU games.[1] Before continuing with the contribution of our paper, it is worthwhile to discuss the relationship between this line of research and our approach. To that aim, we first briefly describe the consistency requirement. Consider a value for TU games, that is, a function that associates a payoff to every player in every game. Starting from a TU game with a set of players N, we can define a reduced game among the players of any $N' \subsetneq N$. The worth of a coalition in the reduced

[1] In this respect, the closest papers to ours are Hart and Mas-Colell (1989) and Oishi et al. (2016).

game takes into account the payoffs that the players in the coalition give, *according to the value*, to the players who are removed, that is, to the players in $N \setminus N'$. Hence, the characteristic function of the reduced game depends on the original characteristic function and the solution in question. The value is consistent if a player in N' obtains the same payoff in the initial game and in the reduced game.

There are several possibilities to define a reduced game depending on how the removed players are compensated. In particular, Hart and Mas-Colell (1989) (*HM*) and Oishi et al. (2016) (*ONHF*) define two different reduced games. They use them to characterize the Shapley value as the only value that is standard for two-person games (that is, it divides the surplus equally between the two players) and consistent.

In contrast to the previous literature on consistency, we study operators that reduce games without reference to any value. We refer to them as *value-free reductions* (*v-f reductions*, for short). For any TU game with a set of players N and any $N' \subseteq N$, a v-f reduction generates a game played by N'. A simple example is the *subgame v-f reduction*, which assigns each coalition in the reduced game the same worth as in the initial game.

Our interest lies in the analysis of the reduction processes, that is, in the v-f reductions. We propose properties that one may ask any such v-f reduction to satisfy. In this paper, we study v-f reductions that satisfy four properties. First, we request that a v-f reduction is "well defined," in the sense that how players in $N \setminus N'$ are removed to arrive at a game with a set of players N' should not matter. The game played by the set N' should be the same if the players in $N \setminus N'$ have been removed one by one, all simultaneously, or in any other sequence. We call this property *path independence*. The second property is the *additivity* of the v-f reduction. Reducing two games through an additive v-f reduction and then summing the corresponding reduced games and directly reducing the sum of the games gives the same result.

The other two properties are related to the presence of null players in the initial game. The contribution of a null player to any coalition is zero. Hence, it seems reasonable that they play no role in a v-f reduction. We require that if a player is a null player in the initial game, he should still be a null player after a v-f reduction. We call this property the *permanent null player*. Moreover, if a null player is removed from the game, then the worth of the coalitions should not change, a property that we call the *null player out* property.

Path independence, additivity, permanent null player, and null player out do not suffice to identify a unique v-f reduction. But, by including alternative "invariance" properties, we characterize several v-f reductions. Each invariance property states how changes in the worth of coalitions of the same size affect the reduction of the game. First, we characterize the subgame v-f reduction using an axiom that requires that an increase in the worth of the grand coalition should not affect the reduction of a game, a property that we call *grand-coalition invariance*.

Second, we consider the four previous properties plus the invariance axiom that states that the reduced game is immune to changes in the players' strategic prospects derived from an identical increase or decrease in all the stand-alone coalitions. The axiom requires that if the worth of each stand-alone coalition, say, increases by the same amount, then this change should not affect how the game is reduced. Interestingly, these five axioms characterize a unique v-f reduction that corresponds to the *HM* v-f reduction, that is, the value-free version of the reduction method proposed by *HM*.

To continue our analysis of the properties of v-f reductions, we propose a duality theory for them. We define the dual of a v-f reduction as the v-f reduction of the dual of the game. We show that the *ONHF* v-f reduction (that is, the value-free version of the *ONHF* reduction method) is dual to that of the *HM* v-f reduction. We also show that our basic properties of path independence, additivity, permanent null player, and null player out are all self-dual properties, in the sense that they are satisfied by a v-f reduction if and only if they are satisfied by the dual of the v-f reduction. We use the duality theory to characterize the *ONHF* v-f reduction by using the invariance axiom that is dual of the one in the characterization of the *HM* v-f reduction. According to this new axiom, the reduction of a game should be immune to an identical increase or decrease in the worth of all the coalitions that include all the players except one.

We note that, given a v-f reduction, then any (initial) game can unambiguously be reduced to a game played by just one player, say player $i \in N$. We can interpret the worth of coalition $\{i\}$ (the only non-empty coalition) in this reduced game as the benefit or cost to be distributed to this player in the initial game. Repeating this process for every player in N allows us to define a value for the initial game. Thus, *a v-f reduction "induces" a value*. We show that the subgame v-f reduction induces the stand-alone value and, as one may expect, the *HM* and the *ONHF* v-f reductions induce the Shapley value. Moreover, we can connect our approach to the previous literature on consistency because any value induced by a path-independent v-f reduction (such as the subgame, the *HM*, and the *ONHF* v-f reductions) is consistent relative to that reduction.

We also link our approach to the theory of implementation. Indeed, we use the players' payoffs obtained at the Pérez-Castrillo–Wettstein bidding mechanism (a mechanism that implements the Shapley value, see Pérez-Castrillo and Wettstein, 2001) to propose another v-f reduction. We characterize the new v-f reduction by an alternative invariance axiom and show that it also induces the Shapley value. Moreover, we apply our duality theory again and characterize the dual of that v-f reduction. The existence of this dual PW v-f reduction prefigures the existence of a new PW-style bidding mechanism (see Chapter 2, for the analysis of such a mechanism). Thus, the connection of our approach to the theory of implementation –by constructing the v-f reduction of an extensive-form game and then finding its dual– could help enrich the literature of the Nash program. It can suggest new mechanisms that are "dual" of existing mechanisms.

Our four basic axioms can lead to characterizations of v-f reductions that induce additive values other than the stand-alone and the Shapley values. We use them as part of the characterization of a v-f reduction that induces the Banzhaf value (Banzhaf, 1964).

Although we do not use them in our characterizations, we discuss the properties of *anonymity* and *linearity*. Anonymity of a v-f reduction requires that a player's name does not matter in the reduction of the game. It has two implications: (a) the worth of the coalitions in the reduced game does not depend on the names of the players in the initial game but only on their contributions to coalitions, and (b) the v-f reduction itself depends not on the names of the removed players but only on their contributions. The notion of anonymity is unrelated to the other axioms. In fact, our basic properties do not imply anonymity. However, all the v-f reductions that we study satisfy anonymity of the process. They also satisfy linearity, which is additivity plus homogeneity.

We have based some of our examples of v-f reductions on existing reduced games, which were introduced to study the internal consistency of values. We also propose the reverse process. That is, given a v-f reduction, we can find a reduced game whose v-f version coincides with the v-f reduction. Through this process, we provide a new characterization of the Banzhaf value as the only value consistent relative to a new reduced game and standard for two-player games. We provide a similar characterization for the stand-alone value.

In addition to Hart and Mas-Colell (1989) and Oishi et al. (2016), several authors have used the consistency principle to characterize values for TU games.[2] Among others, Sobolev (1975) defines a distinct reduced game and axiomatize, together with other axioms, the Shapley value. Noticeabley, Davis and Maschler (1965) de-

[2] For introductions to the consistency principle in general, see Driessen (1991) and Thomson (2011).

fine a reduced game which turns out crucial in Peleg's (1985, 1986) axiomatizations of the core and the prekernel, and Sobolev's (1975) axiomatization of the prenucleolus. Moulin (1985) defines a reduced game and axiomatizes three families of choice methods in the framework of the "quasi-linear social choice problem."[3] Tadenuma (1992) also employs this reduced game to provide another axiomatization of the core.

The analysis of our paper may shed light on the discussion on the use of consistency relative to a reduced game when comparing different solutions for cooperative games. On that matter, Maschler (1990) advocates that the choice between two solution concepts that can be characterized by the same set of basic properties plus consistency relative to a reduced game (reduced games that are different for the two concepts) boils down to the examination of the reduced games. There are two strands of research related to this view. The first strand is pursued by Chang and Hu (2007), who propose a criterion to "distinguish" two different solutions through two different reduced games. The second strand includes Driessen and Radzik (2003), Yanovskaya and Driessen (2002), and Yanovskaya (2004), which characterize reduced games directly. Our approach is closer to the second strand since we adopt a pure axiomatic approach.

The remainder of the paper is organized as follows. In Section 1.2, we recall basic concepts, including the definition of reduced games. In Section 1.3, we introduce our central concept of a value-free reduction, together with a list of properties that a v-f reduction may satisfy. In Section 1.4, we develop a duality theory for v-f reductions. In Section 1.5, we provide an axiomatic characterization of several v-f reductions, we discuss the properties of anonymity and linearity, and we make a comment on non-additive v-f reductions. In Section 1.6, we use our approach to characterize the Banzhaf and the stand-alone values through consistency. Logical independence of each property in the characterization of the *HM* v-f reduction is established in Section 1.7. In Section 1.8, we conclude the paper. All proofs are collected in the Appendix.

1.2 TU games, values, and reduced games

Let an infinite set $\mathcal{U}$ represent the universe of the players. We restrict attention to games where the set of players constitutes a finite subset of $\mathcal{U}$. We denote $\mathcal{P}_{\text{fin}}(\mathcal{U})$ the set of all finite subsets of $\mathcal{U}$.

[3] See also Chang and Hu (2007). Thomson (2011) refers to this type as "complement-reduced games" since the complement of the players that stay in the reduced game is also involved in the reduced game.

A **coalitional game with transferable utility** (abbreviated as a **TU game**) is a vector (N, v) where $N \in \mathcal{P}_{fin}(\mathcal{U})$ is the set of players and $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. For $S \subseteq N$, $v(S)$ represents the worth of the coalition S in the game v. The class of all TU games with N as the set of players is denoted by $\mathcal{G}^N$. Thus, the set of all finite TU games is $\bigcup_{N \in \mathcal{P}_{fin}(\mathcal{U})} \mathcal{G}^N$.

A **subgame** of $(N, v) \in \mathcal{G}^N$ is a game $(N', v|_{N'}) \in \mathcal{G}^{N'}$ for some $N' \subseteq N$, where $v|_{N'}(S) = v(S)$ for all $S \subseteq N'$.

For a fixed set of players N, the set of all TU games $\mathcal{G}^N$ may be viewed as a vector space. The zero vector of $\mathcal{G}^N$ corresponds to the zero game $(N, \mathbf{0}) \in \mathcal{G}^N$. The worth of the coalition $S \subseteq N$ in $(N, \mathbf{0})$ is $\mathbf{0}(S) \equiv 0$. One particular subset of games that we will use as a basis for $\mathcal{G}^N$ is the set of **unanimity games**, which are denoted by $(N, u_T) \in \mathcal{G}^N$, for $T \in 2^N \setminus \{\varnothing\}$. The worth of the coalition $S \subseteq N$ in (N, u_T) is:

$$u_T(S) \equiv \begin{cases} 1 & \text{if } S \supseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

Among the unanimity games, $(N, u_N) \in \mathcal{G}^N$ depicts a particularly simple situation: one unit of transferable utility is generated only when the grand coalition forms.

Cooperative game theory accords particular attention to the search of appealing solution concepts and their characterizations through desirable properties from the mathematics and/or economics points of view. Single-valued solutions for TU games are called **values**. A value allocates a payoff to each player in a game, for every possible game. Thus, a value φ prescribes, for each $N \in \mathcal{P}_{\text{fin}}(\mathcal{U})$, each TU game $(N, v) \in \mathcal{G}^N$, and each $i \in N$, a payoff $\varphi_i(N, v) \in \mathbb{R}$.

The most prominent value is the **Shapley value** (Shapley, 1953), which is denoted by Sh henceforth:[4]

$$Sh_i(N, v) = \sum_{T \subseteq N \setminus \{i\}} \frac{t!(n-t-1)!}{n!} D^i v(T),$$

for any $(N, v) \in \mathcal{G}^N$ and for any $i \in N$, where $D^i v(T) \equiv v(T \cup \{i\}) - v(T)$ denotes the marginal contribution of player i to the coalition $T \subseteq N \setminus \{i\}$.

Another solution concept which we will discuss later is the **Banzhaf value** (see

[4] We follow the convention by using uppercase letters to denote sets of players and letting the corresponding lowercase letters represent their cardinalities. For instance, the cardinality of N, N', and T are n, n', and t.

Banzhaf, 1964, and Owen, 1975) which we henceforth denote by *Ban*:

$$Ban_i(N,v) = \sum_{T \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} D^i v(T).$$

We notice that, in contrast to the Shapley value, the Banzhaf value is not efficient in the sense that the sum of the outcomes obtained by the players need not be $v(N)$.

Two-player TU games constitute the most simple subclass of TU games. Unsurprisingly, several solution concepts for TU games prescribe the same payoff when restricted to this simple subclass. According to this prescription, in the game $(\{i,j\},v) \in \mathcal{G}^{\{i,j\}}$ each player $k \in \{i,j\}$ is assigned, on top of his stand-alone value, half of the surplus generated from the collaboration:

$$\varphi_k(\{i,j\},v) = v(\{k\}) + \frac{1}{2}[v(\{i,j\}) - v(\{i\}) - v(\{j\})]. \tag{1.2.1}$$

This is, in particular, the prescription of the Shapley value and the Banzhaf value for two-player games. Hence, it is commonly said that a value φ is **standard for two-player games** if for each game $(\{i,j\},v) \in \mathcal{G}^{\{i,j\}}$, φ satisfies equation (1.2.1).

For TU games with more than two players, solution concepts may be pinned down by imposing consistency relative to some **reduced games**. In the literature, reduced games are always associated with a solution concept as follows. Given a value φ, a **reduction** Ψ^φ is a function that associates each TU game $(N,v) \in \mathcal{G}^N$ with a reduced game $(N', \Psi^\varphi_{NN'}(v)) \in \mathcal{G}^{N'}$ for any two finite sets of players N, N' such that $N' \subsetneq N$.[5] That is, a reduction applied on a game with a set of players N specifies how to "reduce" the game if it were to be played only by a subset N' of N. Notice that the value φ appears in this function $\Psi^\varphi_{NN'}$ as a parameter, so that different values lead to different ways of "reducing" a game in $\mathcal{G}^N$ to a game in $\mathcal{G}^{N'}$.

Now we can formulate the definition of consistency of a value relative to some reduction:

Definition 1. *The value φ is* ***consistent*** *relative to the reduction Ψ^φ if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, all $(N,v) \in \mathcal{G}^N$, and all $i \in N'$,*

$$\varphi_i\big(N', \Psi^\varphi_{NN'}(v)\big) = \varphi_i(N,v).$$

Consistency of φ means that the prescribed payoff for any player $i \in N'$ in the initial game (N,v) according to the value φ is the same as that in the reduced game

[5] We call the operator Ψ^φ a reduction, even though the previous literature does not address such an operator abstractly. They propose the consistency property using reduced games, which are the images of a concrete reduction.

$(N', \Psi^{\varphi}_{NN'}(v))$ according to this value.

We close this section with two examples of reductions: the *HM* reduction (see Hart and Mas-Colell, 1989) and the *ONHF* reduction (see Oishi et al., 2016).

Definition 2. *Given a value φ, the HM reduction $\Psi^{HM\varphi}$ is defined by:*

$$\Psi^{HM\varphi}_{NN'}(v)(S) \equiv v(S \cup (N \setminus N')) - \sum_{i \in N \setminus N'} \varphi_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')}),$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subsetneq N$ and all $(N, v) \in \mathcal{G}^N$.

The interpretation of the *HM* reduction is as follows. Given a value φ, consider a game $(N, v) \in \mathcal{G}^N$ that is reduced to be played by players in $N' \subsetneq N$. If a coalition $S \subseteq N'$ is formed, then the players in S collaborate with all removed players in $N \setminus N'$, which yields a worth $v(S \cup (N \setminus N'))$. However, each removed player $i \in N \setminus N'$ is entitled to $\varphi_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')})$, his "fair" share of the worth of the coalition $S \cup (N \setminus N')$. Then, the coalition S has a claim to the residual, which defines the worth of coalition S in the *HM* reduced game.

Hart and Mas-Colell (1989) characterize the Shapley value as the unique value that is consistent relative to the *HM* reduction $\Psi^{HM\varphi}$ and that is standard for two-player games.

Oishi et al. (2016) obtain a different characterization of the Shapley value through a reduction à la Hart and Mas-Colell by exploiting the self-duality of the Shapley value. To define the *ONHF* reduction, we first introduce the following notation: given a TU game $(N, v) \in \mathcal{G}^N$ and $S \subsetneq N$, we denote by $(N \setminus S, v^S) \in \mathcal{G}^{N \setminus S}$ the game defined by:

$$v^S(T) \equiv v(T \cup S) - v(S), \tag{1.2.2}$$

for all $T \subseteq N \setminus S$.

Definition 3. *Given a value φ, the ONHF reduction $\Psi^{ONHF\varphi}$ is defined by:*

$$\Psi^{ONHF\varphi}_{NN'}(v)(S) \equiv v(S) - \sum_{i \in N \setminus N'} \varphi_i(N, v) + \sum_{i \in N \setminus N'} \varphi_i(N \setminus S, v^S),$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subsetneq N$ and all $(N, v) \in \mathcal{G}^N$.

In contrast to the *HM* reduction, the intuition of the *ONHF* reduced game (as acknowledged by Oishi et al., 2016) is more involved. To determine the worth of a coalition $S \subseteq N'$ in an *ONHF* reduced game, we consider all the players in S together. Forming the coalition S entitles the players in the coalition to offer their joint collaboration to the rest of the players to play a new TU game $(N \setminus S, v^S) \in$

$\mathcal{G}^{N\setminus S}$. As defined above, in this new game any coalition $T \subseteq N \setminus S$ is formed with the collaboration of S and T, which yields a worth $v(T \cup S)$. The coalition S is entitled to two payments. First, it receives $v(S)$ in forming this game. Second, it makes a swap agreement with the removed players: the coalition S pays $\varphi_i(N, v)$ to each player $i \in N \setminus N'$, which equals the amount i deserves in the initial game, and it collects the sum of what these players receive in $(N \setminus S, v^S)$, which adds up to $\sum_{i \in N\setminus N'} \varphi_i(N \setminus S, v^S)$. The net payoff for S after the two payments corresponds to its worth in the *ONHF* reduced game.

Oishi et al. (2016) show that the Shapley value is the only value that is consistent relative to the *ONHF* reduction $\Psi^{ONHF\varphi}$ and that is standard for two-player games.

1.3 Value-free reductions: Definition and axioms

The existing literature takes the values as the main object of study and considers the reduced games associated with values to characterize particular values. By contrast, our approach takes the reductions as the primitive concept, analyzes properties of the reductions, characterizes some of them through the properties, and eventually uses the reductions to derive values.

To develop our approach, we first formally introduce the concept of a value-free reduction, that is, a reduction that does not make any reference to a value.

Definition 4. *A* ***value-free reduction (v-f reduction*** *for short) Ψ is a function that associates to each finite set of players N, each TU game $(N, v) \in \mathcal{G}^N$, and each subset $N' \subseteq N$, a TU game $(N', \Psi_{NN'}(v)) \in \mathcal{G}^{N'}$.*[6]

Because of the defining feature of v-f reductions, we must forsake the superscript φ from a generic v-f reduction.

To illustrate the concept, we provide a first example of a v-f reduction. Example 1 defines Ψ^{sub}, which we call the subgame v-f reduction.[7] According to this operator, the value of any subset in the reduced game is the same as its value in the initial game.[8]

Example 1. *We define the* ***subgame v-f reduction*** Ψ^{sub} *by:*

$$\Psi^{sub}_{NN'}(v)(S) \equiv v\,|_{N'}\,(S) = v(S),$$

[6] We allow for the possibility that $N' = N$ for convenience.

[7] We refer to all the examples of value-free reductions as "v-f reductions" even though the use of "v-f" is not always necessary.

[8] Myerson (1980) uses the subgame operator to define his famous balanced contributions property of the Shapley value.

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subsetneq N$ *and all* $(N, v) \in \mathcal{G}^N$*.*

Any v-f reduction induces one-player v-f reduced games. That is, a game $(N, v) \in \mathcal{G}^N$ can be reduced to n games $(\{i\}, \Psi_{N\{i\}}(v))$, for $i \in N$. This procedure provides the possibility of identifying the value of a player i in the game (N, v) as the worth of the coalition $\{i\}$ in the v-f reduced game consisting of this player only. We propose the following definition of the value induced by a v-f reduction:

Definition 5. *The value* φ^Ψ *induced by a v-f reduction* Ψ *is, for all* $(N, v) \in \mathcal{G}^N$ *and all* $i \in N$*,*

$$\varphi_i^\Psi(N, v) \equiv \Psi_{N\{i\}}(v)(\{i\}).$$

For instance, the value induced by the subgame v-f reduction is the stand-alone value:

$$\varphi_i^{\Psi^{sub}}(N, v) = \Psi_{N\{i\}}^{sub}(v)(\{i\}) = v(\{i\}),$$

because the prescribed payoff of the value induced by the subgame v-f reduction for all $i \in N$ is $v|_{\{i\}}(\{i\}) = v(\{i\})$.

We now propose and explain some properties that v-f reductions may satisfy. We see v-f reductions as a way to remove players from a game while keeping the remaining players' strategic prospect intact. Thus, we suggest properties that may be coherent with this view.

We first introduce a minimum requirement of a well-behaved v-f reduction, the path-independence property:

Axiom 1. *A v-f reduction* Ψ *is* ***path independent*** *if for all* $N_1, N_2, N_3 \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $N_3 \subseteq N_2 \subseteq N_1$*, then*

$$\Psi_{N_2 N_3} \circ \Psi_{N_1 N_2} = \Psi_{N_1 N_3}.$$[9]

Path independence means that, for any game $(N, v) \in \mathcal{G}^N$, the way players in $N \setminus N'$ are removed to reach the v-f reduced game of (N, v) with N' as the remaining players should be irrelevant. In particular, it should not matter whether a player's removal precedes another player's or if they are removed simultaneously. The only relevant information is the set of players who remain at the end.

Reduced games were introduced in the literature to study the consistency of values. Then, it is natural to ask about the consistency of the value induced by a v-f reduction with respect to that reduction. Although Definition 1 refers to consistency relative to a reduced game (and not to v-f reduced games), the definition can be

[9] The symbol "$\circ$" denotes the composition of two functions: for $f : X \to Y$ and $g : Y \to Z$, $g \circ f(x) = g\big(f(x)\big) \in Z$ for all $x \in X$.

easily accommodated. Proposition 1 shows that the value induced by a v-f reduction is indeed consistent if the v-f reduction is path independent.

Proposition 1. *The value φ^Ψ induced by a path-independent v-f reduction Ψ is consistent relative to Ψ.*

Our second axiom on v-f reductions is additivity:

Axiom 2. *A v-f reduction Ψ is **additive** if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$ and all $(N, v_1), (N, v_2) \in \mathcal{G}^N$, then*

$$\Psi_{NN'}(v_1 + v_2) = \Psi_{NN'}(v_1) + \Psi_{NN'}(v_2).$$

To put it in words, additivity means that if game (N, v) is the sum of two games (N, v_1) and (N, v_2), then directly reducing (N, v), and reducing (N, v_1) and (N, v_2) and then summing the corresponding reduced games, give the same result.

We will use additivity in our characterizations. Since we use the concept of a linear v-f reduction later and in some of the proofs in the Appendix, we introduce linearity here. A v-f reduction is **linear** if it satisfies the axioms of additivity and homogeneity.

Axiom 3. *A v-f reduction Ψ is **homogeneous** if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$, all $(N, v) \in \mathcal{G}^N$, and all $\alpha \in \mathbb{R}$, then*

$$\Psi_{NN'}(\alpha v) = \alpha \Psi_{NN'}(v).$$

Homogeneity of a v-f reduction Ψ means that the scale in which we measure the worth of the coalitions in a TU game does not influence how the game is reduced.

Our next two axioms concern the consequences of the presence of "null players" in the game, that is, players who do not contribute to any coalition, on the reduced game. Before introducing the axioms, we formally define null players.

Definition 6. *A player $i \in N$ is a **null player** in a TU game $(N, v) \in \mathcal{G}^N$ if $D^i v(S) = 0$ for all $S \subseteq N \setminus \{i\}$.*

Given that null players have no impact on the worth of any coalition, it may seem reasonable that they also have no impact on the reduction of games. Thus, we propose the following property:

Axiom 4. *A v-f reduction Ψ satisfies the **null player out** property if for all $N \in \mathcal{P}_{fin}(\mathcal{U})$, all $i \in N$, and all $(N, v) \in \mathcal{G}^N$ such that player i is a null player in (N, v), then*

$$\Psi_{N(N\setminus\{i\})}(v) = v\,|_{N\setminus\{i\}}\,.$$

The null player out property means that if a null player is removed from the game, then his removal has no effect on the worth of coalitions in the game without him. The axiom reflects the idea that given that a null player has no influence on the game, the worth of any coalition should not change if the game is reduced because he is removed.

Moreover, a null player should gain no influence after a reduction:

Axiom 5. *A v-f reduction Ψ satisfies the **permanent null player** property if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$, all $i \in N'$, and all $(N, v) \in \mathcal{G}^N$ such that player i is a null player in (N, v), then player i is also a null player in $(N', \Psi_{NN'}(v))$.*

The interpretation of the permanent null player property is that if a player is a null player in the initial game, then he is still a null player after the removal of some other arbitrary players.

In general, null player out and permanent null player properties reflect the rationale perceiving null players as irrelevant or redundant. Still, they are distinct axioms, as we will show in Section 1.7, where we analyze the logical independence of the axioms.

Our last set of axioms provides alternative views of how the reduction of a game is affected by changes in the worth of coalitions of the same size. Indeed, it is conventional to postulate the monotonocity principle that a player's strategic perspective should be monotonic with respect to the worth of the coalitions containing him (see, e.g., Young, 1985). In line with this principle, if we consider, for example, a symmetric game and we increase the worth of all coalitions of the same size by the same amount, then the enhancing strategic effects for the players may be entirely canceled out. This reasoning is akin to the disagreement convexity in Peters and van Damme (1991) in the context of the bargaining problem: if each player's disagreement point is increased properly, then the solution should not be changed.

Our version of addition invariance properties borrows from ideas developed by Béal et al. (2015). In our formulation, we follow the terminology used in that paper, which we introduce here:

Definition 7. *Given the set of players N, for all $k \in \mathbb{Z}_+$ such that $k \leq n$, and $\alpha \in \mathbb{R}$, the game $(N, w_{(k,\alpha)}) \in \mathcal{G}^N$ is defined as follows: for all $S \subseteq N$,*

$$w_{(k,\alpha)}(S) \equiv \begin{cases} \alpha & \text{if } |S| = k; \\ 0 & \text{otherwise.} \end{cases}$$

The game $(N, w_{(k,\alpha)})$ is a useful tool to express an identical increase or decrease in the worth of all coalitions of size k in a TU game (N, v) as the addition of $(N, w_{(k,\alpha)})$ to (N, v).

We point out that the reduction of a game necessarily leads to losing some information contained in the characteristic function v since the domain of the reduced game is a proper subset of that of the initial game. Our first invariance axiom suggests discarding the information contained in the level of the worth of the coalitions of size one. The reduction may depend on the relative worth of the singletons, that is, whether the stand-alone coalition of one player has a higher or lower worth than the stand-alone coalition of other players. However, the axiom postulates that the reduction cannot depend on whether the worth of all the one-player coalitions is high or low. We can translate this idea to the property that if the worth of every coalition of size one in the unanimity game $(N, u_N) \in \mathcal{G}^N$ is increased or decreased by the same amount, then the reduction of the game u_N should not change.

Axiom 6. *A v-f reduction Ψ satisfies* ***1-addition invariance*** *if for all $\alpha \in \mathbb{R}$ and all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + w_{(1,\alpha)}) = \Psi_{NN'}(u_N).$$

Our second invariance axiom proposes an alternative property, in the same spirit as the previous one. It prescribes what happens after an increase or decrease in the worth of every coalition except the grand coalition, where the change in the worth is proportional to the number of players in the coalition. The axiom requires that the change does not affect the reduction of the unanimity game.

Axiom 7. *A v-f reduction Ψ satisfies* ***proportional addition invariance*** *if for all $\alpha \in \mathbb{R}$ and all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + \sum_{k=1}^{n-1} w_{(k,k\alpha)}) = \Psi_{NN'}(u_N).$$

The previous two axioms share the view that the reduction of the game leads to the loss of information from the worth of coalitions smaller than the grand coalition. Our third invariance axiom takes the opposite view. It postulates that the worth of the grand coalition should not affect the reduction of the unanimity game.

Axiom 8. *A v-f reduction Ψ satisfies* ***grand-coalition invariance*** *if for all $\alpha \in \mathbb{R}$ and all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + w_{(n,\alpha)}) = \Psi_{NN'}(u_N).$$

Before we turn to the characterization of several v-f reductions in Section 1.5, we first propose a duality theory for v-f reductions in Section 1.4. We adapt the approach of Oishi et al. (2016). The main difference of our approach is that we take the v-f reductions as primitive, while Oishi et al. (2016) stick to the conventional view that takes the solution concepts as primitive and uses reduced games to characterize solutions in terms of consistency. We use our duality theory in two characterizations of Section 1.5.

1.4 Duality theory for value-free reductions

We first recall the definition of the dual of a game and the dual of a value. For a TU game $(N, v) \in \mathcal{G}^N$, the dual of (N, v) is the game $(N, v^*) \in \mathcal{G}^N$, defined by:

$$v^*(S) \equiv v(N) - v(N \setminus S), \tag{1.4.1}$$

for all $S \subseteq N$. For a value φ, the dual φ^* of φ is defined by the value:

$$\varphi^*(N, v) \equiv \varphi(N, v^*), \tag{1.4.2}$$

for all $(N, v) \in \mathcal{G}^N$ and all $N \in \mathcal{P}_{fin}(\mathcal{U})$.

A value is self-dual if $\varphi = \varphi^*$. Examples of self-dual values include the Shapley value and the Banzhaf value.

We now define the dual of a v-f reduction:

Definition 8. *The* ***dual*** Ψ^* ***of a v-f reduction*** Ψ *is defined, for all* $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $N' \subseteq N$*, and all* $(N, v) \in \mathcal{G}^N$*, as*

$$\Psi^*_{NN'}(v) \equiv \big(\Psi_{NN'}(v^*)\big)^*.$$

That is, consider a v-f reduction Ψ and a game (N, v). The dual v-f reduction of (N, v) consists in first, applying Ψ to the dual of (N, v), and then taking the dual of the reduced game.

We already know that the dual operator for TU games is reflexive because $(v^*)^* = v$. The dual operator for v-f reductions is also reflexive, that is, $(\Psi^*)^* = \Psi$.[10]

If the v-f reduction is path independent, then we can relate the concepts of duality for values and for v-f reductions. Indeed, by recognizing that a one-player

[10] This property holds because for all $(N, v) \in \mathcal{G}^N$ and all $N' \subseteq N$: $(\Psi^*)^*_{NN'}(v) = (\Psi^*_{NN'}(v^*))^* = ((\Psi_{NN'}((v^*)^*))^*)^* = \Psi_{NN'}(v)$, where the last equality uses twice that the dual operator for TU games is reflexive.

TU game coincides with its dual, we obtain the result that the concept of the dual of a value is compatible with the concept of the dual of a v-f reduction:

Proposition 2. *The value induced by a path-independent v-f reduction is dual to the value induced by the dual v-f reduction:*

$$(\varphi^{\Psi})^* = \varphi^{(\Psi^*)}. \tag{1.4.3}$$

An immediate corollary of Proposition 2 is the following:

Corollary 1. *The value induced by a path-independent v-f reduction is self-dual if and only if it is also induced by the dual of the v-f reduction.*

We also define dual properties, or axioms, of v-f reductions.

Definition 9. *Consider two properties $\mathcal{P}$ and $\mathcal{P}^*$ regarding v-f reductions. We say that property $\mathcal{P}$* ***is dual to property*** *$\mathcal{P}^*$ if for all v-f reductions Ψ,*

$$\Psi \text{ satisfies } \mathcal{P} \iff \Psi^* \text{ satisfies } \mathcal{P}^*.$$

We say that a property is self-dual if it is satisfied by a v-f reduction if and only if it is satisfied by the dual of the v-f reduction:

Definition 10. *$\mathcal{P}$ is* ***self-dual*** *if $\mathcal{P}$ is dual to itself, that is, for all v-f reductions Ψ, Ψ satisfies $\mathcal{P}$ if and only if Ψ^* satisfies $\mathcal{P}$.*

An important result, very helpful in the characterization of v-f reductions, is that the basic axioms that we use are all self-dual, as Proposition 3 states.

Proposition 3. *The axioms of additivity, null player out, permanent null player, and path independence of v-f reductions are all self-dual properties.*

1.5 Characterization of several value-free reductions

In this section, we use the axioms of additivity, null player out, permanent null player, and path independence to characterize several v-f reductions. Each characterization of a v-f reduction uses an additional invariance axiom.

Before presenting our characterizations, we state an intuitive property that is common to the v-f reductions that are path independent and satisfy the axiom of null player out: a game will be unchanged after a reduction where no player is removed. We state this property in Remark 1, which we will use in the proofs of the characterizations.

Remark 1. *If a v-f reduction Ψ satisfies null player out and path independence, then for all $N \in \mathcal{P}_{fin}(\mathcal{U})$ and all $(N, v) \in \mathcal{G}^N$,*

$$\Psi_{NN}(v) = v.$$

1.5.1 Characterization of the subgame value-free reduction

The subgame v-f reduction Ψ^{sub}, defined in Example 1, satisfies our four basic axioms. Moreover, it is characterized with the help of the axiom of grand-coalition invariance (Axiom 8), which postulates that changes in the worth of the grand coalition should not influence the way in which the unanimity game is reduced.

Theorem 1. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and grand-coalition invariance if and only if:*

$$\Psi = \Psi^{sub}.$$

Given that the axiom of grand-coalition invariance emphasizes how difficult coordination is for players striving to achieve the worth of the grand coalition, since the worth of the grand coalition is irrelevant for the reduction, it is reasonable that it leads to the characterization of a v-f reduction where those outside the reduced set of players have no role: the worth of any subgame coincides with that in the initial game.

1.5.2 Characterization of the *HM* value-free reduction

Next, we study the consequences of including the axiom of 1-addition invariance. It requires that an identical increase or decrease in the worth of all the one-player coalitions in a game should not affect the reduction of the game. Interestingly, 1-addition invariance together with our four basic axioms characterize the value-free version of the most popular reduced game, the *HM* reduction (see Definition 2). We call this v-f reduction the *HM* v-f reduction and we denote it by Ψ^{HM}. We construct the *HM* v-f reduction by substituting $\varphi = Sh$ in $\Psi^{HM\varphi}$.

Example 2. *We define the **HM v-f reduction** Ψ^{HM} by:*[11]

$$\begin{aligned}\Psi^{HM}_{NN'}(v)(S) \equiv & v(S \cup (N \setminus N')) - \sum_{i \in N \setminus N'} Sh_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')}) \\ = & \sum_{i \in S} Sh_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')}),\end{aligned}$$

[11] The second equality is implied by the efficiency of the Shapley value.

for all $S, N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and all* $(N, v) \in \mathcal{G}^N$.

Theorem 2 states the characterization. It also stresses that, as one could expect, the *HM* v-f reduction induces the Shapley value.

Theorem 2. *A v-f reduction* Ψ *satisfies additivity, null player out, permanent null player, path independence, and* 1*-addition invariance if and only if:*

$$\Psi = \Psi^{HM}.$$

Moreover, Ψ^{HM} *induces the Shapley value.*

Theorem 2 provides a characterization of Ψ^{HM} that is particularly interesting because it is based on a property (the 1-addition invariance) which seems unrelated to the definition of the reduction. On the one hand, the idea behind the reduction of a game $(N, v) \in \mathcal{G}^N$ to $(N', \Psi^{HM}_{NN'}(v))$ is that the worth of a coalition $S \subseteq N'$ in $(N', \Psi^{HM}_{NN'}(v))$ is computed taking into account that the players in S profit from the collaboration with every removed player $i \in N \backslash N'$, who is entitled to a compensation of $Sh_i(S \cup (N \setminus N'), v \mid_{S \cup (N \setminus N')})$. On the other hand, the notion of 1-addition invariance concerns the effect of identical changes in the worth of the one-player coalitions.[12] Therefore, Theorem 2 highlights that a characteristic property of the *HM* v-f reduction is that it is immune to changes in the strategic prospects of the players derived from the changes in their stand-alone worth, as long as the changes are identical for every player.

1.5.3 Characterization of the *ONHF* value-free reduction

In the previous subsection, we define the value-free version of the *HM* reduction. We can use the same method to define the value-free version of the *ONHF* reduction, Ψ^{ONHF}, which we will refer to as the *ONHF* v-f reduction:

Example 3. *We define the* ***ONHF v-f reduction*** Ψ^{ONHF} *by:*[13]

$$\begin{aligned}\Psi^{ONHF}_{NN'}(v)(S) \equiv & v(S) - \sum_{i \in N \backslash N'} Sh_i(N, v) + \sum_{i \in N \backslash N'} Sh_i(N \setminus S, v^S) \\ = & \sum_{i \in N'} Sh_i(N, v) - \sum_{i \in N' \backslash S} Sh_i(N \setminus S, v^S), \end{aligned} \tag{1.5.1}$$

[12] 1-addition invariance together with additivity imply that $\Psi_{NN'}(v + w_{(1,\alpha)}) = \Psi_{NN'}(v)$ for any $(N, v) \in \mathcal{G}^N$.

[13] The two expressions for Ψ^{ONHF} are equivalent because $\sum_{i \in N \backslash N'} Sh_i(N, v) = v(N) - \sum_{i \in N'} Sh_i(N, v)$, $\sum_{i \in N \backslash N'} Sh_i(N \setminus S, v^S) = v^S(N \setminus S) - \sum_{i \in N' \backslash S} Sh_i(N \setminus S, v^S)$, and $v^S(N \setminus S) = v(N) - v(S)$.

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and all $(N, v) \in \mathcal{G}^N$.

Oishi et al. (2016) construct the *ONHF* reduced game as the dual of the *HM* reduced game. Hence, it is no surprise that Ψ^{ONHF} is the dual v-f reduction of Ψ^{HM}. We state this result as a corollary of the analysis developed by Oishi et al. (2016):

Corollary 2. *The v-f reduction Ψ^{ONHF} is the dual of the v-f reduction Ψ^{HM}.*

As we proved in Section 1.4, additivity, null player out, permanent null player, and path independence are all self-dual properties. Given that they are satisfied by Ψ^{HM}, Ψ^{ONHF} also satisfies these axioms. On the other hand, the property of 1-addition invariance, which is the additional axiom that characterizes Ψ^{HM}, is not self-dual.

Proposition 4 states that the dual property of the 1-addition invariance is the $(n-1)$-addition invariance axiom, defined as follows:

Axiom 9. *A v-f reduction Ψ satisfies **(n-1)-addition invariance** if for all $\alpha \in \mathbb{R}$ and all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$,*

$$\Psi_{NN'}(u_N + w_{(n-1,\alpha)}) = \Psi_{NN'}(u_N).$$

Proposition 4. *The dual of the 1-addition invariance axiom is the $(n-1)$-addition invariance axiom.*

In conjunction with the interpretation of the 1-addition invariance property provided in the previous section, there is a dual interpretation of the $(n-1)$-addition invariance property. This axiom requires discarding the information contained in the level of the worth of the coalitions of size $(n-1)$ (instead of the information contained in the level of the worth of the coalitions of size 1).

Theorem 3 provides our characterization of Ψ^{ONHF}. It can be thought of as a dual theorem to Theorem 2 as it gives a characterization of the dual of Ψ^{HM} through the dual properties of the axioms used in Theorem 2. The theorem also states that the Ψ^{ONHF} v-f reduction induces the Shapley value.

Theorem 3. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and $(n-1)$-addition invariance if and only if*

$$\Psi = \Psi^{ONHF}.$$

Moreover, Ψ^{ONHF} induces the Shapley value.

Theorems 2 and 3 together reveal a distinctive difference between Ψ^{HM} and its dual, Ψ^{ONHF}. Whereas the *HM* v-f reduction postulates that the strategic prospects of the agents should not change after an identical modification in the worth of every stand-alone coalition (the 1-addition invariance property), the *ONHF* v-f reduction considers that the players' strategic prospects should not change after an identical modification in each player's maximum compensation (the $(n-1)$-addition invariance property).

1.5.4 Value-free reductions inspired by the bidding mechanism

We have characterized v-f reductions that bear some relationship to existing reduced games. In the current subsection, we propose and characterize two new v-f reductions. They link our approach to the theory of implementation. Indeed, the first v-f reduction is based on the out-of-equilibrium payoffs obtained at the Pérez-Castrillo–Wettstein (PW) bidding mechanism (see, Pérez-Castrillo and Wettstein, 2001), which implements the Shapley value. The second v-f reduction is the dual of the first. Thus, we start by explaining the bidding mechanism, and its equilibrium.

In the PW bidding mechanism, each player $j \in N$ in a game $(N, v) \in \mathcal{G}^N$ makes a bid $b_i^j \in \mathbb{R}$ to each player $i \neq j$. The player with the highest total net bid (the difference between a player's total bid to the others minus the sum of the bids the others make to him) is chosen as the proposer (let's denote him by α). The proposer α pays the bids to the rest of the players and makes them an offer to join him. If the proposal is accepted, then α pays the offers that he has made to the other players (in addition to the bids that he has already paid), forms the grand coalition, and receives the worth $v(N)$. If the proposal is rejected, then α is removed from the game and obtains the worth of his stand-alone coalition $v(\{\alpha\})$. The rest of the players, that is, the set $N \setminus \{\alpha\}$, keep the bids and play the same game again among them.

At the subgame perfect equilibrium of the bidding mechanism, any player $j \in N$ bids $b_i^j = Sh_i(N, v) - Sh_i(N \setminus \{j\}, v\mid_{N\setminus\{j\}})$ to each player $i \neq j$ and the proposer α makes an offer that is accepted (see, Pérez-Castrillo and Wettstein, 2001). The offer submitted to the players in $N \setminus \{\alpha\}$ makes them indifferent between accepting the offer and playing the new game among them (because this is the continuation outcome of the mechanism in case of rejection). That is, the offer to each player is the payoff that this player would obtain in the "reduced game" where the set of players is $N \setminus \{\alpha\}$. In this reduced game, the assets of any coalition $S \subseteq N \setminus \{\alpha\}$ are composed by two elements: the worth of the coalition and the sum of the bids that the players in S collect from α, that is, $v(S) + \sum_{i \in S} b_i^\alpha = v(S) + \sum_{i \in S}(Sh_i(N, v) - Sh_i(N \setminus \{\alpha\}, v|_{N \setminus \{\alpha\}}))$.

If we continue deleting players, we obtain the extension of the previous formulae for the reduced game played by any $N' \subsetneq N$ (which corresponds to a situation where the players in $N \setminus N'$ were proposers in the bidding mechanism with their proposals being rejected and with their bids being collected). This way, we define the following v-f reduction:

Example 4. *We define the* ***PW v-f reduction*** Ψ^{PW} *by:*

$$\Psi_{NN'}^{PW}(v)(S) \equiv v(S) - \sum_{i \in S} Sh_i(N', v|_{N'}) + \sum_{i \in S} Sh_i(N, v), \tag{1.5.2}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and all* $(N, v) \in \mathcal{G}^N$.

Theorem 4 shows that Ψ^{PW} is characterized in a similar way to Theorems 1, 2, and 3. It uses the alternative property of proportional addition invariance, which we have described in Section 1.3 (see Axiom 7).

Theorem 4. *A v-f reduction* Ψ *satisfies additivity, null player out, permanent null player, path independence, and proportional addition invariance if and only if*

$$\Psi = \Psi^{PW}.$$

Moreover, Ψ^{PW} *induces the Shapley value.*

Theorem 4 also identifies the value $\varphi^{\Psi^{PW}}$ induced by the path-independent v-f reduction Ψ^{PW}. Given that the PW bidding mechanism implements the Shapley value, it is unsurprising that the value induced by the reduction is also the Shapley value. On the other hand, nothing in the bidding mechanism suggests that the equilibrium bids are related to the size of the coalitions. Therefore, the characterization of the PW v-f reduction owing to the axiom of proportional addition invariance provides a new perspective on the out-of-equilibrium payoffs of the players in the bidding mechanism.

We now use the duality theory developed in the previous section to provide and characterize another v-f reduction, the dual of Ψ^{PW}, which we denote by Ψ^{PW^*}. To that end, we first identify the dual of the proportional addition invariance (since the other axioms used in the characterization of Theorem 4 are self-dual). The proportional addition invariance prescribes that a change in the worth of every coalition (except for the grand coalition) that is proportional to the size of the coalition, should not affect the strategic possibilities of the players, hence it should not affect the reduction of the unanimity game either. The reverse-proportional additional invariance axiom proposes that the reduction should not be affected if the worth of every coalition is changed in reverse proportion to their size.

Axiom 10. *A v-f reduction* Ψ *satisfies* ***reverse-proportional addition invariance*** *if for all* $\alpha \in \mathbb{R}$ *and all* $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $N' \subsetneq N$*, then*

$$\Psi_{NN'}(u_N + \sum_{k=1}^{n-1} w_{(k,(n-k)\alpha)}) = \Psi_{NN'}(u_N).$$

Proposition 5. *The dual of the proportional addition invariance axiom is the reverse-proportional addition invariance axiom.*

Theorem 5 provides the characterization of Ψ^{PW^*}, which we formally define in Example 5:[14]

Example 5. *We define the* ***PW* v-f reduction*** Ψ^{PW^*} *by:*

$$\Psi^{PW^*}_{NN'}(v)(S) \equiv v(S \cup (N \setminus N')) - v(N \setminus N') - \sum_{i \in S} Sh_i(N', v^{N \setminus N'}) + \sum_{i \in S} Sh_i(N, v), \tag{1.5.3}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and all* $(N, v) \in \mathcal{G}^N$.

Theorem 5. *A v-f reduction* Ψ *satisfies additivity, null player out, permanent null player, path independence, and reverse-proportional addition invariance if and only if*

$$\Psi = \Psi^{PW^*}.$$

Moreover, Ψ^{PW^*} *induces the Shapley value.*

We note that Example 5 and Theorem 5 suggest the possible existence of a PW-style bidding mechanism such that its subgames on the off-equilibrium path would correspond to the dual PW v-f reduction. In Chapter 2, we will construct such a mechanism and show that it implements the Shapley value.

Taken together, Theorems 2 to 5 provide additional evidence that the Shapley value is a solution concept with strong properties. Indeed, it is induced by v-f reductions that are characterized by very diverse invariance properties. We can use an operator that reduces a game so as to keep the same players' strategic possibilities after an identical change in the worth of all the one-player coalitions or of all maximum possible compensations; or after a change that is proportional to the number of players in any subcoalition, or that is reverse to the number of players in any subcoalition. The Shapley value is attained after any of those different reductions.

[14] See the Appendix for the derivation of the expression for Ψ^{PW^*}.

1.5.5 A value-free reduction inducing the Banzhaf value

The objective of this subsection is to illustrate how to use our approach to characterize v-f reductions that induce solution concepts different from the Shapley value, or the stand-alone value. In particular, we propose a v-f reduction that induces the Banzhaf value, which we introduced in Section 1.2.

Dragan (1996) proposes a reduced game which is implicitly defined by a functional equation to axiomatize the Banzhaf value.[15] In contrast, we propose a v-f reduction that is based on the same basic axioms used in our previous characterizations, to which we add a new axiom that we call the "maximum ignorance" property:

Axiom 11. *A v-f reduction* Ψ *satisfies* ***maximum ignorance*** *if for all* $N \in \mathcal{P}_{\mathit{fin}}(\mathcal{U})$ *such that* $|N| \geq 2$*, all* $i \in N$*, all* $\alpha \in \mathbb{R}$*, and all* $S \subseteq N \setminus \{i\}$*,*

$$\Psi_{N(N\setminus\{i\})}(\alpha u_N)(S) = \frac{\alpha}{2} u_N(S \cup \{i\}).$$

The maximum ignorance property takes the view that when player i is removed from the scene, he is still able to exert influence on the rest of the players, but his influence is uncertain. The resulting reduced game is a game of the remaining players contingent on the removed player's behavior. However, unlike for instance the *HM* v-f reduction, the model analyst is totally ignorant of the removed players' behavior. So the predicted distribution should be the one with the maximum entropy, which is, player i independently chooses to join or leave with equal probability (for an introduction to the principle of maximum entropy, see e.g., chapter 11 of Jaynes, 2003). Then $(N', \Psi_{NN'}(u_N))$ can be interpreted as the resulting expected game.

The reduction that we propose is given in the next example. We call it the Banzhaf v-f reduction.

Example 6. *We define the* ***Banzhaf v-f reduction*** Ψ^{Ban} *by:*

$$\Psi^{Ban}_{NN'}(v)(S) \equiv \sum_{T \subseteq N \setminus N'} \frac{1}{2^{n-n'}} [v(S \cup T) - v(T)], \tag{1.5.5}$$

for all $S, N', N \in \mathcal{P}_{\mathit{fin}}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and all* $(N, v) \in \mathcal{G}^N$*.*

---

[15] The reduced game Ψ^{φ} proposed by Dragan (1996) is implicitly defined as follows: for all $S, N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subsetneq N$ and all $(N, v) \in \mathcal{G}^N$,

$$\sum_{i \in S} Ban_i(S, \Psi^{\varphi}_{NN'}(v)|_S) \equiv \sum_{i \in S \cup (N \setminus N')} Ban_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')}) - \sum_{i \in N \setminus N'} \varphi_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')}). \tag{1.5.4}$$

We can interpret the Banzhaf v-f reduction as follows. Consider a game $(N, v) \in \mathcal{G}^N$ that is reduced to be played by players in $N' \subseteq N$. The players in the coalition $S \subseteq N'$ can collaborate with any subset T of the set of removed players $N \setminus N'$. Then, they obtain a worth of $v(S \cup T)$ but they have to compensate the players in T with the worth of their coalition $v(T)$. Each of the possible coalitions $T \subseteq N \setminus N'$ has the same probability of being available. Therefore, the worth of a coalition $S \subseteq N'$ in $(N', \Psi^{Ban}_{NN'}(v))$ is the simple average of the marginal worth that S can add to the worth of the coalitions $T \subseteq N \setminus N'$.

Theorem 6 provides an axiomatic characterization of Ψ^{Ban}. It also postulates that $\varphi^{\Psi^{Ban}} = Ban$.

Theorem 6. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and the maximum ignorance property if and only if*

$$\Psi = \Psi^{Ban}.$$

Moreover, Ψ^{Ban} induces the Banzhaf value.

1.5.6 The axioms of anonymity and linearity

In this subsection, we discuss two additional properties that v-f reductions can satisfy: anonymity and linearity.

One sensible property that many values satisfy is anonymity, which requires that the players' names are irrelevant for the value they obtain in the game. We can propose an axiom for v-f reductions in the same spirit. The axiom of anonymity for v-f reductions requires that the name of the players does not matter in the reduction of the game. To formally define the axiom, let $\sigma : N \to \mathcal{U}$ be an injection. For $(N, v) \in \mathcal{G}^N$, we define $\sigma v \in \mathcal{G}^{\sigma[N]}$ by $\sigma v(T) \equiv v(\sigma^{-1}(T))$ for all $T \subseteq \sigma[N]$.

Axiom 12. *A v-f reduction Ψ satisfies* ***anonymity*** *if for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$, all $(N, v) \in \mathcal{G}^N$, and all injections $\sigma : N \to \mathcal{U}$, then*

$$\Psi_{\sigma[N]\sigma[N']}(\sigma v)(\sigma[S]) = \Psi_{NN'}(v)(S). \tag{1.5.6}$$

Anonymity of a v-f reduction implies that the contribution of a player in the reduced game depends not on his name but on his contribution in the initial game. It also implies that if two players in the initial game are identical in terms of their contribution, then the reduced game if one of them is removed should be the same if the other is removed.

We notice that although anonymity refers to the way games are reduced according to v-f reductions, it has implications for the prescribed payoff that equal players obtain in the induced value. In fact, if we substitute both N' and S with $\{i\}$ in Axiom 12, we have $\Psi_{N\{i\}}(v)(\{i\}) = \Psi_{\sigma[N]\{\sigma(i)\}}(\sigma v)(\{\sigma(i)\})$, which is, $\varphi_i^{\Psi}(N, v) = \varphi_{\sigma(i)}^{\Psi}(\sigma[N], \sigma v)$. Therefore, anonymity of a v-f reduction Ψ implies anonymity of its induced value φ^{Ψ}. We state this result in Proposition 6.

Proposition 6. *If a v-f reduction Ψ satisfies anonymity, then the induced value φ^{Ψ} satisfies anonymity as well.*

None of the axioms used in the characterizations provided in Theorems 1 to 6 is related to the idea of anonymity. However, Proposition 7, whose proof is immediate, shows that all of the v-f reductions characterized in our paper satisfy the axiom of anonymity.

Proposition 7. *The v-f reductions Ψ^{Sub}, Ψ^{HM}, Ψ^{ONHF}, Ψ^{PW}, Ψ^{PW^*}, and Ψ^{Ban} satisfy anonymity.*

Given that all the characterizations use the axioms of additivity, null player out, permanent null player, and path independence, one may think that these axioms imply anonymity. Moreover, like the aforementioned axioms, we can easily check that anonymity is a self-dual property. However, Example 7 satisfies our four basic properties although it does not satisfy anonymity.

Example 7. *Given $X \subseteq \mathcal{U}$, the v-f reduction Ψ^X is defined by*

$$\Psi^X_{NN'}(v)(S) \equiv \sum_{i\in S} Sh_i(S \cup ((N \setminus N') \cap X), v|_{S\cup((N\setminus N')\cap X)}) - \sum_{i\in S} Sh_i(N' \cup ((N \setminus N') \cap X), v|_{N'\cup((N\setminus N')\cap X)}) + \sum_{i\in S} Sh_i(N, v), \tag{1.5.7}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and all $(N, v) \in \mathcal{G}^N$.

Proposition 8. *The v-f reduction Ψ^X satisfies additivity, null player out, permanent null player, and path independence for any $X \subseteq \mathcal{U}$. However, it does not satisfy anonymity.*

Finally, let us mention that all the v-f reductions that we have characterized also satisfy **linearity**, that is, they are **homogeneous** (see Axiom 3). As anonymity, homogeneity is not implied by our four basic axioms. The construction of an example requires the use of a Hamel basis and we provide it in the Appendix.

1.5.7 A comment on non-additive v-f reductions

Like the classical axiomatization of the Shapley value, our axiomatizations of v-f reductions rely on additivity. It is immediate that the value induced by an additive v-f reduction is necessarily additive as well. Hence, all the values characterized by v-f reductions that satisfy our basic axioms are additive, as is the case for the Shapley, the stand-alone, and the Banzhaf values.

However, we can also consider non-additive v-f reductions. Such reductions can induce non-additive values, such as the prenucleolus $\mathcal{PN}$ (Schmeidler, 1969).[16] We illustrate here this possibility by adapting the Davis-Mascher (DM) reduced game, which allows characterizing the prenucleolus in terms of consistency (Sobolev, 1975). We start by formally defining the prenucleolus.

Denote by $X(N,v)$ the set of preimputations of the game (N,v), that is, $x \in X(N,v)$ if $x \in \mathbb{R}^N$ and $\sum_{i\in N} x_i = v(N)$. For each preimputation $x \in X(N,v)$ and each coalition $S \subseteq N$, we denote $e_S(x) \equiv (S) - v(S)$ the "excess" of coalition S at x. Also, we denote $e(x) \equiv (e_S(x))_{S\in 2^N\setminus\{N,\varnothing\}}$ the vector of excesses, where the entries are arranged in increasing order. Finally, for $x, y \in X(N,v)$, we denote by $e(x) \succ_{lx} e(y)$ if the vector $e(x)$ is lexicographically superior to $e(y)$.[17] We can now define the prenucleolus:

$$\mathcal{PN}(N,v) \equiv \{x \in X(N,v) : \nexists y \in X(N,v) \text{ s.t. } e(y) \succ_{lx} e(x)\}.$$

We construct the v-f version of the DM reduced game as we did for the HM and $ONHF$ v-f reductions: We take the original reduced game, and we substitute the generic value used in that game by the particular value that it helps to characterize. For the DM, we use the prenucleolus:

Example 8. *We define the* ***DM v-f reduction*** Ψ^{DM} *by:*

$$\Psi^{DM}_{NN'}(v)(S) \equiv \begin{cases} v(N) - \sum_{i\in N\setminus N'} \mathcal{PN}_i(N,v) & \text{if } S = N' \subsetneq N, \\ \max_{T\subseteq N\setminus N'} v(S\cup T) - \sum_{i\in T} \mathcal{PN}_i(N,v) & \text{if } \varnothing \neq S \subsetneq N' \subsetneq N, \\ v(S) & \text{if } N' = N, \\ 0 & \text{if } S = \varnothing, \end{cases}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and all $(N,v) \in \mathcal{G}^N$.

[16] Schmeidler (1969) defines the nucleolus for 0-monotonic TU games. Throughout our paper, we focus on v-f reductions defined on an unrestricted domain, so we consider the prenucleolus rather than the nucleolus. The latter is empty for those TU games with an empty set of imputations.

[17] That is, there exists $t' \in \{1, 2, ..., 2^n - 2\}$ such that $e_t(x) \geq e_t(y)$, for $t = 1, \dots, t'$ and $e_{t'}(x) > e_{t'}(y)$.

The v-f reduction Ψ^{DM} is not additive and, as one can expect, it induces the prenucleolus.[18] Moreover, it satisfies null player out and permanent null player, as well as anonymity. However, we do not have a characterization of the *DM* v-f reduction.

1.6 New characterizations of the Banzhaf and the stand-alone values

We have based some of our examples of v-f reductions on existing reduced games, which were introduced to study the internal consistency of values. In this section, we consider the reverse process. We take a v-f reduction Ψ that is defined without reference to an existing reduced game. We look for value-reductions Ψ^{φ} such that $\Psi = \Psi^{\varphi^{\Psi}}$, where $\Psi^{\varphi^{\Psi}}$ results from substituting the value φ^{Ψ} induced by the v-f reduction Ψ in Ψ^{φ}. This process may identify reduction games φ^{Ψ} that would allow the characterization of solution concepts using consistency properties (as in Hart and Mas-Colell, 1989, and Oishi et al., 2016).

We conduct such a reverse process by introducing a new reduced game that, following the terminology used in Lehrer (1988), we call the "amalgamating reduced game." We denote it by $\Psi^{A\varphi}$. It is inspired by the definition of the Banzhaf v-f reduced game Ψ^{Ban}. It satisfies that if we substitute φ for the Banzhaf value *Ban* (which is the value induced by Ψ^{Ban}) in $\Psi^{A\varphi}$, then $\Psi^{A^{Ban}} = \Psi^{Ban}$ (see the Appendix).

The definition of $\Psi^{A\varphi}$ requires some premiliminaries. For $(N, v) \in \mathcal{G}^N$ and $S \in 2^N \setminus \{\varnothing\}$, we may "amalgamate" the coalition S into one player and denoted him by $\bar{S}$. Formally, we define the S-amalgamated game $((N \setminus S) \cup \{\bar{S}\}, v_S)$ (Lehrer, 1988) by:

$$v_S(T) \equiv \begin{cases} v((T \setminus \{\bar{S}\}) \cup S) & \text{if } \bar{S} \in T, \\ v(T) & \text{otherwise.} \end{cases}$$

Then we define the amalgamating reduced game (*A* reduction, for short) $\Psi^{A\varphi}$ as follows:

Definition 11. *Given a value φ, the A reduction $\Psi^{A\varphi}$ is defined by:*

$$\Psi^{A\varphi}_{NN'}(v)(S) \equiv \begin{cases} \varphi_{\bar{S}}((N \setminus N') \cup \{\bar{S}\}, (v|_{S \cup (N \setminus N')})_S) & \textit{if } S \in 2^{N'} \setminus \{\varnothing\}, \\ 0 & \textit{if } S = \varnothing, \end{cases}$$

[18] Indeed, $\varphi^{\Psi^{DM}}_{N\{i\}}(N, v) = \Psi^{DM}_{N\{i\}}(v)(\{i\}) = v(N) - \sum_{j \in N \setminus \{i\}} \mathcal{PN}_j(N, v) = \mathcal{PN}_i(N, v)$, where the last equality follows from efficiency of the prenucleolus.

for all $S, N', N \in \mathcal{P}_{\mathit{fin}}(\mathcal{U})$ *such that* $S \subseteq N' \subsetneq N$ *and all* $(N, v) \in \mathcal{G}^N$*.*

To interpret the A reduction, consider a value φ, a TU game (N, v), and a coalition $N' \subseteq N$. Similar to the *HM* reduced game, every non-empty coalition $S \subseteq N'$ collaborates with all removed players in $N \setminus N'$, and players in $N' \setminus S$ exert no influence. However, in the A reduction, the coalition S is treated as an individual player. Moreover, while in the *HM* reduced game, the worth of S is the residue after paying up those players in $N \setminus N'$, the worth of S in the amalgamating reduced game is what is due for S as a single player according to φ.

The reduced game $\Psi^{A\varphi}$ allows the characterization of the Banzhaf value in a parallel manner as Hart and Mas-Colell (1989) and Oishi et al. (2016) characterize the Shapley value: The Banzhaf value is the only value that is consistent relative to the A reduction and that is standard for two-player games. We establish this result in Theorem 7.

Theorem 7. *Let* φ *be a solution. Then:*

(i) φ *is consistent relative to* $\Psi^{A\varphi}$*; and*

(ii) φ *is standard for two-player games;*

if and only if φ *is the Banzhaf value.*

The Banzhaf value is not the only value that is consistent relative to the A reduction. The stand-alone value is also consistent relative to the amalgamating reduced game. Interestingly, it can also be characterized by consistency plus the behavior of the value in the two-player games.

Theorem 8. *Let* φ *be a solution. Then:*

(i) φ *is consistent relative to* $\Psi^{A\varphi}$*; and*

(ii) φ *coincides with the stand-alone value for two-player games;*

if and only if φ *is the stand-alone value.*

Theorems 7 and 8 provide new characterizations of the Banzhaf and the stand-alone values. They also highlight that, once the "right" consistency requirement is applied, they only differ in their prescriptions for two-player games.

1.7 Logical independence

In this section, we show that our characterization of the *HM* v-f reduction is minimal in the sense that none of the characterizing properties can be deduced from the rest. Each time we leave out one axiom, we can find examples of v-f reductions satisfying the remaining four properties.

First, as we have already shown in Theorems 1, 3 and 4, the subgame v-f reduction, the *ONHF* v-f reduction and the *PW* v-f reduction satisfy all the axioms but 1-addition property. Examples 9, 10, 11, and 12 show that the axioms of null player out, permanent null player, additivity, and path independence are not redundant either.

Example 9 (No null player out). *Let $\Psi^{\neg NPO}$ be the v-f reduction defined by:*

$$\Psi_{NN'}^{\neg NPO}(v)(S) \equiv 0,$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and all $(N, v) \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg NPO}$ satisfies additivity, permanent null player, path independence, and 1-addition invariance, but it does not satisfy null player out.

Example 10 (No permanent null player). *Let $\Psi^{\neg PNP}$ be the v-f reduction defined by:*

$$\Psi_{NN'}^{\neg PNP}(v)(S) \equiv \begin{cases} 0 & S = \varnothing \\ v(S \cup (N \setminus N')) & \text{otherwise,} \end{cases}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and all $(N, v) \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg PNP}$ satisfies additivity, null player out, path independence, and 1-addition invariance, but it does not satisfy permanent null player.

Example 11 (No additivity). *Let $\Psi^{\neg A}$ be the v-f reduction defined by:*

$$\Psi_{NN'}^{\neg A}(v) \equiv \begin{cases} \Psi_{NN'}^{HM}(v) & \text{if } Sh_i(N,v) = 0 \text{ for all } i \in N \setminus N' \\ \Psi_{NN'}^{\neg NPO}(v) & \text{otherwise,} \end{cases}$$

for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$ and all $(N, v) \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg A}$ satisfies null player out, permanent null player, path independence, and 1-addition invariance, but it does not satisfy additivity.

Example 12 (No path independence). *Let $\Psi^{\neg PI}$ be the v-f reduction defined by:*

$$\Psi_{NN'}^{\neg PI}(v)(S) \equiv \begin{cases} 2v(S) & \text{if } n = n' = 1 \\ \Psi_{NN'}^{HM}(v)(S) & \text{otherwise,} \end{cases}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and all* $(N, v) \in \mathcal{G}^N$*. The v-f reduction* $\Psi^{\neg PI}$ *satisfies additivity, null player out, permanent null player, and* 1*-addition invariance, but it does not satisfy path independence.*

1.8 Conclusion

In this paper, we introduce the notion of the value-free reduction of a coalitional game with transferable utility. A v-f reduction of a game describes the change in the worth of the coalitions in a TU game when some players leave the game.[19] Thus, this new concept allows us to study TU games from a new perspective, focusing on the properties that a v-f reduction may or may not satisfy. A v-f reduction induces a value. One may say that the value somehow reflects the properties of the v-f reductions that induce it.

We consider additive v-f reductions that are path independent and satisfy properties that indicate that null players must still be treated as null players when any such reduction is applied. These properties by themselves do not pin down a unique v-f reduction. Moreover, they do not identify a unique value induced by the reductions either. We define v-f reductions that satisfy all the previous properties and induce either the Shapley value, or the Banzhaf value, or the stand-alone value.

To characterize each of the examples of v-f reductions that we have defined, we use an additional axiom that ensures that the players remaining in the reduced game keep the same strategic perspective as in the initial game after a change in the worth of some particular coalitions. These are invariance properties. The exercises suggest that the Shapley value is a resilient value as it is induced by several v-f reductions, each characterized by a different invariance axiom. A duality theory for v-f reductions, which is also developed in this paper, helps in the proof of some of the characterizations. Moreover, we show that that the duality theory can be helpful in the identification of new mechanisms that implement specific values.

We also show that our new approach is a useful tool to provide new characterizations of values in terms of consistency. In this paper, we provide new characterizations of the Banzhaf and the stand-alone values.

[19] Concerning this depiction, v-f reductions may be viewed as generalizations of the subgame operator by allowing the players who leave the game to influence the remaining players. For concepts where subgame plays a role, such as population monotonicity (Sprumont, 1990) and projection consistency (Funaki and Yamato, 2001), one can define and study versions where the subgame is replaced with a distinct v-f reduction.

1.9 Appendix

Proof of Proposition 1. We prove that the value φ^{Ψ} induced by a path-independent v-f reduction Ψ is consistent relative to Ψ. For a given N we have $\Psi_{N'\{i\}} \circ \Psi_{NN'} = \Psi_{N\{i\}}$ for all $N' \subseteq N$ and all $i \in N'$, by path independence. Therefore, for any $(N, v) \in \mathcal{G}^N$, given that $(N', \Psi_{NN'}(v)) \in \mathcal{G}^{N'}$, we have $\varphi_i^{\Psi}\big(N', \Psi_{NN'}(v)\big) = \Psi_{N'\{i\}}\big(\Psi_{NN'}(v)\big)(\{i\}) = \Psi_{N'\{i\}} \circ \Psi_{NN'}(v)(\{i\}) = \Psi_{N\{i\}}(v)(\{i\}) = \varphi_i^{\Psi}(N, v)$. Hence, φ^{Ψ} is consistent relative to Ψ. □

To prove Proposition 3, as well as Propositions 4 and 5 later, several properties of the mapping $v \mapsto v^*$ are useful, which we state in Lemma 1:

Lemma 1. *The mapping $v \mapsto v^*$ is additive. Moreover, if $i \in N$ is a null player in $(N, v) \in \mathcal{G}^N$, then player i is also a null player in (N, v^*).*

Proof of Lemma 1. We check that $v \mapsto v^*$ is additive: for all $(N, v), (N, w) \in \mathcal{G}^N$ and all $S \subseteq N$, then $(v+w)^*(S) = (v+w)(N) - (v+w)(N \setminus S) = (v(N)+w(N)) - (v(N \setminus S) + w(N \setminus S)) = (v(N) - v(N \setminus S)) + (w(N) - w(N \setminus S)) = v^*(S) + w^*(S)$.

To see that if i is a null player in (N, v), then i is also a null player in (N, v^*) we have that for all $S \subseteq N \setminus \{i\}$, $v^*(S \cup \{i\}) - v^*(S) = (v(N) - v(N \setminus (S \cup \{i\}))) - (v(N) - v(N \setminus S)) = v(N \setminus S) - v(N \setminus (S \cup \{i\})) = 0$. □

Proof of Proposition 3. To verify that additivity is self-dual, we show that the mapping $v \mapsto \Psi^*_{NN'}(v)(S)$ is additive if the mapping $v \mapsto \Psi_{NN'}(v)(S)$ is additive. Indeed, $\Psi^*_{NN'}(v+w)(S) = \big(\Psi_{NN'}((v+w)^*)\big)^*(S) = \big(\Psi_{NN'}(v^* + w^*)\big)^*(S) = \big(\Psi_{NN'}(v^*) + \Psi_{NN'}(w^*)\big)^*(S) = \big(\Psi_{NN'}(v^*)\big)^*(S) + \big(\Psi_{NN'}(w^*)\big)^*(S) = \Psi^*_{NN'}(v)(S) + \Psi^*_{NN'}(w)(S)$, where the first equality follows from Definition 8, the second and fourth from the additivity of $v \mapsto v^*$ (Lemma 1 in the Appendix), and the third from the additivity of Ψ. Therefore, additivity is self-dual.

We now check that null player out is self-dual. We show that if Ψ satisfies the null player out axiom, then $\Psi^*_{N(N\setminus\{i\})}(v)(S) = v(S)$ for all $S \subseteq N \setminus \{i\}$ if $i \in N$ is a null player in (N, v). Indeed, $\Psi^*_{N(N\setminus\{i\})}(v)(S) = \big(\Psi_{N(N\setminus\{i\})}(v^*)\big)^*(S) = (v^* \mid_{N\setminus\{i\}})^*(S) = v^* \mid_{N\setminus\{i\}} (N\setminus\{i\}) - v^* \mid_{N\setminus\{i\}} ((N\setminus\{i\})\setminus S) = v^*(N\setminus\{i\}) - v^*(N\setminus(S\cup\{i\})) = v^*(N) - v^*(N \setminus S) = v(S)$, where the first equality follows from Definition 8, the second one holds because i is a null player in (N, v^*) according to Lemma 1, the third from the definition of the dual of a game, and the penultimate equality follows again from the fact that i is a null player in (N, v^*). Therefore, null player out is self-dual.

We verify that the permanent null player property is self-dual by proving that if Ψ satisfies this property and $i \in N'$ is a null player in (N, v), then i is a null player in $(N', \Psi^*_{NN'}(v))$ as well. Let $i \in N'$ be a null player in (N, v). Then, from Lemma 1, $i \in N'$ is a null player in (N, v^*) and, by the permanent null player property of Ψ, he is also a null player in $(N', \Psi_{NN'}(v^*))$. Using Lemma 1 again, i is a null player in $(N', \big(\Psi_{NN'}(v^*)\big)^*)$, that is, in $(N', \Psi^*_{NN'}(v))$. Therefore, the permanent null player property is self-dual.

Finally, we prove that path independence is self-dual by proving $\Psi^*_{N_2N_3}(\Psi^*_{N_1N_2}(v)) = \Psi^*_{N_1N_3}(v)$ if Ψ is path independent: $\Psi^*_{N_2N_3}(\Psi^*_{N_1N_2}(v)) = \big(\Psi_{N_2N_3}(\big(\big(\Psi_{N_1N_2}(v^*)\big)^*\big)^*)\big)^* = \big(\Psi_{N_2N_3}(\Psi_{N_1N_2}(v^*))\big)^* = \big(\Psi_{N_1N_3}(v^*)\big)^* = \Psi^*_{N_1N_3}(v)$, where the first and last equalities follow from Definition 8, the second from $v^{**} = v$, and the third from the assumption of the path-independence of Ψ. Therefore, path independence is self-dual. □

Proof of Remark 1. Given $N \in \mathcal{P}_{fin}(\mathcal{U})$ and $(N, v) \in \mathcal{G}^N$, take any $i \in \mathcal{U} \setminus N$. Define $(N \cup \{i\}, w) \in \mathcal{G}^{N \cup \{i\}}$ by $w(S) \equiv v(S \setminus \{i\})$ for all $S \subseteq N \cup \{i\}$. Notice that player i is a null player in $(N \cup \{i\}, w)$ and that the subgame of $(N \cup \{i\}, w)$ restricted to N is (N, v). Then for any v-f reduction Ψ satisfying null player out and path independence, $\Psi_{NN}(v) = \Psi_{NN}(\Psi_{(N\cup\{i\})N}(w)) = \Psi_{(N\cup\{i\})N}(w) = v$, where the first and the third equality follow from null player out and the second from path independence. Therefore, Ψ_{NN} must be an identity function if Ψ satisfies null player out and path independence. □

Since every v-f reduction we will present satisfies null player out and path independence, we will not repeat the property established in Remark 1 in the proof of their corresponding theorems below.

Proof of Theorem 1. It is immediate that the subgame v-f reduction satisfies all the stated properties.

We now prove that if the v-f reduction Ψ satisfies the five properties, then $\Psi = \Psi^{Sub}$. Notice first that, under path independence, it suffices to show the equality restricted to one-player operators $\big(\Psi_{N(N\setminus\{i\})}\big)$, for all $N \in \mathcal{P}_{fin}(\mathcal{U})$ and all $i \in N$.

Second, by additivity, it suffices to establish the equality for each operator $\Psi_{N(N\setminus\{i\})}$ restricted to the set of all scalar multiples of elements in a basis of $\mathcal{G}^N$. We choose the set of all scalar multiples of all unanimity games $\big(\alpha u_T\big)_{T\in 2^N\setminus\{\varnothing\},\alpha\in\mathbb{R}}$.

We show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \Psi^{Sub}_{N(N\setminus\{i\})}(\alpha u_T)$ for all $T \in 2^N \setminus \{\varnothing\}$, all $\alpha \in \mathbb{R}$, and all $i \in N$ by induction on n. We notice that since $\alpha u_N = w_{(n,\alpha)}$, additivity and grand-coalition invariance imply that $\Psi_{N(N\setminus\{i\})}(\alpha u_N) = \mathbf{0} = \alpha u_N \mid_{N\setminus\{i\}} = \Psi^{Sub}_{N(N\setminus\{i\})}(\alpha u_N)$. Thus, we only need to check the equality of the remaining scalar multiples of elements in the basis, i.e., $\big(\alpha u_T\big)_{T\in 2^N\setminus\{\varnothing,N\},\alpha\in\mathbb{R}}$.

Consider $N = \{i, j\}$, that is, $n = 2$. (a) When $T = \{j\}$, then $\Psi_{\{i,j\}\{j\}}(\alpha u_{\{j\}})(\{j\}) = \alpha u_{\{j\}} \mid_{\{j\}} (\{j\})$ by null player out, since i is a null player in $(\{i, j\}, \alpha u_{\{j\}})$. (b) When

$T = \{i\}$, then $\Psi_{\{i,j\}\{j\}}(\alpha u_{\{i\}})(\{j\}) = 0 = \alpha u_{\{i\}} |_{\{j\}} (\{j\})$ by permanent null player, since j is a null player in $(\{i,j\}, \alpha u_{\{i\}})$. Hence, $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \alpha u_T |_{N\setminus\{i\}} (S) = \Psi^{Sub}_{N(N\setminus\{i\})}(\alpha u_T)(S)$ for all $S \subseteq N \setminus \{i\}$, all T such that $|T| = 1$, all $\alpha \in \mathbb{R}$, and all N such that $|N| = 2$.

Now we proceed to consider any N, and suppose that the induction property holds for any set with fewer than n players. (a) When $i \notin T$ then i is a null player in $(N, \alpha u_T)$, hence $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \alpha u_T|_{N\setminus\{i\}}$ by null player out. (b) We show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \alpha u_T |_{N\setminus\{i\}} (S)$ for all $S \subseteq N \setminus \{i\}$ when $i \in T$ and $T \subsetneq N$. Take any player $j \in N \setminus T$. Then, j is a null player in $(N, \alpha u_T)$. Moreover, by the permanent null player property, j is also a null player in $(N \setminus \{i\}, \Psi_{N(N\setminus\{i\})}(\alpha u_T))$. We consider two possibilities. (b1) First, if $S \subseteq N \setminus \{i, j\}$, then

$$\begin{aligned}\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) &= \Psi_{N(N\setminus\{i\})}(\alpha u_T) |_{N\setminus\{i,j\}} (S) = \Psi_{(N\setminus\{i\})(N\setminus\{i,j\})}\big(\Psi_{N(N\setminus\{i\})}(\alpha u_T)\big)(S) \\ &= \Psi_{(N\setminus\{j\})(N\setminus\{i,j\})}\big(\Psi_{N(N\setminus\{j\})}(\alpha u_T)\big)(S) = \Psi_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T |_{N\setminus\{j\}})(S),\end{aligned} \tag{1.9.1}$$

where the first equality holds because $S \subseteq N \setminus \{i, j\}$; the second by null player out, given that j is a null player in the game $(N \setminus \{i\}, \Psi_{N(N\setminus\{i\})}(\alpha u_T))$; the third by path independence; and the fourth by null player out, given that j is a null player in $(N, \alpha u_T)$. We apply the induction argument to state that the last expression (which involves a reduction from a set of $n-1$ players) is equal to $\Psi^{Sub}_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T |_{N\setminus\{j\}})(S) = \alpha u_T |_{N\setminus\{i,j\}} (S) = \alpha u_T(S)$, where the last equality holds because $S \subseteq N \setminus \{i, j\}$. (b2) Second, if $j \in S$, then $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi_{N(N\setminus\{i\})}(\alpha u_T)(S \setminus \{j\})$ because j is a null player in $(N\setminus\{i\}, \Psi_{N(N\setminus\{i\})}(\alpha u_T))$. Now we apply equation (1.9.1) to $S \setminus \{j\}$ and, by the same argument as in (b1), $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \alpha u_T(S \setminus \{j\})$, which is equal to $\alpha u_T(S)$ since j is a null player in $(N, \alpha u_T)$.

Thus, if a v-f reduction satisfies the five properties, then it is equal to Ψ^{Sub}. □

Proof of Theorem 2. We verify the stated properties of Ψ^{HM}. First, Ψ^{HM} is the composition of three functions: the restriction operator, the Shapley value, and the summation operator. It is easy to check that the three functions are additive. Therefore, Ψ^{HM} is additive.

Second, to verify that Ψ^{HM} satisfies null player out, let $i \in N$ be a null player in $(N, v) \in \mathcal{G}^N$. Then, $\Psi^{HM}_{N(N\setminus\{i\})}(v)(S) = \sum_{j\in S} Sh_j(S \cup (N \setminus (N \setminus \{i\})), v|_{S\cup(N\setminus(N\setminus\{i\}))}) = \sum_{j\in S} Sh_j(S\cup\{i\}, v|_{S\cup\{i\}}) = v(S\cup\{i\}) - Sh_i(S\cup\{i\}, v|_{S\cup\{i\}}) = v(S\cup\{i\}) = v(S)$, where the third equality follows from the efficiency of the Shapley value, the fourth from the null player property of the Shapley value, and the fifth holds because i is a null player in (N, v).

Third, we check that Ψ^{HM} satisfies permanent null player. Let $i \in N'$ be a null player in $(N,v) \in \mathcal{G}^N$. Then, for all $S \subseteq N' \setminus \{i\}$, $D^i(\Psi^{HM}_{NN'}(v))(S) = \Psi^{HM}_{NN'}(v)(S \cup \{i\}) - \Psi^{HM}_{NN'}(v)(S) = [\sum_{j \in S \cup \{i\}} Sh_i((S \cup \{i\}) \cup (N \setminus N'), v|_{(S \cup \{i\}) \cup (N \setminus N')})] - [\sum_{j \in S} Sh_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')})] = [\sum_{j \in S} Sh_i((S \cup \{i\}) \cup (N \setminus N'), v|_{(S \cup \{i\}) \cup (N \setminus N')})] - [\sum_{j \in S} Sh_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')})] = [\sum_{j \in S} Sh_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')})] - [\sum_{j \in S} Sh_i(S \cup (N \setminus N'), v|_{S \cup (N \setminus N')})] = 0$, where the third equality follows from the null player property of the Shapley value, and the fourth from null player out of the Shapley value (see Derks and Haller, 1999).

Fourth, we prove the path independence axiom. For any $T \subseteq S$, we can write $\Psi^{HM}_{(S \cup (N \setminus N'))S}(v|_{S \cup (N \setminus N')})(T) = \sum_{i \in T} Sh_i(T \cup (N \setminus N'), v|_{T \cup (N \setminus N')}) = \Psi^{HM}_{NN'}(v)(T) = \Psi^{HM}_{NN'}(v)|_S(T)$, where the first equality holds because $v|_{S \cup (N \setminus N')}|_{T \cup (S \cup (N \setminus N')) \setminus S)} = v|_{T \cup ((S \cup (N \setminus N')) \setminus S)} = v|_{T \cup (N \setminus N')}$. Therefore:

$$\Psi^{HM}_{NN'}(v)|_S = \Psi^{HM}_{(S \cup (N \setminus N'))S}(v|_{S \cup (N \setminus N')}) \tag{1.9.2}$$

$$\Psi^{HM}_{NN'}(v)(S) = \Psi^{HM}_{(S \cup (N \setminus N'))S}(v|_{S \cup (N \setminus N')})(S). \tag{1.9.3}$$

We now claim that, given equations (1.9.2) and (1.9.3), the verification of path independence, that is, $\Psi^{HM}_{N_2N_3}(\Psi^{HM}_{N_1N_2}(v))(S) = \Psi^{HM}_{N_1N_3}(v)(S)$ for all $N_1, N_2, N_3, S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N_3 \subseteq N_2 \subseteq N_1$, is equivalent to verifying the condition only for $S = N_3$, i.e.,

$$\Psi^{HM}_{N_2N_3}(\Psi^{HM}_{N_1N_2}(v))(N_3) = \Psi^{HM}_{N_1N_3}(v)(N_3). \tag{1.9.4}$$

To prove the equivalence, we use (1.9.3), where we substitute N, N' and v by N_2, N_3 and $\Psi^{HM}_{N_1N_2}(v)$, to obtain

$$\Psi^{HM}_{N_2N_3}(\Psi^{HM}_{N_1N_2}(v))(S) = \Psi^{HM}_{(S \cup (N_2 \setminus N_3))S}(\Psi^{HM}_{N_1N_2}(v)|_{S \cup (N_2 \setminus N_3)})(S). \tag{1.9.5}$$

Similarly, we substitute N, N' and S by N_1, N_2 and $S \cup (N_2 \setminus N_3)$ in (1.9.2), to obtain

$$\Psi^{HM}_{N_1N_2}(v)|_{S \cup (N_2 \setminus N_3)} = \Psi^{HM}_{(S \cup (N_2 \setminus N_3) \cup (N_1 \setminus N_2))(S \cup (N_2 \setminus N_3))}(v|_{S \cup (N_2 \setminus N_3) \cup (N_1 \setminus N_2)}), \text{ i.e.,}$$

$$\Psi^{HM}_{N_1N_2}(v)|_{S \cup (N_2 \setminus N_3)} = \Psi^{HM}_{(S \cup (N_1 \setminus N_3))(S \cup (N_2 \setminus N_3))}(v|_{S \cup (N_1 \setminus N_3)}). \tag{1.9.6}$$

Using (1.9.6) in equation (1.9.5), we have

$$\Psi^{HM}_{N_2N_3}(\Psi^{HM}_{N_1N_2}(v))(S) = \Psi^{HM}_{(S \cup (N_2 \setminus N_3))S}(\Psi^{HM}_{(S \cup (N_1 \setminus N_3))(S \cup (N_2 \setminus N_3))}(v|_{S \cup (N_1 \setminus N_3)}))(S).$$

Then, the worth of coalition $S \subseteq N_3$ in the game resulting from two sequential reductions of (N_1, v) (from N_1 to N_2, then from N_2 to N_3) is equal to the worth of the grand coalition S in the game resulting from two reductions of $(S \cup (N_1 \setminus N_3), v\,|_{S\cup(N_1\setminus N_3)})$ (from $S \cup (N_1 \setminus N_3)$ to $S \cup (N_2 \setminus N_3)$, then from $S \cup (N_2 \setminus N_3)$ to S). This property means that it suffices to verify that the worth of the grand coalition satisfies path independence, that is, that equation (1.9.4) holds for all possible games. To prove (1.9.4), we use the definition of Ψ^{HM}:

$$\Psi^{HM}_{N_2N_3}\big(\Psi^{HM}_{N_1N_2}(v)\big)(N_3) = \sum_{i\in N_3} Sh_i(N_2, \Psi^{HM}_{N_1N_2}(v)) = \sum_{i\in N_3} Sh_i(N_1, v) = \Psi^{HM}_{N_1N_3}(v)(N_3).$$

Therefore, Ψ^{HM} is path independent.

Finally, we verify the 1-invariance property of Ψ^{HM}. The axiom of additivity implies that $\Psi^{HM}_{NN'}(u_N+w_{(1,\alpha)}) = \Psi^{HM}_{NN'}(u_N)$ if and only if $\Psi^{HM}_{NN'}(w_{(1,\alpha)}) = 0$. We show that $\Psi^{HM}_{NN'}(w_{(1,\alpha)})(S) = 0$ for all $S \subseteq N'$. By definition of Ψ^{HM}, $\Psi^{HM}_{NN'}(w_{(1,\alpha)})(S) = \sum_{i\in S} Sh_i(S \cup (N \setminus N'), w_{(1,\alpha)}|_{S\cup(N\setminus N')})$. Notice that $(S \cup (N \setminus N'), w_{(1,\alpha)}|_{S\cup(N\setminus N')}) \in \mathcal{G}^{S\cup(N\setminus N')}$ is a game where each player is symmetric with each other. Then, the Shapley value prescribes an equal share of the worth of the grand coalition $S\cup(N\setminus N')$. Thus, $\sum_{i\in S} Sh_i(S\cup(N\setminus N'), w_{(1,\alpha)}|_{S\cup(N\setminus N')}) = \sum_{i\in S} \frac{1}{|S\cup(N\setminus N')|} w_{(1,\alpha)}(S\cup (N \setminus N')) = 0$. Therefore, the HM v-f reduction satisfies the 1-addition invariance property.

To show the reverse implication of the theorem, we first prove the following lemma:

Lemma 2. *For all $N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $|N| > 2$, the set $\{u_T \mid T \subsetneq N, T \neq \varnothing\} \cup \{w_{(1,1)}\}$ forms a basis of $\mathcal{G}^N$.*

Proof of Lemma 2. Take any $N \in \mathcal{P}_{fin}(\mathcal{U})$. To prove Lemma 2, we start by showing the following equality between games in $\mathcal{G}^N$:

$$(-1)^n n u_N = -w_{(1,1)} + \sum_{S\in 2^N\setminus\{\varnothing,N\}} (-1)^{s-1} s u_S. \tag{1.9.7}$$

We show that the two functions in equation (1.9.7) are equal when evaluated at any $T \subseteq N$, by considering three different cases: (a) If $T \subsetneq N$ and $|T| = 1$, then $-w_{(1,1)}(T) + \sum_{S\in 2^N\setminus\{\varnothing,N\}} (-1)^{s-1} s u_S(T) = -1 + u_T(T) = 0 = u_N(T) = (-1)^n n u_N(T)$. For the other two cases, we use the following formula:

$$\sum_{S\in 2^T\setminus\{\varnothing\}} s(-1)^{s-1} = 0, \tag{1.9.8}$$

for any T such that $|T| > 1$.[20] Then, (b) for $T \subsetneq N$ such that $|T| > 1$, we can write: $-w_{(1,1)}(T) + \sum_{S \in 2^N \setminus \{\varnothing, N\}} (-1)^{s-1} s u_S(T) = \sum_{S \in 2^T \setminus \{\varnothing\}} s(-1)^{s-1} = 0 = u_N(T) = (-1)^n n u_N(T)$. Finally, (c) for $T = N$, $-w_{(1,1)}(T) + \sum_{S \in 2^N \setminus \{\varnothing, N\}} (-1)^{s-1} s u_S(T) = \sum_{S \in 2^N \setminus \{\varnothing, N\}} (-1)^{s-1} s = (-1)^n n + \sum_{S \in 2^N \setminus \{\varnothing\}} (-1)^{s-1} s = (-1)^n n = (-1)^n n u_N(T)$.

Given that equation (1.9.7) holds and the set $\{u_T \mid T \subseteq N, T \neq \varnothing\}$ forms a basis of $\mathcal{G}^N$, then the set resulting from replacing u_N with $w_{(1,1)}$ in this basis spans $\mathcal{G}^N$, which proves Lemma 2. □

We now continue with the reverse implication of Theorem 2. We prove that $\Psi = \Psi^{HM}$ if the v-f reduction Ψ satisfies the five properties, using the same procedure as in the proof of Theorem 1. Because of path independence and additivity, it suffices to show that $\Psi_{N(N\setminus\{i\})}(v) = \Psi^{HM}_{N(N\setminus\{i\})}(v)$ for all $i \in N$ and all $v \in \{\alpha u_T \mid T \subsetneq N, T \neq \varnothing, \alpha \in \mathbb{R}\} \cup \{w_{(1,\alpha)} \mid \alpha \in \mathbb{R}\}$ (see Lemma 2).

First, if $v = w_{(1,\alpha)}$, then additivity and 1-addition invariance imply $\Psi_{N(N\setminus\{i\})}(w_{(1,\alpha)}) = \mathbf{0} = \Psi^{HM}_{N(N\setminus\{i\})}(w_{(1,\alpha)})$ for all $N \in \mathcal{P}_{fin}(\mathcal{U})$ for all $\alpha \in \mathbb{R}$ and all $i \in N$.

Second, we show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)$ for all $T \in 2^N \setminus \{\varnothing, N\}$, all $\alpha \in \mathbb{R}$, and all $i \in N$ by induction on n.

For N such that $n = 2$, the proof is identical to that of Theorem 1 since Ψ^{HM} and Ψ^{Sub} coincide for the proper subsets T of N and we did not use grand-coalition invariance in that part of the proof.

Consider now any N and suppose that the induction property holds for any set with fewer than n players. (a) When $i \notin T$ then i is a null player in $(N, \alpha u_T)$, hence $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \alpha u_T|_{N\setminus\{i\}} = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)$ because both Ψ and Ψ^{HM} satisfy null player out. (b) When $i \in T$ and $T \subsetneq N$, take any $j \in N \setminus T$. Player j is a null player in $(N, \alpha u_T)$ and, under the permanent null player property, also in $(N \setminus \{i\}, \Psi_{N(N\setminus\{i\})}(\alpha u_T))$. Therefore, (b1) if $S \subseteq N \setminus \{i, j\}$, then equation (1.9.1) holds by the same arguments as in the proof of Theorem 1. Using the induction argument, we have $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T|_{N\setminus\{j\}})(S) = \Psi^{HM}_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T|_{N\setminus\{j\}})(S)$. Since j is a null player in $(N, \alpha u_T)$ and Ψ^{HM} satisfies null player out and path independence, we have $\Psi^{HM}_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T|_{N\setminus\{j\}})(S) = \Psi^{HM}_{(N\setminus\{j\})(N\setminus\{i,j\})}\big(\Psi^{HM}_{N(N\setminus\{j\})}(\alpha u_T)\big)(S) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)(S)$. (b2) If $j \in S$, then $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi_{N(N\setminus\{i\})}(\alpha u_T)(S \setminus \{j\})$ because j is a null player in $(N \setminus \{i\}, \Psi_{N(N\setminus\{i\})}(\alpha u_T))$. Now we can apply equation (1.9.1) to $S \setminus \{j\}$ and, by the same argument as in (b1), $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)(S \setminus \{j\}) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)(S)$, where the last equality holds because j is a null player in $(N, \alpha u_T)$.

[20] We check that (1.9.8) holds: $\sum_{S \in 2^T \setminus \{\varnothing\}} s(-1)^{s-1} = \left[\sum_{S \in 2^T \setminus \{\varnothing\}} s x^{s-1}\right]_{x=-1} = \left[\sum_{S \in 2^T \setminus \{\varnothing\}} \frac{dx^s}{dx}\right]_{x=-1} = \left[\frac{d(\sum_{S \in 2^T \setminus \{\varnothing\}} x^s)}{dx}\right]_{x=-1} = \left[\frac{d(\sum_{s=1}^t \binom{t}{s} x^s)}{dx}\right]_{x=-1} = \left[\frac{d((1+x)^t - 1)}{dx}\right]_{x=-1} = \left[t(1+x)^{t-1}\right]_{x=-1} = 0$.

Therefore, if a v-f reduction satisfies the five properties, then it is equal to Ψ^{HM}.

Finally, we notice that $\Psi^{HM}_{N\{i\}}(v)(\{i\}) = Sh_i(\{i\} \cup (N \setminus \{i\}), v|_{\{i\}\cup(N\setminus\{i\})}) = Sh_i(v)$ for all $(N, v) \in \mathcal{G}^N$ and all $i \in N$. Therefore, Ψ^{HM} induces the Shapley value. □

Proof of Corollary 2. We prove that Ψ^{ONHF} is the dual of Ψ^{HM}. Indeed,

$$\begin{aligned}
\left(\Psi^{ONHF}_{NN'}(v^*)\right)^*(S) =& \Psi^{ONHF}_{NN'}(v^*)(N') - \Psi^{ONHF}_{NN'}(v^*)(N' \setminus S) \\
=& \sum_{i \in N'} Sh_i(N, v^*) - \sum_{i \in N' \setminus N'} Sh_i(N \setminus N', (v^*)^{N'}) - \sum_{i \in N'} Sh_i(N, v^*) \\
&+ \sum_{i \in N' \setminus (N' \setminus S)} Sh_i(N \setminus (N' \setminus S), (v^*)^{N' \setminus S}) \\
=& \sum_{i \in S} Sh_i(N \setminus (N' \setminus S), (v^*)^{N' \setminus S}) = \sum_{i \in S} Sh_i(S \cup (N \setminus N'), (v\,|_{S\cup(N\setminus N')})^*) \\
=& \sum_{i \in S} Sh_i(S \cup (N \setminus N'), v\,|_{S\cup(N\setminus N')}),
\end{aligned}$$

where the first equality follows the definition of a dual game, the second one from the defining equation (1.5.1) of Ψ^{ONHF}, and the last equality from the self-duality of the Shapley value. To check the fourth equality, notice that, for all $T \subseteq S \cup (N \setminus N')$, on the one hand, $v^{*N' \setminus S}(T) = v^*(T \cup (N' \setminus S)) - v^*(N' \setminus S) = [v(N) - v(N \setminus (T \cup (N' \setminus S)))] - [v(N) - v(N \setminus (N' \setminus S))] = v(N \setminus (N' \setminus S)) - v(N \setminus (T \cup (N' \setminus S))) = v(S \cup (N \setminus N')) - v((S \cup (N \setminus N')) \setminus T)$; on the other hand, $(v\,|_{S\cup(N\setminus N')})^*(T) = v\,|_{S\cup(N\setminus N')}(S \cup (N \setminus N')) - v\,|_{S\cup(N\setminus N')}((S \cup (N \setminus N')) \setminus T) = v(S \cup (N \setminus N')) - v((S \cup (N \setminus N')) \setminus T)$. Thus $v^{*N' \setminus S} = (v\,|_{S\cup(N\setminus N')})^*$. □

Proof of Proposition 4. Let Ψ be a v-f reduction that satisfies 1-addition invariance. Our aim is to verify that Ψ^* satisfies $(n-1)$-addition invariance.

We first show that the dual of the game $(N, w_{(n-1,\alpha)})$ is $(N, w_{(1,-\alpha)})$. Indeed, $w^*_{(n-1,\alpha)}(S) = w_{(n-1,\alpha)}(N) - w_{(n-1,\alpha)}(N \setminus S) = -w_{(n-1,\alpha)}(N \setminus S) = w_{(1,-\alpha)}(S)$ for all $S \subseteq N$.

Then, for any $\alpha \in \mathbb{R}$, we have $\Psi^*_{NN'}(v + w_{(n-1,\alpha)}) = \left(\Psi_{NN'}((v + w_{(n-1,\alpha)})^*)\right)^* = \left(\Psi_{NN'}(v^* + w^*_{(n-1,\alpha)})\right)^* = \left(\Psi_{NN'}(v^* + w_{(1,-\alpha)})\right)^* = \left(\Psi_{NN'}(v^*)\right)^* = \Psi^*_{NN'}(v)$, where the first and last equalities follow from Definition 8, the second from the additivity of $v \mapsto v^*$ (see Lemma 1), the third equality follows from the fact that the dual of $w_{(n-1,\alpha)}$ is $w_{(1,-\alpha)}$, and the fourth from 1-addition invariance of Ψ. Therefore, $(n-1)$-addition invariance is dual to 1-addition invariance. □

Proof of Theorem 3. The *ONHF* v-f reduction is dual to the *HM* v-f reduction. Then, by Proposition 3, Ψ^{ONHF} satisfies additivity, null player out, permanent null player, and path independence, because they are self-dual properties and Ψ^{HM} satisfies them. Similarly, Ψ^{ONHF} satisfies $(n-1)$-invariance, which is dual to 1-addition invariance (see Proposition 4), because Ψ^{HM} satisfies 1-addition invariance.

For the other direction, consider a v-f reduction Ψ satisfying all the stated axioms. Then, the dual Ψ^* of Ψ satisfies all the axioms stated in Theorem 2, which implies $\Psi^* = \Psi^{HM}$. Hence, the dual v-f reductions of Ψ^* and Ψ^{HM}, i.e., Ψ and Ψ^{ONHF}, must coincide, as we wanted to prove.

Finally, Corollary 1 implies that Ψ^{ONHF} induces the Shapley value since it is a self-dual value. □

Proof of Theorem 4. First, we verify that Ψ^{PW} satisfies all the stated properties. It is linear and hence additive, because it is the composition of linear functions.

To show path independence, linearity ensures that it suffices to verify that the unanimity games satisfy the property. Consider any $T \in 2^N \setminus \{\varnothing\}$, then $\Psi^{PW}_{NN'}(u_T)(S) = u_T(S) - \sum_{i\in S} Sh_i(N', u_T\,|_{N'}) + \sum_{i\in S} Sh_i(N, u_T) = u_T\,|_{N'}\,(S) - \sum_{i\in S} Sh_i(N', u_T\,|_{N'}) + \frac{|T\cap S|}{t}$. Notice that $u_T\,|_{N'} = \mathbf{0}$ if $T \nsubseteq N'$ and $\sum_{i\in S} Sh_i(N', u_T\,|_{N'}) = \frac{|T\cap S|}{t}$ if $T \subseteq N'$. Thus we have, for all $S \subseteq N'$,

$$\Psi^{PW}_{NN'}(u_T)(S) = \begin{cases} u_T\,|_{N'}\,(S) & \text{if } T \subseteq N'; \\ \frac{|T\cap S|}{t} & \text{if } T \nsubseteq N'. \end{cases}$$

The previous expression implies that $\Psi^{PW}_{NN'}(u_T)$ is equal to $\Psi^{Sub}_{NN'}(u_T)$ if $T \subseteq N'$. Otherwise, each player in $N'\setminus T$ is a null player in $(N', \Psi^{PW}_{NN'}(u_T))$ and the rest of the players have a constant marginal contribution $\frac{1}{t}$ to any coalition in $(N', \Psi^{PW}_{NN'}(u_T))$.

Now we verify that Ψ^{PW} satisfies path independence. Take $N_3 \subseteq N_2 \subseteq N_1$. First, if $T \subseteq N_3$, then $\Psi^{PW}_{N_2N_3}(\Psi^{PW}_{N_1N_2}(u_T)) = \Psi^{PW}_{N_1N_3}(u_T) = u_T\,|_{N_3}$ by path independence of Ψ^{Sub}. Second, if $T \nsubseteq N_3$, then $\Psi^{PW}_{N_1N_3}(u_T) = \frac{|T\cap S|}{t}$. There are two possibilities: (a) if $T \subseteq N_2$, it is immediate that $\Psi^{PW}_{N_2N_3}(\Psi^{PW}_{N_1N_2}(u_T)) = \frac{|T\cap S|}{t}$; (b) if $T \nsubseteq N_2$, then for $S \subseteq N_3$, it happens that $\Psi^{PW}_{N_2N_3}(\Psi^{PW}_{N_1N_2}(u_T))(S) = \Psi^{PW}_{N_1N_2}(u_T)(S) - \sum_{i\in S} Sh_i(N_3, \Psi^{PW}_{N_1N_2}(u_T)|_{N_3}) + \sum_{i\in S} Sh_i(N_2, \Psi^{PW}_{N_1N_2}(u_T)) = \frac{|T\cap S|}{t} - \frac{|T\cap S|}{t} + \frac{|T\cap S|}{t} = \frac{|T\cap S|}{t}$, where the first equality follows from equation (1.5.2), and the terms in the second equality follow from (i) the expression of the game $\Psi^{PW}_{N_1N_2}(u_T)(S) = \frac{|T\cap S|}{t}$ and its subgames, (ii) each player $i \in T\cap N_2$ has a constant marginal contribution $\frac{1}{t}$ in $(N_2, \Psi^{PW}_{N_1N_2}(u_T))$, and (iii) the rest of the players are null players in $(N_2, \Psi^{PW}_{N_1N_2}(u_T))$. Therefore, the PW v-f reduction is path independent.

We verify the null player out property, i.e., $\Psi^{PW}_{N(N\setminus\{i\})}(v) = v\,|_{N\setminus\{i\}}$ for all $(N, v) \in \mathcal{G}^N$ such that $i \in N$ is a null player in (N, v). We notice that for all $S \subseteq N \setminus \{i\}$, $\Psi^{PW}_{N(N\setminus\{i\})}(v)(S) = v(S) - \sum_{j\in S} Sh_j(N\setminus\{i\}, v\,|_{N\setminus\{i\}}) + \sum_{j\in S} Sh_j(N, v) = v(S)$, where the first equality follows from (1.5.2) and the second from $Sh_j(N \setminus \{i\}, v\,|_{N\setminus\{i\}}) =$

$Sh_j(N,v)$ if i is a null player in (N,v), i.e., null player out of the Shapley value. Therefore Ψ^{PW} satisfies null player out.

As for permanent null player, let $i \in N'$ be a null player in (N,v). Then, it is the case that for all $S \subseteq N' \setminus \{i\}$, $\Psi^{PW}_{NN'}(v)(S \cup \{i\}) - \Psi^{PW}_{NN'}(v)(S) = v(S \cup \{i\}) - \sum_{j \in S \cup \{i\}} Sh_j(N', v\,|_{N'}) + \sum_{j \in S \cup \{i\}} Sh_j(N,v) - \big(v(S) - \sum_{j \in S} Sh_j(N', v\,|_{N'}) + \sum_{j \in S} Sh_j(N,v)\big) = (v(S \cup \{i\}) - v(S)) - Sh_i(N', v\,|_{N'}) + Sh_i(N,v) = 0$, where the third equality follows from the premise that i is a null player in (N,v) and its subgames and from the null player property of the Shapley value. Therefore Ψ^{PW} satisfies permanent null player.

We check the proportional addition invariance property. By additivity, it suffices to show that $\Psi^{PW}_{NN'}(\sum_{k=1}^{n-1} w_{(k,k\alpha)})(S) = 0$. Indeed, $\Psi^{PW}_{NN'}(\sum_{k=1}^{n-1} w_{(k,k\alpha)})(S) = s\alpha - \sum_{i \in S} Sh_i(N', \sum_{k=1}^{n'} w_{(k,k\alpha)}\,|_{N'}) + \sum_{i \in S} Sh_i(N, \sum_{k=1}^{n-1} w_{(k,k\alpha)}) = s\alpha - \sum_{i \in S} \sum_{k=1}^{n'} Sh_i(N', w_{(k,k\alpha)}\,|_{N'}) + \sum_{i \in S} \sum_{k=1}^{n-1} Sh_i(N, w_{(k,k\alpha)}) = s\alpha - \sum_{i \in S} Sh_i(N', w_{(n',n'\alpha)}\,|_{N'}) = s\alpha - \sum_{i \in S} \frac{n'\alpha}{n'} = 0$, where the first equality follows from (1.5.2), the second from additivity, and the third and fourth equalities hold because the Shapley value of each player in a symmetric game is equal to an equal share of the worth of the grand coalition.

To prove the reverse implication we need a previous lemma:

Lemma 3. *For all $N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $|N| > 2$, the set $\{u_T \mid T \subsetneq N, T \neq \varnothing\} \cup \{\sum_{k=1}^{n-1} w_{(k,k)}\}$ forms a basis of $\mathcal{G}^N$.*

Proof of Lemma 3. Take any $N \in \mathcal{P}_{fin}(\mathcal{U})$. To prove Lemma 3, we first note that:

$$nu_N = \big(\sum_{i \in N} u_{\{i\}}\big) - \big(\sum_{k=1}^{n-1} w_{(k,k)}\big), \tag{1.9.9}$$

which is easily seen by recognizing that $\sum_{i \in N} u_{\{i\}}(S) = s$ for all $S \in 2^N \setminus \{\varnothing\}$.

Given that equation (1.9.9) holds and $\{u_T \mid T \subseteq N, T \neq \varnothing\}$ forms a basis of $\mathcal{G}^N$, then the set resulting from replacing u_N with $\sum_{k=1}^{n-1} w_{(k,k)}$ on this basis spans $\mathcal{G}^N$, which proves Lemma 3. □

The proof that $\Psi = \Psi^{PW}$ if the v-f reduction Ψ satisfies the five properties is very similar to the proof of Theorem 2. The only difference is in the proof that $\Psi_{N(N\setminus\{i\})}(v) = \Psi^{PW}_{N(N\setminus\{i\})}(v)$ for all $i \in N$, when $v = \sum_{k=1}^{n-1} w_{(k,k\alpha)}$. In this case, additivity and proportional addition invariance imply that $\Psi_{N(N\setminus\{i\})}(\sum_{k=1}^{n-1} w_{(k,k\alpha)}) = \mathbf{0} = \Psi^{PW}_{N(N\setminus\{i\})}(\sum_{k=1}^{n-1} w_{(k,k\alpha)})$ for all $N \in \mathcal{P}_{fin}(\mathcal{U})$, all $\alpha \in \mathbb{R}$, and all $i \in N$.

Therefore, if a v-f reduction satisfies the five properties, then it is equal to Ψ^{PW}.

Finally, regarding the value induced by Ψ^{PW}, we notice that, for all $(N,v) \in \mathcal{G}^N$ and all $i \in N$, $\Psi^{PW}_{N\{i\}}(v)(\{i\}) = v(\{i\}) - Sh_i(\{i\}, v\,|_{\{i\}}) + Sh_i(N,v) = Sh_i(N,v)$. Therefore, Ψ^{PW} induces the Shapley value. □

Proof of Proposition 5. We prove that if Ψ satisfies proportional addition invariance then Ψ^* satisfies reverse-proportional addition invariance.

We first show that the dual of the game $(N, w_{(n-k,k\alpha)})$ is $(N, w_{(k,-k\alpha)})$, for $k = 1, 2, \ldots, n-1$. Indeed, $w^*_{(n-k,k\alpha)}(S) = w_{(n-k,k\alpha)}(N) - w_{(n-k,k\alpha)}(N \setminus S) = -w_{(n-k,k\alpha)}(N \setminus S) = w_{(k,-k\alpha)}(S)$ for all $S \subseteq N$.

Then, for all $\alpha \in \mathbb{R}$, $\Psi^*_{NN'}(v+\sum_{k=1}^{n-1} w_{(k,(n-k)\alpha)}) = \big(\Psi_{NN'}((v+\sum_{k=1}^{n-1} w_{(k,(n-k)\alpha)})^*)\big)^* = \big(\Psi_{NN'}(v^* + \sum_{k=1}^{n-1} w^*_{(k,(n-k)\alpha)})\big)^* = \big(\Psi_{NN'}(v^* + \sum_{k=1}^{n-1} w_{(k,-k\alpha)})\big)^* = \big(\Psi_{NN'}(v^*)\big)^* = \Psi^*_{NN'}(v)$, where the first and last equalities follow from Definition 8, the second from the additivity of $v \mapsto v^*$ in Lemma 1, the third from the property that the dual of $(N, w_{(n-k,k\alpha)})$ is $(N, w_{(k,-k\alpha)})$, and the fourth from the proportional addition invariance of Ψ. Therefore, reverse-proportional addition invariance is dual to proportional addition invariance. □

Proof of the expression in Example 5. We prove that the expression for Ψ^{PW^*} corresponds to that provided in Example 5:

$$\begin{aligned}
\Psi^{PW^*}_{NN'}(v)(S) =& \big(\Psi^{PW}_{NN'}(v^*)\big)^*(S) = \Psi^{PW}_{NN'}(v^*)(N') - \Psi^{PW}_{NN'}(v^*)(N' \setminus S) \\
=& [v^*(N') - \sum_{i \in N'} Sh_i(N', v^*|_{N'}) + \sum_{i \in N'} Sh_i(N, v^*)] \\
& - [v^*(N' \setminus S) - \sum_{i \in N' \setminus S} Sh_i(N', v^*|_{N'}) + \sum_{i \in N' \setminus S} Sh_i(N, v^*)] \\
=& v^*(N') - v^*(N' \setminus S) - \sum_{i \in S} Sh_i(N', v^*|_{N'}) + \sum_{i \in S} Sh_i(N, v^*) \\
=& [v(N) - v(N \setminus N')] - [v(N) - v(N \setminus (N' \setminus S))] - \sum_{i \in S} Sh_i(N', v^*|_{N'}) + \sum_{i \in S} Sh_i(N, v^*) \\
=& v(S \cup (N \setminus N')) - v(N \setminus N') - \sum_{i \in S} Sh_i\big(N', (v^*|_{N'})^*\big) + \sum_{i \in S} Sh_i(N, v),
\end{aligned}$$

where the first equality follows from Definition 8, the second from the defining equation (2.2.2) of a dual game, the third from (1.5.2), the fifth from (2.2.2) and the self-duality of the Shapley value, which also leads to the sixth equality.

Finally, we show that $(v^*|_{N'})^* = v^{N \setminus N'}$. Consider $T \subseteq N'$. By repeated application of the definition of dual game, for all $T \subseteq N'$, $(v^*|_{N'})^*(T) = v^*|_{N'}(N') - v^*|_{N'}(N' \setminus T) = v^*(N') - v^*(N' \setminus T) = [v(N) - v(N \setminus N')] - [v(N) - v(N \setminus (N' \setminus T))] = v(T \cup (N \setminus N')) - v(N \setminus N') = v^{N \setminus N'}(T)$. □

Proof of Theorem 5. The proof of this theorem is identical to that of Theorem 3. □

Proof of Theorem 6. First, we verify that Ψ^{Ban} satisfies the properties. It satisfies linearity and hence additivity because it is the composition of linear functions.

To verify the null player out property, let $i \in N$ be a null player in (N, v). Then, for all $S \subseteq N \setminus \{i\}$, $\Psi^{Ban}_{N(N\setminus\{i\})}(v)(S) = \sum_{T\subseteq\{i\}} \frac{1}{2}[v(S \cup T) - v(T)] = \frac{1}{2}v(S) + \frac{1}{2}[v(S \cup \{i\}) - v(\{i\})] = v(S)$, where the first equality follows from the defining equation (1.5.5) and the second holds because i is a null player. Therefore Ψ^{Ban} satisfies null player out.

To verify that Ψ^{Ban} satisfies permanent null player, suppose that $i \in N'$ is a null player in $v \in \mathcal{G}^N$. Then, for all $S \subseteq N' \setminus \{i\}$, $\Psi^{Ban}_{NN'}(v)(S \cup \{i\}) = \sum_{T\subseteq N\setminus N'} \frac{1}{2^{n-n'}}[v((S \cup \{i\}) \cup T) - v(T)] = \sum_{T\subseteq N\setminus N'} \frac{1}{2^{n-n'}}[v(S \cup T) - v(T)] = \Psi^{Ban}_{NN'}(v)(S)$, where the first and last equalities follow from (1.5.5) and the second from the premise that i is a null player in (N, v). Therefore Ψ^{Ban} satisfies permanent null player.

To verify maximum ignorance, we substitute $N' = N \setminus \{i\}$ and $W = N$ in equation (1.5.5), then $\Psi^{Ban}_{N(N\setminus\{i\})}(\alpha u_N)(S) = \sum_{T\subseteq\{i\}} \frac{1}{2}[\alpha u_N(S \cup T) - \alpha u_N(T)] = \frac{1}{2}[\alpha u_N(S) - \alpha u_N(\varnothing)] + \frac{1}{2}[\alpha u_N(S \cup \{i\}) - \alpha u_N(\{i\})] = \frac{\alpha}{2} u_N(S \cup \{i\})$. Therefore Ψ^{Ban} satisfies maximum ignorance.

To verify path independence, we use the following claim:

Claim 1. *For all $N, N', W, S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N', W \in 2^N \setminus \{\varnothing\}$ and $S \subseteq N'$,*

$$\Psi^{Ban}_{NN'}(u_W)(S) = \begin{cases} 2^{-|W\setminus N'|} u_W(S \cup (W \setminus N')) & \text{if } S \cap W \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9.10}$$

To verify the claim, we have,

$$\begin{aligned}
\Psi^{Ban}_{NN'}(v)(u_W)(S) &= \sum_{T\subseteq N\setminus N'} \frac{1}{2^{n-n'}}[u_W(S \cup T) - u_W(T)] \\
&= \sum_{T\subseteq N\setminus N' : T \supseteq W\setminus N'} \frac{1}{2^{n-n'}}[u_W(S \cup T) - u_W(T)] \\
&= \sum_{T'\subseteq (N\setminus N')\setminus W} \frac{1}{2^{n-n'}}[u_W(S \cup (W \setminus N') \cup T') - u_W(W \setminus N')] \\
&= \sum_{T'\subseteq (N\setminus N')\setminus (W\setminus N')} \frac{1}{2^{n-n'}}[u_W(S \cup (W \setminus N')) - u_W(W \setminus N')] \\
&= \sum_{T'\subseteq (N\setminus N')\setminus (W\setminus N')} \frac{2^{n-n'-|W\setminus N'|}}{2^{n-n'}}[u_W(S \cup (W \setminus N')) - u_W(W \setminus N')] \\
&= 2^{-|W\setminus N'|}[u_W(S \cup (W \setminus N')) - u_W(W \setminus N')] \\
&= \begin{cases} 2^{-|W\setminus N'|}[u_W(W \setminus N') - u_W(W \setminus N')] = 0 & \text{if } S \cap W = \varnothing \\ 2^{-|W\setminus N'|} u_W(S \cup (W \setminus N')) & \text{if } S \cap W \neq \varnothing \end{cases}
\end{aligned}$$

where the first equality follows from (1.5.5), the fourth from $u_W(S \cup (W \setminus N') \cup T') =$

$u_W(S \cup (W \setminus N'))$ if $T' \cap W = \varnothing$ and the first case of the seventh from the same reasoning, the second case of the seventh from $u_W(W \setminus N') = 0$ if $S \cap W \neq \varnothing$, which implies $N' \cap W \neq \varnothing$.

Then, to check path independence, first notice that equation (1.9.10) is equivalent to

$$\Psi^{Ban}_{NN'}(u_W) = \begin{cases} 2^{-|W \setminus N'|} u_{W \cap N'}|_{N'} & \text{if } W \cap N' \neq \varnothing \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{1.9.11}$$

Let $N_1, N_2, N_3, T \in \mathcal{P}_{fin}(N)$ such that $N_3 \subseteq N_2 \subseteq N_1$ and $T \subseteq N_1$. To compute $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T))$, there are three different possibilities to consider: (i) If $T \subseteq N_3$, then $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T)) = \Psi^{Ban}_{N_2N_3}(2^{-|T \setminus N_2|} u_{T \cap N_2}|_{N_2}) = 2^{-|T \setminus N_2|} \Psi^{Ban}_{N_2N_3}(u_{T \cap N_2}|_{N_2}) = 2^{-|T \setminus N_2|} \cdot 2^{-|(T \cap N_2) \setminus N_3|} u_{(T \cap N_2) \cap N_3}|_{N_2}|_{N_3} = 2^{-|T \setminus N_3|} u_{T \cap N_3}|_{N_3} = \Psi^{Ban}_{N_1N_3}(u_T)$, where the first, the third and the last equalities follow from equation (1.9.11) and the second from linearity. (ii) If $T \nsubseteq N_3$ and $T \subseteq N_2$, then $T \cap N_2 \nsubseteq N_3$. We have $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T)) = \Psi^{Ban}_{N_2N_3}(2^{-|T \setminus N_2|} u_{T \cap N_2}|_{N_2}) = 2^{-|T \setminus N_2|} \Psi^{Ban}_{N_2N_3}(u_{T \cap N_2}|_{N_2}) = 2^{-|T \setminus N_2|} \cdot \mathbf{0} = \mathbf{0} = \Psi^{Ban}_{N_1N_3}(u_T)$. (iii) If $T \nsubseteq N_2$, then $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T)) = \Psi^{Ban}_{N_2N_3}(\mathbf{0}) = \mathbf{0} = \Psi^{Ban}_{N_1N_3}(u_T)$. Therefore, Ψ^{Ban} satisfies path independence.

We now prove the reverse implication of the theorem by showing that if the v-f reduction Ψ satisfies the five properties, then $\Psi = \Psi^{Ban}$. By path independence and additivity, it suffices to show the equality restricted to one-player operators $(\Psi_{N(N \setminus \{i\})})$, for any $N \in \mathcal{P}_{fin}(\mathcal{U})$ and $i \in N$, restricted to a set of all scalar multiples of elements in a basis of $\mathcal{G}^N$. We choose the set $(\alpha u_T)_{T \in 2^N \setminus \{\varnothing\}, \alpha \in \mathbb{R}}$.

We show that $\Psi_{N(N \setminus \{i\})}(\alpha u_T) = \Psi^{Ban}_{N(N \setminus \{i\})}(\alpha u_T)$ for all $T \in 2^N \setminus \{\varnothing\}$, all $\alpha \in \mathbb{R}$, and all $i \in N$ by induction on the number of players n. We notice that maximum ignorance implies that $\Psi_{N(N \setminus \{i\})}(\alpha u_N) = \frac{\alpha}{2} u_{N \setminus \{i\}}$. Thus, we only need to check the remaining elements in the set, that is, the games $(\alpha u_T)_{T \in 2^N \setminus \{\varnothing, N\}, \alpha \in \mathbb{R}}$. The proof of this part is identical to the corresponding part of the proof of Theorem 1.

Therefore, a v-f reduction that satisfies the five properties coincides with Ψ^{Ban}.

Finally, we show that Ψ^{Ban} induces the Banzhaf value: $\varphi_i^{\Psi^{Ban}}(v) = \Psi_{N\{i\}}(v)(\{i\}) = \sum_{T \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} [v(T \cup \{i\}) - v(T)] = Ban_i(N, v)$, where the second and the third equality follows from the defining equation (1.5.5). Therefore, Ψ induces Ban. □

Proof of Proposition 8. It is easy to see that Ψ^X is additive as a result of the linearity of the Shapley value. Moreover, $\Psi^X = \Psi^{PW}$ if $X = \varnothing$ and $\Psi^X = \Psi^{HM}$ if $X = \mathcal{U}$. Equivalently, $\Psi^X_{NN'} = \Psi^{PW}_{NN'}$ if $(N \setminus N') \cap X = \varnothing$; $\Psi^X_{NN'} = \Psi^{HM}_{NN'}$ if $(N \setminus N') \cap X = N \setminus N'$. Therefore, the reduction of a game from N to $N \setminus \{j\}$ is different depending on whether the removed player j belongs to X or not. Hence, Ψ^X does not satisfy anonymity if $X \neq \varnothing$ and $X \neq \mathcal{U}$. Finally, Ψ^X satisfies null player out and permanent null player if Ψ^X satisfies path independence, which we show next.

For ease of notation, for each $T \subseteq 2^N \setminus \{\varnothing\}$, let us define $(N, e_T) \in \mathcal{G}^N$ by $e_T(S) \equiv \frac{|T\cap S|}{t}$ for all $S \subseteq N$. It is easy to see that

$$\Psi^X_{NN'}(e_T) = e_T\,|_{N'}\,. \tag{1.9.12}$$

By linearity of Ψ^X, it suffices to verify the path independence of Ψ^X operating on a basis $\left(u_T\right)_{T\in 2^N\setminus\{\varnothing\}}$. We need to consider three different cases of T: (i) If $T \subseteq S\cup((N\setminus N')\cap X)$, then $\Psi^X_{NN'}(u_T)(S) = \sum_{i\in S} Sh_i(S\cup((N\setminus N')\cap X), u_T|_{S\cup((N\setminus N')\cap X)}) - \sum_{i\in S} Sh_i(N'\cup((N\setminus N')\cap X), u_T|_{N'\cup((N\setminus N')\cap X)}) + \sum_{i\in S} Sh_i(N, u_T) = \sum_{i\in S} Sh_i(S\cup((N\setminus N')\cap X), u_T\,|_{S\cup((N\setminus N')\cap X)}) - \sum_{i\in S} Sh_i(S\cup((N\setminus N')\cap X), u_T\,|_{S\cup((N\setminus N')\cap X)}) + \sum_{i\in S} Sh_i(N, u_T) = \frac{|T\cap S|}{t}$, where the second equality follows from $Sh_i(S\cup((N\setminus N')\cap X), u_T\,|_{S\cup((N\setminus N')\cap X)}) = Sh_i(N'\cup((N\setminus N')\cap X), u_T\,|_{N'\cup((N\setminus N')\cap X)})$, i.e., the null player out of the Shapley value, and the last from equal treatment of the Shapley value. (ii) If $T \not\subseteq S\cup((N\setminus N')\cap X)$ and $T \subseteq N'\cup((N\setminus N')\cap X)$, then $\Psi^X_{NN'}(u_T)(S) = \sum_{i\in S} Sh_i(S\cup((N\setminus N')\cap X), u_T\,|_{S\cup((N\setminus N')\cap X)}) - \sum_{i\in S} Sh_i(N'\cup((N\setminus N')\cap X), u_T|_{N'\cup((N\setminus N')\cap X)}) + \sum_{i\in S} Sh_i(N, u_T) = -\sum_{i\in S} Sh_i(N'\cup((N\setminus N')\cap X), u_T|_{N'\cup((N\setminus N')\cap X)}) + \sum_{i\in S} Sh_i(N, u_T) = -\sum_{i\in S} Sh_i(N, u_T) + \sum_{i\in S} Sh_i(N, u_T) = 0$, where the second equality follows from the premise $T \not\subseteq S\cup((N\setminus N')\cap X)$, which implies that $u_T|_{S\cup((N\setminus N')\cap X)} = \mathbf{0}$, and the third from null player out of the Shapley value and the premise that $T \subseteq N'\cup((N\setminus N')\cap X)$ which imply that $Sh_i(N'\cup((N\setminus N')\cap X), u_T|_{N'\cup((N\setminus N')\cap X)}) = Sh_i(N, u_T)$. Finally, (iii) if $T \not\subseteq N'\cup((N\setminus N')\cap X)$, then $\Psi^X_{NN'}(u_T)(S) = \sum_{i\in S} Sh_i(S\cup((N\setminus N')\cap X), u_T|_{S\cup((N\setminus N')\cap X)}) - \sum_{i\in S} Sh_i(N'\cup((N\setminus N')\cap X), u_T\,|_{N'\cup((N\setminus N')\cap X)}) + \sum_{i\in S} Sh_i(N, u_T) = \sum_{i\in S} Sh_i(N, u_T) = \frac{|T\cap S|}{t}$, where the second equality follows from the premise, which implies that $u_T|_{S\cup((N\setminus N')\cap X)} = \mathbf{0}$ and $u_T|_{N'\cup((N\setminus N')\cap X)} = \mathbf{0}$.

To sum up, if $T \subseteq N'\cup((N\setminus N')\cap X)$, for all $S \subseteq N'$,

$$\Psi^X_{NN'}(u_T)(S) = \begin{cases} 0 & \text{if } T \not\subseteq S\cup((N\setminus N')\cap X); \\ \frac{|T\cap S|}{t} & \text{if } T \subseteq S\cup((N\setminus N')\cap X). \end{cases}$$

The previous expression means that, if $T \subseteq N'\cup((N\setminus N')\cap X)$,

$$\Psi^X_{NN'}(u_T) = \frac{|T\cap N'|}{t} u_{T\cap N'}\,|_{N'}, \tag{1.9.13}$$

whereas if $T \not\subseteq N'\cup((N\setminus N')\cap X)$,

$$\Psi^X_{NN'}(u_T) = e_T\,|_{N'}\,. \tag{1.9.14}$$

Now we can verify that $\Psi_{N_2N_3}(\Psi_{N_1N_2}(u_T)) = \Psi_{N_1N_3}(u_T)$ for all $N_1, N_2, N_3, S \in$

$\mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N_3 \subseteq N_2 \subseteq N_1$ and all $T \subseteq N_1$. We have three possibilities: (c1) $T \subseteq N_2 \cup ((N_1 \setminus N_2) \cap X)$ and $T \cap N_2 \subseteq N_3 \cup ((N_2 \setminus N_3) \cap X)$; (c2) $T \subseteq N_2 \cup ((N_1 \setminus N_2) \cap X)$ and $T \cap N_2 \nsubseteq N_3 \cup ((N_2 \setminus N_3) \cap X)$; (c3) $T \nsubseteq N_2 \cup ((N_1 \setminus N_2) \cap X)$ and $T \subseteq N_3 \cup ((N_2 \setminus N_3) \cap X)$.

For (c1), $\Psi^X_{N_2N_3}(\Psi^X_{N_1N_2}(u_T)) = \Psi^X_{N_2N_3}(\frac{|T \cap N_2|}{t} u_{T \cap N_2} |_{N_2}) = \frac{|T \cap N_2|}{t} \Psi^X_{N_2N_3}(u_{T \cap N_2} |_{N_2}) = \frac{|T \cap N_2|}{t} \frac{|T \cap N_2 \cap N_3|}{|T \cap N_2|} u_{T \cap N_2 \cap N_3} |_{N_2}|_{N_3} = \frac{|T \cap N_3|}{t} u_{T \cap N_3} |_{N_3} = \Psi^X_{N_1N_3}(u_T)$, where the first and the third equalities follow from equation (1.9.13), the second from linearity of Ψ^X, and the last from the fact that the premise of (c1) implies that $T \subseteq N_3 \cup ((N_1 \setminus N_3) \cap X)$.

For (c2), $\Psi^X_{N_2N_3}(\Psi^X_{N_1N_2}(u_T)) = \Psi^X_{N_2N_3}(\frac{|T \cap N_2|}{t} u_{T \cap N_2} |_{N_2}) = \frac{|T \cap N_2|}{t} \Psi^X_{N_2N_3}(u_{T \cap N_2} |_{N_2}) = \frac{|T \cap N_2|}{t} e_{T \cap N_2} |_{N_3} = e_T |_{N_3} = \Psi^X_{N_1N_3}(u_T)$, where the first equality follows from equation (1.9.13), the second from linearity of Ψ^X, the third from equation (1.9.14), the fifth from the fact that the premise of (c2) implies that $T \nsubseteq N_3 \cup ((N_1 \setminus N_3) \cap X)$.

For (c3), $\Psi^X_{N_2N_3}(\Psi^X_{N_1N_2}(u_T)) = \Psi^X_{N_2N_3}(e_T |_{N_2}) = e_T |_{N_2}|_{N_3} = e_T |_{N_3} = \Psi^X_{N_1N_3}(u_T)$, where the first and second equalities follow from equation (1.9.14), the fourth from the fact that the premise of (c3) implies that $T \nsubseteq N_3 \cup ((N_1 \setminus N_3) \cap X)$.

Therefore, Ψ^X is path independent. □

Example of a v-f reduction that does not satisfy linearity. We construct a v-f reduction that satisfies additivity, null player out, permanent null player, path independence, but not homogeneity.

We can invoke path independence to define $\Psi_{NN'}$ for any $N' \subseteq N$, once we will determine the functions taking the form $\Psi_{N(N \setminus \{k\})}$ such that $k \in N$. Moreover, it suffices to construct a non-homogeneous function $\Psi_{\{i,j\}\{i\}} : \mathcal{G}^{\{i,j\}} \to \mathcal{G}^{\{i\}}$ that satisfies null player out, permanent null player and additivity. For concreteness, we let the rest of functions, i.e., $\Psi_{N(N \setminus \{k\})}$ such that $k \in N$ and $|N| > 2$ coincide with the subgame operator.

Denote by $\mathbb{Q}$ the set of all rational numbers. To define a non-homogeneous additive function, we use the concept of $\mathbb{R}$ as a vector space over $\mathbb{Q}$. A linear basis of this vector space is called a Hamel basis. Let $\mathcal{H}$ be a Hamel basis. Then for each $\gamma \in \mathbb{R}$, we can find a unique finite set of elements $\{x_1, \ldots, x_k\} \subseteq \mathcal{H}$ such that $\gamma = \sum_{j=1}^{k} c_j x_j$ where $c_1, \ldots, c_k \in \mathbb{Q} \setminus \{0\}$. Choose an arbitrary element $y \in \mathcal{H}$. Then for each $\gamma \in \mathbb{R}$, we can determine its corresponding coefficient (which is possibly zero) in the expression of γ, coefficient that we denote $c(\gamma)$. Thus we have a function $c : \mathbb{R} \to \mathbb{Q}$ defined by the projection $\gamma \mapsto c(\gamma)$. This function is additive but not homogeneous. Indeed, choose an arbitrary element $y' \in \mathcal{H} \setminus \{y\}$, then $\alpha c(y) \neq c(\alpha y)$ when $\alpha = \frac{y'}{y}$. Moreover, this function satisfies that $c(0) = 0$.[21]

[21] The construction of a Hamel basis, and hence a non-linear additive function involves the axiom

Before defining $\Psi_{\{i,j\}\{i\}}$, recall that for each $(\{i,j\},v) \in \mathcal{G}^{\{i,j\}}$, v can be expressed by $\alpha u_{\{i\}} + \beta u_{\{j\}} + \gamma u_{\{i,j\}}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Now we define $\Psi_{\{i,j\}\{i\}}(v)$ as follows:

$$\Psi_{\{i,j\}\{i\}}(v)(\{i\}) \equiv \alpha + c(\gamma), \tag{1.9.15}$$

where $\alpha, \gamma \in \mathbb{R}$ are such that $v = \alpha u_{\{i\}} + \beta u_{\{j\}} + \gamma u_{\{i,j\}}$ for some $\beta \in \mathbb{R}$.

Notice that if i is a null player in $(\{i,j\},v)$ then v must take the form of $\beta u_{\{j\}}$ and that if j is a null player in $(\{i,j\},v)$ then v must take the form of $\alpha u_{\{i\}}$. Therefore, $\Psi_{\{i,j\}\{i\}}$ satisfies null player out and permanent null player. Moreover, it is additive but not linear because the function c is additive but not homogeneous. □

Proof of Theorem 7. We check that $\Psi^{A^{Ban}} = \Psi^{Ban}$. Indeed, for $S \in 2^N \setminus \{\varnothing\}$, $\Psi^{A^{Ban}}_{NN'}(v)(S) = Ban_{\bar{S}}((N\backslash N')\cup\{\bar{S}\}, (v\,|_{S\cup(N\backslash N')})_S) = \sum_{T\subseteq((N\backslash N')\cup\{\bar{S}\})\backslash\{\bar{S}\}} \frac{1}{2^{n-n'}} D^{\bar{S}}(v\,|_{S\cup(N\backslash N')})_S)(T) = \sum_{T\subseteq N\backslash N'} \frac{1}{2^{n-n'}}[(v\,|_{S\cup(N\backslash N')})_S)(T\cup\{\bar{S}\}) - (v\,|_{S\cup(N\backslash N')})_S)(T)] = \sum_{T\subseteq N\backslash N'} \frac{1}{2^{n-n'}}[(v(T\cup S) - (v(T)] = \Psi^{Ban}_{NN'}(v)(S)$, where the equalities just follow the definitions of $\Psi^{A^{Ban}}$, Ban, v_S, and Ψ^{Ban}.

Moreover, Ψ^{Ban} is path independent. Therefore, the Banzhaf value is consistent relative to $\Psi^{A^{\varphi}}$. Also, it is immediate that it is standard for two-player games.

For the other direction, we prove that if φ is consistent relative to $\Psi^{A^{\varphi}}$ and standard for two-player games, then $\varphi = Ban$. We do the proof by induction on the number of players $|N|$. It holds for $|N| = 2$ by standardness.

Consider now $(N,v) \in \mathcal{G}^N$ such that $|N| > 2$ and assume that $\varphi = Ban$ for any game with less than $|N|$ players. Take any $i \in N$. We first note that

$$\Psi^{A^{\varphi}}_{N(N\backslash\{i\})}(v)(S) = \varphi_{\bar{S}}(\{i,\bar{S}\},(v\,|_{S\cup\{i\}})_S) = \frac{v(S)}{2} + \frac{v(S\cup\{i\}) - v(\{i\})}{2}, \tag{1.9.16}$$

where the second equality follows from standardness.

To prove that $\varphi_j(N,v) = Ban_j(N,v)$ for any $j \in N$, take any $i \in N$ such that $i \neq j$. Then, $\varphi_j(N,v) = \varphi_j(N\setminus\{i\}, \Psi^{A^{\varphi}}_{N(N\backslash\{i\})}(v)) = Ban_j(N\setminus\{i\}, \Psi^{A^{\varphi}}_{N(N\backslash\{i\})}) = \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-2}}[\Psi^{A^{\varphi}}_{N(N\backslash\{i\})}(v)(S\cup\{j\}) - \Psi^{A^{\varphi}}_{N(N\backslash\{i\})}(v)(S)] = \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-2}}[\frac{v(S\cup\{j\})}{2} + \frac{v(S\cup\{i,j\})-v(\{i\})}{2} - \frac{v(S)}{2} - \frac{v(S\cup\{i\})-v(\{i\})}{2}] = \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-2}}[\frac{v(S\cup\{j\})-v(S)}{2} + \frac{v(S\cup\{i,j\})-v(S\cup\{i\})}{2}] = \sum_{T\subseteq N\backslash\{j\}} \frac{1}{2^{n-1}}[v(T\cup\{j\}) - v(T)] = Ban_j(N,v)$, where the first equality follows from the consistency of φ, the second from the hypothesis that $\varphi = Ban$ for games with $n-1$ players, the third and the last from the definition of the Banzhaf value, and the fourth from the equation (1.9.16). □

Proof of Theorem 8. It is easy to see by substituting the stand-alone value in $\Psi^{A^{\varphi}}$ that it coincides with the subgame v-f reduction Ψ^{sub}. Moreover, Ψ^{sub} is path

of choice. See Herrlich (2006).

independent. Thus, the stand-alone value is consistent relative to $\Psi^{A\varphi}$.

For the other direction, we prove that $\varphi_j(N, v) = v(\{j\})$ for $j \in N$ by induction on the number of players $|N|$. It holds for $|N| = 2$ by condition (ii) of the theorem.

Assume that the induction hypothesis holds for any game with less than n player, with $n > 2$, and consider $(N, v) \in \mathcal{G}^N$ with $|N| = n$. For any $i \in N$, we have $\Psi^{A\varphi}_{N(N\setminus\{i\})}(v)(S) = \varphi_{\bar{S}}(\{i, \bar{S}\}, (v\,|_{S\cup\{i\}})_S) = v(S)$, where the second equality follows from (ii) of the theorem. Thus:

$$(N \setminus \{i\}, \Psi^{A\varphi}_{N(N\setminus\{i\})}(v)) = (N \setminus \{i\}, v). \tag{1.9.17}$$

Consider now any $j \in N$, and take any $i \in N$ such that $i \neq j$. Then, $\varphi_j(N, v) = \varphi_j(N \setminus \{i\}, \Psi^{A\varphi}_{N(N\setminus\{i\})}(v)) = \varphi_j(N \setminus \{i\}, v) = v(\{j\})$, where the first equality follows from consistency, the second from the equation (1.9.17), and the third from the induction hypothesis. □

Chapter 2

Bidding against a Buyout

2.1 Introduction

Game theory is traditionally divided into two branches, cooperative game theory and non-cooperative game theory. Cooperative game theory focuses on the possible payoffs that players may obtain in a game, taking into account the worth of the coalitions of players and abstracting from the actions or decisions that may lead to these payoffs. It often adopts an axiomatic or normative approach to characterize solution concepts: it sets up a number of normative goals and derives their logical implications. By contrast, non-cooperative game theory studies the outcome of the interaction by individual players. A non-cooperative game models the specific details of the interaction among the players and analyzes the final outcome of this interaction. It adopts the strategic approach.

The previous description manifests a gap between the two branches of game theory. To bridge this gap, Nash (1953) initiated the so-called Nash program: assigning each cooperative game with an extensive-form game such that a given cooperative solution of the former coincides with some non-cooperative solution of the later. Since then, the Nash program has been, and still is, pursued by many authors, and has grown into a large body of literature. One of the most important themes in this literature is finding non-cooperative games that lead to the Shapley value of coalitional games as their subgame perfect Nash equilibrium outcome. Notable examples of this theme include Gul (1989), Hart and Mas-Colell (1996), and Pérez-Castrillo and Wettstein (2001). Moreover, some mechanisms proposed in these papers are compatible with the theory of implementation because their rules do not depend on the characteristic function, which only the players are supposed to know.

In this study, we design "natural" non-cooperative games, where players have equal possibilities to propose and to reject offers. At the beginning of the mechanisms, all players are “insiders” (we call them “proposers”), but as the games proceed, some of them may be bought out. The remaining proposers will try to reach a consensus, and if a consensus is not reached, then one of them will be bought out so that the others can try to reach a consensus in the next stage. The inspiration for our mechanisms comes from the phenomenon in corporation management practice of some shareholders buying out those with misaligned interests in order to achieve a consensus among shareholders. To facilitate a buyout, we use the Pérez-Castrillo and Wettstein’s (2001) (PW) multibidding procedure (the PW procedure for short) in which each player submits a bid against every other player and the player with the highest total net bid is selected. However, in contrast to the PW setting, we endow every player with the right to propose allocation plans simultaneously at the beginning, which is reminiscent of the Nash (1953) demand game.[1]

Thus, at each stage of our mechanisms, players have the possibility of reaching an agreement, which requires that all the proposals are identical. In case of a disagreement among proposers one proposer, who is bought out by the PW procedure, will receive the submitted bids from the other players, and his role will be degraded to that of a respondent. This buyout process will continue until the proposers reach a consensus. The players who are bought out may join the coalition of proposers by accepting the consensual allocation plan.

We note that the previous description leaves the procedure of how respondents address a consensual allocation plan unspecified. We consider two different specifications. In the first specification, a representative will be selected to accept or reject the plan on behalf of all respondents. In this case, the respondents’ bargaining power is vested in one of them. In the second specification, each respondent will decide to accept or reject the plan for himself. In this case, the respondents’ bargaining power is dispersed among them. Our main results are that the mechanism with the first specification implements the Shapley value, whereas the mechanism with the second specification implements the equal surplus value.

The Shapley value and the equal surplus value have been compared using the aforementioned axiomatic and strategic approaches. Using the axiomatic approach, Casajus and Huettner (2014) pinned down the difference between the Shapley value and the equal surplus value by one out of four axioms. Roughly speaking, the Shapley value of a player who never collaborates with others is equal to his individual rational payoff, while the equal surplus value of a player who prevents collaboration

[1]Though it enjoys less popularity than the idea of selecting a single proposer, the possibility of submitting simultaneous proposals is equally appealing since it treats every player symmetrically, without referring to a specific probability distribution. See Binmore (1987) and Chatterjee and Samuelson (1990) with regard to the Rubinstein-Stahl bargaining model.

within his coalition is equal to his individual rational payoff. Using the strategic approach, Ju and Wettstein (2009) pinned down the difference between them by constructing two mechanisms differing in the choice of who makes an offer between the rejected proposer and the proposer. Our paper provides a new comparison of the two values based on the strategic approach.

A remarkable by-product of Pérez-Castrillo and Wettstein's (2001) result is revealing a close connection among the Maschler and Owen's (1989) recursive formula of the Shapley value, the Myerson's (1980) balanced contributions property and the PW bidding mechanism. Similarly, we show that our two mechanisms are associated with two new recursive formulae and two new versions of the balanced contributions property respectively. It should be noted that the underlying recursive formula of the first mechanism can be derived from the Maschler and Owen's recursive formula by the self-duality of the Shapley value in response to Ju's (2012) suggestion that the self-duality of the Shapley value be explored in the future research of the Nash program.

Broadly speaking, our mechanisms belong to a class of mechanisms using the PW procedure. In their original PW mechanism, one proposer is selected using the PW procedure. This proposer has the right to make a proposal which will be implemented only if the rest of players accept. Otherwise, the proposer drops out of the game and remains alone. Several papers have considered variants of the PW mechanism where rejected proposers are granted a second chance to return to the coalition of active players. Ju and Wettstein (2009) considered granting only the latest rejected proposer this option. Depending on who makes the offer in the renegotiation, they constructed three mechanisms that implement the Shapley value, the equal surplus value and the consensus value respectively. Ju (2012) allowed all the rejected proposers' return and constructed three mechanisms that all implement the Shapley value. Another line of research generalizes the PW mechanism to different environments. Pérez-Castrillo and Wettstein (2005) constructed a mechanism for pure exchange economies to implement the Pérez-Castrillo and Wettstein's (2006) ordinal Shapley value for three or less players. Macho-Stadler et al. (2006) proposed two mechanisms for TU games with positive externalities and negative externalities which implement two distinct generalizations of the Shapley value respectively. Slikker (2007) presented three mechanims for the Jackson and Wolinsky's (1996) network allocation problem which implement the Myerson value, the position value and the componentwise egalitarian rule respectively. Different from our mechanisms, all the mechanisms mentioned above use the PW procedure to select a single proposer each time.

The rest of this paper is organized as follows. Section 2 provides definitions and proves new results for the Shapley value and the equal surplus value. Section 3 constructs a mechanism and shows that it implements the Shapley value in every subgame perfect Nash equilibrium (**SPNE**). Section 4 constructs a comparable mechanism and shows that it implements the equal surplus value in SPNE. Section 5 concludes the paper with a discussion on possible variations and extensions.

2.2 Preliminaries

Let $N \equiv \{1, \ldots, n\}$ be the set of players. In our context, we refer to each subset $S \subseteq N$ as a coalition. In particular, N is called the grand coalition and any coalition S such that $|S| = 1$ is called a standalone coalition.[2]

A **coalitional game with transferrable utility** (**TU game**) with N as the set of players is a function $v : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. For each coalition $S \subseteq N$, $v(S)$ denotes the worth of coalition S in v. The worth of standalone coalition $\{i\}$ is also called player i's **individual rational payoff**. The class of all TU games with N as the set of players is denoted by $\mathcal{G}^N$.

A TU game $v \in \mathcal{G}^N$ is **0-monotonic** if for each $i \in N$ and each $S \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) - v(S) \geq v(\{i\})$. A 0-monotonic TU game $v \in \mathcal{G}^N$ is **strictly 0-monotonic** if the inequalities are strict. A TU game $v \in \mathcal{G}^N$ is **supper-additive** if for each $S, T \subseteq N$ such that $S \cap T = \varnothing$, $v(S \cup T) \geq v(S) \cup v(T)$. A supper-additive TU game $v \in \mathcal{G}^N$ is **strictly supper-additive** if the inequalities are strict. The classes of all 0-monotonic TU games and all strictly 0-monotonic TU games are denoted by $\mathcal{G}_0^N$ and $\mathcal{G}_{0^*}^N$, respectively. The classes of all supper-additive TU games and all strictly supper-additive TU games are denoted by $\mathcal{G}_s^N$ and $\mathcal{G}_{s^*}^N$, respectively.

A TU game $v \in \mathcal{G}^N$ is **additive** if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. We say that a TU game $v \in \mathcal{G}^N$ is **non-additive** if v is not additive. It is immediate that an additive TU game is 0-monotonic. Furthermore, it is easy to show that a 0-monotonic TU game $v \in \mathcal{G}_0^N$ is additive if and only if $v(N) = \sum_{i \in N} v(\{i\})$. We denote by $\mathcal{G}_a^N$ the class of all additive TU games.

In this paper, we consider several ways to generate new games from a given game $v \in \mathcal{G}^N$ that will be useful for our exposition. A **subgame** of $v \in \mathcal{G}^N$ is a game $v|_{2^{N'}} \in \mathcal{G}^{N'}$ for some $N' \subseteq N$, where

$$v|_{2^{N'}}(S) \equiv v(S) \tag{2.2.1}$$

for all $S \subseteq N'$. The worth of any coalition of N' in the subgame coincides with its

[2]Throughout this paper, $|S|$ represents the number of players in S.

worth in the initial game.

The **dual of TU game** $v \in \mathcal{G}^N$ is the game $v^* \in \mathcal{G}^N$, defined by

$$v^*(S) \equiv v(N) - v(N \setminus S), \tag{2.2.2}$$

for all $S \subseteq N$. The dual game assigns to coalition S the worth that it is lost if S leaves the grand coalition.

Given $T \subsetneq N$, the **aiding game of Type I** $v^T \in \mathcal{G}^{N \setminus T}$ is defined by

$$v^T(S) \equiv v(S \cup T) - v(T) \tag{2.2.3}$$

for all $S \subseteq N \setminus T$. The worth of coalition S in the aiding game v^T corresponds to the surplus that S can generate together with T, once the players in T receive the payment of $v(T)$. The formulation of an aiding game of Type I is due to Oishi et al. (2016).

Given $T \subsetneq N$, the **aiding game of Type II** $v^{\overline{T}} \in \mathcal{G}^{N \setminus T}$ is defined by

$$v^{\overline{T}}(S) \equiv \begin{cases} v(S \cup T) - \sum_{i \in T} v(\{i\}) & \text{for } S = N \setminus T \\ v(S) & \text{otherwise.} \end{cases} \tag{2.2.4}$$

The worth of a coalition in the game $v^{\overline{T}}$ is the same as in the original game v, except if the coalition is the whole set $N \setminus T$. In the latest case, the coalition $N \setminus T$ joins T to generate the surplus $v(N)$ and retains the surplus once the players in T receive their individual rational payoff. The formulation of aiding game of Type II is due to Ju and Wettstein (2009).

A **value** $\varphi : \mathcal{G}^N \to \mathbb{R}^N$ is a function that assigns each game $v \in \mathcal{G}^N$ and each player $i \in N$ a payoff $\varphi_i(v)$. The **dual of value** φ is the value $\varphi^* : \mathcal{G}^N \to \mathbb{R}^N$ defined by $\varphi^*(v) \equiv \varphi(v^*)$. A value φ is **self-dual** if $\varphi = \varphi^*$.

The **Shapley value** Sh is the most popular value for TU games. It is defined by

$$Sh_i(v) \equiv \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \big[v(S \cup \{i\}) - v(S) \big] \tag{2.2.5}$$

for each $v \in \mathcal{G}^N$ and each $i \in N$. Shapley (1953) proved that a value satisfies additivity, null player property, equal treatment, and efficiency if and only if it is representable as equation (2.2.5). It is well-known that the Shapley value is self-dual.

An alternative expression of the Shapley value via a recursive formula was proposed by Maschler and Owen (1989) as follows:

$$Sh_i(v) \equiv \begin{cases} v(N) & \text{if } |N| = 1 \\ \frac{1}{n}\big[v(N) - v(N \setminus \{i\})\big] + \frac{1}{n}\big[\sum_{j \in N \setminus \{i\}} Sh_i(v|_{2^{N\setminus\{j\}}})\big] & \text{if } |N| > 1 \end{cases} \tag{2.2.6}$$

for all $i \in N$ and all $v \in \mathcal{G}^N$.

The Shapley value can be expressed in other ways. The following lemma provides an alternative expression that will be useful for our proofs.

Lemma 4. *A value ϕ is equal to Sh if and only if*

$$\phi_i(v) = \begin{cases} v(N) & \textit{if } |N| = 1 \\ \frac{1}{n}v(\{i\}) + \frac{1}{n}\sum_{j \in N \setminus \{i\}} \phi_i(v^{\{j\}}) & \textit{if } |N| > 1 \end{cases} \tag{2.2.7}$$

for all $i \in N$ and all $v \in \mathcal{G}^N$.

Similarly to the expression (2.2.6), the formula (2.2.7) also expresses the Shapley value of an n-player TU game v as an average: the Shapley value of player i in a game with n players is the average of his individual rational payoff and his Shapley values in $(n-1)$ aiding games of type I.

The **equal surplus value** $ES : \mathcal{G}^N \to \mathbb{R}^N$ is the value defined by

$$ES_i(v) \equiv v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{i\})] \tag{2.2.8}$$

for each $v \in \mathcal{G}^N$ and each $i \in N$. The equal surplus value coincides with many bargaining solutions if we transform a TU game into a bargaining problem by neglecting the worth of every coalition that is neither the grand coalition nor a standalone coalition.

Lemma 5 characterizes the equal surplus value in a similar way as Lemma 4 characterizes the Shapley value. It uses the aiding game of type II instead of the aiding game of type I.[3]

Lemma 5 (Ju and Wettstein, 2009)**.** *A value ϕ is equal to ES if and only if*

$$\phi_i(v) = \begin{cases} v(N) & \textit{if } |N| = 1 \\ \frac{1}{n}v(\{i\}) + \frac{1}{n}\sum_{j \in N \setminus \{i\}} \phi_i(v^{\overline{\{j\}}}) & \textit{if } |N| > 1 \end{cases} \tag{2.2.9}$$

for all $i \in N$ and all $v \in \mathcal{G}^N$.[4]

[3]A proof of Lemma 5 can be found in the proof of Theorem 3.2 in Ju and Wettstein (2009).

[4]In fact, the recursive formula of the equal surplus value in Ju and Wettstein (2009) is slightly

The values can also be characterized through "balanced contributions properties" and, in this paper, we are going to propose and use new balanced contributions properties of the Shapley and the equal sharing values. Myerson (1980) introduced the original balanced contributions property of a value. A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies the **balanced contributions property** if and only if

$$\phi_i(v) - \phi_i(v|_{2^{N\setminus\{j\}}}) = \phi_j(v) - \phi_j(v|_{2^{N\setminus\{i\}}}) \tag{2.2.10}$$

for all $v \in \mathcal{G}^N$ and all $i, j \in N$. This property is a fairness requirement. For each pair of players, their contributions on top of their respective reference points should be equal. The reference point is taken as the value of playing the subgame without the opponent player in the pair. Myerson (1980) showed that the Shapley value is characterized by the balanced contributions property and efficiency.

Theorem 1 (Myerson, 1980)**.** *There exists a unique value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfying the balanced contributions property and efficiency. Moreover, ϕ is equal to the Shapley value.*

While Myerson (1980) defined the balanced contributions property with respect to the subgames, we formulate two analogues of balanced contribution properties of a value using the two types of aiding games defined above.

A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies the **balanced contributions property of Type I** if and only if

$$\phi_i(v) - \phi_i(v^{\{j\}}) = \phi_j(v) - \phi_j(v^{\{i\}}), \tag{2.2.11}$$

for all $v \in \mathcal{G}^N$ and all $i, j \in N$. This property can be interpreted in the same manner as the original balanced contributions property. The difference is in the choice of the reference points. The reference point $\phi_i(v^{\{j\}})$ can be seen as player i's utopia point: what can be achieved by player i with player j's full collaboration at the cost of player j's individual rational payoff. Then $\phi_i(v^{\{j\}}) - \phi_i(v)$ measures the dissatisfaction of player i with $\phi_i(v)$ compared to his utopia point. The balanced contributions property of Type I requires that for each pair of players, their dissatisfaction of the value relative to their respective utopia points should be equalized.

A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies the **balanced contributions property of Type II** if and only if

$$\phi_i(v) - \phi_i(v^{\overline{\{j\}}}) = \phi_j(v) - \phi_j(v^{\overline{\{j\}}}), \tag{2.2.12}$$

different:

$$ES_i(v) = \begin{cases} v(N) & \text{if } |N| = 1 \\ \frac{1}{n}[v(N) - v^{\overline{\{i\}}}(N \setminus \{i\})] + \frac{1}{n}\sum_{j\in N\setminus\{i\}} ES_i(v^{\overline{\{j\}}}) & \text{if } |N| > 1. \end{cases}$$

Their formula is closer to the formula (2.2.6) in spirit.

for all $v \in \mathcal{G}^N$ and for all $i, j \in N$. This property uses yet another reference point, which is a mixture of the features of the aforementioned two choices. For the grand coalition, its worth is the same as that of the aiding game of Type I. For the rest of coalitions, their worth are the same as that of the subgame.

Following Myerson's (1980), we propose two lemmas stating that the Shapley value and the equal surplus value satisfy the balanced contributions properties of Type I and of Type II, respectively.

Lemma 6. *The Shapley value satisfies the modified balanced contributions property of Type I.*

Lemma 7. *The equal surplus value satisfies the balanced contributions property of Type II.*

With the same reasoning as in the proof of Theorem 1, we characterize the Shapley value (resp. the equal surplus value) by efficiency and the balanced contributions property of Type I (resp. Type II).

Proposition 9. *There exists a unique value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfying the balanced contributions property of Type I (resp. Type II) and efficiency. Moreover, ϕ is equal to the Shapley value (resp. the equal surplus value).*

In the next sections, we present two mechanisms that implement the Shapley value and the equal surplus value, respectively. They use Pérez-Castrillo and Wettstein's (2001) technology of ordering players through bidding, which we call the **PW procedure**. For completeness, we describe the PW procedure here as follows:

> Each player $i \in N$ chooses a vector of bids $\mathbf{b}_i^N \equiv (b_{ij}^N)_{j \in N \setminus \{i\}} \in \mathbb{R}^{N \setminus \{i\}}$, which represents that player i submits a bid b_{ij}^N against each player $j \in N \setminus \{i\}$. We denote player i's total net bid by $B_i^N \equiv \sum_{j \in N \setminus \{i\}} (b_{ij}^N - b_{ji}^N)$.[5] Then we arrange the players in N in descending order with respect to their corresponding total net bid and ties are broken through a fair lottery.

In the mechanism proposed by Pérez-Castrillo and Wettstein (2001) (the **PW mechanism**), the player with the highest total net bid is selected as the proposer by the PW procedure. We denote this proposer by α. The proposer α pays the submitted bid $b_{\alpha j}^N$ to each $j \in N \setminus \{\alpha\}$. Then α proposes an allocation plan $\mathbf{y}^N \in \mathbb{R}^{N \setminus \{\alpha\}}$, while the rest of players may respond by accepting or declining it sequentially. The

[5]Note that by construction, the sum of all net bids is zero because $\sum_{i \in N} B_i^N = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} (b_{ij}^N - b_{ji}^N) = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} b_{ij}^N - \sum_{i \in N} \sum_{j \in N \setminus \{i\}} b_{ji}^N = \sum_{(i,j) \in N \times N : i \neq j} b_{ij}^N - \sum_{(j,i) \in N \times N : j \neq i} b_{ji}^N = 0$.

grand coalition will form and the allocation plan $\mathbf{y}^N$ will be implemented only if it is accepted unanimously. It means that each player $j \in N \setminus \{\alpha\}$ receives y_j^N according to the allocation plan, while the proposer α receives the residue $v(N) - \sum_{j \in N \setminus \{\alpha\}} y_j^N$. Compared to the role of the respondents, the role of the proposer is more advantageous since he can propose an allocation plan in his own favor. But this comes at a cost: the proposer's submitted bids have to be paid. Pérez-Castrillo and Wettstein (2001) showed that in any SPNE, each player must be indifferent between the role of proposer and the role of respondent. As a result, each player's SPNE payoff is equal to the mean of the continuation SPNE payoff corresponding to each possibility. They observed that this equation is formula (2.2.6) in disguise.

In the next sections, we construct two mechanisms differing in the technology of negotiation between the coalition of proposers and the coalition of respondents. These mechanisms capture the idea that it is easier for fewer people to reach a consensus than for more people.

2.3 A mechanism that implements the Shapley value

In this section, we present a mechanism that implements the Shapley value. At each stage of this mechanism, active players are divided into proposers and respondents. If the set of respondents is non-empty, then one of the respondents represents all the respondents to negotiate with the proposers. We describe the mechanism informally as follows:

Mechanism A. *Initially, all players are proposers and each of them puts forth an allocation plan that specifies a payoff for each proposer. In case of consensus, the plan will be implemented and the mechanism ends. In case of disagreement, the PW procedure ensues among proposers. The proposer with the lowest total net bid is selected and receives the submitted bids from the rest of proposers. The selected player's role is changed from proposer to respondent. As the only respondent, he automatically becomes the representative of respondents. The mechanism goes to the next stage.*

In any non-initial stage, there are a set of proposers, a set of respondents (with a representative of this set), and possibly a set of dropouts. Each proposer again puts forth an allocation plan for the set of proposers.

In case of consensus among proposers, the plan is submitted to the representative of the respondents. If the representative accepts the plan, the coalition of proposers and the coalition of respondents merge into one coalition. Proposers receive their *payoffs as specified in the plan and the representative receives the residue. If the representative rejects, the coalition of proposers breaks down while the coalition of respondents forms. Each proposer receives his individual rational payoff and the representative receives the worth of the coalition of respondents. The mechanism ends.*

In case of disagreement among proposer, the PW procedure ensues among proposers. The proposer with the lowest total net bid is selected and receives the submitted bids from the rest of proposers. The selected player will offer an amount of money to the incumbent representative in exchange for his role as the representative. If the incumbent representative accepts, the selected player becomes the new representative; otherwise, he quits the mechanism and receives his individual rational payoff. The mechanism is played again with a smaller set of proposers, the same set of respondents and the same representative.

Formally, we define Mechanism A recursively with respect to the number of players as follows:

For $n = 1$, the only player receives his individual rational payoff.

For $n \geq 2$, for ease of exposition, at any stage of the mechanism, the set of players N is partitioned into a set of proposers, a set of respondents, and a set of dropouts. A proposer is a player who has the right to put forth an allocation plan, whereas a respondent is a former proposer who has been bought out. In the set of respondents, there is a player who is the representative of the set and he is entitled to accept or decline a consensual plan. A dropout is a player who quits the game permanently by forming his own standalone coalition. At the initial state, all the players are proposers. The initial state is denoted by $(N, \varnothing, \varnothing)$. A non-initial state is characterized by a vector (P, R, D, β^R), where $P, R, D \subseteq N$ and $\beta^R \in R$ denote the sets of proposers, respondents, dropouts, and a representative of the respondents, respectively. To sum up, the set of all non-initial states is $\{(P, R, D, \beta^R) \in 2^N \times 2^N \times 2^N \times N \mid P, R, D \text{ are mutually disjoint}; |P| \geq 1; |R| \geq 1; P \cup R \cup D = N; \beta^R \in R\}$. In particular, a terminal state is a non-initial state (P, R, D, β^R) such that $|P| = 1$. Mechanism A proceeds as follows:

> We distinguish between the initial state and non-initial states.
>
> At the initial state $(N, \varnothing, \varnothing)$, an order on N is fixed beforehand. Player $m \in N$ is the last player in this order. Each player $i \in N$ proposes an allocation plan $\mathbf{y}_i^N \equiv (y_{ik}^N)_{k \in N} \in \mathbb{R}^N$. Proposals are simultaneous. There are two cases:

Case 1. If the proposers reach a consensus, which means that there exists

$\mathbf{y}^N = \mathbf{y}_i^N$ for all $i \in N$, then the consensual allocation plan $\mathbf{y}^N$ is implemented according to the order: Each player $j \in N \setminus \{m\}$ receives y_j^N, while the last player m receives the residual $v(N) - \sum_{j \in N \setminus \{m\}} y_j^N$. The mechanism ends.

Case 2. If they do not reach a consensus, then the proposers play the PW procedure. The procedure selects $\alpha^n \in \text{argmin}_{i \in N} B_i^N$. Then, every player $i \in N \setminus \{\alpha^n\}$ pays the submitted bid $b_{i\alpha^n}^N$ to the player α^n. Now player α^n is bought out and becomes a respondent. He automatically is the representative of the set of respondents (which only has one member). We rename the new representative α^n by $\beta^{\{\alpha^p\}}$. The state changes to $(N \setminus \{\alpha^n\}, \{\alpha^n\}, \varnothing, \beta^{\{\alpha^n\}})$.

At each non-initial state (P, R, D, β^R), each proposer $i \in P$ puts forth an allocation plan for the set of proposers: $\mathbf{y}_i^P \equiv (y_{ik}^P)_{k \in P} \in \mathbb{R}^P$. Proposals are simultaneous. Again, there are two cases:

Case 1′. If the players in P reach a consensus, which means that there exists $\mathbf{y}^P = \mathbf{y}_i^P$ for all $i \in P$, then the consensual allocation plan $\mathbf{y}^P$ is left up to representative β^R to decide. If β^R accepts, then the coalitions P and R merge and $\mathbf{y}^P$ is implemented as follows: each player $i \in P$ receives y_i^P and representative β^R receives the residual $v(P \cup R) - \sum_{j \in P} y_j^P$. If β^R declines, the coalition P breaks down and the coalition R forms. Each player $i \in P$ receives y_i^P and representative β^R receives $v(R)$. The mechanism ends.

Case 2′. If they do not reach a consensus, a proposer will be bought out by the same PW procedure (Stage I). Then this player negotiates with the incumbent representative to decide who is the new representative (Stage II).

Stage I. By the PW procedure, a player $\alpha^p \in \text{argmin}_{i \in P} B_i^P$ is selected. Once selected, every player $i \in P \setminus \{\alpha^p\}$ pays the submitted bid $b_{i\alpha^p}^P$ to player α^p. Player α^p is bought out.

Stage II. Player α^p makes an offer $z^p \in \mathbb{R}$ to the incumbent representative β^R. If β^R accepts α^p's offer, then α^p becomes the new representative of the set of respondents. The new representative α^p is renamed $\beta^{R \cup \{\alpha^p\}}$, and the state changes to $(P \setminus \{\alpha^p\}, R \cup \{\alpha^p\}, D, \beta^{R \cup \{\alpha^p\}})$; otherwise, α^p becomes a dropout, and the state changes to $(P \setminus \{\alpha^p\}, R, D \cup \{\alpha^p\}, \beta^R)$.

We notice that the mechanism ends in a maximum of n rounds. Indeed, in each round either there is a consensus among players in P, in which case the mechanism goes to Case 1 or Case $1'$ and it ends; or there is a disagreement, in which case the mechanism goes to Case 2 or Case $2'$, and the set of proposers in the next round has one player less. If the game reaches a terminal state, where $|P| = 1$ then the mechanism necessarily goes to Case $1'$ or Case $2'$ and it ends in both cases.

We show in Theorem 2 that the payoff vector of every SPNE of Mechanism A is equal to the Shapley value of v in the set of strictly 0-monotonic TU games.

Theorem 2. *Mechanism A implements the Shapley value for all strictly 0-monotonic TU games $v \in \mathcal{G}_{0^*}^N$ in SPNE.*

We notice that while Theorem 2 states that the payoff vector of every SPNE is the Shapley value, it does not ensure the uniqueness of the SPNE that leads to that payoff vector. In fact, in contrast to the PW mechanism, the SPNE of Mechanism A applied to strictly 0-monotonic TU games is not unique. In the proof of the theorem, we propose a strategy profile (which we denote by σ) which is an SPNE of Mechanism A. We now propose an example of an alternative SPNE: at each state where the set of players is P and the set of respondents is R, each player $i \in P$ proposes an identical allocation plan $\mathbf{y}^P = Sh\big((v|_{P\cup R})^R\big)$. If there exists a representative $\beta^R \in R$, β^R accepts any allocation plan $\mathbf{y}^P$ such that $\sum_{i\in P} y_i^P \leq v(R)$ and declines otherwise. The rest of strategies such as players' bids are the same as σ. In this SPNE, the mechanism ends at the very beginning: each player proposes an identical allocation plan equal to the prescription of the Shapley value and receives what they propose afterwards.

We also notice that Theorem 2 cannot be extended to all 0-monotonic TU games:

Remark 2. *Theorem 2 cannot be extended to all 0-monotonic TU games for $|N| \geq 3$.*

2.4 A mechanism that implements the equal surplus value

In this section, we present a comparable mechanism that implements the equal surplus value. We first describe it informally as follows:

Mechanism B. *The mechanism is identical to Mechanism A except for the following two aspects:*

1. *The role of the representative of the set of respondents is absent and there are no dropouts.*

2. *The proposers' allocation plan specifies a payoff for every player, not just for the proposers. A proposers' consensual allocation plan is left to each respondent to decide sequentially.*

   Formally, let each state be denoted by $P \in 2^N \setminus \{\varnothing\}$.

   For $n = 1$, the only player receives his individual rational payoff.

   For $n \geq 2$, at each state P, an order on N is fixed beforehand. Player $m \in P$ is the last player in this order. Each proposer $i \in P$ puts forth an allocation plan $\mathbf{y}_i^P \equiv (y_{ik}^P)_{k \in N} \in \mathbb{R}^N$. There are two cases:

Case 1. If the proposers in P reach a consensus, which means there exists $\mathbf{y}^P = \mathbf{y}_i^P$ for all $i \in P$, the consensual allocation plan $\mathbf{y}^P$ is left to every respondent $j \in N \setminus P$ to decide sequentially. Each respondent may accept or decline the plan. Let the set of respondents who accept the plan be denoted by Q. Coalitions P and Q merge into one coalition. Then the allocation plan $\mathbf{y}^P$ is implemented according to the order as follows: each player $j \in (P \cup Q) \setminus \{m\}$ receives y_j^P, while the last player m receives the residual $v(P \cup Q) - \sum_{j \in N \setminus \{m\}} y_j^P$. Each player $k \in N \setminus (P \cup Q)$ forms his own standalone coalition and receives his individual rational payoff $v(\{k\})$. The mechanism ends.

Case 2. If they do not reach a consensus, then by the PW procedure, select $\alpha^p \in \operatorname{argmin}_{i \in P} B_i^P$, where in the case of a nonunique minimizer, α^p is randomly chosen among the minimizing indices. Once selected, every player $i \in P \setminus \{\alpha^p\}$ pays his submitted bid $b_{i\alpha^p}^P$ to player α^p. Player α^p is bought out. The state changes to $P \setminus \{\alpha^p\}$.

We show in Theorem 3 that given any 0-monotonic TU game $v \in \mathcal{G}_0^N$, the payoff vector of every SPNE of Mechanism B is equal to the equal surplus value of v.

Theorem 3. *Mechanism B implements the equal surplus value for every 0-monotonic TU game $v \in \mathcal{G}_0^N$ in SPNE.*

Like Mechanism A, the constructed SPNE σ' of Mechanism B is not unique either. An alternative SPNE that the mechanism ends in the beginning can be constructed similarly.

2.5 Conclusion

This paper introduces two new mechanisms that implement the Shapley and the equal surplus values. In these mechanisms, each player is allowed to propose an allocation plan for every player in the beginning. The plan will be implemented only when the proposers' plans are consensual. In case of a disagreement, one player will be bought out by the PW procedure: this player receives the submitted bids from the rest of players and reduces his right to propose a plan to the right to accept or decline a consensual plan. Then the proposing stage repeats. The game will continue until a consensual plan emerges. If the players who are bought are represented by one of them in the negotiation with the proposers, then the mechanism implements the Shapley value. On the other hand, if each player who is bought out negotiates with the proposers for himself, then the mechanism implements the equal surplus value.

Just as the mechanism proposed in Pérez-Castrillo and Wettstein (2001), our two mechanisms are applicable to TU games with externalities. De Clippel and Serrano (2008) pointed out that the PW mechanism implements the Pham Do and Norde's (2007) externality-free value for all 0-monotonic TU games with externalities. Similarly, it is immediate that Mechanism A implements the externality-free Shapley value for all convex TU games with externalities as well.[6] It is worth mentioning that the externality-free value can be expressed as the Shapley value of a derived TU game without externalities. The worth of each coalition of the derived TU game is equal to the worth of the same coalition embedded in the finest coalition structure.

Moreover, we believe that the idea of using the PW procedure to put a player in jeopardy is not only useful for implementing the Shapley value and the equal surplus value. For example, we may consider combining our mechanisms and the PW mechanism. To be precise, we can use the PW procedure to designate the player with the highest total net bid as the proposer. This selected player pays his submitted bid to each of the rest of players and proposes an allocation plan for everyone, which is the same as in the original PW mechanism. The difference appears when the allocation plan of the proposer is rejected by some player. Then the PW procedure follows. But this time, the player with the lowest total net bid is selected. This player receives the submitted bid from the rest of players and drops out of the game with his individual rational payoff. The game restarts with the remaining players. Notice that this mechanism can be viewed as a counterpart of Calvo's (2008) multilateral bargaining model with random removal breakdown. We conclude this paper with the conjecture that this described mechanism also

[6]See Hafalir (2007) for a definition of convexity of TU games with externalities.

implements the Nowak and Radzik's (1994) solidarity value.

2.6 Appendix

Proof of Lemma 4. For $|N| > 1$, the formula (2.2.7) can be derived as follows:

$$\begin{aligned}
&\frac{1}{n}v(\{i\}) + \frac{1}{n}\sum_{j\in N\setminus\{i\}} Sh_i(v^{\{j\}}) \\
=&\frac{1}{n}[v^*(N) - v^*(N\setminus\{i\})] + \frac{1}{n}[\sum_{j\in N\setminus\{i\}} Sh_i(v^*|_{2^{N\setminus\{j\}}}{}^*)] \\
=&\frac{1}{n}[v^*(N) - v^*(N\setminus\{i\})] + \frac{1}{n}[\sum_{j\in N\setminus\{i\}} Sh_i(v^*|_{2^{N\setminus\{j\}}})] \\
=&Sh_i(v^*) = Sh_i(v),
\end{aligned}$$

where the first equality follows from the definition of the dual game and that for all $S \subseteq N\setminus\{j\}$, $v^*|_{2^{N\setminus\{j\}}}{}^*(S) = v^*|_{2^{N\setminus\{j\}}}(N\setminus\{j\}) - v^*|_{2^{N\setminus\{j\}}}((N\setminus\{j\})\setminus S) = v^*|_{2^{N\setminus\{j\}}}(N\setminus\{j\}) - v^*|_{2^{N\setminus\{j\}}}(N\setminus(S\cup\{j\})) = v^*(N\setminus\{j\}) - v^*(N\setminus(S\cup\{j\})) = [v(N) - v(\{j\})] - [v(N) - v(S\cup\{j\})] = v(S\cup\{j\}) - v(\{j\}) = v^{\{j\}}(S)$; the second and fourth from the self-duality of the Shapley value; and the third is an application of the formula (2.2.6). □

Proof of Lemma 6. We derive this property from the balanced contributions property of the Shapley value: $Sh_i(v) - Sh_i(v|_{2^{N\setminus\{j\}}}) = Sh_j(v) - Sh_j(v|_{2^{N\setminus\{i\}}}) \implies Sh_i(v^*) - Sh_i(v^*|_{2^{N\setminus\{j\}}}) = Sh_j(v^*) - Sh_j(v^*|_{2^{N\setminus\{i\}}}) \implies Sh_i(v) - Sh_i(v^*|_{2^{N\setminus\{j\}}}{}^*) = Sh_j(v) - Sh_j(v^*|_{2^{N\setminus\{i\}}}{}^*) \implies Sh_i(v) - Sh_i(v^{\{j\}}) = Sh_j(v) - Sh_j(v^{\{i\}})$, where the first implication follows from substituting v with v^*, the second from the self-duality of the Shapley value, and the third from a property that is included in the proof of Lemma 4: $v^*|_{2^{N\setminus\{j\}}}{}^* = v^{\{j\}}$. □

Proof of Lemma 7. $ES_i(v) - ES_i(v^{\overline{\{j\}}}) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{k\in N} v(\{k\})] - \{v(\{i\}) + \frac{1}{n-1}[v^{\overline{\{j\}}}(N\setminus\{j\}) - \sum_{k\in N\setminus\{j\}} v^{\overline{\{j\}}}(\{k\})]\} = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{k\in N} v(\{k\})] - \{v(\{i\}) + \frac{1}{n-1}[v(N) - v(\{j\}) - \sum_{k\in N\setminus\{j\}} v(\{k\})]\} = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{k\in N} v(\{k\})] - \{v(\{i\}) + \frac{1}{n-1}[v(N) - \sum_{k\in N} v(\{k\})]\} = -\frac{1}{n(n-1)}[v(N) - \sum_{k\in N} v(\{k\})]$, where the first equality follows from the defining formula (2.2.8) of the equal surplus value and the second from the defining formula (2.2.4) of the aiding game of Type II. The expression implies that $ES_i(v) - ES_i(v^{\overline{\{j\}}})$ is independent of i and j. Therefore the equal surplus value satisfies the balanced contributions property of Type II. □

Proof of Proposition 9. We use the same argument as in the proof of the theorem in Myerson (1977). We prove the equivalence using the balanced contributions property of type I; the proof of the result for the balanced contributions property of Type II is similar. By Lemma 6, the Shapley value satisfies efficiency and the modified balanced contributions property of Type I. For the opposite direction, we verify the uniqueness by an induction on the number of players $|N|$. For $|N| = 1$, the value is uniquely pinned down by efficiency. For $|N| > 1$ and two distinct players $i, j \in N$, $\phi_j(v) = \phi_i(v) + [\phi_j(v^{\{i\}}) - \phi_i(v^{\{j\}})] = \phi_i(v) + [Sh_j(v^{\{i\}}) - Sh_i(v^{\{j\}})]$, where the first equality follows from the balanced contributions property of Type I and the second from the induction hypothesis. By efficiency, we have $\sum_{k \in N} \phi_k(v) = v(N)$. Then $n\phi_i(v) + \sum_{j \in N \setminus \{i\}} [Sh_j(v^{\{i\}}) - Sh_i(v^{\{j\}})] = v(N)$. Therefore, $\phi_i(v)$ is unique. □

Proof of Theorem 2. We first construct a strategy profile for Mechanism A, which we denote by σ, and we will verify that it constitutes an SPNE when applied to strictly 0-monotonic TU games. Second, we calculate the payoff vector of σ in the subgame starting at each state. In particular, we show that the payoff vector of σ coincides with the Shapley value. Finally, we show that the payoff vector of every SPNE is equal to that of σ in the subgame starting at each state.

The strategy profile σ is constructed as follows:

1. At the initial state $(N, \varnothing, \varnothing)$, each player $i \in N$ proposes an allocation plan $\mathbf{y}_i^N$ such that $y_{ij}^N = -C$ for all $j \in N \setminus \{i\}$ and $y_{ii}^N = C$, where C is a large enough positive number. Then each player $i \in N$ submits a bid $b_{ij}^N = Sh_i(v) - Sh_i(v^{\{j\}})$ against each $j \in N \setminus \{i\}$.

2. At a terminal state $(\{p\}, R, D, \beta^R)$, player p proposes an allocation plan $y^{\{p\}}$ (which is automatically consensual) such that $y_p^{\{p\}} = v(R \cup \{p\}) - v(\{p\})$. Representative β^R accepts if p's proposal satisfies $y_p^{\{p\}} \leq v(R \cup \{p\}) - v(R)$ and declines otherwise.

3. At any state (P, R, D, β^R) such that $1 < |P| < n$, each $i \in P$ proposes an allocation plan $\mathbf{y}_i^P$ such that $y_{ij}^P = -C$ for all $j \in P \setminus \{i\}$ and $y_{ii}^P = C$. Then each player $i \in P$ submits a bid $b_{ij}^P = Sh_i\big((v|_{2^{P \cup R}})^{R \cup \{j\}}\big) - Sh_i\big((v|_{2^{P \cup R}})^R\big)$ against each $j \in P \setminus \{i\}$. Player $\alpha^p \in P$ who is selected by the PW procedure offers $z^p = v(R)$ to representative β^R. β^R accepts α^p's offer if it satisfies $z^p \geq v(R)$ and declines otherwise.

We verify that the strategy profile σ constitutes an SPNE by an induction on the number of proposers.

Regarding the terminal states, which are states with a single proposer, we prove the following result, which implies in particular that σ constitutes an SPNE if $|P| = 1$.

Claim 2. *In any SPNE, at a terminal state* $(\{p\}, R, D, \beta^R)$*, the player* p *proposes an allocation plan* $y^{\{p\}}$ *such that* $y_p^{\{p\}} = v(R \cup \{p\}) - v(\{p\})$*. The representative* β^R *accepts the plan if* p*'s plan satisfies* $y_p^{\{p\}} \leq v(R \cup \{p\}) - v(R)$ *and declines otherwise.*

To prove Claim 2, notice that given a strictly 0-monotonic TU game $v \in \mathcal{G}_{0^*}^N$, at any terminal state $(\{p\}, R, D, \beta^R)$, in any SPNE, representative β^R declines any allocation plan $y_p^{\{p\}} > v(R \cup \{p\}) - v(R)$ and accepts any allocation plan $y_p^{\{p\}} < v(R \cup \{p\}) - v(R)$ from p. This holds because the payoff of representative β^R is $v(R \cup \{p\}) - y_p^{\{p\}}$ in case of acceptance (where coalitions R and $\{p\}$ merge, player p receives $y_p^{\{p\}}$, and player β^R receives the residue of $v(R \cup \{p\})$), whereas his payoff is $v(R)$ (the worth of the coalition he represents) in case of rejection.

Moreover, a proposal such that $y_p^{\{p\}} < v(R \cup \{p\}) - v(R)$ cannot be part of an equilibrium. Such an offer would be accepted but p would have an incentive to increase $y_p^{\{p\}}$ by a sufficiently small amount and propose another acceptable offer with a higher payoff for himself. Also, a proposal that is rejected cannot be part of an SPNE due to the strict 0-monotonicity of the game: p would have an incentive to propose an acceptable offer close to $v(R \cup \{p\}) - v(R)$. Hence, $y^{\{p\}} > v(R \cup \{p\}) - v(R)$ cannot be part of an SPNE. Therefore, at a terminal state $(\{p\}, R, D, \beta^R)$, in any SPNE, it has to be the case that p's allocation plan is $y_p^{\{p\}} = v(R \cup \{p\}) - v(R)$.

Finally, at equilibrium, representative β^R should accept the allocation plan if $y_p^{\{p\}} = v(R \cup \{p\}) - v(R)$. Otherwise, the proposer would have an incentive to slightly increase the offer. Hence, at equilibrium, the representative accepts any allocation plan that satisfies $y_p^{\{p\}} \leq v(R \cup \{p\}) - v(R)$ and declines otherwise, which ends the proof of Claim 2.

Once we have stated the unique SPNE strategies that can be played at a terminal state, we proceed to consider a non-terminal state with a set of proposers P such that $|P| > 1$.

First, using a similar reasoning as in the proof of Claim 2, we can state the following claim (whose proof is omitted):

Claim 3. *In any SPNE, at any non-initial state* (P, R, D, β^R) *such that* $|P| > 1$*, if the proposers in* P *reach a consensual allocation plan* $\mathbf{y}^P$*, then* β^R *accepts the plan if it satisfies that* $\sum_{i \in P} y_i^P \leq v(P \cup R) - v(R)$ *and declines otherwise. If they don't reach a consensus, then the player* $\alpha^p \in P$ *selected by the PW procedure offers* $z^p = v(R)$ *to representative* β^R*. Moreover,* β^R *accepts any offer* $z^p \geq v(R)$ *and declines otherwise.*

We now verify that the constructed strategy profile σ constitutes an SPNE in the subgame starting at a non-terminal state with a set of proposers P and a set of respondents R.

First, no player $i \in P$ has an incentive to change his proposed allocation plan. This is true because a consensus cannot be reached by a unilateral deviation for $|P| \geq 3$ and, if $|P| = 2$, then a proposer has no incentive to change his proposed allocation plan to match his opponent's proposed allocation.[7]

Second, no player has an incentive to change his submitted bid vector either. Notice that by Lemma 6, $b_{ij}^P - b_{ji}^P = (Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{j\}}\big) - Sh_i\big((v|_{2^{P\cup R}})^R\big)) - (Sh_j\big((v|_{2^{P\cup R}})^{R\cup\{i\}}\big) - Sh_j\big((v|_{2^{P\cup R}})^R\big)) = 0$ for all $i, j \in P$, thus $B_i^P = \sum_{j\in N\setminus\{i\}}(b_{ij}^P - b_{ji}^P) = 0$ for all $i \in P$. Thus, the bids proposed in σ lead to a tie.

To see that a proposer has no incentive to change his submitted bids, we denote the bid vector and the total net bid for each proposer $k \in P$ by $\mathbf{b}_k^P$ and B_k^P, respectively. As we have shown, $B_k^P = 0$ for all $k \in P$. Suppose that proposer $i \in P$ changes his bid $\mathbf{b}_i^P$ to $\bar{\mathbf{b}}_i^P$. We denote the proposer k's total net bid resulting from proposer i's change by $\bar{B}_k^P$ for each $k \in P$. We differentiate two cases depending on whether or not player i himself is selected by the PW procedure after the change in bids. If player i himself is selected, player i receives $\sum_{j\in P\setminus\{i\}} b_{ji}^P$, which is unchanged. If some proposer $j \in P \setminus \{i\}$ is selected, it must be the case that $\bar{B}_j^P \leq 0$ (because if $\bar{B}_j^P > 0$, then there exists $k \in P$ such that $\bar{B}_j^P < 0$ by $\sum_{l\in P} \bar{B}_l^P = 0$ as shown in footnote 5, which implies proposer j cannot be selected). Thus $\bar{B}_j^P - B_j^P \leq 0$ which implies $\bar{b}_{ij}^P - b_{ij}^P \geq 0$. The payment of proposer i cannot increase in this case. In neither case, does a proposer have an incentive to change his bid vector. Therefore, the constructed strategy profile σ constitutes an SPNE.

Now we calculate the payoff vectors of σ in the subgame starting at each state. The result is summarized in the following claim:

Claim 4. *Given a state where the set of proposers is P and the set of respondents is R, the payoff of each proposer $i \in P$ in σ in the subgame starting at this state is $Sh_i\big((v|_{2^{P\cup R}})^R\big)$. If there exists a representative $\beta^R \in R$, the payoff of β^R is $v(R)$.*

We prove this claim by an induction on the number of proposers $|P|$. For $|P| = 1$, it follows from Claim 2 that the payoff of proposer p is $v(R \cup \{p\}) - v(R)$. We have $v(R \cup \{p\}) - v(R) = (v|_{2^{R\cup\{p\}}})^R(\{p\}) = Sh_p\big((v|_{2^{R\cup\{p\}}})^R\big)$. By Claim 2, it is also immediate that the payoff of β^R is $v(R)$.

[7] If $|P| = 2$, a player's payoff if he matches his opponent's proposed allocation plan is at most his individual rational payoff. To see this, we distinguish two cases. First, if it is an initial state, it depends on whether or not the proposer is the last player in the fixed order. If the proposer is the last player, his payoff is the residue $v(N) - C$; otherwise, his payoff is $-C$ according to the plan. Second, if it is at a non-initial state, it depends on whether or not the resulting consensual allocation plan is accepted. If the resulting consensual allocation plan is accepted, his payoff is $-C$ according to the plan; otherwise, his payoff is his individual rational payoff.

Then we consider a state where the set of proposers is P such that $|P| > 1$, the set of respondent is R, and the set of dropouts is D. The induction hypothesis states that the payoff of player $i \in P$ in σ is equal to $Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{j\}}\big)$ in the subgame starting at the state $(P\backslash\{j\}, R\cup\{j\}, D, \beta^{R\cup\{j\}})$, where $\beta^{R\cup\{j\}} = j$ for each $j \in P\backslash\{i\}$ according to σ; and the payoff of player $i \in P$ is equal to $v(R\cup\{i\}) - v(R)$ in σ in the subgame starting at the state $(P \setminus \{i\}, R \cup \{i\}, D, \beta^{R\cup\{i\}})$, where $\beta^{R\cup\{i\}} = i$.

When the players in P play according to σ, there is no consensual plan. Therefore, the payoff of proposer $i \in P$ comes from the subgame played in the PW procedure. The payoff of proposer i in σ in the subgame starting at the state under consideration is $Sh_i\big((v|_{2^{(P\backslash\{j\})\cup(R\cup\{j\})}})^{R\cup\{j\}}\big) - b^P_{ij} = Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{j\}}\big) - b^P_{ij} = Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{j\}}\big) - [Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{j\}}\big) - Sh_i\big((v|_{2^{P\cup R}})^{R}\big)] = Sh_i\big((v|_{2^{P\cup R}})^{R}\big))$ if j is selected by the PW procedure, for each $j \in P \setminus \{i\}$. Similarly, i's payoff is $v(R\cup\{i\}) - v(R) + \sum_{j\in P\backslash\{i\}} b^P_{ji} = v(R\cup\{i\}) - v(R) + \sum_{j\in P\backslash\{i\}}[Sh_j\big((v|_{2^{P\cup R}})^{R\cup\{i\}}\big) - Sh_j\big((v|_{2^{P\cup R}})^{R}\big)] = v(R\cup\{i\}) - v(R) + \sum_{j\in P\backslash\{i\}} Sh_j\big((v|_{2^{P\cup R}})^{R\cup\{i\}}\big) - \sum_{j\in P\backslash\{i\}} Sh_j\big((v|_{2^{P\cup R}})^{R}\big) = v(R\cup\{i\}) - v(R) + [v(P\cup R) - v(R\cup\{i\})] - [v(P\cup R) - v(R) - Sh_i\big((v|_{2^{P\cup R}})^{R}\big)] = Sh_i\big((v|_{2^{P\cup R}})^{R}\big)$ if i is selected by the PW procedure. In both cases, proposer $i \in P$ receives $Sh_i\big((v|_{2^{P\cup R}})^{R}\big)$ in σ in the subgame starting at this state. Moreover, it is immediate that the payoff of β^R is $v(R)$ by Claim 3.

By letting $P = N$ and $R = \varnothing$, Claim 4 implies that Mechanism A leads to the Shapley value if players play the SPNE σ.

Before extending Claim 4 from the particular SPNE σ to every SPNE of Mechanism A, we need the following two properties that are satisfied by every SPNE.

Claim 5. *In any SPNE of Mechanism A, at any state where the set of proposers is P such that $|P| \geq 2$, the total net bid B^P_i is equal to 0 for each $i \in P$.*

To prove Claim 5, we first recall that we have shown that the sum of each proposer's total net bid $\sum_{k\in P} B^P_k$ is equal to 0 in footnote 5. If there exists $i \in P$ such that $B^P_i \neq 0$, then there exist two distinct players $j, k \in P$ such that $B^P_j > 0$ and $B^P_k < 0$. In this case, player j would be strictly better off by lowering each of his bids with an equal small enough amount, which would not change the set of potential winners but would decrease his payment to the winner. Therefore, in any SPNE of the mechanism A, $B^P_i = 0$ for all $v \in \mathcal{G}^N_{0^*}$.

In brief, Claim 5 states that the bids proposed in σ must result in a tie in every SPNE. It implies that in any SPNE, each proposer is indifferent between who is selected by the PW procedure if no consensus is reached. Formally, this property is stated as follows:

Claim 6. *In any SPNE of Mechanism A, at any state where the set of proposers is*

P such that $|P| \geq 2$, if no consensus is reached, proposer i's payoff resulting from proposer j being selected and his payoff resulting from proposer k being selected by the PW procedure are equal for all $i, j, k \in P$.

We prove the claim by contradiction. Suppose that there exist $i, j, k \in P$ such that proposer i's payoff resulting from proposer j being selected is strictly higher than his payoff resulting when proposer k is selected by the PW procedure. Without loss of generality, we assume further that proposer i's payoff when proposer j is selected is not lower than his payoff when any other player in P is selected and that proposer i's payoff when proposer k is selected is not higher than his payoff when any other player in P is selected. We need to consider two cases depending on whether $i = j$. First, if $i = j$, proposer i has an incentive to decrease b_{ih}^P for all $h \in P \backslash \{i\}$ to ensure that he is selected. Second, if $i \neq j$, proposer i has an incentive to increase b_{ij}^P to ensure proposer j is selected and his subsequent payment to j is small enough. In either case, proposer i has a profitable deviation. Therefore, a proposer is indifferent between which proposer is selected in every SPNE.

Now we are ready to extend Claim 4 to every SPNE:

Claim 7. *Given a state where the set of proposers is P and the set of respondents is R, the payoff of each proposer $i \in P$ in **every** SPNE in the the subgame starting at this state is $Sh_i\big((v|_{2^{P\cup R}})^R\big)$. If there exists $\beta^R \in R$, the payoff of β^R is $v(R)$.*

We prove this result by an induction on the number of proposers $|P|$. By Claim 2, it is immediate that the result is true for all terminate states, i.e. the states such that $|P| = 1$.

Consider a state where the set of proposers is P, the set of respondent is R and the set of dropouts is D such that $|P| > 1$. If no consensus is reached, the state changes to $(P\backslash\{j\}, R\cup\{j\}, D, \beta^{R\cup\{j\}})$, where $\beta^{R\cup\{j\}} = j$ by Claim 3, for some $j \in P$. The induction hypothesis states that at the state $(P \setminus \{j\}, R \cup \{j\}, D, \beta^{R\cup\{j\}})$, the payoff for each $i \in P \setminus \{j\}$ is $Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{j\}}\big)$ in every SPNE in the subgame starting at $(P \setminus \{j\}, R \cup \{j\}, D, \beta^{R\cup\{j\}})$ and the payoff for the representative j is $v(R\cup\{j\})$ in every SPNE in the subgame starting from $(P\setminus\{j\}, R\cup\{j\}, D, \beta^{R\cup\{j\}})$.

Any SPNE must fit in one of three cases: (I) each player $i \in P$ proposes an identical allocation plan $\mathbf{y}^P$ such that $\sum_{i\in P} y_i^P \leq v(P \cup R) - v(R)$; (II) each player $i \in P$ proposes an identical allocation plan $\mathbf{y}^P$ such that $\sum_{i\in P} y_i^P > v(P\cup R) - v(R)$; (III) there exist two distinct players $i, j \in P$ such that $\mathbf{y}_i^P \neq \mathbf{y}_j^P$.

Consider Case III. By Claim 16, in any SPNE, it has to be the case that each player $i \in P$ is indifferent between being selected or not by the PW procedure. This means that $v(R \cup \{i\}) - v(R) + \sum_{j\in P\setminus\{i\}} b_{ji}^P = Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{k\}}\big) - b_{ik}^P$ for all

$i \in P$ and $k \in P \setminus \{i\}$. Thus player i is certain to obtain the average of the total payoff $\frac{1}{p}[v(R \cup \{i\}) - v(R) + \sum_{j \in P\setminus\{i\}} b^P_{ji}] + \frac{1}{p}\sum_{k\in P\setminus\{i\}}[Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{k\}}\big) - b^P_{ik}] = \frac{1}{p}[v(R\cup\{i\}) - v(R)] + \frac{1}{p}\sum_{k\in P\setminus\{i\}} Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{k\}}\big) - \frac{1}{p}B^P_i = \frac{1}{p}[v(R\cup\{i\}) - v(R)] + \frac{1}{p}\sum_{k\in P\setminus\{i\}} Sh_i\big((v|_{2^{P\cup R}})^{R\cup\{k\}}\big) = Sh_i\big((v|_{2^{P\cup R}})^{R}\big)$, where the penultimate equality follows from Claim 5 and the last equality from Lemma 4 by letting $v = (v|_{P\cup R})^R$. Therefore, the payoff vector of any SPNE that fits in the Case III is equal to the payoff vector of σ.

Concerning Case II, by Claim 3, β^P will reject the consensual allocation plan $\mathbf{y}^P$. Then the coalition P breaks down. The sum of payoffs of players in P is $\sum_{i\in P} v(\{i\})$. On the other hand, if no consensus were reached, the sum of payoffs of proposers in P would be $\sum_{i\in P} Sh_i\big((v|_{2^{P\cup R}})^R\big) = v(P\cup R) - v(R)$, which is strictly higher than $\sum_{i\in P} v(\{i\})$ by strict 0-monotonicity. Hence, some player in P has an incentive to obstruct this allocation plan by proposing a different allocation plan. Thus, there exists no SPNE that fits in the Case II.

Similarly, for Case I, if the consensual allocation plan is such that $\mathbf{y}^P \neq Sh\big((v|_{P\cup R})^R\big)$ and $\sum_{i\in P} y^P_i \leq v(P\cup R) - v(R)$, then some player in P has an incentive to obstruct the allocation plan. Therefore, the payoff vector of any SPNE that fits the Case I must be equal to that of σ.

Hence, we have proven that the payoff of each proposer $i \in P$ in every SPNE in the subgame starting at this state is $Sh_i\big((v|_{2^{P\cup R}})^R\big)$. By letting $P = N$ and $R = \varnothing$, the payoff vector of every SPNE is equal to the Shapley value of v.

Therefore, given any strictly 0-monotonic TU game v, there exists an SPNE of Mechanism A and the payoff vector of every SPNE is equal to the Shapley value of v. □

Proof of Remark 2. Consider a three-player unanimity game $u_{\{1,2,3\}} \in \mathcal{G}^{\{1,2,3\}}$, which is defined by

$$u_{\{1,2,3\}}(S) = \begin{cases} 1 & \text{if } S = \{1,2,3\}; \\ 0 & \text{otherwise.} \end{cases}$$

It is worth mentioning that $u_{\{1,2,3\}}$ is both 0-monotonic and supper-additive. Let C be a large enough positive number. We construct an SPNE as follows:

1. For the initial state $(\{1,2,3\}, \varnothing, \varnothing)$, player 1 proposes $(C, -C, -C)$; player 2 proposes $(-C, C, -C)$; player 3 proposes $(-C, -C, C)$. Then, each player $i \in \{1,2,3\}$ submits a bid $b^{\{1,2,3\}}_{ij} = 0$ against each $j \in \{1,2,3\} \setminus \{i\}$.

2. For a terminal state $(\{p\}, R, D, \beta^R)$, the player p proposes an allocation plan $y^{\{p\}} = u_{\{1,2,3\}}(R \cup \{p\})$. Representative β^R accepts if p's proposal $y^{\{p\}} \leq u_{\{1,2,3\}}(R \cup \{p\})$ and declines otherwise.

3. For $(\{i,j\},\{k\},\varnothing,k)$ where $i,j,k \in \{i,j,k\}$ are distinct, player i proposes $y_{ii}^{\{i,j\}} = C$ and $y_{ij}^{\{i,j\}} = -C$; player j proposes $y_{ji}^{\{i,j\}} = -C$ and $y_{jj}^{\{i,j\}} = C$. Player i submits a bid $b_{ij}^{\{i,j\}} = 0$ against player j; player j submits a bid $b_{ji}^{\{i,j\}} = 0$ against player i. Let $\alpha^2 \in \{i,j\}$ denote the player who is selected by the PW procedure. α^2 offers $z^2 = 0$ to representative k. Representative k accepts α^2's offer if $z^2 > 0$ and declines otherwise.

It is easy to check that the proposed strategy profile constitutes an SPNE of Mechanism A. Moreover, the payoff vector of the above SPNE is $(0,0,0)$, which is not equal to $Sh(u_{\{1,2,3\}}) = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$. Therefore, it is not the case that Mechanism A implements the Shapley value in every SPNE when $|N| \geq 3$. □

Proof of Theorem 3. As with Mechanism A, we first construct a strategy profile for Mechanism B, denoted by σ', and we verify that it constitutes an SPNE of Mechanism B when applied to 0-monotonic TU games. Second, we calculate the payoff vector of σ' in the subgame starting at each state. In particular, we prove that the payoff vector of σ' coincides with the equal surplus value. Finally, we show that the payoff vector of every SPNE is equal to that of σ' in the subgame starting at each state.

The strategy profile σ' is constructed as follows:

1. At each non-terminal state $P \in 2^N \setminus \{\varnothing\}$, each player $i \in P$ proposes an allocation plan $\mathbf{y}_i^P$ such that $y_{ij}^N = -C$ for all $j \in N \setminus \{i\}$ and $y_{ii}^P = C$, where C is a large enough positive number. Then each player $i \in P$ submits a bid $b_{ij}^P = ES_i(v^{\overline{N\setminus(P\setminus\{j\})}}) - ES_i(v^{\overline{N\setminus P}})$ against each $j \in P \setminus \{i\}$.

2. At each terminal state $\{p\}$, player p proposes an allocation plan $\mathbf{y}^{\{p\}}$ (which is automatically consensual) such that $y_p^{\{p\}} = ES_p(v^{\overline{N\setminus\{p\}}}) = v(N) - \sum_{j\in N\setminus\{p\}} v(\{j\})$ and $y_k^{\{p\}} = v(\{k\})$ for each $k \in N \setminus \{p\}$. Each player $k \in N \setminus \{p\}$ accepts $\mathbf{y}^{\{p\}}$ if $y_k^{\{p\}} \geq v(\{k\})$ and declines otherwise.

We verify that the strategy profile σ' constitutes an SPNE by an induction on the number of proposers.

Regarding the terminal states which are states with a single proposer, we prove the following result that is valid for every SPNE:

Claim 8. *Given a non-additive 0-monotonic TU game $v \in \mathcal{G}_0^N$, in any SPNE, at a terminal state $\{p\}$, the proposer p puts forth an allocation plan $\mathbf{y}^{\{p\}}$ such that $y_p^{\{p\}} = ES_p(v^{\overline{N\setminus\{p\}}}) = v(N) - \sum_{j\in N\setminus\{p\}} v(\{j\})$ and $y_k^{\{p\}} = v(\{k\})$ for each $k \in N \setminus \{p\}$. Each respondent $k \in N\setminus\{p\}$ accepts the plan if it satisfies $y_k^{\{p\}} \geq v(\{k\})$ and declines otherwise.*

The proof of Claim 8 is identical to the proof of Claim 2, and so we omit it. The strategies specified by Claim 8 also constitute part of an SPNE of Mechanism B applied to additive TU games, though the uniqueness is lost. Every SPNE strategies yields the same payoff in any subgame starting at a terminal state.

Claim 8′. *Given an additive TU game $v \in \mathcal{G}_a^N$, in any SPNE, in any subgame starting at a terminal state $\{p\}$, each player $i \in N$ receives his individual rational payoff $v(\{i\})$.*

The sum of every player's payoff in any subgame starting at $\{p\}$ is $v(N) = \sum_{i\in N} v(\{i\})$ by additivity of v. Each respondent $j \in N \setminus \{p\}$ can secure a payoff of $v(\{j\})$ by rejecting any proposal. Proposer p can secure a payoff of $v(\{p\})$ by putting forth an allocation plan specifying C for p himself and $-C$ for each respondent, where C is a large enough positive number. Thus, given an additive TU game, in any SPNE, in any subgame starting at a terminal state, each player receives his individual rational payoff.

Regarding any non-initial state, we prove the following claim.

Claim 9. *Given a non-additive 0-monotonic TU game $v \in \mathcal{G}_0^N$, in any SPNE, at any non-initial state P such that $|P| > 1$, if proposers in P reach a consensual allocation plan $\mathbf{y}^P$, each respondent $i \in N \setminus P$ accepts the plan if it satisfies $y_i^P \geq v(\{i\})$ and declines otherwise.*

The proof of Claim 9 also follows the same logic as that of Claim 2. Again, special attentions should be paid to additive TU games.

Claim 9′. *Given an additive TU game $v \in \mathcal{G}_a^N$, in any SPNE, at any subgame starting at any non-initial state P such that $|P| > 1$, if proposers in P reach a consensual allocation plan $\mathbf{y}^P$, each respondent $i \in N \setminus P$ receives his individual rational payoff $v(\{i\})$.*

The proof of Claim 9′ resembles that of Claim 8′. We consider a non-terminal state P such that $|P| > 1$ and we verify that the constructed strategy profile σ' constitutes an SPNE in the subgame starting at P. First, by the same reasoning in footnote 7, no player $i \in P$ has an incentive to change his proposed allocation plan. Second, we prove that no player has an incentive to change his submitted bid vector either. Notice that by Lemma 7, $b_{ij}^P - b_{ji}^P = (ES_i(v^{\overline{N\setminus(P\setminus\{j\})}}) - ES_i(v^{\overline{N\setminus P}})) - (ES_j(v^{\overline{N\setminus(P\setminus\{i\})}}) - ES_j(v^{\overline{N\setminus P}})) = 0$ for all $i, j \in P$, thus $B_i^P = \sum_{j\in N\setminus\{i\}}(b_{ij}^P - b_{ji}^P) = 0$ for all $i \in P$. It means that the bids proposed in σ' result in a tie. No proposer has an incentive to change his bid vector for the same reason as in the proof of Claim 3. Therefore, σ' constitutes an SPNE.

Now we calculate the payoff vectors of σ' in the subgame starting at each state. The result is summarized in the following claim:

Claim 10. *For each state P, the payoff of each proposer $i \in P$ in σ' in the the subgame starting at this state is $ES_i(v^{\overline{N\setminus P}})$.*

We prove this claim by an induction on the number of proposers $|P|$. For $|P| = 1$, we have proven in both Claim 8 and Claim 8$'$ that the payoff of proposer p is equal to $v(N) - \sum_{j\in N\setminus\{p\}} v(\{j\}) = ES_p(v^{\overline{N\setminus\{p\}}})$.

Consider a state P such that $|P| > 1$. The induction hypothesis states that the payoff of player $i \in P$ is equal to $ES_i(v^{\overline{N\setminus(P\setminus\{j\})}})$ in σ' in the subgame starting at state $P \setminus \{j\}$ for each $j \in P \setminus \{i\}$, and the payoff of player $i \in P$ is equal to $v(\{i\})$ in σ' in the subgame starting at the state $P \setminus \{i\}$.

When the players in P play according to σ', no consensual plan is reached. In this case, the payoff of proposer $i \in P$ comes from the subgame played in the procedure. The payoff of proposer $i \in P$ in σ' in the subgame starting at P is $ES_i(v^{\overline{N\setminus(P\setminus\{j\})}}) - b^P_{ij} = ES_i(v^{\overline{N\setminus(P\setminus\{j\})}}) - [ES_i(v^{\overline{N\setminus(P\setminus\{j\})}}) - ES_i(v^{\overline{N\setminus P}})] = ES_i(v^{\overline{N\setminus P}})$ if j is selected by the PW procedure for each $j \in P \setminus \{i\}$. Similarly, proposer i's payoff is $v(\{i\}) + \sum_{j\in P\setminus\{i\}} b^P_{ji} = v(\{i\}) + \sum_{j\in P\setminus\{i\}}[ES_j(v^{\overline{N\setminus(P\setminus\{i\})}}) - ES_j(v^{\overline{N\setminus P}})] = v(\{i\}) + \sum_{j\in P\setminus\{i\}} ES_j(v^{\overline{N\setminus(P\setminus\{i\})}}) - \sum_{j\in P\setminus\{i\}} ES_j(v^{\overline{N\setminus P}}) = v(\{i\}) + [v(N) - \sum_{k\in N\setminus(P\setminus\{i\})} v(\{k\})] - [v(N) - \sum_{k\in N\setminus P} v(\{k\}) - ES_i(v^{\overline{N\setminus P}})] = ES_i(v^{\overline{N\setminus P}})$ if i is selected by the PW procedure. In both cases, a proposer $i \in P$ receives $ES_i(v^{\overline{N\setminus P}})$ in σ' in the subgame starting at P.

By letting $P = N$, Claim 10 implies that Mechanism B leads to the equal surplus value if players play the SPNE σ'.

Before extending Claim 10 from one particular SPNE σ' to every SPNE, we prove the following two properties.

Claim 11. *In any SPNE of Mechanism B, at any state P such that $|P| \geq 2$, the total net bid B^P_i is equal to 0 for each $i \in P$.*

The proof of Claim 11 is identical to that of Claim 5. Like Claim 5, Claim 11 states that the submitted bids in σ' must result in a tie in every SPNE. It implies Claim 12, whose proof is omitted becasue it is the same as that for Claim 16.

Claim 12. *In any SPNE of Mechanism B, at any state P such that $|P| \geq 2$, if no consensus is reached, proposer i's payoff resulting from proposer j being selected and his payoff resulting from proposer k being selected by the PW procedure are equal for all $i, j, k \in P$.*

Now, we are ready to extend Claim 10 to every SPNE:

Claim 13. *For each state P, the payoff of each proposer $i \in P$ in every SPNE in the the subgame starting at this state is $ES_i(v^{\overline{N\setminus P}})$.*

We prove this claim by an induction on the number of proposers $|P|$. By Claim 8 and Claim 8′, it is immediate that the claim is true for all terminate states.

We consider now a state P such that $|P| > 1$. If no consensus is reached, the state must change to $P \setminus \{j\}$ for some $j \in P$. The induction hypothesis states that at the state $P \setminus \{j\}$, the payoff for each $i \in P \setminus \{j\}$ is $ES_i(v^{\overline{N\setminus(P\setminus\{j\})}})$ in every SPNE in the subgame starting at $P \setminus \{j\}$, and the payoff for the selected player j is $v(\{j\})$ in every SPNE in the subgame starting at $P \setminus \{j\}$.

Any SPNE must fit in one of three cases: (I) each player $i \in P$ proposes an identical allocation plan $\mathbf{y}^P$ such that for all $j \in N \setminus P$, $y_j^P \geq v(\{j\})$; (II) each player $i \in P$ proposes an identical allocation plan $\mathbf{y}^P$ such that there exists $j \in N \setminus P$ with $y_j^P < v(\{j\})$; (III) there exist two distinct players $i, j \in P$ such that $\mathbf{y}_i^P \neq \mathbf{y}_j^P$.

The payoff vector of any SPNE that fits in the Case III must be the same as that of the constructed SPNE. Indeed, by Claim 11, in any SPNE, it has to be the case that each player $i \in P$ is indifferent between being selected or not by the PW procedure. This means that $v(\{i\}) + \sum_{j\in P\setminus\{i\}} b_{ji}^P = ES_i(v^{\overline{N\setminus(P\setminus\{k\})}}) - b_{ik}^P$ for all $i \in P$ and $k \in P \setminus \{i\}$. Thus player i is certain to obtain the average of the total payoff $\frac{1}{p}[v(\{i\}) + \sum_{j\in P\setminus\{i\}} b_{ji}^P] + \frac{1}{p}\sum_{k\in P\setminus\{i\}}[ES_i(v^{\overline{N\setminus(P\setminus\{k\})}}) - b_{ik}^P] = \frac{1}{p}v(\{i\}) + \frac{1}{p}\sum_{k\in P\setminus\{i\}} ES_i(v^{\overline{N\setminus(P\setminus\{k\})}}) - \frac{1}{p}B_i^P = \frac{1}{p}v(\{i\}) + \frac{1}{p}\sum_{k\in P\setminus\{i\}} ES_i(v^{\overline{N\setminus(P\setminus\{k\})}}) = ES_i(v^{\overline{N\setminus P}})$ for all $i \in P$, where the penultimate equality follows from Claim 11 and the last equality from Lemma 5 by letting $v = v^{\overline{N\setminus P}}$. Therefore, the payoff vector of any SPNE that fits in the Case III is equal to the payoff vector of σ'.

As for Case II, by Claim 9 and Claim 9′, given the consensual allocation plan $\mathbf{y}^P$, the respondent $j \in N \setminus P$ such that $y_j^P < v(\{j\})$ will reject the plan $\mathbf{y}^P$. The set of all respondents whose specified payoff is not lower than his individual rational payoff is denoted by T, which is a subset of $N \setminus P$. Then the sum of the proposers' payoffs in P is $v(P \cup T) - \sum_{j\in T} v(\{j\})$. If no consensus is reached, the sum of the payoffs of the proposers in P is $\sum_{i\in P} ES_i(v^{\overline{N\setminus P}}) = v(N) - \sum_{k\in N\setminus P} v(\{k\})$, which is not lower than $v(P \cup T) - \sum_{j\in T} v(\{j\})$ by 0-monotonicity. Then either $y_i^P = ES_i(v^{\overline{N\setminus P}})$ for all $i \in P$ or there exists $i \in P$ such that $y_i^P < ES_i(v^{\overline{N\setminus P}})$. For the second case, some player in P has an incentive to obstruct this allocation plan, i.e. proposing a different plan. Thus, the payoff vector of any SPNE that fits in the Case II is equal to the payoff vector of σ'.

Similarly, for Case I, the consensual allocation plan $\mathbf{y}^P$ satisfies $\sum_{i\in P} y_i^P \leq v(N) - \sum_{j\in N\setminus P} v(\{j\}) = \sum_{i\in P} ES_i(v^{\overline{N\setminus P}})$. Thus for any consensual allocation plan $\mathbf{y}^P \neq ES(v^{\overline{N\setminus P}})$, some player in P has an incentive to obstruct this allocation plan.

Therefore, the payoff vector of every SPNE that fits the Case I is equal to that of σ'.

Hence, the payoff of each proposer $i \in P$ in every SPNE in the the subgame starting at P is $ES_i(v^{\overline{N \setminus P}})$. By letting $P = N$, the payoff vector of every SPNE is equal to the equal surplus value of v.

Therefore, given any 0-monotonic TU game v, there exists an SPNE of Mechanism B and the payoff vector of every SPNE is equal to the equal surplus value of v. $\square$

Chapter 3

The Proportional Ordinal Shapley Solution for Pure Exchange Economies

3.1 Introduction

Economists have long been proposing allocation rules for economic environments and evaluating them by different desiderata. Though no rule is advantageous under every criterion, some allocation rules arise as dominant solution concepts for specific economic environments, such as the Walrasian allocation rule for pure exchange economies and the Shapley value (Shapley, 1953) for coalitional games with transferable utility (TU). A natural question is whether we can extend solution concepts that were initially designed for a specific economic environment to another.

In this paper, we propose a solution concept for pure exchange economies in the spirit of the Shapley value, which satisfies many appealing properties and is characterized by several methods in the class of TU games. Our construction is inspired by Hart and Mas-Colell's (1989) characterization of the Shapley value with the aid of a potential function. This function assigns a number to every TU game with the only condition that the marginal contributions to the potential of all players add up to the worth of the grand coalition. Hart and Mas-Colell (1989) prove the surprising fact that there is only one such potential function and the vector of marginal contributions coincides with the Shapley value. Thus, the Shapley value rewards each player according to his marginal contribution to the potential of the grand coalition.

We follow a similar approach and associate a number to each pure exchange economy, the *potential* of this economy. Due to the absence of a numeraire commodity in these environments, we choose each agent's initial endowment as a yardstick to measure the variation of his welfare in a solution. Moreover, to ensure the feasibility of the proposal, we measure the agent's marginal gain or loss in terms of the ratio of the potential of the economy over the potential of the sub-economy where he does not participate, instead of the difference between the two potentials. Then, the only condition that we impose to the potential function is the existence of an efficient allocation profile in the economy that satisfies that any agent is indifferent between that allocation and his "proportional" marginal contribution to the potential (that is, in terms of the ratio) times his initial endowment. That is, we require that it be possible for each agent to obtain his proportional marginal contribution to the potential through an efficient allocation.

The construction of the potential of a pure exchange economy entails the simultaneous definition of the efficient allocation profiles that are equivalent for all the agents to their proportional marginal contributions. These allocations are our solution for the economy. We name the set of these allocations the *proportional ordinal Shapley* (the *POSh*) solution. We include the word "ordinal" in the name of the solution because its first important characteristic is that, by construction, the *POSh* is an *ordinal solution*, that is, it is invariant to order-preserving transformations of the agents' utilities.

We show that the *POSh* solution is *unique* and *essentially single-valued*[1] in the set of exchange economies where the agents' preferences are reflexive, complete, transitive, strongly monotone, and continuous. It is also *individually rational*. Moreover, the *POSh* inherits several of the appealing properties of the Shapley value. In particular, it is *anonymous*, not only concerning the name of the agents but also the name of the commodities. Additionally, the *POSh* prescribes a zero bundle to any agent with zero endowments (these are "empty-bundle agents," we call them "empty agents" for short); that is, it satisfies the *empty-agent property*. Further, it satisfies the *empty-agent out property*, which requires that the presence of an empty agent does not influence the prescribed bundles for the rest of the agents. These properties are reminiscent of the null player property and the null player out property of the Shapley value (Derks and Haller, 1999).

Similar to the characterization of the Shapley value in terms of the Harsanyi's (1959) coalitional dividends, the *POSh* can be constructed and characterized using *coalitional dividend yield ratios*.

Additionally, we prove that the *POSh* is immune to certain peculiarities suffered by several allocation rules for pure exchange economies, such as the Walrasian equi-

[1] That is, if the *POSh* solution prescribes several allocations to an economy, every agent is indifferent among all these allocations.

librium. First, the *POSh* is *not D-manipulable* (Postlewaite, 1979); that is, an agent cannot be better off by getting rid of part of his endowment. Second, it *does not suffer from the transfer paradox* (Postlewaite and Webb, 1984); that is, the transfer of a portion of his endowment to another individual cannot make an agent better off and the recipient worse off.

Finally, we provide an additional link between the *POSh* for pure exchange economies and the Shapley value for TU games in terms of their non-cooperative foundations. Pérez-Castrillo and Wettstein (2001) propose a bidding mechanism that implements the Shapley value. We adapt their mechanism[2] to our environment and show that it implements the *POSh* in subgame perfect Nash equilibrium (SPNE) for economies with an arbitrary number of agents in environments where the agents' preferences are homothetic.

The closest contribution to ours is the paper by Pérez-Castrillo and Wettstein (2006). They also provide an ordinal solution in the spirit of the Shapley value for pure exchange economies by extending the idea of McLean and Postlewaite (1978). They introduce the notion of Pareto-efficient egalitarian equivalent (PEEE) allocations. A PEEE allocation is Pareto efficient and "fair" because, for each agent, it is equivalent preference-wise to the same fixed bundle. Pérez-Castrillo and Wettstein's (2006) ordinal Shapley value (*OSV*) considers possibly different individual endowments and is constructed so that it satisfies "consistency," in the sense that an agent's payoff is based on what he would obtain according to this value when applied to sub-economies.

An essential difference between the *POSh* and the *OSV* is in the domain of the solutions. We consider economies where the consumption bundles are non-negative, whereas the *OSV* is defined in environments where the consumption of a commodity can be positive or negative. Our set-up is more common in the general equilibrium literature and prevents the consumption of a negative amount of goods, such as apples. Let us note that most of the properties of the *POSh*, such as unicity, essential single-valueness, empty-agent, and empty-agent out, are not satisfied by the *OSV*. In addition, the *OSV* may suffer from the transfer paradox. This is why the version of the bidding mechanism studied in Pérez-Castrillo and Wettstein (2005) can implement the *OSV* in SPNE only for economies with at most three agents.

In addition to Pérez-Castrillo and Wettstein (2006), the early works by Harsanyi (1959), Shapley (1969), and Maschler and Owen (1992) propose extensions of the Shapley value to non-transferable utility environments such as the pure exchange

[2] See also Pérez-Castrillo and Wettstein (2002).

economy that we study. The three proposals are defined in the utility space. They abstract from the physical environment that generates the utilities. However, as Roemer (1986, 1988) discusses, much information is lost when one moves from the economic environment to the utility space. Thus, on the one hand, these proposals are not ordinal since the solutions are not invariant to alternative representations of the agents' utilities. Moreover, Greenberg et al. (2002) make the observation that the von Neumann and Morgenstein stable sets, defined for the economic environment and the utility space, respectively, may not coincide, even though both are ordinal. On the other hand, as Alon and Lehrer (2019) point out, two very different economic environments, whose solution should be different, may lead to the same allocation of utilities and, hence, the same solution.

McLean and Postlewaite (1989) also extend a notion from the class of TU games to the set of pure exchange economies. They provide an ordinal nucleolus, a solution concept proposed by Schmeidler (1969) for TU games. Nicolò and Perea (2005) and Alon and Lehrer (2019) offer ordinal solutions for bargaining problems.

The remainder of the paper is organized as follows. Section 2 describes the economic environment. It also introduces our new solution concept–the proportional ordinal Shapley solution. Section 3 proves the existence and uniqueness of the *POSh*. Several properties of the *POSh* are also stated and proved. Section 4 presents the bidding mechanism that implements the *POSh*. Section 4 concludes the paper and provides several open questions for future research. All the proofs are in the Appendix.

3.2 The environment and the solution concept

We consider a *pure exchange economy.* The set of agents is $N \equiv \{1, \dots, n\}$, with generic agent i. The set of goods is $L \equiv \{1, \dots, l\}$, which is fixed throughout this paper.

Agent i is described by $(\mathbf{w}_i, \succeq_i)$, where $\mathbf{w}_i \equiv (w_{i1}, \dots, w_{il}) \in \mathbb{R}^L_+$ is his commodity bundle, and $\succeq_i$ is his preference relation defined over $\mathbb{R}^L_+$. We assume $\succeq_i$ is reflexive, complete, and transitive for each $i \in N$.[3] We also assume that it is strongly monotone and continuous. Preference $\succeq_i$ is *strongly monotone* if $\mathbf{x} \succ_i \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. Preference $\succeq_i$ is *continuous* if $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \succeq_i \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \preceq_i \mathbf{x}\}$ are closed subsets of $\mathbb{R}^L_+$, for all $\mathbf{x} \in \mathbb{R}^L_+$.

A pure exchange economy is a triplet $(N, \mathbf{w}, \succeq)$, where the vector $\mathbf{w}$ is understood as an endowment profile $(\mathbf{w}_1, \dots, \mathbf{w}_n)$ and $\succeq$ is understood as a preference profile

[3] Agent i's preference $\succeq_i$ is reflexive if $\mathbf{x} \succeq_i \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^L_+$; $\succeq_i$ is complete if either $\mathbf{x} \succeq_i \mathbf{y}$ or $\mathbf{y} \succeq_i \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$; $\succeq_i$ is transitive if $\mathbf{x} \succeq_i \mathbf{y}$ and $\mathbf{y} \succeq_i \mathbf{z}$ imply $\mathbf{x} \succeq_i \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^L_+$.

$(\succeq_1, \ldots, \succeq_n)$. For a fixed set of agents N, the set of all exchange economies where the agents' preferences are reflexive, complete, transitive, strongly monotone, and continuous is denoted by $\mathcal{E}^N$. The set of all such exchange economies with a finite set of agents is denoted by $\mathcal{E}$.

Definition 12. *A* ***feasible*** *allocation for an exchange economy* $(N, \mathbf{w}, \succeq)$ *is a profile* $\mathbf{z} \equiv (\mathbf{z}_1, \ldots, \mathbf{z}_n) \in \mathbb{R}_+^{N \times L}$ *such that* $\sum_{i \in N} \mathbf{z}_i \leq \sum_{i \in N} \mathbf{w}_i$.

We denote by $Z(N, \mathbf{w}, \succeq)$ the set of all feasible allocations for the exchange economy $(N, \mathbf{w}, \succeq)$.

Two feasible allocations are comparable when all agents prefer one to the other in unison. Formally, for $\mathbf{z}, \mathbf{z}' \in Z(N, \mathbf{w}, \succeq)$, we denote $\mathbf{z} \succeq \mathbf{z}'$ if $\mathbf{z}_i \succeq_i \mathbf{z}'_i$ for all $i \in N$. Similarly, $\mathbf{z} \sim \mathbf{z}'$ if $\mathbf{z}_i \sim_i \mathbf{z}'_i$ for all $i \in N$. Then, we can define an efficient allocation.

Definition 13. *A feasible allocation* $\mathbf{z} \in Z(N, \mathbf{w}, \succeq)$ *is* ***efficient*** *if there is no feasible allocation* $\mathbf{z}' \in Z(N, \mathbf{w}, \succeq)$ *such that* $\mathbf{z}' \succeq \mathbf{z}$ *and* $\mathbf{z}'_j \succ_j \mathbf{z}_j$ *for some* $j \in N$.

We denote by $E(N, \mathbf{w}, \succeq)$ the set of all efficient allocations for the exchange economy $(N, \mathbf{w}, \succeq)$.

We now define a solution concept for pure exchange economies.

Definition 14. *A solution is a correspondence* $F : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N \times L}$ *such that* $F(N, \mathbf{w}, \succeq) \subseteq Z(N, \mathbf{w}, \succeq)$ *for all* $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Thus, a solution F assigns a set of feasible allocations to each pure exchange economy. Given two solutions F and F', for simplicity we write $F \subseteq F'$ if $F(N, \mathbf{w}, \succeq) \subseteq F'(N, \mathbf{w}, \succeq)$ for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

A solution F is *single-valued* if F is a function, that is, it prescribes a unique feasible allocation for every economy. A solution F is *essentially single-valued* if $\{\mathbf{y} \in Z(N, \mathbf{w}, \succeq) \mid \mathbf{y} \sim \mathbf{x}\} = F(N, \mathbf{w}, \succeq)$ for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$ and all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$. Thus, an essentially single-valued solution prescribes a $\sim$-equivalence class within the set of all efficient allocations. For an essentially single-valued solution F, we write $F_i(N, \mathbf{w}, \succeq) \succeq_i F_i(N, \mathbf{w}', \succeq)$ for $i \in N$ if player i prefers the profiles in $F_i(N, \mathbf{w}, \succeq)$ to the profiles in $F_i(N, \mathbf{w}', \succeq)$. We write $F(N, \mathbf{w}, \succeq) \succeq F(N, \mathbf{w}', \succeq)$ similarly.

Given that agents have initial private endowments, a reasonable solution should ensure that an agent has an incentive to participate instead of walking away with his endowment. The individual rationality of a solution captures this notion:

Definition 15. *A solution* F *satisfies* ***individual rationality*** *if* $\mathbf{x} \succeq \mathbf{w}$ *for all* $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ *and all* $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Next, we formulate two properties that adapt the ideas of the null player property and the null player out property (Derks and Haller, 1999) to pure exchange economies. We identify a type of agent in pure exchange economies who play a similar role as the null players in coalitional games. They are empty-basket agents; we call them empty agents. An agent $i \in N$ is an *empty agent* in the economy $(N, \mathbf{w}, \succeq)$ if $\mathbf{w}_i = \mathbf{0}$. An economy consisting of empty agents only is called an *empty economy.*

The definition of the second property requires the following notation. Let $\mathbf{x} \in \mathbb{R}_+^{N\times L}$ be an allocation profile. Then, for $N' \subseteq N$, we denote by $\mathbf{x}\,|_{N'} \in \mathbb{R}_+^{N'\times L}$ the profile $\mathbf{x}$ restricted to N', that is, $(\mathbf{x}|_{N'})_i = \mathbf{x}_i$ for all $i \in N'$. The restrictions of the preference profile are denoted analogously.

Definition 16. *A solution F satisfies the* ***empty-agent*** *property if $\mathbf{x}_i = \mathbf{0}$ for each empty agent $i \in N$ in $(N, \mathbf{w}, \succeq)$, all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

Definition 17. *A solution F satisfies the* ***empty-agent out*** *property if $\mathbf{x}\,|_{N\setminus\{i\}} \in F(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ for each empty agent $i \in N$ in $(N, \mathbf{w}, \succeq)$, for all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

The empty-agent and the empty-agent out properties are normative properties. The first one requires that an empty agent be entitled to a zero bundle in any allocation of the solution. In contrast, the empty-agent out property requires that the presence of an empty agent should not influence the allocation of the solution to the rest of the agents. In general, the two properties are logically independent of each other. But, in the presence of efficiency, the empty-agent out property implies the empty-agent property. To see this implication, consider an efficient solution that satisfies the empty-agent out property but does not satisfy the empty-agent property. Then there exists $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ such that $\mathbf{x}_i \neq \mathbf{0}$ for some empty agent i in $(N, \mathbf{w}, \succeq)$. By empty-agent out property, $\mathbf{x}\,|_{N\setminus\{i\}} \in F(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Then we could construct a feasible profile $\mathbf{y} \in Z(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ where $\mathbf{y}_j \equiv \mathbf{x}_j + \frac{\mathbf{x}_i}{n-1}$, which would be strictly preferred by every $j \in N \setminus \{i\}$ by strong monotonicity.

It is worth mentioning that Shafer's (1980) example demonstrates that neither the empty-agent property nor the empty-agent out property is satisfied by Shapley's (1969) NTU value.

We now turn to the property of anonymity. In an exchange economy, anonymity may refer to the agents or the commodities. We will consider both ideas in our definition of this property. We first introduce the notation for bijections of agents and commodities.

For a feasible allocation $\mathbf{z} \in Z(N, \mathbf{w}, \succeq)$ and a bijection $\pi : N \to N'$, we define the allocation $\pi\mathbf{z} \in Z(N', \mathbf{w}, \succeq)$ by $\pi\mathbf{z}_{\pi(i)} \equiv \mathbf{z}_i$ for all $i \in N$. Similarly, for a commodity bundle $\mathbf{x} \in \mathbb{R}^L_+$ and a bijection $\rho : L \to L'$, we define the commodity bundle $\rho\mathbf{x} \in \mathbb{R}^{L'}_+$ by $\rho x_{\rho(h)} \equiv x_h$ for all $h \in L$. We can apply the above two bijections simultaneously. For a pair of bijections (π, ρ), let $\Theta = (\pi, \rho)$. For each economy $(N, \mathbf{w}, \succeq)$ and each Θ, we denote the bijection of the economy by $\Theta(N, \mathbf{w}, \succeq) \equiv (\pi[N], \Theta\mathbf{w}, \succeq^\Theta)$, where $\Theta\mathbf{w}_{\pi(i)} = \rho(\mathbf{w}_i)$ for all $i \in N$, and the preference relation $\succeq^\Theta$ is defined over $\mathbb{R}^{\rho[L]}_+$ by $\rho\mathbf{x} \succeq^\Theta_{\pi(i)} \rho\mathbf{y}$ if $\mathbf{x} \succeq_i \mathbf{y}$, for each $i \in N$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$. That is, the structure of economy $\Theta(N, \mathbf{w}, \succeq)$ is identical to $(N, \mathbf{w}, \succeq)$, but the names of the agents are changed according to π and the names of the commodities are changed according to ρ.

Definition 18. *A solution F is* ***anonymous*** *if for each pair of bijections $\Theta = (\rho, \pi)$ and each $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$, then $\Theta\mathbf{x} \in F\Theta(N, \mathbf{w}, \succeq)$.*

The last two properties that we propose concern the possibility for an agent to "manipulate" the solution outcome via his endowment. Aumann and Peleg (1974) demonstrate that before the Walrasian mechanism is applied to a finite economy, an agent may be better off by getting rid of part of his endowment. In light of this peculiarity, Postlewaite (1979) introduces the following property, which is not implied by efficiency and individual rationality:

Definition 19. *An essentially single-valued solution F is* ***D-manipulable*** *if there exist $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^{N \times L}_+$ such that $\mathbf{w}_i \geq \mathbf{w}'_i$ for some $i \in N$, $\mathbf{w}_j = \mathbf{w}'_j$ for each $j \in N \backslash \{i\}$, and $F_i(N, \mathbf{w}, \succeq) \prec_i F_i(N, \mathbf{w}', \succeq)$.*

An anomaly closely related to D-manipulability is the transfer paradox: a transfer of a portion of his endowment makes the donor better off and the recipient worse off (see, e.g., Postlewaite and Webb, 1984). Definition 20 formally states this paradox.

Definition 20. *An essentially single-valued solution F exhibits the* ***transfer paradox*** *if there exist $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^{N \times L}_+$ and two distinct agents $i, j \in N$ such that $\mathbf{w}_i \geq \mathbf{w}'_i$, $\mathbf{w}_i + \mathbf{w}_j = \mathbf{w}'_i + \mathbf{w}'_j$ and $\mathbf{w}_k = \mathbf{w}'_k$ for each $k \in N \backslash \{i, j\}$, $F_i(N, \mathbf{w}, \succeq) \prec_i F_i(N, \mathbf{w}', \succeq)$, and $F_j(N, \mathbf{w}, \succeq) \succ_i F_j(N, \mathbf{w}', \succeq)$.*

Now we are ready to present our solution concept: the *proportional ordinal Shapley solution* (*POSh*). Similar to the *OSV* (see Pérez-Castrillo and Wettstein, 2006), we define the *POSh* in terms of agents' preferences directly. Thus, it is an ordinal solution.

To define the *POSh*, we use the idea of the potential (Hart and Mas-Colell, 1989), which provides a simple characterization of the Shapley value. Following Hart and Mas-Colell (1989), we define a potential function as follows:

Definition 21. *A potential function $P : \mathcal{E} \to \mathbb{R}_{++}$ is defined inductively on the number of players $|N|$:*

1. $P(\varnothing) \equiv 1$;
2. *for $(N, \mathbf{w}, \succeq) \in \mathcal{E}$, $P(N, \mathbf{w}, \succeq)$ satisfies that there exists $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ such that $\frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in N$.*[4]

A potential function associates each economy with a single number. Moreover, given an economy, the maximum changes from the potentials of each of its one-agent-less sub-economies to the potential of this economy are constrained by its Pareto frontier. The way of representing these changes underlies the critical difference between Definition 21 and the Hart and Mas-Colell's potential: we take the ratio of potentials of economies, while Hart and Mas-Colell (1989) take the difference of potentials of games.

The prescription of the *POSh* is intertwined with our definition of a potential. An allocation is in the *POSh* if it is efficient and each agent i is indifferent between his prescribed bundle and a multiple of his endowment, where the multiple is equal to the change of potential resulting from his entrance. Thus, we have the following definition of a proportional ordinal Shapley solution $POSh : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N\times L}$ in terms of the potential P.

Definition 22. *Given a potential function P, a proportional ordinal Shapley solution $POSh : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N\times L}$ is defined by $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$ if $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and $\frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in N$.*

As we will see in the next section, the *POSh* is an appealing solution that enjoys many properties that echo the properties of the Shapley value, such as the empty-agent property and the empty-agent out property. It is also immune to well-known anomalies of the Walrasian equilibrium, such as the D-manipulability and the transfer paradox.

At last, the *POSh* is often easy to compute owing to its neat definition in terms of the potential. For illustration, we compute the *POSh* for a simple 3-agent economy in Example 13.

[4]If $N = \{i\}$, we let $P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) \equiv P(\varnothing)$.

Example 13. *Fix* $L = \{1, 2\}$. *Consider an exchange economy such that* $N = \{1, 2, 3\}$, $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w}_3 = (1, 1)$, $u_1(x_1, y_1) = x_1$, $u_2(x_2, y_2) = y_2$ *and* $u_3(x_3, y_3) = x_3 y_3$.

To compute $POSh(N, \mathbf{w}, u)$, *we need to find the potential of each subeconomy. First, it is easy to see that* $P(\{i\}, \mathbf{w}_i, u_i) = 1$ *for* $i = 1, 2, 3$, *and* $P(\{1, 2\}, (\mathbf{w}_1, \mathbf{w}_2), (u_1, u_2)) = 2$. *Let* $P(\{1, 3\}, (\mathbf{w}_1, \mathbf{w}_3), (u_1, u_3)) = P(\{2, 3\}, (\mathbf{w}_2, \mathbf{w}_3), (u_2, u_3)) = \lambda$. *Then, it is the case that* $\lambda = \sqrt{2(2 - \lambda)}$, *i.e.,* $\lambda = \sqrt{5} - 1$. *Finally, let* $P(N, \mathbf{w}, u) = \mu$ *and a generic efficient allocation be* $((z, 0), (0, z), (3 - z, 3 - z))$. *Then,* $3 - z = \frac{\mu}{2}$ *and* $z = \frac{\sqrt{5}+1}{4}\mu$. *Therefore,* $P(N, \mathbf{w}, u) = 9 - 3\sqrt{5}$ *and* $POSh(N, \mathbf{w}, u) = ((\frac{3\sqrt{5}-3}{2}, 0), (0, \frac{3\sqrt{5}-3}{2}), (\frac{9-3\sqrt{5}}{2}, \frac{9-3\sqrt{5}}{2})) \approx ((1.85, 0), (0, 1.85), (1.15, 1.15))$.

Remark 3. *It is easy to see that the Walrasian equilibrium allocation and the core for Example 13 coincide, which is* $((2, 0), (0, 2), (1, 1))$. *Therefore, the POSh may not be in the core.*

3.3 Existence and properties of the proportional ordinal Shapley solution

In this section, we establish the existence, uniqueness, and other properties of the proportional ordinal Shapley solution.

Before we state our results regarding the *POSh*, we first prove the existence and uniqueness of the potential function restricted to economies in which each agent is not empty.[5] We use the auxiliary notion of "coalitional dividend yield ratio" ("dividend ratio," for short), which is a multiplicative version of Harsanyi's (1959) coalitional dividend. Parallel to Hart and Mas-Colell (1989), our proof is also based on a simple representation of the potential through the dividend yield ratios.

Denote by $\mathcal{E}'$ the set of all economies with only non-empty agents. We establish in Proposition 10 the existence and uniqueness of the potential function restricted to $\mathcal{E}'$.

Proposition 10. *There exists a unique potential function restricted to* $\mathcal{E}'$.

Proposition 10 states the existence and uniqueness of the potential function if we restrict attention to economies without empty players. Before we use this result to construct a *POSh* for economies without empty agents, and we extend the

[5] We may assign an arbitrary number to an economy consisting of empty agents only, which leads to a multiplicity of potential functions. But the assumption that each agent is not empty is for expository purposes. As we will show, the uniqueness of the proportional ordinal Shapley solution still holds despite the multiplicity of potential functions.

analysis to economies including empty agents, we state two remarks concerning the hypotheses that we use in the proposition.

Remark 4. *Proposition 10 is stated for economies where the agents' preferences satisfy strong monotonicity. We cannot replace this hypothesis by the weaker axiom of strict monotonicity. Recall that player i's preference over commodities* $\succeq_i$ *is* ***strictly monotone*** *if* $\mathbf{x} \succ_i \mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ *such that* $x_h > y_h$ *for all* $h \in L$*. To see that this weaker property is not enough, consider a two-agent economy* $(\{1,2\}, \mathbf{w}, \succeq)$*, where* $\mathbf{w}_1 = (0,1)$*,* $\mathbf{w}_2 = (1,0)$*,* $\succeq_1$ *is represented by* $u_1(x_1, y_1) = x_1$*, and* $\succeq_2$ *is represented by* $u_2(x_2, y_2) = y_2$*. Both agents' preferences satisfy strict monotonicity instead of strong monotonicity. By Definition 21,* $P(\{1\}, \mathbf{w}_1, \succeq_1) = P(\{2\}, \mathbf{w}_2, \succeq_2) = 0$*. Then the denominators of both* $\frac{P(\{1,2\},\mathbf{w},\succeq)}{P(\{2\},\mathbf{w}_2,\succeq_2)}$ *and* $\frac{P(\{1,2\},\mathbf{w},\succeq)}{P(\{1\},\mathbf{w}_1,\succeq_1)}$ *vanish. Therefore, we are unable to assign a number to* $P(\{1,2\}, \mathbf{w}, \succeq)$*. Hence, a potential function does not exist for this economy.*

Remark 5. *The full strength of the property of the continuity of preferences is not necessary for Proposition 10 to hold. The proof only requires lower semi-continuity of the preferences, i.e.,* $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \preceq_i \mathbf{x}\}$ *is closed for all* $\mathbf{x} \in \mathbb{R}^L_+$ *and all* $i \in N$*.*

In the proof of Proposition 10, we construct a system of dividend ratios that allows describing the potential of any economy. Then, Definition 22 together with the proof of Proposition 10 lead to the following representation of the *POSh* solution restricted to $\mathcal{E}'$ in terms of dividend ratios:

Claim 8′. *There exists a unique essentially single-valued proportional ordinal Shapley solution restricted to* $\mathcal{E}'$*.*

Furthermore, for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}'$*, there exists a vector of dividend yield ratios* $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ *such that for all* $N' \in 2^N \setminus \{\varnothing\}$*,* $\mathbf{x} \in POSh(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ *if and only if* $\mathbf{x} \in E(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ *and* $\mathbf{x}_i \sim_i (\prod_{\substack{T \ni i \\ T \subseteq N'}} (1 + d_T))\mathbf{w}_i$ *for all* $i \in N'$*.*

Remark 6. *Pérez-Castrillo and Wettstein (2006) also provide a characterization of the OSV in terms of dividends. However, there is an important difference between their characterization and that of the POSh given in Corollary . For the OSV, the dividends* d_S *and* d'_S *of the same coalition* $S \subseteq N'$ *for an economy* $(N, \mathbf{w}, \succeq)$ *and its subeconomy* $(N', \mathbf{w}|_{N'}, \succeq|_{N'})$*, respectively, are different. By contrast, for the POSh, the dividend ratios of the same coalition of an economy and its sub-economy, respectively, are the same.*

We now proceed to consider the economies including empty agents. We note that the uniqueness of the potential function cannot be extended to the set of economies including empty agents. Indeed, for an empty economy $(N, \mathbf{w}, \succeq)$, the potential of each subeconomy $(S, \mathbf{w}\,|_S, \succeq|_S)$ for $S \in 2^N \setminus \{\varnothing\}$ can be assigned an arbitrary positive number $P(S, \mathbf{w}\,|_S, \succeq|_S)$.

Given that the potential function and the *POSh* exist for economies without empty agents, it is useful to consider, for each economy, the sub-economy that contains only the set of non-empty agents of the original economy. Formally, we define the *support of the economy* $(N, \mathbf{w}, \succeq)$ as the sub-economy where an agent $i \in N$ participates in the support if and only if $\mathbf{w}_i \neq \mathbf{0}$. The support of the economy $(N, \mathbf{w}, \succeq)$ is denoted by $supp(N, \mathbf{w}, \succeq)$. Similarlty, we denote by $0(N, \mathbf{w}, \succeq)$ the subeconomy of $(N, \mathbf{w}, \succeq)$ where only the empty agents participate. Thus, each economy $(N, \mathbf{w}, \succeq)$ can be decomposed into two disjoint subeconomies: $supp(N, \mathbf{w}, \succeq)$ and $0(N, \mathbf{w}, \succeq)$.

Using the notion of the support of an economy, we can propose an extension of the potential function to the unrestricted domain as follows: (a) the potential of an economy consisting of empty agents only is equal to 1, and (b) the potential of an economy with both empty agents and non-empty agents is equal to the potential of its support. Moreover, we will show that this potential is associated with the unique essentially single-valued *POSh* of any economy with empty and non-empty agents. We will state these results in Theorem 9.

To establish the uniqueness of the *POSh*, we will use the relationship between the *POSh* of any pure exchange economy and the *POSh* of the support of that economy. We will also use the properties on empty agents that every *POSh* satisfies and that are stated and proven in Proposition 11.

Proposition 11. *Any proportional ordinal Shapley solution in $\mathcal{E}$ satisfies the empty-agent property and the empty-agent out property.*

Given that every proportional ordinal Shapley solution satisfies the empty-agent property and the empty-agent out property, its prescription for agents in a general economy can distinguish between empty agents and non-empty agents. On the one hand, an empty agent is prescribed a zero bundle by the empty-agent property. On the other hand, a non-empty agent is prescribed a bundle equal to some bundle prescribed by the *POSh* for the support of this economy by the empty-agent out property.

Thus, Proposition 11 indicates that an empty agent can be viewed as a placeholder under any *POSh*. This feature enables us to deduce the uniqueness of the *POSh* for the unrestricted domain from the uniqueness of the *POSh* for the economies without any empty agents.

Theorem 9. *There exists a unique essentially single-valued proportional ordinal Shapley solution in $\mathcal{E}$.*

Furthermore, for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$*, there exists a vector of dividend yield ratios* $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ *such that for all* $N' \in 2^N \setminus \{\varnothing\}$*,* $\mathbf{x} \in POSh(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ *if and only if* $\mathbf{x} \in E(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ *and* $\mathbf{x}_i \sim_i (\prod_{\substack{T \ni i \\ T \subseteq N'}} (1 + d_T))\mathbf{w}_i$ *for all* $i \in N'$*.*

From here onward, a proportional ordinal Shapley solution is referred to as the proportional ordinal Shapley solution since it is unique.

We recall that Theorem 9 establishes the existence and uniqueness of the proportional ordinal Shapley solution for pure exchange economies where preferences are (in addition to reflexivity, completeness, and transitivity) continuous and strongly monotone. The requirements for the existence of the $POSh$ are incomparable with those for Walrasian equilibrium. Indeed, the existence of Walrasian equilibrium requires each agent's preference to be continuous, convex, and non-satiated, and each agent's endowment strictly positive (see Border, 2017). On the one hand, strong monotonicity is a stronger assumption than non-satiation, while neither convex preferences nor strictly positive endowment is needed for the existence of $POSh$.

We have seen that the proportional ordinal Shapley solution exists, and it is unique and essentially single-valued. Moreover, it satisfies the empty-agent and the empty-agent out properties. The last part of the section provides four additional properties of the $POSh$.

First, we show that the $POSh$ is individually rational.

Proposition 12. *The proportional ordinal Shapley solution satisfies individual rationality in* $\mathcal{E}$*.*

Second, we show that the $POSh$ satisfies the property of anonymity. That is, it is immune to changes in the names of the agents and commodities.

Proposition 13. *The proportional ordinal Shapley solution satisfies anonymity in* $\mathcal{E}$*.*

The last two properties of the proportional ordinal Shapley solution that we present state that the $POSh$ is robust against agents' manipulation of their initial endowment. Proposition 14 shows that an agent never has an incentive to throw away any part of his initial endowment, that is, the $POSh$ is not D-manipulable. Finally, Proposition 15 states that an agent is never better-off by transferring part of his initial endowment to another agent. Thus, the $POSh$ does not exhibit the transfer paradox.

Proposition 14. *The proportional ordinal Shapley solution is not D-manipulable in* $\mathcal{E}$*.*

Proposition 15. *The proportional ordinal Shapley solution does not exhibit the transfer paradox in* $\mathcal{E}$*.*

3.4 A mechanism implementing the proportional ordinal Shapley solution

In this section, we propose a new version of the Pérez-Castrillo and Wettstein (2001) and (2002) bidding mechanism to implement the proportional ordinal Shapley solution.

Pérez-Castrillo and Wettstein (2005) also used a variant of the bidding mechanism to implement the ordinal Shapley value. Before we introduce our mechanism, it is worthwhile to recall the one used in that paper informally. First, the agents bid simultaneously to choose the proposer. Each agent's bid consists of an n-tuple of real numbers whose sum must be zero. The number submitted by agent i for agent j is a commitment to forego a commodity bundle (the number times a reference bundle) in case j is chosen as the proposer. The agent for whom the aggregate bid (sum of bids submitted for him by all agents, including himself) is the highest is chosen as the proposer. All the agents pay their "bid" (i.e., the promised commodity bundles) for the proposer. In the second stage, the proposer offers an allocation of the total initial resources. If all the other agents agree, each agent receives the bundle suggested for him in this allocation. Otherwise, all the agents other than the proposer play the same game again where the new initial endowments incorporate the allocations (that is, the "bids") paid and received.

Pérez-Castrillo and Wettstein's (2005) mechanism implements the OSV for economies with at most three agents. The major difficulty for extending their implementation result to any number of agents is, as they pointed out, that the OSV is subject to the transfer paradox. In particular, it can be the case that an agent is better off in a subgame where he has less and the other agents have more initial endowments, which gives him the wrong incentives to bid.

Our proposed mechanism to implement the $POSh$ shares the basic features of previous bidding mechanisms. The implementation is made easier by Proposition 15, which ensures that the $POSh$ solution does not suffer from the transfer paradox. Moreover, given the defining characteristics of the $POSh$, our mechanism differs from previous proposals in two aspects: (i) a bid is interpreted as a promise to transfer a fixed proportion of resources rather than an absolute level of resources since the payment is equal to the bid times the recipient's current bundle; and (ii) in case of a rejection of his allocation plan, the proposer's payments due to his bid are delivered at the very end of the mechanism, rather than just after the rejection.

Finally, let us mention that while our mechanism implements the $POSh$ for any number of agents, we impose three additional constraints or modifications to the set of economies that we consider. (a) The agents' preferences are homothetic.[6] (b) No empty agent is present in the economy. (c) We extend the common domain of each agent i's preference from $\mathbb{R}^L_+$ to $\mathbb{R}^L$ by letting $\mathbf{x} \prec_i \mathbf{y}$ if $\mathbf{x} \in \mathbb{R}^L \setminus \mathbb{R}^L_+$ and $\mathbf{y} \in \mathbb{R}^L_+$ and $\mathbf{x} \sim_i \mathbf{y}$ if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L \setminus \mathbb{R}^L_+$ for all $i \in N$. We extend the domain because, out of equilibrium, there may exist a solvency issue if the rejected proposer does not have enough endowment to pay the bids at the end of the mechanism. We denote by $\mathcal{E}^H$ the subset of economies where preferences are reflexive, complete, transitive, strongly monotone, and continuous, and that satisfy conditions (a)-(c) above.

We now propose the following *proportional bidding mechanism* for $\mathcal{E}^H$:

For $|N| = 1$, for the economy $(\{i\}, \mathbf{w}_i, \succeq_i)$, the only agent i receives his own initial endowment $\mathbf{w}_i$.

For $|N| \geq 2$, we hypothesize that the mechanism has been defined for each economy $(N', \mathbf{w}', \succeq')$ with $|N'| < |N|$. Then the mechanism applied for an economy $(N, \mathbf{w}, \succeq)$ proceeds as follows:

$t = 1$: Each agent $i \in N$ submits a bid $b^N_{ij} \in \mathbb{R}_{++}$ for each agent $j \in N$, with $\prod_{j \in N} b^N_{ij} = 1$.

$t = 2$: Let the cumulative bid for agent $i \in N$ be denoted by $B^N_i \equiv \prod_{j \in N} b^N_{ji}$. An agent $\alpha \in \operatorname{argmax}_{i \in N} B^N_i$ is selected as the proposer by a non-degenerate lottery.[7] Then the proposer α puts forth an allocation plan $\mathbf{x}^N \in \mathbb{R}^{(N \setminus \{\alpha\}) \times L}$ specifying a bundle $\mathbf{x}^N_i \in \mathbb{R}^L_+$ for each agent $i \in N \setminus \{\alpha\}$.

$t = 3$: Each agent $i \in N \setminus \{\alpha\}$ accepts or rejects α's plan sequentially. We distinguish between two cases:

Case I Every agent accepts α's plan. Then the grand coalition N forms, and the plan is implemented. Therefore, the final outcome is that each agent $i \in N \setminus \{\alpha\}$ receives $\mathbf{x}^N_i$ and the proposer α receives the residue $\sum_{j \in N} \mathbf{w}_j - \sum_{i \in N \setminus \{\alpha\}} \mathbf{x}^N_i$.

Case II Some agent rejects α's plan. Then the proposer forms his own standalone coalition $\{\alpha\}$. Moreover, the mechanism is applied to the subeconomy $(N \setminus \{\alpha\}, \mathbf{w}\,|_{N \setminus \{\alpha\}}, \succeq|_{N \setminus \{\alpha\}})$. The final outcome is the following. Let $\mathbf{y}_i \in \mathbb{R}^L$ be the bundle received by agent $i \in N \setminus \{\alpha\}$ from the mechanism played by $(N \setminus \{\alpha\}, \mathbf{w}|_{N \setminus \{\alpha\}}, \succeq|_{N \setminus \{\alpha\}})$. On top of that, the proposer α transfers a commodity bundle $\left(\frac{\sqrt[n]{B^N_\alpha}}{b^N_{i\alpha}} - 1 \right) \mathbf{y}_i$ to each agent $i \in N \setminus \{\alpha\}$.

[6] Agent i's preference $\succeq_i$ is **homothetic** if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ and all $\alpha \in \mathbb{R}_+$, $\mathbf{x} \succeq_i \mathbf{y}$ if and only if $\alpha\mathbf{x} \succeq_i \alpha\mathbf{y}$.

[7] A non-degenerate lottery selects each agent from $\operatorname{argmax}_{i \in N} B^N_i$ with a strictly positive probability.

Therefore, the final commodity bundle is $\left(\frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N}\right)\mathbf{y}_i$ for each $i \in N\backslash\{\alpha\}$, and is $\mathbf{w}_\alpha - \sum_{i\in N\backslash\{\alpha\}}\left(\frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N} - 1\right)\mathbf{y}_i$ for the proposer α.

Before presenting the main result of this section, we provide a characterization of the proportional ordinal Shapley solution in terms of "proportional concessions," which has some resemblance with the original Pérez-Castrillo and Wettstein's (2006) definition of the ordinal Shapley value (OSV). The characterization is interesting by itself. It will also allow us to simplify the proof of the implementation theorem.

Definition 23 proposes a solution for $\mathcal{E}$, and Proposition 16 states that it coincides with the *POSh*.

Definition 23. *The solution $\zeta : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N\times L}$ is defined recursively on the number of agents $|N|$ as follows:*

1. *For $|N| = 1$, i.e., $N = \{i\}$, $\zeta(\{i\}, \mathbf{w}_i, \succeq_i) \equiv \{\mathbf{w}_i\}$.*
2. *For $|N| \geq 2$, we hypothesize that ζ has been defined and is essentially single-valued for each economy $(N', \mathbf{w}', \succeq')$ with $|N'| < |N|$. Then, $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$ if $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and there exists a concession vector $\mathbf{c}_i^N \in \mathbb{R}^{N\backslash\{i\}}$ for each $i \in N$ that satisfies:*
   (a) $\prod_{j\in N\backslash\{i\}} c_{ij}^N = \prod_{j\in N\backslash\{i\}} c_{ji}^N$ for each $i \in N$.
   (b) For each $j \in N \setminus \{i\}$, there exists $a_{ij}^N \in \mathbb{R}$ such that $a_{ij}^N\mathbf{w}_j \sim_j \zeta_j(N \setminus \{i\}, \mathbf{w}|_{N\backslash\{i\}}, \succeq|_{N\backslash\{i\}})$ and $\mathbf{x}_j \sim_j c_{ij}^N a_{ij}^N \mathbf{w}_j$.

We can read part (2b) of Definition 23 as follows. Agent j is indifferent between the bundles that the solution offers to him ($\mathbf{x}_j$) and a bundle ($a_{ij}^N\mathbf{w}_j$) that is equivalent to what he can obtain without agent i according to the solution, boosted by the concession c_{ij}^N of agent i to agent j. Condition (2a) states the "fairness" requirement that the concessions that an agent receives in total (which in our framework corresponds to their product) be the same as the concessions that he makes to the other agents.

Now, we state and prove the characterization of the *POSh* in terms of concessions.

Proposition 16. *The proportional ordinal Shapley solution coincides with the solution ζ.*

An implication of Proposition 16 is that the vector of concessions for an economy $(N, \mathbf{w}, \succeq)$ is unique, given that the *POSh* is essentially single-valued.

Theorem 10 uses Proposition 16 to show that the proportional ordinal bidding mechanism implements *POSh* in subgame perfect Nash equilibrium in pure strategies (SPNE) when the agents' preferences are homothetic. The new characterization of the *POSh* is helpful for the proof of the implementation result because we can relate the equilibrium bids in the mechanism and the concessions in Definition 23.

Theorem 10. *The proportional bidding mechanism implements the proportional ordinal Shapley solution in SPNE in the set of economies with homothetic preferences.*

3.5 Conclusion

We propose a new ordinal solution concept for pure exchange economies, the *POSh* solution. Its construction is inspired by the potential function, which allows a nice characterization of the Shapley value in TU games. The *POSh* solution satisfies properties similar to the Shapley value, such as efficiency, anonymity, and properties related to null players. It is also individually rational and does not suffer from agents' manipulation of their initial endowment.

We further highlight the link between the *POSh* for pure exchange economies and the Shapley value for TU games through their implementation. We show that a variant of a mechanism that implements the Shapley value implements the *POSh* for the particular environments where agents' preferences are homothetic.

One natural avenue for future research is extending our solution concept and its properties to pure exchange economies with a continuum of agents of finite types. It is easy to extend the notions of the potential and the proportional ordinal Shapley solution to these economies. However, the analysis of the properties of the *POSh* in these environments is outside the scope of this paper.

3.6 Appendix

Proof of Proposition 10. First, we show that there exists at most one potential function. Suppose otherwise, that is, suppose that there exist two distinct potential functions P and P'. Then, without loss of generality, assume that $(N, \mathbf{w}, \succeq)$ satisfies $P(N, \mathbf{w}, \succeq) > P'(N, \mathbf{w}, \succeq)$ and $P(S, \mathbf{w}|_S, \succeq|_S) = P'(S, \mathbf{w}|_S, \succeq|_S)$ for all $S \in 2^N \setminus \{N\}$. This implies that there exist two allocations $\mathbf{x}, \mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{x}_k \sim_k \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{k\},\mathbf{w}|_{N\setminus\{k\}},\succeq|_{N\setminus\{k\}})}\mathbf{w}_k \succ_k \frac{P'(N,\mathbf{w},\succeq)}{P'(N\setminus\{i\},\mathbf{w}|_{N\setminus\{k\}},\succeq|_{N\setminus\{k\}})}\mathbf{w}_k \sim_k \mathbf{y}_k$ for all $k \in N$, where $\frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{k\},\mathbf{w}|_{N\setminus\{k\}},\succeq|_{N\setminus\{k\}})}\mathbf{w}_k \succ_k \frac{P'(N,\mathbf{w},\succeq)}{P'(N\setminus\{i\},\mathbf{w}|_{N\setminus\{k\}},\succeq|_{N\setminus\{k\}})}\mathbf{w}_k$ follows from strong monotonicity and the premise on P and P'. However, this contradicts the premise that $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$. Therefore, there exists at most one potential function.

Second, to prove the existence of a potential function, we construct inductively a system of dividend ratios $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ for each economy $(N,\mathbf{w},\succeq)\in\mathcal{E}^N$:

1. For $|S|=1$, $d_S\equiv 0$;

2. for $|S|\geq 2$, we hypothesize that d_T has been defined for each $T\in 2^S\setminus\{\varnothing\}$. Then, we define $d_S\equiv\sup\{d\in[-1,+\infty)\mid \exists\mathbf{x}\in Z(S,\mathbf{w}|_S,\succeq|_S)$ such that $(1+d)(\prod_{\substack{T\ni i\\ T\subsetneq S}}(1+d_T))\mathbf{w}_i\sim_i\mathbf{x}_i\ \forall i\in S\}$.

Notice that d_S is well-defined for $|S|\geq 2$. Indeed, we check that the supremum operates on a non-empty set: $d=-1$ is in the set since $(1-1)(\prod_{\substack{T\ni i\\ T\subsetneq S}}(1+d_T))\mathbf{w}_i\sim_i\mathbf{0}\ \forall i\in S$ and $\mathbf{0}\in Z(S,\mathbf{w}|_S,\succeq|_S)$.

Next, we claim that d_S satisfies that $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1+d_T))\mathbf{w}_i\sim_i\mathbf{x}_i$ for all $i\in S$ and some $\mathbf{x}\in E(S,\mathbf{w}|_S,\succeq|_S)$. Notice that d_S satisfies that there exists $\mathbf{x}\in E(S,\mathbf{w}|_S,\succeq|_S)$ such that $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1+d_T))\mathbf{w}_i\preceq_i\mathbf{x}_i$ for all $i\in S$ because each agent's preference is continuous and $Z(S,\mathbf{w}|_S,\succeq|_S)$ is closed. Then, we prove our claim by contradiction: if there exists $k\in S$ such that $(\prod_{\substack{T\ni k\\ T\subseteq S}}(1+d_T))\mathbf{w}_k\prec_k\mathbf{x}_k$, then we can construct an alternative feasible allocation profile $\mathbf{y}\in Z(S,\mathbf{w}|_S,\succeq|_S)$ such that $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1+d_T))\mathbf{w}_i\prec_i\mathbf{y}_i$ for all $i\in S$. The existence of the profile $\mathbf{y}$ would imply that the supremum was not attained at d_S since d_S could be increased by a sufficiently small amount without violating feasibility. To construct $\mathbf{y}$ from $\mathbf{x}$, first note that $\mathbf{0}\preceq_k(\prod_{\substack{T\ni k\\ T\subseteq S}}(1+d_T))\mathbf{w}_k\prec_k\mathbf{x}_k$, hence there exists $h\in L$ such that $x_{kh}>0$. Define $\mathbf{y}$ by

$$y_{ig}\equiv\begin{cases}x_{ig} & \text{if } i\in S \text{ and } g\in L\setminus\{h\},\\ x_{ig}-\epsilon & \text{if } i=k \text{ and } g=h,\\ x_{ig}+\frac{\epsilon}{|S|-1} & \text{if } i\in S\setminus\{k\} \text{ and } g=h,\end{cases}$$

where $\epsilon\in\mathbb{R}_{++}$ is sufficiently small so that $(\prod_{\substack{T\ni k\\ T\subseteq S}}(1+d_T))\mathbf{w}_k\prec_k\mathbf{y}_k$ and $y_{kh}\geq 0$. By strong monotonicity, we have $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1+d_T))\mathbf{w}_i\prec_i\mathbf{y}_i$ for all $i\in S$.

Therefore, we have proven the existence of dividend ratios $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ that satisfy that, for each $S\in 2^N\setminus\{\varnothing\}$, there exists $\mathbf{x}\in E(S,\mathbf{w}|_S,\succeq|_S)$ such that $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1+d_T))\mathbf{w}_i\sim_i\mathbf{x}_i$ for all $i\in S$.

We can now construct the potential function: $P(N,\mathbf{w},\succeq)\equiv\prod_{S\in 2^N\setminus\{\varnothing\}}(1+d_S)$ for $N\neq\varnothing$ and $P(\varnothing)=1$. The potential function $P(N,\mathbf{w},\succeq)$ satisfies the conditions in Definition 21 given the construction of the dividend ratios. This establishes the existence of a potential function restricted to $\mathcal{E}'$. Therefore, there exists a unique potential function restricted to $\mathcal{E}'$. □

Proof of Corollary . The existence, uniqueness, and essential single-valuedness of $POSh$ restricted to $\mathcal{E}'$ follows from Proposition 10 and Definition 22. The representation in terms of dividend ratios follows from $\frac{P(N',\mathbf{w}|_{N'},\succeq|_{N'})}{P(N'\setminus\{i\},\mathbf{w}|_{N'\setminus\{i\}},\succeq|_{N'\setminus\{i\}})} = \frac{\prod_{T\subseteq N'}(1+d_T)}{\prod_{T\subseteq N'\setminus\{i\}}(1+d_T)} = \prod_{\substack{T\ni i\\ T\subseteq N'}}(1+d_T)$. □

Proof of Proposition 11. The empty-agent property follows from the efficiency implied by Definition 22, once we will prove the empty-agent out property, which we now do.

First, we claim that any potential function satisfies

$$P(N,\mathbf{w},\succeq) = P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq)). \tag{3.6.1}$$

We prove equation (3.6.1) by induction on p by which we denote the number non-empty agents of an economy $(N,\mathbf{w},\succeq)$ with q empty agents (q is an arbitrary fixed positive number). The equation holds trivially for an economy with only q empty agents, i.e, when $p = 0$. Now we consider an economy $(N,\mathbf{w},\succeq)$ with $p \geq 1$ non-empty agents and q empty agents. Denote by $\mathbf{x} \in E(supp(N,\mathbf{w},\succeq))$ an allocation profile satisfying that $\mathbf{x}_i \sim_i \frac{P(supp(N,\mathbf{w},\succeq))}{P(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i$ for all non-empty agent i. The allocation $\mathbf{x}$ satisfies that for each non-empty agent i, $\mathbf{x}_i \sim_i \frac{P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))}{P(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))P(0(N,\mathbf{w},\succeq))}\mathbf{w}_i = \frac{P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))}{P(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))P(0(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i = \frac{P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i$, where the first equality follows from the premise that i is not an empty agent and the second from the induction hypothesis (there exist $p-1$ non-empty agents in $(N\setminus\{i\},\mathbf{w}\,|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})$). Then consider a new allocation profile $\mathbf{y} \in E(N,\mathbf{w},\succeq)$, where $\mathbf{y}_j = \mathbf{x}_j$ for each non-empty agent j and $\mathbf{y}_k = \mathbf{0}$ for each empty agent k. Notice that the constructed profile $\mathbf{y}$ satisfies that $\mathbf{y}_i \sim_i \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i$ for all $i \in N$ where $P(N,\mathbf{w},\succeq) = P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))$. Moreover, by strong monotonicity, $p \geq 1$, and an argument similar to that establishing the uniqueness of potential function restricted to $\mathcal{E}'$ in Proposition 10, we have that the numerical value of $P(N,\mathbf{w},\succeq)$ is unique. Finally, since q is arbitrary, we have proven the equation (3.6.1), which immediately implies the empty-agent out property of any $POSh$. □

Proof of Theorem 9. First, by Proposition 11, any $POSh$ for $\mathcal{E}$ satisfies the empty-agent property and the empty-agent out property. Let us denote $POSh'$ the proportional ordinal Shapley solution restricted to $\mathcal{E}'$. Therefore, for any $POSh$ and any $(N,\mathbf{w},\succeq)$,

$$POSh_i(N,\mathbf{w},\succeq) \equiv \begin{cases} \mathbf{0} & \text{if } \mathbf{w}_i = \mathbf{0}, \\ POSh'_i(supp(N,\mathbf{w},\succeq)) & \text{if } \mathbf{w}_i \neq \mathbf{0}. \end{cases} \tag{3.6.2}$$

Second, the function $POSh$ restricted to $\mathcal{E}'$ is unique and essentially single-valued by Corollary . Hence, if $POSh$ exists for $\mathcal{E}$, it is also unique and essentially single-valued.

Third, let us denote P' the potential associated with $POSh'$ in $\mathcal{E}'$. We now propose the following potential function $P : \mathcal{E} \to \mathbb{R}$:

$$P(N, \mathbf{w}, \succeq) \equiv \begin{cases} 1 & \text{if } \mathbf{w}_i = \mathbf{0} \text{ for all } i \in N \\ P'(supp(N, \mathbf{w}, \succeq)) & \text{otherwise.} \end{cases} \quad (3.6.3)$$

We show that the function P can be associated with the $POSh$ that we constructed above for $\mathcal{E}$. If $\mathbf{w}_i = 0$, then the result is immediate because $POSh_i(N, \mathbf{w}, \succeq) = 0$. Otherwise, consider an economy $(N, \mathbf{w}, \succeq)$ where i is a non-empty agent, and $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$. Then, equation (3.6.2) implies that $\mathbf{x} \in POSh'_i(supp(N, \mathbf{w}, \succeq))$. Therefore, $\mathbf{x}_i \sim_i \frac{P'(supp(N,\mathbf{w},\succeq))}{P'(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i = \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i$.

Moreover, let $N' \subseteq N$ be the set of non-empty agents in $(N, \mathbf{w}, \succeq)$. Then $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and $\mathbf{x}_i \equiv \mathbf{0}$ for all $i \in N \setminus N'$ if and only if $\mathbf{x}|_{N'} \in E(supp(N, \mathbf{w}, \succeq))$. Thus, the constructed $P : \mathcal{E} \to \mathbb{R}$ is a potential function associated with the $POSh$ for $\mathcal{E}$, which means that there exists a $POSh$ for $\mathcal{E}$.

Finally, we show the existence of the vector of dividend ratios. For the coalitions without empty agents, that is, in the $supp(N, \mathbf{w}, \succeq))$, we take the vector found in Corollary . Additionally, we define $d_S \equiv 0$ for each $S \in 2^N \setminus \{\varnothing\}$ if there exists an empty agent in S.

To verify that the previous vector of dividend ratios satisfies the condition stated in the theorem, it suffices to show that $P(N, \mathbf{w}, \succeq) = \prod_{T\in 2^N\setminus\{\varnothing\}}(1+d_T)$ for a general economy $(N, \mathbf{w}, \succeq)$. We prove this by induction on the number of non-empty agents. It is easy to see that $P(N, \mathbf{w}, \succeq) = \prod_{T\in 2^N\setminus\{\varnothing\}}(1 + d_T) = 1$ holds when $(N, \mathbf{w}, \succeq)$ consists of empty agents only. Now consider an economy $(N, \mathbf{w}, \succeq)$ in which i is a non-empty agent. Let $N' \subseteq N$ be the set of all non-empty agents. Then, for any $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$, $\mathbf{x}_i \sim_i \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i = \frac{P'(supp(N,\mathbf{w},\succeq))}{P'(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i = \frac{\prod_{T\in 2^{N'}\setminus\{\varnothing\}}(1+d_T)}{\prod_{T\in 2^{N'\setminus\{i\}}\setminus\{\varnothing\}}(1+d_T)}\mathbf{w}_i = \frac{\prod_{T\in 2^{N}\setminus\{\varnothing\}}(1+d_T)}{\prod_{T\in 2^{N\setminus\{i\}}\setminus\{\varnothing\}}(1+d_T)}\mathbf{w}_i = \frac{\prod_{T\in 2^{N}\setminus\{\varnothing\}}(1+d_T)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i$, where the last equality follows from the induction hypothesis. Thus, it is the case that $P(N, \mathbf{w}, \succeq) = \prod_{T\in 2^N\setminus\{\varnothing\}}(1 + d_T)$. □

Proof of Proposition 12. By Theorem 9, we can verify that $\mathbf{x}_i \sim_i (\prod_{\substack{T\ni i \\ T\subseteq N}}(1+d_T))\mathbf{w}_i \succeq_i \mathbf{w}_i$ for all $i \in N$, all $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$, where $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ is the vector of dividend ratios corresponding to $POSh(N, \mathbf{w}, \succeq)$. We do it by induction on $|N|$. Since $POSh_i(\{i\}, \mathbf{w}_i, \succeq_i) = \{\mathbf{w}_i\}$ for $N = \{i\}$, our assertion trivially holds for $|N| = 1$.

For $|N| > 1$, assume that the property holds for any economy with less than $|N|$

agents. Suppose now that it does not hold for $(N, \mathbf{w}, \succeq)$, that is, there exists $i \in N$ such that $\mathbf{x}_i \sim_i (\sum_{\substack{T \ni i \\ T \subseteq N}} (1 + d_T))\mathbf{w}_i \prec_i \mathbf{w}_i$. Then there must exist $j \in N \setminus \{i\}$ such that $\mathbf{x}_j \succ_j (\prod_{\substack{T \ni j \\ T \subseteq N \setminus \{i\}}} (1 + d_T))\mathbf{w}_j$. The existence of such an agent j follows from $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$, $\mathbf{x}_i \prec_i \mathbf{w}_i$, and the feasibility of the allocation that assigns agent i with $\mathbf{w}_i$ and the rest of agents with a bundle prescribed by $POSh(N \setminus \{i\}, \mathbf{w} |_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}})$, which is individually rational by the induction hypothesis. Therefore, there exists $j \in N \setminus \{i\}$ such that $(\prod_{\substack{T \ni j \\ T \subseteq N}} (1+d_T))\mathbf{w}_j \sim_j \mathbf{x}_j \succ_j (\prod_{\substack{T \ni j \\ T \subseteq N \setminus \{i\}}} (1+d_T))\mathbf{w}_j$. Agent j's strict preference $(\prod_{\substack{T \ni j \\ T \subseteq N}} (1+d_T))\mathbf{w}_j \succ_j (\prod_{\substack{T \ni j \\ T \subseteq N \setminus \{i\}}} (1+d_T))\mathbf{w}_j$ implies that $\prod_{\substack{T \ni i,j \\ T \subseteq N}} (1 + d_T) > 1$ by strong monotonicity.

By the induction hypothesis, we have $(\prod_{\substack{T \ni i \\ T \subseteq N \setminus \{j\}}} (1 + d_T))\mathbf{w}_i \succeq_i \mathbf{w}_i$. Together with $\prod_{\substack{T \ni i,j \\ T \subseteq N}} (1 + d_T) > 1$, it implies that $\mathbf{x}_i \sim_i (\prod_{\substack{T \ni i \\ T \subseteq N}} (1 + d_T))\mathbf{w}_i = (\prod_{\substack{Q \ni i,j \\ Q \subseteq N}} (1 + d_Q))(\prod_{\substack{T \ni i \\ T \subseteq N \setminus \{j\}}} (1 + d_T))\mathbf{w}_i \succeq_i (\prod_{\substack{T \ni i \\ T \subseteq N \setminus \{j\}}} (1 + d_T))\mathbf{w}_i \succeq_i \mathbf{w}_i$, which contradicts our assumption. Therefore, the $POSh$ satisfies individual rationality. □

Proof of Proposition 13. It is easy to see that the efficient allocation correspondence is anonymous, that is, $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ if and only if $\Theta\mathbf{x} \in E\Theta(N, \mathbf{w}, \succeq)$, for all $\Theta = (\pi, \rho)$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Consider the economies $(N, \mathbf{w}, \succeq)$ and $\Theta(N, \mathbf{w}, \succeq)$. Take $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$ and let $(d_T)_{T \in 2^N \setminus \{\varnothing\}}$ be its vector of dividend ratios. We show that $\Theta\mathbf{x} \in POSh\Theta(N, \mathbf{w}, \succeq)$ by proving that $(d'_{\pi[T]})_{T \in 2^N \setminus \{\varnothing\}}$, with $d'_{\pi[T]} = d_T$ for all $T \in 2^N \setminus \{\varnothing\}$, constitutes a vector of dividend ratios for $\Theta\mathbf{x}$.

To see this, for each $N' \in 2^N \setminus \{\varnothing\}$, let $\Theta |_{N'} \equiv (\pi |_{N'}, \rho |_{N'})$. Then for each $\mathbf{y} \in POSh(N', \mathbf{w} |_{N'}, \succeq|_{N'})$, $\mathbf{y}_i \sim_i (\prod_{T \in 2^{N'} \setminus \{\varnothing\}} (1 + d_T))\mathbf{w}_i$ for all $i \in N'$, which is equivalent to $\Theta |_{N'} \mathbf{y}_{\pi(i)} \sim_{\pi(i)}^{\Theta|_{N'}} (\prod_{T \in 2^{\pi[N']} \setminus \{\varnothing\}} (1 + d_T))\mathbf{w}_{\pi(i)}$ for all $i \in N'$, i.e., $\Theta |_{N'} \mathbf{y}_j \sim_j^{\Theta|_{N'}} (\prod_{T \in 2^{\pi[N']} \setminus \{\varnothing\}} (1 + d_T))\mathbf{w}_j$ for all $j \in \pi[N']$. Hence, $(d'_{\pi[T]})_{T \in 2^N \setminus \{\varnothing\}}$ is a vector of dividend ratios for $\Theta(N, \mathbf{w}, \succeq)$, according to Theorem 9. Finally, by letting $N' = N$, we have that $\Theta\mathbf{x} \in POSh\Theta(N, \mathbf{w}, \succeq)$ □

Proof of Proposition 14. Consider two economies $(N, \mathbf{w}, \succeq), (N, \mathbf{w}', \succeq) \in \mathcal{E}$ such that $\mathbf{w}_i > \mathbf{w}'_i$ for $i \in N$ and $\mathbf{w}_j = \mathbf{w}'_j$ for each $j \in N \setminus \{i\}$. By Theorem 9, we denote by $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ and $(d'_S)_{S \in 2^N \setminus \{\varnothing\}}$ their associated vectors of dividend ratios, respectively. We claim that $\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d'_T)$ for all $S \subseteq N$ such that $S \ni i$. We prove the claim by induction on $|S|$. It trivially holds for $|S| = 1$.

For $S \subseteq N$ such that $|S| > 1$, suppose otherwise, i.e., $\prod_{\substack{T \ni i \\ T \subseteq S}} (1+d_T) < \prod_{\substack{T \ni i \\ T \subseteq S}} (1+d'_T)$ and $\prod_{\substack{T \ni i \\ T \subseteq R}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq R}} (1 + d'_T)$ for all $R \subsetneq S$. In particular, $\prod_{\substack{T \ni i \\ T \subseteq S \setminus \{j\}}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq S \setminus \{j\}}} (1 + d'_T)$ for all $j \in S \setminus \{i\}$. Then, for each $j \in S \setminus \{i\}$, $\prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d_T) < \prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d'_T)$ because $\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T) = \prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d_T) \prod_{\substack{T \ni i \\ T \subseteq S \setminus \{j\}}} (1 + d_T)$.

Then, it is also true that $\prod_{\substack{T\ni j\\T\subseteq S}}(1+d'_T) = (\prod_{\substack{T\ni j\\T\subseteq S\setminus\{i\}}}(1+d'_T))(\prod_{\substack{T\ni i,j\\T\subseteq S}}(1+d'_T)) = (\prod_{\substack{T\ni j\\T\subseteq S\setminus\{i\}}}(1+d_T))(\prod_{\substack{T\ni i,j\\T\subseteq S}}(1+d'_T)) > \prod_{\substack{T\ni j\\T\subseteq S}}(1+d_T)$ for all $j \in S \setminus \{i\}$. Thus, $POSh_j(S, \mathbf{w}|_S, \succeq|_S) \prec_j POSh_j(S, \mathbf{w}'\,|_S, \succeq|_S)$ for all $j \in S$ (including i himself by premise), which is impossible. Therefore, the $POSh$ is not D-manipulable. □

Proof of Proposition 15. Consider two economies $(N, \mathbf{w}, \succeq), (N, \mathbf{w}', \succeq) \in \mathcal{E}$ such that $\mathbf{w}_i > \mathbf{w}'_i$, $\mathbf{w}_i + \mathbf{w}_j = \mathbf{w}'_i + \mathbf{w}'_j$ for donor i and recipient j; $\mathbf{w}_k = \mathbf{w}'_k$ for each $k \in N \setminus \{i, j\}$. By Theorem 9, let $(d_T)_{T\in 2^N\setminus\{\varnothing\}}$ and $(d'_T)_{T\in 2^N\setminus\{\varnothing\}}$ are vectors of dividend ratios for economies $(N, \mathbf{w}, \succeq)$ and $(N, \mathbf{w}', \succeq)$, respectively. By considering the subeconomies without player j and without player i, Proposition 14 implies that $\prod_{\substack{T\ni i\\T\subseteq N\setminus\{j\}}}(1+d_T) \geq \prod_{\substack{T\ni i\\T\subseteq N\setminus\{j\}}}(1+d'_T)$ and $\prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d_T) \leq \prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d'_T)$.

Suppose that the donor i is better off in $POSh(N, \mathbf{w}', \succeq)$ than in $POSh(N, \mathbf{w}, \succeq)$, which means that $\prod_{\substack{T\ni i\\T\subseteq N}}(1+d_T) < \prod_{\substack{T\ni i\\T\subseteq N}}(1+d'_T)$. It implies that $\prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d_T) < \prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d'_T)$. Then $\prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d_T)(\prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d_T)) < (\prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d'_T))(\prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d'_T))$, i.e., $\prod_{\substack{T\ni j\\T\subseteq N}}(1+d_T) < \prod_{\substack{T\ni j\\T\subseteq N}}(1+d'_T)$. Thus, recipient j must also be better off in $(N, \mathbf{w}', \succeq)$. Therefore, the transfer paradox is not possible for $POSh$ □

Proof of Proposition 16. To prove $\zeta = POSh$, we first show that $POSh \subseteq \zeta$, and then that ζ is essentially single-valued.

We prove that $POSh \subseteq \zeta$. Recall that there exists a vector of dividend ratios $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ such that $x \in POSh_i(N, \mathbf{w}, \succeq)$ if and only if $x \sim_i (\prod_{\substack{T\ni i\\T\subseteq N}}(1+d_T))\mathbf{w}_i$ for all $i \in N$ and $POSh_j(N \setminus \{k\}, \mathbf{w}\,|_{N\setminus\{k\}}, \succeq|_{N\setminus\{k\}}) \sim_j (\prod_{\substack{T\ni j\\T\subseteq N\setminus\{k\}}}(1+d_T))\mathbf{w}_j$ for all $k \in N$ and all $j \in N \setminus \{k\}$. Take $a_{ij} \equiv \prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d_T)$ and $c_{ij} \equiv \prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d_T)$ for all $j \in N \setminus \{i\}$ and all $i \in N$. Then, x together with the vectors c_i and a_i for all $i \in N$, satisfy part (2b) of Definition 23. Moreover, $c_{ij} = c_{ji}$ for all $j \in N \setminus \{i\}$ and all $i \in N$. Hence, part (2a) of Definition 23 also holds. Therefore, $POSh \subseteq \zeta$.

We prove ζ is essentially single-valued by induction on the number of agents $|N|$. It trivially holds for $|N| = 1$ by definition. For $|N| > 1$, we hypothesize that ζ is essentially single-valued for any economy with $n-1$ agents. It implies that a^N_{ij} in Definition 23 is unique (that is, it is the same for all $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$) for all $i, j \in N$ such that $i \neq j$.

Consider any $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$. According to (2b) in Definition 23, it is the case that $\mathbf{x}_j \sim_j c^N_{ij}a^N_{ij}\mathbf{w}_j$ and $\mathbf{x}_j \sim_j c^N_{kj}a^N_{kj}\mathbf{w}_j$, for all $j \in N$ and all $i, k \in N\setminus\{j\}$ such that $i \neq k$. Then, strong monotonicity implies that $c^N_{ij}a^N_{ij} = c^N_{kj}a^N_{kj}$ for all $i, k \in N \setminus \{j\}$ such that $i \neq k$.

Therefore, we have $|N|(|N|-1)$ equations: $c^N_{i1}n = \frac{a^N_{n1}}{a^N_{i1}}c^N_{n1}$ for all $i \in N \setminus \{1\}$, and

$c_{ij}^N = \frac{a_{(j-1)j}^N}{a_{ij}^N} c_{(j-1)j}^N$ for all $i \in N \setminus \{j\}$ and all $j \in N \setminus \{1\}$. By substituting them in condition $(2a)$, $\prod_{j \in N\setminus\{i\}} c_{ij}^N = \prod_{j \in N\setminus\{i\}} c_{ji}^N$, we have $\frac{a_{n1}^N}{a_{i1}^N} c_{n1}^N [\prod_{j \in N\setminus\{1,i\}} \frac{a_{(j-1)j}^N}{a_{ij}^N} c_{(j-1)j}^N] = \prod_{j \in N\setminus\{i\}} \frac{a_{(i-1)i}^N}{a_{ji}^N} c_{(i-1)i}^N$ for $i \in N \setminus \{1\}$. From this equality we obtain that $c_{(i-1)i}^N = \frac{1}{a_{(i-1)i}^N} \sqrt[n-1]{\frac{a_{n1}^N}{a_{i1}^N} c_{n1}^N [\prod_{j \in N\setminus\{1,i\}} \frac{a_{(j-1)j}^N}{a_{ij}^N} c_{(j-1)j}^N][\prod_{j \in N\setminus\{i\}} a_{ji}^N]}$ for each $i \in N \setminus \{1\}$. It means that $c_{(i-1)i}^N$ can be expressed as an increasing function of $(c_{(k-1)k}^N)_{k \in N\setminus\{1,i\}}$ and c_{n1}^N as its arguments, for each $i \in N \setminus \{1\}$. Moreover, by repeated substitution, we can represent each $c_{(i-1)i}^N$ as an increasing function of c_{n1}^N solely, for all $i = N\setminus\{1\}$. Hence, $c_{(i-1)i}^N$ takes the form $c_{(i-1)i}^N(c_{n1}^N)$ for $i = 2, \ldots, n$. Thus, for all $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$, $\mathbf{x}_1 \sim_1 c_{n1}^N a_{n1}^N \mathbf{w}_1$, and $\mathbf{x}_i \sim_i c_{(i-1)i}^N(c_{n1}^N) a_{(i-1)i}^N \mathbf{w}_i$ for all $i = 2, \ldots, n$. By Pareto efficiency of $\mathbf{x}$, c_{n1} is unique. Therefore, ζ is essentially single-valued, which completes the proof. $\square$

Proof of Theorem 10. To formalize our argument, we introduce the following notation. Applying the mechanism to an economy $(N, \mathbf{w}, \succeq)$ results in an extensive form game, which is denoted by $\Gamma(N, \mathbf{w}, \succeq)$. Denote the SPNE outcome correspondence by $\mathcal{SN}$, which enables us to express the set of all SPNE outcomes of an extensive form game as the value of $\mathcal{SN}$ at this game. For example, the set of all SPNE outcomes of $\Gamma(N, \mathbf{w}, \succeq)$ is $\mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$. Furthermore, if $\mathbf{x} \sim \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$, we may write $\mathcal{SN}_i\Gamma(N, \mathbf{w}, \succeq)$ and compare it with a bundle in terms of $\succeq_i$ for all $i \in N$ without incurring confusion. We may also consider the subgames of $\Gamma(N, \mathbf{w}, \succeq)$. We denote by $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ the subgame starting from the information set after proposer i's allocation plan is rejected and the bids made were $\mathbf{b}^N$. In particular, the bids made for i were b_{ji}^N for all $j \in N$.

The proof comprises three parts: (i) for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$, all $\mathbf{x} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$, and all $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$, if $\mathbf{y} \sim \mathbf{x}$ then $\mathbf{y} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$; (ii) $\mathcal{SN}\Gamma(N, \mathbf{w}, \succeq) \subseteq POSh(N, \mathbf{w}, \succeq)$ for every $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$; (iii) $\mathcal{SN}\Gamma(N, \mathbf{w}, \succeq) \neq \varnothing$ for every $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$. Note that parts (i)-(iii) imply that $\mathcal{SN}\Gamma(N, \mathbf{w}, \succeq) = POSh(N, \mathbf{w}, \succeq)$.

We prove the three parts simultaneously by induction on $|N|$. The case where $|N| = 1$ is trivial, so we restrict attention to the cases where $|N| \geq 2$. We assume the induction hypothesis that (i)-(iii), and consequently the implementation of $POSh$ by the bidding mechanism, hold for all economies with less than n agents.

To prove part (i), we first state and prove two claims. We notice that the set of agents in $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ is N. On the other hand, the set of agents in $\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ is $N \setminus \{i\}$. However, the sets of SPNE of the extensive-form games $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ and $\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ are "similar" in the following sense:

Claim 14. *Given* $\mathbf{b}^N$ *and* $\mathbf{y}' \in Z(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$*, define* $\mathbf{y} \in Z(N, \mathbf{w}, \succeq)$ *by*

$$\mathbf{y}_j = \begin{cases} \left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)\mathbf{y}'_j & \text{if } j \in N \setminus \{i\}; \\ \mathbf{w}_i - \sum_{k\in N\setminus\{i\}}\left(\frac{\sqrt[n]{B_i^N}}{b_{ki}^N} - 1\right)\mathbf{y}'_k & \text{if } j = i. \end{cases}$$

Then $\mathbf{y} \in \mathcal{SN}\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ *if* $\mathbf{y}' \in \mathcal{SN}\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$.

We prove Claim 14. Let $(s_k)_{k\in N\setminus\{i\}}$ be an SPNE strategy profile for $\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ whose outcome is $\mathbf{y}'$. We notice that the final outcome in $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ if the agents play $(s_k)_{k\in N\setminus\{i\}}$ is $\mathbf{y}$ (note that although the game $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ involves all the agents in N, only the agents in $N \setminus \{i\}$ choose a strategy) . We prove that $(s_k)_{k\in N\setminus\{i\}}$ is an SPNE for $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$. Suppose otherwise. Let s'_j be a profitable deviation for $j \in N \setminus \{i\}$ in $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$. After the deviation, j obtains a bundle $\mathbf{z}_j$ which is multiplied by $\left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)$ and such that $\left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)\mathbf{z}_j \succ_j \mathbf{y}_j = \left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)\mathbf{y}'_j$. Since the preferences are homothetic, we have that $\mathbf{z}_j \succ_j \mathbf{y'}_j$. However, if j deviates in $\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ from $(s_k)_{k\in N\setminus\{i\}}$ by choosing s'_j, then he obtains the allocation $\mathbf{z}_j$, which he prefers to $\mathbf{y}'_j$. This is not possible because $\mathbf{y}' \in \mathcal{SN}\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Hence, $(s_k)_{k\in N\setminus\{i\}}$ is an SPNE for $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ and $\mathbf{y} \in \mathcal{SN}\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$, which concludes the proof of Claim 14.

We note that following the same arguments as in the proof of Claim 14, the reverse result also holds. That is, given $\mathbf{b}^N$ and $\mathbf{y} \in \mathcal{SN}\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$, define $\mathbf{y}'$ by $\mathbf{y}'_j = \left(\frac{b_{ji}^N}{\sqrt[n]{B_i^N}}\right)\mathbf{y}_j$ for all $j \in N \setminus \{i\}$. We note that $\mathbf{y}' \in Z(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ because $\mathbf{y}'_j$ is the allocation that player j obtains in the game $\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$ before the rejected proposer i transfers the bundles according to $\mathbf{b}^N$ (see Case II at $t = 3$ as described in the proportional bidding mechanism); hence, $\sum_{k\in N\setminus\{i\}} \mathbf{y}'_k = \sum_{k\in N\setminus\{i\}} \mathbf{w}_k$. Then $\mathbf{y}' \in \mathcal{SN}\Gamma(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ if $\mathbf{y} \in \mathcal{SN}\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq)$.

The induction hypothesis states that $\mathcal{SN}\Gamma(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) = POSh(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Then, Claim 14 and its reverse imply that $\mathcal{SN}_j\Gamma_{\mathbf{b}^N}^{-i}(N, \mathbf{w}, \succeq) = \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}})$ for all $j \in N \setminus \{i\}$. With the aid of this equality, we assert formally that every SPNE outcome can be supported by an SPNE that leads to an immediate agreement in Claim 15.

Claim 15. *For every SPNE outcome* $\mathbf{x} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$*, take an SPNE whose outcome is* $\mathbf{x}$*. Let* $\mathbf{b}^N$ *be the profile of the agents' bid vectors in that SPNE and consider the subgame where agent* $i \in \operatorname{argmax}_{k\in N} B_k^N$ *becomes the proposer. Then,*

a) $\mathbf{x}_j \sim_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ *for all* $j \in N \setminus \{i\}$.

b) There exists an SPNE where:

b1) each agent $j \in N \setminus \{i\}$ accepts any i's allocation plan $\mathbf{z} \in \mathbb{R}_+^{(N\setminus\{i\})\times L}$ if $\mathbf{z}_j \succeq_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ and rejects it otherwise;

b2) the proposer i puts forth an allocation plan $\mathbf{z}$ such that $\mathbf{z}_j \sim_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N\setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ for each agent $j \in N \setminus \{i\}$, so that the resulting allocation $\mathbf{y} \in \mathbb{R}_+^{N\times L}$, which is defined by $\mathbf{y}_j = \mathbf{z}_j$ for all $j \in N \setminus \{i\}$ and $\mathbf{y}_i = \sum_{k\in N} \mathbf{w}_k - \sum_{k\in N\setminus\{i\}} \mathbf{z}_k$, is efficient.

We prove part a) of Claim 15. Notice that, for any economy $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$, given a profile of bid vectors $\mathbf{b}^N$ and a proposer $i \in N$, then agent $j \in N\setminus\{i\}$ accepts at equilibrium any i's allocation plan $\mathbf{z} \in \mathbb{R}_+^{(N\setminus\{i\})\times L}$ if $\mathbf{z}_j \succ_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ and rejects it if $\mathbf{z}_j \prec_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. This holds because agent j obtains a bundle $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ in case of rejection (by the induction hypothesis and Claim 14). Moreover, a proposal $\mathbf{z}$ such that $\mathbf{z}_j \succ_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ cannot be part of an SPNE. Such a proposal would be accepted, but i would have an incentive to lower $\mathbf{z}_j$ by a sufficiently small amount and propose another acceptable offer resulting in a higher residual bundle for himself.

Combining the above observations, agent j cannot obtain at equilibrium a bundle strictly better or strictly worse than $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ in terms of his preference $\succeq_i$. Therefore, j obtains a bundle that makes him indifferent to the bundle $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Hence, the equation in part a) holds.

The previous arguments also prove part b1) of the claim. To prove part b2), consider an efficient allocation plan $\mathbf{z}$ such that $\mathbf{z}_j \sim_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) \sim_j \mathbf{x}_j$ for each agent $j \in N \setminus \{i\}$. Part b1) ensures that the agents in $N \setminus \{i\}$ will accept this proposal. Moreover, given that it is efficient, there is no better allocation for i that would be accepted. Proposing a rejected plan cannot be a profitable deviation for agent i because rejection leads to a feasible allocation where every $j \in N\setminus\{i\}$ obtains a bundle equivalent for him to $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$, which cannot be strictly better for i than $\mathbf{z}$. Hence, Claim 15 is proven.

We now prove part (i) or our induction. Take an allocation $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{y} \sim \mathbf{x}$. Then consider the strategy profile that is identical to that in Claim 15 (including the bids) except that agent i proposes $\mathbf{y}|_{N\setminus\{i\}}$ in b2). Given that the SPNE is an ordinal solution, the strategy profile described in Claim 15 is an SPNE if and only if the new strategy profile is an SPNE. Therefore, for all SPNE outcome $\mathbf{x} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$ and all $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{y} \sim \mathbf{x}$, $\mathbf{y} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$, which completes part (i).

We proceed to prove part (ii). We first establish the following property of the equilibrium bids.

Claim 16. *In any SPNE, $B_i^N = 1$ for all $i \in N$. Moreover, each agent $i \in N$ is indifferent about the identity of the proposer.*

We first show that each $i \in N$ is indifferent about the identity of the proposer among the agents in $\operatorname{argmax}_{j\in N} B_j^N$. Let $|\operatorname{argmax}_{j\in N} B_j^N| = p$. If $p = 1$, then this assertion automatically holds. If $p \geq 2$, then assume that agent i strictly prefers k to z as the proposer, for a pair of agents $k, z \in \operatorname{argmax}_{j\in N} B_j^N$ (where i could possibly be either k or z). In this case, agent i has an incentive to deviate by increasing b_{ik}^N to $(1+\epsilon)b_{ik}^N$ and decreasing b_{iz}^N to $\frac{b_{iz}^N}{1+\epsilon}$, where $\epsilon \in \mathbb{R}_{++}$ is sufficiently small. To see this, note that this deviation would ensure that agent k would become the proposer. There are two cases. On the one hand, if $i \neq k$ then agent i avoids the positive probability of receiving a bundle strictly worse than $\frac{\sqrt[n]{B_k^N}}{b_{ik}^N} POSh_i(N \setminus \{k\}, \mathbf{w}\,|_{N\setminus\{k\}}, \succeq|_{N\setminus\{k\}})$ and ensures receiving a bundle equivalent for him to $\frac{\sqrt[n]{B_k^N}}{(1+\epsilon)b_{ik}^N} POSh_i(N \setminus \{k\}, \mathbf{w}\,|_{N\setminus\{k\}}, \succeq|_{N\setminus\{k\}})$, by Claim 15. This is a profitable deviation if ϵ is small enough. On the other hand, if $i = k$ then agent i becomes the proposer. He can put forth an allocation plan such that each agent $j \in N \setminus \{i\}$ is assigned a bundle $\sim_j$-equivalent to $\frac{\sqrt[n]{(1+\epsilon)B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$, and the plan will be accepted, by Claim 15. As before, by continuity of $\succeq_i$, agent i is strictly better off by switching to the new bid vector for a sufficiently small ϵ. To sum up, in either case, it is profitable for agent i to switch to the new bid vector. Therefore, the converse is true: every agent must be indifferent concerning the identity of the proposer.

Second, suppose that $B_i^N = 1$ does not hold for all $i \in N$, which implies that there is $m \in N \setminus \operatorname{argmax}_{i\in N} B_i^N$. Then, any agent $j \in \operatorname{argmax}_{i\in N} B_i^N$ has an incentive to switch to a new bid vector $\tilde{\mathbf{b}}_j^N$, which is defined by

$$\tilde{b}_{jk}^N \equiv \begin{cases} (1-\epsilon)b_{jk}^N & \text{if } k = j, \\ (1-\epsilon)^2 b_{jk}^N & \text{if } k \in \operatorname{argmax}_{i\in N} B_i^N \setminus \{j\}, \\ (1-\epsilon)^{1-2p} b_{jk}^N & \text{if } k = m, \\ b_{jk}^N & \text{otherwise,} \end{cases}$$

where $\epsilon \in \mathbb{R}_{++}$ is sufficiently small. After this switch, agent j would be the proposer for sure. Notice that $\frac{\sqrt[n]{\tilde{B}_j^N}}{b_{ij}^N} < \frac{\sqrt[n]{B_j^N}}{b_{ij}^N}$ for each $i \in N \setminus \{j\}$ because $\tilde{B}_j^N < B_j^N$ given that $\tilde{b}_{jj}^N < b_{jj}^N$. Then, by Claim 15, agent j can propose an allocation plan that assigns a bundle slightly better for i than $\frac{\sqrt[n]{\tilde{B}_j^N}}{b_{ij}^N} POSh_i(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})$

instead of $\frac{\sqrt[n]{B_j^N}}{b_{ij}^N} POSh_i(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})$, for each agent $i \in N \setminus \{j\}$, and the plan will be accepted. Thus, agent j is strictly better off by switching from $\mathbf{b}_j^N$ to $\tilde{\mathbf{b}}_j^N$. Thus, $\text{argmax}_{i\in N} B_i^N = N$, i.e., $B_i^N = B_j^N$ for all $i, j \in N$. Since $\prod_{i\in N} B_i^N = \prod_{i\in N}\prod_{j\in N} b_{ji}^N = \prod_{j\in N}\prod_{i\in N} b_{ji}^N = 1^n = 1$, then $B_i^N = 1$ for all $i \in N$. This concludes the proof of Claim 16.

To continue with the proof of part (ii), let

$$c_{ij}^N \equiv \frac{\sqrt[n]{B_i^N}}{b_{ji}^N}, \tag{3.6.4}$$

for all $i, j \in N$ such that $i \neq j$. We can verify that $\frac{\sqrt[n]{B_i^N}}{b_{ii}} \prod_{j\in N\setminus\{i\}} c_{ij}^N = \frac{B_i^N}{\prod_{j\in N} b_{ji}^N} = 1$ and $\frac{\sqrt[n]{B_i^N}}{b_{ii}} \prod_{j\in N\setminus\{i\}} c_{ji}^N = \prod_{k\in N} \frac{\sqrt[n]{B_k^N}}{b_{ik}} = \frac{\sqrt[n]{\prod_{k\in N} B_k^N}}{\prod_{k\in N} b_{ik}} = \sqrt[n]{\prod_{k\in N} B_k^N} = \sqrt[n]{\prod_{k\in N}\prod_{j\in N} b_{jk}^N} = \sqrt[n]{\prod_{j\in N}\prod_{k\in N} b_{jk}^N} = 1$. Thus, $\prod_{j\in N\setminus\{i\}} c_{ij}^N = \prod_{j\in N\setminus\{i\}} c_{ji}^N = \frac{b_{ii}}{\sqrt[n]{B_i^N}}$, which satisfies the condition $(2a)$ of Definition 23 of the $POSh$. To check that the concessions that we just defined also satisfy the condition $(2b)$ of Definition 23, we notice that, when agents' preferences are homothetic, this condition is equivalent to:

For each $j \in N \setminus \{i\}$, $\mathbf{x}_j \sim_j c_{ij}^N \zeta_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$.

We can interpret $\zeta_j(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ as $POSh_j(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ since, as we have shown, $\mathcal{SN}_j\Gamma(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) = POSh_j(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Then, condition $(2b)$ is also satisfied, which completes the proof of part (ii).

We now prove part (iii) of our induction. Let us denote $\mathbf{c}_i^{N'}$, for all $i \in N'$, the unique vector of concessions for the subeconomy $(N', \mathbf{w}\,|_{N'}, \succeq|_{N'})$ for all $N' \subseteq N$ such that $n' \geq 2$. We construct the agents' strategy profile as follows. At any subgame where the remaining set of active agents (i.e., agents choose a strategy) is $N' \subseteq N$ and they have to bid, agent $i \in N'$ selects the bid $b_{ij}^{N'} = \frac{1}{c_{ji}^{N'}}$ for player $j \in N' \setminus \{i\}$ and $b_{ii}^{N'} = \prod_{k\in N'\setminus\{i\}} c_{ki}^{N'}$ (hence, the bids are well-defined because $B_i^{N'} = \prod_{j\in N'} b_{ji}^{N'} = \prod_{j\in N'\setminus\{i\}} \frac{1}{c_{ij}^{N'}} \prod_{k\in N'\setminus\{i\}} c_{ki}^{N'} = \prod_{j\in N'\setminus\{i\}} \frac{1}{c_{ji}^{N'}} \prod_{k\in N'\setminus\{i\}} c_{ki}^{N'} = 1$ due to the condition 2a) of Definition 23). Proposers' equilibrium allocation plans and the rest of agents' responses to the proposers' plans at any subgame follow the description in Claim 15 b).

We have shown that no agent has an incentive to deviate once the bids $\mathbf{b}^N$ have been made. It remains to verify that no agent has an incentive to change this bid vector. Suppose that agent $i \in N$ changes his bid from $\mathbf{b}_i^N$ to $\tilde{\mathbf{b}}_i^N$. Then it will not be the case that $B_j^N = 1$ for all $j \in N$. Denote by α the resulting proposer. Given that $\tilde{B}_\alpha^N \equiv \tilde{b}_{i\alpha}^N \prod_{j\in N\setminus\{i\}} b_{j\alpha}^N > \prod_{j\in N} b_{j\alpha}^N = B_\alpha^N$, it is necessarily the case that $\tilde{b}_{i\alpha}^N > b_{i\alpha}^N$. If $\alpha = i$, then each agent $j \in N \setminus \{i\}$ will be allocated a bundle $\mathbf{x}_j \sim_j \frac{\sqrt[n]{\tilde{B}_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w} \mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}}) \succeq_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w} \mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}})$ (see Claim 15 a)), and he will be better off. By Pareto efficiency of the final allocation, agent i, as the residual claimant, cannot be strictly better off. If, on the other hand, $\alpha \neq i$, agent i will be allocated a bundle $\mathbf{x}_i \sim_i \frac{\sqrt[n]{\tilde{B}_\alpha^N}}{\tilde{b}_{i\alpha}^N} POSh_i(N \setminus \{\alpha\}, \mathbf{w} \mid_{N\setminus\{\alpha\}}, \succeq\mid_{N\setminus\{\alpha\}}) \preceq_i \frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N} POSh_i(N \setminus \{\alpha\}, \mathbf{w} \mid_{N\setminus\{\alpha\}}, \succeq\mid_{N\setminus\{\alpha\}})$ because $\tilde{b}_{i\alpha}^N > b_{i\alpha}^N$ implies that $\frac{\sqrt[n]{\tilde{B}_\alpha^N}}{\tilde{b}_{i\alpha}^N} < \frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N}$. Therefore, agent i cannot be strictly better off either. This proves the existence of an SPNE, which concludes the proof of the theorem.

□

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