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GEOMETRIC FUNCTION THEORY IN FLUID MECHANICS

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Dedicated to the loving memory of my brothers,
Arnab and Snigdho

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Contents

Motivation

Between Cauchy-Lipschitz and DiPerna-Lions

This thesis is mainly devoted to the study of *certain planar vector fields and the rotational properties of its flow*. We use strong geometric tools coming from *Geometric Function Theory* to describe these vector fields and their corresponding flows. Let us start with the *Lagrangian method* of describing a vector field. We are given a bounded smooth vector field $\mathbf{b} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. The *flow of* $\mathbf{b}$ is the map $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that the ODE

$$\begin{cases} \frac{d}{dt} X(t,x) = \mathbf{b}\,(t, X(t,x)) \\ X(0,x) = x. \end{cases} \tag{1}$$

holds true. For simplicity, we will assume $\mathbf{b}$ to be time-independent, although most of what we discuss below holds for non-autonomous vector fields $\mathbf{b} = \mathbf{b}(t,x)$. By the classical *Cauchy-Lipschitz* theory it is well known that if the vector field $\mathbf{b}$ is *Lipschitz*, $\mathbf{b} \in Lip(\mathbb{R}^n)$, then the flow X is bi-Lipschitz. Moreover, by the classical *Rademacher-Stepanov* Theorem, this also shows that for every fixed time t the flow map $X(t,\cdot)$ is differentiable at almost every point x. Slightly below Lipschitz vector fields, we have the *Zygmund* class. The Zygmund class $\Lambda_*(\mathbb{R}^n)$ is the space of bounded continuous vector fields $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$|\mathbf{b}(x+h) + \mathbf{b}(x-h) - 2\mathbf{b}(x)| \leq C\|h\| \tag{2}$$

for each $x, h \in \mathbb{R}^n$. This class was introduced by Zygmund in the 40s when he observed that the conjugate of a Lipschitz function in the unit circle needs not be Lipschitz, but rather it is in this particular class. In this sense, $\Lambda_*(\mathbb{R}^n)$ is known to be the natural replacement for $Lip(\mathbb{R}^n)$ in many different circumstances in Harmonic Analysis, due to its Calderón-Zygmund invariance. Also, the following inclusions hold true,

$$Lip(\mathbb{R}^n) \subsetneq \Lambda_*(\mathbb{R}^n) \subsetneq C^\alpha(\mathbb{R}^n), \tag{3}$$

where $C^\alpha(\mathbb{R}^n)$ is the class of *Hölder continuous* vector fields in $\mathbb{R}^n$ with $0 < \alpha < 1$.

Zygmund vector fields are important because they represent the first example of non necessarily Lipschitz vector fields producing well defined flows. Indeed, these vector fields are continuous, with a *modulus of continuity* of type $\delta \log \frac{1}{\delta}$, which ensures existence and uniqueness of a flow of Hölder continuous homeomorphisms, by virtue of *Osgood's theorem*. At the same time, in contrast to the case of Lipschitz vector fields, functions in the Zygmund class may be non-differentiable at any point, whence in general one should not expect the flow $X(t,\cdot)$ of such vector fields to be differentiable either.

Away from Lipschitz and Zygmund vector fields, one has vector fields in the *Sobolev* class $W^{1,p}_{loc}$ for $1 \leq p < \infty$. When $1 \leq p \leq n$, these vector fields need not be continuous in general. For $n < p < \infty$ they are Hölder continous with exponent $1 - \frac{n}{p}$, but they may fail to be Lipschitz or Zygmund. In this setting, there is another option for describing the flow, viz. the *Eulerian formulation*. In this method, we consider the associated PDE to (1), that is

$$\begin{cases} \partial_t u + \mathbf{b} \cdot \nabla u = 0 \\ u(x,0) = u_0. \end{cases} \tag{4}$$

In the literature, this PDE is called the *Linear Transport Equation*. Roughly speaking, well posedness of (4) for $\mathbf{b} \in W^{1,p}_{loc}$ with $p < +\infty$ was proved by Di Perna-Lions in 1989, see [24]. They used the method of *renormalization* from mathematical physics. In 2004 Ambrosio gave a major breakthrough by extending this result to a class of vector fields which are less regular than Sobolev, viz. BV_{loc} vector fields, see [5]. Among the consequences of Di Perna - Lions theory, one obtains for any Sobolev vector field $\mathbf{b}$ the *existence and uniqueness* of a well-defined flow $X(t,x)$, which consists (at any time) of *measurable self-maps* of $\mathbb{R}^n$, and which in general won't be continuous, hence they won't be differentiable either. However, by a result of Le Bris - Lions [40], one may ensure for these flows a much weaker kind of differentiability: they are *differentiable in measure.*

The Gap

There are some particular instances for which the flow X inherits the Sobolev smoothness of the vector field $\mathbf{b}$. This is the case, for instance, when $n = 2$ and the Sobolev vector field mostly points towards a particular direction, as proven by Marconi [41]. More precisely, if there exist $\delta > 0$ and $e \in \mathbb{S}^1$ for which $\mathbf{b} \cdot e > \delta$ a.e. in a ball $B(x,R)$ and $\mathrm{div}(\mathbf{b}) = 0$ then the flow map X has exactly the same degree of Sobolev regularity as $\mathbf{b}$ itself. Unfortunately, the vector fields we are interested in will most likely not satisfy Marconi's assumptions, as happens quite often in Fluid Mechanics.

Trying to avoid any restrictions on the direction of $\mathbf{b}$, it was proven recently that there is a subclasss of Sobolev vector fields which are not Lipschitz but still its flow enjoys some Sobolev smoothness. These are vector fields $\mathbf{b}$ for which its gradient falls into the exponential class. Let us recall that a function u belongs to the local Exponential class $Exp(L)_{loc}$ if there is some $\lambda > 0$ such that

$$\int_B \exp(\lambda |u|) dx < +\infty \qquad\qquad \text{for each ball } B. \tag{5}$$

Functions in $Exp(L)_{loc}$ belong to L^p_{loc} for every finite p. Vector fields with derivatives in $Exp(L)_{loc}$ have $\delta \log \frac{1}{\delta}$ modulus of continuity, and so they admit a well defined flow of Hölder continuous homeomorphisms. In [23], it is shown that if $\mathbf{b} \in W^{1,1}_{loc}$ has gradient $\nabla \mathbf{b} \in Exp(L)_{loc}$ and $\mathrm{div}(\mathbf{b}) \in L^\infty$ then its flow map $X(t,\cdot)$ belongs to the local Sobolev space $W^{1,p}_{loc}$ for each $p \leq \frac{C}{t}$, if $t > 0$ is small enough.

Another interesting example is given in [43]. There it is shown that any vector field with bounded traceless symmetric differential admits a well defined flow of Hölder continuous homeomorphisms which turn out to be Sobolev regular *at any time*. These vector fields are a subclass of Λ_*, so the point here is not the existence and uniqueness of a Hölder continuous flow, but its Sobolev smoothness. Remarkably, these vector fields need not have bounded divergence, and so the classical DiPerna - Lions theory may not be applicable.

In general, though, Sobolev vector fields do not give rise to Sobolev regular flows (not even of fractional order) as shown in [4] and [36]. To be precise, for any finite p there is a vector field $\mathbf{b}$ belonging to $W^{1,p}_{loc}$ such that the DiPerna - Lions flow of $\mathbf{b}$ does not belong to any Sobolev space $W^{\alpha,q}$ even for fractional α. Moreover, one may modify the construction and obtain a vector field $\mathbf{b}$ belonging to the intersection of all Sobolev spaces $W^{1,p}_{loc}$ for finite p for which the flow does not have any degree of Sobolev regularity.

The above lines show the existence of a gap between Cauchy-Lipschitz and DiPerna-Lions theories. In the first one, Lipschitz fields produce Lipchitz flows. In the second, Sobolev fields may produce non-Sobolev flows. In between these two situations, we have many vector fields (for instance, the ones in [23, 15]) for which some Sobolev smoothness can be granted to its flow, yet they are not Lipschitz. This thesis is devoted to vector fields that fall into this gap. As will be clear along the thesis, this gap is very narrow and unstable, as drastic changes in the regularity of X may happen around it.

Most of the examples of vector fields in this gap can be constructed as Riesz potentials of BMO functions. Let us recall that a locally integrable function u in $\mathbb{R}^n$ is said to have bounded mean oscillation, in short $u \in BMO$, if

$$\|u\|_{BMO} = \sup_{\mathbb{B}} \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} |u(x) - u_{\mathbb{B}}|^2 \, dx \right)^{\frac{1}{2}} < +\infty \tag{6}$$

where the supremum is taken over the set of all balls $\mathbb{B}$ in $\mathbb{R}^n$ and

$$u_{\mathbb{B}} = \frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} u(x) dx$$

is the average of u on the balls $\mathbb{B}$. BMO functions are known to have a degree of exponential integrability, due to the well-known John-Nirenberg inequality,

$$|\{x \in Q : |u(x) - u_Q| > \delta\}| \leq C_1 e^{-C_2 \delta / \|u\|_*} |Q|,$$

where $u \in BMO$ and C_1, C_2 are constants depending on the dimension. Q is any cube in $\mathbb{R}^n$ and $\delta > 0$ is any real number. In particular, if $u \in BMO$ then u belongs to the exponential class (5) with $\lambda = \frac{C_2 \delta}{\|u\|_*}$. The space of continuous vector fields such that their distributional derivatives are in BMO is denoted by $I_1(BMO)$. The relation between these three classes of vector fields, viz. Lipschitz, Zygmund and $I_1(BMO)$ is the following:

$$Lip \subsetneq I_1(BMO) \subsetneq \Lambda_*. \tag{7}$$

In contrast to what happens to Zygmund vector fields, if $\mathbf{b} \in I_1(BMO)$ then $\mathbf{b}$ is differentiable in the classical sense at almost every point, and hence asking about the differentiability of X may make sense. When $n = 2$, $I_1(BMO)$ can also be defined by means of the Cauchy-Riemann derivatives,

$$\partial_z \mathbf{b} = \partial \mathbf{b} = \frac{(\partial_x - i\partial_y)(\mathbf{b}^1 + i\mathbf{b}^2)}{2} = \frac{\operatorname{div} \mathbf{b} + i \operatorname{curl} \mathbf{b}}{2}$$

and

$$\partial_{\bar{z}}\mathbf{b} = \overline{\partial}\mathbf{b} = \frac{(\partial_x + i\partial_y)\,(\mathbf{b}^1 + i\mathbf{b}^2)}{2}.$$

So, $\mathbf{b} \in I_1(BMO) \iff \partial\mathbf{b}, \overline{\partial}\mathbf{b} \in BMO$. Examples of this situation are given when either $\partial\mathbf{b} \in L^\infty$ or $\overline{\partial}\mathbf{b} \in L^\infty$, as both conditions force $D\mathbf{b} \in BMO$. Such vector fields are guaranteed all goods from both Zygmund vector fields (they admit a well defined flow, consisting of Hölder continuous homeomorphisms) and DiPerna-Lions theory (the flow is compatible with transport equations). Moreover, they enjoy at small times certain degree of Sobolev regularity. Indeed, in the case $\partial\mathbf{b} \in L^\infty$ this is due to the results at [23, 15], while the conclusion for $\overline{\partial}\mathbf{b} \in L^\infty$ is a consequence of [43]. In the next paragraphs we intend to explain both situations separately.

Reimann's vector fields and the quasiconformal world

According to Reimann [43], a continuous vector field $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ belongs to the Q class iff there exists a constant $C \geq 0$ so that for each $x \in \mathbb{R}^n$ and every $h, k \neq 0$ with $|h| = |k|$ one has

$$\left| \frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), h\rangle}{|h|^2} - \frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), k\rangle}{|k|^2} \right| \leq C \tag{8}$$

The best possible value of C is denoted by $\|\mathbf{b}\|_Q$. It is not hard to see that

$$Lip(\mathbb{R}^n) \subsetneq Q(\mathbb{R}^n) \subsetneq \Lambda_*(\mathbb{R}^n). \tag{9}$$

Moreover, when $n = 1$, $Q = \Lambda_*$. By the classical ODE theory, if $\mathbf{b} \in \Lambda_*$ then the initial value problem (1) has a well-defined, unique flow of time-dependent solutions $X(t, x)$. In particular, this solution is a Hölder continuous homeomorphism in the space variable x. If $\mathbf{b}$ is non-autonomous and also depends on time, the same conclusion holds if one assumes $\sup_t \|\mathbf{b}(\cdot, t)\|_Q < \infty$.

In [43] Reimann was able to identify in differential terms the elements of Q as follows,

$$\mathbf{b} \in Q \iff S\mathbf{b} \in L^\infty(\mathbb{R}^n) \text{ and } \frac{|\mathbf{b}(x)|}{|x|\log(e + |x|)} \leq C, \tag{10}$$

along with its quantitative formulation $\|\mathbf{b}\|_Q \simeq \|S\mathbf{b}\|_\infty$. Here $S\mathbf{b}$ denotes the traceless symmetric differential matrix of $\mathbf{b}$,

$$S\mathbf{b} = \frac{1}{2}\left(D\mathbf{b} + D^t\mathbf{b}\right) - \frac{\operatorname{div}(\mathbf{b})}{n} \cdot \mathcal{I}_n.$$

In the plane, $S\mathbf{b}$ is equivalent to the complex derivative $\overline{\partial}\mathbf{b}$. This explains that the class Q falls into the gap described in the previous section. The Sobolev regularity of the flow associate to any $\mathbf{b} \in Q$ is understood with the help of quasiconformality and quasisymmetry.

Quasiconformality and quasisymmetry

Quasiconformal mapping is the central object of complex function theory. Historically, the discovery of quasiconformal mappings could be thought of as the result of an interesting problem posed by Grötzsch [27]. He asked to find *the best possible nearly conformal mapping* that maps a given square to a given rectangle, while vertices are mapped into vertices. To provide a positive answer, first of all one needs to consider what it means to be nearly conformal. This paved the way towards the generalization of conformal mappings to what would later on become quasiconformal mappings. The term Quasiconformal mappings appeared for the first time in Ahlfors's 1935 famous article on covering spaces [1], for which he received the *Fields Medal* in 1936. Carathedory said that this article opened a new branch in analysis, that could be called *metric topology.*

Given a domain $\Omega \subset \mathbb{R}^n$, and a real number $K \geq 1$, a K-quasiconformal mapping is an orientation-preserving homeomorphism $f : \Omega \to \mathbb{R}^n$ with $f \in W^{1,n}_{loc}(\Omega)$ and such that

$$|Df(z)|^n \leq K \cdot J_f(z) \tag{11}$$

at almost every point $z \in \Omega$. Here $|Df(z)|$ stands for the operator norm of the differential matrix $Df(z)$ at z and $J_f(\cdot) = \det(Df)$ is the Jacobian determinant. The smallest constant $K = K(f)$ for which the distortion inequality (11) holds a.e. is called the distortion of f.

When $n = 2$, a way to construct nontrivial quasiconformal maps is through the so called Beltrami equation,

$$\overline{\partial} f(z) = \mu(z) \partial f(z). \tag{12}$$

Here μ is a bounded measurable function satisfying

$$\|\mu\|_\infty = \frac{K-1}{K+1} < 1,$$

and it is called the *Beltrami coefficient* of f. The classical Measurable Riemann Mapping Theorem asserts that to each such μ one can associate a K-quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ solving (12), and moreover it is unique after choosing its value at three points.

There are other definitions for quasiconformality, whose equivalence was proven by a lot of mathematicians, Ahlfors, Bers and Gehring to name a few, during the 50s and 60s. The different approaches to quasiconformal mappings clearly explains why these maps have played a central role in diverse areas of mathematics such as Harmonic Analysis, Elliptic PDE, Inverse Problems, Complex Dynamics, Differential Geometry and more recently Fluid Mechanics. Among these equivalent definitions, the following metric concept plays an essential role.

Let $\eta : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism. Given a domain $\Omega \subset \mathbb{R}^n$ we call an orientation-preserving homeomorphism $f : \Omega \to \mathbb{R}^n$ to be η-quasisymmetric if for each triple $z, w, y \in \Omega$ we have

$$\frac{|f(z) - f(w)|}{|f(z) - f(y)|} \leq \eta \left(\frac{|z - w|}{|z - y|} \right) \tag{13}$$

Quasiconformality and quasisymmetry are quantitatively equivalent notions. More precisely, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal, then f is η-quasisymmetric, where η depends on K and n. Conversely, η-quasisymmetric mappings can be shown to be K-quasiconformal for some $K = K_\eta$ that depends on η and n.

Sobolev regularity of quasiconformal mappings

By definition, every K-quasiconformal mapping $f : \Omega \to \mathbb{R}^n$ lies in the local Sobolev space $W^{1,n}_{loc}(\Omega)$. The Bojarski-Iwaniec theorem states that f actually belongs to a better Sobolev space $W^{1,p}_{loc}(\Omega)$ for some $p > n$. More precisely, if $f : \Omega \to \mathbb{R}^n$ is a K-quasiconformal mapping, then there exists a number $p_0 = p_0(n, K) > n$ such that $f \in W^{1,p}_{loc}(\Omega)$ for each $p < p_0$. The precise value of $p(n, K)$ remains an open problem, except for $n = 2$. Indeed, the value of $p(2, K)$ can be obtained from Astala's Area Distortion Theorem [3]. It states that for any planar K-quasiconformal mapping f, one has

$$\frac{1}{C_K}\left(\frac{|E|}{|D|}\right)^K \leq \frac{|f(E)|}{|f(D)|} \leq C_K\left(\frac{|E|}{|D|}\right)^{\frac{1}{K}} \tag{14}$$

for any disk $D \subset \mathbb{C}$ and $E \subset D$. Now, if f is a given K-quasiconformal mapping, and we set $E = E_t$ where

$$E_t = \{z \in D : J(z, f) > t\}$$

then one gets from (14) that

$$|E_t| \leq C_K |D|^{\frac{1}{1-K}} \left(\frac{|f(D)|}{t}\right)^{\frac{K}{K-1}}. \tag{15}$$

As a consequence, the Jacobian $J(\cdot, f)$ belongs to the weak Lebesgue space $L^{\frac{K}{K-1},\infty}_{loc}$ and therefore $Df \in L^{\frac{2K}{K-1},\infty}_{loc}$. In particular, $p(2, K) = \frac{2K}{K-1}$, and this is sharp as proven by $f(z) = z|z|^{\frac{1}{K}-1}$.

Reimann's flows are Sobolev regular

One of the main points in Reimann's theory is that if $\mathbf{b} \in Q$ then the solution $x \mapsto X(t, x)$ of (1) consists of quasisymmetric mappings for all $t > 0$. For the reader's convenience, we sketch the proof for autonomous $\mathbf{b}$. Writing $x = X(t, x_0)$ and $x + h = X(t, y_0)$, one immediately sees that

$$\begin{aligned}\frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), h\rangle}{|h|^2} &= \frac{\langle \frac{d}{dt}\left(X(t, y_0) - X(t, x_0)\right), X(t, y_0) - X(t, x_0)\rangle}{|X(t, y_0) - X(t, x_0)|^2} \\ &= \frac{d}{dt}\left(\frac{1}{2}\log|X(t, y_0) - X(t, x_0)|^2\right)\end{aligned}$$

Then repeating with a third point $x + k = X(t, z_0)$, one gets that

$$\frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), k\rangle}{|k|^2} = \frac{d}{dt}\left(\frac{1}{2}\log|X(t, z_0) - X(t, x_0)|^2\right)$$

Then (8) becomes equivalent to

$$\left|\frac{d}{dt}\left(\frac{1}{2}\log\frac{|X(t, y_0) - X(t, x_0)|^2}{|X(t, z_0) - X(t, x_0)|^2}\right)\right| \leq C \tag{16}$$

Roughly speaking, this gives quasisymmetry after integrating in time. By virtue of the equivalence between quasisymmetry and quasiconformality, one then gets that if $\mathbf{b} \in Q$ then its flow is Sobolev regular at all times, at least $W^{1,n}_{loc}$ and for sure something better, according to Bojarski-Iwaniec Theorem. However, only rough bounds can be given for the distortion of $X(t, \cdot)$ at this point: the ones coming from the quasisymmetry modulus η of $X(t, \cdot)$.

In the same way Q is the pointwise version of $S\mathbf{b} \in L^\infty$, and quasisymmetry is the pointwise version of quasiconformality, one can equivalently show that if $S\mathbf{b} \in L^\infty$ then the flow consists of K_t-quasiconformal mappings for all times $t > 0$, and for some $K_t \geq 1$. Here K_t is the best possible quantity for which the inequality

$$|DX(t,x)|^n \leq K_t\, J(x, X_t)$$

holds true at almost every $x \in \mathbb{R}^n$. For the reader's convenience, we sketch below the proof of this fact for $n = 2$. It is based in the fact that the ODE (1) allows to deduce an ODE for the *Beltrami coefficient* $\mu = \mu_t = \frac{\overline{\partial} X_t}{\partial X_t}$ of $X = X_t = X(t,\cdot)$. Indeed, after taking ∂ and $\overline{\partial}$ at (1), one easily gets that

$$\frac{\frac{d}{dt}\mu_t}{1-|\mu_t|^2} = \overline{\partial}\mathbf{b}\,(X_t)\;\frac{\overline{\partial X_t}}{\partial X_t} \tag{17}$$

where $\overline{\partial X}$ is the complex conjugate of ∂X. The most remarkable fact here is that the above ODE can be integrated in time. If we do it, one immediately gets

$$\frac{1}{2}\,\log\left(\frac{1+|\mu_t|}{1-|\mu_t|}\right) \leq \int_0^t \|\overline{\partial}\mathbf{b}(s,\cdot)\|_\infty\, ds$$

Having in mind that $K_t = \frac{1+\|\mu_t\|_\infty}{1-\|\mu_t\|_\infty}$, this is equivalent to say that

$$K_t \leq \exp\left(2\int_0^t \|\overline{\partial}\mathbf{b}(s,\cdot)\|_\infty ds\right). \tag{18}$$

A similar argument in $\mathbb{R}^n$ shows that the optimal bound for K_t is

$$K_t \leq \exp\left(n\int_0^t \|S\mathbf{b}(s,\cdot)\|_\infty ds\right). \tag{19}$$

The advantage of using quasiconformality instead of quasisymmetry is that now the bounds for the distortion of $X(t,\cdot)$ are much more precise. This allows to estimate the best p for which $X(t,\cdot) \in W^{1,p}_{loc}$. Indeed, one can take $p < p(n, K_t)$ where K_t is as in (18) or (19), and $p(n,K)$ is the one in Bojarski-Iwaniec Theorem. In particular, when $n = 2$ one has $p(2, K_t) = \frac{2K_t}{K_t-1}$, and therefore

$$X(t,\cdot) \in W^{1,p}_{loc} \qquad \text{whenever } p < \frac{2}{1-\exp\left(-2\int_0^t \|\overline{\partial}\mathbf{b}(s,\cdot)\|_\infty\right)}. \tag{20}$$

When $n = 2$, the vector field $\mathbf{b}(z) = -z\,\log|z|$ produces the flow $X(t,z) = z|z|^{\frac{1}{K_t}-1}$ with $K_t = e^t$. Since $\|\overline{\partial}\mathbf{b}\|_\infty = \frac{1}{2}$, $\mathbf{b}$ can be used to prove that (20) is sharp, in the sense that the largest value of p at (20) may not be attained.

Flows arising from the incompressible Euler system

When looking for planar vector fields in a similar situation to Reimann's, a natural option consists of replacing the boundedness of the anticonformal derivative $\overline{\partial}\mathbf{b} \in L^\infty$ by its conformal counterpart $\partial\mathbf{b} \in L^\infty$. By doing this, one includes in the discussion certain examples from Fluid Mechanics, as for instance any bounded curl solution to the Euler system of equations. Some solutions to the so-called aggregation system can also be included.

Let us consider the following active scalar model in the plane,

$$\begin{cases} \omega_t + (\mathbf{b}\cdot\nabla)\,\omega = 0 \\ \mathbf{b}(t,\cdot) = \mathcal{K} * \omega(t,\cdot) \\ \omega(0,\cdot) = \omega_0 \end{cases} \tag{21}$$

where $\mathcal{K}(z) = \frac{iz}{2\pi|z|^2}$. This is known as the *planar incompressible Euler equation* in vorticity form. Given a compactly supported $\omega_0 : \mathbb{R}^2 \to \mathbb{R}$ with $\omega_0 \in L^\infty$, Yudovich [48] proved existence and uniqueness of a solution $\omega = \omega(t,z)$ of (21) belonging to $L^\infty((0,\infty)\times\mathbb{R}^2)$. The connection between $\mathbf{b}$ and ω is known as the Biot-Savart law. In particular, this law says that

$$\mathbf{b} = \mathcal{K} * \omega \qquad \Longleftrightarrow \qquad \partial\mathbf{b} = \frac{i}{2}\omega.$$

Since $\omega \in \mathbb{R}$, this means that $\operatorname{div}(\mathbf{b}) = 0$, and $\operatorname{curl}(\mathbf{b}) = \frac{1}{2}\omega$. In conclusion, if $\mathbf{b}$ is the velocity field associated to a Yudovich solution $\omega = \omega(t,z)$ of the above system (21), then $\partial\mathbf{b} \in L^\infty$ and therefore $D\mathbf{b} \in BMO$ so that $\mathbf{b} \in I_1(BMO)$. In particular, $\mathbf{b}$ admits a well defined flow of measure preserving, Hölder continuous homeomorphisms $X(t,\cdot) : \mathbb{C} \to \mathbb{C}$.

A similar situation is given with the kernel choice $\mathcal{K}(z) = \frac{z}{2\pi|z|^2}$. Indeed, now one has $\partial\mathbf{b} = \frac{\omega}{2}$ so that again any solution $\omega \in L^\infty((0,\infty)\times\mathbb{R}^2)$ to this new version of (21) produces another vector field $\mathbf{b}$ with $\partial\mathbf{b} \in L^\infty$. Examples of these solutions were given in [12].

In both examples above, one has $D\mathbf{b} \in Exp_{loc}$ and $\operatorname{div}(\mathbf{b}) \in L^\infty$. This allows to apply [23, Theorem 4], and claim that for small times $X(t,\cdot)$ has L^p_{loc} distributional derivatives, for some p that may vary in time.

Optimal Sobolev regularity of Euler's flow

We may always ask for the best p such that the Euler flow $X(t,\cdot)$ belongs to the local Sobolev space $W^{1,p}_{loc}$. It was conjectured in [23] that

$$X(t,\cdot) \in W^{1,p}_{loc} \qquad \text{whenever} \qquad p < \frac{2}{1-\exp\left(-t\|\omega_0\|_\infty\right)}. \tag{22}$$

If this conjecture holds true, then the Sobolev embedding gives to the Euler flow an optimal Hölder exponent strictly below $\exp\left(-t\|\omega_0\|_\infty\right)$, as proven by Bahouri and Chemin in [9]. In other words, a proof of this conjecture would imply Bahouri-Chemin's theorem. Also in the positive direction, it was proven in [23, Corollary 3] that

$$X(t,\cdot) \in W^{1,p}_{loc} \qquad \text{whenever} \qquad p < \frac{2}{t\,\|\omega_0\|_{L^\infty}}$$

so that Conjecture (22) has the right order as $t \to 0^+$.

It is worth to mention that what we just explained does not only refer to vector fields $\mathbf{b}$ solving the Euler system (21), but indeed to any other vector field $\mathbf{b}$ for which $\partial\mathbf{b}$ is a bounded quantity. In other words, for such $\mathbf{b}$ one can reformulate Conjecture (22) as follows:

$$X(t,\cdot) \in W^{1,p}_{loc} \qquad \text{whenever} \qquad p < \frac{2}{1-\exp\left(-2\int_0^t \|\partial\mathbf{b}(s,\cdot)\|_\infty ds\right)}. \tag{23}$$

This conjecture is partially motivated by the positive and optimal result available for vector fields in the Reimann class, namely (20). Thus, it seems natural to explore if the methods that worked out in proving (20) can be used as well for proving (23). Such a strategy may be faced in two different ways, one of a metric nature and one more geometric. At the metric side, one should start by characterizing in pointwise terms the vector fields $\mathbf{b}$ with bounded $\partial\mathbf{b}$ (just as Reimann did when proving equivalence between $\overline{\partial}\mathbf{b} \in L^\infty$ and Q) and then try to obtain geometric information about the flow from these pointwise conditions (in the same way that quasisymmetry arises from Q). At the geometric side, instead, one should directly work with condition $\partial\mathbf{b} \in L^\infty$, and seek for its geometric effects on the flow, in the same way condition $\overline{\partial}\mathbf{b} \in L^\infty$ guarantees the quasiconformality of the flow.

In this thesis, we got some success in the metric part, and failed in the geometric. We now enter the first one.

Pointwise descriptions of vector fields with bounded curl and divergence

We introduce the class $\overline{Q}$ of vector fields $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ for which there exists a constant $C \geq 0$ such that for all $x \in \mathbb{R}^2$ and every $h, k \neq 0$ with $|h| = |k|$ one has

$$\left| \frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), \overline{h}\rangle}{|h|^2} - \frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), \overline{k}\rangle}{|k|^2} \right| \leq C \tag{24}$$

Here $\overline{h}$ and $\overline{k}$ denote complex conjugates. The best possible value of C is denoted by $\|\mathbf{b}\|_{\overline{Q}}$. Similarly, for a given $\theta \in [0, 2\pi]$, we denote by R_θ the class of vector fields $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ for which there is a constant $C \geq 0$ such that for all $x \in \mathbb{R}^2$, every $h, k \neq 0$ with $|h| = |k|$ one has

$$\left| \frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), e^{i\theta}k\rangle}{|h||k|} - \frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), e^{i\theta}h\rangle}{|h||k|} \right| \leq C \tag{25}$$

As before, $\|\mathbf{b}\|_{R_\theta}$ denotes the best possible C. It should be noted that $\overline{Q} \subset \Lambda_*$ and $R_\theta \subset \Lambda_*$. Finally, we denote $R = \cap_\theta R_\theta$, and call $\|\mathbf{b}\|_R = \sup_\theta \|\mathbf{b}\|_{R_\theta}$. It is not hard to see that $\bar{Q} \subset \Lambda_*$ and also that $R_\theta \subset \Lambda_*$ for each θ. Our first result regarding the equivalence of these two classes, Q and R_θ is the following one.

Theorem (Theorem 2.1.1)**.** *Let* $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a continuous vector field. Then the following are equivalent:*

- $\mathbf{b} \in \overline{Q}$.

- $\mathbf{b} \in R$.
- $\mathbf{b} \in R_0 \cap R_{\pi/2}$
- $\mathbf{b}$ *is differentiable a.e.,* $\partial\mathbf{b} \in L^\infty$ *and* $\frac{|\mathbf{b}(x)|}{|x|\log(e+|x|)} \leq C$.

If one of these holds true, then $\|\mathbf{b}\|_{\bar{Q}} \simeq \|\mathbf{b}\|_R \simeq \max\{\|\mathbf{b}\|_{R_0}, \|\mathbf{b}\|_{R_{\pi/2}}\} \simeq \|\partial\mathbf{b}\|_\infty$.

The above result establishes a very convenient counterpart to Reimann's characterization of the condition $\overline{\partial}\mathbf{b} \in L^\infty$, namely (8). Also, it shows that indeed $R, \overline{Q} \subset I_1(BMO)$, while this may fail for R_θ. It is remarkable that for R only two rotations are needed $\theta = 0$ and $\theta = \pi/2$.

In contrast to Reimann's setting, now it is not immediate to extend the class $\overline{Q}$ to $\mathbb{R}^n$, $n > 2$, due to the presence of complex conjugation in its definition. Also, extending the class R_θ to higher dimensions does not look to be a good idea either, because the set of rotations to be included is not clear. Let us give a brief explanation that why it is not clear. For instance, the rotation factor $e^{i\theta}$ may be replaced by rotations not only in the O_{x_1,x_2} plane, but on any of the coordinate planes O_{x_i,x_j}. To this end, let us introduce the set $\mathcal{J}_n = \{J_{i,j}\}_{1\leq i<j\leq n}$ of matrices $J_{i,j} \in \mathbb{R}^{n\times n}$ defined by

$$\begin{aligned} J_{i,j}\, e_i &= -e_j \\ J_{i,j}\, e_j &= e_i \\ J_{i,j} e_k &= e_k, \quad k \neq i,j \end{aligned}$$

where $e_1, ..., e_n$ is the canonical basis in $\mathbb{R}^n$. A natural extension for the class R in $\mathbb{R}^n$, $n > 2$, would be provided by asking $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ to satisfy

$$\left| \frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), Jk\rangle}{|h||k|} - \frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), Jh\rangle}{|h||k|} \right| \leq C \tag{26}$$

for all matrices $J \in \mathcal{J}_n \cup \{Id\}$, all points x, and all directions $h, k \neq 0$ with $|h| = |k|$. The following Lemma shows that this extension is, indeed, trivial.

Lemma (Lemma 2.12)**.** *Let* $n > 2$ *and* $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ *be a vector field. Moreover, assume that* x *is a differentiability point of* $\mathbf{b}$*. If*

$$\limsup_{|h|=|k|\to 0} \sup_J \left| \frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), Jk\rangle}{|h||k|} - \frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), Jh\rangle}{|h||k|} \right| \leq C \tag{27}$$

then $|D\mathbf{b}(x)| \leq \overline{C}\, C$ *for some dimensional constant* $\overline{C}$.

The above result shows that the class of vector fields $\mathbf{b}$ satisfying (26) consists, indeed, of Lipschitz vector fields when $n > 2$. To the contrary, this class is much larger in the plane. Therefore, it is not a good idea to build higher dimensional counterparts to R_θ in this way.

Nevertheless, one might still get L^∞ estimates by removing all rotations, even in higher dimensions. Formally, one can define the class of vector fields $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ for which there exists a constant $C \geq 0$ such that for all $x \in \mathbb{R}^n$ and every h, k with $|h| = |k| \neq 0$ one has

$$\left| \frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), k\rangle}{|h||k|} - \frac{\langle \mathbf{b}(x+k) - \mathbf{b}(x), h\rangle}{|h||k|} \right| \leq C \tag{28}$$

Above, $\|\mathbf{b}\|_{R_0}$ denotes the best possible constant C. At this point, we state a new theorem on the nature of the vector fields in the class R_0.

Theorem (Theorem 2.1.2). *Let* $\mathbf{b} \in R_0$. *Then* $D\mathbf{b} - D^t\mathbf{b} \in L^\infty(\mathbb{R}^n)$ *in the sense of distributions and*

$$\|D\mathbf{b} - D^t\mathbf{b}\|_\infty \leq C\|b\|_{R_0}$$

for some constant $C > 0$.

Note that the class R_0 is much larger than R even in the plane. So, R_0 may include many elements not differentiable a.e. This prevents us from seeking higher dimensional counterparts to Theorem 2.1.1 for R_0. To overcome this barrier, we need to add conditions on b that guarantee its a.e. differentiability. One option consists of asking $\mathbf{b}$ to be *nearly incompressible*. In other words, $\operatorname{div}\mathbf{b} \in L^\infty$. This motivates us to consider the differential operator $A\mathbf{b}$,

$$A\mathbf{b} = \frac{1}{2}\left(D\mathbf{b} - D^t\mathbf{b}\right) + \frac{\operatorname{div}(\mathbf{b})}{n} \cdot \mathcal{I}_n,$$

which is equivalent to $\partial\mathbf{b}$ in the plane. With the help of $A\mathbf{b}$ we found the following.

Theorem (Theorem 2.1.3). *Let* $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous vector field.*

- *If* $\mathbf{b} \in R_0$ *and is nearly incompressible, then* $\mathbf{b}$ *is differentiable a.e. and*

$$\|A\mathbf{b}\|_\infty \leq C\left(\|\operatorname{div}\mathbf{b}\|_\infty + \|\mathbf{b}\|_{R_0}\right).$$

- *If* $A\mathbf{b} \in L^\infty$ *and* $\frac{|\mathbf{b}(x)|}{|x|\log(e+|x|)} \leq C$, *then* $\mathbf{b} \in R_0$ *and*

$$\|\mathbf{b}\|_{R_0} \leq C\|A\mathbf{b}\|_\infty.$$

As in Reimann's case, one of the main tools used to prove this theorem is that if b is a compactly supported vector field such that $A\mathbf{b} \in L^\infty$ then $\mathbf{b}$ has BMO distributional derivatives, which in turn guarantees that $\mathbf{b}$ is differentiable a.e. One can always relax the assumption $\operatorname{div}\mathbf{b} \in L^\infty$ to $\operatorname{div}\mathbf{b} \in L^p$ for some $p > n$. In this case, one gets $D\mathbf{b} - D^t\mathbf{b} \in L^p$. Then, since the Riesz transforms

$$\mathcal{R}_j\mathbf{b}(x) = \frac{1}{\pi\omega_{n-1}}\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{(x_j - y_j)\,\mathbf{b}(y)}{|x-y|^{n+1}}dy, \quad j = 1, 2, ..., n,$$

boundedly map L^p to L^p, we can say that $\mathbf{b}$ has L^p distributional derivatives, which also ensures that $\mathbf{b}$ is differentiable a.e. because $p > n$. On the other hand, from the applicability point of view, the above result can be used to describe in a pointwise sense, among all the solutions to the Euler equations, the ones with bounded vorticity.

Rotational properties of Mappings of Finite Distortion

The notion of quasiconformality admits a degenerate extension. To be precise, in (11) one can replace the constant K by a measurable function $\mathbb{K}(\cdot, f) \geq 1$, finite almost everywhere but not necessarily bounded. Then one speaks about *mappings of finite distortion* to refer to the class of mappings satisfying this new version of (11). The best possible function $\mathbb{K}(\cdot, f)$ is known as the *distortion function* of f.

Mappings of finite distortion arise in a natural way in Fluid Mechanics. Indeed, it is proven in [23] that if $b = K * \omega$ is the velocity field associate to any solution ω to (21), with compactly supported $\omega_0 \in L^\infty$, and $t > 0$ is small enough, then every flow map $X_t = X(t, \cdot) : \mathbb{C} \to \mathbb{C}$ of $\mathbf{b}$ is a mapping of finite distortion, with distortion function $\mathbb{K}(\cdot, X_t) \in L^p_{loc}(\mathbb{C})$, provided that $p < \frac{C}{t}$, where C depends only on $\|\omega_0\|_\infty$.

In general, if f is a mapping of finite distortion and $\mathbb{K}(\cdot, f) \in L^p_{loc}$, and further f is a homeomorpshism, then f admits lower bounds for compression. In other words, there exists a real valued increasing, onto homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$|f(z) - f(w)| \geq \eta(|z - w|),$$

as proven in [38]. However, η is much weaker than any power function, which means that if $\alpha > 0$ then $\lim_{t\to 0} t^{-\alpha}\, \eta(t) = 0$. That is to say, f should not be expected to have a Hölder continuous inverse. Hence, it is remarkable that each bounded and compactly supported vorticity ω_0 produces solutions ω to the incompressible Euler system (21) for which the corresponding flow X_t and its inverse X_t^{-1} are both Hölder continuous, with a Hölder exponent that decays exponentially in time. This makes Euler flows particularly special within the class of homeomorphisms with distortion in L^p_{loc}.

In the recent years, there has been an increasing interest in understanding the rotational properties of planar mappings of finite distortion. Broadly speaking, given one such map $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$, the main concern is the maximal growth of $|\arg(f(r))|$ as $r \to 0^+$. This represents the number of times that the image $f([r, 1])$ winds around the origin as $r \to 0^+$. It is known that this quantity admits several speeds of growth depending on the class of maps under study. As explained in [7, 31, 32], the local rotational properties go hand in hand with the local stretching behavior. Especially important for the argument are the estimates for the modulus of continuity of the inverse map.

Before entering into our results, we would like to give a vivid description of the earlier works of geometric analysts in this line of research, precisely the study of local pointwise rotation and stretching of planar homeomorphisms. To this end, we start with pointwise stretching and then enter into the details of local rotational properties.

Pointwise stretching

It is well known from the work of Ahlfors [2] that given any K-quasiconformal map $f : \mathbb{C} \to \mathbb{C}$, normalized by $f(0) = 0$ and $f(1) = 1$, we have

$$|f(z)| \geq \frac{1}{c_K}|z|^K, \qquad \forall\, |z| < 1. \tag{29}$$

The K-quasiconformal map

$$f(z) = z|z|^{K-1}$$

shows that the lower bound in (29) is optimal. In analogy to Ahlfors, Herron and Koskela [29] showed that given an arbitrary mapping $f : \mathbb{C} \to \mathbb{C}$ of finite distortion, with $e^{\mathbb{K}(\cdot, f)} \in L^q_{loc}$, and normalized by $f(0)=0$, we have

$$|f(z)| \gtrsim e^{-\frac{c_{f,q}}{q}\log^2\left(\frac{1}{|z|}\right)}, \qquad \text{for small enough } |z|. \tag{30}$$

The radial stretching mapping

$$f(z) = \frac{z}{|z|}\exp\left(-\frac{c}{q}\log^2\frac{1}{|z|}\right)$$

shows that the lower bound in (30) is sharp, in the sense that the exponent 2 of the logarithm on the right hand side of (30) cannot be made smaller. The analog to (29) for mappings f $\mathbb{K}(\cdot, f) \in L^p_{loc}$, $p>1$, has been discovered by Koskela and Takkinen in [38], where they proved that for any such mapping $f: \mathbb{C} \to \mathbb{C}$ normalized by $f(0)=0$, we have

$$|f(z)| \geq e^{-c_{f,p}|z|^{-\frac{2}{p}}}, \qquad \text{for sufficiently small } |z|. \tag{31}$$

Again, the radial map

$$f(z) = \frac{z}{|z|}\log^{-p}\left(1+\frac{1}{|z|}\right)$$

proves the sharpness of (31), that is , the exponent $\frac{2}{p}$ in (31) cannot be made smaller.

Hitruhin in [32] extended the result of Koskela-Takkinen to the borderline case $p=1$. He showed that for any given mapping $f: \mathbb{C} \to \mathbb{C}$ with 1-integrable distortion, normalized by $f(0)=0$, we have

$$|f(z)| \geq e^{-\frac{c_f(|z|)}{|z|^2}}, \qquad \text{for small enough } |z|. \tag{32}$$

Above, $c_f(|z|) \to 0$ as $|z| \to 0$. This result is also sharp in the sense that the exponent 2 cannot be made smaller.

Pointwise rotation

The study of pointwise rotation for mappings of finite distortion classically involves mappings from annulus to annulus. Broadly speaking, one considers mappings that fix some given annulus, keep the outer circle fixed while rotating the inner circle. The case of quasiconformal mappings was studied by Gütlyanskii and Martio in [28]. Balogh, Fässler and Platis in [10] extended this result to annuli with different modulus. Both the works [10] and [28], in spite of considering a fairly general class of mappings of finite distortion, only consider mappings between round annuli.

Astala, Iwaniec, Prause and Saksman came up with an alternative approach in [7] to study pointwise rotation of quasiconformal mappings. They used the technique of holomorphic motion in the plane to measure the maximal pointwise rotation of a general quasiconformal mapping in the entire plane, thus dropping the restriction to annuli as done in the earlier works [10] and [28].

It is proven in [7] that if f is K-quasiconformal then

$$|\arg(f(r))| \leq \frac{1}{2}\left(K - \frac{1}{K}\right)\log\left(\frac{1}{r}\right) + c_K, \qquad \text{for all } 0<r<1, \tag{33}$$

where the branch of the argument is determined by $\arg(1) = 0$. Moreover, there exists a K-quasiconformal mapping that satisfies (33) as an equality with $c_K = 0$.

On the other hand, for homeomorphisms of finite distortion situation changes drastically and the order of spiraling depends on integrability of the distortion function. Namely, Hitruhin discovered in [31] that if $e^{\mathbb{K}(\cdot,f)} \in L^p_{loc}$ for some $p > 0$ then

$$|\arg(f(z))| \leq \frac{c}{p}\log^2\left(\frac{1}{|z|}\right), \qquad \text{for small enough } |z|,$$

and moreover this is sharp up to the constant $c > 0$. In other words, there is a certain payoff to transit from boundedness to exponential integrability of $\mathbb{K}(\cdot, f)$. More precisely, the logarithmic term gets squared in this case. Further optimal results were obtained later on in [32] for homeomorphisms with integrable distortion, that is, when $\mathbb{K}(\cdot, f) \in L^p_{loc}$ for some $p > 1$,

$$|\arg(f(z))| \leq \frac{c}{|z|^{\frac{2}{p}}}, \qquad \text{for small enough } |z| \tag{34}$$

or even if $\mathbb{K}(\cdot, f) \in L^1_{loc}$,

$$\limsup_{|z|\to 0} |z|^2 |\arg(f(z))| = 0. \tag{35}$$

It is clear from the above estimates for the argument that one can allow more spiraling by relaxing the degree of integrability of $\mathbb{K}(\cdot, f)$.

Improved rotational behavior

Being a homeomorphism with distortion in L^p, any Euler flow X_t corresponding to the Euler system (21) is in the assumptions of [32], and therefore the bound (34) can be applied to X_t. In particular, this tells that for any fixed time $t > 0$ the set $X_t([\frac{1}{n}, 1])$ winds around $X_t(0)$ a number of times not exceeding a multiple of

$$n^{2t\|\omega_0\|_\infty} \tag{36}$$

It turns out that the Hölder nature of the inverse map X_t^{-1} results in better rotation bounds. We describe this improvement in our next Theorem.

Theorem (Theorem 3.1.1). *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion such that $f(0) = 0$ and $f(1) = 1$, and assume that $\mathbb{K}(\cdot, f) \in L^p_{loc}$ for some $p > 1$. Suppose also that*

$$|f(x) - f(y)| \geq C\,|x-y|^\alpha, \qquad \text{if } |x-y| \text{ is small,}$$

for some $\alpha > 1$. Then

$$|\arg(f(z))| \leq C\,\sqrt{\alpha}\,|z|^{-\frac{1}{p}}\,\log^{\frac{1}{2}}\left(\frac{1}{|z|}\right) \tag{37}$$

whenever $|z|$ is small enough.

In contrast with (34) and (35), the existence of a Hölder continuous inverse allows the power term exponent to be halved, although then the logarithmic term needs to be included. This improvement is better seen with the particular example of the Euler flow X_t. might be significantly improved for a small time $t > 0$.

Corollary (Corollary 3.1.2)**.** *Given* $\omega_0 \in L^\infty(\mathbb{C};\mathbb{C})$, *let* $\mathbf{b}$ *be the velocity field of Yudovich's solution to* (21) *associated with the Euler Kernel* $\mathcal{K}(z) = \frac{i}{2\pi\bar{z}}$, *and let* X_t *be its flow. Then there is a constant* $C > 0$ *such that*

$$\left|\arg\left(\frac{X_t(z) - X_t(0)}{X_t(1) - X_t(0)}\right)\right| \leq C\, \log^{\frac{1}{2}}\left(\frac{1}{|z|}\right)\, |z|^{-t\|\omega_0\|_\infty}\, \exp\left(Ct\|\omega_0\|_\infty\right)$$

if both $|z|$ *and* $t > 0$ *are small enough.*

In particular, if we fix a time $t_0 > 0$ small enough, then the curve $X_{t_0}([\frac{1}{n}, 1])$ cannot wind around $X_t(0)$ more than a multiple of

$$n^{t_0\|\omega_0\|_\infty}\, (\log n)^{\frac{1}{2}}\, e^{Ct_0\|\omega_0\|_\infty}$$

times. The improvement with respect to (36) is clear. At this point, it is worth mentioning that the rotational behavior of X_t has been object of study in the recent years. For instance, When the initial vorticity ω_0 is *close* to the characteristic function of the unit disk, the work [19] provides bounds for the *winding number* of the trajectories $\{X_t(z)\}_{t>0}$ as $t \to \infty$. However, we wish to emphasize *our results do not refer to the rotational behavior in time, but instead to the rotational behavior as a function of the space variable.* In other words, we provide *spiraling bounds in the space variable for a fixed time* $t > 0$.

Towards the optimality of Theorem 3.1.1, we can show the following.

Theorem (Theorem 3.1.3)**.** *Given an increasing, onto homeomorphism* $h : [0, +\infty) \to [0, +\infty)$, *and a real number* $p > 1$, *there exists a homeomorphism* $g : \mathbb{C} \to \mathbb{C}$ *with the following properties:*

- g *is a mapping of finite distortion, with* $\mathbb{K}(\cdot, g) \in L^p_{loc}$.
- $g(0) = 0$, $g(1) = 1$.
- *If* $\alpha > \frac{3p}{p-1}$, *then* $|g(x) - g(y)| \geq C|x - y|^\alpha$ *whenever* $|x - y| < 1$. *In other words,*

$$g^{-1} \in C^{\frac{1}{3}(1-\frac{1}{p})-\epsilon}, \qquad \forall \epsilon > 0.$$

- *There exists a decreasing sequence* $\{r_n\}$, *with* $r_n \to 0^+$ *as* $n \to \infty$, *for which*

$$|\arg(g(r_n))| \geq r_n^{-\frac{1}{p}}\, \log^{\frac{1}{2}}\left(\frac{1}{r_n}\right)\, h(r_n).$$

Since h can be chosen to approach 0 at any speed, Theorem 3.1.3 shows that the order provided in Theorem 3.1.1 is sharp.

Next, we extend Theorem 3.1.1 to the borderline situation $p = 1$.

Theorem (Corollary 3.1.6)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a homeomorphism of finite distortion such that* $f(0) = 0$ *and* $f(1) = 1$, *and assume that* $\mathbb{K}(\cdot, f) \in L^1_{loc}$. *Moreover, let us suppose that*

$$|f(x) - f(y)| \geq C\,|x - y|^\alpha \qquad \text{if } |x - y| \text{ is small,}$$

for some $\alpha \geq 1$. *Then*

$$\limsup_{|z|\to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{|z|}\right)}}|\arg(f(z))| = 0. \tag{38}$$

Note that in the case $p = 1$ we get an improvement in the form of vanishing limsup compared to the case $p > 1$, which is described by the bound (37). This is analogous to the maximal spiraling bounds (34) and (35), where the exact same improvement happens.

Finally, we prove the optimality of the above result in a strong sense.

Theorem (Theorem 3.1.7)**.** *Given an increasing, onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, an arbitrary $\delta > 0$ and a real number $\beta \geq 1$, there exists a homeomorphism $g : \mathbb{C} \to \mathbb{C}$ with the following properties:*

- *g is a mapping of finite distortion, with $\mathbb{K}(\cdot, g) \in L^1_{loc}$.*
- *$g(0) = 0$, $g(1) = 1$.*
- *If $\alpha \geq 2(\beta + 2) + \delta$, then $|g(x) - g(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$. That is,*

$$g^{-1} \in C^{\frac{1}{2(\beta+2)} - \epsilon}, \qquad \forall \epsilon > 0.$$

- *There exists a decreasing sequence $\{r_n\}$, with limit $r_n \to 0^+$ as $n \to \infty$, for which*

$$|\arg(g(r_n))| \geq \frac{h(r_n)}{r_n} \left(\beta \log\left(\frac{1}{r_n}\right)\right)^{\frac{1}{2}}.$$

As it is clear from the statement, the construction in the proof of Theorem 3.1.3 does not cover the case $\mathbb{K} \in L^1_{loc}$, and thus some modifications in the argument are necessary for proving Theorem 3.1.7. It turns out that these modifications do not only apply to the $p = 1$ setting, and instead work as well when $p > 1$. In this case, the Hölder exponent of the inverse map g^{-1} from Theorem 3.1.3 can be improved from $\frac{1}{3}(1 - \frac{1}{p}) - \epsilon$ to $\frac{p}{(\beta+2)(p+1)} - \epsilon$. In particular, as $p \searrow 1$ this exponent converges to $\frac{1}{2(\beta+2)} - \epsilon$, as one would reasonably expect from Theorem 3.1.7.

Next, we extend our result to a much more general class of mappings of finite distortion. Namely, we continue assuming that $\mathbb{K}(\cdot, f) \in L^p_{loc}$ for some $p \geq 1$, but now we drop the control on the modulus of continuity of f^{-1}, and instead the result is stated in terms of the growth of f. This growth is measured by the quantity $\min_{|\omega|=|z|} |f(\omega)|$ for small values of $|z|$. Note that if f^{-1} is $\frac{1}{\alpha}$-Hölder continuous (as is the case in Theorem 3.1.1 or Corollary 3.1.6 above) then $\min_{|\omega|=|z|} |f(\omega)| \simeq |z|^\alpha$.

Theorem (Theorem 3.1.4)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion such that $f(0) = 0$, $f(1) = 1$, and assume that $\mathbb{K}(\cdot, f) \in L^p_{loc}$; $p > 1$. Then*

$$|\arg(f(z))| \leq C\, |z|^{-\frac{1}{p}} \log^{\frac{1}{2}}\left(\frac{1}{\min_{|\omega|=|z|} |f(\omega)|}\right) \quad \text{when } |z| \text{ is small.} \tag{39}$$

Furthermore, if we assume that $\mathbb{K}(\cdot, f) \in L^1_{loc}$, then

$$\limsup_{|z|\to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{\min_{|\omega|=|z|} |f(\omega)|}\right)}} |\arg(f(z))| = 0. \tag{40}$$

Towards the optimality of Theorem 3.1.4, we can show the following.

Theorem (Theorem 3.1.5)**.** *Let φ be a radially increasing homeomorphism with p-integrable distortion, $p \geq 1$, such that*

$$e^{-m_{\varphi,p}(|z|)|z|^{-\frac{2}{p}}} \leq |\varphi(z)| < |z|^4 \quad \textit{when } |z| \textit{ is small,} \tag{41}$$

where $m_{\varphi,p} : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function with $m(r) \to 0$ when $r \to 0$. Then we can choose an increasing onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, which can converge to zero as slow as we want, and find a radial homeomorphism $g : \mathbb{C} \to \mathbb{C}$ with the following properties:

- *g is a homeomorphism of finite distortion, with $\mathbb{K}(\cdot, g) \in L^p_{loc}$.*
- *$g(0) = 0$, $g(1) = 1$.*
- *There exists a decreasing sequence $\{r_n\}$, such that*

$$|g(r_n)| = |\varphi(r_n)| \tag{42}$$

and

$$|\arg(g(r_n))| \geq r_n^{-\frac{1}{p}} \log^{\frac{1}{2}}\left(\frac{1}{|g(r_n)|}\right) h(r_n). \tag{43}$$

Since h can be chosen to approach zero at any speed, Theorem 3.1.5 shows that the upper bound provided in Theorem 3.1.4 is essentially sharp when we restrict modulus to satisfy (41).

Let us now briefly give some explanation for the bounds (41). The one on the right specifies that we are studying mappings that compress stronger than Hölder maps, and thus have faster maximal spiraling rate than given in (37). On the other hand, the bound on the left is always satisfied when $p = 1$, see [32], and when $p > 1$ it is exact up to the gauge function $m_{\varphi,p}$, see [38]. Studying rotation under extremal compression leads to the extremal pointwise spiraling as shown in [32]. Thus Theorem 3.1.5, together with examples in [32] proving optimality of the extremal spiraling rate (34), show that whenever mapping f is compressing we have essentially sharp spiraling rates.

The Cauchy kernel

In Euler's system of equations

$$\begin{cases} \omega_t + \mathbf{b} \cdot \nabla\omega = 0 \\ \mathbf{b} = \frac{i}{2\pi\bar{z}} * \omega \\ \omega(0, \cdot) = \omega_0 \end{cases} \tag{44}$$

the transport structure of the equation ensures that the solution ω is transported along the flow trajectories $X(t, z)$ of the velocity field $\mathbf{b}$, that is,

$$\omega(t, X(t, z)) = \omega_0(z). \tag{45}$$

Of course, this requires some degree of regularity for $\mathbf{b}$. This degree is certainly attained in the case of Yudovich solutions $\omega \in L^\infty([0,T], L^\infty)$. Indeed, the Biot-Savart law ensures that $\partial \mathbf{b} = \frac{i\omega}{2}$. Equivalently, each Yudovich solution to (44) comes together with an incompressible velocity field with bounded vorticity. As a consequence, *the Lebesgue measure is preserved along trajectories*, and so both $\|\omega(t,\cdot)\|_{L^1}$ and $\|\omega(t,\cdot)\|_{L^\infty}$ are constant in time, thereby making all compactness arguments work in Yudovich's proof.

The rigid structure of (44) is strongly related to the choice of the convolution kernel $\frac{i}{2\pi\bar{z}}$. This means that a change in the kernel may drastically change the nature of solutions. As an example, one may consider the active scalar system of equations

$$\begin{cases} \omega_t + \mathbf{v}\cdot\nabla\omega = 0 \\ \mathbf{v} = \frac{e^{i\theta}}{2\pi z} * \omega \\ \omega(0,\cdot) = \omega_0 \end{cases} \tag{46}$$

Note the only difference between (44) and (46) is on the convolution kernel $\mathcal{K}(z) = \frac{e^{i\theta}}{2\pi z}$, where $\theta \in [0, 2\pi]$ is fixed. $\mathcal{K}$ is indeed a constant multiple of the well known *Cauchy Kernel* from complex analysis. The choice of this new kernel is partially motivated by the fact that now, instead of Biot-Savart law, one is left with the following relation between the unkonw ω and the associate velocity field $\mathbf{v}$,

$$\overline{\partial}\, \mathbf{v} = \frac{e^{i\theta}\omega}{2}. \tag{47}$$

On one hand, this choice tells us that $\mathbf{v}$ is not incompressible anymore. Moreover, $\operatorname{div}(\mathbf{v})$ and $\operatorname{curl}(\mathbf{v})$ may be unbounded functions, even if one assumes that ω_0 is bounded and compactly supported. Thus, the preservation of Lebesgue measure may fail in this case. On the other hand, this new choice of $\mathcal{K}$ suggests that if a solution $\omega \in L^\infty([0,T], L^\infty)$ is to be found then automatically $\overline{\partial}\, \mathbf{v} \in L^\infty$ and therefore $\mathbf{v}$ is an element of Reimann's Q class. Again, the transport structure of the equation makes (45) hold true also in this case, though in contrast to Euler's setting, the flow $X(t,z)$ is not anymore measure-preserving. Thus new arguments are needed to obtain a good control of $\|\omega(t,\cdot)\|_{L^1}$ and $\|\omega(t,\cdot)\|_{L^\infty}$, and these arguments may well rely on the fact that $\mathbf{v} \in Q$.

It has been recently shown in certain linear transport models [20, 21, 22] that their well-posedness do not require the flow to be measure-preserving, rather the preservation of Lebesgue null sets is only needed. In our setting, $\|\overline{\partial}\, \mathbf{v}\|_{L^\infty}$ keeps bounded in time as long as Lebesgue null sets are preserved. However, it was proven in [43] that vector fields in Q produce flows of quasiconformal maps $X(t,z)$ for all $t > 0$, and these maps do preserve Lebesgue null sets. This is the key idea for proving the following result.

Theorem (Theorem 4.1.1)**.** *If the initial datum $\omega_0 \in L^\infty$ is compactly supported, then there exists a solution $\omega \in L^1([0,T], L^\infty)$ for all $T > 0$ of* (46).

Concerning uniqueness of solutions to the above system of equations, it remains an open problem. Uniqueness is available for smooth data (say $\omega_0 \in C^\gamma$, $0 < \gamma < 1$, see [16]). The attempts to show uniqueness for $\omega_0 \in L^\infty$ are based on the following equivalent formulation for the system (46) in terms of the unknown $\mathbf{v}$ and an additional scalar valued unknown q,

$$\begin{cases} \omega_t + \mathbf{v}\cdot\nabla\omega = 0 \\ \mathbf{v} = \frac{e^{i\theta}}{2\pi z} * \omega \\ \omega(0,\cdot) = \omega_0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -e^{i\theta}\,\overline{\nabla q} \\ -\Delta q = \operatorname{div}(\mathbf{v})\ \operatorname{div}(e^{i\theta}\overline{\mathbf{v}}) \\ \overline{\partial}\,\mathbf{v}(0,\cdot)) = \frac{e^{i\theta}\omega_0}{2} \end{cases}$$

As in Euler's case, here q plays the role of a pressure function. It is worth mentioning that solutions $\mathbf{v}$ to the system on the right hand side above do satisfy $e^{i\theta}\,\partial\overline{\mathbf{v}} \in \mathbb{R}$ at any time, and not just at $t = 0$. This is similar to the situation in Euler's equation, where incompressibility at time $t = 0$ implies incompressibility at any $t > 0$.

Open problems

At the geometric side, Reimann noticed that the *time evolution of the distortion function* $\mathbb{K}(\cdot, X_t)$ *of the flow map* $X_t = X(t,\cdot)$ *is controlled by* $\|\overline{\partial}\mathbf{b}\|_\infty$, precisely (18). This came by integrating in time equation (17). In order to find a counterpart for $\|\partial\mathbf{b}\|_{L^\infty}$, then one is expected to replace (17) by

$$\overline{\partial X_t}\,\frac{d}{dt}\partial X_t - \partial\overline{X_t}\,\frac{d}{dt}\overline{\partial}X_t = \partial b(X_t)\,JX_t$$

and then integrate in time. This is better seen by showing both real and imaginary parts of the above equation. Having in mind that for any complex valued function $Z(t)$ one has

$$\frac{d}{dt}Z(t)\,\overline{Z(t)} = |Z(t)|^2\,\log(Z(t))$$

one immediately gets, for the real part,

$$\frac{d}{dt}JX(t,\cdot) = \operatorname{div}\mathbf{b}(X(t,\cdot))\,JX(t,\cdot) \tag{48}$$

and for the imaginary part

$$|\partial X(t,\cdot)|^2\,\frac{d}{dt}\arg(\partial X(t,\cdot)) - |\overline{\partial}X(t,\cdot)|^2\,\frac{d}{dt}\arg(\overline{\partial}X(t,\cdot)) = \operatorname{curl}\mathbf{b}(X(t,\cdot))\,JX(t,\cdot). \tag{49}$$

Equation (48) admits the following well-known equivalent form,

$$JX(t,\cdot) = \exp\left(\int_0^t \operatorname{div}\mathbf{b}(X(s,\cdot))\,ds\right), \tag{50}$$

in particular, $\operatorname{div}(\mathbf{b})$ controls the time evolution of $JX(t,\cdot)$ but also the area expansion rate when thinking $X(t,z)$ as a map from $\mathbb{C}$ onto $\mathbb{C}$. Especially, we get from $\operatorname{div}(\mathbf{b})$ some information for $X(t,z)$ as a function of the space variable. In contrast, integrating (49) in time is not immediate, and we face serious difficulties in finding for $\operatorname{curl}\mathbf{b}$ a counterpart to the role that $JX(t,\cdot)$ plays with respect to $\operatorname{div}\mathbf{b}$ (or the role μ develops for $\overline{\partial}\mathbf{b}$). Among the things one can say, we note that at points z of conformality, i.e. $\overline{\partial}X(t,z) = 0$, $\arg(\partial X(t,z))$ varies in time an exact amount of $\operatorname{curl}\mathbf{b}(t, X(t,z))$, and so *all the rotation effects are included in the conformal derivative.* Away from points of conformality, (49) suggests that the rotation effects due to $\operatorname{curl}\mathbf{b}(t, X(t,z))$ are balanced between $\partial X(t,z)$ and $\overline{\partial} X(t,z)$. Indeed, $|\operatorname{curl} \mathbf{b}(t, X(t,z))|$ bounds from above the imaginary part of the eigenvalues of $D\mathbf{b}(t, X(t,z))$, hence it also bounds the speed of rotation of the solutions *as functions of time*. However, this seems to say not much about the solutions as functions of the space variable. This remains an open problem to date, and is the major obstacle we faced during our work on this thesis. Unfortunately, we were not able to overcome this barrier.

In what concerns the geometric interpretation of the pointwise conditions, we already explained that Q corresponds to the quasisymmetry of the flow in (16). This is so because of the following easy identity,

$$\frac{\langle \frac{d}{dt}\, Z(t), \frac{Z(t)}{|Z(t)|}\rangle}{|Z(t)|} = \frac{d}{dt}\, \log |Z(t)|$$

for $Z(t) = X(t, z_0 + h) - X(t, z_0)$, which allows to integrate in time the inner products on the left hand side above. In contrast, we have not been able to find clear counterparts to quasisymmetry for none of the classes $\overline{Q}$, R_θ or R_0. For instance, $\overline{Q}$ is equivalent to

$$\left| \operatorname{Re} \left(\frac{\frac{d}{dt}(X(t, x_0 + h) - X(t, x_0))}{\overline{X(t, x_0 + h) - X(t, x_0)}} - \frac{\frac{d}{dt}(X(t, x_0 + k) - X(t, x_0))}{\overline{X(t, x_0 + k) - X(t, x_0)}} \right) \right| \leq C$$

Integrating in time the above inequality would require to find primitives in time of the following expression,

$$\frac{\langle \frac{d}{dt}\, Z(t), \frac{\overline{Z(t)}}{|Z(t)|}\rangle}{|Z(t)|}$$

which is not automatic. That is the reason why $\overline{Q}$ seems not to produce a clean geometric condition on $X(t, \cdot)$. The same happens with the alternative R_θ or R classes. In other words, again the same obstruction is found: no information on the flow as a function of the space variable.

Last, we were unable to prove uniqueness of solutions to the system of equations (46), when the datum $\omega_0 \in L^\infty$ is compactly supported. In contrast, the uniqueness of solutions to Euler system of equations (21), when the initial vorticity ω_0 is bounded and compactly supported, was proven in [48]. The divergence-free nature of the vector field b in Euler's case played a significant role in Yudovich's or Bertozzi-Majda's proofs of uniqueness. Indeed, in the latter, incompressibility allows to show that for any two solutions of the same Euler system $\mathbf{v}^1$ and $\mathbf{v}^2$ the quantity $\| \mathbf{v}^1 - \mathbf{v}^2 \|_2^2$ is not just finite at every time, but satisfies an ODE with homogeneous initial conditions and with a unique solution. This ODE comes indeed from the velocity formulation of Euler's system. Unfortunately, in the case of Cauchy kernel, the divergence of $\mathbf{v}$ is not even bounded. Naturally, this makes the whole uniqueness proof in Cauchy's case much more complicated when one is trying to follow similar scheme. As we have shown, a velocity formulation is also available in this case. However, the lack of incompressibility makes it impossible to bound the difference $\| \mathbf{v}^1 - \mathbf{v}^2 \|_2^2$ of any two solutions. In contrast, the appropriate quantity this time seems to be $\| \mathbf{v}^1 - \mathbf{v}^2 \|_p^p$ for any $p > 2$. However, now the homogeneous IVP one obtains does not need to have a unique solution. This suggests new ideas are needed for proving uniqueness.

Chapter 1

Preliminaries

1.1 Preliminaries to Chapter 2

In this section we recall some fundamental facts concerning harmonic functions on the upper half space. We refer the interested reader to [45] for a more detailed review on this. We will be working with functions defined on $\mathbb{R}^{n+1}_+$, where points are represented as (x, y) with $x \in \mathbb{R}^n$ and $y > 0$. Let us recall that a function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is said to be harmonic if

$$\Delta u(x, y) = 0$$

where $\Delta = \Delta_x + \partial^2_{yy} = \sum_{i=1}^n \partial^2_{x_i,x_i} + \partial^2_{yy}$. A typical way of constructing harmonic functions on the upper half space is through the Poisson integral of a function $g : \mathbb{R}^n \to \mathbb{R}$,

$$u(x, y) = P_y * g(x) = \int_{\mathbb{R}^n} P_y(x - z)\, g(z)\, dz$$

where

$$P(z, y) = P_y(z) = \frac{c_n y}{(|z|^2 + y^2)^{\frac{n+1}{2}}}$$

is the Poisson kernel. Above, the constant c_n is chosen so that $\|P_y\|_{L^1(\mathbb{R}^n)} = 1$. For a vector valued $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$, then one interprets $\mathbf{u} = P_y * \mathbf{g} : \mathbb{R}^{n+1}_+ \to \mathbb{R}^m$ componentwise. In either case, one often says that u is the *Poisson integral of* g, and that g represents u's *boundary values.* The following result explains the latter terminology.

Proposition 1.1.1. *If $g \in C_c(\mathbb{R}^n)$ then $u = P_y * g$ is the only bounded solution to the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \mathbb{R}^{n+1}_+ \\ u(\cdot, 0) = g & \mathbb{R}^n. \end{cases}$$

Proof. From

$$\partial^2_{yy} P(z, y) = (n+1) P_y(z)\, \frac{-3|z|^2 + n y^2}{(|z|^2 + y^2)^2} \qquad \Delta_z P_y(z) = (n+1)\, P_y(z)\, \frac{3|z|^2 - n y^2}{(|z|^2 + y^2)^2}$$

it is immediate that $\Delta_z P_y(z) + \partial^2_{yy} P_y(z) = 0$, so $P_y(z)$ is harmonic on $\mathbb{R}^{n+1}_+$.Thus, u is harmonic on $\mathbb{R}^{n+1}_+$. Also, $\|u(\cdot,y)\|_{L^\infty} \le \|P_y\|_{L^1} \|g\|_{L^\infty}$ so u is bounded on $\mathbb{R}^{n+1}_+$. About the boundary condition, it suffices to observe that P_y is an approximation of unity in $\mathbb{R}^n$, so one has $P_y * g \to g$ uniformly as $y \to 0$. In particular, u is continuous on $\overline{\mathbb{R}^{n+1}_+}$ and $u(x,0) = g(x)$ for every $x \in \mathbb{R}^n$. Uniqueness follows from the maximum principle for harmonic functions. □

One may ask if there are other harmonic functions in $\mathbb{R}^{n+1}_+$ that are not representable as $P_y * g$ for some g. The theory of Hardy spaces helps in this direction. Note that one may also define $P_y * g$ even when g is a measure.

Proposition 1.1.2. *Let $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be harmonic.*

- *Given $1 < p \le \infty$, there is $g \in L^p(\mathbb{R}^n)$ such that $u = P_y * g$ if and only if $\sup_y \|u(\cdot,y)\|_{L^p} < \infty$, and moreover in this case one has $\|g\|_{L^p} = \sup_y \|u(\cdot,y)\|_{L^p}$.*
- *There is a finite Borel measure μ on $\mathbb{R}^n$ with $u = P_y * \mu$ if and only if $\sup_y \|u(\cdot,y)\|_1 < \infty$, and moreover in this case one has $\|\mu\| = \sup_y \|u(\cdot,y)\|_1$. Furthermore, if $u > 0$ then μ is non-negative.*

Poisson integrals of BMO functions can also be characterized, but its description involves a completely different quantity, as stated in the following Theorem by Carleson. Let us remind that $g : \mathbb{R}^n \to \mathbb{R}$ belongs to the BMO class if

$$\|g\|_* = \sup \left\{ \frac{1}{|B|} \int_B \left| g - \frac{1}{|B|} \int_B g \right| ; B \subset \mathbb{R}^n \text{ is a ball} \right\} < \infty.$$

Theorem 1.1.3. *Let $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be harmonic. Then $u = P_y * g$ for some $g \in BMO(\mathbb{R}^n)$ if and only if*

$$\|u\|_{**} = \sup_{x_0 \in \mathbb{R}^n, \delta > 0} \frac{1}{|B(x_0,\delta)|} \int_0^\delta \int_{B(x_0,\delta)} (|D_x u(x,y)|^2 + |\partial_y u(x,y)|^2)\, dx\, y\, dy < \infty.$$

Moreover, in case this happens, then $\|u\|_{**} \simeq \|g\|_*$ with universal constants.

For a non continuous function g, calling it to be the *boundary values* of $P_y * g$ requires some explanation. Let us remind that the limit $\lim_{y\to 0} u(x,y) = g(x)$ is said to be taken *nontangentially at the point* x if and only if it happens when (x,y) move within a cone with vertex x.

Proposition 1.1.4. *Let $g \in L^p(\mathbb{R}^n)$.*

- *If $1 \le p \le \infty$, then $P_y * g \to g$ nontangentially at almost every point.*
- *If $1 < p < \infty$, then $\|P_y * g - g\|_{L^p} \to 0$ as $y \to 0$.*
- *If $p = 1$ or $p = \infty$ then there exists $g \in L^p(\mathbb{R}^n)$ such that $\|P_y * g - g\|_{L^p} \nrightarrow 0$ as $y \to 0$.*

In the case of Borel measures, the situation is significantly different. To see this, if δ_0 is the Dirac Delta then $u(\cdot,y) = P_y * \delta_0 = P_y$ so that $u(x,0) \equiv 0$. It turns out the following is true.

Proposition 1.1.5. *If g is a finite Borel measure, singular w.r.t. dx, then $P_y * g$ has nontangential limit 0 almost everywhere.*

Combining propositions 1.1.2 and 1.1.4, one sees that every bounded harmonic function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is precisely of the form $u = P_y * g$ for some $g \in L^\infty(\mathbb{R}^n)$, and moreover $u(\cdot, y)$ converges nontangentially to g at almost every point. It is interesting to note that there is some control as well on the first order derivatives of u.

Lemma 1.1.6. *If $g \in L^\infty(\mathbb{R}^n)$ then*

$$\begin{aligned}\|P_y * g\|_{L^\infty} &\le \|g\|_{L^\infty}\\ \|\partial_y(P_y * g)\|_{L^\infty} &\le n\,\frac{\|g\|_{L^\infty}}{y}\\ \|D_x(P_y * g)\|_{L^\infty} &\le \frac{n+1}{2}\,\frac{\|g\|_{L^\infty}}{y}\end{aligned}$$

Proof. First, one easily sees that $|P_y * g(x)| \le \|P_y\|_1 \cdot \|g\|_{L^\infty} = \|g\|_{L^\infty}$ since $\|P_y\|_{L^1(\mathbb{R}^n)} = 1$. Secondly, direct calculation shows that

$$\partial_y P(z, y) = \frac{P_y(z)}{y}\,\frac{|z|^2 - ny^2}{|z|^2 + y^2} \qquad\qquad D_z P_y(z) = \frac{P_y(z)}{y}\,\frac{-(n+1)\,yz}{|z|^2 + y^2}$$

Thus, $|\partial_y P_y(z)| \le \frac{n\,P_y(z)}{y}$ and hence $|(\partial_y P_y) * g(x)| \le n\,\frac{\|g\|_{L^\infty}}{y}$. The bound for the spatial derivative follows in the same way, after observing that $|D_z P_y(z)| \le \frac{n+1}{2}\,\frac{P_y(z)}{y}$. □

Lemma 1.1.6 motivates the introduction of the class B of *harmonic Bloch functions*, which consists of functions $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ that are harmonic in $\mathbb{R}^{n+1}_+$ and whose gradient blows up as $y \to 0$ like $\frac{1}{y}$, that is,

$$u \in B \qquad \Longleftrightarrow \qquad u \text{ is harmonic and } \|u\|_B = \sup_{\mathbb{R}^{n+1}_+} y(|D_x u(x, y)| + |\partial_y u(x, y)|) < \infty.$$

Vector valued harmonic Bloch functions are defined componentwise. Examples of harmonic Bloch functions are, for instance, Poisson integrals of L^∞ functions, as shown in Lemma 1.1.6. It turns out Poisson integrals of BMO functions also belong to the Bloch class.

Lemma 1.1.7. *If $g \in BMO$ then $P_y * g \in B$, and moreover*

$$\begin{aligned}\|\partial_y(P_y * g)\|_{L^\infty} &\le \frac{C(n)\,\|g\|_*}{y}\\ \|D_x(P_y * g)\|_{L^\infty} &\le \frac{C(n)\,\|g\|_*}{y}\end{aligned}$$

Proof. One can find a proof in [44, p. 86-87] or also in [26, Lemma 1.1]. We sketch the latter here for the reader's convenience. Denote $u(x, y) = P_y * g(x)$. Then u is harmonic in the upper half space, and therefore all its partial derivatives are harmonic as well. By the mean value property, if $r = \frac{y_0}{4}$ then

$$\partial u(x_0, y_0) = \fint_{|x-x_0|^2+|y-y_0|^2<r^2} \partial u(x, y)\,dx\,dy$$

at any point $(x_0, y_0) \in \mathbb{R}^{n+1}_+$. Here ∂ denotes any element of the set $\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_y\}$. We now observe that

$$\begin{aligned}
|\partial u(x_0, y_0)|^2 &= \left| \fint_{|x-x_0|^2+|y-y_0|^2<r^2} \partial u(x,y)\, dx\, dy \right|^2 \\
&\leq \fint_{|x-x_0|^2+|y-y_0|^2<r^2} |\partial u(x,y)|^2\, dx\, dy \\
&\leq \fint_{|x-x_0|^2<r^2} \fint_{[3y_0/4, 5y_0/4]} |\partial u(x,y)|^2\, dy\, dx \\
&\leq \fint_{|x-x_0|^2<r^2} \frac{8}{3y_0^2} \int_{[3y_0/4, 5y_0/4]} |\partial u(x,y)|^2\, y\, dy\, dx \\
&\leq \frac{c}{r^2} \fint_{|x-x_0|^2<(5r)^2} \int_{[0,5r]} |\partial u(x,y)|^2\, y\, dy\, dx \leq \frac{c}{r^2} \|u\|_{**}^2 \leq \frac{c}{r^2} \|f\|_*^2 = \frac{c}{y_0^2} \|f\|_*^2
\end{aligned}$$

as claimed. □

The blow-up at the boundary of higher order derivatives of Poisson integrals is very relevant for this chapter. In this direction, we have the following fact from [45, Appendix].

Lemma 1.1.8. *If $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is harmonic, then*

$$\sup_{(x,y)\in\mathbb{R}^{n+1}_+} \left(\sup_{1\leq i_1 \leq \cdots \leq i_k \leq n+1} y^k \, |\partial^k_{x_{i_1}\cdots x_{i_k}} u(x,y)| \right) \leq C(n,k) \sup_{(x,y)\in\mathbb{R}^{n+1}_+} \sup_{1\leq i\leq n+1} y \, |\partial_{x_i} u(x,y)|.$$

In other words, the blow-up of the first order derivatives roughly determines that of the higher order ones. In particular, if u is a harmonic Bloch function and $Hu(x,y)$ denotes its $(n+1)$-dimensional Hessian,

$$Hu(x,y) = \begin{pmatrix} D_x^2 u(x,y) & D_x \partial_y u(x,y) \\ D_x \partial_y u(x,y) & \partial^2_{yy} u(x,y) \end{pmatrix}$$

then one has

$$y^2 \, |Hu(x,y)| \leq C(n) \, \|u\|_B. \tag{1.1}$$

It turns out that the bound (1.1) may be significantly improved if u is the harmonic extension of a function in the Lipschitz class. Recall that $g : \mathbb{R}^n \to \mathbb{R}$ is Lipszhitz if

$$\|g\|_{Lip} = \inf \{C \geq 0 : |g(x) - g(y)| \leq C|x-y| \text{ for every } x, y \in \mathbb{R}^n\} < \infty.$$

Lipschitz functions are also characterized by having bounded derivatives. Thus, if $g \in Lip$ and $u = P_y * g$ then $D_x u = P_y * Dg$ and therefore combining Lemmas 1.1.6 and 1.1.8 one gets

$$y \, |Hu(x,y)| \leq C(n) \, \|Dg\|_{L^\infty}. \tag{1.2}$$

which certainly improves (1.1). Let us recall that $g : \mathbb{R}^n \to \mathbb{R}$ is an element of Z if and only if

$$\|g\|_Z = \inf \{C \geq 0 : |g(x+h) + g(x-h) - 2g(x)| \leq C|h| \text{ for every } x, h \in \mathbb{R}^n\} < \infty.$$

For instance, if g has distributional derivatives $Dg \in BMO$ then $g \in Z$. The Zygmund class is a little larger than the Lipschitz class *Lip*. Indeed, one may think that Z is to *Lip* what BMO is to L^∞. Thus, the following result has an easy proof for functions in *Lip*, and a more complicate one for functions in Z.

Lemma 1.1.9. *Let $g \in L^\infty(\mathbb{R}^n)$, and $u = P_y * g$. Let $\nabla u = (D_x u, \partial_y u)$ denote the $(n+1)$-dimensional gradient of u. Then $g \in Z$ if and only if $\nabla u \in B$, and moreover*

$$\frac{1}{C}\,\|g\|_Z \le \|\nabla u\|_B \le C\,\|g\|_Z$$

for some constant $C > 0$.

A proof of this fact can be found in [45, p. 146]. As a consequence, if $g \in Z$ and $u = P_y * g$ then Lemma 1.1.9 tells that

$$y\,|Hu(x,y)| \le C\,\|g\|_Z, \tag{1.3}$$

which is better than (1.1). Moreover, one can combine this with Lemma 1.1.8 and obtain that $y^k|\nabla^{k+1}u(x,y)|$ is bounded by a multiple of $\|g\|_Z$, for every $k = 1, 2, \dots$. Inequality (1.3) can be proven, for instance, if g is a function with $Dg \in BMO$. That proof requires the help of the classical $BMO - H^1$ duality (see also [43, p. 263, top Corollary]). However, functions with BMO derivatives belong to the Zygmund class. For this reason, we prefered to state Lemma 1.1.9 and use the notion of harmonic Bloch gradients, which characterizes the class of Zygmund functions and at the same time allows for a more precise constant in the inequality.

Finally, we include in this section the following result, which will be repeatedly used in the second chapter, and whose proof is implicit in [34]. Let us recall that if $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, then one defines the divergence and the curl of $\mathbf{b}$, respectively, as

$$\operatorname{div}\mathbf{b} = \operatorname{Tr}(D\mathbf{b}) \qquad \operatorname{curl}\mathbf{b} = D\mathbf{b} - D^t\mathbf{b}$$

Lemma 1.1.10. *Let $1 < p < \infty$. If $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and compactly supported, and* $\operatorname{curl}\mathbf{b}, \operatorname{div}\mathbf{b} \in L^p(\mathbb{R}^n)$*, then also $D\mathbf{b} \in L^p(\mathbb{R}^n)$, with*

$$\|Db\|_{L^p} \le C\,(\|\operatorname{div}\mathbf{b}\|_{L^p} + \|\operatorname{curl}\mathbf{b}\|_{L^p}).$$

If the assumptions hold with $p = \infty$, then one has $D\mathbf{b} \in BMO$, and

$$\|Db\|_* \le C\,(\|\operatorname{div}\mathbf{b}\|_{L^p} + \|\operatorname{curl}\mathbf{b}\|_{L^p}).$$

Proof. We write the proof for the reader's convenience. When $n = 2$, the assumptions say that $\mathbf{b}$ has complex derivative $\partial\mathbf{b} = \frac{\operatorname{div}\mathbf{b} + i\operatorname{curl}\mathbf{b}}{2}$ in L^p. Since $\mathbf{b}$ is continuous and compactly supported, we can write $\mathbf{b} = \frac{1}{\pi\bar z} * (\partial\mathbf{b})$, whence $\overline{\partial}\mathbf{b} = p.v.\frac{-1}{\pi\bar z^2} * (\partial\mathbf{b})$. But the convolution with $p.v.\frac{-1}{\pi\bar z^2}$ defines a Calderón-Zygmund operator, and thus $\overline{\partial}\mathbf{b} \in L^p$ (or BMO, if $p = \infty$) with $\|\overline{\partial}\mathbf{b}\|_{L^p} \le C\,\|\partial\mathbf{b}\|_{L^p}$ (resp. $\|\overline{\partial}\mathbf{b}\|_* \le C\,\|\partial\mathbf{b}\|_{L^\infty}$) as claimed.

When $n > 2$ the proof is a little bit delicate. We start by reminding that the second derivatives of a function $\mathbf{b}$ vanishing at infinity can be recovered from its laplacian $\Delta\mathbf{b}$ through the second order Riesz transforms,

$$\frac{\partial^2\mathbf{b}}{\partial x_j \partial x_k} = -R_j R_k \Delta\mathbf{b}, \qquad j, k = 1, \dots, n.$$

where $\widehat{R_j\mathbf{b}}(\xi) = -i\frac{\xi_j}{|\xi|}\,\widehat{\mathbf{b}}(\xi)$ at the Fourier side. As Calderón-Zygmund operators, one has again that $R_j : L^p \to L^p$ is bounded if $1 < p < \infty$, and that $R_j : L^\infty \to BMO$ is bounded. We now proceed first with the proof for $p \in (1,\infty)$. Since $\mathbf{b}$ is continuous and compactly supported, the Poisson equation

$$\Delta \mathbf{u} = \mathbf{b}$$

has a unique solution $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ vanishing at infinity. In particular, the distributional Hessian matrix $H\mathbf{u}$ of the solution $\mathbf{u}$ has all its entries in L^s and $\|H\mathbf{u}\|_s \leq C\,\|\mathbf{b}\|_s$, for every $s \in (1,\infty)$. We now decompose $\mathbf{b}$ as follows,

$$\mathbf{b} = \nabla \operatorname{div} \mathbf{u} + \operatorname{div} \operatorname{curl} \mathbf{u} \tag{1.4}$$

where we recall that $\operatorname{curl} \mathbf{u} = D\mathbf{u} - D^t\mathbf{u}$ is a matrix valued field. This is, indeed, the Hodge decomposition of $\mathbf{b}$ as the sum of a curl free vector field (i.e. $\nabla \operatorname{div} \mathbf{u}$) and a divergence free field (i.e. $\operatorname{div} \operatorname{curl} \mathbf{u}$). We now observe that $\operatorname{curl} \mathbf{u}$ solves the following Poisson equation,

$$\Delta(\operatorname{curl} \mathbf{u}) = \operatorname{curl} \mathbf{b} \tag{1.5}$$

because $\Delta(\operatorname{curl} \mathbf{u}) = \operatorname{curl}(\Delta \mathbf{u})$. In particular, if $\operatorname{curl} \mathbf{b} \in L^p$ then the same holds for the hessian $H(\operatorname{curl} \mathbf{u})$, and moreover $\|H(\operatorname{curl} \mathbf{u})\|_{L^p} \leq C\,\|\operatorname{curl} \mathbf{b}\|_{L^p}$. Similarly, $\operatorname{div} \mathbf{u}$ solves the Poisson equation

$$\Delta(\operatorname{div} \mathbf{u}) = \operatorname{div} \mathbf{b} \tag{1.6}$$

because $\Delta(\operatorname{div} \mathbf{u}) = \operatorname{div}(\Delta \mathbf{u})$. This shows that if $\operatorname{div} \mathbf{b}$ belongs to L^p then also the hessian $H(\operatorname{div} \mathbf{u})$ does, and we have the bound $\|H(\operatorname{div} \mathbf{u})\|_{L^p} \leq C\,\|\operatorname{div} \mathbf{b}\|_{L^p}$. Summarizing, if both $\operatorname{curl} \mathbf{b}, \operatorname{div} \mathbf{b} \in L^p$, then both hessians $H(\operatorname{curl} \mathbf{u})$ and $H(\operatorname{div} \mathbf{u})$ have L^p entries, whence both terms in the right hand side of (1.4) belong to the homogeneous Sobolev space $\dot{W}^{1,p}$, and

$$\begin{aligned}
\|\mathbf{b}\|_{\dot{W}^{1,p}} &\leq \|\nabla \operatorname{div} \mathbf{u}\|_{\dot{W}^{1,p}} + \|\operatorname{div} \operatorname{curl} \mathbf{u}\|_{\dot{W}^{1,p}} \\
&\leq \|H(\operatorname{div} \mathbf{u})\|_{L^p} + \|H(\operatorname{curl} \mathbf{u})\|_{L^p} \\
&\leq C\|\operatorname{div} \mathbf{b}\|_{L^p} + C\|\operatorname{curl} \mathbf{b}\|_{L^p}
\end{aligned}$$

so the claim follows if $1 < p < \infty$. In case that $\operatorname{curl} \mathbf{b}, \operatorname{div} \mathbf{b} \in L^\infty$, then the proof follows similarly, with the only difference that now $\operatorname{curl} \mathbf{u}$ and $\operatorname{div} \mathbf{u}$ have distributional hessian in BMO instead, and therefore both terms in (1.4) have first order derivatives in BMO, so $\mathbf{b}$ also does.

It just remains to prove (1.4), which we do by direct calculation,

$$\begin{aligned}
&\nabla \operatorname{div} \mathbf{u} + \operatorname{div} \operatorname{curl} \mathbf{u} = \\
&= \begin{pmatrix} \partial_{x_1} \operatorname{div} \mathbf{u} \\ \vdots \\ \partial_{x_n} \operatorname{div} \mathbf{u} \end{pmatrix} + \operatorname{div} \begin{pmatrix} 0 & \partial_{x_2} u^1 - \partial_{x_1} u^2 & \dots & \partial_{x_n} u^1 - \partial_{x_1} u^n \\ \partial_{x_1} u^2 - \partial_{x_2} u^1 & 0 & \dots & \partial_{x_n} u^2 - \partial_{x_2} u^n \\ \vdots & \vdots & \ddots & \dots \\ \partial_{x_1} u^n - \partial_{x_n} u^1 & \partial_{x_2} u^n - \partial_{x_n} u^2 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sum_j \partial^2_{x_1 x_j} u^j \\ \sum_j \partial^2_{x_2 x_j} u^j \\ \vdots \\ \sum_j \partial^2_{x_n x_j} u^j \end{pmatrix} + \begin{pmatrix} \sum_{j\neq 1} \partial^2_{x_j x_j} u^1 - \partial_{x_1} \sum_{j \neq 1} \partial_{x_j} u^j \\ \sum_{j\neq 2} \partial^2_{x_j x_j} u^2 - \partial_{x_2} \sum_{j \neq 2} \partial_{x_j} u^j \\ \vdots \\ \sum_{j\neq n} \partial^2_{x_j x_j} u^2 - \partial_{x_2} \sum_{j \neq n} \partial_{x_j} u^j \end{pmatrix} = \Delta \mathbf{u}.
\end{aligned}$$

This is legitimate for $\mathbf{u}$ because it has locally integrable second order derivatives. □

1.2 Preliminaries to Chapter 3

A mapping $f : \mathbb{C} \to \mathbb{C}$ is said to be Hölder continuous, or simply Hölder from above, if there exist constants $C > 0$, $d > 0$ and $\alpha > 0$ such that for any two points $x, y \in \mathbb{C}$ and $\alpha \in \mathbb{R}^+ \setminus \{0\}$ with $|x - y| < d$ one has

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

Similarly, we say f is Hölder from below if there are constants $C, \beta > 0$ such that for any two points $x, y \in \mathbb{C}$ and $\beta \in \mathbb{R}^+ \setminus \{0\}$ with $|x - y| < d$ one has

$$|f(x) - f(y)| \geq \bar{C}|x - y|^\beta$$

A mapping $f : \mathbb{C} \to \mathbb{C}$ is called bi-Hölder if it is both Hölder from above and from below.

Let $f : \mathbb{C} \to \mathbb{C}$ be a mapping of finite distortion and fix a point $z_0 \in \mathbb{C}$. In order to study the pointwise rotation of f at the point z_0, one usually fixes an argument $\theta \in [0, 2\pi)$, and then looks at how the quantity

$$\arg(f(z_0 + te^{i\theta}) - f(z_0))$$

changes as the parameter t goes from 1 to a small r. This can also be understood as the winding of the path $f\left([z_0 + re^{i\theta}, z_0 + e^{i\theta}]\right)$ around the point $f(z_0)$. As we are interested in the maximal pointwise spiraling, we need to normalize and then retain the maximum over all directions θ,

$$\sup_{\theta \in [0,2\pi)} |\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|. \tag{1.7}$$

Then, the maximal pointwise rotation is precisely the behavior of the above quantity (1.7) when $r \to 0^+$. In this way, we say that the map f *spirals at the point* z_0 *with a rate* g, where $g : [0, \infty) \to [0, \infty)$ is a decreasing continuous function, if

$$\limsup_{r \to 0^+} \frac{\sup_{\theta \in [0,2\pi)} |\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|}{g(r)} = C \tag{1.8}$$

for some constant $0 < C < \infty$. Finding maximal pointwise rotation for a given class of mappings equals finding the maximal spiraling rate for this class. Note that in (1.8) we must use limit superior as the limit itself might not exist. Furthermore, for a given mapping f there might be many different sequences $r_n \to 0$ along which it has profoundly different rotational behaviour.

Our proof of Theorem 3.1.1 relies heavily on the modulus of path families. We give here the main definitions, and address the interested reader to [46] for a closer look at the topic. The image of a line segment I under a continuous mapping is called a *path*, and we denote by Γ a family of paths. Given a path family Γ, we say that a Borel measurable function ρ is *admissible for* Γ if any rectifiable $\gamma \in \Gamma$ satisfies

$$\int_\gamma \rho(z) dz \geq 1.$$

The *modulus of the path family* Γ is defined by

$$M(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z)\, dA(z),$$

where $dA(z)$ denotes the Lebesgue measure on $\mathbb{C}$. As an intuitive rule, the modulus is big if the family Γ has *lots* of short paths, and it is small if the paths are long and there are *not many* of them.

We will also need a weighted version of the modulus. Any measurable, locally integrable function $\omega : \mathbb{C} \to [0, \infty)$ will be called *a weight function*. In our case, ω will always be the distortion function $\mathbb{K}(\cdot, f)$ of some map f. Then, we define the weighted modulus $M_\omega(\Gamma)$ by

$$M_\omega(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z)\, \omega(z)\, dA(z).$$

Finally, we need the modulus inequality

$$M(f(\Gamma)) \le M_{\mathbb{K}(\cdot, f)}(\Gamma) \tag{1.9}$$

which holds for any mapping f of finite distortion for which the distortion $\mathbb{K}(\cdot, f)$ is locally integrable, proven by Hitruhin in [32].

1.3 Preliminaries to Chapter 4

We denote by C^0 the class of continuous functions. For each $0 < \gamma < 1$, we denote

$$C^\gamma = \{f : \mathbb{R}^2 \to \mathbb{R}; \|f\|_\infty + [f]_\gamma < \infty\},$$

where

$$[f]_\gamma = \sup_{x,y \in \mathbb{C}, x \ne y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$$

and $\|f\|_\infty$ is the classical supremum norm. We also denote by C^1 the class of continuously differentiable functions, that is, such that $Df \in C^0$, and set

$$C^{1,\gamma} = \{f \in C^1 : Df \in C^\gamma\},$$

and set as norms in C^γ and $C^{1,\gamma}$ the following quantities, respectively

$$\begin{aligned} \|f\|_\gamma &= \|f\|_\infty + [f]_\gamma, \\ \|f\|_{1,\gamma} &= |f(0)| + \|\nabla f\|_\gamma = |f(0)| + \|\nabla f\|_\infty + [\nabla f]_\gamma. \end{aligned}$$

Lemma 1.3.1. *[13, Lemmas 4.1] Suppose that $\gamma \in (0, 1]$. Then:*

1. $[fg]_\gamma \le \|f\|_\infty\, [g]_\gamma + \|g\|_\infty\, [f]_\gamma$,
2. $\|fg\|_\gamma \le \|f\|_\gamma \|g\|_\gamma$,
3. $[\frac{1}{f}]_\gamma \le \|\frac{1}{f}\|_\infty^2\, [f]_\gamma$,
4. $\|\frac{1}{f}\|_\gamma \le \|\frac{1}{f}\|_\infty (1 + \|\frac{1}{f}\|_\infty\, [f]_\gamma) \le \max\{1, \|\frac{1}{f}\|_\infty\}^2\, (1 + [f]_\gamma)$.

Proof. The first inequality is trivial. For the second, we use the first and the definition of $\|f\|_\gamma$,

$$\|f\|_\infty\, [g]_\gamma + \|g\|_\infty\, [f]_\gamma \le \|f\|_\infty \|g\|_\gamma + \|g\|_\gamma\, [f]_\gamma = \|g\|_\gamma\, \|f\|_\gamma$$

as claimed. The last two parts of the proof follow since

$$\frac{1}{f(x)} - \frac{1}{f(y)} = -\frac{f(x) - f(y)}{f(x)\, f(y)}.$$

□

In what follows, we write X, Y to denote vector fields. Then $X \in C^\gamma$ means that all the components of X are elements of C^γ. We take the following result from [13, Lemmas 4.2, 4.3].

Lemma 1.3.2. *Assume that* $X : \mathbb{R}^n \to \mathbb{R}^n$ *is smooth, invertible and* $|\det DX(z)| \geq c_0 > 0$ *for some* $0 < c_0 < 1$. *Then for each* $\gamma \in (0,1]$ *there is* $c = c(\gamma, c_0, n)$ *such that*

1. $\|(DX)^{-1}\|_\infty \leq c(n)\, \|\frac{1}{JX}\|_\infty\, \|DX\|_\infty^{n-1}$,
2. $[(DX)^{-1}]_\gamma \leq c(n)\, (1 + \|\frac{1}{JX}\|_\infty\, \|DX\|_\infty^n)\, \|\frac{1}{JX}\|_\infty\, \|DX\|_\infty^{n-2}\, [DX]_\gamma$,
3. $\|(DX)^{-1}\|_\gamma \leq c(n)\, \|\frac{1}{JX}\|_\infty^2\, \|DX\|_\gamma^{2n-1}$,
4. $[f \circ X]_\gamma \leq \|DX\|_\infty^\gamma\, [f]_\gamma$,
5. $\|f \circ X\|_\gamma \leq \max\{1, \|DX\|_\infty^\gamma\}\, \|f\|_\gamma \leq (1 + \|X\|_{1,\gamma}^\gamma)\, \|f\|_\gamma$,
6. $[D(X^{-1})]_\gamma \leq [(DX)^{-1}]_\gamma\, \|(DX)^{-1}\|_\infty^\gamma$,
7. $\|D(X^{-1})\|_\gamma \leq \|(DX)^{-1}\|_\gamma\, \max\{1, \|(DX)^{-1}\|_\infty^\gamma\}$,
8. $\|X^{-1}\|_{1,\gamma} \leq c\, \|X\|_{1,\gamma}^{2n-1}$,
9. $\|f \circ X^{-1}\|_\gamma \leq (1 + c\|X\|_{1,\gamma}^{(2n-1)\gamma})\, \|f\|_\gamma$.

Proof. We remind that

$$(DX)^{-1} = \frac{1}{\det DX}\, (D^\# X)^t$$

so that statements 1,2 and 3 are trivial. Also 4 and 5 follow easily. Statements 6 and 7 are a consequence of the chain rule,

$$D(X^{-1}) = ((DX)(X^{-1}))^{-1}.$$

For 8, we combine the chain rule with 7 and 3, and obtain

$$\begin{aligned}
\|X^{-1}\|_{1,\gamma} &= |X^{-1}(0)| + \|D(X^{-1})\|_\gamma \\
&\leq \|D(X^{-1})\|_\infty\, |X(0)| + \|(DX)^{-1}\|_\gamma\, \max\{1, \|(DX)^{-1}\|_\infty^\gamma\} \\
&\leq \|D(X^{-1})\|_\infty\, |X(0)| + c(n)\, \|1/JX\|_\infty^2\, \|DX\|_\gamma^{2n-1}\, \max\{1, \|(DX)^{-1}\|_\infty^\gamma\} \\
&\leq C(n, \gamma, \|1/JX\|_\infty, \|(DX)^{-1}\|_\infty)\, \|X\|_{1,\gamma}^{2n-1}
\end{aligned}$$

For 9, we combine 5 and 8. □

Even though we primarily deal with the Cauchy Kernel $K(z) = \frac{e^{i\theta}}{2\pi z}$, most of our arguments will work on a much larger class of kernels K. Namely, the kernel $K : \mathbb{C} \to \mathbb{C}$ should satisfy the following conditions,

K1: $|z|\,|K(z)| \leq C$, and

K2: $p.v.DK$ is of Calderón-Zygmund type.

One of the essential points in the transit between the C^γ and the L^∞ theory is an appropriate bound for $\|\,\mathbf{v}(t,\cdot)\|_\infty$. Most of times, we will have

$$\mathbf{v}(t,\cdot) = K * \omega(t,\cdot)$$

for some $\omega \in L^1(0,T;L^\infty)$ with compact support, and for a kernel K satisfying **K1**, **K2**. In this context, the following basic estimate will be very useful.

Lemma 1.3.3. *(a) Let K satisfy* **K1** *and $f \in L^1 \cap L^\infty$. Then $K * f \in L^\infty$ and*

$$\|K * f\|_\infty \leq C(K)\|f\|_1^{\frac{1}{2}}\,\|f\|_\infty^{\frac{1}{2}}, \tag{1.10}$$

for some constant $C(K)$ depending on the kernel K.

(b) Moreover, if $f \in L^\infty$ has compact support, then

$$\|K * f\|_\infty \leq C(K)|\operatorname{supp} f|^{\frac{1}{2}}\,\|f\|_\infty. \tag{1.11}$$

Proof. Let us consider a real number $R > 0$. For each such R and given any two arbitrary points x and y in the plane, we can always divide the plane into two regions, $|x-y| \leq R$ and $|x-y| > R$. Therefore,

$$\begin{aligned}
|K * f(x)| &= \left|\int_{\mathbb{C}} K(x-y)f(y)dA(y)\right| \\
&\leq \int_{\mathbb{C}} |K(x-y)f(y)dA(y)| \\
&= \int_{|x-y|\leq R} |K(x-y)f(y)dA(y)| + \int_{|x-y|>R} |K(x-y)f(y)dA(y)| \\
&\leq \|f\|_\infty \int_{|x-y|\leq R} \frac{C(K)}{|x-y|}dA(y) + \frac{C(K)}{R}\int_{|x-y|>R} |f(y)|dy \\
&\leq C(K)R\,\|f\|_\infty + \frac{C(K)}{R}\|f\|_1
\end{aligned}$$

If we minimize the term on the right hand of the inequality as a function of R, the best possible value attainable is $R = \frac{\|f\|_1^{\frac{1}{2}}}{\|f\|_\infty^{\frac{1}{2}}}$. This gives the bound (1.10).
If f has compact support, we get that $\|f\|_1 \leq |\operatorname{supp} f|\,\|f\|_\infty$, which in turn implies bound (1.11). □

Let us turn our attention to the more delicate Calderón-Zygmund estimates, which affect the convolution with the tempered distribution $p.v.DK$. We recall the formal definition of $p.v.DK$,

$$p.v.DK * f(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} DK(x-y)\,f(y)\,dy$$

The above definition makes sense when $f \in C^\gamma$ is compactly supported. The following result is classical.

Lemma 1.3.4. *Let $0 < \gamma < 1$, and let $f \in C^\gamma$ be compactly supported. Assume that K satisfies* **K1** *and* **K2**. *Then*

(a) $p.v.DK * f$ *is bounded, and* $\|p.v.DK * f\|_\infty \le C(K, \gamma, |\operatorname{supp} f|)\,[f]_\gamma$.

(a') *If* $\epsilon > 0$ *then*

$$\|p.v.DK * f\|_\infty \le C(K,\gamma)\,[f]_\gamma\,\epsilon^\gamma + C(K)\,\|f\|_\infty \left(1 + \log \frac{|\operatorname{supp} f|}{\epsilon}\right) \tag{1.12}$$

(b) $p.v.DK * f$ *is* C^γ *and* $\|p.v.DK * f\|_\gamma \le C(K,\gamma)\,\|f\|_\gamma$.

(c) $K * f \in C^{1,\gamma}$ *and* $D(K * f) = p.v.DK * f$.

(d) $\|K * f\|_{1,\gamma} \le C(K, \gamma, |\operatorname{supp} f|)\,\|f\|_\gamma$.

Lemma 1.3.5. *If f is bounded and compactly supported, and K satisfies* **K1**, **K2** *then $K * f$ belongs to the Zygmund class. Moreover, it has BMO distributional derivatives.*

We omit the proof of the above lemma as it is easy. The interested reader can refer to [16] for a proof.

Chapter 2

Pointwise descriptions of nearly incompressible vector fields with bounded curl

2.1 Introduction

Following [43], we will say that a continuous vector field $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ is of *Reimann's type*, and write $\mathbf{v} \in Q$, if there is a constant $C_0 \geq 0$ such that for each $x, h, k \in \mathbb{R}^n$ with $|h| = |k| \neq 0$ one has

$$\left| \frac{\langle \mathbf{v}(x+h) - \mathbf{v}(x), h\rangle}{|h|^2} - \frac{\langle \mathbf{v}(x+k) - \mathbf{v}(x), k\rangle}{|k|^2} \right| \leq C_0.$$

The best possible value of C_0 is denoted $\| \mathbf{v} \|_Q$. This class of vector fields was introduced by H.M. Reimann in [43]. Even though every Lipschitz vector field belongs to the Q class, there exist many vector fields of Reimann type which are not Lipschitz. Indeed, every element of Q belongs to the Zygmund class. Thus, by the classical ODE theory, the autonomous initial value problem

$$\begin{cases} \frac{d}{dt} X(t,x) = \mathbf{v}(X(t,x)), \\ X(0,x) = x. \end{cases}$$

has a well defined, unique flow of time-dependent solutions $X(t,x)$. Moreover, in the space variable x, this solution is a Hölder continuous homeomorphism. If $\mathbf{v} = \mathbf{v}(t,x)$ is not autonomous and also depends on time, then the same conclusion holds if we assume $\sup_t \| \mathbf{v}(t,\cdot) \|_Q < \infty$.

The relevance of Reimann's vector fields in Geometric Function Theory was first proven in [43] with the quasisymmetry of the flow maps $x \mapsto X(t,x)$. At the same time, it is quite remarkable that these maps enjoy a significant degree of Sobolev regularity in the space variable, as a consequence of the quasisymmetry. This fact puts Reimann's Q class into a very narrow and unstable borderline: the one between the classical ODE theory and a much more recent result by Jabin [36] (see also [4]). Roughly, in the first theory Lipschitz vector fields are proven to produce bilipschitzian flows. The second theory refers to vector fields in the Sobolev space $W^{1,p}$ ($p < \infty$), and asserts that no Sobolev smoothness (even fractional) can be expected for their flow.

Among the tools for proving the Sobolev regularity of the flow of a given $\mathbf{v} \in Q$, there is the following differential characterization from [43, Theorem 3],

$$\mathbf{v} \in Q \qquad \Longleftrightarrow \qquad S\,\mathbf{v} \in L^\infty(\mathbb{R}^n) \text{ and } \frac{|\,\mathbf{v}(x)|}{(|x|+1)\,\log(e+|x|)} \leq C, \tag{2.1}$$

as well as its quantitative formulation $\|\,\mathbf{v}\,\|_Q \simeq \|S\,\mathbf{v}\,\|_{L^\infty}$. Here $S\,\mathbf{v}$ denotes the traceless symmetric differential of $\mathbf{v}$,

$$S\,\mathbf{v} = \frac{D\,\mathbf{v} + D^t\,\mathbf{v}}{2} - \frac{\operatorname{div}\mathbf{v}}{n}\,\mathbf{Id}\,.$$

When $n = 2$, $S\,\mathbf{v}$ reduces to $\overline{\partial}\,\mathbf{v}$, the classical Cauchy-Riemann derivative from complex analysis,

$$\overline{\partial}\,\mathbf{v} = \frac{(\partial_x + i\,\partial_y)(v^1 + i\,v^2)}{2} \equiv \frac{1}{2}\begin{pmatrix} \partial_x v^1 - \partial_y v^2 \\ \partial_x v^2 + \partial_y v^1 \end{pmatrix}.$$

From (2.1), one deduces that if $\mathbf{v} \in Q$ then the flow map $x \mapsto X(t,x)$ is quasiconformal at every time. The interested reader should refer to the monographs [6] or [34] for a self-contained background in quasiconformality. Roughly, quasiconformal maps are a relatively compact class of Sobolev homeomorphisms, and their transcendence goes beyond Geometric Function Theory to many areas in mathematics. In particular, when $n = 2$ their optimal degree of Sobolev regularity can be obtained from Astala's Area Distortion Theorem [3].

It turns out a similar situation occurs in several active scalar models, an apparently disconnected area. For instance, the planar Euler system for incompresible, inviscid fluids, in vorticity form

$$\begin{cases} \omega_t + (\mathbf{v}\cdot\nabla)\omega = 0 \\ \mathbf{v}(t,\cdot) = \frac{i}{2\pi\bar{z}} * \omega(t,\cdot) \\ \omega(0,\cdot) = \omega_0 \end{cases} \tag{2.2}$$

was proven to be well posed by Yudovich [48] in the class of vector fields with bounded curl. More precisely, given a compactly supported $\omega_0 : \mathbb{R}^2 \to \mathbb{R}$ with $\omega_0 \in L^\infty$, Yudovich [48] proved existence and uniqueness of a solution $\omega = \omega(t,z)$ of (2.2) belonging to $L^\infty((0,\infty)\times\mathbb{R}^2)$. This, together with the incompressibility, provides us with a vector field $\mathbf{v} = \mathbf{v}(t,z)$ such that $\partial\,\mathbf{v} \in L^\infty((0,\infty)\times\mathbb{R}^2)$. Here $\partial\,\mathbf{v}$ denotes the complex derivative of the velocity field $\mathbf{v}$,

$$\partial\,\mathbf{v} = \frac{(\partial_x - i\,\partial_y)(v^1 + i\,v^2)}{2} \equiv \frac{1}{2}\begin{pmatrix} \partial_x v^1 + \partial_y v^2 \\ \partial_x v^2 - \partial_y v^1 \end{pmatrix} = \frac{\operatorname{div}\mathbf{v} + i\,\operatorname{curl}\mathbf{v}}{2}.$$

A similar situation is given in the aggregation model (in which the convolution kernel from (2.2) is replaced by $\frac{1}{2\pi\bar{z}}$). In analogy with Reimann, it was recently shown in [23] that, at least for small times, vector fields $\mathbf{v}$ satisfying $\partial\,\mathbf{v} \in L^\infty$ admit a well defined flow which is Sobolev regular in the space variable, with a Sobolev exponent that may vary with time. In [15], this result was improved and obtained a degree of Sobolev regularity for the flow for every time.

Although conditions $\overline{\partial}\,\mathbf{v} \in L^\infty$ and $\partial\,\mathbf{v} \in L^\infty$ may look analytically similar, they have a significant difference. In the first case, for a general non-autonomous $\mathbf{v}$, the flow map $X(t,\cdot)$ belongs to the Sobolev space $W^{1,p}_{loc}$ whenever

$$p < \frac{2}{1 - \exp\left(-2\int_0^t \|\overline{\partial}\,\mathbf{v}(s,\cdot)\|_{L^\infty}\,ds\right)},$$

as a consequence of both Reimann's [43] and Astala's [3] Theorems. In contrast, this remains being an open problem in the second case. In accordance, it was conjectured in [23] that if $\partial\,\mathbf{v} \in L^\infty$ then for each $t > 0$ the flow map $X(t,\cdot)$ belongs to the Sobolev space $W^{1,p}_{loc}$ whenever

$$p < \frac{2}{1 - \exp\left(-2\int_0^t \|\partial\,\mathbf{v}(s,\cdot)\|_{L^\infty}\,ds\right)}.$$

The asymptotic behavior of this conjecture as $t \to 0$ was proven to be the right one in [23]. Moreover, when $\mathbf{v}$ arises from (2.2), this conjecture says that $p < \frac{2}{1-e^{-t\,\|\omega_0\|_{L^\infty}}}$. By the Sobolev embedding, this gives a Hölder exponent strictly below $e^{-t\,\|\omega_0\|_{L^\infty}}$, as shown by Bahouri and Chemin [9].

Geometric Function Theory has proven to be very useful in obtaining the optimal Sobolev regularity in Reimann's case, and therefore it is natural to try to face Euler's case with a similar scheme, as it was done in the works [22, 23]. In this chapter, we continue this line of research by focusing our attention in the pointwise characterization of [43, Theorem 3]. We investigate the existence of similar pointwise characterizations of the condition $\partial\,\mathbf{v} \in L^\infty$, both in the plane and in higher dimensions.

In the plane, we introduce the class $\bar{Q}$ of functions $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ for which there is a constant $C_0 \geq 0$ such that for each $x \in \mathbb{R}^2$ and every $h, k \neq 0$ with $|h| = |k|$ one has

$$\left| \frac{\langle \mathbf{v}(x+h) - \mathbf{v}(x), \bar{h}\rangle}{|h|^2} - \frac{\langle \mathbf{v}(x+k) - \mathbf{v}(x), \bar{k}\rangle}{|k|^2} \right| \leq C_0.$$

Here $\bar{h}$ and $\bar{k}$ mean complex conjugates. By $\|\,\mathbf{v}\,\|_{\bar{Q}}$ we denote the best possible value of C_0. Similarly, we denote by R the set of vector fields $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ for which there is a constant $C_0 \geq 0$ such that for each $x \in \mathbb{R}^2$, every $h, k \neq 0$ with $|h| = |k|$, and every $\theta \in [0, 2\pi]$, one has

$$\left| \frac{\langle \mathbf{v}(x+h) - \mathbf{v}(x), e^{i\theta}k\rangle}{|h||k|} - \frac{\langle \mathbf{v}(x+k) - \mathbf{v}(x), e^{i\theta}h\rangle}{|h||k|} \right| \leq C_0.$$

Again, $\|\,\mathbf{v}\,\|_R$ denotes the best possible constant C_0. Our first result is the following one.

Theorem 2.1.1. *Let $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field. The following are equivalent:*

(a) $\mathbf{v} \in \bar{Q}$.

(b) $\mathbf{v} \in R$.

(c) $\mathbf{v}$ *is differentiable a.e.,* $\partial\mathbf{v} \in L^\infty$*, and* $\frac{|\mathbf{v}(x)|}{(|x|+1)\,\log(e+|x|)} \leq C$.

If one of them holds true, then $\|\mathbf{v}\|_{\bar{Q}} \simeq \|\mathbf{v}\|_R \simeq \|\partial\mathbf{v}\|_{L^\infty}$.

The presence of complex conjugation in the definition of $\bar{Q}$ prevents us from extending it to higher dimensions, at least trivially. Extending the definition of R to $\mathbb{R}^n$, $n \geq 2$, seems not an easy task either, because the set of rotations to be included is not obvious (see Lemma 2.3.6). It turns out that one may still get some L^∞ estimates by removing all rotations, even in higher dimensions. Namely, let us introduce R_0 as the class of vector fields $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ for which there is C_0 such that for each $x \in \mathbb{R}^n$ and each h, k with $|h| = |k| \neq 0$ one has

$$\left| \frac{\langle \mathbf{v}(x+h) - \mathbf{v}(x), k\rangle}{|h||k|} - \frac{\langle \mathbf{v}(x+k) - \mathbf{v}(x), h\rangle}{|h||k|} \right| \leq C_0.$$

As usually, $\|\,\mathbf{v}\,\|_{R_0}$ denotes the best possible constant C_0.

Theorem 2.1.2. *Let* $\mathbf{v} \in R_0$. *Then the distribution* $D\mathbf{v} - D^t\mathbf{v}$ *belongs to* L^∞, *and*

$$\|D\mathbf{v} - D^t\mathbf{v}\|_{L^\infty} \leq C\,\|\mathbf{v}\|_{R_0}.$$

for some constant $C > 0$.

As it was the case for Q, $\bar{Q}$ or R, the elements of R_0 belong as well to the Zygmund class. However, when $n = 2$ the class R_0 is much larger than R, and one cannot guarantee its elements to be differentiable a.e.. This makes it more difficult to find higher dimensional counterparts to Theorem A. In the present chapter we solve this by asking $\mathbf{v}$ to be nearly incompressible, that is, $\operatorname{div}\mathbf{v} \in L^\infty$. This allows to state the above mentioned counterpart, which is based in the differential operator

$$A\,\mathbf{v} = \frac{D\,\mathbf{v} - D^t\,\mathbf{v}}{2} + \frac{\operatorname{div}\mathbf{v}}{n}\,\mathbf{Id}.$$

Note that for $n = 2$ one has $A\,\mathbf{v} \equiv \partial\,\mathbf{v}$.

Theorem 2.1.3. *Let* $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous vector field.*

(a) If $\mathbf{v} \in \mathbb{R}_0$ *and* $\mathbf{v}$ *is nearly incompressible, then* $\mathbf{v}$ *is differentiable a.e. and the estimate*

$$\|A\mathbf{v}\|_{L^\infty} \leq C\,(\|\operatorname{div}\mathbf{v}\|_{L^\infty} + \|\mathbf{v}\|_{R_0})$$

holds.

(b) If $A\mathbf{v} \in L^\infty$ *and* $\frac{|\mathbf{v}(x)|}{|x|\,\log(e+|x|)} \leq C$ *then* $\mathbf{v} \in R_0$ *and*

$$\|\mathbf{v}\|_{R_0} \leq C\,\|A\mathbf{v}\|_{L^\infty}.$$

As in Reimann's setting, one of the main tools here is the fact that if $\mathbf{v}$ is a compactly supported vector fields with $A\,\mathbf{v} \in L^\infty$ then $\mathbf{v}$ has BMO derivatives and, in particular, it is differentiable a.e. (see Lemma 1.1.10). For this reason, here one can relax the assumption $\operatorname{div}\mathbf{v} \in L^\infty$ to $\operatorname{div}\mathbf{v} \in L^p$ for some $p > n$. On the other hand, as a possible application, the above result can be used to describe in a pointwise way, among all the solutions to the Euler system of equations, the ones with bounded curl.

The chapter is structured as follows. In Section 2.2 we prove $(a) \Leftrightarrow (c)$ from Theorem 2.1.1. In Section 2.3 we prove $(b) \Leftrightarrow (c)$ from Theorem 2.1.1. In Section 2.4 we prove Theorems 2.1.2 and 2.1.3.

Notation. Bold letters like $\mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{g}$ denote vector valued functions. After identifying planar vectors with complex numbers, the inner product in $\mathbb{R}^2$ can be represented as $\langle z, w\rangle = \operatorname{Re}(z\bar{w})$, where Re denotes real part and $\bar{w}$ stands for the complex conjugate of w, that is, if $w = (w_1, w_2)$ then $\bar{w} = (w_1, -w_2)$. If $A \simeq B$ then there is a constant $C \geq 0$ such that $\frac{B}{C} \leq A \leq CB$.

2.2 The planar setting: the class $\bar{Q}$

With the spirit of finding a counterpart to Reimann's Q class, we introduce a class $\bar{Q}$ consisting of functions $\mathbf{b}:\mathbb{R}^2\to\mathbb{R}^2$ such that there is $C>0$ with

$$\|\mathbf{b}\|_{\bar{Q}} = \sup_{z\in\mathbb{R}^2}\sup_{|h|=|k|\neq 0}\left|\frac{\langle \mathbf{b}(z+h)-\mathbf{b}(z),\bar{h}\rangle}{|h|^2}-\frac{\langle \mathbf{b}(z+k)-\mathbf{b}(z),\bar{k}\rangle}{|k|^2}\right|<\infty$$

It is not hard to see that Lipszchitz functions are elements of $\bar{Q}$. Also, arguing as in [43], one can show that the elements of $\bar{Q}$ are, at every time t, elements of the Zygmund Z class.

Proposition 2.2.1. *If $\mathbf{b}:\mathbb{R}^2\to\mathbb{R}^2$, then one has*

$$\|\mathbf{b}\|_Z\leq C\,\|\mathbf{b}\|_{\bar{Q}}\leq C\,\|\mathbf{b}\|_{Lip}.$$

In particular, Lipschitz vector fields belong to $\bar{Q}$, and elements of $\bar{Q}$ are Zygmund vector fields. Also, if $\mathbf{b}\in\bar{Q}$ then it holds that

$$\left|\frac{\langle \mathbf{b}(z+h)-\mathbf{b}(z),\bar{h}\rangle}{|h|^2}-\frac{\langle \mathbf{b}(z+k)-\mathbf{b}(z),\bar{k}\rangle}{|k|^2}\right|\leq C\left(1+\left|\log\frac{|h|}{|k|}\right|\right)$$

for all pairs $h,k\neq 0$, and with $C\leq c\,\|\mathbf{b}\|_{\bar{Q}}$, where c is a constant independent of $\mathbf{b}$.

The proof of the above result follows the lines of [43], and therefore we omit it. The interested reader is adressed to Propositon 2.4.3 below, whose proof is very similar. In the following lemma we give a rather descriptive necessary condition for smooth elements of $\bar{Q}$.

Lemma 2.2.2. *Let $\mathbf{b}:\mathbb{R}^2\to\mathbb{R}^2$ be smooth. If $\mathbf{b}\in\bar{Q}$, then $\partial\mathbf{b}\in L^\infty$ and*

$$\|\partial\mathbf{b}\|_{L^\infty}\leq\frac{1}{2}\,\|\mathbf{b}\|_{\bar{Q}}.$$

Proof. In complex coordinates, the Taylor expansion of $\mathbf{b}$ at a differentiability point $z\in\mathbb{R}^2$ looks as follows,

$$\mathbf{b}(z+h)-\mathbf{b}(z)=\partial\mathbf{b}(z)h+\overline{\partial}\mathbf{b}(z)\,\bar{h}+o(|h|).$$

Hence, if we now take inner product with $\bar{h}$, we obtain

$$\begin{aligned}\langle \mathbf{b}(z+h)-\mathbf{b}(z),\bar{h}\rangle &= \langle\partial\mathbf{b}(z)h+\overline{\partial}\mathbf{b}(z)\,\bar{h},\bar{h}\rangle+\langle o(|h|),\bar{h}\rangle\\ &=\mathrm{Re}((\partial\mathbf{b}(z)h+\overline{\partial}\mathbf{b}(z)\,\bar{h}),h)+\langle o(|h|),\bar{h}\rangle\\ &=\mathrm{Re}((\partial\mathbf{b}(z)h^2))+\mathrm{Re}((\overline{\partial}\mathbf{b}(z))\,|h|^2)+\langle o(|h|),\bar{h}\rangle\\ &=\mathrm{Re}((\partial\mathbf{b}(z)h^2))+\mathrm{Re}((\overline{\partial}\mathbf{b}(z))\,|h|^2)+\langle o(|h|),\bar{h}\rangle\end{aligned}$$

whence

$$\begin{aligned}\left|\frac{\langle \mathbf{b}(z+h)-\mathbf{b}(z),\bar{h}\rangle}{|h|^2}-\frac{\langle \mathbf{b}(z+k)-\mathbf{b}(z),\bar{k}\rangle}{|k|^2}\right| &= \mathrm{Re}\left(\partial\mathbf{b}(z)\left(\frac{h^2}{|h|^2}-\frac{k^2}{|k|^2}\right)\right)\\ &\quad+\frac{\langle o(|h|),h\rangle}{|h|^2}+\frac{\langle o(|k|),k\rangle}{|k|^2}\end{aligned}$$

We now choose h, k so that $k = ih$ and $h^2 = \epsilon\,\overline{\partial \mathbf{b}(z)}$, and then let $\epsilon \to 0$. We get

$$\limsup_{|h|=|k|\to 0} \left| \frac{\langle \mathbf{b}(z+h) - \mathbf{b}(z), \bar{h}\rangle}{|h|^2} - \frac{\langle \mathbf{b}(z+k) - \mathbf{b}(z), \bar{k}\rangle}{|k|^2} \right| \geq 2|\partial \mathbf{b}(z)|, \tag{2.3}$$

and therefore $|\partial \mathbf{b}(z)| \leq \frac{1}{2} \|\mathbf{b}\|_{\bar{Q}}$. If $\mathbf{b}$ is differentiable at every point x the claim follows. □

It is a well known fact that Zygmund functions admit a modulus of continuity of the form $\delta \log \frac{1}{\delta}$, but may fail to differentiable almost everywhere. Thus, removing the differentiability assumption in Lemma 2.2.2 does not seem automatic. Our next goal consists of proving this is actually the case.

Theorem 2.2.3. *Let $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ belong to the class $\bar{Q}$. Then, $\mathbf{b}$ is differentiable almost everywhere, has BMO distributional derivatives, and $\partial \mathbf{b} \in L^\infty$. Moreover,*

$$\|\partial \mathbf{b}\|_{L^\infty} \leq C \|\mathbf{b}\|_{\bar{Q}}$$

for some constant $C > 0$.

Proof. We first prove that it is not restrictive to assume that $\mathbf{b}$ has compact support. To do this, let us assume that the theorem is proved under the extra assumption that $\mathbf{b}$ has compact support. Now, let us be given $\mathbf{b} \in \bar{Q}$ non compactly supported. and set $\mathbf{b}_t = g_t\,\mathbf{b}$, where

$$g_t(x) = \begin{cases} 1 & |x| \leq t \\ 1 - \frac{1}{t}\,\log \frac{\log|x|}{\log t} & t \leq |x| \leq t^{e^t} \\ 0 & t^{e^t} \leq |x|. \end{cases} \tag{2.4}$$

Clearly, $\mathbf{b}_t$ has compact support. We now prove that $\mathbf{b}_t \in \bar{Q}$. For proving this, we denote $\tau_h \mathbf{g}(x) = \mathbf{g}(x+h)$ and $\Delta_h \mathbf{g}(x) = \tau_h \mathbf{g}(x) - \mathbf{g}(x)$. Then we observe that

$$\begin{aligned} \frac{\langle \Delta_h \mathbf{b}_t, \bar{h}\rangle}{|h|^2} - \frac{\langle \Delta_k \mathbf{b}_t, \bar{k}\rangle}{|k|^2} &= \tau_h g_t\,\frac{\langle \Delta_h \mathbf{b}, \bar{h}\rangle}{|h|^2} - \tau_k g_t\,\frac{\langle \Delta_k \mathbf{b}, \bar{k}\rangle}{|k|^2} + \Delta_h g_t\,\frac{\langle \mathbf{b}, \bar{h}\rangle}{|h|^2} - \Delta_k g_t\,\frac{\langle \mathbf{b}, \bar{k}\rangle}{|k|^2} \\ &= (\tau_h g_t - \tau_k g_t)\,\frac{\langle \Delta_h \mathbf{b}, \bar{h}\rangle}{|h|^2} + \tau_k g_t \left(\frac{\langle \Delta_h \mathbf{b}, \bar{h}\rangle}{|h|^2} - \frac{\langle \Delta_k \mathbf{b}, \bar{k}\rangle}{|k|^2} \right) + \langle \mathbf{b}, \frac{\bar{h}\Delta_h g_t}{|h|^2} - \frac{\bar{h}\Delta_k g_t}{|k|^2} \rangle \\ &= (\Delta_h g_t - \Delta_k g_t)\,\frac{\langle \Delta_h \mathbf{b}, \bar{h}\rangle}{|h|^2} + \tau_k g_t \left(\frac{\langle \Delta_h \mathbf{b}, \bar{h}\rangle}{|h|^2} - \frac{\langle \Delta_k \mathbf{b}, \bar{k}\rangle}{|k|^2} \right) + \langle \mathbf{b}, \frac{\bar{h}\Delta_h g_t}{|h|^2} - \frac{\bar{h}\Delta_k g_t}{|k|^2} \rangle \end{aligned}$$

Now we use the Mean Value Theorem to deduce that

$$|\Delta_h g_t(x)| \leq \frac{C|h|}{t\,|x|\,\log|x|} \qquad \text{and} \qquad |\Delta_k g_t(x)| \leq \frac{C|k|}{t\,|x|\,\log|x|}$$

We now recall that $\mathbf{b} \in \bar{Q}$ implies that $\mathbf{b} \in Z$, and therefore $\mathbf{b}$ has $|x|\log|x|$ growth at infinity. Having in mind that $|g_t| \leq 1$, we have for $|h| = |k|$ that

$$\left| \frac{\langle \Delta_h \mathbf{b}_t, \bar{h}\rangle}{|h|^2} - \frac{\langle \Delta_k \mathbf{b}_t, \bar{k}\rangle}{|k|^2} \right| \leq \left| \frac{\langle \Delta_h \mathbf{b}, \bar{h}\rangle}{|h|^2} - \frac{\langle \Delta_k \mathbf{b}, \bar{k}\rangle}{|k|^2} \right| + \frac{C}{t}$$

whence $g_t\mathbf{b} \in \bar{Q}$ and $\|g_t\mathbf{b}\|_{\bar{Q}} \leq \|\mathbf{b}\|_{\bar{Q}} + \frac{C}{t}$. We are now in situation to apply the theorem to $g_t\mathbf{b}$ and so $g_t\mathbf{b}$ is differentiable a.e. and moreover we have the bound

$$\|\overline{\partial}(g_t\mathbf{b})\|_{L^\infty} \leq C\,\|g_t\mathbf{b}\|_{\bar{Q}} \leq C\|\mathbf{b}\|_{\bar{Q}} + \frac{C}{t}$$

The proof now finishes easily, as for any fixed x one can always find $t > 0$ large enough so that

$$\overline{\partial}\mathbf{b}(x) = g_t(x)\,\overline{\partial}\mathbf{b}(x) = \overline{\partial}(g_t\mathbf{b})(x) - \mathbf{b}(x)\,\overline{\partial}g_t(x)$$

whence, after enlarging t if needed,

$$|\overline{\partial}\mathbf{b}(x)| \leq \|\overline{\partial}(g_t\mathbf{b})\|_{L^\infty} + |\mathbf{b}(x)\,\overline{\partial}g_t(x)| \leq C\,\|g_t\mathbf{b}\|_{\bar{Q}} + \frac{C}{t} \leq C\|\mathbf{b}\|_{\bar{Q}}$$

as desired. Therefore, we can assume without loss of generality that $\mathbf{b}$ has compact support in $\mathbb{R}^2$.

Through a dilation if needed, we will suppose that $\operatorname{supp}\mathbf{b} \subset \mathbb{D}$, where $\mathbb{D}$ denotes the unit disk on $\mathbb{R}^2$. Then, since $\mathbf{b}$ is continuous, the convolution $\mathbf{u}(z, y) = P_y * \mathbf{b}(z)$ is harmonic on $\mathbb{R}^2 \times (0, +\infty)$ and continuous in $\mathbb{R}^2 \times [0, +\infty)$. Also the complex derivative $\partial\mathbf{u}$ is harmonic in $\mathbb{R}^2 \times (0, \infty)$, and as distributions one has

$$\partial\mathbf{u} = \partial(P_y * \mathbf{b}) = \partial P_y * \mathbf{b} = P_y * \partial\mathbf{b}.$$

In particular, the last convolution is well defined, and from $\operatorname{supp}(\partial\mathbf{b}) \subset \mathbb{D}$ we have

$$|\partial\mathbf{u}(z)| = |\partial P_y * \mathbf{b}(z)| \leq C \int_{\mathbb{D}} \frac{|\mathbf{b}(w)|}{|z-w|^3}\, dA(w) \leq \frac{C}{|z|^3} \qquad \text{for each } z \notin 2\mathbb{D}$$

uniformly for each $y > 0$. In particular, $\partial\mathbf{u} \in L^p(\mathbb{C} \setminus 2\mathbb{D})$ for each $\frac{2}{3} < p < \infty$. From $\mathbf{b} \in \bar{Q}$ and $\|P_y\|_1 = 1$ we have that also $\mathbf{u} \in \bar{Q}$ and $\|\mathbf{u}\|_{\bar{Q}} \leq \|\mathbf{b}\|_{\bar{Q}}$, uniformly in $y > 0$. So by Lemma 2.2.2, one has $2\|\partial\mathbf{u}\|_{L^\infty} \leq \|\mathbf{u}\|_{\bar{Q}} = \|\mathbf{b}\|_{\bar{Q}}$, and this uniformly in y. It then follows that $\partial\mathbf{u} \in L^p(\mathbb{C})$ uniformly in y, for each $1 < p < \infty$. As an element of the harmonic Hardy space $h^p(\mathbb{R}^2 \times (0, +\infty))$, $p > 1$, we know that $\partial\mathbf{u}$ has well defined boundary values $\mathbf{g} \in L^p(\mathbb{R}^2)$, and moreover one necessarily has $\partial\mathbf{u} = P_y * \mathbf{g}$. Since also $P_y * \partial\mathbf{b} = P_y * \mathbf{g}$, and $p > 1$, it then follows that $\partial\mathbf{b} = \mathbf{g}$ and so $\partial\mathbf{b}$ is actually an $L^p(\mathbb{R}^2)$ vector field. By Lemma 1.1.10 we obtain $D\mathbf{b} \in L^p$. This already gives that $\mathbf{b}$ is differentiable a.e., because one can take any $p > 2$ (see for instance [37, Theorem 2.21]). Once we know that $\partial\mathbf{b} \in L^p$ and $P_y * \partial\mathbf{b} \in L^\infty$ we immediately infer that $\partial\mathbf{b} \in L^\infty$ with $\|\partial\mathbf{b}\|_{L^\infty} \leq \|P_y * \partial\mathbf{b}\|_{L^\infty} = \|\partial\mathbf{u}\|_{L^\infty} \leq \frac{1}{2}\|\mathbf{b}\|_{\bar{Q}}$, and this with no dependence on $\operatorname{supp}\mathbf{b}$. Using again Lemma 1.1.10 we get $D\mathbf{b} \in BMO$. In particular, $\mathbf{b}$ is differentiable almost everywhere. ☐

In the converse direction, an extra assumption on the growth of $\mathbf{b}$ is needed.

Theorem 2.2.4. *Let $\mathbf{b} \in W^{1,1}_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ be a continuous vector field such that*

$$\limsup_{|x|\to\infty} \frac{|\mathbf{b}(x)|}{|x|\,\log|x|} < \infty \tag{2.5}$$

and that $\partial\mathbf{b} \in L^\infty$. Then $\mathbf{b} \in \bar{Q}$ and $\|\mathbf{b}\|_{\bar{Q}} \leq C\,\|\partial\mathbf{b}\|_{L^\infty}$.

Proof. This proof follows the scheme of [43, Proposition 12]. So we first assume that $\mathbf{b}$ has compact support. Fix two unit vectors $\alpha, \beta \in \mathbb{R}^2$, and set $a = \alpha h, b = \beta h$ for some $h > 0$. For each vector field $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$, we define

$$\Delta \mathbf{g}(x) = \Delta_{a,b}\mathbf{g}(x) = \langle \mathbf{g}(x+a) - \mathbf{g}(x), \bar{\alpha}\rangle - \langle \mathbf{g}(x+b) - \mathbf{g}(x), \bar{\beta}\rangle.$$

Clearly, $\Delta = \Delta_{a,b}$ is a linear operator in $\mathbf{g}$, and

$$|\Delta \mathbf{g}(x)| \le 4\, \|\mathbf{g}\|_{L^\infty} \tag{2.6}$$

Moreover, $\mathbf{g} \in \bar{Q}$ if and only if $|\Delta \mathbf{g}| \le C\, h$ for some constant C that does not depend on a, b. We can represent $\Delta \mathbf{g}$ in terms of $\partial \mathbf{g}$ and $\overline{\partial} \mathbf{g}$ as follows,

$$\begin{aligned}
\Delta \mathbf{g}(x) &= \int_0^h \frac{d}{ds}\Big(\langle \mathbf{g}(x+\alpha s), \bar{\alpha}\rangle - \langle \mathbf{g}(x+\beta s), \bar{\beta}\rangle \Big)\, ds \\
&= \int_0^h \langle D\mathbf{g}(x+\alpha s)\, \alpha, \bar{\alpha}\rangle - \langle D\mathbf{g}(x+\beta s)\, \beta, \bar{\beta}\rangle\, ds \\
&= \operatorname{Re}\int_0^h (\partial \mathbf{g}(x+\alpha s)\alpha^2 - \partial \mathbf{g}(x+\beta s)\beta^2)\, ds + \operatorname{Re}\int_0^h (\overline{\partial}\mathbf{g}(x+\alpha s) - \overline{\partial}\mathbf{g}(x+\beta s))\, ds \\
&= \Delta_\partial \mathbf{g}(x) + \Delta_{\overline{\partial}}\mathbf{g}(x)
\end{aligned}$$

where we set

$$\begin{aligned}
\Delta_{\overline{\partial}}\mathbf{g}(x) &= \operatorname{Re}\int_0^h (\overline{\partial}\mathbf{g}(x+\alpha s) - \overline{\partial}\mathbf{g}(x+\beta s))\, ds \\
\Delta_{\partial}\mathbf{g}(x) &= \operatorname{Re}\int_0^h (\partial \mathbf{g}(x+\alpha s)\, \alpha^2 - \partial \mathbf{g}(x+\beta s)\, \beta^2)\, ds
\end{aligned}$$

We now proceed with the proof. We denote $\mathbf{u}(x,y) = P_y * \mathbf{b}(x)$, $x \in \mathbb{C}$, $y \ge 0$. We know that $\mathbf{u}$ is harmonic in $\mathbb{R}^3_+$ and continuous up to the boundary, since $\mathbf{b} \in C_c(\mathbb{C})$. For each $y > 0$,

$$\begin{aligned}
\mathbf{b}(x) = \mathbf{u}(x,0) &= \int_0^y t\, \partial^2_{yy}\mathbf{u}(x,t)\, dt - y\, \partial_y \mathbf{u}(x,y) + \mathbf{u}(x,y) \\
&\equiv \int_0^y t\, \mathbf{w}_t(x)\, dt - y\ \mathbf{v}_y(x) + \mathbf{u}_y(x)
\end{aligned}$$

where we wrote $\mathbf{u}_y(x) = \mathbf{u}(x,y)$, $\mathbf{v}_y(x) = \partial_y \mathbf{u}(x,y)$ and $\mathbf{w}_r(x) = \partial^2_{yy}\mathbf{u}(x,r)$. By the linearity of Δ, which acts only on the x variable, one has

$$\Delta \mathbf{b}(x) = \int_0^y t\, \Delta \mathbf{w}_t(x)\, dt - y\, \Delta\, \mathbf{v}_y(x) + \Delta \mathbf{u}_y(x). \tag{2.7}$$

We now bound the three terms in the right hand side. For the first one, we use Lemma 1.1.10 to see that $\partial \mathbf{b} \in L^\infty$ implies $D\mathbf{b} \in BMO$, which in turn guarantees that $\mathbf{b} \in Z$. Now, from Lemma 1.1.9 as well as equation (1.3), we deduce that $\|H\mathbf{u}\|_{L^\infty} \le C\, \frac{\|\mathbf{b}\|_Z}{y}$ which in turn gives us that

$$\|\mathbf{w}_r\|_{L^\infty} \le C\, \frac{\|\mathbf{b}\|_Z}{r}.$$

This fact, together with (2.6), implies for the first term in (2.7) the bound

$$\left|\int_0^y t\,\Delta \mathbf{w}_t(x)\,dt\right| \le \int_0^y t\,4\|\mathbf{w}_t\|_{L^\infty}\,dt = C\,y\,\|\mathbf{b}\|_Z.$$

For the second and third terms in (2.7), we use that $\Delta = \Delta_{\overline{\partial}} + \Delta_\partial$,

$$\begin{gathered} y\,\Delta\,\mathbf{v}_y(x) = y\,\Delta_{\overline{\partial}}\,\mathbf{v}_y(x) + y\,\Delta_\partial\,\mathbf{v}_y(x) \\ \Delta\mathbf{u}_y(x) = \Delta_{\overline{\partial}}\mathbf{u}_y(x) + \Delta_\partial\mathbf{u}_y(x) \end{gathered}$$

and proceed first with the Δ_∂ terms. For each fixed y, Lemma 1.1.6 gives us that

$$\begin{aligned} \partial_{x_i}\,\mathbf{u}_y = \partial_{x_i}\,(P_y * \mathbf{b}) = P_y * (\partial_{x_i}\mathbf{b}) &\Longrightarrow \partial\mathbf{u}_y = P_y * \partial\mathbf{b} \\ &\Longrightarrow \|\partial\mathbf{u}_y\|_{L^\infty} = \|P_y * \partial\mathbf{b}\|_{L^\infty} \le \|\partial\mathbf{b}\|_{L^\infty} \end{aligned}$$

On the other hand, since u is smooth, we can argue similarly to get that

$$\begin{aligned} \partial_{x_i}\,\mathbf{v}_y = \partial^2_{y,x_i}\mathbf{u} = \partial_y\,\,(P_y * \partial_{x_i}\mathbf{b}) &\Longrightarrow \partial\,\mathbf{v}_y = \partial_y(P_y * \partial\mathbf{b}) \\ &\Longrightarrow \|\partial\,\mathbf{v}_y\,\|_{L^\infty} = \|\partial_y(P_y * \partial\mathbf{b})\|_{L^\infty} \le C\,\frac{\|\partial\mathbf{b}\|_{L^\infty}}{y}. \end{aligned}$$

Thus, from $|\Delta_\partial\mathbf{g}(x)| \le 2h\,\|\partial\mathbf{g}\|_{L^\infty}$ one gets that

$$\begin{gathered} |\Delta_\partial\mathbf{u}_y(x)| \le 2h\,\|\partial\mathbf{u}_y\|_{L^\infty} \le C\,h\,\|\partial\mathbf{b}\|_{L^\infty}, \\ |y\,\Delta_\partial\,\mathbf{v}_y(x)| \le 2hy\|\partial\,\mathbf{v}_y\,\|_{L^\infty} \le C\,h\,\|\partial\mathbf{b}\|_{L^\infty}. \end{gathered}$$

Now we proceed with the $\Delta_{\overline{\partial}}$ terms. Calling $\gamma = \frac{\alpha-\beta}{|\alpha-\beta|}$, we see that

$$\begin{aligned} |\Delta_{\overline{\partial}}\mathbf{g}(x)| &= \left|\mathrm{Re}\int_0^h\int_0^{s|\alpha-\beta|}\frac{d}{d\sigma}(\overline{\partial}\mathbf{g}(x+\beta s+\gamma\sigma))\,d\sigma\,ds\right| \\ &= \left|\mathrm{Re}\int_0^h\int_0^{s|\alpha-\beta|}D(\overline{\partial}\mathbf{g}(x+\beta s+\gamma\sigma)\cdot\gamma)\,d\sigma\,ds\right| \le \frac{h^2\,|\alpha-\beta|}{2}\,\|D(\overline{\partial}\mathbf{g})\|_{L^\infty} \end{aligned}$$

After applying this to $\mathbf{g} = \mathbf{u}_y$ and to $\mathbf{g} = \mathbf{v}_y$, and putting all together in (2.7), one obtains

$$|\Delta\mathbf{b}(x)| \le C\,y\,\|\mathbf{b}\|_Z + C\,h\,\|\partial\mathbf{b}\|_{L^\infty} + \frac{h^2\,|\alpha-\beta|}{2}\left(\|D(\overline{\partial}\mathbf{u}_y)\|_{L^\infty} + y\|D(\overline{\partial}\,\mathbf{v}_y)\|_{L^\infty}\right) \tag{2.8}$$

Lemma 1.1.10 tells that from $\partial\mathbf{b} \in L^\infty$ we get $\overline{\partial}\mathbf{b} \in BMO$ and so $P_y * (\overline{\partial}\mathbf{b})$ is harmonic Bloch. This, together with Lemma 1.1.7, implies that

$$\begin{aligned} \mathbf{u}_y = P_y * \mathbf{b} \quad &\Longrightarrow \quad \overline{\partial}\mathbf{u}_y = P_y * \overline{\partial}\mathbf{b} \\ &\Longrightarrow \quad D(\overline{\partial}\mathbf{u}_y) = D(P_y * \overline{\partial}\mathbf{b}) \\ &\Longrightarrow \quad \|D(\overline{\partial}\mathbf{u}_y)\|_{L^\infty} = \|D(P_y * \overline{\partial}\mathbf{b})\|_{L^\infty} \le C\,\frac{\|\overline{\partial}\mathbf{b}\|_*}{y} \le C\,\frac{\|\partial\mathbf{b}\|_{L^\infty}}{y}. \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{v}_y = \partial_y \mathbf{u}_y = \partial_y P_y * \mathbf{b} \quad &\Longrightarrow \quad \overline{\partial}\, \mathbf{v}_y = \partial_y P_y * \overline{\partial}\mathbf{b} \\
&\Longrightarrow \quad D(\overline{\partial}\, \mathbf{v}_y) = D(\partial_y P_y * \overline{\partial}\mathbf{b}) \\
&\Longrightarrow \quad \|D(\overline{\partial}\, \mathbf{v}_y)\|_{L^\infty} = \|D(\partial_y P_y * \overline{\partial}\mathbf{b})\|_{L^\infty} \leq C\, \frac{\|\overline{\partial}\mathbf{b}\|_*}{y^2} \leq C\, \frac{\|\partial\mathbf{b}\|_{L^\infty}}{y^2}
\end{aligned}$$

Plugging the above bounds into (2.8), we get

$$|\Delta\mathbf{b}(x)| \leq C\, y\, \|\mathbf{b}\|_Z + C\, h\, \|\partial\mathbf{b}\|_{L^\infty} + C\, \frac{h^2\, |\alpha - \beta|}{2y}$$

and choose $y = h$ to get $|\Delta\mathbf{b}(x)| \leq C\, h\, \|\partial\mathbf{b}\|_{L^\infty}$. So $\mathbf{b} \in \bar{Q}$ and $\|\mathbf{b}\|_{\bar{Q}} \leq C\, \|\partial\mathbf{b}\|_{L^\infty}$. The claim follows in the case $\mathbf{b} \in C_c(\mathbb{C})$.

In order to remove the assumption on the compact support, we use again Reimann's ideas. So we use the g_t functions introduced at (2.4), and assume that $\partial\mathbf{b} \in L^\infty$ and $|\mathbf{b}(x)| \leq C\,|x|\,\log|x|$ as $|x| \to \infty$. For every fixed $t > 0$, we have that $\partial(g_t\mathbf{b}) = \mathbf{b}\,\partial g_t + g_t\,\partial\mathbf{b}$ and so $\partial(g_t\mathbf{b}) \in L^\infty$. Moreover, $g_t\mathbf{b}$ has compact support. It then follows that $g_t\mathbf{b} \in \bar{Q}$ and $\|g_t\mathbf{b}\|_{\bar{Q}} \leq C\,\|\partial(g_t\mathbf{b})\|_{L^\infty}$. However, from (2.5) we see that

$$\begin{aligned}
|\partial(g_t\mathbf{b})(x)| &\leq |\partial\mathbf{b}(x)| + |\mathbf{b}(x)||\partial g_t(x)| \\
&\leq |\partial\mathbf{b}(x)| + C\,|x|\log|x| \frac{1}{t|x|\,\log|x|} \\
&\leq |\partial\mathbf{b}(x)| + \frac{C}{t}
\end{aligned}$$

Thus, we can always pick $t > 0$ large enough so that $\|g_t\mathbf{b}\|_{\bar{Q}} \leq C\|\partial\mathbf{b}\|_{L^\infty}$. We now fix $x \in \mathbb{R}^2$. For every pair $|h| = |k|$ there is always $t > 0$ large enough and such that $|x|, |x+h|, |x+k| < t^{e^t}$ so that $\mathbf{b} = g_t\mathbf{b}$ at x, $x+h$ and $x+k$. Thus, when evaluating the $\bar{Q}$ norm of $\mathbf{b}$ at $x, x+h$ and $x+k$ one reduces the differences of $\mathbf{b}$ to the differences of $g_t\mathbf{b}$, which are controlled by $\|g_t\mathbf{b}\|_{\bar{Q}}$, which is independent of t, $|h|$ and $|k|$. It follows that $\mathbf{b} \in \bar{Q}$ and $\|\mathbf{b}\|_{\bar{Q}} \leq C\,\|\partial\mathbf{b}\|_{L^\infty}$.

□

In the above proof, among all terms in the right hand side of (2.7), most of them admit the desired key bound precisely because $\mathbf{b} \in Z$, except the two Δ_∂ terms, which are the only ones requiring specifically that $\partial\mathbf{b} \in L^\infty$.

On the other hand, one can deduce from the previous Theorem that $\bar{Q}$ contains many non-trivial, non-Lipschitz vector fields. At least, as many as non-Lipschitz solutions of the planar Euler system with bounded vorticity.

Corollary 2.2.5. *$\bar{Q}$ contains many non-Lipschitzian vector fields.*

Proof. Let us assume that $\omega_0 : \mathbb{C} \to \mathbb{R}$ is a real valued, compactly supported function, such that $\omega_0 \in L^\infty$. It follows from Yudovich Theorem [48] that the associate Euler system, in its vorticity form

$$\begin{cases}\omega_t + (\mathbf{v}\cdot\nabla)\omega = 0\\ \mathbf{v}(t,\cdot) = \frac{1}{2\pi}\,\frac{(y,-x)}{x^2+y^2} * \omega(t,\cdot)\\ \omega(0,\cdot) = \omega_0\end{cases}$$

admits a unique solution ω global in time, belonging to $L^\infty((0,\infty);L^\infty(\mathbb{C}))$, and whose associate velocity field v is such that $\operatorname{curl}\mathbf{v} = \omega$, that is, $2\bar\partial\,\mathbf{v} = i\omega$. In particular, $\partial\,\mathbf{v}(t,\cdot) \in L^\infty$ for every t. Therefore, $\mathbf{v}(t,\cdot)$ is an element of $\bar{Q}$ at every time. However, it is well known that not all bounded vorticities produce Lipschitz vector fields, see for instance the example by Bahouri and Chemin in [9, Theorem 1.3]. □

2.3 An alternative to $\bar{Q}$: the class R

The class $\bar{Q}$ is an appropiate counterpart to Reimann's Q class when $n = 2$, but seems not so convenient if $n > 2$ due to the absence of complex conjugation. The following observation shows that there is another way to recover $|\partial\mathbf{b}(x)|$ from the Taylor development of $\mathbf{b}$ at x that may be more convenient with higher dimensional counterparts.

Lemma 2.3.1. *Let* $\mathbf{b}$ *be a vector field in* $\mathbb{R}^2$*. Assume that* x *is a differentiability point of* $\mathbf{b}$*. Then*

$$\limsup_{|h|,|k|\to 0}\ \sup_{0\le\theta\le 2\pi} \frac{|\langle \mathbf{b}(x+h)-\mathbf{b}(x), e^{i\theta}k\rangle - \langle \mathbf{b}(x+k)-\mathbf{b}(x), e^{i\theta}h\rangle|}{|h|\,|k|} = 2\,|\partial\mathbf{b}(x)|\,.$$

Proof. We first note that

$$\begin{aligned}\langle D\mathbf{b}(x)h, e^{i\theta}k\rangle - \langle D\mathbf{b}(x)k, e^{i\theta}h\rangle &= \mathrm{Re}\Big((\partial\mathbf{b}(x)h + \bar\partial\mathbf{b}(x)\,\bar h)e^{-i\theta}\bar k - (\partial\mathbf{b}(x)k + \bar\partial\mathbf{b}(x)\,\bar k)e^{-i\theta}\bar h\Big)\\ &= \mathrm{Re}\Big(\partial\mathbf{b}(x)e^{-i\theta}(h\bar k - k\bar h)\Big)\\ &= -2\mathrm{Im}\Big(\partial\mathbf{b}(x)e^{-i\theta}\Big)\,\mathrm{Im}(h\bar k)\\ &= \Big(-2\mathrm{Im}(\partial\mathbf{b}(x))\,\cos\theta + 2\,\mathrm{Re}(\partial\mathbf{b}(x))\,\sin\theta\Big)\,\mathrm{Im}(h\bar k)\end{aligned}$$

But since $\mathbf{b}$ is differentiable at x we know that

$$\limsup_{|h|\to 0}\frac{|\langle \mathbf{b}(x+h)-\mathbf{b}(x)-D\mathbf{b}(x)h, e^{i\theta}k\rangle|}{|h|\,|k|} = \limsup_{|k|\to 0}\frac{|\langle \mathbf{b}(x+k)-\mathbf{b}(x)-D\mathbf{b}(x)k, e^{i\theta}h\rangle|}{|h|\,|k|} = 0$$

Thus

$$\begin{aligned}2\mathrm{Im}\Big(\partial\mathbf{b}(x)e^{-i\theta}\Big)\,\frac{\mathrm{Im}(h\bar k)}{|h|\,|k|} &= -\frac{\langle D\mathbf{b}(x)h, e^{i\theta}k\rangle - \langle D\mathbf{b}(x)k, e^{i\theta}h\rangle}{|h|\,|k|}\\ &= -\frac{\langle \mathbf{b}(x+h)-\mathbf{b}(x), e^{i\theta}k\rangle - \langle \mathbf{b}(x+k)-\mathbf{b}(x), e^{i\theta}h\rangle}{|h|\,|k|}\\ &\quad+\frac{\langle o(h),k\rangle}{|h|\,|k|} + \frac{\langle o(k),h\rangle}{|h|\,|k|}\end{aligned}$$

so it is obvious that if we take first supremum in θ and then $\limsup$ in h, k one gets

$$\limsup_{h,k\to 0}\sup_{\theta}\left|\frac{\langle \mathbf{b}(x+h)-\mathbf{b}(x), e^{i\theta}k\rangle - \langle \mathbf{b}(x+k)-\mathbf{b}(x), e^{i\theta}h\rangle}{|h|\,|k|}\right| \le |2\partial\mathbf{b}(x)|\,.$$

For the converse inequality, just choose $k = ih$, then take supremum in θ and let $h \to 0$ then

$$|2\partial\mathbf{b}(x)| \le \limsup_{h,k\to 0}\left|\frac{\langle \mathbf{b}(x+h)-\mathbf{b}(x), e^{i\theta}k\rangle - \langle \mathbf{b}(x+k)-\mathbf{b}(x), e^{i\theta}h\rangle}{|h|\,|k|}\right|.$$

The claim follows. □

Lemma 2.3.1 encourages us to introduce the following definition.

Definition 2.3.2. *We say that a continuous function $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ is an element of the class R if*

$$\sup_{x\in\mathbb{R}^2}\ \sup_{|h|=|k|\neq 0}\ \sup_{0\le\theta\le 2\pi}\frac{|\langle \mathbf{b}(x+h)-\mathbf{b}(x), e^{i\theta}k\rangle - \langle \mathbf{b}(x+k)-\mathbf{b}(x), e^{i\theta}h\rangle|}{|h|\,|k|} \le C.$$

The best constant C will be denoted by $\|\mathbf{b}\|_R$.

It is not hard to see that we have the inequalities

$$\|\mathbf{b}\|_Z \le c\,\|\mathbf{b}\|_R \le c\,\|\mathbf{b}\|_{Lip}.$$

As it was for $\bar{Q}$, these inequalities are actually a direct consequence of Propositon 2.4.3, which will be proven in the next sections. Also, it is not hard to deduce from Lemma 2.3.1 that if $\mathbf{b} \in R$ happens to be smooth then one has the bound

$$\|\partial\mathbf{b}\|_{L^\infty} \le \frac{1}{2}\,\|\mathbf{b}\|_R, \tag{2.9}$$

arguing as we did in Lemma 2.2.2. As in the previous section, the difficulty is in proving that (2.9) also holds true in absence of smoothness.

Theorem 2.3.3. *Let $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ belong to the class R. Then $\mathbf{b}$ is differentiable almost everywhere, it has BMO distributional derivatives, and $\partial\mathbf{b} \in L^\infty$ with $\|\partial\mathbf{b}\|_{L^\infty} \le \frac{1}{2}\|\mathbf{b}\|_R$.*

Proof. The proof of the above result follows the lines of the proof we have given in Theorem 2.2.3, so we omit it. □

The above sufficient condition for belonging to R is also necessary, again with the growth condition.

Theorem 2.3.4. *Let $\mathbf{b} \in W^{1,1}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ be a vector field such that*

$$\limsup_{|x|\to\infty}\frac{|\mathbf{b}(x)|}{|x|\,\log|x|} < \infty$$

and that $\partial\mathbf{b} \in L^\infty$. Then $\mathbf{b} \in R$ and $\|\mathbf{b}\|_R \le C\,\|\partial\mathbf{b}\|_{L^\infty}$.

Proof. Even though the proof is similar to the proof of Theorem 2.2.4, some modifications need to be done. As before, we only do it assuming that $\mathbf{b}$ has compact support (removing this assumption can be done as in Theorem 2.2.4), and start by fixing two unit vectors $\alpha, \beta \in \mathbb{R}^2$, and set $a = \alpha h, b = \beta h$ for some $h > 0$. Given $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$, this time one sets

$$\Delta \mathbf{g}(x) = \Delta_{a,b,\theta} \mathbf{g}(x) = \langle \mathbf{g}(x+a) - \mathbf{g}(x), e^{i\theta}\beta \rangle - \langle \mathbf{g}(x+b) - \mathbf{g}(x), e^{i\theta}\alpha \rangle.$$

Here, $\theta \in \{0, \pi/2\}$. Clearly, $\Delta = \Delta_{a,b,\theta}$ is a linear operator in $\mathbf{g}$, and

$$|\Delta \mathbf{g}(x)| \leq 4 \|\mathbf{g}\|_{L^\infty} \tag{2.10}$$

Moreover, $\mathbf{g} \in R$ if and only if $|\Delta \mathbf{g}| \leq C\, h$ for some constant C that does not depend on a, b or θ. The representation of $\Delta \mathbf{g}$ in terms of $\partial \mathbf{g}$ and $\overline{\partial} \mathbf{g}$ changes a bit with respect to that in Theorem 2.2.4,

$$\begin{aligned}
\Delta \mathbf{g}(x) &= \int_0^h \frac{d}{ds}\Big(\langle \mathbf{g}(x+\alpha s), e^{i\theta}\beta \rangle - \langle \mathbf{g}(x+\beta s), e^{i\theta}\alpha \rangle \Big)\, ds \\
&= \int_0^h \langle D\mathbf{g}(x+\alpha s)\, \alpha, e^{i\theta}\beta \rangle - \langle D\mathbf{g}(x+\beta s)\, \beta, e^{i\theta}\alpha \rangle\, ds \\
&= \Delta_{\overline{\partial}} \mathbf{g}(x) + \Delta_{\partial} \mathbf{g}(x),
\end{aligned}$$

where we have set

$$\begin{aligned}
\Delta_{\overline{\partial}} \mathbf{g}(x) &= \operatorname{Re}\left(e^{-i\theta} \int_0^h (\overline{\partial}\mathbf{g}(x+\alpha s) - \overline{\partial}\mathbf{g}(x+\beta s))\, \bar{\beta}\bar{\alpha}\, ds \right), \\
\Delta_{\partial} \mathbf{g}(x) &= \operatorname{Re}\left(e^{-i\theta} \int_0^h (\partial\mathbf{g}(x+\alpha s)\, \alpha\bar{\beta} - \partial\mathbf{g}(x+\beta s)\, \beta\bar{\alpha})\, ds \right).
\end{aligned}$$

The proof now follows as the one of Theorem 2.2.4. So for $\mathbf{u}(x,y) = P_y * \mathbf{b}(x)$ one knows that $\mathbf{u}$ is harmonic in $\mathbb{R}^3_+$ and continuous up to the boundary, since $\mathbf{b} \in C_c(\mathbb{C})$. For each $t > 0$,

$$\begin{aligned}
\mathbf{b}(x) = \mathbf{u}(x,0) &= \int_0^y t\, \partial^2_{yy}\mathbf{u}(x,t)\, dt - y\, \partial_y \mathbf{u}(x,y) + \mathbf{u}(x,y) \\
&\equiv \int_0^y t\, \mathbf{w}_t(x)\, dt - y\; \mathbf{v}_y(x) + \mathbf{u}_y(x)
\end{aligned}$$

where we wrote $\mathbf{u}_y(x) = \mathbf{u}(x,y)$, $\mathbf{v}_y(x) = \partial_y \mathbf{u}(x,y)$ and $\mathbf{w}_r(x) = \partial^2_{yy}\mathbf{u}(x,r)$. By the linearity of Δ, which acts only on the x variable, one has

$$\Delta \mathbf{b}(x) = \int_0^y t\, \Delta \mathbf{w}_t(x)\, dt - y\, \Delta\, \mathbf{v}_y(x) + \Delta \mathbf{u}_y(x).$$

and now one proceeds term by term. For the $\mathbf{w}$ term, one can use Lemma 1.1.9 to see that

$$\begin{aligned}
\partial \mathbf{b} \in L^\infty \quad &\Longrightarrow \quad D\mathbf{b} \in BMO \\
&\Longrightarrow \quad \mathbf{b} \in Z \\
&\Longleftrightarrow \quad \|H\mathbf{u}\|_{L^\infty} \leq C\, \frac{\|\mathbf{b}\|_Z}{y} \qquad \Rightarrow \|\mathbf{w}_r\|_{L^\infty} \leq C\, \frac{\|\mathbf{b}\|_Z}{r}
\end{aligned}$$

Hence

$$\left| \int_0^y t\, \Delta \mathbf{w}_t(x)\, dt \right| \leq \int_0^y t\, 4 \|\mathbf{w}_t\|_{L^\infty}\, dt = C\, y\, \|\mathbf{b}\|_Z$$

as desired. For the other two terms, we use that $\Delta = \Delta_{\overline{\partial}} + \Delta_\partial$,

$$y\, \Delta\, \mathbf{v}_y(x) = y\, \Delta_{\overline{\partial}}\, \mathbf{v}_y(x) + y\, \Delta_\partial\, \mathbf{v}_y(x)$$
$$\Delta \mathbf{u}_y(x) = \Delta_{\overline{\partial}} \mathbf{u}_y(x) + \Delta_\partial \mathbf{u}_y(x)$$

and proceed first with the Δ_∂ terms. For each fixed y, Lemma 1.1.6 gives us that

$$\begin{aligned} \partial_{x_i} \mathbf{u}_y = \partial_{x_i} (P_y * \mathbf{b}) = P_y * (\partial_{x_i} \mathbf{b}) &\Longrightarrow \partial \mathbf{u}_y = P_y * \partial \mathbf{b} \\ &\Longrightarrow \|\partial \mathbf{u}_y\|_{L^\infty} = \|P_y * \partial \mathbf{b}\|_{L^\infty} \leq C\, \|\partial \mathbf{b}\|_{L^\infty} \end{aligned}$$

On the other hand, since $\mathbf{u}$ is smooth, we can argue similarly to get that

$$\begin{aligned} \partial_{x_i} \mathbf{v}_y = \partial^2_{y,x_i} \mathbf{u} = \partial_y\ (P_y * \partial_{x_i} \mathbf{b}) &\Longrightarrow \partial\, \mathbf{v}_y = \partial_y (P_y * \partial \mathbf{b}) \\ &\Longrightarrow \|\partial\, \mathbf{v}_y\, \|_{L^\infty} = \|\partial_y (P_y * \partial \mathbf{b})\|_{L^\infty} \leq C\, \frac{\|\partial \mathbf{b}\|_{L^\infty}}{y}. \end{aligned}$$

Thus

$$|\Delta_\partial \mathbf{u}_y(x)| \leq 2h\, \|\partial \mathbf{u}_y\|_{L^\infty} \leq C\, h\, \|\partial \mathbf{b}\|_{L^\infty}$$
$$|y\, \Delta_\partial\, \mathbf{v}_y(x)| \leq 2hy \|\partial\, \mathbf{v}_y\, \|_{L^\infty} \leq C\, h\, \|\partial \mathbf{b}\|_{L^\infty}$$

where C is a constant. Concerning the $\Delta_{\overline{\partial}}$ terms, we call $\gamma = \frac{\alpha - \beta}{|\alpha - \beta|}$, and observe that

$$\begin{aligned} |\Delta_{\overline{\partial}} \mathbf{g}(x)| &= \left| \operatorname{Re} \left(e^{-i\theta} \int_0^h \int_0^{s|\alpha-\beta|} \frac{d}{d\sigma} (\overline{\partial} \mathbf{g}(x + \beta s + \gamma \sigma))\, \bar{\beta} \bar{\alpha}\, d\sigma\, ds \right) \right| \\ &= \left| \operatorname{Re} \left(e^{-i\theta} \int_0^h \int_0^{s|\alpha-\beta|} D(\overline{\partial} \mathbf{g}(x + \beta s + \gamma \sigma) \cdot \gamma)\, \bar{\beta} \bar{\alpha}\, d\sigma\, ds \right) \right| \leq \frac{h^2\, |\alpha - \beta|}{2}\, \|D(\overline{\partial} \mathbf{g})\|_{L^\infty} \end{aligned}$$

After applying this to $\mathbf{g} = \mathbf{u}_y$ and to $\mathbf{g} = \mathbf{v}_y$, one obtains

$$|\Delta \mathbf{b}(x)| \leq C\, y\, \|\mathbf{b}\|_Z + C\, h\, \|\partial \mathbf{b}\|_{L^\infty} + \frac{h^2\, |\alpha - \beta|}{2} \left(\|D(\overline{\partial} \mathbf{u}_y)\|_{L^\infty} + y \|D(\overline{\partial}\, \mathbf{v}_y)\|_{L^\infty} \right) \tag{2.11}$$

We now use the first inequality in Lemma 1.1.7 with $\mathbf{g} = \overline{\partial} \mathbf{u}_y$. Indeed, by the linearity of all the involved operators

$$\mathbf{u}_y = P_y * \mathbf{b} \qquad \Rightarrow \qquad \overline{\partial} \mathbf{u}_y = P_y * \overline{\partial} \mathbf{b}$$

Now, since $\partial \mathbf{b} \in L^\infty$ we have $\overline{\partial} \mathbf{b} \in BMO$ and therefore $\mathbf{b} \in Z$, so Lemma 1.1.9 applies,

$$\|D(\overline{\partial} \mathbf{u}_y)\|_{L^\infty} = \|D(P_y * \overline{\partial} \mathbf{b})\|_{L^\infty} \leq C\, \frac{\|\mathbf{b}\|_Z}{y}.$$

For $\mathbf{g} = \overline{\partial}\, \mathbf{v}_y$, we proceed similarly, and observe that

$$\mathbf{v}_y = \partial_y \mathbf{u}_y = \partial_y P_y * \mathbf{b} \qquad \Rightarrow \qquad \overline{\partial}\, \mathbf{v}_y = \partial_y P_y * \overline{\partial} \mathbf{b}.$$

Hence, one may combine Lemmas 1.1.8 and 1.1.9 to get

$$\|D(\overline{\partial}\,\mathbf{v}_y)\|_{L^\infty} = \|D(\partial_y P_y * \overline{\partial}\mathbf{b})\|_{L^\infty} \le C\,\frac{\|\mathbf{b}\|_Z}{y^2}.$$

We now plug the above bounds into (2.11),

$$|\Delta\mathbf{b}(x)| \le C\,y\,\|\mathbf{b}\|_Z + C\,h\,\|\partial\mathbf{b}\|_{L^\infty} + C\,\frac{h^2\,|\alpha-\beta|}{2y}$$

and choose $y = h$ to get $|\Delta\mathbf{b}(x)| \le C\,h\,\|\partial\mathbf{b}\|_{L^\infty}$. So $\mathbf{b} \in R$ and $\|\mathbf{b}\|_R \le C\,\|\partial\mathbf{b}\|_{L^\infty}$. The claim follows in the case $\mathbf{b} \in C_c(\mathbb{C})$. □

The following corollary, Theorem 2.1.1 in the introduction, is a way of putting together Theorems 2.2.3, 2.2.4, 2.3.3 and 2.3.4.

Corollary 2.3.5. *Let $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field. The following conditions are equivalent:*

(a) $\mathbf{b} \in R$

(b) $\mathbf{b} \in \bar{Q}$

(c) $\mathbf{b}$ *is differentiable a.e., $\partial\mathbf{b} \in L^\infty$ and $|\mathbf{b}(x)| \le C|x|\,\log|x|$ as $|x| \to \infty$.*

Moreover, in case this happens, then $\|\mathbf{b}\|_{\bar{Q}} \simeq \|\mathbf{b}\|_R \simeq \|\partial\mathbf{b}\|_{L^\infty} \simeq \|\operatorname{div}\mathbf{b}\|_{L^\infty} + \|\operatorname{curl}\mathbf{b}\|_{L^\infty}$.

As explained at the beginning of this section, the absence of complex conjugation in $\mathbb{R}^n$ when $n > 2$ seems to make the R class more suitable for higher dimensional counterparts. In order to build them, one may replace the rotation factor $e^{i\theta}$ in Definition 2.3.2 by rotations not only in the Ox_1, x_2 plane, but on any of the coordinate planes Ox_i, x_j. For this, let us introduce the set $\mathcal{J}_n = \{J_{i,j}\}_{1\le i<j\le n}$ of matrices $J_{i,j} \in \mathbb{R}^{n\times n}$ defined by

$$\begin{aligned} J_{i,j}e_i &= -e_j \\ J_{i,j}e_j &= e_i \\ J_{i,j}e_k &= e_k\,,\ \ k \ne i,j \end{aligned}$$

where $e_1, \dots, e_n$ is the canonical basis in $\mathbb{R}^n$. When $n = 2$, $\mathcal{J}_n$ contains only the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is nothing but the rotation $e^{-i\frac{\pi}{2}}$. More in general, $\mathcal{J}_n$ contains $\frac{n(n-1)}{2}$ elements.

Lemma 2.3.6. *Suppose that $n \ge 3$. Let $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and assume that x is a differentiability point. If*

$$\limsup_{|h|=|k|\to 0}\ \sup_{J\in\mathcal{J}_n\cup\{\boldsymbol{Id}\}} \left| \frac{\langle \mathbf{b}(x+h)-\mathbf{b}(x), Jk\rangle}{|h|\,|k|} - \frac{\langle \mathbf{b}(x+k)-\mathbf{b}(x), Jh\rangle}{|h|\,|k|} \right| \le C_0 \qquad (2.12)$$

then also $|D\mathbf{b}(x)| \le C\,C_0$ for some dimensional constant C.

Proof. Since x is a differentiability point,

$$\begin{aligned}
&\limsup_{|h|=|k|\to 0}\left|\frac{\langle \mathbf{b}(x+h)-\mathbf{b}(x),Jk\rangle}{|h|\,|k|}-\frac{\langle \mathbf{b}(x+k)-\mathbf{b}(x),Jh\rangle}{|h|\,|k|}\right| \\
&\qquad = \limsup_{|h|=|k|\to 0}\left|\frac{\langle D\mathbf{b}(x)h,Jk\rangle-\langle D\mathbf{b}(x)k,Jh\rangle}{|h|\,|k|}\right| \\
&\qquad = \sup_{|h|=|k|=1}|\langle D\mathbf{b}(x)h,Jk\rangle-\langle D\mathbf{b}(x)k,Jh\rangle| = \sup_{|h|=|k|=1}|\langle h,(D^t\mathbf{b}(x)J-J^tD\mathbf{b}(x))k\rangle|
\end{aligned}$$

When taking $J=\mathbf{Id}$ one recovers the curl matrix $D\mathbf{b}(x)-D^t\mathbf{b}(x)$,

$$\langle D\mathbf{b}(x)h,Jk\rangle-\langle D\mathbf{b}(x)k,Jh\rangle=\langle h,(D^t\mathbf{b}(x)J-J^tD\mathbf{b}(x))k\rangle=\langle h,(D^t\mathbf{b}(x)-D\mathbf{b}(x))k\rangle$$

Let us now take $J=J_{i,j}$ for a given pair $1\le i<j\le n$. We get

$$\begin{aligned}
\langle D\mathbf{b}(x)e_i,Je_j\rangle-\langle D\mathbf{b}(x)e_j,Je_i\rangle &= \langle \partial_i\mathbf{b},e_i\rangle+\langle \partial_j\mathbf{b},e_j\rangle=\partial_i\mathbf{b}_i+\partial_j\mathbf{b}_j \\
\langle D\mathbf{b}(x)e_i,Je_k\rangle-\langle D\mathbf{b}(x)e_k,Je_i\rangle &= \langle \partial_i\mathbf{b},e_k\rangle+\langle \partial_k\mathbf{b},e_j\rangle=\partial_i\mathbf{b}_k+\partial_k\mathbf{b}_j, \qquad k\neq i,j \\
\langle D\mathbf{b}(x)e_j,Je_k\rangle-\langle D\mathbf{b}(x)e_k,Je_j\rangle &= \langle \partial_j\mathbf{b},e_k\rangle-\langle \partial_k\mathbf{b},e_i\rangle=\partial_j\mathbf{b}_k-\partial_k\mathbf{b}_i, \qquad k\neq i,j
\end{aligned}$$

Suming up the second quantity with $-\partial_i\mathbf{b}_k+\partial_k\mathbf{b}_i$, and the third with $-\partial_j\mathbf{b}_k+\partial_k\mathbf{b}_j$ (both of which come from $D\mathbf{b}-D^t\mathbf{b}$), we get that both $\partial_k\mathbf{b}_j+\partial_k\mathbf{b}_i$ and $-\partial_k\mathbf{b}_i+\partial_k\mathbf{b}_j$ are bounded by multiples of C_0, which means that $\partial_k\mathbf{b}_i,\partial_k\mathbf{b}_j$ are bounded by multiples of C_0 whenever $k\neq i,j$. Moving now i,j we obtain the same sort of boundedness for all non-diagonal elements of Db. Also, note that the boundedness of all pairs $\partial_i\mathbf{b}_i+\partial_j\mathbf{b}_j$ implies that of all diagonal elements. This finishes the proof. □

The above result shows that the class of vector fields $\mathbf{b}:\mathbb{R}^n\to\mathbb{R}^n$ satisfying (2.12) reduces, when $n>2$, to Lipschitz vector fields. In contrast, when $n=2$, this class is much larger: this can be deduced from Lemma 2.2.5, together with the fact that in the plane one has $\bar{Q}=R$. This suggests it is not a good idea to build higher dimensional counterparts to R in this way, because the class of vector fields one obtains is included into the Lipschitz ones, which are well understood.

2.4 Extending to higher dimensions: the class R_0

Lemma 2.3.1 gives light to another fact: one may separate $\operatorname{curl}\mathbf{b}$ from $\operatorname{div}\mathbf{b}$ by simply choosing different values of θ. This is the starting point to our following observation. Let us fix an integer $n\ge 2$.

Lemma 2.4.1. *Let $\mathbf{b}:\mathbb{R}^n\to\mathbb{R}^n$ be a vector field, and assume that x is a differentiability point of $\mathbf{b}$. Then*

$$\limsup_{|h|,|k|\to 0}\frac{|\langle \mathbf{b}(x+h)-\mathbf{b}(x),k\rangle-\langle \mathbf{b}(x+k)-\mathbf{b}(x),h\rangle|}{|h|\,|k|}=|D\mathbf{b}(x)-D^t\mathbf{b}(x)|.$$

Proof. We first observe that if $\mathbf{b}$ is differentiable at x, then

$$\langle \mathbf{b}(x+h)-\mathbf{b}(x),k\rangle=\langle D\mathbf{b}(x)h,k\rangle+\langle o(|h|),k\rangle$$

Now, after exchanging the roles of h and k, we also have

$$\langle \mathbf{b}(x+k) - \mathbf{b}(x), h\rangle = \langle D\mathbf{b}(x)k, h\rangle + \langle o(|k|), h\rangle$$

Thus

$$\frac{\langle \mathbf{b}(x+h) - \mathbf{b}(x), k\rangle - \langle \mathbf{b}(x+k) - \mathbf{b}(x), h\rangle}{|h|\,|k|} = \frac{\langle (D\mathbf{b}(x) - D^t\mathbf{b}(x))\, h, k\rangle}{|h|\,|k|} + \frac{\langle o(|h|), k\rangle}{|h|\,|k|} - \frac{\langle o(|k|), h\rangle}{|k|\,|h|}.$$

and therefore one immediately gets

$$\limsup_{|h|,|k|\to 0} \frac{|\langle \mathbf{b}(x+h) - \mathbf{b}(x), k\rangle - \langle \mathbf{b}(x+k) - \mathbf{b}(x), h\rangle|}{|h|\,|k|} \le |D\mathbf{b}(x) - D^t\mathbf{b}(x)|.$$

For the converse inequality, we recall that

$$|D\mathbf{b}(x) - D^t\mathbf{b}(x)| = \sup_{h,k\neq 0} \frac{\langle (D\mathbf{b}(x) - D^t\mathbf{b}(x))\, h, k\rangle}{|h|\,|k|}$$

so we can pick two sequences $h_m, k_m \to 0$ such that

$$\begin{aligned} |D\mathbf{b}(x) - D^t\mathbf{b}(x)| &= \lim_{m\to\infty} \frac{\langle (D\mathbf{b}(x) - D^t\mathbf{b}(x))\, h_m, k_m\rangle}{|h_m|\,|k_m|} \\ &= \lim_{m\to\infty} \frac{|\langle \mathbf{b}(x+h_m) - \mathbf{b}(x), k_m\rangle - \langle \mathbf{b}(x+k_m) - \mathbf{b}(x), h_m\rangle|}{|h_m|\,|k_m|} \\ &\le \limsup_{|h|,|k|\to 0} \frac{|\langle \mathbf{b}(x+h) - \mathbf{b}(x), k\rangle - \langle \mathbf{b}(x+k) - \mathbf{b}(x), h\rangle|}{|h|\,|k|} \end{aligned}$$

and the claim follows. □

The above result motivates the following definition.

Definition 2.4.2. *We say that a continuous function $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ belongs to the class R_0 if*

$$\frac{|\langle \mathbf{b}(x+h) - \mathbf{b}(x), k\rangle - \langle \mathbf{b}(x+k) - \mathbf{b}(x), h\rangle|}{|h|\,|k|} \le C$$

for each pair h, k such that $|h| = |k| \neq 0$. The best constant C will be denoted as $\|\mathbf{b}\|_{R_0}$.

When $n = 1$, the only options are $h = k$ (which gives nothing) or $h = -k$, which gives us that in fact R_0 is exactly the class of Zygmund functions. When $n = 2$, though, the above definition suggests that R_0 is larger than R. For a general $n > 1$, taking $k = -h$ in the definition we obtain that

$$\frac{|\langle \mathbf{b}(x+h) + \mathbf{b}(x-h) - 2\mathbf{b}(x), h\rangle|}{|h|^2} \le C$$

which suggests that there may be some connection between R_0 and Z.

Proposition 2.4.3. *One has $R_0 \subset Z$, and moreover $\|\mathbf{b}\|_Z \le 4\, \|\mathbf{b}\|_{R_0}$.*

Proof. Let us assume for a while that $a, b \in \mathbb{R}^n$ are such that $\langle a, b\rangle = 0$. Then by Pythagoras $|a+b| = |a-b|$ and thus we can use condition R_0 for $x = z + a$, $h = b - a$ and $k = -b - a$. We get

$$\begin{aligned}&|\langle \mathbf{b}(z+b) - \mathbf{b}(z+a), -b-a\rangle - \langle \mathbf{b}(z-b) - \mathbf{b}(z+a), b-a\rangle| \\ &\qquad = |\langle \mathbf{b}(z-b) - \mathbf{b}(z+b), a\rangle - \langle \mathbf{b}(z+b) + \mathbf{b}(z-b), b\rangle + 2\langle \mathbf{b}(z+a), b\rangle| \\ &\qquad \le \|\mathbf{b}\|_{R_0}\, |-b-a|\, |b-a| = \|\mathbf{b}\|_{R_0}\, (|a|^2 + |b|^2)\end{aligned}$$

Similarly, for $x = z - a$, $h = b + a$ and $k = -b + a$,

$$\begin{aligned}&|\langle \mathbf{b}(z+b) - \mathbf{b}(z-a), -b+a\rangle - \langle \mathbf{b}(z-b) - \mathbf{b}(z-a), b+a\rangle| \\ &\qquad = |-\langle \mathbf{b}(z-b) - \mathbf{b}(z+b), a\rangle - \langle \mathbf{b}(z+b) + \mathbf{b}(z-b), b\rangle + 2\langle \mathbf{b}(z-a), b\rangle| \\ &\qquad \le \|\mathbf{b}\|_{R_0}\, |-b+a|\, |b+a| = \|\mathbf{b}\|_{R_0}\, (|a|^2 + |b|^2)\end{aligned}$$

Summing up the above inequalities,

$$|\langle \mathbf{b}(z+a) + \mathbf{b}(z-a), b\rangle - \langle \mathbf{b}(z+b) + \mathbf{b}(z-b), b\rangle| \le \|\mathbf{b}\|_{R_0}\, (|a|^2 + |b|^2)$$

and as a consequence

$$|\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), b\rangle - \langle \mathbf{b}(z+b) + \mathbf{b}(z-b) - 2\mathbf{b}(z), b\rangle| \le \|\mathbf{b}\|_{R_0}\, (|a|^2 + |b|^2)$$

whence

$$\begin{aligned}|\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), b\rangle| &\le |\langle \mathbf{b}(z+b) + \mathbf{b}(z-b) - 2\mathbf{b}(z), b\rangle| + \|\mathbf{b}\|_{R_0}\, (|a|^2 + |b|^2) \\ &= |\langle \mathbf{b}(z-b) - \mathbf{b}(z), b\rangle - \langle \mathbf{b}(z+b) - \mathbf{b}(z), -b\rangle| + \|\mathbf{b}\|_{R_0}\, (|a|^2 + |b|^2) \\ &\le \|\mathbf{b}\|_{R_0}\, (|a|^2 + 2|b|^2)\end{aligned}$$

Let us now take a vector $v \in \mathbb{R}^n$, and decompose it as $v = v_1 + v_2$ with $v_1 = \langle v, a\rangle \frac{a}{|a|^2}$. Then $\langle v_2, a\rangle = 0$ so that taking $b = \frac{v_2}{|v_2|}\, |a|$ we certainly have $\langle a, b\rangle = 0$ and $|a| = |b|$, and so we can apply what we proved before. Namely,

$$\begin{aligned}&|\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), v\rangle| \\ &\quad \le |\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), v_1\rangle| + |\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), v_2\rangle| \\ &\quad = |\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), a\rangle|\, \frac{|\langle v, a\rangle|}{|a|^2} + |\langle \mathbf{b}(z+a) + \mathbf{b}(z-a) - 2\mathbf{b}(z), b\rangle|\, \frac{|v_2|}{|a|} \\ &\quad \le \|\mathbf{b}\|_{R_0}\, |a|^2\, \frac{|\langle v, a\rangle|}{|a|^2} + \|\mathbf{b}\|_{R_0}\, (|a|^2 + 2|b|^2)\, \frac{|v_2|}{|a|} \\ &\quad \le \|\mathbf{b}\|_{R_0}\, |\langle v, a\rangle| + \|\mathbf{b}\|_{R_0}\, 3|a|\, |v_2| \le 4\, \|\mathbf{b}\|_{R_0}\, |a|\, |v|\end{aligned}$$

and the claim follows. □

Remark 2.4.4. *In the above proof, condition R_0 has only been used for precise pairs h and k for which either $h = k$ or $\langle h, k\rangle = 0$ with $|h| = |k|$. It will be clear that the class of vector fields one obtains with this restriction is exactly the same R_0. This can be seen as a consequence of Theorems 2.4.6 and 2.4.9 below.*

Among the consequences, we deduce that each element of R_0 has growth at most $|x|\log|x|$, as $|x|\to\infty$, and also that each element of R_0 has $t\log\frac{1}{t}$ local modulus of continuity. Arguing as in Reimann's Proposition 5 for $n=1$, functions in the R_0 class can be shown to satisfy the following extended version of condition R_0,

$$\frac{|(\mathbf{b}(x+h)-\mathbf{b}(x))k-(\mathbf{b}(x+k)-\mathbf{b}(x))h|}{|hk|}\leq\|\mathbf{b}\|_{R_0}\left(\frac{3}{2}+\frac{1}{2\log 2}\left|\log\frac{|h|}{|k|}\right|\right)\tag{2.13}$$

provided that $h\cdot k>0$ (replace 3/2 by 5/2 in case you want to allow $h\cdot k<0$). The extension of this fact to functions in the higher dimensional R_0 class works as follows.

Proposition 2.4.5. *There exists $C=C(n)\geq 1$ such that if $\mathbf{b}\in R_0$ then*

$$\frac{|\langle\mathbf{b}(x+h)-\mathbf{b}(x),k\rangle-\langle\mathbf{b}(x+k)-\mathbf{b}(x),h\rangle|}{|h|\,|k|}\leq C\,\|\mathbf{b}\|_{R_0}\left(1+\left|\log\frac{|h|}{|k|}\right|\right)$$

whenever $h,k\in\mathbb{R}^n$ are non-zero.

Proof. Let us fix two non-zero vectors $a,b\in\mathbb{R}^n$, choose $y=\frac{a}{|a|}$, and observe that

$$\left|\frac{\langle\mathbf{b}(x+|a|y)-\mathbf{b}(x),|a|\frac{b}{|b|}\rangle}{|a|^2}-\frac{\langle\mathbf{b}(x+|a|\frac{b}{|b|})-\mathbf{b}(x),|a|y\rangle}{|a|^2}\right|\leq\|\mathbf{b}\|_{R_0}$$

while

$$\left|\frac{\langle\mathbf{b}(x+b)-\mathbf{b}(x),a\rangle}{|a||b|}-\frac{\langle\mathbf{b}(x+|a|\frac{b}{|b|})-\mathbf{b}(x),|a|y\rangle}{|a|^2}\right|=\left|\langle\frac{\mathbf{b}(x+b)-\mathbf{b}(x)}{|b|}-\frac{\mathbf{b}(x+|a|\frac{b}{|b|})-\mathbf{b}(x)}{|a|},\frac{a}{|a|}\rangle\right|\tag{2.14}$$

In order to control this quantity, we use the auxiliary function $g:\mathbb{R}\to\mathbb{R}$ defined as $g(t)=\langle\frac{\mathbf{b}(x+tb)}{|b|},\frac{a}{|a|}\rangle$. Since $\mathbf{b}\in R_0$ implies $\mathbf{b}\in Z$, we deduce for each fixed t that

$$\begin{aligned}\left|\frac{g(t+s)+g(t-s)-2g(t)}{s}\right|&=\left|\frac{\mathbf{b}(x+(t+s)b)+\mathbf{b}(x+(t-s)b)-2\mathbf{b}(x+tb),a\rangle}{s|a||b|}\right|\\&\leq\|\mathbf{b}\|_Z\end{aligned}$$

so that $g\in Z$ with $\|g\|_Z\leq\|\mathbf{b}\|_Z\leq 4\|\mathbf{b}\|_{R_0}$. As a consequence, and arguing as in Reimann's proof of Proposition 5 (part C) we get for all $r\in\mathbb{R}$ that if $t,s>0$ then

$$\left|\frac{g(r+t)-g(r)}{t}-\frac{g(r+s)-g(r)}{s}\right|\leq\|g\|_Z\left(\frac{3}{2}+\frac{1}{2\log 2}\left|\log\frac{t}{s}\right|\right)$$

In particular, if $r=0$ and $s<0<t$,

$$\begin{aligned}\left|\frac{g(t)-g(0)}{t}-\frac{g(s)-g(0)}{s}\right|&\leq\left|\frac{g(t)-g(0)}{t}-\frac{g(-s)-g(0)}{-s}\right|+\left|\frac{g(-s)-g(0)}{-s}-\frac{g(s)-g(0)}{s}\right|\\&\leq\|g\|_Z\left(\frac{3}{2}+\frac{1}{2\log 2}\left|\log\frac{t}{-s}\right|\right)+\|g\|_Z\\&=\|g\|_Z\left(\frac{5}{2}+\frac{1}{2\log 2}\left|\log\frac{t}{-s}\right|\right).\end{aligned}$$

We now go back to (2.14), and apply the above estimate with $s = 1$, $t = \frac{|a|}{|b|}$. We obtain

$$\left| \langle \frac{\mathbf{b}(x+b) - \mathbf{b}(x)}{|b|}, \frac{a}{|a|} \rangle - \langle \frac{\mathbf{b}(x + \frac{|a|}{|b|} b) - \mathbf{b}(x)}{|a|}, \frac{a}{|a|} \rangle \right| = \left| \frac{g(t) - g(0)}{t} - \frac{g(s) - g(0)}{s} \right|$$
$$\leq 4 \|\mathbf{b}\|_{R_0} \left(\frac{5}{2} + \frac{1}{2 \log 2} \left| \log \frac{|a|}{|b|} \right| \right)$$

and the claim follows. □

As it was done in the previous sections for the classes $\bar{Q}$ and R, we are interested in a differential characterization of the class R_0. It is clear from Lemma 2.4.1 that if $\mathbf{b}$ is a smooth element of R_0 then

$$\|D\mathbf{b} - D^t \mathbf{b}\|_{L^\infty} \leq \|\mathbf{b}\|_{R_0}$$

However, this time the situation for a non necessarily smooth $\mathbf{b} \in R_0$ is more delicate than in the previous sections, because differentiability points may not even exist. Indeed, with $D\mathbf{b} - D^t\mathbf{b}$ there is not enough information to control all of $D\mathbf{b}$. Observe also that if $n = 2$ then R_0 is strictly larger than R.

Theorem 2.4.6. *Let $\mathbf{b} \in R_0$. Then the distribution $D\mathbf{b} - D^t\mathbf{b}$ is an element of $L^\infty(\mathbb{R}^n)$, and*

$$\|D\mathbf{b} - D^t \mathbf{b}\|_{L^\infty} \leq C(n) \, \|\mathbf{b}\|_{R_0}$$

for some constant $C(n)$ that depends only on n.

The proof of this result is very similar to that of Theorem 2.2.3. However, some special attention is needed to stop the argument at an earlier point.

Proof. We will first assume that $\mathbf{b}$ has compact support. Let us call $\mathbf{u} = P_y * \mathbf{b}$. We will write $\mathbf{u} = (u^1, \dots, u^n)$ and similarly $\mathbf{b} = (b^1, \dots, b^n)$. One immediately sees that $\partial_{x_i} b^j$ is a well defined distribution, because $\mathbf{b}$ has compact support. Moreover, this distribution can be easily extended to act against testing functions with polynomial decay, as for instance Poisson extensions of smooth compactly supported functions. So the action $\langle \partial_{x_i} b^j, P_y * \varphi \rangle$ is well defined whenever $\varphi \in C_c^\infty$. One has

$$\begin{aligned} \langle \partial_{x_i} u^j - \partial_{x_j} u^i, \varphi \rangle &= -\langle u^j, \partial_{x_i} \varphi \rangle + \langle u^i, \partial_{x_j} \varphi \rangle \\ &= -\langle P_y * b^j, \partial_{x_i} \varphi \rangle + \langle P_y * b^i, \partial_{x_j} \varphi \rangle \\ &= -\langle b^j, P_y * \partial_{x_i} \varphi \rangle + \langle b^i, P_y * \partial_{x_j} \varphi \rangle \\ &= -\langle b^j, \partial_{x_i} (P_y * \varphi) \rangle + \langle b^i, \partial_{x_j} (P_y * \varphi) \rangle \\ &= \langle \partial_{x_i} b^j, P_y * \varphi \rangle - \langle \partial_{x_j} b^i, P_y * \varphi \rangle \\ &= \langle \partial_{x_i} b^j - \partial_{x_j} b^i, P_y * \varphi \rangle \end{aligned}$$

In particular, we have the following equality of distributions,

$$\partial_{x_i} u^j - \partial_{x_j} u^i = \partial_{x_i} (P_y * b^j) - \partial_{x_j} (P_y * b^i) = P_y * (\partial_{x_i} b^j - \partial_{x_j} b^i) \tag{2.15}$$

which is equivalent to say that $D\mathbf{u} - D^t\mathbf{u} = P_y * (D\mathbf{b} - D^t\mathbf{b})$. The convolution operator $P_y *$ commutes with translations. Therefore it is not hard to see that

$$\mathbf{b} \in R_0 \qquad \Rightarrow \qquad \mathbf{u}(\cdot, y) \in R_0, \text{ and } \|\mathbf{u}(\cdot, y)\|_{R_0} \leq \|P_y\|_{L^1(\mathbb{R}^n)} \, \|\mathbf{b}\|_{R_0} = \|\mathbf{b}\|_{R_0}$$

where we used that $\|P_y\|_{L^1(\mathbb{R}^n)} = 1$. However, $\mathbf{u}$ is smooth. Thus, every point x is a differentiability point of $\mathbf{u}(\cdot, y)$, and therefore by Lemma 2.4.1

$$|D\mathbf{u}(x,y) - D^t\mathbf{u}(x,y)| \leq \|\mathbf{u}(\cdot,y)\|_{R_0} \leq \|\mathbf{b}\|_{R_0}. \tag{2.16}$$

In particular, this shows that each slice of $\partial_{x_i} u^j - \partial_{x_j} u^i$ belongs to $L^\infty(\mathbb{R}^n)$, and this happens uniformly in $y > 0$. We now show that one also has $\partial_i u^j - \partial_j u^i \in L^p(\mathbb{R}^n)$ for some $p \in (1,\infty)$. Indeed, from $\mathbf{u} = P_y * \mathbf{b}$ we see that $\partial_{x_i}\mathbf{u} = (\partial_{x_i} P_y) * \mathbf{b}$ and therefore

$$\begin{aligned}
|D\mathbf{u}(x,y)| &\leq C \int_{\operatorname{supp}\mathbf{b}} |DP_y(x-z)|\,|\mathbf{b}(z)|\,dz \\
&= C \int_{\operatorname{supp}\mathbf{b}} \frac{|(n+1)c_n\, y(x-z)|}{(y^2+|x-z|^2)^{\frac{n+3}{2}}}\,|\mathbf{b}(z)|\,dz \\
&= C \int_{\operatorname{supp}\mathbf{b}} \frac{|(n+1)c_n\, y(x-z)|}{y^2+|x-z|^2}\,\frac{|\mathbf{b}(z)|}{(y^2+|x-z|^2)^{\frac{n+1}{2}}}\,dz \leq c_n \int \frac{|\mathbf{b}(z)|}{|x-z|^{n+1}}\,dz
\end{aligned}$$

As a consequence, if $\operatorname{supp}\mathbf{b} \subset B(0,R)$ and $|x| > 2R$ then

$$|\partial_{x_i} u^j(x,y) - \partial_{x_j} u^i(x,y)| \leq \frac{c_n\,\|\mathbf{b}\|_{L^\infty}}{|x|^{n+1}}$$

From this, if $p > \frac{n}{n+1}$ then $\|\partial_{x_i} u^j(\cdot,y) - \partial_{x_j} u^i(\cdot,y)\|_{L^p(\mathbb{R}^n\setminus B(0,2R))}$ is bounded uniformly in y. Combining this fact with (2.16), one gets that

$$\sup_{y>0} \|\partial_{x_i} u^j(\cdot,y) - \partial_{x_j} u^i(\cdot,y)\|_{L^p(\mathbb{R}^n)} \leq C(n,R).$$

As a consequence, $\partial_{x_i} u^j - \partial_{x_j} u^i$ belongs to the Hardy space of harmonic functions $h^p(\mathbb{R}^{n+1}_+)$. As such, we can infer that there is $v_{i,j} \in L^p(\mathbb{R}^n)$ such that $\partial_{x_i} u^j - \partial_{x_j} u^i = P_y * v_{i,j}$ and moreover

$$\lim_{y\to 0} \|(\partial_{x_i} u^j - \partial_{x_j} u^i) - (v_{i,j})\|_{L^p(\mathbb{R}^n)} = 0.$$

In particular, there is a subsequence of heights $y_n \to 0$ for which the converge is pointwise,

$$\lim_{n\to\infty} \partial_{x_i} u^j - \partial_{x_j} u^i = v_{i,j} \qquad a.e. \tag{2.17}$$

which combined with (2.16) gives us that $v_{i,j} \in L^\infty(\mathbb{R}^n)$. Finally, since (2.15) holds for all testing functions $\varphi \in L^{p'}(\mathbb{R}^n)$, we see that

$$\lim_{y\to 0} \|P_y * (\partial_{x_i} b^j - \partial_{x_j} b^i) - v_{i,j}\|_{L^p(\mathbb{R}^n)} = 0$$

This forces $\|P_y * (\partial_{x_i} b^j - \partial_{x_j} b^i)\|_{L^p}$ to remain bounded as $y \to 0$, which in turn forces the distribution $\partial_{x_i} b^j - \partial_{x_j} b^i$ to belong to $L^p(\mathbb{R}^n)$, and therefore by Fatou's Theorem $v_{i,j} = \partial_{x_i} b^j - \partial_{x_j} b^i$ almost everywhere. Moreover, since $v_{i,j} \in L^\infty(\mathbb{R}^n)$ we also have $\partial_{x_i} b^j - \partial_{x_j} b^i \in L^\infty(\mathbb{R}^n)$, and

$$\|\partial_{x_i} b^j - \partial_{x_j} b^i\|_{L^\infty} = \|v_{i,j}\|_{L^\infty} \leq \sup_{y>0} \|\partial_{x_i} u^j - \partial_{x_j} u^i\|_{L^\infty} \leq \|P_y\|_1\,\|\mathbf{b}\|_{R_0}$$

so the claim follows.

In order to remove the assumption on $\operatorname{supp}\mathbf{b}$, we proceed as in Theorem 2.2.3. So we start by recalling that $\mathbf{b} \in R_0$ implies $\mathbf{b} \in Z$, whence

$$L = \limsup_{|x|\to\infty} \frac{|\mathbf{b}(x)|}{|x|\,\log|x|} < \infty.$$

Setting $\Delta_h\varphi(x) = \varphi(x+h) - \varphi(x)$ and $\tau_h\varphi(x) = \varphi(x+h)$ and taking $g = g_t$ as in (2.4) one has

$$\begin{aligned}
&\langle \Delta_h(g\mathbf{b}), k\rangle - \langle \Delta_k(g\mathbf{b}), h\rangle \\
&\qquad = \tau_h g\,\langle \Delta_h\mathbf{b}, k\rangle - \tau_k g\,\langle \Delta_k\mathbf{b}, h\rangle + \langle \mathbf{b}, k\rangle\,\Delta_h g - \langle \mathbf{b}, h\rangle\,\Delta_k g \\
&\qquad = \tau_h g\,(\langle \Delta_h\mathbf{b}, k\rangle - \langle \Delta_k\mathbf{b}, h\rangle) + (\tau_h g - \tau_k g)\,\langle \Delta_k\mathbf{b}, h\rangle + \langle \mathbf{b}, k\rangle\,\Delta_h g - \langle \mathbf{b}, h\rangle\,\Delta_k g
\end{aligned}$$

If $|x|$ is large, from the mean value theorem we see that

$$|\Delta_k g(x)| \le \frac{C|k|}{t|x|\,\log|x|}$$

$$|\Delta_h g(x)| \le \frac{C|h|}{t|x|\,\log|x|}$$

$$|\tau_h g(x) - \tau_k g(x)| \le \frac{C|h-k|}{t|x|\,\log|x|}$$

This, together with the growth of $\mathbf{b}$ at infinity, gives

$$\left|\frac{\Delta_h(g\mathbf{b}), k\rangle - \langle \Delta_k(g\mathbf{b}), h\rangle}{|h|\,|k|}\right| \le \|\mathbf{b}\|_{R_0} + \frac{C}{t}$$

and so $g\mathbf{b} \in R_0$ and has compact support. From the first part of the proof, we deduce that $D(g\mathbf{b}) - D^t(g\mathbf{b}) \in L^\infty$, with norm les than $\|g\mathbf{b}\|_{R_0}$. But from

$$D(g\mathbf{b}) - D^t(g\mathbf{b}) = \mathbf{b} \otimes \nabla g - \nabla g \otimes \mathbf{b} + g\,(D\mathbf{b} - D^t\mathbf{b})$$

one gets at points $|x| \le t$ that

$$D(g\mathbf{b}) - D^t(g\mathbf{b}) = D\mathbf{b} - D^t\mathbf{b}$$

whence for $|x| \le t$ one has

$$\begin{aligned}
|D\mathbf{b}(x) - D^t\mathbf{b}(x)| &\le \|D(g\mathbf{b}) - D^t(g\mathbf{b})\|_{L^\infty} \\
&\le \|g\mathbf{b}\|_{R_0} \\
&\le \|\mathbf{b}\|_{R_0} + \frac{C}{t} \le C(n)\|D\mathbf{b} - D^t\mathbf{b}\|_{L^\infty} + \frac{C}{t}.
\end{aligned}$$

The proof finishes by letting $t \to \infty$. $\square$

Theorem 2.4.6 provides a sufficient condition for L^∞ bounds for the distributional curl $D\mathbf{b} - D^t\mathbf{b}$. It says nothing about the differentiability of $\mathbf{b}$, nor the total pointwise differential $D\mathbf{b}$. For instance, if $u \in W^{1,1}(\mathbb{R}^n)$ and $\mathbf{b} = \nabla u$, then $D\mathbf{b} - D^t\mathbf{b} = 0$ in the sense of distributions, but $\mathbf{b}$ may not be differentiable almost everywhere. That is, zero curl does not imply pointwise differentiability a.e.. Hence, at this point it is not clear why should any $\mathbf{b} \in R_0$ be differentiable almost everywhere. This absence of regularity makes it harder to state Theorem 2.4.6 in the same terms we stated Theorems 2.2.3 and 2.3.3 above. We solve this obstruction in the following result, which is a slight modification of Theorem 2.4.6. It refers to a slightly smaller subclass of R_0, given in terms of the divergence $\operatorname{div}\mathbf{b}$.

Theorem 2.4.7. *Let $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ belong to the class R_0. Assume that $\operatorname{div}\mathbf{b} \in L^p_{loc}(\mathbb{R}^n)$.*

- *If $1 < p < \infty$, then $\mathbf{b}$ has $L^p_{loc}(\mathbb{R}^n)$ distributional derivatives, and $\operatorname{curl}\mathbf{b} \in L^\infty(\mathbb{R}^n)$.*
- *If $p > n$, then one further has that $\mathbf{b}$ is differentiable almost everywhere.*

Proof. Let us first assume that $\mathbf{b}$ has compact support. If $\mathbf{b} \in R_0$ then we know from Theorem 2.4.6 that $D\mathbf{b} - D^t\mathbf{b} \in L^\infty$. Then, from the compact support we deduce that $D\mathbf{b} - D^t\mathbf{b} \in L^p$, and from $\operatorname{div}\mathbf{b} \in L^p$ and Lemma 1.1.10 we get that $D\mathbf{b} \in L^p$. The rest is standard real analysis. If $\mathbf{b}$ has not compact support, then using the functions g_t from (2.4) we see that $g_t\mathbf{b}$ is an element of R_0 with L^p divergence, so using again Lemma 1.1.10 we get that $g_t\mathbf{b}$ has $L^p(\mathbb{R}^n)$ derivatives, which in turn ensures $D\mathbf{b} \in L^p_{loc}$. The differentiability a.e. is a well known result of classical real analysis, see for instance [25]. □

We also obtain the following counterpart to Theorem 2.4.7 in $\mathbb{R}^n$ for the case $p = \infty$. It states that vector fields in R_0 with bounded divergence must necessarily have BMO derivatives and bounded curl. Let us recall that

$$A\mathbf{b} = \frac{D\mathbf{b} - D^t\mathbf{b}}{2} + \frac{\operatorname{div}\mathbf{b}}{n}\ \mathbf{Id}\,.$$

It can also be seen as a counterpart to [43, Proposition 15], as well as to Theorems 2.2.3 and 2.3.3 above.

Corollary 2.4.8. *Let $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ belong to the class R_0. Assume that $\operatorname{div}\mathbf{b} \in L^\infty(\mathbb{R}^n)$. Then $\mathbf{b}$ is differentiable almost everywhere, has $BMO(\mathbb{R}^n)$ distributional derivatives, and $A\mathbf{b} \in L^\infty(\mathbb{R}^n)$.*

Proof. We first assume that that $\mathbf{b}$ has compact support. Having also that $\mathbf{b} \in R_0$, we proved in Theorem 2.4.6 that also $\operatorname{curl}\mathbf{b} \in L^\infty$. It then follows from Lemma 1.1.10 that $D\mathbf{b} \in BMO$, and so the differentiability a.e. is automatic. The boundedness of $A\mathbf{b}$ is immediate. The proof for non compactly supported $\mathbf{b}$ goes similarly, since $g_t\mathbf{b}$ is compactly supported and also $g_t\mathbf{b} \in R_0$. □

In the converse direction, we have the following result, which establishes a much better counterpart to [43, Proposition 12] or Theorems 2.2.4 or 2.3.4.

Theorem 2.4.9. *Let $\mathbf{b} \in W^{1,1}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ be a vector field. Assume that $\mathbf{b}$ is continuous, and that $|\mathbf{b}(x)| \leq O(|x|\log|x|)$ as $|x| \to \infty$. If there exists a constant $C > 0$ such that*

$$\|A\mathbf{b}\|_{L^\infty} \leq C$$

then $\mathbf{b}$ belongs to the R_0 class, and $\|\mathbf{b}\|_{R_0} \leq C'$ for some constant C' depending only on C.

Proof. Let us remind that $A\mathbf{b} \in L^\infty$ gives us bounds for $\operatorname{div}\mathbf{b}$ and $\operatorname{curl}\mathbf{b}$ in the L^∞ norm. Again, we follow the steps in the proof of Theorem 2.2.4, so we will first assume that $\mathbf{b}$ has compact support, and later on will remove this assumption. Given $\alpha, \beta \in \mathbb{R}^n$, $|\alpha| = |\beta| = 1$, set $a = \alpha h, b = \beta h$ for some $h > 0$. For each $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$, define

$$\Delta\mathbf{g}(x) = \Delta_{a,b}\mathbf{g}(x) = \langle \mathbf{g}(x+a) - \mathbf{g}(x), \beta\rangle - \langle \mathbf{g}(x+b) - \mathbf{g}(x), \alpha\rangle.$$

Clearly, $\Delta = \Delta_{a,b}$ is a linear operator in $\mathbf{g}$, and

$$|\Delta\mathbf{g}(x)| \leq 4\, \|\mathbf{g}\|_{L^\infty} \tag{2.18}$$

Moreover, $\mathbf{g}$ belongs to the class R_0 if and only if $|\Delta\mathbf{g}| \leq C\, h$ for some constant C that does not depend on a or b. Using that $D\mathbf{g} = S\mathbf{g} + A\mathbf{g}$, we can represent $\Delta\mathbf{g}$ as follows,

$$\begin{aligned}\Delta\mathbf{g}(x) &= \int_0^h \frac{d}{ds}\Big(\langle \mathbf{g}(x+\alpha s), \beta\rangle - \langle \mathbf{g}(x+\beta s), \alpha\rangle \Big)\, ds \\ &= \int_0^h \langle D\mathbf{g}(x+\alpha s)\, \alpha, \beta\rangle - \langle D\mathbf{g}(x+\beta s)\, \beta, \alpha\rangle\, ds = \Delta_S\mathbf{g}(x) + \Delta_A\mathbf{g}(x)\end{aligned}$$

with

$$\begin{aligned}\Delta_S\mathbf{g}(x) &= \int_0^h \langle S\mathbf{g}(x+\alpha s)\, \alpha, \beta\rangle - \langle S\mathbf{g}(x+\beta s)\, \beta, \alpha\rangle\, ds \\ \Delta_A\mathbf{g}(x) &= \int_0^h \langle A\mathbf{g}(x+\alpha s)\, \alpha, \beta\rangle - \langle A\mathbf{g}(x+\beta s)\, \beta, \alpha\rangle\, ds\end{aligned}$$

By construction, $\mathbf{u}(x,y) = P_y * \mathbf{b}(x)$ is harmonic in $\mathbb{R}^{n+1}_+$ and continuous up to the boundary, since $\mathbf{b} \in C_c(\mathbb{R}^n)$. For each $t > 0$,

$$\begin{aligned}\mathbf{b}(x) = \mathbf{u}(x,0) &= \int_0^y t\, \partial^2_{yy}\mathbf{u}(x,t)\, dt - y\, \partial_y\mathbf{u}(x,y) + \mathbf{u}(x,y) \\ &\equiv \int_0^y t\, \mathbf{w}_t(x)\, dt - y\;\mathbf{v}_y(x) + \mathbf{u}_y(x)\end{aligned}$$

where we wrote $\mathbf{u}_y(x) = \mathbf{u}(x,y)$, $\mathbf{v}_y(x) = \partial_y\mathbf{u}(x,y)$ and $\mathbf{w}_r(x) = \partial^2_{yy}\mathbf{u}(x,r)$. By the linearity of Δ, which acts only on the x variable, one has

$$\Delta\mathbf{b}(x) = \int_0^y t\, \Delta\mathbf{w}_t(x)\, dt - y\, \Delta\,\mathbf{v}_y(x) + \Delta\mathbf{u}_y(x).$$

We now proceed term by term. First, from Lemma 1.1.10 we know that $A\mathbf{b} \in L^\infty$ implies $D\mathbf{b} \in BMO$, which in turn gives $\mathbf{b} \in Z$. Hence, from Lemma 1.1.9,

$$\left| \int_0^y t\, \Delta\mathbf{w}_t(x)\, dt \right| \leq \int_0^y t\, 4\|\mathbf{w}_t\|_{L^\infty}\, dt \leq \int_0^y t\, 4\, \frac{C(n)\, \|\mathbf{b}\|_Z}{t}\, dt = C(n)\, y\, \|\mathbf{b}\|_Z.$$

For the second term, we use that $\Delta = \Delta_S + \Delta_A$,

$$\begin{aligned}y\, \Delta\, \mathbf{v}_y(x) &= y\, \Delta_S\, \mathbf{v}_y(x) + y\, \Delta_A\, \mathbf{v}_y(x) \\ \Delta\mathbf{u}_y(x) &= \Delta_S\mathbf{u}_y(x) + \Delta_A\mathbf{u}_y(x)\end{aligned}$$

and proceed first with the Δ_A terms. For each fixed y, Lemma 1.1.6 gives us that

$$\begin{aligned}\partial_{x_i}\mathbf{u}_y = \partial_{x_i}(P_y * \mathbf{b}) = P_y * (\partial_{x_i}\mathbf{b}) &\Longrightarrow A\mathbf{u}_y = P_y * A\mathbf{b} \\ &\Longrightarrow \|A\mathbf{u}_y\|_{L^\infty} = \|P_y * A\mathbf{b}\|_{L^\infty} \le \|P_y\|_1 \|A\mathbf{b}\|_{L^\infty} = \|A\mathbf{b}\|_{L^\infty}\end{aligned}$$

On the other hand, since $\mathbf{u}$ is smooth, we can argue similarly to get that

$$\begin{aligned}\partial_{x_i}\mathbf{v}_y = \partial^2_{y,x_i}\mathbf{u} = \partial_y\,(P_y * \partial_{x_i}\mathbf{b}) &\Longrightarrow A\,\mathbf{v}_y = \partial_y(P_y * A\mathbf{b}) \\ &\Longrightarrow \|A\,\mathbf{v}_y\|_{L^\infty} \le C(n)\,\frac{\|A\mathbf{b}\|_{L^\infty}}{y}.\end{aligned}$$

Thus

$$\begin{aligned}|\Delta_A\mathbf{u}_y(x)| &\le 2h\,\|A\mathbf{u}_y\|_{L^\infty} \le 2h\,\|A\mathbf{b}\|_{L^\infty} \\ |y\,\Delta_A\,\mathbf{v}_y(x)| &\le 2hy\|A\,\mathbf{v}_y\,\|_{L^\infty} \le C(n)\,h\,\|A\mathbf{b}\|_{L^\infty}\end{aligned}$$

for some dimensional constant $C(n)$. Now is time to proceed with the Δ_S terms. For any function $\mathbf{g}$, set

$$(S\mathbf{g})_{\alpha,\beta}(x) = \langle S\mathbf{g}(x)\cdot\alpha, \beta\rangle.$$

Using that $S\mathbf{g}$ is a symmetric matrix, and calling $\gamma = \frac{\alpha-\beta}{|\alpha-\beta|}$,

$$\begin{aligned}\langle S\mathbf{g}(x+\alpha s)\,\alpha,\beta\rangle - \langle S\mathbf{g}(x+\beta s)\,\beta,\alpha\rangle &= \langle\alpha,(S\mathbf{g}(x+\alpha s) - S\mathbf{g}(x+\beta s))\,\beta\rangle \\ &= \langle\alpha, \left(\int_0^{s|\alpha-\beta|}\frac{d}{d\sigma}(S\mathbf{g}(x+\beta s+\sigma\gamma))\,d\sigma\right)\beta\rangle \\ &= \int_0^{s|\alpha-\beta|}\frac{d}{d\sigma}\Big(\langle\alpha, S\mathbf{g}(x+\beta s+\sigma\gamma)\,\beta\rangle\Big)d\sigma\end{aligned} \tag{2.19}$$

Therefore

$$\begin{aligned}|\Delta_S\mathbf{g}(x)| &\le \int_0^h\int_0^{s|\alpha-\beta|}\left|\frac{d}{d\sigma}(S\mathbf{g})_{\alpha,\beta}(x+\beta s+\sigma\gamma)\right|d\sigma\,ds \\ &\le \int_0^h\int_0^{s|\alpha-\beta|}|D((S\mathbf{g})_{\alpha,\beta})(x+\beta s+\sigma\gamma)|\,d\sigma\,ds \\ &\le \|D((S\mathbf{g})_{\alpha,\beta})\|_{L^\infty}\int_0^h\int_0^{s|\alpha-\beta|}d\sigma\,ds = \|D((S\mathbf{g})_{\alpha,\beta})\|_{L^\infty}\,\frac{h^2\,|\alpha-\beta|}{2}\end{aligned}$$

After applying this to $\mathbf{g} = \mathbf{u}_y$ and to $\mathbf{g} = \mathbf{v}_y$, one obtains

$$|\Delta\mathbf{b}(x)| \le C(n)\,y\,\|\mathbf{b}\|_Z + C(n)\,h\,\|A\mathbf{b}\|_{L^\infty} + \frac{h^2\,|\alpha-\beta|}{2}\,(\|D((S\mathbf{u}_y)_{\alpha,\beta})\|_{L^\infty} + y\|D((S\,\mathbf{v}_y)_{\alpha,\beta})\|_{L^\infty}) \tag{2.20}$$

Next, we see that

$$\begin{aligned}\mathbf{u}_y = P_y * \mathbf{b} \quad &\Rightarrow \quad D\mathbf{u}_y = P_y * D\mathbf{b} \\ &\Rightarrow \quad S\mathbf{u}_y = P_y * S\mathbf{b} \\ &\Rightarrow \quad (S\mathbf{u}_y)_{\alpha,\beta} = P_y * (S\mathbf{b})_{\alpha,\beta}.\end{aligned}$$

Now, since $D\mathbf{b} \in BMO$ we have in particular that $(S\mathbf{b})_{\alpha,\beta} \in BMO$, in particular $(S\mathbf{u}_y)_{\alpha,\beta}$ is harmonic Bloch. Lemma 1.1.7 with $\mathbf{g} = (S\mathbf{u}_y)_{\alpha,\beta}$ gives us that

$$\|D((S\mathbf{u}_y)_{\alpha,\beta})\|_{L^\infty} = \|D(P_y * (S\mathbf{b})_{\alpha,\beta})\|_{L^\infty} \leq \frac{C(n)\,\|(S\mathbf{b})_{\alpha,\beta}\|_*}{y} \leq \frac{C(n)\,\|S\mathbf{b}\|_*}{y}. \tag{2.21}$$

For $\mathbf{g} = (S\,\mathbf{v}_y)_{\alpha,\beta}$, we proceed similarly and note that

$$\begin{aligned} \mathbf{v}_y = \partial_y \mathbf{u}_y = \partial_y P_y * \mathbf{b} \qquad &\Rightarrow \qquad D\,\mathbf{v}_y = \partial_y P_y * D\mathbf{b} \\ &\Rightarrow \qquad S\,\mathbf{v}_y = \partial_y P_y * S\mathbf{b} \\ &\Rightarrow \qquad (S\,\mathbf{v}_y)_{\alpha,\beta} = \partial_y P_y * (S\mathbf{b})_{\alpha,\beta}. \end{aligned}$$

Therefore one can combine Lemma 1.1.7 and Lemma 1.1.8 and obtain

$$\|D((S\,\mathbf{v}_y)_{\alpha,\beta})\|_{L^\infty} = \|D(\partial_y P_y * (S\mathbf{b})_{\alpha,\beta})\|_{L^\infty} \leq \frac{C(n)\,\|(S\mathbf{b})_{\alpha,\beta}\|_*}{y^2} \leq \frac{C(n)\|S\mathbf{b}\|_*}{y^2} \tag{2.22}$$

It is worth mentioning here that both in (2.21) and (2.22) one could replace the constant $\|S\mathbf{b}\|_*$ by $\|\mathbf{b}\|_Z$ (note that $\|\mathbf{b}\|_Z \leq C\,\|Sb\|_*$). To do this, one only needs to use Lemma 1.1.9 instead of Lemma 1.1.7. We now plug the above bounds for $\|D((S\mathbf{u}_y)_{\alpha,\beta})\|_{L^\infty}$ and $\|D((S\,\mathbf{v}_y)_{\alpha,\beta})\|_{L^\infty}$ into (2.11), and then take $h = y$. This finishes the proof in the case $\mathbf{b} \in C_c(\mathbb{R}^n)$.
In order to remove the assumption on the compact support, we use once more the auxiliary function $g = g_t$ introduced at (2.4). We have

$$D(g\mathbf{b}) - D^t(g\mathbf{b}) = \mathbf{b} \otimes \nabla g - \nabla g \otimes \mathbf{b} + g\,(D\mathbf{b} - D^t\mathbf{b})$$

so

$$\begin{aligned} \|D(g\mathbf{b}) - D^t(g\mathbf{b})\|_{L^\infty} &\leq \|D\mathbf{b} - D^t\mathbf{b}\|_{L^\infty} + \sup_{t \leq |x| \leq t^{e^t}} |\mathbf{b}(x)||\nabla g(x)| \\ &\leq \|D\mathbf{b} - D^t\mathbf{b}\|_{L^\infty} + \sup_{t \leq |x| \leq t^{e^t}} C\,|x|\log|x| \frac{1}{t|x|\,\log|x|} \\ &\leq \|D\mathbf{b} - D^t\mathbf{b}\|_{L^\infty} + \frac{C}{t} \end{aligned}$$

Now the claim follows since for every $x \in \mathbb{R}^n$ we can pick $t > 0$ large enough and such that $|x| < t^{e^t}$ so that $\mathbf{b} = g\mathbf{b}$ in a neighbourhood of x, and therefore $D\mathbf{b} - D^t\mathbf{b} = D(g\mathbf{b}) - D^t(g\mathbf{b})$. □

Chapter 3

Rotational bounds for homeomorphisms with integrable distortion and Hölder continuous inverse

3.1 Introduction

Recently there has been a growing interest in understanding the rotational properties of planar homeomorphisms, see [7, 14, 30, 31, 32, 33]. Special attention has been devoted to the spiraling rate of these maps. More precisely, given a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ normalized by $f(0) = 0$ and $f(1) = 1$, one is interested in the growth of $|\arg(f(r))|$ as $r \to 0$. This growth represents the number of times that the image $f([r, 1])$ winds around the origin as $r \to 0$. This quantity has been proven to admit several speeds of growth which depend on the class of maps under study. In this way, it was proven in [7] that if f is K-quasiconformal then

$$|\arg(f(r))| \leq \frac{1}{2}\left(K - \frac{1}{K}\right)\log\left(\frac{1}{r}\right) + c_K, \qquad \text{for all } 0 < r < 1. \tag{3.1}$$

In contrast, if the maps under study are homeomorphisms of finite distortion, the situation changes and the order of growth depends on the integrability of the distortion function. Namely, Hitruhin discovered in [31] that if $e^{\mathbb{K}(\cdot,f)} \in L^p_{loc}$ for some $p > 0$ then

$$|\arg(f(z))| \leq \frac{c}{p}\log^2\left(\frac{1}{|z|}\right), \qquad \text{for small enough } |z|,$$

and moreover this is sharp up to the value of the constant $c > 0$. In other words, the transition between boundedness and exponential integrability of $\mathbb{K}(\cdot, f)$ results in a larger power of the logarithmic term. Further optimal results were obtained later on in [32], in the case of integrable distortion, that is, when $\mathbb{K}(\cdot, f) \in L^p_{loc}$ for some $p > 1$,

$$|\arg(f(z))| \leq \frac{c}{|z|^{\frac{2}{p}}}, \qquad \text{for small enough } |z| \tag{3.2}$$

or even if $\mathbb{K}(\cdot, f) \in L^1_{loc}$,

$$\lim_{|z|\to 0} |z|^2 \, |\arg(f(z))| = 0. \tag{3.3}$$

The moral here is that more spiraling is allowed at the cost of relaxing the integrability properties of $\mathbb{K}(\cdot, f)$. As explained in [7, 31, 32], the local rotational properties go hand in hand with the local stretching behavior. Especially important for the argument are the estimates for the modulus of continuity of the inverse map.

It turns out mappings of finite distortion also have a role in fluid mechanics. To be precise, let us think of the planar incompressible Euler system of equations in vorticity form,

$$\begin{cases} \frac{d}{dt}\omega + (\mathbf{v} \cdot \nabla)\omega = 0 \\ \operatorname{div}(\mathbf{v}) = 0 \\ \omega(0, \cdot) = \omega_0. \end{cases} \tag{3.4}$$

Here $\omega = \omega(t, z) : [0, T] \times \mathbb{C} \to \mathbb{C}$ is the unknown, $\omega_0 \in L^\infty(\mathbb{C}; \mathbb{C})$ is given, and $\mathbf{v}$ is the velocity field. The Biot-Savart law,

$$\mathbf{v} = \frac{i}{2\pi \bar{z}} * \omega$$

makes more precise the relation between $\mathbf{v}$ and ω. As it is well known, Yudovich [48] proved existence and uniqueness of a solution $\omega \in L^\infty([0, T]; L^\infty(\mathbb{C}; \mathbb{C}))$ for any given ω_0. In particular, the corresponding velocity field $\mathbf{v}$ belongs to the Zygmund class, and therefore the classical Cauchy-Lipschitz theory guarantees for the ODE

$$\begin{cases} \frac{d}{dt} X(t, z) = \mathbf{v}(t, X(t, z)) \\ X(0, z) = z \end{cases}$$

both existence and uniqueness of a flow map $X : [0, T] \times \mathbb{C} \to \mathbb{C}$. It was proven in [23] that, for small enough $t > 0$, each of the flow homeomorphisms $X_t = X(t, \cdot) : \mathbb{C} \to \mathbb{C}$ is indeed a mapping of finite distortion. Moreover, for each small value $t > 0$ there is a number $p(t) > 1$ such that the distortion function $\mathbb{K}(\cdot, X_t)$ belongs to L^p_{loc} whenever $p < p(t)$.

As mappings with L^p distortion, the mappings X_t are a bit special because both X_t and X_t^{-1} are Hölder continuous, as shown in [47], with a Hölder exponent that decays exponentially in time. This is not true in general, and mappings of L^p distortion need not have a Hölder continuous inverse, as shown in [38]. Therefore, it is a question of interest to find out if the Hölder nature of the inverse map results in better rotation bounds. Indeed, even though the bounds obtained in [32] can be applied to X_t, the Hölder continuous nature of X_t^{-1} provides a significant improvement to (3.2). We describe this improvement in our next Theorem.

Theorem 3.1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion such that $f(0) = 0$ and $f(1) = 1$, and assume that $\mathbb{K}(\cdot, f) \in L^p_{loc}$ for some $p > 1$. Suppose also that*

$$|f(x) - f(y)| \geq C\,|x - y|^\alpha, \qquad \text{if } |x - y| \text{ is small,}$$

for some $\alpha > 1$. Then

$$|\arg(f(z))| \leq C\,\sqrt{\alpha}\,|z|^{-\frac{1}{p}}\, \log^{\frac{1}{2}}\left(\frac{1}{|z|}\right) \tag{3.5}$$

whenever $|z|$ is small enough.

In contrast with (3.2) and (3.3), the existence of a Hölder continuous inverse allows the power term exponent to be halved, although then the logarithmic term needs to be included.

As an application, we can estimate the spiraling rate of X_t for small times. The rotational behavior of X_t is nowadays studied a lot. For instance, in the case of ω_0 being *close* to the characteristic function of the unit disk, the article [19] provides bounds for the winding number of most of the trajectories $\{X_t(z)\}_{t>0}$ as $t \to \infty$. Here, instead, we do not evaluate the rotational behavior at large times, but look instead at spiraling bounds *in the space variable* for a fixed and small enough time.

Corollary 3.1.2. *Given $\omega_0 \in L^\infty(\mathbb{C};\mathbb{C})$, let $\mathbf{v}$ be the velocity field of Yudovich's solution to* (3.4)*, and let X_t be its flow. Then there is a constant $C > 0$ such that*

$$\left|\arg\left(\frac{X_t(z) - X_t(0)}{X_t(1) - X_t(0)}\right)\right| \leq C\, \log^{\frac{1}{2}}\left(\frac{1}{|z|}\right)\, |z|^{-t\|\omega_0\|_\infty}\, \exp\left(Ct\|\omega_0\|_\infty\right)$$

if both $|z|$ and $t > 0$ are small enough.

In particular, if one fixes a time $t_0 > 0$ small enough, then the curve $X_{t_0}([\frac{1}{n}, 1])$ cannot wind around $X_t(0)$ more than an integral multiple of

$$n^{t_0\|\omega_0\|_\infty}\, (\log n)^{\frac{1}{2}}\, e^{Ct_0\|\omega_0\|_\infty}$$

times. Towards the optimality of Theorem 3.1.1, we can show the following.

Theorem 3.1.3. *Given an increasing, onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, and a real number $p > 1$, there exists a homeomorphism $\bar{f} : \mathbb{C} \to \mathbb{C}$ with the following properties:*

- *$\bar{f}$ is a mapping of finite distortion, with $\mathbb{K}(\cdot, \bar{f}) \in L^p_{loc}$.*
- *$\bar{f}(0) = 0$, $\bar{f}(1) = 1$.*
- *If $\alpha > \frac{3p}{p-1}$, then $|\bar{f}(x) - \bar{f}(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$.*
- *There exists a decreasing sequence $\{r_n\}$, with $r_n \to 0+$ as $n \to \infty$, for which*

$$|\arg(\bar{f}(r_n))| \geq r_n^{-\frac{1}{p}}\, \log^{\frac{1}{2}}\left(\frac{1}{r_n}\right)\, h(r_n).$$

Since h can be chosen to approach 0 at any speed, Theorem 3.1.3 shows that the order provided in Theorem 3.1.1 is sharp.

Next, we extend Theorem 3.1.1 and Theorem 3.1.3 to a more general class of homeomorphisms, which have L^p_{loc} distortion for $p \geq 1$ and the inverse having predetermined modulus of continuity.

Theorem 3.1.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion such that $f(0) = 0$, $f(1) = 1$, and assume that $\mathbb{K}(\cdot, f) \in L^p_{loc}$; $p > 1$. Then*

$$|\arg\left(f(z)\right)| \leq C\, |z|^{-\frac{1}{p}}\, \log^{\frac{1}{2}}\left(\frac{1}{\min_{|\omega|=|z|} |f(\omega)|}\right) \quad \text{when } |z| \text{ is small.} \tag{3.6}$$

Furthermore, if we assume that $\mathbb{K}(\cdot, f) \in L^1_{loc}$, then

$$\limsup_{|z|\to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{\min\limits_{|\omega|=|z|}|f(\omega)|}\right)}} |\arg(f(z))| = 0. \tag{3.7}$$

Towards the optimality of Theorem 3.1.4, we can show the following.

Theorem 3.1.5. *Let φ be a radially increasing homeomorphism with p-integrable distortion, $p \geq 1$, such that*

$$e^{-g_{\varphi,p}(|z|)|z|^{-\frac{2}{p}}} \leq |\varphi(z)| < |z|^4 \quad \text{when } |z| \text{ is small,} \tag{3.8}$$

where $g_{\varphi,p} : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function with $g(r) \to 0$ when $r \to 0$. Then we can choose an increasing onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, which can converge to zero as slow as we want, and find a homeomorphism $\bar{f} : \mathbb{C} \to \mathbb{C}$ with the following properties:

- *$\bar{f}$ is a homeomorphism of finite distortion, with $\mathbb{K}(\cdot, \bar{f}) \in L^p_{loc}$.*
- *$\bar{f}(0) = 0$, $\bar{f}(1) = 1$.*
- *There exists a decreasing sequence $\{r_n\}$, such that*

$$|\bar{f}(r_n)| = |\varphi(r_n)| \tag{3.9}$$

and

$$\left|\arg\left(\bar{f}(r_n)\right)\right| \geq r_n^{-\frac{1}{p}} \log^{\frac{1}{2}}\left(\frac{1}{|\bar{f}(r_n)|}\right) h(r_n). \tag{3.10}$$

Note that the homeomorphism $\bar{f}$ in Theorem 3.1.5 is radial and hence $\min\limits_{|\omega|=|z|}|\bar{f}(\omega)| = |\bar{f}(z)|$. Since h can be chosen to approach zero at any speed and sequence r_n can be chosen freely, Theorem 3.1.5 shows that the upper bound provided in Theorem 3.1.4 is essentially sharp when we restrict modulus to satisfy (3.8).

At this point, we provide some brief explanation for the bounds (3.8). The one on the right specifies that we are studying mappings that compress stronger than Hölder maps, and thus have faster maximal spiraling rate than given in (3.5). On the other hand, the bound on the left is always satisfied when $p = 1$, see [32], and when $p > 1$ it is exact up to the gauge function $g_{\varphi,p}$, see [38]. Studying rotation under extremal compression leads to the extremal pointwise spiraling as shown in [32]. Thus Theorem 3.1.5, together with examples in [32] proving optimality of the extremal spiraling rate (3.2), show that whenever mapping f is compressing we have essentially sharp spiraling rates.

As a Corollary to Theorem 3.1.4 we can extend Theorem 3.1.1 to borderline situation $p = 1$.

Corollary 3.1.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion such that $f(0) = 0$ and $f(1) = 1$, and assume that $\mathbb{K}(\cdot, f) \in L^1_{loc}$. Moreover, let us suppose that*

$$|f(x) - f(y)| \geq C\,|x - y|^{\alpha} \qquad \text{if } |x - y| \text{ is small,}$$

for some $\alpha \geq 1$. Then

$$\limsup_{|z|\to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{|z|}\right)}} |\arg(f(z))| = 0. \tag{3.11}$$

Note that in the case $p = 1$ we get an improvement in the form of vanishing limsup compared to the case $p > 1$, which is described by the bound (3.5). This is analogous to the maximal spiraling bounds (3.2) and (3.3), where the exact same improvement happens.

Finally, we prove the optimality of the above result in a strong sense.

Theorem 3.1.7. *Given an increasing, onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, an arbitrary $\epsilon > 0$ and a real number $\beta \geq 1$, there exists a homeomorphism $\bar{f} : \mathbb{C} \to \mathbb{C}$ with the following properties:*

(a) $\bar{f}$ is a mapping of finite distortion, with $\mathbb{K}(\cdot, \bar{f}) \in L^1_{loc}$

(b) $\bar{f}(0) = 0$, $\bar{f}(1) = 1$

(c) If $\alpha \geq 2(\beta + 2) + \epsilon$, then $|\bar{f}(x) - \bar{f}(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$.

(d) There exists a decreasing sequence $\{r_n\}$, with limit $r_n \to 0^+$ as $n \to \infty$, for which

$$|\arg(\bar{f}(r_n))| \geq \frac{h(r_n)}{r_n} \left(\beta \log\left(\frac{1}{r_n}\right)\right)^{\frac{1}{2}}.$$

Towards the proof of Theorem 3.1.7, we modify the construction from Theorem 3.1.3 giving optimality for the bound (3.5). However, this construction as written in the case of Theorem 3.1.3 does not cover the case $\mathbb{K} \in L^1_{loc}$, and thus some changes in the argument are necessary. Also we note that Corollary 3.1.6 is extremely sharp as the homeomorphism h in Theorem 3.1.7 can go to zero as slow as we wish.

The chapter is structured as follows. In Section 3.2 we prove the positive theorems and the optimal results in Section 3.3.

3.2 Spiraling bounds

We will write Theorem 3.1.1 in the following, clearly equivalent, form.

Theorem 3.2.1. *Let f be a homeomorphism of finite distortion with distortion $\mathbb{K}(\cdot, f) \in L^p(\mathbb{C})$, $p > 1$, normalized by $f(0) = 0$ and $f(1) = 1$. Assume that it satisfies the following condition,*

$$|f(x) - f(y)| \geq C|x - y|^\alpha$$

whenever $|x - y|$ is small. Then the winding number $n(z_0)$ of the image of the line segment $\left[z_0, \frac{z_0}{|z_0|}\right]$ around the image of the origin is bounded from above by

$$n(z_0) \leq C\sqrt{\alpha}\, |z_0|^{-\frac{1}{p}} \log^{\frac{1}{2}}\left(\frac{1}{|z_0|}\right)$$

Proof. We would like to prove this theorem using the modulus inequality for homeomorphisms of finite distortion (1.9) following the presentation in [32]. At first, we would like to estimate the modulus term $M_{\mathbb{K}(\cdot,f)}(\Gamma)$ from above. To this end, let us choose an arbitrary point $z_0 \in \mathbb{C} \setminus \{0\}$ such that $|z_0| < 1$. Without loss of generality, we may assume that z_0 lies on the positive side of the real axis. Next, let us fix the line segments $E = [z_0, 1]$ and $F = (-\infty, 0]$, and Γ be the family of paths connecting a point in E to a point in F. Also, let us fix balls $B_j = (2^j z_0, 2^j z_0)$, $j \in \{0, 1, ..., n\}$ and let n be the smallest positive integer such that $2^n z_0 \geq 1$. Define

$$\rho_0(z) = \begin{cases} \frac{2}{r(B_0)} & \text{if } \ z \in B_0 \\ \frac{2}{r(B_1)} & \text{if } \ z \in B_1 \setminus B_0 \\ \vdots & \\ \frac{2}{r(B_n)} & \text{if } \ z \in B_n \setminus B_{n-1} \\ 0 & \text{otherwise} \end{cases}$$

Note that any $z \in E$ belongs to some ball $\frac{1}{2}B_j$ and that $\rho_0(z) \geq \frac{2}{r(B_j)}$, whenever $z \in B_j$. This implies, since $B_j \cap F = \emptyset$ for every j, that $\rho_0(z)$ is admissible with respect to Γ. Hence we can estimate the modulus from above by

$$\begin{aligned} M_{\mathbb{K}(\cdot,f)}(\Gamma) &= \inf_{\rho \ admissible} \int_{\mathbb{C}} \mathbb{K}(\cdot, f)\rho^2(z)\, dA(z) \\ &\leq \int_{\mathbb{C}} \mathbb{K}(\cdot, f)\rho_0^2(z)\, dA(z) \\ &\leq \|\mathbb{K}(\cdot, f)\|_{L^p(B(0,4))} \left(\int_{B(0,4)} \rho_0^{\frac{2p}{p-1}}(z)\, dA(z) \right)^{\frac{p-1}{p}} \\ &\leq c_{f,p} \left(\int_{B(0,4)} \rho_0^{\frac{2p}{p-1}}(z)\, dA(z) \right)^{\frac{p-1}{p}} \end{aligned}$$

Let us now estimate the integral term by using the definition of ρ_0.

$$\begin{aligned} \int_{B(0,4)} \rho_0^{\frac{2p}{p-1}}(z)\, dA(z) &\leq \sum_{j=0}^{n} \int_{B_j} \left(\frac{2}{r(B_j)} \right)^{\frac{2p}{p-1}} dA(z) \\ &= \sum_{j=0}^{n} |B_j| \left(\frac{2}{r(B_j)} \right)^{\frac{2p}{p-1}} \\ &= c_p \sum_{j=0}^{n} \frac{(r(B_j))^2}{(r(B_j))^{\frac{2p}{p-1}}} \\ &= c_p \sum_{j=0}^{n} \frac{1}{z_0^{\frac{2}{p-1}}} \frac{1}{2^{\frac{2j}{p-1}}} \\ &= c_p z_0^{-\frac{2}{p-1}} \sum_{j=0}^{n} \frac{1}{2^{\frac{2j}{p-1}}} \end{aligned}$$

The series $\sum_{j=0}^{n} \frac{1}{2^{\frac{2j}{p-1}}}$ converges to a constant depending on p for any fixed $p > 1$. Therefore,

$$M_{\mathbb{K}(\cdot,f)}(\Gamma) \leq c_{f,p} z_0^{-\frac{2}{p}} \tag{3.12}$$

Next, we would like to estimate the modulus term $M\left(f(\Gamma)\right)$ from below. Let us start with the definition of $M\left(f(\Gamma)\right)$ in polar coordinates

$$\begin{aligned} M\left(f(\Gamma)\right) &= \inf_{\rho\ admissible} \int_{\mathbb{C}} \rho^2(z)\, dA(z) \\ &= \inf_{\rho\ admissible} \int_0^{2\pi} \int_0^{\infty} \rho^2(r,\theta) r\ drd\theta \end{aligned}$$

and provide a lower bound for

$$\int_0^{\infty} \rho^2(r,\theta) r dr$$

for an arbitrary direction $\theta \in [0, 2\pi)$ and an arbitrary admissible ρ. To this end, we fix a direction θ and consider the half line L_θ starting from origin in the direction θ. One might assume that the image set $f(E)$ winds once around the origin when z moves from a point t_0 to a point t_2 along E and $f(t_0) \in L_\theta$. Moreover, let there exists a point $\bar{z} \in F$ such that $f(\bar{z}) \in L_\theta$ and $|f(\bar{z})| > |f(t_0)|$. Now, the origin and the point $f(\bar{z})$ are contained in the image set $f(F)$ and by our assumption f is a homeomorphism. This implies that $f(F)$ must intersect the line segment $(f(t_2), f(t_0))$ at least once, say at a point $f(t_1)$, with $t_1 \in F$. One could choose t_1 in such a way that either the line segment $(f(t_1), f(t_0))$ or the line segment $(f(t_2), f(t_1))$ belongs to the path family $f(\Gamma)$. It is evident that $f(E)$ cycles around the origin $n(z_0) = \left\lfloor \frac{|\arg(f(z_0)) - \arg(f(1))|}{2\pi} \right\rfloor$ times. So, it is possible to find at least

$$n(z_0) = \left\lfloor \frac{|\arg\left(f(z_0)\right) - \arg\left(f(1)\right)|}{2\pi} \right\rfloor - 1$$

disjoint line segments belonging to the path family $f(\Gamma)$, when t_0 is sufficienty close to the origin. Note that $n(z_0)$ does not depend on the direction θ. Since we are interested in extremal rotation, it can be assumed that $f(E)$ winds around the origin at least once, which makes it clear that $n(z_0)$ is non-negative. Now, the $n(z_0)$ disjoint line segments can be written in the form $\left(x_j e^{i\theta}, y_j e^{i\theta}\right) \subset L_\theta$, where $j \in \{1, 2, ..., n(z_0)\}$ and x_j,y_j are positive real numbers satisfying

$$0 < r_f \leq x_1 < y_1 < ... < x_{n(z_0)} < y_{n(z_0)} \leq c_f$$

where $c_f = \sup_{z\in E} |f(z)|$ and $r_f = \min_{z\in E} |f(z)|$. Here, neither c_f nor r_f depends on θ or z_0. So, one could write

$$\int_0^{\infty} \rho^2(r,\theta) r dr \geq \sum_{j=1}^{n(z_0)} \int_{x_j}^{y_j} \rho^2(r,\theta) r dr.$$

Next, let us consider the Hölder inequality with the functions $f(r) = \rho\sqrt{r}$ and $g(r) = \frac{1}{\sqrt{r}}$, which after squaring both sides gives

$$\int_{x_j}^{y_j} \rho^2(r,\theta) r dr \geq \left(\int_{x_j}^{y_j} \rho(r,\theta) dr\right)^2 \left(\int_{x_j}^{y_j} \frac{1}{r} dr\right)^{-1} \geq \frac{1}{\log\left(\frac{y_j}{x_j}\right)}.$$

The last inequality holds true as ρ is admissible with respect to $f(\Gamma)$ where the line segments $(x_je^{i\theta}, y_je^{i\theta})$ belong to the path family $f(\Gamma)$. Therefore,

$$\int_0^\infty \rho^2(r,\theta)rdr \geq \sum_{j=1}^{n(z_0)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)}.$$

It is quite clear from the definition of c_f that

$$\sum_{j=1}^{n(z_0)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)} \geq \sum_{j=1}^{n(z_0)-1} \frac{1}{\log\left(\frac{x_{j+1}}{x_j}\right)} + \frac{1}{\log\left(\frac{c_f}{x_{n(z_0)}}\right)}.$$

Next, let us consider the arithmetic mean - harmonic mean inequality, which states that for every positive real number a_j,

$$\sum_{j=1}^{n} a_j \geq \frac{n^2}{\sum_{j=1}^{n}\frac{1}{a_j}}.$$

At this point, we would like to use the above inequality with the precise choices

$$a_j = \frac{1}{\log\left(\frac{x_{j+1}}{x_j}\right)} \qquad \text{if } j \in \{1,2,...,n(z_0)-1\}, \text{ and} \qquad a_{n(z_0)} = \frac{1}{\log\left(\frac{c_f}{x_{n(z_0)}}\right)},$$

which gives

$$\sum_{j=1}^{n(z_0)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)} \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{x_1}\right)} \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{r_f}\right)}.$$

Therefore,

$$\int_0^\infty \rho^2(r,\theta)rdr \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{r_f}\right)}.$$

The constant c_f can be defined as $\max_{z\in\mathbb{D}}|f(z)|$, which is finite and does not depend on either θ or z_0, and thus it is irrelevant at the limit $z_0 \to 0$. On the other hand, the constant r_f must be estimated using the Hölder modulus of continuity assumption on our mapping f, that is

$$|f(z_0)| \geq C|z_0|^\alpha$$

for sufficiently small z_0. This combined with the estimate above gives that

$$M\left(f(\Gamma)\right) \geq \frac{n^2(z_0)}{C\alpha\log\left(\frac{1}{|z_0|}\right)}$$

Now, using the modulus inequality (1.9) and (3.12) we get

$$\frac{n^2(z_0)}{C\alpha\log\left(\frac{1}{|z_0|}\right)} \leq c_{f,p}\left(\frac{1}{|z_0|}\right)^{\frac{2}{p}}$$

which implies the desired estimate. □

Proof of Corollary 3.1.2. Corollary 3.1.2 follows immediately after noting that one can take $f = X_t$ in Theorem 3.1.1. Indeed, from [23, Corollary 3] we know that X_t belongs to $W^{1,p}$ for any $p < \frac{2}{t\|\omega_0\|_\infty}$, provided that $0 < t < \frac{2}{\|\omega_0\|_\infty}$. Since $J(\cdot, X_t) = 1$ due to the incompressibility, it then follows X_t is a homeomorphism with finite distortion, and moreover $\mathbb{K}(\cdot, X_t) \in L^p_{\text{loc}}$ for $p < \frac{1}{t\|\omega_0\|_\infty}$. Especially, if t is so small that $0 < t < \frac{1}{\|\omega_0\|_\infty}$ then one may take $p > 1$. Also, we recall from [47] (see also [9]) that X_t^{-1} is α-Hölder continuous with some exponent $\alpha \geq e^{-ct\,\|\omega_0\|_\infty}$ for some $c > 0$. Hence, Theorem 3.1.1 applies to $f = X_t$ and the claim follows. $\square$

Proof of Theorem 3.1.4. Let f satisfy the hypothesis of Theorem 3.1.4, and let $z \in \mathbb{C} \setminus \{0\}$ be such that $|z| < 1$. Our goal is to estimate the *winding number* $n(z)$ of the image set $f\left(\left[z, \frac{z}{|z|}\right]\right)$ around the origin (recall that $f(0) = 0$). We will bound $n(z)$ using the modulus inequality (1.9). More precisely, we will prove that

$$n(z) \leq C\,|z|^{-\frac{1}{p}}\,\log^{\frac{1}{2}}\left(\frac{1}{\min\limits_{|z_0|=|z|}|f(z_0)|}\right)$$

which is equivalent to Theorem 3.1.4 when $p > 1$, and

$$\limsup_{|z|\to 0}\frac{|z|}{\sqrt{\log\left(\frac{1}{\min\limits_{|z_0|=|z|}|f(z_0)|}\right)}}n(z) = 0$$

for $p = 1$.

Let us first prove $p > 1$ case. To this end, choose an arbitrary point $z_0 \in \mathbb{C} \setminus \{0\}$ such that $|z_0| < 1$. Without loss of generality we can assume that z_0 is real and positive. Next, fix line segments $E = [z_0, 1]$ and $F = (-\infty, 0]$, and let Γ be the family of paths connecting them. Then we can estimate the modulus term $M_{\mathbb{K}(\cdot,f)}(\Gamma)$ from above as in (3.12).
Next, we would like to estimate the modulus term $M\left(f(\Gamma)\right)$ from below for $p \geq 1$. Let us recall that $f(0) = 0$ and define $M\left(f(\Gamma)\right)$ in polar coordinates as follows:

$$\begin{aligned} M\left(f(\Gamma)\right) &= \inf_{\rho\ admissible}\int_{\mathbb{C}}\rho^2(z)\,dA(z) \\ &= \inf_{\rho\ admissible}\int_0^{2\pi}\int_0^{\infty}\rho^2(r,\theta)r\ drd\theta \end{aligned}$$

and provide a lower bound for

$$\int_0^{\infty}\rho^2(r,\theta)rdr$$

for an arbitrary direction $\theta \in [0, 2\pi)$ and an arbitrary admissible ρ. To this end, let us fix a direction θ and consider the half line L_θ starting from the origin to the direction θ. Assume that the image set $f(E)$ winds once around the origin when z moves from a point t_0 to a point t_2 along E and $f(t_0), f(t_2) \in L_\theta$. Since f is a homeomorphism and the image $f(F)$ contains the origin and points with big modulus, we can deduce that $f(F)$ must intersect the line segment $(f(t_2), f(t_0))$ at least once, say at a point $f(t_1)$, with $t_1 \in F$. Moreover, we can choose t_1 in such a way that either the line segment $(f(t_1), f(t_0))$ or the line segment $(f(t_2), f(t_1))$ belongs to the path family $f(\Gamma)$. It is evident that $f(E)$ cycles around the origin $n(z_0) = \left\lfloor \frac{|\arg(f(z_0))-\arg(f(1))|}{2\pi} \right\rfloor$ times. So, it is possible to find at least

$$n(z_0) = \left\lfloor \frac{|\arg(f(z_0)) - \arg(f(1))|}{2\pi} \right\rfloor - 1$$

disjoint line segments belonging to the path family $f(\Gamma)$ using this argument. Note that $n(z_0)$ does not depend on the direction θ. Since we are interested in extremal rotation, it can be assumed that $f(E)$ winds around the origin at least once, which makes it clear that $n(z_0)$ is non-negative. Now, the $n(z_0)$ disjoint line segments can be written in the form $\left(x_j e^{i\theta}, y_j e^{i\theta}\right) \subset L_\theta$, where $j \in \{1, 2, ..., n(z_0)\}$ and x_j,y_j are positive real numbers satisfying

$$0 < r_f \le x_1 < y_1 < ... < x_{n(z_0)} < y_{n(z_0)} \le c_f$$

where $c_f = \sup_{z\in E} |f(z)|$ and $r_f = \inf_{z\in E} |f(z)|$. Hence we can write

$$\int_0^\infty \rho^2(r,\theta) r dr \ge \sum_{j=1}^{n(z_0)} \int_{x_j}^{y_j} \rho^2(r,\theta) r dr.$$

Next, let us consider the Hölder inequality with the functions $f(r) = \rho\sqrt{r}$ and $g(r) = \frac{1}{\sqrt{r}}$, which after squaring both sides gives

$$\int_{x_j}^{y_j} \rho^2(r,\theta) r dr \ge \left(\int_{x_j}^{y_j} \rho(r,\theta) dr\right)^2 \left(\int_{x_j}^{y_j} \frac{1}{r} dr\right)^{-1} \ge \frac{1}{\log\left(\frac{y_j}{x_j}\right)}.$$

The last inequality holds true as ρ is admissible with respect to $f(\Gamma)$ and the line segments $(x_j e^{i\theta}, y_j e^{i\theta})$ belong to the path family $f(\Gamma)$. Therefore,

$$\int_0^\infty \rho^2(r,\theta) r dr \ge \sum_{j=1}^{n(z_0)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)}.$$

The definition of c_f makes it clear that

$$\sum_{j=1}^{n(z_0)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)} \ge \sum_{j=1}^{n(z_0)-1} \frac{1}{\log\left(\frac{x_{j+1}}{x_j}\right)} + \frac{1}{\log\left(\frac{c_f}{x_{n(z_0)}}\right)}.$$

Next, we consider the arithmetic-harmonic means inequality. For every positive integer a_j,

$$\sum_{j=1}^{n} a_j \ge \frac{n^2}{\sum_{j=1}^{n} \frac{1}{a_j}}. \tag{3.13}$$

We use (3.13) with the precise choices

$$a_j = \frac{1}{\log\left(\frac{x_{j+1}}{x_j}\right)} \quad \text{if } j \in \{1, 2, ..., n(z_0) - 1\}, \text{ and} \quad a_{n(z_0)} = \frac{1}{\log\left(\frac{c_f}{x_{n(z_0)}}\right)},$$

that gives

$$\sum_{j=1}^{n(z_0)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)} \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{x_1}\right)} \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{r_f}\right)}.$$

Therefore,

$$\int_0^\infty \rho^2(r,\theta) r dr \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{r_f}\right)}. \tag{3.14}$$

The constant c_f is finite and does not depend on either θ or z_0, at least for small z_0. So, it is irrelevant at the limit $z_0 \to 0$. Hence the estimate (3.14) implies that

$$M\left(f(\Gamma)\right) \geq \frac{cn^2(z_0)}{\log\left(\frac{1}{\min\limits_{|z|=|z_0|}|f(z)|}\right)}. \tag{3.15}$$

Next, use the modulus inequality (1.9) and (3.12) to get

$$\frac{n^2(z_0)}{\log\left(\frac{1}{\min\limits_{|z|=|z_0|}|f(z)|}\right)} \leq c_{f,p} z_0^{-\frac{2}{p}},$$

which implies the desired estimate (3.6).

To prove $p = 1$ **case**, we will again use the modulus inequality (1.9). Note that we have already lower bound for $M\left(f(\Gamma)\right)$ from (3.15) for any $p \geq 1$. Therefore, we just need to estimate modulus term $M_{\mathbb{K}(\cdot,f)}(\Gamma)$ from above. To this end, let us define the function

$$\rho_0(z) = \begin{cases} \frac{1}{z_0} & \text{if } dist(z,E) < z_0 \\ 0 & otherwise \end{cases}$$

Note that ρ_0 is admissible with respect to the path family Γ. Therefore,

$$\begin{aligned} M_{\mathbb{K}(.,f)}(\Gamma) &\leq \int_{\mathbb{C}} \mathbb{K}(.,f)\rho_0^2(z) dA(z) \\ &= \frac{1}{z_0^2} \int_{\{z:dist(z,E)<z_0\}} \mathbb{K}(.,f)(z) dA(z). \end{aligned}$$

Denote

$$\int_{\{z:dist(z,E)<z_0\}} \mathbb{K}(.,f)(z) dA(z) = C_f(z_0)$$

and note that since $\mathbb{K}(.,f)(z) \in L^1_{loc}(\mathbb{C})$ and

$$|\{z : dist(z,E) < z_0\}| \to 0$$

it follows that $C_f(z_0) \to 0$ as $z_0 \to 0$, and thus

$$M_{\mathbb{K}(.,f)}(\Gamma) \leq \frac{C_f(z_0)}{z_0^2}. \tag{3.16}$$

Next, we use the modulus inequality (1.9), bounds (3.15) and (3.16) to get

$$\frac{n^2(z_0)}{\log\left(\frac{1}{\min\limits_{|z|=|z_0|}|f(z)|}\right)} \leq \frac{C_f(z_0)}{z_0^2}$$

which implies the desired estimate (3.7). Hence, Theorem 3.1.4 is proved. □

Proof of Corollary 3.1.6. As in Theorem 3.1.4, our aim is to estimate the *winding number* $n(z)$ of the image set $f\left(\left[z, \frac{z}{|z|}\right]\right)$ around the origin (recall that $f(0) = 0$), when f satisfies the hypothesis of Corollary 3.1.6 and $z \in \mathbb{C} \setminus \{0\}$ such that $|z| < 1$. We will estimate $n(z)$ using the modulus inequality for homeomorphisms with integrable distortion (1.9). More precisely, we will show that

$$\limsup_{|z|\to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{|z|}\right)}} n(z) = 0,$$

which is equivalent to Corollary 3.1.6.

To this end, let us choose an arbitrary point $z_0 \in \mathbb{C} \setminus \{0\}$ such that $|z_0| < 1$ and which we can again assume to be positive and real. Next, as in the proof of Theorem 3.1.4, we fix line segments $E = [z_0, 1]$ and $F = (-\infty, 0]$, and let Γ be the family of paths connecting them. We have already estimated modulus term $M_{\mathbb{K}(\cdot,f)}(\Gamma)$ in (3.16) and thus we can concentrate on estimating $M(f(\Gamma))$ from below.

To this end we use the exact same steps as in the proof of Theorem 3.1.4 until the lower bound (3.14), where we now estimate the constant r_f using the Hölder modulus of continuity assumption on the inverse of our map f. That is, we estimate

$$|f(z_0)| \geq C\,|z_0|^{\alpha}$$

for sufficiently small z_0, and obtain

$$M(f(\Gamma)) \geq \frac{n^2(z_0)}{\alpha \log\left(\frac{1}{|z_0|}\right)}.$$

The estimates for moduli combined with the modulus inequality (1.9) results in

$$\frac{n^2(z_0)}{\alpha \log\left(\frac{1}{|z_0|}\right)} \leq \frac{C_f(z_0)}{z_0^2},$$

which provides the desired estimate (3.11). Hence Corollary 3.1.6 is proved. □

3.3 Optimality of spiraling

Proof of Theorem 3.1.3. We will get Theorem 3.1.3 in two steps. In the first step, we will construct a map which *only rotates*. This map will already give us the optimal result (in the power scale). In the second step, we will strengthen this up with a second map, that *both rotates and stretches.* This second map is going to be the optimal one.

Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ we define the corresponding rotation map as

$$\phi_A(z) = \begin{cases} z & |z| > R \\ z\, e^{i\alpha \log\left|\frac{z}{R}\right|} & r \leq |z| \leq R \\ z\, e^{i\alpha\, \log \frac{r}{R}} & |z| < r \end{cases}$$

Here $0 < r < R$, and $\alpha \in \mathbb{R}$. One must note that $\phi_A : \mathbb{C} \to \mathbb{C}$ is bilipschitz (i.e. both ϕ_A and its inverse are Lipschitz), hence quasiconformal (its quasiconformality constant depends only on α), and moreover it is conformal outside the annulus A. Note also that ϕ_A leaves fixed all circles centered at 0, since $|\phi_A(te^{i\theta})| = t$ for each $t > 0$ and $\theta \in \mathbb{R}$. Finally, a direct calculation shows for the jacobian determinant that $J(z, \phi_A) = 1$ for each z.

Next, we fix a sequence $\{r_n\}$ such that $0 < r_{n+1} < \frac{r_n}{2e}$ and $r_1 < \frac{1}{e}$. Also, let $R_n = er_n$. These assumptions make sure that $2r_{n+1} < R_{n+1} < \frac{r_n}{2}$. Let us now construct disjoint annuli $A_n = B(0,R_n) \setminus B(0,r_n)$, and set $\{f_n\}_n$ to be a sequence of maps, constructed in an iterative way as follows. For $n = 1$, we set

$$f_1(z) = \phi_{A_1}(z) = \begin{cases} z & |z| > R_1 \\ z\, e^{i\alpha_1\, \log \frac{|z|}{R_1}} & r_1 \leq |z| \leq R_1 \\ z\, e^{-i\alpha_1} & |z| < r_1 \end{cases}$$

where $\alpha_1 \in \mathbb{R}$, $\alpha_1 \geq 1$, is to be determined later. We then define f_n for $n \geq 2$ as

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

again for some values $\alpha_n \in \mathbb{R}$, $\alpha_n \geq 1$, to be determined later. Clearly, each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli A_i, $i = 1, \ldots, n$. It is also clear that $f_n(z) = f_{n-1}(z)$ on the unbounded component of $\mathbb{C} \setminus f_{n-1}(A_n)$ (i.e. outside of $B(0,R_n)$). This proves that the sequence f_n is uniformly Cauchy and hence it converges to a map f, that is,

$$f = \lim_{n\to\infty} f_n$$

which is again a homeomorphism by construction. Now, since f_n is quasiconformal for every n and $f_n(z) = f_{n-1}(z)$ everywhere except inside the ball $B(0,R_n)$, where $R_n \to 0$ as $n \to \infty$, the limit map f is absolutely continuous on almost every line parallel to the coordinate axes and differentiable almost everywhere.

It is helpful to note that each f_n leaves fixed all circles centered at the origin, so in particular we have $f_n(A_j) = A_j$ for each j, and therefore $\phi_{f_{n-1}(A_n)} = \phi_{A_n}$. Direct calculation shows that

$$|D\phi_{A_n}(z)| = |\partial\phi_{A_n}(z)| + |\overline{\partial}\phi_{A_n}(z)| = \begin{cases} 1 & |z| > R_n \\ \frac{|2+i\alpha_n| + |\alpha_n|}{2} & r_n \leq |z| \leq R_n \\ 1 & |z| < r_n \end{cases}$$

which allows us to estimate that

$$|\partial f(z)| + |\bar{\partial} f(z)| \leq 2\alpha_n \qquad \text{whenever } z \in A_n,$$

and $|Df(z)| \leq 1$ otherwise. Therefore, in order to have $Df(z) \in L^1_{loc}(\mathbb{C})$ it suffices that

$$\sum_n \alpha_n \, r_n^2 < +\infty. \tag{3.17}$$

This, together with the absolute continuity, guarantees $f \in W^{1,1}_{loc}(\mathbb{C})$. Also, since f is a homeomorphism, we have that $J_f(z) \in L^1_{loc}(\mathbb{C})$, and in fact $J(z,f) = 1$ at almost every $z \in \mathbb{C}$. Therefore, f is a homeomorphism of finite distortion, with distortion function

$$\mathbb{K}(z,f) = \frac{|Df(z)|^2}{J(z,f)} \leq \begin{cases} 4\alpha_n^2 & z \in A_n, \\ 1 & \text{otherwise.} \end{cases}$$

Especially, in order to have $\mathbb{K}(\cdot, f) \in L^p_{loc}$, it suffices to ensure the convergence of the series

$$\sum_{n=1}^{\infty} |A_n| (4\alpha_n^2)^p \simeq \sum_{n=1}^{\infty} \alpha_n^{2p} \, r_n^2 \tag{3.18}$$

which can be done by choosing α_n properly. Note that if (3.18) holds, then also (3.17) holds, because our choice of α_n will guarantee $\alpha_n \geq 1$. The last restriction to choose our α_n comes from rotational behavior of f. It is clear from the above construction that $f(0) = 0$, $f(1) = 1$ and

$$|\arg\left(f(r_n)\right)| \geq \left| \arg\left(\left(\frac{1}{e}\right)^{1+i\alpha_n} \right) \right| = \alpha_n$$

for every r_n. Since we want our map to be optimal for Theorem 3.1.1, we may be tempted to choose $\alpha_n = r_n^{-1/p} \log^{1/2}(1/r_n)$. Unfortunately such a choice does not meet the requirement (3.18). The same problem occurs if we simply choose $\alpha_n = r_n^{-1/p}$. So we choose

$$\alpha_n = h(r_n) \, r_n^{-1/p}.$$

Here $h : [0,\infty) \to [0,\infty)$ is any monotonically decreasing gauge function such that $\lim_{r\to 0^+} h(r) = 0$. With this choice, (3.18) is fulfilled if the series

$$\sum_{n=1}^{\infty} h(r_n)^{2p} < +\infty.$$

But this can always be done by simply reducing the already chosen values of r_n, for instance if $h(r_n) < \frac{1}{n^{1/2}}$. Note that this does not provide full optimality for Theorem 3.1.1, but it already gives the right order (in the power scale).

We now show that f is Hölder continuous with exponent $1 - \frac{1}{p}$. For this, let us recall that our map f is a limit of iterates of logarithmic spiral maps inside the annuli $A_n = B(0, R_n) \setminus B(0, r_n)$. In particular, as shown in [7], if $\gamma \in \mathbb{R}$ then the basic logarithmic spiral map $g(z) = z|z|^{i\alpha} = ze^{i\gamma \log|z|}$

is L-bilipschitz, for a constant L such that $|\gamma| = L - \frac{1}{L}$. When $|\gamma|$ is large, L is large as well and so one roughly has $|\gamma| \simeq L$. Since our f_n behaves on the annulus A_n as a spiral map with $|\gamma| = \alpha_n$, we deduce the bilipschitz constant of f_n on A_n is

$$L \simeq |\gamma| = \alpha_n = h(r_n)\, r_n^{-1/p}.$$

Let us now start the proof. To this end, let us consider two arbitrary points x and y in $\mathbb{D} \setminus \{0\}$. We first consider the case where $x, y \in A_n$. In this case, $f(x) = f_n(x)$ and $f(y) = f_n(y)$. Since $r_n > C|x-y|$, we have

$$\begin{aligned}|f(x) - f(y)| = |f_n(x) - f_n(y)| &\lesssim h(r_n)\, r_n^{-1/p} |x-y| \\ &\leq h(r_n) \left(\frac{C}{|x-y|} \right)^{\frac{1}{p}} |x-y| \\ &\leq C|x-y|^{1-\frac{1}{p}}\end{aligned}$$

where we have used the bilipschitz nature of f_n on A_n.

We now assume that $x, y \in D_n = B(0, r_n) \setminus B(0, R_{n+1})$. On that set f is of the form $ze^{i\beta}$, where $\beta \in \mathbb{R} \setminus \{0\}$, which is clearly an isometry.

Next, we take $x \in A_n$ and $y \in D_n$. In particular, $|x| \geq |y|$. Then let w be any point on the outer boundary of D_n joining x and y. We have

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f(w)| + |f(w) - f(y)| \\ &\leq C|x-w|^{1-\frac{1}{p}} + |w-y| \\ &\leq 2C|x-y|^{1-\frac{1}{p}}\end{aligned}$$

The same happens if $x \in D_{n-1}$ and $y \in A_n$.

So it just remains to see what happens when $x \in A_n = B(0, R_n) \setminus B(0, r_n)$ and $y \in B(0, R_{n+1})$. Let L be the line joining x and y. We divide it into three parts, viz., L_1, L_2 and L_3. L_1 connects x to a point a on the inner boundary of A_n, so that

$$|f(x) - f(a)| = |f_n(x) - f_n(a)| \leq C|x-a|^{1-\frac{1}{p}}$$

Next, L_2 connects a to b, which is the closest point to y where the line L crosses the inner boundary of D_n. From $2R_{n+1} < r_n < \frac{R_n}{2}$ we get that $|f(a)| > 2|f(b)|$. Also, since $a, b \in D_n$ and f is an isometry there, we get

$$|f(b) - f(y)| \leq 2|f(b)| \leq 2|f(a) - f(b)| = 2|a-b|$$

Summarizing

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f(a)| + |f(a) - f(b)| + |f(b) - f(y)| \\ &\leq C|x-a|^{1-\frac{1}{p}} + |a-b| + 2|a-b| \\ &\leq C|x-y|^{1-\frac{1}{p}}\end{aligned}$$

The case $x \in D_n$ and $y \in B(0, r_{n+1})$ can be proved in a similar manner. Therefore, we have covered all the possible cases. Since the set $\mathbb{D} \setminus \{0\}$ is partitioned by separated annuli A_n and D_n, it is clear that we have proved that f is Hölder continuous with exponent $1 - \frac{1}{p}$. At this point, it is worth noting that this regularity could also be proven by means of the Sobolev embedding. Indeed, we proved above that $\mathbb{K}(\cdot, f) \in L^p_{loc}$, and also that the Jacobian determinant is constantly 1. This together implies that $Df \in L^{2p}_{loc}$.

Now we show that also f^{-1} is Hölder continuous. Indeed, let us recall that f is the limit of iterates of logarithmic spiral maps inside the annuli and conformal outside. Now, f^{-1} can be constructed using the same building blocks as f itself, just changing the sign of α_n at each step. This is possible because the inverse of a logarithmic spiral map is the same spiral map, just the direction of rotation is opposite of the original map. Since it is clear that the direction of rotation does not play any role in the proof of Hölder continuity of f, this implies that f^{-1} is also Hölder from above. Thus f is Hölder from below as well.

As we said before, the above example approaches the borderline stated in Theorem 3.1.1, but it does not attain full optimality yet. To this end, we have to modify it by adding to our building blocks a stretching factor. This is done by replacing, at each iterate, the logarithmic spiral map $z|z|^{i\alpha} = ze^{i\alpha \log|z|}$ by a complex power $z|z|^{q+i\alpha} = z|z|^q\, e^{i\alpha\log|z|}$. We now proceed with the details.

So, similarly as in the previous construction, we fix a rapidly decreasing sequence $\{r_n\}$ such that $r_{n+1} < \frac{r_n}{2e}$ and $r_1 < \frac{1}{e}$. Also, let $R_n = er_n$. Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ we define the corresponding radial stretching combined with rotation map as follows:

$$\phi_A(z) = \begin{cases} z & |z| > R \\ z\left|\frac{z}{R}\right|^{q-1} e^{i\alpha\log\frac{|z|}{R}} & r \le |z| \le R \\ z\left(\frac{r}{R}\right)^{q-1} e^{i\alpha\log\frac{r}{R}} & |z| < r \end{cases} \tag{3.19}$$

Note that this time we will have $q \ge 1$. Direct calculation shows that

$$|\partial\phi_A(z)| + |\overline{\partial}\phi_A(z)| = \begin{cases} 1 & |z| > R \\ R^{1-q}|z|^{q-1}\frac{|q+1+i\alpha|+|q-1+i\alpha|}{2} & r \le |z| \le R \\ R^{1-q}r^{q-1} & |z| < r \end{cases}$$

and also that

$$J(z,\phi_A) = \begin{cases} 1 & |z| > R \\ q\,\left(\frac{|z|}{R}\right)^{2(q-1)} & r \le |z| \le R \\ \left(\frac{r}{R}\right)^{2(q-1)} & |z| < r \end{cases}$$

whence

$$\mathbb{K}(z,\phi_A) = \begin{cases} 1 & |z| > R \\ \frac{(|q+1+i\alpha|+|q-1+i\alpha|)^2}{4q} & r \le |z| \le R \\ 1 & |z| < r \end{cases}$$

In particular, if $2 \le q+1 < \alpha$ then one may estimate $\|\mathbb{K}(\cdot,\phi_A)\|_\infty \le \frac{4\alpha^2}{q}$. Next, let us construct the sequence of maps f_n in an iterative way as follows. For $n=1$, we set

$$f_1(z) = \phi_{A_1}(z) = \begin{cases} z & |z| < R_1 \\ z\left|\frac{z}{R_1}\right|^{q_1-1} e^{i\alpha_1 \log\frac{|z|}{R_1}} & r_1 \le |z| \le R_1 \\ z\left(\frac{1}{e}\right)^{q_1-1} e^{-i\alpha_1} & |z| < r_1 \end{cases}$$

where q_1 and α_1 are to be determined later. Next, assuming we have $f_1, \ldots, f_{n-1}$, we define f_n for $n \ge 2$ as:

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

Note that $\phi_{f_{n-1}(A_n)}$ is determined by the inner and outer radii of $\phi_{f_{n-1}(A_n)}$ (which are already available since $f_1, \ldots, f_{n-1}$ are known) as well as for the parameters q_n and α_n, which will be determined later. Clearly, each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli A_i, $i \in \{1, ..., n\}$. Moreover, one can easily show that

$$\mathbb{K}(\cdot, f_n) = \prod_{j=1}^{n} \mathbb{K}(\cdot, f_{n-j} \circ \phi_{f_{n-j}(A_{n-j+1})}) = \prod_{j=1}^{n} \mathbb{K}(\cdot, \phi_{A_{n-j+1}})$$

so that $\mathbb{K}(z, f_n) \le C\frac{\alpha_j^2}{q_j}$ whenever $z \in A_j$, $j = 1 \ldots n$ while $\mathbb{K}(\cdot, f_n) = 1$ otherwise. In a similar way, we can use that $|D\phi_A(z)| \le C\alpha$ when $z \in A$ (and $|D\phi_A(z)| \le 1$ at all other points) to obtain that $|Df_n| \le C\alpha_j$ on A_j, $j = 1 \ldots n$, and $|Df_n| \le 1$ otherwise.

By construction, we have $f_n(z) = f_{n-1}(z)$ whenever $z \notin B(0, R_n)$. Thus $\{f_n\}_n$ converges uniformly to a map $\bar{f}(z)$, that is,

$$\bar{f} = \lim_{n\to\infty} f_n$$

which is again a homeomorphism by construction. A similar argument to the one before shows that $\bar{f}$ is absolutely continuous on almost every line parallel to the coordinate axis. For almost every fixed $z_0 \neq 0$ there is a neighbourhood of z_0 such that the sequence $\{f_n(z)\}_n$ remains constant for n very large and z in that neighbourhood. Therefore the same happens to the sequences $Df_n(z)$, $J(z, f_n)$ and $\mathbb{K}(z, f_n)$, and so their limits are precisely $D\bar{f}(z)$, $J(z, \bar{f})$ and $\mathbb{K}(z, \bar{f})$. Especially, in order to have $D\bar{f} \in L^1_{loc}$ it suffices that

$$\sum_{n=1}^{\infty} |A_n|\, \alpha_n < +\infty \tag{3.20}$$

In case this holds true, then $\bar{f}$ is a homeomorphism in $W^{1,1}_{loc}$, and as a consequence its jacobian determinant $J(\cdot, \bar{f}) \in L^1_{loc}$. Moreover, in order to have $\mathbb{K}(\cdot, \bar{f}) \in L^p_{loc}$ one needs to require that

$$\sum_{n=1}^{\infty} |A_n|\, \frac{\alpha_n^{2p}}{q_n^p} < +\infty \tag{3.21}$$

Again, as it was the case for f, (3.21) implies (3.20) when $q_n^{\frac{p}{2p-1}} < \alpha_n$ and so our parameters α_n and q_n need to be chosen according to (3.21) as well as the purpose of $\bar{f}$ to be optimal for Theorem 3.1.1. For this, again as before, we have $\bar{f}(0) = 0$, $\bar{f}(1) = 1$ and

$$\left|\arg\left(\bar{f}(r_n)\right)\right| \ge \left|\arg\left(\left(\frac{1}{e}\right)^{q_n + i\alpha_n}\right)\right| = |\alpha_n|$$

which motivates us to choose

$$\alpha_n = h(r_n) \left(\log \frac{1}{r_n}\right)^{1/2} r_n^{-\frac{1}{p}} \qquad\qquad q_n = \log \frac{1}{r_n},$$

where h is any gauge function such that $h(r) \to 0$ as $r \to 0$ and the condition $q_n^{\frac{p}{2p-1}} < \alpha_n$ is satisfied. Indeed, with these choices (3.21) becomes

$$\sum_n h(r_n)^{2p} < +\infty$$

which, as before, may always be granted by choosing smaller r_n, if needed. Having (3.21) fulfilled, our map $\bar{f}$ is a mapping of finite distortion with $\mathbb{K}(\cdot, \bar{f}) \in L^p_{loc}$. Also, the resulting map $\bar{f}$ attains the optimal rotational behavior stated at Theorem 3.1.1 modulo the gauge function h which can be chosen to converge to 0 as slowly as desired.

Therefore, Theorem 3.1.3 will be proven if we are able to show that $\bar{f}$ is Hölder from below. Furthermore, we also show that $\bar{f}$ is Hölder from above, highlighting regularity of our mappings.

To do this, we first observe that the composition of $z \mapsto ze^{i\alpha \log|z|}$ followed by $z \mapsto z|z|^{q-1}$ is precisely $z \mapsto z|z|^{q-1}e^{i\alpha \log|z|}$. This observation suggests us to decompose $\bar{f} = g \circ f$, where f is essentially the first example in this section (with different choice of α_n) and g is constructed by building blocks (3.19) with $\alpha = 0$ at each step. Morally, f leaves fixed all circles centered at 0, and only rotates the annuli A_n, while g conveniently stretches each A_n.

For any $p > 1$, the bi-Hölder nature of f has already been proven when $\alpha_n = h(r_n)\, r_n^{-1/p}$. Hence we can directly use the same proof there after we estimate

$$h(r_n) \left(\log \frac{1}{r_n}\right)^{1/2} r_n^{-\frac{1}{p}} \le h(r_n)\, r_n^{-1/(p-\epsilon)}$$

for all small r_n and $\epsilon = (p-1)/2$. Therefore, it only remains to show that g is bi-Hölder as well. To this end, we first show that g is Hölder from above using the fundamental theorem of calculus.

Let $x, y \in B(0,1)$ be given. Without loss of generality let us assume that $|y| \ge |x|$ and let w be the point for which $|w| = |x|$ and $\arg(w) = \arg(y)$. Now

$$|g(x) - g(y)| \le |g(w) - g(x)| + |g(y) - g(w)|, \tag{3.22}$$

and we will show that both of these are Hölder. First, since g maps circles centered at the origin radially to similar circles with equal or smaller radius (as $q_n \ge 1$) it is clear that

$$|g(x) - g(w)| \le |x - w| \le |x - y|.$$

Let us then concentrate of the second part. First we note, that we can without loss of generality assume that y and w are real numbers as g is a radial mapping. From our construction we see that the line segments $[r_n, R_n]$, (R_{n+1}, r_n) and $(R_1, 1]$ partition the line segment $(0, 1]$. Furthermore, from (3.19) it is clear that the differential is bounded from above by 1 in the segments (R_{n+1}, r_n) and $(R_1, 1]$. On the other hand, in segments $[r_n, R_n]$ we can estimate

$$|g'(t)| \leq \log\left(\frac{1}{r_n}\right) \leq \frac{C}{\sqrt{t}}$$

for any $t \in [r_n, R_n]$ with fixed C that does not depend on n or t. This is so because of our choice of q_n. Combining these two estimates we have

$$|g'(t)| \leq \frac{C}{\sqrt{t}}$$

for any $t \in (0, 1)$. Thus we can use fundamental theorem of calculus to estimate

$$\begin{aligned}
|g(y) - g(w)| &= \int_w^y |g'(t)| dt \\
&\leq \int_w^y \frac{C}{\sqrt{t}} dt \\
&= 2C\left(\sqrt{y} - \sqrt{w}\right) \\
&\leq 2C\sqrt{y - w}.
\end{aligned}$$

This proves that also the second part in (3.22) is Hölder from above, and thus we obtain

$$|g(y) - g(x)| \leq |g(y) - g(w)| + |g(w) - g(x)| \leq C\sqrt{|y - w|} + \sqrt{|x - w|} \leq 2C\sqrt{|x - y|},$$

which shows g is Hölder from above.

Let us next prove that g is Hölder from below. To this end, given any two points $x, y \in B(0, 1)$ we again without loss of generality assume that $|y| \geq |x|$ and let w be the point for which $|w| = |x|$ and $\arg(w) = \arg(y)$. Now, as g is a radial homeomorphism, it follows that

$$|g(x) - g(y)| \geq \max\{|g(x) - g(w)|, |g(y) - g(w)|\}$$

Moreover,

$$\max\{|x - w|, |y - w|\} \geq \frac{1}{2}|x - y|$$

Therefore, it is enough to show that both $|g(x) - g(w)|$ and $|g(y) - g(w)|$ satisfy Hölder bounds from below. Note that if $x = 0$ then clearly $w = 0$ and we have only the radial part $|g(y) - g(w)|$.

Let us first check the term $|g(x) - g(w)|$. Since g maps radially circles centered at the origin to similar circles we see that $|g(x) - g(w)|$ gets contracted the same amount as the modulus $|g(w)|$ is contracted under g. Now we must consider two possibilities, either $x, w \in A_n$ or $x, w \in D_n$ for some n. Let us first assume $x, w \in A_n = B(0, R_n) \setminus B(0, r_n)$ for some n. Here we impose an additional assumption that

$$r_n < \left(\frac{1}{e}\right)^{q_{n-1} + q_{n-2} + \ldots + q_1 - (n-1)}, \tag{3.23}$$

which we can do as the radii r_n can be assumed to decrease as fast as we want. Then we can estimate

$$
\begin{aligned}
|g(x)| &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot |x| \left(\frac{|x|}{R_n}\right)^{q_n-1} \\
&\geq r_n \cdot |x| \left(\frac{|x|}{R_n}\right)^{q_n-1} \\
&\geq r_n \cdot |x| \left(\frac{1}{e}\right)^{q_n-1} = e \cdot r_n^2 \cdot |x|
\end{aligned}
$$

for any $x \in A_n$. Therefore,

$$
\begin{aligned}
|g(x) - g(w)| &\geq e \cdot r_n^2 \cdot |x - w| \\
&\geq C \cdot |x - w|^3
\end{aligned}
$$

since $|x - w| < C \cdot r_n$ for some fixed constant $C > 0$ when $x, w \in A_n$.

Next, let $x, w \in D_n = B(0, r_n) \setminus B(0, R_{n+1})$ for some n. Using (3.23) we get

$$
\begin{aligned}
|g(x)| &\geq c \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot r_n \cdot |x| \\
&\geq c \cdot r_n^2 \cdot |x|.
\end{aligned}
$$

Thus we can use a similar argument as in the previous case to estimate

$$
\begin{aligned}
|g(x) - g(w)| &\geq c \cdot r_n^2 \cdot |x - w| \\
&\geq c \cdot |x - w|^3
\end{aligned}
$$

since $|x - w| < c \cdot r_n$ for some fixed constant $c > 0$ when $x, w \in D_n$.

Since the set $\mathbb{D} \setminus \{0\}$ is partitioned by separated annuli A_n and D_n we have thus proven that $|g(x) - g(w)|$ satisfies Hölder estimates from below.

Finally, let us prove the Hölder estimates from below for the term $|g(y) - g(w)|$. As the mapping g is radial, we can again assume that y and w are real. We aim to use again the Fundamental Theorem of Calculus, and thus have to estimate the differential from below. Using (3.23), as well as the fact that $q_n > 1$, we can estimate for any real number $t \in [r_n, R_n]$ that

$$
\begin{aligned}
g'(t) &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot q_n \cdot \left(\frac{t}{R_n}\right)^{q_n-1} \\
&\geq r_n \, q_n \cdot \left(\frac{r_n}{R_n}\right)^{q_n-1} \\
&= e \, q_n \, r_n^2 \\
&\geq c \cdot t^2 \, \log \frac{1}{t}.
\end{aligned}
$$

Next, if $t \in [R_{n+1}, r_n]$, we have

$$
\begin{aligned}
g'(t) &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot \left(\frac{1}{e}\right)^{q_n-1} \\
&\geq e \cdot r_n^2 \\
&\geq c \cdot t^2
\end{aligned}
$$

Thus, as before, since $(0,1)$ is partitioned by the intervals $[r_n, R_n]$, $[R_{n+1}, r_n]$ and $[R_1, 1)$, we end up getting that

$$g'(t) \geq c \cdot t^2$$

for every $t \in (0,1)$. Now, we use the fundamental theorem of calculus to get

$$
\begin{aligned}
|g(y) - g(w)| &= \int_w^y g'(t)dt \\
&\geq \int_w^y c \cdot t^2 dt \\
&= C\left(y^3 - w^3\right) \\
&\geq C|y-w|^3
\end{aligned}
$$

This proves that the second term is Hölder from below as well, which in turn proves that g is Hölder from below. This finishes the proof of Theorem 3.1.3. □

Proof of Theorem 3.1.5. We prove Theorem 3.1.5 in two steps. In the first step, we construct a map which *only rotates*. This map will have the correct spiraling rate but the distortion of the map will not belong to the desired space. To overcome this barrier we compose it with a radial stretching map, which gives us better control over distortion.

Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ let us define the corresponding rotation map as

$$\phi_A(z) = \begin{cases} z & |z| > R \\ z\, e^{i\alpha \log\left|\frac{z}{R}\right|} & r \leq |z| \leq R \\ z\, e^{i\alpha\ \log\frac{r}{R}} & |z| < r \end{cases} \tag{3.24}$$

Here $0 < r < R$, and $\alpha \in \mathbb{R}$. It is clear that $\phi_A : \mathbb{C} \to \mathbb{C}$ is bilipschitz, hence quasiconformal (its quasiconformality constant depends only on α), and moreover it is conformal outside the annulus A. Moreover, $|\phi_A(te^{i\theta})| = t$ for each $t > 0$ and $\theta \in \mathbb{R}$. This means that ϕ_A leaves fixed all circles centered at 0. It is easy to check that the jacobian determinant $J(z, \phi_A) = 1$ for each z.

Next, let us consider sequence $\{r_n\}$ such that $0 < r_{n+1} < \frac{r_n}{2e}$ and $r_1 < \frac{1}{e}$. Also, let $R_n = er_n$, which ensures that $2r_{n+1} < R_{n+1} < \frac{r_n}{2}$. Let us now construct disjoint annuli $A_n = B(0,R_n) \setminus B(0,r_n)$, and set $\{f_n\}_n$ to be a sequence of maps, constructed in an iterative way as follows. For $n = 1$, we set

$$f_1(z) = \phi_{A_1}(z) = \begin{cases} z & |z| > R_1 \\ z\, e^{i\alpha_1\ \log\frac{|z|}{R_1}} & r_1 \leq |z| \leq R_1 \\ z\, e^{-i\alpha_1} & |z| < r_1 \end{cases} \tag{3.25}$$

where $\alpha_1 \in \mathbb{R}$, $\alpha_1 \geq 1$, is to be determined later. We then define f_n for $n \geq 2$ as

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

again for some values $\alpha_n \in \mathbb{R}$, $\alpha_n \geq 1$, to be determined later. Clearly, each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli A_i, $i = 1, \ldots, n$. It is also clear that $f_n(z) = f_{n-1}(z)$ on the unbounded component of $\mathbb{C} \setminus f_{n-1}(A_n)$ (i.e. outside of $B(0, R_n)$). This proves that the sequence f_n is uniformly Cauchy and hence it converges to a map f, that is,

$$f = \lim_{n\to\infty} f_n$$

which is again a homeomorphism by construction. Now, since f_n is quasiconformal for every n and $f_n(z) = f_{n-1}(z)$ everywhere except inside the ball $B(0, R_n)$, where $R_n \to 0$ as $n \to \infty$, the limit map f is absolutely continuous on almost every line parallel to the coordinate axes and differentiable almost everywhere.

It is helpful to note that each f_n leaves fixed all circles centered at the origin, so in particular we have $f_n(A_j) = A_j$ for each j, and therefore $\phi_{f_{n-1}(A_n)} = \phi_{A_n}$. Direct calculation shows that

$$|D\phi_{A_n}(z)| = |\partial\phi_{A_n}(z)| + |\bar{\partial}\phi_{A_n}(z)| = \begin{cases} 1 & |z| > R_n \\ \frac{|2+i\alpha_n|+|\alpha_n|}{2} & r_n \leq |z| \leq R_n \\ 1 & |z| < r_n \end{cases}$$

which allows us to estimate that

$$|\partial f(z)| + |\bar{\partial} f(z)| \leq 2\alpha_n \qquad \text{whenever } z \in A_n,$$

and $|Df(z)| \leq 1$ otherwise. Therefore, in order to have $Df(z) \in L^1_{loc}(\mathbb{C})$ it suffices that

$$\sum_n \alpha_n \, r_n^2 < +\infty. \tag{3.26}$$

This, together with the absolute continuity, guarantees $f \in W^{1,1}_{loc}(\mathbb{C})$. Also, since f is a homeomorphism, we have that $J_f(z) \in L^1_{loc}(\mathbb{C})$, and in fact $J(z, f) = 1$ at almost every $z \in \mathbb{C}$. Therefore, f is a homeomorphism of finite distortion, with distortion function

$$\mathbb{K}(z, f) = \frac{|Df(z)|^2}{J(z, f)} \leq \begin{cases} 4\alpha_n^2 & z \in A_n, \\ 1 & \text{otherwise.} \end{cases}$$

Especially, in order to have $\mathbb{K}(\cdot, f) \in L^p_{loc}$, it suffices to ensure the convergence of the series

$$\sum_{n=1}^{\infty} |A_n| (4\alpha_n^2)^p \simeq \sum_{n=1}^{\infty} \alpha_n^{2p} \, r_n^2 \tag{3.27}$$

which can be done by choosing α_n properly. Note that if (3.27) holds, then also (3.26) holds, because our choice of α_n will guarantee $\alpha_n \geq 1$. The last restriction to choose our α_n comes from rotational behavior of f. It is clear from the above construction that $f(0) = 0$, $f(1) = 1$ and

$$|\arg(f(r_n))| \geq \left| \arg\left(\left(\frac{1}{e}\right)^{1+i\alpha_n} \right) \right| = \alpha_n$$

for every r_n. Let us choose $\alpha_n = r_n^{-1/p}\ \log^{1/2}(1/\varphi(r_n))$. This implies that

$$|\arg(f(r_n))| \geq r_n^{-1/p}\ \log^{1/2}(1/\varphi(r_n)),$$

which shows that this map would be optimal for Theorem 3.1.4.
However, with this particular choice of α_n,

$$\sum_{n=1}^{\infty} \alpha_n^{2p}\, r_n^2 = \sum_{n=1}^{\infty} \log^p(1/\varphi(r_n))$$

which is certainly not finite. This means that $\mathbb{K}(\cdot, f) \notin L^p_{loc}$.

Hence we need to modify the construction by adding a stretching factor to our building blocks, which lets us reduce the distortion while preserving spiraling rate. This is precisely done by substituting the logarithmic spiral map $z|z|^{i\alpha} = ze^{i\alpha\log|z|}$ by a complex power $z|z|^{q+i\alpha} = z|z|^q\, e^{i\alpha\log|z|}$ at each iterate. Let us explain this process in detail.

Similarly as in the previous construction, we consider a rapidly decreasing sequence $\{r_n\}$ such that $r_{n+1} < \frac{r_n}{2e}$, $r_1 < \frac{1}{e}$ and set $R_n = er_n$. Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ we define the corresponding composition map as follows:

$$\phi_A(z) = \begin{cases} z & |z| > R \\ z\left|\frac{z}{R}\right|^{q-1}\, e^{i\alpha\log\frac{|z|}{R}} & r \leq |z| \leq R \\ z\left(\frac{r}{R}\right)^{q-1}\, e^{i\alpha\log\frac{r}{R}} & |z| < r \end{cases} \tag{3.28}$$

Note that we will always choose $q \geq 1$. Direct calculation shows that

$$|\partial\phi_A(z)| + |\overline{\partial}\phi_A(z)| = \begin{cases} 1 & |z| > R \\ R^{1-q}|z|^{q-1}\frac{|q+1+i\alpha|+|q-1+i\alpha|}{2} & r \leq |z| \leq R \\ R^{1-q}r^{q-1} & |z| < r \end{cases} \tag{3.29}$$

and also that

$$J(z,\phi_A) = \begin{cases} 1 & |z| > R \\ q\left(\frac{|z|}{R}\right)^{2(q-1)} & r \leq |z| \leq R \\ \left(\frac{r}{R}\right)^{2(q-1)} & |z| < r \end{cases} \tag{3.30}$$

whence

$$\mathbb{K}(z,\phi_A) = \begin{cases} 1 & |z| > R \\ \frac{(|q+1+i\alpha|+|q-1+i\alpha|)^2}{4q} & r \leq |z| \leq R \\ 1 & |z| < r \end{cases} \tag{3.31}$$

In particular, if $2 \leq q+1 \leq \alpha$, which will be satisfied for our choices of α and q, then one may estimate $\|\mathbb{K}(\cdot,\phi_A)\|_\infty \leq \frac{4\alpha^2}{q}$. Next, let us construct the sequence of maps f_n in an iterative way as follows. For $n = 1$, we set

$$f_1(z) = \phi_{A_1}(z) = \begin{cases} z & |z| < R_1 \\ z\left|\frac{z}{R_1}\right|^{q_1-1}\, e^{i\alpha_1\log\frac{|z|}{R_1}} & r_1 \leq |z| \leq R_1 \\ z\left(\frac{1}{e}\right)^{q_1-1}\, e^{-i\alpha_1} & |z| < r_1 \end{cases} \tag{3.32}$$

where q_1 and α_1 are to be determined later. Next, assuming we have $f_1, \ldots, f_{n-1}$, we define f_n for $n \geq 2$ as:

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

Note that $\phi_{f_{n-1}(A_n)}$ is determined by the inner and outer radii of $\phi_{f_{n-1}(A_n)}$ (which are already available since $f_1, \ldots, f_{n-1}$ are known) as well as for the parameters q_n and α_n, which will be determined later. Clearly, each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli A_i, $i \in \{1, ..., n\}$. Moreover, one can easily show that

$$\mathbb{K}(\cdot, f_n) = \prod_{j=1}^{n} \mathbb{K}(\cdot, f_{n-j} \circ \phi_{f_{n-j}(A_{n-j+1})}) = \prod_{j=1}^{n} \mathbb{K}(\cdot, \phi_{A_{n-j+1}})$$

so that $\mathbb{K}(z, f_n) \leq C \frac{\alpha_j^2}{q_j}$ whenever $z \in A_j$, $j = 1 \ldots n$ while $\mathbb{K}(\cdot, f_n) = 1$ otherwise. In a similar way, we can use that $|D\phi_A(z)| \leq C\alpha$ when $z \in A$ (and $|D\phi_A(z)| \leq 1$ at all other points) to obtain that $|Df_n| \leq C\alpha_j$ on A_j, $j = 1 \ldots n$, and $|Df_n| \leq 1$ otherwise.

By construction, we have $f_n(z) = f_{n-1}(z)$ whenever $z \notin B(0, R_n)$. Thus $\{f_n\}_n$ converges uniformly to a map $\bar{f}$, that is,

$$\bar{f} = \lim_{n\to\infty} f_n$$

which is again a homeomorphism by construction. A similar argument to the one before shows that $\bar{f}$ is absolutely continuous on almost every line parallel to the coordinate axis. For almost every fixed z_0 there is a neighbourhood of z_0 such that the sequence $\{f_n(z)\}_n$ remains constant for n very large and z in that neighbourhood. Therefore the same happens to the sequences $Df_n(z)$, $J(z, f_n)$ and $\mathbb{K}(z, f_n)$, and so their limits are precisely $D\bar{f}(z)$, $J(z, \bar{f})$ and $\mathbb{K}(z, \bar{f})$. Especially, in order to have $D\bar{f} \in L^1_{loc}$ it suffices that

$$\sum_{n=1}^{\infty} |A_n|\, \alpha_n < +\infty. \tag{3.33}$$

In case this holds true, then $\bar{f}$ is a homeomorphism in $W^{1,1}_{loc}$, and as a consequence its jacobian determinant $J(\cdot, \bar{f}) \in L^1_{loc}$. Moreover, in order to have $\mathbb{K}(\cdot, \bar{f}) \in L^p_{loc}$; $(p \geq 1)$ one needs to require that

$$\sum_{n=1}^{\infty} |A_n|\, \frac{\alpha_n^{2p}}{q_n^p} < +\infty. \tag{3.34}$$

Again, as it was the case for the pure rotation example, when $p > 1$ condition (3.34) implies (3.33) if $q_n^{\frac{p}{2p-1}} \leq \alpha_n$, and for $p = 1$ case we must verify $q_n \leq \alpha_n$. So, our parameters α_n and q_n need to be chosen according to these constraints as well as the purpose of $\bar{f}$ to be optimal for Theorem 3.1.4. To this end, note that $\bar{f}(0) = 0$, $\bar{f}(1) = 1$ and

$$\left|\arg\left(\bar{f}(r_n)\right)\right| \geq \left|\arg\left(\left(\frac{1}{e}\right)^{q_n + i\alpha_n}\right)\right| = |\alpha_n|, \tag{3.35}$$

which motivates us to choose

$$q_n = \begin{cases} \log \frac{e\, r_1}{|\varphi(r_1)|} & n = 1 \\ \log \left(\frac{e \cdot r_n \cdot \left(\frac{1}{e}\right)^{q_{n-1} + q_{n-2} + \ldots + q_1 - (n-1)}}{|\varphi(r_n)|} \right) & n \geq 2 \end{cases} \tag{3.36}$$

and

$$\alpha_n = h(r_n)\left(\log\frac{1}{|\varphi(r_n)|}\right)^{1/2} r_n^{-\frac{1}{p}} \tag{3.37}$$

where h is a monotone non-increasing gauge function such that $h(r) \to 0$ as $r \to 0$ which we specify later.

Next, we show that $q_n \leq \alpha_n$ for all $p \geq 1$, from which $q_n^{\frac{p}{2p-1}} \leq \alpha_n$ then also follows. At this point, we impose an ansatz on r_n:

$$r_n < \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)}, \tag{3.38}$$

which is feasible as the radii r_n can be assumed to decrease as fast as we want. Let us then recall our assumption on φ to satisfy compression bound:

$$|\varphi(z)| \geq e^{-g_{\varphi,p}(|z|)|z|^{-\frac{2}{p}}}, \tag{3.39}$$

where $g_{\varphi,p} : \mathbb{R} \to \mathbb{R}$ is some increasing gauge function such that $|g_{\varphi,p}| \to 0$ as $|z| \to 0$. Now, let us proceed with the calculations.

$$\begin{aligned} q_n &= \log\left(\frac{e\cdot r_n\cdot\left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)}}{|\varphi(r_n)|}\right) \\ &\leq \log\frac{1}{|\varphi(r_n)|} \\ &= \log^{\frac{1}{2}}\frac{1}{|\varphi(r_n)|}\cdot\log^{\frac{1}{2}}\frac{1}{|\varphi(r_n)|} \\ &\leq \log^{\frac{1}{2}}\frac{1}{|\varphi(r_n)|}\cdot\sqrt{g_{\varphi,p}(r_n)}\frac{1}{r_n^{\frac{1}{p}}} \\ &\leq \log^{\frac{1}{2}}\frac{1}{|\varphi(r_n)|}\cdot\frac{h(r_n)}{r_n^{\frac{1}{p}}} = \alpha_n \end{aligned}$$

where the last inequality holds for h converging to zero slowly enough. Note that, from [32, Theorem 1.6] we see that if $p = 1$ then the compression bound (3.39) is always satisfied with some g_φ. Thus our choices for q_n and α_n satisfy technical constrains.

Next, we show that estimate (3.34) governing integrability of the distortion holds true for $p \geq 1$. We start by estimating

$$\begin{aligned} |A_n|\frac{\alpha_n^{2p}}{q_n^p} &= C\,r_n^2\,\frac{h^{2p}\,r_n^{-2}\,\log^p\left(\frac{1}{|\varphi(r_n)|}\right)}{\log^p\left(\frac{e\cdot r_n\cdot\left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)}}{|\varphi(r_n)|}\right)} \\ &\leq C\,h^{2p}\,\frac{\log^p\left(\frac{1}{|\varphi(r_n)|}\right)}{\log^p\left(\frac{r_n^2}{|\varphi(r_n)|}\right)}. \end{aligned}$$

It is easy to check that $1 < \frac{\log^p\left(\frac{1}{|\varphi(r_n)|}\right)}{\log^p\left(\frac{r_n^2}{|\varphi(r_n)|}\right)} \leq 2^p$, using the condition (3.8), and therefore, up to constants, condition (3.34) is equivalent to

$$\sum_n h(r_n)^{2p} < +\infty$$

which we can always satisfy by choosing r_n small enough. Having (3.34) fulfilled, our map $\bar{f}$ is a mapping of finite distortion with $\mathbb{K}(\cdot, \bar{f}) \in L^p_{loc}$.

Next we must show that our mapping f has right compression and spiraling behaviour. Let us start with modulus and show that $|\bar{f}(r_n)| = |\varphi(r_n)|$ by calculating

$$\begin{aligned} |\bar{f}(r_n)| &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot r_n \left(\frac{r_n}{R_n}\right)^{q_n-1} \\ &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot r_n \left(\frac{1}{e}\right)^{q_n-1} \\ &= |\varphi(r_n)|, \end{aligned}$$

where the last line follows from the penultimate due to our choice of q_n.

For the spiraling part we must show that the rotation bound (3.10) holds true. But this follows directly from (3.35), (3.37) and from the above modulus equation. This concludes the proof of Theorem 3.1.5. □

Proof of Theorem 3.1.7. We prove Theorem 3.1.7 in two steps similarly to Theorem 3.1.5. In the first step we construct a map which *only rotates*. This map already provides the optimal result in the exponent scale. Then, as in the previous construction, we compose this map with radial stretching mapping and finish the proof.

Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ we define the corresponding rotation map ϕ_A as in (3.24). It is clear that $\phi_A : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and moreover it is conformal outside the annulus A. Furthermore, ϕ_A leaves fixed all circles centered at 0, and the Jacobian determinant $J(z, \phi_A) = 1$ for each z.

Next, we again consider sequence $\{r_n\}$ such that $0 < r_{n+1} < \frac{r_n}{2e}$, $r_1 < \frac{1}{e}$, and fix $R_n = er_n$. We then construct disjoint annuli $A_n = B(0, R_n) \setminus B(0, r_n)$ and a sequence of maps $\{f_n\}_n$ iteratively as before. That is, set f_1 as in (3.25) and define f_n for $n \geq 2$ as

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z)$$

for some $\alpha_n \geq 1$, to be determined later. We can use word by word the same arguments as before to deduce that the limit

$$f = \lim_{n\to\infty} f_n$$

is a homeomorphism with integrable distortion if

$$\sum_n \alpha_n \, r_n^2 < +\infty \tag{3.40}$$

and

$$\sum_{n=1}^{\infty} |A_n| 4\alpha_n^2 \simeq \sum_{n=1}^{\infty} \alpha_n^2\, r_n^2 < +\infty. \tag{3.41}$$

Moreover, as we will choose $\alpha_n > 1$, we see that in fact (3.41) implies (3.40). Hence we only need to choose α_n so that (3.41) is satisfied.
Furthermore, it is clear from the above construction that $f(0) = 0$, $f(1) = 1$ and

$$|\arg\left(f(r_n)\right)| \geq \left|\arg\left(\left(\frac{1}{e}\right)^{1+i\alpha_n}\right)\right| = \alpha_n \tag{3.42}$$

for every r_n. Since we want our map to be optimal for Corollary 3.1.6, we may be tempted to choose $\alpha_n = \frac{\log^{1/2}(1/r_n)}{r_n}$. Unfortunately such a choice does not meet the requirement (3.41) and instead we are forced to choose

$$\alpha_n = \frac{h(r_n)}{r_n},$$

where $h : [0, \infty) \to [0, \infty)$ is a monotonically decreasing gauge function such that $\lim_{r\to 0^+} h(r) = 0$. With this choice, (3.41) is fulfilled if

$$\sum_{n=1}^{\infty} \left(h(r_n)\right)^2 < +\infty,$$

which we can ensure by choosing small enough r_n. Note that this does not provide optimality for Corollary 3.1.6 in full generality, but it already gives the right order in the exponent scale.

Finally, we show that f^{-1} is Hölder continuous with exponent $\frac{1}{2}$. To this end, let us recall that our map f is actually a limit of iterates of logarithmic spiral maps inside the annuli $A_n = B(0, R_n) \setminus B(0, r_n)$. In particular, as shown in [7], if $\gamma \in \mathbb{R}$ then the basic logarithmic spiral map $g(z) = ze^{i\gamma \log|z|}$ is L-bilipschitz for a constant L such that $|\gamma| = L - \frac{1}{L}$. And thus for large $|\gamma|$ one roughly has $|\gamma| \simeq L$. Since our f_n behaves in the annulus A_n as a spiral map with $|\gamma| = \alpha_n$, we deduce that the bilipschitz constant of f_n on A_n is

$$L \simeq |\gamma| = \alpha_n = \frac{h(r_n)}{r_n}.$$

Let us now start the proof. We first consider the case where $x, y \in A_n$, and hence $f(x) = f_n(x)$, $f(y) = f_n(y)$. Since $r_n > C|x - y|$, we have

$$\begin{aligned} |f(x) - f(y)| = |f_n(x) - f_n(y)| &\gtrsim \frac{r_n}{h(r_n)}|x - y| \\ &\geq \frac{C}{h(r_n)}|x - y|^2 \\ &\geq C|x - y|^2 \end{aligned}$$

where we have used the bilipschitz nature of f_n on A_n. The fact that f is Hölder from below inside the annuli A_n with exponent 2 implies that in these sets f^{-1} is Hölder continuous with exponent $\frac{1}{2}$. Here we note, that f and f^{-1} are essentially the same mapping, only the direction of rotation is changed, and hence f is also Hölder continuous with exponent $\frac{1}{2}$ inside A_n.

Then we assume that $x, y \in D_n = B(0, r_n) \setminus B(0, R_{n+1})$. In this case f is of the form $ze^{i\beta}$, where $\beta \in \mathbb{R} \setminus \{0\}$, which is clearly an isometry and hence Hölder estimate inside D_n is trivial.

Next, we take $x \in A_n$ and $y \in D_n$. In particular, $|x| \geq |y|$. Then let w be the point on the outer boundary of D_n joining x and y. We have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(w)| + |f(w) - f(y)| \\ &\leq C|x - w|^{\frac{1}{2}} + |w - y| \\ &\leq 2C|x - y|^{\frac{1}{2}} \end{aligned}$$

The same happens if $x \in D_{n-1}$ and $y \in A_n$.

So it just remains to see what happens when points are further apart from each other. Let us first cover the case $x \in A_n = B(0, R_n) \setminus B(0, r_n)$ and $y \in B(0, R_{n+1})$. Let L be the line joining x and y. We divide it into three parts, viz., L_1, L_2 and L_3. Fix L_1 so that it connects x to a point a on the inner boundary of A_n, giving estimate

$$|f(x) - f(a)| = |f_n(x) - f_n(a)| \leq C|x - a|^{\frac{1}{2}}$$

Next, L_2 connects a to the crossing point of the line L and the inner boundary of D_n, which we denote by b. And since f is an isometry in D_n an estimate for line segment L_2 is trivial.

For L_3 part we note that from $2R_{n+1} < r_n < \frac{R_n}{2}$ we get that $|f(a)| > 2|f(b)|$ and hence

$$|f(b) - f(y)| \leq 2|f(b)| \leq 2|f(b) - f(a)| = 2|b - a|.$$

Combining these estimates we get

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(a)| + |f(a) - f(b)| + |f(b) - f(y)| \\ &\leq C|x - a|^{\frac{1}{2}} + |a - b| + 2|a - b| \\ &\leq C|x - y|^{\frac{1}{2}} \end{aligned}$$

The case $x \in D_n$ and $y \in B(0, r_{n+1})$ can be proved in a similar manner. Thus f is Hölder continuous with exponent $\frac{1}{2}$.

Here we again note that f and f^{-1} are essentially the same mapping modulo the direction of rotation, and hence f^{-1} is also Hölder continuous with the exponent $\frac{1}{2}$. Thus f is Hölder from below with the exponent 2.

As we discussed before, the above example approaches the borderline stated in Corollary 3.1.6, but it does not attain full optimality yet. To this end, we need to modify it by adding a stretching factor to our building blocks, which lets us increase rotation without increasing the distortion. This is done by replacing, at each iterate, the logarithmic spiral map $z|z|^{i\alpha} = ze^{i\alpha \log |z|}$ by a complex power $z|z|^{q+i\alpha} = z|z|^q \, e^{i\alpha \log |z|}$. Let us proceed with the details.

So, similarly as in the previous construction, we consider a rapidly decreasing sequence $\{r_n\}$ such that $r_{n+1} < \frac{r_n}{2e}$, $r_1 < \frac{1}{e}$ and fix $R_n = er_n$. Given an arbitrary annulus $A = B(0,R) \setminus B(0,r)$ we define the corresponding radial stretching combined with rotation map as in (3.28). As before we will choose $q \geq 1$.

The values of the differential, Jacobian and distortion of ϕ_A are already known from (3.29), (3.30) and (3.31). In particular, if $2 \leq q+1 < \alpha$ then one may estimate $\|\mathbb{K}(\cdot,\phi_A)\|_\infty \leq \frac{4\alpha^2}{q}$. Next, we construct the sequence of maps f_n in an iterative way as before. Let us set f_1 as in (3.32) and f_n for $n \geq 2$ as:

$$f_n(z) = \phi_{f_{n-1}(A_n)} \circ f_{n-1}(z).$$

Each $f_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, and conformal outside the annuli A_i, $i \in \{1,...,n\}$. Moreover, we still calculate distortion by

$$\mathbb{K}(\cdot, f_n) = \prod_{j=1}^{n} \mathbb{K}(\cdot, f_{n-j} \circ \phi_{f_{n-j}(A_{n-j+1})}) = \prod_{j=1}^{n} \mathbb{K}(\cdot, \phi_{A_{n-j+1}})$$

so that $\mathbb{K}(z, f_n) \leq C\frac{\alpha_j^2}{q_j}$ whenever $z \in A_j$, $j = 1 \dots n$ while $\mathbb{K}(\cdot, f_n) = 1$ otherwise. As before we also use $|D\phi_A(z)| \leq C\alpha$ when $z \in A$ (and $|D\phi_A(z)| \leq 1$ at all other points) to obtain that $|Df_n| \leq C\alpha_j$ on A_j, $j = 1 \dots n$, and $|Df_n| \leq 1$ otherwise.

Using the exact same arguments as before we see that for the limit

$$\bar{f} = \lim_{n\to\infty} f_n$$

to be a homeomorphism of integrable distortion it is enough to check that

$$\sum_{n=1}^{\infty} |A_n|\, \alpha_n < +\infty \tag{3.43}$$

and

$$\sum_{n=1}^{\infty} |A_n|\, \frac{\alpha_n^2}{q_n} < +\infty. \tag{3.44}$$

Note that as in the case of f, (3.44) implies (3.43) when $q_n < \alpha_n$ and so our parameters α_n and q_n need to be chosen such that (3.44) is satisfied as well as the purpose of $\bar{f}$ to be optimal for Corollary 3.1.6. Thus we choose

$$\alpha_n = \frac{h(r_n)}{r_n}\left(\beta \log\frac{1}{r_n}\right)^{1/2} \qquad\qquad q_n = \beta\log\frac{1}{r_n}, \qquad \beta \geq 1 \tag{3.45}$$

where h is any gauge function such that $h(r) \to 0$ as $r \to 0$ and the condition $q_n < \alpha_n$ is satisfied. Indeed, with these choices (3.44) becomes

$$\sum_n (h(r_n))^2 < +\infty$$

which, as before, may always be satisfied by choosing small enough r_n. Having (3.44) fulfilled, our map $\bar{f}$ is a mapping of finite distortion with $\mathbb{K}(\cdot, \bar{f}) \in L^1_{loc}$. Furthermore, since we can bound spiraling from below by α_n at the points r_n using the same estimate (3.42) as before, the resulting map $\bar{f}$ attains the optimal rotational behavior stated at Corollary 3.1.6 modulo the gauge function h which can be chosen to converge to 0 as slowly as desired.

Therefore, Theorem 3.1.7 will be proven once we show that $\bar{f}$ is Hölder from below.

To this end, we first observe that the composition of $z \mapsto ze^{i\alpha \log|z|}$ followed by $z \mapsto z|z|^{q-1}$ is precisely $z \mapsto z|z|^{q-1}e^{i\alpha \log|z|}$. This observation suggests us to decompose $\bar{f} = g \circ f$, where f is essentially the first example in this section (with slightly different choices for the constants α_n) and g is constructed by building blocks (3.28) with $\alpha = 0$ at each step. Morally, f leaves fixed all circles centered at 0 and only rotates inside the annuli A_n, while g conveniently stretches each A_n.

The Hölder nature of f^{-1} has already been proven when $\alpha_n = \frac{h(r_n)}{r_n}$. We need to show that our map f^{-1} is still Hölder continuous with our new choices for α_n, which we can estimate by

$$\alpha_n = \frac{h(r_n)}{r_n} \left(\beta \log \frac{1}{r_n}\right)^{1/2} \leq \sqrt{\beta} h(r_n)\, r_n^{-1/(1-\epsilon)} \tag{3.46}$$

for an arbitrary $\epsilon > 0$ and small enough r_n. This can be done by exactly the same proof as before once we check that f is Hölder from below inside the annuli A_n. To this end, let us consider two points $x, y \in A_n$ and note that $f(x) = f_n(x)$ and $f(y) = f_n(y)$. Since $r_n > C|x-y|$, using the estimate (3.46) gives

$$\begin{aligned} |f(x) - f(y)| = |f_n(x) - f_n(y)| &\gtrsim \frac{r_n^{1/(1-\epsilon)}}{h(r_n)} |x-y| \\ &\geq \frac{C}{h(r_n)} |x-y|^{1+\frac{1}{1-\epsilon}} \\ &\geq C|x-y|^{2+\epsilon} \end{aligned}$$

where we are using the bilipschitz nature of f_n in A_n. Therefore, in order to prove Theorem 3.1.7 it remains to prove that g is Hölder from below.

To this end, given any two points $x, y \in B(0,1)$, we can without loss of generality assume that $|y| \geq |x|$ and let w be the point for which $|w| = |x|$ and $\arg(w) = \arg(y)$. Now, as g is a radial stretching map, it follows that

$$|g(x) - g(y)| \geq \max\{|g(x) - g(w)|, |g(y) - g(w)|\}.$$

Moreover,

$$\max\{|x-w|, |y-w|\} \geq \frac{1}{2}|x-y|.$$

Therefore, it is enough to show that both $|g(x) - g(w)|$ and $|g(y) - g(w)|$ satisfy Hölder bounds from below. Note that if $x = 0$ then clearly $w = 0$ and we have only the radial part $|g(y) - g(w)|$.

Let us first check the term $|g(x) - g(w)|$. Since g maps radially circles centered at the origin to similar circles we see that $|g(x)-g(w)|$ gets contracted the same amount as the modulus $|g(x)|$ is contracted under g. Now we must consider two possibilities, either $x, w \in A_n$ or $x, w \in D_n$ for some n. Let us first assume $x, w \in A_n = B(0, R_n) \setminus B(0, r_n)$ for some n. Here we recall the ansatz (3.38) on r_n. Then we can estimate

$$\begin{aligned}|g(x)| &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot |x| \left(\frac{|x|}{R_n}\right)^{q_n-1} \\ &\geq r_n \cdot |x| \left(\frac{|x|}{R_n}\right)^{q_n-1} \\ &\geq r_n \cdot |x| \left(\frac{1}{e}\right)^{q_n-1} = e \cdot r_n^{1+\beta} \cdot |x|\end{aligned}$$

for any $x \in A_n$, where in the last step we use (3.45). Therefore,

$$\begin{aligned}|g(x)-g(w)| &\geq e \cdot r_n^{1+\beta} \cdot |x-w| \\ &\geq C \cdot |x-w|^{2+\beta}\end{aligned}$$

since $|x-w| < C \cdot r_n$ for some fixed constant $C > 0$ when $x, w \in A_n$.

Next, let $x, w \in D_n = B(0, r_n) \setminus B(0, R_{n+1})$ for some n. Using (3.38) we get

$$\begin{aligned}|g(x)| &\geq c \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot r_n^{\beta} \cdot |x| \\ &\geq c \cdot r_n^{1+\beta} \cdot |x|.\end{aligned}$$

Thus we can use similar argument as in the previous case to estimate

$$\begin{aligned}|g(x)-g(w)| &\geq c \cdot r_n^{1+\beta} \cdot |x-w| \\ &\geq c \cdot |x-w|^{2+\beta}\end{aligned}$$

since $|x-w| < c \cdot r_n$ for some fixed constant $c > 0$ when $x, w \in D_n$.

Since the set $\mathbb{D} \setminus \{0\}$ is partitioned by separated annuli A_n and D_n we have thus proven that $|g(x)-g(w)|$ satisfies Hölder estimates from below.

Finally, let us prove the Hölder estimates from below for the term $|g(y)-g(w)|$. As the mapping g is radial, we can assume that y and w are real. We intend to use the Fundamental Theorem of Calculus, and thus have to estimate the differential from below. Using (3.38), as well as the facts that $q_n > 1$ and $R_n = e r_n$, we can estimate for any real number $t \in [r_n, R_n]$ that

$$\begin{aligned}g'(t) &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot q_n \cdot \left(\frac{t}{R_n}\right)^{q_n-1} \\ &\geq r_n\, q_n \cdot \left(\frac{r_n}{R_n}\right)^{q_n-1} \\ &= e\, q_n\, r_n^{1+\beta} \\ &\geq c\beta \cdot t^{1+\beta} \log\left(e + \frac{1}{t}\right).\end{aligned}$$

Next, if $t \in [R_{n+1}, r_n]$, we have

$$\begin{aligned} g'(t) &= \left(\frac{1}{e}\right)^{q_{n-1}+q_{n-2}+\ldots+q_1-(n-1)} \cdot \left(\frac{1}{e}\right)^{q_n-1} \\ &\geq e \cdot r_n^{1+\beta} \\ &\geq c \cdot t^{1+\beta} \end{aligned}$$

Thus, as before, since $(0,1)$ is partitioned by the intervals $[r_n, R_n]$, $[R_{n+1}, r_n]$ and $[R_1, 1)$, we end up getting that

$$g'(t) \geq c \cdot t^{1+\beta}$$

for every $t \in (0,1)$. Now, we use the fundamental theorem of calculus to get

$$\begin{aligned} |g(y) - g(w)| &= \int_w^y g'(t)dt \\ &\geq \int_w^y c \cdot t^{1+\beta} dt \\ &= C(\beta)\left(y^{2+\beta} - w^{2+\beta}\right) \\ &\geq C|y-w|^{2+\beta} \end{aligned}$$

This proves that the second term is Hölder from below as well, which in turn proves that g is Hölder from below with exponent $(2+\beta)$. This finishes the proof of Theorem 3.1.7. □

Chapter 4

Nonlinear transport equations and quasiconformal mappings

4.1 Introduction

In this chapter we prove existence of global in time solutions to the following active scalar equation,

$$\begin{cases} \frac{d}{dt}\omega + \mathbf{v}\cdot\nabla\omega = 0, \\ \mathbf{v}(t,\cdot) = K * \omega(t,\cdot), \\ \omega(0,\cdot) = \omega_0. \end{cases} \tag{4.1}$$

In the above system, one has

$$K(z) = \frac{e^{i\theta}}{2\pi z} = \frac{1}{2\pi}\,\frac{(x\,\cos\theta + y\,\sin\theta, x\,\sin\theta - y\,\cos\theta)}{x^2+y^2}$$

and $\theta \in [0, 2\pi]$ is fixed, while $\omega_0 \in L^\infty$ is a given compactly supported and real valued function. This model arises as a natural counterpart to the classical *planar Euler system of equations in vorticity form*, which is given also by (4.1) but with a different choice for the kernel K, namely

$$K(z) = \frac{i}{2\pi\,\bar{z}} = \frac{1}{2\pi}\,\frac{(-y,x)}{x^2+y^2}$$

In both cases, the quantity $\partial_t + \mathbf{v}\cdot\nabla$ is called the *material derivative* of the unknown $\omega : [0,\infty)\times\mathbb{C} \to \mathbb{R}$, and $\mathbf{v}$ is called the *velocity*. In Euler system, $\mathbf{v}$ represents the velocity field of a fluid, and ω is known as the *vorticity of the fluid*.

In Euler's setting, the Biot-Savart law $\mathbf{v} = K * \omega$ can be written in terms of complex derivatives as

$$\partial\,\mathbf{v} = \frac{i\omega}{2}.$$

where $\partial = \frac{\partial_x - i\partial_y}{2}$ denotes the classical complex derivative. Since ω is real valued, this ensures that $\operatorname{div}\mathbf{v} = 0$ as well as $\operatorname{curl}\mathbf{v} = \omega$. The first condition says that the fluid under consideration is *incompressible*, and so the flow map $X(t,z) = X_t(z)$ produced by $\mathbf{v}$ does not distort area, i.e. $|X_t(E)| = |E|$ for any Lebesgue measurable set $E \subset \mathbb{C}$. On the other hand, and under enough regularity, the transport structure of the equation guarantees for the solution the following representation formula,

$$\omega(t, X(t,x)) = \omega_0(x). \tag{4.2}$$

Having X to be a measure preserving map, Lebesgue null sets are also preserved and so both L^1 and L^∞ norms of $\omega(t,\cdot)$ are constant in time, which helps in making all compactness arguments work. This is the reason why both the incompressibility of the fluid and the boundedness of curl $\mathbf{v}$ were essential for Yudovich [48] to prove existence and uniqueness of global in time solutions to the Euler system under the assumption $\omega_0 \in L^\infty$.

In contrast, in our setting (4.1), the kernel ensures instead that

$$\overline{\partial}\,\mathbf{v} = \frac{e^{i\theta}\,\omega}{2}$$

where now $\overline{\partial} = \frac{\partial_x + i\partial_y}{2}$ denotes the anticonformal complex derivative. Especially, div $\mathbf{v}$ needs not be identically 0, so the vector field $\mathbf{v}$ is not anymore incompressible. Moreover, classical Calderón-Zygmund Theory can be used to show that now div $\mathbf{v}$ and curl $\mathbf{v}$ may be unbounded functions, even for bounded and compactly supported ω_0. Thus the preservation of Lebesgue null sets, or of L^∞ and L^1 norms of $\omega(t,\cdot)$ may seem unclear or even be false. Nevertheless, still the transport structure of the equation tells that the solution $\omega(t,\cdot)$ admits again the representation formula (4.2), although now the flow $X(t,\cdot)$ needs not be measure preserving.

Recently it has been shown in certain linear transport models [20, 21, 22] that their well-posedness do not depend on the measure-preservation property of the flow and, instead, the preservation of Lebesgue null sets is the only requirement. Such models already show that this is possible even if the velocity field has unbounded divergence.

In the same way $\|\partial\,\mathbf{v}\|_{L^\infty}$ keeps bounded in time for any Yudovich solution to the Euler system, in our setting (4.1) the quantity $\|\overline{\partial}\,\mathbf{v}\|_{L^\infty}$ keeps bounded in time as long as one is able to show the preservation of Lebesgue null sets, rather than the preservation of Lebesgue measure through the flow. Having uniform in time bounds for $\|\overline{\partial}\,\mathbf{v}\|_{L^\infty}$ immediately drives our attention to H.M. Reimann's paper [43]. There it was shown that such vector fields produce flows $X(t,x)$ with the very special property of being quasiconformal for every $t > 0$. Quasiconformal maps are known to Geometric Function Theory experts to be a very well understood class of homeomorphisms preserving Lebesgue null sets, and their compactness properties make them specially suitable for solving certain elliptic PDE problems. This time, though, we will use them for a purely hyperbolic PDE. Our main result is as follows.

Theorem 4.1.1. *If the initial datum $\omega_0 \in L^\infty$ has compact support, there exists a solution $\omega \in L^1([0,T], L^\infty)$ of (4.1) for every $T > 0$.*

This chapter is structured as follows. In Section 4.2 we remind a proof of the smooth theory for a general kernel K. In Section 4.3 we prove the L^∞ theorem for the particular case $K(z) = \frac{e^{i\theta}}{2\pi z}$. In Section 4.4 we prove the equivalence of the velocity field formulation and the vorticity formulation of the Cauchy system. Finally, in Section 4.5 we discuss about the hindrance one faces while trying to prove uniqueness of solutions to Cauchy using similar techniques to the one used for proving uniqueness in Euler's setting.

4.2 Existence theory for $\omega_0 \in C^\gamma$

In this section, we follow the lines of [13] to prove that the system (4.1) can be proven, for any $\omega_0 \in C^\gamma$, to have a unique solution ω which is global-in-time. As specified at [16], this scheme works for many kernels $K(z)$, and not just Euler's or Cauchy's. We sketch the details for the reader's convenience.

As it happens to any transport equation under enough smoothness, if $\frac{d}{dt}\omega + \mathbf{v}\cdot\nabla\omega = 0$ then the solution ω is obtained from the initial value $\omega(0,\cdot) = \omega_0$ through the trajectories of the defining velocity field $\mathbf{v}$,

$$\omega(t, X(t,\alpha)) = \omega_0(\alpha)$$

where

$$\begin{cases} \frac{d}{dt} X(t,\alpha) = \mathbf{v}(t, X(t,\alpha)) \\ X(0,\alpha) = \alpha \end{cases} \tag{4.3}$$

provided these trajectories do exist and are nice enough. In our setting, though, the equation is nonlinear as $\mathbf{v}$ depends on the unknown. We assume that $\mathbf{v} = K * \omega$ at every point and every time. Here K is a Kernel satisfying conditions **K1, K2** as in the previous section. In particular, on the trajectories,

$$\begin{aligned} \mathbf{v}(t, X(t,\alpha)) &= \int \omega(t,w)\, K(X(t,\alpha) - w)\, dA(w) \\ &= \int \omega(t, X(t,\beta)\, K(X(t,\alpha) - X(t,\beta))\, |\det(DX(t,\beta))|\, dA(\beta) \\ &= \int \omega_0(\beta)\, K(X(t,\alpha) - X(t,\beta))\, |\det(DX(t,\beta))|\, dA(\beta). \end{aligned} \tag{4.4}$$

This establishes a direct connection between ω_0 and $\mathbf{v}$ through X. That is to say, to each ω_0 we can associate a map $F = F_{\omega_0}$ that sends a given flow X to a new map $F(X)$, which is defined by

$$F(X)(\alpha) = \int \omega_0(\beta)\, K(X(\alpha) - X(\beta))\, |\det(DX(\beta))|\, dA(\beta). \tag{4.5}$$

This map is constructed as $F : \mathbf{E} \to \mathbf{E}$ in an autonomous way (i.e. time-independent) , on a Banach space $\mathbf{E}$ to be defined later. Symbolically,

$$F(X) = (K * (\omega_0 \circ X^{-1})) \circ X.$$

This representation allows us to see F as an application on a set of maps X. Once F is constructed, we are then led to look for solutions $X = X(t,\cdot)$ of the following Banach valued ODE,

$$\begin{cases} \frac{d}{dt} X(t,\cdot) = F(X(t,\cdot)), \\ X(0,\cdot) = \mathbf{Id} \end{cases} \tag{4.6}$$

The strategy for finding solutions of (4.1) consists of solving (4.6). This is done in two steps: first, one finds local-in-time solutions, and then one shows secondly that these local-in-time solutions are indeed defined for every time.

4.2.1 Local-in-time solutions

This is done with the help of the following Banach space version of the classical Picard Theorem. We follow the scheme of [13], and write the details for the reader's convenience.

Theorem 4.2.1. *[13, Theorem 4.1] Let $\mathcal{O} \subset \mathbf{E}$ be an open subset of the Banach space $\mathbf{E}$. Let $F : \mathcal{O} \to \mathbf{E}$ be locally Lipschitz continuous, that is,*

$$\limsup_{y \to x} \frac{\|F(y) - F(x)\|_{\mathbf{E}}}{\|y - x\|_{\mathbf{E}}} < \infty.$$

For each $x_0 \in \mathcal{O}$ there is a real number $T = T(x_0) > 0$ such that the ODE

$$\begin{cases} \dot{X} = F(X), \\ X(0) = x_0, \end{cases}$$

admits a unique classical solution $X \in C^1((-T, T); \mathcal{O})$.

The job consist of finding appropriate $\mathcal{O}$ and $\mathbf{E}$ so that the map F given in (4.5) is in the hypothesis of the above theorem. To this end, we set

$$\mathbf{E} = \{X : \mathbb{C} \to \mathbb{C}; \|X\|_{1,\gamma} < \infty\}$$

and for every fixed $M > 0$

$$\mathcal{O}_M = \left\{ X \in \mathbf{E} : \|X\|_{1,\gamma} < M, \inf_{z \in \mathbb{C}} \det DX(z) > \frac{1}{M} \right\}.$$

Clearly, $X(z) = z$ belongs to $\mathcal{O}_M$, and all elements in $\mathcal{O}_M$ are local homeomorphisms by the inverse function theorem. As in [13, Lemma 4.4] one can show that they are actually global homeomorphisms, as a consequence of Hadamard's theorem, which can also be relaxed by means of a result by John which asserts that local homeomorphisms $X : \mathbb{C} \to \mathbb{C}$ satisfying

$$\liminf_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} > \epsilon > 0, \qquad \text{for all } x_0 \in \mathbb{C}$$

are automatically global homeomorphisms, see [39, Theorem 8]. In the following result, dA denotes the area measure in the plane (that is, the 2-dimensional Lebesgue measure).

Lemma 4.2.2. *For each $M > 1$ the set $\mathcal{O}_M \subset \mathbf{E}$ is non-empty, open, and consists of homeomorphisms $X : \mathbb{C} \to \mathbb{C}$. Moreover, the image measures $X_\# A$ and $X^{-1}_\# A$ are absolutely continuous with bounded densities. Both bounds depend only on M.*

Proof. Non-emptiness is clear since $X(z) = z$ belongs to $\mathcal{O}_M$. Its openness comes as $X \mapsto \|X\|_{1,\gamma}$ is continuous on $\mathbf{E}$, while $X \mapsto \inf_{z \in \mathbb{C}} \det(DX)(z)$ is lower semi-continuous on $\mathbf{E}$. The fact that the elements of $\mathcal{O}_M$ are global homeomorphisms follows both by Hadamard's or by John's lemma. To finish the proof, just note that if $X \in \mathcal{O}_M$ and A denotes the area measure then $X_\# A$ is absolutely continuous with respect to dA, with bounded density,

$$d(X_\# A) = (\det DX^{-1})\, dA = \frac{1}{\det DX(X^{-1})}\, dA \leq M\, dA$$

and in particular if $|E| = 0$ then $|X^{-1}(E)| = 0$. Similarly, $X^{-1}_\# A$ is also absolutely continuous with bounded density, as $X \in C^{1,\gamma}$ and so in particular $DX \in L^\infty$. □

Proposition 4.2.3. *Let ω_0 be compactly supported and such that $[\omega_0]_\gamma < \infty$ for some $0 < \gamma < 1$. Let $M \geq 1$ be fixed. For each $X \in \mathcal{O}_M$, define*

$$F(X) = (K * (\omega_0 \circ X^{-1})) \circ X.$$

Then:

(a) $F : \mathcal{O}_M \to \mathbf{E}$ is well defined.

(b) F is locally Lipschitz continuous on $\mathcal{O}_M$.

Proof. The first part is very easy. Indeed, after the change of variables $\alpha = X^{-1}(x)$ we can represent

$$F(X)(X^{-1}(x)) = \int K(x-y)\, \omega_0(X^{-1}(y))\, dA(y).$$

It is worth mentioning that $X \in \mathcal{O}_M$ is a sufficient condition to legitimate the change of variables formula. From this, combined with Lemma 1.3.2, one gets

$$\begin{aligned} \|F(X)\|_{1,\gamma} &\leq C\, \|(F(X))(X^{-1})\|_{1,\gamma}\, \|X^{-1}\|_{1,\gamma} \\ &= C\, \|K * (\omega_0(X^{-1}))\|_{1,\gamma}\, \|X^{-1}\|_{1,\gamma} \end{aligned}$$

and we are left to estimate the first factor on the right hand side. To this end, we use Lemmas 1.3.4 and 4.2.2 and obtain

$$\begin{aligned} \|K * (\omega_0(X^{-1}))\|_{1,\gamma} &\leq C(K, \gamma, |\operatorname{supp}(\omega_0(X^{-1}))|)\, \|\omega_0(X^{-1})\|_\gamma \\ &\leq C(K, \gamma, |\operatorname{supp}(\omega_0)|, M)\, \|\omega_0\|_\gamma \end{aligned}$$

Hence

$$\|F(X)\|_{1,\gamma} \leq C(K, \gamma, |\operatorname{supp}(\omega_0)|, M)\, \|\omega_0\|_\gamma$$

In particular, if $X \in \mathcal{O}_M$ then $F(X) \in \mathbf{E}$ whence (a) follows.

To see the bound for the difference quotients, we will prove that if $X \in \mathcal{O}_M$ then the differential $F'(X) : \mathbf{E} \to \mathbf{E}$ is a bounded linear operator, with the following bound,

$$\|F'(X)Y\|_{1,\gamma} \leq C(M)\, \|\omega_0\|_\gamma\, \|Y\|_{1,\gamma}. \tag{4.7}$$

After this, one can immediately deduce that

$$\begin{aligned} \|F(X_2) - F(X_1)\|_{1,\gamma} &= \left\| \int_0^1 \frac{d}{d\epsilon} F(X_1 + \epsilon(X_2 - X_1))\, d\epsilon \right\|_{1,\gamma} \\ &\leq \int_0^1 \left\| \frac{d}{d\epsilon} F(X_1 + \epsilon(X_2 - X_1)) \right\|_{1,\gamma} d\epsilon \\ &\leq \int_0^1 \|F'(X_1 + \epsilon(X_2 - X_1)) \cdot (X_2 - X_1)\|_{1,\gamma}\, d\epsilon \leq\; C(M)\, \|\omega_0\|_\gamma\, \|X_2 - X_1\|_{1,\gamma} \end{aligned}$$

and so local Lipschitz continuity is automatic. To do this, we start by finding appropriate bounds for all directional derivatives. So, let us fix a direction in the tangent space $Y \in E$. From the definition of F, we have

$$
\begin{aligned}
F(X+\epsilon Y)(\alpha) &= \int \omega_0(\beta) K((X+\epsilon Y)(\alpha) - (X+\epsilon Y)(\beta)) J(\beta, X+\epsilon Y)\, dA(\beta) \\
&= \int \omega_0(\beta) K((X(\alpha) - X(\beta)) + \epsilon(Y(\alpha) - Y(\beta))) J(\beta, X+\epsilon Y)\, dA(\beta)
\end{aligned}
$$

We now have to differentiate the above integral in ϵ, and evaluate at $\epsilon = 0$. The first term can be formally handled as follows,

$$
\frac{d}{d\epsilon} \left(K((X(\alpha) - X(\beta)) + \epsilon(Y(\alpha) - Y(\beta))) \right) |_{\epsilon=0} = DK((X(\alpha) - X(\beta))) \cdot (Y(\alpha) - Y(\beta))
$$

Integrals with this integrand inside might look delicate, as the singularity of DK may not be locally integrable. However, this is not a problem since the smoothness of both X, Y prevents us to have such integrability problems. Yet still one may simply consider this factor in the principal value sense, to be at the safe side. Concerning the second factor, let us remind now the classical Jacobi formula, which states for smooth invertible matrix valued functions $A(\epsilon)$ that

$$
\frac{d}{d\epsilon} \det A(\epsilon) = \det A(\epsilon) \ \mathrm{Tr}(A(\epsilon)^{-1}\, A'(\epsilon))
$$

As a consequence,

$$
\frac{d}{d\epsilon} J(\beta, X+\epsilon Y) = J(\beta, X+\epsilon Y) \ \mathrm{Tr}(D(X+\epsilon Y)(\beta)^{-1}\, DY(\beta))
$$

whence at $\epsilon = 0$

$$
\frac{d}{d\epsilon} J(\beta, X+\epsilon Y)|_{\epsilon=0} = J(\beta, X) \ \mathrm{Tr}(DX(\beta)^{-1}\, DY(\beta))
$$

hence

$$
\begin{aligned}
F'(X)(Y)(\alpha) &= \lim_{\epsilon\to 0} \frac{F(X+\epsilon Y)(\alpha) - F(X)(\alpha)}{\epsilon} = \frac{d}{d\epsilon} F(X+\epsilon Y)(\alpha)|_{\epsilon=0} \\
&= \int \omega_0(\beta) K(X(\alpha) - X(\beta))\, \mathrm{Tr}(DX(\beta)^{-1}\, DY(\beta))\, J(\beta, X)\, dA(\beta) \\
&+ p.v. \int \omega_0(\beta) DK(X(\alpha) - X(\beta)) \cdot (Y(\alpha) - Y(\beta))\, J(\beta, X)\, dA(\beta) \\
&= G_1(X)Y(\alpha) + G_2(X)Y(\alpha)
\end{aligned}
$$

Now the job consists of finding bounds

$$
\|G_i(X)(Y)\|_{1,\gamma} \leq C(M)\, \|\omega_0\|_\gamma \|Y\|_{1,\gamma}, \quad i = 1, 2,
$$

and the claim will follow. For this, we start with G_1. We proceed as follows,

$$
\begin{aligned}
\|G_1(X)(Y)\|_{1,\gamma} &\leq C(M) \|G_1(X)(Y) \circ X^{-1}\|_{1,\gamma} \\
&= C(M) \|K * (\omega_0(X^{-1}) \ \mathrm{Tr}(DX(X^{-1})^{-1}\, DY(X^{-1})))\|_{1,\gamma} \\
&\leq C(K, \gamma, |\operatorname{supp} \omega_0(X^{-1})|)\, \|\omega_0(X^{-1}) \ \mathrm{Tr}(DX(X^{-1})^{-1}\, DY(X^{-1}))\|_\gamma \\
&\leq C(K, \gamma, |\operatorname{supp} \omega_0(X^{-1})|, M)\, \|\omega_0(X^{-1})\|_\gamma \, \| \mathrm{Tr}(DX(X^{-1})^{-1}\, DY(X^{-1}))\|_\gamma \\
&\leq C(K, \gamma, |\operatorname{supp} \omega_0|, M)\, \|\omega_0\|_\gamma \, \| \mathrm{Tr}((DX)^{-1}\, DY)\|_\gamma \\
&\leq C(K, \gamma, |\operatorname{supp} \omega_0|, M)\, \|\omega_0\|_\gamma \, \|Y\|_{1,\gamma}
\end{aligned}
$$

For G_2, the argument is a little more delicate since derivatives of the kernel K appear. However, this is not a problem since we end up getting a commutator between the Calderón-Zygmund operator $p.v.DK$ and the pointwise multiplier $Y \circ X^{-1}$, and therefore all the integrals are absolutely convergent. Namely,

$$\begin{aligned} G_2(X)(Y)(X^{-1}(x)) &= \int \omega_0(X^{-1}(y))\, DK(x-y) \cdot (Y(X^{-1}(x)) - Y(X^{-1}(y)))\, dA(y) \\ &= [p.v.DK, Y \circ X^{-1}](\omega_0 \circ X^{-1})(x) \end{aligned}$$

As a consequence,

$$\begin{aligned} \|G_2(X)(Y)\|_{1,\gamma} &\leq C(M)\, \|G_2(X)(Y) \circ X^{-1}\|_{1,\gamma} \\ &= C(M)\, \|[p.v.DK, Y \circ X^{-1}](\omega_0 \circ X^{-1})\|_{1,\gamma} \\ &\leq C(M)\, \|[p.v.DK, Y \circ X^{-1}]\|_{C^\gamma \to C^{1,\gamma}}\, \|(\omega_0 \circ X^{-1})\|_\gamma \end{aligned}$$

Now, classical results from harmonic analysis allow us to state that

$$\|[p.v.DK, Y \circ X^{-1}]\|_{C^\gamma \to C^{1,\gamma}} \leq C(K,\gamma)\, \|Y \circ X^{-1}\|_{1,\gamma} \leq C(K,\gamma,M)\, \|Y\|_{1,\gamma}$$

whence

$$\|G_2(X)(Y)\|_{1,\gamma} \leq C(K,\gamma,M)\, \|Y\|_{1,\gamma}\, \|\omega_0\|_\gamma$$

as desired. The claim follows. □

From the local existence theorem, to each initial condition in $\mathcal{O}_M$ we can associate a unique trajectory, well defined on a maximal time interval that depends on the Lipschitz constant as well as on the norm of the initial condition. In our setting, this means that for each fixed $M \geq 1$, there is a time $T^* = T^*(M) > 0$ such that the O.D.E.

$$\begin{cases} \dot{X} = F_{\omega_0}(X) \\ X(0) = \mathbf{Id} \end{cases}$$

has a unique solution $X \in C^1((-T^*, T^*), \mathcal{O}_M)$, and T^* is the largest possible with this property. This quantity T^* is called the maximal time of existence, and depends on the local Lipschitz constant of F on $\mathbf{Id}$ (bounded by $C(M)\, \|\omega_0\|_\gamma$, as proven in (4.7)) as well as on $\|\,\mathbf{Id}\,\|_{\mathbf{E}}$.

Once such X is proven to exist and be unique locally in time, then it is immediate to check that $\omega(t,z)$ defined by

$$\omega(t, X(t,z)) = \omega_0(z)$$

solves (4.1) in the weak sense with kernel K satisfying **K1** and **K2**, and initial data ω_0, at least in $[0,T^*) \times \mathbb{C}$. That is,

$$-\int \omega_0(z)\, \varphi(0,z)\, dA(z) - \iint \omega(t,z)\, \partial_t \varphi(t,z)\, dA(z)\, dt - \iint \omega(t,z)\, \operatorname{div}(\mathbf{v}\, \varphi)(t,\cdot)\, dA(z) dt = 0$$

for each $\varphi \in C_c^\infty([0,T^*) \times \mathbb{C})$. The magnitude of T^* is the following question to be addressed.

4.2.2 Global-in-time solutions

In this section we want to connect the quantities $\|X\|_{L^\infty((0,T),C^{1,\gamma})}$ and $\|\omega\|_{L^1((0,T),L^\infty)}$ and see to which extent their finiteness determines global-in-time existence. We start with the following three lemmas.

Lemma 4.2.4. *If $\omega_0 \in C^\gamma$, and $\omega \in L^1(0,T;L^\infty)$ solves (4.1) with $\mathbf{v}$ given by (4.4), then*

$$\|\omega(t,\cdot)\|_\gamma \le \|\omega_0\|_\gamma \exp\left(\gamma \int_0^t \|D\mathbf{v}(s,\cdot)\|_\infty ds\right).$$

Proof. From (4.2) we see that

$$\omega(t,x) = \omega_0(X^{-1}(t,x))$$

where $X(t,x)$ denotes the flow of v, and $X^{-1}(t,\cdot)$ denotes the inverse map of $X(t,\cdot)$. Thus

$$\begin{aligned}
|\omega(t,x) - \omega(t,y)| &= |\omega_0(X^{-1}(t,x)) - \omega_0(X^{-1}(t,y))| \\
&\le [\omega_0]_\gamma |X^{-1}(t,x) - X^{-1}(t,y)|^\gamma \\
&\le [\omega_0]_\gamma \, \|DX^{-1}(t,\cdot)\|_\infty^\gamma |x-y|^\gamma
\end{aligned}$$

Now, keeping in mind that

$$\frac{d}{dt} X^{-1}(t,x) = -\,\mathbf{v}(t, X^{-1}(t,x))$$

we obtain

$$\|DX^{-1}(t,\cdot)\|_\infty \le \exp\left(\int_0^t \|D\,\mathbf{v}(s,\cdot)\|_\infty \, ds\right)$$

whence

$$[\omega(t,\cdot)]_\gamma \le [\omega_0]_\gamma \exp\left(\gamma \int_0^t \|D\,\mathbf{v}(s,\cdot)\|_\infty \, ds\right).$$

Therefore

$$\begin{aligned}
\|\omega(t)\|_\gamma &= \|\omega(t)\|_\infty + [\omega(t)]_\gamma \\
&\le \|\omega_0\|_\infty + [\omega_0]_\gamma \exp\left(\gamma \int_0^t \|D\,\mathbf{v}(s,\cdot)\|_\infty \, ds\right) \\
&\le \|\omega_0\|_\gamma \exp\left(\gamma \int_0^t \|D\,\mathbf{v}(s,\cdot)\|_\infty \, ds\right)
\end{aligned}$$

as claimed. □

Lemma 4.2.5. *Let $\mathbf{v}, \omega, X$ be linked as above. Then $\|X(t,\cdot)\|_{1,\gamma}$ can be bounded in terms of $\|D\mathbf{v}\|_{L^1((0,t),L^\infty)}$.*

Proof. The ODE $\dot{X} = \mathbf{v}(t,X)$ implies at $z=0$ that

$$|X(t,0)| \le \int_0^t |\,\mathbf{v}(s,X(s,0))|ds \le \int_0^t \|\,\mathbf{v}(s,\cdot)\|_\infty ds. \tag{4.8}$$

In order to bound $\|DX(t,\cdot)\|_\infty$, we use the first variational equation $\dot{DX} = D\,\mathbf{v}(X)\,DX$ to deduce that

$$|DX(t,z)| \le \exp\int_0^t |D\,\mathbf{v}(s,X(s,z))|ds \le \exp\int_0^t \|D\,\mathbf{v}(s,\cdot)\|_\infty \, ds \tag{4.9}$$

Bounding the C^γ norm of $DX(t,\cdot)$ is a little bit rigorous. One can easily see that

$$\begin{aligned}\frac{d}{dt}(DX(t,x)-DX(t,y)) &= D\,\mathbf{v}(t,X(t,x))DX(t,x)-D\,\mathbf{v}(t,X(t,y))DX(t,y)\\ &= D\,\mathbf{v}(t,X(t,x))(DX(t,x)-DX(t,y))+(D\,\mathbf{v}(t,X(t,x))-D\,\mathbf{v}(t,X(t,y)))DX(t,y)\end{aligned}$$

whence

$$\begin{aligned}\frac{d}{dt}\left(\frac{|DX(t,x)-DX(t,y)|}{|x-y|^\gamma}\right) &\le \frac{1}{|x-y|^\gamma}\left|\frac{d}{dt}\left(DX(t,x)-DX(t,y)\right)\right|\\ &\le |D\,\mathbf{v}(t,X(t,x))|\,\frac{|DX(t,x)-DX(t,y)|}{|x-y|^\gamma}+\frac{|D\,\mathbf{v}(t,X(t,x))-D\,\mathbf{v}(t,X(t,y))|}{|x-y|^\gamma}\,|DX(t,y)|\\ &\le \|D\,\mathbf{v}(t,\cdot)\|_\infty\,[DX(t,\cdot)]_\gamma+[D\,\mathbf{v}(t,\cdot)]_\gamma\,\|DX(t,\cdot)\|_\infty^{1+\gamma}\end{aligned}$$

and therefore

$$\frac{d}{dt}[DX(t,\cdot)]_\gamma \le \|D\,\mathbf{v}(t,\cdot)\|_\infty\,[DX(t,\cdot)]_\gamma+[D\,\mathbf{v}(t,\cdot)]_\gamma\,\|DX(t,\cdot)\|_\infty^{1+\gamma}$$

We now infer that

$$\begin{aligned}\frac{d}{dt}\|DX(t,\cdot)\|_\gamma &= \frac{d}{dt}\|DX(t,\cdot)\|_\infty+\frac{d}{dt}[DX(t,\cdot)]_\gamma\\ &\le \|D\,\mathbf{v}(t,\cdot)\|_\infty\,\|DX(t,\cdot)\|_\gamma+\|D\,\mathbf{v}(t,\cdot)\|_\gamma\,\|DX(t,\cdot)\|_\infty^{1+\gamma}\end{aligned}\tag{4.10}$$

The second term needs to be bounded carefully. First, from Lemma 1.3.4 and Lemma 4.2.4

$$\begin{aligned}\|D\,\mathbf{v}(t,\cdot)\|_\gamma &= \|D(K*\omega(t,\cdot))\|_\gamma\\ &= \|p.v.DK*\omega(t,\cdot)\|_\gamma\\ &\le \|p.v.DK\|_\gamma\|\omega(t,\cdot)\|_\gamma\\ &\le \|p.v.DK\|_\gamma\|\omega_0\|_\gamma\,\exp\left(\gamma\int_0^t\|D\,\mathbf{v}(s,\cdot)\|_\infty\,ds\right)\end{aligned}$$

Hence, together with (4.9) and (4.10), we obtain

$$\begin{aligned}\frac{d}{dt}\|DX(t,\cdot)\|_\gamma &\le \|D\,\mathbf{v}(t,\cdot)\|_\infty\,\|DX(t,\cdot)\|_\gamma+\|D\,\mathbf{v}(t,\cdot)\|_\gamma\,\|DX(t,\cdot)\|_\infty^{1+\gamma}\\ &\le \|D\,\mathbf{v}(t,\cdot)\|_\infty\,\|DX(t,\cdot)\|_\gamma+\|p.v.DK\|_\gamma\|\omega_0\|_\gamma\,\exp\left((1+2\gamma)\int_0^t\|D\,\mathbf{v}(s,\cdot)\|_\infty\,ds\right)\end{aligned}$$

Setting now $a(t)=\exp\left(-\int_0^t\|D\,\mathbf{v}(s,\cdot)\|_\infty\,ds\right)$, this means that

$$\frac{d}{dt}\|DX(t,\cdot)\|_\gamma \le \frac{-a'(t)}{a(t)}\,\|DX(t,\cdot)\|_\gamma+\|p.v.DK\|_\gamma\|\omega_0\|_\gamma\,a(t)^{1+2\gamma}$$

From $DX(0,\cdot)=\mathbf{Id}$ we end up getting that

$$\begin{aligned}\|DX(t,\cdot)\|_\gamma &\le \|p.v.DK\|_\gamma\|\omega_0\|_\gamma\int_0^t\frac{a(s)^{2+2\gamma}}{a(t)}\,ds\\ &= \|p.v.DK\|_\gamma\|\omega_0\|_\gamma\int_0^t\frac{a(s)^{2+2\gamma}}{a(t)}\,ds\end{aligned}$$

as desired. □

Lemma 4.2.6. *One can bound $\|D\mathbf{v}\|_{L^1(0,t;L^\infty)}$ in terms of $\|\omega\|_{L^1(0,t;L^\infty)}$.*

Proof. By (1.12), we have for any $\epsilon > 0$

$$\begin{aligned}\|D\,\mathbf{v}\,\|_\infty &= \|p.v.DK * \omega\|_\infty \\ &\leq C(K,\gamma)[\omega]_\gamma\,\epsilon^\gamma + C(K)\,\|\omega\|_\infty\left(1+\log\frac{|\operatorname{supp}\omega|}{\epsilon}\right)\end{aligned}$$

Then choosing $\epsilon = |\operatorname{supp}\omega|\,[\omega]_\gamma^{-1/\gamma}\,\|\omega\|_\infty^{1/\gamma}$ one gets

$$\begin{aligned}\|D\,\mathbf{v}\,\|_\infty &\leq C(K,\gamma)\|\omega\|_\infty\,|\operatorname{supp}\omega|^\gamma + C(K)\,\|\omega\|_\infty\left(1+\frac{1}{\gamma}\log\frac{[\omega_0]_\gamma}{\|\omega_0\|_\infty}+\int_0^t\|D\,\mathbf{v}(s,\cdot)\|_\infty ds\right)\\ &\leq C(K,\gamma,|\operatorname{supp}\omega|)\,\|\omega\|_\infty\left(1+\frac{1}{\gamma}\log\frac{[\omega_0]_\gamma}{\|\omega_0\|_\infty}+\int_0^t\|D\,\mathbf{v}(s,\cdot)\|_\infty ds\right)\end{aligned}$$

Thus

$$\frac{d}{dt}\log\left(1+\frac{1}{\gamma}\log\frac{[\omega_0]_\gamma}{\|\omega_0\|_\infty}+\int_0^t\|D\,\mathbf{v}(s,\cdot)\|_\infty ds\right)\leq C(K,\gamma,|\operatorname{supp}\omega|)\,\|\omega\|_\infty$$

The claim follows. □

We now recall a general result from Cauchy-Lipschitz theory in Banach spaces.

Theorem 4.2.7. *[13, Theorem 4.4] Let $\mathcal{O} \subsetneq \mathbf{E}$ be an open subset of a Banach space. Let $F : \mathcal{O} \to \mathbf{E}$ be locally Lipschitz, and let $X \in C^1([0,T),\mathcal{O})$ be the unique solution to the autonomous ODE* (4.6). *Then either $T = \infty$ or $T < \infty$ and $X(t,\cdot)$ leaves $\mathcal{O}$ as $t \nearrow T$.*

The following is our main result in this section.

Theorem 4.2.8. *Assume that ω_0 is compactly supported, real valued, and $[\omega_0]_\gamma < \infty$ for some $0 < \gamma < 1$. Let $\omega(t,\cdot)$ be a solution of* (4.1) *with kernel K satisying* **K1** *and* **K2***, initial data ω_0 and $\mathbf{v}$ given by* (4.4).

(a) Assume that for each $T > 0$ one has $\omega \in L^1((0,T),L^\infty)$. Then for each $T > 0$ there is $M = M_T > 0$ such that $X \in C^1([0,T),\mathcal{O}_{M_T})$.

(b) Assume that for each $M > 0$ there is a finite maximal time $0 < T = T_M < \infty$ such that $X \in C^1([0,T_M),\mathcal{O}_M)$ and $\lim_{M\to\infty} T_M < \infty$. Then necessarily

$$\lim_{M\to\infty}\int_0^{T_M}\|\omega(t,\cdot)\|_\infty\,dt = \infty.$$

Proof. Let us begin with (a). From the lemmas 4.2.5 and 4.2.6, the finiteness of $\|\omega\|_{L^1(0,T;L^\infty)}$ implies that of $\|X(T,\cdot)\|_{1,\gamma}$, and so for each T there is M_T such that $X(t) \in \mathcal{O}_{M_T}$ for each $0 < t < T$. Of course, M_T depends on

$$\int_0^T\|\omega(t)\|_\infty dt.$$

Now, since the trajectories X are obtained by $\dot{X} = F(X)$, we automatically get that $X \in C^1([0,T);\mathcal{O}_{M_T})$, and (a) is proven.

For (b), let us now assume that for each $M > 0$ there is T_M such that $X \in C^1([0,T_M);\mathcal{O}_M)$, that is,

$$\|X(t)\|_{1,\gamma} < M \qquad \text{for each } t \in [0,T_M),$$

and $\lim_{M\to\infty} T_M = T^* < \infty$. By Theorem 4.2.7, $X(t)$ must escape from $\mathcal{O}_M$ as t increasingly approaches T_M. But we know that $\|X(t)\|_{1,\gamma}$ can be bounded in terms of $\|\omega\|_{L^1(0,t;L^\infty)}$, and moreover the finiteness of the latter implies the finiteness of the first. Hence the only option is that

$$\lim_{\epsilon\to 0}\int_0^{T^*-\epsilon} \|\omega(t)\|_\infty \, dt = \infty$$

Indeed, if the above limit was finite, then also $\|X(t)\|_{1,\gamma}$ would remain finite as $t \nearrow T^*$, and this would contradict Theorem 4.2.7. □

The main application is the following corollary.

Corollary 4.2.9. *Assume that* ω_0 *is compactly supported, real valued, and* $[\omega_0]_\gamma < \infty$ *for some* $0 < \gamma < 1$. *Then the ODE* (4.6) *admits a unique solution* $X \in C^1(\mathbb{R}, \mathbf{E})$.

Proof. The local-in-time existence of a unique solution X of (4.6) is granted by Theorem 4.2.1. Thus, for each M there is a maximal time T_M of existence of trajectories $X \in C^1([0,T_M),\mathcal{O}_M)$ solving (4.6). As a consequence, we can solve (4.1) with kernel K satisying **K1** and **K2**, and initial data ω_0 by setting

$$\omega(t, X(t,\alpha)) = \omega_0(\alpha).$$

From the smoothness of $X(t,\cdot)$ we know that it preserves Lebesgue-null sets, and so $\|\omega(t,\cdot)\|_\infty = \|\omega_0\|_\infty$ whenever $0 < t < T_M$. Assume now that for each $M > 0$ one has $T_M < \infty$. Then

$$\int_0^{T_M} \|\omega(t,\cdot)\|_\infty \, dt = \int_0^{T_M} \|\omega_0\|_\infty \, dt = \|\omega_0\|_\infty \, T_M$$

By part (b) of Theorem 4.2.8, we cannot have $\lim_{M\to\infty} T_M < \infty$. Thus either $\lim_{M\to\infty} T_M = \infty$ (and so we get global-in-time existence) or for some finite $M > 0$ one has $T_M = \infty$ (thus giving global existence in time again). □

4.3 Existence theory for $\omega_0 \in L^\infty$

Given compactly supported $\omega_0 \in L^\infty(\mathbb{C})$, we look for scalar-valued functions $\omega : \mathbb{R}\times\mathbb{C} \to \mathbb{R}$ belonging to $L^1(\mathbb{R}, L^\infty(\mathbb{C}))$ that solve the problem (4.1). The arguments in this section only work for the particular kernel $K(z) = \frac{e^{i\theta}}{2\pi z}$. Our goal is to prove that a weak solution exists and can be represented by

$$\omega(t, X(t,z)) = \omega_0(z)$$

where X are the trajectories of the vector field $\mathbf{v}$. To this end, we start by mollifying to $\omega_0^\epsilon \in C^\infty$ in such a way that

$$\|\omega_0^\epsilon\|_\infty \leq \|\omega_0\|_\infty$$
$$\|\omega_0^\epsilon\|_1 \leq \|\omega_0\|_1$$

and moreover $\|\omega_0^\epsilon - \omega_0\|_1 \to 0$, as $\epsilon \to 0$. Then, by virtue of previous section, to each ω_0^ϵ we can associate its unique solution ω^ϵ to

$$\begin{cases} \frac{d}{dt}\,\omega^\epsilon + \mathbf{v}^\epsilon \cdot \nabla \omega^\epsilon = 0, \\ \mathbf{v}^\epsilon(t,\cdot) = K * \omega^\epsilon(t,\cdot), \\ \omega^\epsilon(0,\cdot) = \omega_0^\epsilon \end{cases} \tag{4.11}$$

with $K(z) = \frac{e^{i\theta}}{2\pi z}$. This solution is obtained by translating the datum ω_0^ϵ along the trajectories $X^\epsilon(t,x)$ of $\mathbf{v}^\epsilon$, that is, $\omega^\epsilon(t, X^\epsilon(t,z)) = \omega_0^\epsilon(z)$. Of course, $X^\epsilon(t,\cdot)$ preserves Lebesgue null sets. In particular, $\omega^\epsilon \in L^1(\mathbb{R}, L^\infty(\mathbb{C}))$ as $\omega_0^\epsilon \in L^\infty(\mathbb{C})$.

Theorem 4.3.1. *If $\omega_0 \in L^\infty$ has compact support, there exists a solution $\omega \in L^1([0,T], L^\infty)$ for each $T > 0$.*

The proof of Theorem 4.3.1 consists of proving convergence of the solutions ω^ϵ and $\mathbf{v}^\epsilon$ in an appropriate sense. As usually, the most delicate point is the following L^1 bound,

$$\|\omega(t,\cdot)\|_1 \le e^{2t\,\|\omega_0\|_\infty}\,\|\omega_0\|_\infty\,|\operatorname{supp}\omega_0|.$$

This, combined with the preservation of $\|\omega(t,\cdot)\|_\infty$ and Lemma 1.3.3(b), gives uniform bounds for the velocity field, and so Ascoli's Theorem allows to find limit trajectories.

In Euler's setting, that is for the kernel $K(z) = \frac{i}{2\pi \bar{z}}$, the L^1 control comes from the L^∞ bounds of the jacobian, which in turn comes from the fact $\operatorname{div}\mathbf{v}^\epsilon \in L^\infty$. Now, for $K(z) = \frac{e^{i\theta}}{2\pi z}$ unfortunately $\operatorname{div}\mathbf{v}^\epsilon \notin L^\infty$, but still the same L^1 control is possible, and comes as a consequence of the fact that the flow consists of principal quasiconformal maps which are conformal outside of the support of ω_0, and moreover with uniformly bounded distortion . To show this, step by step, we first need to recall the following result, due to H.M. Reimann [43]. We only state it on the plane, although it holds also in higher dimensions.

Theorem 4.3.2. *Let $\mathbf{v} : [0,T] \times \mathbb{C} \to \mathbb{C}$ be a continuous vector field, such that for each t one has*

$$\limsup_{|z|\to+\infty} \frac{|\mathbf{v}(t,z)|}{|z|\,\log|z|} < +\infty.$$

Suppose that the distributional derivatives $\partial\mathbf{v}(t,\cdot)$ and $\overline{\partial}\mathbf{v}(t,\cdot)$ are locally integrable functions of $z \in \mathbb{C}$, and moreover suppose that

$$\sup_{t\in[0,T]} \|\overline{\partial}\mathbf{v}(t,\cdot)\|_\infty \le C_0 < \infty.$$

Then, $\mathbf{v}$ admits a unique flow $X(t,z)$ of K_t-quasiconformal maps $X(t,\cdot) : \mathbb{C} \to \mathbb{C}$, and

$$K_t \le \exp\left(2\int_0^\infty \|\overline{\partial}\mathbf{v}(s,\cdot)\|_\infty\, ds\right).$$

Next, in order to proceed with the proof of Theorem 4.3.1, we start with a Lemma.

Lemma 4.3.3. *Let $T > 0$ be fixed. Then:*

(a) $X^\epsilon(t,\cdot)$ *is* $\mathcal{K}_t$*-quasiconformal, with* $1 \le \mathcal{K}_t \le e^{|t|\|\omega_0\|_\infty}$*, where* $-T \le t \le T$.

(b) *There exists a constant* $C = C(\mathcal{K}_t)$ *such that*

$$\frac{1}{C}\left(\frac{|z-z_0|}{|z-w_0|}\right)^{\mathcal{K}_t} \le \frac{|X^\epsilon(t,z)-X^\epsilon(t,z_0)|}{|X^\epsilon(t,z)-X^\epsilon(t,w_0)|} \le C\left(\frac{|z-z_0|}{|z-w_0|}\right)^{\frac{1}{\mathcal{K}_t}}$$

for any $z, z_0, w_0 \in \mathbb{C}$ *and any time* $t \in [-T,T]$.

(c) *There exists a constant* $C = C(\mathcal{K}_t)$ *such that*

$$\frac{|X^\epsilon(t,E)|}{|X^\epsilon(t,D)|} \le C(\mathcal{K}_t)\left(\frac{|E|}{|D|}\right)^{\frac{1}{\mathcal{K}_t}}.$$

whenever $D \subset \mathbb{C}$ *is a disk and* $E \subset D$ *is measurable.*

Proof. The structure of the Cauchy Kernel makes it clear that

$$2\|\overline{\partial}\mathbf{v}^\epsilon(t,\cdot)\|_\infty = \|\omega^\epsilon(t,\cdot)\|_\infty = \|\omega_0^\epsilon\|_\infty \le \|\omega_0\|_\infty.$$

Moreover, from Lemma 1.3.5 we know that $\mathbf{v}^\epsilon(t,\cdot)$ belongs to the Zygmund class as $p.v.DK$ is bounded in BMO and $w^\epsilon \in L^1(\mathbb{R}, L^\infty(\mathbb{C}))$. It is a classical fact that Zygmund functions are Log-Lipschitz. Therefore,

$$\begin{aligned}\limsup_{|z|\to\infty}\frac{|\,\mathbf{v}^\epsilon(t,z)|}{|z|\log(e+|z|)} &\le C\|\,\mathbf{v}^\epsilon(t,\cdot)\|_{\Lambda_*}\\ &\le C'\|\overline{\partial}\,\mathbf{v}^\epsilon(t,\cdot)\|_\infty\\ &\le C''\|\omega_0\|_\infty.\end{aligned}$$

Thus, quasiconformality follows from Reimann's Theorem 4.3.2, with constant

$$\mathcal{K}_t \le \exp\left(2\int_0^t \|\overline{\partial}\,\mathbf{v}^\epsilon(s,\cdot)\|_\infty ds\right) \le e^{t\|\omega_0\|_\infty}$$

and by definition, $\mathcal{K}_t \ge 1$. Therefore, part (a) is clear. Part (b) says that quasiconformal maps are quantitatively quasisymmetric. The interested reader should check [6, Corollary 3.10.4] for a detailed proof. Part (c) follows from [6, Theorem 13.1.5] and the classical area distortion estimates for $\mathcal{K}_t$-quasiconformal maps. □

Next, we would like to find an *accumulation point* $X(t,\cdot)$ of the *trajectories* $X^\epsilon(t,\cdot)$. As always, this will be done by using the *control in time* of the L^1 norm of ω^ϵ. However, this control will be obtained in a completely different way. As a first step, let us note that *compactness of the flow* will be a direct consequence of *local boundedness*.

Lemma 4.3.4. *Assume that* $X^\epsilon(t,\cdot)$ *is uniformly bounded on compact sets. Then:*

(a) $\{X^\epsilon(t,\cdot)\}_\epsilon$ *is pointwise equicontinuous.*

(b) $\{X^\epsilon(t,\cdot)\}_\epsilon$ *accumulate to a* $\mathcal{K}_t$*-quasiconformal map* $X(t,\cdot)$.

Proof. To prove the claim (a), let us remind from Lemma 4.3.3 that $X^\epsilon(t,\cdot)$ is quasisymmetric. That is, given any three points $z_0, z, w \in \mathbb{C}$ we have

$$\frac{|X^\epsilon(t,z) - X^\epsilon(t,z_0)|}{|X^\epsilon(t,w) - X^\epsilon(t,z_0)|} \le \eta_{\mathcal{K}_t}\left(\frac{|z-z_0|}{|w-z_0|}\right).$$

As a consequence

$$\begin{aligned} |X^\epsilon(t,z) - X^\epsilon(t,z_0)| &\le \eta_{\mathcal{K}_t}(|z-z_0|/|w-z_0|)|X^\epsilon(t,w) - X^\epsilon(t,z_0)| \\ &\le \eta_{\mathcal{K}_t}(|z-z_0|/|w-z_0|)(|X^\epsilon(t,w)| + |X^\epsilon(t,z_0)|) \\ &\le \eta_{\mathcal{K}_t}(|z-z_0|/|w-z_0|)(C(t,|w|) + C(t,|z_0|)) \end{aligned}$$

In particular, by leaving w fixed one can easily get that $X^\epsilon(t,\cdot)$ is equicontinuous at z_0.
The family of maps, $\{X^\epsilon(t,\cdot)\}_\epsilon$ is pointwise equicontinuous and locally uniformly bounded. Therefore, Arzela-Ascoli theorem ensures the existence of a locally uniform accumulation point $X(t,\cdot)$. It is worth mentioning that by classical tools in *Geometric Function Theory* [8, Theorem 3.1.3] it can only be either $\mathcal{K}_t$-quasiconformal or constant. To see that it cannot be constant, one must observe that the quasisymmetry bounds are preserved by uniform limits. Being two sided, these quasisymmetry bounds guarantee bijectivity. Therefore, the accumulation point $X(t,\cdot)$ is $\mathcal{K}_t$-quasiconformal. □

In order to get the local boundedness of the flow, the key point is the following elementary fact.

Lemma 4.3.5. *Let $X^\epsilon(t,\cdot)$ be as before, and assume that ω_0^ϵ has compact support. Then*

$$\omega_0^\epsilon(z) = 0 \qquad \Longrightarrow \qquad \overline{\partial} X^\epsilon(t,z) = 0,$$

in other words $X^\epsilon(t,\cdot)$ is conformal outside of $\operatorname{supp}\omega_0^\epsilon$.

Proof. The $\mathcal{K}_t$-quasiconformality of $X^\epsilon(t,\cdot)$ ensures the existence of a well-defined, uniformly elliptic Beltrami coefficient $\mu^\epsilon(t,\cdot) = \frac{\overline{\partial} X^\epsilon(t,\cdot)}{\partial X^\epsilon(t,\cdot)}$, and moreover we know that

$$\|\mu^\epsilon(t,\cdot)\|_\infty \le \frac{\mathcal{K}_t - 1}{\mathcal{K}_t + 1}.$$

The smoothness in time of $\partial X^\epsilon(t,z)$ and $\overline{\partial} X^\epsilon(t,z)$ guarantees that $t \mapsto \mu^\epsilon(t,z)$ is also smooth. From the equation for the flow $\dot{X}^\epsilon(t,z) = \mathbf{v}^\epsilon(t, X^\epsilon(t,z))$ and the chain rule we get that

$$\begin{aligned} \overline{\partial}\,\mathbf{v}^\epsilon(t, X^\epsilon(t,z)) &= \frac{\frac{d}{dt}\overline{\partial} X^\epsilon(t,z)\,\partial X^\epsilon(t,z) - \overline{\partial} X^\epsilon(t,z)\,\frac{d}{dt}\partial X^\epsilon(t,z)}{J^\epsilon(t,z)} \\ &= \frac{\frac{d}{dt}\mu^\epsilon(t,z)\ (\partial X^\epsilon(t,z))^2}{J^\epsilon(t,z)} \\ &= \frac{\frac{d}{dt}\mu^\epsilon(t,z)}{1 - |\mu^\epsilon(t,z)|^2}\,\frac{\partial X^\epsilon(t,z)}{\overline{\partial X^\epsilon(t,z)}} \end{aligned}$$

On the other hand, from the kernel structure we have

$$2|\overline{\partial}\,\mathbf{v}^\epsilon(t, X^\epsilon(t,z))| = |\omega^\epsilon(t, X^\epsilon(t,z))| = |\omega_0^\epsilon(z)|.$$

Thus

$$\frac{\frac{d}{dt}|\mu^\epsilon(t,z)|}{1-|\mu^\epsilon(t,z)|^2} \le \frac{\left|\frac{d}{dt}\mu^\epsilon(t,z)\right|}{1-|\mu^\epsilon(t,z)|^2} = \frac{1}{2}\,|\omega_0^\epsilon(z)|$$

Now, given any time $t>0$, we can integrate on $(0,t)$ the above inequality to obtain that

$$\log\left(\frac{1+|\mu^\epsilon(t,z)|}{1-|\mu^\epsilon(t,z)|}\right) \le t\,|\omega_0^\epsilon(z)|, \tag{4.12}$$

since $X^\epsilon(0,z)=z$ implies $\mu^\epsilon(0,z)=0$. Now, if $\omega_0^\epsilon(z)=0$ then necessarily $\mu^\epsilon(t,z)=0$ and hence $\overline{\partial}X^\epsilon(t,z)=0$. The claim follows. □

Remark 4.3.6. *The above proof also shows that, at time $t=0$,*

$$\frac{\omega_0^\epsilon(z)}{2} = \frac{1}{2}\,\omega^\epsilon(0,\cdot)(z) = \overline{\partial}\mathbf{v}^\epsilon(0,z) = \frac{d}{dt}\left[\mu^\epsilon(t,z)\right]_{t=0}.$$

That is, the initial vorticity is determined by the time derivative of the Beltrami coefficient at time $t=0$. Thus, it is natural to ask for the dependence of $X^\epsilon(t,\cdot)$ under second-order perturbations of $\mu^\epsilon(t,z)$.

Now, it just remains to observe that $\mathbf{v}^\epsilon(t,\cdot)$ cannot grow without control as $|z|\to\infty$. This, together with the conformality of the flow outside of $\operatorname{supp}\omega_0^\epsilon$, provides improved area estimates which are essential for the control of $\|\omega^\epsilon(t,\cdot)\|_1$.

Lemma 4.3.7. *Let $X^\epsilon(t,\cdot)$ be as before, and assume that ω_0 has compact support.*

(a) For each t,ϵ there exists $b^\epsilon(t)\in\mathbb{C}$ such that $\lim_{|z|\to\infty}|X^\epsilon(t,z)-z-b^\epsilon(t)|=0$.

(b) One has $|X^\epsilon(t,E)|\le\mathcal{K}_t\,|E|$ for each set $E\supset\operatorname{supp}\omega_0^\epsilon$.

Proof. From Lemma 1.3.3 (b) and the integral representation of $X^\epsilon(t,\cdot)$, we know that

$$\begin{aligned}
\frac{|X^\epsilon(t,z)-z|}{|z|} &= \frac{\left|\int_0^t \mathbf{v}^\epsilon(s,X^\epsilon(s,z))\right|}{|z|}ds \\
&\le \int_0^t \frac{|\mathbf{v}^\epsilon(s,X^\epsilon(s,z))|}{|z|}ds \\
&\le \int_0^t \frac{C(K)\,\|\omega^\epsilon(s,\cdot)\|_\infty\,|\operatorname{supp}\omega^\epsilon(s,\cdot)|^{\frac12}}{|z|}ds \\
&\le \int_0^t \frac{C(K)\,\|\omega_0^\epsilon\|_\infty\,|X^\epsilon(s,\operatorname{supp}\omega_0^\epsilon)|^{\frac12}}{|z|}ds \\
&\le \frac{C(K)\,\|\omega_0^\epsilon\|_\infty}{|z|}\int_0^t |X^\epsilon(s,\operatorname{supp}\omega_0^\epsilon)|^{\frac12}\,ds \\
&\le \frac{C(K)\,\|\omega_0^\epsilon\|_\infty}{|z|}\,t\,|\operatorname{supp}\omega_0^\epsilon|^{\frac12}\max_{0\le s\le t}\|J^\epsilon(s,\cdot)^{\frac12}\|_{L^\infty(\operatorname{supp}\omega_0^\epsilon)}.
\end{aligned}$$

Above, the maximum term on the right hand side (even depending on t and ϵ) is finite and stays bounded as $|z|\to\infty$, due to the smoothness in t and z of $X^\epsilon(t,z)$. Thus, for every fixed t and $\epsilon>0$ one has

$$\lim_{|z|\to\infty}\frac{|X^\epsilon(t,z)-z|}{|z|}=0. \tag{4.13}$$

However, by Lemma 4.3.5 we know that $X^\epsilon(t,\cdot)$ is conformal on a neighborhood of ∞. Therefore, it has around ∞ a Laurent series development whose higher order term is linear,

$$X^\epsilon(t,z) = a^\epsilon(t)z + b^\epsilon(t) + \frac{c^\epsilon(t)}{z} + ...$$

Now, (4.13) tells us that necessarily $a^\epsilon(t) = 1$, and so (a) follows. To see (b), we observe that $X^\epsilon(t,\cdot) - b^\epsilon(t)$ is a *principal* $\mathcal{K}_t$-quasiconformal map, because

$$\lim_{|z|\to\infty} |X^\epsilon(t,z) - b^\epsilon(t) - z| = 0.$$

Moreover, it is conformal outside of $\operatorname{supp}\omega_0^\epsilon$ by Lemma 4.3.5. Hence, by [8, Theorem 13.1.2], we have the following area distortion estimates,

$$|X^\epsilon(t,E)| = |X^\epsilon(t,E) - b^\epsilon(t)| \leq \mathcal{K}_t\,|E|$$

$\forall E \supset \operatorname{supp}(\omega_0^\epsilon)$, as claimed. □

We are now in position of getting the L^∞ bounds for $\mathbf{v}^\epsilon$.

Proposition 4.3.8. *Assume that* $K(z) = \frac{e^{i\theta}}{2\pi z}$. *If* ω_0 *is compactly supported, then*

(a) $\|\omega^\epsilon(t,\cdot)\|_\infty \leq \|\omega_0\|_\infty$

(b) $\|\omega^\epsilon(t,\cdot)\|_1 \leq \|\omega_0\|_\infty\, e^{t\|\omega_0\|_\infty}\,|\operatorname{supp}\omega_0^\epsilon|$.

(c) $\|\,\boldsymbol{v}^\epsilon(t,\cdot)\|_\infty \leq C(K)\, e^{\frac{t}{2}\|\omega_0\|_\infty}\,\|\omega_0\|_\infty\,|\operatorname{supp}\omega_0^\epsilon|^{\frac{1}{2}}$.

Proof. Claim (a) can be proved by recalling that $\omega^\epsilon(t,\cdot)\circ X^\epsilon(t,\cdot) = \omega_0^\epsilon(\cdot)$ and the facts that $X^\epsilon(t,\cdot)$ preserves Lebesgue-null sets and $\|\omega_0^\epsilon\|_\infty \leq \|\omega_0\|_\infty$. For (b), we use Lemmas 4.3.3 (a), 4.3.5 and 4.3.7 (b) to obtain

$$\begin{aligned}
\|\omega^\epsilon(t,\cdot)\|_1 &= \int_{\mathbb{C}} |\omega^\epsilon(t,z)|\,dA(z)\\
&= \int_{\mathbb{C}} |\omega_0^\epsilon(\zeta)|\,J^\epsilon(t,\zeta)\,dA(\zeta)\\
&\leq \|\omega_0^\epsilon\|_\infty \int_{\operatorname{supp}\omega_0^\epsilon} J^\epsilon(t,\zeta)\,dA(\zeta)\\
&= \|\omega_0^\epsilon\|_\infty\,|X^\epsilon(t,\operatorname{supp}\omega_0^\epsilon)|\\
&\leq \|\omega_0^\epsilon\|_\infty\,\mathcal{K}_t\,|\operatorname{supp}\omega_0^\epsilon|\\
&\leq \|\omega_0\|_\infty\,e^{t\|\omega_0\|_\infty}\,|\operatorname{supp}\omega_0^\epsilon|
\end{aligned}$$

as desired. Estimate (c) follows from Lemma 1.3.3 (b). □

The control on $\|\,\mathbf{v}^\epsilon\,\|_{L^1(\mathbb{R},L^\infty)}$ allows for local boundeness of $X^\epsilon(t,\cdot)$, since

$$|X^\epsilon(t,z) - z| \leq \int_0^t |\,\mathbf{v}^\epsilon(s,X^\epsilon(s,z))|\,ds \leq \|\,\mathbf{v}^\epsilon\,\|_{L^1((0,t),L^\infty)} \leq C(K)\,|\operatorname{supp}\omega_0|^{\frac{1}{2}}\,e^{\frac{t}{2}\|\omega_0\|_\infty}$$

and so Lemma 4.3.4 guarantees the existence of a limit flow map $X(t,\cdot):\mathbb{C}\to\mathbb{C}$ which is $\mathcal{K}_t$-quasiconformal at each time t. Setting then $\omega(t,\cdot) = \omega_0(X(-t,\cdot))$, we obtain a well defined $L^1((0,t),L^\infty)$ function. We also define $\mathbf{v}(t,\cdot) = K * \omega(t,\cdot)$.

Theorem 4.3.9. *With the above notation,*

(a) $\|\omega^\epsilon(t,\cdot) - \omega(t,\cdot)\|_1 \to 0$*, and*

(b) $\|\mathbf{v}^\epsilon(t,\cdot) - \mathbf{v}(t,\cdot)\|_\infty \to 0$.

Proof. One has

$$\|\omega^\epsilon(t,\cdot) - \omega(t,\cdot)\|_1 \leq \|\omega_0^\epsilon(X^\epsilon(-t,\cdot)) - \omega_0(X^\epsilon(-t,\cdot)\|_1 + \|\omega_0(X^\epsilon(-t,\cdot) - \omega_0(X(-t,\cdot)\|_1$$

At the first term, we consider a disk D such that $\operatorname{supp}\omega_0^\epsilon, \operatorname{supp}\omega_0 \subset D$, and use the higher integrability of quasiconformal jacobians. If $1 < p < \frac{\mathcal{K}_t}{\mathcal{K}_t - 1}$,

$$\begin{aligned}
\|\omega_0^\epsilon(X^\epsilon(-t,\cdot)) - \omega_0(X^\epsilon(-t,\cdot)\|_1 &= \int |\omega_0^\epsilon - \omega_0|\, J^\epsilon(t,\cdot) \\
&\leq \|\omega_0^\epsilon - \omega_0\|_{L^{p'}(D)}\, \|J^\epsilon(t,\cdot)\|_{L^p(D)} \\
&\leq \|\omega_0^\epsilon - \omega_0\|_{L^1(D)}^{\frac{1}{p'}}\, \|\omega_0^\epsilon - \omega_0\|_{L^\infty(D)}^{\frac{1}{p}}\, \|J^\epsilon(t,\cdot)\|_{L^p(D)}
\end{aligned}$$

Above, $\|\omega_0^\epsilon - \omega_0\|_{L^1(D)}$ converges to 0, while $\|J^\epsilon(t,\cdot)\|_{L^p(D)}$ is bounded in terms of $\mathcal{K}_t$ and $|D|$,

$$\begin{aligned}
\|J^\epsilon(t,\cdot)\|_{L^p(D)} &= \left(\int_D J^\epsilon(t,\cdot)^p\right)^{\frac{1}{p}} \\
&\leq C(p,\mathcal{K}_t)\, |D|^{\frac{1}{p}-1} \int_D J^\epsilon(t,\cdot) \\
&\leq C(p,\mathcal{K}_t)\, |\operatorname{supp}\omega_0^\epsilon|^{\frac{1}{p}} \leq C(p,\mathcal{K}_t)\, |\operatorname{supp}\omega_0|^{\frac{1}{p}}
\end{aligned}$$

by Lemma 4.3.7 (b) and the reverse Hölder property of quasiconformal jacobians [8], and provided that $\epsilon > 0$ is small enough. Concerning the second term, let us choose $\omega_0^n \in C_0$ such that $\|\omega_0^n - \omega_0\|_1 \leq 1/n$ and $\|\omega_0^n\|_\infty \leq \|\omega_0\|_\infty$. Then

$$\begin{aligned}
\|\omega_0(X^\epsilon(-t,\cdot)) - \omega_0(X(-t,\cdot))\|_1 \leq\; & \|\omega_0(X^\epsilon(-t,\cdot)) - \omega_0^n(X^\epsilon(-t,\cdot))\|_1 \\
&+ \|\omega_0^n(X^\epsilon(-t,\cdot)) - \omega_0^n(X(-t,\cdot))\|_1 \\
&+ \|\omega_0^n(X(-t,\cdot)) - \omega_0(X(-t,\cdot))\|_1
\end{aligned}$$

Above, again because of the higher integrability of quasiconformal jacobians,

$$\begin{aligned}
\|\omega_0(X^\epsilon(-t,\cdot)) - \omega_0^n(X^\epsilon(-t,\cdot))\|_1 &= \int |\omega_0 - \omega_0^n|\, J^\epsilon(t,\cdot) \\
&= \|\omega_0 - \omega_0^n\|_{L^{p'}(D)}\, \|J^\epsilon(t,\cdot)\|_{L^p(D)} \\
&\leq \|\omega_0 - \omega_0^n\|_{L^1(D)}^{\frac{1}{p'}}\, \|\omega_0^n - \omega_0\|_\infty^{\frac{1}{p}}\, \|J^\epsilon(t,\cdot)\|_{L^p(D)} \\
&\leq n^{\frac{-1}{p'}}\, 2^{\frac{1}{p}}\, \|\omega_0\|_\infty^{\frac{1}{p}}\, \|J^\epsilon(t,\cdot)\|_{L^p(D)}
\end{aligned}$$

and similarly for $\|\omega_0^n(X(-t,\cdot)) - \omega_0(X(-t,\cdot))\|_1$. Thus each of these two terms can be made smaller than $\delta/3$ if n is chosen large enough. The control of the second term comes by continuity. Precisely, as $X^\epsilon(-t,\cdot) \to X(-t,\cdot)$ and ω_0^n is continuous, there is $\epsilon > 0$ such that $\|\omega_0^n(X^\epsilon(-t,\cdot) - \omega_0^n(X(-t,\cdot))\|_\infty < \delta/3$. Thus (a) follows. For the proof of (b), use (a) and Lemma 1.3.3 (b). □

The above convergence result suffices to prove that ω is a weak solution to the desired nonlinear transport equation. Existence is proved.

There exist extensions to Reimann's Theorem. To mention one, it was proven in [23, Theorem 1] that if $\mathbf{v}$ is a planar Sobolev vector field with certain control on its growth at infinity, such that

$$\overline{\partial}\,\mathbf{v} + \lambda\,\mathrm{Im}(\partial\,\mathbf{v}) \in L^\infty$$

then the flow $X(t,\cdot)$ of (4.3) consists of quasiconformal mappings. Above, one may choose λ to be a constant with $|\lambda| < 1$ if $\lambda \in \mathbb{C}$, or also a smooth, compactly supported function with $\|\lambda\|_{L^\infty(\mathbb{R}^2)} < 1$. This makes it reasonable to extend Theorem 4.3.1 to other kernels $K(z)$ different than the one we used here $K(z) = \frac{e^{i\theta}}{2\pi z}$. The new kernels K we have in mind are multiples of the fundamental solution of the operator $\overline{\partial}\,\mathbf{v} + \lambda\,\mathrm{Im}(\partial\,\mathbf{v})$.

4.4 The governing equations

Let us recall that the Euler's system of equations is given, in its original formulation, in terms of the velocity field $\mathbf{v}$. Namely, one has the following equivalence

$$\begin{cases} \omega_t + \mathbf{v}\cdot\nabla\omega = 0, \\ \mathbf{v} = \frac{i}{2\pi\bar{z}} * \omega, \\ \omega|_{t=0} = \omega_0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\,\mathbf{v} = -\nabla p, \\ \mathrm{div}\,\mathbf{v} = 0, \\ \mathrm{curl}\,\mathbf{v}\,|_{t=0} = \frac{1}{2}\omega_0. \end{cases}$$

where p is the scalar valued *pressure* function. Especially, the equation $\mathrm{div}\,\mathbf{v} = 0$ on the right hand side is superfluous and can be replaced by $\mathrm{div}\,\mathbf{v}\,|_{t=0} = 0$. It turns out that a similar equivalent formulation can be provided for (4.1), and this is our goal in the present section. From now on, we denote

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and set $\mathbf{M}_\theta z = e^{i\theta}\,\mathbf{C}\,z = e^{i\theta}\,\bar{z}$. Thus, indeed $\mathbf{M}_\theta$ is the $\mathbb{R}$-linear map with matrix

$$\mathbf{M}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

To avoid formalities, we reduce ourselves to the smooth setting, and assume the datum $\omega_0 : \mathbb{C} \to \mathbb{R}$ is smooth and compactly supported. Let us remind that $K(z) = K_\theta(z) = \frac{e^{i\theta}}{2\pi z}$.

Proposition 4.4.1. *The scalar-valued function $\omega : [0,T] \times \mathbb{C} \to \mathbb{R}$ is a weak solution of*

$$\begin{cases} \omega_t + \mathbf{v}\cdot\nabla\omega = 0 \\ \mathbf{v} = K * \omega \\ \omega|_{t=0} = \omega_0 \end{cases} \tag{4.14}$$

if and only if $\mathbf{v} : [0,T] \times \mathbb{C} \to \mathbb{C}$ and $q : [0,T] \times \mathbb{C} \to \mathbb{R}$ solve

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\mathbf{M}_\theta\nabla Q \\ -\Delta Q = \mathrm{div}(\mathbf{v})\ \mathrm{div}(\mathbf{M}_\theta\mathbf{v}) \\ \mathrm{curl}(\mathbf{M}_\theta\mathbf{v})|_{t=0} = 0 \\ \mathrm{div}(\mathbf{M}_\theta\mathbf{v})|_{t=0} = \omega_0 \end{cases} \tag{4.15}$$

also in the weak sense.

Proof. We first go from (4.15) to (4.14). We identify $\mathbb{R}^2 \equiv \mathbb{C}$, and write the system (4.15) in complex notation,

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\,\partial\,\mathbf{v} + \bar{\mathbf{v}}\,\overline{\partial}\,\mathbf{v} = -e^{i\theta}\overline{\nabla Q} \\ -\Delta Q = \operatorname{div}(\mathbf{v})\,\operatorname{div}(e^{i\theta}\bar{\mathbf{v}}) \\ \operatorname{Im}(\partial(e^{i\theta}\bar{\mathbf{v}}))|_{t=0} = 0 \\ \operatorname{Re}(\partial(e^{i\theta}\bar{\mathbf{v}}))|_{t=0} = \omega_0 \end{cases}$$

Now, taking $\overline{\partial}$ on the first equation, and obtain

$$(\overline{\partial}\,\mathbf{v})_t + \mathbf{v}\;\partial(\overline{\partial}\,\mathbf{v}) + \bar{\mathbf{v}}\,\overline{\partial}(\overline{\partial}\,\mathbf{v}) + \overline{\partial}\,\mathbf{v}(\partial\,\mathbf{v} + \overline{\partial}\bar{\mathbf{v}}) = -\overline{\partial}(e^{i\theta}\overline{\nabla Q})$$

or equivalently,

$$(\overline{\partial}\,\mathbf{v})_t + \mathbf{v}\cdot\nabla(\overline{\partial}\,\mathbf{v}) + \overline{\partial}\,\mathbf{v}\;\operatorname{div}\mathbf{v} = -\frac{1}{2}\,e^{i\theta}\,\Delta Q.$$

We now multiply by $e^{-i\theta}$, and use the $\mathbb{C}$-linearity of the transport operator $\frac{d}{dt} + \mathbf{v}\cdot\nabla$ to get

$$(e^{-i\theta}\,\overline{\partial}\,\mathbf{v})_t + \mathbf{v}\cdot\nabla(e^{-i\theta}\,\overline{\partial}\,\mathbf{v}) + e^{-i\theta}\,\overline{\partial}\,\mathbf{v}\;\operatorname{div}\mathbf{v} = -\frac{1}{2}\,\Delta Q.$$

After taking real and imaginary parts,

$$\begin{cases} \operatorname{Re}((e^{-i\theta}\overline{\partial}\,\mathbf{v})_t + \mathbf{v}\cdot\nabla(e^{-i\theta}\overline{\partial}\,\mathbf{v})) + \operatorname{Re}(e^{-i\theta}\overline{\partial}\,\mathbf{v})\;\operatorname{div}\mathbf{v} = -\frac{1}{2}\,\Delta Q \\ \operatorname{Im}((e^{-i\theta}\overline{\partial}\,\mathbf{v})_t + \mathbf{v}\cdot\nabla(e^{-i\theta}\overline{\partial}\,\mathbf{v})) + \operatorname{Im}(e^{-i\theta}\overline{\partial}\,\mathbf{v})\;\operatorname{div}\mathbf{v} = 0. \end{cases} \tag{4.16}$$

The above equations may be seen as scalar conservation laws for $\operatorname{Re}(e^{-i\theta}\,\overline{\partial}\,\mathbf{v})$ and $\operatorname{Im}(e^{-i\theta}\,\overline{\partial}\,\mathbf{v})$. The second one is homogeneous, and so from the initial condition

$$2\operatorname{Im}(e^{-i\theta}\overline{\partial}\,\mathbf{v})|_{t=0} = -\operatorname{curl}(\mathbf{M}_\theta\,\mathbf{v})|_{t=0} = 0$$

we deduce that at any time $t > 0$

$$2\operatorname{Im}(e^{-i\theta}\overline{\partial}\,\mathbf{v}) = -\operatorname{curl}(\mathbf{M}_\theta\,\mathbf{v}) = 0.$$

To see this, simply call $\rho = 2\operatorname{Im}(e^{-i\theta}\overline{\partial}\,\mathbf{v})$ and note it satisfies the following initial value problem,

$$\begin{cases} \frac{d}{dt}\rho + \operatorname{div}(\rho\;\mathbf{v}) = 0 \\ \rho(0,\cdot) = 0 \end{cases}$$

which has $\rho = 0$ as its unique solution, due to the smoothness of $\mathbf{v}$. As a consequence, $e^{-i\theta}\overline{\partial}\,\mathbf{v} \in \mathbb{R}$ and so if we now denote $\omega = 2\operatorname{Re}(e^{-i\theta}\overline{\partial}\,\mathbf{v})$, then

$$\omega = \operatorname{div}(e^{i\theta}\bar{\mathbf{v}}).$$

Thus the first equation at (4.16) implies that

$$\omega_t + \mathbf{v}\cdot\nabla\omega + \omega\;\operatorname{div}\mathbf{v} = -\Delta Q.$$

Now, since the second equation at (4.15) tells us that $\omega \operatorname{div} \mathbf{v} = -\Delta Q$, we necessarily have for ω a homogeneous transport equation

$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0$$

together with the initial condition $\omega|_{t=0} = \operatorname{div}(e^{i\theta}\bar{\mathbf{v}})|_{t=0} = \omega_0$ as claimed.

For the converse implication, we start by noting that our choice of the kernel K and the second equation in (4.14) tell us that $2e^{-i\theta}\overline{\partial}\mathbf{v} = \omega$, which by assumption is real valued. We now use the first equation in (4.14), together with the $\mathbb{C}$-linearity of the complex operator, to get

$$\overline{\partial}\,\mathbf{v}_t + \mathbf{v} \cdot \partial(\overline{\partial}\,\mathbf{v}) + \overline{\mathbf{v}} \cdot \overline{\partial}(\overline{\partial}\,\mathbf{v}) = 0$$

or equivalently

$$\overline{\partial}(\mathbf{v}_t + \mathbf{v} \cdot \partial\,\mathbf{v} + \overline{\mathbf{v}} \cdot \overline{\partial}\,\mathbf{v}) = \overline{\partial}\,\mathbf{v} \operatorname{div} \mathbf{v}$$

We now complex conjugate at both sides of the equality, multiply by $e^{i\theta}$, and use $\mathbb{C}$-linearity of the transport operator, and obtain

$$\partial(e^{i\theta}(\overline{\mathbf{v}_t + \mathbf{v} \cdot \partial\,\mathbf{v} + \overline{\mathbf{v}} \cdot \overline{\partial}\,\mathbf{v}})) = \frac{\omega}{2} \operatorname{div} \mathbf{v} \tag{4.17}$$

By assumption, the right hand side above is real, whence $e^{i\theta}(\overline{\mathbf{v}_t + \mathbf{v} \cdot \partial\,\mathbf{v} + \overline{\mathbf{v}} \cdot \overline{\partial}\,\mathbf{v}})$ is a conservative vector field. Thus there exists a scalar valued potential Q such that

$$e^{i\theta}(\overline{\mathbf{v}_t + \mathbf{v} \cdot \partial\,\mathbf{v} + \overline{\mathbf{v}} \cdot \overline{\partial}\,\mathbf{v}}) = -\nabla Q$$

This automatically gives the first equation at (4.15). Moreover, if we take real parts at (4.17),

$$-\frac{1}{2}\Delta Q = \frac{1}{2}\operatorname{div}(e^{i\theta}(\overline{\mathbf{v}_t + \mathbf{v} \cdot \partial\,\mathbf{v} + \overline{\mathbf{v}} \cdot \overline{\partial}\,\mathbf{v}})) = \frac{\omega}{2} \operatorname{div} \mathbf{v}$$

or equivalently

$$-\Delta Q = \operatorname{div}(\mathbf{v}) \operatorname{div}(\mathbf{M}_\theta\, \mathbf{v})$$

as claimed. The third and fourth equations in (4.15) are automatic from the second and fourth in (4.14). □

Let us mention that the relation between $\mathbf{v}$ and ω is precisely $\omega = \operatorname{Re}(2e^{-i\theta}\overline{\partial}\mathbf{v})$. This comes directly from $\mathbf{v} = K * \omega$ and the fact that ω is real valued.

4.5 Comments about uniqueness

In this section, we try to adapt the proof of uniqueness of solutions to Euler's system to the case of (4.1). To this end, we will use the previous section Proposition 4.4.1. First, some estimates are needed.

Lemma 4.5.1. *Let $\omega_0 \in L^\infty$ be compactly supported, and assume that $\mathbf{v}(0,\cdot) = K * \omega_0$ where $K(z) = \frac{e^{i\theta}}{2\pi z}$. Let $\omega \in L^\infty(0,T; L^1 \cap L^\infty)$ be a real valued weak solution of*

$$\begin{cases} \omega_t + \mathbf{v} \cdot \nabla \omega = 0 \\ 2e^{-i\theta}\,\overline{\partial}\mathbf{v} = \omega \\ \omega(0,\cdot) = \omega_0. \end{cases}$$

Finally, let Q be such that

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\mathbf{M}_\theta \nabla Q \\ -\Delta Q = \operatorname{div}(\mathbf{v})\,\omega \end{cases}$$

Then there exists a constant $C(\omega_0)$ depending only on ω_0 such that:

(a) $\mathbf{v}(t,\cdot)\in L^\infty(\mathbb{C})$ *and*

$$\|\mathbf{v}(t,\cdot)\|_\infty \le e^{\frac{1}{2}\,t\|\omega_0\|_\infty}\,C(\omega_0) \tag{4.18}$$

(b) If $1<q<\infty$ and $q^*=\max\{q,\frac{q}{q-1}\}$ then $\operatorname{div}(\mathbf{v})\in L^q(\mathbb{C})$ and

$$\|\operatorname{div}\mathbf{v}(t,\cdot)\|_{L^q(\mathbb{C})} \le e^{\frac{1}{q}\,t\|\omega_0\|_\infty}\,C(\omega_0)\,(q^*-1) \tag{4.19}$$

(c) If $2<p<\infty$ then $\nabla Q\in L^p(\mathbb{C})$ and

$$\|\nabla Q\|_{L^p(\mathbb{C})} \le e^{\frac{p+2}{2p}\,t\|\omega_0\|_\infty}\,C(\omega_0)\,C_p \tag{4.20}$$

for some constant C_p depending only on p.

Proof. If $\omega\in L^\infty(0,T;L^1\cap L^\infty)$ is a weak solution, and $K(z)=\frac{e^{i\theta}}{2\pi z}$, then $\mathbf{v}=K*\omega$ belongs to the Zygmund class, and satisfies the Osgood condition, so that it admits a unique well defined flow of homeomorphisms $X_t(\cdot)=X(t,\cdot):\mathbb{R}\times\mathbb{C}\to\mathbb{C}$ solving the ODE

$$\begin{cases} \frac{d}{dt}X(t,z)=\mathbf{v}(t,X(t,z)), \\ X(0,z)=z \end{cases}$$

From $\|2\overline{\partial}\,\mathbf{v}(t,\cdot)\| = \|\omega(t,\cdot)\|_\infty = \|\omega_0\|_\infty$ it follows from Reimann's theorem [43] that X_t are $\mathcal{K}_t$-quasiconformal maps, with $\mathcal{K}_t\le e^{t\,\|\omega_0\|_\infty}$. In particular, Lebesgue null sets are preserved by X_t. As a consequence, $\omega(t,X_t(z))=\omega_0(z)$ and therefore $\omega(t,\cdot)$ has compact support at each time $t>0$. Also, $\|\omega(t,\cdot)\|_\infty=\|\omega_0\|_\infty$. Moreover, we can use Lemma 4.3.7 (b) to see that

$$|\operatorname{supp}\omega(t,\cdot)| = |X_t(\operatorname{supp}\omega_0)| \le e^{t\,\|\omega_0\|_\infty}\,|\operatorname{supp}\omega_0|.$$

Hence, from Lemma 1.3.3 (b),

$$\|\,\mathbf{v}(t,\cdot)\|_\infty \le C\,|\operatorname{supp}\omega(t,\cdot)|^{\frac{1}{2}}\,\|\omega(t,\cdot)\|_\infty \le C\,e^{\frac{t\|\omega_0\|_\infty}{2}}\,|\operatorname{supp}\omega_0|^{\frac{1}{2}}\,\|\omega_0\|_\infty,$$

where C is a constant that depends only on the size of the kernel K. So (a) follows.
For proving (b), we observe that $\mathcal{B}(\overline{\partial}\,\mathbf{v})=\partial\,\mathbf{v}$, because $\mathbf{v}$ belongs to each global Sobolev space $W^{1,q}(\mathbb{C})$, for any $1<q<\infty$. Here $\mathcal{B}$ is the Beurling-Ahlfors singular integral operator,

$$\mathcal{B}f(z)=\lim_{\epsilon\to 0}-\frac{1}{\pi}\int_{|w-z|>\epsilon}\frac{f(w)}{(w-z)^2}\,dA(w).$$

Thus we can write

$$\operatorname{div}(\mathbf{v}) = 2\mathrm{Re}(\partial\,\mathbf{v}) = 2\mathrm{Re}\,\mathcal{B}(\overline{\partial}\,\mathbf{v}) = \mathrm{Re}(e^{i\theta}\,\mathcal{B}\omega).$$

From [42] we know that $\|\mathrm{Re}\mathcal{B}(e^{i\theta}\omega)\|_q \le (q^*-1)\,\|\omega\|_q$ and using it deduce that

$$\begin{aligned}
\|\operatorname{div}\mathbf{v}(t,\cdot)\|_{L^q(\mathbb{C})} &\le \|\mathrm{Re}\mathcal{B}(e^{i\theta}\omega(t,\cdot))\|_{L^q(\mathbb{C})}\\
&\le (q^*-1)\,\|\omega(t,\cdot)\|_{L^q(\mathbb{C})}\\
&\le (q^*-1)\,\|\omega(t,\cdot)\|_\infty\,|\operatorname{supp}\omega(t,\cdot)|^{\frac{1}{q}}\\
&\le (q^*-1)\,\|\omega_0\|_\infty\, e^{\frac{t\|\omega_0\|_\infty}{q}}\,|\operatorname{supp}\omega_0|^{\frac{1}{q}}\\
&\le (q^*-1)\, e^{\frac{t\|\omega_0\|_\infty}{q}}\,C(\omega_0).
\end{aligned}$$

as claimed.
For the proof of (c), we observe that Q is determined modulo constants. Indeed, if $L(z)=\frac{1}{4\pi}\,\log|z|^2$ denotes the fundamental solution of Δ, then the difference $Q-L*(\operatorname{div}(\mathbf{v})\,\omega)$ is constant. Indeed, the gradient

$$\nabla(Q-L*(\operatorname{div}(\mathbf{v})\,\omega))=\nabla Q-\frac{1}{4\pi\bar{z}}*(\operatorname{div}(\mathbf{v})\,\omega)$$

is antiholomorphic. Moreover, it vanishes at infinity, because ∇Q does (due to the equation for $\mathbf{v}$) and ω has compact support. As a consequence, it follows that

$$\nabla Q=\frac{1}{4\pi\bar{z}}*(\operatorname{div}(\mathbf{v})\omega).$$

But it is also clear that the convolution with $\frac{1}{4\pi\bar{z}}$ continuously maps $L^{\frac{2p}{p+2}}(\mathbb{C})$ into $L^p(\mathbb{C})$ for any $2<p<\infty$. Thus,

$$\begin{aligned}
\|\nabla Q\|_{L^p(\mathbb{C})} &\le C(p)\,\|\operatorname{div}(\mathbf{v})\,\omega\|_{L^{\frac{2p}{p+2}}(\mathbb{C})}\\
&\le C(p)\,\|\omega_0\|_\infty\,\|\operatorname{div}(\mathbf{v})\|_{L^{\frac{2p}{p+2}}(\mathbb{C})}
\end{aligned}$$

and now we can just use (b). The claim follows. □

In Euler's setting, one is given two solutions $\mathbf{v}^1,\mathbf{v}^2$ to the same initial value problem. The first task consists of proving that one actually has $\mathbf{v}^1-\mathbf{v}^2\in L^2$. This is a consequence of the incompressibility, together with the fact that $\mathbf{v}^1(0,\cdot)=\mathbf{v}^1(0,\cdot)$. Having this in mind, then one finds an ODE for $E(t)=\|\mathbf{v}^1-\mathbf{v}^2\|_2$. Under the assumptions of (4.1), that is with the kernel $K(z)=\frac{e^{i\theta}}{2\pi z}$, it is not possible to control $\|\mathbf{v}^1-\mathbf{v}^2\|_2$ anymore, and instead one needs to look for the L^p norms, $p>2$.

Lemma 4.5.2. *Let $\omega_0\in L^\infty$ be given, and assume that* $\operatorname{supp}(\omega_0)\subset\mathbb{D}$. *Let $\omega^i\in L^\infty(0,T;L^1\cap L^\infty)$, $i=1,2$, be two real valued weak solutions of*

$$\begin{cases}\omega_t+\mathbf{v}\cdot\nabla\omega=0\\ \mathbf{v}=K*\omega\\ \omega(0,\cdot)=\omega_0.\end{cases}$$

Let $\mathbf{v}^i=K*\omega^i$, and let $p>2$ be fixed. Then for each $q>1$

$$\frac{d}{dt}E(t)\le C_0\,E(t)+C_1\,q\,E(t)^{1-\frac{1}{q}}+C_2\,E(t)^{\frac{1}{p'}},\tag{4.21}$$

where $E(t)=\|\mathbf{v}^1-\mathbf{v}^2\|_p$, and C_0,C_1,C_2 are constants that do not depend on q.

Proof. From the equivalent formulation (4.15), we know there exist two functions Q^i, $i = 1, 2$, such that

$$\begin{cases} \mathbf{v}^i_t + \mathbf{v}^i \cdot \nabla \mathbf{v}^i = -\mathbf{M}_\theta \nabla Q^i \\ -\Delta Q^i = \operatorname{div}(\mathbf{v}^i)\, \omega^i \\ \mathbf{v}^i(0, \cdot) = K * \omega_0, \end{cases}$$

where $K(z) = \frac{e^{i\theta}}{2\pi z}$. Set $\mathbf{v} = \mathbf{v}^1 - \mathbf{v}^2$, $\omega = \omega^1 - \omega^2$ and $Q = Q^1 - Q^2$. This gives an equation for $\mathbf{v}$,

$$\mathbf{v}_t + \mathbf{v}^1 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}^2 = -\mathbf{M}_\theta \nabla Q.$$

We fix now a number $p \in (2, \infty)$, and take dot product by $|\,\mathbf{v}\,|^{p-2}\, \mathbf{v}$ at both sides of the above equality. Using the chain rule and observing that $(\mathbf{v} \cdot \nabla \mathbf{v}^2) \cdot \mathbf{v} = \mathbf{v}^t \cdot D\,\mathbf{v}^2 \cdot \mathbf{v}$ (here $\mathbf{v}^t$ is the row obtained by transposing the column $\mathbf{v}$) one gets

$$\frac{d}{dt}\left(\frac{|\,\mathbf{v}\,|^p}{p}\right) = -\,\mathbf{v}^1 \cdot \nabla \left(\frac{|\,\mathbf{v}\,|^p}{p}\right) - |\,\mathbf{v}\,|^{p-2}\; \mathbf{v}^t \cdot D\,\mathbf{v}^2 \cdot \mathbf{v} - |\,\mathbf{v}\,|^{p-2}\; \mathbf{v}^t \cdot \mathbf{M}_\theta \nabla Q. \tag{4.22}$$

It is clear that each $\mathbf{v}^i(t, \cdot) = K * \omega^i(t, \cdot)$ belongs to $L^p(\mathbb{C})$, since $\omega^i(t, \cdot)$ is compactly supported and the kernel K decays linearly at ∞. Thus we are legitimate to introduce the quantity

$$E(t) = E_p(t) = \int \frac{|\,\mathbf{v}(t, z)|^p}{p}\, dA(z)$$

which may be infinite if $p = 2$, but is certainly finite if $2 < p < \infty$. We now integrate with respect to $dA(z)$ at (4.22), and after an integration by parts we get

$$\frac{d}{dt}E(t) = -\int \mathbf{v}^1 \cdot \nabla \left(\frac{|\,\mathbf{v}\,|^p}{p}\right) - \int |\,\mathbf{v}\,|^{p-2}\; \mathbf{v}^t \cdot D\,\mathbf{v}^2 \cdot \mathbf{v} - \int |\,\mathbf{v}\,|^{p-2}\; \mathbf{v}^t \cdot \mathbf{M}_\theta \nabla Q. \tag{4.23}$$

and take care of each term separately. To do this, we will need an exponent $q \in (1, \infty)$, that will be chosen large enough later on. First, an integration by parts gives that

$$\begin{aligned} \left| -\int \mathbf{v}^1 \cdot \nabla \left(\frac{|\,\mathbf{v}\,|^p}{p}\right)\right| &\leq \int |\operatorname{div}(\mathbf{v}^1)|\, \frac{|\,\mathbf{v}\,|^p}{p} \\ &\leq \left(\int |\operatorname{div}(\mathbf{v}^1)|^q\right)^{\frac{1}{q}} \left(\int \left(\frac{|\,\mathbf{v}\,|^p}{p}\right)^{q'} dA(z)\right)^{\frac{1}{q'}} \\ &\leq \left(\int |\operatorname{div}(\mathbf{v}^1)|^q\right)^{\frac{1}{q}} \left\|\frac{|\,\mathbf{v}\,|^p}{p}\right\|_\infty^{\frac{1}{q}} E(t)^{\frac{1}{q'}} \end{aligned} \tag{4.24}$$

Second, it is even easier to see that

$$\begin{aligned} \left| -\int |\,\mathbf{v}\,|^{p-2}\; \mathbf{v}^t \cdot D\,\mathbf{v}^2 \cdot \mathbf{v} \right| &\leq \int |D\,\mathbf{v}^2\,|\,|\,\mathbf{v}\,|^p \\ &\leq \left(\int |D\,\mathbf{v}^2\,|^q\right)^{\frac{1}{q}} \left(\int |\,\mathbf{v}\,|^{pq'}\right)^{\frac{1}{q'}} \end{aligned} \tag{4.25}$$

and from here one could proceed similarly. However, in the above bound, one can replace $|D\,\mathbf{v}^2\,|$ in the right hand term by $|\operatorname{div}(\mathbf{v}^2)|$. Indeed, from

$$\begin{aligned}\mathbf{v}^t \cdot D\,\mathbf{v}^2 \cdot \mathbf{v} &= \mathbf{v}^t \left(\frac{D\,\mathbf{v}^2 + D^t\,\mathbf{v}^2}{2} - \frac{\operatorname{div}(\mathbf{v}^2)}{2}\ \mathbf{Id}\right) \cdot \mathbf{v} + \frac{\operatorname{div}(\mathbf{v}^2)}{2}\,|\,\mathbf{v}\,|^2\\ &= \mathbf{v}^t\left(\overline{\partial}\,\mathbf{v}^2\right)\cdot \mathbf{v} + \frac{\operatorname{div}(\mathbf{v}^2)}{2}\,|\,\mathbf{v}\,|^2\end{aligned}$$

we obtain

$$\begin{aligned}\left|-\int |\,\mathbf{v}\,|^{p-2}\ \mathbf{v}^t \cdot D\,\mathbf{v}^2\cdot\mathbf{v}\right| &\le \int |\,\mathbf{v}\,|^p\,|\overline{\partial}\,\mathbf{v}^2\,| + \frac{1}{2}\int |\operatorname{div}(\mathbf{v}^2)|\,|\,\mathbf{v}\,|^p\\ &\le p\,\|\overline{\partial}\,\mathbf{v}^2\,\|_\infty\,E(t) + \frac{p}{2}\left(\int |\operatorname{div}(\mathbf{v}^2)|^q\right)^{\frac{1}{q}}\ \left\|\frac{|\,\mathbf{v}\,|^p}{p}\right\|_\infty^{\frac{1}{q}}\ E(t)^{\frac{1}{q'}}\end{aligned} \tag{4.26}$$

This shows that the control of $\operatorname{div}(\mathbf{v}^2)$ suffices when $\overline{\partial}\,\mathbf{v}^2 \in L^\infty$, instead of controlling all of $D\,\mathbf{v}^2$. We prefer this argument since the proof becomes better.

Finally, for the third term we use that $\mathbf{M}_\theta$ is an isometry, and obtain

$$\begin{aligned}\left|-\int |\,\mathbf{v}\,|^{p-2}\ \mathbf{v}^t \cdot \mathbf{M}_\theta \nabla Q\right| &\le \int |\,\mathbf{v}\,|^{p-1}\,|\nabla Q|\\ &\le E(t)^{\frac{p-1}{p}}\ p^{\frac{p-1}{p}}\ \left(\int |\nabla Q|^p\right)^{\frac{1}{p}}.\end{aligned} \tag{4.27}$$

We now plug (4.24), (4.26) and (4.27) into (4.23) and obtain

$$\begin{aligned}\frac{d}{dt}E(t) \le E(t)^{\frac{1}{q'}}\ &\left\|\frac{|\,\mathbf{v}\,|^p}{p}\right\|_\infty^{\frac{1}{q}} \left(\|\operatorname{div}(\mathbf{v}^1)\|_{L^q(\mathbb{C})} + \frac{p}{2}\|\operatorname{div}(\mathbf{v}^2)\|_{L^q(\mathbb{C})}\right)\\ &+ p\,\|\overline{\partial}\,\mathbf{v}^2\,\|_\infty\,E(t) + E(t)^{\frac{1}{p'}}\ p^{\frac{1}{p'}}\ \|\nabla Q\|_{L^p(\mathbb{C})}\end{aligned} \tag{4.28}$$

We now take $q > p > 2$ and use plug (4.18), (4.19) and (4.20) into (4.28), and obtain

$$\begin{aligned}\frac{d}{dt}E(t) &\le E(t)\,C(\omega_0)\,C_p\\ &\quad + q\,E(t)^{\frac{1}{q'}}\ C(\omega_0)\,C_p\,e^{\frac{p+2}{2q}\ t\|\omega_0\|_\infty}\\ &\quad + E(t)^{\frac{1}{p'}}\ e^{\frac{p+2}{2p}\ t\|\omega_0\|_\infty}\ C(\omega_0)\,C_p.\end{aligned}$$

Let us now fix $T > 0$ to be chosen, and restrict to $0 < t < T$. Then

$$\frac{d}{dt}E(t) \le C_0\,E(t) + C_1\,q\,E(t)^{\frac{1}{q'}} + C_2\,E(t)^{\frac{1}{p'}}, \tag{4.29}$$

for constants C_0, C_1, C_2 that depend only on p, ω_0 and T, but not on q or t. □

As we mentioned before, in Euler's setting, namely (4.1) with the kernel $K(z) = \frac{i}{2\pi\bar{z}}$, one is allowed to take $p = 2$, so that $E(t) = \|\mathbf{v}^1 - \mathbf{v}^2\|_2$. Under these circumstances, the inequality (4.21) improves to

$$\frac{d}{dt}E(t) \le q\,M\,E(t)^{1-\frac{1}{q}}, \tag{4.30}$$

where $M = C(\|\omega_0\|_\infty)$. In that particular setting, $E(t) = \|\mathbf{v}^1 - \mathbf{v}^2\|_2$ is a solution to (4.30). Although, (4.30) does not need to have a unique solution. Indeed, the maximal solution $\hat{E}(t)$ to (4.30) is $\hat{E}(t) = (Mt)^q$, and so any other solution satisfies $E(t) \leq \hat{E}(t)$. Let us consider an interval $[0, T^*]$ such that $MT^* \leq \frac{1}{2}$. Therefore, as $q \to \infty$, one has $E(t) \leq (\frac{1}{2})^q \to 0$. This implies $E(t) = 0$, $\forall\, 0 \leq t \leq T^*$. Repeating these techniques, we conclude that $E(t) = 0$, $\forall\, 0 \leq t \leq T$, which automatically leds to $\mathbf{v}^1 = \mathbf{v}^2$, proving uniqueness for the Euler system.

Unfortunately the system (4.1) we are looking at is a bit more delicate, and the arguments in the above paragraph do not seem to work. Remarkably, instead of (4.30) we get (4.21),

$$\frac{d}{dt}E(t) \leq C_0\, E(t) + C_1\, q\, E(t)^{1-\frac{1}{q}} + C_2\, E(t)^{\frac{1}{p'}},$$

which includes *two additional terms on the right hand side* that identically vanish under Euler (due to incompressibility). The first of these terms $C_0 E(t)$ is not a problem, and could be easily reabsorved by means of an integrating factor. However, the second of these two terms (i.e. $E(t)^{\frac{1}{p'}}$) produces an ODE for which uniqueness certainly fails. To see this, note that the problem

$$\begin{cases} F'(t) = F(t)^{\frac{1}{p'}} \\ F(0) = 0 \end{cases}$$

admits as solution

$$F(t) = \begin{cases} 0 & 0 \leq t \leq t_0 \\ \left(\frac{t-t_0}{p}\right)^p & t > t_0 \end{cases}$$

for all $t_0 > 0$. This explains that, in order to prove uniqueness for (4.1) for measurable datum, better estimates are needed for the Q term at (4.20).

Last but not least, we would like to mention some recent progresses on the study of *smooth patches* in the context of the Cauchy Kernel. To this end, let us remind that in the planar Euler system (4.1), with kernel $K(z) = \frac{i}{2\pi\bar{z}}$, solutions consisting of characteristic functions of domains are usually refered to as patches. Remarkably, if the initial datum is $\omega_0 = \chi_{D_0}$ for some domain D_0, and since the vorticity equation is a transport equation, the solution $\omega(t, \cdot)$ is forced to have the form $\omega(t, z) = \chi_{D_t}(z)$, for some domain D_t, because the vorticity is to be conserved along the trajectories. The most natural question is, under this setting, the preservation in time of the boundary smoothness. That is, one asks if the regularity of ∂D_0 keeps stable in time and ∂D_t has the same regularity, or instead some singularities, cusps, etc may appear. In [18], Chemin proved that this is indeed the case for the Euler equation, and if $\partial D_0 \in C^{1+\gamma}$ then $\partial D_t \in C^{1+\gamma}$ for every time $t > 0$. His proof is based in *paradifferential calculus*. Bertozzi and Constantin in [11] provided an alternative proof of the same result using techniques from *classical analysis*. Very recently, Cantero, Mateu, Orobitg and Verdera were the first to study at [17] the same question with Euler's kernel replaced by a much more general one (including Cauchy's kernel into their discussion, among others). They were able to prove global uniqueness of solutions to the patch problem and assuming for the datum $\omega_0 = \chi_{D_0}$ a boundary regularity of class $C^{1+\gamma}$, $0 < \gamma < 1$. Moreover, in the particular case of the Cauchy kernel $K(z) = \frac{1}{2\pi z}$ and an elliptic domain D_0,

$$D_0 = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\},$$

they proved that the unique solution is of the form $\omega(t,z) = \chi_{D_t}(z)$, where D_t is the domain enclosed by an ellipse with semiaxes $a(t)$ and $b(t)$ collapsing to a line segment on the horizontal axis as t becomes large enough. We wish to mention that uniqueness for the patch problem needs not imply uniqueness for a general datum $\omega_0 \in L^\infty$, thouhgh [17] is the first positive result about global uniqueness.

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