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On fractional caloric capacities in several function spaces

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Certifiquem que la present Memòria
ha estat realitzada per Joan Hernández García, sota la direcció
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A mi abuela.
Estés donde estés, estás conmigo.

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Contents

Introduction

The foundation of parabolic potential theory was established in the late 1950s. At the core of this theory lies a paradigmatic partial differential equation (PDE) that serves as a model: the heat equation, defined by the heat operator

$$\Theta := (-\Delta_x) + \partial_t, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Authors such as Gehring [**Ge**] and Friedman [**Fr**] introduced the family of subparabolic functions (analogous to continuous subharmonic functions) and investigated their properties. Through their work, they derived uniqueness and representation theorems for solutions of the heat equation, as well as the existence of solutions of general parabolic equations using an analog of superharmonic functions, as presented in [**Fr**]. Later, in the 1970s, Constantinescu and Cornea [**ConCor**], Watson [**Wa1, Wa2, Wa3**] or Lanconelli [**Lan**], expanded on these ideas, developing a heat potential theory inspired by the harmonic potential theory of Helms [**He**]. During this period, methods and properties for subharmonic functions were adapted to the caloric context, including weak and strong maximum principles, continuity and semicontinuity principles, the existence of caloric majorants, and various uniqueness theorems. These contributions generalized the works of Gehring and Friedman, relaxing continuity assumptions to upper semicontinuity.

In [**Wa3**], Watson introduced the concept of caloric capacity, along with a theory of balayage and polar sets. Kaiser and Müller [**KMü**] later formalized the notion of a set being removable for bounded solutions of the heat equation in terms of its capacity. For a compact set $E \subset \mathbb{R}^{n+1}$, its caloric capacity is defined as

$$\gamma_{\Theta,+}(E) := \sup \mu(E),$$

with μ being a nonnegative Borel measure supported on E satisfying that its heat potential is bounded by 1, that is $\|W * \mu\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1$. Here, W is the heat kernel,

$$W(x,t) = c_n t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}.$$

Watson proved that this capacity is subadditive and outer regular. In the late 1980s, following the work of Gariepy and Zimmer [**GaZ**] and Maeda [**Mae**], Watson also proved the comparability of $\gamma_{\Theta,+}$ and its adjoint version $\gamma_{\overline{\Theta},+}$, which is the one associated with the adjoint operator $\overline{\Theta} := (-\Delta_x) - \partial_t$ [**Wa4, Theorem 7.46**].

More recently, significant advancements in parabolic theory on time-varying domains have been made, with major contributions from Hofmann, Lewis, Nyström and Strömqvist [**Ho1, HoL, NySt**], for example. With it, as expected, there has also been an increasing interest in understanding the properties of the above caloric capacity. For instance, Mourgoglou and Puliatti [**MoPu**] studied properties of $\gamma_{\Theta,+}$ related to a capacity density condition on a particular scale which allow proving several PDE estimates around the boundary, essential for their blow-up-type arguments.

Building on this extensive body of work, this dissertation addresses a closely related problem motivated by recent results from Mateu, Prat, and Tolsa [**MatP, MatPT**]. Specifically, the main goal of this dissertation is to focus on capacities associated not only with the heat equation but also with its fractional variants, i.e., the PDE's associated with the pseudo-differential operator

$$\Theta^s := (-\Delta_x)^s + \partial_t, \qquad s \in (0,1],$$

where for $s=1$ we recover the classical heat equation and $(-\Delta_x)$ is the usual Laplacian, computed with respect to the spatial variables. If $s<1$, $(-\Delta_x)^s$ is known as the s-fractional Laplacian or s-Laplacian, and has to be defined alternatively. Typically, one defines it via its Fourier transform,

$$\widehat{(-\Delta_x)^s f}(\xi,t) = |\xi|^{2s}\widehat{f}(\xi,t),$$

or by the following integral representation

$$\begin{aligned}(-\Delta_x)^s f(x,t) &= c_{n,s}\ \text{p.v.}\int_{\mathbb{R}^n}\frac{f(x,t)-f(y,t)}{|x-y|^{n+2s}}\,\mathrm{d}y\\ &= c'_{n,s}\int_{\mathbb{R}^n}\frac{f(x+y,t)-2f(x,t)+f(x-y,t)}{|y|^{n+2s}}\,\mathrm{d}y.\end{aligned}$$

These representations are equivalent and highlight that $(-\Delta_x)^s$ is no longer a local operator and that as $s\to 1$, the constants $c_{n,s}$ and $c'_{n,s}$ should behave in a way so that one recovers the expression of $(-\Delta_x)$. For further properties of $(-\Delta_x)^s$, the reader may consult [**DPaV, §3**] or [**Ste**], as well as the works of Ros-Oton and Serra in [**RoSe1**], and even in a more general setting in [**RoSe2**], for discussions on the regularity theory for these fractional operators.

In recent years, fractional diffusion processes have gained significant attention due to their applications in modeling physical phenomena such as anomalous diffusion, quasi-geostrophic flows, turbulence, molecular dynamics, population dynamics, and relativistic quantum mechanics (see the book of Chen, Li and Ma [**ChLMa**] and the references therein for more details). Another important application has to do with the modelling of stock prices, since the non-local nature of the fractional Laplacian accounts for potential market volatility. In such cases, the valuation of financial options is often framed as an optimal stopping problem, leading naturally to the so-called parabolic obstacle problem, a topic studied by Ros-Oton and Torres-Latorre [**RoTo**].

In the sequel, we focus on a specific function associated with the operator Θ^s, known as its fundamental solution and denoted by P_s. The key feature of P_s is that it satisfies the distributional identity $\Theta^s P_s = \delta_0$, where δ_0 stands for the Dirac mass at $0 \in \mathbb{R}^{n+1}$. The function P_s is defined as the inverse spatial Fourier transform of $e^{-4\pi^2 t|\xi|^{2s}}$ for $t > 0$, and equals 0 when $t \leq 0$. For $s = 1$, we recover the heat kernel and write $P_1 := W$. When $0 < s < 1$, although the explicit expression of P_s is not available, Blumenthal and Getoor [**BlG, Theorem 2.1**] proved that

$$P_s(x,t) \approx_{n,s} \frac{t}{\left(|x|^2 + t^{1/s}\right)^{(n+2s)/2}}\chi_{t>0},$$

where the symbol $\approx_{n,s}$ indicates that P_s is bounded above and below by the previous quotient up to constant factors only depending on n and s. To make notation more efficient, we shall write points of $\mathbb{R}^{n+1}$ as $\overline{x} := (x,t) \in \mathbb{R}^{n+1}$. We also define the s-parabolic distance for $0 < s \leq 1$ as,

$$\operatorname{dist}_{p_s}\big((x,t),(y,\tau)\big) := \max\left\{|x-y|^2, |t-\tau|^{1/s}\right\} \approx_{n,s} \left(|x-y|^2 + |t-\tau|^{1/s}\right)^{1/2}.$$

From the latter, one defines s-parabolic cubes and balls in the natural way. We denote the s-parabolic norm as $|\overline{x}|_{p_s} := \operatorname{dist}_{p_s}(\overline{x}, 0)$. Therefore, P_s satisfies

$$P_s(\overline{x}) \approx_{n,s} \frac{t}{|\overline{x}|_{p_s}^{n+2s}}\chi_{t>0}.$$

For $s = 1/2$, the s-parabolic distance is comparable to the usual Euclidean distance in $\mathbb{R}^{n+1}$, and we simply put $|\cdot| := |\cdot|_{p_{1/2}}$. In fact, if $s = 1/2$ the above relation becomes an equality [**Va**]. For this case, we write

$$P(\overline{x}) := P_{1/2}(\overline{x}) = c_n \frac{t}{|\overline{x}|^{n+1}}\chi_{t>0}.$$

In [**MatP, §3, §4, §5**], Mateu and Prat study the analogous caloric capacity to that of Watson but in the above 1/2-caloric context. That is,

$$\gamma_{\Theta^{1/2},+}(E) := \sup \mu(E),$$

with μ a nonnegative Borel measure supported on E and such that $\|P * \mu\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1$. In fact, they work with a more general capacity

$$\gamma_{\Theta^{1/2}}(E) := \sup |\langle T, 1\rangle|,$$

the supremum taken among distributions supported on E with $\|P * T\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1$. With this framework, they study removable sets for bounded solutions of the $\Theta^{1/2}$-equation and determine that the critical dimension of $\gamma_{\Theta^{1/2}}$ is n in $\mathbb{R}^{n+1}$. This raises the question of the comparability between the 1/2-caloric capacity and analytic or Newtonian capacities in the planar setting, since they all share the same critical dimension.

Let us briefly recall some important properties of Newtonian and analytic capacities. Given $E \subset \mathbb{C}$ compact set, its Newtonian capacity $\operatorname{Cap}(E)$ is defined similarly to the previous discussion, but now with the normalization condition $\||z|^{-1} * \mu\|_{L^\infty(\mathbb{C})} \leq 1$.

Analytic capacity γ is defined requiring $\|z^{-1} * \mu\|_{L^\infty(\mathbb{C})} \leq 1$. We have deliberately written μ instead of T, as it suffices to consider positive Borel measures rather than general distributions. This fact can be proved rather straightforwardly for Newtonian capacity, see for example [**Ve2, §3**][1], while for analytic capacity it is a deeper result established by Tolsa in [**T3**], where he proved the semi-additivity of γ. There are many equivalent definitions of γ that turn out to be very surprising. A notable breakthrough which played a key role in a better understanding of γ is the Menger curvature, discovered by Mel'nikov. In [**Me2**] he found

$$\sum_{\sigma\in\mathfrak{S}_3} \frac{1}{z_{\sigma_1}-z_{\sigma_2}} \overline{\frac{1}{z_{\sigma_1}-z_{\sigma_3}}} = \frac{1}{R(z_1,z_2,z_3)^2}, \qquad z_1,z_2,z_3\in\mathbb{C},$$

where $\mathfrak{S}_3$ is the group of permutations of three elements and R is the radius of the circle determined by z_1, z_2 and z_3. The inverse of R is known as the Menger curvature of z_1, z_2 and z_3. This identity allowed Mel'nikov and Verdera [**MeVe**] to prove that, for any finite measure μ with linear growth and all $\varepsilon > 0$,

$$\int \bigg| \int_{|z-w|>\varepsilon} \frac{1}{w-z}\,\mathrm{d}\mu(w) \bigg|^2 \mathrm{d}\mu(z) \\ = \frac{1}{6}\iiint_{T_\varepsilon} \frac{1}{R(z_1,z_2,z_3)^2}\,\mathrm{d}\mu(z_1)\,\mathrm{d}\mu(z_2)\,\mathrm{d}\mu(z_3) + O(\mu(\mathbb{C})),$$

where T_ε is the collection of triplets (z_1,z_2,z_3) satisfying $|z_i - z_j| > \varepsilon$ if $i\neq j$. The previous identity plays a fundamental role in proving the L^2 boundedness of the Cauchy transform (of the arc length) on Lipschitz graphs and regular curves. While the above curvature method works well for the Cauchy kernel, it does not generalize to most convolution kernels K, since the sum

$$\sum_{\sigma\in\mathfrak{S}_3} K(z_{\sigma_1}-z_{\sigma_2})\overline{K(z_{\sigma_1}-z_{\sigma_3})},$$

may attain negative values [**Fa**]. However, for certain kernels, it remains nonnegative. A particular case of interest, studied by Chousionis, Mateu, Prat and Tolsa [**ChouMatPT**], involves the kernel in $\mathbb{C}$,

$$K_n(z) = \frac{(\mathrm{Im} z)^{2n-1}}{|z|^{2n}}, \qquad \text{for each } n\geq 1.$$

From this fact, and choosing $n = 1$, the authors are able to compare γ with the capacity defined via K_1. We are interested in this particular definition of γ because,

[1] The arguments of [**Ve2**] can be adapted to more general settings. For instance, let X be a locally compact Hausdorff space and $K(\cdot,\cdot) : X\times X \to [0,+\infty]$ be symmetric, positive, finite outside the diagonal and continuous in the extended sense. Then, if K satisfies the so-called *first maximum principle* and the capacity associated to it is outer regular, the comparability result also follows. The reader may consult [**La, Ch.VI, §2**] for a detailed analysis and review of the work done by Ugaheri [**U**], Ninomiya [**Ni**], Cartan and Deny [**CaD**], Choquet [**Cho**], Ohtsuka [**O**] or Kishi [**Ki**], the latter covering the non-symmetric case.

in the planar setting, when the kernel P is expressed in complex notation and $\overline{x}$ is identified with z, one gets

$$P(\overline{x}) = \frac{t}{|\overline{x}|^2}\chi_{t>0} = \frac{\text{Im}z}{|z|^2}\chi_{\text{Im}z>0} = K_1(z)\chi_{\text{Im}z>0}.$$

Therefore, from the previous identity, one easily deduces that $\gamma_{\Theta^{1/2},+}$ is greater than Newtonian capacity. In fact, in Chapter 4 we provide the tools to prove that $\gamma_{\Theta^{1/2},+} \lesssim \gamma$[2]. Consequently, we conclude that the $(1/2,+)$-caloric capacity lies between Newtonian capacity and analytic capacity. This is consistent with [**MatP, Proposition 6.1**], which states that the 1/2-caloric capacity of a horizontal line segment behaves as its length, while that of a vertical line segment is null. This is generalized in [**Her, Proposition 3.3**], where it is established that any non-horizontal line segment has null capacity and even further in §5.5 for the multi-dimensional setting. In fact, in Chapter 5 we are able to give a precise description of the capacity of closed rectangles with sides parallel to the coordinate axes. Namely, if its respective side lengths are $\ell_x > 0$, $\ell_t > 0$, we show that

Theorem.

$$\gamma_{\Theta^{1/2},+}(R) \approx \ell_t \left[\frac{1}{2}\ln\left(1+\frac{\ell_t^2}{\ell_x^2}\right) + \frac{\ell_t}{\ell_x}\arctan\left(\frac{\ell_x}{\ell_t}\right)\right]^{-1}.$$

An interesting open question is whether the relationship

$$\gamma_{\Theta^{1/2},+}(E) \lesssim \gamma(E)$$

and the rectangle capacity formula also hold for $\gamma_{\Theta^{1/2}}$. Another intriguing problem is determining a multidimensional analog of the expression for $\gamma_{\Theta^{1/2},+}(R)$, where $R \subset \mathbb{R}^{n+1}$ is a general parallelepiped. Nonetheless, some positive comparability results have been achieved, not for $\gamma_{\Theta^{1/2}}$, but for $\widetilde{\gamma}_{\Theta^{1/2}}$. The latter, called 1/2-symmetric caloric capacity, is the main object of study in Chapter 5 and it is defined as $\gamma_{\Theta^{1/2}}$ but requiring both conditions

$$\|P*T\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1 \qquad \text{and} \qquad \|P^* * T\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1,$$

where $P^*(\overline{x}) := P(-\overline{x})$ is the adjoint kernel associated with P. We prove the following:

Theorem. *The following hold:*

1. *For any compact set $E \subset \mathbb{R}^2$,*

$$\widetilde{\gamma}_{\Theta^{1/2}}(E) \approx \gamma_{\Theta^{1/2},+}(E).$$

2. *In $\mathbb{R}^{n+1}$, if $E := R_1 \cup R_2 \cup \cdots \cup R_N$ is a finite union of disjoint closed parallelepipeds with sides parallel to the coordinate axes,*

$$\widetilde{\gamma}_{\Theta^{1/2}}(E) \approx_n \gamma_{\Theta^{1/2},+}(E).$$

[2] More precisely, we prove that assuming $\|P*\mu\|_{L^\infty(\mu)}$ we get the inclusion $P^* * \mu \in \text{BMO}_\rho(\mu)$ for some $\rho \geq 2$. Then, since $K_1 = P - P^*$, applying a $T1$ theorem, for example [**T5, Theorem 9.41**], we obtain the result.

These are the main results regarding the 1/2-caloric capacity which are covered in this text. However, many additional results are presented in Chapters 1 and 4, where we examine capacities defined by imposing that the potentials belong to other function spaces, not only to L^∞. In fact, this study is carried out in a general s-fractional setting for any $0 < s \leq 1$.

For instance, we study the capacity obtained by imposing that $P_s * T$ (T supported on a previously fixed compact set) belongs to the s-parabolic BMO space, shortly BMO_{p_s}. As expected, we say that a function $f \in L^1_{\text{loc}}$ belongs to BMO_{p_s} if

$$\|f\|_{*,p_s} := \sup_Q \frac{1}{|Q|} \int_Q |f(\overline{x}) - f_Q| \, \mathrm{d}\overline{x} < \infty,$$

where the supremum is taken among all s-parabolic cubes and f_Q is the average of f over Q with respect to the Lebesgue measure. This approach will lead us to the capacity $\gamma_{\Theta^s,*}$.

We will also be interested in the capacity obtained by imposing an s-parabolic Lip_α, $0 < \alpha < 1$, normalization over $P_s * T$. This will lead us to the capacity $\gamma_{\Theta^s,\alpha}$. Observe that a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ will be s-parabolic Lip_α for some $0 < \alpha < 1$, shortly Lip_{α,p_s}, if

$$\|f\|_{\text{Lip}_\alpha,p_s} := \sup_{\overline{x},\overline{y} \in \mathbb{R}^{n+1}} \frac{|f(\overline{x}) - f(\overline{y})|}{|\overline{x} - \overline{y}|_{p_s}^\alpha} < \infty.$$

This way, writing $\mathcal{H}^d_{\infty,p_s}$ the d-dimensional s-parabolic Hausdorff content (defined using the s-parabolic diameter), we get:

Proposition. *Given $s \in (0,1]$, $\alpha \in (0,1)$ and $E \subset \mathbb{R}^{n+1}$ a compact set,*

1. $\gamma_{\Theta^s,*}(E) \approx_{n,s} \mathcal{H}^n_{\infty,p_s}(E)$.
2. *If $\alpha < 2s$, then* $\gamma_{\Theta^s,\alpha}(E) \approx_{n,s,\alpha} \mathcal{H}^{n+\alpha}_{\infty,p_s}(E)$.

Inspired by the capacities studied in [**MatP, §7**], we generalize even further these results by introducing additional parameters. Given $s \in (0,1]$, $\sigma \in [0,s)$ and $E \subset \mathbb{R}^{n+1}$ compact set, we define the capacities

$$\gamma^\sigma_{\Theta^s,*}(E) := \sup |\langle T, 1\rangle|, \qquad \gamma^\sigma_{\Theta^s,\alpha}(E) := \sup |\langle S, 1\rangle|,$$

where distributions T and S are supported on E and satisfy

$$\|(-\Delta_x)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1,$$

in the BMO case, or

$$\|(-\Delta_x)^\sigma P_s * S\|_{\text{Lip}_\alpha,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * S\|_{\text{Lip}_\alpha,p_s} \leq 1,$$

in the Lip_α case. Here, ∂_t^β is the fractional time derivative of order $\beta \in (0,1)$,

$$\partial_t^\beta f(x,t) := \int_{\mathbb{R}} \frac{f(x,\tau) - f(x,t)}{|\tau - t|^{1+\beta}} \, \mathrm{d}\tau.$$

One of the main results in Chapter 1 reads as follows:

THEOREM. *Let $s \in (0,1]$, $\alpha \in (0,1)$ and $\sigma \in [0,s)$. For any compact set $E \subset \mathbb{R}^{n+1}$,*

1. $\gamma^{\sigma}_{\Theta^s,*}(E) \approx_{n,s,\sigma} \mathcal{H}^{n+2\sigma}_{\infty,p_s}(E)$.
2. *If $\alpha < 2s-2\sigma$, then* $\gamma^{\sigma}_{\Theta^s,\alpha}(E) \approx_{n,s,\sigma,\alpha} \mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E)$.

In addition to the comparability results previously discussed, we have established an equivalence between the nullity of both above capacities and the removability of compact sets for the Θ^s-equation for solutions satisfying certain s-parabolic BMO or Lip_α conditions. Let us clarify what we mean by this equivalence. A compact set $E \subset \mathbb{R}^{n+1}$ is said to be removable for s-caloric functions with BMO_{p_s}-$(\sigma,\sigma/s)$-Laplacian if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ with

$$\|(-\Delta)^{\sigma} f\|_{*,p_s} < \infty, \qquad \|\partial_t^{\sigma/s} f\|_{*,p_s} < \infty,$$

satisfying the Θ^s-equation in $\Omega\setminus E$, also satisfies the previous equation in the whole Ω. If the same property holds but with respect to Lip_{α,p_s} seminorms, i.e.,

$$\|(-\Delta)^{\sigma} f\|_{\text{Lip}_\alpha,p_s} < \infty, \qquad \|\partial_t^{\sigma/s} f\|_{\text{Lip}_\alpha,p_s} < \infty,$$

we say that the compact set is removable for s-caloric functions with Lip_{α,p_s}-$(\sigma,\sigma/s)$-Laplacian. If $\sigma=0$, we will also say that E is removable for BMO_{p_s} (resp. Lip_{α,p_s}) s-caloric functions. Bearing in mind the previous definitions, we prove the following:

THEOREM. *Let $s \in (0,1]$, $\alpha \in (0,1)$ and $\sigma \in [0,s)$. Let $E \subset \mathbb{R}^{n+1}$ be a compact set. Then,*

1. *E is removable for s-caloric functions with* BMO_{p_s}-$(\sigma,\sigma/s)$*-Laplacian if and only if* $\gamma^{\sigma}_{\Theta^s,*}(E)=0$.
2. *If $\alpha < 2s-2\sigma$, E is removable for s-caloric functions with* Lip_{α,p_s}-$(\sigma,\sigma/s)$*-Laplacian if and only if* $\gamma^{\sigma}_{\Theta^s,\alpha}(E)=0$.

To end the study of the previous s-caloric capacities, in Chapter 4 we provide a geometric characterization of the value of $\gamma_{\Theta^s,+}$ for the usual corner-like Cantor sets in $\mathbb{R}^{n+1}$ made up of s-parabolic cubes. Here, $\gamma_{\Theta^s,+}$ is the capacity obtained by considering positive Borel measures satisfying $\|P_s*\mu\|_{L^\infty(\mathbb{R}^{n+1})}\leq 1$ and supported on the desired compact set. Such Cantor sets are defined as follows: start from the unit cube Q^0 of $\mathbb{R}^{n+1}$ and apply to it a contraction ratio $\lambda_1 \in (0,1/2)$. Then, we keep 2^{n+1} copies of this contracted cube, placed at the vertices of Q^0. We repeat the same splitting for each of the cubes of the first generation with another contraction ratio $\lambda_2 \in (0,1/2)$, and so on. Iterating this process indefinitely, for a given sequence $\lambda=(\lambda_1,\lambda_2,\ldots)$ with the property $0<\lambda_j\leq\tau_0<1/2$, one constructs a Cantor set $E=E(s,\lambda)$ that satisfies:

THEOREM. *For $s\in(0,1]$, if $\ell_j := \lambda_1\cdots\lambda_j$ is the length of the cubes in the j-th generation of the Cantor set,*

$$\gamma_{\Theta^s,+}(E) \approx_{n,s,\tau_0} \left(\sum_{j=0}^{\infty}\theta_j\right)^{-1}, \qquad \textit{where} \quad \theta_j := \frac{\ell_j^{-n}}{2^{j(n+1)}}.$$

The study of s-caloric capacities inspired by Watson and later on by Mateu and Prat leads to consider the problem of removable singularities for bounded solutions of the Θ^s-equation, $s \in (0,1]$. However, one could ask the same question for functions that satisfy some other normalization conditions, as seen above by requiring BMO or Lip_α estimates. This problem was already studied for the BMO variant of analytic capacity by Kaufman [**Ka**] or Verdera [**Ve1**], and also for the Lip_α variant of the same capacity by Mel'nikov [**Me1**] or O'Farrell [**O'**]. It is precisely the work done by the previous authors that motivated the author to investigate these problems posed in the s-caloric case.

However, what if the solution of the Θ^s-equation satisfies a usual Lipschitz property? Is there any characterization of removable singularities for s-caloric solutions? In [**MatPT**], Mateu, Prat and Tolsa address this question for the Θ-equation. There, when imposing a Lipschitz-type restriction to the solution, the authors note that the appropriate bounds that a solution of the heat equation should satisfy are

$$\|\nabla_x f\|_{L^\infty(\mathbb{R}^{n+1})} < \infty, \qquad \|\partial_t^{1/2} f\|_{*,p_1} < \infty.$$

That is, a usual Lipschitz condition for the spatial variables and a parabolic BMO estimate for the half time-derivative. Using the results of Hofmann and Lewis [**HoL, Ho1**], one can prove that above bounds imply that f is such that $\|f\|_{\text{Lip}_{1/2,t}} < \infty$.

That is, the functions of interest are Lipschitz in space and 1/2-Lipschitz in time. We call them $(1,1/2)$-Lipschitz, and they were already studied by Nyström and Strömqvist in [**NySt**]. The main motivations that led the authors in [**MatPT**] to impose such a regularity property come from the results of Hofmann, Lewis, Murray and Silver [**LeSi, LeMu, HoL, Ho2**], where the connection between parabolic singular integral operators and caloric layer potentials on graphs is studied. Following their analysis, it becomes clear that the *right* graphs to consider are indeed those of $(1,1/2)$-Lipschitz functions.

Having conveyed the latter, Mateu, Prat and Tolsa define the $(1,1/2)$-Lipschitz caloric capacity Γ_Θ of a compact set $E \subset \mathbb{R}^{n+1}$ by considering the supremum of expressions of the form $|\langle T, 1\rangle|$, where T is distribution supported on E such that

$$\|\nabla_x W * T\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1, \qquad \|\partial_t^{1/2} W * T\|_{*,p_1} \leq 1.$$

They establish the equivalence between the nullity of Γ_Θ and the removability of compact sets for $(1,1/2)$-Lipschitz solutions of the heat equation. Moreover, the authors are also able to determine the critical (parabolic) Hausdorff dimension of Γ_Θ, that in $\mathbb{R}^{n+1}$ turns out to be $n+1$, and construct a fractal set with positive and finite $\mathcal{H}_p^{n+1}$ measure that is removable.

In Chapter 2 we focus on obtaining the above results in the fractional case for $1/2 < s \leq 1$. We extend a certain result of Hofmann and Lewis [**HoL, Theorem 7.4**] and prove the following:

THEOREM. *For* $1/2 < s \leq 1$*, a function satisfying*

$$\|\nabla_x f\|_{L^\infty(\mathbb{R}^{n+1})} < \infty, \qquad \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty,$$

is $(1, \frac{1}{2s})$*-Lipschitz.*

With this, we define a $(1, \frac{1}{2s})$-Lipschitz caloric capacity Γ_{Θ^s} as in [**MatPT**], requiring

$$\|\nabla_x P_s * T\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Using localization technics for these potentials, we are able to prove:

THEOREM. *Let $1/2 < s \leq 1$ and $E \subset \mathbb{R}^{n+1}$ be a compact set. Then,*

1. *E is removable for $(1, \frac{1}{2s})$-Lipschitz s-caloric functions if and only if $\Gamma_{\Theta^s}(E) = 0$.*
2. *$\Gamma_{\Theta^s}(E) \leq C\, \mathcal{H}^{n+1}_{\infty,p_s}(E)$, for some constant $C(n,s) > 0$.*
3. *If $\dim_{\mathcal{H}_{p_s}}(E) > n+1$, then $\Gamma_{\Theta^s}(E) > 0$.*

Therefore, the critical (s-parabolic) Hausdorff dimension of Γ_{Θ^s} is $n+1$.

Regarding the critical dimension, it is worth noting that the fractional parameter s only appears in the definition of the s-parabolic distance, which determines the corresponding s-parabolic Hausdorff dimension. This suggests that, as s approaches $1/2$ and the metric approaches the usual Euclidean one, the critical dimension should equal that of the ambient space. Inspired by the work of Uy [**Uy**] on analytic capacity, if one considered the capacity $\Gamma_{\Theta^{1/2}}$[3] of a compact set $E \subset \mathbb{R}^{n+1}$, it should be comparable to its Lebesgue measure $\mathcal{L}^{n+1}(E)$. However, this remains an open question.

In Chapter 2 we have also constructed a fractal set, for each $1/2 < s \leq 1$, that presents positive and finite $\mathcal{H}^{n+1}_{p_s}$ measure and that has null Γ_{Θ^s} capacity. Such set E_{p_s} is a enlarged version of the typical corner-like Cantor set of $\mathbb{R}^{n+1}$, inspired by that introduced in [**MatPT**]. The set E_{p_s} grows denser in the unit cube as s approaches $1/2$, pointing towards the comparability between $\mathcal{L}^{n+1}$ and $\Gamma_{\Theta^{1/2}}$ in $\mathbb{R}^{n+1}$.

In Chapter 4 we generalize the Cantor set E_{p_s} of Chapter 2 via a sequence of contraction ratios $(\lambda_j)_j$ satisfying some restrictions depending on s. By the spatial antisymmetry of the kernel $\nabla_x P_s$, and drawing on the work of Mateu and Tolsa on Riesz kernels in [**MatT, T4**], we have been able to obtain the next partial result:

THEOREM. *Let $1/2 < s \leq 1$ and $(\lambda_j)_j$ be such that $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j, where δ is a parameter that depends on s. Let E_{p_s} be its associated s-parabolic Cantor set. Then, if $\ell_j := \lambda_1 \cdots \lambda_j$ is the length of the cubes conforming the j-th generation,*

$$\Gamma_{\Theta^s}(E_{p_s}) \gtrsim_{n,s,\tau_0} \left(\sum_{j=0}^{\infty} \theta_{j,p_s}^2 \right)^{-1/2}, \qquad \textit{where} \quad \frac{\ell_j^{-(n+1)}}{(\delta+1)^j \delta^{nj}}.$$

The reverse inequality remains an open problem.

In Chapter 1, we also explore the s-parabolic BMO and Lip_α variants of the $(1, 1/2)$-Lipschitz caloric capacity, denoted by $\Gamma_{\Theta^s,*}$ and $\Gamma_{\Theta^s,\alpha}$ respectively. For the first we require the normalization conditions

$$\|\nabla_x P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

[3] The Lipschitz capacity for the $\Theta^{1/2}$ equation, i.e. that obtained by imposing $\|\nabla P * T\|_{L^\infty(\mathbb{R}^{n+1})}$, where now $\nabla = (\nabla_x, \partial_t)$ is a full gradient.

while for the second

$$\|\nabla_x P_s * T\|_{\mathrm{Lip}_\alpha, p_s} \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{\mathrm{Lip}_\alpha, p_s} \leq 1.$$

We have obtained the following comparability results:

THEOREM. *Let $s \in (1/2, 1]$, $\alpha \in (0,1)$ and $E \subset \mathbb{R}^{n+1}$ a compact set. Then,*

1. $\Gamma_{\Theta^s,*}(E) \approx_{n,s} \mathcal{H}^{n+1}_{\infty,p_s}(E)$.
2. *If $\alpha < 2s-1$, then $\Gamma_{\Theta^s,\alpha}(E) \approx_{n,s,\alpha} \mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E)$.*

Moreover, the nullity of these capacities is equivalent to the removability of the corresponding compact set for solutions satisfying $(1, \frac{1}{2s})$-gradient estimates in either s-parabolic BMO *or* Lip_α, *assuming $\alpha < 2s-1$ in the latter case.*

Finally, in Chapter 3, motivated by a question posed by X. Tolsa, we compare a certain Lipschitz caloric capacity with another one similar to those introduced in Chapter 1. We have been able to do it only in $\mathbb{R}^2$. Namely, we consider the capacity Γ_Θ, the $(1, 1/2)$-Lipschitz caloric capacity studied in [**MatPT**]; and that obtained by imposing the normalization conditions

$$\|(-\Delta_x)^{1/2} W * T\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1, \qquad \|\partial_t^{1/2} W * T\|_{*,p_1} \leq 1,$$

and we name it $\gamma_\Theta^{1/2}$. We establish that these two capacities share the same critical parabolic Hausdorff dimension in $\mathbb{R}^{n+1}$. However, the main result of Chapter 3 is:

THEOREM. *The capacities $\gamma_\Theta^{1/2}$ and Γ_Θ are not comparable in $\mathbb{R}^2$.*

To prove this, we show that, although Γ_Θ assigns a positive capacity to a vertical line segment, its $\gamma_\Theta^{1/2}$ capacity is zero.

To conclude this introduction, we briefly sum up the different contents covered in this text. In Chapter 1 we study different BMO and Lip_α s-caloric capacities. We present them in a rather general form so that they include the corresponding analogous capacities introduced by Watson [**Wa3**] and Mateu, Prat and Tolsa [**MatPT, MatP**]. The main results obtained in this chapter involve the comparability of the previous capacities to a certain s-parabolic Hausdorff content.

In Chapter 2 we generalize the parabolic Lipschitz caloric capacity studied in [**MatPT**], for the cases $1/2 < s \leq 1$. We give a criteria for the removability of compact sets in terms of such capacity and we establish that the critical s-parabolic Hausdorff dimension in $\mathbb{R}^{n+1}$ is $n+1$. Also, we are able construct a set with positive and finite $\mathcal{H}^{n+1}_{p_s}$ measure that is removable. In addition, it is the first time where the concept of symmetric caloric capacity appears in this text. We establish that such capacity is comparable to another one for which we require an $L^2(\mu)$-operator bound in its definition. This indicates that, working with such symmetric capacities, one should be able to obtain similar results to those obtained when working with odd kernels.

In Chapter 3 we compare the Lipschitz caloric capacity to that where instead of asking for solutions with $\|\nabla_x f\|_{L^\infty(\mathbb{R}^{n+1})} < \infty$, they satisfy $\|(-\Delta_x)^{1/2} f\|_{L^\infty(\mathbb{R}^{n+1})} < \infty$. We obtain that these two capacities are not comparable in $\mathbb{R}^2$.

In Chapter 4 we estimate some s-caloric capacities of Cantor sets in $\mathbb{R}^{n+1}$. For the s-Lipschitz capacities we are only able to obtain a lower bound, similar to the estimate obtained by Mateu and Tolsa for vectorial Riesz kernels in [**MatT, T4**]. For the s-fractional version of Watson's caloric capacity, we obtain a full geometric characterization, that turns out to be analogous to that studied by Eiderman in [**E**] for radial nonnegative kernels.

Finally, in Chapter 5 we fix $s = 1/2$ and study the comparability in $\mathbb{R}^2$ between the 1/2-symmetric caloric capacity and the 1/2-caloric capacity studied by Mateu and Prat in [**MatP**] when computed only with respect to positive Borel measures. Moreover, we are able to estimate these capacities for rectangles. The results obtained illustrate the non-isotropic behavior of caloric capacities regarding spatial and temporal directions.

About the notation:

We use an upper bar to denote generic points in $\mathbb{R}^{n+1}$, as for example $\overline{x}, \overline{y}, \overline{z} \ldots$ We will also write $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^{n+1})}$.

The symbol χ_A, where $A \subset \mathbb{R}^{n+1}$ is Borel measurable, means the indicator function that equals 1 on A, and is 0 otherwise.

For any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, the symbol f^* will denote $f^*(\overline{x}) := f(-\overline{x})$. Hence f being an even function is equivalent to $f = f^*$. Fourier transforms of smooth functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$ will be always taken with respect to spatial variables. For a fixed $t \in \mathbb{R}$, we put $g_t(x) := f(x,t)$ and write $\widehat{f}(\cdot, t) := \widehat{g_t}$.

Since Laplacian operators (fractional or not) will frequently appear in our discussion and will always be taken with respect to spatial variables, we will adopt the notation:

$$(-\Delta)^s := (-\Delta_x)^s, \qquad s \in (0,1], \quad \text{and we convey } (-\Delta)^0 := \mathrm{Id}.$$

Constants appearing in the sequel may depend on the dimension of the ambient space and the parameter s, and their value may change at different occurrences. They will frequently be denoted by the letters c or C. The notation $A \lesssim B$ means that there exists C, so that $A \leq CB$. Moreover, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$, while $A \simeq B$ will mean $A = CB$. If the reader finds expressions of the form $\lesssim_\beta$ or $\approx_\beta$, for example, it will mean that the implicit constants depend on n, s and β. We are not concerned with the behavior of constants as s approaches limit values. We warn the reader that some of the constants and parameters depending on s in Chapter 2 or the first section of Chapter 4 may blow up as $s \to 1/2$.

An important parameter which will play a fundamental role in Chapter 1 is

$$2\zeta := \min\{1, 2s\}.$$

A cube in $\mathbb{R}^{n+1}$ centered at $\overline{x}$ with side length $\ell(Q)$ will be denoted $Q(\overline{x}, \ell(Q))$ and usually named $Q, R, S \ldots$ Cubes will always have sides parallel to the coordinate axes. If we write λQ we mean a dilation of Q of factor λ. That is, λQ will be the cube concentric with Q of side length $\lambda\ell(Q)$ (we follow analogous conventions with balls, as well as with s-parabolic dilations instead of usual ones).

Given $E \subset \mathbb{R}^{n+1}$ a compact set, $\Sigma_d^s(E)$ will denote the collection of positive Borel measures supported on E with upper s-parabolic growth of degree d with constant 1 (the concept of upper s-parabolic growth will be defined later in this text).

Finally, denote by $\mathrm{D} := \{(\overline{x},\overline{y}) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \, : \, \overline{x}=\overline{y}\}$ the diagonal of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$. Consider the kernel function $K : (\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\setminus \mathrm{D} \to \mathbb{C}$, say continuous in its domain. In our discussion, such kernel functions will be usually of convolution type and smooth outside the diagonal. We will denote by K^* its conjugate, defined as $K^*(\overline{x},\overline{y}) = K(\overline{y},\overline{x})$. Moreover, for a given μ finite Radon measure in $\mathbb{R}^{n+1}$, we will be typically interested in the operator acting on elements of $L^1_{\text{loc}}(\mu)$ as

$$\mathcal{K}_\mu f(\overline{x}) := \int_{\mathbb{R}^{n+1}} K(\overline{x},\overline{y})f(\overline{y})\mathrm{d}\mu(\overline{y}), \qquad \overline{x}\notin \operatorname{supp}(\mu).$$

In the particular case in which f is the constant function 1 we write

$$\mathcal{K}\mu(\overline{x}) := \mathcal{K}_\mu 1(\overline{x}).$$

It is clear that the previous expression is defined pointwise on $\mathbb{R}^{n+1}\setminus \operatorname{supp}(\overline{x})$. We also introduce the truncated version of $\mathcal{K}_\mu$,

$$\mathcal{K}_{\mu,\varepsilon} f(\overline{x}) := \int_{|\overline{x}-\overline{y}|>\varepsilon} K(\overline{x},\overline{y})f(\overline{y})\mathrm{d}\mu(\overline{y}), \qquad \overline{x}\in \mathbb{R}^{n+1}, \; \varepsilon>0.$$

For a given $1\leq p\leq\infty$, we will say that $\mathcal{K}_\mu f$ belongs to $L^p(\mu)$ if the $L^p(\mu)$-norm of the truncations $\|\mathcal{K}_{\mu,\varepsilon} f\|_{L^p(\mu)}$ is uniformly bounded on ε, and we write

$$\|\mathcal{K}_\mu f\|_{L^p(\mu)} := \sup_{\varepsilon>0} \|\mathcal{K}_{\mu,\varepsilon} f\|_{L^p(\mu)}$$

We will say that the operator $\mathcal{K}_\mu$ is bounded on $L^p(\mu)$ if the operators $\mathcal{K}_{\mu,\varepsilon}$ are bounded on $L^p(\mu)$ uniformly on ε, and we equally set

$$\|\mathcal{K}_\mu\|_{L^p(\mu)\to L^p(\mu)} := \sup_{\varepsilon>0} \|\mathcal{K}_{\mu,\varepsilon}\|_{L^p(\mu)\to L^p(\mu)}.$$

We also introduce the notation used for maximal operators

$$\mathcal{K}_{*,\mu} f(\overline{x}) := \sup_{\varepsilon>0} |\mathcal{K}_{\mu,\varepsilon} f(\overline{x})|.$$

We also have the same definitions for the conjugate operator $\mathcal{K}^*$. Notice the difference in position of the asterisk with $\mathcal{K}_*$, Then, if the reader encounters expressions of the form $\mathcal{K}^*_*$, they refer to the maximal operator associated with the conjugate kernel K^*.

Chapter 1

The s-parabolic BMO and Lip_α caloric capacities

The aim of this chapter is to characterize different s-caloric capacities under s-parabolic BMO and Lip_α normalization conditions for $0 < s \leq 1$. We begin by recalling the notion of *s-parabolic distance* between two points $\overline{x} := (x,t)$, $\overline{y} := (y,\tau)$ in $\mathbb{R}^{n+1}$,

$$|\overline{x}-\overline{y}|_{p_s} = \mathrm{dist}_{p_s}(\overline{x},\overline{y}) := \max\big\{|x-y|, |t-\tau|^{\frac{1}{2s}}\big\}, \qquad \text{for } 0<s\leq 1,$$

which is equivalent to

$$\mathrm{dist}_{p_s}(\overline{x},\overline{y}) \approx \big(|x-y|^2 + |t-\tau|^{1/s}\big)^{1/2}.$$

Using this metric, s-parabolic cubes and s-parabolic balls arise naturally. We convey that $B(\overline{x},r)$ will be the *s-parabolic ball* centered at $\overline{x}$ with radius r, and Q will be an *s-parabolic cube* of side length ℓ. For an s-parabolic ball $B(\overline{x},r)$, the spatial coordinates are contained in a Euclidean ball B_1 of radius r, while the temporal coordinate lies in a real interval I of length $(2r)^{2s}$. On the other hand, an s-parabolic cube Q is a set of the form

$$I_1 \times \cdots \times I_n \times I_{n+1},$$

where $I_1, \ldots, I_n$ are intervals of length ℓ, while I_{n+1} is another interval of length ℓ^{2s}. We write $\ell(Q)$ to refer to the particular side length of Q.

The *s-parabolic dilation* of factor $\lambda > 0$, written δ_λ, is given by

$$\delta_\lambda(x,t) = \big(\lambda x, \lambda^{2s} t\big).$$

To ease notation, since we will always work with s-parabolic distances, we will write λQ to denote $\delta_\lambda(Q)$, the s-parabolic cube concentric with Q of side length $\lambda\ell(Q)$.

As expected, the notion of *s-parabolic BMO space*, BMO_{p_s}, refers to the space of usual BMO functions (strictly, equivalence classes of functions where constants are identified as 0) obtained by replacing Euclidean cubes by s-parabolic ones. Namely, a function $f \in L^1_{\mathrm{loc}}$ will belong to BMO_{p_s} if

$$\|f\|_{*,p_s} := \sup_Q \frac{1}{|Q|}\int_Q |f(\overline{x}) - f_Q|\,\mathrm{d}\overline{x} < \infty,$$

where the supremum is taken among all s-parabolic cubes and f_Q is the average of f over Q with respect to the Lebesgue measure.

Similarly, a function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is said to be *s-parabolic Lip_α* for some $0<\alpha<1$, shortly Lip_{α,p_s}, if

$$\|f\|_{\mathrm{Lip}_\alpha,p_s} := \sup_{\overline{x},\overline{y}\in\mathbb{R}^{n+1}} \frac{|f(\overline{x})-f(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \lesssim 1.$$

Returning to the original problem we wished to address, let us first precise its statement with the following question: can we characterize removable sets for solutions of the Θ^s-equation, provided these solutions satisfy certain s-parabolic BMO or Lip_α conditions? For the moment, the reader who is not familiar with the notion of removability may conceive removable sets as those which "do not matter" when solving the Θ^s-equation. This has to be understood in the sense that, if a function is a solution on their complement, actually satisfies the Θ^s-equation throughout the entire domain, including the set itself.

One of the main results of this chapter characterizes removability in terms of two different capacities: one requiring the $(1,\frac{1}{2s})$-gradient of solutions of the Θ^s-equation satisfy s-parabolic BMO estimates, and another one requiring s-parabolic Lip_α bounds. These capacities, denoted by $\Gamma_{\Theta^s,*}$ and $\Gamma_{\Theta^s,\alpha}$ respectively, are related to certain s-parabolic Hausdorff contents, defined as in the Euclidean case (see [**Matti**], for instance), just replacing the Euclidean distance by the s-parabolic distance introduced above. Our first main result reads as follows:

THEOREM. *Let $s\in(1/2,1]$, $\alpha\in(0,1)$ and $E\subset\mathbb{R}^{n+1}$ compact set. Then,*

1. $\Gamma_{\Theta^s,*}(E)\approx\mathcal{H}^{n+1}_{\infty,p_s}(E)$.
2. *If $\alpha<2s-1$, then* $\Gamma_{\Theta^s,\alpha}(E)\approx_\alpha\mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E)$.

Moreover, the nullity of these capacities is equivalent to the removability of the corresponding compact set for solutions satisfying $(1,\frac{1}{2s})$-gradient estimates in either s-parabolic BMO *or* Lip_α*, assuming $\alpha<2s-1$ in the latter case.*

We further study the same type of question for a generalization of the capacities presented by Mateu and Prat in [**MatP, §4 & §7**]. That is, we will ask for a characterization of removable sets for solutions of the Θ^s-equation now such that

$$\|(-\Delta)^\sigma f\|<\infty, \qquad \|\partial_t^{\sigma/s} f\|<\infty, \qquad s\in(0,1] \text{ and } \sigma\in[0,s).$$

Symbols $\|\cdot\|$ can refer both to s-parabolic BMO norms or both to s-parabolic Lip_α seminorms, giving rise to the capacities $\gamma^\sigma_{\Theta^s,*}$ and $\gamma^\sigma_{\Theta^s,\alpha}$ respectively. Our second main result is the following:

THEOREM. *For any $s\in(0,1]$, $\sigma\in[0,s)$, $\alpha\in(0,1)$ and $E\subset\mathbb{R}^{n+1}$ compact set,*

1. $\gamma^\sigma_{\Theta^s,*}(E)\approx_\sigma\mathcal{H}^{n+2\sigma}_{\infty,p_s}(E)$.
2. *If $\alpha<2s-2\sigma$, then* $\gamma^\sigma_{\Theta^s,\alpha}(E)\approx_{\sigma,\alpha}\mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E)$.

The nullity of these capacities is equivalent to the removability of the corresponding compact set for solutions satisfying $(\sigma, \sigma/s)$-Laplacian estimates in either s-parabolic BMO *or* Lip_α, *assuming $\alpha < 2s - 2\sigma$ in the latter case.*

This study extends similar analyses carried out for the BMO variant of analytic capacity in earlier works by Kaufman [**Ka**] and Verdera [**Ve1**] (for a brief overview the reader may consult [**AsIM, §13.5.1**]); and also those for the Lip_α variant of the same capacity in the direction presented by Mel'nikov [**Me1**] or O'Farrell [**O'**]. Our results also generalize those in [**Her, §5 & §6**].

In sections §1.1 and §1.2 we focus on obtaining some technical results regarding kernel estimates as well as growth estimates for *admissible* functions. Moreover, in §1.3 we deduce some important properties regarding potentials defined against positive Borel measures with some growth restrictions. It is in §1.4 that we define all the different capacities and characterize them in terms of s-parabolic Hausdorff contents.

1.1 Some kernel estimates

Let us fix $s \in (0,1]$. The fundamental solution $P_s(x,t)$ of the Θ^s-equation, defined via the operator

$$\Theta^s := (-\Delta)^s + \partial_t,$$

is the inverse spatial Fourier transform of $e^{-4\pi^2 t|\xi|^{2s}}$ for $t > 0$, and it equals 0 if $t \leq 0$. For the special case $s = 1$, we retrieve the classical *heat kernel*, given by:

$$W(\overline{x}) := P_1(\overline{x}) = c_n t^{-\frac{n}{2}} \phi_{n,1}(|x| t^{-\frac{1}{2}}), \qquad \text{if } \; t > 0,$$

with $\phi_{n,1}(\rho) := e^{-\rho^2/4}$. Although the expression of P_s is not explicit in general, Blumenthal and Getoor [**BlG, Theorem 2.1**] established that for $s < 1$,

$$P_s(\overline{x}) = c_{n,s}\, t^{-\frac{n}{2s}} \phi_{n,s}\big(|x| t^{-\frac{1}{2s}}\big) \chi_{t>0}, \tag{1.1.1}$$

Here, $\phi_{n,s}$ is a smooth function, radially decreasing and satisfying, for $0 < s < 1$,

$$\phi_{n,s}(\rho) \approx \big(1 + \rho^2\big)^{-(n+2s)/2}, \tag{1.1.2}$$

being an exact equality if $s = 1/2$ [**Va**]. Therefore,

$$P_s(\overline{x}) \approx \frac{t}{|\overline{x}|_{p_s}^{n+2s}} \chi_{t>0}.$$

The function $\phi_{n,s}$ is tightly related to the Fourier transform of $e^{-4\pi^2|\xi|^{2s}}$. Indeed, taking the spatial Fourier transform in both sides of identity (1.1.1),

$$e^{-4\pi^2 t|\xi|^{2s}} = c_{n,s}\, t^{-\frac{n}{2s}} \big[\phi_{n,s}\big(|\cdot|\, t^{-\frac{1}{2s}}\big)\big]^{\wedge}(\xi),$$

Recall that for $\lambda > 0$, the dilation $f_\lambda := f(\lambda x)$ satisfies $\widehat{f_\lambda}(\xi) = \lambda^{-n} \widehat{f}(\lambda^{-1}\xi)$. Then,

$$e^{-4\pi^2 t|\xi|^{2s}} = c_{n,s}\, \widehat{\phi_{n,s}(|\cdot|)}\big(\xi t^{\frac{1}{2s}}\big), \quad \text{ that implies } \quad e^{-4\pi^2|\xi|^{2s}} \simeq \widehat{\phi_{n,s}(|\cdot|)}(\xi).$$

The above relations will allow us to obtain explicit bounds for the derivatives of $\phi_{n,s}$. Let us briefly recall that $(-\Delta)^s$ is the operator with the following Fourier multiplier,

$$\widehat{(-\Delta)^s} f(\xi,t) = |\xi|^{2s}\widehat{f}(\xi,t),$$

that also admits the following integral representation

$$\begin{aligned}(-\Delta)^s f(x,t) &\simeq \text{p.v.}\int_{\mathbb{R}^n}\frac{f(x,t)-f(y,t)}{|x-y|^{n+2s}}\,\mathrm{d}y\\ &\simeq \int_{\mathbb{R}^n}\frac{f(x+y,t)-2f(x,t)+f(x-y,t)}{|y|^{n+2s}}\,\mathrm{d}y.\end{aligned}$$

Let us present our first lemma. Although it can be deduced straightforwardly from [**GrTe, Theorem 1.1**], we shall give a detailed proof for the sake of clarity and completeness.

Lemma 1.1.1. *Let $s\in(0,1]$ and $\beta\in(0,1)$. We define the following function in $\mathbb{R}^n$:*

$$\psi_{n,s}^{(\beta)}(x) := (-\Delta)^\beta \phi_{n,s}(|x|).$$

Then,

1. $\phi_{n,s}'(\rho)\simeq -\rho\,\phi_{n+2,s}(\rho)$.
2. $|\psi_{n,s}^{(\beta)}(x)| \lesssim_\beta \big(1+|x|^2\big)^{-(n+2\beta)/2}$.
3. $\nabla\psi_{n,s}^{(\beta)}(x)\simeq -x\,\psi_{n+2,s}^{(\beta)}(x)$.

Proof. We begin by proving *1* for $s<1$ (the case $s=1$ is trivial). To do so, we will use the explicit integral representation for the inverse Fourier transform of a radial function in [**Gr, §B.5**] or [**SteW, §IV.I**]. Applying it to the Fourier transform $e^{-4\pi^2|\xi|^{2s}}$ we get

$$\phi_{n,s}(|z|) = 2\pi|z|^{1-n/2}\int_0^\infty e^{-4\pi^2r^{2s}}r^{n/2}J_{n/2-1}(2\pi r|z|)\,\mathrm{d}r,\quad \text{for any } z\in\mathbb{R}^n\setminus\{0\},$$

where J_k is the classical Bessel function of order k [**AS, §9**]. Since we are interested in the derivatives of $\phi_{n,s}$ as a radial real variable function, let us rewrite the previous expression in terms of $\rho\in(0,\infty)$ so that it reads as

$$\phi_{n,s}(\rho) = 2\pi\rho^{1-n/2}\int_0^\infty e^{-4\pi^2r^{2s}}r^{n/2}J_{n/2-1}(2\pi r\rho)\,\mathrm{d}r. \tag{1.1.3}$$

Therefore, to estimate the derivatives of $\phi_{n,s}$ we need to determine first if we can differentiate under the integral sign. To that end, we use the following recurrence relation for classical Bessel functions [**AS, §9.1.27**],

$$J_k'(x) = \frac{k}{x}J_k(x) - J_{k+1}(x).$$

This recurrence formula together with (1.1.3) are also valid for the case $k = -1/2$, conveying that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$. In our case these imply

$$\partial_\rho J_{n/2-1}(2\pi r\rho) = \left(\frac{n}{2} - 1\right)\rho^{-1} J_{n/2-1}(2\pi r\rho) - 2\pi r\, J_{n/2}(2\pi r\rho).$$

Differentiating under the integral sign in (1.1.3), we would get integrands of the form

$$e^{-r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho), \qquad e^{-r^{2s}} r^{n/2+1} J_{n/2}(2\pi r\rho).$$

Notice that both are bounded by integrable functions in the domain of integration, locally for each $\rho > 0$ (by the boundedness of the functions J_k for $n > 1$, and by that of $\cos x$ if $n = 1$). Hence, we can indeed differentiate under the integral sign to compute $\phi'_{n,s}$, obtaining the desired result:

$$\begin{aligned}
\phi'_{n,s}(\rho) = 2\pi\Bigg[\left(1 - \frac{n}{2}\right)\rho^{-n/2} &\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\,\mathrm{d}r \\
&+ \rho^{1-n/2}\partial_\rho\left(\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\,\mathrm{d}r\right)\Bigg] \\
= 2\pi\Bigg[\left(1 - \frac{n}{2}\right)\rho^{-n/2} &\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(r\rho)\,\mathrm{d}r \\
\rho^{1-n/2}\left(\frac{n}{2} - 1\right)\rho^{-1}&\left(\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\,\mathrm{d}r\right) \\
- 2\pi\rho\,\rho^{1-(n+2)/2} &\int_0^\infty e^{-4\pi^2 r^{2s}} r^{(n+2)/2} J_{(n+2)/2-1}(r\rho)\,\mathrm{d}r\Bigg] = -2\pi\rho\,\phi_{n+2,s}(\rho).
\end{aligned}$$

Next we prove statement *2*. Observe that for $s \in (0, 1]$ and $\beta \in (0, 1)$, we have $\widehat{\psi^{(\beta)}_{n,s}}(\xi) = |\xi|^{2\beta} e^{-4\pi^2|\xi|^{2s}}$, which is an integrable function, and thus $\psi^{(\beta)}_{n,s}$ is bounded (in fact, since the product of $\widehat{\psi^{(\beta)}_{n,s}}$ by any polynomial is also integrable, we infer that $\psi^{(\beta)}_{n,s}$ is smooth). By the integral representation formula for inverse Fourier transforms of radial functions,

$$\psi^{(\beta)}_{n,s}(x) = 2\pi|x|^{1-n/2} \int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|)\,\mathrm{d}r, \quad x \in \mathbb{R}^n \setminus \{0\}. \quad (1.1.4)$$

Now, we apply [**PrTa, Lemma 1**] to deduce the desired decaying property $|\psi^{(\beta)}_{n,s}(x)| = O\big(|x|^{-n-2\beta}\big)$, for $|x|$ large. Hence, since $\psi^{(\beta)}_{n,s}$ is bounded, we deduce the desired bound $|\psi^{(\beta)}_{n,s}(x)| \lesssim_\beta \big(1 + |x|^2\big)^{-(n+2\beta)/2}$.

We are left to control the norm of $\nabla\psi^{(\beta)}_{n,s}$, provided the latter is well-defined. We claim that this is the case, since we can differentiate under the integral sign in (1.1.4). Indeed, by the recurrence relation satisfied by the derivatives of J_k we get

$$|\nabla_x J_{n/2-1}(r|x|)| = \left|\left(\frac{n}{2} - 1\right)\frac{1}{|x|} J_{n/2-1}(2\pi r|x|) - 2\pi r J_{n/2}(2\pi r|x|)\right|.$$

So the resulting integrands to study are terms of the form

$$e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} |J_{n/2-1}(2\pi r|x|)|, \qquad e^{-4\pi^2 r^{2s}} r^{n/2+2\beta+1} |J_{n/2}(2\pi r|x|)|,$$

both bounded by the integrable functions $C_1 e^{-r^{2s}} r^{n/2+2\beta}$ and $C_2 e^{-r^{2s}} r^{n/2+2\beta+1}$ for some constants C_1, C_2 depending on n, s and β, and locally for each $x \in \mathbb{R}^n$ with $|x| > 0$. Hence, we can differentiate under the integral sign in (1.1.4) and obtain

$$\begin{aligned}\nabla \psi_{n,s}^{(\beta)}(x) = 2\pi \bigg[& \left(1 - \frac{n}{2}\right) \frac{x}{|x|^{n/2+1}} \int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|) \,\mathrm{d}r \\ & + \left(\frac{n}{2} - 1\right) \frac{x}{|x|^{n/2+1}} \int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|) \,\mathrm{d}r \\ & - 2\pi \frac{x}{|x|^{n/2}} \int_0^\infty e^{-4\pi^2 r^{2s}} r^{(n+2)/2+2\beta} J_{(n+2)/2-1}(2\pi r|x|) \,\mathrm{d}r \bigg] \\ = -2\pi x \, \psi_{n+2,s}^{(\beta)}(x). \end{aligned}$$

□

Using the above lemma together with (1.1.2) we can estimate the derivatives of $\phi_{n,s}$ and $\psi_{n,s}^{(\beta)}$. In particular, the following relations hold:

$$\text{If } s < 1, \quad \phi_{n,s}'(\rho) \approx \frac{-\rho}{(1+\rho^2)^{(n+2s+2)/2}}, \qquad \phi_{n,s}''(\rho) \approx \frac{-1 + (2\pi - 1)\rho^2}{(1+\rho^2)^{(n+2s+4)/2}}, \tag{1.1.5}$$

$$|\nabla \psi_{n,s}^{(\beta)}(x)| \lesssim_\beta \frac{|x|}{(1+|x|^2)^{(n+2\beta+2)/2}}. \tag{1.1.6}$$

1.1.1 Estimates for $\nabla_x P_s$ and $(-\Delta)^\beta P_s$

We shall now present some growth estimates for kernels associated with P_s, namely $\nabla_x P_s$ and $(-\Delta)^\beta P_s$, $\beta \in (0,1)$. The bounds obtained for the latter will imply that they satisfy being Calderón-Zygmund kernels of a certain dimension. Recall that if $\mathrm{D} := \{(\overline{x}, \overline{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,:\, \overline{x} \neq \overline{y}\}$,

Definition 1.1.1. A function $K : (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \setminus \mathrm{D} \to \mathbb{C}$ is an *m-dimensional Calderón-Zygmund (C-Z) kernel* if there are $C_1 > 0$ and $m > 0$ such that,

$$|K(\overline{x}, \overline{y})| \leq \frac{C_1}{|\overline{x} - \overline{y}|^m}.$$

Moreover, for any $\overline{x}'$ with $|\overline{x} - \overline{x}'| \leq |\overline{x} - \overline{y}|/2$, there exist $C_2 > 0, \eta > 0$ so that

$$|K(\overline{x}, \overline{y}) - K(\overline{x}', \overline{y})| + |K(\overline{y}, \overline{x}) - K(\overline{y}, \overline{x}')| \leq \frac{C_2 |\overline{x} - \overline{x}'|^\eta}{|\overline{x} - \overline{y}|^{m+\eta}}.$$

If there exists $k : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ with $K(\overline{x}, \overline{y}) = k(\overline{x} - \overline{y})$, we say that K is a *C-Z convolution kernel.*

Our first result provides bounds for $\nabla_x P_s$, $s \in (0,1)$. These estimates are analogous to those of [**MatPT, Lemma 5.4**] which cover the case $s = 1$. In the forthcoming results, the parameter $2\zeta := \min\{1, 2s\}$ will play an important role.

THEOREM 1.1.2. *The following hold for any $\overline{x} = (x,t) \neq 0$ and $s \in (0,1)$:*

$$|\nabla_x P_s(\overline{x})| \lesssim \frac{|xt|}{|\overline{x}|_{p_s}^{n+2s+2}}, \qquad |\Delta P_s(\overline{x})| \lesssim \frac{|t|}{|\overline{x}|_{p_s}^{n+2s+2}}, \qquad |\partial_t \nabla_x P_s(\overline{x})| \lesssim \frac{|x|}{|\overline{x}|_{p_s}^{n+2s+2}}.$$

The rightmost bound is only valid for points with $t \neq 0$. Also, if $\overline{x}'$ is such that $|\overline{x} - \overline{x}'|_{p_s} \leq |\overline{x}|_{p_s}/2$,

$$|\nabla_x P_s(\overline{x}) - \nabla_x P_s(\overline{x}')| \lesssim \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}.$$

Therefore, $\nabla_x P_s$ is an $(n+1)$-dimensional C-Z convolution kernel.

Proof. To simplify the arguments below, we specify the dependence of P_s with respect to n. Let us write $P_{s,n+1}$ to refer to the fundamental solution of the Θ^s-equation in $\mathbb{R}^{n+1}$ and use the following abuse of notation: given $\overline{x} = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1}$, write

$$\begin{aligned} P_{s,n+3}(\overline{x}) &:= P_{s,n+3}(x_1, \ldots, x_n, 0, 0, t), \\ P_{s,n+5}(\overline{x}) &:= P_{s,n+5}(x_1, \ldots, x_n, 0, 0, 0, 0, t). \end{aligned}$$

This way, we directly apply relations (1.1.1) and (1.1.5) to obtain for each $t > 0$,

$$|\nabla_x P_s(\overline{x})| \simeq t^{-\frac{n+1}{2s}} |\phi'_{n,s}(|x| t^{-\frac{1}{2s}})| \simeq |x P_{s,n+3}(\overline{x})| \approx \frac{|xt|}{|\overline{x}|_{p_s}^{n+2s+2}}.$$

The bounds for $\Delta P_{s,n+1}$ and $\partial_t \nabla_x P_{s,n+1}$ can be obtained from the previous result and (1.1.1). Indeed,

$$\begin{aligned} |\Delta P_{s,n+1}(\overline{x})| &\simeq P_{s,n+3}(\overline{x}) + |x|^2 P_{s,n+5}(\overline{x}) \lesssim \frac{|t|}{|\overline{x}|_{p_s}^{n+2s+2}}, \\ |\partial_t \nabla_x P_{s,n+1}(\overline{x})| &\lesssim \frac{|x|}{t} \Big(P_{s,n+3}(\overline{x}) + |x|^2 P_{s,n+5}(\overline{x}) \Big) \lesssim \frac{|x|}{|\overline{x}|_{p_s}^{n+2s+2}}. \end{aligned}$$

For the final estimate, we recover the notation $P_s := P_{s,n+1}$. Let $\overline{x}' = (x', t') \in \mathbb{R}^{n+1}$ with $|\overline{x} - \overline{x}'|_{p_s} \leq |\overline{x}|_{p_s}/2$ and use the definition of dist_{p_s} to obtain

$$|\overline{x}|_{p_s} \leq 2|\overline{x}'|_{p_s} \quad \text{and} \quad |x'| \geq |x| - \frac{|\overline{x}|_{p_s}}{2}. \tag{1.1.7}$$

Put $\widehat{x} = (x', t)$ and write

$$|\nabla_x P_s(\overline{x}) - \nabla_x P_s(\overline{x}')| \leq |\nabla_x P_s(\overline{x}) - \nabla_x P_s(\widehat{x})| + |\nabla_x P_s(\widehat{x}) - \nabla_x P_s(\overline{x}')|.$$

We observe that the first term in the above inequality satisfies the desired bound,

$$|x-x'|\sup_{\xi\in[x,x']}|\Delta P_s(\xi,t)|\lesssim\frac{|x-x'|}{|\overline{x}|_{p_s}^{n+2}}\leq\frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}\left(\frac{|\overline{x}-\overline{x}'|_{p_s}}{|\overline{x}|_{p_s}}\right)^{1-2\zeta}\leq\frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}.$$

Regarding the second term, assume without loss of generality $t>t'$. If $t'>0$, use $|\overline{x}'|_{p_s}\geq|\overline{x}|_{p_s}/2$ so that we also have

$$|t-t'|\sup_{\tau\in[t,t']}|\partial_t\nabla_xP_s(x',\tau)|\lesssim\frac{|t-t'|}{|\overline{x}|_{p_s}^{n+2s+1}}\leq\frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}\left(\frac{|\overline{x}-\overline{x}'|_{p_s}}{|\overline{x}|_{p_s}}\right)^{2s-2\zeta}\lesssim\frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}},$$

If $t<0$ then $|\nabla_xP_s(\widehat{x})-\nabla_xP_s(\overline{x}')|=0$, and the estimate becomes trivial. Then, we are left to study the case $t>0$ and $t'<0$. These two conditions imply that the p_s-ball

$$B(\overline{x}):=\left\{\overline{y}\in\mathbb{R}^{n+1}\,:\,|\overline{x}-\overline{y}|_{p_s}\leq\frac{|\overline{x}|_{p_s}}{2}\right\}\ni\overline{x}'$$

intersects the hyperplane $\{t=0\}$. Since the radius of $B(\overline{x})$ also depends on $\overline{x}$, the previous property imposes the following condition over $\overline{x}$,

$$t^{1/s}\leq\frac{x_1^2+\cdots+x_n^2}{3},\quad\text{that is}\quad t^{\frac{1}{2s}}\leq\frac{|x|}{\sqrt{3}},$$

which is attained if the point $(x,0)$ belongs to $\partial B(\overline{x})$. Therefore $|\overline{x}|_{p_s}:=\max\left\{|x|,t^{\frac{1}{2s}}\right\}=|x|$, so by (1.1.7) we get $|x'|\geq|x|/2$, and this in turn implies

$$\frac{|\overline{x}|_{p_s}}{2}\leq|\overline{x}'|_{p_s}\leq|\overline{x}-\overline{x}'|_{p_s}+|\overline{x}|_{p_s}\leq\frac{3|x|}{2}\leq3|x'|. \tag{1.1.8}$$

Using this last inequality we can finally conclude:

$$\begin{aligned}|\nabla_xP_s(\widehat{x})-\nabla_xP_s(\overline{x}')|&=|\nabla_xP_s(x',t)-\nabla_xP_s(x',0)|\lesssim|t|\sup_{\tau\in(0,t]}|\partial_t\nabla_xP_s(x',\tau)|\\&\lesssim\frac{|t|}{|x'|^{n+2s+1}}\lesssim\frac{|t|}{|\overline{x}|_{p_s}^{n+2s+1}}\leq\frac{|t-t'|}{|\overline{x}|_{p_s}^{n+2s+1}}\lesssim\frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}.\end{aligned}$$

□

THEOREM 1.1.3. *Let $s\in(0,1]$ and $\beta,\gamma\in[0,1)$. Then, for any $\overline{x}\neq0$ we have,*

1. $|(-\Delta)^\beta P_s(\overline{x})|\lesssim_\beta\dfrac{1}{|\overline{x}|_{p_s}^{n+2\beta}}$,

2. $|(-\Delta)^\gamma(-\Delta)^\beta P_s(\overline{x})|\lesssim_{\beta,\gamma}\dfrac{1}{|\overline{x}|_{p_s}^{n+2\beta+2\gamma}}$, 3. $|\nabla_x(-\Delta)^\beta P_s(\overline{x})|\lesssim_\beta\dfrac{|x|}{|\overline{x}|_{p_s}^{n+2\beta+2}}$.

Moreover, for any $\overline{x}\neq(x,0)$,

4. $|\partial_t(-\Delta)^\beta P_s(\overline{x})|\lesssim_\beta\dfrac{1}{|\overline{x}|_{p_s}^{n+2\beta+2s}}$.

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ *is such that* $|\overline{x} - \overline{x}'|_{p_s} \le |\overline{x}|_{p_s}/2$,

$$5.\ \ |(-\Delta)^\beta P_s(\overline{x}) - (-\Delta)^\beta P_s(\overline{x}')| \lesssim_\beta \frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+2\beta+2\zeta}}.$$

Therefore, $(-\Delta)^\beta P_s$ *is an* $(n+2\beta)$*-dimensional C-Z convolution kernel.*

Proof. We shall also assume $\beta > 0$, since the case $\beta = 0$ is already covered in [**MatP, Lemma 2.2**]. For the sake of notation, in this proof we will write $\phi := \phi_{n,s}$ and $\psi := \psi_{n,s}^{(\beta)}$, and we also set $K_\beta := (-\Delta)^\beta P_s$. Let us begin by applying the integral representation of K_β together with relation (1.1.1) to obtain for $t > 0$,

$$\begin{aligned} K_\beta(x,t) := (-\Delta)^\beta P_s(x,t) &\simeq_\beta \text{p.v.} \int_{\mathbb{R}^n} \frac{P_s(x,t) - P_s(y,t)}{|x-y|^{n+2\beta}}\,\mathrm{d}y \\ &= t^{-\frac{n}{2s}}\,\text{p.v.} \int_{\mathbb{R}^n} \frac{\phi\big(|x|t^{-\frac{1}{2s}}\big) - \phi\big(|y|t^{-\frac{1}{2s}}\big)}{|x-y|^{n+2\beta}}\,\mathrm{d}y \\ &= t^{-\frac{n+2\beta}{2s}}\,\text{p.v.} \int_{\mathbb{R}^n} \frac{\phi\big(|x|t^{-\frac{1}{2s}}\big) - \phi\big(|z|\big)}{|xt^{-\frac{1}{2s}} - z|^{n+2\beta}}\,\mathrm{d}z \\ &= t^{-\frac{n+2\beta}{2s}} (-\Delta)^\beta \phi\big(|x|t^{-\frac{1}{2s}}\big) = t^{-\frac{n+2\beta}{2s}} \psi\big(xt^{-\frac{1}{2s}}\big). \end{aligned}$$

Using the estimate proved in Lemma 1.1.1 for ψ we deduce the desired bound:

$$\big|K_\beta(x,t)\big| \lesssim_\beta \frac{t^{-\frac{n+2\beta}{2s}}}{\big(1+|x|^2t^{-1/s}\big)^{(n+2\beta)/2}} = \frac{1}{\big(t^{1/s}+|x|^2\big)^{(n+2\beta)/2}} \approx \frac{1}{|\overline{x}|_{p_s}^{n+2\beta}}.$$

We shall continue by studying estimate *2* in a similar way. Indeed,

$$\begin{aligned} (-\Delta)^\gamma K_\beta(x,t) &\simeq_\gamma \text{p.v.} \int_{\mathbb{R}^n} \frac{K_\beta(x,t) - K_\beta(y,t)}{|x-y|^{n+2\gamma}}\,\mathrm{d}y \\ &\simeq_\beta t^{-\frac{n+2\beta}{2s}}\,\text{p.v.} \int_{\mathbb{R}^n} \frac{\psi\big(xt^{-\frac{1}{2s}}\big) - \psi\big(yt^{-\frac{1}{2s}}\big)}{|x-y|^{n+2\gamma}}\,\mathrm{d}y \\ &= t^{-\frac{n+2\beta+2\gamma}{2s}}\,\text{p.v.} \int_{\mathbb{R}^n} \frac{\psi\big(xt^{-\frac{1}{2s}}\big) - \psi(z)}{|xt^{-\frac{1}{2s}} - z|^{n+2\gamma}}\,\mathrm{d}z = t^{-\frac{n+2\beta+2\gamma}{2s}} (-\Delta)^\gamma \psi\big(xt^{-\frac{1}{2s}}\big). \end{aligned}$$

Set $\Psi := (-\Delta)^\gamma \psi(\,\cdot\,)$ and notice that

$$\widehat{\Psi}(\xi) = |\xi|^{2\gamma}|\xi|^{2\beta}e^{-4\pi^2|\xi|^{2s}} = |\xi|^{2\beta+2\gamma}e^{-4\pi^2|\xi|^{2s}}.$$

Thus, since $\widehat{\Psi}$ is integrable, Ψ is the radial bounded function in $\mathbb{R}^n$ given by

$$\Psi(z) = 2\pi|z|^{1-n/2}\int_0^\infty e^{-4\pi^2r^{2s}}r^{n/2+2\beta+2\gamma}J_{n/2-1}(2\pi r|z|)\,\mathrm{d}r,$$

By [**PrTa, Lemma 1**] Ψ decays as

$$|\Psi(z)| = O\big(|z|^{-n-2\beta-2\gamma}\big), \quad \text{for } |z| \text{ large.}$$

Therefore
$$|\Psi(z)| \lesssim_{\beta,\gamma} \big(1+|z|^2\big)^{-(n+2\beta+2\gamma)/2}.$$
So analogously to the proof of *1*, we deduce the desired result:
$$\big|(-\Delta)^\gamma K_\beta(x,t)\big| \lesssim_{\beta,\gamma} \frac{t^{-\frac{n+2\beta+2\gamma}{2s}}}{\big(1+|x|^2t^{-1/s}\big)^{(n+2\beta+2\gamma)/2}} \approx \frac{1}{|\overline{x}|_{p_s}^{n+2\beta+2\gamma}}.$$

Regarding estimate *3*, notice that
$$|\nabla_x K_\beta(x,t)| \simeq_\beta \Big|\nabla_x\Big(t^{-\frac{n+2\beta}{2s}}\psi\big(xt^{-\frac{1}{2s}}\big)\Big)\Big| = t^{-\frac{n+2\beta+1}{2s}}\big|\nabla\psi\big(xt^{-\frac{1}{2s}}\big)\big|.$$
Therefore, applying the bound obtained for $\nabla\psi$ in (1.1.6) we deduce
$$\big|\nabla_x K_\beta(x,t)\big| \lesssim_\beta t^{-\frac{n+2\beta+1}{2s}} \frac{|x|t^{-\frac{1}{2s}}}{\big(1+|x|^2t^{-1/s}\big)^{(n+2\beta+1)/2}} \approx \frac{|x|}{|\overline{x}|_{p_s}^{n+2\beta+2}}.$$
We move on to estimate *4*, that is, the one concerning $\partial_t K_\beta(\overline{x})$ at points of the form $\overline{x}\neq(x,0)$. Observe that the previous derivative is well defined if $t>0$, since the expression of K_β can be written as
$$\begin{aligned} K_\beta(x,t) &\simeq t^{-\frac{n+2\beta}{2s}}(-\Delta)^\beta\phi\big(|x|t^{-\frac{1}{2s}}\big)\\ &\simeq_\beta |x|^{1-n/2}\bigg(\frac{1}{t^{\frac{n+4\beta+2}{4s}}}\int_0^\infty e^{-4\pi^2r^{2s}}r^{n/2+2\beta}J_{n/2-1}\big(2\pi r|x|t^{-\frac{1}{2s}}\big)\,\mathrm{d}r\bigg), \end{aligned}$$
so differentiating under the integral sign, it is clear that temporal derivatives of any order exist in $\mathbb{R}^{n+1}\setminus\{t=0\}$. We claim now that the operators ∂_t and $(-\Delta)^\beta$ commute when applied to P_s. To prove this, let us first observe that for each $t_0>0$ fixed we have
$$\big[(-\Delta)^\beta(\partial_t P_s)\big]^\wedge(\xi,t_0) = |\xi|^{2\beta}\widehat{\partial_t P_s}(\xi,t_0) = |\xi|^{2\beta}\int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle}\partial_t P_s(x,t_0)\,\mathrm{d}x.$$
If we can bound $\partial_t P_s$ by an integrable function on $\mathbb{R}^n$ in a neighborhood of t_0, we will be able to locally differentiate outside the integral sign for each t_0. If $0<s<1$, this is a consequence of [**Va, Equation 2.6**] and (1.1.2). Indeed,
$$|\partial_t P_s(x,t_0)| \lesssim \frac{1}{t_0}|P_s(x,t_0)| \lesssim \frac{1}{t_0^{\frac{n+2s}{2s}}}\Bigg[\frac{1}{\big(1+|x|^2t_0^{-1/s}\big)^{(n+2s)/2}}\Bigg].$$
On the other hand, if $s=1$ by definition we have
$$|\partial_t W(x,t_0)| \lesssim \bigg(1+\frac{|x|^2}{t_0}\bigg)\frac{1}{t_0^{n/2+1}}e^{-|x|^2/(4t_0)}.$$
In both cases we obtain a bounded function of x that decreases like $|x|^{-n-2}$ at infinity (for the case $s=1$, see [**MatPT, Lemma 2.1**]) and thus it is integrable on $\mathbb{R}^n$. Therefore, differentiating outside the integral sign we have
$$\big[(-\Delta)^\beta(\partial_t P_s)\big]^\wedge(\xi,t_0) = \partial_t\big[(-\Delta)^\beta P_s\big]^\wedge(\xi,t_0), \qquad \forall t_0>0.$$

So we are left to check whether we can enter ∂_t inside the previous Fourier transform, that is, whether the following holds

$$\partial_t\big[(-\Delta)^\beta P_s\big]^\wedge(\xi,t_0) = \big[\partial_t(-\Delta)^\beta P_s\big]^\wedge(\xi,t_0).$$

Again, the latter is just a matter of being able to bound $|\partial_t(-\Delta)^\beta P_s| = |\partial_t K_\beta|$ locally for each $t_0 > 0$ by an integrable function, so that we can differentiate under the integral defining the Fourier transform. We know that

$$|\partial_t K_\beta(x,t_0)| = \Big|\partial_t\Big[t^{-\frac{n+2\beta}{2s}}\psi\big(xt^{-\frac{1}{2s}}\big)\Big]_{t=t_0}\Big| \\ \lesssim_\beta C_1(t_0)\big|\psi\big(xt_0^{-\frac{1}{2s}}\big)\big| + C_2(t_0)|x|\big|\nabla\psi\big(xt_0^{-\frac{1}{2s}}\big)\big|.$$

For the first summand, using that $|\psi|$ is bounded and decays as $|x|^{-n-2\beta}$, we deduce the desired integrability condition. For the second summand we can argue exactly in the same manner, using that $|\nabla\psi|$ is bounded and decays as $|x|^{-n-2\beta-1}$. Hence, we conclude that ∂_t and $(-\Delta)^\beta$ commute.

The previous commutativity relation and [**MatP, Eq. 2.5**] yield the following for $t>0$,

$$\begin{aligned}\partial_t K_\beta(x,t) &= \partial_t\big[(-\Delta)^\beta P_s\big](x,t) = (-\Delta)^\beta\big(\partial_t P_s\big)(x,t) \\ &= (-\Delta)^\beta\Big[-(-\Delta)^s P_s\Big](x,t) = -(-\Delta)^s K_\beta(x,t),\end{aligned}$$

where we have commuted the operators $(-\Delta)^s$ and $(-\Delta)^\beta$, that can be easily checked via their Fourier transform. Then, applying *2* with $\gamma = s$ we are done.

Finally, regarding estimate *5*, we can follow the same proof to that presented for the last estimate in Theorem 1.1.2, using estimates *3* and *4* from above. □

1.1.2 Estimates for $\partial_t^\beta P_s$

In this subsection we obtain similar estimates now for the kernel $\partial_t^\beta P_s$, with $\beta \in (0,1)$. Recall that the β*-temporal derivative* of $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is defined, provided it exists, as

$$\partial_t^\beta f(x,t) := \int_{\mathbb{R}} \frac{f(x,\tau)-f(x,t)}{|\tau-t|^{1+s}}\,\mathrm{d}\tau.$$

The study below considers the cases $s<1$ and $s=1$ separately. In the following lemma, which generalizes [**MatP, Lemma 2.2**], we get dimensional restrictions that in the end will not matter for our purposes.

Lemma 1.1.4. *Let $\beta, s\in(0,1)$. The following hold for any $\overline{x}=(x,t)\neq(0,t)$:*

1. *If $n>1$,* $\qquad |\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}.$

2. *If $n=1$ and $\beta > 1-\frac{1}{2s}$,*

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{1-2s+\alpha}|\overline{x}|_{p_s}^{2s(1+\beta)-\alpha}}, \quad \forall\alpha\in(2s-1,4s).$$

Moreover, for every n,

3. $|\nabla_x\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}}$,

4. $|\partial_t\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n}|\overline{x}|_{p_s}^{2s(1+\beta)}}$, *for* $t\neq 0$.

Finally, if $\overline{x}'\in\mathbb{R}^{n+1}$ is such that $|\overline{x}-\overline{x}'|_{p_s}\leq |x|/2$,

5. $|\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\overline{x}')| \lesssim_\beta \dfrac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}$.

Proof. To prove *1*, we use [**MatP, Equation 2.9**] and deduce the existence of a function F_s such that for $t>0$,

$$P_s(x,t) = \frac{1}{|x|^n}F_s\bigg(\frac{t}{|x|^{2s}}\bigg), \tag{1.1.9}$$

and such that

$$F_s(u)\approx \frac{u}{\big(1+u^{1/s}\big)^{(n+2s)/2}}. \tag{1.1.10}$$

We extend continuously $F_s(u):=0$ for $u\leq 0$, so that (1.1.9) is verified for any value of t. The existence of F_s is clear, since for $t>0$ the function P_s can be written as

$$P_s(x,t) = \frac{1}{|x|^n}\bigg(\frac{t}{|x|^{2s}}\bigg)^{-\frac{n}{2s}}\phi_{n,s}\Bigg[\bigg(\frac{t}{|x|^{2s}}\bigg)^{-\frac{1}{2s}}\Bigg],$$

and defining for $u>0$, $F_s(u):=u^{-\frac{n}{2s}}\phi_{n,s}\big(u^{-\frac{1}{2s}}\big)$, we are done. Notice that F_s is a bounded continuous function, null for negative values of u, smooth in the domain $u>0$ and vanishing at ∞. Moreover, using the bounds obtained for ϕ' and ϕ'' we obtain the following estimates for $u>0$,

$$|F_s'(u)|\lesssim \frac{1}{\big(1+u^{1/s}\big)^{(n+2s)/2}}, \qquad |F_s''(u)|\lesssim \frac{1}{u\big(1+u^{1/s}\big)^{(n+2s)/2}}. \tag{1.1.11}$$

Let us argue that, in fact, $|F_s''(u)|$ is also a bounded function. Notice that, by definition,

$$\partial_\tau^2 P_s(x,\tau) = \frac{1}{|x|^{n+4s}}F_s''\bigg(\frac{\tau}{|x|^{2s}}\bigg) \quad \Leftrightarrow \quad \bigg|F_s''\bigg(\frac{\tau}{|x|^{2s}}\bigg)\bigg| = |x|^{n+4s}\big|\partial_\tau^2P_s(x,\tau)\big|,$$

and using that P_s is the fundamental solution of the Θ^s-equation and that $\tau>0$, we have

$$\partial_\tau^2 P_s(x,\tau) = \partial_\tau\big[-(-\Delta)^s P_s(x,\tau)\big].$$

By the commutativity of ∂_τ and $(-\Delta)^s$, we deduce

$$\big|\partial_\tau^2P_s(x,\tau)\big| = \big|(-\Delta)^s\big[\partial_\tau P_s(x,\tau)\big]\big| = \big|(-\Delta)^s\big[-(-\Delta)^sP_s(x,\tau)\big]\big| \lesssim \frac{1}{|\overline{x}|_{p_s}^{n+4s}}.$$

Therefore,

$$\left|F_s''\left(\frac{\tau}{|x|^{2s}}\right)\right| \lesssim \frac{|x|^{n+4s}}{|\overline{x}|_{p_s}^{n+4s}} = \frac{1}{\max\left\{1, \left(\tau/|x|^{2s}\right)^{1/(2s)}\right\}^{n+4s}} \approx \frac{1}{\left[1+\left(\tau/|x|^{2s}\right)^{1/s}\right]^{(n+4s)/2}},$$

that implies the following (improved) bound for F_s'',

$$|F_s''(u)| \lesssim \frac{1}{\left(1+u^{1/s}\right)^{(n+4s)/2}} \leq 1, \qquad u > 0. \tag{1.1.12}$$

We continue by observing that by a change of variables the following holds,

$$\partial_t^\beta P_s(x,t) = \frac{1}{|x|^n}\left[\partial_t^\beta F_s\left(\frac{\cdot}{|x|^{2s}}\right)\right](t) = \frac{1}{|x|^{n+2s\beta}}\partial^\beta F_s\left(\frac{t}{|x|^{2s}}\right). \tag{1.1.13}$$

We shall prove the following inequality,

$$\left|\partial^\beta F_s(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}, \tag{1.1.14}$$

where for $u=0$ is just asking for $\left|\partial^\beta F_s(0)\right|$ to be bounded. To verify (1.1.14) we distinguish whether if $u=0$, $u<0$ or $u>0$. For $u=0$ observe that by definition and relation (1.1.10),

$$\begin{aligned}
\left|\partial^\beta F_s(0)\right| &\leq \int_{\mathbb{R}} \frac{|F_s(0)-F_s(w)|}{|0-w|^{1+\beta}}\,\mathrm{d}w = \int_0^\infty \frac{|F_s(w)|}{w^{1+\beta}}\,\mathrm{d}w \\
&\lesssim_\beta \int_0^\infty \frac{1}{w^{1+\beta}}\frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w \\
&= \int_0^1 \frac{\mathrm{d}w}{w^\beta\left(1+w^{1/s}\right)^{(n+2s)/2}} + \int_1^\infty \frac{\mathrm{d}w}{w^\beta\left(1+w^{1/s}\right)^{(n+2s)/2}} \\
&\approx \int_0^1 \frac{\mathrm{d}w}{w^\beta} + \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}+1+\beta}} \lesssim (1-\beta)^{-1} + \left(\frac{n}{2s}+\beta\right)^{-1} \lesssim_\beta 1,
\end{aligned}$$

so case $u=0$ is done. Let us assume $u<0$, so that

$$\left|\partial^\beta F_s(u)\right| \leq \int_{\mathbb{R}} \frac{|F_s(w)|}{||u|+w|^{1+\beta}}\,\mathrm{d}w \lesssim \int_0^\infty \frac{1}{(|u|+w)^{1+\beta}}\frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w.$$

On the one hand notice that the since $|u|+w>w$, the previous expression is bounded by a constant depending on n, s and β (by the same arguments given for the case $u=0$). On the other hand, observe that

$$\begin{aligned}
\left|\partial^\beta F_s(u)\right| &\lesssim \frac{1}{|u|^{1+\beta}}\int_0^\infty \frac{1}{(w/|u|+1)^{1+\beta}}\frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w \\
&= \frac{1}{|u|^{1+\beta}}\left[\int_0^1 \frac{1}{(w/|u|+1)^{1+\beta}}\frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w\right. \\
&\qquad\left.+\int_1^\infty \frac{1}{(w/|u|+1)^{1+\beta}}\frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w\right] =: \frac{1}{|u|^{1+\beta}}(\mathrm{I}+\mathrm{II}).
\end{aligned}$$

Regarding I, since the denominators are bigger than 1, we directly have

$$\mathrm{I} \lesssim \int_0^1 w \,\mathrm{d}w \leq 1. \tag{1.1.15}$$

Turning to II, we similarly obtain

$$\mathrm{II} \leq \int_1^\infty \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w \leq \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}}} = w^{-\frac{n}{2s}+1}\Big|_1^\infty = 1, \tag{1.1.16}$$

where notice that $-\frac{n}{2s}+1<0$ because $n>1$ and $s<1$. Therefore, we also have $\left|\partial^\beta F_s(u)\right| \lesssim |u|^{-1-\beta}$ and we conclude that for $u\leq 0$,

$$\left|\partial^\beta F_s(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}.$$

Let us finally assume $u>0$. Begin by writing

$$\begin{aligned}\left|\partial^\beta F_s(u)\right| \leq \int_{|w|\leq u/2} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w + \int_{u/2\leq |w|\leq 2u} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w \\ + \int_{|w|>2u} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w =: \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

We study each of the previous integrals separately. Concerning the first, notice that in its domain of integration $u/2\leq |w-u|\leq 3u/2$, i.e. $|w-u|\approx u$. We split it as follows

$$\mathrm{I} = \int_{-u/2}^0 \frac{|F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w + \int_0^{u/2} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w =: \mathrm{I}_1+\mathrm{I}_2.$$

Observe that I_1 can be estimated by

$$\mathrm{I}_1 \lesssim \frac{u}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \int_{-u/2}^0 \frac{\mathrm{d}w}{|u|^{1+\beta}} \simeq_\beta \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}}.$$

The expression of the right, viewed as a continuous function of u, tends to zero as $u\to 0$ and decays as $|u|^{-\beta-\frac{n}{2s}}$ as $u\to\infty$. Hence, it is bounded by a constant (depending on n,s and β) and so $\mathrm{I}_1\lesssim_\beta 1$. On the other hand, to prove that $\mathrm{I}_1 \lesssim_\beta |u|^{-1-\beta}$ it suffices to check that the following expression is bounded by a constant,

$$\frac{u^2}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \approx uF_s(u).$$

Again, it is clear it that tends to zero as $u\to 0$, but observe that it behaves as $|u|^{-\frac{n}{2s}+1}$ as $u\to\infty$, which vanishes only if $n>2s$, that is, only if $n>1$, since $s<1$. But this is satisfied by hypothesis. Therefore we deduce $\mathrm{I}_1 \lesssim_\beta \min\{1,|u|^{-1-\beta}\}$. Regarding I_2 proceed in a similar manner to obtain

$$\mathrm{I}_2 \lesssim_\beta \frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w + \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}}.$$

The second summand has already been studied in I_1. Regarding the first, notice that

$$\int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w \leq \int_0^1 \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w + \int_1^\infty \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w$$
$$\leq \int_0^1 w\,\mathrm{d}w + \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}}} \lesssim 1,$$

where we have applied the same arguments as in (1.1.15) and (1.1.16). On the other hand, by applying the following inequality for $w > 0$,

$$(1+w^{1/s})^{(n+2s)/2} > w^{1-\beta},$$

that can be checked by a direct computation, we deduce

$$\int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\,\mathrm{d}w < \int_0^{u/2} w^\beta\,\mathrm{d}w \lesssim_\beta u^{1+\beta}.$$

Therefore we conclude

$$\mathrm{I}_2 \lesssim_\beta \frac{1}{u^{1+\beta}} \min\left\{1, u^{1+\beta}\right\} + \min\left\{1, \frac{1}{u^{1+\beta}}\right\} = 2\min\left\{1, \frac{1}{u^{1+\beta}}\right\},$$

that implies the desired estimate for I.

Moving on to II, we split it as follows

$$\mathrm{II} = \int_{-2u}^{-u/2} \frac{|F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w + \int_{u/2}^{2u} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w =: \mathrm{II}_1 + \mathrm{II}_2.$$

The study of II_1 is exactly the same as the one presented for I_1, so we focus on II_2. Apply the mean value theorem to obtain

$$\mathrm{II}_2 \leq \sup_{\nu\in[u/2,2u]} |F_s'(\nu)| \int_{u/2}^{2u} \frac{\mathrm{d}w}{|w-u|^\beta} \lesssim_\beta \sup_{\nu\in[u/2,2u]} |F_s'(\nu)|\, u^{1-\beta}.$$

Therefore, if we are able to bound $|F_s'|$ by $u^{\beta-1}$ and u^{-2} we will be done. But recalling relation (1.1.11), this is equivalent to proving that the following functions are bounded by a constant:

$$\frac{u^{\beta-1}}{\left(1+u^{1/s}\right)^{(n+2s)/2}}, \qquad \frac{u^2}{\left(1+u^{1/s}\right)^{(n+2s)/2}},$$

that has already been done in I_1. Therefore, we are only left to study III,

$$\mathrm{III} = \int_{-\infty}^{-2u} \frac{|F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w + \int_{2u}^{\infty} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w =: \mathrm{III}_1 + \mathrm{III}_2.$$

To deal with III_1 we first notice that in the domain of integration $|w-u| \approx |w|$, implying

$$\mathrm{III}_1 \approx \frac{u}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \int_{-\infty}^{-2u} \frac{\mathrm{d}w}{|w|^{1+\beta}} \lesssim_\beta \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \lesssim_\beta \min\left\{1, \frac{1}{u^{1+\beta}}\right\}.$$

We study III_2 by splitting it as

$$\mathrm{III}_2 \leq \int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}}\,\mathrm{d}w + \int_{2u}^{\infty} \frac{|F_s(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w.$$

The second summand is tackled in exactly the same way as III_1, so we focus on the first one. Using that $|w-u| \approx |w| \gtrsim u$, we have

$$\begin{aligned}\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}}\,\mathrm{d}w &\lesssim \frac{1}{u^{1+\beta}} \int_{2u}^{\infty} \frac{w}{\big(1+w^{1/s}\big)^{(n+2s)/2}}\,\mathrm{d}w \\ &\leq \frac{1}{u^{1+\beta}}\left[\int_0^1 w\;\mathrm{d}w + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n}{2s}}}\right] \lesssim \frac{1}{u^{1+\beta}},\end{aligned}$$

by the same arguments used in (1.1.15) and (1.1.16). On the other hand, we also have

$$\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}}\,\mathrm{d}w \lesssim \int_{2u}^{\infty} \frac{1}{w^{1+\beta}} \frac{w}{\big(1+w^{1/s}\big)^{(n+2s)/2}}\,\mathrm{d}w \leq \int_0^1 \frac{\mathrm{d}w}{w^{\beta}} + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n}{2s}}}.$$

We already know that the second integral is bounded by a constant for $n>1$, while the first one is also bounded, since $0<\beta<1$. So we conclude that $\mathrm{III}_2 \lesssim_\beta \min\{1, |u|^{-1-\beta}\}$ and we obtain the desired bound for III and thus for $|\partial^\beta F_s(u)|$ if $u>0$.

All in all, returning to (1.1.13), we finally have

$$\begin{aligned}|\partial_t^\beta P_s(x,t)| &= \frac{1}{|x|^{n+2s\beta}}\left|\partial^\beta F_s\left(\frac{t}{|x|^{2s}}\right)\right| \lesssim_\beta \frac{1}{|x|^{n+2s\beta}} \min\left\{1, \frac{|x|^{2s(1+\beta)}}{|t|^{1+\beta}}\right\} \\ &= \frac{1}{|x|^{n-2s}} \min\left\{\frac{1}{|x|^{2s(1+\beta)}}, \frac{1}{|t|^{\frac{2s(1+\beta)}{2s}}}\right\} = \frac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}},\end{aligned}$$

that is estimate *1* in the statement of the lemma.

In order to prove *2*, we follow the same scheme. Indeed, the desired estimate follows once we prove

$$\big|\partial^\beta F_s(u)\big| \lesssim_{\beta,\alpha} \min\left\{1, \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}}\right\}, \qquad \text{for } 2s-1<\alpha<4s.$$

If one followed the same arguments used to prove *1*, in the regime $u<0$ one already encounters a first bound for which dimension $n=1$ is troublesome, namely when trying to obtain $|\partial^\beta F_s(u)| \lesssim |u|^{-1-\beta+\frac{\alpha}{2s}}$. However, in our current setting we observe that

$$\begin{aligned}&\int_0^{\infty} \frac{1}{(w+|u|)^{1+\beta}} \frac{w}{\big(1+w^{1/s}\big)^{(n+2s)/2}}\,\mathrm{d}w \\ &\quad\lesssim \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}} \int_0^{\infty} \frac{1}{(w/|u|+1)^{1+\beta-\frac{\alpha}{2s}}} \frac{1}{(w+|u|)^{\frac{\alpha}{2s}}} \frac{w}{\big(1+w^{1/s}\big)^{(n+2s)/2}}\,\mathrm{d}w \\ &\quad\lesssim \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}}\left(\int_0^1 w^{1-\frac{\alpha}{2s}}\,\mathrm{d}w + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n+\alpha}{2s}}}\right) \lesssim_{\beta,\alpha} \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}},\end{aligned}$$

since $2s-1<\alpha<4s$, so the desired bound for $|\partial^\beta F_s(u)|$ follows. For the case $u>0$ we also proceed analogously. Let us comment those steps where the hypotheses on α and β come into play. In I, using the same notation as for the case $n>1$, we obtain the estimates

$$\mathrm{I}_1 \lesssim_\beta \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \qquad \text{and} \qquad \mathrm{I}_2 \lesssim_\beta \frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\, \mathrm{d}w,$$

expression that we already know to be bounded by a constant. To prove that $\mathrm{I}_1 \lesssim_{\beta,\alpha} |u|^{-1-\beta+\frac{\alpha}{2s}}$ observe that the function

$$\frac{u^{2-\frac{\alpha}{2s}}}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \approx u^{1-\frac{\alpha}{2s}} F_s(u)$$

tends to zero as $u\to 0$, since $\alpha<4s$. Moreover, it behaves as $|u|^{-\frac{n+\alpha}{2s}+1}$ as $u\to\infty$, which also tends to 0 because $\alpha<2s-1$. Thus, $\mathrm{I}_1 \lesssim_{\beta,\alpha} \min\{1,|u|^{-1-\beta+\frac{\alpha}{2s}}\}$. On the other hand, since the following holds

$$\left(1+w^{1/s}\right)^{(n+2s)/2} > w^{2-\frac{\alpha}{2s}},$$

we obtain

$$\frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\, \mathrm{d}w < \frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{\mathrm{d}w}{w^{1-\frac{\alpha}{2s}}} \lesssim_{\beta,\alpha} \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}}.$$

Therefore, $\mathrm{I}_2 \lesssim_{\beta,\alpha} \min\{1,|u|^{-1-\beta+\frac{\alpha}{2s}}\}$, hence I satisfies the same estimate. The study of II is completely analogous to that of $n>1$. Therefore we are only left to study III. The arguments can be carried out analogously up to the point of estimating

$$\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}}\, \mathrm{d}w.$$

Using that $|w-u|\approx|w|\gtrsim u$, we have

$$\begin{aligned}\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}}\, \mathrm{d}w &\lesssim \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}} \int_{2u}^{\infty} \frac{w^{1-\frac{\alpha}{2s}}}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\, \mathrm{d}w \\ &\le \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}} \left[\int_0^1 w^{1-\frac{\alpha}{2s}}\, \mathrm{d}w + \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n+\alpha}{2s}}}\right] \lesssim_{\beta,\alpha} \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}},\end{aligned}$$

since $2s-1<\alpha<4s$. Therefore, $\mathrm{III}_2 \lesssim_{\beta,\alpha} \min\{1,|u|^{-1-\beta+\frac{\alpha}{2s}}\}$, and with this we get the desired bound for III and the completion of the proof for the case $n=1$.

Moving on to estimate *3*, we begin by defining for $u>0$ the real variable function

$$G_s(u) := u^{-\frac{n+1}{2s}} \phi_n'\left(u^{-\frac{1}{2s}}\right),$$

so that in light of relation (1.1.1) we have

$$\nabla_x P_s(x,t) \simeq \frac{x}{|x|^{n+2}} G_s\Big(\frac{t}{|x|^{2s}}\Big), \qquad \text{for} \quad t>0, x\neq 0.$$

By (1.1.5) it is clear that

$$|G_s(u)| \approx \frac{u}{\big(1+u^{1/s}\big)^{(n+2s+2)/2}}. \tag{1.1.17}$$

Hence, as done for F_s, we can extend continuously the definition of G_s by zero for negative values of u. Notice also that the previous estimate implies that G_s is bounded on $\mathbb{R}$.

Our next claim is that the operators ∇_x and ∂_t^β commute when applied to P_s. To prove this, it suffices to check that the following integral is locally well-defined for every x and t,

$$\int_{\mathbb{R}} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \, \mathrm{d}w.$$

Split the domain of integration as

$$\int_{|t-w|<1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \, \mathrm{d}w + \int_{|t-w|\geq 1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \, \mathrm{d}w.$$

The second integral is clearly well-defined, since $\nabla_x P_s(x,t) \simeq x/|x|^{n+2} G_s(t/|x|^{2s})$ and we know that G_s is bounded. Thus, directly applying the triangle inequality in the numerator and using that $\beta>0$, we deduce that, indeed, the second integral is finite. For the first one, we need some more work. We shall distinguish four possibilities:

Case 1: $t\leq -1$. For such values of t the integral becomes null, since $\nabla_x P_s(x,t)$ and $\nabla_x P(x,w)$ are zero.

Case 2: $t\in(-1,0]$. Observe that in this setting the integral can be rewritten as

$$\begin{aligned}\int_0^{1-|t|} \frac{|\nabla_x P_s(x,w)|}{|w-t|^{1+\beta}} \, \mathrm{d}w &= \int_0^{1-|t|} \frac{|\nabla_x P_s(x,w) - \nabla_x P_s(x,0)|}{|w-t|^{1+\beta}} \, \mathrm{d}w \\ &\lesssim \frac{1}{|x|^{n+2s+1}} \int_0^{1-|t|} \frac{|G_s'(\tau/|x|^{2s})|}{|w|^\beta} \, \mathrm{d}w,\end{aligned}$$

for some $\tau\in(0,w)$. By definition, there are constants C_1, C_2 so that for $u>0$

$$G_s'(u) = C_1\, u^{-(n+2s+1)/(2s)} \phi_{n,s}'\big(u^{-\frac{1}{2s}}\big) + C_2\, u^{-(n+2s+2)/(2s)} \phi_{n,s}''\big(u^{-\frac{1}{2s}}\big),$$

so using the estimates for ϕ_n' and $\phi_{n,s}''$ in (1.1.5) we deduce

$$|G_s'(u)| \approx \frac{1}{\big(1+u^{1/s}\big)^{(n+2s+2)/2}}, \tag{1.1.18}$$

which is a bounded function. Therefore

$$\frac{1}{|x|^{n+2s+1}} \int_0^{1-|t|} \frac{|G_s'(\tau/|x|^{2s})|}{|w|^\beta} \, \mathrm{d}w \lesssim_\beta \frac{1}{|x|^{n+2s+1}} < \infty, \qquad \text{for every } x\neq 0.$$

Case 3: $t \in (0,1]$. The integral we were initially studying can be written as

$$\int_{t-1}^{0} \frac{|\nabla_x P_s(x,t)|}{|t-w|^{1+\beta}}\,\mathrm{d}w + \int_{0}^{t+1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}}\,\mathrm{d}w.$$

The second integral can be tackled in exactly the same way as the integral in *Case 2*. Regarding the first one, estimate it as follows

$$\begin{aligned}\int_{t-1}^{0} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,0)|}{|t-w|^{1+\beta}}\,\mathrm{d}w &\leq \frac{1}{|x|^{n+2s+1}} \int_{t-1}^{0} \frac{|G_s'(\tau/|x|^{2s})||t|}{|t-w|^{1+\beta}}\,\mathrm{d}w \\ &\lesssim \frac{1}{|x|^{n+2s+1}} \int_{t-1}^{0} \frac{|G_s'(\tau/|x|^{2s})|}{|w|^{\beta}}\,\mathrm{d}w < \infty,\end{aligned}$$

where we have used $|t| \leq |t-w| + |w|$ and also that $|t-w| = (t+|w|) \geq |w|$. The last inequality follows by the same arguments used in *Case 2*.

Case 4: $t > 1$. For this final case, the integral can be estimated as

$$\int_{t-1}^{t+1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}}\,\mathrm{d}w \lesssim \frac{1}{|x|^{n+2s+1}} \int_{t-1}^{t+1} \frac{|G_s'(\tau/|x|^{2s})|}{|t-w|^{\beta}}\,\mathrm{d}w \lesssim_\beta \frac{1}{|x|^{n+2s+1}}.$$

Thus, we have obtained the desired commutativity between ∂_t^β and ∇_x, which yields

$$\begin{aligned}&\nabla_x \partial_t^\beta P_s(x,t) \\ &\quad = \partial_t^\beta [\nabla_x P_s](x,t) = \frac{x}{|x|^{n+2}} \bigg[\partial_t^\beta G_s\Big(\frac{\cdot}{|x|^{2s}}\Big)\bigg](t) = \frac{x}{|x|^{n+2s\beta+2}} \partial^\beta G_s\Big(\frac{t}{|x|^{2s}}\Big).\end{aligned}$$

Now it is a matter of showing that the following inequality holds

$$\big|\partial^\beta G_s(u)\big| \lesssim_\beta \min\bigg\{1, \frac{1}{|u|^{1+\beta}}\bigg\}, \tag{1.1.19}$$

The proof of (1.1.19) is essentially identical to the one given for (1.1.14), using the bounds for G_s and G_s' ((1.1.17) and (1.1.18) respectively) instead of those for F_s and F_s'. The faster decay of G_s and its derivative implies that one does not find any obstacles in (1.1.16). In fact, the integral that appears in the current analysis is $\int_1^\infty w^{-\frac{n+1}{2s}}\,\mathrm{d}w$, which also converges for $n=1$. So using the previous estimate we deduce, for any $n > 0$,

$$\begin{aligned}|\nabla_x \partial_t^\beta P_s(x,t)| &= \frac{1}{|x|^{n+2s\beta+1}} \bigg|\partial_t^\beta G_s\Big(\frac{t}{|x|^{2s}}\Big)\bigg| \lesssim_\beta \frac{1}{|x|^{n+2s\beta+1}} \min\bigg\{1, \frac{|x|^{2s(1+\beta)}}{|t|^{1+\beta}}\bigg\} \\ &= \frac{1}{|x|^{n-2s+1}} \min\bigg\{\frac{1}{|x|^{2s(1+\beta)}}, \frac{1}{|t|^{\frac{2s(1+\beta)}{2s}}}\bigg\} = \frac{1}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}},\end{aligned}$$

which proves the statement *3* in our lemma.

We continue by estimating $\partial_t \partial_t^\beta P_s(x,t)$ for $x \neq 0$ and $t \neq 0$. Using (1.1.13) we rewrite it as

$$\partial_t \partial_t^\beta P_s(\overline{x}) = \frac{1}{|x|^{n+2s(1+\beta)}} \partial^\beta F_s'\Big(\frac{t}{|x|^{2s}}\Big),$$

and we claim that the following inequality holds for $u\neq 0$,

$$\big|\partial^\beta F_s'(u)\big| \lesssim_\beta \min\bigg\{1,\frac{1}{|u|^{1+\beta}}\bigg\}.$$

Let us also recall that we had the following estimates for $u>0$,

$$|F_s'(u)|\lesssim \frac{1}{\big(1+u^{1/s}\big)^{(n+2s)/2}},\qquad |F_s''(u)|\lesssim \frac{1}{\big(1+u^{1/s}\big)^{(n+4s)/2}}\leq \frac{1}{u\big(1+u^{1/s}\big)^{(n+2s)/2}}.$$

Observe that, on the one hand,

$$\begin{aligned}\big|\partial^\beta F_s'(u)\big| &\leq \int_{\mathbb{R}}\frac{|F_s'(u)-F_s'(w)|}{|u-w|^{1+\beta}}\,\mathrm{d}w\\ &\leq \sup_{\nu\in\mathbb{R}}|F_s''(\nu)|\int_{|u-w|<1}\frac{\mathrm{d}w}{|u-w|^{\beta}}+2\sup_{\nu\in\mathbb{R}}|F_s'(\nu)|\int_{|u-w|\geq 1}\frac{\mathrm{d}w}{|u-w|^{1+\beta}}\lesssim_\beta 1,\end{aligned}$$

by the boundedness of F_s' and F_s'', and the fact that $\beta\in(0,1)$. Therefore we are left to verify $\big|\partial^\beta F_s'(u)\big|\lesssim_\beta |u|^{-1-\beta}$. If $u<0$, since F_s' is supported on $(0,\infty)$ and $|u-w|>|u|$ for $w\geq 0$, we have

$$\begin{aligned}\big|\partial^\beta F_s'(u)\big| &\leq \int_0^\infty \frac{|F_s'(w)|}{|u-w|^{1+\beta}}\,\mathrm{d}w\lesssim \int_0^1\frac{\mathrm{d}w}{|u-w|^{1+\beta}}+\int_1^\infty\frac{|F_s'(w)|}{|u-w|^{1+\beta}}\,\mathrm{d}w\\ &\lesssim \frac{1}{|u|^{1+\beta}}\bigg(1+\int_1^\infty\frac{\mathrm{d}w}{\big(1+w^{1/s}\big)^{(n+2s)/2}}\bigg)\leq \frac{1}{|u|^{1+\beta}}\bigg(1+\int_1^\infty\frac{\mathrm{d}w}{w^{\frac{n}{2s}+1}}\bigg)\lesssim\frac{1}{|u|^{1+\beta}},\end{aligned}$$

and we are done. If on the other hand $u>0$, we estimate $|\partial^\beta F_s'|$ in a similar way as $|\partial^\beta F_s|$ in the proof of point *1* of this lemma. Namely, we write

$$\begin{aligned}\big|\partial^\beta F_s'(u)\big| \leq &\int_{|w|\leq u/2}\frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w+\int_{u/2\leq|w|\leq 2u}\frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w\\ &+\int_{|w|>2u}\frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w=:\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Regarding I, notice that in the domain of integration we have $|w-u|\approx u$, so

$$\mathrm{I}\lesssim \int_{-u/2}^0\frac{|F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w+\int_0^{u/2}\frac{|F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w+\int_0^{u/2}\frac{|F_s'(w)|}{|w-u|^{1+\beta}}\,\mathrm{d}w.$$

The first two integrals can be directly bounded by

$$\frac{1}{|u|^{1+\beta}}|F_s'(u)|\int_0^{u/2}\mathrm{d}w\leq\frac{1}{|u|^{1+\beta}}\bigg(\frac{u}{(1+u^{1/s})^{(n+2s)/2}}\bigg)\leq\frac{1}{|u|^{1+\beta}}.$$

For the third,

$$\begin{aligned}\int_0^{u/2}\frac{|F_s'(w)|}{|w-u|^{1+\beta}}\,\mathrm{d}w &\lesssim\frac{1}{|u|^{1+\beta}}\bigg(\int_0^1|F_s'(w)|\,\mathrm{d}w+\int_1^\infty|F_s'(w)|\,\mathrm{d}w\bigg)\\ &\lesssim\frac{1}{|u|^{1+\beta}}\bigg(1+\int_1^\infty\frac{\mathrm{d}w}{w^{\frac{n}{2s}+1}}\bigg)\lesssim\frac{1}{|u|^{1+\beta}},\end{aligned}$$

and we are done with I. Moving on to II, we split it as follows

$$\mathrm{II} = \int_{-2u}^{-u/2} \frac{|F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w + \int_{u/2}^{2u} \frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\,\mathrm{d}w =: \mathrm{II}_1 + \mathrm{II}_2.$$

The study of II_1 can be carried analogously to that of I, since in that domain of integration one has $|w-u| \geq 3|u|/2$, so we focus on II_2. Applying the mean value theorem and the bound for $|F_s''|$ of (1.1.11) as well as relation (1.1.10) we get

$$\begin{aligned}\mathrm{II}_2 \leq \sup_{\nu\in[u/2,2u]} |F_s''(\nu)| \int_{u/2}^{2u} \frac{\mathrm{d}w}{|w-u|^{\beta}} &\lesssim_\beta \sup_{\nu\in[u/2,2u]} |F_s''(\nu)|\, u^{1-\beta} \\ &\lesssim \frac{u^{1-\beta}}{u(1+u^{1/s})^{(n+2s)/2}} \approx F_s(u)u^{-1-\beta} \leq u^{-1-\beta}.\end{aligned}$$

So we are left to study III. Since in its domain of integration we have $|w-u| \gtrsim w$, we get

$$\begin{aligned}\mathrm{III} &\lesssim \int_{-\infty}^{-2u} \frac{|F_s'(u)|}{|w|^{1+\beta}}\,\mathrm{d}w + \int_{2u}^{\infty} \frac{|F_s'(w)|+|F_s'(u)|}{|w|^{1+\beta}}\,\mathrm{d}w \\ &\lesssim \frac{3}{(1+u^{1/s})^{(n+2s)/2}} \int_{2u}^{\infty} \frac{\mathrm{d}w}{|w|^{1+\beta}} \lesssim_\beta \frac{u^{-\beta}}{(1+u^{1/s})^{(n+2s)/2}} \leq u^{-1-\beta},\end{aligned}$$

that allows us to finally conclude

$$\left|\partial^\beta F_s'(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}, \qquad u \neq 0.$$

So using the previous estimate get, for $x \neq 0$ and $t \neq 0$,

$$\begin{aligned}|\partial_t \partial_t^\beta P_s(x,t)| &= \frac{1}{|x|^{n+2s(1+\beta)}} \left|\partial_t^\beta F_s'\left(\frac{t}{|x|^{2s}}\right)\right| \lesssim_\beta \frac{1}{|x|^{n+2s(1+\beta)}} \min\left\{1, \frac{|x|^{2s(1+\beta)}}{|t|^{1+\beta}}\right\} \\ &= \frac{1}{|x|^n} \min\left\{\frac{1}{|x|^{2s(1+\beta)}}, \frac{1}{|t|^{\frac{2s(1+\beta)}{2s}}}\right\} = \frac{1}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}}.\end{aligned}$$

Finally, the proof of estimate *5* is analogous to that of *5* in Theorem 1.1.3. Indeed, let $\overline{x}' = (x', t') \in \mathbb{R}^{n+1}$ such that $|\overline{x} - \overline{x}'|_{p_s} \leq |x|/2$, which is a stronger assumption than that of Theorem 1.1.3. In fact, it can be checked by a direct computation that this already implies $|\overline{x}|_{p_s} \leq 2|\overline{x}'|_{p_s}$ and $|x| \leq 2|x'|$. Write again $\widehat{x} = (x', t)$ and consider

$$|\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\overline{x}')| \leq |\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\widehat{x})| + |\partial_t^\beta P_s(\widehat{x}) - \partial_t^\beta P_s(\overline{x}')|.$$

By estimate *2*, the first term in the above inequality now satisfies

$$\begin{aligned}|x-x'| \sup_{\xi\in[x,x']} |\nabla_x \partial_t^\beta P_s(\xi,t)| \lesssim_\beta \frac{|x-x'|}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}} &\leq \frac{|\overline{x}-\overline{x}'|_{p_s}}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}} \\ &\lesssim \frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}},\end{aligned}$$

where we have used that $1-2\zeta \geq 0$ and that condition $|\overline{x}-\overline{x}'|_{p_s} \leq |x|/2$ implies that the line segment joining $\overline{x}$ with $\overline{x}'$ is at a distance of the time axis comparable to $|x|$. Regarding the second term, assume $t>t'$. If t and t' share sign we apply estimate *3* to directly deduce

$$|t-t'| \sup_{\tau\in[t,t']} |\partial_t \partial_t^\beta P_s(x',\tau)| \lesssim_\beta \frac{|t-t'|}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}} \leq \frac{|\overline{x}-\overline{x}'|_{p_s}^{2s}}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}} \lesssim \frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s} |\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

If on the other hand $t>0$ and $t'<0$, we use relation (1.1.8), valid also in this case, together with $|x'|\geq |x|/2$ to finally obtain

$$\begin{aligned}
&|\partial_t^\beta P_s(\widehat{x}) - \partial_t^\beta P_s(\overline{x}')| \\
&\qquad \leq |\partial_t^\beta P_s(x',t) - \partial_t^\beta P_s(x',0)| + |\partial_t^\beta P_s(x',0) - \partial_t^\beta P_s(x',t')| \\
&\qquad \lesssim t \sup_{\tau\in(0,t)} |\partial_t\partial_t^\beta P_s(x',\tau)| + |t'| \sup_{\tau\in(t',0)} |\partial_t\partial_t^\beta P_s(x',\tau)| \\
&\qquad \lesssim_\beta \frac{t+|t'|}{|x'|^n |x'|^{2s(1+\beta)}} \lesssim \frac{|t-t'|}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}} \lesssim \frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s} |\overline{x}|_{p_s}^{2s(1+\beta)}}.
\end{aligned}$$

□

We will now carry out the same study for the case $s=1$. First, we prove the following auxiliary lemma:

LEMMA 1.1.5. *Let $f_1, f_2, f_3 : \mathbb{R}\to\mathbb{R}$ be defined as*

$$f_1(t) := \frac{e^{-1/t}}{t^{n/2}}\,\chi_{t>0}, \qquad f_2(t) := \frac{e^{-1/t}}{t^{n/2+1}}\,\chi_{t>0}, \qquad f_3(t) := \frac{e^{-1/t}}{t^{n/2+2}}\,\chi_{t>0}.$$

Then, if $\beta\in(0,1)$, the following estimates hold:

$$\begin{aligned}
&\textit{if}\ \ n>2, && |\partial_t^\beta f_1(t)| \lesssim_\beta \min\{1, |t|^{-1-\beta}\}, \\
&\textit{if}\ \ n=2 \textit{ and } \beta>\frac{1}{2}, && |\partial_t^\beta f_1(t)| \lesssim_{\beta,\alpha} \min\{1, |t|^{-1-\beta+\alpha/2}\}, \ \ \forall \alpha\in(0,2+2\beta], \\
&\textit{if}\ \ n=1, && |\partial_t^\beta f_1(t)| \lesssim_\beta 1.
\end{aligned}$$

In addition, for every n,

$$|\partial_t^\beta f_2(t)| \lesssim_\beta \min\{1,|t|^{-1-\beta}\}, \qquad |\partial_t^\beta f_3(t)| \lesssim_\beta \min\{1,|t|^{-1-\beta}\}.$$

For $t=0$ the previous estimates have to be understood simply as a bound by a constant depending on n and β.

The above result will imply the following estimates for $\partial_t^\beta W$:

LEMMA 1.1.6. *For any $\overline{x} = (x, t) \neq (0, t)$ and $\beta \in (0, 1)$, the following hold:*

1. *For* $n > 2$, $\qquad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n-2}|\overline{x}|_{p_1}^{2+2\beta}},$
2. *For* $n = 2$, $\qquad |\partial_t^\beta W(\overline{x})| \lesssim_{\beta,\alpha} \dfrac{1}{|x|^{\alpha}|\overline{x}|_{p_1}^{2+2\beta-\alpha}}, \quad \forall \alpha \in (0, 2+2\beta],$
3. *For* $n = 1$, $\qquad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \dfrac{1}{|\overline{x}|_{p_1}^{1+2\beta}}.$

Moreover, for every n,

4. $|\nabla_x \partial_t^\beta W(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}, \qquad$ 5. $|\partial_t \partial_t^\beta W(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n}|\overline{x}|_{p_1}^{2+2\beta}}.$

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ is such that $|\overline{x} - \overline{x}'|_{p_1} \leq |x|/2$, then

6. $|\partial_t^\beta W(\overline{x}) - \partial_t^\beta W(\overline{x}')| \lesssim_\beta \dfrac{|\overline{x} - \overline{x}'|_{p_1}}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}.$

Proof of Lemma 1.1.5. We deal first with the estimate concerning $\partial_t^\beta f_1$ for $n > 2$. We distinguish whether if $t = 0$, $t < 0$ or $t > 0$. If $t = 0$ we are done because,

$$|\partial_t^\beta f_1(0)| \leq \int_{\mathbb{R}} \frac{|f_1(u) - f_1(0)|}{|u - 0|^{1+\beta}}\, \mathrm{d}u = \int_0^\infty \frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\, \mathrm{d}u = \Gamma\left(\frac{n+2\beta}{2}\right) \lesssim_\beta 1,$$

where Γ denotes the usual gamma function.

Let us continue by assuming $t < 0$. By definition,

$$|\partial_t^\beta f_1(t)| \leq \int_{\mathbb{R}} \frac{|f_1(u)|}{|u + |t||^{1+\beta}}\, \mathrm{d}u = \int_0^\infty \frac{e^{-1/u}}{u^{n/2}(u + |t|)^{1+\beta}}\, \mathrm{d}u.$$

Observe that on the one hand, since $|u + |t|| \geq u$,

$$|\partial_t^\beta f_1(t)| \leq \int_0^\infty \frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\, \mathrm{d}u \lesssim_\beta 1.$$

On the other hand, since $n > 2$,

$$|\partial_t^\beta f_1(t)| \leq \frac{1}{|t|^{1+\beta}} \int_0^\infty \frac{e^{-1/u}}{u^{n/2}(u/|t| + 1)^{1+\beta}}\, \mathrm{d}u \leq \frac{1}{|t|^{1+\beta}} \int_0^\infty \frac{e^{-1/u}}{u^{n/2}}\, \mathrm{d}u \lesssim \frac{1}{|t|^{1+\beta}},$$

Therefore, $|\partial_t^\beta f_1(t)| \lesssim_\beta \min\{1, |t|^{-1-\beta}\}$ and we are done.

If $t > 0$, we split the integral as follows

$$\begin{aligned}|\partial_t^\beta f_1(t)| \leq &\int_{|u| \leq t/2} \frac{|f_1(u) - f_1(t)|}{|u - t|^{1+\beta}}\, \mathrm{d}u + \int_{t/2 \leq |u| \leq 2t} \frac{|f_1(u) - f_1(t)|}{|u - t|^{1+\beta}}\, \mathrm{d}u \\ &+ \int_{|u| \geq 2t} \frac{|f_1(u) - f_1(t)|}{|u - t|^{1+\beta}}\, \mathrm{d}u =: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

In I we have $t/2 \le |u-t| \le 3t/2$. Therefore,

$$\mathrm{I} := \int_{-t/2}^{0} \frac{|f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u + \int_0^{t/2} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u \lesssim \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_0^{t/2} \frac{|f_1(u)-f_1(t)|}{t^{1+\beta}}\,\mathrm{d}u.$$

By the definition of f_1, the last term can be bound by

$$\frac{1}{t^{1+\beta}}\int_0^{t/2} \frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u + \frac{1}{t^{1+\beta}}\int_0^{t/2} \frac{e^{-1/t}}{t^{n/2}}\,\mathrm{d}u \simeq \frac{1}{t^{1+\beta}}\int_0^{t/2} \frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u + \frac{e^{-1/t}}{t^{(n+2\beta)/2}}. \tag{1.1.20}$$

We split the remaining integral as follows

$$\int_0^{t/2} \frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u = \int_0^{1} \frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u + \int_1^{t/2} \frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u \le e^{-\frac{1}{2t}}\int_0^1 \frac{e^{-\frac{1}{2u}}}{u^{n/2}}\,\mathrm{d}u + e^{-2/t}\int_1^{t/2}\frac{1}{u^{n/2}}\,\mathrm{d}u \lesssim e^{-\frac{1}{2t}} + \frac{e^{-\frac{1}{2t}}}{t^{n/2-1}},$$

where in the first inequality we have used $e^{-1/u} \le e^{-\frac{1}{2u}}\, e^{-\frac{1}{2t}}$, which is true for $0 \le u \le t/2$; and in the second the general inequality $e^{-2/t} \le e^{-\frac{1}{2t}}$. In addition, observe that in the last step we have used that $n \neq 2$ in order to compute the corresponding integral. Thus, returning to (1.1.20), we obtain

$$\mathrm{I} \lesssim \frac{e^{-\frac{1}{2t}}}{t^{1+\beta}} + \frac{e^{-\frac{1}{2t}}}{t^{(n+2\beta)/2}}.$$

Notice that for $t>0$

$$e^{-\frac{1}{2t}} \le 3\min\big\{1, t^{1+\beta}\big\}, \qquad e^{-\frac{1}{2t}} \le C\min\big\{t^{(n+2\beta)/2}, t^{(n-2)/2}\big\}, \tag{1.1.21}$$

where C depends only on n and β, and the second estimate only holds for $n>1$ (if $n=1$, $e^{-\frac{1}{2t}} \le Ct^{(n+2\beta)/2}$ still holds). Therefore, we finally get

$$\mathrm{I} \lesssim_\beta \frac{\min\big\{1, t^{1+\beta}\big\}}{t^{1+\beta}} + \frac{\min\big\{t^{(n+2\beta)/2}, t^{(n-2)/2}\big\}}{t^{(n+2\beta)/2}} \simeq \min\bigg\{1, \frac{1}{t^{1+\beta}}\bigg\}.$$

Let us turn to II. Write

$$\mathrm{II} := \int_{-2t}^{-t/2} \frac{|f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u + \int_{t/2}^{2t} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u \lesssim \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{t/2}^{2t} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u, \tag{1.1.22}$$

where in the first integral we have used that $3t/2 \le |u-t| \le 3t$. For the second integral observe that

$$|f_1(u)-f_1(t)| \le \sup_{\xi\in[s,t]} |f_1'(\xi)||u-t|, \quad \text{where} \quad f_1'(\xi) = \bigg(1-\frac{n}{2}\xi\bigg)\frac{e^{-1/\xi}}{\xi^{n/2+2}}\,\chi_{\xi>0}.$$

Since $t/2 \le \xi \le 2t$, we have

$$|f_1'(\xi)| \lesssim (1+t)\frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{t>0} = (1+t)\frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{t>1} + (1+t)\frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{0<t\le 1}$$
$$\lesssim \frac{e^{-\frac{1}{2t}}}{t^{n/2+1}}\,\chi_{t>1} + \frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{0<t\le 1}.$$

Combining the last two estimates we can bound the remaining integral of (1.1.22) by

$$\left(\frac{e^{-\frac{1}{2t}}}{t^{n/2+1}}\,\chi_{t>1} + \frac{2e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{0<t\le 1}\right)\int_{t/2}^{2t}\frac{\mathrm{d}u}{|u-t|^{\beta}} \lesssim_\beta \frac{e^{-\frac{1}{2t}}}{t^{(n+2\beta)/2}}\,\chi_{t>1} + \frac{e^{-\frac{1}{2t}}}{t^{(n+2+2\beta)/2}}\,\chi_{0<t\le 1}.$$

Thus,

$$\mathrm{II} \lesssim_\beta \frac{2e^{-\frac{1}{2t}}}{t^{(n+2\beta)/2}} + \frac{e^{-\frac{1}{2t}}}{t^{(n+2+2\beta)/2}}.$$

If we now apply estimates

$$e^{-\frac{1}{2t}} \le C_1 \min\left\{t^{(n+2\beta)/2}, t^{(n-2)/2}\right\}, \qquad e^{-\frac{1}{2t}} \le C_2 \min\left\{t^{(n+2+2\beta)/2}, t^{n/2}\right\},$$

for some constants C_1, C_2 depending on n and β, we conclude

$$\mathrm{II} \lesssim_\beta \frac{\min\left\{t^{(n+2\beta)/2}, t^{(n-2)/2}\right\}}{t^{(n+2\beta)/2}} + \frac{\min\left\{t^{(n+2+2\beta)/2}, t^{n/2}\right\}}{t^{(n+2+2\beta)/2}} \simeq \min\left\{1, \frac{1}{t^{1+\beta}}\right\}.$$

Finally, for III, since $|u|/2 \le |u-t| \le 3|u|/2$, we have

$$\mathrm{III} := \int_{-\infty}^{-2t}\frac{|f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u + \int_{2t}^{\infty}\frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u \simeq \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{2t}^{\infty}\frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u$$
$$\le \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\,\mathrm{d}u + \int_{2t}^{\infty}\frac{e^{-1/t}}{t^{n/2}u^{1+\beta}}\,\mathrm{d}u$$
$$\simeq_\beta \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\,\mathrm{d}u \lesssim_\beta \min\left\{1, \frac{1}{t^{1+\beta}}\right\} + \int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\,\mathrm{d}u.$$

For the remaining integral observe that on the one hand

$$\int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\,\mathrm{d}u \le \Gamma\left(\frac{n+2\beta}{2}\right) \lesssim_\beta 1,$$

while on the other hand, since $u > 2t$,

$$\int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\,\mathrm{d}u \lesssim \frac{1}{t^{1+\beta}}\int_{2t}^{\infty}\frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u \le \frac{1}{t^{1+\beta}}\int_{0}^{\infty}\frac{e^{-1/u}}{u^{n/2}}\,\mathrm{d}u \lesssim \frac{1}{t^{1+\beta}},$$

where the last inequality holds since $n > 2$. Therefore, combining the previous estimates we conclude that for $n > 2$, $|\partial_t^\beta f_1(t)| \lesssim_\beta \min\left\{1, t^{-1-\beta}\right\}$.

Before approaching the case $n = 2$, let us comment that the case $n = 1$ also follows from the above arguments. We also notice that the bounds for $|\partial_t^\beta f_2|$ and

$|\partial_t^\beta f_3|$ are obtained by exactly the same computations. The fact that the exponent of the denominator of the previous functions is increased at least by one unit, makes the result independent of the dimension n. Indeed, observe that the critical steps where we had to use that $n > 2$ appeared for the case $t > 0$ in the study of I, when dealing with the integral

$$\int_1^{t/2} \frac{\mathrm{d}u}{u^{n/2}}.$$

Now, with f_2 and f_3, the previous integral is respectively written as

$$\int_1^{t/2} \frac{\mathrm{d}u}{u^{n/2+1}} \quad \text{and} \quad \int_{\mathrm{d}u}^{t/2} \frac{\mathrm{d}u}{u^{n/2+2}},$$

which are proportional to $t^{-n/2}$ and $t^{-n/2-1}$ respectively, and thus we are able to apply similar estimates as in (1.1.21) (that in this setting hold for any dimension n), deducing from the latter the desired estimate for I. The second crucial step has comes up analyzing the possible divergence of the integral,

$$\int_{2t}^{\infty} \frac{e^{-1/u}}{u^{n/2}}\, \mathrm{d}u,$$

that appears for any value of t. When working with f_2 and f_3 the previous expression becomes, respectively,

$$\int_{2t}^{\infty} \frac{e^{-1/u}}{u^{n/2+1}}\, \mathrm{d}u \quad \text{and} \quad \int_{2t}^{\infty} \frac{e^{-1/u}}{u^{n/2+2}}\, \mathrm{d}u,$$

which converge for every dimension n. Apart from the previous facts, when carrying out the study for $t > 0$ one also encounters new terms of the form

$$\frac{e^{-\frac{1}{2t}}}{t^{(n+4+2\beta)/2}}, \qquad \frac{e^{-\frac{1}{2t}}}{t^{(n+6+2\beta)/2}},$$

for which the following estimates can be equally applied

$$e^{-\frac{1}{2t}} \leq C_3 \min\left\{t^{(n+4+2\beta)/2}, t^{(n+2)/2}\right\}, \qquad e^{-\frac{1}{2t}} \leq C_4 \min\left\{|t|^{(n+6+2\beta)/2}, |t|^{(n+4)/2}\right\},$$

for some constants C_3, C_4 depending only on n and β. All in all, we conclude that the results $|\partial_t^\beta f_2(t)| \lesssim_\beta \min\left\{1, |t|^{-1-\beta}\right\}$ and $|\partial_t^\beta f_3(t)| \lesssim_\beta \min\left\{1, |t|^{-1-\beta}\right\}$ hold for every n.

So we are left to verify the following estimate

$$|\partial_t^\beta f_1(t)| \lesssim_{\beta,\alpha} \min\left\{1, |t|^{-1-\beta+\alpha/2}\right\}, \quad \forall \alpha \in (0, 2+2\beta], \quad n = 2.$$

We fix any $\alpha \in (0, 2+2\beta]$ and follow the same scheme of proof. If $t < 0$, we now obtain the following inequality

$$|\partial_t^\beta f_1(t)| \leq \frac{1}{|t|^{1+\beta}} \int_0^{\infty} \frac{e^{-1/u}}{u(u/|t|+1)^{1+\beta}}\, \mathrm{d}u,$$

that we bound as follows,

$$\frac{1}{|t|^{1+\beta}}\int_0^\infty \frac{e^{-1/u}}{u(u/|t|+1)^{1+\beta}}\,\mathrm{d}u \le \frac{1}{|t|^{1+\beta}}\int_0^\infty \frac{e^{-1/u}}{u(u/|t|+1)^{\alpha/2}}\,\mathrm{d}u$$
$$\le \frac{1}{|t|^{1+\beta}}\int_0^\infty \frac{e^{-1/u}}{u(u/|t|)^{\alpha/2}}\,\mathrm{d}u = \frac{1}{|t|^{1+\beta-\alpha/2}}\int_0^\infty \frac{e^{-1/u}}{u^{1+\alpha/2}}\,\mathrm{d}u \lesssim_\alpha \frac{1}{|t|^{1+\beta-\alpha/2}},$$

which implies the desired inequality in this case. Observe that the convergence of the last integral is ensured by the fact that $\alpha > 0$. For $t > 0$, we begin by estimating I exactly as we have done for the case $n > 2$, yielding

$$\mathrm{I} := \int_{-t/2}^{0} \frac{|f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u + \int_0^{t/2} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u \lesssim \frac{e^{-1/t}}{t^{1+\beta}} + \int_0^{t/2}\frac{|f_1(u)|}{t^{1+\beta}}\,\mathrm{d}u$$
$$\le \frac{e^{-1/t}}{t^{1+\beta}} + \frac{1}{t^{1+\beta}}\left[e^{-\frac{1}{2t}}\int_0^1 \frac{e^{-\frac{1}{2u}}}{u}\,\mathrm{d}u + e^{-2/t}\int_1^{t/2}\frac{\mathrm{d}u}{u}\right] \lesssim \frac{e^{-\frac{1}{2t}}}{t^{1+\beta}} + \frac{e^{-2/t}\ln(t)}{t^{1+\beta}}.$$

Apply the following bounds:

$$e^{-\frac{1}{2t}} \le D_1(\alpha,\beta)\,\min\{t^{1+\beta}, t^{\alpha/2}\},$$
$$e^{-2/t}\,\ln t \le D_2(\alpha,\beta)\,\min\{t^{1+\beta}, t^{\alpha/2}\}.$$

Observe also that for the second bound to hold we need $\alpha > 0$ so that the term $t^{\alpha/2}$ does not become constant. The previous estimates imply $\mathrm{I} \lesssim \min\{1, t^{-1-\beta+\alpha/2}\}$, the desired estimate. Concerning II, we have

$$\mathrm{II} \lesssim_\beta \frac{2e^{-\frac{1}{2t}}}{t^{1+\beta}} + \frac{e^{-\frac{1}{2t}}}{t^{(3+2\beta)/2}},$$

and using one of the previous bounds together with

$$e^{-\frac{1}{2t}} \le D_3(\alpha,\beta)\,\min\{t^{(3+2\beta)/2}, t^{1+\alpha/2}\},$$

we also deduce $\mathrm{II} \lesssim \min\{1, t^{-(3-\alpha)/2}\}$. Finally, for III we now have

$$\mathrm{III} \lesssim \frac{e^{-1/t}}{t^{1+\beta}} + \int_{2t}^\infty \frac{e^{-1/u}}{u^{2+\beta}}\,\mathrm{d}u.$$

This last integral is clearly bounded by a constant and since $u > 2t$ it also satisfies

$$\int_{2t}^\infty \frac{e^{-1/u}}{u^{2+\beta}}\,\mathrm{d}u = \int_{2t}^\infty \frac{e^{-1/u}}{u^1 u^{1+\beta-\alpha/2}u^{\alpha/2}}\,\mathrm{d}u \lesssim \frac{1}{t^{1+\beta-\alpha/2}}\int_{2t}^\infty \frac{e^{-1/u}}{u^{1+\alpha/2}}\,\mathrm{d}u \lesssim_\alpha \frac{1}{t^{1+\beta-\alpha/2}},$$

and thus $\mathrm{III} \lesssim_{\beta,\alpha} \min\{1, t^{-1-\beta-\alpha/2}\}$ holds, and we obtain the desired result. □

Proof of Lemma 1.1.6. We write $K_\beta(\overline{x}) := \partial_t^\beta W(\overline{x})$. Regarding estimate *1*, by the same reasoning presented at the beginning of the proof of [**MatPT, Lemma 2.1**] we get

$$K_\beta(\overline{x}) \simeq \frac{1}{|x|^{n+2\beta}}\partial_t^\beta f_1\left(\frac{4t}{|x|^2}\right).$$

Hence, if $n>2$, by Lemma 1.1.5 we get

$$|K_\beta(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n+2\beta}}\min\left\{1,\frac{|x|^{2+2\beta}}{|t|^{1+\beta}}\right\} = \frac{1}{|x|^{n-2}|\overline{x}|_{p_1}^{2+2\beta}}.$$

For estimates *2* and *3* we follow the same procedure.

We move on to estimate *4*. First, observe that the expression $\nabla_x K$ is well-defined. Indeed, let us check if can let the operator ∇_x inside the integral of ∂_t^β and study if the resulting term is bounded by an integrable function locally for each $x\neq 0$. More explicitly, we aim to prove that for each $i=1,\ldots,n$, and each $x\neq 0$

$$\int_{\mathbb{R}} \frac{|\partial_{x_i}W(x,u)-\partial_{x_i}W(x,t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u<\infty.$$

Split the latter in two domains of integration

$$\int_{|u-t|<1} \frac{|\partial_{x_i}W(x,u)-\partial_{x_i}W(x,t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u + \int_{|u-t|>1} \frac{|\partial_{x_i}W(x,u)-\partial_{x_i}W(x,t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u =: \mathrm{I}+\mathrm{II}.$$

For the first, apply the mean value theorem with respect to the time variable and [**MatPT, Lemma 5.4**] to obtain that

$$\begin{aligned}\mathrm{I} &\lesssim \int_{|u-t|<1} \frac{\big|\partial_t\big(\partial_{x_i}W(x,\widetilde{u}\,)\big)\big|}{|u-t|^{\beta}}\,\mathrm{d}u \leq \int_{|u-t|<1}\frac{1}{|(x,\widetilde{u}\,)|_{p_1}^{n+3}|u-t|^\beta}\,\mathrm{d}u \\ &\leq \frac{1}{|x|^{n+3}}\int_{|u-t|<1}\frac{\mathrm{d}u}{|u-t|^\beta}<\infty, \end{aligned}\tag{1.1.23}$$

where $\widetilde{u}$ is a point in the line segment joining u and t. Regarding II, observe that the explicit expression for $\partial_{x_i}W(x,t)$, where here (x,t) is a generic point, satisfies

$$|\partial_{x_i}W(x,t)| = C|x_i|\,\frac{e^{-|x|^2/(4t)}}{t^{n/2+1}}\,\chi_{t>0}, \qquad \text{for some } C(n)>0. \tag{1.1.24}$$

For every fixed $x\neq 0$, the previous expression as a function of one variable attains a global maximum for $t=|x|^2/(2n+4)$, implying that

$$|\partial_{x_i}W(x,t)| \leq C\frac{[2(n+2)]^{n/2+1}}{e^{2(n+2)}}\frac{|x_i|}{|x|^{n+2}} \lesssim \frac{1}{|x|^{n+1}}.$$

Therefore

$$\mathrm{II}\lesssim \frac{1}{|x|^{n+1}}\int_{|u-t|>1}\frac{\mathrm{d}u}{|u-t|^{1+\beta}}<\infty,$$

and with this we conclude that the expression for $\nabla_x K(\overline{x})$ is well-defined and, in fact, our previous argument shows that the operators ∇_x and ∂_t^β commute when applied to W.

Notice that (1.1.24) implies that there is a constant C such that

$$\nabla_x W(x,t) = C\frac{x}{(4t)^{n/2+1}}\, e^{-|x|^2/(4t)}\,\chi_{t>0} = C\frac{x}{|x|^{n+2}}\left(\frac{|x|^2}{4t}\right)^{n/2+1} e^{-|x|^2/(4t)}\,\chi_{t>0},$$

so we can write

$$\nabla_x W(x,t) = C\frac{x}{|x|^{n+2}}\, f_2\left(\frac{4t}{|x|^2}\right),$$

with f_2 defined in Lemma 1.1.5. Since ∇_x and ∂_t^β commute,

$$\nabla_x K(x,t) = C\frac{x}{|x|^{n+2}}\,\partial_t^\beta\left[f_2\left(\frac{4\,\cdot}{|x|^2}\right)\right](t).$$

The previous fractional derivative can be written as follows

$$\partial_t^\beta\left[f_2\left(\frac{4\,\cdot}{|x|^2}\right)\right](t) = \int_{\mathbb{R}}\frac{f_2(4u/|x|^2)-f_2(4t/|x|^2)}{|u-t|^{1+\beta}}\,\mathrm{d}u \simeq \frac{1}{|x|^{2\beta}}\partial_t^\beta f_2\left(\frac{4t}{|x|^2}\right),$$

yielding the final equality

$$\nabla_x K(\overline{x}) = C\frac{x}{|x|^{n+2+2\beta}}\,\partial_t^\beta f_2\left(\frac{4t}{|x|^2}\right).$$

Applying Lemma 1.1.5 we finally deduce *3*:

$$|\nabla_x K(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n+1+2\beta}}\min\left\{1,\frac{|x|^{2+2\beta}}{|t|^{1+\beta}}\right\} = \frac{1}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}.$$

Concerning inequality *4*, since the operators ∂_t^β and ∂_t commute, we directly have

$$\partial_t K(\overline{x}) = \int_{\mathbb{R}}\frac{\partial_t W(x,u)-\partial_t W(x,t)}{|u-t|^{1+\beta}}\,\mathrm{d}u,$$

provided this last integral makes sense. Let us see that this is the case. Indeed, proceeding in the same way as we have done when studying $\nabla_x K$, we reach the following expression

$$\int_{|u-t|<1}\frac{|\partial_t W(x,u)-\partial_t W(x,t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u + \int_{|u-t|>1}\frac{|\partial_t W(x,u)-\partial_t W(x,t)|}{|u-t|^{1+\beta}}\,\mathrm{d}u = \mathrm{I}+\mathrm{II}.$$

For I we apply the mean value theorem and bound the resulting second derivative $\partial_t^2 W$ with analogous techniques to those used in the proof of [**MatPT, Lemma 2.1**]. Indeed, observe that there exist dimensional constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned}|\partial_t^2 W(x,t)| &\leq c_1\left|\frac{e^{-|x|^2/(4t)}}{t^{n/2+2}}\right| + c_2|x|^2\left|\frac{e^{-|x|^2/(4t)}}{t^{n/2+3}}\right| + c_3|x|^4\left|\frac{e^{-|x|^2/(4t)}}{t^{n/2+4}}\right| \\ &\leq \frac{C_1}{|\overline{x}|_{p_1}^{n+4}} + \frac{C_2}{|\overline{x}|_{p_1}^{n+4}} + \frac{C_3}{|\overline{x}|_{p_1}^{n+4}} \lesssim \frac{1}{|\overline{x}|_{p_1}^{n+4}},\end{aligned}$$

where here C_1, C_2, C_3 are dimensional constants chosen so that

$$\begin{aligned} e^{-|y|} &\leq C_1 \min\left\{1, |y|^{-n/2-2}\right\}, \\ e^{-|y|} &\leq C_2 \min\left\{|y|^{-1}, |y|^{-n/2-3}\right\}, \\ e^{-|y|} &\leq C_3 \min\left\{|y|^{-3}, |y|^{-n/2-4}\right\}. \end{aligned}$$

Hence, we can proceed as in equation (1.1.23) to estimate I. To deal with II we use that

$$|\partial_t W(x,t)| = \left| C_1 \frac{e^{-|x|^2/(4t)}}{t^{n/2+1}} + C_2|x|^2 \frac{e^{-|x|^2/(4t)}}{t^{n/2+2}} \right| \chi_{t>0} \lesssim \frac{1}{|\overline{x}|_{p_1}^{n+2}}, \tag{1.1.25}$$

by the same arguments used to bound (1.1.24). So, indeed, the expression for $\partial_t K$ is well-defined.

As we have done for $\nabla_x W$, the identity obtained in (1.1.25) can be rewritten for $t > 0$ in the form

$$\begin{aligned} \partial_t W(x,t) &= C_1 \frac{e^{-|x|^2/(4t)}}{t^{n/2+1}} + C_2|x|^2 \frac{e^{-|x|^2/(4t)}}{t^{n/2+2}} \\ &= \left[\frac{C_1'}{|x|^{n+2}} \left(\frac{|x|^2}{4t} \right)^{n/2+1} + \frac{C_2'}{|x|^{n+2}} \left(\frac{|x|^2}{4t} \right)^{n/2+2} \right] e^{-|x|^2/(4t)} \\ &= \frac{C_1'}{|x|^{n+2}} f_2\left(\frac{4t}{|x|^2} \right) + \frac{C_2'}{|x|^{n+2}} f_3\left(\frac{4t}{|x|^2} \right), \end{aligned}$$

where f_3 is defined in Lemma 1.1.5. By exactly the same change of variables as the one performed when studying $\nabla_x K$, we reach the identity

$$\partial_t K(\overline{x}) = \frac{C_1'}{|x|^{n+2+2\beta}} \partial_t^\beta f_2\left(\frac{4t}{|x|^2} \right) + \frac{C_2'}{|x|^{n+2+2\beta}} \partial_t^\beta f_3\left(\frac{4t}{|x|^2} \right).$$

By Lemma 1.1.5, we get inequality *4*:

$$|\partial_t K(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n+2+2\beta}} \min\left\{ 1, \frac{|x|^{2+2\beta}}{|t|^{1+\beta}} \right\} = \frac{1}{|x|^n |\overline{x}|_{p_1}^{2+2\beta}}.$$

Finally, regarding *5*, we follow exactly the same proof as that of estimate *4* in Lemma 1.1.4. □

Therefore, we can sum up both of the previous lemmas in the following result:

THEOREM 1.1.7. *Let $\beta \in (0,1)$ and $s \in (0,1]$. Then, the following hold for any $\overline{x} \neq (0,t)$:*

1. *If $n = 1$, whenever $s < 1$ we get*

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{1-2s+\alpha} |\overline{x}|_{p_s}^{2s(1+\beta)-\alpha}}, \quad \forall \alpha \in (2s-1, 4s) \;\; \text{and} \;\; \beta > 1 - \frac{1}{2s}.$$

Moreover, if $s = 1$,

$$|\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|\overline{x}|_{p_s}^{1+2\beta}}.$$

2. *If $n = 2$, whenever $s < 1$ we have*

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{2-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}},$$

and if $s = 1$,

$$|\partial_t^\beta W(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{\alpha}|\overline{x}|_{p_1}^{2+2\beta-\alpha}}, \quad \forall \alpha \in (0, 2+2\beta].$$

3. *If $n > 2$,*

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

4. *For every n,*

$$|\nabla_x \partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}}, \qquad |\partial_t \partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n}|\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

If $s < 1$, the bound of $\partial_t\partial_t^\beta P_s$ is only valid for points with $t \neq 0$.

5. *Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ is such that $|\overline{x} - \overline{x}'|_{p_s} \leq |x|/2$,*

$$|\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\overline{x}')| \lesssim_\beta \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

1.2 Growth estimates for admissible functions

We will say that a positive Borel measure μ in $\mathbb{R}^{n+1}$ has upper s-parabolic *growth of degree ρ* (*with constant C*) or simply *s-parabolic ρ-growth* if there is some constant $C(n,s) > 0$ such that for any s-parabolic ball $B(\overline{x}, r)$ in $\mathbb{R}^{n+1}$,

$$\mu\big(B(\overline{x}, r)\big) \leq Cr^\rho.$$

It is clear that this property is invariant if formulated using cubes instead of balls. We will be interested in a generalized version of such growth that can be defined not only for measures, but also for general distributions. To introduce such notion we present the concept of admissible function:

Definition 1.2.1. Let $s \in (0,1)$. Given $\phi \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, we will say that it is an *admissible* function for an s-parabolic cube Q in $\mathbb{R}^{n+1}$ if $\text{supp}(\phi) \subset Q$ and

$$\|\phi\|_\infty \leq 1, \qquad \|\nabla_x \phi\|_\infty \leq \ell(Q)^{-1}, \qquad \|\partial_t \phi\|_\infty \leq \ell(Q)^{-2s}, \qquad \|\Delta\phi\|_\infty \leq \ell(Q)^{-2}.$$

Remark 1.2.1. If ϕ is a $\mathcal{C}^2$ function supported on Q s-parabolic cube with $\|\phi\|_\infty \leq 1$, $\|\nabla_x\phi\|_\infty \leq \ell(Q)^{-1}$ and $\|\Delta\phi\|_\infty \leq \ell(Q)^{-2}$, then it also satisfies

$$\|(-\Delta)^s\phi\|_\infty \lesssim \ell(Q)^{-2s}.$$

Indeed, begin by observing that translations in $\mathbb{R}^n$ commute with ∇_x and $(-\Delta)^s$. From it, it is clear that we may assume Q to be centered at the origin. Assuming this, let us fix $t \in \mathbb{R}$ and compute

$$\begin{aligned}(-\Delta)^s\phi(x,t) &:= c_{n,s}\int_{\mathbb{R}^n}\frac{\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)}{|y|^{n+2s}}\,\mathrm{d}y\\ &= c_{n,s}\int_{2Q}\frac{\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)}{|y|^{n+2s}}\,\mathrm{d}y\\ &\qquad c_{n,s}\int_{\mathbb{R}^n\setminus 2Q}\frac{\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)}{|y|^{n+2s}}\,\mathrm{d}y =: \mathrm{I}+\mathrm{II}.\end{aligned}$$

Regarding II, integration in polar coordinates yields

$$|\mathrm{II}| \leq 4c_{n,s}\int_{\mathbb{R}^n\setminus 2Q}\frac{\mathrm{d}y}{|y|^{n+2s}} \lesssim \ell(Q)^{-2s}.$$

For I, we apply twice the mean value theorem so that it can be also bounded as follows

$$c_{n,s}\int_{2Q}\frac{|\langle\nabla_x\phi(x+\eta_1 y,t),y\rangle+\langle\nabla_x\phi(x-\eta_2 y,t),y\rangle|}{|y|^{n+2s-1}} \lesssim \int_{2Q}\frac{\|\Delta\phi\|_\infty}{|y|^{n+2s-2}}\,\mathrm{d}y \lesssim \ell(Q)^{-2s}.$$

Definition 1.2.2. We will say that a distribution T in $\mathbb{R}^{n+1}$ has s-parabolic *n-growth* if there exists some constant $C=C(n,s)>0$ such that, given any s-parabolic cube $Q\subset\mathbb{R}^{n+1}$ and any function ϕ admissible for Q, we have

$$|\langle T,\phi\rangle| \leq C\ell(Q)^n.$$

In the end, the results below will help us estimate the growth of distributions of the form φT, for some particular choices of T and a fixed admissible function φ, associated with a fixed s-parabolic cube.

In any case, let us clarify that in the following Theorems 1.2.1, 1.2.2, 1.2.3 and 1.2.4, we will fix $s\in(0,1]$ and Q and R will be s-parabolic cubes in $\mathbb{R}^{n+1}$ with $Q\cap R\neq\varnothing$. We will write $Q:=Q_1\times I_Q\subset\mathbb{R}^n\times\mathbb{R}$ and analogously for R. Moreover, φ and ϕ will denote $\mathcal{C}^1$ functions with $\mathrm{supp}(\varphi)\subset Q$, $\mathrm{supp}(\phi)\subset R$ and such that $\|\varphi\|_\infty\leq 1$ and $\|\phi\|_\infty\leq 1$.

Theorem 1.2.1. *Let $\beta\in(0,1)$, $\alpha\in(0,1)$ and $f:\mathbb{R}^{n+1}\to\mathbb{R}$. Assume $\|\partial_t\varphi\|_\infty\leq\ell(Q)^{-2s}$ and $\|\partial_t\phi\|_\infty\leq\ell(R)^{-2s}$. Then, if $\ell(R)\leq\ell(Q)$,*

1. *If $f\in\mathrm{BMO}_{p_s}$,*

$$|\langle f,\partial_t(\varphi\phi)*_t|t|^{-\beta}\rangle| \lesssim_\beta \|f\|_{*,p_s}\ell(R)^{n+2s(1-\beta)}.$$

2. *If $f\in\mathrm{Lip}_{\alpha,p_s}$ and $\alpha<2s\beta$,*

$$|\langle f,\partial_t(\varphi\phi)*_t|t|^{-\beta}\rangle| \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}}\ell(R)^{n+2s(1-\beta)+\alpha}.$$

Proof. Set $g := \partial_t(\varphi\phi) *_t |t|^{-\beta}$ and begin by proving that g is integrable. Firstly, observe that if $c_{Q\cap R}$ is the center of $I_Q \cap I_R$, then for each $t \notin 2(I_Q \cap I_R)$ we get

$$\begin{aligned}
|g(x,t)| &= \left| \int_{I_Q\cap I_R} \frac{\partial_t(\varphi\phi)(x,u)}{|t-u|^\beta} \, \mathrm{d}u \right| \\
&\le \int_{I_Q\cap I_R} |\partial_t(\varphi\phi)(x,u)| \left| \frac{1}{|t-u|^\beta} - \frac{1}{|t-c_{Q\cap R}|^\beta} \right| \mathrm{d}u \\
&\lesssim \frac{\ell(I_Q\cap I_R)}{|t-c_{Q\cap R}|^{1+\beta}} \int_{I_Q\cap I_R} |\partial_t(\varphi\phi)(x,u)| \, \mathrm{d}u \\
&\lesssim_\beta \frac{\ell(I_Q\cap I_R)}{|t-c_{Q\cap R}|^{1+\beta}} \left(\frac{1}{\ell(Q)^{2s}} + \frac{1}{\ell(R)^{2s}} \right) \ell(I_Q\cap I_R) \lesssim \frac{\ell(I_Q\cap I_R)}{|t-c_{Q\cap R}|^{1+\beta}}. \qquad (1.2.1)
\end{aligned}$$

That is, $|g|$ decays as $|t|^{-1-\beta}$ for large values of t. Hence, since $\operatorname{supp}(g) \subset (Q_1 \cap R_1) \times \mathbb{R}$, this implies $g \in L^1(\mathbb{R}^{n+1})$. Then, for any constant $c \in \mathbb{R}$ we have

$$|\langle f, g\rangle| = \left| \int (f-c) g \right| \le \int_{2R} |f-c||g| + \int_{\mathbb{R}^{n+1}\setminus 2R} |f-c||g| =: \mathrm{I} + \mathrm{II},$$

where we have used that g has null integral (it can be easily checked taking the Fourier transform, for example). To study I, observe that for $t \in 4I_R$ we get

$$|g(x,t)| \le \|\partial_t(\varphi\phi)\|_\infty \int_{-5\ell(I_R)}^{5\ell(I_R)} \frac{\mathrm{d}u}{|u|^\beta} \lesssim_\beta \left(\frac{1}{\ell(R)^{2s}} + \frac{1}{\ell(Q)^{2s}} \right) \ell(I_R)^{1-\beta} \lesssim \ell(R)^{-2s\beta},$$

since $\ell(R) \le \ell(Q)$. Therefore,

$$\mathrm{I} \lesssim_\beta \frac{1}{\ell(R)^{2s\beta}} \int_{2R} |f-c|.$$

If $f \in \mathrm{BMO}_{p_s}$, pick $c := f_{2R}$, the average of f over $2R$, so that

$$\mathrm{I} \lesssim_\beta \ell(R)^{n+2s(1-\beta)} \|f\|_{*,p_s}.$$

If $f \in \mathrm{Lip}_{\alpha,p_s}$, pick $c := f(\overline{x}_R)$, where $\overline{x}_R$ is the center of $2R$, so that

$$\mathrm{I} \lesssim_{\beta,\alpha} \ell(R)^{n+2s(1-\beta)+\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}},$$

and we are done with I. To study II, define the s-parabolic annuli $A_j := 2^j R \setminus 2^{j-1} R$ for $j \ge 2$. Then, since $\operatorname{supp}(g) \subset (Q_1 \cap R_1) \times \mathbb{R}$ applying 1.2.1 we have

$$\begin{aligned}
\mathrm{II} &= \sum_{j=2}^\infty \int_{A_j \cap \operatorname{supp}(g)} |f(\overline{x}) - c||g(\overline{x})| \, \mathrm{d}\overline{x} \\
&\lesssim_\beta \frac{1}{\ell(R)^{2s\beta}} \sum_{j=2}^\infty \frac{1}{2^{2s(1+\beta)j}} \int_{A_j\cap\operatorname{supp}(g)} |f(\overline{x}) - c| \, \mathrm{d}\overline{x}. \qquad (1.2.2)
\end{aligned}$$

If $f \in \mathrm{BMO}_{p_s}$, pick again $c := f_{2R}$ and observe that the previous expression can be bounded by

$$\frac{1}{\ell(R)^{2s\beta}} \sum_{j=2}^{\infty} \frac{1}{2^{2s(1+\beta)j}} \bigg(\int_{A_i\cap \mathrm{supp}(g)} |f(\overline{x}) - f_{2^jR}| \, \mathrm{d}\overline{x} + \int_{A_j \cap \mathrm{supp}(g)} |f_{2^jR} - f_{2R}| \, \mathrm{d}\overline{x} \bigg),$$

Regarding the first integral, apply Hölder's inequality (with q, to be fixed later on) and John-Nirenberg's, so that

$$\begin{aligned} \int_{A_i\cap \mathrm{supp}(g)} |f(\overline{x}) - f_{2^jR}| \, \mathrm{d}\overline{x} &\leq \bigg(\int_{A_j\cap \mathrm{supp}(g)} |f(\overline{x}) - f_{2^jR}|^q \, \mathrm{d}\overline{x} \bigg)^{\frac{1}{q}} |\mathrm{supp}(g) \cap 2^jR|^{\frac{1}{q'}} \\ &\leq \|f\|_{*,p_s} (2^j\ell(R))^{\frac{n+2s}{q}} \big[2^{2sj}\ell(R)^{n+2s}\big]^{\frac{1}{q'}} = \|f\|_{*,p_s} 2^{j(\frac{n}{q}+2s)} \ell(R)^{n+2s}. \end{aligned}$$

For the second integral we apply [**Gar2, Ch.VI, Lemma 1.1**] to deduce $|f_{2^jR} - f_{2R}| \lesssim j\|f\|_{*,p_s} \leq j$, so that

$$\int_{A_j\cap \mathrm{supp}(g)} |f_{2^jR} - f_{2R}| \, \mathrm{d}\overline{x} \lesssim j \, |\mathrm{supp}(g) \cap 2^jR| = j\|f\|_{*,p_s} \, 2^{2sj}\ell(R)^{n+2s}.$$

Therefore, choosing $q > \frac{n}{2s\beta}$,

$$\mathrm{II} \lesssim_\beta \frac{\|f\|_{*,p_s}}{\ell(R)^{2s\beta}} \sum_{j=2}^{\infty} \frac{1}{2^{2s(1+\beta)j}} (2^{j(\frac{n}{q}+2s)} + j2^{2sj}) \ell(R)^{n+2s} \lesssim \|f\|_{*,p_s} \ell(R)^{n+2s(1-\beta)}.$$

If on the other hand $f \in \mathrm{Lip}_{\alpha,p_s}$, pick $c := f(\overline{x}_R)$ so that Hölder's inequality in (1.2.2) yields

$$\begin{aligned} \mathrm{II} &\lesssim_\beta \frac{\|f\|_{\mathrm{Lip}_{\alpha,p_s}}}{\ell(R)^{2s\beta}} \sum_{j=2}^{\infty} \frac{1}{2^{2s(1+\beta)j}} \big(2^j\ell(R)\big)^\alpha |\mathrm{supp}(g) \cap 2^jR| \\ &\lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \ell(R)^{n+2s(1-\beta)+\alpha} \sum_{j=2}^{\infty} \frac{1}{2^{(2s\beta-\alpha)j}}, \end{aligned}$$

being this last sum convergent because $\alpha < 2s\beta$, so we are done. □

THEOREM 1.2.2. *Let $\alpha \in (0,1)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume $\|\nabla_x \varphi\|_\infty \leq \ell(Q)^{-1}$ and $\|\nabla_x \phi\|_\infty \leq \ell(R)^{-1}$. Then, if $\ell(R) \leq \ell(Q)$, for each $i = 1, \dots, n$ we have*

1. *If $f \in \mathrm{BMO}_{p_s}$,*
$$|\langle f, \partial_{x_i}(\varphi\phi)\rangle| \lesssim_\beta \|f\|_{*,p_s} \ell(R)^{n+2s-1}.$$

2. *If $f \in \mathrm{Lip}_{\alpha,p_s}$,*
$$|\langle f, \partial_{x_i}(\varphi\phi)\rangle| \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \ell(R)^{n+2s-1+\alpha}.$$

Proof. First, observe that for any real constant c, we have the identity

$$\langle f, \partial_{x_i}(\varphi\phi)\rangle = \langle f - c, \partial_{x_i}(\varphi\phi)\rangle,$$

Therefore,

$$\begin{aligned}
\langle f, \partial_{x_i}(\varphi\phi)\rangle &= \Big|\int_{Q\cap R} f(\overline{x})\partial_{x_i}(\varphi\phi)(\overline{x})\,\mathrm{d}\overline{x}\Big| \leq \int_{Q\cap R} |f(\overline{x}) - c|\big|\partial_{x_i}(\varphi\phi)(\overline{x})\big|\,\mathrm{d}\overline{x}\\
&\leq \Big(\int_R |f(\overline{x}) - c|^2\,\mathrm{d}\overline{x}\Big)^{1/2}\Big(\int_{Q\cap R} |\partial_{x_i}(\varphi\phi)(\overline{x})|^2\,\mathrm{d}\overline{x}\Big)^{1/2}\\
&\lesssim \Big(\int_R |f(\overline{x}) - c|^2\,\mathrm{d}\overline{x}\Big)^{1/2}\Big(\int_{Q\cap R} \Big[\|\nabla_x\varphi\|_\infty^2\|\phi\|_\infty^2 + \|\varphi\|_\infty^2\|\nabla_x\phi\|_\infty^2\Big]\,\mathrm{d}\overline{x}\Big)^{1/2}\\
&\leq \Big(\int_R |f(\overline{x}) - c|^2\,\mathrm{d}\overline{x}\Big)^{1/2} |Q\cap R|^{1/2}\Big(|Q|^{-\frac{1}{n+2s}} + |R|^{-\frac{1}{n+2s}}\Big)\\
&\leq \Big(\int_R |f(\overline{x}) - c|^2\,\mathrm{d}\overline{x}\Big)^{1/2}\Big(|Q\cap R|^{\frac{n+2s-2}{2(n+2s)}}\frac{|Q\cap R|^{\frac{1}{n+2s}}}{\ell(Q)} + \ell(R)^{\frac{n+2s}{2}-1}\Big)\\
&\leq \Big(\int_R |f(\overline{x}) - c|^2\,\mathrm{d}\overline{x}\Big)^{1/2}\ell(R)^{\frac{n+2s}{2}-1}.
\end{aligned}$$

Now, if $f \in \mathrm{BMO}_{p_s}$, choose $c := f_R$ and apply an s-parabolic version of John-Nirenberg's inequality (that admits an analogous proof) to deduce estimate *1.* On the other hand, if $f \in \mathrm{Lip}_{\alpha,p_s}$, choose $c := f(\overline{x}_R)$ to obtain estimate *2.* □

THEOREM 1.2.3. *Let $\beta \in (0,1)$, $\alpha \in (0,1)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume that φ and ϕ are $\mathcal{C}^2$ with $\|\nabla_x\varphi\|_\infty \leq \ell(Q)^{-1}$, $\|\Delta\varphi\|_\infty \leq \ell(Q)^{-2}$ and $\|\nabla_x\phi\|_\infty \leq \ell(R)^{-1}$, $\|\Delta\phi\|_\infty \leq \ell(R)^{-2}$. Then, if $\ell(R) \leq \ell(Q)$,*

1. *If $f \in \mathrm{BMO}_{p_s}$,*

$$|\langle f, (-\Delta)^\beta(\varphi\phi)\rangle| \lesssim_\beta \|f\|_{*,p_s}\ell(R)^{n+2(s-\beta)}.$$

2. *If $f \in \mathrm{Lip}_{\alpha,p_s}$ and $\alpha < 2\beta$,*

$$|\langle f, (-\Delta)^\beta(\varphi\phi)\rangle| \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}}\ell(R)^{n+2(s-\beta)+\alpha}.$$

Proof. Observe that for any real constant c,

$$\begin{aligned}
|\langle f, (-\Delta)^\beta(\varphi\phi)\rangle| &= |\langle f - c, (-\Delta)^\beta(\varphi\phi)\rangle|\\
&\leq \int_{2R_1\times(I_Q\cap I_R)} |f(\overline{x}) - c|\big|(-\Delta)^\beta(\varphi\phi)(\overline{x})\big|\,\mathrm{d}\overline{x}\\
&\quad + \int_{(\mathbb{R}^n\setminus 2R_1)\times(I_Q\cap I_R)} |f(\overline{x}) - c|\big|(-\Delta)^\beta(\varphi\phi)(\overline{x})\big|\,\mathrm{d}\overline{x} =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Regarding I, observe that for any $\overline{x}\in\mathbb{R}^{n+1}$ by Remark 1.2.1 we have $|(-\Delta)^{\beta}(\varphi\phi)(\overline{x})|\lesssim_\beta \ell(R)^{-2\beta}$. Therefore,

$$\mathrm{I}\lesssim_\beta \frac{1}{\ell(R)^{2\beta}}\int_{2R_1\times(I_Q\cap I_R)}|f(\overline{x})-c|\,\mathrm{d}\overline{x}.$$

Let $\overline{x}_0$ be the center of $2R_1\times(I_Q\cap I_R)$. Choosing $c:=f_{2R}$ or $c:=f(\overline{x}_0)$ for $f\in\mathrm{BMO}_{p_s}$ or $f\in\mathrm{Lip}_{\alpha,p_s}$ respectively, we obtain the desired estimates.

Let us turn to II. We first notice that, taking the Fourier transform, the operator $(-\Delta)^\beta$ can be rewritten as

$$(-\Delta)^\beta(\cdot)\simeq_\beta\sum_{j=1}^n\partial_{x_j}\bigg(\frac{1}{|x|^{n+2\beta-2}}\bigg)*_n\partial_{x_j}(\cdot),$$

where the notation $*_n$ is used to stress that the convolution is taken with respect the first n spatial variables. With this, if $x_0\in\mathbb{R}^n$ denotes the center of $Q_1\cap R_1$, for any $\overline{x}\in(\mathbb{R}^n\setminus 2R_1)\times(I_Q\cap I_R)$ we get

$$\begin{aligned}\big|(-\Delta)^\beta(\varphi\phi)(\overline{x})\big|&\lesssim_\beta\sum_{j=1}^n\bigg|\int_{Q_1\cap R_1}\partial_j(\varphi\phi)(z,t)\frac{z_j-x_j}{|z-x|^{n+2\beta}}\,\mathrm{d}z\bigg|\\&=\sum_{j=1}^n\bigg|\int_{Q_1\cap R_1}\partial_j(\varphi\phi)(z,t)\bigg(\frac{z_j-x_j}{|z-x|^{n+2\beta}}-\frac{x_{0,j}-x_j}{|x_0-x|^{n+2\beta}}\bigg)\,\mathrm{d}z\bigg|\\&\lesssim_\beta\sum_{j=1}^n\frac{\ell(R)}{|x_0-x|^{n+2\beta}}\|\nabla_x(\varphi\phi)\|_\infty\ell(R)^n\lesssim\frac{\ell(R)^n}{|x_0-x|^{n+2\beta}},\end{aligned}\tag{1.2.3}$$

by the mean value theorem. So, defining the cylinders $C_j:=2^jR_1\times(I_Q\cap I_R)$ for $j\geq 1$, relation (1.2.3) implies

$$\mathrm{II}\lesssim_\beta\frac{1}{\ell(R)^{2\beta}}\sum_{j=1}^\infty\frac{1}{2^{j(n+2\beta)}}\int_{C_{j+1}\setminus C_j}|f(\overline{x})-c|\,\mathrm{d}\overline{x},$$

If $f\in\mathrm{BMO}_{p_s}$, we choose $c:=f_{2R}$ and proceed as in Theorem 1.2.1,

$$\begin{aligned}\mathrm{II}&\lesssim_\beta\frac{1}{\ell(R)^{2\beta}}\sum_{j=1}^\infty\frac{1}{2^{j(n+2\beta)}}\bigg(\int_{C_{j+1}\setminus C_j}|f(\overline{x})-f_{2^jR}|\,\mathrm{d}\overline{x}+\int_{C_{j+1}\setminus C_j}|f_{2R}-f_{2^jR}|\,\mathrm{d}\overline{x}\bigg)\\&\lesssim\frac{\|f\|_{*,p_s}}{\ell(R)^{2\beta}}\sum_{j=1}^\infty\frac{1}{2^{j(n+2\beta)}}\Big[\big(2^j\ell(R)\big)^{\frac{n+2s}{q}}|C_{j+1}\setminus C_j|^{\frac{1}{q'}}+j|C_{j+1}\setminus C_j|\Big]\\&\lesssim\frac{\|f\|_{*,p_s}}{\ell(R)^{2\beta}}\sum_{j=1}^\infty\frac{1}{2^{j(n+2\beta)}}\Big[\ell(R)^{n+2s}2^{j(n+\frac{2s}{q})+\frac{2s}{q'}}+j\ell(R)^{n+2s}\,2^{jn+2s}\Big]\\&\lesssim\|f\|_{*,p_s}\ell(R)^{n+2(s-\beta)}\bigg(1+\sum_{j=1}^\infty\frac{2^{\frac{2}{q'}}}{2^{j(2\beta-\frac{2s}{q})}}\bigg).\end{aligned}$$

Fixing $q > s/\beta$ so that this last sum is convergent, proves the result.

On the other hand, if $f \in \mathrm{Lip}_{\alpha,p_s}$ let $c := f(\overline{x}_0)$ and also proceed as in Theorem 1.2.1 to deduce

$$\mathrm{II} \lesssim_{\beta,\alpha} \frac{\|f\|_{\mathrm{Lip}_{\alpha,p_s}}}{\ell(R)^{2\beta}} \sum_{j=1}^{\infty} \frac{(2^j \ell(R))^{\alpha}}{2^{j(n+2\beta)}} |C_{j+1} \setminus C_j| \lesssim \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \ell(R)^{n+2(s-\beta)+\alpha} \sum_{j=1}^{\infty} \frac{1}{2^{(2\beta-\alpha)j}},$$

that is a convergent sum since $\alpha < 2\beta$ by hypothesis. □

Recall that given $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $\beta \in (0,n)$, we define its n-dimensional β-Riesz transform (whenever it makes sense) as

$$\mathcal{I}_\beta^n f(\cdot,t) := \frac{1}{|x|^{n-\beta}} * f(\cdot,t),$$

for each t, where the convolution is thought in a principal value sense. Let us observe that for a test function f, for example, the operators $\mathcal{I}_\beta^n$ and ∂_{x_i} commute.

Theorem 1.2.4. *Let $\beta \in (0,1)$, $\alpha \in (0, 1-\beta)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume $\|\nabla_x \varphi\|_\infty \leq \ell(Q)^{-1}$ and $\|\nabla_x \phi\|_\infty \leq \ell(R)^{-1}$. Then, if $\ell(R) \leq \ell(Q)$, for each $i = 1,\ldots,n$ we have*

1. *If $f \in \mathrm{BMO}_{p_s}$,*
$$|\langle f, \partial_{x_i}[\mathcal{I}_\beta^n(\varphi\phi)]\rangle| \lesssim_\beta \|f\|_{*,p_s} \ell(R)^{n+2s+\beta-1}.$$
2. *If $f \in \mathrm{Lip}_{\alpha,p_s}$,*
$$|\langle f, \partial_{x_i}[\mathcal{I}_\beta^n(\varphi\phi)]\rangle| \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \ell(R)^{n+2s+\beta+\alpha-1}.$$

Proof. Notice that for any $c \in \mathbb{R}$,

$$\begin{aligned}
|\langle f, \partial_{x_i}[\mathcal{I}_\beta^n(\varphi\phi)]\rangle| &= |\langle f - c, \partial_{x_i}[\mathcal{I}_\beta^n(\varphi\phi)]\rangle| \\
&\leq \int_{2R_1 \times (I_Q \cap I_R)} |f(\overline{x}) - c| \big|\partial_{x_i}[\mathcal{I}_\beta^n(\varphi\phi)](\overline{x})\big| \,\mathrm{d}\overline{x} \\
&\quad + \int_{(\mathbb{R}^n \setminus 2R_1) \times (I_Q \cap I_R)} |f(\overline{x}) - c| \big|\partial_{x_i}[\mathcal{I}_\beta^n(\varphi\phi)](\overline{x})\big| \,\mathrm{d}\overline{x} =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Regarding I, we have for some conjugate exponents q, q' to be fixed later on,

$$\begin{aligned}
\mathrm{I} &\lesssim \Big(\int_{2R} |f(\overline{x}) - c|^{q'} \,\mathrm{d}\overline{x} \Big)^{\frac{1}{q'}} \Big(\int_{I_Q \cap I_R} \int_{2R_1} \big|\mathcal{I}_\beta^n[\partial_{x_i}(\varphi\phi)](x,t)\big|^q \,\mathrm{d}x \,\mathrm{d}t \Big)^{\frac{1}{q}} \\
&\lesssim \Big(\int_{2R} |f(\overline{x}) - c|^{q'} \,\mathrm{d}\overline{x} \Big)^{\frac{1}{q'}} \Big(\int_{I_Q \cap I_R} \|\mathcal{I}_\beta^n[\partial_{x_i}(\varphi\phi)(\cdot,t)]\|_q^q \,\mathrm{d}t \Big)^{\frac{1}{q}}.
\end{aligned}$$

Choosing $q > \frac{n}{n-\beta}$, we shall apply [**Gr, Theorem 6.1.3**] and obtain

$$\begin{aligned}
\mathrm{I} &\lesssim_\beta \bigg(\int_{2R} |f(\overline{x})-c|^{q'}\,\mathrm{d}\overline{x} \bigg)^{\frac{1}{q'}} \bigg(\int_{I_Q\cap I_R} \|\partial_{x_i}(\varphi\phi)(\cdot,t)\|^q_{\frac{qn}{n+q\beta}}\,\mathrm{d}t \bigg)^{\frac{1}{q}} \\
&\lesssim \bigg(\int_{2R} |f(\overline{x})-c|^{q'}\,\mathrm{d}\overline{x} \bigg)^{\frac{1}{q'}} \ell(R)^{\frac{n+q\beta+2s}{q}-1}.
\end{aligned}$$

If we assume $f \in \mathrm{BMO}_{p_s}$, we choose $c := f_{2R}$ and apply a s-parabolic version of John-Nirenberg's inequality to deduce

$$\mathrm{I} \lesssim_\beta \|f\|_{*,p_s} \ell(R)^{\frac{n+2s}{q'}} \ell(R)^{\frac{n+q\beta+2s}{q}-1} = \|f\|_{*,p_s}\ell(R)^{n+2s+\beta-1}.$$

If we assume $f \in \mathrm{Lip}_{\alpha,p_s}$, we choose $c := f(\overline{x}_R)$, being $\overline{x}_R$ the center of R, and obtain

$$\mathrm{I} \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_\alpha,p_s} \ell(R)^{\frac{n+2s}{q'}+\alpha} \ell(R)^{\frac{n+q\beta+2s}{q}-1} = \|f\|_{*,p_s}\ell(R)^{n+2s+\beta+\alpha-1}.$$

To study II, we proceed as in Theorem 1.2.3. For any $\overline{x} \in (\mathbb{R}^n \setminus 2R_1)\times(I_Q\cap I_R)$, if $x_0 \in \mathbb{R}^n$ denotes the center of $Q_1 \cap R_1$, by the mean value theorem we get

$$\begin{aligned}
\big|\mathcal{I}^n_\beta[\partial_{x_i}(\varphi\phi)](\overline{x})\big| &= \bigg| \int_{Q_1\cap R_1} \partial_{x_i}(\varphi\phi)(z,t)\frac{1}{|z-x|^{n-\beta}}\,\mathrm{d}z\bigg| \\
&= \bigg| \int_{Q_1\cap R_1} \partial_{x_i}(\varphi\phi)(z,t)\bigg(\frac{1}{|z-x|^{n-\beta}} - \frac{1}{|x_0-x|^{n-\beta}}\bigg)\,\mathrm{d}z\bigg| \\
&\lesssim_\beta \sum_{j=1}^n \frac{\ell(R)}{|x_0-x|^{n-\beta+1}}\|\nabla_x(\varphi\phi)\|_\infty\,\ell(R)^n \lesssim \frac{\ell(R)^n}{|x_0-x|^{n-\beta+1}}.
\end{aligned}$$

This way, putting $C_j := 2^jR_1\times(I_Q\cap I_R)$ for $j\geq 1$, as in Theorem 1.2.3,

$$\mathrm{II} \lesssim_\beta \frac{1}{\ell(R)^{-\beta+1}}\sum_{j=1}^\infty \frac{1}{2^{j(n-\beta+1)}}\int_{C_{j+1}\setminus C_j} \big|f(\overline{x})-c\big|\,\mathrm{d}\overline{x}.$$

The case $f\in\mathrm{BMO}_{p_s}$ is dealt with analogously as in Theorem 1.2.3, obtaining

$$\mathrm{II} \lesssim_\beta \|f\|_{*,p_s}\ell(R)^{n+2s+\beta-1}\sum_{j=1}^\infty \frac{1}{2^{j(n-\beta+1)}}\Big[2^{j(n+\frac{2s}{q})} + j\,2^{jn}\Big],$$

so choosing $q > \frac{2s}{1-\beta}$ we are done. Observe that we also need $\beta<1$ in order for the above sum to converge. The case $f\in\mathrm{Lip}_{\alpha,p_s}$ can be dealt with as follows

$$\mathrm{II} \lesssim_{\beta,\alpha} \frac{\|f\|_{\mathrm{Lip}_{\alpha,p_s}}}{\ell(R)^{-\beta+1}}\sum_{j=1}^\infty \frac{(2^j\ell(R))^\alpha}{2^{j(n-\beta+1)}}|C_{j+1}\setminus C_j| \lesssim \|f\|_{\mathrm{Lip}_{\alpha,p_s}}\sum_{j=1}^\infty \frac{\ell(R)^{n+2s+\beta+\alpha-1}}{2^{(1-\beta-\alpha)j}},$$

and this sum is convergent by the hypothesis $\alpha < 1-\beta$. $\square$

1.3 Potentials of positive measures with growth restrictions

The main goal of this section is to deduce some important BMO_{p_s} and $\mathrm{Lip}_{\alpha,p_s}$ estimates of potentials of the form $\partial_t^\beta P_s * \mu$, where μ is a finite positive Borel measure with some upper s-parabolic growth. We begin by proving a generalization of [**MatPT, Lemma 4.2**] and [**MatP, Lemma 7.2**].

LEMMA 1.3.1. *Let $s \in (0,1]$, $\eta \in (0,1)$ and μ be a positive measure in $\mathbb{R}^{n+1}$ which has upper s-parabolic growth of degree $n+2s\eta$. Then*

$$\|P_s * \mu\|_{\mathrm{Lip}_{\eta,t}} \lesssim_\eta 1.$$

Proof. Let $\overline{x} := (x,t), \widehat{x} := (x,\tau)$ be fixed points in $\mathbb{R}^{n+1}$ with $t \neq \tau$, and set $\overline{x}_0 := (\overline{x}+\widehat{x})/2$. Writing $\overline{y} := (y,u)$ and $B_0 := B(\overline{x}_0, |\overline{x}-\widehat{x}|_{p_s}) = B(\overline{x}_0, |t-\tau|^{\frac{1}{2s}})$, we split

$$\begin{aligned}
&|P_s * \mu(\overline{x}) - P_s * \mu(\widehat{x})| \\
&\qquad \leq \int_{\mathbb{R}^{n+1}\setminus 2B_0} |P_s(x-y,t-u) - P_s(x-y,\tau-u)| \mathrm{d}\mu(\overline{y}) \\
&\qquad\qquad + \int_{2B_0} |P_s(x-y,t-u) - P_s(x-y,\tau-u)| \mathrm{d}\mu(\overline{y}) =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Defining the s-parabolic annuli $A_j := 2^{j+1}B_0 \setminus 2^j B_0$ for $j \geq 1$ and arguing as in the last estimate of Theorem 1.1.2 we get

$$\begin{aligned}
\mathrm{I} &\lesssim \sum_{j\geq 1} \int_{A_j} \frac{|t-\tau|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s}} \mathrm{d}\mu(\overline{y}) \lesssim |t-\tau| \sum_{j\geq 1} \frac{\mu(2^{j+1}B_0)}{\big(2^j|t-\tau|^{\frac{1}{2s}}\big)^{n+2s}} \\
&\lesssim |t-\tau|^\eta \sum_{j\geq 1} \frac{1}{2^{2s(1-\eta)}} \simeq_\eta |t-\tau|^\eta,
\end{aligned}$$

that is the desired estimate. Regarding II, observe that

$$\mathrm{II} \leq P_s * (\chi_{2B_0}\mu)(\overline{x}) + P_s * (\chi_{2B_0}\mu)(\widehat{x}).$$

Notice now that

$$P_s * (\chi_{2B_0}\mu)(\overline{x}) \lesssim \int_{2B_0} \frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^n} \leq \int_{|\overline{x}-\overline{y}|_{p_s} \leq 5|t-\tau|^{\frac{1}{2s}}} \frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^n} \lesssim_\eta |t-\tau|^\eta,$$

where we have split the latter domain of integration into (decreasing) s-parabolic annuli. Since this also holds replacing $\overline{x}$ by $\widehat{x}$, we also have $\mathrm{II} \lesssim |t-\tau|^\eta$ and we are done. □

The above result allows us to prove that, given a positive measure as in the above statement, we can ensure that the potential $\partial_t^\beta P_s * \mu$ already belongs to BMO_{p_s}.

Lemma 1.3.2. *Let $s \in (0,1]$, $\beta \in (0,1)$. Let μ be a finite positive Borel measure in $\mathbb{R}^{n+1}$ with upper s-parabolic growth of degree $n+2s\beta$. Then,*

$$\big\|\partial_t^\beta P_s * \mu\big\|_{*,p_s} \lesssim_\beta 1.$$

Proof. Fix $\overline{x}_0 \in \mathbb{R}^{n+1}$ and $r > 0$. Consider the s-parabolic ball $B := B(\overline{x}_0, r) = B_0 \times I_0 \subset \mathbb{R}^n \times \mathbb{R}$ and a certain constant c_B to be determined later. We want to show that there exists c_B such that

$$\frac{1}{|B|}\int_B |\partial_t^\beta P_s * \mu(\overline{y}) - c_B| \,\mathrm{d}\overline{y}.$$

To that end, begin by considering the following sets, which define a partition of $\mathbb{R}^{n+1}$:

$$R_1 := 5B, \qquad R_2 := \mathbb{R}^{n+1} \setminus (5B_0 \times \mathbb{R}), \qquad R_3 := (5B_0 \times \mathbb{R}) \setminus 5B,$$

as well as their corresponding characteristic functions χ_1, χ_2 and χ_3. Bearing in mind the estimates proved in Theorems 1.1.4 and 1.1.6 for $\partial_t^\beta P_s$ and the fact that μ is finite, it is clear that the quantity $|\partial_t^\beta P_s * (\chi_2\mu)(\overline{x}_0)|$ is also finite. Moreover, notice that $|\partial_t^\beta P_s|$ is bounded by s-parabolically homogeneous functions of degree $-n-2s\beta$ for any dimension. In fact, we deduce the following estimates: given any $\varepsilon, \alpha > 0$, we obtain if $n > 2$,

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}} \leq \frac{1}{|x|^{n-\varepsilon_1}|t|^{\frac{\varepsilon_1+2s\beta}{2s}}}, \qquad \text{if } \varepsilon_1 < 2s(1-\beta).$$

For $n = 2$,

$$\text{if } s<1, \quad |\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{2-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}} \leq \frac{1}{|x|^{2-\varepsilon}|t|^{\frac{\varepsilon+2s\beta}{2s}}}, \qquad \text{if } \varepsilon < 2s(1-\beta),$$

$$\text{if } s=1, \quad |\partial_t^\beta W(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{\alpha}|\overline{x}|_{p_s}^{2+2\beta-\alpha}} \leq \frac{1}{|x|^{\alpha}|t|^{1+\beta-\frac{\alpha}{2}}}, \qquad \text{if } 2\beta < \alpha < 2.$$

And for $n = 1$, if $s < 1$,

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{1-2s+\alpha}|\overline{x}|_{p_s}^{2s(1+\beta)-\alpha}} \leq \frac{1}{|x|^{1-2s+\alpha}|t|^{1+\beta-\frac{\alpha}{2s}}}, \qquad \text{if } 2s\beta < \alpha < 2s,$$

while if $s = 1$,

$$|\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|\overline{x}|_{p_s}^{1+2\beta}} \leq \frac{1}{|x|^{\varepsilon}|t|^{\frac{1+2\beta-\varepsilon}{2}}}, \qquad \text{if } 2\beta-1 < \varepsilon < 1.$$

In light of the above inequalities, and using that $\beta < 1$, it is clear that $\partial_t^\beta P_s$ defines a $\mathcal{L}^{n+1}$-locally integrable function in $\mathbb{R}^{n+1}$ once endowed with the s-parabolic distance. Hence, there exists some $\overline{\xi}_0 \in B$ (that we may think as close as we need to $\overline{x}_0$) such that $|\partial_t^\beta P_s * (\chi_3\mu)(\overline{\xi}_0)|$ is finite. Bearing all these observations in mind, we choose c_B to be

$$c_B := \partial_t^\beta P_s * (\chi_2\mu)(\overline{x}_0) + \partial_t^\beta P_s * (\chi_3\mu)(\overline{\xi}_0).$$

Therefore, we are interested in bounding by a constant the following quantity:

$$\begin{aligned}
&\frac{1}{|B|}\int_B |\partial_t^\beta P_s * \mu(\overline{y}) - c_B|\,\mathrm{d}\overline{y}\\
&\qquad\le \frac{1}{|B|}\int_B |\partial_t^\beta P_s * (\chi_1\mu)(\overline{y})|\,\mathrm{d}\overline{y}\\
&\qquad\quad + \frac{1}{|B|}\int_B |\partial_t^\beta P_s * (\chi_2\mu)(\overline{y}) - \partial_t^\beta P_s * (\chi_2\mu)(\overline{x}_0)|\,\mathrm{d}\overline{y}\\
&\qquad\quad + \frac{1}{|B|}\int_B |\partial_t^\beta P_s * (\chi_3\mu)(\overline{y}) - \partial_t^\beta P_s * (\chi_3\mu)(\overline{\xi}_0)|\,\mathrm{d}\overline{y} =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For I, simply notice that

$$\mathrm{I} \le \frac{1}{|B|}\int_{5B}\bigg(\int_B |\partial_t^\beta P_s(\overline{y}-\overline{z})|\,\mathrm{d}\overline{y}\bigg)\mathrm{d}\mu(\overline{z}).$$

Using any of the bounds above for $\partial_t^\beta P_s$, depending on n and s, integration in polar coordinates yields

$$\mathrm{I} \lesssim_\beta \frac{1}{|B|} r^{2s(1-\beta)}\mu(5B) \lesssim_\beta 1.$$

Regarding II, write

$$\mathrm{II} \le \frac{1}{|B|}\int_B \bigg(\int_{R_2} |\partial_t^\beta P_s(\overline{y}-\overline{z}) - \partial_t^\beta P_s(\overline{x}_0-\overline{z})|\mathrm{d}\mu(\overline{z})\bigg)\,\mathrm{d}\overline{y}.$$

If we name $\overline{x} := \overline{x}_0 - \overline{z}$ and $\overline{x}' := \overline{y} - \overline{z}$, we have in particular

$$|\overline{x} - \overline{x}'|_{p_s} = |\overline{x}_0 - \overline{y}|_{p_s} \le r < \frac{|x_0 - z|}{2} = \frac{|x|}{2},$$

where the second inequality holds because $\overline{z} \in R_2$. Therefore, by the last estimate of Theorems 1.1.4 and 1.1.6, writing $2\zeta := \min\{1, 2s\}$ we get

$$\begin{aligned}
\mathrm{II} &\lesssim_\beta \frac{1}{|B|}\int_B\bigg(\int_{R_2} \frac{|\overline{y}-\overline{x}_0|_{p_s}^{2\zeta}}{|x_0 - z|^{n+2\zeta-2s}|\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}}\mathrm{d}\mu(\overline{z})\bigg)\,\mathrm{d}\overline{y}\\
&\qquad\qquad\lesssim r^{2\zeta}\int_{R_2}\frac{\mathrm{d}\mu(\overline{z})}{|x_0 - z|^{n+2\zeta-2s}|\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}}.
\end{aligned}$$

Let us split R_2 into proper disjoint pieces. Take the cylinders given by $C_j := 5^j B_0 \times \mathbb{R}$, $j \in \mathbb{Z}$, $j \ge 1$, as well as the annular cylinders $\widehat{C}_j := C_{j+1} \setminus C_j$, $j \ge 1$. The partition of R_2 we are interested in is given by the disjoint union of all the sets $\widehat{C}_j$, $j \ge 1$, which clearly cover R_2. Therefore

$$\mathrm{II} \lesssim_\beta r^{2\zeta}\sum_{j=1}^\infty \int_{\widehat{C}_j}\frac{\mathrm{d}\mu(\overline{z})}{|x_0 - z|^{n+2\zeta-2s}|\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}}. \tag{1.3.1}$$

At the same time, for each $j\geq 1$, we shall consider a proper partition of $\widehat{C}_j$. Denote $A_k=5^{k+1}B\setminus 5^kB$ for every positive integer k and define $\widehat{C}_{j,k}:=\widehat{C}_j\cap A_k$, $k\geq 1$. Let us make some observations about the sets $\widehat{C}_{j,k}$. First, notice that by definition, for each $j\geq 1$,

$$\widehat{C}_{j,k}=\big[(5^{j+1}B_0\setminus 5^jB_0)\times\mathbb{R}\big]\cap\big(5^{k+1}B\setminus 5^kB\big).$$

Hence, using that

$$\big[(5^{j+1}B_0\setminus 5^jB_0)\times\mathbb{R}\big]\cap\big(5^{k+1}B\setminus 5^kB\big)=\varnothing,\qquad\text{for}\qquad k<j,$$

we have that, in fact, $\widehat{C}_j$ can be covered by $\widehat{C}_{j,k}$ for $k\geq j$, that is

$$\widehat{C}_j=\bigcup_{k=1}^{\infty}\widehat{C}_{j,k}=\bigcup_{k=j}^{\infty}\widehat{C}_{j,k}.$$

Secondly, in order to estimate $\mu(\widehat{C}_{j,k})$, observe that for any $k\geq j$, by definition, the set $\widehat{C}_{j,k}$ can be written explicitly as follows:

$$\begin{aligned}\widehat{C}_{j,k}&=\big[(5^{j+1}B_0\setminus 5^jB_0)\times\mathbb{R}\big]\cap\big(5^{k+1}B\setminus 5^kB\big)\\&=\big[(5^{j+1}B_0\setminus 5^jB_0)\times\mathbb{R}\big]\\&\qquad\cap\Big\{\big[(5^{k+1}B_0\setminus 5^kB_0)\times 5^{2s(k+1)}I_0\big]\cup\big[5^kB_0\times\big(5^{2s(k+1)}I_0\setminus 5^{2k}I_0\big)\big]\Big\}.\end{aligned}$$

Continue by observing that if $k=j$, the intersection with the second element of the union is empty, so

$$\widehat{C}_{j,j}=\big(5^{j+1}B_0\setminus 5^jB_0\big)\times 5^{2s(j+1)}I_0,$$

while if $k>j$ one has the contrary, that is, the intersection with the first element is empty, and therefore, since $5^{j+1}B_0\setminus 5^jB_0\subset 5^kB_0$,

$$\widehat{C}_{j,k}=\big(5^{j+1}B_0\setminus 5^jB_0\big)\times\big[5^{2s(k+1)}I_0\setminus 5^{2sk}I_0\big].$$

Observe that $\widehat{C}_{j,j}\subset 5^{j+1}B$, which implies $\mu(\widehat{C}_{j,j})\leq\mu(5^{j+1}B)\lesssim(5^{j+1}r)^{n+2s\beta}$. On the other hand, for $k>j$, notice that the set $\widehat{C}_{j,k}$ can be covered by disjoint temporal translates of $\widehat{C}_{j,j}$, and the number needed to do it is proportional to the ratio between their respective time lengths, that is

$$\frac{2\big(5^{2s(k+1)}-5^{2sk}\big)}{5^{2s(j+1)}}\simeq\frac{5^{2sk}}{5^{2sj}}.$$

Therefore, since this last ratio is also valid for the case $k=j$, for every $k\geq j$ we have

$$\mu(\widehat{C}_{j,k})\simeq\frac{5^{2sk}}{5^{2sj}}\mu(\widehat{C}_{j,j})\lesssim_\beta\frac{5^{2sk}}{5^{2sj}}\big(5^{j+1}r\big)^{n+2s\beta}.$$

All in all, we finally obtain

$$\begin{aligned}
\mathrm{II} &\lesssim_\beta r^{2\zeta}\sum_{j=1}^{\infty}\sum_{k\geq j}\int_{\widehat{C}_{j,k}}\frac{\mathrm{d}\mu(\overline{z})}{|x_0-z|^{n+2\zeta-2s}|\overline{x}_0-\overline{z}|_{p_s}^{2s(1+\beta)}}\\
&\lesssim r^{2\zeta}\sum_{j=1}^{\infty}\sum_{k\geq j}\frac{\mu(\widehat{C}_{j,k})}{(5^j r)^{n+2\zeta-2s}(5^k r)^{2s(1+\beta)}}\lesssim_\beta \sum_{j=1}^{\infty}\sum_{k\geq j}\frac{1}{5^{j(2\zeta-2s\beta)}5^{2s\beta k}}\\
&=\sum_{k=1}^{\infty}\frac{1}{5^{2s\beta k}}\sum_{j=1}^{k}\frac{1}{5^{j(2\zeta-2s\beta)}}\lesssim\sum_{k=1}^{\infty}\frac{1}{5^{2s\beta k}}\left(1+\frac{1}{5^{(2\zeta-2s\beta)k}}\right)\lesssim_\beta 1.
\end{aligned}$$

Finally, let us study III. Notice that the estimate we want to check is deduced if we prove

$$|\partial_t^\beta P_s*(\chi_3\mu)(\overline{y})-\partial_t^\beta P_s*(\chi_3\mu)(\overline{\xi}_0)|\lesssim_\beta 1,$$

that at the same time, can be obtained if we show that for any $\overline{x},\overline{y}\in B$ we have

$$|\partial_t^\beta P_s*(\chi_3\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_3\mu)(\overline{y})|\lesssim_\beta 1. \tag{1.3.2}$$

It is clear that it suffices to check the latter estimate in two particular cases: when $\overline{x}$ and $\overline{y}$ share their time coordinate, and when they share their spatial coordinate.

Case 1: $\overline{x}=(x,t)$ and $\overline{y}=(y,t)$ points of B. Let us begin by observing that

$$\begin{aligned}
&|\partial_t^\beta P_s*(\chi_3\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_3\mu)(\overline{y})|\\
&=\left|\int\frac{P_s*(\chi_3\mu)(x,\tau)-P_s*(\chi_3\mu)(x,t)}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau-\int\frac{P_s*(\chi_3\mu)(y,\tau)-P_s*(\chi_3\mu)(y,t)}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau\right|\\
&\leq\int_{|\tau-t|\leq(2r)^{2s}}\frac{|P_s*(\chi_3\mu)(x,\tau)-P_s*(\chi_3\mu)(x,t)|}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau\\
&\quad+\int_{|\tau-t|\leq(2r)^{2s}}\frac{|P_s*(\chi_3\mu)(y,\tau)-P_s*(\chi_3\mu)(y,t)|}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau\\
&\quad+\int_{|\tau-t|>(2r)^{2s}}\frac{|P_s*(\chi_3\mu)(x,\tau)-P_s*(\chi_3\mu)(x,t)-P_s*(\chi_3\mu)(y,\tau)+P_s*(\chi_3\mu)(y,t)|}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau\\
&\quad=:\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

First, we estimate I. Arguing as in the proof of the last estimate of Theorem 1.1.2, we obtain

$$|P_s*(\chi_3\mu)(x,\tau)-P_s*(\chi_3\mu)(x,t)|\leq|\tau-t|\int_{R_3}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+2s}}\lesssim_\beta\frac{|t-\tau|}{r^{2s(1-\beta)}},$$

where the last inequality can be obtained by splitting the domain of integration into s-parabolic annuli and by the s-parabolic growth of degree $n+2s\beta$ of μ. Thus,

$$\mathrm{I}\lesssim_\beta\frac{1}{r^{2s(1-\beta)}}\int_{|\tau-t|\leq(2r)^{2s}}\frac{\mathrm{d}\tau}{|\tau-t|^\beta}\lesssim_\beta\frac{(r^{2s})^{(1-\beta)}}{r^{2s(1-\beta)}}=1.$$

The arguments to obtain II $\lesssim_\beta 1$ are analogous. Concerning III, we split it as follows

$$\mathrm{III} \leq \int_{|\tau-t|>(2r)^{2s}} \frac{|P_s*(\chi_3\mu)(x,\tau)-P_s*(\chi_3\mu)(y,\tau)|}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau + \int_{|\tau-t|>(2r)^{2s}} \frac{|P_s*(\chi_3\mu)(x,t)-P_s*(\chi_3\mu)(y,t)|}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau =: \mathrm{III}_1+\mathrm{III}_2.$$

First, let us deal with integral III_2. Since $(x,t),(y,t)\in B$,

$$|P_s*(\chi_3\mu)(x,t)-P_s*(\chi_3\mu)(y,t)| \leq |x-y|\,\|\nabla_x P_s*(\chi_3\mu)\|_{\infty,B}.$$

Notice that for any $\overline{z}\in B$, by Theorem 1.1.2 and the fact that $s\beta<1$, we have

$$|\nabla_x P_s*(\chi_3\mu)(\overline{z})| \lesssim \int_{R_3} \frac{|z-w|}{|\overline{z}-\overline{w}|_{p_s}^{n+2}}\mathrm{d}\mu(\overline{w}) \lesssim r\int_{\mathbb{R}^{n+1}\setminus 5B} \frac{\mathrm{d}\mu(\overline{w})}{|\overline{z}-\overline{w}|_{p_s}^{n+2}} \lesssim_\beta r^{2s\beta-1}.$$

Therefore, since $|x-y|\leq r$,

$$\mathrm{III}_2 \lesssim_\beta r^{2s\beta}\int_{|\tau-t|>(2r)^{2s}} \frac{\mathrm{d}\tau}{|\tau-t|^{1+\beta}} \lesssim_\beta r^{2s\beta}\frac{1}{(r^{2s})^\beta} = 1. \tag{1.3.3}$$

Regarding III_1, observe that for each τ the points (x,τ) and (y,τ) belong to a temporal translate of B that does not intersect B, since $|\tau-t|>(2r)^{2s}$ and $t\in I_0$. We call it B_τ. Hence, bearing in mind the first estimate of [**MatPT, Lemma 2.1**] we deduce

$$\begin{aligned}
|P_s*(\chi_3\mu)(x,\tau)-P_s*(\chi_3\mu)(y,\tau)| &\leq \int_{2B_\tau} |P_s((x,\tau)-\overline{w})-P_s((y,\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\
&\quad + \int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} |P_s((x,\tau)-\overline{w})-P_s((y,\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\
&\lesssim \int_{2B_\tau}\frac{\mathrm{d}\mu(\overline{w})}{|(x,\tau)-\overline{w}|_{p_s}^n} + \int_{2B_\tau}\frac{\mathrm{d}\mu(\overline{w})}{|(y,\tau)-\overline{w}|_{p_s}^n} \qquad (1.3.4) \\
&\quad + |x-y|\int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} |\nabla_x P_s((\widetilde{x},\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\
&\lesssim_\beta r^{2s\beta} + r\int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} \frac{|\widetilde{x}-w|}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}}\mathrm{d}\mu(\overline{w}) \\
&\lesssim r^{2s\beta} + r^2\int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} \frac{\mathrm{d}\mu(\overline{w})}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}} \\
&\leq r^{2s\beta} + r^2\int_{\mathbb{R}^{n+1}\setminus 2B_\tau} \frac{\mathrm{d}\mu(\overline{w})}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}} \lesssim_\beta r^{2s\beta},
\end{aligned}$$

where for both integrals in (1.3.4) we have split the domain of integration into (decreasing) s-parabolic annuli; while in the remaining term, $\widetilde{x}$ belongs to the segment joining x and y and we have split the domain of integration into s-parabolic annuli centered at $(x_0,t+s)$. Hence, similarly to (1.3.3) we get $\mathrm{III}_1\lesssim_\beta 1$ and we are done with *Case 1*.

Case 2: $\overline{x} = (x,t)$ and $\overline{y} = (x,u)$ points of B. Write

$$\begin{aligned}
&|\partial_t^\beta P_s * (\chi_3\mu)(\overline{x}) - \partial_t^\beta P_s * (\chi_3\mu)(\overline{y})| \\
&= \Bigg| \int \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)}{|\tau - t|^{1+\beta}} \, \mathrm{d}\tau \\
&\qquad\qquad - \int \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,u)}{|\tau - u|^{1+\beta}} \, \mathrm{d}\tau \Bigg| \\
&\leq \int_{|\tau - t| \leq (2r)^{2s}} \frac{|P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)|}{|\tau - t|^{1+\beta}} \, \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| \leq (2r)^{2s}} \frac{|P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,u)|}{|\tau - u|^{1+\beta}} \, \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| > (2r)^{2s}} \Bigg| \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)}{|\tau - t|^{1+\beta}} \\
&\qquad\qquad - \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,u)}{|\tau - u|^{1+\beta}} \Bigg| \, \mathrm{d}\tau =: \mathrm{I}' + \mathrm{II}' + \mathrm{III}'.
\end{aligned}$$

The expressions corresponding to $\mathrm{I}', \mathrm{II}'$ can be tackled in the same way as I, II. Hence, $\mathrm{I}' \lesssim_\beta 1$ and $\mathrm{II}' \lesssim_\beta 1$. Finally, for III', adding and subtracting $P_s * (\chi_3\mu)(x,t)/|\tau - u|^{1+\beta}$,

$$\begin{aligned}
\mathrm{III}' \leq &\int_{|\tau - t| > (2r)^{2s}} \left| \frac{1}{|\tau - t|^{1+\beta}} - \frac{1}{|\tau - u|^{1+\beta}} \right| |P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)| \, \mathrm{d}\tau \\
&+ \int_{|\tau - t| > (2r)^{2s}} \frac{1}{|\tau - u|^{1+\beta}} |P_s * (\chi_3\mu)(x,t) - P_s * (\chi_3\mu)(x,u)| \, \mathrm{d}\tau.
\end{aligned}$$

Since $|\tau - t| > (2r)^{2s}$ we can apply the mean value theorem to deduce

$$\left| \frac{1}{|\tau - t|^{1+\beta}} - \frac{1}{|\tau - u|^{1+\beta}} \right| \lesssim_\beta \frac{|t-u|}{|\tau - t|^{2+\beta}} \lesssim \frac{r^{2s}}{|\tau - t|^{2+\beta}}.$$

In addition, since μ has upper s-parabolic growth of degree $n + 2s\beta$, by Lemma 1.3.1, with $\eta := \beta$, the time function $P_s * (\chi_3\mu)(x, \cdot)$ is Lip-β. Therefore,

$$\mathrm{III}' \lesssim_\beta \int_{|\tau - t| > (2r)^{2s}} \frac{r^{2s}}{|\tau - t|^{2+\beta}} |\tau - t|^\beta \, \mathrm{d}\tau + \int_{|\tau - t| > (2r)^{2s}} \frac{1}{|\tau - u|^{1+\beta}} |t - u|^\beta \, \mathrm{d}\tau \lesssim_\beta 1.$$

Therefore estimate (1.3.2) is satisfied and we are done with III and also with the proof. □

In the same spirit, if we ask the positive measure for an extra α growth, the potential $\partial_t^\beta P_s * \mu$ will satisfy a $\mathrm{Lip}_{\alpha, p_s}$ property. Recall that $2\zeta := \min\{1, 2s\}$.

LEMMA 1.3.3. *Let $s \in (0,1]$, $\beta \in (0,1)$ and $\alpha \in (0, 2\zeta)$ such that $2s\beta + \alpha < 2$. Let μ be a positive measure in $\mathbb{R}^{n+1}$ which has upper s-parabolic growth of degree $n + 2s\beta + \alpha$. Then,*

$$\|\partial_t^\beta P_s * \mu\|_{\mathrm{Lip}_{\alpha, p_s}} \lesssim_{\beta, \alpha} 1.$$

Proof. Fix any $\overline{x},\overline{y}\in\mathbb{R}^{n+1},\overline{x}\neq\overline{y}$. We have to check if the following holds

$$|\partial_t^\beta P_s*\mu(\overline{x})-\partial_t^\beta P_s*\mu(\overline{y})|\lesssim_{\beta,\alpha}|\overline{x}-\overline{y}|_{p_s}^\alpha.$$

Begin by choosing the following partition of $\mathbb{R}^{n+1}$

$$\begin{aligned}R_1:&=\big\{\overline{z}\;:\;|\overline{x}-\overline{y}|_{p_s}\leq|x-z|/5\big\}\cup\big\{\overline{z}\;:\;|\overline{y}-\overline{x}|_{p_s}\leq|y-z|/5\big\},\\ R_2:=\mathbb{R}^{n+1}\setminus R_1&=\big\{\overline{z}\;:\;|\overline{x}-\overline{y}|_{p_s}>|x-z|/5\big\}\cap\big\{\overline{z}\;:\;|\overline{y}-\overline{x}|_{p_s}>|y-z|/5\big\},\end{aligned}$$

and their corresponding characteristic functions χ_1,χ_2. From the latter we have

$$\begin{aligned}&\frac{|\partial_t^\beta P_s*\mu(\overline{x})-\partial_t^\beta P_s*\mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\\ &\qquad\leq\frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{|\overline{x}-\overline{y}|_{p_s}\leq|x-z|/5}|\partial_t^\beta P_s(\overline{x}-\overline{z})-\partial_t^\beta P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\\ &\qquad\quad+\frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{|\overline{y}-\overline{x}|_{p_s}\leq|y-z|/5}|\partial_t^\beta P_s(\overline{x}-\overline{z})-\partial_t^\beta P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\\ &\qquad\quad+\frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\big|\partial_t^\beta P_s*(\chi_2\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_2\mu)(\overline{y})\big|=:\mathrm{I}_{\overline{x}}+\mathrm{I}_{\overline{y}}+\mathrm{II}.\quad(1.3.5)\end{aligned}$$

Regarding $\mathrm{I}_{\overline{x}}$, name $\overline{\xi}:=\overline{x}-\overline{z}$, $\overline{\xi}':=\overline{y}-\overline{z}$ and observe that, in particular, one has

$$|\overline{\xi}-\overline{\xi}'|_{p_s}=|\overline{x}-\overline{y}|_{p_s}<\frac{|x-z|}{2}=\frac{|\xi|}{2},$$

Applying the last estimate either of Theorem 1.1.4 or Theorem 1.1.6, we deduce

$$\mathrm{I}_{\overline{x}}\lesssim_\beta\frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-2\zeta}}\int_{|\overline{x}-\overline{y}|_{p_s}\leq|x-z|/5}\frac{\mathrm{d}\mu(\overline{z})}{|x-z|^{n+2\zeta-2s}|\overline{x}-\overline{z}|_{p_s}^{2s(1+\beta)}}.$$

Let us split the domain of integration into proper disjoint pieces. For $\overline{x}=(x,t)$, we denote

$$B_{\overline{x}}:=B(\overline{x},|\overline{x}-\overline{y}|_{p_s})=B_1(x,|\overline{x}-\overline{y}|_{p_s})\times J_{\overline{x}},$$

where $B_1(x,|\overline{x}-\overline{y}|_{p_s})$ is an Euclidean ball in $\mathbb{R}^n$ and $J_{\overline{x}}$ is a real interval centered at t with length $2|\overline{x}-\overline{y}|_{p_s}^{2s}$. As in Lemma 1.3.2, take cylinders $C_{j,\overline{x}}:=5^jB_1(\overline{x},|\overline{x}-\overline{y}|_{p_s})\times\mathbb{R}$ for $j\geq1$, as well as the annular cylinders $\widehat{C}_{j,\overline{x}}:=C_{j+1,\overline{x}}\setminus C_{j,\overline{x}}$, for $j\geq1$. We express $\{\overline{z}\,:\,|\overline{x}-\overline{y}|_{p_s}\leq|x-z|/5\}$ as the disjoint union of the sets $\widehat{C}_{j,\overline{x}}$, so that

$$\mathrm{I}_{\overline{x}}\lesssim_\beta\frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-2\zeta}}\sum_{j=1}^\infty\int_{\widehat{C}_{j,\overline{x}}}\frac{\mathrm{d}\mu(\overline{z})}{|x-z|^{n+2\zeta-2s}|\overline{x}-\overline{z}|_{p_s}^{2s(1+\beta)}}.$$

The above integral can be studied as that appearing in (1.3.1), in the study of the term II of Lemma 1.3.2 (centering now the cylinders in $\overline{x}$ and interchanging the roles of r and $|\overline{x}-\overline{y}|_{p_s}$). Doing so, and taking into account the $n+2s\beta+\alpha$ growth of μ, one obtains

$$\begin{aligned}
\mathrm{I}_{\overline{x}} &\lesssim_{\beta,\alpha} \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-2\zeta}} \sum_{j=1}^{\infty}\sum_{k\geq j} \frac{(5^{j+1}|\overline{x}-\overline{y}|_{p_s})^{n+2s\beta+\alpha}}{(5^j|\overline{x}-\overline{y}|_{p_s})^{n+2\zeta-2s}(5^k|\overline{x}-\overline{y}|_{p_s})^{2s(1+\beta)}} \frac{5^{2sk}}{5^{2sj}} \\
&= \sum_{j=1}^{\infty}\sum_{k\geq j} \frac{5^{j(n+2s\beta+\alpha)}}{5^{j(n+2\zeta-2s)}5^{2s(1+\beta)k}} \frac{5^{2sk}}{5^{2sj}} \simeq \sum_{j=1}^{\infty}\sum_{k\geq j} \frac{5^{j(2s\beta+\alpha-2\zeta)}}{5^{2s\beta k}} \\
&= \sum_{k=1}^{\infty} \frac{1}{5^{2s\beta k}} \sum_{j=1}^{k} \frac{1}{5^{j(2\zeta-2s\beta-\alpha)}} \lesssim \sum_{k=1}^{\infty} \frac{1}{5^{2s\beta k}} \left(1+\frac{1}{5^{(2\zeta-2s\beta-\alpha)k}}\right) \lesssim_{\beta,\alpha} 1, \quad \text{if } \alpha<2\zeta.
\end{aligned}$$

The study of $\mathrm{I}_{\overline{y}}$ is analogous, interchanging the roles of $\overline{x}$ and $\overline{y}$. Finally we deal with II. We claim that the following estimate holds

$$\left|\partial_t^\beta P_s*(\chi_2\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_2\mu)(\overline{y})\right| \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{\alpha}.$$

The general case will follows from the following two cases: whether $\overline{x}$ and $\overline{y}$ share their time coordinate, or if they share their spatial coordinate. Indeed, write $\overline{x}=(x,t), \overline{y}=(y,\tau)$ and set $\widehat{x}:=(x,\tau)$ so that

$$\begin{aligned}
&\left|\partial_t^\beta P_s*(\chi_2\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_2\mu)(\overline{y})\right| \\
&\leq \left|\partial_t^\beta P_s*(\chi_2\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_2\mu)(\widehat{x})\right| + \left|\partial_t^\beta P_s*(\chi_2\mu)(\widehat{x})-\partial_t^\beta P_s*(\chi_2\mu)(\overline{y})\right| \\
&\lesssim_{\beta,\alpha} |\overline{x}-\widehat{x}|_{p_s}^{\alpha}+|\widehat{x}-\overline{y}|_{p_s}^{\alpha} = |t-\tau|^{\alpha/2}+|x-y|^{\alpha} \leq 2|\overline{x}-\overline{y}|_{p_s}^{\alpha}, \text{ and we are done.}
\end{aligned}$$

Case 1: $\overline{x}=(x,t)$ and $\overline{y}=(x,u)$. Write $\mu_2:=\chi_2\mu$ and estimate $|\partial_t^\beta P_s*\mu_2(\overline{x})-\partial_t^\beta P_s*\mu_2(\overline{y})|$ as follows

$$\begin{aligned}
&\left|\int \frac{P_s*\mu_2(x,\tau)-P_s*\mu_2(x,t)}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau - \int \frac{P_s*\mu_2(x,\tau)-P_s*\mu_2(x,u)}{|\tau-u|^{1+\beta}}\,\mathrm{d}\tau\right| \\
&\leq \int_{|\tau-t|\leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s*\mu_2(x,\tau)-P_s*\mu_2(x,t)|}{|\tau-t|^{1+\beta}}\,\mathrm{d}\tau \\
&\quad + \int_{|\tau-t|\leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s*\mu_2(x,\tau)-P_s*\mu_2(x,u)|}{|\tau-u|^{1+\beta}}\,\mathrm{d}\tau \\
&\quad + \int_{|\tau-t|> 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \left|\frac{P_s*\mu_2(x,\tau)-P_s*\mu_2(x,t)}{|\tau-t|^{1+\beta}}\right. \\
&\qquad\qquad \left. - \frac{P_s*\mu_2(x,\tau)-P_s*\mu_2(x,u)}{|\tau-u|^{1+\beta}}\right|\,\mathrm{d}\tau =: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

By a direct application of Lemma 1.3.1 we are able to obtain, straightforwardly,

$$\begin{aligned}
\mathrm{I} &\lesssim_{\beta,\alpha} \int_{|\tau-t|\leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{\mathrm{d}\tau}{|\tau-t|^{1-\frac{\alpha}{2s}}} \lesssim_\alpha |\overline{x}-\overline{y}|_{p_s}^{\alpha} \quad \text{and} \\
\mathrm{II} &\lesssim_{\beta,\alpha} \int_{|\tau-t|\leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{\mathrm{d}\tau}{|\tau-u|^{1-\frac{\alpha}{2s}}} \lesssim_\alpha |\overline{x}-\overline{y}|_{p_s}^{\alpha}.
\end{aligned}$$

For III, adding and subtracting the term $P_s*\mu_2(x,t)/|\tau-u|^{1+\beta}$ we get

$$\begin{aligned}\mathrm{III} \leq &\int_{|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \left|\frac{1}{|\tau-t|^{1+\beta}}-\frac{1}{|\tau-u|^{1+\beta}}\right| |P_s*\mu_2(x,\tau)-P_s*\mu_2(x,t)|\,\mathrm{d}\tau \\ &+\int_{|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{1}{|\tau-u|^{1+\beta}}|P_s*\mu_2(x,t)-P_s*\mu_2(x,u)|\,\mathrm{d}\tau.\end{aligned}$$

Since $|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}$ we can apply the mean value theorem to deduce

$$\left|\frac{1}{|\tau-t|^{1+\beta}}-\frac{1}{|\tau-u|^{1+\beta}}\right| \lesssim_\beta \frac{|t-u|}{|\tau-t|^{2+\beta}} \lesssim \frac{|\overline{x}-\overline{y}|_{p_s}^{2s}}{|\tau-t|^{2+\beta}}.$$

Therefore, by Lemma 1.3.1 with $\eta:=\beta+\frac{\alpha}{2s}$, we finally have

$$\begin{aligned}\mathrm{III} \lesssim_{\beta,\alpha} &\int_{|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|\overline{x}-\overline{y}|_{p_s}^{2s}}{|\tau-t|^{2+\beta}}|\tau-t|^{\beta+\frac{\alpha}{2s}}\,\mathrm{d}\tau \\ &+\int_{|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{1}{|\tau-u|^{1+\beta}}|t-u|^{\beta+\frac{\alpha}{2s}}\,\mathrm{d}\tau \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{\alpha}.\end{aligned}$$

Therefore $|\partial_t^\beta P_s*\mu_2(\overline{x})-\partial_t^\beta P_s*\mu_2(\overline{y})|\leq \mathrm{I}+\mathrm{II}+\mathrm{III}\lesssim_{\beta,\alpha}|\overline{x}-\overline{y}|_{p_s}^{\alpha}$, and this ends the study of *Case 1*.

Case 2: $\overline{x}=(x,t)$ and $\overline{y}=(y,t)$. To tackle this case, let us first rewrite the set R_2 as

$$R_2=\Big[5B_1(x,|\overline{x}-\overline{y}|_{p_s})\times\mathbb{R}\Big]\cap\Big[5B_1(y,|\overline{y}-\overline{x}|_{p_s})\times\mathbb{R}\Big]=(5B_{1,x}\times\mathbb{R})\cap(5B_{1,y}\times\mathbb{R}),$$

Continue rewriting R_2 as follows

$$\begin{aligned}R_2&=\Big\{5B_{\overline{x}}\cup\big[(5B_{1,x}\times\mathbb{R})\setminus 5B_{\overline{x}}\big]\Big\}\cap\Big\{5B_{\overline{y}}\cup\big[(5B_{1,y}\times\mathbb{R})\setminus 5B_{\overline{y}}\big]\Big\}\\ &=(5B_{\overline{x}}\cap 5B_{\overline{y}})\cup\Big\{5B_{\overline{x}}\cap\big[(5B_{1,y}\times\mathbb{R})\setminus 5B_{\overline{y}}\big]\Big\}\\ &\qquad\cup\Big\{5B_{\overline{y}}\cap\big[(5B_{1,x}\times\mathbb{R})\setminus 5B_{\overline{x}}\big]\Big\}\\ &\qquad\cup\Big\{\big[(5B_{1,x}\times\mathbb{R})\setminus 5B_{\overline{x}}\big]\cap\big[(5B_{1,y}\times\mathbb{R})\setminus 5B_{\overline{y}}\big]\Big\}\\ &=:R_{21}\cup R_{22}\cup R_{23}\cup R_{24}.\end{aligned}$$

Observe that in *Case 2* intervals $J_{\overline{x}}$ and $J_{\overline{y}}$ coincide. We name them J. Therefore,

$$\begin{aligned}R_{22}&:=5B_{\overline{x}}\cap\big[(5B_{1,y}\times\mathbb{R})\setminus 5B_{\overline{y}}\big]=(5B_{1,x}\times J)\cap\big[5B_{1,y}\times(\mathbb{R}\setminus J)\big]=\varnothing,\\ R_{23}&:=5B_{\overline{y}}\cap\big[(5B_{1,y}\times\mathbb{R})\setminus 5B_{\overline{y}}\big]=(5B_{1,y}\times J)\cap\big[5B_{1,x}\times(\mathbb{R}\setminus J)\big]=\varnothing,\end{aligned}$$

meaning that, in fact, $R_2=R_{21}\cup R_{24}$. Observe also that R_{24} can be rewritten as

$$\begin{aligned}R_{24}:&=\big[(5B_{1,x}\times\mathbb{R})\setminus 5B_{\overline{x}}\big]\cap\big[(5B_{1,y}\times\mathbb{R})\setminus 5B_{\overline{y}}\big]\\ &=(5B_{1,x}\cap 5B_{1,y})\times(\mathbb{R}\setminus J).\end{aligned}$$

Therefore, if χ_{21} and χ_{24} are the characteristic functions of R_{21} and R_{24}, we have, naming $\mu_{21} := \chi_{21}\mu$ and $\mu_{24} := \chi_{24}\mu$,

$$\mathrm{II} \leq \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \big|\partial_t^{\beta} P_s * \mu_{21}(\overline{x}) - \partial_t^{\beta} P_s * \mu_{21}(\overline{y})\big| \\ + \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \big|\partial_t^{\beta} P_s * \mu_{24}(\overline{x}) - \partial_t^{\beta} P_s * \mu_{24}(\overline{y})\big| =: \mathrm{II}_1 + \mathrm{II}_4.$$

Hence, fixing $j \in \{1,4\}$, begin by establishing the following estimate

$$\begin{aligned}
&|\partial_t^{\beta} P_s * \mu_{2j}(\overline{x}) - \partial_t^{\beta} P_s * \mu_{2j}(\overline{y})| \\
&\quad = \bigg| \int \frac{P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(x,t)}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau - \int \frac{P_s * \mu_{2j}(y,\tau) - P_s * \mu_{2j}(y,t)}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau \bigg| \\
&\quad \leq \int_{|\tau - t| \leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(x,t)|}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau \\
&\quad\quad + \int_{|\tau - t| \leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(y,\tau) - P_s * \mu_{2j}(y,t)|}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau \\
&\quad\quad + \int_{|\tau - t| > 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(x,t) - P_s * \mu_{2j}(y,\tau) + P_s * \mu_{2j}(y,t)|}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau \\
&\quad\quad =: C_1 + C_2 + C_3.
\end{aligned}$$

Lemma 1.3.1 with $\eta = \beta$ yields $C_1 \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{\alpha}$ and $C_2 \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{\alpha}$, so we focus on C_3. Split it as follows

$$C_3 \leq \int_{|\tau - t| > 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(y,\tau)|}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau \\ + \int_{|\tau - t| > 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,t) - P_s * \mu_{2j}(y,t)|}{|\tau - t|^{1+\beta}}\, \mathrm{d}\tau =: C_{31} + C_{32}.$$

First, let us deal with integral C_{32}. On the one hand, if $j = 1$, observe that for any $\overline{z} \in 2B_{\overline{x}}$, since $2B_{\overline{x}} \subset R_{21} \subset 5B_{\overline{x}}$, we can contain R_{21} into s-parabolic annuli centered at $\overline{z}$ and (exponentially decreasing) radii proportional to $|\overline{x}-\overline{y}|_{p_s}$. Hence, by [**MatP, Lemma 2.2**] and the upper s-parabolic growth of degree $n + 2s\beta + \alpha$ of μ, we deduce

$$|P_s * \mu_{21}(\overline{z})| \lesssim \int_{5B_{\overline{x}} \cap 5B_{\overline{y}}} \frac{\mathrm{d}\mu(\overline{w})}{|\overline{z}-\overline{w}|_{p_s}^{n}} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha}.$$

If $j = 4$, observe that $|P_s * \mu_{24}(x,t) - P_s * \mu_{2j}(y,t)| \leq |x-y|\, \|\nabla_x P_s * \mu_{24}\|_{\infty, 2B_{\overline{x}}}$. So for any $\overline{z} \in 2B_{\overline{x}}$, by Theorem 1.1.2 we obtain

$$\begin{aligned}
|\nabla_x P_s * \mu_{24}(\overline{z})| &\lesssim \int_{(5B_{\overline{x}} \cap 5B_{\overline{y}}) \times (\mathbb{R} \setminus J)} \frac{|z-w|}{|\overline{z}-\overline{w}|_{p_s}^{n+2}} \mathrm{d}\mu(\overline{w}) \\
&\lesssim |\overline{x}-\overline{y}|_{p_s} \int_{\mathbb{R}^{n+1} \setminus (5B_{p,\overline{x}} \cap 5B_{p,\overline{y}})} \frac{\mathrm{d}\mu(\overline{w})}{|\overline{z}-\overline{w}|_{p_s}^{n+2}} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha-1},
\end{aligned}$$

since $2s\beta+\alpha<2$. For the last inequality we can split, for example, the domain of integration into s-parabolic annuli centered at $\overline{z}$ with (exponentially increasing) radii proportional to $2|\overline{x}-\overline{y}|_{p_s}$. Then,

$$C_{32} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha} \int_{|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{\mathrm{d}\tau}{|\tau-t|^{1+\beta}} \lesssim_\beta |\overline{x}-\overline{y}|_{p_s}^{\alpha}. \tag{1.3.6}$$

Regarding C_{31}, the points (x,τ) and (y,τ) belong to a temporal translate of $2B_{\overline{x}}\cap 2B_{\overline{y}}$ that does not intersect $2B_{\overline{x}}\cap 2B_{\overline{y}}$, since $|\tau-t|>2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}$. We call it $2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau}$. For each $j\in\{1,4\}$ and τ (and bearing in mind Theorem 1.1.2) we deduce

$$\begin{aligned}
|P_s*&\mu_{2j}(x,\tau)-P_s*\mu_{2j}(y,\tau)| \\
&\leq \int_{2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau}} |P_s((x,\tau)-\overline{w})-P_s((y,\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\
&\qquad\qquad + \int_{R_{2j}\setminus(2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau})} |P_s((x,\tau)-\overline{w})-P_s((y,\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\
&\lesssim \int_{2B_{\overline{x}}^{\tau}} \frac{\mathrm{d}\mu(\overline{w})}{|(x,\tau)-\overline{w}|_{p_s}^{n}} + \int_{2B_{\overline{y}}^{\tau}} \frac{\mathrm{d}\mu(\overline{w})}{|(y,\tau)-\overline{w}|_{p_s}^{n}} \qquad\qquad (1.3.7)\\
&\qquad\qquad + |x-y| \int_{R_{2j}\setminus(2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau})} |\nabla_x P_s((\widetilde{x},\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\
&\lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha} + |\overline{x}-\overline{y}|_{p_s}^{2} \int_{\mathbb{R}^{n+1}\setminus(2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau})} \frac{\mathrm{d}\mu(\overline{w})}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}}, \qquad\qquad (1.3.8)
\end{aligned}$$

where for both integrals of (1.3.7) we have split the domain of integration into (exponentially decreasing) s-parabolic annuli; while in the remaining term $\widetilde{x}$ belongs to the segment joining x and y. Observe also that in the last inequality we have used that the spatial distance between any two points of $R_{21}\setminus(2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau})$ and $R_{24}\setminus(2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau})$ is bounded by a multiple of $|x-y|$ and thus of $|\overline{x}-\overline{y}|_{p_s}$. Observe now that, if $\xi:=(x+y)/2$, we have

$$\begin{aligned}
2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau} &= B\big((x,t+\tau),2|\overline{x}-\overline{y}|_{p_s}\big)\cap B\big((y,t+\tau),2|\overline{x}-\overline{y}|_{p_s}\big) \\
&\supset B\big((\xi,t+\tau),|\overline{x}-\overline{y}|_{p_s}\big) =: \widehat{B}^{\tau},
\end{aligned}$$

meaning that

$$\mathbb{R}^{n+1}\setminus(2B_{\overline{x}}^{\tau}\cap 2B_{\overline{y}}^{\tau})\subset \mathbb{R}^{n+1}\setminus\widehat{B}^{\tau}.$$

Return to (1.3.8) and estimate the remaining integral by another one with the same integrand, but over the enlarged domain $\mathbb{R}^{n+1}\setminus\widehat{B}^{\tau}$. Afterwards, split the latter into s-parabolic annuli centered at $(\widetilde{x},\tau)$ and (exponentially increasing) radii proportional to $|\overline{x}-\overline{y}|_{p_s}/2$ and use that $2s\beta+\alpha<2$ so that

$$|P_s*\mu_{2j}(x,\tau)-P_s*\mu_{2j}(y,\tau)| \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha} + \frac{|\overline{x}-\overline{y}|_{p_s}^{2}}{|\overline{x}-\overline{y}|_{p_s}^{2-2s\beta-\alpha}} \simeq |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha}.$$

Hence, similarly to (1.3.6) we deduce $C_{31} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^\alpha$, which means $\text{II} \leq \text{II}_1+\text{II}_4 \lesssim_{\beta,\alpha} 1$ and we are done with *Case 2*. This last estimate finally implies

$$\left|\partial_t^\beta P_s * (\chi_2\mu)(\overline{x}) - \partial_t^\beta P_s * (\chi_2\mu)(\overline{y})\right| \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^\alpha,$$

which means $\text{II} \lesssim_{\beta,\alpha} 1$. So applying it to (1.3.5) we conclude that

$$\frac{|\partial_t^\beta P_s * \mu(\overline{x}) - \partial_t^\beta P_s * \mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^\alpha} \leq \text{I}_{\overline{x}} + \text{I}_{\overline{y}} + \text{II} \lesssim_{\beta,\alpha} 1,$$

and the desired s-parabolic Lip_α condition follows. □

1.4 The s-parabolic BMO and Lip_α caloric capacities

We are finally ready to introduce the s-parabolic BMO and Lip_α variants of the caloric capacities presented in [**MatPT, MatP**]. This section generalizes the concept to include a broader set of variants. The principal result will be that, in any case, such capacities will turn out to be comparable to a certain s-parabolic Hausdorff content. Moreover, we will be able to characterize removable sets for BMO_{p_s} and Lip_{α,p_s} solutions of the Θ^s-equation in terms of the nullity of the respective capacities. In order to do so, we will need a fundamental lemma that we present before introducing the different capacities. The result below will characterize distributions supported on a compact set with finite d-dimensional Hausdorff measure that satisfy some growth property only for small enough s-parabolic cubes.

Lemma 1.4.1. *Let $d>0$ and $E\subset\mathbb{R}^{n+1}$ be a compact set with $\mathcal{H}_{p_s}^d(E)<\infty$. Let T be a distribution supported on E with the property that there exists $0<\ell_0\leq\infty$ such that for any $R\subset\mathbb{R}^{n+1}$ s-parabolic cube with $\ell(R)\leq\ell_0$,*

$$|\langle T,\phi\rangle| \lesssim \ell(R)^d, \qquad \forall\phi \text{ admissible for } R.$$

Then, T is a signed measure satisfying

$$|\langle T,\psi\rangle| \lesssim \mathcal{H}_{p_s}^d(E)\|\psi\|_\infty, \qquad \forall\psi\in\mathcal{C}_c^\infty(\mathbb{R}^{n+1}).$$

Proof. We follow the proof of [**MatPT, Lemma 6.2**]. Let $\psi\in\mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ and $0<\varepsilon\leq\ell_0/4$. Let $Q_i, i\in I_\varepsilon$ be a collection of s-parabolic cubes with $F\subset\bigcup_{i\in I_\varepsilon} Q_i$ with $\ell(Q_i)\leq\varepsilon$ and

$$\sum_{i\in I_\varepsilon}\ell(Q_i)^d \leq C\mathcal{H}_{p_s}^d(E)+\varepsilon.$$

Now cover each Q_i by a bounded number (depending on the dimension) of dyadic s-parabolic cubes $R_i^1,\ldots,R_i^m$ with $\ell(R_i^j)\leq\ell(Q_i)/8$ and apply an s-parabolic version of Harvey-Polking's lemma (that admits an analogous proof, see [**HPo, Lemma 3.1**]) to obtain a collection of non-negative functions $\{\varphi_i\}_{i\in I_\varepsilon}$ with $\operatorname{supp}(\varphi_i)\subset 2Q_i$, $c\varphi_i$ admissible for $2Q_i$ and satisfying $\sum_{i\in I_\varepsilon}\varphi_i\equiv 1$ on $\bigcup_{i\in I_\varepsilon} Q_i\supset E$. Now we write

$$|\langle T,\psi\rangle|\leq\sum_{i\in I_\varepsilon}|\langle T,\varphi_i\psi\rangle|.$$

Proceeding as in [**MatPT, Lemma 6.2**] it can be shown that

$$\eta_i:=\frac{\varphi_i\psi}{\|\psi\|_\infty+\ell(Q_i)\|\nabla_x\psi\|_\infty+\ell(Q_i)^{2s}\|\partial_t\psi\|_\infty+\ell(Q_i)^2\|\Delta\psi\|_\infty}$$

is an admissible function for $2Q_i$ (up to a dimensional constant), with $\ell(2Q_i)\leq\ell_0/2$. Therefore, by the growth assumptions on T,

$$\begin{aligned}|\langle T,\psi\rangle|&\lesssim\sum_{i\in I_\varepsilon}\ell(Q_i)^d\big(\|\psi\|_\infty+\ell(Q_i)\|\nabla_x\psi\|+\ell(Q_i)^2\|\partial_t\psi\|_\infty+\ell(Q_i)^2\|\Delta\psi\|_\infty\big)\\&\lesssim\big(\mathcal{H}^d_{p_s}(E)+\varepsilon\big)\big(\|\psi\|_\infty+\varepsilon\|\nabla_x\psi\|+\varepsilon^2\|\partial_t\psi\|_\infty+\varepsilon^2\|\Delta\psi\|_\infty\big),\end{aligned}$$

and making ε tend to 0, we deduce the result. □

1.4.1 The capacity $\Gamma_{\Theta^s,*}$

The first capacity we introduce is the BMO_{p_s} variant of the caloric capacity first defined in [**MatPT**] for the usual heat equation.

Definition 1.4.1. Given $s\in(1/2,1]$ and $E\subset\mathbb{R}^{n+1}$ compact set, define its BMO_{p_s}*-caloric capacity* as

$$\Gamma_{\Theta^s,*}(E):=\sup|\langle T,1\rangle|,$$

where the supremum is taken among all distributions T with $\operatorname{supp}(T)\subset E$ and satisfying

$$\|\nabla_x P_s*T\|_{*,p_s}\leq 1,\qquad\|\partial_t^{\frac{1}{2s}}P_s*T\|_{*,p_s}\leq 1.\tag{1.4.1}$$

Such distributions will be called *admissible for* $\Gamma_{\Theta^s,*}(E)$.

Let us also introduce what we will understand as removable sets in this context:

Definition 1.4.2. A compact set $E\subset\mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with* BMO_{p_s}*-*$(1,\frac{1}{2s})$*-derivatives* if for any open subset $\Omega\subset\mathbb{R}^{n+1}$, any function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ with

$$\|\nabla_x f\|_{*,p_s}<\infty,\qquad\|\partial_t^{\frac{1}{2s}}f\|_{*,p_s}<\infty,$$

satisfying the Θ^s-equation in $\Omega\setminus E$, also satisfies the previous equation in the whole Ω.

First, we shall prove that if T satisfies (1.4.1), then T has upper s-parabolic growth of degree $n+1$. In fact, we shall prove a stronger result:

THEOREM 1.4.2. *Let $s \in (1/2, 1]$ and T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|\nabla_x P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim \ell(R)^{n+1}$.

Proof. Let T, Q and φ be as above. Let R be an s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$ (if not, the result is trivial) and ϕ admissible function for R. Since P_s is the fundamental solution of the Θ^s-equation,

$$|\langle \varphi T, \phi\rangle| = |\langle \Theta^s P_s * T, \varphi\phi\rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi\rangle| + |\langle P_s * T, \partial_t(\varphi\phi)\rangle| =: \mathrm{I} + \mathrm{II}.$$

Regarding II, observe that defining $\beta := 1 - \frac{1}{2s} \in (0, 1/2]$ we get

$$\partial_t(\varphi\phi) = c\, \partial_t^{1-\beta}\Big(\partial_t(\varphi\phi) *_t |t|^{-\beta}\Big),$$

for some constant c. The latter can be checked via the Fourier transform with respect to the t variable. Therefore, applying Theorem 1.2.1 we get

$$\mathrm{II} \simeq c|\langle \partial_t^{1-\beta} P_s * T, \partial_t(\varphi\phi) *_t |t|^{-\beta}\rangle| \lesssim \ell(R)^{n+2s(1-\beta)} = \ell(R)^{n+1}.$$

To study I we distinguish whether if $s = 1$ or $s < 1$. If $s = 1$, Theorem 1.2.2 yields

$$\mathrm{I} = |\langle \Delta W * T, \varphi\phi\rangle| = |\langle \nabla_x W * T, \nabla_x(\varphi\phi)\rangle| \lesssim \ell(R)^{n+1}.$$

Recall that the operator $(-\Delta)^s$ can be rewritten as

$$(-\Delta)^s(\cdot) \simeq \sum_{i=1}^n \partial_{x_i}\left(\frac{1}{|x|^{n+2s-2}}\right) *_n \partial_{x_i}(\cdot),$$

where $*_n$ indicates that the convolution is taken with respect the first n spatial variables. Therefore, by Theorem 1.2.4, since $s \in (1/2, 1)$, we have

$$\begin{aligned} \mathrm{I} &\lesssim \sum_{i=1}^n \left|\left\langle \partial_{x_i} P_s * T, \partial_{x_i}\left(\frac{1}{|x|^{n+2s-2}}\right) *_n (\varphi\phi)\right\rangle\right| \\ &= \sum_{i=1}^n |\langle \partial_{x_i} P_s * T, \partial_{x_i}[\mathcal{I}^n_{2-2s}(\varphi\phi)]\rangle| \lesssim \ell(R)^{n+1}, \end{aligned}$$

and we are done. □

REMARK 1.4.1. Let us observe that in the particular case in which T is compactly supported, we may simply convey that $Q := \mathbb{R}^{n+1}$ and $\varphi \equiv 1$ so that we deduce

$$|\langle T, \phi\rangle| \lesssim \ell(R)^{n+1},$$

for any R s-parabolic cube and ϕ admissible function for R. Therefore, bearing in mind Lemma 1.4.1, if $E \subset \mathbb{R}^{n+1}$ is a compact set with $\mathcal{H}^{n+1}_{p_s}(E) = 0$ and T is a distribution supported on E and satisfying the BMO_{p_s} estimates of Theorem 1.4.2, choosing $\ell_0 := \infty$ we get $T \equiv 0$.

THEOREM 1.4.3. *For any $s\in(1/2,1]$ and $E\subset\mathbb{R}^{n+1}$ compact set,*

$$\Gamma_{\Theta^s,*}(E)\approx\mathcal{H}^{n+1}_{\infty,p_s}(E).$$

Proof. Let us first prove

$$\Gamma_{\Theta^s,*}(E)\lesssim\mathcal{H}^{n+1}_{\infty,p_s}(E). \tag{1.4.2}$$

Proceed by fixing $\varepsilon>0$ and $\{A_k\}_k$ a collection of sets in $\mathbb{R}^{n+1}$ that cover E such that

$$\sum_{k=1}^{\infty}\mathrm{diam}_{p_s}(A_k)^{n+1}\leq\mathcal{H}^{n+1}_{\infty,p_s}(E)+\varepsilon.$$

Now, for each k let Q_k an open s-parabolic cube centered at some point $a_k\in A_k$ with side length $\ell(Q_k)=\mathrm{diam}_{p_s}(A_k)$, so that $E\subset\bigcup_k Q_k$. Apply the compactness of E and [**HPo, Lemma 3.1**] to consider $\{\varphi_k\}_{k=1}^N$ a collection of smooth functions satisfying, for each k: $0\leq\varphi_k\leq 1$, $\mathrm{supp}(\varphi_k)\subset 2Q_k$, $\sum_{k=1}^N\varphi_k=1$ in $\bigcup_{k=1}^N Q_k$ and also $\|\nabla_x\varphi_k\|_\infty\leq\ell(2Q_k)^{-1}$, $\|\partial_t\varphi_k\|\leq\ell(2Q_k)^{-2s}$. Hence, by Theorem 1.4.2, if T is any distribution admissible for $\Gamma_{\Theta^s,*}(E)$,

$$|\langle T,1\rangle|=\left|\sum_{k=1}^N\langle T,\varphi_k\rangle\right|\lesssim\sum_{k=1}^N\ell(2Q_k)^{n+1}\simeq\sum_{k=1}^N\mathrm{diam}_{p_s}(A_k)^{n+1}\leq\mathcal{H}^{n+1}_{\infty,p_s}(E)+\varepsilon.$$

Since this holds for any T and $\varepsilon>0$ can be arbitrarily small, (1.4.2) follows.

For the lower bound we will apply (an s-parabolic version of) Frostman's lemma [**Matti, Theorem 8.8**], which can be proved using an s-parabolic dyadic lattice, as it is presented in the proof of [**MatPT, Lemma 5.1**]. Assume then $\mathcal{H}^{n+1}_{\infty,p_s}(E)>0$ and consider a non trivial positive Borel regular measure μ supported on E with $\mu(E)\geq c\mathcal{H}^{n+1}_{\infty,p_s}(E)$ and $\mu(B(\overline{x},r))\leq r^{n+1}$ for all $\overline{x}\in\mathbb{R}^{n+1}$, $r>0$. If we prove that

$$\|\nabla_x P_s*\mu\|_{*,p_s}\lesssim 1\quad\text{ and }\quad\|\partial_t^{\frac{1}{2s}}P_s*\mu\|_{*,p_s}\lesssim 1,$$

we will be done, since this will imply $\Gamma_{\Theta^s,*}(E)\gtrsim\langle\mu,1\rangle=\mu(E)\gtrsim\mathcal{H}^{n+1}_{\infty,p_s}$. But by Lemma 1.3.2 we already have $\|\partial_t^{\frac{1}{2s}}P_s*\mu\|_{*,p_s}\lesssim 1$, so we are only left with the BMO_{p_s} norm of $\nabla_x P_s*\mu$. Thus, let us fix an s-parabolic ball $B(\overline{x}_0,r)$ and consider the characteristic function χ_{2B} associated to $2B$. Denote also $\chi_{2B^c}=1-\chi_{2B}$. In this setting, we pick

$$c_B:=\nabla_x P_s*(\chi_{2B^c}\mu)(\overline{x}_0).$$

Using Theorem 1.1.2 it easily follows that this last expression is well-defined. Let us now estimate $\|\nabla_x P_s*\mu\|_{*,p_s}$,

$$\begin{aligned}\frac{1}{|B|}\int_B&|\nabla_x P_s*\mu(\overline{y})-c_B|\,\mathrm{d}\overline{y}\\ &\leq\frac{1}{|B|}\int_B\left(\int_{2B}|\nabla_x P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\right)\mathrm{d}\overline{y}\\ &\qquad+\frac{1}{|B|}\int_B\left(\int_{\mathbb{R}^{n+1}\setminus 2B}|\nabla_x P_s(\overline{y}-\overline{z})-\nabla_x P_s(\overline{x}_0-\overline{z})|\mathrm{d}\mu(\overline{z})\right)\mathrm{d}\overline{y}=:\mathrm{I}+\mathrm{II}.\end{aligned}$$

To deal with I we first notice that by Theorem 1.1.2 and Tonelli's theorem we have

$$\mathrm{I} \lesssim \frac{1}{|B|}\int_{2B}\bigg(\int_B \frac{1}{|\overline{y}-\overline{z}|_{p_s}^{n+1}}\,\mathrm{d}\overline{y}\bigg)\mathrm{d}\mu(\overline{z}).$$

Writing $B = B_0\times I_0 \subset \mathbb{R}^n\times\mathbb{R}$, $\overline{y}=(y,t)$, $\overline{z}=(z,u)$ and choosing $0<\varepsilon<2s-1$, integration in polar coordinates yields

$$\mathrm{I} \lesssim \frac{1}{|B|}\int_{2B}\bigg(\int_{B_0}\frac{\mathrm{d}y}{|y-z|^{n-\varepsilon}}\int_{I_0}\frac{dt}{|t-u|^{\frac{1+\varepsilon}{2s}}}\bigg)\mathrm{d}\mu(\overline{z}) \lesssim \frac{1}{|B|}\big(r^{\varepsilon}(r^{2s})^{1-\frac{1+\varepsilon}{2s}}\big)\mu(2B)\lesssim 1.$$

Regarding II, we name $\overline{x} := \overline{x}_0-\overline{z}$ and $\overline{x}' := \overline{y}-\overline{z}$, and observe that $|\overline{x}-\overline{x}'|_{p_s}\leq |\overline{x}|_{p_s}/2$. Hence, we apply the fourth estimate in Theorem 1.1.2 with $2\zeta = 1$ since $s>1/2$, and obtain

$$\begin{aligned}\mathrm{II} &\lesssim \frac{1}{|B|}\int_B\bigg(\int_{\mathbb{R}^{n+1}\setminus 2B}\frac{|\overline{y}-\overline{x}_0|_{p_s}}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2}}\mathrm{d}\mu(\overline{z})\bigg)\,\mathrm{d}\overline{y} \leq r\int_{\mathbb{R}^{n+1}\setminus 2B}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2}}\\ &= r^{2\zeta}\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2}} \lesssim r\sum_{j=1}^{\infty}\frac{(2^{j+1}r)^{n+1}}{(2^jr)^{n+2}}\lesssim \sum_{j=1}^{\infty}\frac{1}{2^j}\lesssim 1,\end{aligned}$$

and we are done. □

THEOREM 1.4.4. *Let $s\in(1/2,1]$. A compact set $E\subset\mathbb{R}^{n+1}$ is removable for s-caloric functions with* BMO_{p_s}*-$(1,\frac{1}{2s})$-derivatives if and only if* $\Gamma_{\Theta^s,*}(E)=0$.

Proof. Fix $E\subset\mathbb{R}^{n+1}$ compact set and begin by assuming that is removable. Now pick T admissible for $\Gamma_{\Theta^s,*}(E)$ and observe that defining $f:=P_s*T$, we have $\|\nabla_x f\|_{*,p_s}<\infty$, $\|\partial_t^{\frac{1}{2s}}f\|_{*,p_s}<\infty$ and $\Theta^s f=0$ on $\mathbb{R}^{n+1}\setminus E$. So by hypothesis $\Theta^s f = 0$ in $\mathbb{R}^{n+1}$ and therefore $T\equiv 0$. Since T was an arbitrary admissible distribution for $\Gamma_{\Theta^s,*}(E)$, we deduce that $\Gamma_{\Theta^s,*}(E)=0$.

Let us now assume $\Gamma_{\Theta^s,*}(E)=0$ and prove the removability of E. Notice that by Theorem 1.4.3 we get $\mathcal{H}^{n+1}_{\infty,p_s}(E)=0$ and thus, by [**Matti, Lemma 4.6**], we have $\mathcal{H}^{n+1}_{p_s}(E)=0$. With this in mind, fix $\Omega\supset E$ any open set and $f:\mathbb{R}^{n+1}\to\mathbb{R}$ any function with $\|\nabla_x f\|_{*,p_s}<\infty$, $\|\partial_t^{\frac{1}{2s}}f\|_{*,p_s}<\infty$ and $\Theta^s f=0$ on $\Omega\setminus E$. We will assume $\Theta^s f\neq 0$ in Ω and reach a contradiction. Define the distribution

$$T:=\frac{\Theta^s f}{\|\nabla_x f\|_{*,p_s}+\|\partial_t^{\frac{1}{2s}}f\|_{*,p_s}},$$

which is such that $\|\nabla_x P_s*T\|_{*,p_s}\leq 1$, $\|\partial_t^{\frac{1}{2s}}P_s*T\|_{*,p_s}\leq 1$ and $\text{supp}(T)\subset E\cup\Omega^c$. Since $T\neq 0$ in Ω, there exists Q s-parabolic cube with $4Q\subset\Omega$ so that $T\neq 0$ in Q. Observe that $Q\cap E\neq\varnothing$. Then, by definition, there is φ test function supported on Q with $\langle T,\varphi\rangle>0$. Consider

$$\widetilde{\varphi}:=\frac{\varphi}{\|\varphi\|_\infty+\ell(Q)\|\nabla_x\varphi\|_\infty+\ell(Q)^{2s}\|\partial_t\varphi\|_\infty+\ell(Q)^2\|\Delta\varphi\|_\infty},$$

so that $\widetilde{\varphi}$ is admissible for Q. Apply Theorem 1.4.2 to deduce that $\widetilde{\varphi}T$ has upper s-parabolic growth of degree $n+1$ for cubes R with $\ell(R)\leq \ell(Q)$. Apply Lemma 1.4.1 to $\widetilde{\varphi}T$ with the compact set $\overline{Q}\cap E$, $\ell_0:=\ell(Q)$ and $d:=n+1$. Then,

$$|\langle \widetilde{\varphi}T,\psi\rangle| = 0, \qquad \forall \psi\in\mathcal{C}_c^\infty(\mathbb{R}^{n+1}),$$

since $\mathcal{H}^{n+1}_{p_s}(\overline{Q}\cap E)=0$. This would imply $\widetilde{\varphi}T\equiv 0$, which is impossible, since $\langle\varphi,T\rangle>0$. Therefore $\Theta^s f=0$ in Ω, and by the arbitrariness of Ω and f we are done. □

1.4.2 The capacity $\Gamma_{\Theta^s,\alpha}$

We shall now present an s-parabolic Lip_α variant of the caloric capacity presented above.

Definition 1.4.3. Given $s\in(1/2,1]$, $\alpha\in(0,1)$ and $E\subset\mathbb{R}^{n+1}$ compact set, define its *Lip_{α,p_s}-caloric capacity* as

$$\Gamma_{\Theta^s,\alpha}(E):=\sup|\langle T,1\rangle|,$$

where the supremum is taken among all distributions T with $\mathrm{supp}(T)\subset E$ and satisfying

$$\|\partial_{x_i}P_s*T\|_{\mathrm{Lip}_{\alpha},p_s}\leq 1,\;\; \forall i=1,\ldots,n, \qquad \|\partial_t^{\frac{1}{2s}}P_s*T\|_{\mathrm{Lip}_{\alpha,p_s}}\leq 1.$$

Such distributions will be called *admissible for* $\Gamma_{\Theta^s,\alpha}(E)$.

Definition 1.4.4. A compact set $E\subset\mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with Lip_{α,p_s}-$(1,\frac{1}{2s})$-derivatives* if for any open subset $\Omega\subset\mathbb{R}^{n+1}$, any function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ with

$$\|\nabla_x f\|_{\mathrm{Lip}_{\alpha,p_s}}<\infty, \qquad \|\partial_t^{\frac{1}{2s}}f\|_{\mathrm{Lip}_{\alpha,p_s}}<\infty,$$

satisfying the Θ^s-equation in $\Omega\setminus E$, also satisfies the previous equation in the whole Ω.

As in the s-parabolic BMO case, if T is a compactly supported distribution satisfying the required normalization conditions, T will present upper s-parabolic growth of degree $n+1+\alpha$. In fact, the following result holds:

Theorem 1.4.5. *Let $s\in(1/2,1]$, $\alpha\in(0,2s-1)$ and T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|\partial_{x_i}P_s*T\|_{\mathrm{Lip}_{\alpha},p_s}\leq 1,\;\; \forall i=1,\ldots,n, \qquad \|\partial_t^{\frac{1}{2s}}P_s*T\|_{\mathrm{Lip}_{\alpha,p_s}}\leq 1.$$

Let Q be a fixed s-parabolic cube and φ admissible for Q. Then, if R is any s-parabolic cube with $\ell(R)\leq\ell(Q)$ and ϕ is admissible for R, we have $|\langle\varphi T,\phi\rangle|\lesssim_\alpha \ell(R)^{n+1+\alpha}$.

Proof. Let T, Q and φ be as above. Let us also consider R s-parabolic cube with $\ell(R)\leq\ell(Q)$ and $R\cap Q\neq\varnothing$ and ϕ admissible function for R. We proceed as in the proof of Theorem 1.4.2 to obtain

$$|\langle\varphi T,\phi\rangle|\leq|\langle(-\Delta)^s P_s*T,\varphi\phi\rangle|+|\langle P_s*T,\partial_t(\varphi\phi)\rangle| =: \mathrm{I}+\mathrm{II}.$$

Regarding II, we now define define $\beta := 1 - \frac{1}{2s}$ and observe that $2s\beta = 2s - 1 > \alpha$, so applying Theorem 1.2.1 we get $\mathrm{II} \lesssim_\alpha \ell(R)^{n+1+\alpha}$. The study of I is also analogous to that done in Theorem 1.4.2. The case $s = 1$ follows in exactly the same way by Theorem 1.2.2, and if $s \in (1/2, 1)$ we also have

$$\mathrm{I} \lesssim \sum_{i=1}^{n} \big|\big\langle \partial_{x_i} P_s * T, \partial_{x_i}[\mathcal{I}^n_{2-2s}(\varphi\phi)]\big\rangle\big|.$$

So by Theorem 1.2.4 and condition $\alpha < 2s - 1$ we deduce the desired result. □

THEOREM 1.4.6. *For any $s \in (1/2, 1]$, $\alpha \in (0, 2s-1)$ and $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\Gamma_{\Theta^s,\alpha}(E) \approx_\alpha \mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E).$$

Proof. For the upper bound we proceed analogously as we have done in the proof of Theorem 1.4.3, using now the growth restriction given by Theorem 1.4.5. So we focus on the lower bound, which will also rely on Frostman's lemma. Assume then $\mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E) > 0$ and consider a non trivial positive Borel measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E)$ and $\mu(B(\overline{x}, r)) \leq r^{n+1+\alpha}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. It is enough to check

$$\|\partial_{x_i} P_s * \mu\|_{\mathrm{Lip}_\alpha, p_s} \lesssim_\alpha 1, \;\; \forall i = 1, \ldots, n \qquad \text{and} \qquad \|\partial_t^{\frac{1}{2s}} P_s * \mu\|_{\mathrm{Lip}_\alpha, p_s} \lesssim_\alpha 1.$$

Notice that the right inequality follows directly from Lemma 1.3.3 with $\beta := \frac{1}{2s}$, so we just focus on controlling the s-parabolic Lip_α seminorm of the spatial derivatives of $P_s * \mu$. Fix $i = 1, \ldots, n$ and choose any $\overline{x}, \overline{y} \in \mathbb{R}^{n+1}$ with $\overline{x} \neq \overline{y}$. Consider the following partition

$$R_1 := \big\{\overline{z} \;:\; |\overline{x} - \overline{y}|_{p_s} \leq |\overline{x} - \overline{z}|_{p_s}/2\big\} \cup \big\{z \;:\; |\overline{y} - \overline{x}|_{p_s} \leq |\overline{y} - \overline{z}|_{p_s}/2\big\},$$
$$R_2 := \mathbb{R}^{n+1} \setminus R_1 = \big\{z \;:\; |\overline{x} - \overline{y}|_{p_s} > |\overline{x} - \overline{z}|_{p_s}/2\big\} \cap \big\{z \;:\; |\overline{y} - \overline{x}|_{p_s} > |\overline{y} - \overline{z}|_{p_s}/2\big\},$$

with their corresponding characteristic functions χ_1, χ_2 respectively. This way, we have

$$\begin{aligned}
&\frac{|\partial_{x_i} P_s * \mu(\overline{x}) - \partial_{x_i} P_s * \mu(\overline{y})|}{|\overline{x} - \overline{y}|^\alpha_{p_s}} \\
&\qquad \leq \frac{1}{|\overline{x} - \overline{y}|^\alpha_{p_s}} \int_{|\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2} |\partial_{x_i} P_s(\overline{x} - \overline{z}) - \partial_{x_i} P_s(\overline{y} - \overline{z})| \mathrm{d}\mu(\overline{z}) \\
&\qquad\quad + \frac{1}{|\overline{x} - \overline{y}|^\alpha_{p_s}} \int_{|\overline{y}-\overline{x}|_{p_s} \leq |\overline{y}-\overline{z}|_{p_s}/2} |\partial_{x_i} P_s(\overline{x} - \overline{z}) - \partial_{x_i} P_s(\overline{y} - \overline{z})| \mathrm{d}\mu(\overline{z}) \\
&\qquad\quad + \frac{1}{|\overline{x} - \overline{y}|^\alpha_{p_s}} \int_{R_2} |\partial_{x_i} P_s(\overline{x} - \overline{z}) - \partial_{x_i} P_s(\overline{y} - \overline{z})| \mathrm{d}\mu(\overline{z}) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Regarding I, apply the fourth estimate of Theorem 1.1.2 to obtain

$$\mathrm{I} \lesssim \frac{1}{|\overline{x} - \overline{y}|^\alpha_{p_s}} \int_{|\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2} \frac{|\overline{x} - \overline{y}|_{p_s}}{|\overline{x} - \overline{z}|^{n+2}_{p_s}} \mathrm{d}\mu(\overline{z}).$$

Split the previous domain of integration into the s-parabolic annuli

$$A_j := 2^{j+1}B\big(\overline{x},|\overline{x}-\overline{y}|_{p_s}\big)\setminus 2^jB\big(\overline{x},|\overline{x}-\overline{y}|_{p_s}\big), \quad \text{for} \quad j\geq 1,$$

and use that μ has upper parabolic growth of degree $n+1+\alpha$ to deduce

$$\begin{aligned}\mathrm{I} &\lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-1}}\sum_{j=1}^\infty\int_{A_j}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+2}} \lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-1}}\sum_{j=1}^\infty\frac{(2^{j+1}|\overline{x}-\overline{y}|_{p_s})^{n+1+\alpha}}{(2^j|\overline{x}-\overline{y}|_{p_s})^{n+2}}\\ &\lesssim \sum_{j=1}^\infty\frac{1}{2^{(1-\alpha)j}}\lesssim_\alpha 1.\end{aligned}$$

The study of II is analogous interchanging the roles of $\overline{x}$ and $\overline{y}$. Finally, for III, we apply the first estimate of Theorem 1.1.2 so that

$$\begin{aligned}\mathrm{III} &\lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\int_{R_2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+1}}+\frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\int_{R_2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|_{p_s}^{n+1}}\\ &\leq \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\int_{|\overline{x}-\overline{y}|_{p_s}>|\overline{x}-\overline{z}|_{p_s}/2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+1}}+\frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\int_{|\overline{y}-\overline{x}|_{p_s}>|\overline{y}-\overline{z}|_{p_s}/2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|_{p_s}^{n+1}}\\ &=: \mathrm{III}_1+\mathrm{III}_2.\end{aligned}$$

Concerning III_1, split the domain of integration into the s-parabolic annuli

$$\widetilde{A}_j := 2^{-j}B\big(\overline{x},|\overline{x}-\overline{y}|_{p_s}\big)\setminus 2^{-j-1}B\big(\overline{x},|\overline{x}-\overline{y}|_{p_s}\big), \quad \text{for} \quad j\geq -1.$$

Thus, in this case we have

$$\begin{aligned}\mathrm{III}_1 &\lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\sum_{j=-1}^\infty\int_{\widetilde{A}_j}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+1}} \lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\sum_{j=-1}^\infty\frac{(2^{-j}|\overline{x}-\overline{y}|_{p_s})^{n+1+\alpha}}{(2^{-j-1}|\overline{x}-\overline{y}|_{p_s})^{n+1}}\\ &\lesssim \sum_{j=-1}^\infty\frac{1}{2^{\alpha j}}\lesssim_\alpha 1.\end{aligned}$$

On the other hand, for III_2 we apply the same reasoning but using the partition

$$\widetilde{A}_j' := 2^{-j}B_p\big(\overline{y},|\overline{y}-\overline{x}|_{p_s}\big)\setminus 2^{-j-1}B_p\big(\overline{y},|\overline{y}-\overline{x}|_{p_s}\big), \quad \text{for} \quad j\geq -1,$$

so that $\mathrm{III}_2\lesssim 1$. Thus, combining the estimates obtained for I,II and III we deduce

$$\frac{|\partial_{x_i}P_s*\mu(\overline{x})-\partial_{x_i}P_s*\mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}}\lesssim_\alpha 1,$$

and since the (different) points $\overline{x}$ and $\overline{y}$ were arbitrarily chosen, we deduce the desired s-parabolic Lip_α condition. ☐

THEOREM 1.4.7. *Let $s\in(1/2,1]$ and $\alpha\in(0,2s-1)$. A compact set $E\subset\mathbb{R}^{n+1}$ is removable for s-caloric functions with $\mathrm{Lip}_{\alpha,p_s}$-$(1,\frac{1}{2s})$-derivatives if and only if $\Gamma_{\Theta^s,\alpha}(E)=0$.*

Proof. The proof is completely analogous to that of Theorem 1.4.4, now using Theorems 1.4.5 and 1.4.6, as well as Lemma 1.4.1 with $d:=n+1+\alpha$. ☐

1.4.3 The capacity $\gamma^\sigma_{\Theta^s,*}$

Now, we shall present the BMO_{p_s} variant of the capacities presented in [**MatP, §4 & §7**]. To be precise, in the aforementioned reference, Mateu and Prat work with the normalization conditions

$$\|(-\Delta)^{s-\frac{1}{2}}P_s * T\|_\infty \leq 1, \qquad \|\partial_t^{1-\frac{1}{2s}}P_s * T\|_{*,p_s} \leq 1,$$

allowing $s \in [1/2, 1)$. In our case we will deal with its s-parabolic BMO variant and we define it more generally as follows:

Definition 1.4.5. Given $s \in (0,1]$, $\sigma \in [0,s)$ and $E \subset \mathbb{R}^{n+1}$ compact set, define its Δ^σ-BMO_{p_s}*-caloric capacity* as

$$\gamma^\sigma_{\Theta^s,*}(E) := \sup |\langle T, 1\rangle|,$$

where the supremum is taken among all distributions T with $\mathrm{supp}(T) \subset E$ and satisfying

$$\|(-\Delta)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1.$$

Such distributions will be called *admissible for* $\gamma^\sigma_{\Theta^s,*}(E)$.

Definition 1.4.6. Let $s \in (0,1]$ and $\sigma \in [0,s)$. A compact set $E \subset \mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with* BMO_{p_s}*-*$(\sigma, \sigma/s)$*-Laplacian* if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|(-\Delta)^\sigma f\|_{*,p_s} < \infty, \qquad \|\partial_t^{\sigma/s} f\|_{*,p_s} < \infty,$$

satisfying the Θ^s-equation in $\Omega \setminus E$, also satisfies the previous equation in the whole Ω. If $\sigma = 0$, we will also say that E is *removable for* BMO_{p_s} *s-caloric functions.*

Firstly, we shall prove that if T is a compactly supported distribution satisfying the expected normalization conditions, then T has upper s-parabolic growth of degree $n + 2s - 2\sigma$. In fact, we prove a stronger result:

Theorem 1.4.8. *Let $s \in (0,1]$, $\sigma \in [0,s)$ and T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|(-\Delta)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim_\sigma \ell(R)^{n+2\sigma}$.

Proof. Let T, Q and φ be as above, as well as R s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$, and ϕ admissible function for R. We already know, in light of the proof of Theorem 1.4.2,

$$|\langle \varphi T, \phi\rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi\rangle| + |\langle P_s * T, \partial_t(\varphi\phi)\rangle| =: \mathrm{I} + \mathrm{II}.$$

For I, simply apply Theorem 1.2.3 with $\beta := s - \sigma$ so that

$$\mathrm{I} = |\langle (-\Delta)^\sigma P_s * T, (-\Delta)^{s-\sigma}(\varphi\phi)\rangle| \lesssim_\sigma \ell(R)^{n+2\sigma}.$$

Regarding II, if $\sigma > 0$, observe that defining $\beta := 1 - \sigma/s \in (0,1)$ we get

$$\partial_t(\varphi\phi) \simeq_\sigma \partial_t^{1-\beta}\Big(\partial_t(\varphi\phi) *_t |t|^{-\beta}\Big),$$

so by Theorem 1.2.1 we are done. If $\sigma = 0$, we simply have

$$\begin{aligned}\mathrm{II} = |\langle P_s * T - (P_s * T)_R, \partial_t(\varphi\phi)\rangle| &\leq \int_{Q\cap R} \big|P_s * T(\overline{x}) - (P * T)_R\big|\big|\partial_t(\varphi\phi)(\overline{x})\big|\,\mathrm{d}\overline{x} \\ \leq \ell(R)^{-2s}\int_R \big|P * T(\overline{x}) - (P*T)_R\big|\,\mathrm{d}\overline{x} &\leq \ell(R)^{-2s}\ell(R)^{n+2s}\|P_s * T\|_{*,p_s} \leq \ell(R)^n.\end{aligned}$$

□

Theorem 1.4.9. *For any $s \in (0,1]$, $\sigma \in [0,s)$ and $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\gamma^\sigma_{\Theta^s,*}(E) \approx_\sigma \mathcal{H}^{n+2\sigma}_{\infty,p_s}(E).$$

Proof. Again, for the upper bound we proceed analogously as in the proof of Theorem 1.4.3, using now Theorem 1.4.8. For the lower bound, we apply Frostman's lemma. Assume then $\mathcal{H}^{n+2\sigma}_{\infty,p_s}(E) > 0$ and consider a non trivial positive Borel measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+2\sigma}_{\infty,p_s}(E)$ and $\mu(B(\overline{x},r)) \leq r^{n+2\sigma}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. We have to prove

$$\|(-\Delta)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1,$$

If $\sigma > 0$, by Lemma 1.3.2 with $\beta := \sigma/s$ we already have $\|\partial_t^{\sigma/s} P_s * \mu\|_{*,p_s} \lesssim_\sigma 1$. So we are left to control the BMO_{p_s} norm of $(-\Delta)^\sigma P_s * \mu$ for $\sigma \in [0,s)$. Thus, let us fix an s-parabolic ball $B(\overline{x}_0, r)$ and consider the characteristic function χ_{2B} associated to $2B$. Set also $\chi_{2B^c} = 1 - \chi_{2B}$. In this setting, we pick

$$c_B := (-\Delta)^\sigma P_s * (\chi_{2B^c}\mu)(\overline{x}_0).$$

Using Theorem 1.1.3 it easily follows that this last expression is well-defined. We estimate $\|(-\Delta)^\sigma P_s * \mu\|_{*,p_s}$ using the previous constant:

$$\begin{aligned}&\frac{1}{|B|}\int_B |(-\Delta)^\sigma P_s * \mu(\overline{y}) - c_B|\,\mathrm{d}\overline{y} \\ &\quad\leq \frac{1}{|B|}\int_B \bigg(\int_{2B} |(-\Delta)^\sigma P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\bigg)\,\mathrm{d}\overline{y} \\ &\qquad + \frac{1}{|B|}\int_B \bigg(\int_{\mathbb{R}^{n+1}\setminus 2B} |(-\Delta)^\sigma P_s(\overline{y}-\overline{z}) - (-\Delta)^\sigma P_s(\overline{x}_0-\overline{z})|\mathrm{d}\mu(\overline{z})\bigg)\,\mathrm{d}\overline{y} =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

To deal with I we first notice that by Theorem 1.1.3 and arguing as in Theorem 1.4.3, choosing $0<\varepsilon<2(s-\sigma)$ we have

$$\mathrm{I} \lesssim_\sigma \frac{1}{|B|}\int_{2B}\bigg(\int_B \frac{\mathrm{d}\overline{y}}{|\overline{y}-\overline{z}|_{p_s}^{n+2\sigma}}\bigg)\mathrm{d}\mu(\overline{z}) \lesssim \frac{1}{|B|}\big(r^\varepsilon (r^{2s})^{1-\frac{\varepsilon+2\sigma}{2s}}\big)\mu(2B)\lesssim 1,$$

by the $n+2\sigma$ growth of μ. Regarding II, notice that naming $\overline{x}:=\overline{x}_0-\overline{z}$ and $\overline{x}':=\overline{y}-\overline{z}$, we have $|\overline{x}-\overline{x}'|_{p_s}\leq |\overline{x}|_{p_s}/2$ so we can apply the fifth estimate of Theorem 1.1.3, that implies

$$\mathrm{II} \lesssim_\sigma \frac{1}{|B|}\int_B\bigg(\int_{\mathbb{R}^{n+1}\setminus 2B}\frac{|\overline{y}-\overline{x}_0|_{p_s}^{2\zeta}}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2\sigma+2\zeta}}\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\overline{y} \leq r^{2\zeta}\int_{\mathbb{R}^{n+1}\setminus 2B}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2\sigma+2\zeta}}\lesssim_\sigma 1,$$

again by the by the $n+2\sigma$ growth of μ. □

THEOREM 1.4.10. *Let $s\in(0,1]$ and $\sigma\in[0,s)$. A compact set $E\subset\mathbb{R}^{n+1}$ is removable for s-caloric functions with* BMO_{p_s}*-$(\sigma,\sigma/s)$-Laplacian if and only if $\gamma^\sigma_{\Theta^s,*}(E)=0$.*

Proof. The proof is analogous to that of Theorem 1.4.4, applying Theorems 1.4.8, 1.4.9 and Lemma 1.4.1 with $d:=n+2\sigma$. □

1.4.4 The capacity $\gamma^\sigma_{\Theta^s,\alpha}$

We define now a capacity with an s-parabolic Lip_α normalization condition.

DEFINITION 1.4.7. Given $\alpha\in(0,1)$, $s\in(0,1]$, $\sigma\in[0,s)$ and $E\subset\mathbb{R}^{n+1}$ compact set, define its Δ^σ-$\mathrm{Lip}_{\alpha,p_s}$*-caloric capacity* as

$$\gamma^\sigma_{\Theta^s,\alpha}(E):=\sup|\langle T,1\rangle|,$$

where the supremum is taken among all distributions T with $\mathrm{supp}(T)\subset E$ and satisfying

$$\|(-\Delta)^\sigma P_s * T\|_{\mathrm{Lip}_\alpha,p_s}\leq 1,\qquad \|\partial_t^{\sigma/s}P_s * T\|_{\mathrm{Lip}_\alpha,p_s}\leq 1.$$

Such distributions will be called *admissible for* $\gamma^\sigma_{\Theta^s,\alpha}(E)$.

DEFINITION 1.4.8. Let $\alpha\in(0,1)$, $s\in(0,1]$ and $\sigma\in[0,s)$. A compact set $E\subset\mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with* $\mathrm{Lip}_{\alpha,p_s}$*-$(\sigma,\sigma/s)$-Laplacian* if for any open subset $\Omega\subset\mathbb{R}^{n+1}$, any function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ with

$$\|(-\Delta)^\sigma f\|_{\mathrm{Lip}_\alpha,p_s}<\infty,\qquad \|\partial_t^{\sigma/s}f\|_{\mathrm{Lip}_\alpha,p_s}<\infty,$$

satisfying the Θ^s-equation in $\Omega\setminus E$, also satisfies the previous equation in the whole Ω. If $\sigma=0$, we will also say that E is *removable for* $\mathrm{Lip}_{\alpha,p_s}$ *s-caloric functions*.

If T is a compactly supported distribution satisfying the above properties, then T presents upper s-parabolic growth of degree $n+2\sigma+\alpha$. As in §1.4.2, the following result will only be valid for a certain range of values of α, dependent on s and σ.

THEOREM 1.4.11. *Let $s \in (0,1]$, $\sigma \in [0,s)$ and $\alpha \in (0,1)$ with $\alpha < 2s - 2\sigma$. Let T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|(-\Delta)^\sigma P_s * T\|_{\mathrm{Lip}_\alpha, p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{\mathrm{Lip}_\alpha, p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim \ell(R)^{n+2\sigma+\alpha}$.

Proof. Let T, Q and φ be as above, as well as R s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$, and ϕ admissible function for R. Again,

$$|\langle \varphi T, \phi\rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi\rangle| + |\langle P_s * T, \partial_t(\varphi\phi)\rangle| =: \mathrm{I} + \mathrm{II}.$$

For I, simply apply Theorem 1.2.3 with $\beta := s - \sigma$ so that

$$\mathrm{I} = |\langle (-\Delta)^\sigma P_s * T, (-\Delta)^{s-\sigma}(\varphi\phi)\rangle| \lesssim_{\sigma,\alpha} \ell(R)^{n+2\sigma+\alpha}.$$

Regarding II, if $\sigma > 0$, we define $\beta := 1 - \sigma/s \in (0,1)$ and apply Theorem 1.2.1. If $\sigma = 0$, let $\overline{x}_R$ be the center of R and notice that

$$\mathrm{II} = |\langle P_s * T - P_s * T(\overline{x}_R), \partial_t(\varphi\phi)\rangle| \leq \ell(R)^{-2s} \int_R |\overline{x} - \overline{x}_R|_{p_s}^\alpha \, \mathrm{d}\overline{x} \lesssim \ell(R)^{n+\alpha}.$$

□

THEOREM 1.4.12. *Let $s \in (0,1]$, $\sigma \in [0,s)$ and $\alpha \in (0,1)$ with $\alpha < 2s - 2\sigma$. Then, for $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\gamma^\sigma_{\Theta^s,\alpha}(E) \approx_{\sigma,\alpha} \mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E).$$

Proof. For the upper bound we argue again as in Theorem 1.4.3, using now Theorem 1.4.11. For the lower bound, assume $\mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E) > 0$ and apply Frostman's lemma to consider a non trivial positive Borel measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E)$ and $\mu(B(\overline{x},r)) \leq r^{n+2\sigma+\alpha}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. It suffices to verify

$$\|(-\Delta)^\sigma P_s * T\|_{\mathrm{Lip}_\alpha, p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{\mathrm{Lip}_\alpha, p_s} \leq 1.$$

If $\sigma > 0$, by Lemma 1.3.3 with $\beta := \sigma/s$ we already have $\|\partial_t^{\sigma/s} P_s * \mu\|_{\mathrm{Lip}_\alpha, p_s} \lesssim_{\sigma,\alpha} 1$. So we are left to estimate $\|(-\Delta)^\sigma P_s * \mu\|_{\mathrm{Lip}_\alpha, p_s}$ for $\sigma \in [0,s)$, and we do it as in Theorem 1.4.6. Let $\overline{x}, \overline{y} \in \mathbb{R}^{n+1}$ with $\overline{x} \neq \overline{y}$ and consider the following partition of $\mathbb{R}^{n+1}$,

$$\begin{aligned} R_1 :&= \big\{\overline{z} \,:\, |\overline{x} - \overline{y}|_{p_s} \leq |\overline{x} - \overline{z}|_{p_s}/2\big\} \cup \big\{\overline{z} \,:\, |\overline{y} - \overline{x}|_{p_s} \leq |\overline{y} - \overline{z}|_{p_s}/2\big\}, \\ R_2 := \mathbb{R}^{n+1} \setminus R_1 &= \big\{\overline{z} \,:\, |\overline{x} - \overline{y}|_{p_s} > |\overline{x} - \overline{z}|_{p_s}/2\big\} \cap \big\{\overline{z} \,:\, |\overline{y} - \overline{x}|_{p_s} > |\overline{y} - \overline{z}|_{p_s}/2\big\}, \end{aligned}$$

with their corresponding characteristic functions χ_1, χ_2 respectively. This way, we have

$$\begin{aligned}
&\frac{|(-\Delta)^\sigma P_s * \mu(\overline{x}) - (-\Delta)^\sigma P_s * \mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\\
&\qquad \leq \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{|\overline{x}-\overline{y}|_{p_s}\leq |\overline{x}-\overline{z}|_{p_s}/2} |(-\Delta)^\sigma P_s(\overline{x}-\overline{z}) - (-\Delta)^\sigma P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\\
&\qquad\quad + \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{|\overline{y}-\overline{x}|_{p_s}\leq |\overline{y}-\overline{z}|_{p_s}/2} |(-\Delta)^\sigma P_s(\overline{x}-\overline{z}) - (-\Delta)^\sigma P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\\
&\qquad\quad + \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{R_2} |(-\Delta)^\sigma P_s(\overline{x}-\overline{z}) - (-\Delta)^\sigma P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z}) =: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

Regarding I, the fifth estimate of Theorem 1.1.3 yields

$$\mathrm{I} \lesssim_\sigma \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{|\overline{x}-\overline{y}|_{p_s}\leq |\overline{x}-\overline{z}|_{p_s}/2} \frac{|\overline{x}-\overline{y}|_{p_s}^{2\zeta}}{|\overline{x}-\overline{z}|_{p_s}^{n+2\sigma+2\zeta}}\mathrm{d}\mu(\overline{z}).$$

Split the previous domain of integration into s-parabolic annuli centered at $\overline{x}$ with exponentially increasing radii proportional to $|\overline{x}-\overline{y}|_{p_s}$, and deduce as in Theorem 1.4.6 that $\mathrm{I} \lesssim_{\sigma,\alpha} 1$, using now that μ has $n+2\sigma+\alpha$ growth. For II, we argue as in I just interchanging the roles of $\overline{x}$ and $\overline{y}$. Finally, for III, the first estimate of Theorem 1.1.3 yields

$$\begin{aligned}
\mathrm{III} &\leq \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{R_2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+2\sigma}} + \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\int_{R_2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|_{p_s}^{n+2\sigma}}\\
&\leq \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\bigg(\int_{|\overline{x}-\overline{y}|_{p_s}> |\overline{x}-\overline{z}|_{p_s}/2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+2\sigma}} + \int_{|\overline{y}-\overline{x}|_{p_s}> |\overline{y}-\overline{z}|_{p_s}/2}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|_{p_s}^{n+2\sigma}}\bigg)\\
&=: \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha}(\mathrm{III}_1+\mathrm{III}_2).
\end{aligned}$$

III_1 and III_2 can be dealt with by splitting the domain of integration into exponentially decreasing annuli, centered at $\overline{x}$ and $\overline{y}$ respectively, and using that μ has growth of degree strictly bigger than $n+2\sigma$. Thus, we obtain $\mathrm{III}_1+\mathrm{III}_2 \lesssim_{\sigma,\alpha} 1$ and we are done □

THEOREM 1.4.13. *Let $s\in(0,1]$, $\sigma\in[0,s)$ and $\alpha\in(0,1)$ with $\alpha<2s-2\sigma$. A compact set $E\subset\mathbb{R}^{n+1}$ is removable for s-caloric functions with* $\mathrm{Lip}_{\alpha,p_s}$-$(\sigma,\sigma/s)$-*Laplacian if and only if $\gamma^\sigma_{\Theta^s,\alpha}(E)=0$.*

Proof. The proof is analogous to that of Theorem 1.4.4, applying Theorems 1.4.11, 1.4.12 and Lemma 1.4.1 with $d:=n+2\sigma+\alpha$. □

Chapter 2

The Lipschitz s-caloric capacity

The concept of $(1,1/2)$-Lipschitz caloric capacity, introduced by Mateu, Prat, and Tolsa, has proven to be a valuable tool for characterizing removable sets in the context of the classical heat equation. This chapter extends their framework to the s-fractional setting. Our goal is to define and study the $(1,\frac{1}{2s})$-Lipschitz s-caloric capacity, denoted by Γ_{Θ^s} and explore its role in determining the critical dimension of removable sets for $(1,\frac{1}{2s})$-Lipschitz s-caloric functions. These are functions such that their s-parabolic gradient $(\nabla_x, \partial_t^{\frac{1}{2s}})$ is bounded and that satisfy the Θ^s-equation, that is, that associated with the operator

$$\Theta^s := (-\Delta)^s + \partial_t, \quad s\in(0,1].$$

Our study, however, will only consider the cases $s\in(1/2,1]$. In fact, most of the arguments that we develop below are inspired by those found in [**MatPT**], and they break down if one considers the limit $s\to 1/2$ (in the sense that most implicit constants blow up). Hence, the study of Lipschitz s-caloric functions and their associated capacity remains still an open problem in the regime $s\in(0,1/2]$. The author conjectures that in the particular case $s=1/2$, where the s-parabolic gradient simplifies to the usual gradient and the s-parabolic distance is nothing but the Euclidean one, the associated capacity of a compact set should be comparable to the Lebesgue measure of the ambient space restricted to the latter. The results obtained in §2.3.1 regarding Cantor sets together with the work carried out by Uy in [**Uy**] for analytic capacity point towards this precise comparability.

Let us begin by generalizing the notion of (1,1/2)-Lipschitz function of [**MatPT**]. These are functions that exhibit Lipschitz continuity in the spatial variables and a fractional Hölder-type regularity in time, specifically of order 1/2. That is, functions $f:\mathbb{R}^{n+1}\to\mathbb{R}$ with $\|\nabla_x f\|_\infty\lesssim 1$ and $\|f\|_{\text{Lip}_{1/2,t}}\lesssim 1$. Nevertheless, in [**MatPT**] the authors are mainly interested in functions such that

$$\|\nabla_x f\|_\infty \lesssim 1 \qquad \text{and} \qquad \|\partial_t^{1/2} f\|_{*,p_1}\lesssim 1.$$

In fact, the previous restrictions motivated those imposed in §1.4.1 and §1.4.2, choosing $s=1$. In [**MatPT**], the authors invoke [**Ho1, Lemma 1**] and [**HoL, Theorem 7.4**] to justify that the above bounds are more general and imply the desired $(1, 1/2)$-Lipschitz regularity. Functions satisfying these regularity conditions naturally arise in the study of the heat equation (see the works of Nyström, Strömqvist, Hofmann, Lewis, Murray and Silver, for example [**NySt, LeSi, LeMu, HoL, Ho2**]) and serve as the basis for defining its s-fractional variant. More precisely, we aim at introducing an analogous capacity for solutions of the Θ^s-equation, $1/2 < s < 1$, now with $(1, \frac{1}{2s})$-Lipschitz regularity, i.e.

$$\|\nabla_x f\|_\infty \lesssim 1 \qquad \text{and} \qquad \|f\|_{\mathrm{Lip}_{\frac{1}{2s},t}} \lesssim 1.$$

In §2.1 we begin our analysis by noting that the results of Hofmann and Lewis can be adapted to the fractional caloric case. In §2.2 we localize $(1, \frac{1}{2s})$-Lipschitz s-caloric potentials and with this, in §2.3 we prove our main results:

Theorem. *Let $s \in (1/2, 1]$ and $E \subset \mathbb{R}^{n+1}$ compact set. Then,*

1. *The critical s-parabolic Hausdorff dimension of Γ_{Θ^s} is $n+1$.*
2. *E is removable for Lipschitz s-caloric functions if and only if $\Gamma_{\Theta^s}(E) = 0$.*

From this point on and until the end of this chapter, we fix $s \in (1/2, 1)$. The forthcoming results for the 1-parabolic case are already covered in [**MatPT**].

2.1 The Γ_{Θ^s} capacity. Generalizing the $(1, \frac{1}{2s})$-Lipschitz condition

Let us begin by introducing what will be an equivalent definition of the s-parabolic norm, already presented in [**Ho1**]. For $\overline{x} := (x, t) \in \mathbb{R}^{n+1}$, the quantity $\|\overline{x}\|_{p_s}$ is defined to be the only positive solution of

$$1 = \sum_{j=1}^n \frac{x_j^2}{\|\overline{x}\|_{p_s}^2} + \frac{t^2}{\|\overline{x}\|_{p_s}^{4s}}. \tag{2.1.1}$$

If $s = 1$, one recovers the proper parabolic norm defined in [**Ho1**], which admits the explicit expression

$$\|\overline{x}\|_{p_1}^2 = \frac{1}{2}\Big(|x|^2 + \sqrt{|x|^4 + 4t^2}\Big).$$

This quantity is comparable to the parabolic norm we have been working with throughout this text, $|\overline{x}|_{p_1}^2 := |x|^2 + |t|$, which in turn is also the chosen expression in [**MatPT**]. In [**MatP**], Mateu and Prat study the case $s < 1$ and introduce the following quantity comparable to $\|\overline{x}\|_{p_s}^2$,

$$|\overline{x}|_{p_s}^2 := |x|^2 + |t|^{1/s},$$

that is precisely the expression we will use for the s-parabolic norm in this text. However, the choice of $\|\cdot\|_{p_s}$ instead of $|\cdot|_{p_s}$ presents some advantages regarding some Fourier representation formulae of certain operators, as we will see in the sequel.

As in [**Ho1**] we define the s-parabolic fractional integral operator of order 1 as

$$(I_{p_s} f)^\wedge(\overline{\xi}) := \|\overline{\xi}\|_{p_s}^{-1} \widehat{f}(\overline{\xi}),$$

and its inverse

$$(D_{p_s} f)^\wedge(\overline{\xi}) := \|\overline{\xi}\|_{p_s} \widehat{f}(\overline{\xi}).$$

Then, multiplying both sides of (2.1.1) by $\|x\|_{p_s}$ and taking the Fourier transform we obtain the following identity between operators

$$D_{p_s} \simeq \sum_{j=1}^{n} R_{j,s}\partial_j + R_{n+1,s}D_{n+1,s}, \tag{2.1.2}$$

where

$$(R_{j,s})^\wedge := \frac{\xi_j}{\|\overline{\xi}\|_{p_s}}, \qquad j = 1, \dots, n,$$

are the s-parabolic Riesz transforms. As observed in the comments that follow [**HoL, Eq.(2.10)**], the operators $R_{j,s}$ satisfy analogous properties to those of the usual Riesz transforms. Indeed, observe that their symbols are antisymmetric and s-parabolically homogeneous of degree 0. Thus, $R_{j,s}$ are defined via convolution (in a principal value sense) against odd kernels, s-parabolically homogeneous of degree $-n-2s$, which are bounded in L^p and BMO_{p_s} (see [**Pe, Remark 1.3**]). We have also set $\overline{\xi} := (\xi, \tau)$ and defined

$$(R_{n+1,s})^\wedge(\overline{\xi}) := \frac{\tau}{\|\overline{\xi}\|_{p_s}^{2s}}, \qquad (D_{n+1,s})^\wedge(\overline{\xi}) := \frac{\tau}{\|\overline{\xi}\|_{p_s}^{2s-1}}.$$

Observe that if $s = 1/2$, the operator $D_{n+1,s}$ is nothing but a temporal derivative. The result [**HoL, Theorem 7.4**] establishes the comparability between $\|D_{n+1,1}f\|_{*,p_1}$ and $\|\partial_t^{1/2} f\|_{*,p_1}$ in the 1-parabolic case under the assumption $\|\nabla_x f\|_\infty \lesssim 1$. For our purposes, we will only be interested in the estimate

$$\|D_{n+1,s}f\|_{*,p_s} \lesssim \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} + \|\nabla_x f\|_\infty. \tag{2.1.3}$$

To prove such a bound we simply follow the proof of [**HoL, Theorem 7.4**] and make dimensional adjustments. First, we observe that

$$(D_{n+1,s}f)^\wedge(\overline{\xi}) = \Big[|\tau|^{\frac{1}{2s}} m(\overline{\xi})\Big]\widehat{f}, \qquad \text{where} \qquad m(\overline{\xi}) := \frac{\tau}{|\tau|^{\frac{1}{2s}}\|\overline{\xi}\|_{p_s}^{2s-1}}.$$

Now, since m is not smooth in $\mathbb{R}^{n+1} \setminus \{0\}$, we introduce the auxiliary function $\phi \in \mathcal{C}^\infty(\mathbb{R})$, which is even, equals 1 on $(2/K, \infty)$ and is supported on $(-\infty, -1/K) \cup (1/K, \infty)$, for some $K \geq 2$. We also choose ϕ so that $\|\partial^l \phi\|_\infty \lesssim K^l$, for $0 \leq l \leq n+5$. Let us introduce the following auxiliary functions,

$$m^+(\overline{\xi}) := m(\overline{\xi})\phi\bigg(\frac{\tau}{|\xi|^{2s}}\bigg), \qquad m^{++}(\overline{\xi}) := \frac{|\tau|^{\frac{1}{2s}}\|\overline{\xi}\|_{p_s}}{|\xi|^2} m(\overline{\xi})(1-\phi)\bigg(\frac{\tau}{|\xi|^{2s}}\bigg),$$

$$m_j^{++}(\overline{\xi}) := \frac{\xi_j}{\|\overline{\xi}\|_{p_s}} m^{++}(\overline{\xi}).$$

Now we proceed as follows:

$$\begin{aligned}(D_{n+1,s}f)^\wedge(\bar\xi) &= \Big[m^+(\bar\xi)|\tau|^{\frac{1}{2s}} + m(\bar\xi)(1-\phi)\Big(\frac{\tau}{|\xi|^{2s}}\Big)|\tau|^{\frac{1}{2s}}\Big]\widehat{f}(\bar\xi)\\ &= \Big[m^+(\bar\xi)|\tau|^{\frac{1}{2s}} + \sum_{j=1}^n \frac{\xi_j}{\|\bar\xi\|_{p_s}} m^{++}(\bar\xi)\xi_j\Big]\widehat{f}(\bar\xi)\\ &= \Big[m^+(\bar\xi)|\tau|^{\frac{1}{2s}} + \sum_{j=1}^n m_j^{++}(\bar\xi)\xi_j\Big]\widehat{f}(\bar\xi). \end{aligned} \tag{2.1.4}$$

Regarding m^+, we observe that is an odd multiplier, smooth in $\mathbb{R}^{n+1}\setminus\{0\}$ and s-parabolically homogeneous of degree 0. Therefore, by [**Gr, Proposition 2.4.7**] (which admits a direct adaptation to the s-parabolic case), m^+ is associated to a convolution kernel L^+, odd, s-parabolically homogeneous of degree $-n-2s$ and bounded on L^p and BMO_{p_s} (see again [**Pe, Remark 1.3**]).

Let us turn to m_j^{++}, which in particular lacks the smoothness properties of m^+. Our first goal will be to prove that there exists L_j^{++} convolution kernel associated to m_j^{++} and that is bounded from L^∞ to BMO_{p_s}. We begin by noticing that

$$\mathrm{supp}(m_j^{++}) \subset \Big\{(\xi,\tau)\,:\, 0\le|\tau|\le\frac{2|\xi|^{2s}}{K}\Big\},$$

meaning that in the support of m_j^{++} we have $\|\xi\|_{p_s}\approx|\xi|$. Moreover,

$$|m_j^{++}(\bar\xi)| = \frac{|\tau|^{\frac{1}{2s}}|\xi_j|}{|\xi|^2}(1-\phi)\Big(\frac{\tau}{|\xi|^{2s}}\Big)|m(\bar\xi)| \lesssim |m(\bar\xi)| \le \frac{|\tau|^{1-\frac{1}{2s}}}{\|\bar\xi\|_{p_s}^{2s-1}} \lesssim 1.$$

In fact, more generally we have for $\alpha\in\{0,1,2,\ldots\}$ and $\beta\in\{0,1,2\ldots\}^n$,

$$|\partial_\tau^\alpha\partial_\xi m_j^{++}(\bar\xi)| \lesssim |\tau|^{1-\frac{1}{2s}-\alpha}\|\bar\xi\|_{p_s}^{-2s+1-|\beta|}, \tag{2.1.5}$$

$$|\partial_\tau^\alpha\partial_\xi(\xi_j m_j^{++})(\bar\xi)| \lesssim |\tau|^{1-\frac{1}{2s}-\alpha}\|\bar\xi\|_{p_s}^{-2s+2-|\beta|}, \tag{2.1.6}$$

$$|\partial_\tau^\alpha\partial_\xi(\tau m_j^{++})(\bar\xi)| \lesssim |\tau|^{2-\frac{1}{2s}-\alpha}\|\bar\xi\|_{p_s}^{-2s+1-|\beta|}. \tag{2.1.7}$$

We will use the above inequalities to justify the existence of L_j^{++}. To do so, let $\{g_i\}$ be a smooth partition of unity of $(0,\infty)$ with $g_i\equiv 1$ on $(2^{-i},2^{-i+1})$ and $\mathrm{supp}(g_i)\subset(2^{-i-1},2^{-i+2})$, for $i\in\mathbb{Z}$. Let $\{L_{j,i}^{++}\}$ be the kernels corresponding to

$$m_{j,i}^{++}(\bar\xi) := m_j^{++}(\bar\xi)\cdot g_i(\|\bar\xi\|_{p_s}),$$

so that

$$\mathrm{supp}(m_{j,i}^{++}) \subset \Big\{(\xi,\tau)\,:\, 0\le\frac{\tau}{|\xi|^{2s}}\le\frac{2}{K},\ \frac{1}{2^{i+1}}\le\|\bar\xi\|_{p_s}\le\frac{1}{2^{i-2}}\Big\}.$$

We observe that

$$\|L_{j,i}^{++}\|_\infty \lesssim \|m_{j,i}^{++}\|_1 \lesssim 2^{-i(n+2s)}, \qquad \text{since} \quad \|m_j^{++}\|_\infty \lesssim 1.$$

Similarly, for any $\beta \in \{0, 1, 2, \ldots\}^n$ with $|\beta| = n+3$ we have by (2.1.5),

$$|\xi^\beta L_{j,i}^{++}(\xi,\tau)| \lesssim \|\partial_\xi^\beta m_{j,i}^{++}\|_1 \lesssim 2^{i(3-2s)}.$$

Moreover, for any $\beta \in \{0, 1, 2, \ldots\}^n$ with $|\beta| = n$,

$$\big||\tau|^{\frac{1}{4s}} \tau \xi^\beta L_{j,i}^{++}(\xi,\tau)\big| \lesssim \|\partial_\tau^{\frac{1}{4s}} (\partial_\tau \partial_\xi^\beta m_{j,i}^{++})\|_1.$$

Using again (2.1.5) and the definition of $\partial_\tau^{\frac{1}{4s}}$ we get, setting $\sigma := \partial_\tau \partial_\xi^\beta m_{j,i}^{++}$,

$$\Big| \int_{\mathbb{R}} \frac{\sigma(\xi,\tau) - \sigma(\xi,r)}{|t-r|^{1+\frac{1}{4s}}} \, \mathrm{d}r \Big| \begin{cases} = 0, & \text{if } |\xi| \geq C2^{-i}, \\ \lesssim \dfrac{2^{in}}{|\tau|^{1+\frac{1}{4s}}}, & \text{if } |\tau| > \dfrac{4|\xi|^{2s}}{K}, \\ \lesssim \dfrac{2^{i(n+2s-1)}}{|\tau|^{\frac{3}{4s}}}, & \text{if } |\tau| \leq \dfrac{8|\xi|^{2s}}{K} \leq C' 2^{-2si}, \end{cases}$$

where C, C' are constants large enough, depending on n and s. Then,

$$\|\partial_\tau^{\frac{1}{4s}} (\partial_\tau \partial_\xi^\beta m_{j,i}^{++})\|_1 \lesssim 2^{i/2}.$$

Therefore, we have obtained

$$|L_{j,i}^{++}(\xi,\tau)| \lesssim \min\big\{2^{-i(n+2s)}, 2^{i(3-2s)}|\xi|^{-(n+3)}, 2^{i/2}|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n}\big\}.$$

Proceeding analogously using (2.1.6) and (2.1.7), one can obtain the bounds

$$\begin{aligned} |\nabla_\xi L_{j,i}^{++}(\xi,\tau)| &\lesssim \min\big\{2^{-i(n+2s+1)}, 2^{i(3-2s)}|\xi|^{-(n+4)}, 2^{i/2}|\tau|^{-1-\frac{1}{4s}}|\xi|^{-(n+1)}\big\}, \\ |\partial_\tau L_{j,i}^{++}(\xi,\tau)| &\lesssim \min\big\{2^{-i(n+4s)}, 2^{i(5-4s)}|\xi|^{-(n+5)}, 2^{i/2}|\tau|^{-1-\frac{1}{4s}}|\xi|^{-(n+2)}\big\}. \end{aligned}$$

Now let $L_j^{++} := \sum_i L_{j,i}^{++}$ whenever the sum converges absolutely. Let us distinguish two cases: if $|\xi|^{2s+1/2} \gtrsim |\tau|^{1+\frac{1}{4s}}$ we get

$$\begin{aligned} |L_j^{++}(\xi,\tau)| &\leq \sum_i |L_{j,i}^{++}(\xi,\tau)| \\ &\lesssim \sum_{\{i:\, |\xi| \leq 2^i\}} 2^{-i(n+2s)} + \sum_{\{i:\, |\xi| > 2^i\}} 2^{i(3-2s)}|\xi|^{-(n+3)} \lesssim |\xi|^{-(n+2s)}. \end{aligned}$$

If $|\xi|^{2s+1/2} \lesssim |\tau|^{1+\frac{1}{4s}}$ and $\lambda > 0$ we have

$$\begin{aligned} |L_j^{++}(\xi,\tau)| &\leq \sum_i |L_{j,i}^{++}(\xi,\tau)| \\ &\leq \sum_{\{i:\, 2^{-i} \leq \lambda\}} C_1 2^{-i(n+2s)} + C_2|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n} \sum_{\{i:\, 2^{-i} > \lambda\}} 2^{i/2} \\ &\leq C_1 \lambda^{n+2s} + C_2 |\tau|^{-1-\frac{1}{4s}}|\xi|^{-n}\lambda^{-1/2}. \end{aligned}$$

This last expression is minimized for

$$\lambda \simeq \Big(|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n}\Big)^{\frac{1}{n+2s+1/2}},$$

and therefore

$$|L_j^{++}(\xi,\tau)| \lesssim \Big(|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n}\Big)^{\frac{n+2s}{n+2s+1/2}},$$

that, under condition $|\xi|^{2s+1/2} \lesssim |\tau|^{1+\frac{1}{4s}}$, is smaller than $|\xi|^{-(n+2s)}$. So we have obtained that L_j^{++} is well-defined in $\mathbb{R}^{n+1}\setminus\{0\}$ and satisfies

$$|L_j^{++}(\xi,\tau)| \lesssim \min\Big\{|\xi|^{-(n+2s)}, \Big(|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n}\Big)^{\frac{n+2s}{n+2s+1/2}}\Big\}.$$

Again, following an analogous procedure one gets

$$|\nabla_\xi L_j^{++}(\xi,\tau)| \lesssim \min\Big\{|\xi|^{-(n+2s+1)}, \Big(|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n-1}\Big)^{\frac{n+2s}{n+2s+1/2}}\Big\},$$
$$|\partial_\tau L_j^{++}(\xi,\tau)| \lesssim \min\Big\{|\xi|^{-(n+4s)}, \Big(|\tau|^{-1-\frac{1}{4s}}|\xi|^{-n-2}\Big)^{\frac{n+2s}{n+2s+1/2}}\Big\}.$$

Let us also notice that for any $s\in(0,1)$ and any dimension n,

$$\bigg(1+\frac{1}{4s}\bigg)\bigg(\frac{n+2s}{n+2s+1/2}\bigg) \geq \frac{17}{16} > 1.$$

From this point on, we are able to follow the arguments of [**HoL, p. 407**] to deduce that the principal value convolution operator associated to L_j^{++} exists and maps L^∞ to BMO_{p_s} (the arguments are, in fact, those already given in [**Pe**]). So returning to (2.1.4) we have

$$D_{n+1,s}f(\overline{x}) = C_1(L^+ * \partial_t^{\frac{1}{2s}}f)(\overline{x}) + C_2\sum_{j=1}^n (L_j^{++} * \partial_j f)(\overline{x}),$$

and therefore

$$\|D_{n+1,s}f\|_{*,p_s} \lesssim \|\partial_t^{\frac{1}{2s}}f\|_{*,p_s} + \|\nabla_x f\|_\infty,$$

that is what we wanted to prove. Let us also mention that the roll of the parameter $K\geq 2$ introduced in the above arguments becomes clear if one follows its dependence in the different inequalities previously established. Doing so, and adapting the arguments of [**HoL, Theorem 7.4**], one is able to prove a reverse inequality of the form $\|\partial_t^{\frac{1}{2s}}f\|_{*,p_s} \lesssim \|D_{n+1,s}f\|_{*,p_s} + \|\nabla_x f\|_\infty$, although we have not presented the details since it is not necessary in our context.

In any case, now is just a matter of applying [**Ho1, Lemma 1**] to a function f satisfying $\|\nabla_x f\|_\infty \lesssim 1$ and $\|\partial_t^{\frac{1}{2s}}f\|_{*,p_s} \lesssim 1$. Then, returning to (2.1.2),

$$\begin{aligned}\|D_{p_s}f\|_{*,p_s} &\lesssim \sum_{j=1}^n \|R_{j,s}\partial_j f\|_{*,p_s} + \|R_{n+1,s}D_{n+1,s}f\|_{*,p_s}\\ &\lesssim \|\nabla_x f\|_\infty + \|D_{n+1,s}f\|_{*,p_s} \lesssim \|\nabla_x f\|_\infty + \|\partial_t^{\frac{1}{2s}}f\|_{*,p_s} \lesssim 1,\end{aligned}$$

where this last inequality is due to (2.1.3). Therefore, by [**Ho1, Lemma 1**], which also holds in the fractional parabolic context (since we also have $H^1 - \text{BMO}_{p_s}$ duality (see [**CoW, §2, Theorem B**], for example) and because s-parabolic Riesz kernels enjoy the expected regularity properties a mentioned above), we finally get the desired result:

THEOREM 2.1.1. *Let $s \in (1/2, 1]$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that $\|\nabla_x f\|_\infty \lesssim 1$ and $\|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} \lesssim 1$. Then, f is $(1, \frac{1}{2s})$-Lipschitz.*

REMARK 2.1.1. In the above arguments it is not clear why the restriction $s > 1/2$ is necessary. In fact, relation (2.1.3) becomes trivial if $s = 1/2$, since the operator $D_{n+1,s}$ is just the ordinary derivative ∂_t. The necessity of $s > 1/2$ is required in order for estimate [**Ho1, Equation 14**] to hold, that in our current setting reads as follows:

$$\iint \left| \frac{1}{|(y, u-t)|_{p_s}^{n+2s-1}} - \frac{1}{|(y,u)|_{p_s}^{n+2s-1}} \right| \mathrm{d}y\, \mathrm{d}u \leq C|t|^{\frac{1}{2s}}.$$

It is not difficult to prove that if $s \leq 1/2$, such finite constant C does not exist and we could not proceed. In fact, the result for $s = 1/2$ *must* be false since, in general, it is not true that a function f satisfying $\|\nabla_x f\|_\infty \lesssim 1$ and $\|\partial_t f\|_* \lesssim 1$ is Lipschitz.

In light of the above theorem, we are able to define the so-called Lipschitz s-caloric capacity and the notion of removability in a more general manner as follows:

DEFINITION 2.1.1. For $E \subset \mathbb{R}^{n+1}$ compact set, its *Lipschitz s-caloric capacity* is

$$\Gamma_{\Theta^s}(E) := \sup |\langle T, 1\rangle|,$$

where the supremum is taken among all distributions T with $\text{supp}(T) \subset E$ and satisfying

$$\|\nabla_x P_s * T\|_\infty \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Such distributions will be called *admissible for* $\Gamma_{\Theta^s}(E)$. If the supremum is taken only among positive Borel measures supported on E satisfying the same normalization conditions, we obtain the smaller capacity $\Gamma_{\Theta^s,+}(E)$.

DEFINITION 2.1.2. A compact set $E \subset \mathbb{R}^{n+1}$ is said to be *removable for Lipschitz s-caloric functions* if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|\nabla_x f\|_\infty < \infty, \qquad \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty,$$

satisfying the Θ^s-equation in $\Omega \setminus E$, also satisfies such equation in Ω.

Let us finally observe that, as expected, admissible distributions for Γ_{Θ^s} satisfy a certain upper s-parabolic growth property analogous to that of Theorem 1.4.2.

THEOREM 2.1.2. *Let T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|\nabla_x P_s * T\|_\infty \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim \ell(R)^{n+1}$.

Proof. Is a direct consequence of Theorem 1.4.2. □

2.2 Localization of potentials

Our next goal will be to prove a so-called localization result. This type of result will ensure that if a distribution T satisfies

$$\|\nabla_x P_s * T\|_\infty \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1,$$

then

$$\|\nabla_x P_s * \varphi T\|_\infty \lesssim 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * \varphi T\|_{*,p_s} \lesssim 1,$$

where φ is an admissible function for some s-parabolic cube. That is, φT, the *localized* distribution, is admissible for $\Gamma_{\Theta^s}(Q)$. We begin by estimating $\|\nabla_x P_s * \varphi T\|_\infty$ in the following result (stated in a more general way taking into account Theorem 2.1.1).

LEMMA 2.2.1. *Let T be a distribution in $\mathbb{R}^{n+1}$ satisfying*

$$\|\nabla_x P_s * T\|_\infty \leq 1, \qquad \|P_s * T\|_{\text{Lip}_{\frac{1}{2s},t}} \leq 1.$$

Let Q be an s-parabolic cube and φ admissible function for Q. Then,

$$\|\nabla_x P_s * \varphi T\|_\infty \lesssim 1.$$

Proof. Let us begin by recalling the following product rule that can be deduced directly from the definition of $(-\Delta)^s$ (see [**RoSe1, Eq.(4.1)**], for example)

$$(-\Delta)^s(fg) = f(-\Delta)^s g + g(-\Delta)^s f - I_s(f,g),$$

where I_s is defined as

$$I_s(f,g) \simeq \int_{\mathbb{R}^n} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+2s}}\,\mathrm{d}y.$$

Then, for some constant c to be fixed later on we have

$$\begin{aligned}\Theta^s\big(\varphi(P_s * T - c)\big) &= \varphi\Theta^s(P_s * T - c) + (P_s * T - c)\Theta^s\varphi - I_s(\varphi, P_s * T - c)\\ &= \varphi T + (P_s * T - c)\Theta^s\varphi - I_s(\varphi, P_s * T),\end{aligned}$$

and therefore

$$\begin{aligned}\nabla_x P_s * \varphi T &= \nabla_x\big(\varphi(P_s * T - c)\big) - \nabla_x P_s * \big((P_s * T - c)\Theta^s\varphi\big) + \nabla_x P_s * I_s(\varphi, P_s * T)\\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

To estimate the L^∞ norm of the previous terms write $Q := Q_1 \times I_Q$, where $Q_1 \subset \mathbb{R}^n$ is an euclidean n-dimensional cube of side $\ell(Q)$, and $I_Q \subset \mathbb{R}$ is an interval of length $\ell(Q)^{2s}$. Choose $c := P_s * T(\overline{x}_Q)$, with $\overline{x}_Q := (x_Q, t_Q)$ the center of Q. This way, since $P_s * T$ satisfies a $(1, \frac{1}{2s})$-Lipschitz property, for any $\overline{x} = (x,t) \in Q$ we have

$$\begin{aligned}&|P_s * T(\overline{x}) - P_s * T(\overline{x}_Q)|\\ &\qquad \leq |P_s * T(x,t) - P_s * T(x_Q,t)| + |P_s * T(x_Q,t) - P_s * T(x_Q,t_Q)| \lesssim \ell(Q).\end{aligned} \tag{2.2.1}$$

Using this estimate, we are able to bound term I as follows:

$$\begin{aligned}&\|\nabla_x(\varphi(P_s * T - c))\|_\infty \\ &\qquad \leq \|\nabla_x\varphi\|_\infty \|P_s * T - P_s * T(\overline{x}_Q)\|_{\infty,Q} + \|\varphi\|_\infty \|\nabla_x P_s * T\|_\infty \lesssim 1.\end{aligned}$$

Let us move on by studying II. We name $g_1 := (P_s * T - P_s * T(\overline{x}_Q))\Theta^s\varphi$ and observe that by relation (2.2.1) and the admissibility of φ (see Remark 2.2.1), we have $\|g_1\|_\infty \lesssim \ell(Q)^{-2s+1}$. Let us now proceed by computing,

$$\begin{aligned}|\mathrm{II}(\overline{x})| &\leq \int_{2Q} |\nabla_x P_s(\overline{x}-\overline{y})||g_1(\overline{y})|\,\mathrm{d}\overline{y} + \int_{\mathbb{R}^{n+1}\setminus 2Q} |\nabla_x P_s(\overline{x}-\overline{y})||g_1(\overline{y})|\,\mathrm{d}\overline{y} \\ &=: \mathrm{II}_1(\overline{x}) + \mathrm{II}_2(\overline{x}).\end{aligned}$$

Let R be the s-parabolic cube centered at $\overline{x}$ with length $\ell(Q)$. We begin by studying II_1. Assume $\overline{x} \in 4Q$ so that by the first estimate of Theorem 1.1.2 we get

$$\begin{aligned}\mathrm{II}_1(\overline{x}) &\lesssim \|g_1\|_\infty \int_{2Q} \frac{|x-y||t-u|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+2}}\,\mathrm{d}\overline{y} \lesssim \ell(Q)^{-2s+1}\int_{8R}\frac{\mathrm{d}y}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \\ &\lesssim \ell(Q)^{-2s+1}\bigg(\int_{8R_1}\frac{\mathrm{d}y}{|x-y|^{n-\varepsilon}}\bigg)\bigg(\int_{8^{2s}I_R}\frac{\mathrm{d}u}{|t-u|^{\frac{\varepsilon+1}{2s}}}\bigg) \lesssim 1,\end{aligned}$$

where we have chosen $0<\varepsilon<2s-1$. On the other hand, if $\overline{x}\notin 4Q$,

$$\mathrm{II}_1(\overline{x}) \lesssim \ell(Q)^{-2s+1}\int_{2Q}\frac{\mathrm{d}y}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \leq \ell(Q)^{-2s+1}\frac{|2Q|}{\ell(Q)^{n+1}} \lesssim 1.$$

We move on by studying II_2. Let us first consider the case in which $\overline{x} \in \mathbb{R}^n \times 2I_Q$. Since $\mathrm{supp}(g_1) \subset \mathbb{R}^n \times I_Q$,

$$\begin{aligned}|\mathrm{II}_2(\overline{x})| &\lesssim \ell(Q)^{-2s+1}\int_{(\mathbb{R}^n\times I_Q)\setminus 2Q}\frac{|t-u|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}}\,\mathrm{d}\overline{y} \\ &= \ell(Q)^{-2s+1}\int_{[(\mathbb{R}^n\times I_Q)\setminus 2Q]\setminus 8R}\frac{|t-u|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}}\,\mathrm{d}\overline{y} \\ &\qquad\qquad + \ell(Q)^{-2s+1}\int_{[(\mathbb{R}^n\times I_Q)\setminus 2Q]\cap 8R}\frac{|t-u|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}}\,\mathrm{d}\overline{y} \\ &\lesssim \ell(Q)\int_{\mathbb{R}^{n+1}\setminus 8R}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}} + \ell(Q)^{-2s+1}\int_{8R}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \lesssim 1.\end{aligned}$$

Now assume that $\overline{x}\notin\mathbb{R}^n\times 2I_Q$. in this case, set $\ell_t := \mathrm{dist}_{p_s}(\overline{x},\mathbb{R}^n\times I_q) \gtrsim \ell(Q)$. Then, if $A_j := Q(\overline{x},2^j\ell_t)\setminus Q(\overline{x},2^{j-1}\ell_t)$ for $j\geq 0$,

$$\begin{aligned}|\mathrm{II}_2(\overline{x})| &\lesssim \ell(Q)^{-2s+1}\sum_{j=0}^\infty\int_{A_j\cap[(\mathbb{R}^n\times I_Q)\setminus 2Q]}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \\ &\lesssim \ell(Q)^{-2s+1}\sum_{j=0}^\infty\frac{\ell(Q)^{2s}\big[2^j\ell(Q)\big]^n}{\big[2^j\ell(Q)\big]^{n+1}} \lesssim 1,\end{aligned}$$

and we are done with II. Finally, we study III, and we notice that if we prove that the function

$$g_2(\overline{x}) := I_s(\varphi, P_s * T)(\overline{x}) = c_{n,s}\int_{\mathbb{R}^n} \frac{(\varphi(x,t)-\varphi(y,t))(P_s*T(x,t)-P*T(y,t))}{|x-y|^{n+2s}}\,\mathrm{d}y$$

is such that $\|g_2\|_\infty \lesssim \ell(Q)^{-2s+1}$ we will be done, since we would be able to repeat the same arguments done for II. To do so, we distinguish two cases: if $\overline{x}\notin 2Q$, then

$$\begin{aligned}|g_2(\overline{x})| &\lesssim \int_Q \frac{|\varphi(y,t)||P_s*T(x,t)-P_s*T(y,t)|}{|x-y|^{n+2s}}\,\mathrm{d}y\\ &\lesssim \|\varphi\|_\infty \int_Q \frac{\|\nabla_x P_s * T(\cdot,t)\|_\infty}{|x-y|^{n+2s-1}}\,\mathrm{d}y \lesssim \frac{\ell(Q)^n}{\ell(Q)^{n+2s-1}} = \ell(Q)^{-2s+1}.\end{aligned}$$

If $\overline{x}\in 2Q$, then $|g_2(\overline{x})|$ is bounded by

$$\begin{aligned}&\int_{4Q} \frac{|\varphi(x,t)-\varphi(y,t)||P_s*T(x,t)-P*T(y,t)|}{|x-y|^{n+2s}}\,\mathrm{d}y\\ &\qquad\qquad + \int_{\mathbb{R}^n\setminus 4Q} \frac{|\varphi(x,t)-\varphi(y,t)||P_s*T(x,t)-P*T(y,t)|}{|x-y|^{n+2s}}\,\mathrm{d}y\\ &\quad\le \int_{Q(x,8\ell(Q))} \frac{\|\nabla_x\varphi\|_\infty\,\mathrm{d}y}{|x-y|^{n+2s-2}} + \int_{\mathbb{R}^n\setminus Q(x,\ell(Q)/2)} \frac{2\|\varphi\|_\infty\,\mathrm{d}y}{|x-y|^{n+2s-1}} \lesssim \ell(Q)^{-2s+1},\end{aligned}$$

and we are done. □

The next goal is to obtain an analogous result for the potential $\partial_t^{\frac{1}{2s}} P_s * \varphi T$. Our arguments are inspired by those in [**MatPT, §3**]. To this end, we first prove an auxiliary result that generalizes [**MatPT, Lemma 3.5**].

LEMMA 2.2.2. *Let $Q = Q_1\times I_Q \subset \mathbb{R}^{n+1}$ be an s-parabolic cube and g a function supported on $\mathbb{R}^n\times I_Q$ such that $\|g\|_\infty \lesssim \ell(Q)^{-2s+1}$. Then,*

$$\|P_s * g\|_{\mathrm{Lip}_{\frac{1}{2s},t}} \lesssim 1.$$

Proof. Fix $\overline{x} = (x,t), \widetilde{x} = (x,r)$ as well as a function g with $\|g\|\lesssim \ell(Q)^{-2s+1}$, supported on $\mathbb{R}^n\times I_Q$. If $\ell^{2s} := |t-r|$ and R is the s-parabolic cube centered at $\overline{x}$ with side length $\max\{\ell(Q),\ell\}$,

$$\begin{aligned}&|P_s*g(\overline{x}) - P_s*g(\widetilde{x})|\\ &\quad\lesssim \frac{1}{\ell(Q)^{2s-1}}\int_{(\mathbb{R}^n\times I_Q)\cap 4R} |P_s(x-z,t-u)-P_s(x-z,r-u)|\,\mathrm{d}z\,\mathrm{d}u\\ &\qquad + \frac{1}{\ell(Q)^{2s-1}}\int_{(\mathbb{R}^n\times I_Q)\setminus 4R} |P_s(x-z,t-u)-P_s(x-z,r-u)|\,\mathrm{d}z\,\mathrm{d}u =: \mathrm{I}+\mathrm{II}.\end{aligned}$$

Let us first deal with II:

$$\begin{aligned}\mathrm{II} &\lesssim \frac{\ell^{2s}}{\ell(Q)^{2s-1}} \int_{(\mathbb{R}^n\times I_Q)\setminus 4R} \frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,t-u)|_{p_s}^{n+2s}} \\ &\le \frac{\ell^{2s}}{\ell(Q)^{2s-1}} \int_{\mathbb{R}^n\setminus 4R_1} \frac{\mathrm{d}z}{|x-z|^{n+1}} \int_{I_Q\setminus 4^{2s}I_R} \frac{\mathrm{d}u}{|t-u|^{1-\frac{1}{2s}}}.\end{aligned}$$

For the spatial integral we simply integrate using polar coordinates for example, and for the temporal integral we use that in $I_Q\setminus 4^{2s}I_R$, $|t-u|\ge \max\{\ell(Q),\ell\}^{2s}$, and then it can be bounded by

$$\frac{\ell^{2s}}{\ell(Q)^{2s-1}}\cdot\frac{1}{\max\{\ell(Q),\ell\}}\cdot\frac{\ell(Q)^{2s}}{\max\{\ell(Q),\ell\}^{2s-1}} \le \frac{\ell^{2s}\ell(Q)}{\ell(Q)\ell^{2s-1}} = |t-r|^{\frac{1}{2s}}.$$

Regarding I, let S be the s-parabolic cube centered at $\widetilde{x}$ with side length $\max\{\ell(Q),\ell\}$, so that $4R\subset 8S$. If we assume $\ell\ge\ell(Q)$,

$$\mathrm{I}\le \frac{1}{\ell(Q)^{2s-1}}\bigg[\int_{(\mathbb{R}^n\times I_Q)\cap 4R}\frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,t-u)|_{p_s}^{n}} + \int_{(\mathbb{R}^n\times I_Q)\cap 8S}\frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,r-u)|_{p_s}^{n}}\bigg].$$

We study the first integral, the second can be estimated analogously. We compute,

$$\begin{aligned}\int_{(\mathbb{R}^n\times I_Q)\cap 4R}\frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,t-u)|_{p_s}^{n+2s}} &\le \int_{4R_1}\frac{\mathrm{d}z}{|x-z|^{n-1}}\int_{I_Q\cap 4^{2s}I_R}\frac{\mathrm{d}u}{|t-u|^{\frac{1}{2s}}} \\ &\lesssim \max\{\ell(Q),\ell\}[\ell(Q)^{2s}]^{1-\frac{1}{2s}} = \ell\cdot\ell(Q)^{2s-1},\end{aligned}$$

and with this we conclude $\mathrm{I}\lesssim\ell$. If $\ell\le\ell(Q)$, denote R' and S' be s-parabolic cubes of side length ℓ centered at $\overline{x}$ and $\widetilde{x}$ respectively, so that

$$\begin{aligned}\mathrm{I}\le \frac{1}{\ell(Q)^{2s-1}}&\bigg[\int_{(\mathbb{R}^n\times I_Q)\cap 2R'}\frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,t-u)|_{p_s}^{n}} + \int_{(\mathbb{R}^n\times I_Q)\cap 4S'}\frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,r-u)|_{p_s}^{n}}\bigg] \\ &+\frac{1}{\ell(Q)^{2s-1}}\int_{(\mathbb{R}^n\times I_Q)\cap(4R\setminus 2R')}|P_s(x-z,t-u)-P_s(x-z,r-u)|\,\mathrm{d}z\,\mathrm{d}u.\end{aligned}$$

The first two summands satisfy being controlled by above by ℓ by an analogous argument to that given for the case $\ell\ge\ell(Q)$. The last term it can be estimated as follows:

$$\begin{aligned}&\frac{\ell^{2s}}{\ell(Q)^{2s-1}}\int_{(\mathbb{R}^n\times I_Q)\cap(4R\setminus 2R')}\frac{\mathrm{d}z\,\mathrm{d}u}{|(x-z,t-u)|_{p_s}^{n+2s}} \\ &\qquad\le \frac{\ell^{2s}}{\ell(Q)^{2s-1}}\int_{\mathbb{R}^n\setminus 2R_1'}\frac{\mathrm{d}z}{|x-z|^{n+2s-1}}\int_{I_Q\cap(4^{2s}I_R\setminus 2^2 sI_{R'})}\frac{\mathrm{d}u}{|t-u|^{\frac{1}{2s}}} \\ &\qquad\lesssim \frac{\ell}{\ell(Q)^{2s-1}}\int_{4^{2s}I_R}\frac{\mathrm{d}u}{|t-u|^{\frac{1}{2s}}} \lesssim \frac{\ell}{\ell(Q)^{2s-1}}[\ell(Q)^{2s}]^{1-\frac{1}{2s}} = |t-r|^{\frac{1}{2s}},\end{aligned}$$

and we are done. $\square$

The previous lemma allows us to prove a weaker localization-type result, where we ask for potentials to satisfy, explicitly, a $(1,\frac{1}{2s})$-Lipschitz property, instead of asking for a BMO_{p_s} estimate over its $\partial_t^{\frac{1}{2s}}$ derivative.

LEMMA 2.2.3. *Let $Q\subset\mathbb{R}^{n+1}$ be an s-parabolic cube and φ admissible for Q. Let T be a distribution with $\|\nabla_x P_s * T\|_\infty \leq 1$ and $\|P_s*T\|_{\text{Lip}_{\frac{1}{2s},t}}\leq 1$. Then*

$$\|P_s*\varphi T\|_{\text{Lip}_{\frac{1}{2s},t}}\lesssim 1.$$

Proof. We already know from the proof of Lemma 2.2.1 that the following identity holds:

$$\Theta^s\big(\varphi(P_s*T-c)\big)=\varphi T+(P_s*T-c)\Theta^s\varphi - I_s(\varphi,P_s*T),$$

and then

$$P_s*\varphi T=\varphi(P_s*T-c)-P_s*\big((P_s*T-c)\Theta^s\varphi\big)+P_s*I_s(\varphi,P_s*T).$$

Set $\overline{x}=(x,t),\widetilde{x}=(x,r)$. Then,

$$\begin{aligned} P_s*\varphi T(\overline{x})-P_s*\varphi T(\widetilde{x})& \\ =\varphi(\overline{x})(P_s*T(\overline{x})-c)-\varphi(\widetilde{x})(P_s*T(\widetilde{x})-c) & \qquad (2.2.2)\\ +P_s*I_s(\varphi,P_s*T)(\overline{x})-P_s*I_s(\varphi,P_s*T)(\widetilde{x}) & \qquad (2.2.3)\\ -P_s*\big((P_s*T-c)\Theta^s\varphi\big)(\overline{x})+P_s*\big((P_s*T-c)\Theta^s\varphi\big)(\widetilde{x}). & \qquad (2.2.4)\end{aligned}$$

We study (2.2.2) as follows: if $\overline{x},\widetilde{x}\notin Q$ we have that the previous difference is null. If $\overline{x}\in Q$, choosing $c:=P_s*T(\overline{x}_Q)$ with $\overline{x}_Q$ the center of Q, we define $\widetilde{x}'$ as follows:

i. if $\widetilde{x}\in Q$, $\widetilde{x}':=\widetilde{x}$,

ii. if $\widetilde{x}\notin Q$, consider $\widetilde{x}':=(x,r')\in 2Q\setminus Q$ with the property $|\widetilde{x}'-\overline{x}|_{p_s}\leq|\widetilde{x}-\overline{x}|_{p_s}$.

Either way we get $\varphi(\widetilde{x})(P_s*T(\widetilde{x})-c)=\varphi(\widetilde{x}')(P_s*T(\widetilde{x}')-c)$. So by (2.2.1) and the $\frac{1}{2s}$-Lipschitz property with respect to t of P_s*T to obtain,

$$\begin{aligned} &\big|\varphi(\overline{x})(P_s*T(\overline{x})-c)-\varphi(\widetilde{x})(P_s*T(\widetilde{x})-c)\big| \\ &\qquad\leq|\varphi(\widetilde{x}')-\varphi(\overline{x})||P_s*T(\widetilde{x}')-c|+|\varphi(\overline{x})||P_s*T(\widetilde{x}')-P_s*T(\overline{x})| \\ &\qquad\lesssim\frac{|r'-t|}{\ell(Q)^{2s}}\ell(Q)+|r'-t|^{\frac{1}{2s}}=\left[\frac{|r'-t|^{\frac{2s-1}{2s}}}{\ell(Q)^{2s-1}}+1\right]|r'-t|^{\frac{1}{2s}} \\ &\qquad\lesssim|r'-t|^{\frac{1}{2s}}\leq|r-t|^{\frac{1}{2s}}.\end{aligned}$$

In order to estimate the remaining differences (2.2.3) and (2.2.4) we name $g_1:=(P_s*T-c)\Theta^s\varphi$ and $g_2:=I_s(\varphi,P_s*T)$. Such functions have already appeared in the proof of Lemma 2.2.1 and satisfy $\text{supp}(g_j)\subset\mathbb{R}^n\times I_Q$ and $\|g_j\|_\infty\lesssim\ell(Q)^{-2s+1}$, for $j=1,2$. By a direct application of Lemma 2.2.2 we deduce that both differences are bounded by $|r-t|^{\frac{1}{2s}}$ up to a constant, and we are done. □

We move on by proving two additional auxiliary lemmas which will finally allow us to deduced the desired localization theorem.

Lemma 2.2.4. *Let T be a distribution in $\mathbb{R}^{n+1}$ such that $\|\nabla_x P_s * T\|_\infty \leq 1$ and $\|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1$. Let $Q, R \subset \mathbb{R}^{n+1}$ be s-parabolic cubes such that $Q \subset R$. Then, if φ is admissible for Q,*

$$\int_R \big|\partial_t^{\frac{1}{2s}} P_s * \varphi T(\overline{x})\big| \,\mathrm{d}\overline{x} \lesssim \ell(R)^{n+2s}.$$

Proof. We know that

$$P_s * \varphi T = \varphi(P_s * T - c) - P_s * \big((P_s * T - c)\Theta^s \varphi\big) + P_s * I_s(\varphi, P_s * T),$$

and we choose $c := P_s * T(\overline{x}_Q)$, with $\overline{x}_Q$ the center of Q. Name $g_1 := (P_s * T - c)\Theta^s\varphi$ and $g_2 := I_s(\varphi, P_s * T)$ so that

$$\partial_t^{\frac{1}{2s}} P_s * \varphi T = \partial_t^{\frac{1}{2s}}\big[\varphi(P_s * T - c)\big] - \partial_t^{\frac{1}{2s}} P_s * g_1 + \partial_t^{\frac{1}{2s}} P_s * g_2. \tag{2.2.5}$$

Begin by noticing that $\forall \overline{x} \in \mathbb{R}^{n+1}$ and $i = 1, 2$, if $Q_{\overline{x}} = Q_{1,\overline{x}} \times I_{Q,\overline{x}}$ is the s-parabolic cube of side length $\ell(Q)$ and center $\overline{x}$,

$$\begin{aligned}\big|\partial_t^{\frac{1}{2s}} P_s * g_i(\overline{x})\big| &\lesssim \ell(Q)^{-2s+1} \int_{\mathbb{R}^n \times I_Q} \big|\partial_t^{\frac{1}{2s}} P_s(\overline{x} - \overline{y})\big| \,\mathrm{d}\overline{y} \\ &= \ell(Q)^{-2s+1}\bigg[\int_{(\mathbb{R}^n \times I_Q) \cap 4Q_{\overline{x}}} \big|\partial_t^{\frac{1}{2s}} P_s(\overline{x} - \overline{y})\big| \,\mathrm{d}\overline{y} + \int_{(\mathbb{R}^n \times I_Q) \setminus 4Q_{\overline{x}}} \big|\partial_t^{\frac{1}{2s}} P_s(\overline{x} - \overline{y})\big| \,\mathrm{d}\overline{y}\bigg].\end{aligned}$$

Regarding the first integral, assuming $n > 1$ and applying Lemma 1.1.4 with $\beta := \frac{1}{2s}$, we have

$$\begin{aligned}\int_{(\mathbb{R}^n \times I_Q) \cap 4Q_{\overline{x}}} \big|\partial_t^{\frac{1}{2s}} P_s(\overline{x} - \overline{y})\big| \,\mathrm{d}\overline{y} &\lesssim \int_{(\mathbb{R}^n \times I_Q) \cap 4Q_{\overline{x}}} \frac{\mathrm{d}\overline{y}}{|x - y|^{n-2s} |\overline{x} - \overline{y}|_{p_s}^{2s+1}} \\ &\leq \int_{4Q_{1,\overline{x}}} \frac{\mathrm{d}y}{|x - y|^{n-2s+\varepsilon}} \int_{4^{2s} I_{Q,\overline{x}}} \frac{\mathrm{d}r}{|t - r|^{\frac{2s+1-\varepsilon}{2s}}} \\ &\lesssim \ell(Q)^{2s-\varepsilon} \ell(Q)^{-1+\varepsilon} = \ell(Q)^{2s-1},\end{aligned}$$

where we have chosen $1 < \varepsilon < 2s$. If $n = 1$, we know that for any $\alpha \in (2s - 1, 4s)$,

$$\int_{(\mathbb{R}^n \times I_Q) \cap 4Q_{\overline{x}}} \big|\partial_t^{\frac{1}{2s}} P_s(\overline{x} - \overline{y})\big| \,\mathrm{d}\overline{y} \lesssim \int_{(\mathbb{R}^n \times I_Q) \cap 4Q_{\overline{x}}} \frac{\mathrm{d}\overline{y}}{|x - y|^{1-2s+\alpha} |\overline{x} - \overline{y}|_{p_s}^{2s+1-\alpha}}.$$

So choosing $1 - \alpha < \varepsilon < 2s - \alpha$ we can carry out a similar argument and deduce the desired bound. For the second integral we also distinguish whether if $n > 1$ or $n = 1$ (we will only give the details for the case $n > 1$). Defining $A_j := \{\overline{y} : 2^j \ell(Q) \leq \operatorname{dist}_{p_s}(\overline{x}, \overline{y}) \leq 2^{j+1}\ell(Q)\}$ for $j \geq 1$, we have

$$\begin{aligned}\int_{(\mathbb{R}^n \times I_Q) \setminus 4Q_{\overline{x}}} \big|\partial_t^{\frac{1}{2s}} P_s(\overline{x} - \overline{y})\big| \,\mathrm{d}\overline{y} &\lesssim \sum_{j=1}^\infty \int_{(\mathbb{R}^n \times I_Q) \cap A_j} \frac{\mathrm{d}\overline{y}}{|x - y|^{n-2s} |\overline{x} - \overline{y}|_{p_s}^{2s+1}} \\ &\leq \ell(Q)^{2s} \sum_{j=1}^\infty \int_{(\mathbb{R}^n \times I_Q) \cap A_j} \frac{\mathrm{d}y}{|x - y|^{n+1}} \lesssim \ell(Q)^{2s} \sum_{j=1}^\infty \frac{(2^j \ell(Q))^n}{(2^j \ell(Q))^{n+1}} = \ell(Q)^{2s-1}.\end{aligned}$$

Therefore we obtain

$$\big|\partial_t^{\frac{1}{2s}} P_s * g_i(\overline{x})\big| \lesssim 1, \qquad \forall \overline{x}\in\mathbb{R}^{n+1} \text{ and } i=1,2.$$

Now, returning to (2.2.5) and integrating both sides over R, we get

$$\int_R \big|\partial_t^{\frac{1}{2s}} P_s * \varphi T(\overline{x})\big| \,\mathrm{d}\overline{x} = \int_R \big|\partial_t^{\frac{1}{2s}} \big[\varphi(P_s*T-c)\big](\overline{x})\big| \,\mathrm{d}\overline{x} + \ell(R)^{n+2s}.$$

So we are left to study the integral

$$\int_R \big|\partial_t^{\frac{1}{2s}} \big[\varphi(P_s*T-P_s*T(\overline{x}_Q))\big](\overline{x})\big| \,\mathrm{d}\overline{x}.$$

To this end, take ψ_Q test function with $\chi_Q\leq\psi_Q\leq\chi_{2Q}$, with $\|\nabla_x\psi_Q\|_\infty\leq\ell(Q)^{-1}$, $\|\partial_t\psi_Q\|_\infty\leq\ell(Q)^{-2s}$ and $\|\Delta\psi_Q\|_\infty\leq\ell(Q)^{-2}$. We also write for locally integrable function F,

$$m_{\psi_Q}(F) := \frac{\int F\psi_Q}{\int\psi_Q}.$$

Using the product rule we also know that

$$\partial_t^{\frac{1}{2s}}(fg)(t) = g\partial_t^{\frac{1}{2s}}f(t) + f\partial_t^{\frac{1}{2s}}g(t) + c_s\int_{\mathbb{R}} \frac{\big(f(r)-f(t)\big)\big(g(r)-g(t)\big)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r.$$

So choosing $f:=\varphi(x,\cdot)$ and $g:=P_s*T(x,\cdot)-P_s*T(\overline{x}_Q)$,

$$\begin{aligned}
&\big|\partial_t^{\frac{1}{2s}}\big[\varphi(P_s*T-P_s*T(\overline{x}_Q))\big](x,t)\big|\\
&\qquad= \big(P_s*T(x,t)-P_s*T(\overline{x}_Q)\big)\partial_t^{\frac{1}{2s}}\varphi(x,t) + \varphi(x,t)\partial_t^{\frac{1}{2s}}P_s*T(x,t)\\
&\qquad\qquad + \int_{\mathbb{R}} \frac{\big(\varphi(x,r)-\varphi(x,t)\big)\big(P_s*T(x,r)-P_s*T(x,t)\big)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r\\
&\qquad= \big(P_s*T(x,t)-P_s*T(\overline{x}_Q)\big)\partial_t^{\frac{1}{2s}}\varphi(x,t)\\
&\qquad\qquad + \varphi(x,t)\big(\partial_t^{\frac{1}{2s}}P_s*T(x,t) - m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}}P_s*T\big)\big)\\
&\qquad\qquad + \int_{\mathbb{R}} \frac{\big(\varphi(x,r)-\varphi(x,t)\big)\big(P_s*T(x,r)-P_s*T(x,t)\big)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r\\
&\qquad\qquad + \varphi(x,t)m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}}P_s*T\big) =: \mathrm{I}(\overline{x})+\mathrm{II}(\overline{x})+\mathrm{III}(\overline{x})+\mathrm{IV}(\overline{x}).
\end{aligned}$$

To study I, observe that $\operatorname{supp}(\partial_t^{\frac{1}{2s}}\varphi)\subset Q_1\times\mathbb{R}$. If $(x,t)\in Q_1\times 2^{2s}I_Q$ we have

$$\begin{aligned}
|\partial_t^{\frac{1}{2s}}\varphi(x,t)| &\leq \int_{\mathbb{R}}\frac{|\varphi(x,r)-\varphi(x,t)|}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r\\
&\lesssim \frac{1}{\ell(Q)^{2s}}\int_{4^{2s}I_Q}\frac{|r-t|}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r + \int_{\mathbb{R}\setminus 4^{2s}I_Q}\frac{\mathrm{d}r}{|r-t|^{1+\frac{1}{2s}}} \lesssim \frac{1}{\ell(Q)}.
\end{aligned}$$

If on the other hand $(x,t) \notin Q_1 \times 2^{2s} I_Q$,

$$|\partial_t^{\frac{1}{2s}} \varphi(x,t)| \leq \int_{I_Q} \frac{|\varphi(x,r)|}{|r-t|^{1+\frac{1}{2s}}} \, \mathrm{d}r \lesssim \frac{\ell(Q)^{2s}}{|t-t_Q|^{1+\frac{1}{2s}}}.$$

So in any case we are able to deduce

$$|\partial_t^{\frac{1}{2s}} \varphi(x,t)| \lesssim \frac{\ell(Q)^{2s}}{\ell(Q)^{2s+1} + |t-t_Q|^{1+\frac{1}{2s}}}, \qquad \forall \overline{x} \in Q_1 \times \mathbb{R}.$$

Then, we infer that for any $x \in Q_1$, by the $(1, \frac{1}{2s})$-Lipschitz property of $P_s * T$,

$$|\mathrm{I}(\overline{x})| \lesssim \frac{\ell(Q)^{2s}\big(\ell(Q) + |t-t_Q|^{\frac{1}{2s}}\big)}{\ell(Q)^{2s+1} + |t-t_Q|^{1+\frac{1}{2s}}},$$

and therefore

$$\begin{aligned}
\int_R |\mathrm{I}(\overline{x})| \, \mathrm{d}\overline{x} &\lesssim \int_{Q_1} \int_{|t-t_Q|^{\frac{1}{2s}} \leq 2\ell(R)} \frac{\ell(Q)^{2s}\big(\ell(Q) + |t-t_Q|^{\frac{1}{2s}}\big)}{\ell(Q)^{2s+1} + |t-t_Q|^{1+\frac{1}{2s}}} \, \mathrm{d}t \, \mathrm{d}x \\
&= \ell(Q)^{n+2s} \int_{|u| \leq \left(2\frac{\ell(R)}{\ell(Q)}\right)^{2s}} \frac{1 + |u|^{\frac{1}{2s}}}{1 + |u|^{1+\frac{1}{2s}}} \, \mathrm{d}u \\
&\lesssim \ell(Q)^{n+2s} \left(1 + \int_{1 \leq |u| \leq \left(2\frac{\ell(R)}{\ell(Q)}\right)^{2s}} \frac{1 + |u|^{\frac{1}{2s}}}{1 + |u|^{1+\frac{1}{2s}}} \, \mathrm{d}u \right) \\
&\lesssim \ell(Q)^{n+2s} \left(1 + \left(\frac{\ell(R)}{\ell(Q)} \right)^{2s} + \log \frac{\ell(R)}{\ell(Q)} \right) \\
&\lesssim \ell(R)^{n+2s} \left(1 + \left(\frac{\ell(Q)}{\ell(R)} \right)^{2s} \log \frac{\ell(R)}{\ell(Q)} \right) \lesssim \ell(R)^{n+2s},
\end{aligned}$$

where in the last step we have used that the function $x \mapsto x \log(1/x)$ is bounded for $0 < x \leq 1$. We move on to II. As discussed in [**MatPT**], one has

$$\big| m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big) - \big(\partial_t^{\frac{1}{2s}} P_s * T\big)_Q \big| \lesssim \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1. \tag{2.2.6}$$

One way to see this is to observe that

$$\begin{aligned}
&\big| m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big) - \big(\partial_t^{\frac{1}{2s}} P_s * T\big)_Q \big| \\
&\qquad = \frac{1}{|Q|} \left| \int_Q \big(\partial_t^{\frac{1}{2s}} P_s * T(\overline{x}) - m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big)\big) \, \mathrm{d}\overline{x} \right| \\
&\qquad \leq \frac{\|\psi_Q\|_1}{|Q|} \cdot \frac{1}{\|\psi_Q\|_1} \int_{2Q} \big| \partial_t^{\frac{1}{2s}} P_s * T(\overline{x}) - m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big) \big| \psi_Q(\overline{x}) \, \mathrm{d}\overline{x} \\
&\qquad \leq c_{n,s} \|\partial_t^{\frac{1}{2s}} P_s * T\|_{\mathrm{BMO}_{p_s}(\psi_Q)},
\end{aligned}$$

being the latter a weighted s-parabolic BMO space defined via the regular weight ψ_Q. The latter clearly belongs to the Muckenhoupt class A_∞ and therefore, by a well-known classical result [**MucWh**] which admits an extension to this in the current s-parabolic setting (just take into account s-parabolic cubes in the definition of the bounded mean oscillation space), we have the continuous inclusion $\text{BMO}_{p_s} \hookrightarrow \text{BMO}_{p_s}(\psi_Q)$ and we deduce (2.2.6). Therefore,

$$\begin{aligned}\int_R |\text{II}(\overline{x})|\,\mathrm{d}\overline{x} &= \int_Q |\varphi(x,t)|\big|\partial_t^{\frac{1}{2s}} P_s * T(x,t) - m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big)\big|\,\mathrm{d}\overline{x} \\ &\lesssim \ell(Q)^{n+2s} + \int_Q \big|\partial_t^{\frac{1}{2s}} P_s * T(x,t) - m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big)\big| \\ &\leq \ell(Q)^{n+2s} + \ell(Q)^{n+2s}\|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \lesssim \ell(R)^{n+2s}.\end{aligned}$$

Let us now deal with III:

$$\begin{aligned}\text{III}(\overline{x}) :&= \int_{\mathbb{R}} \frac{\big(\varphi(x,r)-\varphi(x,t)\big)\big(P_s * T(x,r) - P_s * T(x,t)\big)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \\ &= \int_{|r-t|\leq \ell(Q)^{2s}} \frac{\big(\varphi(x,r)-\varphi(x,t)\big)\big(P_s * T(x,r) - P_s * T(x,t)\big)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \\ &\qquad + \int_{|r-t|> \ell(Q)^{2s}} \frac{\big(\varphi(x,r)-\varphi(x,t)\big)\big(P_s * T(x,r) - P_s * T(x,t)\big)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \\ &=: \text{III}_1(\overline{x}) + \text{III}_2(\overline{x}).\end{aligned}$$

By the $\text{Lip}_{\frac{1}{2s},t}$ property of $P_s * T$ (Theorem 2.1.1) and $\|\partial_t \varphi\|_\infty \leq \ell(Q)^{-2s}$,

$$|\text{III}_1(\overline{x})| \lesssim \int_{|r-t|\leq \ell(Q)^{2s}} \frac{\ell(Q)^{-2s}|r-t||r-t|^{\frac{1}{2s}}}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \lesssim 1,$$

so that $\int_R |\text{III}_1(\overline{x})|\,\mathrm{d}\overline{x} \lesssim \ell(R)^{n+2s}$. For III_2 we have

$$\begin{aligned}\text{III}_2(\overline{x}) &= \int_{|r-t|> \ell(Q)^{2s}} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\varphi(x,r)\,\mathrm{d}r \\ &\qquad - \varphi(x,t)\int_{|r-t|> \ell(Q)^{2s}} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r =: \text{III}_{21}(\overline{x}) - \text{III}_{22}(\overline{x}).\end{aligned}$$

Using again the Lipschitz property and the fact that $|\varphi(x,\cdot)| \leq \chi_{I_Q}$ we obtain

$$|\text{III}_{21}(\overline{x})| \lesssim \int_{|r-t|> \ell(Q)^{2s}} \frac{|\varphi(x,r)|}{|r-t|}\,\mathrm{d}r < \frac{1}{\ell(Q)^{2s}}\int |\varphi(x,r)|\,\mathrm{d}r \lesssim 1,$$

so that we also have $\int_R |\text{III}_{21}(\overline{x})|\,\mathrm{d}\overline{x} \lesssim \ell(R)^{n+2s}$. Then, the only remaining term to study is $|-\text{III}_{22} + \text{IV}|$, III_{22} and IV being both supported on Q. So to conclude the proof it suffices to show for $\overline{x} \in Q$:

$$\left| m_{\psi_Q}\big(\partial_t^{\frac{1}{2s}} P_s * T\big) - \int_{|r-t|> \ell(Q)^{2s}} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \right| \lesssim 1. \tag{2.2.7}$$

Let us turn our attention to $m_{\psi_Q}(\partial_t^{\frac{1}{2s}} P_s * T)$ and begin by noticing

$$\partial_t^{\frac{1}{2s}} P_s * T(y,u) = \int_{|r-u|\le \ell(Q)^{2s}} \frac{P_s * T(y,r) - P_s * T(y,u)}{|r-u|^{1+\frac{1}{2s}}}\,\mathrm{d}r + \int_{|r-u|> \ell(Q)^{2s}} \frac{P_s * T(y,r) - P_s * T(y,u)}{|r-u|^{1+\frac{1}{2s}}}\,\mathrm{d}r =: F_1(\overline{y}) + F_2(\overline{y}).$$

Observe that the kernel

$$K_y(r,u) := \chi_{|r-u|\le \ell(Q)^{2s}} \frac{P_s * T(y,r) - P_s * T(y,u)}{|r-u|^{1+\frac{1}{2s}}}$$

is antisymmetric, and therefore

$$\begin{aligned} m_{\psi_Q} F_1 &= \frac{1}{\|\psi_Q\|_1} \iiint K_y(r,u)\psi_Q(y,u)\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}u \\ &= -\frac{1}{\|\psi_Q\|_1} \iiint K_y(r,u)\psi_Q(y,r)\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}r \\ &= \frac{1}{2\|\psi_Q\|_1} \iiint K_y(r,u)\big(\psi_Q(y,u) - \psi_Q(y,r)\big)\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}u. \end{aligned}$$

Then,

$$\begin{aligned} &|m_{\psi_Q} F_1| \\ &\le \frac{1}{2\|\psi_Q\|_1} \iint \left(\int_{|r-u|\le \ell(Q)^{2s}} \frac{|P_s * T(y,r) - P_s * T(y,u)|}{|r-u|^{1+\frac{1}{2s}}} \big|\psi_Q(y,u) - \psi_Q(y,r)\big|\,\mathrm{d}u \right) \mathrm{d}y\,\mathrm{d}r \\ &\lesssim \frac{1}{\ell(Q)^{n+2s}} \int_{2Q_1} \int_{2^{2s} I_Q} \int_{|r-u|\le \ell(Q)^{2s}} \frac{|r-u|^{\frac{1}{2s}} |r-u|}{|r-u|^{1+\frac{1}{2s}} \ell(Q)^{2s}}\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}r \lesssim 1. \end{aligned}$$

So returning to (2.2.7) we are left to show

$$\left| m_{\psi_Q} F_2 - \int_{|r-t|>\ell(Q)^{2s}} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \right| \lesssim 1, \qquad (x,t)\in Q,$$

that in turn is implied by the inequality

$$\left| F_2(\overline{y}) - \int_{|r-t|>\ell(Q)^{2s}} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \right| = |F_2(\overline{y}) - F_2(\overline{x})| \lesssim 1,$$

where $(y,u)\in 2Q$. Write $A_t := \{r : |r-t| > \ell(Q)^{2s}\}$ and $A_u := \{r : |r-u| > \ell(Q)^{2s}\}$, so that

$$\begin{aligned} &|F_2(\overline{y}) - F_2(\overline{x})| \\ &\quad \le \int_{A_u\setminus A_t} \frac{P_s * T(y,r) - P_s * T(y,u)}{|r-u|^{1+\frac{1}{2s}}}\,\mathrm{d}r + \int_{A_t\setminus A_u} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \\ &\qquad + \int_{A_u\cap A_t} \left| \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}} - \frac{P_s * T(y,r) - P_s * T(y,u)}{|r-u|^{1+\frac{1}{2s}}} \right| \mathrm{d}r \\ &\quad =: \mathrm{I}' + \mathrm{II}' + \mathrm{III}'. \end{aligned}$$

Using the $\mathrm{Lip}_{\frac{1}{2s},t}$ property of $P_s * T$ and that $|r-u| \approx \ell(Q)^{2s}$ in $A_u \setminus A_t$, and $|r-t| \approx \ell(Q)^{2s}$ in $A_t \setminus A_u$, it is easily checked $\mathrm{I}' + \mathrm{II}' \lesssim 1$. Concerning III' we split

$$\begin{aligned}\mathrm{III}' \leq &\int_{A_u \cap A_t} \left| \frac{1}{|r-t|^{1+\frac{1}{2s}}} - \frac{1}{|r-u|^{1+\frac{1}{2s}}} \right| |P_s * T(x,r) - P_s * T(x,t)| \,\mathrm{d}r \\ &+ \int_{A_u \cap A_t} \frac{|P_s * T(x,r) - P_s * T(x,t) - P_s * T(y,r) + P_s * T(y,u)|}{|r-u|^{1+\frac{1}{2s}}} \,\mathrm{d}r \\ &=: \mathrm{III}_1' + \mathrm{III}_2'.\end{aligned}$$

For III_1' notice that, in the domain of integration, by the mean value theorem we have

$$\left| \frac{1}{|r-t|^{1+\frac{1}{2s}}} - \frac{1}{|r-u|^{1+\frac{1}{2s}}} \right|| \lesssim \frac{|t-u|}{|r-t|^{2+\frac{1}{2s}}}.$$

Therefore, by the $\mathrm{Lip}_{\frac{1}{2s},t}$ property of $P_s * T$,

$$\mathrm{III}_1' \lesssim \int_{|r-t|>\ell(Q)^{2s}} \frac{|t-u|}{|r-t|^{2+\frac{1}{2s}}} |r-t|^{\frac{1}{2s}} \,\mathrm{d}r \lesssim 1.$$

Regarding III_2', apply the $(1, \frac{1}{2s})$-Lipschitz property of $P_s * T$ and obtain

$$\begin{aligned}\mathrm{III}_2' &\leq \int_{|r-u|>\ell(Q)^{2s}} \frac{|P_s * T(x,r) - P_s * T(y,r)| + |P_s * T(y,u) + P_s * T(x,t)|}{|r-u|^{1+\frac{1}{2s}}} \,\mathrm{d}r \\ &\lesssim \int_{|r-u|>\ell(Q)^{2s}} \frac{\ell(Q)}{|r-u|^{1+\frac{1}{2s}}} \,\mathrm{d}r \lesssim 1.\end{aligned}$$

This finally shows $|F_2(\overline{y}) - F_2(\overline{x})| \lesssim 1$, for all $\overline{x} \in Q$ and $\overline{y} \in 2Q$ and we are done. $\square$

LEMMA 2.2.5. *Let $Q \subset \mathbb{R}^{n+1}$ be an s-parabolic cube and let T be a distribution supported in $\mathbb{R}^{n+1} \setminus 4Q$ with upper s-parabolic growth of degree $n+1$ and such that $\|\nabla_x P_s * T\|_\infty \leq 1$ and $\|P_s * T\|_{\mathrm{Lip}_{\frac{1}{2s},t}} \leq 1$. Then,*

$$\int_Q \left|\partial_t^{\frac{1}{2s}} P_s * T(\overline{x}) - \left(\partial_t^{\frac{1}{2s}} P_s * T\right)_Q\right| \mathrm{d}\overline{x} \lesssim \ell(Q)^{n+2s}.$$

Proof. Observe that the hypotheses of the lemma imply that $P_s * T$ is a uniformly continuous function. Let us fix Q s-parabolic cube. To obtain the result it suffices to verify

$$\left|\partial_t^{\frac{1}{2s}} P_s * T(\overline{x}) - \partial_t^{\frac{1}{2s}} P_s * T(\overline{y})\right| \lesssim 1, \qquad \text{for } \overline{x}, \overline{y} \in Q.$$

We distinguish to cases:

i. $\overline{x} = (x,t)$ and $\overline{y} = (y,t)$,

ii. or that $\overline{x} = (x,t)$ and $\overline{y} = (x,u)$.

Let first tackle case i. We compute:

$$
\begin{aligned}
&\big|\partial_t^{\frac{1}{2s}} P_s * T(x,t) - \partial_t^{\frac{1}{2s}} P_s * T(y,t)\big| \\
&\quad = \bigg| \int_{\mathbb{R}} \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\, \mathrm{d}r - \int_{\mathbb{R}} \frac{P_s * T(y,r) - P_s * T(y,t)}{|r-t|^{1+\frac{1}{2s}}}\, \mathrm{d}r \bigg| \\
&\quad \leq \int_{|r-t|\leq 2^{2s}\ell(Q)^{2s}} \frac{|P_s * T(x,r) - P_s * T(x,t)|}{|r-t|^{1+\frac{1}{2s}}}\, \mathrm{d}r \\
&\qquad + \int_{|r-t|\leq 2^{2s}\ell(Q)^{2s}} \frac{|P_s * T(y,r) - P_s * T(y,t)|}{|r-t|^{1+\frac{1}{2s}}}\, \mathrm{d}r \\
&\qquad + \int_{|r-t|> 2^{2s}\ell(Q)^{2s}} \frac{|P_s * T(x,r) - P_s * T(x,t) - P_s * T(y,r) + P_s * T(y,t)|}{|r-u|^{1+\frac{1}{2s}}}\, \mathrm{d}r \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Let us estimate I. In order to avoid possible t-differentiability obstacles regarding the kernel P_s, we resort a standard regularization process: take ψ test function supported on the unit s-parabolic ball $B(0,1)$ such that $\int \psi = 1$ and set $\psi_\varepsilon := \varepsilon^{-n-2s}\psi(\cdot/\varepsilon)$ and the regularized kernel $P_s^\varepsilon := \psi_\varepsilon * P_s$. As it already mentioned in [**MatP**], the previous kernel satisfies the same growth estimates as P_s (see Theorem 1.1.2 or [**MatP, Lemma 2.2**]). For r, t such that $|r-t| \leq 2^{2s}\ell(Q)^{2s}$, we write

$$
|P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(x,t)| \leq |r-t| \|\partial_t P_s^\varepsilon * T\|_{\infty,3Q}.
$$

We claim that

$$
\|\partial_t P_s^\varepsilon * T\|_{\infty,3Q} \lesssim \frac{1}{\ell(Q)^{2s-1}}, \quad \text{uniformly on } \varepsilon > 0. \tag{2.2.8}
$$

If this holds, applying $\partial_t P_s^\varepsilon * T$ to test functions and making $\varepsilon \to 0$, we deduce by duality the bound also for $\partial_t P_s * T$. Notice that if (2.2.8) holds we will be done, since

$$
\mathrm{I} \lesssim \int_{|r-t|\leq 2^{2s}\ell(Q)^{2s}} \frac{|r-t|}{\ell(Q)^{2s-1}|r-t|^{1+\frac{1}{2s}}}\, \mathrm{d}r \lesssim \frac{\big(\ell(Q)^{2s}\big)^{1-\frac{1}{2s}}}{\ell(Q)^{2s-1}} = 1.
$$

Analogously, interchanging the roles of $\overline{x}$ and $\overline{y}$, we deduce II $\lesssim 1$. Regarding III,

$$
\begin{aligned}
\mathrm{III} \leq \int_{|r-t|>2^{2s}\ell(Q)^{2s}} &\frac{|P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(y,r)|}{|r-u|^{1+\frac{1}{2s}}}\, \mathrm{d}r \\
&+ \int_{|r-t|>2^{2s}\ell(Q)^{2s}} \frac{|P_s^\varepsilon * T(x,t) - P_s^\varepsilon * T(y,t)|}{|r-u|^{1+\frac{1}{2s}}}\, \mathrm{d}r.
\end{aligned}
$$

Then $|P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(y,r)| \leq \|\nabla_x P_s^\varepsilon * T\|_\infty |x-y| \leq \|\psi_\varepsilon\|_1 \|\nabla_x P_s * T\|_\infty |x-y| \lesssim \ell(Q)$. The same holds replacing (x,r) and (y,r) by (x,t) and (y,t). Hence,

$$
\mathrm{III} \lesssim \int_{|r-t|>2^{2s}\ell(Q)^{2s}} \frac{\ell(Q)}{|r-t|^{1+\frac{1}{2s}}}\, \mathrm{d}r \lesssim \frac{\ell(Q)}{\big(\ell(Q)^{2s}\big)^{\frac{1}{2s}}} = 1.
$$

So once (2.2.8) is proved, case i is done. To prove it, we split $\mathbb{R}^{n+1}\setminus 4Q$ into s-parabolic annuli $A_k := 2^{k+1}Q\setminus 2^kQ$ and consider $\mathcal{C}^\infty$ functions $\widetilde{\chi}_k$ such that $\chi_{A_k}\leq \widetilde{\chi}_k\leq \chi_{\frac{11}{10}A_k}$, as well as $\|\nabla_x\widetilde{\chi}_k\|_\infty \lesssim (2^k\ell(Q))^{-1}$, $\|\partial_t\widetilde{\chi}_k\|_\infty\lesssim (2^k\ell(Q))^{-2s}$ and

$$\sum_{k\geq 2}\widetilde{\chi}_k = 1, \qquad \text{on } \mathbb{R}^{n+1}\setminus 4Q.$$

Let us fix $\overline{z}=(z,v)\in 3Q$ and observe

$$|\partial_t P_s^\varepsilon * T(\overline{z})| \leq \sum_{k\geq 2}|\partial_t P_s^\varepsilon * \widetilde{\chi}_k T(\overline{z})|.$$

We prove $|\partial_t P_s^\varepsilon * \widetilde{\chi}_k T(\overline{z})|\lesssim (2^k\ell(Q))^{-2s+1}$, and with it we will be done. Write

$$|\partial_t P_s^\varepsilon * \widetilde{\chi}_k T(\overline{z})| = |\langle T, \widetilde{\chi}_k\partial_t P_s^\varepsilon(\overline{z}-\cdot)\rangle| =: |\langle T,\phi_{k,\overline{z}}^\varepsilon\rangle|.$$

We study $\|\nabla_x\phi_{k,\overline{z}}^\varepsilon\|_\infty$ and $\|\partial_t\phi_{k,\overline{z}}^\varepsilon\|_\infty$ in order to apply the upper s-parabolic growth of T. On the one hand we have, for each $\overline{x}=(x,t)\in A_k$ with $t\neq v$, by [**MatP, Lemma 2.2**] and Theorem 1.1.2,

$$\begin{aligned}|\nabla_x\phi_{k,\overline{z}}^\varepsilon(\overline{x})| &\leq |\nabla_x\widetilde{\chi}_k\cdot\partial_t P_s^\varepsilon(\overline{z}-\overline{x})| + |\widetilde{\chi}_k\cdot\nabla_x\partial_t P_s^\varepsilon(\overline{z}-\overline{x})|\\ &\lesssim (2^k\ell(Q))^{-1}\frac{1}{|\overline{x}-\overline{z}|_{p_s}^{n+2s}} + \frac{1}{|\overline{x}-\overline{z}|_{p_s}^{n+2s+1}}\lesssim (2^k\ell(Q))^{-(n+2s+1)},\end{aligned}$$

and then $\|\nabla_x\phi_{k,\overline{z}}^\varepsilon\|\lesssim (2^k\ell(Q))^{-(n+2s+1)}$. Similarly, for each $\overline{x}=(x,t)\in A_k$ with $t\neq v$,

$$\begin{aligned}|\partial_t\phi_{k,\overline{z}}^\varepsilon(\overline{x})| &\leq |\partial_t\widetilde{\chi}_k\cdot\partial_t P_s^\varepsilon(\overline{z}-\overline{x})| + |\widetilde{\chi}_k\cdot\partial_t^2 P_s^\varepsilon(\overline{z}-\overline{x})|\\ &\lesssim (2^k\ell(Q))^{-2s}\frac{1}{|\overline{x}-\overline{z}|_{p_s}^{n+2s}} + \frac{1}{|\overline{x}-\overline{z}|_{p_s}^{n+4s}}\lesssim (2^k\ell(Q))^{-(n+4s)},\end{aligned}$$

where the bound for $\partial_t^2 P_s^\varepsilon$ can be argued with same arguments to those of [**MatP, Lemma 2.2**], for example. Then, $\|\partial_t\phi_{k,\overline{z}}^\varepsilon\|\lesssim (2^k\ell(Q))^{-(n+4s)}$ and with this we deduce that $(2^k\ell(Q))^{n+2s}\phi_{k,\overline{z}}^\varepsilon$ is a function to which we can apply Theorem 2.1.2. This way,

$$|\partial_t P_s^\varepsilon * \widetilde{\chi}_k T(\overline{z})| =: |\langle T,\phi_{k,\overline{z}}^\varepsilon\rangle| \lesssim \frac{1}{(2^k\ell(Q))^{n+2s}}(2^k\ell(Q))^{n+1} = (2^k\ell(Q))^{-2s+1},$$

that is what we wanted to prove, and this ends case i.

We move on to case *ii*, where $\overline{x} = (x,t)$ and $\overline{y} = (x,u)$. As in *i* we write

$$\begin{aligned}
&\big|\partial_t^{\frac{1}{2s}} P_s^\varepsilon * T(x,t) - \partial_t^{\frac{1}{2s}} P_s^\varepsilon * T(x,u)\big| \\
&\quad = \bigg| \int_{\mathbb{R}} \frac{P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(x,t)}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r - \int_{\mathbb{R}} \frac{P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(x,u)}{|r-u|^{1+\frac{1}{2s}}}\,\mathrm{d}r \bigg| \\
&\quad \leq \int_{|r-t|\leq 2^{2s}\ell(Q)^{2s}} \frac{|P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(x,t)|}{|r-t|^{1+\frac{1}{2s}}}\,\mathrm{d}r \\
&\qquad + \int_{|r-t|\leq 2^{2s}\ell(Q)^{2s}} \frac{|P_s^\varepsilon * T(x,r) - P_s^\varepsilon * T(x,u)|}{|r-u|^{1+\frac{1}{2s}}}\,\mathrm{d}r \\
&\qquad + \int_{|r-t|> 2^{2s}\ell(Q)^{2s}} \bigg| \frac{P_s * T(x,r) - P_s * T(x,t)}{|r-t|^{1+\frac{1}{2s}}} - \frac{P_s * T(x,r) - P_s * T(x,u)}{|r-u|^{1+\frac{1}{2s}}} \bigg|\,\mathrm{d}r \\
&\quad =: \mathrm{I}' + \mathrm{II}' + \mathrm{III}'.
\end{aligned}$$

I' and II' can be dealt with as I and II appearing in case *i*. Then $\mathrm{I}' + \mathrm{II}' \lesssim 1$. Concerning III' we have, by the $\mathrm{Lip}_{\frac{1}{2s},t}$ property of $P_s * T$ (and thus of $P_s^\varepsilon * T$, since ψ_ε integrates 1),

$$\begin{aligned}
\mathrm{III}' &\leq \int_{|r-t|> 2^{2s}\ell(Q)^{2s}} \bigg| \frac{1}{|r-t|^{1+\frac{1}{2s}}} - \frac{1}{|r-u|^{1+\frac{1}{2s}}} \bigg| |P_s * T(x,r) - P_s * T(x,t)|\,\mathrm{d}r \\
&\qquad\qquad + \int_{|r-t|> 2^{2s}\ell(Q)^{2s}} \frac{1}{|r-u|^{1+\frac{1}{2s}}} |P_s * T(x,t) - P_s * T(x,u)|\,\mathrm{d}r \\
&\lesssim \int_{|r-t|> 2^{2s}\ell(Q)^{2s}} \frac{\ell(Q)^{2s}}{|r-t|^{2+\frac{1}{2s}}} |r-t|^{\frac{1}{2s}}\,\mathrm{d}r + \int_{|r-t|> 2^{2s}\ell(Q)^{2s}} \frac{|t-u|^{\frac{1}{2s}}}{|r-u|^{1+\frac{1}{2s}}}\,\mathrm{d}r \lesssim 1.
\end{aligned}$$

□

Applying the above results we are able to deduce the following final lemma:

LEMMA 2.2.6. *Let T be a distribution in $\mathbb{R}^{n+1}$ satisfying*

$$\|\nabla_x P_s * T\|_\infty \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be an s-parabolic cube and φ admissible function for Q. Then,

$$\|\partial_t^{\frac{1}{2s}} P_s * \varphi T\|_{*,p_s} \lesssim 1.$$

Proof. We fix $\widetilde{Q}$ any s-parabolic cube and prove that there exists $c_{\widetilde{Q}}$ constant so that

$$\int_{\widetilde{Q}} \big|\partial_t^{\frac{1}{2s}} P_s * \varphi T(\overline{x}) - c_{\widetilde{Q}}\big|\,\mathrm{d}\overline{x} \lesssim \ell(\widetilde{Q})^{n+2s}.$$

To do so, pick a $\mathcal{C}^\infty$ bump function $\phi_{5\widetilde{Q}}$ with $\chi_{5\widetilde{Q}} \leq \phi_{5\widetilde{Q}} \leq \chi_{6\widetilde{Q}}$ and satisfying

$$\|\nabla_x \phi_{5\widetilde{Q}}\|_\infty \lesssim \ell(\widetilde{Q})^{-1}, \qquad \|\Delta \phi_{5\widetilde{Q}}\|_\infty \lesssim \ell(\widetilde{Q})^{-2}, \qquad \|\partial_t \phi_{5\widetilde{Q}}\|_\infty \lesssim \ell(\widetilde{Q})^{-2s}.$$

We also write $\phi_{5\widetilde{Q}^c} := 1 - \phi_{5\widetilde{Q}}$. Then we split

$$\begin{aligned}\int_{\widetilde{Q}} &\big|\partial_t^{\frac{1}{2s}} P_s * \varphi T(\overline{x}) - c_{\widetilde{Q}}\big| \,\mathrm{d}\overline{x} \\ &\leq \int_{\widetilde{Q}} \big|\partial_t^{\frac{1}{2s}} P_s * \phi_{5\widetilde{Q}}\varphi T(\overline{x})\big| \,\mathrm{d}\overline{x} + \int_{\widetilde{Q}} \big|\partial_t^{\frac{1}{2s}} P_s * \phi_{5\widetilde{Q}^c}\varphi T(\overline{x}) - c_{\widetilde{Q}}\big| \,\mathrm{d}\overline{x} =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Let us estimate II applying Lemma 2.2.5. Notice that $\operatorname{supp}(\phi_{5\widetilde{Q}^c}\varphi T) \subset \overline{(5\widetilde{Q})^c} \cap Q$. We claim that

$$\|\nabla_x P_s * \phi_{5\widetilde{Q}^c}\varphi T\|_\infty \lesssim 1 \qquad \text{and} \qquad \|P_s * \phi_{5\widetilde{Q}^c}\varphi T\|_{\mathrm{Lip}_{\frac{1}{2s},t}} \lesssim 1. \tag{2.2.9}$$

To check this, we write $P_s * \phi_{5\widetilde{Q}^c}\varphi T = P_s * \varphi T - P_s * \phi_{5\widetilde{Q}}\varphi T$ and since φ is admissible for Q we already have

$$\|\nabla_x P_s * \varphi T\|_\infty \lesssim 1 \qquad \text{and} \qquad \|P_s * \varphi T\|_{\mathrm{Lip}_{\frac{1}{2s},t}} \lesssim 1,$$

by Lemmas 2.2.1 and 2.2.3. Let us observe that, if $\ell(\widetilde{Q}) \leq \ell(Q)$, there exists some constant that makes $\phi_{5\widetilde{Q}}\varphi$ admissible for $5\widetilde{Q}$. On the other hand, if $\ell(\widetilde{Q}) > \ell(Q)$, then there is another constant making $\phi_{5\widetilde{Q}}\varphi$ admissible for Q. So in any case, also by Lemmas 2.2.1 and 2.2.3, we have

$$\|\nabla_x P_s * \phi_{5\widetilde{Q}}\varphi T\|_\infty \lesssim 1 \qquad \text{and} \qquad \|P_s * \phi_{5\widetilde{Q}}\varphi T\|_{\mathrm{Lip}_{\frac{1}{2s},t}} \lesssim 1,$$

and (2.2.9) follows. With this in mind and the fact that $\phi_{5\widetilde{Q}}\varphi T$ has upper s-parabolic growth of degree $n+1$ (use that $\phi_{5\widetilde{Q}}\varphi T$ is either admissible for Q or $\widetilde{Q}$ and apply Theorem 2.1.2), we choose

$$c_{\widetilde{Q}} := \big(\partial_t^{\frac{1}{2s}} P_s * \phi_{5\widetilde{Q}}\varphi T\big)_{\widetilde{Q}}$$

and apply Lemma 2.2.5 to obtain $\mathrm{II} \lesssim \ell(\widetilde{Q})^{n+2s}$.

To study I, we shall assume $Q \cap 6\widetilde{Q} \neq \varnothing$. Now, if $\ell(\widetilde{Q}) \leq \ell(Q)$, we have that for some constant $\phi_{5\widetilde{Q}}\varphi$ is admissible for $5\widetilde{Q}$. Applying Lemma 2.2.4 with both cubes of its statement equal to $5\widetilde{Q}$, we get

$$\mathrm{I} \leq \int_{5\widetilde{Q}} \big|\partial_t^{\frac{1}{2s}} P_s * \phi_{5\widetilde{Q}}\varphi T(\overline{x})\big| \,\mathrm{d}\overline{x} \lesssim \ell(\widetilde{Q})^{n+2s}.$$

If $\ell(\widetilde{Q}) > \ell(Q)$, since $Q \cap 6\widetilde{Q} \neq \varnothing$ we deduce $Q \subset 8\widetilde{Q}$ and hence $\phi_{5\widetilde{Q}}\varphi$ is admissible for Q for some constant. Applying again Lemma 2.2.4 now for the cubes Q and $8\widetilde{Q}$, we have

$$\mathrm{I} \leq \int_{8\widetilde{Q}} \big|\partial_t^{\frac{1}{2s}} P_s * \phi_{5\widetilde{Q}}\varphi T(\overline{x})\big| \,\mathrm{d}\overline{x} \lesssim \ell(\widetilde{Q})^{n+2s},$$

and we are done. $\square$

Combining Lemmas 2.2.1 and 2.2.6 we are able to finally state the main theorem of this subsection:

THEOREM 2.2.7. *Let T be a distribution in $\mathbb{R}^{n+1}$ satisfying*

$$\|\nabla_x P_s * T\|_\infty \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be an s-parabolic cube and φ admissible function for Q. Then, φT is an admissible distribution, up to a constant depending on n and s, for $\Gamma_{\Theta^s}(Q)$.

2.3 Removable singularities. The critical dimension of Γ_{Θ^s}

The main reason to prove the localization result of the previous section is to obtain a characterization of removable sets for solutions of the Θ^s-equation satisfying a $(1, \frac{1}{2s})$-Lipschitz property.

THEOREM 2.3.1. *A compact set $E \subset \mathbb{R}^{n+1}$ is removable for Lipschitz s-caloric functions if and only if $\Gamma_{\Theta^s}(E) = 0$.*

Proof. Assume $s < 1$, since the case $s = 1$ is covered in [**MatPT, Theorem 5.3**]. Let $E \subset \mathbb{R}^{n+1}$ be compact and assume that is removable for Lipschitz s-caloric functions. Let T be admissible for $\Gamma_{\Theta^s}(E)$ and define $f := P_s * T$, so that $\|\nabla_x f\|_\infty < \infty$, $\|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty$ and $\Theta^s f = 0$ on $\mathbb{R}^{n+1} \setminus E$. By hypothesis $\Theta^s f = 0$ in $\mathbb{R}^{n+1}$ so $T \equiv 0$, and then $\Gamma_{\Theta^s}(E) = 0$.

Assume now that E is not removable for Lipschitz s-caloric functions. Then, there exists $\Omega \supset E$ open set and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|\nabla_x f\|_\infty < \infty, \qquad \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty$$

such that $\Theta^s f = 0$ on $\Omega \setminus E$, but $\Theta^s f \neq 0$ on Ω. Define the distribution

$$T := \frac{\Theta^s f}{\|\nabla_x f\|_\infty + \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s}},$$

that is such that $\|\nabla_x P_s * T\|_\infty \leq 1$, $\|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1$ and $\operatorname{supp}(T) \subset E \cup \Omega^c$. Since $T \neq 0$ in Ω, there exists Q s-parabolic cube with $4Q \subset \Omega$ so that $T \neq 0$ in Q. Observe that $Q \cap E \neq \varnothing$. Then, by definition, there is φ test function supported on Q with $\langle T, \varphi \rangle > 0$. Consider

$$\widetilde{\varphi} := \frac{\varphi}{\|\varphi\|_\infty + \ell(Q)\|\nabla_x \varphi\|_\infty + \ell(Q)^{2s}\|\partial_t \varphi\|_\infty + \ell(Q)^2\|\Delta_x \varphi\|_\infty},$$

so that $\widetilde{\varphi}$ is admissible for Q. Apply Theorem 2.2.7 to deduce that $\widetilde{\varphi}T$ is admissible for $\Gamma_{\Theta^s}(E)$ (up to a constant) and therefore

$$\Gamma_{\Theta^s}(E) \gtrsim \frac{1}{\|\varphi\|_\infty + \ell(Q)\|\nabla_x \varphi\|_\infty + \ell(Q)^{2s}\|\partial_t \varphi\|_\infty + \ell(Q)^2\|\Delta_x \varphi\|_\infty} |\langle \varphi T, 1 \rangle| > 0.$$

□

Let us prove now that, given $E\subset\mathbb{R}^{n+1}$, its removability for Lipschitz s-caloric functions will be tightly related to its s-parabolic Hausdorff dimension:

THEOREM 2.3.2. *For every compact set $E\subset\mathbb{R}^{n+1}$ the following hold:*

1. $\Gamma_{\Theta^s}(E)\leq C\,\mathcal{H}^{n+1}_{\infty,p_s}(E)$, *for some constant $C(n,s)>0$.*

2. *If $\dim_{\mathcal{H}_{p_s}}(E)>n+1$, then $\Gamma_{\Theta^s}(E)>0$.*

Therefore, the critical (s-parabolic Hausdorff) dimension of Γ_{Θ^s} is $n+1$.

Proof. Again, let us restrict ourselves to $s<1$, since the case $s=1$ is already covered in [**MatPT, Lemma 5.1**].

To prove *1* we proceed analogously as we have done in the proof of Theorem 1.4.3. In order to prove *2* we apply an s-parabolic version of Frostman's lemma. Let us name $d:=\dim_{\mathcal{H}_{p_s}}(E)$ and assume $0<\mathcal{H}^d_{p_s}(E)<\infty$ without loss of generality. Indeed, if $\mathcal{H}^d_{p_s}(E)=\infty$, apply an s-parabolic version of [**F, Theorem 4.10**] to construct a compact set $\widetilde{E}\subset\mathbb{R}^{n+1}$ with $\widetilde{E}\subset E$ and $0<\mathcal{H}^d_{p_s}(\widetilde{E})<\infty$. On the other hand, if $\mathcal{H}^d_{p_s}(E)=0$ apply the same reasoning with $d':=(d+n+1)/2$. In any case, by Frostman's lemma we shall then consider a non trivial positive measure μ with $\text{supp}(\mu)\subset E$ satisfying $\mu(B(\overline{x},r))\leq r^d$, for all $\overline{x}\in\mathbb{R}^{n+1}$ and all $r>0$. Observe that if we prove

$$\big\|\nabla_x P_s*\mu\big\|_\infty\lesssim 1\qquad\text{and}\qquad\big\|\partial_t^{\frac{1}{2s}}P_s*\mu\big\|_{*,p_s}\lesssim 1,$$

we will be done, since this would imply $\Gamma_{\Theta^s}(E)\geq\Gamma_{\Theta^s}(\widetilde{E})\gtrsim\langle\mu,1\rangle=\mu(\widetilde{E})>0$. The bound for $\partial_t^{\frac{1}{2s}}P_s*\mu$ follows directly from Lemma 1.3.2 with $\beta:=\frac{1}{2s}$. To prove that of $\nabla_x P_s*\mu$, use [**MatP, Lemma 2.2**] to deduce that for any $\overline{x}\in\mathbb{R}^{n+1}$,

$$|\nabla_x P_s*\mu(\overline{x})|\lesssim\int_E\frac{\text{d}\mu(\overline{z})}{|\overline{x}-\overline{y}|^{n+1}_{p_s}}\lesssim\text{diam}_{p_s}(E)^{d-(n+1)}\lesssim 1,$$

where we have split the previous domain into annuli and used that μ presents upper s-parabolic growth of degree $d>n+1$. ☐

We proceed by providing a result regarding the capacity of subsets of $\mathcal{H}^{n+1}_{p_s}$-positive measure of regular $\text{Lip}(1,\frac{1}{2s})$ graphs. To proceed, let us introduce the following operator: for a given μ, a real compactly supported Borel regular measure with upper s-parabolic growth of degree $n+1$, we define the operator $\mathcal{P}^s_\mu$ acting on elements of $L^1_{\text{loc}}(\mu)$ as

$$\mathcal{P}^s_\mu f(\overline{x}):=\int_{\mathbb{R}^{n+1}}\nabla_x P_s(\overline{x}-\overline{y})f(\overline{y})\text{d}\mu(\overline{y}),\qquad\overline{x}\notin\text{supp}(\mu).$$

Together with $\mathcal{P}^s_\mu$, we also have its truncated version $\mathcal{P}^s_{\mu,\varepsilon}$ for $\varepsilon>0$, as well as all the notation presented in the introductory chapter regarding L^p norms, maximal operators $\mathcal{P}^s_{*,\mu}$ and conjugate operators $\mathcal{P}^{s,*}_\mu$.

Recall also that $\Sigma^s_{n+1}(E)$ is the collection of positive Borel measures supported on E with upper s-parabolic growth of degree $n+1$ with constant 1 and define the auxiliary capacity:

$$\widetilde{\Gamma}_{\Theta^s,+}(E) := \sup \mu(E), \tag{2.3.1}$$

where the supremum is taken over all measures $\mu \in \Sigma^s_{n+1}(E)$ such that

$$\|\mathcal{P}^s\mu\|_\infty \leq 1, \qquad \|\mathcal{P}^{s,*}\mu\|_\infty \leq 1.$$

We aim at proving the following result, analogous to [**MatPT, Theorem 5.5**]:

THEOREM 2.3.3. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set. Then, for each $s \in (1,2,1]$,*

$$\widetilde{\Gamma}_{\Theta^s,+}(E) \lesssim \Gamma_{\Theta^s,+}(E) \approx \sup\big\{\mu(E)\,:\,\mu \in \Sigma^s_{n+1}(E),\, \|\mathcal{P}^s\mu\|_\infty \leq 1\big\}.$$

Moreover,

$$\widetilde{\Gamma}_{\Theta^s,+}(E) \approx \sup\big\{\mu(E)\,:\,\mu \in \Sigma^s_{n+1}(E),\, \|\mathcal{P}^s_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1\big\}.$$

Previous to that, let us verify the following lemma, which follows from the growth estimates proved for the kernel $\nabla_x P_s$ in Theorem 1.1.2 and analogous arguments to those in [**MattiPar, Lemma 5.4**], for example:

LEMMA 2.3.4. *Let μ be a real Borel measure with compact support and upper s-parabolic growth of degree $n+1$ with $\|\mathcal{P}^s\mu\|_\infty \leq 1$. Then, there is $\kappa > 0$, constant depending on n and s, so that*

$$\mathcal{P}^s_*\mu(\overline{x}) \leq \kappa, \qquad \forall\, \overline{x} \in \mathbb{R}^{n+1}.$$

Proof. Let $|\mu|$ denote the variation of μ. which corresponds to $|\mu| = \mu^+ + \mu^-$, where μ^+, μ^- are the positive and negative variations of μ respectively, defined as the set functions

$$\mu^+(B) = \sup\{\mu(A)\;:\; A \in \mathcal{B}(\mathbb{R}^{n+1}), A \subset B\},$$
$$\mu^-(B) = -\inf\{\mu(A)\;:\; A \in \mathcal{B}(\mathbb{R}^{n+1}), A \subset B\}.$$

It is known that $|\mu|$ is a positive measure [**Ru2, Theorem 6.2**] with $\mu \ll |\mu|$. In fact, there exists an $L^1_{\text{loc}}(\mathbb{R}^{n+1})$ function g with $g(\overline{x}) = \pm 1$ such that $\mu = g|\mu|$ [**Ru2, Theorem 6.12**]. It is clear that $|\mu|$ still satisfies the same upper s-parabolic growth condition as μ.

Let us fix $0 < \varepsilon < 1$ and $\overline{x} = (x,t) \in \mathbb{R}^{n+1}$ and notice that for some $0 < \delta < 2s-1$,

$$\begin{aligned}
&\int_{B(\overline{x},\varepsilon/4)} \bigg(\int_{B(\overline{x},\varepsilon)} \frac{\mathrm{d}|\mu|(\overline{y})}{|\overline{z}-\overline{y}|^{n+1}_{p_s}} \bigg) \mathrm{d}\overline{z} \leq \int_{B(\overline{x},\varepsilon)} \bigg(\int_{B(\overline{y},2\varepsilon)} \frac{\mathrm{d}\overline{z}}{|\overline{z}-\overline{y}|^{n+1}_{p_s}} \bigg) \mathrm{d}|\mu|(\overline{y}) \\
&\leq \int_{B(\overline{x},\varepsilon)} \bigg(\int_{B_1(y,2\varepsilon)} \frac{\mathrm{d}z}{|z-y|^{n-\delta}} \int_{-2^{2s}\varepsilon^{2s}}^{2^{2s}\varepsilon^{2s}} \frac{du}{u^{\frac{1+\delta}{2s}}} \bigg) \mathrm{d}|\mu|(\overline{y}) \lesssim \varepsilon^{\delta}\varepsilon^{2s-1-\delta}|\mu|(B(\overline{x},\varepsilon)) \lesssim \varepsilon^{n+2s}.
\end{aligned}$$

By the first estimate of Theorem 1.1.2 this implies, in particular, that we can find some $\overline{z} \in B(\overline{x}, \varepsilon/4)$ such that $|\mathcal{P}^s\mu(\overline{z})| \leq \|\mathcal{P}^s\mu\|_\infty \leq 1$ and satisfying

$$|\mathcal{P}^s_\mu \chi_{B(\overline{x},\varepsilon)}(\overline{z})| \leq C_2,$$

for some positive constant C_2. Therefore, we obtain

$$\begin{aligned}\big|\mathcal{P}^s_\varepsilon\mu(\overline{x}) - \mathcal{P}^s\mu(\overline{z})\big| &\leq \big|\mathcal{P}^s_\varepsilon\mu(\overline{x}) - \mathcal{P}^s_\varepsilon\mu(\overline{z})\big| + C_2 \\ &\leq \int_{|\overline{x}-\overline{y}|_{p_s} > \frac{\varepsilon}{2}} \big|\nabla_x P_s(\overline{x}-\overline{y}) - \nabla_x P_s(\overline{z}-\overline{y})\big| \mathrm{d}|\mu|(\overline{y}) + C_2.\end{aligned}$$

Applying the last estimate of Theorem 1.1.2 we get

$$\int_{|\overline{x}-\overline{y}|_{p_s} > \frac{\varepsilon}{2}} \big|\nabla_x P_s(\overline{x}-\overline{y}) - \nabla_x P_s(\overline{z}-\overline{y})\big| \mathrm{d}|\mu|(\overline{y}) \lesssim \varepsilon \int_{|\overline{x}-\overline{y}|_{p_s} > \frac{\varepsilon}{2}} \frac{\mathrm{d}|\mu|(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2}}.$$

Splitting the domain of integration into s-parabolic annuli:

$$A_j := \{\overline{y} \,:\, 2^{j-1}\varepsilon \leq |\overline{x}-\overline{y}|_{p_s} \leq 2^j\varepsilon\}, \quad j \geq 0,$$

and using the growth of $|\mu|$ we obtain, for some positive constant C_1,

$$\varepsilon \int_{|\overline{x}-\overline{y}|_{p_s} > \frac{\varepsilon}{2}} \frac{\mathrm{d}|\mu|(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2}} \leq \varepsilon \sum_{j=0}^\infty \int_{A_j} \frac{\mathrm{d}|\mu|(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2}} \lesssim \varepsilon \sum_{j=0}^\infty \frac{(2^j\varepsilon)^{n+1}}{(2^{j-1}\varepsilon)^{n+2}} \leq C_1.$$

Thus, we have established the bound

$$|\mathcal{P}^s_\varepsilon\mu(\overline{x}) - \mathcal{P}^s\mu(\overline{z})| \leq C_1 + C_2,$$

so setting $\kappa := C_1 + C_2 + \|\mathcal{P}^s\mu\|_\infty$ and using that by hypothesis $\|\mathcal{P}^s\mu\|_\infty \leq 1$, we get the desired inequality. □

Proof of Theorem 2.3.3. Denote

$$\begin{aligned}\Gamma_1 &:= \sup\{\mu(E) : \mu \in \Sigma^s_{n+1}(E),\, \|\mathcal{P}^s\mu\|_\infty \leq 1\}, \\ \Gamma_2 &:= \sup\{\mu(E) : \mu \in \Sigma^s_{n+1}(E),\, \|\mathcal{P}^s_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1\}.\end{aligned}$$

It is clear that $\widetilde{\Gamma}_{\Theta^s,+}(E) \leq \Gamma_1$. It is also clear that Theorem 2.1.2 implies $\Gamma_1 \gtrsim \Gamma_{\Theta^s,+}(E)$, and that the converse estimate follows from Lemma 1.3.2 applied with $\beta := \frac{1}{2s}$.

To prove that $\widetilde{\Gamma}_{\Theta^s,+}(E) \approx \Gamma_2$ we argue as in [**MatPT, Theorem 5.5**]. Take $\mu \in \Sigma^s_{n+1}(E)$ satisfying $\widetilde{\Gamma}_{\Theta^s,+}(E) \leq 2\mu(E)$ and $\|\mathcal{P}^s\mu\|_\infty \leq 1$, $\|\mathcal{P}^{s,*}\mu\|_\infty \leq 1$. By Lemma 2.3.4 we get, uniformly on $\varepsilon > 0$,

$$\|\mathcal{P}^s_\varepsilon\mu\|_{L^\infty(\mu)} \lesssim 1, \qquad \|\mathcal{P}^{s,*}_\varepsilon\mu\|_{L^\infty(\mu)} \lesssim 1, \tag{2.3.2}$$

The boundedness in $L^2(\mu)$ of $\mathcal{P}^s_\mu$, that by Theorem 1.1.2 is associated with an $(n+1)$-dimensional C-Z convolution kernel, will follow from a Tb theorem of Hytönen and Martikainen [**HyMar, Theorem 2.3**] for non-doubling measures in geometrically doubling spaces, such as the s-parabolic space. We shall apply the previous theorem with $b = 1$. Taking into account (2.3.2), to ensure that $\mathcal{P}^s_\mu$ is bounded in $L^2(\mu)$, by [**HyMar, Theorem 2.3**] it is enough to verify that the weak boundedness property holds for s-parabolic balls with thin boundary. An s-parabolic ball of radius r_B has A-thin boundary if

$$\mu\big\{\overline{x} \in 2B : \operatorname{dist}_{p_s}(\overline{x}, \partial B) \leq \lambda r_B\big\} \leq A\,\lambda\mu(2B) \qquad \forall \lambda > 0. \tag{2.3.3}$$

Hence, we need to prove that, for some fixed $A > 0$ and any $B \subset \mathbb{R}^{n+1}$ s-parabolic ball with A-thin boundary,

$$|\langle \mathcal{P}^s_{\mu,\varepsilon}\chi_B, \chi_B\rangle| \leq C\mu(2B), \qquad \text{uniformly on } \varepsilon > 0. \tag{2.3.4}$$

To prove (2.3.4), consider a smooth function φ compactly supported on $2B$ with $\varphi \equiv 1$ on B and write

$$|\langle \mathcal{P}^s_{\mu,\varepsilon}\chi_B, \chi_B\rangle| \leq \int_B |\mathcal{P}^s_{\mu,\varepsilon}\varphi|\,\mathrm{d}\mu + \int_B |\mathcal{P}^s_{\mu,\varepsilon}(\varphi - \chi_B)|\,\mathrm{d}\mu.$$

Since μ presents $n+1$ upper s-parabolic growth and $\|\mathcal{P}^s\mu\|_\infty \leq 1$, by Lemma 1.3.2 and the localization Theorem 2.2.7 we have $\|\mathcal{P}^s_\mu\varphi\|_\infty \leq 1$, that is $\|\mathcal{P}^s_{\mu,\varepsilon}\varphi\|_\infty \leq 1$ uniformly on $\varepsilon > 0$. Therefore, the first integral on the right side of the above inequality is bounded by $C\mu(B)$. To estimate the second term we will use that B has a thin boundary and the growth estimates of Theorem 1.1.2. We compute:

$$\begin{aligned}\int_B |\mathcal{P}^s_{\mu,\varepsilon}(\varphi - \chi_B)|\,\mathrm{d}\mu &\lesssim \int_{2B\setminus B}\int_B \frac{\mathrm{d}\mu(y)}{|x-y|_{p_s}^{n+1}}\,\mathrm{d}\mu(x)\\ &\leq \sum_{j\geq 1}\int_{\left\{\frac{r_B}{2^j}\leq \operatorname{dist}_{p_s}(x,\partial B)\leq \frac{r_B}{2^{j-1}}\right\}\setminus B}\int_B \frac{\mathrm{d}\mu(y)}{|x-y|_{p_s}^{n+1}}\,\mathrm{d}\mu(x).\end{aligned}$$

Given $j \geq 1$ and $x \notin B$ with $2^{-j}r_B \leq \operatorname{dist}_{p_s}(x, \partial B) \leq 2^{-j+1}r_B$, since $\mu \in \Sigma^s_{n+1}(E)$ we get

$$\int_B \frac{d\mu(y)}{|x-y|_{p_s}^{n+1}} \leq \sum_{k=-2}^{j}\int_{\frac{r_B}{2^{k+1}}\leq |x-y|_{p_s}\leq \frac{r_B}{2^k}} \frac{\mathrm{d}\mu(y)}{|x-y|_{p_s}^{n+1}} \lesssim \sum_{k=-2}^{j}\frac{\mu(B(x, 2^{-k}r_B))}{(2^{-k}r_B)^{n+1}} \lesssim j+3.$$

Therefore, by (2.3.3)

$$\begin{aligned}\int_B |\mathcal{P}^s_{\mu,\varepsilon}(\varphi - \chi_B)|\,\mathrm{d}\mu &\lesssim \sum_{j\geq 1}(j+3)\,\mu(\{x : 2^{-j}r_B \leq \operatorname{dist}_{p_s}(x,\partial B) \leq 2^{-j+1}r(B)\})\\ &\lesssim \sum_{j\geq 0}\frac{j+3}{2^j}\mu(2B) \lesssim \mu(2B).\end{aligned}$$

So (2.3.4) holds and $\mathcal{P}^s_\mu$ satisfies $\|\mathcal{P}^s_\mu\|_{L^2(\mu)\to L^2(\mu)} \lesssim 1$. Therefore $\Gamma_2 \gtrsim \widetilde{\Gamma}_{\Theta^s,+}(E)$.

To prove the upper estimate, take $\mu \in \Sigma^s_{n+1}(E)$ with $\|\mathcal{P}^s_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1$ and $\Gamma_2 \leq 2\mu(E)$. The $L^2(\mu)$ boundedness of $\mathcal{P}^s_\mu$ ensures that $\mathcal{P}^s$ and $\mathcal{P}^{s,*}$ are bounded from the space of finite signed measures $M(\mathbb{R}^{n+1})$ to $L^{1,\infty}(\mu)$. In other words, there exists a constant $C>0$ so that, given any $\nu \in M(\mathbb{R}^{n+1})$, for any $\varepsilon > 0$ and $\lambda > 0$,

$$\mu\big(\{x\in\mathbb{R}^{n+1} : |\mathcal{P}^s_\varepsilon \nu(x)| > \lambda\}\big) \leq C\,\frac{\|\nu\|}{\lambda},$$

and the same replacing $\mathcal{P}^s_\varepsilon$ by $\mathcal{P}^{s,*}_\varepsilon$. The reader can consult a proof of the latter in [**T5, Theorem 2.16**] (to apply the previous arguments, it is used that the Besicovitch covering theorem with respect to s-parabolic balls is valid. Otherwise, the reader may also consult [**NTrVo1, Theorem 5.1**].) From this point on, by a well known dualization of these estimates (essentially due to Davie and Øksendal [**DaØ**], see [**Chr, Ch.VII, Theorem 23**] for a precise statement) and an application of Cotlar's inequality (see [**T5, Theorem 2.18**], for example), we deduce the existence of a function $h : E \to [0,1]$ so that

$$\mu(E) \leq C\int h\,\mathrm{d}\mu, \qquad \|\mathcal{P}^s_\mu h\|_\infty \leq 1, \qquad \|\mathcal{P}^{s,*}_\mu h\|_\infty \leq 1.$$

Therefore, $\widetilde{\Gamma}_{\Theta^s,+}(E) \geq \int h\,\mathrm{d}\mu \approx \mu(E) \approx \Gamma_2$, and we are done with the proof. □

Applying the previous result and arguing as in [**MatPT, Example 5.6**], we get that any subset of $\mathcal{H}^{n+1}_{p_s}$-positive measure of the graph of a $\text{Lip}(1,\frac{1}{2s})$ function is not removable. It is not clear, however, if for $s<1$ such an object exists.

2.3.1 Existence of removable sets with positive $\mathcal{H}^{n+1}_{p_s}$ measure

We would like to carry out a similar study to that done for the capacity introduced in [**MatP, §4**], that with our notation would be $\gamma^0_{\Theta^{1/2}}$ (the normalization being $\|P * T\|_\infty \leq 1$), in the current s-Lipschitz context, for $s\in(1/2,1]$. Let us remark that the critical dimension of the previous capacity presented in [**MatP**] in $\mathbb{R}^{n+1}$ is n and that the $1/2$-parabolic distance is, in fact, the usual Euclidean distance.

The first question to ask is if it suffices to consider the same corner-like Cantor set of the aforementioned reference, but now consisting on the intersection of successive families of s-parabolic cubes. If the reader is familiar with [**MatPT, §6**], he or she might anticipate that the answer to the previous question is negative. Let us motivate why this is the case.

Recall that for a given sequence $\lambda := (\lambda_k)_k$, $0<\lambda_k<1/2$, we define its associated Cantor set $E\subset\mathbb{R}^{n+1}$ by the following algorithm: set $Q^0 := [0,1]^{n+1}$ the unit cube of $\mathbb{R}^{n+1}$ and consider 2^{n+1} disjoint (Euclidean) cubes inside Q^0 of side length $\ell_1 := \lambda_1$, with sides parallel to the coordinate axes and such that each cube contains a vertex of Q^0. Continue this same process now for each of the 2^{n+1} cubes from the previous step, but now using a contraction factor λ_2. That is, we end up with $2^{2(n+1)}$ cubes with side length $\ell_2 := \lambda_1\lambda_2$. Proceeding inductively we have that at the k-th step of the iteration we encounter $2^{k(n+1)}$ cubes, that we denote Q^k_j for $1\leq j\leq 2^{k(n+1)}$, with side length $\ell_k := \prod_{j=1}^k \lambda_j$. We will refer to them as cubes of the k-*th generation*. We define

$$E_k = E(\lambda_1, \dots, \lambda_k) := \bigcup_{j=1}^{2^{k(n+1)}} Q_j^k,$$

and from the latter we obtain the Cantor set associated with λ,

$$E = E(\lambda) := \bigcap_{k=1}^{\infty} E_k.$$

If we chose $\lambda_j = 2^{-(n+1)/n}$ for every j, we would recover the particular Cantor set presented in [**MatP, §5**]. The previous choice is so particular that ensures

$$0 < \mathcal{H}^n(E) = \mathcal{H}^n_{p_{1/2}}(E) < \infty.$$

If $\#(E_k)$ is the number of cubes of E_k, the above property followed, in essence, from

$$\#(E_k) \cdot \ell_k^n = 2^{k(n+1)} \cdot \ell_k^n = 1, \quad \forall k \geq 1. \tag{2.3.5}$$

If we were to obtain such critical value of λ_j in the s-parabolic setting, taking into account the critical dimension of Γ_{Θ^s}, it should be such that

$$0 < \mathcal{H}^{n+1}_{p_s}(E) < \infty.$$

So if we directly consider the analog of the previous corner-like Cantor set, but made up of s-parabolic cubes, we should rewrite (2.3.5) as

$$2^{k(n+1)} \cdot \ell_k^{n+1} = 1, \quad \forall k \geq 1,$$

meaning that the corresponding critical value of λ_j has to be $1/2$, that is not admissible. Another reason that suggests that working with an s-parabolic version of the corner-like Cantor set could not be the best choice, is that it becomes too *small* in an s-parabolic Hausdorff-dimensional sense. Indeed, if we assume that there is τ_0 so that $\lambda_j \leq \tau_0 < 1/2$, $\forall j$, for any fixed $0 < \varepsilon \ll 1$ we may choose a generation $k \gg 1$ so that

$$\mathcal{H}^{n+1}_{\varepsilon,p_s}(E) \leq \mathcal{H}^{n+1}_{\varepsilon,p_s}(E_k) \lesssim 2^{k(n+1)}\ell_k^{n+1} \leq (2\tau_0)^{k(n+1)} \xrightarrow[k\to\infty]{} 0,$$

that implies $\Gamma_{\Theta^s}(E) = 0$, by Theorem 2.3.2.

Hence, it is clear that in order to obtain a potentially non-removable Cantor set E, one has to *enlarge* it. One way to do it (motivated by [**MatPT, §6**]) is as follows: let us fix $s \in (1/2, 1]$ and choose what we call the *non-self-intersection* parameter $\delta \in \mathbb{Z}_+$, the minimum integer $\delta = \delta(s) \geq 2$ satisfying

$$s > \frac{\log_\delta(\delta+1)}{2}, \qquad \text{that is} \qquad \delta + 1 < \delta^{2s}.$$

Let $Q^0 := [0,1]^{n+1}$ be the unit cube of $\mathbb{R}^{n+1}$ and consider $(\delta+1)\delta^n$ disjoint s-parabolic cubes Q_i^1, $1 \leq i \leq (\delta+1)\delta^n$, contained in Q^0, with sides parallel to the coordinate axes, side length $0<\lambda_1<1/\delta$, and disposed as follows: first, we consider the first n intervals of the cartesian product $Q^0:=[0,1]^{n+1}$ (that is, those contained in spatial directions) and we divide, each one, into δ equal subintervals $I_1,\ldots,I_\delta$. For each subinterval I_j, we contain another one J_j of length λ_1. Now, we distribute $J_1,\ldots,J_\delta$ in an equispaced way, fixing J_1 to start at 0 and J_δ to end at 1. More precisely, if for each interval $[0,1]$ we name

$$l_\delta:=\frac{1-\delta\lambda_1}{\delta-1},\qquad J_j:=\big[(j-1)(\lambda_1+l_\delta),j\lambda_1+(j-1)l_\delta\big],\quad j=1,\ldots,\delta,$$

we keep the following union of δ closed disjoint intervals of length λ_1

$$T_\delta:=\bigcup_{j=1}^{\delta}J_j.$$

Finally, for the remaining temporal interval $[0,1]$, we do the same splitting but in $\delta+1$ intervals of length λ_1^{2s}. That is, we name

$$\widetilde{l}_\delta:=\frac{1-(\delta+1)\lambda_1^{2s}}{\delta},\qquad \widetilde{J}_j:=\big[(j-1)(\lambda_1^{2s}+\widetilde{l}_\delta),j\lambda_1^{2s}+(j-1)\widetilde{l}_\delta\big],\quad j=1,\ldots,\delta+1.$$

Now we keep the union of $\delta+1$ closed disjoint intervals of length λ_1^{2s}

$$\widetilde{T}_\delta:=\bigcup_{j=1}^{\delta+1}\widetilde{J}_j.$$

From the above, we define the first generation of the Cantor set as $E_{1,p_s}:=(T_\delta)^n\times\widetilde{T}_\delta$, that is conformed by $(\delta+1)\delta^n$ disjoint s-parabolic cubes (see Figure 2.1).

This procedure continues inductively, i.e. the next generation E_{2,p_s} will be the family of $(\delta+1)^2\delta^{2n}$ disjoint s-parabolic cubes of side length $\ell_2:=\lambda_1\lambda_2$, $\lambda_2<1/\delta$, obtained from applying the previous construction to each of the cubes of E_{1,p_s}. More generally, the k-th generation E_{k,p_s} will be formed by $(\delta+1)^k\delta^{nk}$ disjoint s-parabolic cubes with side length $\ell_k:=\lambda_1\cdots\lambda_k$, $\lambda_k<1/\delta$, and with locations determined by the above iterative process. We name such cubes Q_j^k, with $j=1,\ldots,(\delta+1)^k\delta^{nk}$. The resulting *$s$-parabolic* Cantor set is

$$E_{p_s}=E_{p_s}(\lambda):=\bigcap_{k=1}^{\infty}E_{k,p_s}. \tag{2.3.6}$$

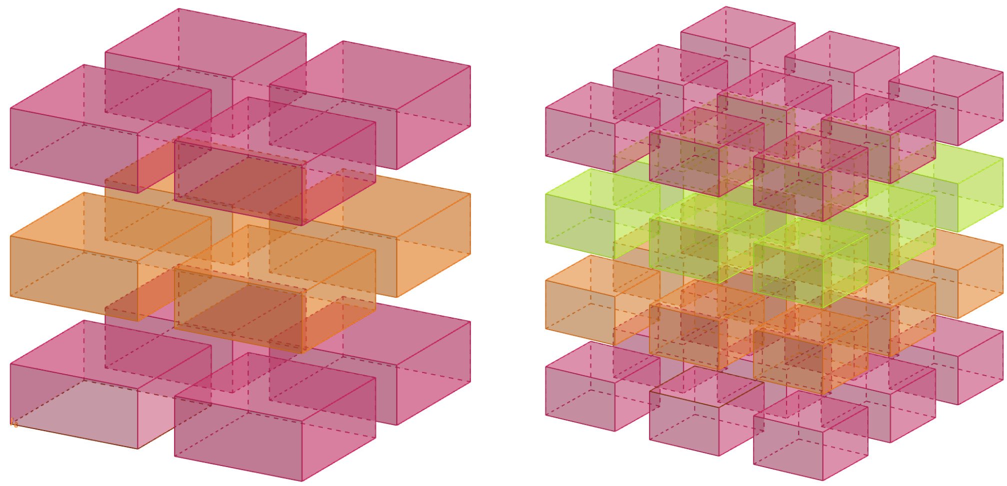
This way, the critical value becomes $((\delta+1)\delta^n)^{-1/(n+1)}<1/\delta$. For instance, if

$$\lambda_j:=((\delta+1)\delta^n)^{-1/(n+1)},\qquad \text{for every }\; j, \tag{2.3.7}$$

then

$$\#(E_{k,p_s})\cdot\ell_k^{n+1}=(\delta+1)^k\delta^{nk}\cdot\ell_k^{n+1}=1,\;\;\forall k\geq 1.$$

Using the previous fact one can deduce $\mathcal{H}^{n+1}_{p_s}(E_{p_s})>0$. Indeed, consider the probability measure μ defined on E_{p_s} such that for each generation k, $\mu(Q_j^k):=(\delta+$



(a) Depiction of E_{1,p_1} with $\delta = 2$ (b) Depiction of $E_{1,p_{2/3}}$ with $\delta = 3$.

Figure 2.1: First iterates involved in the construction of E_{p_1} and $E_{p_{2/3}}$ in $\mathbb{R}^3$. For $s = 1$ we have chosen $\lambda_1 := 12^{-1/3}$, and for $s = 2/3$ we have chosen $\lambda_1 := 1/4$.

$1)^{-k}\delta^{-nk}$, $1 \leq j \leq (\delta+1)^k\delta^{nk}$. Let Q be any s-parabolic cube, that we may assume to be contained in Q^0, and pick k with the property $\ell_{k+1} \leq \ell(Q) \leq \ell_k$, so that Q can meet, at most, $(\delta+1)\delta^n$ cubes Q_j^k. Thus $\mu(Q) \leq ((\delta+1)\delta^n)^{-(k-1)}$ and we deduce

$$\mu(Q) \lesssim \frac{1}{((\delta+1)\delta^n)^{(k+1)}} = \ell_{k+1}^{n+1} \leq \ell(Q)^{n+1}, \tag{2.3.8}$$

meaning that μ presents upper s-parabolic $n+1$-growth. Therefore, by [**Gar2, Chapter IV, Lemma 2.1**], which follows from Frostman's lemma, we get $\mathcal{H}_{p_s}^{n+1}(E) \geq \mathcal{H}_{p_s,\infty}^{n+1}(E) > 0$. Moreover, observe that for a fixed $0 < \varepsilon \ll 1$, there is k large enough so that $\operatorname{diam}_{p_s}(Q_j^k) \leq \varepsilon$. Thus, as E_{k,p_s} defines a covering of E_{p_s} admissible for $\mathcal{H}_{p_s,\varepsilon}^{n+1}$, we get

$$\mathcal{H}_{p_s,\varepsilon}^{n+1}(E) \leq \sum_{j=1}^{((\delta+1)\delta^n)^k} \operatorname{diam}_{p_s}(Q_j^k)^{n+1} \simeq \ell_k^{n+1}\,((\delta+1)\delta^n)^k = 1.$$

Since this procedure can be done for any ε, we also have $\mathcal{H}_{p_s}^{n+1}(E) < \infty$ and thus

$$0 < \mathcal{H}_{p_s}^{n+1}(E_{p_s}) < \infty.$$

Remark 2.3.1. Let us observe that as $s \to 1/2$, the value of δ grows arbitrarily. That is, as s approaches the value $1/2$, the Cantor set E_{p_s} becomes progressively more and more dense in the unit cube. This suggests that for $s = 1/2$, that corresponds to a space-time usual Lipschitz condition in $\mathbb{R}^{n+1}$, the capacity one would obtain should be comparable to the Lebesgue measure in $\mathbb{R}^{n+1}$, as in [**Uy**] for analytic capacity.

THEOREM 2.3.5. *Given $s\in(1/2,1]$, the Cantor set E_{p_s} defined in (2.3.6) with the choice (2.3.7) is removable for Lipschitz s-caloric functions.*

Proof. We follow an analogous proof to that of [**MatPT, Theorem 6.3**]. We will assume that E_{p_s} is not removable and reach a contradiction. By Theorem 2.3.1, there is ν a distribution with $\text{supp}(\nu)\subset E_{p_s}$ satisfying $|\langle\nu,1\rangle|>0$ and

$$\|\nabla_x P_s*\nu\|_\infty\leq 1, \qquad \|\partial_t^{\frac{1}{2s}}P_s*\nu\|_{*,p_s}\leq 1.$$

By [**MatPT, Theorem 6.1**], that admits an almost identical proof in the s-parabolic context, ν admits the following representation:

$$\nu=f\,\mu, \qquad \|f\|_{L^\infty(\mu)}\lesssim 1,$$

for some $f:E\to\mathbb{R}$ Borel function, and where $\mu:=\mathcal{H}^{n+1}_{p_s}|_{E_{p_s}}$ coincides, up to a constant (depending on n and s), with the probability measure supported on E_{p_s} that satisfies $\mu(Q_j^k)=((\delta+1)\delta^n)^{-k}$ for all j,k. By (2.3.8) μ (and also $|\nu|$) has upper s-parabolic growth of degree $n+1$. By analogous arguments to those of Lemma 2.3.4, there must exist a constant $\kappa(n,s)$ so that

$$\mathcal{P}^s_*\nu(\overline{x})\leq\kappa, \quad \overline{x}\in\mathbb{R}^{n+1}. \tag{2.3.9}$$

For $\overline{x}\in E_{p_s}$, let $Q^k_{\overline{x}}$ be the cube Q^k_i containing $\overline{x}$. Define the auxiliary operator

$$\widetilde{\mathcal{P}}^s_*\nu(\overline{x})=\sup_{k\geq 0}|\mathcal{P}^s_\nu\chi_{\mathbb{R}^{n+1}\setminus Q^k_{\overline{x}}}(\overline{x})|.$$

Since the cubes Q^k_i are separated, applying the growth of $|\nu|$ as well as estimate (2.3.9),

$$\widetilde{\mathcal{P}}^s_*\nu(\overline{x})\leq\kappa', \quad \overline{x}\in E, \tag{2.3.10}$$

for some constant κ'. We shall contradict the previous bound.

To do so, pick $\overline{x}_0\in E_{p_s}$ a Lebesgue point (with respect to μ and to s-parabolic cubes) of the Radon-Nikodym derivative $f=\frac{d\nu}{d\mu}$ such that $f(\overline{x}_0)>0$. This can be done since $\nu(E)>0$. Given $\varepsilon>0$ small enough to be chosen below, consider a parabolic cube Q^k_i containing $\overline{x}_0$ such that

$$\frac{1}{\mu(Q^k_i)}\int_{Q^k_i}|f(\overline{y})-f(\overline{x}_0)|\,\text{d}\mu(y)\leq\varepsilon.$$

Let us begin by choosing ε so that $f(\overline{x}_0)>\varepsilon$. This last condition implies that for a given $m\gg 1$ and any $k\leq h\leq k+m$, if Q^h_j is contained in $Q^k_{\overline{x}_0}$,

$$\begin{aligned}\frac{\nu(Q^h_j)}{\mu(Q^h_j)}=\frac{1}{\mu(Q^h_j)}\int_{Q^h_j}f(\overline{y})\,\text{d}\mu(\overline{y})\geq f(\overline{x}_0)-\left|\frac{1}{\mu(Q^h_j)}\int_{Q^h_j}\big(f(\overline{y})-f(\overline{x}_0)\big)\,\text{d}\mu(\overline{y})\right|\\ \geq f(\overline{x}_0)-\varepsilon\frac{\mu(Q^k_{\overline{x}_0})}{\mu(Q^h_j)}=f(\overline{x}_0)-((\delta+1)\delta^n)^{h-k}\varepsilon.\end{aligned} \tag{2.3.11}$$

For each $m \gg 1$, we fix the value of ε (and therefore also the value of k) to satisfy

$$\varepsilon < \frac{((\delta+1)\delta^n)^{-m}}{2} f(\overline{x}_0),$$

which implies $\nu(Q_j^h) \geq f(\overline{x}_0)\mu(Q_j^h)/2$ and hence, in particular, $\nu(Q_j^h) \geq 0$. Therefore,

$$\frac{\nu(Q_j^h)}{\mu(Q_j^h)} = \left| \frac{1}{\mu(Q_j^h)} \int_{Q_j^h} f(\overline{y}) \, \mathrm{d}\mu(\overline{y}) \right| \leq \frac{3}{2} f(\overline{x}_0). \tag{2.3.12}$$

All in all, the previous estimates ensure that choosing $\varepsilon > 0$ small enough (depending on m), every cube Q_j^h contained in $Q_{\overline{x}_0}^k$ with $k \leq h \leq k+m$ satisfies

$$\frac{1}{2} f(\overline{x}_0)\mu(Q_j^h) \leq \nu(Q_j^h) \leq \frac{3}{2} f(\overline{x}_0)\mu(Q_j^h). \tag{2.3.13}$$

Notice also that if we decompose ν in terms of its positive and negative variations, that is $\nu = \nu^+ - \nu^-$, using $f(\overline{x}_0) > 0$ we deduce

$$\begin{aligned} \nu^-(Q_{\overline{x}_0}^k) &= \int_{Q_{\overline{x}_0}^k} f^-(\overline{y}) \, \mathrm{d}\mu(\overline{y}) = \frac{1}{2} \int_{Q_{\overline{x}_0}^k} \big(|f(\overline{y})| - f(\overline{y})\big) \, \mathrm{d}\mu(\overline{y}) \\ &\leq \frac{1}{2} \int_{Q_{\overline{x}_0}^k} |f(\overline{y}) - f(\overline{x}_0)| \, \mathrm{d}\mu(\overline{y}) - \frac{1}{2} \int_{Q_{\overline{x}_0}^k} \big(f(\overline{y}) - f(\overline{x}_0)\big) \, \mathrm{d}\mu(\overline{y}) \leq \varepsilon \mu(Q_{\overline{x}_0}^k). \end{aligned} \tag{2.3.14}$$

Consider $\overline{z} = (z_1, \ldots, z_n, u)$ one of the upper leftmost corners of $Q_{\overline{x}_0}^k$ (that is, with z_1 minimal and u maximal in $Q_{\overline{x}_0}^k$). Since $\overline{z} \in E_{p_s}$ and by definition $|\mathcal{P}_\nu^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})| = |\mathcal{P}_\nu^s \chi_{\mathbb{R}^{n+1} \setminus Q_{\overline{z}}^{k+m}}(\overline{z}) - \mathcal{P}_\nu^s \chi_{\mathbb{R}^{n+1} \setminus Q_{\overline{z}}^{k}}(\overline{z})| \leq 2\widetilde{\mathcal{P}}_*^s \nu(\overline{z})$, we have

$$\widetilde{\mathcal{P}}_*^s \nu(\overline{z}) \geq \frac{1}{2} |\mathcal{P}_\nu^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})| \geq \frac{1}{2} |\mathcal{P}_{\nu^+}^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})| - \frac{1}{2} |\mathcal{P}_{\nu^-}^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})|.$$

Observe that $\mathrm{dist}_{p_s}(\overline{z}, Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}) \gtrsim \ell(Q_{\overline{z}}^{k+m})$ (with implicit constants depending on $\delta = \delta(s)$), so we can estimate $|\mathcal{P}_{\nu^-}^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})|$ from above in the following way

$$\begin{aligned} |\mathcal{P}_{\nu^-}^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})| &\leq \int_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}} |\nabla_x P_s(\overline{z} - \overline{y})| \, \mathrm{d}\nu^-(\overline{y}) \leq \int_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}} \frac{\mathrm{d}\nu^-(\overline{y})}{|\overline{z} - \overline{y}|_{p_s}^{n+1}} \\ &\lesssim \frac{\nu^-(Q_{\overline{z}}^k)}{\ell(Q_{\overline{z}}^{k+m})^{n+1}} \leq \varepsilon \frac{\mu(Q_{\overline{z}}^k)}{\ell(Q_{\overline{z}}^{k+m})^{n+1}} = \varepsilon \frac{((\delta+1)\delta^n)^{-k}}{((\delta+1)\delta^n)^{-m} \ell(Q_{\overline{z}}^k)^{n+1}} = ((\delta+1)\delta^n)^m \varepsilon, \end{aligned}$$

where we have used Theorem 1.1.2 and (2.3.14).

To estimate $|\mathcal{P}_{\nu^+}^s \chi_{Q_{\overline{z}}^k \setminus Q_{\overline{z}}^{k+m}}(\overline{z})|$ from below, we refer the reader to the proof of Theorem 1.1.2 to check that the first component of $\nabla_x P_s$, for $s < 1$, satisfies

$$(\nabla_x P_s)_1(\overline{x}) \approx c_0 \frac{-x_1 t}{|\overline{x}|_{p_s}^{n+2s+2}} \chi_{t>0},$$

for some constant $c_0>0$. Then, by the choice of $\overline{z}$, it follows that

$$(\nabla_x P_s)_1(\overline{z}-\bar{y})\geq 0 \quad \text{ for all } \bar{y}\in Q_{\overline{z}}^k\setminus Q_{\overline{z}}^{k+m}. \tag{2.3.15}$$

We write

$$\begin{aligned}|\mathcal{P}^s_{\nu^+}\chi_{Q_{\overline{z}}^k\setminus Q_{\overline{z}}^{k+m}}(\overline{z})| &\geq \int_{Q_{\overline{z}}^k\setminus Q_{\overline{z}}^{k+m}}(\nabla_x P_s)_1(\overline{z}-\bar{y})\,\mathrm{d}\nu^+(\bar{y})\\ &= \sum_{h=k}^{k+m-1}\int_{Q_{\overline{z}}^h\setminus Q_{\overline{z}}^{h+1}}(\nabla_x P_s)_1(\overline{z}-\bar{y})\,\mathrm{d}\nu^+(\bar{y}).\end{aligned}$$

By relation (2.3.15) and that for $k\leq h\leq k+m-1$, the set $Q_{\overline{z}}^h\setminus Q_{\overline{z}}^{h+1}$ contains a cube Q_j^{h+1} such that for all $\bar{y}=(y_1,\ldots,y_n,\tau)$,

$$0<y_1-z_1\approx|\bar{y}-\overline{z}|\approx\ell(Q_j^{h+1}),\qquad 0<u-\tau\approx\ell(Q_j^{h+1})^{2s}.$$

Indeed, we might just consider the lower rightmost cube of the $h+1$ generation that is contained in $Q_{\overline{z}}^h$ and take advantage of the corner choice of $\overline{z}$ and the fact that $\overline{z}\notin Q_{\overline{z}}^k\setminus Q_{\overline{z}}^{k+m}$. Now, using also (2.3.13), we deduce

$$\begin{aligned}\int_{Q_{\overline{z}}^h\setminus Q_{\overline{z}}^{h+1}}(\nabla_x P_s)_1(\overline{z}-\overline{y})\,\mathrm{d}\nu^+(\overline{y}) &\geq \int_{Q_j^{h+1}}(\nabla_x P_s)_1(\overline{z}-\overline{y})\,\mathrm{d}\nu^+(\overline{y})\gtrsim\frac{\nu^+(Q_j^{h+1})}{\ell(Q_j^{h+1})^{n+1}}\\ &\gtrsim f(\overline{x}_0)\frac{\mu(Q_j^{h+1})}{\ell(Q_j^{h+1})^{n+1}}=f(\overline{x}_0).\end{aligned}$$

Therefore, $|\mathcal{P}^s_{\nu^+}\chi_{Q_{\overline{z}}^k\setminus Q_{\overline{z}}^{k+m}}(\overline{z})|\gtrsim(m-1)\,f(\overline{x}_0)$, and combining this with the previous estimate obtained for $|\mathcal{P}^s_{\nu^-}\chi_{Q_{\overline{z}}^k\setminus Q_{\overline{z}}^{k+m}}(\overline{z})|$ we get

$$\widetilde{\mathcal{P}}^s_*\nu(\overline{z})\gtrsim(m-1)f(\overline{x}_0)-C((\delta+1)\delta^n)^m\varepsilon,$$

for some constant $C>0$. So choosing m big enough and then ε small enough, depending on m, relation (2.3.10) cannot hold and we reach a contradiction. □

Chapter 3

The non-comparability of Γ_Θ and $\gamma_\Theta^{1/2}$ in the plane

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In this brief chapter we would like to introduce a capacity tightly related to the capacities $\gamma^\sigma_{\Theta^s,*}$ presented in §1.4.3 and compare it with the Lipschitz caloric capacity Γ_Θ studied in [**MatPT**]. Recall that, given $E\subset\mathbb{R}^{n+1}$ compact set, we defined

$$\Gamma_\Theta(E):=\sup|\langle T,1\rangle|,$$

with the supremum taken among all distributions in $\mathbb{R}^{n+1}$ supported on E such that

$$\|\nabla_x W*T\|_\infty\leq 1 \qquad \text{and} \qquad \|\partial_t^{1/2}W*T\|_{*,p_1}\leq 1.$$

Bearing in mind the above definition, we introduce the capacity $\gamma_\Theta^{1/2}$. It is defined analogously, but instead of requiring a $(1,1/2)$-Lipschitz property over the potentials, we ask for

$$\|(-\Delta)^{1/2}W*T\|_\infty\leq 1 \qquad \text{and} \qquad \|\partial_t^{1/2}W*T\|_{*,p_1}\leq 1.$$

We will prove that Γ_Θ and $\gamma_\Theta^{1/2}$ share the same critical dimension but, at least in $\mathbb{R}^2$, they do not share the same removable sets. This is remarkable since the operators ∇_x and $(-\Delta)^{1/2}$ present Fourier symbols with shared homogeneity.

3.1 Basic definitions and properties

Let us recall that given $s\in(0,1]$, $\sigma\in[0,s)$ and $E\subset\mathbb{R}^{n+1}$ compact set, we defined its Δ^σ-BMO$_{p_s}$-caloric capacity as

$$\gamma^\sigma_{\Theta^s,*}(E):=\sup|\langle T,1\rangle|,$$

the supremum being taken among distributions T supported on E and satisfying $\|(-\Delta)^\sigma P_s*T\|_{*,p_s}\leq 1$ and $\|\partial_t^{\sigma/s}P_s*T\|_{*,p_s}\leq 1$. In this chapter, we will fix $s=1$ and choose $\sigma:=1/2$ and study the following capacity:

Definition 3.1.1. Given $E\subset\mathbb{R}^{n+1}$ compact set, its $\Delta^{1/2}$-*caloric capacity* is

$$\gamma_\Theta^{1/2}(E) := \sup |\langle T,1\rangle|,$$

where the supremum is taken among all distributions T with $\operatorname{supp}(T)\subseteq E$ and satisfying

$$\big\|(-\Delta)^{1/2}W*T\big\|_\infty\leq 1, \qquad \big\|\partial_t^{1/2}W*T\big\|_{*,p}\leq 1.$$

Such distributions will be called *admissible for* $\gamma_\Theta^{1/2}(E)$. Let us observe that the particular choice of $\sigma:=1/2$ allows the operator $(-\Delta)^{1/2}$ to be represented as

$$(-\Delta)^{1/2}\varphi(x,t)\simeq \text{ p.v.}\int_{\mathbb{R}^n}\frac{\varphi(x,t)-\varphi(y,t)}{|x-y|^{n+1}}\simeq\sum_{j=1}^n R_j\partial_j\varphi,$$

where R_j are the usual n-dimensional Riesz transforms, with Fourier multiplier an absolute multiple of $\xi_j/|\xi|$. Let us first observe that as a direct consequence of Theorem 1.4.8 we obtain the following growth result for admissible distributions for $\gamma_\Theta^{1/2}$.

Theorem 3.1.1. *Let $E\subset\mathbb{R}^{n+1}$ be compact and T be an admissible distribution for $\gamma_\Theta^{1/2}$. Then, T presents upper 1-parabolic growth of degree $n+1$, that is,*

$$|\langle T,\varphi\rangle|\lesssim \ell(Q)^{n+1}, \qquad \textit{for } Q\subset\mathbb{R}^{n+1} \textit{ any 1-parabolic cube and } \varphi \textit{ admissible for } Q.$$

Let us clarify that we could have stated the above result in a more general manner as in Theorem 1.4.8, but we have chosen to do it as above for the sake of clarity.

Hence, given such growth, one of the first questions that arises is if Γ_Θ and $\gamma_\Theta^{1/2}$ are comparable. Indeed, let us observe that we are able to prove an analogous result to Theorem 2.3.2 in the current setting, which implies that Γ_Θ and $\gamma_\Theta^{1/2}$ both share critical dimension:

Theorem 3.1.2. *For every compact set $E\subset\mathbb{R}^{n+1}$ the following hold:*

1. $\gamma_\Theta^{1/2}(E)\leq C\,\mathcal{H}^{n+1}_{\infty,p_1}(E)$, for some dimensional constant $C>0$.

2. If $\dim_{\mathcal{H}_{p_1}}(E)>n+1$, then $\gamma_\Theta^{1/2}(E)>0$.

Proof. To prove *1* we proceed analogously as we have done in the proof of Theorem 1.4.3, using now the growth restriction given by Lemma 3.1.1. To prove *2* we argue as in Theorem 2.3.2. We name $d:=\dim_{\mathcal{H}_{p_1}}(E)$ and assume $0<\mathcal{H}^d_{p_1}(E)<\infty$. Apply Frostman's lemma and pick a non-zero positive measure μ supported on E with $\mu(B(\overline{x},r))\leq r^d$, being $B(\overline{x},r)$ any 1-parabolic ball. If we prove

$$\big\|(-\Delta)^{1/2}W*\mu\big\|_\infty\lesssim 1 \quad\text{ and }\quad \big\|\partial_t^{1/2}W*\mu\big\|_{*,p}\lesssim 1,$$

we will be done, because then we would have $\gamma_\Theta^{1/2}(E)\gtrsim\langle\mu,1\rangle>0$. To estimate $\partial_t^{1/2}W*\mu$ we apply Lemma 1.3.2 with $\beta:=1/2$. To study the bound of $(-\Delta)^{1/2}W*\mu$, apply [**MatP, Lemma 2.2**] to deduce that for any $\overline{x}\in\mathbb{R}^{n+1}$,

$$|(-\Delta)^{1/2}W*\mu(\overline{x})|\lesssim\int_E\frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{y}|_p^{n+1}}\lesssim \operatorname{diam}_p(E)^{d-(n+1)}\lesssim 1,$$

and we are done. □

3.2 The non-comparability in $\mathbb{R}^2$

To study whether if Γ_Θ is comparable to $\gamma_\Theta^{1/2}$ we notice that, as a consequence of Theorem 2.3.3, any subset of positive $\mathcal{H}_{p_1}^{n+1}$ measure of a non-horizontal hyperplane (that is, not parallel to $\mathbb{R}^n\times\{0\}$) is not Lipschitz caloric removable. Therefore, in the planar case ($n=1$), the vertical line segment

$$E := \{0\}\times[0,1], \qquad \text{that is such that } \dim_{\mathcal{H}_{p_1}}(E) = n+1 = 2,$$

is a first candidate to consider. To proceed, let us define a series of operators analogous to those presented in §2.3. Given μ, a real compactly supported Borel regular measure with upper 1-parabolic growth of degree $n+1$, let $\mathcal{T}_\mu$ be acting on elements of $L^1_{\text{loc}}(\mu)$ as

$$\mathcal{T}_\mu f(\overline{x}) := \int_{\mathbb{R}^{n+1}} (-\Delta)^{1/2} W(\overline{x}-\overline{y}) f(\overline{y}) \mathrm{d}\mu(\overline{y}), \qquad \overline{x}\notin \operatorname{supp}(\mu),$$

as well as its truncated version $\mathcal{T}_{\mu,\varepsilon}$ and the maximal operator $\mathcal{T}_{*,\mu}$, defined analogously as in §2.3.

Notice that comparing [**MatPT, Lemma 5.4**] and Theorem 1.1.3 with the particular choice of $s=1$ and $\beta := 1/2$, the growth-like behavior of the kernels $\nabla_x W$ and $(-\Delta)^{1/2}W$ is analogous. From this observation, the following result admits an analogous proof to that of Lemma 2.3.4.

Lemma 3.2.1. *Let μ be a real Borel measure with compact support and $n+1$ upper 1-parabolic growth with $\|\mathcal{T}\mu\|_\infty \leq 1$. Then, there is $\kappa>0$ absolute constant so that*

$$\mathcal{T}_*\mu(\overline{x}) \leq \kappa, \qquad \forall\, \overline{x}\in\mathbb{R}^{n+1}.$$

Using the above lemma we are able to prove the following:

Theorem 3.2.2. *The vertical segment $E := \{0\}\times[0,1]\subset\mathbb{R}^2$ satisfies $\gamma_\Theta^{1/2}(E)=0$. Therefore, Γ_Θ and $\gamma_\Theta^{1/2}$ are not comparable in $\mathbb{R}^2$.*

Proof. We will prove it by contradiction, i.e. by assuming $\gamma_\Theta^{1/2}(E)>0$. Begin by noticing that, under this last hypothesis, we would be able to find a distribution ν supported on E such that

$$\|(-\Delta)^{1/2}W*\nu\|_\infty \leq 1, \qquad \|\partial_t^{1/2}W*\nu\|_{*,p}\leq 1 \qquad \text{and} \qquad |\langle \nu, 1\rangle|>0.$$

By Theorem 3.1.1 the distribution ν has upper 1-parabolic growth of degree $n+1=2$. Therefore, since $0<\mathcal{H}^2_{p_1}(E)<\infty$, by [**MatPT, Lemma 6.2**], we deduce that $\nu = f\mathcal{H}^2_{p_1}|_E$, where $f:E\to\mathbb{R}$ is Borel function with $\|f\|_{L^\infty(\mathcal{H}^2_{p_1}|_E)}\lesssim 1$. In addition, we will also assume, without loss of generality, that $\nu(E)>0$.

We shall contradict the estimate presented in Lemma 3.2.1 by finding a point $\overline{x}_0\in E$ such that $\mathcal{T}_*\nu(\overline{x}_0)$ is arbitrarily big. To this end, firstly, we properly choose such point by a similar argument to that of the proof of [**MatPT, Theorem 6.3**]. Pick $\overline{x}_0 = (0,t_0)\in E$ with $0<t_0<1$ a Lebesgue point for the density $f = \mathrm{d}\nu/\mathrm{d}\mathcal{H}^2_{p_1}|_E$

satisfying $f(\overline{x}_0) > 0$, that can be done since $\nu(E) > 0$. Hence, for $\varepsilon > 0$ small enough, which will be fixed later on, there is an integer $k > 0$ big enough so that if $B_{k,0}$ is the 1-parabolic ball centered at $\overline{x}_0$ with radius $r(B_{k,0}) = 2^{-k}$,

$$\frac{1}{\mathcal{H}^2_{p_1}(B_{k,0}\cap E)}\int_{B_{k,0}\cap E}|f(\overline{y})-f(\overline{x}_0)|\mathrm{d}\mathcal{H}^2_{p_1}(\overline{y}) \leq \varepsilon. \tag{3.2.1}$$

In addition, we may assume, without loss of generality, that k is big enough so that we have the inclusion $B_{k,0}\cap(\{0\}\times\mathbb{R})\subset E$. This way, the above estimate can be simply reformulated as

$$2^{2k}\int_{t_0-2^{-2k}}^{t_0+2^{-2k}}|f(0,t)-f(0,t_0)|\,\mathrm{d}t \leq \varepsilon.$$

In any case, let us still work with (3.2.1). Begin by fixing the value of ε so that $f(\overline{x}_0) > \varepsilon$. Proceeding as in the proof of Theorem 2.3.5, choosing

$$\varepsilon < 2^{-2m-1}f(\overline{x}_0),$$

by analogous arguments to those presented in (2.3.11) and (2.3.12) (interchanging the role of μ in this case by $\mathcal{H}^2_{p_1}|_{B_h}$), we get that for any given $m \gg 1$, if $\varepsilon < 2^{-2m-1}f(\overline{x}_0)$,

$$\frac{1}{2}f(\overline{x}_0)\mathcal{H}^2_{p_1}(B_h\cap E) \leq \nu(B_h) \leq \frac{3}{2}f(\overline{x}_0)\mathcal{H}^2_{p_1}(B_h\cap E),$$

which is an analogous bound to (2.3.13). Notice also that if we decompose ν in terms of its positive and negative variations, that is $\nu = \nu^+ - \nu^-$, using $f(\overline{x}_0) > 0$ we deduce as in (2.3.14)

$$\nu^-(B_{k,0}) \leq \varepsilon\mathcal{H}^2_{p_1}(B_{k,0}\cap E). \tag{3.2.2}$$

Having fixed $\overline{x}_0$ and the value of ε, we shall proceed with the proof. Observe that since ν is supported on E we have

$$|\mathcal{T}_\nu\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)| = |\mathcal{T}_\nu\chi_{E\setminus B_{k+m,0}}(\overline{x}_0) - \mathcal{T}_\nu\chi_{E\setminus B_{k,0}}(\overline{x}_0)| \lesssim 2\mathcal{T}_*\nu(\overline{x}_0).$$

Therefore,

$$\begin{aligned}\mathcal{T}_*\nu(\overline{x}_0) &\gtrsim \frac{1}{2}|\mathcal{T}_\nu\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)|\\ &\geq \frac{1}{2}|\mathcal{T}_{\nu^+}\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)| - \frac{1}{2}|\mathcal{T}_{\nu^-}\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)|.\end{aligned}$$

Since $\mathrm{dist}_p(\overline{x}_0, B_{k,0}\setminus B_{k+m,0}) \geq r(B_{k+m,0})$ we estimate $|\mathcal{T}_{\nu^-}\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)|$ as follows

$$\begin{aligned}|\mathcal{T}_{\nu^-}\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)| &\leq \int_{B_{k,0}\setminus B_{k+m,0}}|(-\Delta)^{1/2}W(\overline{x}_0-\overline{y})|d\nu^-(\overline{y})\\ &\lesssim \int_{B_{k,0}\setminus B_{k+m,0}}\frac{d\nu^-(\overline{y})}{|\overline{x}_0-\overline{y}|_p^2} \leq \frac{\nu^-(B_{k,0})}{r(B_{k+m,0})^2} \leq \varepsilon\frac{\mathcal{H}^2_{p_1}(B_{k,0}\cap E)}{r(B_{k+m,0})^2} \leq 2^{2m}\varepsilon,\end{aligned}$$

where we have used Theorem 1.1.3 and relation (3.2.2). So we are left to study the quantity $|\mathcal{T}_{\nu^+}\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)|$. Defining $f:\mathbb{R}^n\to\mathbb{R}$ as $f(z):=e^{-|z|^2/4}$, we have that

$$(-\Delta)^{1/2}f(0)\simeq \text{p.v.}\int_{\mathbb{R}^n}\frac{1-e^{-|y|^2}}{|y|^{n+1}}\,\mathrm{d}y\simeq\int_0^\infty\frac{1-e^{-r^2}}{r^2}\,\mathrm{d}r=\sqrt{\pi}. \tag{3.2.3}$$

Relation (3.2.3) and the fact that for each $t>0$ we have

$$\begin{aligned}(-\Delta)^{1/2}W(x,t)&\simeq t^{-n/2}(-\Delta)^{1/2}\Big[e^{-|\cdot|^2/(4t)}\Big](x)\\&=t^{-\frac{n+1}{2}}(-\Delta)^{1/2}e^{-|x|^2/(4t)}=:t^{-\frac{n+1}{2}}(-\Delta)^{1/2}f\big(x|t|^{-1/2}\big),\end{aligned}$$

implies that, in $\mathbb{R}^2$,

$$(-\Delta)^{1/2}W(0,t)\simeq t^{-1}\chi_{t>0}.$$

Hence,

$$\begin{aligned}|\mathcal{T}_{\nu^+}\chi_{B_{k,0}\setminus B_{k+m,0}}(\overline{x}_0)|&=\Big|\int_{B_{k,0}\setminus B_{k+m,0}\cap E}(-\Delta)^{1/2}W(0,t_0-t)\,\mathrm{d}\nu^+(0,t)\Big|\\
&=\Big|\int_{\left[t_0-\frac{1}{2^{2k}},\,t_0+\frac{1}{2^{2k}}\right]\setminus\left[t_0-\frac{1}{2^{2(k+m)}},\,t_0+\frac{1}{2^{2(k+m)}}\right]}(-\Delta)^{1/2}W(0,t_0-t)\,\mathrm{d}\nu^+(0,t)\Big|\\
&\simeq\Big|\int_{\left[t_0-\frac{1}{2^{2k}},\,t_0+\frac{1}{2^{2k}}\right]\setminus\left[t_0-\frac{1}{2^{2(k+m)}},\,t_0+\frac{1}{2^{2(k+m)}}\right]}\frac{1}{|t_0-t|}\chi_{\{t_0-t>0\}}\,\mathrm{d}\nu^+(0,t)\Big|\\
&=\int_{\left[t_0-\frac{1}{2^{2k}},\,t_0-\frac{1}{2^{2(k+m)}}\right]}\frac{1}{t_0-t}\,\mathrm{d}\nu^+(0,t)\\
&=\sum_{h=0}^{2m+1}\int_{\left[t_0-\frac{1}{2^{2(k+m)-h-1}},\,t_0-\frac{1}{2^{2(k+m)-h}}\right]}\frac{1}{t_0-t}\,\mathrm{d}\nu^+(0,t)\\
&\geq\sum_{h=0}^{2m+1}2^{2(k+m)-h}\cdot\nu^+\left(\left[t_0-\frac{1}{2^{2(k+m)-h-1}},\,t_0-\frac{1}{2^{2(k+m)-h}}\right]\right)\\
&\geq\sum_{h=0}^{2m+1}2^{2(k+m)-h}\cdot\frac{1}{2}f(\overline{x}_0)\mathcal{H}^2_{p_1}\big(B_{k+m-\frac{h}{2}}\cap E\big)\simeq(2m+1)f(\overline{x}_0),\end{aligned}$$

where for the last inequality we have used the left estimate of (3.2.2). All in all, we get

$$\mathcal{T}_*\nu(\overline{x}_0)\gtrsim(2m+1)f(\overline{x}_0)-2^{2m}\varepsilon,$$

so choosing m big enough and then ε small enough, we are able to reach the desired contradiction. □

Chapter 4

Caloric capacities of Cantor sets

In the present chapter we present a lower estimate for the Lipschitz s-caloric capacity of the Cantor sets defined in §2.3.1 and a complete characterization of another capacity $\gamma_{\Theta^s,+}$ (that we will simply refer to as $(s,+)$-*caloric capacity*) of the usual corner-like Cantor set of $\mathbb{R}^{n+1}$. The way we present the latter, it generalizes [**Her, Corollary 4.3**].

In any case, the result obtained for the Lipschitz s-caloric capacity in §4.1 resembles that of [**MatT, T4**] for s-dimensional Riesz transforms. To be precise, consider $1/2 < s \leq 1$ and $(\lambda_j)_j$ a sequence that determines the length $\ell_j := \lambda_1 \cdots \lambda_j$ of the s-parabolic cubes conforming the j-th generation of the Cantor set E_{p_s} as in (2.3.6). Assume $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j, where δ is a parameter that depends on s defined in §2.3.1. Then,

Theorem.

$$\Gamma_{\Theta^s}(E_{p_s}) \gtrsim_{\tau_0} \left(\sum_{j=0}^{\infty} \theta_{j,p_s}^2 \right)^{-1/2}, \qquad \textit{where} \quad \frac{\ell_j^{-(n+1)}}{(\delta+1)^j \delta^{nj}}.$$

The reverse inequality remains an open problem.

The particular feature that allows us to follow the arguments of Mateu and Tolsa is essentially that, although the kernel $\nabla_x P_s$ is not fully anti-symmetric, it satisfies this property with respect to spatial variables. In fact, as the arguments below suggest, it is enough to ask for anti-symmetry with respect to a single variable, provided the kernel is a locally integrable function in $\mathbb{R}^{n+1}$, endowed with a proper metric.

On the other hand, the results obtained for $\gamma_{\Theta^s,+}$ are essentially different, since the kernel involved in the definition of the latter capacity is simply P_s. The nonnegativity of this kernel suggests that the value of this capacity for Cantor sets should be similar to that described by Eiderman in [**E**], where he studied radial nonnegative kernels. In our setting, however, the kernel P_s will be nonnegative *but not radial*. To circumvent such inconvenience, the author will compare $\gamma_{\Theta^s,+}$ with an auxiliary capacity, defined through a nonnegative symmetric kernel and deduce, in a rather straightforward manner, the expected estimate. For instance, let $s \in (0,1]$ and $E \subset \mathbb{R}^{n+1}$ be the typical corner-like Cantor set, now made up of s-parabolic cubes and with associated sequence $(\lambda_j)_j$. If $0 < \lambda_j \leq \tau_0 < 1/2$ for each j, then

THEOREM.

$$\gamma_{\Theta^s,+}(E) \approx_{\tau_0} \left(\sum_{j=0}^{\infty} \theta_j\right)^{-1}, \qquad \textit{where} \quad \theta_j := \frac{\ell_j^{-n}}{2^{j(n+1)}}.$$

In fact, the study conducted for $\gamma_{\Theta^s,+}$ will also yield the comparability of the latter capacity with the analogous one associated with the *conjugate* operator (and thus conjugate equation), $\overline{\Theta}^{1/2} := (-\Delta)^{1/2} - \partial_t$.

Finally, let us stress that in both studies of Γ_{Θ^s} and $\gamma_{\Theta^s,+}$ we will always assume the existence of an absolute constant τ_0 so that $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j. In our computations *we have not carefully followed* the dependence of implicit constants with respect to τ_0. Thus, the reader must understand symbols such as $\lesssim, \gtrsim, \simeq$ and $\approx$ possibly as $\lesssim_{\tau_0}, \gtrsim_{\tau_0}, \simeq_{\tau_0}$ and $\approx_{\tau_0}$ respectively.

4.1 A lower bound for the Γ_{Θ^s} capacity of Cantor sets

Let us briefly recall the construction presented in §2.3.1 of the particular Cantor set associated with the capacity Γ_{Θ^s}, for a fixed $s \in (1/2, 1]$. We began by choosing a positive integer $\delta \geq 2$ that was such that

$$\delta + 1 < \delta^{2s}.$$

By convention we fixed the minimum value of $\delta \geq 2$ that satisfied the above condition so that $\delta = \delta(s)$. Then, we considered $Q^0 := [0,1]^{n+1}$ the unit cube of $\mathbb{R}^{n+1}$ and $(\delta+1)\delta^n$ disjoint s-parabolic cubes Q_i^1, contained in Q^0, with sides parallel to the coordinate axes, side length $0 < \lambda_1 < 1/\delta$, and with the following locations: for each of the first n intervals $[0,1]$ of the cartesian product defining Q^0, we set

$$l_\delta := \frac{1-\delta\lambda_1}{\delta - 1}, \qquad J_j := \big[(j-1)(\lambda_1 + l_\delta), j\lambda_1 + (j-1)l_\delta\big], \quad j = 1, \ldots, \delta,$$

and take $T_\delta := \bigcup_{j=1}^{\delta} J_j$. The remaining temporal interval $[0,1]$ is split in $\delta+1$ intervals of length λ_1^{2s} in an analogous way. That is, we set

$$\widetilde{l}_\delta := \frac{1-(\delta+1)\lambda_1^{2s}}{\delta}, \qquad \widetilde{J}_j := \big[(j-1)(\lambda_1^{2s} + \widetilde{l}_\delta), j\lambda_1^{2s} + (j-1)\widetilde{l}_\delta\big], \quad j = 1, \ldots, \delta+1,$$

and keep the subset $\widetilde{T}_\delta := \bigcup_{j=1}^{\delta+1} \widetilde{J}_j$. This way, the first generation of the Cantor set is

$$E_{1,p_s} := (T_\delta)^n \times \widetilde{T}_\delta.$$

This procedure continues inductively, so that the k-th generation E_{k,p_s} will be formed by $(\delta+1)^k\delta^{nk}$ disjoint s-parabolic cubes with side length $\ell_k := \lambda_1 \cdots \lambda_k$, $0 < \lambda_k < 1/\delta$, and with locations determined by the above iterative process. We name such cubes Q_j^k, with $j = 1, \ldots, (\delta+1)^k\delta^{nk}$. The resulting *s-parabolic* Cantor set is

$$E_{p_s} = E_{p_s}(\lambda) := \bigcap_{k=1}^{\infty} E_{k,p_s}. \tag{4.1.1}$$

Now, from the resulting Cantor set we may consider, for a fixed generation k, the probability measure

$$\mu_k := \frac{1}{|E_{k,p_s}|}\mathcal{L}^{n+1}|_{E_{k,p_s}}, \qquad \theta_{j,p_s} := \frac{\mu_k(Q_i^j)}{\ell_j^{n+1}} = \frac{1}{(\delta+1)^j\delta^{nj}\ell_j^{n+1}}, \ \text{ for } \ 0 \leq j \leq k.$$

In §2.3.1 we argue that if one chooses

$$\lambda_j := ((\delta+1)\delta^n)^{-1/(n+1)}, \qquad \text{for every } \ j,$$

then $0 < \mathcal{H}_{p_s}^{n+1}(E_{p_s}) < \infty$. Observe that, as a consequence, this would imply in the current setting $\theta_{j,p_s} = 1$, for all j.

Assume that there exists $\kappa > 0$ so that $\theta_{j,p_s} \leq \kappa$ for every j. We claim that under this assumption μ_k presents upper s-parabolic growth of degree $n+1$ with constant depending only on n, s and κ. Indeed, fix $Q \subset \mathbb{R}^{n+1}$ any s-parabolic cube contained in Q^0 and distinguish two cases: whether if $\ell(Q) \leq \ell_k$ or not. If $\ell(Q) \leq \ell_k$, notice that $|E_{k,p_s}| = (\delta+1)^k\delta^{nk}\ell_k^{n+2s}$, so we have

$$\mu_k(Q) \leq \frac{1}{(\delta+1)^k\delta^{nk}\ell_k^{n+2s}}\,\ell(Q)^{n+2s} = \theta_{k,p_s}\frac{\ell(Q)^{2s-1}}{\ell_k^{2s-1}}\ell(Q)^{n+1} \leq \kappa\ell(Q)^{n+1}.$$

If $\ell(Q) > \ell_k$, there exists $0 \leq m \leq k-1$ such that $\ell_{m+1} < \ell(Q) \leq \ell_m$ and, in this case, the number of cubes of the m-th generation that Q can intersect is bounded by $(\delta+1)\delta^n$ (the latter is not the best bound, but it suffices for our computations). Therefore

$$\begin{aligned}\mu_k(Q) &\leq (\delta+1)\delta^n\mu_k(Q_0^m) \\ &\simeq ((\delta+1)\delta^n)^{-m} \simeq ((\delta+1)\delta^n)^{-m-1} = \theta_{m+1,p_s}\,\ell_{m+1}^{n+1} \leq \kappa\ell(Q)^{n+1},\end{aligned}$$

and we deduce the desired growth of μ_k.

Similarly, and proceeding as in §2.3.1, if one considers the probability measure μ defined on E_{p_s} such that for each generation k, $\mu(Q_j^k) := (\delta+1)^{-k}\delta^{-nk}$, $1 \leq j \leq (\delta+1)^k\delta^{nk}$, the assumption $\theta_{j,p_s} \leq \kappa$ implies that

$$\mathcal{H}_{p_s}^{n+1}(E_{p_s}) > 0, \quad \text{and then} \quad \dim_{\mathcal{H}_{p_s}}(E_{p_s}) \geq n+1.$$

4.1.1 Preliminary results

In this subsection, we fix $s \in (1/2, 1]$ and work with the following assumptions and notation:

1. As we have already pointed out, k will be a fixed positive integer associated with the generation E_{k,p_s}. We will also write $|E_{k,p_s}| := \mathcal{L}^{n+1}(E_{k,p_s})$.

2. For each generation $j \geq 0$, we introduce the set $\mathcal{Q}^j := \{Q_i^j : 1 \leq i \leq (\delta+1)^j \delta^{nj}\}$ and $\mathcal{Q} := \bigcup_{j=0}^k \mathcal{Q}^j$. We may write Q^j to emphasize that Q is an arbitrary cube of $\mathcal{Q}^j$.

3. For a fixed $\overline{x} \in E_{k,p_s}$, we write the chain of s-parabolic cubes $\overline{x} \in \Delta_k \subset \Delta_{k-1} \subset \cdots \subset \Delta_1 \subset \Delta_0 = Q^0$, where Δ_j is the only s-parabolic cube of the j-th generation that contains $\overline{x}$.

4. We assume that there is an absolute constant τ_0 so that $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j.

We mention that the results and observations that will follow in this subsection are inspired by those presented in [**T1, §3**] for the s-dimensional Riesz transform.

Remark 4.1.1. Begin by observing that given any s-parabolic cube $Q^k \in \mathcal{Q}^k$ and $R \in \mathcal{Q}$,

$$\begin{aligned}\int_{Q^k}\int_R \frac{\mathrm{d}\mu_k(\overline{y})\,\mathrm{d}\mu_k(\overline{x})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} &= \int_{Q^k}\int_{R\cap E_{k,p_s}} \frac{\mathrm{d}\mu_k(\overline{y})\,\mathrm{d}\mu_k(\overline{x})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \\ &= \int_{Q^k}\int_{(R\cap E_{k,p_s})\setminus Q^k} \frac{\mathrm{d}\mu_k(\overline{y})\,\mathrm{d}\mu_k(\overline{x})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \\ &\qquad + \int_{Q^k}\int_{Q^k} \frac{\mathrm{d}\mu_k(\overline{y})\,\mathrm{d}\mu_k(\overline{x})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Observe that the points of Q^k and $(R \cap E_{k,p_s}) \setminus Q^k$ are separated, at least, by an s-parabolic distance comparable to

$$\min\left\{\ell_{k-1}\left(\frac{1-\delta\lambda_k}{\delta-1}\right), \ell_{k-1}\left(\frac{1-(\delta+1)\lambda_1^{2s}}{\delta}\right)^{\frac{1}{2s}}\right\} \gtrsim \ell_{k-1},$$

where the previous implicit constant depends on τ_0 and s. Therefore, it is clear that $\mathrm{I} \lesssim 1/\ell_{k-1}^{n+1} < \infty$. On the other hand, to deal with II, for each $\overline{y} \in Q^k$ we shall contain Q^k in the s-parabolic cube $\widetilde{Q}$ centered at $\overline{y}$ with side length $2\,\mathrm{diam}_{p_s}(Q^k)$, and split the previous set into the s-parabolic annuli

$$A_j := Q\big(\overline{y}, 2^{-j}\mathrm{diam}_{p_s}(Q^k)\big) \setminus Q\big(\overline{y}, 2^{-j-1}\mathrm{diam}_{p_s}(Q^k)\big), \quad j \geq -1.$$

Hence, by definition of μ_k we get, for each $\overline{y} \in Q^k$,

$$\begin{aligned}\int_{Q^k} \frac{\mathrm{d}\mu_k(\overline{x})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} &\leq \frac{1}{|E_{k,p_s}|}\sum_{j=-1}^{\infty}\int_{A_j}\frac{\mathrm{d}\overline{x}}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \\ &\lesssim \frac{1}{|E_{k,p_s}|}\sum_{j=-1}^{\infty}\frac{\big(2^{-j}\ell(Q^k)\big)^{n+2s}}{\big(2^{-j-1}\ell(Q^k)\big)^{n+1}} \lesssim \frac{1}{(\delta+1)^k\delta^{nk}\ell_k^{n+1}} = \theta_{k,p_s} < \infty.\end{aligned}$$

Since $Q^k \in \mathcal{Q}^k$ and $R \in \mathcal{Q}$ were arbitrary, combining the previous estimates and the fourth estimate of [**MatP, Lemma 2.2**], we deduce that

$$\mathcal{P}^s_{\mu_k}\chi_R := \nabla_x P_s * (\chi_R \mu_k) \in L^1_{\text{loc}}(\mu_k), \qquad \text{for any } R \subset \mathbb{R}^{n+1} \;\; s\text{-parabolic cube.}$$

Notice that this implies, in particular,

$$\mathcal{P}^s \mu_k = \mathcal{P}^s_{\mu_k}\chi_{Q^0} \in L^1_{\text{loc}}(\mu_k).$$

REMARK 4.1.2. Fix any $R \subset \mathbb{R}^{n+1}$ s-parabolic cube and notice that $R \cap E_{k,p_s}$ can be written as a translated copy of a cartesian product of the form $\mathcal{X}_{R,k} \times \mathcal{T}_{R_k} \subset \mathbb{R}^n \times \mathbb{R}$, where $\mathcal{X}_{R,k}$ and $\mathcal{T}_{R,k}$ are contained in some generations involved in the construction of a Cantor set in $\mathbb{R}^n$ and $\mathbb{R}$ respectively. Let us also observe that by (1.1.1) and Lemma 1.1.1, for any $(x,t) \neq (0,t)$ we have

$$\nabla_x P_s(x,t) \simeq t^{-\frac{n+1}{2s}} \frac{x}{|x|} \phi'_{n,s}\big(|x|t^{-\frac{1}{2s}}\big)\chi_{t>0} \simeq x t^{-\frac{n+2}{2s}} \phi_{n+2,s}\big(|x|t^{-\frac{1}{2s}}\big)\chi_{t>0},$$

so for each $t \in \mathbb{R}$, the kernel $\nabla_x P_s(\cdot, t)$ is anti-symmetric. Then, by Fubini's theorem (that can be applied by Remark 4.1.1),

$$\begin{aligned}
\int_R \mathcal{P}^s_{\mu_k}\chi_R(\overline{x})\,\mathrm{d}\mu_k(\overline{x}) &= \int_R \int_R \nabla_x P_s(\overline{x} - \overline{y})\,\mathrm{d}\mu_k(\overline{y})\,\mathrm{d}\mu_k(\overline{x}) \\
&= \frac{1}{|E_{k,p_s}|^2} \int_{R \cap E_{k,p_s}} \int_{R \cap E_{k,p_s}} \nabla_x P_s(\overline{x} - \overline{y})\,\mathrm{d}\overline{y}\,\mathrm{d}\overline{x} \\
&= \frac{1}{|E_{k,p_s}|^2} \int_{\mathcal{T}_{R,k}} \int_{\mathcal{T}_{R,k}} \left(\int_{\mathcal{X}_{R,k}} \int_{\mathcal{X}_{R,k}} \nabla_x P_s(x-y, t-s)\,\mathrm{d}y\,\mathrm{d}x \right) \mathrm{d}s\,\mathrm{d}t \\
&= \frac{-1}{|E_{k,p_s}|^2} \int_{\mathcal{T}_{R,k}} \int_{\mathcal{T}_{R,k}} \left(\int_{\mathcal{X}_{R,k}} \int_{\mathcal{X}_{R,k}} \nabla_x P_s(y-x, t-s)\,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}s\,\mathrm{d}t \\
&= -\int_R \mathcal{P}^s_{\mu_k}\chi_R(\overline{x})\,\mathrm{d}\mu_k(\overline{x}).
\end{aligned}$$

That is, we have deduced

$$\int_R \mathcal{P}^s_{\mu_k}\chi_R\,\mathrm{d}\mu_k = 0, \qquad \forall R \subset \mathbb{R}^{n+1} \; s\text{-parabolic cube.}$$

In the sequel we will present a series of auxiliary results and a final main theorem, which will be essential when proving the final estimate. But first, we define the following functions: given any $Q \in \mathcal{Q}$ and $f \in L^1_{\text{loc}}(\mu_k)$,

$$S_Q f(\overline{x}) := \left(\frac{1}{\mu_k(Q)} \int_Q f \;\mathrm{d}\mu_k \right) \chi_Q(\overline{x}), \qquad S_j f(\overline{x}) := \sum_{Q \in \mathcal{Q}^j} S_Q f(\overline{x}), \quad 0 \leq j \leq k.$$

In addition, for any $Q \in \mathcal{Q} \setminus \mathcal{Q}^k$, if $\mathcal{F}(Q)$ is the set of s-parabolic cubes *children* of Q, we write

$$D_Q f(\overline{x}) := \Bigg(\sum_{P \in \mathcal{F}(Q)} S_P f(\overline{x}) \Bigg) - S_Q f(\overline{x}),$$
$$D_j f(\overline{x}) := \sum_{Q \in \mathcal{Q}^j} D_Q f(\overline{x}), \quad 0 \leq j \leq k-1.$$

Notice that $|\int D_Q f \, \mathrm{d}\mu_k| = 0$, as well as

$$D_j f(\overline{x}) = S_{j+1} f(\overline{x}) - S_j f(\overline{x}), \quad 0 \leq j \leq k-1.$$

REMARK 4.1.3. Let $Q_1, Q_2 \in \mathcal{Q} \setminus \mathcal{Q}^k$ with disjoint support. Then it is clear that

$$\langle D_{Q_1} f, D_{Q_2} f \rangle := \int D_{Q_1} f \cdot D_{Q_2} f \, \mathrm{d}\mu_k = 0.$$

If on the contrary $Q_1 \subsetneq Q_2$, write $\widetilde{Q}_2$ the only son of Q_2 such that $Q_1 \subseteq \widetilde{Q}_2$, and observe

$$\begin{aligned}
&D_{Q_1} f(\overline{x}) \cdot D_{Q_2} f(\overline{x}) \\
&\quad = \Bigg[\Bigg(\sum_{P \in \mathcal{F}(Q_1)} S_P f(\overline{x}) \Bigg) - S_{Q_1} f(\overline{x}) \Bigg] \cdot \Bigg[\Bigg(\sum_{R \in \mathcal{F}(Q_2)} S_P f(\overline{x}) \Bigg) - S_{Q_2} f(\overline{x}) \Bigg] \\
&\quad = \Bigg(\frac{1}{\mu_k(\widetilde{Q}_2)} \int_{\widetilde{Q}_2} f \, \mathrm{d}\mu_k \Bigg) \cdot \sum_{P \in \mathcal{F}(Q_1)} S_p f(\overline{x}) \\
&\qquad\qquad - \Bigg(\frac{1}{\mu_k(Q_2)} \int_{Q_2} f \, \mathrm{d}\mu_k \Bigg) \cdot \sum_{P \in \mathcal{F}(Q_1)} S_p f(\overline{x}) \\
&\qquad\qquad - \Bigg(\frac{1}{\mu_k(Q_2)} \int_{Q_2} f \, \mathrm{d}\mu_k \Bigg) \cdot \Bigg(\frac{1}{\mu_k(\widetilde{Q}_2)} \int_{\widetilde{Q}_2} f \, \mathrm{d}\mu_k \Bigg) \chi_{Q_1}(\overline{x}) \\
&\qquad\qquad + \Bigg(\frac{1}{\mu_k(Q_2)} \int_{Q_2} f \, \mathrm{d}\mu_k \Bigg) \cdot \Bigg(\frac{1}{\mu_k(Q_2)} \int_{Q_2} f \, \mathrm{d}\mu_k \Bigg) \chi_{Q_1}(\overline{x}) \\
&\quad = \Bigg(\frac{1}{\mu_k(\widetilde{Q}_2)} \int_{\widetilde{Q}_2} f \, \mathrm{d}\mu_k \Bigg) \cdot D_{Q_1} f(\overline{x}) - \Bigg(\frac{1}{\mu_k(Q_2)} \int_{Q_2} f \, \mathrm{d}\mu_k \Bigg) \cdot D_{Q_1} f(\overline{x}).
\end{aligned}$$

Then, in particular, $\langle D_{Q_1} f, D_{Q_2} f \rangle = 0$. So we deduce that, in general, the functions $D_{Q_1} f$ and $D_{Q_2} f$ are orthogonal in an $L^2(\mu_k)$-sense if $Q_1 \neq Q_2$ belong to $\mathcal{Q} \setminus \mathcal{Q}^k$.

REMARK 4.1.4. Notice that

$$\|S_k f\|_{L^2(\mu_k)}^2 = \Bigg\| \sum_{j=0}^{k-1} D_j f + S_0 f \Bigg\|_{L^2(\mu_k)}^2 = \Bigg\| \sum_{j=0}^{k-1} D_j f + \Bigg(\int f \, \mathrm{d}\mu_k \Bigg) \chi_{Q^0} \Bigg\|_{L^2(\mu_k)}^2,$$

meaning that if $\int f\,\mathrm{d}\mu_k = 0$, by Remark 4.1.3,

$$\|S_k f\|_{L^2(\mu_k)}^2 = \bigg\| \sum_{j=0}^{k-1} D_j f \bigg\|_{L^2(\mu_k)}^2 = \sum_{Q\in\mathcal{Q}\setminus\mathcal{Q}^k} \|D_Q f\|_{L^2(\mu_k)}^2.$$

From this point on let us fix $f := \mathcal{P}^s\mu_k$, that by Remark 4.1.1 belongs to $L^1_{\text{loc}}(\mu_k)$, by Remark 4.1.2 is such that $\int \mathcal{P}^s\mu_k \,\mathrm{d}\mu_k = 0$, and thus by Remark 4.1.4 satisfies

$$\|S_k(\mathcal{P}^s\mu_k)\|_{L^2(\mu_k)}^2 = \sum_{Q\in\mathcal{Q}\setminus\mathcal{Q}^k} \|D_Q(\mathcal{P}^s\mu_k)\|_{L^2(\mu_k)}^2.$$

Lemma 4.1.1. *Let $Q\in\mathcal{Q}^j$, for any $0\leq j\leq k$, and $\overline{x},\overline{x}'\in Q$. Then,*

$$\big|\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}) - \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}')\big| \lesssim_{\tau_0} p(Q),$$

where $p(R) := \sum_{r=0}^j \theta_{r,p_s}\frac{\ell_j}{\ell_r}$, for $R\in\mathcal{Q}^j$.

Proof. It is clear that if $j=0$ each term of the above difference is null, so let us assume $j>0$. Let $\widehat{Q}$ be the parent of Q and write $\overline{x}=(x,t), \overline{x}'=(x',t')$ and $\widehat{x} := (x',t)$. Then, applying the mean value theorem component-wise similarly as in Theorem 1.1.2,

$$\begin{aligned}
&\big|\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}) - \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}')\big| \\
&\qquad\leq \int_{\mathbb{R}^{n+1}\setminus Q} \big|\nabla_x P_s(\overline{x}-\overline{y}) - \nabla_x P_s(\widehat{x}-\overline{y})\big|\,\mathrm{d}\mu_k(\overline{y}) \\
&\qquad\qquad\qquad + \int_{\mathbb{R}^{n+1}\setminus Q} \big|\nabla_x P_s(\widehat{x}-\overline{y}) - \nabla_x P_s(\overline{x}'-\overline{y})\big|\,\mathrm{d}\mu_k(\overline{y}) \\
&\qquad\lesssim \int_{\mathbb{R}^{n+1}\setminus Q} \frac{|x-x'|}{|\overline{x}-\overline{y}|_{p_s}^{n+2}}\,\mathrm{d}\mu_k(\overline{y}) + \int_{\mathbb{R}^{n+1}\setminus Q} \frac{|t-t'|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}}\,\mathrm{d}\mu_k(\overline{y}) \\
&\qquad\lesssim \ell(Q)\sum_{r=1}^{j}\int_{\Delta_{r-1}\setminus\Delta_r} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2}} + \ell(Q)^{2s}\sum_{r=1}^{j}\int_{\Delta_{r-1}\setminus\Delta_r} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}} \\
&\qquad\lesssim \ell(Q)\sum_{r=0}^{j-1}\frac{\mu_k(\Delta_r)}{\ell_r^{n+2}} + \ell(Q)^{2s}\sum_{r=0}^{j-1}\frac{\mu_k(\Delta_r)}{\ell_r^{n+2s+1}} \\
&\qquad\lesssim \frac{\ell(Q)}{\ell(\widehat{Q})}\sum_{r=0}^{j-1}\theta_{r,p_s}\frac{\ell_{j-1}}{\ell_r} + \frac{\ell(Q)^{2s}}{\ell(\widehat{Q})^{2s}}\sum_{r=0}^{j-1}\theta_{r,p_s}\frac{\ell_{j-1}^{2s}}{\ell_r^{2s}} \leq \bigg[\frac{\ell(Q)}{\ell(\widehat{Q})} + \frac{\ell(Q)^{2s}}{\ell(\widehat{Q})^{2s}}\bigg]p(\widehat{Q}) \\
&\qquad\lesssim \frac{\ell(Q)}{\ell(\widehat{Q})}p(\widehat{Q}).
\end{aligned}$$

Finally observe that

$$\begin{aligned}
p(\widehat{Q}) &= \theta_{j-1,p_s} + \big(\theta_{j-2,p_s}\lambda_{j-1}\big) + \cdots + \big(\theta_{1,p_s}\lambda_{j-1}\cdots\lambda_2\big) + \big(\lambda_{j-1}\cdots\lambda_1\big) \\
&= \frac{1}{\lambda_j}\big(p(Q)-\theta_{j,p_s}\big) \leq \frac{1}{\lambda_j}p(Q) = \frac{\ell(\widehat{Q})}{\ell(Q)}p(Q),
\end{aligned}$$

and the result follows. □

Lemma 4.1.2. *If $Q \in \mathcal{Q}^j$ with $j < k$, then*

$$\Big|S_Q(\mathcal{P}^s\mu_k) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s\mu_k)\Big| \lesssim_{\tau_0} p(Q).$$

Proof. Notice that by Remark 4.1.2, for any $P \in \mathcal{F}(Q)$ we have $S_P(\mathcal{P}^s_{\mu_k}\chi_P) = 0$ and $S_Q(\mathcal{P}^s_{\mu_k}\chi_Q) = 0$. Hence,

$$\begin{aligned}\Big|S_Q(\mathcal{P}^s\mu_k) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s\mu_k)\Big| &= \Big|S_Q(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus P})\Big|\\ &\leq \sum_{P\in\mathcal{F}(Q)}|S_P(\mathcal{P}^s_{\mu_k}\chi_{Q\setminus P})| + \Big|S_Q(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q})\Big|.\end{aligned}$$

It is clear that $|\mathcal{P}^s_{\mu_k}\chi_{Q\setminus P}(\overline{x})| \lesssim \mu_k(Q)/\ell(Q)^{n+1} = \theta_{j,p_s} \leq p(Q)$, for each $\overline{x} \in P$. So the first sum satisfies

$$\sum_{P\in\mathcal{F}(Q)}|S_P(\mathcal{P}^s_{\mu_k}\chi_{Q\setminus P})| \lesssim \sum_{P\in\mathcal{F}(Q)}\theta_{j,p_s}\chi_P \leq \theta_{j,p_s} \leq p(Q).$$

For the remaining term write

$$\begin{aligned}&\Big|S_Q(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q})\Big|\\ &= \Big|\sum_{P\in\mathcal{F}(Q)}\Big(\frac{1}{\mu_k(P)}\int_P \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\,\mathrm{d}\mu_k(\overline{x}) - \frac{1}{\mu_k(Q)}\int_Q \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\Big)\chi_P\Big|\\ &= \Big|\sum_{P\in\mathcal{F}(Q)}\sum_{P'\in\mathcal{F}(Q)}\Big(\frac{1}{(\delta+1)\delta^n\mu_k(P)}\int_P \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\\ &\qquad\qquad - \frac{1}{\mu_k(Q)}\int_{P'} \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\Big)\chi_P\Big|\\ &= \Big|\sum_{P\in\mathcal{F}(Q)}\sum_{P'\in\mathcal{F}(Q)}\frac{1}{\mu_k(Q)}\Big(\int_P \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\,\mathrm{d}\mu_k(\overline{x}) - \int_{P'} \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\Big)\chi_P\Big|\\ &\leq \sum_{P\in\mathcal{F}(Q)}\sum_{P'\in\mathcal{F}(Q)}\frac{1}{\mu_k(Q)}\Big(\int_Q \big|\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}) - \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\tau_{P\to P'}(\overline{x}))\big|\,\mathrm{d}\mu_k(\overline{x})\Big)\chi_P,\end{aligned}$$

where $\tau_{P\to P'}$ is the translation of $\mathbb{R}^{n+1}$ satisfying $\tau_{P\to P'}(P) = P'$. Thus, by Lemma 4.1.1,

$$\begin{aligned}&\Big|S_Q(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q})\Big|\\ &\qquad\lesssim \sum_{P\in\mathcal{F}(Q)}\sum_{P'\in\mathcal{F}(Q)} p(Q)\chi_P = (\delta+1)\delta^n p(Q)\chi_Q \lesssim p(Q).\end{aligned}$$

□

Remark 4.1.5. Observe that as an immediate consequence of the previous lemma we have

$$\|D_Q(\mathcal{P}^s\mu_k)\|^2_{L^2(\mu_k)} = \int_Q \Big| S_Q(\mathcal{P}^s\mu_k) - \sum_{P\in\mathcal{F}(Q)} S_P(\mathcal{P}^s\mu_k)\Big|^2 \mathrm{d}\mu_k \lesssim_{\tau_0} p(Q)^2\mu_k(Q).$$

Lemma 4.1.3. *Let $M\geq 0$ be an integer and $Q^j\in\mathcal{Q}^j$, for $0\leq j\leq M$. Then,*

$$\sum_{j=0}^M p(Q^j)^2 \lesssim \sum_{j=0}^M \theta_{j,p_s}^2.$$

Proof. It follows from Cauchy-Schwarz's inequality and the following computation:

$$\begin{aligned}\sum_{j=0}^M p(Q^j)^2 &= \sum_{j=0}^M \bigg(\sum_{r=0}^j \theta_{r,p_s}\frac{\ell_j}{\ell_r}\bigg)^2 = \sum_{j=0}^M \ell_j^2\bigg(\sum_{r=0}^j \frac{\theta_{r,p_s}}{\sqrt{\ell_r}}\frac{1}{\sqrt{\ell_r}}\bigg)^2\\ &\leq \sum_{j=0}^M \ell_j^2\bigg(\sum_{r=0}^j\frac{\theta_{r,p_s}^2}{\ell_r}\bigg)\bigg(\sum_{r=0}^j\frac{1}{\ell_r}\bigg) \leq \sum_{j=0}^M \bigg(\sum_{r=0}^j \theta_{r,p_s}^2\frac{\ell_j}{\ell_r}\bigg)\bigg(\sum_{r=0}^j\frac{1}{\delta^r}\bigg)\\ &\lesssim \sum_{j=0}^M\sum_{r=0}^j \theta_{r,p_s}^2\frac{\ell_j}{\ell_r} = \sum_{r=0}^M \theta_{r,p_s}^2\sum_{j=r}^M\frac{\ell_j}{\ell_r} \lesssim \sum_{r=0}^M \theta_{r,p_s}^2.\end{aligned}$$

□

The previous three lemmas will be used to prove the below auxiliary estimate, analogous to [**T1, Theorem 3.1**]. In essence, once proved, it will imply Lemma 4.1.7, the main result of this subsection.

Lemma 4.1.4. *The following estimate holds:*

$$\|\mathcal{P}^s\mu_k\|^2_{L^2(\mu_k)} \lesssim_{\tau_0} \sum_{j=0}^k \theta_{j,p_s}^2.$$

Proof. Begin by noticing that Remarks 4.1.4 and 4.1.5 imply

$$\begin{aligned}\|S_k(\mathcal{P}^s\mu_k)\|^2_{L^2(\mu_k)} &= \sum_{Q\in\mathcal{Q}\setminus\mathcal{Q}^k}\|D_Q(\mathcal{P}^s\mu_k)\|^2_{L^2(\mu_k)} \lesssim \sum_{Q\in\mathcal{Q}\setminus\mathcal{Q}^k} p(Q)^2\mu(Q)\\ &= \sum_{j=0}^{k-1}\sum_{Q^j\in\mathcal{Q}^j} p(Q^j)^2\mu(Q^j) = \sum_{j=0}^{k-1} p(Q^j)^2.\end{aligned}$$

Moreover, by Remark 4.1.2 and Lemma 4.1.1 we also have for each $Q\in\mathcal{Q}^k$ and $\overline{x}\in Q$,

$$\begin{aligned}&\big|S_Q(\mathcal{P}^s\mu_k)(\overline{x}) - \chi_Q\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\big|\\ &\qquad= \frac{1}{\mu(Q)}\int_Q \big|\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{y}) - \mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})\big|\,\mathrm{d}\mu_k(\overline{y}) \lesssim p(Q).\end{aligned}$$

In addition, for each $Q \in \mathcal{Q}^k$ and $\overline{x} \in Q$, [**MatP, Lemma 2.2**] and polar integration yield,

$$|\mathcal{P}^s_{\mu_k}\chi_Q(\overline{x})| \lesssim \frac{1}{|E_{k,p_s}|}\int_Q \frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \lesssim \frac{\ell_k}{|E_{k,p_s}|} = \theta_{k,p_s}.$$

Notice the need of $s > 1/2$ in the previous estimate. Combining the three above computations and Lemma 4.1.3 we finally conclude:

$$\begin{aligned}
\|\mathcal{P}^s\mu_k\|^2_{L^2(\mu_k)} &= \sum_{Q\in\mathcal{Q}^k} \|\chi_Q\mathcal{P}^s\mu_k\|^2_{L^2(\mu_k)} \\
&\leq 2\sum_{Q\in\mathcal{Q}^k}\Big(\|\chi_Q\mathcal{P}^s_{\mu_k}\chi_Q\|^2_{L^2(\mu_k)} + \|\chi_Q\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q}\|^2_{L^2(\mu_k)}\Big) \\
&\lesssim \sum_{Q\in\mathcal{Q}^k}\Big(\|\chi_Q\mathcal{P}^s_{\mu_k}\chi_Q\|^2_{L^2(\mu_k)} + \|\chi_Q\mathcal{P}^s_{\mu_k}\chi_{\mathbb{R}^{n+1}\setminus Q} - S_Q(\mathcal{P}^s\mu_k)\|^2_{L^2(\mu_k)} \\
&\qquad\qquad + \|S_Q(\mathcal{P}^s\mu_k)\|^2_{L^2(\mu_k)}\Big) \\
&\lesssim \sum_{Q\in\mathcal{Q}^k}\theta^2_{k,p_s}\mu_k(Q) + \sum_{Q\in\mathcal{Q}^k} p(Q)^2\mu_k(Q) + \|S_k(\mathcal{P}^s\mu_k)\|^2_{L^2(\mu_k)} \\
&\lesssim \theta^2_{k,p_s} + p(Q^k)^2 + \sum_{j=0}^{k-1} p(Q^j)^2 \lesssim \sum_{j=0}^{k} p(Q^j)^2 \lesssim \sum_{j=0}^{k}\theta^2_{j,p_s}.
\end{aligned}$$

□

Notice that we have just proved an $L^2(\mu_k)$-bound for $\mathcal{P}^s\mu_k = \mathcal{P}^s_{\mu_k}\chi_{Q^0}$. Our next goal is to obtain a bound for $\mathcal{P}^s_{\mu_k}\chi_{Q^m}$ for any cube of the m-th generation, $0 < m \leq k$, so that it generalizes the estimate of Lemma 4.1.4 if $m = 0$. But as it is pointed out in [**T1, §3**], the procedure to obtain the estimate for $\mathcal{P}^s_{\mu_k}\chi_{Q^0}$ already illustrates the computations one has to carry out to deduce the corresponding estimate for $\mathcal{P}^s_{\mu_k}\chi_{Q^m}$. This is due to the *self-similarity* of the Cantor set we are dealing with.

Let us tackle first the case $0 < m < k$ (the arguments that will follow will be general enough so that we could also assume $m = 0$ and recover all of the previous results). Fix a cube Q^m of the m-th generation and consider the following truncation of μ_k,

$$\mu_{k,m} := \frac{1}{\mu_k(Q^m)}\mu_k|_{Q^m}.$$

It is clear that Remarks 4.1.1 and 4.1.2 are also valid in this setting. More precisely, performing essentially the same computations we get $\mathcal{P}^s_{\mu_{k,m}}\chi_R \in L^1_{\text{loc}}(\mu_{k,m})$ as well as

$$\int_R \mathcal{P}^s_{\mu_{k,m}}\chi_R \,\mathrm{d}\mu_{k,m} = 0, \qquad \forall R\subset\mathbb{R}^{n+1}\ s\text{-parabolic cube.} \tag{4.1.2}$$

We also consider the set (analogous to $\mathcal{Q}$)

$$\mathcal{Q}(m) := \bigcup_{j=m}^{k}\mathcal{Q}^j\cap Q^m,$$

and the functions (analogous to S_Q, S_j, D_Q and D_j) defined for $f \in L^1_{\text{loc}}(\mu_{k,m})$,

$$\begin{aligned}
S_Q^m f(\overline{x}) &:= \left(\frac{1}{\mu_{k,m}(Q)}\int_Q f \,\mathrm{d}\mu_{k,m}\right)\chi_Q(\overline{x}), && Q \in \mathcal{Q}(m),\\
S_j^m f(\overline{x}) &:= \sum_{Q\in\mathcal{Q}^j\cap Q^m} S_Q^m f(\overline{x}), && m \le j \le k,\\
D_Q^m f(\overline{x}) &:= \left(\sum_{P\in\mathcal{F}(Q)} S_P^m f(\overline{x})\right) - S_Q^m f(\overline{x}), && Q \in \mathcal{Q}(m)\setminus(\mathcal{Q}^k\cap Q^m),\\
D_j^m f(\overline{x}) &:= \sum_{Q\in\mathcal{Q}^j\cap Q^m} D_Q^m f(\overline{x}) = S_{j+1}^m f(\overline{x}) - S_j^m f(\overline{x}), && m \le j \le k-1.
\end{aligned}$$

It is clear that $\int D_Q^m f\,\mathrm{d}\mu_{k,m} = 0$ for any $Q \in \mathcal{Q}(m)\setminus(\mathcal{Q}^k\cap Q^m)$. So analogously to the case $m=0$, $D_{Q_1}^m f$ and $D_{Q_2}^m f$ are orthogonal in an $L^2(\mu_{k,m})$-sense if $Q_1 \neq Q_2$ belong to $\mathcal{Q}(m)\setminus(\mathcal{Q}^k\cap Q^m)$. Thus, Remark 4.1.3 also admits a generalization in the current setting. Moreover, if $f := \mathcal{P}^s\mu_{k,m}$, by (4.1.2) we have a Q^m-truncated version of Remark 4.1.4,

$$\big\|S_k^m(\mathcal{P}^s\mu_{k,m})\big\|^2_{L^2(\mu_{k,m})} = \sum_{Q\in\mathcal{Q}(m)\setminus(\mathcal{Q}^k\cap Q^m)} \big\|D_Q^m(\mathcal{P}^s\mu_{k,m})\big\|^2_{L^2(\mu_{k,m})}. \tag{4.1.3}$$

The previous relations allow to generalize Lemmas 4.1.1 and 4.1.2. We will only give the details of the proof of the former since they will suffice to illustrate that the methods of proof are analogous to those presented for the aforementioned lemmas.

Lemma 4.1.5. *Let $Q \in \mathcal{Q}^j\cap Q^m$ for $m \le j \le k$, and $\overline{x}, \overline{x}' \in Q$. Then,*

$$\big|\mathcal{P}^s_{\mu_{k,m}}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}) - \mathcal{P}^s_{\mu_{k,m}}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}')\big| \lesssim_{\tau_0} \frac{1}{\mu_k(Q^m)}p_m(Q),$$

where now $p_m(R) := \sum_{r=m}^j \theta_{r,p_s}\frac{\ell_j}{\ell_r}$, for $R\in\mathcal{Q}^j$.

Proof. The proof is analogous to that of Lemma 4.1.1, but taking into account the support of $\mu_{k,m}$. Indeed, assume $j>m$ and notice that

$$\begin{aligned}
&\big|\mathcal{P}^s_{\mu_{k,m}}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}) - \mathcal{P}^s_{\mu_{k,m}}\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x}')\big|\\
&\quad\lesssim \ell(Q)\sum_{r=m+1}^j\int_{\Delta_{r-1}\setminus\Delta_r}\frac{\mathrm{d}\mu_{k,m}(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2}} + \ell(Q)^2\sum_{r=m+1}^j\int_{\Delta_{r-1}\setminus\Delta_r}\frac{\mathrm{d}\mu_{k,m}(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}}\\
&\quad= \frac{\ell(Q)}{\mu_k(Q^m)}\sum_{r=m+1}^j\int_{\Delta_{r-1}\setminus\Delta_r}\frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2}} + \frac{\ell(Q)^2}{\mu_k(Q^m)}\sum_{r=m+1}^j\int_{\Delta_{r-1}\setminus\Delta_r}\frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+2s+1}}\\
&\quad\lesssim \frac{1}{\mu(Q^m)}\frac{\ell(Q)}{\ell(\widehat{Q})}p_m(\widehat{Q}) \lesssim \frac{1}{\mu_k(Q^m)}p_m(Q).
\end{aligned}$$

□

LEMMA 4.1.6. *If $Q \in \mathcal{Q}^j \cap Q^m$ with $m \leq j < k$, then*

$$\Big| S_Q^m(\mathcal{P}^s \mu_{k,m}) - \sum_{P \in \mathcal{F}(Q)} S_P^m(\mathcal{P}^s \mu_{k,m}) \Big| \lesssim_{\tau_0} \frac{1}{\mu_k(Q^m)} p_m(Q).$$

As a direct consequence of Lemma 4.1.6 we also have

$$\big\| D_Q^m(\mathcal{P}^s \mu_{k,m}) \big\|_{L^2(\mu_{k,m})}^2 \lesssim_{\tau_0} \frac{1}{\mu_k(Q^m)^2} p_m(Q)^2 \mu_{k,m}(Q), \tag{4.1.4}$$

analogous to the estimate of Remark 4.1.5. Combining all of the above generalized results and observations, we finally deduce the result we were interested in

LEMMA 4.1.7. *The following estimate holds for any $0 < m < k$:*

$$\|\mathcal{P}^s \mu_{k,m}\|_{L^2(\mu_{k,m})}^2 \lesssim_{\tau_0} \frac{1}{\mu_k(Q^m)^2} \sum_{j=0}^{k} \theta_{j,p_s}^2.$$

Proof. By relations (4.1.3) and (4.1.4) we now have

$$\begin{aligned}
\big\| S_k^m(\mathcal{P}^s \mu_{k,m}) \big\|_{L^2(\mu_{k,m})}^2 &= \sum_{Q \in \mathcal{Q}(m) \setminus (\mathcal{Q}^k \cap Q^m)} \big\| D_Q^m(\mathcal{P}^s \mu_{k,m}) \big\|_{L^2(\mu_{k,m})}^2 \\
&\lesssim \frac{1}{\mu_k(Q^m)^2} \sum_{Q \in \mathcal{Q}(m) \setminus (\mathcal{Q}^k \cap Q^m)} p_m(Q)^2 \mu_{k,m}(Q) \\
&= \frac{1}{\mu_k(Q^m)^2} \sum_{j=m}^{k-1} \sum_{Q^j \in \mathcal{Q}^j \cap Q^m} p_m(Q^j)^2 \mu_{k,m}(Q^j) = \frac{1}{\mu_k(Q^m)^2} \sum_{j=m}^{k-1} p_m(Q^j)^2.
\end{aligned}$$

Moreover, (4.1.2) and Lemma 4.1.5 imply that for each $Q \in \mathcal{Q}^k \cap Q^m$ and $\overline{x} \in Q$,

$$\big| S_Q^m(\mathcal{P}^s \mu_{k,m})(\overline{x}) - \chi_Q \mathcal{P}_{\mu_{k,m}}^s \chi_{\mathbb{R}^{n+1} \setminus Q}(\overline{x}) \big| \lesssim \frac{1}{\mu_k(Q^m)} p_m(Q).$$

It is also clear that for $Q \in \mathcal{Q}^k \cap Q^m$ and $\overline{x} \in Q$, we have $|\mathcal{P}_{\mu_{k,m}}^s \chi_Q(\overline{x})| \lesssim \theta_{k,p_s} / \mu_k(Q^m)$. All in all, we finally conclude:

$$\begin{aligned}
&\|\mathcal{P}_{\mu_{k,m}}^s 1\|_{L^2(\mu_{k,m})}^2 \\
&\quad\lesssim \sum_{Q \in \mathcal{Q}^k \cap Q^m} \Big(\|\chi_Q \mathcal{P}_{\mu_{k,m}}^s \chi_Q\|_{L^2(\mu_{k,m})}^2 + \|\chi_Q \mathcal{P}_{\mu_{k,m}}^s \chi_{\mathbb{R}^{n+1} \setminus Q} - S_Q^m(\mathcal{P}^s \mu_{k,m})\|_{L^2(\mu_{k,m})}^2 \\
&\qquad\qquad + \|S_Q^m(\mathcal{P}^s \mu_{k,m})\|_{L^2(\mu_{k,m})}^2 \Big) \\
&\quad\lesssim \sum_{Q \in \mathcal{Q}^k \cap Q^m} \frac{\theta_{k,p_s}^2}{\mu_k(Q^m)^3} \mu_k(Q) + \sum_{Q \in \mathcal{Q}^k \cap Q^m} \frac{p_m(Q)^2}{\mu_k(Q^m)^3} \mu_k(Q) + \|S_k^m(\mathcal{P}^s \mu_{k,m})\|_{L^2(\mu_k)}^2 \\
&\quad\lesssim \frac{1}{\mu_k(Q^m)^2} \left(\theta_{k,p_s}^2 + p_m(Q^k)^2 + \sum_{j=m}^{k-1} p_m(Q^j)^2 \right) \lesssim \frac{1}{\mu_k(Q^m)^2} \sum_{j=m}^{k} p_m(Q^j)^2 \\
&\quad\leq \frac{1}{\mu_k(Q^m)^2} \sum_{j=0}^{k} p(Q^j)^2 \lesssim \frac{1}{\mu_k(Q^m)^2} \sum_{j=0}^{k} \theta_{j,p_s}^2,
\end{aligned}$$

where for the last inequality we have used Lemma 4.1.3. $\square$

Observe that we can rewrite the previous $L^2(\mu_{k,m})$ norm as

$$\|\mathcal{P}^s\mu_{k,m}\|_{L^2(\mu_{k,m})} = \frac{1}{\mu_k(Q^m)^{3/2}}\|\mathcal{P}^s_{\mu_k}\chi_{Q^m}\|_{L^2(\mu_k|_{Q^m})},$$

and the previous result can be restated as

$$\|\mathcal{P}^s_{\mu_k}\chi_{Q^m}\|_{L^2(\mu_k|_{Q^m})} \lesssim_{\tau_0} \left(\sum_{j=0}^k \theta^2_{j,p_s}\right)^{1/2} \mu_k(Q^m)^{1/2}. \tag{4.1.5}$$

In fact, bearing in mind Lemma 4.1.4, (4.1.5) is also valid for $0 \leq m < k$. For the case $m = k$ simply notice that for any $Q^k \in \mathcal{Q}^k$ and $\overline{x} \in Q$, polar integration yields

$$|\mathcal{P}^s_{\mu_k}\chi_{Q^k}(\overline{x})| \lesssim \frac{1}{|E_{k,p_s}|}\int_{Q^k}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|^{n+1}_{p_s}} \lesssim \frac{\ell_k}{|E_{k,p_s}|} = \theta_{k,p_s},$$

so (4.1.5) also holds if $m = k$. Again, we need $s > 1/2$ in the above estimate.

Finally, as it is remarked in [**MatT, §3**] and [**T1, §3**], since the support of μ_k is $\mathcal{Q}^k$, relation (4.1.5) suffices to deduce the same result not only for Q^m, but also for any s-parabolic cube $Q \subset \mathbb{R}^{n+1}$. Moreover, by the arguments used to prove (4.1.5), it is clear that such estimate is also valid for the operator $\mathcal{P}^{s,*}_{\mu_k}$, associated with the kernel $(\nabla_x P_s)^*(\overline{x}) := \nabla_x P_s(-\overline{x})$. With this, we are finally ready to state the main theorem of this subsection:

THEOREM 4.1.8. *Let $Q \subset \mathbb{R}^{n+1}$ be any s-parabolic cube. Then,*

$$\|\mathcal{P}^s_{\mu_k}\chi_Q\|_{L^2(\mu_k|_Q)} + \|\mathcal{P}^{s,*}_{\mu_k}\chi_Q\|_{L^2(\mu_k|_Q)} \lesssim_{\tau_0} \left(\sum_{j=0}^k \theta^2_{j,p_s}\right)^{1/2} \mu_k(Q)^{1/2}.$$

4.1.2 Statement and proof of the estimate

Bearing in mind [**T5, Theorem 3.21**], it may seem that from Theorem 4.1.8 we could directly obtain the desired estimate for the Γ_{Θ^s} capacity of the s-parabolic Cantor set. However, such result cannot be applied in our case, since we have to take into account that our ambient space $\mathbb{R}^{n+1}$ is not endowed with the usual Euclidean distance. Nevertheless, there are Tb-like theorems which are also valid in a greater variety of settings. More precisely, we may apply [**HyMar, Theorem 2.3**], since $\mathbb{R}^{n+1}$ is geometrically doubling once endowed with the s-parabolic distance. In essence, the latter result adapted to our context implies that, for a fixed generation k, to control the boundedness of $\mathcal{P}^s_{\mu_k}$ as an $L^2(\mu_k)$-operator, it suffices to verify that

i. $\mathcal{P}^s\mu_k$ and $\mathcal{P}^{s,*}\mu_k$ belong to a certain s-parabolic BMO space (that is precised below),

ii. and that $\mathcal{P}^s\mu_k$ is μ_k-weakly bounded.

In fact, the following observations will also be important to simplify our proof:

1. By Remark 4.1.2, the weak boundedness property follows trivially, since any pairing of the form $\langle\chi_R, \mathcal{P}^s_{\mu_k}\chi_R\rangle$ is null, for any $R \subset \mathbb{R}^{n+1}$ s-parabolic cube.

2. By the second point of [**HyMar, Remark 2.4**], since the s-parabolic distance is a proper distance (and not a *quasi-distance*, as in the statement of [**HyMar, Theorem 2.3**]), it suffices to show that $\mathcal{P}^s\mu_k$ and $\mathcal{P}^{s,*}\mu_k$ belong to some s-parabolic $\text{BMO}_{\rho,p_s}(\mu_k)$ space, for some $\rho > 1$. Recall:

   Definition 4.1.1. Given $\rho > 1$ and $f \in L^1_{\text{loc}}(\mu_k)$, we say that f belongs to the $BMO_{\rho,p_s}(\mu_k)$ space if for some constant $c > 0$,
   $$\sup_Q \frac{1}{\mu_k(\rho Q)} \int_Q |f(\overline{x}) - f_{Q,\mu_k}| \mathrm{d}\mu_k(\overline{x}) \leq c,$$
   where the supremum is taken among all s-parabolic cubes such that $\mu_k(Q) \neq 0$, and f_{Q,μ_k} is the average of f in Q with respect to μ_k. The infimum over all values c satisfying the above inequality is the so-called $BMO_{\rho,p_s}(\mu_k)$ *norm of* f.

3. As it is verified in Theorem 1.1.2, the operator defined through the kernel $\nabla_x P_s$ defines a $(n+1)$-dimensional C-Z convolution operator in the s-parabolic space $\mathbb{R}^{n+1}$. With this we mean that it satisfies the required bounds of an $(n+1)$-dimensional C-Z convolution kernel but changing the usual distance $|\cdot|$ by $|\cdot|_{p_s}$.

4. In light of the previous observation, we should impose that for each generation k, the measure μ_k presents upper s-parabolic growth of degree $n+1$. To satisfy such property, we will assume that there exists an absolute constant $\kappa > 0$ so that $\theta_{j,p_s} \leq \kappa$ for every $j \geq 0$. Recall that such condition implied the desired growth restriction for μ_k with a constant C depending only on n, s and κ. Renormalizing μ with such constant, we shall assume $C = 1$. With this, and borrowing the notation of [**HyMar, §2.2**], μ_k is *upper doubling* with dominating function $\lambda(r) = r^{n+1}$.

5. Recall that given $A > 0$ and μ Borel measure on $\mathbb{R}^{n+1}$, we say that an s-parabolic cube $Q \subset \mathbb{R}^{n+1}$ has A-*small boundary* (*with respect to* μ) if
   $$\mu\big(\{\overline{x} \in 2Q \,:\, \text{dist}_{p_s}(\overline{x}, \partial Q) \leq \lambda\, \ell(Q)\}\big) \leq A\lambda\, \mu(2Q), \qquad \forall \lambda > 0.$$

Previous to the main lemma, we prove two additional preliminary results. The first is an s-parabolic version of [**T5, Lemma 9.43**]. Let us remark that in some of the forthcoming statements, the reader will encounter expressions of the form αQ, for some $\alpha > 0$ and Q a s-parabolic cube. Recall that this has to be understood as an *s-parabolic dilation*: i.e. if $Q = Q_1 \times I_Q \subset \mathbb{R}^n \times \mathbb{R}$, then $\alpha Q = (\alpha Q_1) \times (\alpha^{2s} I_Q)$.

Lemma 4.1.9. *Let μ be a real finite Borel measure on $\mathbb{R}^{n+1}$ and $A(n,s) > 0$ some constant big enough. Let $Q \subset \mathbb{R}^{n+1}$ be any fixed s-parabolic cube. Then, there exists a concentric s-parabolic cube Q' with $Q \subset Q' \subset 1.1Q$ which has A-small boundary with respect to μ.*

Proof. The proof will be almost identical to that of [**T5, Lemma 9.43**]. Assume that Q is centered at the origin and write $\sigma := \mu|_{2Q}$. For $a \in \mathbb{R}$ and $1 \leq j \leq n+1$, let $H_j(a)$ be the hyperplane
$$H_j(a) := \{\overline{x} \in \mathbb{R}^{n+1} \,:\, x_j = a\},$$

where we convey $x_{n+1} := t$. For $\delta > 0$, write U_δ the (Euclidean) δ-neighborhood of a set. The existence of Q' will follow from the existence of some $a \in [\ell(Q), 1.05\ell(Q)]$ such that

$$\frac{1}{\eta\ell(Q)}\sigma\Big(U_{\eta\ell(Q)}\big(H_j(\pm a)\big)\Big) \leq A\frac{\|\sigma\|}{\ell(Q)}, \qquad \forall\eta > 0,\ j = 1,\dots,n, \tag{4.1.6}$$

$$\frac{1}{\eta\ell(Q)}\sigma\Big(U_{\eta^{2s}\ell(Q)^{2s}}\big(H_{n+1}(\pm a)\big)\Big) \leq A\frac{\|\sigma\|}{\ell(Q)}, \qquad \forall\eta > 0. \tag{4.1.7}$$

Recall that $\|\sigma\| := |\sigma|(\mathbb{R}^{n+1})$, where $|\sigma|$ is the variation of σ. Let $\pi_j, \widetilde{\pi}_j : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projections defined by $\pi_j(\overline{x}) := x_j$, $\widetilde{\pi}_j(\overline{x}) := -x_j$, for $j = 1, \dots n$; as well as $\pi'_{n+1} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ given by $\pi'_{n+1}(\overline{x}) := x_{n+1}^{\frac{1}{2s}}$, and $\widetilde{\pi}'_{n+1} : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ given by $\widetilde{\pi}'_{n+1}(\overline{x}) := (-x_{n+1})^{\frac{1}{2s}}$. Consider the image measures

$$\nu_j := \pi_j\#\sigma, \qquad \widetilde{\nu}_j := \widetilde{\pi}_j\#\sigma, \qquad j = 1,\dots,n,$$
$$\nu_{n+1} := \pi'_{n+1}\#\sigma, \qquad \widetilde{\nu}_{n+1} := \widetilde{\pi}'_{n+1}\#\sigma,$$

where $f\#\mu(\cdot) := \mu(f^{-1}(\cdot))$. This way, conditions (4.1.6) and (4.1.7) can be simply rewritten as

$$\frac{1}{\eta\ell(Q)}\nu_j\Big(I\big(a,\eta\ell(Q)\big)\Big) \leq A\frac{\|\sigma\|}{\ell(Q)}, \qquad \frac{1}{\eta\ell(Q)}\widetilde{\nu}_j\Big(I\big(a,\eta\ell(Q)\big)\Big) \leq A\frac{\|\sigma\|}{\ell(Q)}, \qquad \forall\eta > 0,$$

where now $1 \leq j \leq n+1$ and $I(y,\ell)$ denotes the real interval centered at y with length 2ℓ. In fact, the above condition can be rephrased as

$$M\nu_j(a) \leq A\frac{\|\sigma\|}{\ell(Q)}, \qquad M\widetilde{\nu}_j(a) \leq A\frac{\|\sigma\|}{\ell(Q)}, \qquad j = 1,\dots,n+1, \tag{4.1.8}$$

where $M \equiv M_{\mathcal{L}^1}$ is maximal Hardy-Littlewood operator in $\mathbb{R}$. We now define the measure $\nu := \sum_{j=1}^{n+1} \nu_j + \widetilde{\nu}_j$. Observe that $\|\nu_j\| = \|\widetilde{\nu}_j\| = \|\sigma\|$ for $j = 1,\dots,n$, and $\|\nu_{n+1}\| + \|\widetilde{\nu}_{n+1}\| = \|\sigma\|$. Therefore $\|\nu\| = (2n+1)\|\sigma\|$. Notice that if we prove

$$M\nu(a) \leq A\frac{\|\sigma\|}{\ell(Q)} = A\frac{\|\nu\|}{(2n+1)\ell(Q)},$$

condition (4.1.8) will hold. But due to [**T5, Theorem 2.5**] (a standard result concerning the weak boundedness of M in a general non-doubling setting),

$$\mathcal{L}^1\left(\left\{a \in \mathbb{R} \,:\, M\nu(a) > A\frac{\|\nu\|}{(2n+1)\ell(Q)}\right\}\right) \leq C\frac{(2n+1)\ell(Q)}{A}.$$

So for A big enough there is $a \in [\ell(Q), 1.05\ell(Q)]$ with $M\nu(a) \leq A\frac{\|\nu\|}{(2n+1)\ell(Q)}$. □

The second preliminary result refers to the existence of *large* doubling balls. It can be understood as a direct consequence of [**Hy, Lemma 3.2**]. Recall that for a given real Borel measure μ in $\mathbb{R}^{n+1}$ and $\alpha, \beta > 1$, a s-parabolic cube $Q \subset \mathbb{R}^{n+1}$ is said to be (α,β)-*doubling* (*with respect to* μ) if $\mu(\alpha Q) \leq \beta\mu(Q)$. For the sake of completeness, let us comment that following the same argument presented at the beginning of [**T5, §2.4**], one could also prove a generalization of the below result for the s-parabolic setting.

LEMMA 4.1.10. *Let $Q \subset \mathbb{R}^{n+1}$ be an s-parabolic cube and μ a real Borel measure that has upper s-parabolic growth of degree $n+1$ with constant 1. Then, there exists $j_0 \in \mathbb{N}$ such that $Q_0 := 3^{j_0}Q$ is $(3, 3^{n+2})$-doubling.*

Proof. Apply [**Hy, Lemma 3.2**] with $C_\lambda := 2^{n+1}$, $\alpha = 3$ and $\beta = 3^{n+2}$. $\square$

We are now ready to prove the result we were initially interested in:

LEMMA 4.1.11. *Let $Q \subset \mathbb{R}^{n+1}$ be any s-parabolic cube and μ a compactly supported positive Borel measure with upper s-parabolic growth of degree $n+1$ with constant 1. Assume that $|\langle \chi_R, \mathcal{P}^s_\mu \chi_R \rangle| \lesssim 1$ for any $R \subset \mathbb{R}^{n+1}$ s-parabolic cube with A-small boundary, $A = A(n,s)$. Then,*

$$\|\mathcal{P}^s\mu\|_{\mathrm{BMO}_{3,p_s}(\mu)} + \|\mathcal{P}^{s,*}\mu\|_{\mathrm{BMO}_{3,p_s}(\mu)} \lesssim 1 + \|\mathcal{P}^s_\mu \chi_Q\|_{L^2(\mu|_Q)}\mu(Q)^{-1/2}$$

Proof. We give the details to estimate $\|\mathcal{P}^s\mu\|_{\mathrm{BMO}_{3,p_s}(\mu)}$, since the arguments can be directly adapted for $\|\mathcal{P}^{s,*}\mu\|_{\mathrm{BMO}_{3,p_s}(\mu)}$. We clarify that the arguments below are inspired by those given for [**T5, Proposition 9.45**].

Let $A = A(n,s) > 0$ be big enough (as in Lemma 4.1.9) and consider a s-parabolic cube Q with A-small boundary. By Lemma 4.1.10 let $Q_0 := 3^{j_0}Q$ be a $(3, 3^{n+2})$-doubling s-parabolic cube (with respect to μ) with the minimal $j_0 \in \mathbb{N}$ such that this property is satisfied. That is, we require that $\mu(3^jQ) > 3^{n+2}\mu(3^{j-1}Q)$, for $j = 1, \dots, j_0 - 1$. Iterating the previous inequality we also deduce

$$\mu(3^jQ) \leq \frac{\mu(3^{j_0-1}Q)}{3^{(n+2)(j_0-1-j)}}, \qquad \text{for } j = 1, \dots, j_0 - 1. \tag{4.1.9}$$

By Lemma 4.1.9 we can take $\widehat{Q}$ with A-small boundary concentric with Q_0 such that $Q_0 \subset \widehat{Q} \subset 1.1Q_0$. Since $2\widehat{Q} \subset 3Q_0$, it is clear that $\widehat{Q}$ is $(2, 3^{n+2})$-doubling. Assume that for any s-parabolic cube Q with A-small boundary we prove the estimate

$$\left| \left(\int_Q \left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right|^2 \mathrm{d}\mu \right)^{1/2} - \left(\int_Q \left| \mathcal{P}^s_\mu \chi_Q \right|^2 \mathrm{d}\mu \right)^{1/2} \right| \leq C\mu(2Q)^{1/2}, \tag{4.1.10}$$

for some constant $C(n,s)$ say bigger than 1, where recall that $(\mathcal{P}^s\mu)_{\widehat{Q},\mu}$ is the average of $\mathcal{P}^s\mu$ in $\widehat{Q}$ with respect to μ. Then, by Cauchy-Schwarz's inequality we infer that for any s-parabolic cube Q with A-small boundary,

$$\int_Q \left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right| \mathrm{d}\mu \lesssim \left(C + \frac{\|\mathcal{P}^s_\mu\chi_Q\|_{L^2(\mu|_Q)}}{\mu(Q)^{1/2}} \right)\mu(2Q) \lesssim \left(1 + \frac{\|\mathcal{P}^s_\mu\chi_Q\|_{L^2(\mu|_Q)}}{\mu(Q)^{1/2}} \right)\mu(2Q).$$

Now observe for an arbitrary s-parabolic cube P, we can take Q with A-small boundary concentric with P and such that $P \subset Q \subset 1.1P$. Hence,

$$\begin{aligned}
\int_P \left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{P,\mu} \right| \mathrm{d}\mu &\leq \int_P \left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right| \mathrm{d}\mu + \left| (\mathcal{P}^s\mu)_{P,\mu} - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right| \mu(P) \\
&\leq \int_Q \left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right| \mathrm{d}\mu + \left(\left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right| \right)_{P,\mu} \mu(P) \\
&\leq 2\int_Q \left| \mathcal{P}^s\mu - (\mathcal{P}^s\mu)_{\widehat{Q},\mu} \right| \mathrm{d}\mu \lesssim \left(1 + \frac{\|\mathcal{P}^s_\mu\chi_Q\|_{L^2(\mu|_Q)}}{\mu(Q)^{1/2}} \right)\mu(3P)
\end{aligned}$$

Therefore, it suffices to prove (4.1.10) in order to deduce the desired result. Begin by noticing that the triangle inequality applied to the left-hand side of (4.1.10) yields

$$\begin{aligned}\bigg|\bigg(\int_Q\Big|\mathcal{P}^s\mu-(\mathcal{P}^s\mu)_{\widehat{Q},\mu}\Big|^2\,\mathrm{d}\mu\bigg)^{1/2}-\bigg(\int_Q\big|\mathcal{P}^s_\mu\chi_Q\big|^2\,\mathrm{d}\mu\bigg)^{1/2}\bigg|&\\ \leq\bigg(\int_Q\Big|\mathcal{P}^s\mu-(\mathcal{P}^s\mu)_{\widehat{Q},\mu}-\mathcal{P}^s_\mu\chi_Q\Big|^2\,\mathrm{d}\mu\bigg)^{1/2}&\\ =\bigg(\int_Q\Big|\mathcal{P}^s_\mu\chi_{\mathbb{R}^{n+1}\setminus Q}-(\mathcal{P}^s\mu)_{\widehat{Q},\mu}\Big|^2\,\mathrm{d}\mu\bigg)^{1/2}&.\end{aligned}$$

For each $\overline{x}\in Q$ write the previous integrand as follows:

$$\begin{aligned}\mathcal{P}^s_\mu\chi_{\mathbb{R}^{n+1}\setminus Q}(\overline{x})-(\mathcal{P}^s\mu)_{\widehat{Q},\mu}=\mathcal{P}^s_\mu\chi_{2Q\setminus Q}(\overline{x})+\mathcal{P}^s_\mu\chi_{2\widehat{Q}\setminus 2Q}(\overline{x})&\\ -\Big(\mathcal{P}^s_\mu\chi_{\widehat{Q}}\Big)_{\widehat{Q},\mu}-\Big(\mathcal{P}^s_\mu\chi_{2\widehat{Q}\setminus\widehat{Q}}\Big)_{\widehat{Q},\mu}&\\ +\bigg[\mathcal{P}^s_\mu\chi_{\mathbb{R}^{n+1}\setminus 2\widehat{Q}}(\overline{x})-\Big(\mathcal{P}^s_\mu\chi_{\mathbb{R}^{n+1}\setminus 2\widehat{Q}}\Big)_{\widehat{Q},\mu}\bigg]&.\end{aligned}\tag{4.1.11}$$

Let us begin by estimating the second term of the right-hand side. Since $2\widehat{Q}\subset 3^{j_0+1}Q$ and μ satisfies an upper s-parabolic growth condition, we have

$$\begin{aligned}\Big|\mathcal{P}^s_\mu\chi_{2\widehat{Q}\setminus 2Q}(\overline{x})\Big|&\lesssim\int_{3^{j_0+1}Q\setminus Q}\frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}}=\sum_{j=1}^{j_0+1}\int_{3^jQ\setminus 3^{j-1}Q}\frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}}\\ &\lesssim\sum_{j=1}^{j_0+1}\frac{\mu(3^jQ)}{\big(3^j\ell(Q)\big)^{n+1}}\leq 2\frac{\mu(3^{j_0+1}Q)}{\big(3^{j_0}\ell(Q)\big)^{n+1}}+\sum_{j=1}^{j_0-1}\frac{\mu(3^jQ)}{\big(3^j\ell(Q)\big)^{n+1}}\\ &\lesssim 1+\sum_{j=1}^{j_0-1}\frac{\mu(3^jQ)}{\big(3^j\ell(Q)\big)^{n+1}}.\end{aligned}$$

For the remaining sum, relation (4.1.9) implies

$$\begin{aligned}\sum_{j=1}^{j_0-1}\frac{\mu(3^jQ)}{\big(3^j\ell(Q)\big)^{n+1}}&\leq\frac{\mu(3^{j_0-1}Q)}{\big(3^{j_0-1}\ell(Q)\big)^{n+1}}\sum_{j=1}^{j_0-1}\frac{1}{3^{(n+2)(j_0-1-j)}3^{(-j_0+1+j)(n+1)}}\\ &\lesssim\sum_{j=1}^{j_0-1}\frac{1}{3^{j_0-1-j}}=\sum_{j=0}^{j_0-2}\frac{1}{3^j}\lesssim 1,\end{aligned}$$

so indeed $|\mathcal{P}^s_\mu\chi_{2\widehat{Q}\setminus 2Q}(\overline{x})|\lesssim 1$ for $\overline{x}\in Q$. The modulus of the third term of (4.1.11) is bounded by a constant depending on n and s by hypothesis, since $\langle\chi_R,\mathcal{P}^s_\mu\chi_R\rangle\lesssim 1$ for any $R\subset\mathbb{R}^{n+1}$ s-parabolic cube with A-small boundary. Observe that the fourth term satisfies

$$\Big(\mathcal{P}^s_\mu\chi_{2\widehat{Q}\setminus\widehat{Q}}\Big)_{\widehat{Q},\mu}\leq\frac{1}{\mu(\widehat{Q})}\int_{\widehat{Q}}\bigg(\int_{2\widehat{Q}\setminus\widehat{Q}}\frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}}\bigg)\,\mathrm{d}\mu(\overline{x}).$$

Such expression can be dealt with as in [**T5, Lemma 9.44**]. Indeed, the above domains of integration imply $|\overline{x}-\overline{y}|_{p_s} \geq \text{dist}_{p_s}(\overline{x},\partial\widehat{Q})$. Then, defining

$$Q_j := Q\big(\overline{x}, 2^j\text{dist}_{p_s}(\overline{x},\partial\widehat{Q})\big), \qquad 0\leq j\leq \left\lceil \log_2\left(\frac{4\ell(\widehat{Q})}{\text{dist}_{p_s}(\overline{x},\partial\widehat{Q})}\right)\right\rceil =: N,$$

integration over annuli and the upper s-parabolic growth of degree $n+1$ of μ yield

$$\begin{aligned}\int_{2\widehat{Q}\setminus\widehat{Q}} \frac{\text{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} &\leq \int_{\text{dist}_{p_s}(\overline{x},\partial\widehat{Q})\,\leq\,|\overline{x}-\overline{y}|_{p_s}\,\leq\,4\ell(\widehat{Q})} \frac{\text{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \leq \sum_{j=0}^N \int_{Q_{j+1}\setminus Q_j} \frac{\text{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}} \\ &\leq \sum_{j=0}^N \frac{\mu(Q_{j+1})}{\ell(Q_j)^{n+1}} \lesssim N = \left\lceil \log_2\left(\frac{4\ell(\widehat{Q})}{\text{dist}_{p_s}(\overline{x},\partial\widehat{Q})}\right)\right\rceil.\end{aligned}$$

For $j\geq 0$ let

$$V_j := \Big\{\overline{x}\in\widehat{Q}\,:\, 2^{-j-1}\ell(\widehat{Q}) < \text{dist}_{p_s}(\overline{x},\partial\widehat{Q}) \leq 2^{-j}\ell(\widehat{Q})\Big\},$$

and observe that the A-small boundary property of $\widehat{Q}$ implies $\mu(V_j)\leq A2^{-j}\mu(2\widehat{Q})$. Therefore,

$$\begin{aligned}\frac{1}{\mu(\widehat{Q})}\int_{\widehat{Q}}\bigg(\int_{2\widehat{Q}\setminus\widehat{Q}} \frac{\text{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}}\bigg)\text{d}\mu(\overline{x}) &\lesssim \frac{1}{\mu(\widehat{Q})}\sum_{j\geq 0}\int_{V_j}\Big\lceil \log_2\big(4\cdot 2^{j+1}\big)\Big\rceil\,\text{d}\mu(\overline{x}) \\ &\leq A\frac{\mu(2\widehat{Q})}{\mu(\widehat{Q})}\sum_{j\geq 0}\frac{\big\lceil \log_2\big(4\cdot 2^{j+1}\big)\big\rceil}{2^j} \lesssim 1,\end{aligned}$$

where in the last step we have used that $\widehat{Q}$ is $(2,3^{n+2})$-doubling. Finally, applying [**T5, Lemma 9.12**] (that admits a straightforward generalization to the s-parabolic setting) with $f := \chi_{\mathbb{R}^{n+1}\setminus 2\widehat{Q}}$, we also deduce

$$\Big|\mathcal{P}^s_\mu\chi_{\mathbb{R}^{n+1}\setminus 2\widehat{Q}}(\overline{x}) - \Big(\mathcal{P}^s_\mu\chi_{\mathbb{R}^{n+1}\setminus 2\widehat{Q}}\Big)_{\widehat{Q},\mu}\Big| \lesssim 1$$

Therefore, the left-hand side of (4.1.10) is bounded above by

$$\bigg(\int_Q \Big|\mathcal{P}^s_\mu\chi_{2Q\setminus Q}\Big|^2\text{d}\mu\bigg)^{1/2} + C\mu(Q)^{1/2},$$

for some $C(n,s)$. Notice that the remaining integral is such that

$$\int_Q \Big|\mathcal{P}^s_\mu\chi_{2Q\setminus Q}\Big|^2\text{d}\mu \leq \int_Q\bigg(\int_{2Q\setminus Q}\frac{\text{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^{n+1}}\bigg)^2\text{d}\mu(\overline{x}).$$

As Q has A-small boundary, we can proceed as we have done for the fourth term of the right-hand side of (4.1.11) (again, see [**T5, Lemma 9.44**] for more details), and deduce

$$\int_Q \Big|\mathcal{P}^s_\mu\chi_{2Q\setminus Q}\Big|^2\text{d}\mu \lesssim \mu(2Q).$$

All in all, we get that the left-hand side of (4.1.10) is bounded by $\mu(2Q)^{1/2}$, up to a multiplicative constant depending only on n and s, that implies the desired result. □

Let $(\lambda_j)_j$ be such that $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j, and denote by E_{p_s} its associated s-parabolic Cantor set as in (4.1.1). Assume that there exists an absolute constant $\kappa > 0$ so that $\theta_{j,p_s} \leq \kappa$ for every $j \geq 0$. Fix a generation k and let μ_k be the usual uniform probability measure of E_{k,p_s}. Then, by Theorem 4.1.8, relation (4.1.2), the fact that μ_k satisfies an upper s-parabolic growth condition and that $\theta_{0,p_s} = 1$,

$$\|\mathcal{P}^s\mu_k\|_{\mathrm{BMO}_{3,p_s}(\mu_k)} + \|\mathcal{P}^{s,*}\mu_k\|_{\mathrm{BMO}_{3,p_s}(\mu_k)} \lesssim_{\tau_0,\kappa} 1 + \left(\sum_{j=0}^{k}\theta_{j,p_s}^2\right)^{1/2} \leq \left(\sum_{j=0}^{k}\theta_{j,p_s}^2\right)^{1/2}.$$

Lemma 4.1.11 allows us to deduce the desired estimate for $\Gamma_\Theta(E_{k,p_s})$:

THEOREM 4.1.12. *Let $(\lambda_j)_j$ be such that $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j, and denote by E_{p_s} its associated s-parabolic Cantor set as in (4.1.1). Assume that there exists an absolute constant $\kappa > 0$ so that $\theta_{j,p_s} \leq \kappa$ for each $j \geq 0$. Then, for every generation k,*

$$\Gamma_{\Theta^s}(E_{k,p_s}) \gtrsim_{\tau_0,\kappa} \left(\sum_{j=0}^{k}\theta_{j,p_s}^2\right)^{-1/2}.$$

Proof. By a direct application of Lemma 4.1.11 and [**HyMar, Theorem 2.3**] we deduce

$$\|\mathcal{P}^s_{\mu_k}\|_{L^2(\mu_k)\to L^2(\mu_k)} \leq C\left(\sum_{j=0}^{k}\theta_{j,p_s}^2\right)^{1/2},$$

for some $C = C(n,s,\tau_0,\kappa)$. Then, by Theorem 2.3.3, $C^{-1}\big(\sum_{j=0}^k \theta_{j,p_s}^2\big)^{-1/2}\mu_k$ becomes an admissible measure for $\widetilde{\Gamma}_{\Theta^s,+}(E_{k,p_s})$, defined in (2.3.1). Thus,

$$\Gamma_{\Theta^s}(E_{k,p_s}) \geq \widetilde{\Gamma}_{\Theta^s,+}(E_{k,p_s}) \gtrsim C^{-1}\left(\sum_{j=0}^{k}\theta_{j,p_s}^2\right)^{-1/2}.$$

□

The next lemma will allow us to extend the result to the final s-parabolic Cantor set E_{p_s}. We prove it for $\Gamma_{\Theta^s,+}$ for simplicity, but it can also be proved to hold for $\widetilde{\Gamma}_{\Theta^s,+}$.

LEMMA 4.1.13. *If $(E_k)_k$ is a nested sequence of compact sets of $\mathbb{R}^{n+1}$ that decreases to $\mathcal{E} := \cap_{k=1}^\infty E_k$,*

$$\lim_{k\to\infty}\Gamma_{\Theta^s,+}(E_k) = \Gamma_{\Theta^s,+}(\mathcal{E}).$$

Proof. It is clear that $\Gamma_{\Theta^s,+}(\mathcal{E}) \leq \lim_{k\to\infty}\Gamma_{\Theta^s,+}(E_k)$, so we are left to prove the converse inequality. For each k consider an admissible measure μ_k for $\Gamma_{\Theta^s,+}(E_k)$ with

$$\Gamma_{\Theta^s,+}(E_k) - \frac{1}{k} \leq \mu_k(E_k) \leq \Gamma_{\Theta^s,+}(E_k),$$

We shall verify that there exists an admissible measure μ for $\Gamma_{\Theta^s,+}(\mathcal{E})$ so that

$$\limsup_{k\to\infty} \mu_k(E_k) \leq \mu(\mathcal{E}). \tag{4.1.12}$$

If this is the case,

$$\lim_{k\to\infty} \Gamma_{\Theta^s,+}(E_k) \leq \limsup_{k\to\infty} \mu_k(E_k) \leq \mu(\mathcal{E}) \leq \Gamma_{\Theta^s,+}(\mathcal{E}),$$

and we would be done. To construct such μ, notice that Theorem 2.1.2 implies that each μ_k has upper s-parabolic growth of degree $n+1$ with an absolute constant C. Then $\mu_k(\mathbb{R}^{n+1}) \leq C\,\text{diam}_{p_s}(E_1)^{n+1}$, $\forall k \geq 0$, so by [**Matti, Theorem 1.23**] there exists a positive Radon measure μ on $\mathbb{R}^{n+1}$ such that $\mu_k \xrightarrow{w} \mu$. Arguing by contradiction it is not difficult to verify that $\text{supp}(\mu) \subseteq \mathcal{E}$, and it is also clear that (4.1.12) is satisfied (in fact, taking $\varphi \in \mathcal{C}_0(\mathbb{R}^{n+1})$ with $\varphi \equiv 1$ on a neighborhood of E_1, (4.1.12) holds with a proper limit and an equality). So we are left to estimate the quantities $\|\nabla_x P_s * \mu\|_\infty$ and $\|\partial_t^{\frac{1}{2s}} P_s * \mu\|_{*,p_s}$.

By assumption $\nabla_x P_s * \mu_k$ belongs to the unit ball of $L^\infty(\mathbb{R}^{n+1}) \cong L^1(\mathbb{R}^{n+1})^*$, and it is clear that $L^1(\mathbb{R}^{n+1})$ is separable. Then, by the sequential version of Banach-Alaoglu's theorem there exists some $S \in L^\infty(\mathbb{R}^{n+1})$ with $\|S\|_\infty \leq 1$ and $\nabla_x P_s * \mu_k \to S$ as $k\to\infty$ in a weak*-L^∞ sense. Now take $\psi \in \mathcal{C}_c^\infty(B(0,1))$ positive and radial with $\int \psi = 1$ and set $\psi_\varepsilon := \varepsilon^{-(n+2s)}\psi(\cdot/\varepsilon)$. Since $\nabla_x P_s * \mu_k$ converges to S in a weak*-L^∞ sense and by construction $\|\psi_\varepsilon\|_{L^1(\mathbb{R}^{n+1})} = 1$,

$$\lim_{k\to\infty} \big(\psi_\varepsilon * \nabla_x P_s * \mu_k\big)(\overline{x}) = \psi_\varepsilon * S(\overline{x}), \qquad \overline{x} \in \mathbb{R}^{n+1}.$$

In addition, since $\psi_\varepsilon * \nabla_x P_s \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ and μ_k converges to μ in the weak topology of (compactly supported) real Radon measures, we have

$$\lim_{k\to\infty} \big(\psi_\varepsilon * \nabla_x P_s * \mu_k\big)(\overline{x}) = \big(\psi_\varepsilon * \nabla_x P_s * \mu\big)(\overline{x}), \qquad \overline{x} \in \mathbb{R}^{n+1}.$$

Hence $\psi_\varepsilon * S = \psi_\varepsilon * \nabla_x P_s * \mu$ for every $\varepsilon > 0$, so $S = \nabla_x P_s * \mu$ and in particular $\|\nabla_x P_s * \mu\|_\infty \leq 1$. Finally, applying Lemma 1.3.2 with $\beta := \frac{1}{2s}$ we deduce $\|\partial_t^{\frac{1}{2s}} P_s * \mu\|_{*,p_s} \lesssim 1$ and the proof is complete. □

Theorem 4.1.14. *Let $(\lambda_j)_j$ be such that $0 < \lambda_j \leq \tau_0 < 1/\delta$, for every j, and denote by E_{p_s} its associated s-parabolic Cantor set as in (4.1.1). If $\theta_* := \sup_j \theta_{j,p_s}$, there exists a constant $C = C(n,s,\tau_0,\theta_*)$, with $C = 0$ if $\theta_* = \infty$, such that*

$$\Gamma_{\Theta^s}(E_{p_s}) \geq C\left(\sum_{j=0}^{\infty} \theta_{j,p_s}^2\right)^{-1/2}.$$

Proof. We assume, without loss of generality, that the sum involved in the estimate is convergent. If this is the case, it is clear that there exists some $\kappa^2 > 0$ such that $\theta^2_{j,p_s} \leq \kappa^2$ for every j. Therefore, we are under the hypothesis of Theorem 4.1.12. Applying Lemma 4.1.13 we conclude:

$$\Gamma_{\Theta^s}(E_{p_s}) \geq \Gamma_{\Theta^s,+}(E_{p_s}) = \lim_{k\to\infty} \Gamma_{\Theta^s,+}(E_{k,p_s}) \geq \lim_{k\to\infty} \widetilde{\Gamma}_{\Theta^s,+}(E_{k,p_s})$$
$$\gtrsim_{\tau_0,\kappa} \left(\sum_{j=0}^{\infty} \theta^2_{j,p_s} \right)^{-1/2}.$$

□

4.2 The $\gamma_{\Theta^s,+}$ capacity of Cantor sets

As in Chapter 3, we will introduce a capacity related to $\gamma^\sigma_{\Theta^s,*}$ for which we will choose a particular value σ. More precisely, we will be interested in the smaller variant obtained by taking the supremum only among positive Borel measures instead of general distributions.

Definition 4.2.1. Given $s \in (0,1]$ and $E \subset \mathbb{R}^{n+1}$ a compact set, define its *s-caloric capacity* as

$$\gamma_{\Theta^s}(E) := \gamma^0_{\Theta^s}(E) := \sup |\langle T, 1\rangle|,$$

where the supremum is taken among all admissible distributions T, i.e. those supported on E and satisfying

$$\|P_s * T\|_\infty \leq 1.$$

We define the $(s,+)$-*caloric capacity* just as γ_{Θ^s}, writing

$$\gamma_{\Theta^s,+}(E) := \sup \mu(E),$$

where the supremum is taken over positive Borel regular measures μ with $\text{supp}(\mu) \subseteq E$ and such that $\|P_s * \mu\|_\infty \leq 1$.

As a consequence of Theorem 1.4.8 we obtain the following growth result for admissible distributions for γ_{Θ^s}.

Theorem 4.2.1. *Let $E \subset \mathbb{R}^{n+1}$ be compact and T be an admissible distribution for γ_{Θ^s}. Then, T presents upper s-parabolic growth of degree n, that is,*

$$|\langle T, \varphi\rangle| \lesssim \ell(Q)^n, \qquad \textit{for any } Q \subset \mathbb{R}^{n+1} \textit{ s-parabolic cube and } \varphi \textit{ admissible for } Q.$$

Remark 4.2.1. Let us observe that the previous growth property, by an argument analogous to that of the proof of Theorem 3.1.2, implies that in $\mathbb{R}^{n+1}$ the capacity γ_{Θ^s} has critical s-parabolic Hausdorff dimension n.

We recall that for the particular cases $s = 1/2$ and $s = 1$, the kernel P_s is explicit and is given by

$$P(\overline{x}) := P_{1/2}(\overline{x}) = c_n \frac{t}{|\overline{x}|^{n+1}} \chi_{t>0}, \qquad W(\overline{x}) := P_1(\overline{x}) = c_n t^{-n/2} e^{-\frac{|x|^2}{4t}} \chi_{t>0},$$

where in the expression of P we have written $|\overline{x}|$ and not $|\overline{x}|_{p_{1/2}}$ simply because it is not necessary, since the choice $s = 1/2$ implies that the s-parabolic metric is, in fact, the usual Euclidean one. Notice that the above explicit expressions are nonnegative and continuous in $\mathbb{R}^{n+1} \setminus \{0\}$, although that P is not differentiable at any point of the hyperplane $\{t = 0\}$. Let us argue that, in fact, for any $s \in (0, 1]$, the expression P_s is nonnegative. We present a short argument that relies on Bochner's theorem [**Ru1, §1.4.3**]. We know (see §2.1) that for $t > 0$, the Fourier transform of $f_t := P_s(\cdot, t)$ is

$$\widehat{f_t}(\xi) = c_{n,s} e^{-4\pi^2 t|\xi|^{2s}},$$

which is a radial function. By Schoenberg's theorem [**Sc, Theorem 3**], if we check that the function $g_t(r) := e^{-4\pi t r^s}$ (that is such that $g_t(|\xi|^2) = \widehat{f_t}(\xi)$) is completely monotonic, it will imply that the Fourier transform of $\widehat{f_t}$ is nonnegative, and thus $P_s(\cdot, t) \geq 0$ for any $t > 0$. Recall that a function h with domain $\mathbb{R}_+$ is said to be *completely monotonic* if it is smooth and

$$(-1)^k h^{(k)}(r) \geq 0, \qquad \forall k,\, r > 0.$$

The particular case of $g_t(r) = e^{-4\pi^2 t r^s}$ is an example of completely monotonic function [**MiSa, Equation 1.13**], so that we are able to deduce that $P_s \geq 0$.

Another fundamental function that will frequently appear is P_s^*,

$$P_s^*(\overline{x}) := P_s(-\overline{x}) = P_s(x, -t) = c_{n,s}\, |t|^{-\frac{n}{2s}} \phi_{n,s}\big(|xt|^{-\frac{1}{2s}}\big) \chi_{t<0}.$$

More generally, given T any distribution in $\mathbb{R}^{n+1}$, we define T^* to be the distribution acting on test functions as

$$\langle T^*, \varphi \rangle := \langle T, \varphi^* \rangle, \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}).$$

We will say that a distribution is *even* if $T = T^*$. It is not hard to check (approximating via test functions and using the associativity of [**C, Theorem 8.15**]) that for any T distribution and S distribution with compact support,

$$\langle T * S, \varphi \rangle = \langle S, T^* * \varphi \rangle, \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}).$$

With this notions we observe that, on the one hand, for any $\overline{x} = (x, t) \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned}(-\Delta)^s P_s^*(\overline{x}) &\simeq \int_{\mathbb{R}^n} \frac{P_s^*(x+y,t) - 2P_s^*(x,t) + P_s^*(x-y,t)}{|y|^{n+2s}}\,\mathrm{d}y \\ &\simeq \int_{\mathbb{R}^n} \frac{P_s(-x-y,-t) - 2P_s(-x,-t) + P_s(-x+y,-t)}{|y|^{n+2s}}\,\mathrm{d}y = (-\Delta)^s P(-\overline{x}).\end{aligned}$$

That is, we have

$$(-\Delta)^s P_s^* = \big[(-\Delta)^s P_s\big]^*.$$

On the other hand, understanding $\partial_t P_s$ distributionally (that makes sense, since P_s is a locally integrable function in $\mathbb{R}^{n+1}$ and thus a distribution) we have, for any φ test function in $\mathbb{R}^{n+1}$,

$$\langle (\partial_t P_s)^*, \varphi \rangle := \langle \partial_t P_s, \varphi^* \rangle := -\langle P_s, \partial_t \varphi^* \rangle = \langle P_s, (\partial_t \varphi)^* \rangle = \langle P_s^*, \partial_t \varphi \rangle =: -\langle \partial_t P_s^*, \varphi \rangle.$$

Therefore, the following distributional identity holds

$$(\partial_t P_s)^* = -\partial_t P_s^*.$$

Hence, if we define the operator

$$\overline{\Theta}^s := (-\Delta)^s - \partial_t,$$

we have that

$$\overline{\Theta}^s P_s^* = \big[\Theta^s P_s\big]^* = \delta_0^* = \delta_0,$$

implying that P_s^* is the fundamental solution of $\overline{\Theta}^s$. P_s^* and $\overline{\Theta}^s$ will be the *conjugates* of P_s and Θ^s respectively. With this, we introduce the conjugate s-caloric capacities

$$\gamma_{\overline{\Theta}^s}, \quad \gamma_{\overline{\Theta}^s,+},$$

which are defined as γ_{Θ^s} and $\gamma_{\Theta^s,+}$, respectively, but using the conjugate kernel P_s^*.

We shall also introduce yet another variant of γ_{Θ^s}, namely $\widetilde{\gamma}_{\Theta^s,+}$, that will be referred to as $(s,+)$-*symmetric caloric capacity* and will be the main object of study of the forthcoming chapter. As in (2.3.1), admissible measures for $\widetilde{\gamma}_{\Theta^s,+}$ must also satisfy $\|P_s^* * \mu\|_\infty \leq 1$, and we shall assume $\mu \in \Sigma_n^s(E)$, the collection of positive Borel measures supported on E with upper s-parabolic n-growth with constant 1.

4.2.1 Properties of $\gamma_{\Theta^s,+}$

In the sequel we will be concerned with estimating the $(s,+)$-caloric capacity of a family of Cantor sets of $\mathbb{R}^{n+1}$. Previous to that, we shall present some important features of $\gamma_{\Theta^s,+}$ that we consider of their own interest and that will turn out to be useful in the next chapter.

Theorem 4.2.2. *Let $E \subset \mathbb{R}^{n+1}$ be a Borel subset and $\lambda > 0, \tau \in \mathbb{R}^{n+1}$. Set $\tau(E) := E + \tau$ and denote by λE the s-parabolic dilation of E by λ. The following identities hold:*

1. *Translation invariance: $\gamma_{\Theta^s,+}(E) = \gamma_{\Theta^s,+}(\tau(E))$.*
2. *$\gamma_{\Theta^s,+}(\lambda E) = \lambda^n \gamma_{\Theta^s,+}(E)$.*
3. *Outer regularity: If $(E_k)_k$ is a nested sequence of compact sets of $\mathbb{R}^{n+1}$ that decreases to $\mathcal{E} := \cap_{k=1}^\infty E_k$,*

$$\lim_{k\to\infty} \gamma_{\Theta^s,+}(E_k) = \gamma_{\Theta^s,+}(\mathcal{E}).$$

4. *Countable subadditivity: if $E_1, E_2, \ldots$ are disjoint Borel subsets of $\mathbb{R}^{n+1}$,*

$$\gamma_{\Theta^s,+}\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \gamma_{\Theta^s,+}(E_j).$$

Proof. During this proof we shall write $\gamma_+ := \gamma_{\Theta^s,+}$ to ease notation. To verify *1*, we pick $E \subset \mathbb{R}^{n+1}$ compact and prove that for any μ admissible for $\gamma_+(E)$ there exists a measure μ_τ, admissible for $\gamma_+(\tau(E))$, such that $\mu(E) = \mu_\tau(\tau(E))$. It is clear that once this property is verified, the result will follow. Let μ be admissible for $\gamma_+(E)$ and define $\mu_\tau(X) := \mu(X-\tau)$ for any $X \subseteq \mathbb{R}^{n+1}$ that is μ-measurable. This way μ_τ is clearly a positive Borel regular measure supported on $\tau(E)$ with $\mu_\tau(\tau(E)) = \mu(E)$. In addition, for any $\overline{x} \in \mathbb{R}^{n+1}$,

$$|P_s * \mu_\tau(\overline{x})| = \left| \int_{\tau(E)} P_s(\overline{x}-\overline{y})\mathrm{d}\mu_\tau(\overline{y}) \right| = \left| \int_E P_s(\overline{x}-\tau-\overline{u})\mathrm{d}\mu(\overline{u}) \right| = |P_s * \mu(\overline{x}-\tau)| \\ \leq 1,$$

implying that μ_τ is admissible for $\gamma_+(\tau(E))$ and we are done. To deal with E an arbitrary Borel subset of $\mathbb{R}^{n+1}$ just notice that by Theorem 4.2.1 admissible measures for γ_+ are locally finite and Borel regular, and thus Radon [**Matti, Corollary 1.11**]. So the quantity $\mu(E)$ can be computed as the limit $\lim_{k\to\infty} \mu(E_k)$, where E_k is a proper sequence of compact sets that approximates E.

The proof of *2* is analogous. Indeed, take the measure $\mu_\lambda(X) := \lambda^n \mu(\lambda^{-1}X)$ supported on λE and just notice that for any $\overline{x} \in \mathbb{R}^{n+1}$, by the explicit expression of P_s given in (1.1.1) and the fact that the dilation is s-parabolic,

$$|P_s * \mu_\lambda(\overline{x})| = \left| \int_{\lambda E} P_s(\overline{x}-\overline{y})\mathrm{d}\mu_\lambda(\overline{y}) \right| = \lambda^n \left| \int_E P_s(\overline{x}-\lambda\overline{u})\mathrm{d}\mu(\overline{u}) \right| = |P_s * \mu(\lambda^{-1}\overline{x})| \\ \leq 1.$$

Moving on to *3*, we proceed as in Lemma 4.1.13 and observe that it suffices to prove that there exists an admissible measure μ for $\gamma_+(\mathcal{E})$ so that for each test function φ, $\lim_{k\to\infty}\langle \mu_k, \varphi\rangle = \langle \mu, \varphi \rangle$. To construct such μ, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ and observe that $\langle \mu_k, \varphi \rangle = \langle \Theta^s P_s * \mu_k, \varphi \rangle = \langle P_s * \mu_k, \overline{\Theta}^s \varphi \rangle$. By assumption $P_s * \mu_k$ belongs to the unit ball of $L^\infty(\mathbb{R}^{n+1}) \cong L^1(\mathbb{R}^{n+1})^*$ and moreover, proceeding as in [**MatP, §3**], for example, and using that

$$(-\Delta)^s \varphi = c_{n,s} \sum_{i=1}^n \partial_{x_i} \left(\frac{1}{|x|^{n+2s-2}} \right) *_n \partial_{x_i} \varphi,$$

it is clear that $\overline{\Theta}^s \varphi \in L^1(\mathbb{R}^{n+1})$. By Banach-Alaoglu's theorem we may assume that there exists some $S \in L^\infty(\mathbb{R}^{n+1})$ with $\|S\|_\infty \leq 1$ and $P_s * \mu_k \to S$ as $k \to \infty$ in a weak*-L^∞ sense. Therefore,

$$\lim_{k\to\infty} \langle \mu_k, \varphi \rangle = \langle S, \overline{\Theta}^s \varphi \rangle, \qquad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}).$$

Let us define a distribution (a priori) μ acting on test functions as $\langle \mu, \varphi \rangle := \langle S, \overline{\Theta}^s \varphi \rangle$, so that we have $\lim_{k\to\infty}\langle \mu_k, \varphi \rangle = \langle \mu, \varphi \rangle$ for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$. Observe that by the latter identity, for any $\varphi \geq 0$ we have $\langle \mu, \varphi \rangle \geq 0$. It is not difficult to prove that such property implies that μ is a distribution of order 0 (we refer the reader to the proof of [**C, Theorem 2.7**], for example), so applying [**C, Theorem 2.5**] and Riesz's representation theorem, we deduce that in fact μ is a positive Radon measure. In addition, since the supports of μ_k are contained in E_k and $E_k \downarrow \mathcal{E}$, it follows that $\text{supp}(\mu) \subseteq \mathcal{E}$. Therefore, if we prove that $\|P_s * \mu\|_\infty \leq 1$ we will be done, since μ would become an admissible measure for $\gamma_+(\mathcal{E})$. Such estimate follows from the equality $P_s * \mu = S$, that can be verified exactly as in Lemma 4.1.13.

Finally we prove *4*. Abusing notation, let us set $E := \bigcup_{j=1}^\infty E_j$, which is also a Borel subset of $\mathbb{R}^{n+1}$, and fix $K \subset E \subset \mathbb{R}^{n+1}$ compact. Let μ be admissible for $\gamma_+(K)$. Observe that for any $X \subseteq \mathbb{R}^{n+1}$ μ-measurable, one has

$$\mu(X) = \mu\left(\bigcup_{j=1}^\infty (K \cap E_j) \cap X\right) = \sum_{j=1}^\infty \mu|_{K\cap E_j}(X),$$

so in particular, since K is also a Borel set and thus μ-measurable,

$$\mu(K) = \sum_{j=1}^\infty \mu|_{E_j}(K).$$

If we take the supremum over all admissible measures for $\gamma_+(K)$ on both sides of the previous inequality, we have

$$\gamma_+(K) \leq \sum_{j=1}^\infty \sup_{\substack{\text{supp}(\mu)\subseteq K \\ \|P_s*\mu\|_\infty \leq 1}} \mu|_{E_j}(K).$$

We claim that for each $j \geq 1$ the following is true:

$$\sup_{\substack{\text{supp}(\mu)\subseteq K \\ \|P_s*\mu\|_\infty \leq 1}} \mu|_{E_j}(K) \leq \gamma_+(E_j \cap K). \tag{4.2.1}$$

To verify such estimate we assume that it does not hold and reach a contradiction. So suppose that there exists μ admissible for $\gamma_+(K)$ with

$$\mu|_{E_j}(K) > \gamma_+(E_j \cap K).$$

Then, for any compact subset $F \subseteq E_j \cap K$ we have $\mu|_{E_j}(K) > \gamma_+(F)$. Clearly $\mu|_F$ is admissible for $\gamma_+(F)$. Indeed, for any $\overline{x} \in \mathbb{R}^{n+1}$, since $P_s \geq 0$,

$$|P_s * \mu|_F(\overline{x})| = \int_F P_s(\overline{x} - \overline{y})\text{d}\mu(\overline{y}) \leq \int_K P_s(\overline{x} - \overline{y})\text{d}\mu(\overline{y}) \leq \|P_s * \mu\|_\infty \leq 1,$$

and the Borel regularity follows from that of μ and [**Matti, Theorem 1.9**], that can be applied by the n-growth of μ. Thus $\gamma_+(F) \geq \mu(F)$. Hence, by hypothesis,

$$\mu(E_j \cap K) > \mu(F), \qquad \forall F \subseteq E_j \cap K \text{ with } F \text{ compact},$$

which contradicts that μ enjoys an inner regularity property, since it is a Radon measure on $\mathbb{R}^{n+1}$. Therefore, (4.2.1) must hold, which implies

$$\gamma_+(K) \leq \sum_{j=1}^\infty \gamma_+(E_j \cap K) \leq \sum_{j=1}^\infty \gamma_+(E_j).$$

Then, since K was any compact set contained in E, the desired estimate follows. $\square$

Remark 4.2.2. The argument we have presented for properties *1*, *2* and *3* can be easily adapted for general distributions. So, in particular, it can be checked that γ_{Θ^s} also enjoys the outer regularity property.

The next result (that can be also generalized for distributions) describes the behavior of $\gamma_{\Theta^s,+}$ under canonical reflections of $\mathbb{R}^{n+1}$.

Theorem 4.2.3. *Let $E \subset \mathbb{R}^{n+1}$ be a Borel set and for each $i \in \{1,\ldots,n\}$ denote by $\mathcal{R}_i$ the reflection with respect to the hyperplane $\{x_i = 0\}$, and by $\mathcal{R}_t$ the reflection with respect to $\{t = 0\}$. Then,*

$$\gamma_{\Theta^s,+}(E) = \gamma_{\Theta^s,+}(\mathcal{R}_i(E)), \qquad 1 \leq i \leq n,$$

and moreover,

$$\gamma_{\Theta^s,+}(E) = \gamma_{\overline{\Theta}^s,+}(\mathcal{R}_t(E)).$$

Proof. Fix $i \in \{1,\ldots,n\}$ and check, as in the proofs of properties *1* and *2* of Theorem 4.2.2, that for any μ admissible for and $\gamma_{\Theta^s,+}(E)$, there is μ_i, admissible for $\gamma_{\Theta^s,+}(\mathcal{R}_i(E))$, such that $\mu(E) = \mu_i(\mathcal{R}_i(E))$. So we fix μ admissible for $\gamma_{\Theta^s,+}(E)$ and define

$$\mu_i(X) := \mu\big(\mathcal{R}_i^{-1}(X)\big), \qquad \forall X \subseteq \mathbb{R}^{n+1} \; \mu\text{-measurable}.$$

Again, μ_i is a positive Borel regular measure supported on $\mathcal{R}_i(E)$ such that $\mu(E) = \mu_i(\mathcal{R}_i(E))$. Finally, to verify the admissibility of μ_i, notice that for any $\overline{x} \in \mathbb{R}^{n+1}$,

$$|P_s * \mu_i(\overline{x})| = \int_{\mathcal{R}_i(E)} P_s(\overline{x}-\overline{y})\mathrm{d}\mu_i(\overline{y}) = \int_E P_s\big(\overline{x}-\mathcal{R}_i(\overline{u})\big)\mathrm{d}\mu(\overline{u}).$$

Observe that $\mathcal{R}_i(\overline{u}) = (u_1,\ldots,-u_i,\ldots,u_{n+1})$, so by the definition of P_s in (1.1.1),

$$\int_E P_s\big(\overline{x}-\mathcal{R}_i(\overline{u})\big)\mathrm{d}\mu(\overline{u}) = \int_E P_s\big(\mathcal{R}_i(\overline{x})-\overline{u}\big)\mathrm{d}\mu(\overline{u}) = P_s * \mu(\mathcal{R}_i(\overline{x})) \leq 1,$$

that is what we wanted to prove. On the other hand, if $i = n+1$, that is, if $x_i = t$, the computations are similar, but the role of the indicator function is responsible for a change of Θ^s into $\overline{\Theta}^s$:

$$|P_s^* * \mu_{n+1}(\overline{x})| = \int_E P_s^*\big(\overline{x}-\mathcal{R}_t(\overline{u})\big)\mathrm{d}\mu(\overline{u}) = \int_E P_s\big(\mathcal{R}_t(\overline{x})-\overline{u}\big)\mathrm{d}\mu(\overline{u}) = P_s * \mu(\mathcal{R}_t(\overline{x})) \leq 1,$$

that implies the desired result. □

Remark 4.2.3. Observe that combining the translation invariance and Theorem 4.2.3, the latter result also holds for any affine canonical reflection. That is, any reflection with respect to hyperplanes of the form $\{x_i = c\}$ or $\{t = c\}$, for any $c \in \mathbb{R}$. This implies, in particular, that if E presents any temporal axis of symmetry, then its $\gamma_{\Theta^s,+}$ and $\gamma_{\overline{\Theta}^s,+}$ capacities coincide.

Remark 4.2.4. Notice that we have also obtained that if the measure μ satisfies the condition $\|P_s * \mu\|_\infty \leq 1$, then

$$\|P_s * \mu_\tau\|_\infty \leq 1 \quad \text{and} \quad \|P_s * \mu_i\|_\infty \leq 1, \;\; \text{for} \;\; i = 1,2,\ldots,n.$$

4.2.2 Comparability between $\gamma_{\Theta^s,+}$ and $\widetilde{\gamma}_{\Theta^s,+}$

One of the main characteristics of the kernels P_s and P_s^* is the presence of an indicator function with respect to the t-variable. Such fact seems to endow the temporal axis with a distinct feature when it comes to constructing removable sets for the Θ^s-equation, as it is exemplified in [**MatP, Proposition 6.1**] with the vertical line segment $\{0\} \times [0,1]$ for the case $s = 1/2$. And what about the time-reflected line segment $\{0\} \times [0,-1]$? It is clear, by the translation invariance of $\gamma_{\Theta^{1/2},+}$, that its capacity is equally 0.

When trying to find a subset $E \subset \mathbb{R}^{n+1}$ with non-comparable $\gamma_{\Theta^s,+}$ and $\gamma_{\overline{\Theta}^s,+}$ capacities, the above trivial observation suggests that it may be not possible. In fact, the following result was a first motivation to carry out the study of the present subsection:

THEOREM 4.2.4. *The $\gamma_{\Theta^{1/2},+}$ capacity of any non-horizontal line segment is null.*

Proof. It is clear that we may assume $n = 1$, that is, the ambient space is $\mathbb{R}^2$. Denote by E the unit segment with one of its end-points at the origin and with angle $\alpha \in (0,\pi)$ between the positive direction of the x-axis and E. We shall follow the same method of proof given for [**MatP, Proposition 6.1**], that is: we will assume $\gamma_{\Theta^{1/2},+}(E) > 0$ and reach a contradiction.

Under the previous assumption, there exists an admissible measure μ for $\gamma_{\Theta^{1/2},+}(E)$ with $\mu(E) > 0$. Let us parameterize E as $u \mapsto (u\cos\alpha, u\sin\alpha)$, $u \in [0,1]$, and note that since μ has linear growth, given $\eta > 0$ we can take $c \in (0,1)$ such that

$$\mu\big(\{(u\cos\alpha, u\sin\alpha) \;:\; c \leq u \leq 1\}\big) < \eta.$$

Writing explicitly the normalization condition $\|P * \mu\|_\infty \leq 1$, we have

$$P * \mu(\overline{x}) = \int_0^1 \frac{t - u\sin\alpha}{\big(x - u\cos\alpha\big)^2 + \big(t - u\sin\alpha\big)^2}\, \chi_{\{t-u\sin\alpha>0\}} \mathrm{d}\mu(u) \leq 1,$$

for $\mathcal{L}^2$-a.e. $\overline{x} \in \mathbb{R}^2$. Therefore, if we set $F := \{(u\cos\alpha, u\sin\alpha) \;:\; 0 \leq u < c\}$ and choose $\overline{x} = (u_0\cos\alpha, u_0\sin\alpha) \in F$, we get

$$\sin\alpha \int_0^{u_0} \frac{\mathrm{d}\mu(u)}{u_0 - u} \leq 1.$$

So for any $\overline{x} \in F$ there exists $\ell = \ell(\overline{x}) > 0$ such that

$$\sin\alpha \int_{u_0-\ell}^{u_0} \frac{\mathrm{d}\mu(u)}{u_0 - u} \leq \eta.$$

Hence, since F is an interval, there exists a finite number of almost disjoint intervals I_j with $|I_j| = \ell_j = \ell(\overline{x}_j)$ such that $F \subset \bigcup_{j=1}^N I_j$ and

$$\mu(I_j) = \int_{u_{0,j}-\ell_j}^{u_{0,j}} \mathrm{d}\mu(u) \leq \int_{u_{0,j}-\ell_j}^{u_{0,j}} \frac{\ell_j}{u_{0,j} - u} \mathrm{d}\mu(u) \leq \ell_j \frac{\eta}{\sin\alpha}.$$

All in all,

$$\mu(E) < \mu(F) + \eta \lesssim \sum_{j=1}^{N} \mu(I_j) + \eta \leq \eta\left(\frac{1}{\sin\alpha}\sum_{j=1}^{N}\ell_n + 1\right) \lesssim \eta\left(\frac{c}{\sin\alpha} + 1\right),$$

and this leads to a contradiction, since η can be chosen arbitrarily small. Therefore, $\gamma_{\Theta^{1/2},+}(E) = 0$ and this, together with the first and fourth properties of Theorem 4.2.2 suffices to generalize the result for any other line segment. □

REMARK 4.2.5. We have proved this result only for $\gamma_{\Theta^{1/2},+}$ just for the sake of simplicity and to focus our study on its properties. However, by exactly the same method of proof of [**MatP, Proposition 6.1**] (involving the approximation of distributions by signed measures) one can obtain the same result for $\gamma_{\Theta^{1/2}}$.

The second aspect that motivated the study of the comparability between $\gamma_{\Theta^s,+}$ and $\widetilde{\gamma}_{\Theta^s,+}$ is related to the different equivalent definitions admitted by the latter capacity. We stress that in the proofs given for the forthcoming results, we will exploit the fact that P_s *is a nonnegative kernel*. To ease notation, let us simply set

As it is pointed out in [**MatP, §4**] for the case $s = 1/2$, one of the main advantages of working with $\widetilde{\gamma}_{\Theta^s,+}$ instead of just $\gamma_{\Theta^s,+}$ is that it can be characterized in a similar manner as those capacities defined through anti-symmetric kernels, by means of the L^2-bound of a particular operator. To make such property explicit, we shall first introduce some notation.

For a given real compactly supported Borel regular measure μ with n-growth, we define the operator $\mathscr{P}^s_\mu$ acting on elements of $L^1_{\text{loc}}(\mu)$ as

$$\mathscr{P}^s_\mu f(\overline{x}) := \int_{\mathbb{R}^{n+1}} P_s(\overline{x}-\overline{y})f(\overline{y})\mathrm{d}\mu(\overline{y}), \qquad \overline{x} \notin \operatorname{supp}(\mu),$$

together with its truncated version $\mathscr{P}^s_{\mu,\varepsilon}$ and all the corresponding definitions relative to $L^p(\mu)$-bounds, maximal and conjugate operators presented in the introductory chapter. Let us state a crucial property of the capacity $\widetilde{\gamma}_{\Theta^s,+}$. Recall that $\mu \in \Sigma^s_n(E)$ denoted the set of positive Borel measures supported on E with upper s-parabolic n-growth with constant 1.

THEOREM 4.2.5. *For any $E \subset \mathbb{R}^{n+1}$ compact set and $s \in (0,1]$,*

$$\widetilde{\gamma}_{\Theta^s,+}(E) \approx \gamma_{2,+}(E) := \sup\Big\{\mu(E) \;:\; \mu \in \Sigma^s_n(E),\; \|\mathscr{P}^s_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1\Big\}.$$

Proof. The argument is analogous to that of Theorem 2.3.3. Applying Theorem 1.1.3 with $\beta = 0$ we get that P_s is an n-dimensional C-Z convolution kernel in the s-parabolic space $\mathbb{R}^{n+1}$. The latter combined with the n-growth of admissible measures for $\widetilde{\gamma}_{\Theta^s,+}(E)$ allows to carry out the same proof. □

Observe that in the argument of Lemma 2.3.4, checked for the operator $\mathcal{P}^s_\mu$ and also valid with an analogous proof for $\mathscr{P}^s_\mu$, it is proved that if $\mu \in \Sigma^s_n(E)$,

$$\text{if} \quad \|P_s * \mu\|_\infty \leq 1, \quad \text{then} \quad \|P_s * \mu\|_{L^\infty(\mu)} \lesssim 1, \tag{4.2.2}$$

and similarly for P_s^*.

We move on with our study introducing, for any $\overline{x}\neq 0$, the following kernel

$$P_{s,\text{sy}}(\overline{x}) := \frac{1}{2}\big[P_s(\overline{x})+P_s(-\overline{x})\big] = c_{n,s}|t|^{-\frac{n}{2s}}\phi_{n,s}\big(|xt|^{-\frac{1}{2s}}\big).$$

Notice that for the particular case $s=1/2$ we get, for $\overline{x}\neq 0$,

$$P_{\text{sy}}(\overline{x}) := P_{1/2,\text{sy}}(\overline{x}) = c_{n,1/2}\frac{|t|}{|\overline{x}|^{n+1}}.$$

LEMMA 4.2.6. *Let $E\subset\mathbb{R}^{n+1}$ be compact and define*

$$\gamma_{\text{sy},+}(E) := \sup_{\mu}\Big\{\mu(E)\ :\ \mu\in\Sigma_n^s(E),\ \|P_{s,\text{sy}}*\mu\|_\infty\leq 1\Big\}.$$

Then,

$$\frac{1}{2}\gamma_{\text{sy},+}(E)\leq\widetilde{\gamma}_{\Theta^s,+}(E)\leq\gamma_{\text{sy},+}(E).$$

Proof. Take μ any admissible measure for $\widetilde{\gamma}_{\Theta^s,+}(E)$ and observe that

$$\|P_{s,\text{sy}}*\mu\|_\infty\leq\frac{1}{2}\Big[\|P_s*\mu\|_\infty+\|P_s^**\mu\|_\infty\Big]\leq 1,$$

that yields $\widetilde{\gamma}_{\Theta^s,+}(E)\leq\gamma_{\text{sy},+}(E)$. Conversely, if we consider any $\mu\in\Sigma(E)$ with $\|P_{s,\text{sy}}*\mu\|_\infty\leq 1$, since P_s is nonnegative, $P_s\leq 2P_{s,\text{sy}}$ and $P_s^*\leq 2P_{s,\text{sy}}$, and therefore

$$\|P_s*\mu\|_\infty\leq 2,\qquad\|P_s^**\mu\|_\infty\leq 2.$$

So $\mu/2$ becomes admissible for $\widetilde{\gamma}_{\Theta^s,+}(E)$ and we deduce the remaining inequality. □

Therefore, we can encapsulate the different ways to understand $\widetilde{\gamma}_{\Theta^s,+}$ in the following corollary:

COROLLARY 4.2.7. *For $E\subset\mathbb{R}^{n+1}$ compact set and $s\in(0,1]$,*

$$\widetilde{\gamma}_{\Theta^s,+}(E)\approx\gamma_{2,+}(E)\approx\gamma_{\text{sy},+}(E).$$

To be able to compare $\gamma_{\Theta^s,+}$ and $\widetilde{\gamma}_{\Theta^s,+}$, we recall Definition 4.1.1 and observe that if $f\in L^\infty(\mu)$, then $f\in\text{BMO}_\rho(\mu),\forall\rho>1$. Also, given $a\in\mathbb{R}$ and $Q\subset\mathbb{R}^{n+1}$ cube,

$$\begin{aligned}\int_Q&|f(\overline{x})-f_{Q,\mu}|\text{d}\mu(\overline{x})\\&\leq\bigg[\int_Q\big|f(\overline{x})-a\big|\text{d}\mu(\overline{x})+\int_Q\big|a-f_{Q,\mu}\big|\text{d}\mu(\overline{x})\bigg]\leq 2\int_Q\big|f(\overline{x})-a\big|\text{d}\mu(\overline{x}).\end{aligned}$$

Therefore, if for each cube Q with $\mu(Q)\neq 0$ we are able to find c_Q so that

$$\frac{1}{\mu(\rho Q)}\int_Q\big|f(\overline{x})-c_Q\big|\text{d}\mu(\overline{x})\leq c,$$

where c is constant independent of Q, we deduce that f belongs to $\mathrm{BMO}_\rho(\mu)$.

We clarify that in the forthcoming Definition 4.2.2 and Lemma 4.2.8, μ will be a positive compactly supported Borel regular measure on $\mathbb{R}^{n+1}$ with upper s-parabolic n-growth. The previous conditions ensure that the degree of growth of μ is the same as the degree of homogeneity of P_s. Moreover, as we have pointed out in the proof of Theorem 4.2.2, μ is locally finite and therefore becomes a Radon measure.

Definition 4.2.2. We will say that the operator $\mathscr{P}^s_\mu$ is *μ-weakly bounded* if for any s-parabolic cube $Q \subset \mathbb{R}^{n+1}$,

$$|\langle \mathscr{P}^s_{\mu,\varepsilon}\chi_Q, \chi_Q\rangle| := \left| \int_Q \left(\int_{Q\cap\{|\overline{x}-\overline{y}|>\varepsilon\}} P(\overline{x}-\overline{y})\mathrm{d}\mu(\overline{y}) \right) \mathrm{d}\mu(\overline{x}) \right| \leq c\mu(2Q),$$

uniformly on $\varepsilon > 0$.

Lemma 4.2.8. *Let $s \in (0,1]$ and assume that $\|P_s * \mu\|_{L^\infty(\mu)} \leq 1$. Then, the operator $\mathscr{P}^s_\mu$ is μ-weakly bounded and $P^*_s * \mu \in \mathrm{BMO}_\rho(\mu)$ for $\rho \geq 2$.*

Proof. Condition $\|P_s * \mu\|_{L^\infty(\mu)} \leq 1$ can be simply rewritten as $\sup_{\varepsilon>0} \|\mathscr{P}^s_\varepsilon \mu\|_{L^\infty(\mu)} \leq 1$; so for any $\varepsilon > 0$, by the nonnegativity of P_s and μ,

$$\left| \int_Q \left(\int_{Q\cap\{|\overline{x}-\overline{y}|>\varepsilon\}} P_s(\overline{x}-\overline{y})\mathrm{d}\mu(\overline{y}) \right) \mathrm{d}\mu(\overline{x}) \right| \leq \|\mathscr{P}^s_\varepsilon \mu\|_{L^\infty(\mu)}\mu(Q) \leq \mu(2Q).$$

Hence $\mathscr{P}^s_\mu$ is μ-weakly bounded. Finally, notice that by Tonelli's theorem,

$$\int_{\mathbb{R}^{n+1}} P^*_s * \mu(\overline{x})\mathrm{d}\mu(\overline{x}) = \int_{\mathbb{R}^{n+1}} P_s * \mu(\overline{y})\mathrm{d}\mu(\overline{y}) \leq \mu(\mathbb{R}^{n+1}) < \infty,$$

so $P^*_s * \mu \in L^1(\mu) \subset L^1_{\mathrm{loc}}(\mu)$, and $P^*_s * \mu$ is indeed a candidate to belong to $\mathrm{BMO}_\rho(\mu)$. To estimate its $\mathrm{BMO}_\rho(\mu)$ norm, fix an s-parabolic cube $Q = Q(\overline{x}_0, \ell(Q))$, $\mu(Q) \neq 0$ and consider the characteristic function χ_{2Q} associated with $2Q$. Denote also $\chi_{2Q^c} := 1 - \chi_{2Q}$. Consider the constant

$$c_Q := P^*_s * (\chi_{2Q^c}\mu)(\overline{x}_0),$$

that is an expression pointwise well-defined, since $\overline{x}_0 \notin \mathrm{supp}(\chi_{2Q^c}\mu)$. Indeed,

$$c_Q = \int_{\mathbb{R}^{n+1}\setminus 2Q} P^*_s(\overline{x}_0 - \overline{z})\mathrm{d}\mu(\overline{z}) \leq \int_{\mathbb{R}^{n+1}\setminus 2Q} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}_0 - \overline{z}|^n} \leq \frac{\mu(\mathbb{R}^{n+1})}{(2\ell(Q))^n} < \infty.$$

Observe that the following estimate holds

$$\begin{aligned}
&\frac{1}{\mu(\rho Q)}\int_Q |P_s^* * \mu(\overline{y}) - c_Q|\mathrm{d}\mu(\overline{y}) \\
&\qquad \leq \frac{1}{\mu(\rho Q)}\int_Q P_s^* * (\chi_{2Q}\mu)(\overline{y})\mathrm{d}\mu(\overline{y}) \\
&\qquad\qquad + \frac{1}{\mu(\rho Q)}\int_Q |P_s^* * (\chi_{2Q^c}\mu)(\overline{y}) - P_s^* * (\chi_{2Q^c}\mu)(\overline{x}_0)|\mathrm{d}\mu(\overline{y}) \\
&\qquad \leq \frac{1}{\mu(\rho Q)}\int_Q \bigg(\int_{2Q} P_s^*(\overline{y}-\overline{z})\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\mu(\overline{y}) \\
&\qquad\qquad + \frac{1}{\mu(\rho Q)}\int_Q \bigg(\int_{\mathbb{R}^{n+1}\setminus 2Q} |P_s^*(\overline{y}-\overline{z}) - P_s^*(\overline{x}_0-\overline{z})|\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\mu(\overline{y}) =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

For I, by Tonelli's theorem and the nonnegativity of P_s together with the assumptions $\|P_s * \mu\|_{L^\infty(\mu)} \leq 1$ and $\rho \geq 2$,

$$\mathrm{I} \leq \frac{1}{\mu(\rho Q)}\int_{2Q} P_s * \mu(\overline{z})\mathrm{d}\mu(\overline{z}) \leq \frac{\mu(2Q)}{\mu(\rho Q)} \leq 1.$$

For II, apply the fifth estimate of Theorem 1.1.3 with $\beta := 0$, Theorem 4.2.1 and split the domain of integration into annuli to obtain

$$\begin{aligned}
\mathrm{II} &\lesssim \frac{1}{\mu(\rho Q)}\int_Q \bigg(\int_{\mathbb{R}^{n+1}\setminus 2Q} \frac{|\overline{y}-\overline{x}_0|^{2\zeta}}{|\overline{z}-\overline{x}_0|^{n+2\zeta}}\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\mu(\overline{y}) \\
&\leq \ell(Q)^{2\zeta}\frac{\mu(Q)}{\mu(\rho Q)}\int_{\mathbb{R}^{n+1}\setminus 2Q}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|^{n+2\zeta}} \leq \ell(Q)^{2\zeta}\sum_{j=1}^\infty \int_{2^{j+1}Q\setminus 2^jQ}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|^{n+2\zeta}} \\
&\lesssim \ell(Q)^{2\zeta}\sum_{j=1}^\infty \frac{(2^{j+1}\ell(Q))^n}{(2^j\ell(Q))^{n+2\zeta}} \lesssim \sum_{j=1}^\infty \frac{1}{2^j} = 1,
\end{aligned}$$

and the desired result follows. □

The previous lemma allows us to prove the main result of this subsection:

THEOREM 4.2.9. *Let $s \in (0,1]$ and $E \subset \mathbb{R}^{n+1}$ be a compact set. Then,*

$$\gamma_{\Theta^s,+}(E) \approx \widetilde{\gamma}_{\Theta^s,+}(E).$$

Proof. It suffices to prove $\gamma_{\Theta^s,+}(E) \lesssim \widetilde{\gamma}_{\Theta^s,+}(E)$. To this end, consider μ an admissible measure for $\gamma_{\Theta^s,+}(E)$, i.e. μ is a positive Borel regular measure supported on E such that $\|P_s * \mu\|_\infty \leq 1$. We know by Theorem 4.2.1 that μ has upper s-parabolic n-growth with an absolute constant $C > 0$. Hence, up to such constant, $\mu \in \Sigma_n^s(E)$. Moreover, by relation (4.2.2) and Lemma 4.2.8 we have, for any $\rho \geq 2$,

1. $\mathscr{P}^s\mu \in L^\infty(\mu)$ and thus $\mathscr{P}^s\mu \in \mathrm{BMO}_\rho(\mu)$,
2. $\mathscr{P}^{s,*}\mu \in \mathrm{BMO}_\rho(\mu)$,
3. $\mathscr{P}^s\mu$ is μ-weakly bounded.

Applying a suitable $T1$-theorem [**HyMar, Theorem 2.3**], we deduce $\|\mathscr{P}^s_\mu\|_{L^2(\mu)\to L^2(\mu)} \lesssim 1$. This implies (again, up to constant factor) that μ is admissible for $\gamma_{2,+}(E)$; and by the arbitrariness of μ we have $\gamma_{\Theta^s,+}(E) \lesssim \gamma_{2,+}(E)$. So we conclude, by Theorem 4.2.5,

$$\gamma_{\Theta^s,+}(E) \lesssim \gamma_{2,+}(E) \approx \widetilde{\gamma}_{\Theta^s,+}(E).$$

□

A particular consequence of the last result is

$$\gamma_{\Theta^s,+}(E) \approx \gamma_{\text{sy},+}(E).$$

Since the same proof of Theorem 4.2.3 yields that $\gamma_{\text{sy},+}$ is invariant under temporal reflections, we also obtain

$$\gamma_{\Theta^s,+}(E) \approx \gamma_{\text{sy},+}(E) = \gamma_{\text{sy},+}(\mathcal{R}_t(E)) \approx \gamma_{\Theta^s,+}(\mathcal{R}_t(E)) = \gamma_{\overline{\Theta}^s,+}(E).$$

COROLLARY 4.2.10. *For $s \in (0,1]$, the capacities $\gamma_{\Theta^s,+}, \gamma_{\overline{\Theta}^s,+}$ and $\gamma_{\text{sy},+}$ are comparable.*

4.2.3 The $\gamma_{\Theta^s,+}$ capacity of Cantor sets

Let us fix $s \in (0,1]$ and finally estimate the $(s,+)$-caloric capacity of a certain family of Cantor sets, which generalize the example given in [**MatP, §5**]. In such example we are presented with a Cantor set E inspired by the one constructed in [**Gar1, Chapter IV, §2**] with positive $\mathcal{H}^n$-measure and removable for the $\Theta^{1/2}$-equation, meaning that $\gamma_{\Theta^{1/2}}(E) = 0$. Our goal will be to generalize the above example and study its $\gamma_{\Theta^s,+}$ capacity.

Let $\lambda = (\lambda_j)_j$ be a sequence of real numbers satisfying $0 < \lambda_j < 1/2$. We shall define its usual corner-like Cantor set $E \subset \mathbb{R}^{n+1}$ by the following algorithm (it will be *simpler* version of the Cantor set defined in §2.3.1). Set $Q^0 := [0,1]^{n+1}$ the unit cube of $\mathbb{R}^{n+1}$ and consider 2^{n+1} disjoint s-parabolic cubes inside Q^0 of side length $\ell_1 := \lambda_1$, with sides parallel to the coordinate axes and such that each cube contains a vertex of Q^0. Repeat this splitting for each of the 2^{n+1} cubes from the previous step, now with a contraction ratio λ_2. This way we obtain $2^{2(n+1)}$ s-parabolic cubes with side length $\ell_2 := \lambda_1\lambda_2$. Proceeding inductively, at the k-th step of the iteration we encounter $2^{k(n+1)}$ s-parabolic cubes, denoted Q^k_j for $1 \leq j \leq 2^{k(n+1)}$, with side length $\ell_k := \prod_{j=1}^k \lambda_j$. We will refer to them as cubes of the *k-th generation*. We define

$$E_k = E(\lambda_1, \ldots, \lambda_k) := \bigcup_{j=1}^{2^{k(n+1)}} Q^k_j,$$

and from the latter we obtain the Cantor set associated with λ,

$$E = E(\lambda) := \bigcap_{k=1}^{\infty} E_k. \tag{4.2.3}$$

If we chose $s = 1/2$ and $\lambda_j = 2^{-(n+1)/n}$ for every j, we would recover the particular Cantor set presented in [**MatP, §5**]. We are first concerned with studying the Hausdorff dimension of E in terms of λ, which we want it to be n, the critical dimension for γ_{Θ^s} (see Remark 4.2.1). An example of restriction one imposes to $(\lambda_j)_j$ to ensure the latter, as it is done in [**MatT, T4**], is the following

$$\lim_{k\to\infty} \ell_k\, 2^{k(n+1)/n} = 1.$$

Observe that a particular consequence of the previous equality is that there exists a constant $C > 0$ depending on λ so that

$$\ell_k 2^{k(n+1)/n} \geq C, \qquad \forall k \geq 1.$$

Using such property one deduces $\mathcal{H}^n_{p_s}(E) > 0$. Indeed, consider the probability measure μ defined on E such that for each generation k, $\mu(Q_j^k) := 2^{-k(n+1)}$, $1 \leq j \leq 2^{k(n+1)}$. Let Q be any s-parabolic cube, that we may assume to be contained in Q^0, and pick k with $\ell_{k+1} \leq \ell(Q) \leq \ell_k$, so that Q can meet, at most, 2^{n+1} cubes Q_j^k. Thus $\mu(Q) \leq 2^{-(k-1)(n+1)}$ and we deduce

$$\mu(Q) \leq \frac{2^{2(n+1)}}{2^{(k+1)(n+1)}} = 2^{2(n+1)}\Big(\ell_{k+1}2^{(k+1)(n+1)/n}\Big)^{-n}\ell_{k+1}^n \leq \frac{2^{2(n+1)}}{C^n}\ell(Q)^n \simeq \ell(Q)^n, \tag{4.2.4}$$

meaning that μ presents upper s-parabolic n-growth. Therefore, by [**Gar1, Chapter IV, Lemma 2.1**] we get $\mathcal{H}^n_{p_s}(E) \geq \mathcal{H}^n_{\infty,p_s}(E) > 0$. Moreover, observe that for a fixed $0 < \varepsilon \ll 1$, there is k large enough so that $\text{diam}_{p_s}(Q_j^k) \leq \varepsilon$ and $\ell_k\, 2^{k(n+1)/n} \leq 2$. Thus, as E_k defines a covering of E admissible for $\mathcal{H}^n_{\varepsilon,p_s}$, we get

$$\mathcal{H}^n_{\varepsilon,p_s}(E) \leq \sum_{j=1}^{2^{k(n+1)}} \text{diam}_{p_s}(Q_j^k)^n \simeq \ell_k^n\, 2^{k(n+1)} = \big(\ell_k\, 2^{k(n+1)/n}\big)^n \leq 2^n.$$

Since this procedure can be done for any ε, we also have $\mathcal{H}^n_{p_s}(E) < \infty$ and thus $\dim_{\mathcal{H}_{p_s}}(E) = n$ as wished.

In order to state in a more compact way the results we are interested in, we introduce the following density for each $k \geq 1$:

$$\theta_k := \frac{1}{\ell_k^n\, 2^{k(n+1)}} = \frac{\mu(Q^k)}{\ell_k^n},$$

where μ is the probability measure defined above and Q^k is any s-parabolic cube of the k-th generation. We also set $\theta_0 := 1$.

Theorem 4.2.11. *Let $(\lambda_j)_j$ be a sequence of real numbers satisfying $0 < \lambda_j \leq \tau_0 < 1/2$ for every j, and E its associated Cantor set as in (4.2.3). Then, for each generation k,*

$$\gamma_{\Theta^s,+}(E_k) \lesssim_{\tau_0} \left(\sum_{j=0}^k \theta_j\right)^{-1}.$$

Proof. Fix a positive integer k and μ the probability measure supported on E_k defined as

$$\mu_k := \frac{1}{|E_k|}\mathcal{L}^{n+1}|_{E_k} := \frac{1}{\mathcal{L}^{n+1}(E_k)}\mathcal{L}^{n+1}|_{E_k}.$$

Observe that for every Q_i^j of the j-th generation, with $0 \leq j \leq k$ and $1 \leq i \leq 2^{j(n+1)}$,

$$\mu_k(Q_i^j) = \frac{1}{2^{k(n+1)}\ell_k^{n+2s}}2^{(k-j)(n+1)}\ell_k^{n+2s} = 2^{-j(n+1)}.$$

Fix any $\overline{x} = (x,t) \in E_k$ and consider the corresponding chain of s-parabolic cubes associated with $\overline{x}$,

$$\overline{x} \in \Delta_k \subset \Delta_{k-1} \subset \cdots \subset \Delta_1 \subset \Delta_0 =: Q^0,$$

where Δ_j is the unique s-parabolic cube of the family E_j that contains $\overline{x}$. Observe that

$$\begin{aligned}\int_{E_k} P_{s,\text{sy}}(\overline{y}-\overline{x})\mathrm{d}\mu_k(\overline{y}) &= \int_{\Delta_0} P_{s,\text{sy}}(\overline{y}-\overline{x})\mathrm{d}\mu_k(\overline{y})\\ &= \sum_{j=0}^{k-1}\int_{\Delta_j\setminus\Delta_{j+1}} P_{s,\text{sy}}(\overline{y}-\overline{x})\mathrm{d}\mu_k(\overline{y}) + \int_{\Delta_k} P_{s,\text{sy}}(\overline{y}-\overline{x})\mathrm{d}\mu_k(\overline{y}) =: \sum_{j=1}^{k-1} I_j + I_k.\end{aligned}$$

If for each $0 \leq j \leq k-1$ we write $\widetilde{\Delta}_{j+1}$ the cube of E_{j+1} contained in Δ_j diagonally opposite to Δ_{j+1}, we have for $0 < s < 1$,

$$\begin{aligned}I_j &\simeq \int_{\Delta_j\setminus\Delta_{j+1}} |t-\tau|^{-\frac{n}{2s}}\phi_{n,s}\big(|x-y||t-\tau|^{-\frac{1}{2s}}\big)\,\mathrm{d}\mu_k(\overline{y}) \approx \int_{\widetilde{\Delta}_{j+1}} \frac{|s-t|}{|\overline{y}-\overline{x}|_{p_s}^{n+2s}}\mathrm{d}\mu_k(\overline{y})\\ &\gtrsim \frac{\ell_j^{2s}-2\ell_{j+1}^{2s}}{\ell_j^{n+2s}}\mu_k(\widetilde{\Delta}_{j+1}) = \frac{1}{\ell_j^n}(1-2\lambda_{j+1}^{2s})2^{-(j+1)(n+1)} \gtrsim \theta_j(1-2\tau_0^{2s}) \simeq \theta_j.\end{aligned}$$

where we have applied relations (1.1.1) and (1.1.2). If $s=1$ we similarly get

$$I_j \simeq \int_{\Delta_j\setminus\Delta_{j+1}} |t-\tau|^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4|t-\tau|}}\,\mathrm{d}\mu_k(\overline{y}) \gtrsim \frac{e^{-C(n,\tau_0)}}{\ell_j^n}\mu_k(\widetilde{\Delta}_{j+1}) \gtrsim \theta_j.$$

Regarding I_k, consider the s-parabolic cube $Q_{\overline{x}} := Q(\overline{x}, 2\,\text{diam}_{p_s}(\Delta_k))$, that clearly contains Δ_k, and for each positive integer j write

$$F_j := Q_{\overline{x}} \cap \big\{(y,\tau)\,:\,|t-\tau|^{\frac{1}{2s}} > 2^{-j}\text{diam}_{p_s}(\Delta_k)\big\},$$

as well as $F_0 := \varnothing$. Set $\widehat{F}_j := F_{j+1}\setminus F_j$, and notice that $\{\widehat{F}_j\}_{j\geq 0}$ is a disjoint open covering of $Q_{\overline{x}}$. Therefore, since $\{\widehat{F}_j \cap \Delta_k\}_{j\geq 0}$ is also a disjoint covering of Δ_k, for $s<1$ we have

$$\begin{aligned}I_k &\approx \frac{1}{|E_k|}\int_{\Delta_k}\frac{|t-s|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s}}\,\mathrm{d}\overline{y} \gtrsim \frac{1}{|E_k|\ell_k^{n+2s}}\sum_{j=0}^{\infty}\int_{\widehat{F}_j\cap\Delta_k}|t-s|\,\mathrm{d}\overline{y}\\ &\gtrsim \frac{1}{|E_k|\ell_k^n}\sum_{j=0}^{\infty}\frac{|\widehat{F}_j\cap\Delta_k|}{2^{2sj}} \simeq \frac{\ell_k^{2s}}{|E_k|}\sum_{j=0}^{\infty}\frac{1}{2^{4sj}} \simeq \frac{1}{2^{k(n+1)}\ell_k^n} = \theta_k.\end{aligned}$$

If $s = 1$ we similarly have

$$I_k \simeq \frac{1}{|E_k|}\int_{\Delta_k} |t-\tau|^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4|t-\tau|}} \mathrm{d}\overline{y} \gtrsim \frac{1}{|E_k|\ell_k^n}\sum_{j=0}^{\infty}\int_{\widehat{F}_j\cap\Delta_k} e^{-\frac{1}{2^{2sj}}} \mathrm{d}\overline{y}$$
$$\simeq \frac{\ell_k^{2s}}{|E_k|}\sum_{j=0}^{\infty}\frac{e^{-\frac{1}{2^{2sj}}}}{2^{2sj}} \simeq \frac{1}{2^{k(n+1)}\ell_k^n} = \theta_k.$$

Thus, there exists some positive constant $C = C(n, s, \tau_0)$ such that

$$\int_{E_k} P_{s,\mathrm{sy}}(\overline{y}-\overline{x})\mathrm{d}\mu_k(\overline{y}) \geq C^{-1}\sum_{j=0}^{k}\theta_j, \qquad \forall \overline{x}\in E_k.$$

The last inequality can be rewritten as

$$1 \leq C\left(\sum_{j=0}^{k}\theta_j\right)^{-1}\int_{E_k} P_{s,\mathrm{sy}}(\overline{y}-\overline{x})\mathrm{d}\mu_k(\overline{y}), \qquad \forall \overline{x}\in E_k. \tag{4.2.5}$$

Take ν any admissible measure for $\gamma_{\mathrm{sy},+}(E_k)$. Integrating on both sides of equation (4.2.5) and applying Tonelli's theorem together with $\|P_{s,\mathrm{sy}} * \nu\|_\infty \leq 1$,

$$\nu(E_k) \leq C\left(\sum_{j=0}^{k}\theta_j\right)^{-1}\int_{E_k} P_{s,\mathrm{sy}} * \nu(\overline{y})\mathrm{d}\mu_k(\overline{y}) \leq C\left(\sum_{j=0}^{k}\theta_j\right)^{-1}.$$

Hence, since ν was arbitrary, by Corollary 4.2.10, we finally conclude

$$\gamma_{\Theta^s,+}(E_k) \lesssim \gamma_{\mathrm{sy},+}(E_k) \leq C\left(\sum_{j=0}^{k}\theta_j\right)^{-1}.$$

□

THEOREM 4.2.12. *Let $(\lambda_j)_j$ be a sequence of real numbers satisfying $0 < \lambda_j \leq \tau_0 < 1/2$, for every j. Then, for any fixed generation k,*

$$\gamma_{\Theta^s,+}(E_k) \gtrsim_{\tau_0} \left(\sum_{j=0}^{k}\theta_j\right)^{-1}.$$

Proof. Fix a generation k as well as the measure introduced in the proof of Theorem 4.2.11, $\mu_k := \frac{1}{|E_k|}\mathcal{L}^{n+1}|_{E_k}$.

Recall that $\mu_k(Q_i^j) = 2^{-j(n+1)}$ for any s-parabolic cube of the j-th generation, with $0 \leq j \leq k$ and $1 \leq i \leq 2^{j(n+1)}$. Let us fix any $\overline{x}\in\mathbb{R}^{n+1}$ and proceed in an inductive way as follows:

1. If $\text{dist}_{p_s}(\overline{x}, Q^0) \geq 1$, it is clear that $P_s * \mu_k(\overline{x}) \leq 1 = \theta_0$. If $\text{dist}_{p_s}(\overline{x}, Q^0) < 1$, denote by Δ_1 one of the s-parabolic cubes of the first generation E_1 that is closest to $\overline{x}$. Observe that $\text{dist}_{p_s}(\overline{x}, E_1 \setminus \Delta_1) \gtrsim 1 - 2\lambda_1 \geq 1 - 2\tau_0$. Then,

$$\begin{aligned} P_s * \mu_k(\overline{x}) &= \int_{E_k} P_s(\overline{x} - \overline{y}) \mathrm{d}\mu_k(\overline{y}) \leq \int_{E_1 \setminus \Delta_1} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n} + \int_{\Delta_1} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n} \\ &\lesssim \frac{\theta_0}{(1 - 2\tau_0)^n} + \int_{\Delta_1} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n}. \end{aligned}$$

2. If $\text{dist}_{p_s}(\overline{x}, \Delta_1) \geq \ell_1$, it is clear that the above remaining integral satisfies

$$\int_{\Delta_1} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n} \leq \frac{\mu_k(\Delta_1)}{\ell_1^n} = \theta_1,$$

and therefore $P_s * \mu_k(\overline{x}) \lesssim \theta_0 + \theta_1$. On the other hand, if $\text{dist}_{p_s}(\overline{x}, \Delta_1) < \ell_1$, we repeat the process of step *1* and pick Δ_2 one of the cubes of E_2 that is closest to $\overline{x}$. In the current setting notice that $\text{dist}_{p_s}(\overline{x}, E_2 \setminus \Delta_2) \gtrsim (1 - 2\tau_0)\ell_1$, which implies

$$\begin{aligned} P_s * \mu_k(\overline{x}) &\lesssim \frac{\theta_0}{(1 - 2\tau_0)^n} + \int_{\Delta_1 \setminus \Delta_2} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n} + \int_{\Delta_2} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n} \\ &\lesssim \frac{1}{(1 - 2\tau_0)^n}(\theta_0 + \theta_1) + \int_{\Delta_2} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n}. \end{aligned}$$

In general, for $1 \leq m < k - 1$, the $(m+1)$-th step of the above process we would begin by dealing with an estimate of the form

$$P_s * \mu_k(\overline{x}) \lesssim \frac{1}{(1 - 2\tau_0)^n}(\theta_0 + \theta_1 + \cdots + \theta_{m-1}) + \int_{\Delta_m} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n},$$

and we would distinguish whether if $\text{dist}_{p_s}(\overline{x}, \Delta_j) \geq \ell_j$ or not. If the inequality is satisfied, it follows that $P_s * \mu_k(\overline{x}) \lesssim \sum_{j=0}^m \theta_j$. If on the other hand $\text{dist}_{p_s}(\overline{x}, \Delta_j) < \ell_j$, write Δ_{m+1} one of the s-parabolic cubes of E_{m+1} that is closest to $\overline{x}$ and notice that $\text{dist}_{p_s}(\overline{x}, E_{m+1} \setminus \Delta_{m+1}) \gtrsim (1 - 2\tau_0)\ell_m$, and move on to step $m + 2$. The previous process can carry on a maximum of k steps (this is the case, for example, if $\overline{x} \in E_k$), and in this situation $\overline{x}$ satisfies $\text{dist}_{p_s}(\overline{x}, \Delta_k) < \ell_k$ as well as the estimate

$$P_s * \mu_k(\overline{x}) \lesssim \frac{1}{(1 - 2\tau_0)^n} \sum_{j=0}^{k-1} \theta_j + \int_{\Delta_k} \frac{\mathrm{d}\mu_k(\overline{y})}{|\overline{x} - \overline{y}|_{p_s}^n},$$

that cannot be dealt with the same iterative method. We name the remaining integral I_k and write $Q_k := Q\big(\overline{x}, 5\ell_k\big)$ so that $\Delta_k \subset Q_k$. We split the latter into the s-parabolic annuli $A_j := Q\big(\overline{x}, 5\ell_k 2^{-j}\big) \setminus Q\big(\overline{x}, 5\ell_k 2^{-j-1}\big)$, for $j \geq 0$ integer. Therefore,

$$I_k \leq \frac{1}{|E_k|} \sum_{j=0}^{\infty} \int_{A_j} \frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|_{p_s}^n} \leq \frac{1}{|E_k|} \sum_{j=0}^{\infty} \frac{(5\,\ell_k 2^{-j})^{n+2s}}{(5\ell_k 2^{-j-1})^n} \simeq \frac{\ell_k^{2s}}{|E_k|} \sum_{j=0}^{\infty} \frac{1}{2^j} \simeq \theta_k.$$

With this we conclude that, in general, there exists a constant $C = C(n, s, \tau_0) > 0$ so that

$$P_s * \mu_k(\overline{x})\left(C\sum_{j=0}^{k}\theta_j\right)^{-1} \leq 1, \qquad \forall \overline{x} \in \mathbb{R}^{n+1},$$

so the measure $\big(C\sum_{j=0}^{k}\theta_k\big)^{-1}\mu_k$ is admissible for $\gamma_{\Theta^s,+}(E_k)$, and the result follows. □

Combining both Theorems 4.2.11 and 4.2.12, we obtain the following result:

COROLLARY 4.2.13. *Let $s \in (0,1]$ and $(\lambda_j)_j$ be a sequence of real numbers satisfying $0 < \lambda_j \leq \tau_0 < 1/2$, for every j. Then, if E denotes the associated Cantor set as in (4.2.3),*

$$\gamma_{\Theta^s,+}(E) \approx_{\tau_0} \left(\sum_{j=0}^{\infty}\theta_j\right)^{-1}.$$

Proof. The estimate follows from Theorem 4.2.11 and the monotonicity of $\gamma_{\Theta^s,+}$; and from Theorem 4.2.12 combined with the third point of Theorem 4.2.2 (outer regularity). □

Chapter 5

The semi-additivity of $\widetilde{\gamma}_{\Theta^{1/2}}$ in the plane

In this final chapter we aim at obtaining a more precise description of a smaller variant of $\gamma_{\Theta^{1/2}}$, already defined in [**MatP, §4**]. Named 1/2-*symmetric caloric capacity* and denoted by $\widetilde{\gamma}_{\Theta^{1/2}}$, it has been already introduced in §5.2 and it is obtained by computing the supremum of expressions $|\langle T, 1\rangle|$, where distributions T are supported on a compact set and satisfy

$$\|P * T\|_\infty \leq 1 \quad \text{and} \quad \|P^* * T\|_\infty \leq 1.$$

We write $\widetilde{\gamma}_{\Theta^{1/2},+}$ to refer to its smaller version, which considers only positive Borel measures. The main result of this chapter is found in §5.5 and it states that, in fact, the latter capacity is not that small:

Theorem. *For $E \subset \mathbb{R}^2$ compact,*

$$\widetilde{\gamma}_{\Theta^{1/2}}(E) \approx \gamma_{\Theta^{1/2},+}(E).$$

Such result implies, bearing in mind the fourth property in Theorem 4.2.2, the semi-additivity of $\widetilde{\gamma}_{\Theta^{1/2}}$ in $\mathbb{R}^2$ (Theorem 5.5.8). We also give a result in a general $\mathbb{R}^{n+1}$ setting, also proved in §5.5, where the same type of estimate is established for the typical multidimensional corner-like Cantor set. In §5.1, §5.2, §5.3 and §5.4 we present notation and preliminary results which are necessary to prove the above estimates. More precisely, in §5.1 we introduce basic terminology and properties, in §5.2 we prove Theorem 5.2.5, which encapsulates all the different ways to define, equivalently, $\gamma_{\Theta^{1/2},+}$. In §5.3 we use a particular characterization of the latter via a variational approach. With it, we construct a Whitney decomposition of a certain family of compact sets, so that we gain control of their $\widetilde{\gamma}_{\Theta^{1/2}}$ capacity. Finally, in §5.4, we prove Theorem 5.4.19, a general comparability result between $\widetilde{\gamma}_{\Theta^{1/2}}$ and $\gamma_{\Theta^{1/2}}$ if an additional assumption $\mathbf{A_3}$ is satisfied. In general, the arguments carried out are influenced by those of Tolsa [**T5, Ch.5**] and Volberg [**Vo**], where the same type of comparability results are studied for analytic and Lipschitz harmonic capacity respectively.

In our setting, taking into account that P is not an anti-symmetric kernel, such arguments can be applied with some modifications inspired by the arguments presented to prove general Tb-theorems such as those found in [**NTrVo2**] and [**HyMar**].

The fact that P is harmonic outside the hyperplane $\{t=0\}$, an $\mathcal{L}^{n+1}$-null set, and that satisfies being an n-dimensional Calderón-Zygmund kernel [**MatP, Lemma 2.1**] are the essential features used to carry out similar arguments. Finally, §5.6 is devoted to the computation of the $\widetilde{\gamma}_{\Theta^{1/2}}$ capacity of a rectangle $R\subset\mathbb{R}^2$, obtaining:

$$\widetilde{\gamma}_{\Theta^{1/2}}(R)\approx \ell_t\left[\frac{1}{2}\ln\left(1+\frac{\ell_t^2}{\ell_x^2}\right)+\frac{\ell_t}{\ell_x}\arctan\left(\frac{\ell_x}{\ell_t}\right)\right]^{-1},$$

where ℓ_x and ℓ_t are the horizontal and vertical side lengths of R. The above behavior differs substantially, for example, with that obtained for $\gamma(R)$, the analytic capacity of a rectangle. For the latter one has $\gamma(R)\approx \operatorname{diam}(R)$ (see, for example, [**T5, Proposition 1.5**]), which is a significantly simpler expression.

We highlight one last feature of the above capacities associated to the 1/2-fractional heat equations: the kernels P and P^* are both *nonnegative*. This suggests that, possibly, using classical arguments of potential theory (see [**La, Ch.I & II**], [**R, Ch.3**] or [**Ki**]), the comparability between capacities defined through positive measures and distributions should follow by a simpler argument similar to that of [**Ve2, p.10**], provided that one is able to prove the existence of an *equilibrium measure*. The author has not yet been able to deduce its existence for $\gamma_{\Theta^{1/2},+}$. The primary obstacle lies in the inability to derive an equivalent formulation of this capacity as an (inverted) infimum over energies. The following simple example already illustrates some problems that arise by proceeding this way: pick $E\subset\mathbb{R}^2$ the horizontal line segment $[0,1]\times\{0\}$ and $\mu:=\mathcal{H}^1|_E$. In [**MatP, §6**] the authors obtain $P*\mu\leq\pi$, which implies $\gamma_{\Theta^{1/2},+}(E)\geq\pi^{-1}$. However, by the definition of P and the choice of μ, it is also clear that $P*\mu(x,0)=0$, for any $x\in[0,1]$. Therefore, if we were to compute the energy of μ we would obtain

$$I[\mu]:=\int P*\mu\,\mathrm{d}\mu=\int_0^1 P*\mu(x,0)\,\mathrm{d}\mathcal{H}^1(x)=0,$$

which would imply $\gamma_{\Theta^{1/2},+}(E)=+\infty$. This simple example already shows some of the difficulties encountered in pursuing such a formulation and, in fact, also reveals that the potentials $P*\mu$ do not obey the so called *maximum principle*, i.e. they do not attain their maximum values at $\operatorname{supp}(\mu)$. To avoid this problem, the author has tried to work with the auxiliary potentials $U_\mu(x,t):=\limsup_{(y,s)\to(x,t)}P*\mu(y,s)$. For these the maximum principle holds (this follows, essentially, from the fact that P is subharmonic in $\mathbb{R}^{n+1}\setminus\{0\}$). However, the potentials U_μ still lack the *continuity principle*, i.e. if U_μ restricted to $\operatorname{supp}(\mu)$ is continuous, then U_μ should be continuous everywhere else (it already fails in the same example of above); which is an essential tool in order to carry out the construction of possible equilibrium measures. In any case, the above classical methods were finally discarded and we chose to follow arguments similar to those of [**T5, Ch.5**] and [**Vo**].

5.1 Notation and basic definitions

Begin by observing that the choice $s = 1/2$ implies that the ambient space $\mathbb{R}^{n+1}$ will be endowed with the usual Euclidean metric. To compactify notation, we set

$$B_r(\overline{x}), \quad \text{the ball in } \mathbb{R}^{n+1} \text{ of center } \overline{x} \text{ and radius } r > 0.$$

By Theorem 4.2.1 we know that admissible distributions for $\gamma_{\Theta^{1/2}}$ present n-growth with a constant C that depends only on n. This motivates the following redefinition of $\gamma_{\Theta^{1/2}}$, that will be the one used throughout the whole chapter. We will also omit the subscript $\Theta^{1/2}$ in order to ease notation.

Definition 5.1.1. For a compact set $E \subset \mathbb{R}^{n+1}$, define its 1/2-*caloric capacity* as

$$\gamma(E) = \gamma_{\Theta^{1/2}}(E) := \sup |\langle T, 1\rangle|,$$

the supremum taken over distributions satisfying $\|P * T\|_\infty \leq 1$ and belonging to $\mathcal{T}(E)$, the set of distributions in $\mathbb{R}^{n+1}$ with $\text{supp}(T) \subset E$ and n-growth with constant 1.

We also redefine the $(1/2, +)$-*caloric capacity*, in the same way as γ, but with the supremum only taken with respect to positive Borel measures. More precisely,

$$\gamma_+(E) = \gamma_{\Theta^{1/2},+}(E) := \sup \big\{\mu(E) \,:\, \mu \in \Sigma(E),\, \|P * \mu\|_\infty \leq 1\big\},$$

where $\Sigma(E) := \Sigma_n^{1/2}(E)$ is the collection of positive Borel measures supported on E with n-growth with constant 1. Analogous definitions are associated with the operator $\overline{\Theta}^{1/2}$, giving rise to the objects

$$\overline{\gamma} := \gamma_{\overline{\Theta}^{1/2}} \quad \text{and} \qquad \overline{\gamma}_+ := \gamma_{\overline{\Theta}^{1/2},+}.$$

In some of the results presented in [**MatP, §3**], one encounters expressions of the form $\langle T, \varphi\rangle$, where φ is $\mathcal{C}^1$ and compactly supported, and T is a compactly supported distribution satisfying $\|P * T\|_\infty \leq 1$. This is due to the definition of admissible function given by Mateu and Prat, in which they only require $\mathcal{C}^1$ regularity. It is because $\|P * T\|_\infty \leq 1$ and $(-\Delta)^{1/2}\varphi \in L^1(\mathbb{R}^{n+1})$ (proved in [**MatP, §3**]) that we can define $\langle T, \varphi\rangle$, that a priori may not have sense since φ is not $\mathcal{C}^\infty$. Indeed, observe that

$$\langle T, \varphi\rangle := \big\langle P * T, \overline{\Theta}^{1/2}\varphi\big\rangle = \big\langle P * T, (-\Delta)^{1/2}\varphi + \partial_t\varphi\big\rangle.$$

We need to give meaning to $\langle T, \varphi\rangle$ for a slightly wider class of functions. We do not claim that such class will be the largest where $\langle T, \varphi\rangle$ can be defined if T is a compactly supported distribution with $\|P * T\|_\infty \leq 1$, but it will suffice for our purposes.

Definition 5.1.2. Let $Q \subset \mathbb{R}^{n+1}$ be a cube, $\mathcal{N} \subset \mathbb{R}^{n+1}$ a $\mathcal{L}^{n+1}$-null set, and $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ a function. For $m \in \mathbb{N} \cup \{\infty\}$ we will write

$$\varphi \in \mathcal{C}_c(Q) \cap \big(\mathcal{C}_x^m(Q), \mathcal{C}_t^m(Q \setminus \mathcal{N})\big),$$

or simply
$$\varphi \in \mathcal{C}^m_{c,\mathcal{N}}(Q),$$
if φ is continuous and compactly supported on Q, m times continuously differentiable with respect to the spatial variables, and m times continuously differentiable with respect to the temporal variable t except for a null set $\mathcal{N}$ (where it may not even be differentiable). We also write
$$\varphi \in \mathcal{C}(Q) \cap \big(\mathcal{C}^m_x(Q), \mathcal{C}^m_t(Q\setminus \mathcal{N})\big) = \mathcal{C}^m_{\mathcal{N}}(Q),$$
if φ satisfies the above regularity properties on Q, but we do not require it to be compactly supported there (in fact, it may even have a larger domain of definition). In the sequel we will only be interested in the case $m=\infty$.

For $\varphi \in \mathcal{C}^m_{c,\mathcal{N}}(Q)$ one can define $\langle T, \varphi\rangle$ as above, assuming $\|P*T\|_\infty \leq 1$ and T compactly supported. Indeed, observe that we only have to give meaning to $\langle P * T, (-\Delta)^{1/2}\varphi\rangle$ and $\langle P*T, \partial_t\varphi\rangle$. The former term can be defined as in [**MatP, §3**], where the integrability of $(-\Delta)^{1/2}\varphi$ is proved using only regularity assumptions over the spatial variables. Regarding the term $\langle P*T, \partial_t\varphi\rangle$, since $P*T\in L^\infty(\mathbb{R}^{n+1})$, it can be simply defined as
$$\langle P*T, \partial_t\varphi\rangle := \int_{\text{supp}(\varphi)\setminus\mathcal{N}} P*T(\overline{x})\partial_t\varphi(\overline{x})\,\mathrm{d}\overline{x},$$
since the set $\mathcal{N}$ has null $\mathcal{L}^{n+1}$-measure.

Remark 5.1.1. We notice that if for $\varphi\in\mathcal{C}^1_{c,\mathcal{N}}(Q)$ it also happens that $\|\nabla\varphi\|_\infty \leq \ell(Q)^{-1}$, then [**MatP, Corollary 3.3**] and [**MatP, Lemma 3.5**] also hold with exactly the same proofs for such φ; and therefore also do [**MatP, Theorem 3.1**] (localization of potentials) and [**MatP, Lemma 3.4**] (n-growth of admissible distributions). Here, condition $\|\nabla\varphi\|_\infty\leq \ell(Q)^{-1}$ must be understood as: for every test function ψ,
$$|\langle\nabla\varphi,\psi\rangle| = \left|\int_Q \varphi(\nabla\psi)\;\mathrm{d}\mathcal{L}^{n+1}\right| = \left|\int_{Q\setminus\mathcal{N}}(\nabla\varphi)\psi\;\mathrm{d}\mathcal{L}^{n+1}\right| \leq \ell(Q)^{-1}\|\psi\|_1.$$

To end this breif introduction, we redefine more capacities that are already presented in [**MatP**] and some other auxiliary ones. Recall from §5.2 the so called 1/2-*symmetric caloric capacity*, where the normalization conditions for the potentials against P and P^* are both required:
$$\widetilde{\gamma}(E) = \widetilde{\gamma}_{\Theta^{1/2}}(E) := \sup\big\{|\langle T, 1\rangle| \,:\, T\in\mathcal{T}(E),\, \|P*T\|_\infty\leq 1, \|P^* * T\|_\infty \leq 1\big\},$$
$$\widetilde{\gamma}_+(E) = \widetilde{\gamma}_{\Theta^{1/2},+}(E) := \sup\big\{\mu(E) \,:\, \mu\in\Sigma(E),\, \|P*\mu\|_\infty\leq 1, \|P^* * \mu\|_\infty \leq 1\big\}.$$
We also have the following auxiliary capacities, that we only consider in the context of positive measures:
$$\gamma_{\text{sy},+}(E) := \sup\big\{\mu(E) \,:\, \mu\in\Sigma(E),\, \|P_{\text{sy}}*\mu\|_\infty\leq 1\big\},$$

where recall that P_{sy} is the symmetric part of P, that is,

$$P_{\text{sy}}(\overline{x}) := \frac{P(\overline{x}) + P(-\overline{x})}{2} = \frac{|t|}{2|\overline{x}|^{n+1}}.$$

Finally, we also consider the following capacities defined via a normalization condition that involves an $L^2(\mu)$ operator bound:

$$\begin{aligned}\gamma_{\text{op}}(E) = \gamma_{\Theta^{1/2},\text{op}}(E) &:= \sup\big\{\mu(E) \,:\, \mu \in \Sigma(E),\; \|\mathscr{P}_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1\big\},\\ \gamma_{\text{sy,op}}(E) &:= \sup\big\{\mu(E) \,:\, \mu \in \Sigma(E),\; \|\mathscr{P}_{\text{sy},\mu}\|_{L^2(\mu)\to L^2(\mu)} \leq 1\big\},\end{aligned}$$

where $\mathscr{P}_\mu$ and $\mathscr{P}_{\text{sy},\mu}$ are the convolution operators associated to P and P_{sy} with respect to the measure μ.

From the results of §5.2 one can deduce an important property of the above different capacities: they are all comparable.

Theorem 5.1.1. *The previous capacities defined through positive measures are all comparable. That is, for any compact set $E \subset \mathbb{R}^{n+1}$,*

$$\begin{aligned}\gamma_+(E) \approx \overline{\gamma}_+(E) \approx \widetilde{\gamma}_+(E) \approx \gamma_{\text{sy},+}(E)&\\ \approx \gamma_{\text{op}}(E) \approx \gamma_{\text{sy,op}}(E).&\end{aligned}$$

Remark 5.1.2. To be precise, the capacity $\gamma_{\text{sy,op}}$ is not introduced in §5.2. To justify its comparability to all the rest, it suffices to check $\gamma_{\text{sy,op}} \lesssim \gamma_{\text{op}}$, since the reverse inequality holds trivially. Fix μ any admissible measure for $\gamma_{\text{sy,op}}(E)$ with $\gamma_{\text{sy,op}}(E) \leq 2\mu(E)$. Notice that since $\mathscr{P}_\mu$ and $\mathscr{P}_\mu^*$ are n-dimensional C-Z convolution operators, then $\mathscr{P}_{\text{sy},\mu}$ also satisfies such property. Therefore, following an analogous proof to [**T5, Theorem 2.16**] one deduces, from the $L^2(\mu)$-boundedness of $\mathscr{P}_{\text{sy},\mu}$, that there exists a constant $C>0$ such that for any ν Borel finite signed measure,

$$\mu\big(\big\{|\mathscr{P}_{\text{sy},\varepsilon}\nu(\overline{x})| > \lambda\big\}\big) \leq C\frac{\|\nu\|}{\lambda}, \qquad \forall \varepsilon > 0,\, \forall \lambda > 0.$$

With this, and using that $\mathscr{P}_{\text{sy}} = \mathscr{P}_{\text{sy}}^*$, applying for example [**Chr, Ch.VII, Theorem 23**], we can find a function $h : E \to [0,1]$ such that $\int_E h\mathrm{d}\mu \geq \mu(E)/2$ and $\|\mathscr{P}_{\text{sy}}(h\mu)\|_\infty < 4C$. Then,

$$\gamma_{\text{sy},+}(E) \geq \frac{1}{4C}\int_E h\mathrm{d}\mu \geq \frac{\mu(E)}{8C} \geq \frac{1}{16C}\,\gamma_{\text{sy,op}}(E),$$

that implies $\gamma_{\text{sy,op}}(E) \lesssim \gamma_{\text{sy},+}(E) \approx \gamma_{\text{op}}(E)$, and we are done.

5.2 Even more capacities

The main goal of this section is to add three additional ways of understanding 1/2-caloric capacities: one via an $L^2(\mu)$ normalization condition, another using a uniform bound at *all* points of the compact set, and a last one formulated in a variational way. All the results proved in this section are summarized in Theorem 5.2.5.

5.2.1 $L^2(\mu)$ normalization. Suppressed kernels.

Given a convolution kernel one may define its *suppressed version* in terms of a general nonnegative 1-Lipschitz function (with constant 1). However, for our purposes, we do not need to work with such generality, and we will focus on a particular type of function. Given a closed set $F \subset \mathbb{R}^{n+1}$, we will typically have

$$\Lambda(\overline{x}) := \text{dist}(\overline{x}, F), \quad \forall \overline{x} \in \mathbb{R}^{n+1}.$$

The above Λ is indeed a 1-Lipschitz function: take any $\overline{x}, \overline{y} \in \mathbb{R}^{n+1}$ and observe that for any $\overline{z} \in F$ we have $\Lambda(\overline{x}) \leq |\overline{x}-\overline{z}| \leq |\overline{x}-\overline{y}| + |\overline{y}-\overline{z}|$. Since $\overline{z}$ is arbitrary, $\Lambda(\overline{x}) \leq |\overline{x}-\overline{y}| + \Lambda(\overline{y}) \leftrightarrow \Lambda(\overline{x}) - \Lambda(\overline{y}) \leq |\overline{x}-\overline{y}|$. Repeating the same argument changing the roles of $\overline{x}$ and $\overline{y}$ we deduce the desired Lipschitz property. We define the suppressed kernel associated to P, that depends on Λ and a previously fixed closed set F, as:

$$P_\Lambda(\overline{x}, \overline{y}) := \frac{P(\overline{x}-\overline{y})}{1 + P(\overline{x}-\overline{y})^2 \Lambda(\overline{x})^n \Lambda(\overline{y})^n}, \tag{5.2.1}$$

for all the pair of points where such expression makes sense. Notice that it is also a nonnegative kernel and differentiable, with respect to $\overline{x}$ or $\overline{y}$, out of a set of null Lebesgue measure. We may also define P_Λ^* and $P_{\text{sy},\Lambda}$ in an analogous way. In fact, for the sake of notation, we shall prove three basic properties only for P_Λ, but they can be checked to hold also for P_Λ^* and $P_{\text{sy},\Lambda}$ with almost exactly the same proofs. Essentially, the convolution kernels P_Λ, P_Λ^* and $P_{\text{sy},\Lambda}$ in $\mathbb{R}^{n+1}$ will satisfy being n-dimensional C-Z operators with an additional property.

LEMMA 5.2.1. *Let $\Lambda : \mathbb{R}^{n+1} \to [0,\infty)$ be a 1-Lipschitz function. Then, kernels P_Λ, P_Λ^* and $P_{\text{sy},\Lambda}$ define n-dimensional C-Z operators which satisfy being "well suppressed" at the points where $\Lambda > 0$. Namely, if we fix any $\overline{x} \neq \overline{y}$ points in $\mathbb{R}^{n+1}$, the precise estimates that P_Λ satisfies (as well as P_Λ^* and $P_{\text{sy},\Lambda}$) are*

1. $|P_\Lambda(\overline{x}, \overline{y})| \leq |\overline{x}-\overline{y}|^{-n}$.

2. *For any $\overline{x}' \in \mathbb{R}^{n+1}$ satisfying $|\overline{x}-\overline{x}'| \leq |\overline{x}-\overline{y}|/2$,*

$$\big|P_\Lambda(\overline{x}, \overline{y}) - P_\Lambda(\overline{x}', \overline{y})\big| \lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}} \quad \text{and} \quad \big|P_\Lambda(\overline{y}, \overline{x}) - P_\Lambda(\overline{y}, \overline{x}')\big| \lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}}.$$

   In particular, where the differentiation is well-defined,

$$\big|\nabla_{\overline{x}} P_\Lambda(\overline{x}, \overline{y})\big| + \big|\nabla_{\overline{y}} P_\Lambda(\overline{x}, \overline{y})\big| \lesssim \frac{1}{|\overline{x}-\overline{y}|^{n+1}}.$$

3. $|P_\Lambda(\overline{x}, \overline{y})| \lesssim \min\big\{\Lambda(\overline{x})^{-n}, \Lambda(\overline{y})^{-n}\big\}$.

Proof. The proof of *1* is trivial, since $|P_\Lambda(\overline{x}, \overline{y})| \leq P(\overline{x}-\overline{y})$. We move on by proving the first inequality in *2*, and we will do it in a way that the arguments will also be valid for P^*, implying the validity of the second estimate of *2*. Let us first assume $\Lambda(\overline{x}') \leq \Lambda(\overline{x})$ and define

$$g(\overline{x},\overline{y}) := 1 + P(\overline{x}-\overline{y})^2\Lambda(\overline{x})^n\Lambda(\overline{y})^n.$$

Using [**MatP, Lemma 2.1**] (that is also valid for P^* and hence also for P_{sy}) we get

$$\begin{aligned}
&|P_\Lambda(\overline{x},\overline{y}) - P_\Lambda(\overline{x}',\overline{y})| \\
&\quad\leq \frac{|P(\overline{x}-\overline{y}) - P(\overline{x}'-\overline{y})|}{g(\overline{x},\overline{y})} + P(\overline{x}'-\overline{y})\frac{|P(\overline{x}-\overline{y})^2\Lambda(\overline{x})^n - P(\overline{x}'-\overline{y})^2\Lambda(\overline{x}')^n|\Lambda(\overline{y})^n}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})} \\
&\quad\lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}} + P(\overline{x}'-\overline{y})\frac{|P(\overline{x}-\overline{y})^2 - P(\overline{x}'-\overline{y})^2|\Lambda(\overline{x}')^n\Lambda(\overline{y})^n}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})} \\
&\qquad\qquad + P(\overline{x}'-\overline{y})\frac{P(\overline{x}-\overline{y})^2|\Lambda(\overline{x})^n - \Lambda(\overline{x}')^n|\Lambda(\overline{y})^n}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})} =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

We only have to deal with II and III. For the first, we rewrite it as

$$\mathrm{II} = |P(\overline{x}-\overline{y}) - P(\overline{x}'-\overline{y})|\,\mathrm{IV},$$

where

$$\mathrm{IV} := \frac{P(\overline{x}'-\overline{y})|P(\overline{x}-\overline{y}) + P(\overline{x}'-\overline{y})|\Lambda(\overline{x}')^n\Lambda(\overline{y})^n}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})}.$$

It is clear that by the definition of g

$$\mathrm{IV} \leq 1 + \frac{P(\overline{x}'-\overline{y})P(\overline{x}-\overline{y})\Lambda(\overline{x}')^n\Lambda(\overline{y})^n}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})}.$$

For the remaining fraction, if $P(\overline{x}-\overline{y}) \leq P(\overline{x}'-\overline{y})$ we obtain the same bound by 1. If on the other hand $P(\overline{x}-\overline{y}) > P(\overline{x}'-\overline{y})$, using that $\Lambda(\overline{x}') \leq \Lambda(\overline{x})$ we also obtain the same estimate and we are done with II. Regarding III, observe that there exists a point $\overline{\xi}$ in the line segment joining $\overline{x}$ and $\overline{x}'$ such that

$$\mathrm{III} \leq |\overline{x}-\overline{x}'|P(\overline{x}'-\overline{y})\frac{n\Lambda(\overline{\xi})^{n-1}\Lambda(\overline{y})^nP(\overline{x}-\overline{y})^2}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})}.$$

We distinguish two cases: if $\Lambda(\overline{\xi}) \leq |\overline{x}-\overline{y}|$. By the Lipschitz property of Λ, $\Lambda(\overline{y}) \leq 2|\overline{x}-\overline{y}|$. Thus,

$$\begin{aligned}
\mathrm{III} &\leq |\overline{x}-\overline{x}'|\frac{n\Lambda(\overline{\xi})^{n-1}\Lambda(\overline{y})^nP(\overline{x}-\overline{y})^2}{g(\overline{x},\overline{y})g(\overline{x}',\overline{y})}\Big[P(\overline{x}-\overline{y}) + \big|P(\overline{x}-\overline{y}) - P(\overline{x}'-\overline{y})\big|\Big] \\
&\lesssim |\overline{x}-\overline{x}'|\frac{|\overline{x}-\overline{y}|^{2n-1}}{|\overline{x}-\overline{y}|^{2n}}\left[\frac{1}{|\overline{x}-\overline{y}|^n} + \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}}\right] = \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}}\left[1 + \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|}\right] \\
&\lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}},
\end{aligned}$$

that is the desired estimate. If on the other hand we had $\Lambda(\overline{\xi}) > |\overline{x}-\overline{y}|$, then notice that by the Lipschitz property of Λ,

$$\Lambda(\overline{\xi}) \leq \Lambda(\overline{x}) + |\overline{x}-\overline{\xi}| \leq \Lambda(\overline{x}) + |\overline{x}-\overline{x}'| \leq \Lambda(\overline{x}) + \frac{|\overline{x}-\overline{y}|}{2} < \Lambda(\overline{x}) + \frac{\Lambda(\overline{\xi})}{2},$$

that is $\Lambda(\overline{x}) > \Lambda(\overline{\xi})/2$, and in particular $g(\overline{x},\overline{y}) \gtrsim g(\overline{\xi},\overline{y})$. Therefore,

$$\begin{aligned} \text{III} &\lesssim P(\overline{x}'-\overline{y})\frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|}\frac{P(\overline{x}-\overline{y})^2\Lambda(\overline{\xi})^n\Lambda(\overline{y})^n}{g(\overline{\xi},\overline{y})g(\overline{x}',\overline{y})} \leq P(\overline{x}'-\overline{y})\frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|} \\ &\leq \frac{1}{2}\big|P(\overline{x}'-\overline{y}) - P(\overline{x}-\overline{y})\big| + \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}} \lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}-\overline{y}|^{n+1}}, \end{aligned}$$

that is what we wanted to prove. With this, we have proved *2* for the case $\Lambda(\overline{x}') \leq \Lambda(\overline{x})$. The case $\Lambda(\overline{x}) \leq \Lambda(\overline{x}')$ can be treated analogously, using symmetric considerations with respect to $\overline{x}$ and $\overline{x}'$.

To prove the gradient estimates, let us fix $j \in \{1,\ldots,n+1\}$ and consider $\partial_{x_j}P_\Lambda(\overline{x},\overline{y})$ at the points where this expression is well defined. Observe that since Λ is a Lipschitz function and P can be differentiated except for null Lebesgue sets, $\partial_{x_j}P_\Lambda$ makes sense out of sets of null Lebesgue measure. In any case, choose $0<|h|\ll 1$ so that $|h| \leq |\overline{x}-\overline{y}|/2$, where $|\overline{x}-\overline{y}|$ is strictly positive since $\overline{x}\neq\overline{y}$ are two fixed different points. Hence, applying the already proved estimates of *2* we get

$$\frac{|P_\Lambda(\overline{x}+h\widehat{e}_j,\overline{y}) - P_\Lambda(\overline{x},\overline{y})|}{|h|} \lesssim \frac{1}{|h|}\frac{|h|}{|\overline{x}-\overline{y}|^{n+1}} = \frac{1}{|\overline{x}-\overline{y}|^{n+1}},$$

so taking the limit as $h\to 0$ the result follows.

Finally we prove *3*. Assume that $\Lambda(\overline{x}) = \max\{\Lambda(\overline{x}),\Lambda(\overline{y})\}$. In this setting,

$$\big|P_\Lambda(\overline{x},\overline{y})\big| = \frac{1}{P(\overline{x}-\overline{y})}\left[\frac{1}{P(\overline{x}-\overline{y})^{-2}+\Lambda(\overline{x})^n\Lambda(\overline{y})^n}\right] \leq \frac{1}{P(\overline{x}-\overline{y})}\frac{1}{\Lambda(\overline{x})^n\Lambda(\overline{y})^n}. \tag{5.2.2}$$

Assume that

$$\frac{1}{P(\overline{x}-\overline{y})} \leq \frac{\Lambda(\overline{x})^n}{2^n}. \tag{5.2.3}$$

In this case, $|\overline{x}-\overline{y}|^n \leq P(\overline{x}-\overline{y})^{-1} \leq \Lambda(\overline{x})^n/2^n$, that is $|\overline{x}-\overline{y}| \leq \Lambda(\overline{x})/2$. By the Lipschitz property we also have $\Lambda(\overline{y}) \geq \Lambda(\overline{x}) - |\overline{x}-\overline{y}|$, so $\Lambda(\overline{y}) \geq \Lambda(\overline{x})/2$, and applying this estimate to (5.2.2) the result would follow. If (5.2.3) did not hold, we would simply have

$$|P_\Lambda(\overline{x}-\overline{y})| \leq |P(\overline{x}-\overline{y})| \leq \frac{2^n}{\Lambda(\overline{x})^n},$$

and we would be also done. Moreover, it is clear that if $\Lambda(\overline{y}) = \max\{\Lambda(\overline{x}),\Lambda(\overline{y})\}$, the above proof can be carried out in an analogous way, obtaining the desired result. □

For a fixed $E\subset\mathbb{R}^{n+1}$ compact set, we will use the above suppressed kernels to study the capacity

$$\gamma_{\text{sy},2}(E) := \sup\big\{\mu(E)\,:\,\mu\in\Sigma(E),\,\|P_{\text{sy}}*\mu\|_{L^2(\mu)} \leq \mu(E)^{1/2}\big\}.$$

By (5.2.2), it is clear that $\gamma_{\mathrm{sy},+}(E) \leq \gamma_{\mathrm{sy},2}(E)$. The fundamental lemma that will imply the reverse inequality is the following:

Lemma 5.2.2. *For $E \subset \mathbb{R}^{n+1}$ compact, the following inequality holds:*

$$\gamma_{\mathrm{sy},2}(E) \lesssim \gamma_{\mathrm{sy},\mathrm{op}}(E).$$

Proof. Let μ be an admissible measure for $\gamma_{\mathrm{sy},2}(E)$. Then, by definition,

$$\int_E |\mathscr{P}_{\mathrm{sy},\varepsilon}\mu(\overline{x})|^2 \mathrm{d}\mu(\overline{x}) \leq \mu(E), \quad \forall \varepsilon > 0.$$

Consider the maximal operator

$$\mathscr{P}_{\mathrm{sy},*}\mu(\overline{x}) := \sup_{\varepsilon>0} |\mathscr{P}_{\mathrm{sy},\varepsilon}\mu(\overline{x})|,$$

that by the nonnegativity of P_{sy} and μ is such that

$$\mathscr{P}_{\mathrm{sy},*}\mu(\overline{x}) = \lim_{\varepsilon\to 0} \mathscr{P}_{\mathrm{sy},\varepsilon}\mu(\overline{x}),$$

which is well-defined for every point $\overline{x} \in \mathbb{R}^{n+1}$. In fact, by the monotone convergence theorem, it is clear that the above $L^2(\mu)$ uniform estimate with respect to $\varepsilon > 0$, can be rewritten as

$$\int_E |\mathscr{P}_{\mathrm{sy},*}\mu(\overline{x})|^2 \mathrm{d}\mu(\overline{x}) \leq \mu(E).$$

Now we define

$$F := \Big\{\overline{x} \in E \,:\, \mathscr{P}_{\mathrm{sy},*}\mu(\overline{x}) \leq \sqrt{2}\Big\},$$

and apply Chebyshev's inequality to obtain

$$\mu(E \setminus F) \leq \frac{1}{2}\int_E |\mathscr{P}_{\mathrm{sy},*}\mu(\overline{x})|^2 \mathrm{d}\mu(\overline{x}) \leq \frac{\mu(E)}{2} \quad \text{and then} \quad \mu(F) \geq \frac{\mu(E)}{2}.$$

If we are able to prove

$$\|\mathscr{P}_{\mathrm{sy},\mu|_F}\|_{L^2(\mu|_F)\to L^2(\mu|_F)} \lesssim 1$$

we will be done, since the latter would mean $\gamma_{\mathrm{sy},\mathrm{op}}(E) \gtrsim \mu(F) \geq \mu(E)/2$, and by the arbitrariness of μ the result would follow.

To proceed, let us observe that $\mathscr{P}_{\mathrm{sy},*}\mu$ can be understood as a non-decreasing limit of continuous functions. Indeed, we may rewrite, for each $\varepsilon > 0$ and $\overline{x} \in \mathbb{R}^{n+1}$,

$$\mathscr{P}_{\mathrm{sy},\varepsilon}\mu(\overline{x}) = \int_{\mathbb{R}^{n+1}} P_{\mathrm{sy}}(\overline{x}-\overline{y})\chi_{\{|\overline{x}-\overline{y}|>\varepsilon\}}\,\mathrm{d}\mu(\overline{y}) =: P_{\mathrm{sy},\chi_\varepsilon} * \mu(\overline{x}),$$

where $P_{\mathrm{sy},\chi_\varepsilon}(\cdot) := P_{\mathrm{sy}}(\cdot)\chi_{\{|\cdot|>\varepsilon\}}$. Now, let us fix ψ any test function such that $0 \leq \psi \leq 1$, $\psi \equiv 0$ in $B_1(0)$ and $\psi \equiv 1$ in $\mathbb{R}^{n+1} \setminus B_2(0)$. Assume also that $\|\nabla\psi\|_\infty \leq 1$. Write $\psi_\varepsilon(\cdot) := \psi(\cdot/\varepsilon)$ and define, for each $\overline{x} \in \mathbb{R}^{n+1}$,

$$P_{\mathrm{sy},\psi_\varepsilon} * \mu(\overline{x}) := \int_{\mathbb{R}^{n+1}} P_{\mathrm{sy}}(\overline{x}-\overline{y})\psi_\varepsilon(\overline{x}-\overline{y})\mathrm{d}\mu(\overline{y}).$$

Observe that since $P_{\text{sy}} \geq 0$,

$$P_{\text{sy},\chi_{2\varepsilon}} * \mu(\overline{x}) \leq P_{\text{sy},\psi_\varepsilon} * \mu(\overline{x}) \leq P_{\text{sy},\chi_\varepsilon} * \mu(\overline{x}), \quad \forall \overline{x} \in \mathbb{R}^{n+1},$$

which implies, by definition,

$$\lim_{\varepsilon\to 0} P_{\text{sy},\psi_\varepsilon} * \mu(\overline{x}) = \mathscr{P}_{\text{sy},\varepsilon}\mu(\overline{x}), \quad \forall \overline{x} \in \mathbb{R}^{n+1}.$$

Now, since μ is in particular a finite measure, it is not difficult to prove using Fatou's lemma and its reverse version, that for each $\varepsilon > 0$ fixed, $P_{\text{sy},\psi_\varepsilon} * \mu$ is a continuous function. Moreover, since μ and P_{sy} are nonnegative, it is clear that for $\varepsilon_1 < \varepsilon_2$, then $P_{\text{sy},\psi_{\varepsilon_2}} * \mu \leq P_{\text{sy},\psi_{\varepsilon_1}} * \mu$. Hence, indeed, $\mathscr{P}_{\text{sy},*}\mu$ can be defined as a non-decreasing limit of continuous functions. Therefore, $\mathscr{P}_{\text{sy},*}\mu$ is lower semicontinuous [**B, Ch.4**] and thus F is a closed set. Now consider B a large open ball containing E inside and define $G := B \setminus F$. For any $\overline{z} \in G \cap E = E \setminus F$ we define

$$\varepsilon(\overline{z}) := \operatorname{dist}(\overline{z}, F) = \Lambda(\overline{z}),$$

recalling the notation used when defining the suppressed kernels. First we estimate for any $\varepsilon \geq \varepsilon(\overline{z})$:

$$\mathscr{P}_{\text{sy},\varepsilon}\mu(\overline{z}) = \int_{|\overline{y}-\overline{z}|\geq 2\varepsilon} P_{\text{sy}}(\overline{z}-\overline{y})\,\mathrm{d}\mu(\overline{y}) + \int_{\varepsilon<|\overline{y}-\overline{z}|\leq 2\varepsilon} P_{\text{sy}}(\overline{z}-\overline{y})\,\mathrm{d}\mu(\overline{y}) =: \mathrm{I} + \mathrm{II}.$$

Notice that by the n-growth of μ we have

$$|\mathrm{II}| \leq \frac{\mu(B_{2\varepsilon}(\overline{z}))}{\varepsilon^n} \lesssim 1.$$

To deal with I, let $\overline{x}_0$ be a closest point to $\overline{z}$ on F, so that $\varepsilon(\overline{z}) = |\overline{x}_0 - \overline{z}|$. Then

$$|\mathrm{I}| \leq \int_{|\overline{y}-\overline{z}|\geq 2\varepsilon} P_{\text{sy}}(\overline{x}_0-\overline{y})\,\mathrm{d}\mu(\overline{y}) + \int_{|\overline{y}-\overline{z}|\geq 2\varepsilon} |P_{\text{sy}}(\overline{x}_0-\overline{y}) - P_{\text{sy}}(\overline{z}-\overline{y})|\,\mathrm{d}\mu(\overline{y}) =: \mathrm{III} + \mathrm{IV}.$$

It is clear that $\mathrm{III} \leq \sqrt{2}$, since $\overline{x}_0 \in F$. On the other hand, since $|(\overline{z}-\overline{y}) - (\overline{x}_0-\overline{y})| \leq \varepsilon \leq |\overline{z}-\overline{y}|/2$, integration over annuli yields

$$|\mathrm{IV}| \lesssim \varepsilon \int_{|\overline{y}-\overline{z}|\geq\varepsilon} \frac{\mathrm{d}\mu(\overline{y})}{|\overline{z}-\overline{y}|^{n+1}} \lesssim 1.$$

So we have $|\mathscr{P}_{\text{sy},\varepsilon}\mu(\overline{z})| \lesssim 1$, for any $\varepsilon \geq \varepsilon(\overline{z})$. Now we shall prove

$$A := \left|\mathscr{P}_{\text{sy},\Lambda,\varepsilon}\mu(\overline{z}) - \mathscr{P}_{\text{sy},\varepsilon}\mu(\overline{z})\right| \lesssim 1, \quad \forall \varepsilon \geq \varepsilon(\overline{z}),$$

where $\mathscr{P}_{\text{sy},\Lambda,\varepsilon}\mu(\overline{z}) := \int_{|\overline{z}-\overline{y}|>\varepsilon} P_{\text{sy},\Lambda}(\overline{z},\overline{y})\,\mathrm{d}\mu(\overline{y})$. Since $\varepsilon(\overline{z}) = \Lambda(\overline{z})$ by definition,

$$A \leq \int_{|\overline{y}-\overline{z}|\geq\Lambda(\overline{z})} \left|P_{\text{sy},\Lambda}(\overline{z},\overline{y}) - P_{\text{sy}}(\overline{z}-\overline{y})\right|\mathrm{d}\mu(\overline{y}).$$

Using the definition of $P_{\mathrm{sy},\Lambda}$ we know

$$\big|P_{\mathrm{sy},\Lambda}(\overline{z},\overline{y}) - P_{\mathrm{sy}}(\overline{z}-\overline{y})\big| \leq P_{\mathrm{sy}}(\overline{z}-\overline{y})\left[\frac{P_{\mathrm{sy}}(\overline{z}-\overline{y})^2\Lambda(\overline{z})^n\Lambda(\overline{y})^n}{1+P_{\mathrm{sy}}(\overline{z}-\overline{y})^2\Lambda(\overline{z})^n\Lambda(\overline{y})^n}\right]. \tag{5.2.4}$$

In the region of integration $\Lambda(\overline{z}) \leq |\overline{y}-\overline{z}|$, then $\Lambda(\overline{y}) \leq 2|\overline{y}-\overline{z}|$, by the Lipschitz property. Therefore, returning to (5.2.4) we get

$$\begin{aligned}\big|P_{\mathrm{sy},\Lambda}(\overline{z},\overline{y}) - P_{\mathrm{sy}}(\overline{z}-\overline{y})\big| &\leq P_{\mathrm{sy}}(\overline{z}-\overline{y})\Big[P_{\mathrm{sy}}(\overline{z}-\overline{y})^2\Lambda(\overline{z})^n\Lambda(\overline{y})^n\Big]\\ &\leq 2^n P_{\mathrm{sy}}(\overline{z}-\overline{y})^2\Lambda(\overline{z})^n \lesssim \frac{\Lambda(\overline{z})^n}{|\overline{z}-\overline{y}|^{2n}},\end{aligned}$$

so by the n-growth of μ we obtain (again, integrating over annuli)

$$A \lesssim \int_{|\overline{y}-\overline{z}|\geq\Lambda(\overline{z})} \frac{\Lambda(\overline{z})^n}{|\overline{z}-\overline{y}|^{2n}}\,\mathrm{d}\mu(\overline{y}) \lesssim 1,$$

that is what we wanted to prove. Hence, combining this last estimate with $|\mathscr{P}_{\mathrm{sy},\varepsilon}\mu(\overline{z})| \lesssim 1$, for any $\varepsilon \geq \varepsilon(\overline{z})$ we deduce

$$|\mathscr{P}_{\mathrm{sy},\Lambda,\varepsilon}\mu(\overline{z})| \lesssim 1, \quad \forall\varepsilon \geq \varepsilon(\overline{z}).$$

Now, fixing $\eta \in (0,\varepsilon(\overline{z}))$, using property *3* in Lemma 5.2.1 together with $\eta < \Lambda(\overline{z})$, we get

$$\bigg|\int_{|\overline{y}-\overline{z}|\leq\eta} P_{\mathrm{sy},\Lambda}(\overline{z},\overline{y})\,\mathrm{d}\mu(\overline{y})\bigg| \leq \frac{1}{\Lambda(\overline{z})^n}\mu(B_\eta(\overline{z})) \leq 1.$$

All in all, we have proved

$$\big|\mathscr{P}_{\mathrm{sy},\Lambda,*}\mu(\overline{z})\big| \lesssim 1, \quad \forall \overline{z} \in E\setminus F.$$

In fact, this last estimate also holds for $\overline{z} \in F$, since in this case $\Lambda(\overline{z}) = 0$ and $P_{\mathrm{sy},\Lambda} = P_{\mathrm{sy}}$, implying

$$\begin{aligned}\big|\mathscr{P}_{\mathrm{sy},\Lambda,*}\mu(\overline{z})\big| &= \sup_{\eta>0}\bigg|\int_{|\overline{y}-\overline{z}|\geq\eta} P_{\mathrm{sy},\Lambda}(\overline{z},\overline{y})\,\mathrm{d}\mu(\overline{y})\bigg|\\ &= \sup_{\eta>0}\bigg|\int_{|\overline{y}-\overline{z}|\geq\eta} P_{\mathrm{sy}}(\overline{z}-\overline{y})\,\mathrm{d}\mu(\overline{y})\bigg| = \mathscr{P}_{\mathrm{sy},*}\mu(\overline{z}) \leq \sqrt{2},\end{aligned}$$

by definition of F. Hence, we get $\|\mathscr{P}_{\mathrm{sy},\Lambda}\mu\|_{L^\infty(\mu)} \lesssim 1$, where this estimate has to be understood as $\|\mathscr{P}_{\mathrm{sy},\Lambda,\varepsilon}\mu\|_{L^\infty(\mu)} \lesssim 1$, uniformly on $\varepsilon > 0$. The same estimate also holds for $\mathscr{P}^*_{\mathrm{sy},\Lambda}\mu$, since $P_{\mathrm{sy},\Lambda}(\overline{x},\overline{y}) = P_{\mathrm{sy},\Lambda}(\overline{y},\overline{x})$, by the symmetry of P_{sy}. Moreover, it is also clear that for any cube $Q \subset \mathbb{R}^{n+1}$ and any $\varepsilon > 0$,

$$\begin{aligned}|\langle\mathscr{P}_{\mathrm{sy},\Lambda,\mu,\varepsilon}\chi_Q,\chi_Q\rangle| &= \int_Q\bigg(\int_{Q\cap\{|\overline{y}-\overline{z}|>\varepsilon\}} P_{\mathrm{sy},\Lambda}(\overline{z},\overline{y})\,\mathrm{d}\mu(\overline{y})\bigg)\,\mathrm{d}\mu(\overline{z})\\ &\leq \int_Q \mathscr{P}_{\mathrm{sy},\Lambda,\varepsilon}\mu(\overline{z})\,\mathrm{d}\mu(\overline{z}) \lesssim \mu(Q),\end{aligned}$$

by the nonnegativity of $P_{\text{sy},\Lambda}$. So bearing in mind Remark Lemma 5.2.1, we may apply a suitable $T1$-theorem, namely [**T2, Theorem 1.3**], to deduce $\|\mathscr{P}_{\text{sy},\Lambda,\mu}\|_{L^2(\mu)\to L^2(\mu)} \lesssim 1$. But for $f,g\in L^2(\mu)$ supported on F one has $\langle \mathscr{P}_{\text{sy},\Lambda}f,g\rangle = \langle \mathscr{P}_{\text{sy}}f,g\rangle$, meaning that

$$\|\mathscr{P}_{\text{sy},\mu|_F}\|_{L^2(\mu|_F)\to L^2(\mu|_F)} \lesssim 1,$$

and we are done. □

Remark 5.2.1. Due to the above Lemma 5.2.2, we may add $\gamma_{\text{sy},2}$ to the statement of Theorem 5.1.1. In fact, defining

$$\widetilde{\gamma}_2(E) := \sup\big\{\mu(E)\,:\,\mu\in\Sigma(E),\,\|P*\mu\|_{L^2(\mu)}\leq \mu(E)^{1/2}, \|P^**\mu\|_{L^2(\mu)}\leq\mu(E)^{1/2}\big\},$$

it is clear that $\widetilde{\gamma}_2(E)\leq\gamma_{\text{sy},2}(E)$ and that $\widetilde{\gamma}_+(E)\leq\widetilde{\gamma}_2(E)$, by (5.2.2). That is, $\widetilde{\gamma}_2$ can also be added to Theorem 5.1.1.

5.2.2 Normalization by a uniform bound at all points of the support

For a fixed $E\subset\mathbb{R}^{n+1}$ compact set, let us consider now the capacity

$$\gamma_+^\star(E) := \sup\big\{\mu(E)\,:\,\mu\in\Sigma(E),\,|P*\mu(\overline{x})|\leq 1,\,\forall\overline{x}\in E\big\},$$

as well as all the variants $\overline{\gamma}_+^\star, \widetilde{\gamma}_+^\star$ and $\gamma_{\text{sy},+}^\star$. It is clear that $\gamma_+^\star(E)\leq\gamma_+(E)\approx\gamma_{\text{op}}(E)$. We claim that the following holds:

$$\gamma_{\text{op}}(E)\lesssim\gamma_+^\star(E).$$

Indeed, given $\mu\in\Sigma(E)$ admissible for $\gamma_{\text{op}}(E)$ with $\gamma_{\text{op}}(E)\leq 2\mu(E)$, proceeding as in Remark 5.1.2 we can find a function $h:E\to[0,1]$ such that $\int_E h\mathrm{d}\mu\geq\mu(E)/2$ and $\|\mathscr{P}(h\mu)\|_\infty\leq C$, for some $C>0$ dimensional constant. We know that the latter estimate implies $\|\mathscr{P}(h\mu)\|_{L^\infty(\mu)}\leq C'$ for some other dimensional constant $C'>0$ (by a Cotlar type inequality analogous to that of [**MattiPar, Lemma 5.4**]). Applying now Cotlar's inequality of [**T5, Theorem 2.18**], for example, we have

$$\sup_{\varepsilon>0}\big|\mathscr{P}_\varepsilon(h\mu)(\overline{x})\big|\leq C''\big(\widetilde{M}_\mu|\mathscr{P}(h\mu)|^\delta\big)^{1/\delta}(\overline{x})+C'''\big(\widetilde{M}_\mu h\big)(\overline{x}),\quad\forall\overline{x}\in\mathbb{R}^{n+1},$$

where

$$\widetilde{M}_\mu h(\overline{x}) := \sup_{r>0}\frac{1}{\mu(B_{3r}(\overline{x}))}\int_{B_r(\overline{x})}|h(\overline{y})|\,\mathrm{d}\mu(\overline{y}),$$

$\delta\in(0,1)$ is arbitrarily fixed and C'',C''' are positive constants that depend on the dimension, δ and the $L^2(\mu)$-norm of $\mathscr{P}_\mu$, that equals 1 by hypothesis. Since $0\leq h\leq 1$, it is clear that $\widetilde{M}_\mu h\leq 1$; and since $\|\mathscr{P}(h\mu)\|_{L^\infty(\mu)}\leq C'$, we deduce for $\delta=1/2$

$$\widetilde{M}_\mu|\mathscr{P}(h\mu)|^{1/2}(\overline{x}) = \sup_{r>0}\frac{1}{\mu(B_{3r}(\overline{x}))}\int_{B_r(\overline{x})}\big|\mathscr{P}(h\mu)(\overline{x})\big|^{1/2}\,\mathrm{d}\mu(\overline{y})\leq(C')^{1/2}.$$

Therefore, setting $\widetilde{C} := \max\{1, C'C'' + C'''\}$ we get

$$\big|\mathscr{P}(\widetilde{C}^{-1}h\mu)(\overline{x})\big| = \sup_{\varepsilon>0}\big|\mathscr{P}_\varepsilon(\widetilde{C}^{-1}h\mu)(\overline{x})\big| \leq 1, \quad \forall \overline{x}\in\mathbb{R}^{n+1}.$$

So in particular $\widetilde{C}^{-1}h\mu$ is an admissible measure (up to a dimensional factor that makes it an n-growth measure with constant 1) for $\gamma_+^\star(E)$. Thus,

$$\gamma_+^\star(E) \gtrsim \frac{1}{\widetilde{C}}\int_E h\,\mathrm{d}\mu \geq \frac{1}{2\widetilde{C}}\mu(E),$$

and by the arbitrariness of μ we conclude that $\gamma_{\mathrm{op}}(E) \lesssim \gamma_+^\star(E)$.

Remark 5.2.2. Therefore, we conclude that the capacity $\gamma_+^\star$ can be added to the statement of Theorem 5.1.1. In fact, since the operator $L^2(\mu)$-norm of $\mathscr{P}_\mu^*$ is the same as that of $\mathscr{P}_\mu$, following an analogous argument we may also add $\overline{\gamma}_+^\star$ as well as $\widetilde{\gamma}_+^\star$. Further, the same arguments can also be followed to compare $\gamma_{\mathrm{sy},+}^\star$ with $\gamma_{\mathrm{sy,op}}$, allowing us to also consider $\gamma_{\mathrm{sy},+}^\star$ in Theorem 5.1.1.

5.2.3 An equivalent variational capacity

We shall consider a construction already presented in the proof of Lemma 5.2.2: let ψ be a radial test function with $0\leq\psi\leq 1$, $\psi\equiv 0$ in $B_{1/2}(0)$, $\psi\equiv 1$ in $\mathbb{R}^{n+1}\setminus B_1(0)$ and such that $\|\nabla\psi\|_\infty\leq 1$. Fix $E\subset\mathbb{R}^{n+1}$ compact set and $\mu\in\Sigma(E)$. For each $\tau>0$, write $\psi_\tau(\cdot) := \psi(\cdot/\tau)$ and define, for each $\overline{x}\in\mathbb{R}^{n+1}$ and $f\in L^1_{\mathrm{loc}}(\mu)$.

$$\mathscr{P}_{\mathrm{sy},\psi_\tau}(f\mu)(\overline{x}) := P_{\mathrm{sy}}\psi_\tau * (f\mu)(\overline{x}) = \int_{\mathbb{R}^{n+1}} P_{\mathrm{sy}}(\overline{x}-\overline{y})\psi_\tau(\overline{x}-\overline{y})f(\overline{y})\mathrm{d}\mu(\overline{y}).$$

Let us prove, using [**MatP, Lemma 2.1**] (also valid for P^* and P_{sy}) , that the regularized (continuous) kernel $P_{\mathrm{sy}}\psi_\tau$ defines a C-Z convolution operator with constants not depending on τ. It is clear that $|P_{\mathrm{sy}}\psi_\tau(\overline{x})|\leq|\overline{x}|^{-n}$. Then, by the symmetry of $P_{\mathrm{sy}}\psi_\tau$, it suffices to check that for any $\overline{x},\overline{x}'\in\mathbb{R}^{n+1}$ with $\overline{x}\neq 0$ and $|\overline{x}-\overline{x}'|\leq|\overline{x}|/2$,

$$\big|P_{\mathrm{sy}}\psi_\tau(\overline{x}) - P_{\mathrm{sy}}\psi_\tau(\overline{x}')\big| \leq C\frac{|\overline{x}-\overline{x}'|}{|\overline{x}|^{n+1}}, \tag{5.2.5}$$

where $C>0$ is a dimensional constant independent of τ. To prove this, we distinguish two cases: if $\tau\geq|\overline{x}|/4$,

$$\begin{aligned}\big|P_{\mathrm{sy}}\psi_\tau(\overline{x}) - P_{\mathrm{sy}}\psi_\tau(\overline{x}')\big| &\leq \big|P_{\mathrm{sy}}(\overline{x}) - P_{\mathrm{sy}}(\overline{x}')\big|\psi_\tau(\overline{x}') + \big|\psi_\tau(\overline{x})-\psi_\tau(\overline{x}')\big|P_{\mathrm{sy}}(\overline{x})\\ &\lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}|^{n+1}} + \frac{|\overline{x}-\overline{x}'|}{\tau}\frac{1}{|\overline{x}|^n} \lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}|^{n+1}},\end{aligned}$$

where we have applied [**MatP, Lemma 2.1**] and $\|\nabla\psi\|_\infty\leq 1$. If on the other hand $\tau<|\overline{x}|/4$, by definition of ψ_τ we have $\psi_\tau(\overline{x})=1$. In addition, by the triangle inequality,

$$|\overline{x}'| \geq \big||\overline{x}-\overline{x}'| - |\overline{x}|\big|.$$

If $||\overline{x}-\overline{x}'|-|\overline{x}||=|\overline{x}-\overline{x}'|-|\overline{x}|$, then $|\overline{x}-\overline{x}'|\geq|\overline{x}|$, so $|\overline{x}-\overline{x}'|\approx|\overline{x}|$. So in this case,

$$\begin{aligned}\big|P_{\text{sy}}\psi_\tau(\overline{x})-P_{\text{sy}}\psi_\tau(\overline{x}')\big| &\leq \big|P_{\text{sy}}(\overline{x})-P_{\text{sy}}(\overline{x}')\big|\psi_\tau(\overline{x}')+\big|1-\psi_\tau(\overline{x}')\big|P_{\text{sy}}(\overline{x})\\ &\lesssim \frac{|\overline{x}-\overline{x}'|}{|\overline{x}|^{n+1}}+\frac{2}{|\overline{x}|^n}\leq\frac{3|\overline{x}-\overline{x}'|}{|\overline{x}|^{n+1}},\end{aligned}$$

and we are done. If on the other hand $||\overline{x}-\overline{x}'|-|\overline{x}||=|\overline{x}|-|\overline{x}-\overline{x}'|$, then $|\overline{x}'|\geq|\overline{x}|-|\overline{x}-\overline{x}'|\geq|\overline{x}|/2$ which implies $\psi_\tau(\overline{x}')=1$. Then, in this case,

$$\big|P_{\text{sy}}\psi_\tau(\overline{x})-P_{\text{sy}}\psi_\tau(\overline{x}')\big|=\big|P_{\text{sy}}(\overline{x})-P_{\text{sy}}(\overline{x}')\big|\lesssim\frac{|\overline{x}-\overline{x}'|}{|\overline{x}|^{n+1}},$$

and the proof of (5.2.5) is complete.

Let us also observe that for any $f\in L^1_{\text{loc}}(\mu)$ any $\tau>0$ and any $\overline{x}\in E$,

$$\begin{aligned}\big|\mathscr{P}_{\text{sy},\tau}(f\mu)(\overline{x})-\mathscr{P}_{\text{sy},\psi_\tau}(f\mu)(\overline{x})\big| &\leq \int_{\tau/2<|\overline{x}-\overline{y}|<\tau}P_{\text{sy}}(\overline{x}-\overline{y})\big|1-\psi_\tau(\overline{x}-\overline{y})\big||f(\overline{y})|\,\mathrm{d}\mu(\overline{y})\\ &\lesssim \sup_{\tau>0}\frac{1}{\tau^n}\int_{|\overline{x}-\overline{y}|<\tau}|f(\overline{y})|\,\mathrm{d}\mu(\overline{y}).\end{aligned}$$

In particular, choosing $f\equiv 1$, by the n-growth of μ we get

$$\big|\mathscr{P}_{\text{sy},\tau}\mu(\overline{x})-\mathscr{P}_{\text{sy},\psi_\tau}\mu(\overline{x})\big|\lesssim 1,\quad\forall\overline{x}\in E,\,\tau>0. \tag{5.2.6}$$

Let us define the auxiliary capacity, for each $\tau>0$,

$$\gamma^{\star}_{\text{sy},\psi_\tau,+}(E):=\sup\big\{\mu(E)\,:\,\mu\in\Sigma(E),\,|P_{\text{sy}}\psi_\tau*\mu(\overline{x})|\leq 1,\,\forall\overline{x}\in E\big\}.$$

Lemma 5.2.3. *The following estimates hold:*

1. $\limsup_{\tau\to 0}\gamma^{\star}_{\text{sy},\psi_\tau,+}(E)\lesssim\gamma^{\star}_{\text{sy},+}(E)$.
2. $\liminf_{\tau\to 0}\gamma^{\star}_{\text{sy},\psi_\tau,+}(E)\gtrsim\gamma^{\star}_{\text{sy},+}(E)$.

Proof. We begin by proving *1*. Let $(\tau_k)_k$ be a monotonically decreasing sequence to 0, and let $\mu_k\in\Sigma(E)$ be admissible for $\gamma^{\star}_{\text{sy},\psi_{\tau_k},+}(E)$ and such that $\gamma^{\star}_{\text{sy},\psi_{\tau_k},+}(E)\leq 2\mu_k(E)$. Passing to a subsequence if necessary, assume $\mu_k\rightharpoonup\mu_0$ weakly [**Matti, Theorem 1.23**], i.e.

$$\lim_{k\to\infty}\int\varphi\,\mathrm{d}\mu_k=\int\varphi\,\mathrm{d}\mu_0,\quad\forall\varphi\in\mathcal{C}_c(\mathbb{R}^{n+1}).$$

It is not hard to prove that $\mu_0\in\Sigma(E)$. Now fix $m>0$ integer and $\tau\in[\tau_{m+1},\tau_m]$, and also assume $k>m$. By (5.2.6) we get that for $\overline{x}\in E$,

$$\big|\mathscr{P}_{\text{sy},\psi_\tau}\mu_k(\overline{x})\big|\leq\big|\mathscr{P}_{\text{sy},\psi_{\tau_k}}\mu_k(\overline{x})\big|+\big|\mathscr{P}_{\text{sy},\psi_\tau}\mu_k(\overline{x})-\mathscr{P}_{\text{sy},\psi_{\tau_k}}\mu_k(\overline{x})\big|\leq 1+C,$$

for some finite dimensional constant $C>0$. Since $\mu_k\rightharpoonup\mu_0$ we obtain $\big|\mathscr{P}_{\text{sy},\psi_\tau}\mu_0\big|\leq 1+C$ on E and for each $\tau>0$. In addition, using (5.2.6) again we get

$$\big|\mathscr{P}_{\text{sy},\tau}\mu_0(\overline{x})\big|\leq 1+2C,\quad\forall\overline{x}\in E,\,\tau>0,$$

that is $|\mathscr{P}_{\text{sy}}\mu_0(\overline{x})| \leq 1+2C$ for all $\overline{x}\in E$. Thus, by the semicontinuity properties of weak convergence of [**Matti, Theorem 1.24**] we get

$$\gamma^{\star}_{\text{sy},+}(E) \geq \frac{\mu_0(E)}{1+2C} \geq \frac{1}{1+2C}\limsup_{k\to\infty}\mu_k(E) \gtrsim \limsup_{\tau\to 0}\gamma^{\star}_{\text{sy},\psi_{\tau_k},+}(E),$$

and *1* follows. To prove *2*, take μ admissible for $\gamma^{\star}_{\text{sy},+}(E)$. Using (5.2.6) we deduce

$$\big|\mathscr{P}_{\text{sy},\psi_\tau}\mu(\overline{x})\big| \leq 1+C, \quad \forall \overline{x}\in E,\, \tau>0,$$

implying that $\gamma^{\star}_{\text{sy},\psi_\tau,+}(E) \geq \frac{\mu(E)}{1+C}$, and by the arbitrariness of μ we have

$$\inf_{\tau>0}\gamma^{\star}_{\text{sy},\psi_\tau,+}(E) \gtrsim \gamma^{\star}_{\text{sy},+}(E),$$

which implies the desired result. □

Remark 5.2.3. Let us observe that, for each $\tau>0$, using the regularized (continuous) kernel $P_{\text{sy}}\psi_\tau$ we may also define the rest of corresponding capacities, requiring different normalization conditions over the potentials. In fact, the following chain of estimates holds for any compact set $E\subset\mathbb{R}^{n+1}$:

$$\gamma^{\star}_{\text{sy},\psi_\tau,+}(E) \underset{(1)}{\leq} \gamma_{\text{sy},\psi_\tau,2}(E) \underset{(2)}{\lesssim} \gamma_{\text{sy},\psi_\tau,\text{op}}(E) \underset{(3)}{\lesssim} \gamma_{\text{sy},\psi_\tau,+}(E) \underset{(4)}{\leq} \gamma^{\star}_{\text{sy},\psi_\tau,+}(E).$$

Inequality (1) is trivial. To verify (2), proceed, for example, as in the proof of $\gamma_{\text{sy},2}(E)\lesssim\gamma_{\text{sy},\text{op}}(E)$ in Lemma 5.2.2 (although the arguments could be possibly simplified using the continuity of $P_{\text{sy}}\psi_\tau$). The proof of (3) goes as in Remark 5.1.2, and it relies on [**T5, Theorem 2.16**] and [**Chr, Ch.VII, Theorem 23**]. it is important to notice that a fundamental property that ensures the validity of (2) and (3) is that the C-Z constants of $P_{\text{sy}}\psi_\tau$ are independent of τ. Finally, (4) holds by the continuity of $\mathscr{P}_{\text{sy},\psi_\tau}\mu$ for each $\tau>0$, that has already been argued in the proof of Lemma 5.2.2. Observe that the previous chain of estimates also holds changing every γ_{sy} for $\widetilde{\gamma}$, where we ask the respective normalization conditions in each case for both kernels $P\psi_\tau$ and $P^*\psi_\tau$.

Therefore, in particular, we shall restate Lemma 5.2.3 as

1. $\limsup_{\tau\to 0}\gamma_{\text{sy},\psi_\tau,2}(E)\lesssim\gamma_{\text{sy},2}(E)$,

2. $\liminf_{\tau\to 0}\gamma_{\text{sy},\psi_\tau,2}(E)\gtrsim\gamma_{\text{sy},2}(E)$.

Now, bearing in mind that we have found

- ⋆ $\limsup_{\tau\to 0}\gamma_{\text{sy},\psi_\tau,2}(E)\leq C_1\,\gamma_{\text{sy},2}(E)$,
- ⋆ $\gamma_{\text{sy},2}(E)\leq C_2\,\gamma_{\text{sy},\text{op}}(E)$,

for some positive dimensional constants C_1, C_2, we define the variational capacity:

DEFINITION 5.2.1. Let $E \subset \mathbb{R}^{n+1}$ be a compact set and choose τ_0 small enough so that

$$\gamma_{\mathrm{sy},\psi_{\tau_0},2}(E) \leq 2C_1C_2\,\gamma_{\mathrm{sy,op}}(E).$$

Let $\mathcal{S}$ be the convolution operator associated to the kernel $P_{\mathrm{sy}}\psi_{\tau_0}$ (that depends on τ_0 and thus on E). We define the *variational capacity of* E as

$$\gamma_{\mathrm{var}}(E) := \sup\left\{\frac{\mu(E)^2}{\mu(E)+\int_E |\mathcal{S}\mu|^2\,\mathrm{d}\mu}\;:\;\mu\in\Sigma(E)\right\}.$$

We convey that expressions of the form $\frac{0}{0}$ equal 0.

LEMMA 5.2.4. *The supremum in* $\gamma_{\mathrm{var}}(E)$ *is attained. Moreover,* $\gamma_{\mathrm{var}}(E) \approx \gamma_{\mathrm{sy,op}}(E)$*, where we remark that the implicit constants do not depend on* E*.*

Proof. For each $\mu \in \Sigma(E)$ define

$$F(\mu) := \frac{\mu(E)^2}{\mu(E)+\int_E |\mathcal{S}\mu|^2\,\mathrm{d}\mu}.$$

Recalling Lemma 5.2.3, it is not difficult to prove that $\Sigma(E)$ is closed and sequentially compact with respect to weak convergence of measures. In fact, $\Sigma(E)$ is compact (since it is contained in the space of finite signed Radon measures on E, which is metrizable, thought as the dual of the separable space $\mathcal{C}_c(\mathbb{R}^{n+1})$). Moreover, it is clear that if $\mu_k \rightharpoonup \mu_0$ on $\Sigma(E)$, then $\mu_k(E) \to \mu_0(E)$. In this setting, we also claim that

$$\lim_{k\to\infty}\int_E |\mathcal{S}\mu_k|^2\,\mathrm{d}\mu_k = \int_E |\mathcal{S}\mu_0|^2\,\mathrm{d}\mu_0. \tag{5.2.7}$$

To prove the claim (5.2.7) we argue as follows: given continuous functions φ_i, ψ_i for $i = 1,\dots N$ in $\mathcal{C}_c(\mathbb{R}^{n+1})$, by definition of weak convergence of measures we get

$$\begin{aligned}
&\lim_{k\to\infty}\int_E\bigg(\int_E\sum_{i=1}^N \varphi_i(\overline{x})\psi_i(\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2\,\mathrm{d}\mu_k(\overline{y}) \qquad (5.2.8)\\
&\quad=\lim_{k\to\infty}\bigg\{\sum_{i=1}^N\int_E\bigg(\int_E \varphi_i(\overline{x})\psi_i(\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2\,\mathrm{d}\mu_k(\overline{y})\\
&\qquad+2\sum_{1\leq i<j\leq N}\int_E\bigg(\int_E \varphi_i(\overline{x})\psi_i(\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)\bigg(\int_E \varphi_j(\overline{x})\psi_j(\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)\,\mathrm{d}\mu_k(\overline{y})\bigg\}\\
&\quad=\lim_{k\to\infty}\bigg\{\sum_{i=1}^N\bigg(\int_E \varphi_i(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2\bigg(\int_E \psi_i^2(\overline{y})\,\mathrm{d}\mu_k(\overline{y})\bigg)\\
&\qquad+2\sum_{1\leq i<j\leq N}\bigg(\int_E \varphi_i(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\bigg)\bigg(\int_E \varphi_j(\overline{x})\,\mathrm{d}\mu_k(\overline{x})\bigg)\bigg(\int_E \psi_i(\overline{y})\psi_j(\overline{y})\,\mathrm{d}\mu_k(\overline{y})\bigg)\bigg\}\\
&\quad=\int_E\bigg(\int_E\sum_{i=1}^N \varphi_i(\overline{x})\psi_i(\overline{y})\,\mathrm{d}\mu_0(\overline{x})\bigg)^2\,\mathrm{d}\mu_0(\overline{y}).
\end{aligned}$$

Observe that the collection of continuous functions on the compact set $E \times E$ of the form $\sum_{i=1}^N \varphi_i(\overline{x})\psi_i(\overline{y})$ is an algebra that contains the constant functions (on E) and separates points. Therefore, by the Stone-Weierstrass theorem, every continuous function $f : E \times E \to \mathbb{R}$ can be uniformly approximated by such sums. Namely, consider

$$\big(g_j(\overline{x},\overline{y})\big)_j := \bigg(\sum_{i=1}^{N_j} \varphi_{i,j}(\overline{x})\psi_{i,j}(\overline{y})\bigg)_j,$$

so that $\|f - g_j\|_\infty := \|f - g_j\|_{L^\infty(E\times E)} \to 0$ as $j \to \infty$. Observe:

$$\begin{aligned}
&\bigg| \int_E \bigg(\int_E f(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2 \mathrm{d}\mu_k(\overline{y}) - \int_E \bigg(\int_E f(\overline{x},\overline{y})\,\mathrm{d}\mu_0(\overline{x})\bigg)^2 \mathrm{d}\mu_0(\overline{y}) \bigg| \\
&\quad \leq \bigg| \int_E \bigg(\int_E f(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2 \mathrm{d}\mu_k(\overline{y}) - \int_E \bigg(\int_E g_j(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2 \mathrm{d}\mu_k(\overline{y}) \bigg| \\
&\qquad + \bigg| \int_E \bigg(\int_E g_j(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2 \mathrm{d}\mu_k(\overline{y}) - \int_E \bigg(\int_E g_j(\overline{x},\overline{y})\,\mathrm{d}\mu_0(\overline{x})\bigg)^2 \mathrm{d}\mu_0(\overline{y}) \bigg| \\
&\qquad + \bigg| \int_E \bigg(\int_E g_j(\overline{x},\overline{y})\,\mathrm{d}\mu_0(\overline{x})\bigg)^2 \mathrm{d}\mu_0(\overline{y}) - \int_E \bigg(\int_E f(\overline{x},\overline{y})\,\mathrm{d}\mu_0(\overline{x})\bigg)^2 \mathrm{d}\mu_0(\overline{y}) \bigg| \\
&\qquad =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Choose j large enough so that $\|g_j\|_\infty \leq 2\|f\|_\infty$. We estimate I as follows

$$\begin{aligned}
\mathrm{I} &= \bigg| \int_E \bigg(\int_E (f+g_j)(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)\bigg(\int_E (f-g_j)(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg) \mathrm{d}\mu(\overline{y}) \bigg| \\
&\leq 3\|f\|_\infty \cdot \operatorname{diam}(E)^{3n} \|f-g_j\|_\infty \xrightarrow{j\to\infty} 0.
\end{aligned}$$

Term III can be treated the same way. For II use that $\mu_k \rightharpoonup \mu_0$ as $k \to \infty$ and apply (5.2.8) to deduce that II also tends to 0 as $j \to \infty$. The latter can be done since the supports of the measures μ_k are all contained in E, so we may understand each factor $\varphi_{i,j}$ and $\psi_{i,j}$ of the summands that define g_j once extended continuously onto $\mathcal{C}_c(\mathbb{R}^{n+1})$. Hence, by the arbitrariness of ε we deduce

$$\lim_{k\to\infty} \int_E \bigg(\int_E f(\overline{x},\overline{y})\,\mathrm{d}\mu_k(\overline{x})\bigg)^2 \mathrm{d}\mu_k(\overline{y}) = \int_E \bigg(\int_E f(\overline{x},\overline{y})\,\mathrm{d}\mu_0(\overline{x})\bigg)^2 \mathrm{d}\mu_0(\overline{y}),$$

for any f continuous on $E \times E$. Therefore, applying this result to $f := P_{\mathrm{sy}}\psi_{\tau_0}|_E \geq 0$ we get (5.2.7). All in all, we have proved that F defines a continuous functional on a compact space, meaning that it attains its maximum and thus the supremum that defines γ_{var} is indeed attained.

In order to prove $\gamma_{\mathrm{var}}(E) \approx \gamma_{\mathrm{sy,op}}(E)$, we claim that it suffices to prove

$$\gamma_{\mathrm{var}}(E) \approx \gamma_{\mathrm{sy},\psi_{\tau_0},2}(E).$$

Indeed, since $0 \leq P_{\mathrm{sy}}\psi_{\tau_0} \leq P_{\mathrm{sy}}$, we trivially have $\gamma_{\mathrm{sy,op}}(E) \leq \gamma_{\mathrm{sy},\psi_{\tau_0},2}(E)$; and by the choice of τ_0, we also have $\gamma_{\mathrm{sy},\psi_{\tau_0},2}(E) \leq 2C_1C_2\, \gamma_{\mathrm{sy,op}}(E)$, so the claim follows.

Further, observe that we can restrict ourselves to the case that there exists $\mu \in \Sigma(E)$ such that $F(\mu) > 0$. If this was not the case, since $\int_E |\mathcal{S}\mu| \,\mathrm{d}\mu \lesssim \operatorname{diam}(E)^{2n}/\tau_0^n < \infty$, we would have, necessarily, $\mu(E) = 0$, implying, by definition, $\gamma_{\mathrm{var}}(E) = 0 = \gamma_{\mathrm{sy},\psi_{\tau_0},2}(E)$.

So let us assume that there exists $\mu \in \Sigma(E)$ such that $F(\mu) > 0$, meaning $\gamma_{\mathrm{var}}(E) > 0$. Let us begin by proving that any extremal μ_0 for F in this setting satisfies

$$\int_E |\mathcal{S}\mu_0|^2 \,\mathrm{d}\mu_0 \leq \mu_0(E). \tag{5.2.9}$$

If this was not the case, then $\int_E |\mathcal{S}\mu_0|^2 \,\mathrm{d}\mu_0 = M\mu_0(E)$ for some $M > 1$. Define $\mu_1 := M^{-1/2}\mu_0 \in \Sigma(E)$ and notice that

$$\begin{aligned} F(\mu_1) &= \frac{M^{-1}\mu_0(E)^2}{M^{-1/2}\mu_0(E) + \int_E |\mathcal{S}\mu_1|^2 \,\mathrm{d}\mu_1} \\ &= \frac{M^{-1}\mu_0(E)^2}{M^{-1/2}\mu_0(E) + M^{-1/2}\mu_0(E)} = \frac{\mu_0(E)}{2M^{1/2}} > \frac{\mu_0(E)}{1+M} = F(\mu_0), \end{aligned}$$

that is a contradiction. Then, (5.2.9) holds, and it implies, by definition, $\mu_0(E) \leq \gamma_{\mathrm{sy},\psi_{\tau_0},2}(E)$. So we have

$$\gamma_{\mathrm{var}}(E) = F(\mu_0) < \mu_0(E) \leq \gamma_{\mathrm{sy},\psi_{\tau_0},2}(E).$$

On the other hand, for any μ admissible for $\gamma_{\mathrm{sy},\psi_{\tau_0},2}(E)$,

$$\frac{\mu(E)}{2} = \frac{\mu(E)^2}{\mu(E) + \mu(E)} \leq \frac{\mu(E)^2}{\mu(E) + \int_E |\mathcal{S}\mu|^2 \,\mathrm{d}\mu} \leq \gamma_{\mathrm{var}}(E),$$

and the proof is complete. □

Remark 5.2.4. Let us notice that (5.2.9) holds in general for any extremal measure μ_0 for F. Indeed, in the case that for any $\mu \in \Sigma(E)$ one had $F(\mu) = 0$, then $\mu(E) = 0$. So any measure would be extremal and (5.2.9) would hold trivially.

Finally we give a new version of Theorem 5.1.1 that relates the capacities presented in this section:

Theorem 5.2.5. *All of the above capacities defined via positive Borel measures are comparable. More precisely, for $E \subset \mathbb{R}^{n+1}$ compact,*

$$\gamma_+(E) \approx \gamma_+^\star(E),$$

where the respective bounds of the potentials can be taken indifferently and independently for P, P^*, P_{sy} or both P and P^* simultaneously. Moreover, they are also comparable to

$$\widetilde{\gamma}_2(E),\ \gamma_{\mathrm{sy},2}(E),\ \gamma_{\mathrm{op}}(E),\ \gamma_{\mathrm{sy,op}}(E) \quad \textit{and} \quad \gamma_{\mathrm{var}}(E).$$

In addition, for each $\tau > 0$ the following holds

$$\gamma_{\mathrm{sy},\psi_\tau,+}(E) \approx \gamma^\star_{\mathrm{sy},\psi_\tau,+}(E) \approx \gamma_{\mathrm{sy},\psi_\tau,2}(E) \approx \gamma_{\mathrm{sy},\psi_\tau,\mathrm{op}}(E)$$

as well as

$$\limsup_{\tau\to 0} \gamma_{\mathrm{sy},\psi_\tau,+}(E) \lesssim \gamma_{\mathrm{sy},+}(E) \lesssim \liminf_{\tau\to 0} \gamma_{\mathrm{sy},\psi_\tau,+}(E).$$

The relations for the capacities depending on ψ_τ also hold changing each γ_{sy} for $\widetilde{\gamma}$.

5.3 The construction of the cubes

The goal of this section is to use Theorem 5.2.5 to carry out the construction done in [**Vo, §5.2**] for Riesz kernels. We aim to prove the following result:

THEOREM 5.3.1. *Let $E \subset B_1(0) \subset \mathbb{R}^{n+1}$ be a compact set consisting of a finite union of closed cubes, with sides parallel to the axes. Then, there exists a finite collection of dyadic cubes $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_N\}$ that cover E and such that $\frac{1}{2}\mathcal{Q}_1, \ldots, \frac{1}{2}\mathcal{Q}_N$ have disjoint interiors. Moreover, if $\mathcal{F} := \cup_{i=1}^N \mathcal{Q}_i$,*

- $\mathbf{P_1}$. *$\frac{5}{8}\mathcal{Q}_i \cap E \neq \varnothing$, for each $i = 1, \ldots, N$.*
- $\mathbf{P_2}$. *$\widetilde{\gamma}_+(\mathcal{F}) \leq C_0 \widetilde{\gamma}_+(E)$.*
- $\mathbf{P_3}$. *$\sum_{i=1}^N \widetilde{\gamma}_+(2\mathcal{Q}_i \cap E) \leq C_1 \widetilde{\gamma}_+(E)$.*
- $\mathbf{P_4}$. *If $\widetilde{\gamma}_+(E) \leq C_+ \text{diam}(E)^n$, then $\text{diam}(\mathcal{Q}_i) \leq \frac{1}{10}\text{diam}(E)$, for each $i = 1, \ldots, N$.*
- $\mathbf{P_5}$. *The family $\{5\mathcal{Q}_1, \ldots, 5\mathcal{Q}_N\}$ has bounded overlap with constant C_2.*

Constants C_0, C_1, C_2, C_+ are dimensional.

Before proceeding, let us clarify the from this point on $E \subset \mathbb{R}^{n+1}$ will be a fixed compact set and μ_0 will always denote a maximizer of $\gamma_{\text{var}}(E)$. Let us also write explicitly the following expression, that will appear repeatedly, for the sake of clarity: for μ, ν say positive finite Borel measures supported on E, we have, in light of Definition 5.2.1,

$$\mathcal{S}_\mu(\mathcal{S}\nu)(\overline{x}) = \int_E \int_E P_{\text{sy}}(\overline{x}-\overline{y}) P_{\text{sy}}(\overline{y}-\overline{z}) \psi_{\tau_0}(\overline{x}-\overline{y}) \psi_{\tau_0}(\overline{y}-\overline{z})\, \mathrm{d}\nu(\overline{z})\, \mathrm{d}\mu(\overline{y}).$$

LEMMA 5.3.2. *Let H be a positive Borel measure supported on E such that for any $\lambda_0 > 0$ and any $\lambda \in [0, \lambda_0]$, $\mu_\lambda := \mu_0 + \lambda H \in \Sigma(E)$. Then,*

$$H(E)\mu_0(E)\bigg(\mu_0(E) + 2\int_E |\mathcal{S}\mu_0|^2\, \mathrm{d}\mu_0\bigg) \leq \mu_0(E)^2 \int_E \Big(|\mathcal{S}\mu_0|^2 + 2\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)\Big)\, \mathrm{d}H.$$

Proof. Let us begin by noticing that

$$F(\mu_\lambda) = \frac{\big[\mu_0(E) + \lambda H(E)\big]^2}{\mu_0(E) + \lambda H(E) + \int_E |\mathcal{S}\mu_\lambda|^2\, \mathrm{d}\mu_\lambda},$$

where the integral in the denominator can be expanded as

$$\begin{aligned}\int_E |\mathcal{S}\mu_\lambda|^2\, \mathrm{d}\mu_\lambda = \int_E (\mathcal{S}\mu_0)^2\, \mathrm{d}\mu_0 + \lambda \int_E (\mathcal{S}\mu_0)^2\, \mathrm{d}H + 2\lambda \int_E (\mathcal{S}\mu_0)(\mathcal{S}H)\, \mathrm{d}\mu_0 \\ + 2\lambda^2 \int_E (\mathcal{S}\mu_0)(\mathcal{S}H)\, \mathrm{d}H + \lambda^2 \int_E (\mathcal{S}H)^2\, \mathrm{d}\mu_0 + \lambda^3 \int_E (\mathcal{S}H)^2\, \mathrm{d}H.\end{aligned}$$

Observe that for $\nu \in \{\mu_0, H\}$, since P_{sy} and ψ_{τ_0} are symmetric functions,

$$\begin{aligned}
&\int_E \mathcal{S}\mu_0 1(\overline{x})\mathcal{S}H(\overline{x})\,\mathrm{d}\nu(\overline{x})\\
&\quad= \int_E \bigg(\int_E P_{\text{sy}}(\overline{x}-\overline{y})\psi_{\tau_0}(\overline{x}-\overline{y})\,\mathrm{d}\mu_0(\overline{y})\bigg)\bigg(\int_E P_{\text{sy}}(\overline{x}-\overline{z})\psi_{\tau_0}(\overline{x}-\overline{z})\,\mathrm{d}H(\overline{z})\bigg)\,\mathrm{d}\nu(\overline{x})\\
&\quad= \int_E \bigg(\int_E \bigg(\int_E P_{\text{sy}}(\overline{x}-\overline{y})\psi_{\tau_0}(\overline{x}-\overline{y})\,\mathrm{d}\mu_0(\overline{y})\bigg)P_{\text{sy}}(\overline{z}-\overline{x})\psi_{\tau_0}(\overline{z}-\overline{x})\,\mathrm{d}\nu(\overline{x})\bigg)\,\mathrm{d}H(\overline{z})\\
&\quad= \int_E \mathcal{S}_\nu(\mathcal{S}\mu_0)(\overline{z})\,\mathrm{d}H(\overline{z}).
\end{aligned}$$

Therefore, we may rewrite $\int_E |\mathcal{S}\mu_\lambda|^2\,\mathrm{d}\mu_\lambda$ as

$$\begin{aligned}
\int_E |\mathcal{S}\mu_\lambda|^2\,\mathrm{d}\mu_\lambda = \int_E (\mathcal{S}\mu_0)^2\,\mathrm{d}\mu_0 + \lambda\bigg[\int_E (\mathcal{S}\mu_0)^2\,\mathrm{d}H + 2\int_E \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)\,\mathrm{d}H\bigg]&\\
+\lambda^2\bigg[\int_E (\mathcal{S}H)^2\,\mathrm{d}\mu_0 + 2\int_E \mathcal{S}_H(\mathcal{S}\mu_0)\,\mathrm{d}H\bigg] + \lambda^3\int_E (\mathcal{S}H)^2\,\mathrm{d}H.&
\end{aligned}$$

So condition $F(\mu_0) \geq F(\mu_\lambda)$ can be written as

$$\begin{aligned}
&\frac{\mu_0(E)^2}{\mu_0(E) + \int_E (\mathcal{S}\mu_0)^2\,\mathrm{d}\mu_0}\\
&\qquad\geq \frac{\mu_0(E)^2 + 2\lambda\mu_0(E)H(E) + \lambda^2 H(E)^2}{\mu_0(E) + \lambda H(E) + \int_E (\mathcal{S}\mu_0)^2\,\mathrm{d}\mu_0 + \lambda[\cdots] + \lambda^2[\cdots] + \lambda^3[\cdots]},
\end{aligned}$$

or equivalently

$$\begin{aligned}
\lambda\bigg\{\mu_0(E)^2\bigg(-H(E) + \int_E (\mathcal{S}\mu_0)^2\,\mathrm{d}H + 2\int_E \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)\,\mathrm{d}H\bigg)&\\
-2\mu_0(E)H(E)\int_E (\mathcal{S}\mu_0)^2\,\mathrm{d}\mu_0\bigg\} + \lambda^2[\cdots] + \lambda^3\int_E (\mathcal{S}H)^2\,\mathrm{d}H \geq 0.&
\end{aligned}$$

Assume that $\lambda > 0$ and divide both sides of the previous inequality by this factor and make $\lambda \to 0$ to obtain the desired estimate. Let us notice that the kernel associated to $\mathcal{S}$ is continuous, meaning that the terms in $[\cdots]$ and $\int_E (\mathcal{S}H)^2\,\mathrm{d}H$ are finite, since $H(E) < \infty$ due to the fact that $\mu_\lambda \in \Sigma(E)$ for any $\lambda \in [0, \lambda_0]$ with $\lambda_0 > 0$. □

We remark that in the particular case we will apply this lemma, E will be a finite union of cubes, so $\mu_0(E) > 0$. Moreover, we will choose a specific measure H so that $H(E) > 0$. Therefore, in this setting, the inequality in Lemma 5.3.3 can be rewritten as

$$\frac{\mu_0(E) + 2\int_E |\mathcal{S}\mu_0|^2\,\mathrm{d}\mu_0}{\mu_0(E)} \leq \frac{1}{H(E)}\int_E \Big(|\mathcal{S}\mu_0|^2 + 2\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)\Big)\,\mathrm{d}H, \tag{5.3.1}$$

which is a similar estimate that resembles that of [**Vo, Lemma 5.6**].

We shall apply (5.3.1) to prove an auxiliary lemma, but first let us introduce the following notation for a certain maximal function applied to a measure:

$$M\nu(\overline{x}) := \sup_{r>0} \frac{\nu(B_r(\overline{x}))}{r^n}.$$

LEMMA 5.3.3. *Let E be a finite union of cubes. Then, for the potential*

$$\mathcal{U}^{\mu_0} := M\mu_0 + \mathcal{S}\mu_0 + \mathcal{S}_{\mu_0}\mathcal{S}\mu_0,$$

there is a dimensional constant $\alpha_0 > 0$ such that

$$\mathcal{U}^{\mu_0}(\overline{x}) \geq \alpha_0, \quad \forall \overline{x} \in E.$$

Proof. Choose $\alpha_1 \in (0, 80^{-n})$. If $\overline{x}_0 \in E$ is such that $M\mu_0(\overline{x}_0) \geq \alpha_1$ then we are done at $\overline{x}_0$. So assume $M\mu_0(\overline{x}_0) < \alpha_1$. Pick $\varepsilon_1 \leq \tau_0, R_0 := \varepsilon_1/10$ and $B_0 := B_{R_0}(\overline{x}_0)$. Set $\mu_{00} := \mu_0|_{2B_0}$ and define

$$G := \{\overline{y} \in B_0 \cap E \,:\, M\mu_0(\overline{y}) \leq 4^{-n}\}.$$

We shall prove that the complement of G in $B_0 \cap E$ is small. If $\overline{y} \in (B_0 \cap E) \setminus G$, then there exists $r = r(\overline{y}) > 0$ so that

$$\frac{\mu_0(B_r(\overline{y}))}{r^n} > 4^{-n}.$$

If we had $r > \varepsilon_1/20 = R_0/2$, then

$$\frac{1}{80^n} \leq \frac{\mu_0(B_r(\overline{y}))}{20^n r^n} \leq \frac{\mu_0(B_{20r}(\overline{x}_0))}{(20r)^n} < \alpha_1, \quad \text{since } M\mu_0(\overline{x}_0) < \alpha_1,$$

and this cannot be. Then $r \leq \varepsilon_1/20$, implying $B_r(\overline{y}) \cap (2B_0) = B_r(\overline{y})$, which in turn implies $\mu_{00}(B_r(\overline{y})) = \mu_0(B_r(\overline{y}))$. So we have found

$$\forall \overline{y} \in (B_0 \cap E) \setminus G,\, \exists r(\overline{y}) > 0 \text{ such that } \frac{\mu_{00}(B_r(\overline{y}))}{r^n} > 4^{-n}. \tag{5.3.2}$$

We continue by choosing ε_1 (that was already smaller than τ_0) small enough so that

$$\mathcal{L}^{n+1}(B_0 \cap E) \geq a(n)R_0^{n+1} =: aR_0^{n+1},$$

for some dimensional constant $a > 0$. This can be done since E is a finite union of cubes. Notice also that the dependence on E of the previous estimate is in R_0, since ε_1 depends on τ_0, that depends on E. Using (5.3.2) and [**Matti, Theorem 2.1**] we obtain a countable covering $\{B_{5r_j}(\overline{y}_j)\}$ of $(B_0 \cap E) \setminus G$ with $\{B_{r_j}(\overline{y}_j)\}$ disjoint and also satisfying (5.3.2). Now,

$$\begin{aligned}
\mathcal{L}^{n+1}\big((B_0 \cap E) \setminus G\big) &\leq 5^{n+1} \sum_j r_j^{n+1} \leq 2 \cdot 5^{n+1} R_0 \sum_j r_j^n \\
&< 10 \cdot 4^n \cdot 5^{n+1} R_0 \sum_j \mu_{00}(B_{r_j}(\overline{y}_j)) = 10 \cdot 20^n R_0\, \mu_{00}\Big(\bigcup_j B_{r_j}(\overline{y}_j)\Big) \\
&\leq 10 \cdot 20^n R_0\, \mu_0(2B_0) \leq 10 \cdot 40^n R_0^{n+1}\alpha_1 =: A(n)R_0^{n+1}\alpha_1 \\
&\leq \frac{A}{a}\alpha_1 \mathcal{L}^{n+1}(B_0 \cap E),
\end{aligned}$$

where we have applied the hypothesis $M\mu_0(\overline{x}_0) < \alpha_1$. Therefore, by definition of G,

$$\mathcal{L}^{n+1}(G) \geq \left(\frac{a}{A\alpha_1} - 1\right)\mathcal{L}^{n+1}\big((B_0 \cap E) \setminus G\big).$$

We pick $\alpha_1 \in (0, 80^{-n})$ depending only on n so that

$$\mathcal{L}^{n+1}(G) := \mathcal{L}^{n+1}\big(\{\overline{y} \in B_0 \cap E \,:\, M\mu_0(\overline{y}) \leq 4^{-n}\}\big) > 0.$$

We will now prove that for all sufficiently small $\lambda > 0$, $\mu_0 + \lambda\mathcal{L}^{n+1}|_G$ belongs to $\Sigma(E)$, i.e.

$$M\big(\mu_0 + \lambda\mathcal{L}^{n+1}|_G\big) \leq 1, \quad \text{on } E \text{ if } \lambda \in (0, \lambda_0).$$

Let us fix $\overline{z} \in E, r > 0$ and write $\mu_\lambda := \mu_0 + \lambda\mathcal{L}^{n+1}|_G$. Distinguish three cases: first, if $B_r(\overline{z}) \cap G = \varnothing$ then it is clear that

$$\frac{\mu_\lambda(B_r(\overline{z}))}{r^n} = \frac{\mu_0(B_r(\overline{z}))}{r^n} \leq 1,$$

that is the desired estimate. If $B_r(\overline{z}) \cap G \neq \varnothing$ and $\overline{z} \in G$, then

$$\frac{\mu_\lambda(B_r(\overline{z}))}{r^n} = \frac{\mu_0(B_r(\overline{z}))}{r^n} + \lambda\frac{\mathcal{L}^{n+1}(B_r(\overline{z}) \cap G)}{r^n} \leq \frac{1}{4^n} + \lambda\frac{\min\{r^{n+1}, R_0^{n+1}\}}{r^n}.$$

If $r \leq R_0$, then

$$\frac{\mu_\lambda(B_r(\overline{z}))}{r^n} \leq \frac{1}{4^n} + \lambda r \leq \frac{1}{4^n} + \lambda R_0 < 1, \quad \text{for } \lambda \text{ small enough.}$$

If on the other hand $r > R_0$, we also have

$$\frac{\mu_\lambda(B_r(\overline{z}))}{r^n} \leq \frac{1}{4^n} + \lambda\frac{R_0^{n+1}}{r^n} \leq \frac{1}{4^n} + \lambda R_0 < 1, \quad \text{for } \lambda \text{ small enough.}$$

Finally, for the third case, that is, if $B_r(\overline{z}) \cap G \neq \varnothing$ and $\overline{z} \notin G$, take $\overline{y} \in B_r(\overline{z}) \cap G$ so that

$$\begin{aligned}\frac{\mu_\lambda(B_r(\overline{z}))}{r^n} &= \frac{\mu_0(B_r(\overline{z}))}{r^n} + \lambda\frac{\min\{r^{n+1}, R_0^{n+1}\}}{r^n} \leq 2^n\frac{\mu_0(B_{2r}(\overline{y}))}{(2r)^n} + \lambda R_0 \\ &\leq \frac{1}{2^n} + (2^n + 1)\lambda R_0,\end{aligned}$$

where for the last inequality we have applied the reasoning of the second case. Hence, in general, we deduce that for $\lambda > 0$ small enough, $\mu_\lambda \in \Sigma(E)$.

Now we argue as follows: choose $\varepsilon_1^{(k)} \leq \varepsilon_1$ a sequence converging to 0. For each k set $R_0^{(k)} := \varepsilon_1^{(k)}/10$ and build a set G_k in $B_{R_0^{(k)}}(\overline{x}_0)$ as above. Let $H_k := \mathcal{L}^{n+1}|_{G_k}$ and apply Lemma 5.3.2, or in this particular case (5.3.1), to get

$$1 \leq \frac{\mu_0(E) + 2\int_E |\mathcal{S}\mu_0|^2 \,\mathrm{d}\mu_0}{\mu_0(E)} \leq \frac{1}{H_k(E)}\int_E \Big(|\mathcal{S}\mu_0|^2 + 2\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)\Big)\,\mathrm{d}H_k.$$

Since the measures $H_k/H_k(E)$ converge weakly to a point mass probability measure at $\overline{x}_0$, by the continuity of the kernel associated to $\mathcal{S}$ we have

$$1 \leq |\mathcal{S}\mu_0|^2(\overline{x}_0) + 2\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}_0) \quad \to \quad \frac{1}{2} \leq |\mathcal{S}\mu_0|^2(\overline{x}_0) + \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}_0).$$

Therefore, either $|\mathcal{S}\mu_0|^2(\overline{x}_0) \geq 1/4$ or $\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}_0) \geq 1/4$, so in any case we get

$$\mathcal{S}\mu_0(\overline{x}_0) + \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}_0) \geq \frac{1}{4},$$

by the nonnegativity of each term. Since this last estimate follows if $M\mu_0(\overline{x}_0)$ is smaller than α_1, chosen to depend only on n, setting $\alpha_0 := \min\{\alpha_1, 1/4\}$ the result follows. □

5.3.1 The construction

Both of the above lemmas are enough to carry out the construction of the desired cubes in a similar way to [**Vo, pp. 38-42**]. Fix $E \subset B_1(0) \subset 2Q_0$ compact consisting of a finite union of cubes, and define the auxiliary potential

$$\widetilde{\mathcal{U}}^{\mu_0}(\overline{x}) := \mathcal{U}^{\mu_0}(\overline{x}) + M(\mathcal{S}\mu_0\,\mathrm{d}\mu_0)(\overline{x}).$$

Apply Lemma 5.3.3 to E and obtain its corresponding constant α_0, and pick $0 < \beta \ll 1$ depending only on n, to be fixed later on, and set

$$\mathcal{G} := \big\{\overline{y} \in \mathbb{R}^{n+1} \,:\, \widetilde{\mathcal{U}}^{\mu_0}(\overline{y}) > \beta\alpha_0\big\} \supset E.$$

Consider $\{Q_j\}_j$ a Whitney decomposition of $\mathcal{G}$ (see [**Ste, pp. 167-169**], for example), that is: a countable family of closed dyadic cubes with disjoint interiors that cover $\mathcal{G}$ and such that for some dimensional constant $A > 0$,

- $20Q_j \subset \mathcal{G}$ for each j,
- $(AQ_j) \cap \mathcal{G}^c \neq \varnothing$ for each j,
- The family $\{10Q_j\}_j$ has bounded overlap. Moreover, if $10Q_j \cap 10Q_i \neq \varnothing$, then $\ell(Q_j) \approx \ell(Q_i)$.

From those Q_j satisfying $\frac{5}{4}Q_j \cap E \neq \varnothing$, choose a finite subcovering and enumerate them as $Q_{j_1}, \ldots, Q_{j_N}$. We shall check that

$$\mathcal{Q}_i := 2Q_{j_i}, \quad i = 1, \ldots, N$$

is the desired family of cubes that satisfies the properties in Theorem 5.3.1. It is clear that properties $\mathbf{P_1}$ and $\mathbf{P_5}$ are satisfied by construction, so we are only left to verify $\mathbf{P_2}$, $\mathbf{P_3}$ and $\mathbf{P_4}$. The first two will be checked for $\gamma_{\text{sy},+}$ instead of $\widetilde{\gamma}_+$, which is enough due to the comparability of both capacities.

Proof of $\mathbf{P_2}$ *in Theorem 5.3.1.* We shall prove that property $\widetilde{\mathcal{U}}^{\mu_0}(\overline{x}) \geq \beta\alpha_0$ on $\mathcal{F}$, implies

$$\gamma_{\text{sy,op}}(\mathcal{F}) \leq \frac{C}{\beta\alpha_0}\mu_0(E),$$

where μ_0 is an extremal measure for $\gamma_{\text{var}}(E)$ and $C>0$ is dimensional. If this is the case, we are done, since we have

$$\begin{aligned}\gamma_{\text{sy},+}(\mathcal{F}) \approx \gamma_{\text{sy,op}}(\mathcal{F}) \leq \frac{C}{\beta\alpha_0}\gamma_{\text{var}}(E) &\leq \frac{C}{\beta\alpha_0}\gamma_{\text{sy},\psi_{\tau_0},2}(E)\\ &\leq \frac{C}{\beta\alpha_0}2C_1C_2\gamma_{\text{sy,op}}(E) \approx \gamma_{\text{sy},+}(E),\end{aligned}$$

where for the third estimate recall Remark 5.2.4 and the fact that relation (5.2.9) holds in general, and in the fourth inequality we have used the definition of τ_0.

So let us choose μ admissible for $\gamma_{\text{sy,op}}(\mathcal{F})$ with $\gamma_{\text{sy,op}}(\mathcal{F}) \leq 2\mu(\mathcal{F})$ and observe that

$$\|\mathcal{S}_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq \|\mathscr{P}_{\text{sy},\mu}\|_{L^2(\mu)\to L^2(\mu)} \leq 1,$$

since P_{sy} and the measure μ are nonnegative. Therefore, $\mathcal{S}_\mu$ becomes a C-Z convolution operator (with continuous kernel), so that it satisfies the weak estimate

$$\mu\big(\{\mathcal{S}\mu_0 \geq \lambda\}\big) \leq C'\frac{\mu_0(E)}{\lambda},$$

where $C'>0$ is dimensional. By definition, for each $\overline{x} \in \mathcal{F}$,

$$\begin{aligned}\widetilde{\mathcal{U}}^{\mu_0}(\overline{x}) &:= M\mu_0(\overline{x}) + \mathcal{S}\mu_0(\overline{x}) + \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}) + M(\mathcal{S}\mu_0\,\mathrm{d}\mu_0)(\overline{x})\\ &=: \mathrm{I}(\overline{x}) + \mathrm{II}(\overline{x}) + \mathrm{III}(\overline{x}) + \mathrm{IV}(\overline{x}),\end{aligned}$$

so at least one of the four terms is larger than $\frac{\beta\alpha_0}{4} =: a$. In fact, if we name

$$\begin{aligned}\mathcal{F}_1 &:= \big\{\overline{x} \in F \,:\, \mathrm{I}(\overline{x}) \geq a\big\}, &\quad \mathcal{F}_2 &:= \big\{\overline{x} \in F \,:\, \mathrm{II}(\overline{x}) \geq a\big\},\\ \mathcal{F}_3 &:= \big\{\overline{x} \in F \,:\, \mathrm{III}(\overline{x}) \geq a\big\}, &\quad \mathcal{F}_4 &:= \big\{\overline{x} \in F \,:\, \mathrm{IV}(\overline{x}) \geq a\big\},\end{aligned}$$

we have $\mathcal{F} = \cup_{i=1}^4 \mathcal{F}_i$. So in particular $\mu(\mathcal{F}) \leq \sum_{i=1}^4 \mu(\mathcal{F}_i)$ and then, necessarily, there exists $i \in \{1,2,3,4\}$ such that $\mu(\mathcal{F}_i) \geq \mu(\mathcal{F})/4$. Let us study separately each possibility:

1. If $i=1$, then $M\mu_0 \geq a$ on $\mathcal{F}_1$ with $\mu(\mathcal{F}_1) \geq \mu(\mathcal{F})/4$. For each $\overline{x} \in \mathcal{F}_1$ choose a ball $B_{\overline{x}} := B_{r(\overline{x})}(\overline{x})$ so that $\mu_0(B_{\overline{x}}) \geq ar(\overline{x})^n$. Apply Besicovitch's covering theorem [**Matti, Theorem 2.7**] to extract a countable collection of balls $\{B_{\overline{x}_j}\}_j$ with bounded multiplicity of overlapping (with constant depending only on n) covering $\mathcal{F}_1$. Then,

$$\frac{\mu(\mathcal{F})}{4} \leq \mu(\mathcal{F}_1) \leq \sum_j \mu(B_{\overline{x}_j}) \leq \sum_j r^n(\overline{x}_j) \leq \frac{1}{a}\sum_j \mu_0(B_{\overline{x}_j}) \lesssim \frac{\mu_0(\mathcal{F}_1)}{a} \leq \frac{\mu_0(E)}{a},$$

since μ_0 is supported on E.

2. If $i=2$, $\mathcal{S}\mu_0 \geq a$ on $\mathcal{F}_2$ with $\mu(\mathcal{F}_2) \geq \mu(\mathcal{F})/4$. Using the weak estimate presented above we get the desired estimate

$$\frac{\mu(\mathcal{F})}{4} \leq \mu(\mathcal{F}_2) = \mu\big(\{\mathcal{S}\mu_0 \geq a\}\big) \leq \frac{C'}{a}\mu_0(E).$$

3. If $i=3$ we get $\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0) \geq a$ on $\mathcal{F}_3$ with $\mu(\mathcal{F}_3) \geq \mu(\mathcal{F})/4$. By (5.2.9) we obtain

$$\frac{\mu(\mathcal{F})}{4} \leq \mu(\mathcal{F}_3) = \mu\big(\{\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0) \geq a\}\big) \leq \frac{C'}{a}\int_E \mathcal{S}\mu_0 \, \mathrm{d}\mu_0$$
$$\leq \frac{C'}{a}\mu_0(E)^{1/2}\bigg(\int_E (\mathcal{S}\mu_0)^2 \, \mathrm{d}\mu_0\bigg)^2 \leq \frac{C'}{a}\mu_0(E).$$

4. If $i=4$, $M(\mathcal{S}\mu_0 \, \mathrm{d}\mu_0) \geq a$ on $\mathcal{F}_4$ with $\mu(\mathcal{F}_4) \geq \mu(\mathcal{F})/4$. Proceeding analogously as in the first case, we deduce

$$\frac{\mu(\mathcal{F})}{4} \lesssim \frac{1}{a}\int_E \mathcal{S}\mu_0 \, \mathrm{d}\mu_0,$$

and arguing as in the third case we obtain the desired result.

□

To prove $\mathbf{P_3}$ we will need the following lemma:

Lemma 5.3.4. *For $i=1,\ldots,N$ define $\mu_i := \mu_0|_{5Q_{j_i}}$ and consider the auxiliary potential*

$$\mathcal{W}^{\mu_i} := M\mu_i + \mathcal{S}_{\mu_i}1 + \mathcal{S}_{\mu_i}(\mathcal{S}\mu_0) + M(\mathcal{S}\mu_0 \, \mathrm{d}\mu_i).$$

Then, there exists a dimensional constant $\alpha_0' > 0$ such that for each $i=1,\ldots,N$,

$$\mathcal{W}^{\mu_i}(\overline{x}) \geq \alpha_0'\alpha_0, \qquad \forall \overline{x} \in 4Q_{j_i} \cap E.$$

Proof. For the sake of notation, in this proof we rename $Q_i := Q_{j_i}$. Fix $i=1,\ldots,N$, $\overline{x} \in 4Q_i \cap E$ and $\overline{z} \in (AQ_i) \cap \mathcal{G}^c$, which is non-empty by construction. So in particular

$$M\mu_0(\overline{z}) \leq \beta\alpha_0, \quad \mathcal{S}\mu_0(\overline{z}) \leq \beta\alpha_0. \tag{5.3.3}$$

Now choosing, for example, $R := 2(4+A)\mathrm{diam}(Q_i)$ it is clear that we have

$$|\overline{x}-\overline{z}| \leq (4+A)\mathrm{diam}(Q_i) \leq \frac{R}{2}.$$

Therefore, using the C-Z estimates for $P_{\mathrm{sy}}\psi_{\tau_0}$, the nonnegativity of the latter kernel and relations (5.3.3) we have

$$\mathcal{S}_R\mu_0(\overline{x}) := \int_{|\overline{x}-\overline{y}|>R} P_{\mathrm{sy}}(\overline{x}-\overline{y})\psi_{\tau_0}(\overline{x}-\overline{y}) \, \mathrm{d}\mu_0(\overline{y})$$
$$\leq \beta\alpha_0 + \int_{|\overline{x}-\overline{y}|>R} \Big| P_{\mathrm{sy}}(\overline{x}-\overline{y})\psi_{\tau_0}(\overline{x}-\overline{y}) - P_{\mathrm{sy}}(\overline{z}-\overline{y})\psi_{\tau_0}(\overline{z}-\overline{y}) \Big| \, \mathrm{d}\mu_0(\overline{y})$$
$$\lesssim \beta\alpha_0 + |\overline{x}-\overline{z}| \int_{|\overline{x}-\overline{y}|>R} \frac{\mathrm{d}\mu_0(\overline{y})}{|\overline{x}-\overline{y}|^{n+1}} \lesssim \beta\alpha_0 + |\overline{x}-\overline{z}| \int_{|\overline{z}-\overline{y}|>R/4} \frac{\mathrm{d}\mu_0(\overline{y})}{|\overline{z}-\overline{y}|^{n+1}}$$
$$\leq \beta\alpha_0 + \frac{R}{2}\sum_{j=0}^{\infty} \frac{\mu_0(B_{2^{j+1}R}(\overline{z}))}{(2^jR)^{n+1}} \leq \beta\alpha_0 + \frac{R}{2}\sum_{j=0}^{\infty} \frac{\beta\alpha_0(2^{j+1}R)^n}{(2^jR)^{n+1}} \lesssim \beta\alpha_0. \tag{5.3.4}$$

We also notice that there exists $\widetilde{A} = \widetilde{A}(n) > 0$ such that

$$M\mu_0(\overline{x}) \leq \max\big\{\widetilde{A}\,\beta\alpha_0, M\mu_i(\overline{x})\big\}. \tag{5.3.5}$$

Indeed, if $r \geq \ell(Q_i)/20$ then

$$\frac{\mu_0(B_r(\overline{x}))}{r^n} \leq \frac{\mu_0(B_{40\sqrt{n}Ar}(\overline{z}))}{r^n} \leq \widetilde{A}(n)\beta\alpha_0.$$

If $r < \ell(Q_i)/20$ use that $\overline{x} \in 4Q_i$ and $\mu_i := \mu_0|_{5Q_i}$ to simply have

$$\frac{\mu_0(B_r(\overline{x}))}{r^n} = \frac{\mu_j(B_r(\overline{x}))}{r^n},$$

so indeed $M\mu_0(\overline{x}) \leq \max\big\{\widetilde{A}\,\beta\alpha_0, M\mu_i(\overline{x})\big\}$. Having made this observations, we recall that by Lemma 5.3.3 we always have

$$M\mu_0 + \mathcal{S}\mu_0 + \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0) \geq \alpha_0, \quad \text{on } E.$$

Now we distinguish three cases:

1. If $M\mu_0(\overline{x}) \geq \alpha_0/3$. Then (5.3.5) implies

$$\frac{\alpha_0}{3} \leq \max\big\{\widetilde{A}\,\beta\alpha_0, M\mu_i(\overline{x})\big\},$$

so choosing $\beta \leq (6\widetilde{A})^{-1}$ we deduce $M\mu_i(\overline{x}) \geq \alpha_0/6$ and we would be done.

2. If $\mathcal{S}\mu_0(\overline{x}) \geq \alpha_0/3$, by (5.3.4) there exists $C > 0$ dimensional such that

$$\mathcal{S}\big(\mu|_{B_R(\overline{x})}\big)(\overline{x}) \geq \frac{\alpha_0}{3} - C\beta\alpha_0 \geq \frac{\alpha_0}{4},$$

for β small enough. In addition, by definition of R,

$$\mathcal{S}\big(\mu|_{B_R(\overline{x})\setminus 5Q_i}\big)(\overline{x}) \leq C' \frac{\mu_0(B_{2R}(\overline{z}))}{\ell(Q_i)^n} \leq 2^n C'\beta\alpha_0 \frac{R^n}{\ell(Q_i)^n} = C''\beta\alpha_0.$$

Hence

$$\mathcal{S}\mu_i(\overline{x}) = \mathcal{S}\big(\mu_0|_{5Q_i}\big)(\overline{x}) \geq \frac{\alpha_0}{4} - C''\beta\alpha_0 \geq \frac{\alpha_0}{5},$$

for β small enough, and we are done.

3. If $\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}) \geq \alpha_0/3$. Let us observe that for $\overline{z} \in (AQ_i) \cap \mathcal{G}^c$, by definition of $\mathcal{G}$ we also have

$$M(\mathcal{S}\mu_0\,\mathrm{d}\mu_0)(\overline{z}) \leq \beta\alpha_0, \qquad \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{z}) \leq \beta\alpha_0.$$

So choosing again $R := 2(4+A)\mathrm{diam}(Q_i)$ we would be able to prove analogously as in (5.3.4) that

$$\mathcal{S}_{\mu_0,R}(\mathcal{S}\mu_0)(\overline{x}) \leq C\beta\alpha_0,$$

for some $C > 0$ dimensional. Moreover,

$$\mathcal{S}_{\mu_0|_{B_R(\overline{x})\setminus 5Q_i}}(\mathcal{S}\mu_0)(\overline{x}) \le C' \frac{\int_{B_{2R}(\overline{z})} \mathcal{S}\mu_0 \,\mathrm{d}\mu_0}{\ell(Q_i)^n} \le C''\beta\alpha_0,$$

where we have used, again, the definition of R and that $M(\mathcal{S}\mu_0 \,\mathrm{d}\mu_0)(\overline{z}) \le \beta\alpha_0$. Therefore,

$$\begin{aligned}\mathcal{S}_{\mu_i}(\mathcal{S}\mu_0)(\overline{x}) &= \mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}) - \mathcal{S}_{\mu_0|_{B_R(\overline{x})\setminus 5Q_i}}(\mathcal{S}\mu_0)(\overline{x}) - \mathcal{S}_{\mu_0,R}(\mathcal{S}\mu_0)(\overline{x})\\ &\ge \frac{\alpha_0}{3} - C''\beta\alpha_0 - C\beta\alpha_0 \ge \frac{\alpha_0}{5},\end{aligned}$$

for β small enough, and the proof is completed.

□

Proof of $\mathbf{P_3}$ *in Theorem 5.3.1.* Repeating the same arguments presented for the proof of $\mathbf{P_2}$, one deduces that property $\mathcal{W}^{\mu_i}(\overline{x}) \ge \alpha_0'\alpha$, for every $\overline{x} \in 4Q_{j_i} \cap E$ and every $i = 1, \ldots, N$, implies

$$\gamma_{\mathrm{sy,op}}(4Q_{j_i} \cap E) \le \frac{C}{\alpha_0'\alpha_0}\bigg(\mu_0(5Q_{j_i}) + \int_{5Q_{j_i}} \mathcal{S}\mu_0 \,\mathrm{d}\mu_0\bigg).$$

But notice that the bounded multiplicity of overlapping of the family $\{10Q_j\}_j$ and the fact that μ_0 is supported on E yield

$$\sum_{i=1}^N \bigg(\mu_0(Q_{j_i}) + \int_{5Q_{j_i}} \mathcal{S}\mu_0 \,\mathrm{d}\mu_0\bigg) \lesssim \mu_0(E) + \int_E \mathcal{S}\mu_0 \,\mathrm{d}\mu_0 \le 2\mu_0(E),$$

where for the last inequality we have used (5.2.9). Therefore, by the latter relation and using the definition of τ_0 we get

$$\begin{aligned}\sum_{i=1}^N \gamma_{\mathrm{sy},+}(4Q_{j_i} \cap E) &\lesssim \sum_{i=1}^N \gamma_{\mathrm{sy,op}}(4Q_{j_i} \cap E) \lesssim \mu_0(E)\\ &= \gamma_{\mathrm{var}}(E) \le \gamma_{\mathrm{sy},\psi_{\tau_0},2}(E) \le 2C_1C_2\gamma_{\mathrm{sy,op}}(E) \lesssim \gamma_{\mathrm{sy},+}(E),\end{aligned}$$

that is the desired estimate. □

Finally, we prove $\mathbf{P_4}$. Let us first observe that the assumption

$$\widetilde{\gamma}_+(E) \le C_+ \mathrm{diam}(E)^n, \qquad \text{for } C_+ > 0 \text{ dimensional to be fixed below},$$

is not at all restrictive. Indeed, if it failed, [**MatP, Lemma 4.1**] would imply

$$\widetilde{\gamma}(E) \le \gamma(E) \lesssim \mathcal{H}^n_\infty(E) \le \mathrm{diam}(E)^n < \frac{1}{C_+}\widetilde{\gamma}_+(E),$$

and we would be done.

Proof of $\mathbf{P_4}$ *in Theorem 5.3.1.* Fix $\overline{x} \notin E$ and observe that the following estimates hold

$$\begin{aligned}
\mathcal{S}\mu_0(\overline{x}) &\leq \frac{\mu_0(E)}{\operatorname{dist}(\overline{x},E)^n}, \\
\mathcal{S}_{\mu_0}(\mathcal{S}\mu_0)(\overline{x}) &\leq \frac{1}{\operatorname{dist}(\overline{x},E)^n}\int_E\int_E P_{\text{sy}}(\overline{y}-\overline{z})\psi_{\tau_0}(\overline{y}-\overline{z})\,\mathrm{d}\mu_0(\overline{z})\,\mathrm{d}\mu_0(\overline{y}) \\
&= \frac{1}{\operatorname{dist}(\overline{x},E)^n}\int_E \mathcal{S}\mu_0(\overline{y})\,\mathrm{d}\mu_0(\overline{y}) \leq \frac{\mu_0(E)}{\operatorname{dist}(\overline{x},E)^n},
\end{aligned}$$

where for the last inequality we have applied (5.2.9). Since $\operatorname{supp}(\mu_0) \subset E$, we have

$$\sup_{r>0}\frac{\mu_0(B_r(\overline{x}))}{r^n} = \sup_{r>\operatorname{dist}(\overline{x},E)}\frac{\mu_0(B_r(\overline{x}))}{r^n} \leq \frac{\mu_0(E)}{\operatorname{dist}(\overline{x},E)^n},$$

and moreover, due to (5.2.9) again,

$$\begin{aligned}
\sup_{r>0}\frac{1}{r^n}\int_{B_r(\overline{x})}\mathcal{S}\mu_0(\overline{y})\,\mathrm{d}\mu_0(\overline{y}) &= \sup_{r>\operatorname{dist}(\overline{x},E)}\frac{1}{r^n}\int_{B_r(\overline{x})}\mathcal{S}\mu_0(\overline{y})\,\mathrm{d}\mu_0(\overline{y}) \\
&\leq \frac{1}{\operatorname{dist}(\overline{x},E)^n}\int_E \mathcal{S}\mu_0(\overline{y})\,\mathrm{d}\mu_0(\overline{y}) \leq \frac{\mu_0(E)}{\operatorname{dist}(\overline{x},E)^n}.
\end{aligned}$$

All in all, we get

$$\widetilde{\mathcal{U}}^{\mu_0}(\overline{x}) \leq \frac{4\mu_0(E)}{\operatorname{dist}(\overline{x},E)}.$$

Let us pick $\overline{x} \in \mathcal{G}\cap E^c$ (which is non empty, since $E\subset\mathcal{G}$ with $\mathcal{G}$ open, by the lower semicontinuity of the potential $\widetilde{\mathcal{U}}^{\mu_0}$) so that $\widetilde{\mathcal{U}}^{\mu_0}(\overline{x}) > \beta\alpha_0$ and observe that using the definition of μ_0 and τ_0 as in the previous proofs, we get

$$\begin{aligned}
\operatorname{dist}(\overline{x},E)^n &\leq \frac{8C_1C_2}{\alpha_0\beta}\gamma_{\text{sy,op}}(E) \leq \frac{8CC_1C_2}{\alpha_0\beta}\gamma_{\text{sy},+}(E) \\
&\leq \frac{16CC_1C_2}{\alpha_0\beta}\widetilde{\gamma}_+(E) \leq \frac{16CC_1C_2}{\alpha_0\beta}C_+\operatorname{diam}(E)^n.
\end{aligned}$$

Choosing C_+ appropriately to neglect the effect of all the above constants that have already been fixed, the result follows. Indeed, this is because the previous estimate is valid for all $\overline{x}\in\mathcal{G}\cap E^c$, implying that one can make $\partial\mathcal{G}$ say $\frac{\operatorname{diam}(E)}{1000}$ close to E. Since we also know that $20Q_j \subset \mathcal{G}$ we have

$$\operatorname{diam}(2Q_j) \leq \frac{1}{10}\operatorname{dist}(\partial\mathcal{G},Q_j),$$

where $\operatorname{dist}(\partial\mathcal{G},Q_j)$ is comparable to $\operatorname{dist}(\partial\mathcal{G},E)$ since the cubes Q_j have been also chosen so that $\frac{5}{4}Q_j\cap E\neq\varnothing$. $\square$

5.4 A comparability result under an additional assumption

Let $E \subset \mathbb{R}^{n+1}$ be a compact set. As it is stated in $\mathbf{P_4}$ in Theorem 5.3.1, we shall work under the following additional assumption:

$\mathbf{A_1}$: $\widetilde{\gamma}_+(E) \leq C_+ \text{diam}(E)^n$, with $C_+ > 0$ dimensional.

By the translation invariance and $\widetilde{\gamma}(\lambda E) = \lambda^n \widetilde{\gamma}(E)$, $\widetilde{\gamma}_+(\lambda E) = \lambda^n \widetilde{\gamma}_+(E)$ (see Theorem 4.2.2), it is clear that we may assume $E \subset B_1(0)$ without loss of generality. In fact, in order to apply Theorem 5.3.1 let us check that we can assume:

$\mathbf{A_2}$: *E is contained in the unit ball and consists of a finite union of dyadic cubes belonging to a dyadic grid in $\mathbb{R}^{n+1}$ (with sides parallel to the coordinate axes), all with the same size and with disjoint interiors.*

Let us verify that if we deduce the comparability between $\widetilde{\gamma}$ and $\widetilde{\gamma}_+$ for E satisfying $\mathbf{A_2}$, we obtain the same result for a general E. So fix $E \subset \mathbb{R}^{n+1}$ any compact set and let $\overline{\mathcal{U}_\delta(E)}$ be the closed δ-neighbourhood of E. Consider a grid of dyadic cubes in $\mathbb{R}^{n+1}$ with sides parallel to the axes and diameter smaller than $\delta/2$. Name E_0 the collection of dyadic cubes that intersect E and notice that $E \subset E_0 \subset \overline{\mathcal{U}_\delta(E)}$. Now we would have

$$\widetilde{\gamma}(E) \leq \widetilde{\gamma}(E_0) \lesssim \widetilde{\gamma}_+(E_0) \leq \widetilde{\gamma}_+(\overline{\mathcal{U}_\delta(E)}) \leq \gamma_+(\overline{\mathcal{U}_\delta(E)}).$$

Letting $\delta \to 0$ and using the outer regularity of γ_+ we deduce, by Theorem 5.2.5,

$$\widetilde{\gamma}(E) \lesssim \gamma_+(E) \lesssim \widetilde{\gamma}_+(E),$$

and the result follows.

Finally, we present the final additional assumption that motivates the title of this section:

$\mathbf{A_3}$. *Let $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_N\}$ be the family of cubes provided by Theorem 5.3.1 and C_1 the constant appearing in point $\mathbf{P_3}$ of the same theorem. We will assume that*

$$\widetilde{\gamma}(E) \geq C_1^{-1} \sum_{i=1}^{N} \widetilde{\gamma}(2\mathcal{Q}_i \cap E).$$

The aim of this section is to prove the comparability between $\widetilde{\gamma}$ and $\widetilde{\gamma}_+$ under assumption $\mathbf{A_3}$. Later on, we will get rid of these hypotheses in some particular cases, for which we will obtain the desired comparability in full generality.

5.4.1 Basic definitions and properties

Let $E \subset \mathbb{R}^{n+1}$ be a compact set verifying $\mathbf{A_1}, \mathbf{A_2}$ and $\mathbf{A_3}$. We begin our argument by fixing a distribution T_0 admissible for $\widetilde{\gamma}(E)$ so that $|\langle T_0, 1\rangle| = \widetilde{\gamma}(E)/2$. We know that T_0 has n-growth with constant 1 and

$$\|P * T_0\|_\infty \leq 1, \qquad \|P^* * T_0\|_\infty \leq 1.$$

Let $\mathcal{F} := \cup_{i=1}^N \mathcal{Q}_i$ be the covering by cubes provided by Theorem 5.3.1. Recall that $\mathcal{Q}_i = 2Q_i$, where $\{Q_i\}_{i=1}^N$ is the finite subset of dyadic cubes with disjoint interiors of a Whitney decomposition of $\mathcal{G} \supset E$, with the property $\frac{5}{4}Q_i \cap E \neq \varnothing$. We write

$$F := \bigcup_{i=1}^N Q_i \subset \mathcal{F}.$$

Since $E \subset F$, by monotonicity and the properties in Theorem 5.3.1 we also have

P$'_2$. $\widetilde{\gamma}_+(F) \leq C_0 \widetilde{\gamma}_+(E)$,

P$'_3$. $\sum_{i=1}^N \widetilde{\gamma}_+(2Q_i \cap E) \leq C_1 \widetilde{\gamma}_+(E)$ and by $\mathbf{A_3}$ we also have $\sum_{i=1}^N \widetilde{\gamma}(2Q_i \cap E) \leq C_1 \widetilde{\gamma}(E)$.

P$'_4$. $\operatorname{diam}(Q_i) \leq \frac{1}{20} \operatorname{diam}(E)$, for each $i = 1, \dots, N$,

P$'_5$. The family $\{10Q_1, \dots, 10Q_N\}$ has bounded overlap. Moreover, if $10Q_j \cap 10Q_i \neq \varnothing$, then $\ell(Q_j) \approx \ell(Q_i)$.

In order to simplify the arguments, in this section we will work with the cubes $Q_1, \dots, Q_N$ instead of $\mathcal{Q}_1, \dots, \mathcal{Q}_N$, and with F instead of $\mathcal{F}$. Let us choose for each Q_i a ball $B_i \subset \frac{1}{2}Q_i$ concentric with Q_i with radius r_i comparable to $\widetilde{\gamma}(2Q_i \cap E)^{1/n}$. Notice that

$$\widetilde{\gamma}(2Q_i \cap E)^{1/n} \leq \gamma(2Q_i)^{1/n} \approx \ell(Q_i).$$

Therefore, we may choose $r_i \approx \widetilde{\gamma}(2Q_i \cap E)^{1/n}$ with constants depending at most on the dimension and still have $B_i \subset \frac{1}{2}Q_i$. Notice that $\operatorname{dist}(B_i, B_j) \geq \frac{1}{2}\min\{\ell(Q_i), \ell(Q_j)\}$, for $i \neq j$. We define the positive measure

$$\mu := \sum_{i=1}^N \mu_i := \sum_{i=1}^N \frac{r_i^n}{\mathcal{L}^{n+1}(B_i)} \mathcal{L}^{n+1}|_{B_i}. \tag{5.4.1}$$

Apply [**HPo, Lemma 3.1**] to pick test functions $\varphi_i \in \mathcal{C}_c^\infty(2Q_i), 0 \leq \varphi_i \leq 1$, satisfying $\|\nabla\varphi_i\|_\infty \leq \ell(2Q_i)^{-1}$ and with $\sum_{i=1}^N \varphi_i \equiv 1$ in F. We also define the signed measure

$$\nu := \sum_{i=1}^N \frac{\langle T_0, \varphi_i \rangle}{\mathcal{L}^{n+1}(B_i)} \mathcal{L}^{n+1}|_{B_i}, \qquad \text{that is} \qquad \nu = \sum_{i=1}^N \frac{\langle T_0, \varphi_i \rangle}{r_i^n} \mu_i. \tag{5.4.2}$$

Since $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu) \subset \mathcal{F}$ and $\nu \ll \mu$, have $\nu = b\mu$, where b is the Radon-Nikodym derivative.

Lemma 5.4.1. *The following hold for some dimensional constants c_1, c_3 and c_4:*

1. $\|b\|_\infty \leq c_1$.
2. $|\nu(F)| = \widetilde{\gamma}(E)/2$.
3. $\widetilde{\gamma}(E) \leq c_3 \mu(F)$.
4. $|\nu(Q)| \leq c_4 \ell(Q)^n$ *for any cube* $Q \subset \mathbb{R}^{n+1}$.

Proof. Let us prove *1*. By the localization result [**MatP, Theorem 3.1**] we get $\|P*(\varphi_i T_0)\|_\infty \leq A_1$, for every i, where $A_1 = A_1(n) > 0$. Analogously, the same localization result holds for P^*, so there also exists $A_2 = A_2(n) > 0$ with $\|P^* * (\varphi_i T_0)\|_\infty \leq A_2$, $\forall i$. Therefore, since $\text{supp}(\varphi_i T_0) \subset 2Q_i \cap E$ we have

$$|\langle T_0, \varphi_i\rangle| = |\langle \varphi_i T_0, 1\rangle| \lesssim \widetilde{\gamma}(2Q_i \cap E) \approx r_i^n, \quad \forall i = 1, \ldots, n, \tag{5.4.3}$$

which yields the desired result.

To prove *2* we simply use the definition of ν, $B_i \cap F = B_i$ and $\text{supp}(T_0) \subset E \subset F$:

$$|\nu(F)| = \left| \sum_{i=1}^N \frac{\langle T_0, \varphi_i\rangle}{\mathcal{L}^{n+1}(B_i)} \int_{B_i} \mathrm{d}\mathcal{L}^{n+1} \right| = \left| \left\langle T_0, \sum_{i=1}^N \varphi_i \right\rangle \right| = |\langle T_0, 1\rangle| = \frac{1}{2}\widetilde{\gamma}(E).$$

Finally, *3* follows by the choice of radii r_i and the admissibility of $\varphi_i T_0$ for $\widetilde{\gamma}(2Q_i \cap E)$,

$$\widetilde{\gamma}(E) = 2|\langle T_0, 1\rangle| \leq 2\sum_{i=1}^N |\langle T_0, \varphi_i\rangle| \lesssim \sum_{i=1}^N \widetilde{\gamma}(2Q_i \cap E) \approx \sum_{i=1}^N r_i^n = \mu(F).$$

The proof of inequality *4* can be followed analogously to that given for property (g) of [**T5, Lemma 6.8**]. We refer the reader to this last reference for the detailed arguments. □

Remark 5.4.1. In the proof of inequality *4* it is used that for a Borel signed measure ν satisfying $\|P*\nu\|_\infty \leq 1$ one has $|\nu(Q)| \leq C\ell(Q)^n$, for some dimensional constant $C > 0$ and any cube $Q \subset \mathbb{R}^{n+1}$. In order to justify that this holds in our context we will prove the following: given $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ such that $\|P*\varphi\|_\infty \leq 1$, then $|\int_Q \varphi \, \mathrm{d}\mathcal{L}^{n+1}| \leq C\ell(Q)^n$, for some $C > 0$ dimensional. If this is the case, we will be done by mollifying ν to be $\nu_\varepsilon := \nu * \psi_\varepsilon$ with $(\psi_\varepsilon)_\varepsilon$ a proper approximation of the identity and applying, for example, [**Matti, Theorems 1.24 & 1.26**]. Let us estimate $|\int_Q \varphi \, \mathrm{d}\mathcal{L}^{n+1}|$ as follows:

$$\left| \int_Q \varphi(\overline{x}) \, \mathrm{d}\overline{x} \right| \leq \left| \int_Q \partial_t (P*\varphi)(\overline{x}) \, \mathrm{d}\overline{x} \right| + \left| \int_Q (-\Delta)^{1/2} (P*\varphi)(\overline{x}) \, \mathrm{d}\overline{x} \right|$$

For the first term of the right-hand side we set $Q = Q_1 \times I_Q \subset \mathbb{R}^n \times \mathbb{R}$ where I_Q is the *temporal* interval, that we write as $I_Q = [a,b]$ for some $a, b \in \mathbb{R}$ with $a < b$. This way, by Fubini's theorem,

$$\begin{aligned} \left| \int_Q \partial_t (P*\varphi)(\overline{x}) \, \mathrm{d}\overline{x} \right| &= \left| \int_{Q_1} \left(\int_a^b \partial_t (P*\varphi)(x,t) \, \mathrm{d}t \right) \mathrm{d}x \right| \\ &= \left| \int_{Q_1} \Big(P*\varphi(x,b) - P*\varphi(x,a) \Big) \, \mathrm{d}x \right| \lesssim \|P*\varphi\|_\infty \mathcal{L}^n(Q_1) \leq \ell(Q)^n. \end{aligned}$$

For the second term we recall that we have the following representation

$$(-\Delta)^{1/2} \approx \sum_{j=1}^n \partial_j R_j,$$

where R_j, $1 \leq j \leq n$ are the usual Riesz transforms with Fourier multiplier $\xi_j/|\xi_j|$. Notice that if we prove that $P * (R_j\varphi) \in L^\infty(\mathbb{R}^{n+1})$ for each $j = 1, \dots, n$, we are done, since we may argue as above decomposing the cube Q as a cartesian product of an interval and an n-dimensional cube. To study $P * (R_j\varphi)$ we consider its Fourier transform $\frac{\xi_j}{|\xi|}\widehat{P}\widehat{\varphi}$. Notice that $P \in (L^1 + L^2)(\mathbb{R}^{n+1})$ (just decompose it as $P\chi_{|\overline{x}|\leq 1} + P\chi_{|\overline{x}|>1}$), and therefore $\widehat{P} \in (L^\infty + L^2)(\mathbb{R}^{n+1})$, and we write its decomposition as $\widehat{P}_\infty + \widehat{P}_2$. Now it is immediate to check that $\frac{\xi_j}{|\xi|}\widehat{P}_\infty\widehat{\varphi} \in L^1(\mathbb{R}^{n+1})$ and we also have $\frac{\xi_j}{|\xi|}\widehat{P}_2\widehat{\varphi} \in L^1(\mathbb{R}^{n+1})$ using Cauchy-Schwarz inequality. Therefore the Fourier transform of $P * (R_j\varphi)$ is integrable, implying $P * (R_j\varphi) \in L^\infty(\mathbb{R}^{n+1})$ that is what we wanted to see.

5.4.2 The exceptional sets $H_{\mathcal{D}}$ and $T_{\mathcal{D}}$

In light of the third property in Lemma 5.4.1, assume we find a compact set $G \subset F$ such that

1. $\mu(F) \lesssim \mu(G)$,
2. $\mu|_G$ has n-growth,
3. $\mathscr{P}_{\mu|_G}$ is a bounded operator in $L^2(\mu|_G)$.

Then, $\mu|_G$ is admissible for $\gamma_{\text{op}}(G)$, and we are done, since we have

$$\widetilde{\gamma}(E) \lesssim \mu(F) \lesssim \mu(G) \leq \gamma_{\text{op}}(G) \leq \gamma_{\text{op}}(F) \approx \widetilde{\gamma}_+(F) \lesssim \widetilde{\gamma}_+(E), \tag{5.4.4}$$

where the first inequality is due to Lemma 5.4.1 and the last is by property $\mathbf{P_2'}$ above.

The construction of G is inspired by the Tb-theorems found in [**T5, Theorem 5.1**] and [**Vo, Theorem 7.1**], which are presented for the particular case of Cauchy and Riesz potentials respectively. To apply this kind of results we need to introduce the notions of Ahlfors ball, Ahlfors radius and Ahlfors point in our context. Due to [**MatP, Lemma 4.1**] there are positive dimensional constants a_1, a_2 such that

$$\widetilde{\gamma}(E) \leq a_1 \operatorname{diam}(E)^n,$$

and also

$$a_2^{-1} r_i^n \leq \widetilde{\gamma}(2Q_i \cap E) \leq a_2 r_i^n,$$

for each $i = 1, \dots, N$, where r_i is the radius of the ball $B_i \subset Q_i$ constructed at the beginning of §5.4.1. Let

$$L := 100^n a_1 a_2 C_1,$$

where C_1 and C_2 are the constants appearing in Theorem 5.3.1. The factor 100 is arbitrarily chosen; we may pick, for convenience, any other *large* constant. Assumption $\mathbf{A_3}$ implies

$$\mu(F) = \sum_{i=1}^N r_i^n \leq a_2 \sum_{i=1}^N \widetilde{\gamma}(2Q_i \cap E) \leq a_2 C_1 \widetilde{\gamma}(E) \leq a_1 a_2 C_1 \operatorname{diam}(E)^n.$$

Therefore:

$$\forall \overline{x} \in F, \forall R > \frac{1}{100}\mathrm{diam}(F) : \quad \mu(B_R(\overline{x})) \leq LR^n.$$

A ball $B_R(\overline{x})$, $\overline{x} \in F$, is an *Ahlfors ball* if precisely the estimate $\mu(B_R(\overline{x})) \leq LR^n$ holds. Notice that if $R > \frac{1}{100}\mathrm{diam}(F)$, then the ball is an Ahlfors ball. We also define the *Ahlfors radius*,

$$\mathcal{R}(\overline{x}) := \sup\big\{r > 0 : B_r(\overline{x}) \text{ is non-Ahlfors}\big\}. \tag{5.4.5}$$

Ahlfors points are those for which $\mathcal{R}(\overline{x}) = 0$. For every $\overline{x} \in F$, we have $\mathcal{R}(\overline{x}) \leq \frac{1}{100}\mathrm{diam}(F)$. Set

$$H' := \bigcup_{\overline{x}\in F} B_{\mathcal{R}(\overline{x})}(\overline{x}),$$

where we convey $B_0(\overline{x}) := \varnothing$, and apply a $5r$-covering theorem [**Matti, Theorem 2.1**] to choose a countable family of disjoint balls $\{B_{\mathcal{R}_k}\}_k := \{B_{\mathcal{R}(\overline{x}_k)}(\overline{x}_k)\}_k$ such that

$$H' \subset H := \bigcup_k B_{5\mathcal{R}_k}, \tag{5.4.6}$$

so that all non-Ahlfors balls are contained in H. Observe that since $\mu(B_{\mathcal{R}_k}) \geq L\mathcal{R}_k^n$ and the balls $B_{\mathcal{R}_k}$ are disjoint, we have

$$\sum_k \mathcal{R}_k^n \leq \frac{1}{L}\sum_k \mu(B_{\mathcal{R}_k}) \leq \frac{1}{L}\mu(F).$$

Let $\mathcal{D}^0$ be the usual dyadic lattice in $\mathbb{R}^{n+1}$ and for $\overline{x} \in \mathbb{R}^{n+1}$ we write

$$\mathcal{D}(\overline{x}) := \overline{x} + \mathcal{D}^0,$$

the translation of $\mathcal{D}^0$ by the vector $\overline{x}$. Let $\mathcal{D} = \mathcal{D}(\overline{x})$ be any fixed dyadic lattice from the family of lattices $\{\mathcal{D}(\overline{y})\}_{\overline{y}\in\mathbb{R}^{n+1}}$. Consider the subcollection of dyadic cubes $\mathcal{D}_H \subset \mathcal{D}$ defined as follows: $Q \in \mathcal{D}_H$ if there is a ball $B_{5\mathcal{R}_k}(\overline{x}_k)$ satisfying

$$B_{5\mathcal{R}_k}(\overline{x}_k) \cap Q \neq \varnothing \qquad \text{and} \qquad 10\mathcal{R}_k < \ell(Q) \leq 20\mathcal{R}_k. \tag{5.4.7}$$

Then, it is clear that

$$\bigcup_k B_{5\mathcal{R}_k}(\overline{x}_k) \subset \bigcup_{Q\in\mathcal{D}_H} Q.$$

Pick a subfamily of disjoint maximal cubes $\{Q_k\}_k$ from $\mathcal{D}_H$ so that we may define the exceptional set $H_{\mathcal{D}}$ as follows

$$\bigcup_{Q\in\mathcal{D}_H} Q = \bigcup_k Q_k =: H_{\mathcal{D}} = H_{\mathcal{D}(\overline{x})}.$$

By construction one has

$$H \subset H_{\mathcal{D}(\overline{x})}, \quad \forall \overline{x} \in \mathbb{R}^{n+1},$$

and since for each ball $B_{5\mathcal{R}_k}(\overline{x}_k)$ conditions (5.4.7) can only occur for a number of cubes in $\mathcal{D}_H$ which is bounded by a dimensional constant, there is $c_H = c_H(n) > 0$ so that

$$\sum_k \ell(Q_k)^n \leq c_H \sum_k \mathcal{R}_k^n \leq \frac{c_H}{L}\mu(F). \tag{5.4.8}$$

We define the exceptional set $T_{\mathcal{D}}$ in a similar manner but imposing a proper *accretivity condition* on the Radon-Nikodym derivative b. Consider the family $\mathcal{D}_T \subset \mathcal{D}$ of cubes satisfying

$$\mu(Q) \geq c_T|\nu(Q)|, \tag{5.4.9}$$

with $c_T(n) > 0$ big constant to be fixed below. Let $\{Q_k\}_k$ be the subcollection of maximal (and thus disjoint) dyadic cubes from $\mathcal{D}_T$. We define the exceptional set $T_{\mathcal{D}} = T_{\mathcal{D}(\overline{x})}$ as

$$T_{\mathcal{D}} := \bigcup_k Q_k.$$

The next result shows that $\mu(F \setminus (H_{\mathcal{D}} \cup T_{\mathcal{D}}))$ is comparable to $\mu(F)$. The reader can find its proof in [**T5, Lemma 6.12**].

LEMMA 5.4.2. *If L and c_T are chosen big enough, then*

$$|\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})| \leq \frac{1}{2}|\nu(F)| \qquad \textit{and} \qquad \mu(H_{\mathcal{D}} \cup T_{\mathcal{D}}) \leq \delta_0 \mu(F),$$

with $\delta_0 < 1$ dimensional.

5.4.3 Verifying the hypotheses of a Tb-theorem

The goal of this subsection is to prove the lemma below, analogous to [**T5, Lemma 6.8 (Main Lemma)**]. We remark that in order to apply the Tb-theorem of [**NTrVo2**] we will need to check an additional weak boundedness property for a particular suppressed kernel, due to the lack of anti-symmetry of P. Following the proof of the aforementioned reference or that of [**T5, Theorem 5.1**], one notices that such condition is already satisfied, since the Cauchy kernel is anti-symmetric.

LEMMA 5.4.3. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set verifying $\mathbf{A_1}, \mathbf{A_2}$ and $\mathbf{A_3}$. Let $\mathcal{F} := \cup_{i=1}^N \mathcal{Q}_i$ be the covering provided by Theorem 5.3.1, and $F := \cup_{i=1}^N Q_i = \cup_{i=1}^N \frac{1}{2}\mathcal{Q}_i$, that still satisfies $E \subset F$. Let μ and ν be the measures defined in (5.4.1) and (5.4.2). Then, there exists a subset $H \subset F$, such that*

1. *$\nu = b\mu$ for some b with $\|b\|_\infty \leq c_1$.*
2. *$|\nu(F)| = \widetilde{\gamma}(E)/2$.*
3. *$c_3^{-1}\widetilde{\gamma}(E) \leq \mu(F) \leq c_3\widetilde{\gamma}(E)$.*
4. *For any cube $Q \subset \mathbb{R}^{n+1}$, $|\nu(Q)| \leq c_4\ell(Q)^n$.*

5. The following hold

$$\int_{F\setminus H} \mathscr{P}_* \nu(\overline{x})\, \mathrm{d}\mu(\overline{x}) \leq c_5\, \mu(F), \qquad \int_{F\setminus H} \mathscr{P}_*^* \nu(\overline{x})\, \mathrm{d}\mu(\overline{x}) \leq c_5\, \mu(F).$$

6. If $\mu(B_r(\overline{x})) > Lr^n$ (for some big constant L), then $B_r(\overline{x}) \subset H$. In particular, for any $\overline{x} \in F \setminus H$ and $r > 0$, one has $\mu(B_r(\overline{x})) \leq Lr^n$.

7. H is of the form $\bigcup_{k\in I_H} B_{r_k}(\overline{x}_k)$, for some countable set of indices I_H. Moreover $\sum_{k\in I_H} r_k^n \leq \frac{5^n}{L}\mu(F)$.

Constants c_1, c_3, c_4 and c_5 are fixed and dimensional, while L can be chosen arbitrarily large.

Let us check that many of the hypothesis in Lemma 5.4.3 are verified for our particular choice of μ and ν. Properties *1*, *2* and *4* are proved in Lemma 5.4.1, and *3* follows from the third property of the same lemma, and by assumption $\mathbf{A_3}$ combined with $r_i^n \approx \widetilde{\gamma}(2Q_i \cap E)$. Moreover, taking $H \subset F$ to be as in (5.4.6), properties *6* and *7* are also guaranteed by definition. So it remains to check hypothesis *5*. To do so, let us begin by presenting a series of auxiliary results.

Lemma 5.4.4. *Let T be a distribution supported on a compact set $E \subset \mathbb{R}^{n+1}$ with $\|P * T\|_\infty \leq 1$ or $\|P^* * T\|_\infty \leq 1$. Let $Q \subset \mathbb{R}^{n+1}$ be a cube and $\varphi \in \mathcal{C}^1_{c,\mathcal{N}}(2Q)$ such that $\|\varphi\|_\infty \leq D\,\ell(Q)$ and $\|\nabla\varphi\|_\infty \leq A\,D$. Then*

$$|\langle T, \varphi\rangle| \lesssim D\,\ell(Q)^{n+1}.$$

Proof. Recall Remark 5.1.1 to notice that the proof is simply an application of [**MatP, Corollary 3.3**] to $\psi := \big[2AD\ell(Q)\big]^{-1}\varphi \in \mathcal{C}^1_{c,\mathcal{N}}(2Q)$ that is such that $\|\nabla\psi\|_\infty \leq \ell(2Q)^{-1}$. Observe that [**MatP, Corollary 3.3**] is also valid for P^*, which can be proved analogously using that P^* is fundamental solution of the conjugate operator $\overline{\Theta}^{1/2}$. □

Lemma 5.4.5. *Let $E \subset \mathbb{R}^{n+1}$ be compact set and $Q \subset \mathbb{R}^{n+1}$ a cube. If T is a distribution supported on $E \subset Q$, then the following holds:*

1. If $\|P * T\|_\infty \leq 1$ or $\|P^* * T\|_\infty \leq 1$ and $\varphi \in \mathcal{C}^1_{\mathcal{N}}(2Q)$ satisfies $\|\varphi\|_{L^\infty(2Q)} \leq D\,\ell(Q)$ and $\|\nabla\varphi\|_{L^\infty(2Q)} \leq D$, then

$$|\langle T, \varphi\rangle| \lesssim D\,\ell(Q)^{n+1}.$$

2. If $\|P * T\|_\infty \leq 1$, $\|P^* * T\|_\infty \leq 1$ and φ is as above, then

$$|\langle T, \varphi\rangle| \lesssim D\,\ell(Q) \cdot \widetilde{\gamma}(E).$$

Proof. We prove *1* under the assumption $\|P * T\|_\infty \leq 1$ (the proof is analogous for P^*). Consider a test function $\psi \in \mathcal{C}^\infty_c(2Q)$ with $\|\psi\|_\infty \leq 1, \|\nabla\psi\|_\infty \leq \ell(2Q)^{-1}$

and $\psi|_Q \equiv 1$. Notice that $\eta := \varphi\psi$ belongs to $\mathcal{C}^1_{c,\mathcal{N}}(2Q)$ and $\|\eta\|_\infty \leq D\,\ell(Q)$ and $\|\nabla\eta\|_\infty \leq 2D$. Then, by Lemma 5.4.4 and the fact that $\text{supp}(T) \subset E \subset Q$,

$$|\langle T, \varphi\rangle| = |\langle T, \eta\rangle| \lesssim D\,\ell(Q)^{n+1},$$

To prove *2*, assume that $\|P * T\|_\infty \leq 1$ and $\|P^* * T\|_\infty \leq 1$. Consider ψ as before and define the auxiliary function

$$\eta := \big[4D\ell(Q)\big]^{-1}\varphi\psi \in \mathcal{C}^1_{c,\mathcal{N}}(2Q),$$

that is such that $\|\nabla\eta\|_\infty \leq \ell(2Q)^{-1}$. By Remark 5.1.1, we shall apply [**MatP, Theorem 3.1**] to deduce $\|P * \eta T\|_\infty \lesssim 1$ and $\|P^* * \eta T\|_\infty \lesssim 1$. Hence $|\langle \eta T, 1\rangle| \lesssim \widetilde{\gamma}(E)$, that by definition means

$$|\langle T, \varphi\rangle| \lesssim D\,\ell(Q)\cdot\widetilde{\gamma}(E).$$

□

Let us now consider a distribution T supported on $E \subset Q = Q(c_Q, \ell(Q))$ with $\|P * T\|_\infty \leq 1$ and $\|P^* * T\|_\infty \leq 1$. Fix any $\overline{z} \in \mathbb{R}^{n+1} \setminus 3Q$ and set

$$\varphi_{\overline{z}}(\cdot) := P(\overline{z} - \cdot) - P(\overline{z} - c_Q), \qquad \varphi^*_{\overline{z}}(\cdot) := P^*(\overline{z} - \cdot) - P^*(\overline{z} - c_Q).$$

Write $\overline{z} = (z, \tau)$ and let $\overline{y} = (y, s)$ be a generic point in $\mathbb{R}^{n+1}$. We define the null set $\mathcal{N} = \mathcal{N}(\overline{z})$ to be the horizontal hyperplane

$$\mathcal{N} := \{\overline{y} \in \mathbb{R}^{n+1} \,:\, s = \tau\}.$$

This way $\varphi_{\overline{z}}, \varphi^*_{\overline{z}} \in \mathcal{C}^\infty_{\mathcal{N}}(2Q)$. Moreover, for any $\overline{y} \in 2Q$, by [**MatP, Lemma 2.1**],

$$|\varphi_{\overline{z}}(\overline{y})| = |P(\overline{z} - \overline{y}) - P(\overline{z} - c_Q)| \lesssim \frac{|\overline{y} - c_Q|}{|\overline{z} - c_Q|^{n+1}} \lesssim \frac{\ell(Q)}{\text{dist}(\overline{z}, E)^{n+1}},$$

and for any $\overline{y} \in 2Q \setminus \mathcal{N}$,

$$|\nabla\varphi_{\overline{z}}(\overline{y})| = |\nabla P(\overline{z} - \overline{y})| \lesssim \frac{1}{|\overline{z} - \overline{y}|^{n+1}} \leq \frac{3^{n+1}}{|\overline{z} - c_Q|^{n+1}} \lesssim \frac{1}{\text{dist}(\overline{z}, E)^{n+1}}.$$

and the same bounds clearly also hold for $\varphi^*_{\overline{z}}$. Therefore we have

$$\|\varphi_{\overline{z}}\|_{L^\infty(2Q)} \lesssim \frac{\ell(Q)}{\text{dist}(\overline{z}, E)^{n+1}}, \qquad \|\varphi^*_{\overline{z}}\|_{L^\infty(2Q)} \lesssim \frac{\ell(Q)}{\text{dist}(\overline{z}, E)^{n+1}},$$
$$\|\nabla\varphi_{\overline{z}}\|_{L^\infty(2Q)} \lesssim \frac{\ell(Q)}{\text{dist}(\overline{z}, E)^{n+1}}, \qquad \|\nabla\varphi^*_{\overline{z}}\|_{L^\infty(2Q)} \lesssim \frac{1}{\text{dist}(\overline{z}, E)^{n+1}}.$$

Hence, using the identities

$$\big|P * T(\overline{z}) - \langle T, 1\rangle P(\overline{z} - c_Q)\big| = |\langle T, \varphi_{\overline{z}}\rangle|,$$
$$\big|P^* * T(\overline{z}) - \langle T, 1\rangle P^*(\overline{z} - c_Q)\big| = |\langle T, \varphi^*_{\overline{z}}\rangle|,$$

a direct application of the second statement in Lemma 5.4.5 yields

Corollary 5.4.6. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set contained in the cube $Q = Q(c_Q, \ell(Q))$, and T a distribution admissible for $\widetilde{\gamma}(E)$. Then, for any $\overline{z} \in \mathbb{R}^{n+1} \setminus 3Q$,*

$$\big|P * T(\overline{z}) - \langle T, 1\rangle P(\overline{z} - c_Q)\big| \lesssim \frac{\ell(Q)}{\mathrm{dist}(\overline{z}, E)^{n+1}}\widetilde{\gamma}(E),$$
$$\big|P^* * T(\overline{z}) - \langle T, 1\rangle P^*(\overline{z} - c_Q)\big| \lesssim \frac{\ell(Q)}{\mathrm{dist}(\overline{z}, E)^{n+1}}\widetilde{\gamma}(E).$$

Lemma 5.4.7. *Let*

$$\widetilde{\varphi_i T_0} := \frac{\langle T_0, \varphi_i\rangle}{\mathcal{L}^{n+1}(B_i)}\mathcal{L}^{n+1}|_{B_i}, \quad i = 1, \dots, N.$$

For each $i = 1, \dots, N$, we have

1. *$\|P * \varphi_i T_0\|_\infty \lesssim 1$ and $\|P * \widetilde{\varphi_i T_0}\|_\infty \lesssim 1$.*
2. *For any $\overline{z} \in \mathbb{R}^{n+1} \setminus 3Q_i$,*

$$\big|P * \varphi_i T_0(\overline{z}) - P * \widetilde{\varphi_i T_0}(\overline{z})\big| \lesssim \frac{\ell(2Q_i)}{\mathrm{dist}(\overline{z}, 2Q_i)^{n+1}}\widetilde{\gamma}(2Q_i \cap E).$$

The same results hold changing P by P^*.

Proof. The first estimate in *1* is just a consequence of the localization result [**MatP, Theorem 3.1**]. Regarding the second, fix $\overline{x} \in \mathbb{R}^{n+1}$ and compute:

$$|P * \widetilde{\varphi_i T_0}|(\overline{x}) \leq \frac{|\langle T_0, \varphi_i\rangle|}{\mathcal{L}^{n+1}(B_i)}\int_{B_i}\frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|^n} \lesssim \frac{1}{r_i}\int_{B_i}\frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|^n},$$

where we have used that $|\langle T_0, \varphi_i\rangle| \lesssim r_i^n$, that has already been argued in (5.4.3). We deal with remaining integral as follows:

$$\begin{aligned}\int_{B_i}\frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|^n} &= \int_{B_i \cap \{|\overline{x} - \overline{y}| \geq r_i\}}\frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|^n} + \int_{B_i \cap \{|\overline{x} - \overline{y}| < r_i\}}\frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|^n} \\ &\leq \frac{\mathcal{L}^{n+1}(B_i)}{r_i^n} + \int_{B_{4r_i}(\overline{x})}\frac{\mathrm{d}\overline{y}}{|\overline{x} - \overline{y}|^n} \lesssim r_i.\end{aligned}$$

Let us prove *2*: fix $\overline{z} \in \mathbb{R}^{n+1} \setminus 3Q_i$ and notice

$$\big|P * \varphi_i T_0(\overline{z}) - P * \widetilde{\varphi_i T_0}(\overline{z})\big| = \left|P * \varphi_i T_0(\overline{z}) - \frac{\langle \varphi_i T_0, 1\rangle}{r_i^n}P * \mu_i(\overline{z})\right|,$$

where

$$\frac{1}{r_i^n}P * \mu_i(\overline{z}) = \int_{B_i}\big(P(\overline{z} - \overline{y}) - P(\overline{z} - c_{B_i})\big)\frac{\mathrm{d}\mu_i(\overline{y})}{r_i^n} + P(\overline{z} - c_{B_i}).$$

Therefore

$$\begin{aligned}\big|P*\varphi_iT_0(\overline{z})-P*\widetilde{\varphi_iT_0}(\overline{z})\big|&\\ \leq \big|P*\varphi_iT_0(\overline{z})-\langle\varphi_iT_0,1\rangle P(\overline{z}-c_{B_i})\big|&\\ +\frac{|\langle\varphi_iT_0,1\rangle|}{\mathcal{L}^{n+1}(B_i)}\int_{B_i}\big|P(\overline{z}-\overline{y})-P(\overline{z}-c_{B_i})\big|\,\mathrm{d}\overline{y}.&\qquad(5.4.10)\end{aligned}$$

Apply Corollary 5.4.6 to $T:=\varphi_iT_0$, admissible for $\widetilde{\gamma}(2Q_i\cap E)$ to deduce

$$\begin{aligned}\big|P*\varphi_iT_0(\overline{z})-\langle\varphi_iT_0,1\rangle P(\overline{z}-c_{B_i})\big|&\lesssim\frac{\ell(2Q_i)}{\operatorname{dist}(\overline{z},2Q_i\cap E)^{n+1}}\widetilde{\gamma}(2Q_i\cap E)\\ &\leq\frac{\ell(2Q_i)}{\operatorname{dist}(\overline{z},2Q_i)^{n+1}}\widetilde{\gamma}(2Q_i\cap E).\qquad(5.4.11)\end{aligned}$$

By [**MatP, Lemma 2.1**] and estimates in the proof of Lemma 5.4.1 we also have

$$\begin{aligned}\frac{|\langle\varphi_iT_0,1\rangle|}{\mathcal{L}^{n+1}(B_i)}&\int_{B_i}\big|P(\overline{z}-\overline{y})-P(\overline{z}-c_{B_i})\big|\,\mathrm{d}\overline{y}\\ &\lesssim\frac{1}{r_i}\frac{1}{|\overline{z}-c_{B_i}|^{n+1}}\int_{B_i}|\overline{y}-c_{B_i}|\,\mathrm{d}\overline{y}\\ &\lesssim\frac{\mathcal{L}^{n+1}(B_i)}{\operatorname{dist}(\overline{z},2Q_i)^{n+1}}\approx\frac{r_i}{\operatorname{dist}(\overline{z},2Q_i)^{n+1}}\widetilde{\gamma}(2\mathcal{Q}_i\cap E)\\ &\leq\frac{\ell(2Q_i)}{\operatorname{dist}(\overline{z},2Q_i)^{n+1}}\widetilde{\gamma}(2Q_i\cap E).\qquad(5.4.12)\end{aligned}$$

Hence, using (5.4.11) and (5.4.12) in (5.4.10) we finish the proof of *2*. □

Our next goal will be to obtain a regularized version of statement *2* in Lemma 5.4.7. Consider ψ a smooth radial function ψ supported on $B_1(0)$ with $0\leq\psi\leq 1$, $\int\psi=1$ and $\|\nabla\psi\|_\infty\leq 1$, and set

$$\Psi_\varepsilon(\overline{x}):=\frac{1}{\varepsilon^{n+1}}\psi\Big(\frac{\overline{x}}{\varepsilon}\Big),\quad \varepsilon>0.$$

Notice that $\int\Psi_\varepsilon=1$ for every $\varepsilon>0$. Define the regularized kernels

$$R_\varepsilon:=\Psi_\varepsilon*P,\qquad R_\varepsilon^*:=\Psi_\varepsilon*P^*,$$

as well as $\mathcal{R}_{\mu,\varepsilon}$ and $\mathcal{R}^*_{\mu,\varepsilon}$, its associated convolution operators with respect to the finite Borel measure μ.

Remark 5.4.2. Observe that, in particular, for a point $\overline{x}_0=(x_0,t_0)$ with $|\overline{x}_0|>\varepsilon$ and $t_0\neq 0$ we have

$$R_\varepsilon(\overline{x}_0)=P(\overline{x}_0)\quad\text{ and }\quad R^*_\varepsilon(\overline{x}_0)=P^*(\overline{x}_0).$$

This follows as in [**Vo, Equation 3.32**], using the harmonicity of P and P^* in $\mathbb{R}^{n+1}\setminus\{t=0\}$ and the mean value property in that domain. Notice that we need the spherical symmetry of Ψ_ε and $\int\Psi_\varepsilon=1$ to ensure this.

On the other hand, for any $\overline{x}_0$ such that $|\overline{x}_0|<\varepsilon$ we have

$$|R_\varepsilon(\overline{x}_0)|\lesssim\frac{1}{\varepsilon^n}\qquad\text{and}\qquad|R^*_\varepsilon(\overline{x}_0)|\lesssim\frac{1}{\varepsilon^n}.$$

Indeed, using the definition of Ψ and the fact that $0\leq\psi\leq 1$, we get

$$|R_\varepsilon(\overline{x}_0)|\leq\frac{1}{\varepsilon^{n+1}}\int_{B_\varepsilon(0)}\frac{\mathrm{d}\overline{x}}{|\overline{x}_0-\overline{x}|^n}\leq\frac{1}{\varepsilon^{n+1}}\int_{2B_\varepsilon(\overline{x}_0)}\frac{\mathrm{d}\overline{x}}{|\overline{x}_0-\overline{x}|^n}\lesssim\frac{1}{\varepsilon^n},$$

and analogously for $R^*_\varepsilon(\overline{x}_0)$.

LEMMA 5.4.8. *For any $\overline{z}\in\mathbb{R}^{n+1}\setminus 4Q_i$, any $\varepsilon>0$ and each $i=1,\dots,N$,*

$$\big|R_\varepsilon*\varphi_iT_0(\overline{z})-R_\varepsilon*\widetilde{\varphi_iT_0}(\overline{z})\big|\lesssim\frac{\ell(2Q_i)}{\mathrm{dist}(\overline{z},2Q_i)^{n+1}}\widetilde{\gamma}(2Q_i\cap E).$$

The same result holds changing R_ε by R^*_ε.

Proof. We will proof the result only for R_ε, since the arguments for R^*_ε are analogous. Fix $\overline{z}\in\mathbb{R}^{n+1}\setminus 4Q_i$ and $\varepsilon<\frac{1}{2}\mathrm{dist}(\overline{z},2Q_i)$. This way, since for any $\overline{y}\in B_\varepsilon(\overline{z})$ we have $\mathrm{dist}(\overline{y},2Q_i)\approx\mathrm{dist}(\overline{z},2Q_i)$, by Lemma 5.4.7 and the fact that $\int_{B_\varepsilon(\overline{z})}\Psi_\varepsilon(\overline{z}-\overline{y})\,\mathrm{d}\overline{y}=1$ we deduce

$$\begin{aligned}\big|R_\varepsilon*\varphi_iT_0(\overline{z})-R_\varepsilon*\widetilde{\varphi_iT_0}(\overline{z})\big|&\leq\int_{B_\varepsilon(\overline{z})}\Psi_\varepsilon(\overline{z}-\overline{y})\big|P*\varphi_iT_0(\overline{y})-P*\widetilde{\varphi_iT_0}(\overline{y})\big|\,\mathrm{d}\overline{y}\\&\lesssim\frac{\ell(2Q_i)}{\mathrm{dist}(\overline{z},2Q_i)^{n+1}}\widetilde{\gamma}(2Q_i\cap E),\end{aligned}$$

the desired inequality. Hence, we are only left to study the case $\varepsilon\geq\frac{1}{2}\mathrm{dist}(\overline{z},2Q_i)$. Observe that in this setting, since $\overline{z}\notin 4Q_i$ we have $\varepsilon\gtrsim\ell(Q_i)$. Write

$$\alpha_i:=\varphi_iT_0-\widetilde{\varphi_iT_0},$$

so that

$$|R_\varepsilon*\alpha_i(\overline{z})|\leq\int_{\mathrm{supp}(\Psi_\varepsilon*\alpha_i)}P(\overline{z}-\overline{y})\big|\Psi_\varepsilon*\alpha_i(\overline{y})\big|\,\mathrm{d}\overline{y}.$$

Observe that $\mathrm{supp}(\widetilde{\varphi_iT_0})\subset B_i$, $\mathrm{supp}(\varphi_iT_0)\subset 2Q_i\cap E$ and $\mathrm{supp}(\Psi_\varepsilon)\subset B_\varepsilon(0)$. Then, the support of $\Psi_\varepsilon*\alpha_i$ is contained in $\mathcal{U}_\varepsilon(2Q_i)$, an open ε-neighborhood of $2Q_i$. This implies

$$|R_\varepsilon*\alpha_i(\overline{z})|\leq\|\Psi_\varepsilon*\alpha_i\|_\infty\int_{\mathcal{U}_\varepsilon(2Q_i)}\frac{\mathrm{d}\overline{y}}{|\overline{z}-\overline{y}|^n}\lesssim(\ell(Q_i)+\varepsilon)\,\|\Psi_\varepsilon*\alpha_i\|_\infty\lesssim\varepsilon\,\|\Psi_\varepsilon*\alpha_i\|_\infty,$$

by integrating over decreasing annuli centered at $\overline{z}$, and using that $\text{dist}(\overline{z}, 2Q_i) \leq 2\varepsilon$ and $\varepsilon \gtrsim \ell(Q_i)$. On the other hand, if $\eta_i \in \mathcal{C}_c^\infty(3Q_i)$ with $0 \leq \eta_i \leq 1$, $\eta_i \equiv 1$ on $2Q_i$ (where the support of α_i is contained) and $\|\nabla \eta_i\|_\infty \leq \ell(Q_i)^{-1}$, we have

$$\begin{aligned}\Psi_\varepsilon * \alpha_i(\overline{w}) &= \langle \Psi_\varepsilon(\overline{w} - \cdot), \alpha_i \rangle = \langle \Psi_\varepsilon(\overline{w} - \cdot) - \Psi_\varepsilon(\overline{w} - c_{Q_i}), \alpha_i \rangle \\ &= \big\langle \big(\Psi_\varepsilon(\overline{w} - \cdot) - \Psi_\varepsilon(\overline{w} - c_{Q_i}) \big) \eta_i, \alpha_i \big\rangle,\end{aligned}$$

where we have used $\langle \varphi_i T_0, 1 \rangle = \langle \widetilde{\varphi_i T_0}, 1 \rangle$.

We claim that for each $\overline{w} \in \mathbb{R}^{n+1}$, the function

$$\varphi(\overline{\xi}) := \big(\Psi_\varepsilon(\overline{w} - \xi) - \Psi_\varepsilon(\overline{w} - c_{Q_i}) \big) \eta_i(\xi),$$

satisfies $\|\varphi\|_\infty \lesssim \varepsilon^{-(n+2)} \ell(Q_i)$ and $\|\nabla \varphi\|_\infty \lesssim \varepsilon^{-(n+2)}$. Then, using that statement *1* in Lemma 5.4.7 implies that φT_0 is admissible for $\widetilde{\gamma}(2Q_i \cap E)$ and that $\widehat{\varphi T_0}$ is admissible for $\widetilde{\gamma}(B_i)$, we have, by Lemma 5.4.5,

$$\begin{aligned}|\Psi_\varepsilon * \alpha_i| = |\langle \varphi, \alpha_i \rangle| &\lesssim \frac{\ell(Q_i)}{\varepsilon^{n+2}} \big\{ \widetilde{\gamma}(2Q_i \cap E) + \widetilde{\gamma}(B_i) \big\} \\ &\lesssim \frac{\ell(Q_i)}{\varepsilon^{n+2}} \big\{ \widetilde{\gamma}(2Q_i \cap E) + r_i^n \big\} \approx \frac{\ell(Q_i)}{\varepsilon^{n+2}} \widetilde{\gamma}(2Q_i \cap E).\end{aligned}$$

Therefore,

$$|R_\varepsilon * \alpha_i(\overline{z})| \lesssim \varepsilon \cdot \|\Psi_\varepsilon * \alpha_i\|_\infty \lesssim \frac{\ell(Q_i)}{\varepsilon^{n+1}} \widetilde{\gamma}(2Q_i \cap E) \lesssim \frac{\ell(2Q_i)}{\text{dist}(\overline{z}, 2Q_i)^{n+1}} \widetilde{\gamma}(2Q_i \cap E),$$

where we have used $\text{dist}(\overline{z}, 2Q_i) \leq 2\varepsilon$, and we deduce the desired estimate. Hence, we are left to prove the claim. Let us fix $\overline{w} \in \mathbb{R}^{n+1}$ and compute: on the one hand

$$\|\varphi\|_\infty = \|\varphi\|_{L^\infty(3Q_i)} \leq \frac{1}{\varepsilon^{n+2}} \sup_{\overline{\xi} \in 3Q_i} |\overline{\xi} - c_{Q_i}| \cdot \|\nabla \psi\|_\infty \lesssim \frac{\ell(Q_i)}{\varepsilon^{n+2}},$$

while on the other hand

$$\begin{aligned}\|\nabla \varphi\|_\infty &= \|\nabla \varphi\|_{L^\infty(3Q_i)} \\ &\leq \frac{1}{\ell(Q_i)} \sup_{\overline{\xi} \in 3Q_i} \big| \Psi_\varepsilon(\overline{w} - \overline{\xi}) - \Psi_\varepsilon(\overline{w} - c_{Q_i}) \big| + \frac{1}{\varepsilon^{n+2}} \|\nabla \psi\|_\infty \lesssim \frac{1}{\varepsilon^{n+2}}.\end{aligned}$$

Hence, we are done. □

Let us finally prove property *5* in Lemma 5.4.3:

LEMMA 5.4.9. *The following estimates hold:*

$$\int_{F \setminus H} \mathscr{P}_* \nu(\overline{y}) \, \mathrm{d}\mu(\overline{y}) \lesssim \mu(F), \qquad \int_{F \setminus H} \mathscr{P}_*^* \nu(\overline{y}) \, \mathrm{d}\mu(\overline{y}) \lesssim \mu(F).$$

Proof. We only prove the first estimate, since the proof of the second is analogous. Begin by noticing that

$$\int_{F\setminus H} \mathscr{P}_*\nu(\overline{y})\,\mathrm{d}\mu(\overline{y}) = \int_{F\setminus H} \sup_{\varepsilon>0} \big|\mathscr{P}_\varepsilon\nu(\overline{y})\big|\,\mathrm{d}\mu(\overline{y})$$
$$\leq \int_{F\setminus H} \sup_{\varepsilon>0} \big|\mathscr{P}_\varepsilon\nu(\overline{y}) - \mathcal{R}_\varepsilon\nu(\overline{y})\big|\,\mathrm{d}\mu(\overline{y}) + \int_{F\setminus H} \sup_{\varepsilon>0} \big|\mathcal{R}_\varepsilon\nu(\overline{y})\big|\,\mathrm{d}\mu(\overline{y}) =: \mathrm{I} + \mathrm{II}.$$

For I, observe that for each $\varepsilon > 0$ and $\overline{y} = (y, s) \in F \setminus H$, since $\nu \ll \mu \ll \mathcal{L}^{n+1}$,

$$\nu\big(\{(z,\tau) \in \mathbb{R}^{n+1} \,:\, \tau = s\}\big) = 0,$$

so applying Remark 5.4.2 we deduce

$$\big|\mathscr{P}_\varepsilon\nu(\overline{y}) - \mathcal{R}_\varepsilon\nu(\overline{y})\big|$$
$$= \left| \int_{|\overline{y}-\overline{z}|\leq\varepsilon} R_\varepsilon(\overline{y}-\overline{z})\,\mathrm{d}\nu(\overline{z}) + \int_{|\overline{y}-\overline{z}|>\varepsilon} R_\varepsilon(\overline{y}-\overline{z})\,\mathrm{d}\nu(\overline{z}) - \int_{|\overline{y}-\overline{z}|>\varepsilon} P(\overline{y}-\overline{z})\,\mathrm{d}\nu(\overline{z}) \right|$$
$$= \left| \int_{|\overline{y}-\overline{z}|\leq\varepsilon} R_\varepsilon(\overline{y}-\overline{z})\,\mathrm{d}\nu(\overline{z}) \right| \leq \frac{1}{\varepsilon^n}|\nu|(B_\varepsilon(\overline{y})) \lesssim \frac{1}{\varepsilon^n}\mu(B_\varepsilon(\overline{y})) \leq M\mu(\overline{y}).$$

Then, using that $\overline{y} \in F \setminus H$ we get $|\mathrm{I}| \lesssim \mu(F)$. So we are left to estimate II. We introduce the notation

$$R_* * \nu(\overline{y}) := \sup_{\varepsilon>0} |R_\varepsilon * \nu(\overline{y})|,$$

so that

$$\mathrm{II} := \int_{F\setminus H} R_* * \nu(\overline{y})\,\mathrm{d}\mu(\overline{y}) \leq \int_{F\setminus H} R_* * T_0(\overline{y})\,\mathrm{d}\mu(\overline{y}) + \int_{F\setminus H} R_* * (\nu - T_0)(\overline{y})\,\mathrm{d}\mu(\overline{y}).$$

Since $\|P * T_0\|_\infty \leq 1$ by construction, we also have $\|R_\varepsilon * T_0\|_\infty \leq 1$, uniformly on $\varepsilon > 0$. For the second integral in II,

$$\int_{F\setminus H} R_* * (\nu - T_0)(\overline{y})\,\mathrm{d}\mu(\overline{y}) \leq \sum_{i=1}^N \int_{F\setminus H} R_* * \big(\widetilde{\varphi_i T_0} - \varphi_i T_0\big)(\overline{y})\,\mathrm{d}\mu(\overline{y})$$
$$= \sum_{i=1}^N \bigg(\int_{4Q_i} R_* * \big(\widetilde{\varphi_i T_0} - \varphi_i T_0\big)(\overline{y})\,\mathrm{d}\mu(\overline{y}) + \int_{F\setminus(4Q_i\cup H)} R_* * \big(\widetilde{\varphi_i T_0} - \varphi_i T_0\big)(\overline{y})\,\mathrm{d}\mu(\overline{y}) \bigg)$$
$$=: \sum_{i=1}^N I_{i,1} + I_{i,2},$$

where in the first inequality we have used that $\sum_i \varphi_i \equiv 1$ on F and $\nu = \sum_i \widetilde{\varphi_i T_0}$. For each $i = 1, \ldots, N$ we set $\alpha_i := \widetilde{\varphi_i T_0} - \varphi_i T_0$ and apply the first statement in Lemma 5.4.7 to deduce $\|P * \alpha_i\|_\infty \lesssim 1$. Hence, $I_{i,1} \lesssim \mu(4Q_i)$. To estimate $I_{i,2}$ we will use Lemma 5.4.8. Let N be the smallest integer such that

$$A_N := \big(4^{N+1}Q_i \setminus 4^N Q_i\big) \setminus H \neq \varnothing.$$

Then,

$$I_{i,2} \lesssim \ell(2Q_i)\,\widetilde{\gamma}(2Q_i \cap E) \sum_{k=N}^{\infty} \int_{A_N} \frac{\mathrm{d}\mu(\overline{y})}{\mathrm{dist}(\overline{y}, 2Q_i)^{n+1}}$$
$$\lesssim \ell(2Q_i)\,\widetilde{\gamma}(2Q_i \cap E) \sum_{k=N}^{\infty} \frac{\mu(4^{k+1}Q_i)}{(4^k \ell(Q_i))^{n+1}}.$$

Observe that for any $\overline{x} \in A_N$, we have for each $k \geq N$

$$\mu(4^{k+1}Q_i) \leq \mu(B_{2\cdot\mathrm{diam}(4^{k+1}Q_i)}(\overline{x})) \lesssim \big(4^k \ell(Q_i)\big)^n,$$

where we have used that $\overline{x} \notin H$. Therefore,

$$I_{i,2} \lesssim \ell(2Q_i)\,\widetilde{\gamma}(2Q_i \cap E) \sum_{k=N}^{\infty} \frac{1}{4^k \ell(Q_i)} \lesssim \widetilde{\gamma}(2Q_i \cap E) \approx r_i^n = \mu(B_i).$$

Since the cubes $10Q_1, \ldots, 10Q_N$ have bounded overlap we conclude that

$$\int_{F\setminus H} R_* * (\nu - T_0)(\overline{y})\,\mathrm{d}\mu(\overline{y}) \lesssim \sum_{i=1}^{N} \mu(4Q_i) \lesssim \mu(F).$$

□

The proof of *5* concludes that of Lemma 5.4.3. In order to apply the Tb-theorem of [**NTrVo2**] or [**T5, Theorem 5.1**] we still need to check an additional weak boundedness property for our n-dimensional C-Z kernel, which fails being anti-symmetric.

5.4.4 The exceptional sets $\mathcal{S}$ and $\boldsymbol{W}_{\mathcal{D}}$. The additional eighth property

Assume that $F \subset B_{2^{N-3}}(0)$ for some integer N large enough. Observe that assumption $\mathbf{A_2}$ together with properties *1* and *4* in Theorem 5.3.1 imply that we can take $N = 4$, for example. We write

$$\Omega := \big[-2^{N-1}, 2^{N-1}\big)^{n+1},$$

and consider the *random* cube $Q^0(\overline{w}) := \overline{w} + [-2^N, 2^N)^{n+1}$, with $\overline{w} \in \Omega$. Observe that $F \subset Q^0(\overline{w})$ for any $\overline{w} \in \Omega$. Let $\mathbb{P}$ be the uniform probability measure on Ω, that is, the normalized Lebesgue measure on the cube Ω.

When carefully reviewing the proofs of the the non-homogeneous Tb-theorems of [**T5, §5**] and [**Vo**], one encounters expressions that, in our context, would be of the form

$$\langle \mathscr{P}_{\Lambda,\mu}(\chi_\Delta b), (\chi_\Delta b)\rangle := \int_\Delta \int_\Delta P_\Lambda(\overline{x}, \overline{y})\,\mathrm{d}\nu(\overline{y})\,\mathrm{d}\nu(\overline{x}),$$

where Λ is a certain 1-Lipschitz function associated to the suppressed kernel P_Λ, $\Delta = P \cap S$ is a parallelepiped obtained as the intersection of two cubes $P \in \mathcal{D}(\overline{w}_1) =:$

$\mathcal{D}_1$, $S \in \mathcal{D}(\overline{w}_2) =: \mathcal{D}_2$, with $\overline{w}_1, \overline{w}_2 \in \Omega$, and where $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dyadic lattices which are reciprocally *good* to one another. Moreover, such cubes P, S satisfy $P \subset F \setminus (H_{\mathcal{D}_1} \cup T_{\mathcal{D}_1})$ and $S \subset F \setminus (H_{\mathcal{D}_2} \cup T_{\mathcal{D}_2})$. The latter inclusions imply that P is a *transit cube* with respect to $\mathcal{D}_1$ and we write $P \in \mathcal{D}_1^{\mathrm{tr}}$, and analogously for S with respect to $\mathcal{D}_2$ (we will specify the precise meaning of the terms *good* and *transit* in Remark 5.4.3). P, S and all their dyadic children have *M-thin boundaries* (with respect to μ), for some $M > 0$ dimensional constant. In this setting, a set $X \subset \mathbb{R}^{n+1}$ has M-thin boundary if

$$\mu\big(\{\overline{x} \in \mathbb{R}^{n+1} : \mathrm{dist}(\overline{x}, \partial X) \leq \tau\}\big) \leq M\tau^n, \qquad \text{for all } \ \tau > 0.$$

As one may expect, expressions of the form $\langle \mathscr{P}_{\Lambda,\mu}(\chi_\Delta b), (\chi_\Delta b)\rangle$ are null if the kernel associated to the operator $\mathscr{P}$ is anti-symmetric, as it occurs in [**T5**] with the Cauchy kernel, or in [**Vo**] with the vector Riesz kernel. If this is not the case, following the proofs of general non-homogeneous Tb-theorems and, more precisely, the arguments of [**NTrVo3, §10.2**] or [**HyMar, §9**], the latter expressions can be dealt with if we have $\|\mathscr{P}_{\Lambda,\varepsilon}\nu\|_{L^\infty(\mu)} \lesssim 1$, $\|\mathscr{P}^*_{\Lambda,\varepsilon}\nu\|_{L^\infty(\mu)} \lesssim 1$ uniformly on $\varepsilon > 0$, as well as the following weak boundedness property

$$|\langle \mathscr{P}_{\Lambda,\mu}(\chi_Q b), (\chi_Q b)\rangle| \lesssim \mu(\lambda Q), \quad \lambda > 1,$$

that suffices to hold for cubes Q with M-thin boundary and contained in parallelepipeds Δ of the form $P \cap S$, where P and S are as above. Such kind of restricted weak boundedness property for cubes with thin boundary is already checked in the proofs of [**MatPT, Theorem 5.5**] and [**MatP, Theorem 4.3**].

Our goal in this subsection will be to verify the above conditions by choosing a proper 1-Lipschitz function Λ which will depend on an additional exceptional set $\mathcal{S}$. If such additional properties hold, the proofs of the Tb-theorems found in [**T5, Ch.5**] and [**Vo**] can be adapted to our non anti-symmetric setting. To define the exceptional set $\mathcal{S}$, we will follow an analogous construction to that of [**T5, §5.2**]. We begin by writing

$$\mathcal{S}_1' := \big\{\overline{x} \in F \, : \, \mathscr{P}_*\nu(\overline{x}) > \alpha\big\}, \qquad \mathcal{S}_2' := \big\{\overline{x} \in F \, : \, \mathscr{P}^*_{\nu,*}1(\overline{x}) > \alpha\big\},$$

for a large constant $\alpha > 0$ to be chosen below. For the moment, let us say that $\alpha \gg c_1 L$. We also write,

$$\begin{aligned} &\text{for } \ \overline{x} \in \mathcal{S}_1' : \quad e_1(\overline{x}) := \sup\big\{\varepsilon > 0 \, : \, |\mathscr{P}_\varepsilon\nu(\overline{x})| > \alpha\big\}, \\ &\text{for } \ \overline{x} \in \mathcal{S}_2' : \quad e_2(\overline{x}) := \sup\big\{\varepsilon > 0 \, : \, |\mathscr{P}^*_{\nu,\varepsilon}1(\overline{x})| > \alpha\big\}. \end{aligned}$$

If $\overline{x} \in F \setminus (\mathcal{S}_1' \cup \mathcal{S}_2')$ we convey $e_1(\overline{x}) = e_2(\overline{x}) := 0$. We define also

$$\mathcal{S}_1 := \bigcup_{\overline{x} \in \mathcal{S}_1'} B_{e_1(\overline{x})}(\overline{x}) \qquad \text{and} \qquad \mathcal{S}_2 := \bigcup_{\overline{x} \in \mathcal{S}_2'} B_{e_2(\overline{x})}(\overline{x}),$$

as well as the exceptional set

$$\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2. \tag{5.4.13}$$

Let us first show that for any $\overline{x} \in \mathbb{R}^{n+1}$, $\mu(\mathcal{S} \setminus H_{\mathcal{D}(\overline{x})})$ is small if α is taken big enough. The proof of the following lemma is analogous to that of [**T5, Lemma 5.2**].

LEMMA 5.4.10. *Let $\overline{x}_0 \in \mathbb{R}^{n+1}$ and $\mathcal{D} := \mathcal{D}(\overline{x}_0)$. Then, for $\alpha > 0$ big enough,*

$$\mu(\mathcal{S} \setminus H_{\mathcal{D}}) \leq \frac{4c_5}{\alpha}\mu(F).$$

Proof. Let $\overline{y} \in \mathcal{S} \setminus H_{\mathcal{D}}$ and assume, for example, $\overline{y} \in \mathcal{S}_1$. Then $\overline{y} \in B_{e_1(\overline{x})}(\overline{x})$ for some $\overline{x} \in \mathcal{S}_1'$. Let $\varepsilon_0(\overline{x})$ be such that $|\mathscr{P}_{\varepsilon_0(\overline{x})}\nu| > \alpha$ and $\overline{y} \in B_{\varepsilon_0(\overline{x})}(\overline{x})$. Observe that

$$\begin{aligned}\big|\mathscr{P}_{\varepsilon_0(\overline{x})}\nu(\overline{x}) - \mathscr{P}_{\varepsilon_0(\overline{x})}\nu(\overline{y})\big| &\leq \big|\mathscr{P}_{\varepsilon_0(\overline{x})}\big(\chi_{B_{2\varepsilon_0(\overline{x})}(\overline{y})}\nu\big)\overline{x})\big| + \big|\mathscr{P}_{\varepsilon_0(\overline{x})}\big(\chi_{B_{2\varepsilon_0(\overline{x})}(\overline{y})}\nu\big)(\overline{y})\big| \\ &\quad + c_1\int_{\mathbb{R}^{n+1}\setminus B_{2\varepsilon_0(\overline{x})}(\overline{y})} |P(\overline{x}-\overline{z}) - P(\overline{y}-\overline{z})|\,\mathrm{d}\mu(\overline{z}).\end{aligned}$$

Since $\overline{y} \notin H_{\mathcal{D}}$, the first two terms are bounded above by

$$c_1\frac{\mu\big(B_{2\varepsilon_0(\overline{x})}(\overline{y})\big)}{\varepsilon_0(\overline{x})^n} \leq 2^n Lc_1.$$

Regarding the third term, if A is the C-Z constant of P from property (c) of [**MatP, Lemma 2.1**], integration over annuli and using again that $\overline{y} \notin H_{\mathcal{D}}$ yield

$$c_1\int_{\mathbb{R}^{n+1}\setminus B_{2\varepsilon_0(\overline{x})}(\overline{y})} |P(\overline{x}-\overline{z}) - P(\overline{y}-\overline{z})|\,\mathrm{d}\mu(\overline{z}) \lesssim 2^n ALc_1.$$

Therefore, naming $\kappa := 2^n Lc_1(2+A)$, we have $\big|\mathscr{P}_{\varepsilon_0(\overline{x})}\nu(\overline{x}) - \mathscr{P}_{\varepsilon_0(\overline{x})}\nu(\overline{y})\big| \leq \kappa$, and thus

$$\big|\mathscr{P}_{\varepsilon_0(\overline{x})}\nu(\overline{y})\big| > \alpha - \kappa,$$

which implies, in particular, $\mathscr{P}_*\nu(\overline{y}) > \alpha - \kappa$. If we have had $\overline{y} \in \mathcal{S}_2$ we would have obtained the same bound for $\mathscr{P}_*^*\nu(\overline{y})$, since P and P^* share the same C-Z constants. In any case, pick $\alpha \geq 2\kappa$ and observe that

$$\begin{aligned}\mu(\mathcal{S}\setminus H_{\mathcal{D}}) &\leq \int_{\mathcal{S}_1\setminus H_{\mathcal{D}}}\mathrm{d}\mu + \int_{\mathcal{S}_2\setminus H_{\mathcal{D}}}\mathrm{d}\mu \\ &\leq \frac{2}{\alpha}\int_{F\setminus H_{\mathcal{D}}}\mathscr{P}_*\nu\,\mathrm{d}\mu + \frac{2}{\alpha}\int_{F\setminus H_{\mathcal{D}}}\mathscr{P}_*^*\nu\,\mathrm{d}\mu \leq \frac{4c_5}{\alpha}\mu(F).\end{aligned}$$

□

One of the implications of the above lemma is the following: by setting $\delta_1 := (1+\delta_0)/2$ (where δ_0 is the parameter appearing in Lemma 5.4.2), we have $\delta_0 < \delta_1 < 1$, and choosing $\alpha := \max\{2\kappa, 8c_5/(1-\delta_0)\}$, we get

$$\mu\big(H_{\mathcal{D}(\overline{x})} \cup T_{\mathcal{D}(\overline{x})}\big) + \mu\big(\mathcal{S}\setminus H_{\mathcal{D}(\overline{x})}\big) \leq \delta_1\mu(F), \qquad \forall \overline{x} \in \mathbb{R}^{n+1}.$$

With this in mind, and defining the *total exceptional set*

$$W_{\mathcal{D}(\overline{x})} := H_{\mathcal{D}(\overline{x})} \cup T_{\mathcal{D}(\overline{x})} \cup \mathcal{S},$$

one obtains

$$\mu(F \setminus W_{\mathcal{D}(\overline{x})}) \geq (1-\delta_1)\mu(F), \tag{5.4.14}$$

which is a necessary inequality in order to carry out the final probabilistic argument in the proof of the Tb-theorem (see [**T5, §5.11.2**]).

The exceptional set $\mathcal{S}$ exhibits additional important properties regarding suppressed kernels (see §5.2.1), provided their associated 1-Lipschitz function Λ satisfies certain conditions. In order to prove them, we present two preliminary results that admit almost identical proofs to [**Vo, Lemma 8.3**] and [**T5, Lemma 5.5**] respectively, and we will precise the arguments. Fix σ any finite Borel measure in $\mathbb{R}^{n+1}$ and $\Lambda : \mathbb{R}^{n+1} \to [0,\infty)$ a 1-Lipschitz function. The following result will be useful in the sequel:

LEMMA 5.4.11. *For any $\overline{x} \in \mathbb{R}^{n+1}$ and any $\varepsilon \geq \Lambda(\overline{x})$,*

$$\big|\mathscr{P}_{\varepsilon}\sigma(\overline{x}) - \mathscr{P}_{\Lambda,\varepsilon}\sigma(\overline{x})\big| \lesssim \sup_{r\geq\Lambda(\overline{x})} \frac{|\sigma|(B_r(\overline{x}))}{r^n}.$$

The same result holds changing $\mathscr{P}$ by $\mathscr{P}^*$.

In light of the previous result, the following lemma also follows:

LEMMA 5.4.12. *Let $\overline{x} \in \mathbb{R}^{n+1}$ and $r_0 > 0$ such that $\mu(B_r(\overline{x})) \leq Lr^n$ for $r \geq r_0$, as well as $|\mathscr{P}_{\varepsilon}\nu(\overline{x})| \leq \alpha$ and $|\mathscr{P}^*_{\varepsilon}\nu(\overline{x})| \leq \alpha$ for $\varepsilon \geq r_0$. If $\Lambda(\overline{x}) \geq r_0$, then*

$$\big|\mathscr{P}_{\Lambda,\varepsilon}\nu(\overline{x})\big| \lesssim \alpha + c_1 L \qquad \text{and} \qquad \big|\mathscr{P}^*_{\Lambda,\varepsilon}\nu(\overline{x})\big| \lesssim \alpha + c_1 L,$$

uniformly on $\varepsilon > 0$.

Bearing in mind the above result, we choose our 1-Lipschitz function $\Lambda : \mathbb{R}^{n+1} \to [0,\infty)$ to satisfy, for some $\overline{x}_0 \in \mathbb{R}^{n+1}$,

$$\Lambda(\overline{x}) \geq \operatorname{dist}\big(\overline{x}, \mathbb{R}^{n+1} \setminus \big(H_{\mathcal{D}(\overline{x}_0)} \cup \mathcal{S}\big)\big).$$

This choice is consistent with the Lipschitz functions Λ that appear in the proofs of the Tb-theorems presented in [**T5, Ch.5**] and [**Vo**], in the sense that they are constructed to ensure that the previous inequality is satisfied. This way, since $H_{\mathcal{D}(\overline{x}_0)} \cup \mathcal{S}$ contains all non-Ahlfors balls and all the balls $B_{e_1(\overline{x})}(\overline{x})$, $B_{e_2(\overline{x})}(\overline{x})$ for $\overline{x} \in F$, we have

$$\Lambda(\overline{x}) \geq \max\{\mathcal{R}(\overline{x}), e_1(\overline{x}), e_2(\overline{x})\},$$

where $\mathcal{R}(\overline{x})$ is the Ahlfors radius defined in (5.4.5). So by the definition of $\mathcal{S}$ and choosing $r_0 := \max\{\mathcal{R}(\overline{x}), e_1(\overline{x}), e_2(\overline{x})\}$, Lemma 5.4.2 directly yields

LEMMA 5.4.13. *Let $\overline{x}_0 \in \mathbb{R}^{n+1}$ and $\Lambda : \mathbb{R}^{n+1} \to [0,\infty)$ a 1-Lipschitz function such that $\Lambda(\overline{x}) \geq \operatorname{dist}\big(\overline{x}, \mathbb{R}^{n+1} \setminus \big(H_{\mathcal{D}(\overline{x}_0)} \cup \mathcal{S}\big)\big)$ for all $\overline{x} \in \mathbb{R}^{n+1}$. Then,*

$$\mathscr{P}_{\Lambda,*}\nu(\overline{x}) \leq c_\Lambda \quad \text{and} \quad \mathscr{P}^*_{\Lambda,*}\nu(\overline{x}) \leq c_\Lambda, \quad \forall \overline{x} \in F,$$

with c_Λ depending only on c_1, c_5, L and δ_0.

Hence, if we choose such a Λ, we have $\|\mathscr{P}_{\Lambda,\varepsilon}\nu\|_{L^\infty(\mu)}\lesssim 1$ and $\|\mathscr{P}^*_{\Lambda,\varepsilon}\nu\|_{L^\infty(\mu)}\lesssim 1$ uniformly on $\varepsilon>0$. So we are left to verify the weak boundedness property for cubes with M-thin boundary contained in the parallelepipeds presented at the beginning of this subsection. To do so, we will need two auxiliary results, similar to those found in [**MatP, §3**].

LEMMA 5.4.14. *Let φ be a function supported on a cube $Q\subset\mathbb{R}^{n+1}$ with $\|\varphi\|_\infty\leq 1$. Then, $\mathscr{P}_\Lambda(\varphi\nu)$ is a locally integrable function. Moreover, if $\mu(Q)\lesssim\ell(Q)^n$, there exists $\overline{x}_0\in\frac{1}{4}Q$ and a dimensional constant c_0 such that*

$$\big|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x}_0)\big|\leq c_0.$$

Proof. In [**MatP, Lemma 3.5**], the authors deal with general distributions with n-growth. In our statement, however, ν is a specified signed measure and the cubes Q satisfy the additional growth condition $\mu(Q)\lesssim\ell(Q)^n$. To prove the local integrability of $\mathscr{P}_\Lambda(\varphi\nu)$, let us fix $\overline{x}\in\mathbb{R}^{n+1}$ and name $I_Q:=\{j=1,\ldots,N\,:\,Q\cap B_j\neq\varnothing\}$. We compute, bearing in mind Lemma 5.2.1:

$$\begin{aligned}\big|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})\big| &= \left|\sum_{j\in I_Q}\frac{\langle T_0,\varphi_j\rangle}{\mathcal{L}^{n+1}(B_j)}\int_{Q\cap B_j}P_\Lambda(\overline{x},\overline{y})\varphi(\overline{y})\,\mathrm{d}\overline{y}\right|\\ &\lesssim\sum_{j\in I_Q}\frac{1}{r_j}\int_{Q\cap B_j}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|^n}=:\sum_{j\in I_Q}\frac{1}{r_j}I_j(\overline{x}).\end{aligned}$$

To study $I_j(\overline{x})$ we split the integral into the domain

$$D_{1,j}:=Q\cap B_j\cap\big\{\overline{y}\,:\,2|\overline{x}-\overline{y}|\geq\operatorname{diam}(B_j\cap Q)\big\},$$

and its complementary $D_{2,j}:=(Q\cap B_j)\setminus D_1$. For the first domain we directly have

$$\int_{D_{1,j}}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|^n}\lesssim\frac{\mathcal{L}^{n+1}(D_{1,j})}{\operatorname{diam}(B_j\cap Q)^n}\leq\operatorname{diam}(B_j\cap Q)\leq r_j.$$

For the second, notice that $D_{2,j}=Q\cap B_j\cap\{\overline{y}\,:\,2|\overline{x}-\overline{y}|<\operatorname{diam}(B_j\cap Q)\}\subset B_{3\operatorname{diam}(B_j\cap Q)}(\overline{x})$. Then, writing $A_k:=B_{2^{-k}3\operatorname{diam}(B_j\cap Q)}(\overline{x})\setminus B_{2^{-k-1}3\operatorname{diam}(B_j\cap Q)}(\overline{x})$ for $k\geq 0$ we obtain

$$\int_{D_{2,j}}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|^n}\leq\sum_{k\geq 0}\int_{A_k}\frac{\mathrm{d}\overline{y}}{|\overline{x}-\overline{y}|^n}\lesssim\operatorname{diam}(B_j\cap Q)\sum_{k\geq 0}\frac{1}{2^k}\leq r_j.$$

Hence

$$\big|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})\big|\lesssim|I_Q|<N<\infty,$$

and thus the local integrability of $\mathscr{P}_\Lambda(\varphi\nu)$ follows. To prove the second assertion, we use $\mu(Q)\lesssim\ell(Q)^n$ together with Tonelli's theorem and Lemma 5.2.1 to obtain

$$\begin{aligned}\int_Q\big|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})\big|\,\mathrm{d}\overline{x} &= \int_Q\left|\int_{F\cap Q}P_\Lambda(\overline{x},\overline{y})\varphi(\overline{y})\,\mathrm{d}\nu(\overline{y})\right|\mathrm{d}\overline{x}\\ &\leq\int_{F\cap Q}\bigg(\int_Q\frac{\mathrm{d}\overline{x}}{|\overline{x}-\overline{y}|^n}\bigg)|\varphi(\overline{y})|\,\mathrm{d}\mu(\overline{y})\leq\ell(Q)\int_{F\cap Q}|\varphi(\overline{y})|\,\mathrm{d}\mu(\overline{y})\\ &\leq\ell(Q)\mu(Q)\lesssim\ell(Q)^{n+1}=\mathcal{L}^{n+1}(Q),\end{aligned}$$

Therefore,

$$\frac{1}{\mathcal{L}^{n+1}(\frac{1}{4}Q)} \int_{\frac{1}{4}Q} |\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})| \,\mathrm{d}\overline{x} \lesssim 1,$$

and the desired result follows. □

With the above lemma, we are able to prove a weaker *localization-type* result for $\mathscr{P}_\Lambda$, analogous to [**MatP, Theorem 3.1**]. It will be valid for our particular signed measure ν and cubes contained in $F \setminus H$. Recall that the latter inclusion implies, by property *6* in Lemma 5.4.3, that for any $\overline{x} \in Q$, if $R(\overline{x})$ is the cube centered at $\overline{x}$ with side length $\ell(R)$, then $\mu(\lambda R(\overline{x})) \leq \sqrt{n+1}L\,\ell(\lambda R)^n \simeq \ell(\lambda R)^n$, for any $\lambda > 0$.

LEMMA 5.4.15. *Let $Q \subset F \setminus H$ be a cube and φ a test function with $0 \leq \varphi \leq 1$, $\|\nabla\varphi\|_\infty \leq \ell(Q)^{-1}$ and such that $\varphi \equiv 1$ on Q and $\varphi \equiv 0$ on $(2Q)^c$. Then,*

$$|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})| \lesssim 1, \qquad \forall \overline{x} \in Q.$$

Proof. Let us fix any $\overline{x} \in Q$ and consider $\overline{x}_0 \in \frac{1}{2}Q$ the point obtained in Lemma 5.4.14. Observe that $\varphi(\overline{x}) = \varphi(\overline{x}_0) = 1$. We rewrite $\mathscr{P}_\Lambda(\varphi\nu)(\cdot)$ as follows

$$\begin{aligned}\mathscr{P}_\Lambda(\varphi\nu)(\cdot) = &\int_{\mathbb{R}^{n+1}\setminus 4Q} P_\Lambda(\cdot,\overline{y})(\varphi(\overline{y}) - \varphi(\overline{x}))\,\mathrm{d}\nu(\overline{y}) \\ &+ \int_{4Q} P_\Lambda(\cdot,\overline{y})(\varphi(\overline{y}) - \varphi(\overline{x}))\,\mathrm{d}\nu(\overline{y}) + \varphi(\overline{x})\int P_\Lambda(\cdot,\overline{y})\,\mathrm{d}\nu(\overline{y}),\end{aligned}$$

and we apply this decomposition to $\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})$ and $\mathscr{P}_\Lambda(\varphi\nu)(\overline{x}_0)$. Therefore

$$\begin{aligned}&|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x}) - \mathscr{P}_\Lambda(\varphi\nu)(\overline{x}_0)| \\ &\quad\leq \left|\int_{\mathbb{R}^{n+1}\setminus 4Q} (P_\Lambda(\overline{x},\overline{y}) - P_\Lambda(\overline{x}_0,y))(\varphi(\overline{y}) - \varphi(\overline{x}))\,\mathrm{d}\nu(\overline{y})\right| \\ &\quad+ \left|\int_{4Q} P_\Lambda(\overline{x},\overline{y})(\varphi(\overline{y}) - \varphi(\overline{x}))\,\mathrm{d}\nu(\overline{y})\right| + \left|\int_{4Q} P_\Lambda(\overline{x}_0,\overline{y})(\varphi(\overline{y}) - \varphi(\overline{x}))\,\mathrm{d}\nu(\overline{y})\right| \\ &\quad+ \left|\varphi(\overline{x})\int P_\Lambda(\overline{x},\overline{y})\,\mathrm{d}\nu(\overline{y})\right| + \left|\varphi(\overline{x})\int P_\Lambda(\overline{x}_0,\overline{y})\,\mathrm{d}\nu(\overline{y})\right| \\ &\quad=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.\end{aligned}$$

Regarding IV, observe that Lemma 5.4.13 implies

$$|\varphi(\overline{x})|\left|\int P_\Lambda(\overline{x},\overline{y})\,\mathrm{d}\nu(\overline{y})\right| = |\mathcal{P}_\Lambda\nu(\overline{x})| \lesssim 1.$$

The same estimate holds for V. To study II observe that

$$\left|\int_{4Q} P_\Lambda(\overline{x},\overline{y})(\varphi(\overline{y}) - \varphi(x))\,\mathrm{d}\nu(\overline{y})\right| \leq c_1\|\nabla\varphi\|_\infty \int_{4Q} \frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|^{n-1}} \lesssim 1,$$

where we have used that $\overline{x}\in Q\subset F\setminus H$ and integrated over decreasing annuli using the n-growth of μ for cubes centered at $\overline{x}$. The same holds for III because $\overline{x}_0\in\frac{1}{2}Q$ and $\varphi(\overline{x})=\varphi(\overline{x}_0)$, so III $\lesssim 1$. For I we apply property *2* of Lemma 5.2.1 and obtain

$$\mathrm{I}\lesssim 2\ell(Q)\|\varphi\|_\infty\int_{\mathbb{R}^{n+1}\setminus 4Q}\frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|^{n+1}}\lesssim 1,$$

where we have integrated over increasing annuli. Therefore, we finally have

$$|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})|\leq|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})-\mathscr{P}_\Lambda(\varphi\nu)(\overline{x}_0)|+|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x}_0)|\lesssim 1.$$

□

The previous lemma implies the main result of this subsection, which concerns a particular weak boundedness property for $\mathscr{P}_\Lambda$. We call it *property 8*, since it is the additional property needed in Lemma 5.4.3 in order to apply a Tb-theorem.

COROLLARY 5.4.16 (*Property 8*). *Let $Q\subset\mathbb{R}^{n+1}$ be a cube contained in $F\setminus H$ with M-thin boundary. Then, for some $c_8>0$ dimensional constant,*

$$|\langle\mathscr{P}_{\Lambda,\mu}(\chi_Q b),(\chi_Q b)\rangle|\leq c_8\mu(2Q).$$

Proof. Fix a cube $Q\subset F\setminus H$. We shall assume that Q is open, since the involved measures μ and ν are null on sets of zero Lebesgue measure. Observe that since the center of Q does not belong to H, in particular we have $\mu(2Q)\lesssim L\ell(2Q)^n$. Take a test function φ with $0\leq\varphi\leq 1$, $\|\nabla\varphi\|_\infty\leq\ell(Q)^{-1}$ and such that $\varphi\equiv 1$ on Q and $\varphi\equiv 0$ on $(2Q)^c$. Then,

$$|\langle\mathscr{P}_{\Lambda,\mu}(\chi_Q b),(\chi_Q b)\rangle|\leq|\langle\mathscr{P}_{\Lambda,\mu}(\varphi b),(\chi_Q b)\rangle|+|\langle\mathscr{P}_{\Lambda,\mu}((\varphi-\chi_Q)b),(\chi_Q b)\rangle|=:A+B.$$

To estimate A, observe that $\mathscr{P}_{\Lambda,\mu}(\varphi b)=\mathscr{P}_\Lambda(\varphi\nu)$ and apply Lemma 5.4.15 to directly deduce

$$A\leq c_1\int_Q|\mathscr{P}_\Lambda(\varphi\nu)(\overline{x})|\,\mathrm{d}\mu(\overline{x})\lesssim\mu(Q).$$

To estimate B, set $\varphi_1:=(\varphi-\chi_Q)b$ and $\varphi_2:=\chi_Q b$. Observe that $\operatorname{supp}(\varphi_1)\subset\overline{2Q}\setminus Q\subset\mathbb{R}^{n+1}\setminus Q$ and $\operatorname{supp}(\varphi_2)\subset\overline{Q}$. We apply [**T5, Lemma 5.23**] with $\Omega_1:=Q$ and $\Omega_2:=\mathbb{R}^{n+1}\setminus\overline{\Omega_1}$. The proof of the previous result is almost identical in our context, just change the function $d(\overline{x})^{-1/2}$ of the previous reference by $d(\overline{x})^{-n/2}$. Thus, we finally get

$$B\lesssim\|\varphi_1\|_{L^2(\mu)}\,\|\varphi_2\|_{L^2(\mu)}\leq c_1\,\mu\big(\overline{2Q}\setminus Q\big)^{1/2}\mu\big(\overline{Q}\big)^{1/2}\leq c_1\,\mu(2Q).$$

□

The above corollary suffices to prove an analogous Tb-theorem to that of [**T5, Ch.5**], since the weak boundedness property needs only to be applied to cubes contained in parallelepipeds $\Delta:=P\cap S$ with P,S cubes having M-thin boundary (and all their dyadic children too) that belong to $\mathcal{D}_1:=\mathcal{D}(\overline{w}_1)$ and $\mathcal{D}_2:=\mathcal{D}(\overline{w}_2)$ respectively for some $\overline{w}_1,\overline{w}_2\in\Omega$. Moreover, $P\subset F\setminus(H_{\mathcal{D}_1}\cup T_{\mathcal{D}_1})$ and $S\subset F\setminus(H_{\mathcal{D}_2}\cup T_{\mathcal{D}_2})$. Therefore, since in particular $H\subset H_{\mathcal{D}_1}\cap H_{\mathcal{D}_2}$, the cubes contained in Δ do not intersect H. Hence, Corollary 5.4.16 can be applied, yielding the following result:

Theorem 5.4.17. *Let μ be a positive finite Borel measure on $\mathbb{R}^{n+1}$ supported on a compact set $F \subset \mathbb{R}^{n+1}$. Assume there is a finite measure ν and, for each $\overline{w} \in \Omega$, two subsets $H_{\mathcal{D}(\overline{w})}, T_{\mathcal{D}(\overline{w})} \subset \mathbb{R}^{n+1}$ consisting of dyadic cubes in $\mathcal{D}(\overline{w})$ so that:*

1. *$\nu = b\mu$ for some b with $\|b\|_\infty \leq c_b$.*
2. *Every ball satisfying $\mu(B_r) > Lr^n$ is contained in $\bigcap_{\overline{w}\in\Omega} H_{\mathcal{D}(\overline{w})}$.*
3. *If $Q \in \mathcal{D}(\overline{w})$ is such that $Q \not\subset T_{\mathcal{D}(\overline{w})}$, then $\mu(Q) \leq c_T|\nu(Q)|$.*
4. *$\mu\big(H_{\mathcal{D}(\overline{w})} \cup T_{\mathcal{D}(\overline{w})}\big) \leq \delta_0\mu(F)$, for some $\delta_0 \in (0,1)$.*
5. *For each $\overline{w} \in \Omega$,*
$$\int_{F\setminus H_{\mathcal{D}(\overline{w})}} \mathscr{P}_*\nu(\overline{y})\,\mathrm{d}\mu(\overline{y}) \leq c_*\mu(F), \qquad \int_{F\setminus H_{\mathcal{D}(\overline{w})}} \mathscr{P}_*^*\nu(\overline{y})\,\mathrm{d}\mu(\overline{y}) \leq c_*\mu(F).$$
6. *Let $Q \subset \mathbb{R}^{n+1}$ be a cube contained in $F \setminus \bigcap_{\overline{w}\in\Omega} H_{\mathcal{D}(\overline{w})}$ with M-thin boundary, and $\Lambda : \mathbb{R}^{n+1} \to [0,\infty)$ a 1-Lipschitz function satisfying*
$$\Lambda(\overline{x}) \geq \operatorname{dist}\big(\overline{x}, \mathbb{R}^{n+1} \setminus \big(H_{\mathcal{D}(\overline{w})} \cup \mathcal{S}\big)\big), \qquad \forall \overline{x} \in \mathbb{R}^{n+1},$$
where $\mathcal{S}$ is the exceptional set defined in (5.4.13). Then,
$$|\langle \mathscr{P}_{\Lambda,\mu}(\chi_Q b), (\chi_Q b)\rangle| \leq c_W\mu(2Q),$$
where $\mathscr{P}_\Lambda$ is the operator associated to the suppressed kernel P_Λ defined in (5.2.1).

Then, there is $G \subset F \setminus \bigcap_{\overline{w}\in\mathbb{R}^{n+1}} \big(H_{\mathcal{D}(\overline{w})} \cup T_{\mathcal{D}(\overline{w})}\big)$ compact, and dimensional constants A_1, A_2 and A_3 so that

1. *$\mu(F) \leq A_1\mu(G)$,*
2. *$\mu|_G(B_r(\overline{x})) \leq A_2 r^n$, for every ball $B_r(\overline{x})$,*
3. *$\|\mathscr{P}_{\mu|_G}\|_{L^2(\mu|_G)\to L^2(\mu|_G)} \leq A_3$.*

Constants $c_b, L, c_T, \delta_0, c_*, c_W$ and M are dimensional.

Remark 5.4.3. Let us give some details on how to prove Theorem 5.4.17, although the arguments to follow are just those given in the proofs of [**T5, Theorem 5.1**] or [**Vo, Theorem 7.1**], using, essentially, the weak boundedness property instead of the anti-symmetry of the Cauchy and Riesz kernels. Let us recall that we wrote, for $N \geq 4$ integer,
$$\Omega := \big[-2^{N-1}, 2^{N-1}\big)^{n+1}.$$
Fix $\overline{w} \in \Omega$. Recall $F \subset Q^0(\overline{w}) := \overline{w} + [-2^N, 2^N)^{n+1}$. A dyadic cube $Q \in Q^0(\overline{w})$ with $\mu(Q) \neq 0$ is called *terminal* if $Q \subset H_{\mathcal{D}(\overline{w})} \cup T_{\mathcal{D}(\overline{w})}$ and we write $Q \in \mathcal{D}^{\text{term}}(\overline{w})$. Otherwise is called *transit* and we write $Q \in \mathcal{D}^{\text{tr}}(\overline{w})$. With this, one considers a

martingale decomposition of a function $f \in L^1_{\text{loc}}(\mu)$ in terms of $Q^0(\overline{w})$. For any cube $Q \subset \mathbb{R}^{n+1}$ with $\mu(Q) \neq 0$ one sets

$$\langle f \rangle_Q := \frac{1}{\mu(Q)} \int_Q f \, \mathrm{d}\mu$$

and defines the operator

$$\Xi f := \frac{\langle f \rangle_{Q^0(\overline{w})}}{\langle b \rangle_{Q^0(\overline{w})}} b.$$

It is clear that $\Xi f \in L^2(\mu)$ if $f \in L^2(\mu)$ and $\Xi^2 = \Xi$. Moreover, the definition of Ξ does not depend on the choice of $\overline{w} \in \Omega$.

If $Q \in \mathcal{D}(\overline{w})$, the set of at most 2^{n+1} dyadic children of Q whose μ-measure is not null is denoted by $\mathcal{CH}(Q)$. For any $Q \in \mathcal{D}^{\text{tr}}(\overline{w})$ and $f \in L^1_{\text{loc}}(\mu)$ we define the function $\Delta_Q f$ as

$$\Delta_Q f := \begin{cases} 0 & \text{in } \mathbb{R}^{n+1} \setminus \bigcup_{R \in \mathcal{CH}(Q)} R, \\ \Big(\frac{\langle f \rangle_R}{\langle b \rangle_R} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \Big) b & \text{in } R \text{ if } R \in \mathcal{CH}(Q) \cap \mathcal{D}^{\text{tr}}(\overline{w}), \\ f - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} b & \text{in } R \text{ if } R \in \mathcal{CH}(Q) \cap \mathcal{D}^{\text{term}}(\overline{w}). \end{cases}$$

The fundamental properties of the operators Ξ and Δ_Q are proved in [**T5, Lemmas 5.10, 5.11**] and, essentially, allow to decompose $f \in L^2(\mu)$ as

$$f = \Xi f + \sum_{Q \in \mathcal{D}^{\text{tr}}} \Delta_Q f,$$

where the sum is unconditionally convergent in $L^2(\mu)$ and, in addition,

$$\|f\|^2_{L^2(\mu)} \approx \|\Xi f\|^2_{L^2(\mu)} + \sum_{Q \in \mathcal{D}^{\text{tr}}} \|\Delta_Q f\|^2_{L^2(\mu)}.$$

At this point, one of the fundamental steps of the proof of the Tb-theorem consists in using the above decomposition to estimate the $L^2(\mu)$ norm of the suppressed operator $\mathcal{P}_\Lambda$ when applied to the so called *good functions*. To define them, we need to introduce first *good* and *bad* cubes. Let $\overline{w}_1, \overline{w}_2 \in \Omega$ and consider $\mathcal{D}_1 := \mathcal{D}(\overline{w}_1)$, $\mathcal{D}_2 := \mathcal{D}(\overline{w}_2)$ two dyadic lattices. We consider as in [**NTrVo3, Definition 6.2**] the parameter

$$\alpha := \frac{1}{2(n+1)},$$

and we will say that $Q \in \mathcal{D}_1^{\text{tr}}$ is *bad with respect to* $\mathcal{D}_2$ if either

1. there is $R \in \mathcal{D}_2^{\text{tr}}$ such that $\text{dist}(Q, \partial R) \leq \ell(Q)^\alpha \ell(R)^{1-\alpha}$ and $\ell(R) \geq 2^m \ell(Q)$, for some positive integer m to be fixed later, or

2. there is $R \in \mathcal{D}_2^{\text{tr}}$ such that $2^{-m}\ell(Q) \leq \ell(R) \leq 2^m \ell(Q)$, $\text{dist}(Q, R) \leq 2^m \ell(Q)$ and, at least, one of the children of R does not have M-thin boundary.

If Q is not bad, then we say that it is *good with respect to* $\mathcal{D}_2$. An important property regarding bad cubes is that they do not appear very often in dyadic lattices. More precisely, given $\varepsilon_b > 0$ arbitrarily small, if m and M are chosen big enough, then for each fixed $Q \in \mathcal{D}_1$, the probability that it is bad with respect to $\mathcal{D}_2$ is not larger than ε_b. That is,

$$\mathbb{P}\big(\{\overline{w}_2 \in \Omega \,:\, Q \in \mathcal{D}_1 \text{ is bad with respect to } \mathcal{D}(\overline{w}_2)\}\big) \leq \varepsilon_b.$$

In light of the above notions, we say that a function $f \in L^2(\mu)$ is $\mathcal{D}_1$*-good with respect to* $\mathcal{D}_2$ if $\Delta_Q f = 0$ for all bad cubes $Q \in \mathcal{D}_1^{\text{tr}}$ (with respect to $\mathcal{D}_2$). Now one proceeds as follows: define the 1-Lipschitz function

$$\Lambda_{\mathcal{D}(\overline{w})}(\overline{x}) := \operatorname{dist}\big(\overline{x}, \mathbb{R}^{n+1} \setminus W_{\mathcal{D}(\overline{w})}\big), \qquad \overline{w} \in \Omega.$$

It satisfies $\Lambda_{\mathcal{D}(\overline{w})}(\overline{x}) \geq \operatorname{dist}\big(\overline{x}, \mathbb{R}^{n+1} \setminus \big(H_{\mathcal{D}(\overline{w})} \cup \mathcal{S}\big)\big)$, so that Lemma 5.4.13 can be applied to $\Lambda_{\mathcal{D}(\overline{w})}$, and then we have:

LEMMA 5.4.18. ([**T5, Lemma 5.13**]) *Let* $\mathcal{D}_1 = \mathcal{D}(\overline{w}_1)$ *and* $\mathcal{D}_2 = \mathcal{D}(\overline{w}_2)$ *with* $\overline{w}_1, \overline{w}_2 \in \Omega$. *Given* $\varepsilon > 0$, *let* $\Lambda : \mathbb{R}^{n+1} \to [\varepsilon, \infty)$ *be a 1-Lipschitz function such that*

$$\Lambda(\overline{x}) \geq \max\big\{\Lambda_{\mathcal{D}_1}(\overline{x}), \Lambda_{\mathcal{D}_2}(\overline{x})\big\}, \qquad \forall \overline{x} \in \mathbb{R}^{n+1}.$$

Then, if $f \in L^2(\mu)$ *is* $\mathcal{D}_1$*-good with respect to* $\mathcal{D}_2$ *and* $g \in L^2(\mu)$ *is* $\mathcal{D}_2$*-good with respect to* $\mathcal{D}_1$,

$$|\langle \mathcal{P}_\Lambda(f\mu), g\rangle| \lesssim \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)},$$

where the implicit constant depends on $c_b, L, c_T, \delta_0, c_*, c_W$ *and* ε_b, *but not on* ε.

The proof of this result uses the above martingale decomposition of the functions f, g, so that one is left to estimate:

$$\langle \mathcal{P}_\Lambda(f\mu), g\rangle = \langle \mathcal{P}_\Lambda(\Xi f\mu), \Xi g\rangle + \langle \mathcal{P}_\Lambda(\Xi f\mu), g\rangle + \langle \mathcal{P}_\Lambda(f\mu), \Xi g\rangle + \sum_{Q\in\mathcal{D}_1^{\text{tr}}, R\in\mathcal{D}_2^{\text{tr}}} \langle \mathcal{P}_\Lambda((\Delta_Q f)\mu), \Delta_R g\rangle.$$

If our kernel were anti-symmetric, the first term of the right-hand side would be null. Although this is not our case, it can still be estimated as follows (notice that the weak boundedness property will not be used in the arguments below):

$$\langle \mathcal{P}_\Lambda(\Xi f\mu), \Xi g\rangle \leq \|\mathcal{P}_\Lambda(\Xi f\mu)\|_{L^2(\mu)} \|\Xi g\|_{L^2(\mu)}.$$

Observe that $\operatorname{supp}(\mu) \subset F \subset Q^0(\overline{w}_1) \cap Q^0(\overline{w}_2)$ and by definition

$$\Xi g := \frac{\langle g\rangle_{Q^0(\overline{w}_2)}}{\langle b\rangle_{Q^0(\overline{w}_2)}} b.$$

This implies, since $Q^0(\overline{w}_2)$ is always a transit cube (this is easy to see just arguing by contradiction and using assumption *4* in Theorem 5.4.17),

$$\begin{aligned}\|\Xi g\|_{L^2(\mu)} &\leq \frac{\mu(Q^0(\overline{w}_2))^{1/2}}{|\nu(Q^0(\overline{w}_2))|} \bigg(\int_F |b|^2 \,\mathrm{d}\mu\bigg)^{1/2} \|g\|_{L^2(\mu)} \\ &\leq c_T^{1/2} \bigg(\frac{1}{|\nu(Q^0(\overline{w}_2))|} \int_F |b|^2 \,\mathrm{d}\mu\bigg)^{1/2} \|g\|_{L^2(\mu)} \leq c_1 c_T \|g\|_{L^2(\mu)}.\end{aligned}$$

Moreover, by Lemma 5.4.13 and using that $Q^0(\overline{w}_1)$ is a transit cube, we deduce

$$\|\mathscr{P}_\Lambda(\Xi f\mu)\|_{L^2(\mu)} = \frac{|\langle f\rangle_{Q^0(\overline{w}_1)}|}{|\langle b\rangle_{Q^0(\overline{w}_1)}|}\|\mathscr{P}_\Lambda(b\mu)\|_{L^2(\mu)} \leq c_\Lambda \frac{|\langle f\rangle_{Q^0(\overline{w}_1)}|}{|\langle b\rangle_{Q^0(\overline{w}_1)}|}\mu(Q^0(\overline{w}_1))^{1/2}$$
$$\leq c_\Lambda c_T |\langle f\rangle_{Q^0(\overline{w}_1)}|\mu(Q^0(\overline{w}_1))^{1/2} \leq c_\Lambda c_T \|f\|_{L^2(\mu)}.$$

Hence

$$|\langle \mathcal{P}_\Lambda(\Xi f\mu), \Xi g\rangle| \lesssim \|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}.$$

The terms $\langle \mathcal{P}_\Lambda(\Xi f\mu), g\rangle$ and $\langle \mathcal{P}_\Lambda(f\mu), \Xi g\rangle$ can be estimated similarly (see [**T5, p. 155**]), and it is important to notice that to do so one uses Lemma 5.4.13 for the adjoint suppressed operator $\mathscr{P}_\Lambda^*$. Hence, the only term left is

$$\sum_{Q\in\mathcal{D}_1^{\text{tr}}, R\in\mathcal{D}_2^{\text{tr}}} \langle \mathcal{P}_\Lambda((\Delta_Q f)\mu), \Delta_R g\rangle.$$

The above sum is studied in [**T5, §5.6, §5.7, §5.8 & §5.9**], and the arguments can be followed analogously up to [**T5, §5.9**]. Obviously, there are changes that need to be done regarding the dimensionality. Such modifications were already done in a general multidimensional setting in the study carried out in [**Vo**] for Riesz kernels. In any case, it is in [**T5, §5.9**] where the weak boundedness property needs to be invoked in order to deal with expressions of the form

$$\langle \mathscr{P}_{\Lambda,\mu}(\chi_\Delta b), (\chi_\Delta b)\rangle,$$

where Δ is a certain parallelepiped introduced at the beginning of §5.4.4. Such expressions were already tackled in [**NTrVo3**] or [**HyMar**], and they can be deduced from the estimate

$$|\langle \mathscr{P}_{\Lambda,\mu}(\chi_Q b), (\chi_Q b)\rangle| \lesssim \mu(2Q),$$

where in our setting we may assume Q to be contained in $F \setminus H$ and with M-thin boundary. This precise bound is covered by assumption *6* in the statement of Theorem 5.4.17. From this point on, the rest of the proof can be followed as in the remaining sections of [**T5, Ch.5**] to obtain the desired result. Let us also remark that the proof can be followed almost identically (apart from some dimensional changes) taking into account that we have constructed unique measures μ and ν so that assumption *5* holds for the operators $\mathscr{P}$ and $\mathscr{P}^*$ *simultaneously*. This enables to obtain relation (5.4.14), which is essential to carry out the final probabilistic arguments in the proof of the Tb-theorem found in [**T5, Ch.5**] or [**Vo**].

In any case, in light of Theorem 5.4.17 and also bearing in mind the argument of (5.4.4) and that assumptions $\mathbf{A_1}$ and $\mathbf{A_2}$ are superfluous, we have proved the following:

THEOREM 5.4.19. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set satisfying $\mathbf{A_3}$. Then,*

$$\widetilde{\gamma}(E) \approx \widetilde{\gamma}_+(E).$$

5.5 General comparability results and the semi-additivity of $\widetilde{\gamma}$ in $\mathbb{R}^2$

The main goal of this section is to obtain similar results to Theorem 5.4.19 but removing assumption $\mathbf{A_3}$. We will be able to do this for any compact set in $\mathbb{R}^2$ and for a certain family of sets in $\mathbb{R}^{n+1}$, which in particular contains the usual corner-like Cantor sets presented in §5.2

5.5.1 The capacity of some parallelepipeds in $\mathbb{R}^{n+1}$

First, let us present some preliminary results that extend that of [**MatP, Proposition 6.1**]. Let us remark that throughout the forthcoming discussion, any parallelepiped will be closed and will have sides parallel to the coordinate axes.

Lemma 5.5.1. *Let $a\in\mathbb{R}$ and $R\subset\mathbb{R}^n\times\{a\}$ be a parallelepiped contained in the affine hyperplane $\{t=a\}$. Then*

$$\widetilde{\gamma}_+(R)\gtrsim\mathcal{H}^n(R).$$

Proof. Let $\mu:=\mathcal{H}^n|_R$ and pick any $\overline{x}=(x,t)$ with $|t-a|>0$. Observe that

$$P_{\text{sy}}*\mu(x,t)=\int_R\frac{|t-a|}{\big[(t-a)^2+|x-y|^2\big]^{\frac{n+1}{2}}}\mathrm{d}\mathcal{H}^n(y)=\int_{R-x}\frac{|t-a|}{\big[(t-a)^2+|u|^2\big]^{\frac{n+1}{2}}}\mathrm{d}\mathcal{H}^n(u),$$

where $u:=y-x$ and $R-x$ denotes a translation of R with respect to the vector $(-x,0)\in\mathbb{R}^{n+1}$. Let us pick D_{r_0} an n-dimensional ball embedded in $\mathbb{R}^n\times\{a\}$, centered at $(0,a)\in\mathbb{R}^n\times\{a\}$ and with radius $r_0=r_0(\overline{x})$ big enough so that $R-x\subset D_{r_0}$. Then, there exists a dimensional constant $C>0$ so that

$$\begin{aligned}P_{\text{sy}}*\mu(x,t)&\leq\int_{D_{r_0}}\frac{|t-a|}{\big[(t-a)^2+|u|^2\big]^{\frac{n+1}{2}}}\mathrm{d}\mathcal{H}^n(u)=C\int_0^{r_0}\frac{|t-a|}{\big[(t-a)^2+r^2\big]^{\frac{n+1}{2}}}r^{n-1}\mathrm{d}r\\&=C\int_0^1\frac{|t-a|r_0^{-1}}{\big[(t-a)^2r_0^{-2}+\rho^2\big]^{\frac{n+1}{2}}}\rho^{n-1}\mathrm{d}\rho.\end{aligned}$$

where in the last step we have introduced the change of variables $r_0\rho=r$. Naming $\tau:=|t-a|r_0^{-1}$ the previous integral can be finally written as

$$\int_0^1\frac{\tau}{\big[\tau^2+\rho^2\big]^{\frac{n+1}{2}}}\rho^{n-1}\mathrm{d}\rho,$$

which admits an explicit representation in terms of Gauss's (or Kummer's) hypergeometric function (use [**Bu, §1.4, eq.(13)**], for example), obtaining the estimate

$$\begin{aligned} P_{\text{sy}} * \mu(x,t) &\leq \frac{C}{n}\left(\frac{\rho}{\tau}\right)^n {}_2F_1\left(\frac{n}{2},\frac{n+1}{2};\frac{n+2}{2};-\left(\frac{\rho}{\tau}\right)^2\right)\Bigg|_{\rho=0}^{\rho=1} \\ &= \frac{C}{n}\cdot\frac{1}{\tau^n}\,{}_2F_1\left(\frac{n}{2},\frac{n+1}{2};\frac{n+2}{2};-\frac{1}{\tau^2}\right) \\ &= \frac{C}{n}\cdot\frac{1}{\tau(1+\tau^2)^{(n-1)/2}}\,{}_2F_1\left(1,\frac{1}{2};\frac{n+2}{2};-\frac{1}{\tau^2}\right). \end{aligned}$$

In the last step we have applied [**AS, Eq.15.3.3**]. This last expression, thought as a function of $\tau > 0$, is bounded uniformly on $(0,+\infty)$ with respect to a dimensional constant (and in fact attains its maximum when $\tau \to 0$). In other words, we deduce that $P_{\text{sy}} * \mu(x,t) \lesssim 1$ whenever $|t-a| > 0$. On the other hand, it is clear that if $t = a$, then $P_{\text{sy}} * \mu(x,t) = 0$, meaning that in general

$$P_{\text{sy}} * \mu(x,t) \lesssim 1, \qquad \forall (x,t) \in \mathbb{R}^{n+1}.$$

Therefore we conclude

$$\gamma_{\text{sy},+}(R) \gtrsim \mu(R) = \mathcal{H}^n(R),$$

and using Theorem 5.2.5 we obtain the desired result. □

The above lemma characterizes the $\widetilde{\gamma}_+$ capacity of parallelepipeds contained in affine hyperplanes of the form $\{t = a\}$. We will refer to such hyperplanes as *horizontal* hyperplanes. Observe that the above bound combined with [**MatP, Lemma 4.1**] implies

$$\mathcal{H}^n(R) \lesssim \widetilde{\gamma}_+(E) \leq \gamma_+(R) \leq \gamma(R) \lesssim \mathcal{H}^n_\infty(R) \leq \mathcal{H}^n(R),$$

for any parallelepiped contained in a horizontal hyperplane. Hence, for such objects,

$$\mathcal{H}^n(R) \approx \gamma_+(R) \approx \gamma(R).$$

Our next goal will be to study the capacity associated to parallelepipeds contained in *vertical* hyperplanes, that is, sets of the form $\{x_i = a\}$ for some $a \in \mathbb{R}$ and some $i = 1,\ldots,n$. Previous to that, we need to make an auxiliary construction: let us assume that we have a compact set $E \subset \mathbb{R}^{n+1}$ contained, for example, in $\{x_1 = 0\}$. Consider T a distribution in $\mathbb{R}^{n+1}$ with $\text{supp}(T) \subseteq E$ as well as the maps

$$\begin{aligned} \pi:\quad \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^n, \\ (x_1,x) &\longmapsto x \end{aligned} \qquad\qquad \begin{aligned} \iota:\mathbb{R}^n &\longrightarrow \quad \mathbb{R}^{n+1}, \\ x &\longmapsto (0,x) \end{aligned}$$

the canonical projection onto the last n coordinates and the canonical inclusion into the hyperplane $\{x_1 = 0\}$. Let us take $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ a smooth function and define β_M, for some $M > 0$ (a positive parameter that we may take as large as we need), a smooth bump function in $\mathbb{R}^{n+1}$ that equals 1 in an open M-neighborhood of E, i.e. $\beta_M \equiv 1$ in $\mathcal{U}_M(E)$. Also, we require that $\beta_M \equiv 0$ in $\mathbb{R}^{n+1} \setminus \mathcal{U}_{2M}(E)$ and $0 \leq \beta_M \leq 1$. We set

$$\widetilde{\varphi} := (\varphi \circ \pi) \cdot \beta_M,$$

that is a smooth extension of $\varphi|_E$ to $\mathbb{R}^{n+1}$, with compact support contained in $\mathcal{U}_{2M}(E)$. With this, we define the following distribution in $\mathbb{R}^n$ associated to T:

$$\langle \widetilde{T}, \varphi \rangle := \langle T, \widetilde{\varphi} \rangle, \qquad \varphi \in \mathcal{C}^\infty(\mathbb{R}^n),$$

which is a definition independent of the choice of β_M, since $\text{supp}(T) \subseteq E$. Observe that $\text{supp}(\widetilde{T}) \subseteq \pi(E)$ (a compact set of $\mathbb{R}^n$) and also that for any $\psi \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$,

$$\langle T, \psi \rangle = \langle \widetilde{T}, \psi \circ \iota \rangle.$$

Notice that the previous identity is accurate because, even though ψ and $\psi \circ \iota \circ \pi$ may not coincide in $\mathbb{R}^{n+1} \setminus \{x_1 = 0\}$, we have $\text{supp}(T) \subseteq E \subset \{x_1 = 0\}$, that ensures the validity of the above equality. Let us take $(\Phi_\varepsilon)_\varepsilon$ an approximation of the identity in $\mathbb{R}^n$ and set

$$\widetilde{T}_\varepsilon := \widetilde{T} * \Phi_\varepsilon,$$

for $0 < \varepsilon \ll M$, so that the following inclusions hold

$$\text{supp}(\widetilde{T}_\varepsilon) \subseteq \mathcal{V}_{2\varepsilon}(\pi(E)) \subset \pi(\mathcal{U}_M(E)),$$

where the notation $\mathcal{V}$ is used to emphasize that we are considering an open neighborhood in $\mathbb{R}^n$ and not $\mathbb{R}^{n+1}$. So $(\widetilde{T}_\varepsilon)_\varepsilon$ defines a collection of signed measures in $\mathbb{R}^n$ that approximates $\widetilde{T}$ in a distributional sense:

$$\lim_{\varepsilon \to 0} \langle \widetilde{T}_\varepsilon, \varphi \rangle = \langle \widetilde{T}, \varphi \rangle = \langle T, \widetilde{\varphi} \rangle, \qquad \varphi \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Let us stress (although the reader may have already noticed), that the spaces of functions we have been considering are of the form $\mathcal{C}^\infty$ and not $\mathcal{C}^\infty_c$. This can be done since our distributions are compactly supported. Let us proceed by naming $\Sigma_\varepsilon := \overline{\mathcal{V}_{2\varepsilon}(\pi(E))}$, and observe that for any $\psi \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$

$$\begin{aligned}
\langle T, \psi \rangle = \langle \widetilde{T}, \psi \circ \iota \rangle &= \lim_{\varepsilon \to 0} C \int_{\Sigma_E} \widetilde{T}_\varepsilon(x)\, \psi \circ \iota(x)\, \mathrm{d}\mathcal{H}^n(x) \\
&= \lim_{\varepsilon \to 0} C \int_{\Sigma_E} \widetilde{T}_\varepsilon \circ \pi(0,x)\, \psi(0,x)\, \mathrm{d}\mathcal{H}^n(x) \\
&= \lim_{\varepsilon \to 0} C \int_{\mathbb{R}^{n+1}} \widetilde{T}_\varepsilon \circ \pi(\overline{x})\, \psi(\overline{x})\, \mathrm{d}\mathcal{H}^n|_{\iota(\Sigma_E)}(\overline{x}) \\
&= \lim_{\varepsilon \to 0} \langle C(\widetilde{T}_\varepsilon \circ \pi) \mathcal{H}^n|_{\iota(\Sigma_\varepsilon)}, \psi \rangle,
\end{aligned}$$

where $\widetilde{T}_\varepsilon \circ \pi$ is smooth and supported on $\mathbb{R} \times \Sigma_\varepsilon \subset \mathbb{R} \times \mathbb{R}^n$, and $C > 0$ is some dimensional constant. Therefore, the previous construction implies the following remark:

Remark 5.5.1. Given a distribution T in $\mathbb{R}^{n+1}$ supported on $E \subset \{x_1 = 0\}$, there is a family of signed measures $(\nu_\varepsilon)_\varepsilon$ with $\text{supp}(\nu_\varepsilon) \subset \iota(\Sigma_\varepsilon) =: \iota\big(\overline{\mathcal{V}_{2\varepsilon}(\pi(E))}\big)$ of the form

$$\nu_\varepsilon = \psi_\varepsilon\, \mathcal{H}^n|_{\iota(\Sigma_\varepsilon)},$$

where ψ_ε is a smooth function that satisfies $\psi_\varepsilon \circ \iota = \widetilde{T}_\varepsilon$ (a function of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ supported on Σ_ε), that is such that

$$\langle T, \psi\rangle = \lim_{\varepsilon\to 0}\langle \nu_\varepsilon, \psi\rangle, \qquad \psi \in \mathcal{C}^\infty(\mathbb{R}^n).$$

The reader may think of the above result as a construction that exploits the fact that the support of T is contained in a hyperplane where x_1 is constant, so that we may approximate T via an *approximation of the identity with respect to the remaining variables* $x_2, \ldots, x_{n+1}$ (here $x_{n+1} = t$). Moreover, notice that if one starts assuming a condition of the form $\|P * T\|_\infty \leq 1$, by choosing ε small enough we can assume, for example, $\|P * \nu_\varepsilon\|_\infty \leq 2$. Indeed, just fix any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ and observe that

$$\langle P * T, \psi\rangle = \langle T, P^* * \psi\rangle = \lim_{\varepsilon\to 0}\langle \nu_\varepsilon, P^* * \psi\rangle = \lim_{\varepsilon\to 0}\langle P * \nu_\varepsilon, \psi\rangle,$$

meaning that $\lim_{\varepsilon\to 0}|\langle P * \nu_\varepsilon, \psi\rangle| \leq \|\psi\|_{L^1(\mathbb{R}^{n+1})}$, which implies the desired estimate. In the previous argument we have used that $P^* * \psi$ is a smooth function, which can be argued thinking of P^* as an $L^1_{\text{loc}}(\mathbb{R}^{n+1})$ function and thus as a regular distribution. This allows us to prove the following lemma:

LEMMA 5.5.2. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set contained in an affine hyperplane of the form $\{x_i = a\}$, for some $a \in \mathbb{R}$ and $i = 1, \ldots, n$. Then*

$$\gamma(E) = 0.$$

Proof. We shall assume that E is contained in $\{x_1 = 0\}$ for the sake of simplicity. Consider D_{r_0} an n-dimensional ball in $\{x_1 = 0\}$ centered at the origin and with radius r_0. Let us assume that $\gamma(D_{r_0}) > 0$ and reach a contradiction. If $\gamma(D_{r_0}) > 0$, there exists a distribution T admissible for $\gamma_{\Theta^{1/2}}(D_{r_0})$ with $|\langle T, 1\rangle| > 0$. By Remark 5.5.1 we obtain a family of signed measures that approximate T of the form

$$\nu_\varepsilon = \psi_\varepsilon \cdot \mathcal{H}^n|_{D_{r_0+2\varepsilon}},$$

for some ψ_ε smooth, such that $\psi_\varepsilon \circ \iota \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\text{supp}(\psi_\varepsilon \circ \iota) \subset \pi(D_{r_0+2\varepsilon})$. We may also choose ε small enough so that $\|P * \nu_\varepsilon\|_\infty \leq 2$. Since $|\langle T, 1\rangle| > 0$, we are able to pick some $\overline{x}_0 = (x_0, t_0) \in D_{r_0+2\varepsilon}$ such that $|\psi_\varepsilon(\overline{x}_0)| > 0$, as well as an n-dimensional ball centered at $\overline{x}_0$ with radius η satisfying $D_\eta(\overline{x}_0) \subset D_{r_0+2\varepsilon}$ and $|\psi_\varepsilon| \geq A > 0$ there, for some constant $A > 0$ (this can be done by the continuity of $\psi_\varepsilon \circ \iota$).

Proceed by fixing $Q = Q(\overline{x}_0)$ a cube in $\mathbb{R}^{n+1}$ centered at $\overline{x}_0$ with $4\ell(Q) \leq \eta$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \varphi \leq 1$, $\varphi|_Q \equiv 1, \varphi|_{\mathbb{R}^{n+1}\setminus 2Q} \equiv 0$ and $\|\nabla\varphi\|_\infty \leq \ell(Q)^{-1}$. Also take $B_\xi(\overline{x}_0)$ a ball in $\mathbb{R}^{n+1}$ centered at $\overline{x}_0$ with radius ξ so that $4\xi \leq \ell(Q)$, and name $D_\xi(\overline{x}_0) := B_\xi(\overline{x}_0) \cap \{x_1 = 0\}$, that is such that

$$D_\xi(\overline{x}_0) \subset Q(\overline{x}_0) \cap \{x_1 = 0\} \subset D_\eta(\overline{x}_0).$$

We finally define the positive measure

$$\mu' := A \cdot \mathcal{H}^n|_{D_\xi(\overline{x}_0)},$$

and observe that by the choice of φ and the non-negativity of P, we have

$$\begin{aligned}\|P*\mu'\|_\infty = \left\|P*A\,\mathcal{H}^n|_{D_\xi(\overline{x}_0)}\right\|_\infty = \left\|P*\varphi A\,\mathcal{H}^n|_{D_\xi(\overline{x}_0)}\right\|_\infty \\ \leq \left\|P*\varphi\,\psi_\varepsilon\mathcal{H}^n|_{D_{r_0+2\varepsilon}}\right\|_\infty = \|P*\varphi\nu_\varepsilon\|_\infty \lesssim 1,\end{aligned}$$

where the last inequality is due to the localization estimate [**MatP, Theorem 3.1**]. Therefore, we have constructed $D_\xi(\overline{x}_0)$ an n-dimensional ball that admits a positive measure μ supported on it, proportional to $\mathcal{H}^n|_{D_\xi(\overline{x}_0)}$ and with $\|P*\mu\|_\infty\leq 1$. Let us prove that this last condition is not possible. Indeed, for each $\overline{x}=(0,x_2,\dots,x_n,t)\in D_\xi(\overline{x}_0)$ pick an n-dimensional ball $D_\rho(\overline{x})$ centered at $\overline{x}$ and with radius ρ small enough so that $D_\rho(\overline{x})\subset D_\xi(\overline{x}_0)$. We write $\Delta_{\downarrow,\rho}(\overline{x})$ its *lower temporal half*, which is obtained from the intersection $D_\rho(\overline{x})\cap\{t-s>0\}$; as well as $\Delta_{\uparrow,\rho}(\overline{x})$ the *upper temporal half*, obtained from $D_\rho(\overline{x})\cap\{t-s<0\}$. Now, integration in ($n$-dimensional) spherical coordinates yields

$$\begin{aligned}P*\mu(\overline{x}) &\geq A\int_{\Delta_{\downarrow,\rho}(\overline{x})}\frac{t-s}{\big[(t-s)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2\big]^{\frac{n+1}{2}}}\mathrm{d}\mathcal{H}^n(y_2,\dots,y_n,s)\\ &\simeq A\int_{\Delta_{\uparrow,\rho}(0)}\frac{\tau}{\big[\tau^2+u_2^2+\cdots+u_n^2\big]^{\frac{n+1}{2}}}\mathrm{d}u_2\cdots\mathrm{d}u_n\mathrm{d}\tau = A\cdot L\int_0^\rho\frac{\mathrm{d}r}{r}=+\infty,\end{aligned}$$

where L is a positive dimensional constant obtained from integration in the angular domain (notice that $\tau>0$ in $\Delta_{\uparrow,\rho}(0)$). So by the arbitrariness of $\overline{x}$ we get that $\|P*\mu\|_{L^\infty(\mu)}=+\infty$, since $\mu(D_\eta(\overline{x}_0))\simeq A\,\mathcal{H}^n(D_\eta(\overline{x}_0))>0$. But this is contradictory with $\|P*\mu\|_\infty\leq 1$, because such condition implies, in particular, $\|P*\mu\|_{L^\infty(\mu)}\lesssim 1$ (use a Cotlar type inequality analogous to that of [**MattiPar, Lemma 5.4**]). Therefore we conclude that $\gamma(D_{r_0})=0$ for any radius r_0, implying the desired result, by the monotonicity of γ. $\square$

Corollary 5.5.3. *Let $R\subset\mathbb{R}^{n+1}$ be a parallelepiped and $\ell_1,\dots,\ell_n,\ell_t$ its side lengths. Let $R_\uparrow$ denote the upper face of R (contained in a horizontal affine hyperplane),*

1) if $\gamma(R_\uparrow)=0$, then $\gamma(R)=0$,

2) if $\ell_t\leq\min\{\ell_1,\dots,\ell_n\}$, then $\gamma(R)\approx\gamma_+(R_\uparrow)$.

Proof. To prove *1*, begin by noticing that $\gamma(R_\uparrow)=0$ implies $\mathcal{H}^n(R_\uparrow)=0$, by the comments made after Lemma 5.5.1. Embedding $R_\uparrow$ into $\mathbb{R}^n$ via $\pi_x:\mathbb{R}^{n+1}\to\mathbb{R}^n$, $\pi_x(\overline{x})=x$, the canonical projection onto the spatial coordinates; we deduce that $Q=\pi_x(R)$ is such that $\mathcal{L}^n(Q)=0$. But Q is itself a parallelepiped of $\mathbb{R}^n$, i.e.

$$Q=[a_1,b_1]\times\cdots\times[a_n,b_n],\qquad\text{for some }\;a_i,b_i\in\mathbb{R},\,a_i\leq b_i.$$

So there must exist some $[a_j,b_j]$ such that $a_j=b_j$. Hence,

$$R=[a_1,b_1]\times\cdots\times[a_{j-1},b_{j-1}]\times\{a_j\}\times[a_{j+1},b_{j+1}]\times\cdots\times[a_{n+1},b_{n+1}],$$

and thus $R\subset\{x_j=a_j\}$, a vertical hyperplane. Applying Lemma 5.5.2 the result follows.

Moving on to *2*, let us assume $\ell_t > 0$ (if not the result is trivial) and notice that by [**MatP, Lemma 4.1**] and Lemma 5.5.1 we have

$$\gamma(R) \lesssim \mathcal{H}^n_\infty(R) \lesssim \left[\frac{\ell_1}{\ell_t}\cdots\frac{\ell_n}{\ell_t}\cdot 1\right]\cdot \ell_t^n \simeq \mathcal{H}^n(R_\uparrow) \lesssim \gamma_+(R_\uparrow),$$

and we are done. □

5.5.2 General comparability for parallelepipeds in $\mathbb{R}^{n+1}$

In light of the above results, we shall obtain a first estimate analogous to Theorem 5.4.19 for parallelepipeds, where assumption $\mathbf{A_3}$ will not be needed.

Lemma 5.5.4. *Let $R\subset\mathbb{R}^{n+1}$ be a closed parallelepiped with sides parallel to the coordinate axes. Then,*

$$\widetilde{\gamma}(R)\approx\widetilde{\gamma}_+(R).$$

Proof. Let us denote by $\ell_1,\ldots,\ell_n,\ell_t$ the side lengths of R. Notice that by Lemmas 5.5.1 and 5.5.2 we already know the above result if any ℓ_i is null. So let us assume that

$$\ell_i > 0, \qquad \text{for each } \; i=1,\ldots,n,t. \tag{5.5.1}$$

Moreover, we will also assume without loss of generality that R is contained in $B_1(0)$, the unit ball of $\mathbb{R}^{n+1}$. The argument that follows is inspired by that presented in [**Vo, Ch.6**]. Let us begin by applying Theorem 5.3.1 to the compact set $E := R$ to obtain a first family of cubes $\{\mathcal{Q}_1,\ldots,\mathcal{Q}_{N_1}\}$ satisfying properties $\mathbf{P_1}$ to $\mathbf{P_5}$. We observe that:

- Regarding property $\mathbf{P_4}$, if $\ell_{\max} := \max\{\ell_1,\ldots,\ell_n,\ell_t\}$, we get for each $i_1 = 1,\ldots,N_1$,

$$\operatorname{diam}(\mathcal{Q}_{i_1}) \leq \frac{1}{10}\operatorname{diam}(R) \leq \frac{\sqrt{n+1}}{10}\ell_{\max} \quad \text{and then} \quad \ell(\mathcal{Q}_{i_1}) \leq \frac{1}{10}\ell_{\max}.$$

- Regarding $\mathbf{P_1}$, we get $\frac{5}{8}\mathcal{Q}_{i_1}\cap R\neq\varnothing$ for each $i_1 = 1,\ldots,N_1$. We wish to study the intersections $2\mathcal{Q}_{i_1}\cap R$ using the previous property. Notice that $\frac{5}{8}\mathcal{Q}_{i_1}\cap R\neq\varnothing$ implies that the sets $2\mathcal{Q}_{i_1}\cap R$ are parallelepipeds such that their maximal side length lies between $\frac{11}{16}\ell(\mathcal{Q}_{i_1})$ and $2\ell(\mathcal{Q}_{i_1})$. On the other hand, we distinguish two cases for the sides with minimal length, depending on $\ell_{\min} := \min\{\ell_1,\ldots,\ell_n,\ell_t\}$:

  A) If it happens

  $$\ell_{\min} \geq \frac{11}{16}\max_{i_1=1,\ldots,N_1}\ell(Q_{i_1}),$$

  then the parallelepipeds $2\mathcal{Q}_{i_1}\cap R$ have minimal side length between $\frac{11}{16}\ell(\mathcal{Q}_{i_1})$ and $2\ell(\mathcal{Q}_{i_1})$.

  B) If on the other hand

  $$\ell_{\min} < \frac{11}{16}\max_{i_1=1,\ldots,N_1}\ell(\mathcal{Q}_{i_1}),$$

the parallelepipeds $2\mathcal{Q}_{i_1} \cap R$ have minimal side length bigger or equal than the quantity $\min\{\ell_{\min}, \frac{11}{16}\ell(\mathcal{Q}_{i_1})\}$ and smaller or equal than $\min\{\ell_{\min}, 2\ell(\mathcal{Q}_{i_1})\}$, for each $i_1 = 1, \dots, N_1$.

For each $i_1 = 1, \dots, N_1$ we call $R_{i_1} := 2\mathcal{Q}_{i_1} \cap R$ and consider two cases

1) Assumption $\mathbf{A_3}$ is satisfied for $E := R$.

2) Assumption $\mathbf{A_3}$ is not satisfied for $E := R$.

If **1)** occurs we are done, by Theorem 5.4.19. If **2)** occurs, notice that if **A)** also occurs we get that $R_1, \dots R_{N_1}$ are parallelepipeds such that all of their respective sides have comparable lengths. In other words, there exists a cube $S_{i_1} \subset R_{i_1}$ so that $\widetilde{\gamma}(R_{i_1}) \approx \widetilde{\gamma}(S_{i_1})$, for each $i_1 = 1, \dots, N_1$. Then, using that $\mathbf{A_3}$ is not satisfied together with point *2* in Corollary 5.5.3, we get

$$\begin{aligned} \widetilde{\gamma}(R) < C_1^{-1} \sum_{i=1}^{N_1} \widetilde{\gamma}(R_{i_1}) &\lesssim C_1^{-1} \sum_{i=1}^{N_1} \widetilde{\gamma}_+(R_{i_1}) \\ &\leq C_1^{-1} C_1 \widetilde{\gamma}_+(R) = \widetilde{\gamma}_+(R), \end{aligned} \tag{5.5.2}$$

where we have also used property $\mathbf{P_3}$ in Theorem 5.3.1. So the only case left to study is when **2)** and **B)** happen simultaneously.

In this case we apply again, for each $i_1 = 1, \dots, N_1$, the splitting given by Theorem 5.3.1 to all the parallelepipeds R_{i_1}. This way, we obtain a second family $\{\mathcal{Q}_{i_1 1}, \dots, \mathcal{Q}_{i_1 N_{i_1}}\}$ associated to each R_{i_1}, satisfying properties $\mathbf{P_1}$ to $\mathbf{P_5}$. Let us fix $i_1 = 1, \dots, N_1$ and observe that:

- Regarding property $\mathbf{P_4}$, now we have for each $i_2 = 1, \dots, N_{i_1}$,

$$\operatorname{diam}(\mathcal{Q}_{i_1 i_2}) \leq \frac{1}{10}\operatorname{diam}(R_{i_1}) \leq \frac{\sqrt{n+1}}{10} \cdot 2\ell(\mathcal{Q}_{i_1}) \quad \text{and then} \quad \ell(\mathcal{Q}_{i_1 i_2}) \leq \frac{1}{5}\ell(\mathcal{Q}_{i_1}).$$

  This implies, in particular,

$$2\ell(\mathcal{Q}_{i_1 i_2}) \leq \frac{2}{5}\ell(\mathcal{Q}_{i_1}) < \frac{11}{16}\ell(\mathcal{Q}_{i_1}). \tag{5.5.3}$$

- Property $\mathbf{P_1}$ now reads $\frac{5}{8}\mathcal{Q}_{i_1 i_2} \cap R_{i_1} \neq \varnothing$. Therefore, $2\mathcal{Q}_{i_1 i_2} \cap R_{i_1}$ are now parallelepipeds such that their maximal side length is between $\frac{11}{16}\ell(\mathcal{Q}_{i_1 i_2})$ and $2\ell(\mathcal{Q}_{i_1 i_2})$ (where we have applied (5.5.3)). For their side with minimal length, we distinguish:

  A) If it happens

$$\ell_{\min} \geq \frac{11}{16} \max_{i_2 = 1, \dots, N_{i_1}} \ell(\mathcal{Q}_{i_1 i_2}),$$

  then the parallelepipeds $2\mathcal{Q}_{i_1 i_2} \cap R_{i_1}$ have minimal side length between $\frac{11}{16}\ell(\mathcal{Q}_{i_1 i_2})$ and $2\ell(\mathcal{Q}_{i_1 i_2})$.

B) If on the other hand

$$\ell_{\min} < \frac{11}{16} \max_{i_2=1,\dots,N_{i_1}} \ell(\mathcal{Q}_{i_1 i_2}),$$

now the parallelepipeds $2\mathcal{Q}_{i_1 i_2} \cap R_{i_1}$ have minimal side length bigger or equal than $\min\{\ell_{\min}, \frac{11}{16}\ell(\mathcal{Q}_{i_1 i_2})\}$ and smaller or equal than $\min\{\ell_{\min}, 2\ell(\mathcal{Q}_{i_1 i_2})\}$.

For each $i_1 = 1, \dots, N_1$ and $i_2 = 1, \dots, N_{i_1}$ we call $R_{i_1 i_2} := 2\mathcal{Q}_{i_1 i_2} \cap R_{i_1}$. Two cases may occur for each i_1:

1) Assumption $\mathbf{A_3}$ is satisfied for $E := R_{i_1}$, that is

$$\widetilde{\gamma}(R_{i_1}) \geq C_1^{-1} \sum_{i_2=1}^{N_{i_1}} \widetilde{\gamma}(R_{i_1 i_2}).$$

2) Assumption $\mathbf{A_3}$ is not satisfied for $E := R_{i_1}$, that is

$$\widetilde{\gamma}(R_{i_1}) < C_1^{-1} \sum_{i_2=1}^{N_{i_1}} \widetilde{\gamma}(R_{i_1 i_2}).$$

We are interested in proving $\widetilde{\gamma}(R_{i_1}) \approx \widetilde{\gamma}_+(R_{i_1})$ for every i_1, since if this is the case, arguing as in (5.5.2) we are done. The only indices i_1 where we would not be able to deduce $\widetilde{\gamma}(R_{i_1}) \approx \widetilde{\gamma}_+(R_{i_1})$ are, again, those for which **2)** and **B)** occur simultaneously. For such indices i_1 we would again apply the splitting provided by Theorem 5.3.1 to every $R_{i_1 1}, \dots, R_{i_1 N_{i_1}}$ and construct for each $R_{i_1 i_2}$ a third family of cubes $\{\mathcal{Q}_{i_1 i_2 1}, \dots, \mathcal{Q}_{i_1 i_2 N_{i_1 i_2}}\}$. Now, for each $i_2 = 1, \dots N_{i_1}$ one similarly obtains

$$\ell(\mathcal{Q}_{i_1 i_2 i_3}) \leq \frac{1}{5}\ell(\mathcal{Q}_{i_1 i_2}), \qquad \forall i_3 = 1, \dots, N_{i_1 i_2},$$

and that $2\mathcal{Q}_{i_1 i_2 i_3} \cap R_{i_1 i_2}$ are now parallelepipeds such that their maximal side length is between $\frac{11}{16}\ell(\mathcal{Q}_{i_1 i_2 i_3})$ and $2\ell(\mathcal{Q}_{i_1 i_2 i_3})$ for each $i_3 = 1, \dots, N_{i_1 i_2}$. Regarding their minimal side lengths we again distinguish cases **A)** and **B)** in the current setting. Finally, for each i_2 one calls $R_{i_1 i_2 i_3} := 2\mathcal{Q}_{i_1 i_2 i_3} \cap R_{i_1 i_2}$ and studies two cases:

1) If assumption $\mathbf{A_3}$ is satisfied for $E := R_{i_1 i_2}$,

2) or if assumption $\mathbf{A_3}$ is not satisfied for $E := R_{i_1 i_2}$.

Combinations **1)A)**, **1)B)** and **2)A)** lead to the estimate $\widetilde{\gamma}(R_{i_1 i_2}) \approx \widetilde{\gamma}_+(R_{i_1 i_2})$. If one obtained such result for every i_2, proceeding as in (5.5.2) it would yield $\widetilde{\gamma}(R_{i_1}) \approx \widetilde{\gamma}_+(R_{i_1})$, and we would be done. However, **2)** and **B)** may occur simultaneously for some indices i_2. In this setting, we would repeat the above splitting argument for the families $R_{i_1 i_2 1}, \dots, R_{i_1 i_2 N_{i_1 i_2}}$ associated to those $R_{i_1 i_2}$ where **2)** and **B)** occur.

We repeat this processes iteratively and we notice that after a number of steps large enough, say $S \geq 1$ steps, **B)** will no longer be satisfied. This is due to the fact that relation (5.5.1) ensures $\ell_{\min} \geq \kappa > 0$, for some positive constant κ depending only on R; and also because the size of the cubes at each step strictly decreases. Indeed, at step S of the iteration, property *4* in Theorem 5.3.1 and the fact that $R_{i_1 i_2 \cdots i_{S-1}}$ has maximal side length bounded by $2\ell(\mathcal{Q}_{i_1 i_2 \cdots i_{S-1}})$ imply

$$\ell(\mathcal{Q}_{i_1 i_2 \cdots i_S}) \leq \frac{1}{5}\ell(\mathcal{Q}_{i_1 i_2 \cdots i_{S-1}}) \leq \cdots \leq \frac{1}{5^{S-1}}\frac{1}{10}\ell_{\max}, \qquad \forall i_S = 1, \ldots, N_{i_1 i_2 \cdots i_{S-1}}.$$

So choosing S large enough, depending on κ, it is clear that **A)** will be satisfied instead of **B)**. Therefore, whether if **1)** occurs, or if **2)** and **A)** occur, one equally deduces (arguing as in (5.5.1) in the latter setting),

$$\widetilde{\gamma}(R_{i_1 i_2 \cdots i_{S-1}}) \approx \widetilde{\gamma}_+(R_{i_1 i_2 \cdots i_{S-1}}).$$

So tracing back all the steps of the iteration one gets, in general, $\widetilde{\gamma}(R) \approx \widetilde{\gamma}_+(R)$. In Remark 5.5.2 we argue why the implicit constants in $\widetilde{\gamma}(R) \approx \widetilde{\gamma}_+(R)$ do not depend on $\ell_{\min}$ (essentially because the constant of assumption $\mathbf{A_3}$ is carefully fixed to be C_1, the same constant of $\mathbf{P_3}$ in Theorem 5.3.1.) □

Corollary 5.5.5. *Let $E := R_1 \cup R_2 \cup \cdots \cup R_N$ be a finite union of disjoint closed parallelepipeds with sides parallel to the coordinate axes. Then,*

$$\widetilde{\gamma}(E) \approx \widetilde{\gamma}_+(E).$$

Proof. The iterative scheme of the proof we have given for Lemma 5.5.4 can be also applied in this case, but now taking into account the parameter

$$\delta := \min_{i \neq j} \{\text{dist}(R_i, R_j)\} > 0.$$

More precisely, for example, at the first step of the iteration it may happen:

A) $E_{i_1} := 2\mathcal{Q}_{i_1} \cap E$ has one connected component (and thus it is a parallelepiped) for every $i_1 = 1, \ldots, N_1$.

B) Or there exists an index i_1 such that E_{i_1} presents more than one connected component.

We would also distinguish whether if:

1) Assumption $\mathbf{A_3}$ is satisfied for E.

2) Assumption $\mathbf{A_3}$ is not satisfied for E.

If **1)** happens, we are done. If **2)** happens, observe that if in turn **A)** occurred for every i_1, applying Lemma 5.5.4 and the same estimates of (5.5.2) we would also be done. So we are left to study the case where **2)** occurs and there exist indices i_1 so that E_{i_1} presents more than one connected component. In this setting, we would repeat the above argument for the every compact set $E_1, \ldots, E_{N_1}$, based on the splitting given by Theorem 5.3.1. We repeat this process iteratively, that is: at step $S \geq 1$ of the iteration we would obtain (for a set of the form $E_{i_1 i_2 \cdots i_{S-1}}$ (where we convey $E_{i_0} := E$), with multiple connected components consisting of parallelepipeds) a family of cubes $\{\mathcal{Q}_{i_1 i_2 \cdots i_{S-1} 1}, \ldots, \mathcal{Q}_{i_1 i_2 \cdots i_{S-1} N_{i_1 i_2 \cdots i_{S-1}}}\}$ with diameters comparable to $5^{-S}\text{diam}(E)$. Now we would distinguish the cases

A) If $E_{i_1 i_2 \cdots i_S} := 2\mathcal{Q}_{i_1 i_2 \cdots i_S} \cap E_{i_1 i_2 \cdots i_{S-1}}$ has one connected component for every i_S.

B) Or if there exists an index i_S so that $E_{i_1 i_2 \cdots i_S}$ presents more than one connected component.

As well as

1) Assumption $\mathbf{A_3}$ is satisfied for $E_{i_1 i_2 \cdots i_{S-1}}$.

2) Assumption $\mathbf{A_3}$ is not satisfied for $E_{i_1 i_2 \cdots i_{S-1}}$.

The possibilities **1)A)**, **1)B)** and **2)A)** are *good* in the sense that lead to the estimate $\widetilde{\gamma}(E_{i_1 i_2 \cdots i_{S-1}}) \lesssim \widetilde{\gamma}_+(E_{i_1 i_2 \cdots i_{S-1}})$. If on the other hand **2)** and **B)** occur simultaneously, we would move on to the next step of the iteration, by applying again Theorem 5.3.1 to each $E_{i_1 i_2 \cdots i_{S-1} 1}, \ldots, E_{i_1 i_2 \cdots i_{S-1} N_{i_1 i_2 \cdots i_{S-1}}}$. The key point, however, is that for a finite number of steps S large enough (depending on δ) we would have $5^{-S}\text{diam}(E) \lll \delta$, so that **B)** can no longer happen, and thus we obtain the desired estimate tracing back all the steps of the iteration as in Lemma 5.5.4. □

Remark 5.5.2. Observe that in the proofs of Lemma 5.5.4 and Corollary 5.5.5 the constants appearing in $\widetilde{\gamma}(R) \approx \widetilde{\gamma}_+(R)$ and $\widetilde{\gamma}(E) \approx \widetilde{\gamma}_+(E)$ do not depend on the minimal side length ℓ_{min} of the parallelepiped R in the first case, nor on $\delta > 0$ in the second; both being strictly positive parameters. Such quantities, however, do determine the number of steps needed to carry out the iterative argument of the proofs. But this is not an issue, since the constant appearing in assumption $\mathbf{A_3}$ is precisely taken to be C_1, the same constant of property $\mathbf{P_3}$ in Theorem 5.3.1. This enables to carry out the estimates of (5.5.2) and avoid any possible dependence on ℓ_{min} or δ when tracing back each step of the iteration, since C_1 cancels itself out with its own inverse. Again, let us remark that this type of argument has been inspired by that of [**Vo, §6**].

Let us apply Corollary 5.5.5 to the usual family of corner-like Cantor sets of $\mathbb{R}^{n+1}$. Recall that for a sequence of real numbers $\lambda = (\lambda_j)_j$ satisfying $0 < \lambda_j < 1/2$ we defined its associated Cantor set $E \subset \mathbb{R}^{n+1}$ as in (5.2.3). If we chose $\lambda_j = 2^{-(n+1)/n}$ for every j we would recover the particular Cantor set presented in [**MatP, §5**]. Let us introduce the following *density* for each $k \geq 1$,

$$\theta_k := \frac{2^{-k(n+1)}}{\ell_k^n},$$

where Q^k is any cube of the k-th generation. We also set $\theta_0 := 1$. Combining Theorem 5.2.5, Corollary 4.2.13 and Corollary 5.5.5 we deduce:

Corollary 5.5.6. *Let $(\lambda_j)_j$ be a sequence of real numbers satisfying $0 < \lambda_j \leq \tau_0 < 1/2$, for every j. Then,*

$$\widetilde{\gamma}(E_k) \approx \widetilde{\gamma}_+(E_k) \approx \left(\sum_{j=0}^{k} \theta_j \right)^{-1},$$

where the implicit constants only depend on n and τ_0. Moreover,

$$\widetilde{\gamma}(E) \approx \widetilde{\gamma}_+(E) \approx \left(\sum_{j=0}^{\infty} \theta_j\right)^{-1}.$$

Proof. The result for E_k is a direct application of Corollary 4.2.13, Theorem 5.2.5 and Corollary 5.5.5. To obtain the result for E notice that for each generation k we have,

$$\widetilde{\gamma}(E) \leq \widetilde{\gamma}(E_k) \approx \widetilde{\gamma}_+(E_k) \leq \gamma_+(E_k) \approx \left(\sum_{j=0}^{k} \theta_j\right)^{-1}.$$

Therefore, using the outer regularity of γ_+ (Theorem 4.2.2) we get

$$\widetilde{\gamma}(E) \lesssim \gamma_+(E) \approx \left(\sum_{j=0}^{\infty} \theta_j\right)^{-1},$$

and by Theorem 5.2.5 we are done. □

5.5.3 General comparability for compact sets in $\mathbb{R}^2$

This subsection deals with the proof of the following result:

THEOREM 5.5.7. *Let $E \subset \mathbb{R}^2$ be a compact set. Then,*

$$\widetilde{\gamma}(E) \approx \widetilde{\gamma}_+(E).$$

Proof. Let $E \subset \mathbb{R}^2$ be a compact set, that we assume without loss of generality that satisfies assumptions $\mathbf{A_1}$ and $\mathbf{A_2}$. So in particular E is contained in the unit ball and consists of a finite union of dyadic cubes belonging to a dyadic grid in $\mathbb{R}^2$ (with sides parallel to the coordinate axes), all of the same size and with disjoint interiors. We denote by

$$\delta := \text{diameter of the cubes of the dyadic grid.}$$

Again, we repeat the iterative scheme of Lemma 5.5.4: in general, at step $S \geq 1$ of the iteration we would obtain for a set of the form

$$\begin{aligned} E_{i_1 i_2 \cdots i_{S-1}} &= E, \qquad &\text{if} \quad S = 1,\\ E_{i_1 i_2 \cdots i_{S-1}} &= 2\mathcal{Q}_{i_1 i_2 \cdots i_{S-1}} \cap E_{i_1 i_2 \cdots i_{S-2}}, \qquad &\text{if} \quad S > 1, \end{aligned}$$

a family of cubes $\{\mathcal{Q}_{i_1 i_2 \cdots i_{S-1} 1}, \ldots, \mathcal{Q}_{i_1 i_2 \cdots i_{S-1} N_{i_1 i_2 \cdots i_{S-1}}}\}$ with diameters comparable to $5^{-S}\mathrm{diam}(E)$. Now we would distinguish the cases

- **A)** If $E_{i_1 i_2 \cdots i_S} := 2\mathcal{Q}_{i_1 i_2 \cdots i_S} \cap E_{i_1 i_2 \cdots i_{S-1}}$ has diameter smaller than $\delta/4$ for every $i_S = 1, \ldots, N_{i_1 i_2 \cdots i_{S-1}}$.
- **B)** Or if there exists an index i_S so that $\mathrm{diam}(E_{i_1 i_2 \cdots i_S}) \geq \delta/4$.

As well as

1) Assumption $\mathbf{A_3}$ is satisfied for $E_{i_1 i_2\cdots i_{S-1}}$.

2) Assumption $\mathbf{A_3}$ is not satisfied for $E_{i_1 i_2\cdots i_{S-1}}$.

If **1)** occurs we are done, in the sense that we can deduce the estimate

$$\widetilde{\gamma}(E_{i_1 i_2\cdots i_{S-1}}) \lesssim \widetilde{\gamma}_+(E_{i_1 i_2\cdots i_{S-1}}).$$

If **2)** occurs, we move on to the next step of the iteration up to the point where S is large enough (depending on δ) so that option **B)** is no longer possible. That is, the iteration stops once the only two possible scenarios are: **1)** or **2)A)**. In this setting, we are left to study the case where **2)** and **A)** occur simultaneously. To deal with this case, we write the precise definition of $E_{i_1 i_2\cdots i_S}$, which is

$$E_{i_1 i_2\cdots i_S} := \big(2\mathcal{Q}_{i_1 i_2\cdots i_S} \cap 2\mathcal{Q}_{i_1 i_2\cdots i_{S-1}} \cap \cdots \cap 2\mathcal{Q}_{i_1 i_2} \cap 2\mathcal{Q}_{i_1}\big) \cap E =: 2R_{i_1 i_2\cdots i_S} \cap E,$$

where $R_{i_1 i_2\cdots i_S}$ is a rectangle with diameter comparable to $5^{-S}\text{diam}(E) \ll \delta$. In fact, applying properties $\mathbf{P_1}$ and $\mathbf{P_4}$ in Theorem 5.3.1 at each step, one easily deduces the inclusions

$$\frac{11}{16}\mathcal{Q}_{i_1 i_2\cdots i_S} \subset 2R_{i_1 i_2\cdots i_S} \subset 2\mathcal{Q}_{i_1 i_2\cdots i_S}. \tag{5.5.4}$$

The latter is trivial for $S = 1$. If $S > 1$, the argument is similar to one that is presented in the proof of Lemma 5.5.4. For example, if $S = 2$, $\mathbf{P_1}$ yields, for each $i_2 = 1, \dots, N_{i_1}$,

$$\frac{5}{8}\mathcal{Q}_{i_1 i_2} \cap (2\mathcal{Q}_{i_1} \cap E) \neq \varnothing, \qquad \text{so in particular} \qquad \frac{5}{8}\mathcal{Q}_{i_1 i_2} \cap 2Q_{i_1} \neq \varnothing.$$

On the other hand, $\mathbf{P_4}$ yields $\text{diam}(\mathcal{Q}_{i_2 i_2}) \leq \frac{1}{10}\text{diam}(2\mathcal{Q}_{i_1} \cap E)$, which implies

$$\ell(\mathcal{Q}_{i_1 i_2}) \leq \frac{1}{5}\ell(\mathcal{Q}_{i_1}), \qquad \text{for each} \quad i_2 = 1, \dots, N_{i_1}.$$

All in all, one easily gets $\frac{11}{16}\mathcal{Q}_{i_1 i_2} \subset 2\mathcal{Q}_{i_1 i_2} \cap 2\mathcal{Q}_{i_1} =: R_{i_1 i_2}$, which are the desired inclusions. Such scheme can be repeated at each step to obtain, in general, relation (5.5.4). The latter inclusions imply that the rectangle $R_{i_1 i_2\cdots i_S}$ *behaves as a square*, in the sense that its side lengths are comparable and thus, by the second statement in Corollary 5.5.3,

$$\widetilde{\gamma}(R_{i_1 i_2\cdots i_S}) \leq 2\,\gamma(\mathcal{Q}_{i_1 i_2\cdots i_S}) \approx \gamma_+(\mathcal{Q}_{i_1 i_2\cdots i_S}) \leq \frac{16}{11}\,\gamma_+(R_{i_1 i_2\cdots i_S}) \approx \widetilde{\gamma}_+(R_{i_1 i_2\cdots i_S}).$$

That is, $\widetilde{\gamma}(R_{i_1 i_2\cdots i_S}) \approx \widetilde{\gamma}_+(R_{i_1 i_2\cdots i_S})$. In addition, since **A)** occurs, by the definition of δ we get that $2R_{i_1 i_2\cdots i_S}$ can only intersect one connected component of E. Moreover, if $2R_{i_1 i_2\cdots i_S}$ intersects one of the dyadic squares that conform E, it can only intersect, in addition, those squares which are adjacent to it (a maximum of four). Therefore, there is only a finite number of possible compact sets that $E_{i_1 i_2\cdots i_S} := 2R_{i_1 i_2\cdots i_S} \cap E$

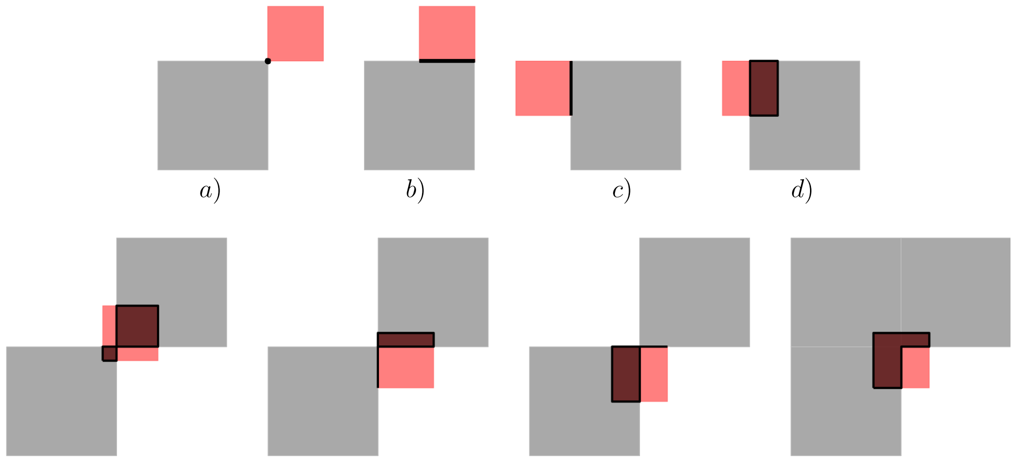

Figure 5.1: In black, the eight possible intersections that can be obtained from $E_{i_1i_2\cdots i_S} := 2R_{i_1i_2\cdots i_S} \cap E$, for S large enough so that $5^{-S}\text{diam}(E) \ll \delta$. In red the rectangle $2R_{i_1i_2\cdots i_S}$ (assumed to be a square by relation (5.5.4)), and in gray some of the dyadic squares conforming E. Let us remark that in $d)$ we also contemplate the case where $2R_{i_1i_2\cdots i_S} \cap E = 2R_{i_1i_2\cdots i_S}$.

can be. We represent them in Figure 5.1. There, $2R_{i_1i_2\cdots i_S}$ is assumed, without loss of generality by (5.5.4), to be a (red) square; and some of the dyadic squares conforming E are depicted in gray.

It is clear that in cases $a)$, $b)$, $c)$ and $d)$, a direct application of Lemma 5.5.4 yields

$$\widetilde{\gamma}(E_{i_1i_2\cdots i_S}) \lesssim \widetilde{\gamma}_+(E_{i_1i_2\cdots i_S}).$$

Regarding $f)$, $g)$ and $h)$, we write $2\ell^H_{i_1i_2\cdots i_S}$ the horizontal side length of $2R_{i_1i_2\cdots i_S}$, and $I^H_{i_1i_2\cdots i_S}$ the horizontal side of $E_{i_1i_2\cdots i_S}$ with such length. Then,

$$\widetilde{\gamma}(E_{i_1i_2\cdots i_S}) \leq \widetilde{\gamma}(2R_{i_1i_2\cdots i_S}) \approx \widetilde{\gamma}_+(I^H_{i_1i_2\cdots i_S}) \leq \widetilde{\gamma}_+(E_{i_1i_2\cdots i_S}),$$

where for the second inequality we have used the second statement in Corollary 5.5.3 and relation (5.5.4), and the third follows simply by the monotonicity of the capacity. Hence we also obtain the desired estimate. Finally, case $e)$ can be dealt with in a similar way, just noticing that at least one of the diagonally opposed rectangles $R^{\uparrow}_{i_1i_2\cdots i_S}$ and $R^{\uparrow}_{i_1i_2\cdots i_S}$, obtained in $E_{i_1i_2\cdots i_S}$, presents horizontal side length half of $2\ell^H_{i_1i_2\cdots i_S}$. Assume it is $R^{\uparrow}_{i_1i_2\cdots i_S}$. Then,

$$\widetilde{\gamma}(E_{i_1i_2\cdots i_S}) \leq \widetilde{\gamma}(2R_{i_1i_2\cdots i_S}) \approx \widetilde{\gamma}_+(I^H_{i_1i_2\cdots i_S}) \leq 2\,\widetilde{\gamma}_+(R^{\uparrow}_{i_1i_2\cdots i_S}) \leq 2\,\widetilde{\gamma}(E_{i_1i_2\cdots i_S}).$$

Therefore, in all possible scenarios we get

$$\widetilde{\gamma}(E_{i_1i_2\cdots i_S}) \lesssim \widetilde{\gamma}_+(E_{i_1i_2\cdots i_S}),$$

so applying the same type of estimates of (5.5.2) (since we are assuming that $\mathbf{A_3}$ is not satisfied) we deduce

$$\widetilde{\gamma}(E_{i_1 i_2 \cdots i_{S-1}}) \lesssim \widetilde{\gamma}_+(E_{i_1 i_2 \cdots i_{S-1}}),$$

that is the necessary estimate to trace back the iterations and obtain the desired result. □

The previous result combined with Theorem 4.2.2 yields:

THEOREM 5.5.8. *The capacity $\widetilde{\gamma}$ is semi-additive in $\mathbb{R}^2$. That is, there is an absolute constant $C > 0$ so that for any $E_1, E_2, \ldots$ disjoint compact sets of $\mathbb{R}^2$,*

$$\widetilde{\gamma}\bigg(\bigcup_{j=1}^{\infty} E_j\bigg) \leq C \sum_{j=1}^{\infty} \widetilde{\gamma}(E_j).$$

5.6 Further results in $\mathbb{R}^2$. The $\widetilde{\gamma}$ capacity of rectangles

In this section we compute the $\widetilde{\gamma}$ capacity of a closed rectangle $R \subset \mathbb{R}^2$ with sides parallel to the coordinate axes and respective side lengths $\ell_x > 0$, $\ell_t > 0$. We obtain the following result:

THEOREM 5.6.1.

$$\widetilde{\gamma}(R) \approx \ell_t \left[\frac{1}{2} \ln\left(1 + \frac{\ell_t^2}{\ell_x^2}\right) + \frac{\ell_t}{\ell_x} \arctan\left(\frac{\ell_x}{\ell_t}\right) \right]^{-1}.$$

Proof. Let $R \subset \mathbb{R}^2$ be such a rectangle and assume, without loss of generality, that its lower left corner coincides with the origin. To simplify the computations, we also normalize its temporal side length ℓ_t to be 1 by dilating R by the factor $\lambda := \ell_t^{-1}$. We name the resulting rectangle R_0, that is such that

$$\widetilde{\gamma}(R) = \lambda^{-1} \widetilde{\gamma}(R_0) = \ell_t \, \widetilde{\gamma}(R_0).$$

We introduce the parameter

$$r := \frac{\ell_x}{\ell_t},$$

that is nothing but the spatial side length of R_0, as well as the measure

$$\mu := \mathcal{L}^2|_{R_0}.$$

By a direct computation, one obtains that the potential $P * \mu$ at a point $\overline{x} = (x, t)$ is given by the following explicit expression:

$$P * \mu(\overline{x}) = \begin{cases} \text{if } t \leq 0, & 0, \\ \text{if } t \in (0,1], & \dfrac{x}{2} \ln\left(1 + \dfrac{t^2}{x^2}\right) + \dfrac{r-x}{2} \ln\left(1 + \dfrac{t^2}{(r-x)^2}\right) \\ & \qquad + t\left[\dfrac{\pi}{2}\operatorname{sgn}(x) - \arctan\left(\dfrac{t}{x}\right) + \arctan\left(\dfrac{r-x}{t}\right)\right], \\ \text{if } t > 1, & \dfrac{x}{2} \ln\left[\dfrac{x^2+t^2}{x^2+(t-1)^2}\right] + \dfrac{r-x}{2} \ln\left[\dfrac{(x-r)^2+t^2}{(x-r)^2+(t-1)^2}\right] \\ & \quad + 1\left[\arctan\left(\dfrac{r-x}{t-1}\right) + \arctan\left(\dfrac{x}{t-1}\right)\right] \\ & \quad + t\left[\arctan\left(\dfrac{t-1}{x}\right) - \arctan\left(\dfrac{t}{x}\right) \right. \\ & \qquad \left. + \arctan\left(\dfrac{r-x}{t}\right) - \arctan\left(\dfrac{r-x}{t-1}\right)\right]. \end{cases}$$

In the previous formula we have written the factor 1 in the fifth line just to emphasize that such factor would be ℓ_t if R was not normalized to be R_0. It is not difficult to prove that the above expression defines a continuous function in the whole $\mathbb{R}^2$ (once extended to $x = r$ and $x = 0$ when $t > 0$ by taking limits). Also, using the following identity

$$\operatorname{sgn}(x) = \frac{2}{\pi}\left[\arctan(x) + \arctan\left(\frac{1}{x}\right)\right],$$

it follows that for any fixed t,

$$P * \mu(x,t) = P * \mu(r-x,t),$$

or in other words, the one variable function $x \mapsto P * \mu(x,t)$ is symmetric with respect to the point $x = r/2$, for any $t \in \mathbb{R}$. Moreover, it is clear that it is nonnegative, tends to 0 as $x \to \pm\infty$ and in fact, for $t > 0$, it attains its maximum precisely at $x = r/2$. This last property can be argued as follows: begin by fixing $t \in (0,1]$ and noticing that

$$\lim_{x\to 0} P * \mu(x,t) = \lim_{x \to r} P * \mu(x,t) = \frac{r}{2} \ln\left(1 + \frac{t^2}{r^2}\right) + t \arctan\left(\frac{r}{t}\right) =: C_t.$$

Compute the derivative with respect to x of $P * \mu$ at points $x \neq 0, x \neq r$, that is given by

$$\partial_x P * \mu(x,t)' = \frac{1}{2}\left[\ln\left(1 + \frac{t^2}{x^2}\right) - \ln\left(1 + \frac{t^2}{(r-x)^2}\right)\right],$$

and that satisfies $\partial_x P * \mu(\cdot,t) < 0$ in $(r/2,r) \cup (r,\infty)$, $\partial_x P * \mu(\cdot,t) > 0$ in $(-\infty,0) \cup (0,r/2)$ and $\partial_x P * \mu(r/2,t) = 0$. Therefore, $P * \mu(\cdot,t)$ may attain its global maximum at $x = 0$, $x = r/2$ or $x = r$. The value of $P * \mu(\cdot,t)$ at $x = r/2$ is

$$P * \mu\left(\frac{r}{2},t\right) = \frac{r}{2} \ln\left(1 + \frac{4t^2}{r^2}\right) + 2t \arctan\left(\frac{r}{2t}\right) > C_t.$$

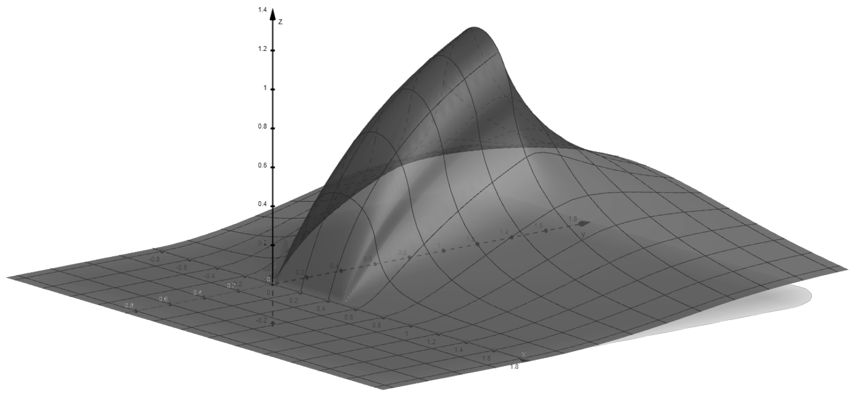

Figure 5.2: Graph of $P * \mu$ for $r = 1/2$.

Hence, for $t \in (0,1)$, $P * \mu(\cdot, t)$ attains its global maximum at $x = r/2$. For $t > 1$ a similar study can be carried out, yielding the same conclusion.

Now, by restricting $P * \mu$ to the vertical line $x = r/2$ we obtain a nonnegative piece-wise continuous function of t given by the expression

$$P*\mu\left(\frac{r}{2}, t\right) = \begin{cases} \text{if } t \leq 0, & 0, \\ \text{if } t \in (0,1], & \dfrac{r}{2} \ln\left(1 + \dfrac{4t^2}{r^2}\right) + 2t \arctan\left(\dfrac{r}{2t}\right), \\ \text{if } t > 1, & \dfrac{r}{2} \ln\left[\dfrac{r^2+4t^2}{r^2+4(t-1)^2}\right] + 2\arctan\left(\dfrac{r}{2(t-1)}\right) \\ & \qquad - 2t\left[\dfrac{\pi}{2} - \arctan\left(\dfrac{2(t-1)}{r}\right) - \arctan\left(\dfrac{r}{2t}\right)\right]. \end{cases}$$

which also tends to 0 as $t \to \pm\infty$ and it can be proved that it attains its maximum for $t = 1$, for any value of $r > 0$. Combining the above computations, we have obtained

$$P * \mu(\overline{x}) \leq P * \mu(r/2, 1), \qquad \forall \overline{x} \in \mathbb{R}^2.$$

For the sake of clarity, Figure 5.2 depicts the graph of the potential $P * \mu$ for a particular value of r (that already represents its qualitative behavior).

Using the particular value of $M(r) := P * \mu(r/2, 1)$ we are able to obtain a lower bound for γ_+:

$$\gamma_+(R_0) \geq M(r)^{-1}\mu(R_0) = M(r)^{-1} r = \left[\frac{1}{2}\ln\left(1 + \frac{4}{r^2}\right) + \frac{2}{r}\arctan\left(\frac{r}{2}\right)\right]^{-1}.$$

To obtain an upper bound we shall work with the capacity $\gamma_{\text{sy},+}$ (comparable to γ_+

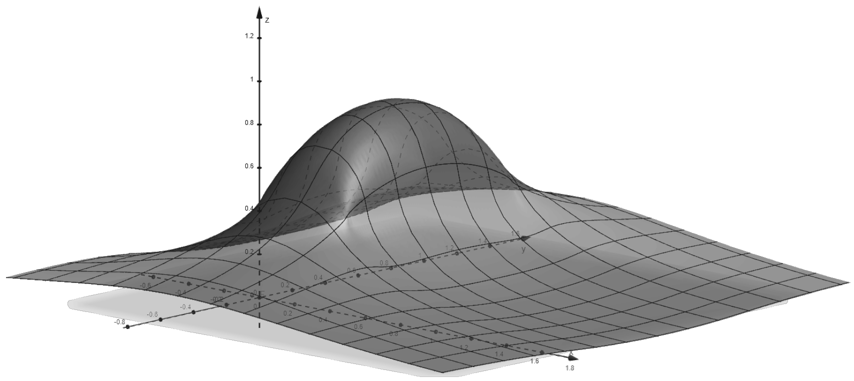

Figure 5.3: Graph of $P_{\rm sy} * \mu$ for $r = 1/2$.

by Theorem 5.2.5). By definition of $P_{\rm sy}$ we get

$$P_{\rm sy} * \mu(x,t) := \frac{1}{2}\big(P * \mu(x,t) + P^* * \mu(x,t)\big) = \frac{1}{2}\big(P * \mu(x,t) + P * \mu(x,1-t)\big),$$

where the last equality can be easily deduced from the definition of P^* and the symmetry of R with respect to the horizontal line $t = 1/2$. By a similar study to the one done for $P * \mu$, it can be proved that $(P_{\rm sy} * \mu)|_{R_0}$ attains its minimum at the vertices of R_0. Just proceed by fixing $t \in [0,1]$ and studying the one variable function $x \mapsto P_{\rm sy} * \mu(x,t)$ restricted to the domain $[0,r]$; and by fixing $x \in [0,1]$ and studying $t \mapsto P_{\rm sy} * \mu(x,t)$ once restricted to $[0,1]$. The former is again symmetric with respect to the point $x = r/2$ and attains its minimum for $x = 0$ and $x = r$, while the latter is symmetric with respect $t = 1/2$ and attains its minimum for $t = 0$ and $t = 1$. In Figure 5.3 the reader may visualize the graph of $P_{\rm sy} * \mu$ for a particular value of r.

Therefore, for any $(x,t) \in R_0$ we have

$$P_{\rm sy} * \mu(x,t) \geq \lim_{(x,t)\to(0,0)} P_{\rm sy} * \mu(x,t) = \frac{r}{4}\ln\left(1 + \frac{1}{r^2}\right) + \frac{1}{2}\arctan(r) =: \frac{1}{2}m(r).$$

Now take any admissible measure ν for $\gamma_{{\rm sy},+}(R_0)$ and observe that

$$\langle \nu, 1\rangle \leq 2\,m(r)^{-1}\langle \nu, P_{\rm sy} * \mu\rangle = 2\,m(r)^{-1}\langle P_{\rm sy} * \nu, \mu\rangle \leq 2\,m(r)^{-1}\mu(R_0) = 2\,m(r)^{-1}r,$$

where we have applied Tonelli's theorem, the symmetry of $P_{\rm sy}$ and the fact that $\mu \ll \mathcal{L}^2$. So by the arbitrariness of ν and the comparability of γ_+ with $\gamma_{{\rm sy},+}$ we deduce that there exists an absolute constant $C > 0$ so that

$$\gamma_+(R_0) \leq C\,m(r)^{-1}r = C\left[\frac{1}{2}\ln\left(1 + \frac{1}{r^2}\right) + \frac{\arctan(r)}{r}\right]^{-1}.$$

Finally, using that $M(r)-4m(r)\leq 0$ for any $r>0$, we get the following estimate

$$\gamma_+(R_0)\approx m(r)^{-1}r=\left[\frac{1}{2}\ln\left(1+\frac{1}{r^2}\right)+\frac{\arctan(r)}{r}\right]^{-1}.$$

Therefore, regarding the original rectangle R and applying Theorems 5.2.5 and 5.5.7, Theorem 5.6.1 follows. □

Remark 5.6.1. Let us check how the above relation extends to cases $\ell_x=0$ (i.e. $r=0$) and $\ell_t=0$ (i.e. $r=+\infty$). For the first case, notice that if $r\leq 1/2$ we have

$$\frac{1}{3|\ln(r)|}\leq\left[\frac{1}{2}\ln\left(1+\frac{1}{r^2}\right)+\frac{\arctan(r)}{r}\right]^{-1}\leq\frac{1}{|\ln(r)|},$$

where the hypothesis of $r\leq 1/2$ is used in the first bound. Therefore, we deduce

$$\widetilde{\gamma}(R)\approx\ell_t\left|\ln\left(\frac{\ell_x}{\ell_t}\right)\right|^{-1},\qquad\text{if}\quad\ell_x\leq\ell_t/2,$$

which is a result consistent with the outer regularity of γ_+ and the fact that a vertical line segment has null γ capacity (see [**MatP, Proposition 6.1**]). On the other hand, for the regime $r\to+\infty$, the following holds if $r\geq 1$

$$\frac{r}{2}\leq\left[\frac{1}{2}\ln\left(1+\frac{1}{r^2}\right)+\frac{\arctan(r)}{r}\right]^{-1}\leq r,$$

(in fact, the lower bound holds for any $r>0$) so in this case we deduce

$$\widetilde{\gamma}(R)\approx\ell_x,\qquad\text{if}\quad\ell_x\geq\ell_t,$$

that is what we expected by point *2* in Corollary 5.5.3.

Open problems

Let us present some of the questions and problems that have remained open in this dissertation.

P1 : Study the Lipschitz caloric capacity introduced in Chapter 2 for $s = 1/2$. In this setting, the ambient space is endowed with the usual Euclidean metric and the corresponding Lipschitz restriction for admissible distributions is simply written as

$$\|\nabla P * T\|_\infty \leq 1, \quad \text{where } \nabla = (\nabla_x, \partial_t).$$

Inspired by the work carried out by Uy in [**Uy**] for analytic capacity, we conjecture that the capacity one would obtain for a compact set $E \subset \mathbb{R}^{n+1}$, should be comparable to the Lebesgue measure in $\mathbb{R}^{n+1}$ restricted to E.

P2 : In Chapter 2 we have established for $s > 1/2$ that the critical s-parabolic Hausdorff dimension of the $(1, \frac{1}{2s})$-Lipschitz caloric capacity is $n+1$. Moreover, we have constructed a set that has such dimension and that is removable. However, the question of finding a set with critical dimension that is not removable remains open if $1/2 < s < 1$. Notice that the case $s = 1$ is solved by Theorem 2.3.3 in [**MatPT, Example 5.6**], since in this case one can choose any subset of positive $\mathcal{H}^{n+1}_p$-measure of a $(1, 1/2)$-Lipschitz graph and it will present positive capacity. However, for $1/2 < s < 1$, it is not clear if a set of positive $\mathcal{H}^{n+1}_{p_s}$-measure in a $(1, \frac{1}{2s})$-Lipschitz graph exists.

Another fact that might help convince the reader that the case $1/2 < s < 1$ is different from $s = 1$ is the following example. Let us fix $n = 1$, so that our ambient space is $\mathbb{R}^2$, and consider E a vertical line segment of unit length. We endow the plane with the s-parabolic distance so that E has s-parabolic Hausdorff dimension $2s$. Observe that $2s$ is strictly smaller than the critical dimension except for the case $s = 1$. Therefore, if $s < 1$, the set E is removable. However, if $s = 1$, E is an example of set with positive $\mathcal{H}^{n+1}_p$-measure contained in a $(1, 1/2)$-Lipschitz graph and hence it is not removable.

Then, possibly, one should look for more exotic candidates for non-removable sets, such as the Cantor sets presented in the first section of Chapter 4. There, we have been only able to prove that those which are not removable present

s-parabolic Hausdorff dimension bigger or equal than $n+1$, but were not capable of finding an upper bound. Another possibility, maybe more sensible, might be to first analyze the removability of sets with critical dimension and finite $\mathcal{H}^{n+1}_{p_s}$-measure. In this case, since an analogous version of [**MatPT, Lemma 6.2**] should hold, the study would become more accessible.

P3 : Extend the non-comparability result of Chapter 3 to a general multi-dimensional setting. Moreover, extend such result to an s-fractional context for $1/2 < s < 1$, comparing $\gamma_{\Theta^s}^{1/2}$ and Γ_{Θ^s}.

P4 : Study with more detail the capacities defined via the normalization condition $\|P_s * T\|_\infty \leq 1$, for $0 < s \leq 1$ (see Definition 4.2.1). The case $s = 1$ was already studied by Watson [**Wa3**] and Kaiser and Müller [**KMü**], while the case $s = 1/2$ was covered by Mateu and Prat in [**MatP**]. More specifically, it remains open to localize these potentials and prove the equivalence between the nullity of the capacity and the corresponding removability of compact sets for bounded solutions of the Θ^s-equation.

P5 : Prove the corresponding upper bound for the Γ_{Θ^s} capacity of the Cantor sets defined in the first section of Chapter 4.

P6 : Extend the comparability result (Theorem 5.5.7) to a multi-dimensional setting and find a closed expression for the capacity of a parallelepiped in $\mathbb{R}^{n+1}$. Study also the comparability for the genuine 1/2-caloric capacity $\gamma_{\Theta^{1/2}}$, where one only demands the normalization condition over P, instead of imposing it also over its conjugate P^*. Moreover, if possible, try to extend any of the above results to a more general s-fractional caloric context or even for any of the Lipschitz capacities introduced in this text.

P7 : Develop a fractional heat potential theory in a similar way as Watson does in [**Wa3**] for the heat equation. Obtain results regarding the balayage of measures, the existence of equilibrium measures, maximum and continuity principles of potentials, etc.

Bibliography

[**AS**] Abramowitz, M., Stegun, I. A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover Publications, New York (1964).

[**AsIM**] Astala, K., Iwaniec, T., Martin, G.: Elliptic partial differential equations and quasiconformal mappings in the plane (PMS-48). Princeton University Press, Princeton (2009).

[**B**] Baire, R.: Leçons sur les fonctions discontinues professées au Collège de France. Gauthier-Villars, Paris (1905).

[**BlG**] Blumenthal, R.M., Getoor, R. K.: Some theorems on stable processes. Transactions of the American Mathematical Society **95**(2), 263–273 (1960).

[**Bu**] Buchholz, H.: The Confluent Hypergeometric Function. Springer-Verlag, Berlin (1969).

[**C**] Carlsson, H.: Lecture notes on distributions. Department of Mathematics, Chalmers University of Technology, Gothenburg (2011).

[**CaD**] Cartan, H., Deny, J.: Le principe du maximum en théorie du potentiel et la notion de fonction surharmonique. Acta Scientiarum Mathematicarum **12A**, 81–100 (1950).

[**ChLMa**] Chen, W. X., Li, Y., Ma, P.-E.: The Fractional Laplacian. World Scientific, Hackensack (2020).

[**Cho**] Choquet, G. (1956). Les royaux réguliers en théorie du potentiel. Comptes Rendus de l'Académie des Sciences **243**, 635–638 (1956).

[**ChouMatPT**] Chousionis, V., Mateu, J., Prat, L., Tolsa, X.: Calderón-Zygmund kernels and rectifiability in the plane. Advances in Mathematics **231**(1), 535–568 (2012).

[**Chr**] Christ, M.: Lectures on Singular Integral Operators. CBMS Regional Conference Series in Mathematics, Number 77. American Mathematical Society, Providence, Rhode Island (1990).

[**CoW**] Coifman, R. R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. Bulletin of the American Mathematical Society **83**(4), 569–645 (1977).

[**ConCor**] Constantinescu, C., Cornea, A.: Potential theory in harmonic spaces. Springer-Verlag, Berlin (1972).

[**DPaV**] Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bulletin of Mathematical Sciences **136**(5), 521–573 (2012).

[**DaØ**] Davie, A. M., Øksendal, B.: Analytic capacity and differentiability properties of finely harmonic functions. Acta Mathematica **149**(1–2), 127–152 (1982).

[**E**] Eiderman, V. Y.: Estimates for potentials and δ-subharmonic functions outside exceptional sets. Izvestiya: Mathematics **61**(6), 1293–1329 (1997).

[**F**] Falconer, K. Fractal geometry. Mathematical foundations and applications. John Wiley & Sons Ltd., West Sussex (2003).

[**Fa**] Farag, H. M.: The Riesz kernels do not give rise to higher-dimensional analogues of the Menger-Melnikov curvature. Publicacions Matemàtiques **43**(1), 251–260 (1999).

[**Fr**] Friedman, A.: Parabolic equations of the second order. Transactions of the American Mathematical Society **93**, 509–530 (1959).

[**GaZ**] Gariepy, R., Ziemer, W. P.: Removable sets for quasilinear parabolic equations. Journal of the London Mathematical Society **21**, 311–318 (1980).

[**Gar1**] Garnett, J.: Analytic capacity and measure. Lecture Notes in Mathematics, Vol. 297. Springer-Verlag, Berlin (1972).

[**Gar2**] Garnett, J.: Bounded analytic functions. Springer New York, New York (2007).

[**Ge**] Gehring, F. W.: On solutions of the heat equation of heat conduction. Michigan Mathematical Journal **5**, 191–202 (1958).

[**Gr**] Grafakos, L.: Classical Fourier Analysis, Second Edition, Graduate Texts in Mathematics, n. 249. Springer New York, New York (2008).

[**GrTe**] Grafakos, L., Teschl, G.: On Fourier transforms of radial functions and distributions. Journal of Fourier Analysis and Applications **19**(1), 167–179 (2013).

[**HPo**] Harvey, R., Polking, J.: Removable singularities of solutions of linear partial differential equations. Acta Mathematica **125**, 39–56 (1970).

[**He**] Helms, L. L.: Introduction to potential theory. Wiley-Interscience, New York (1969).

[**Her**] Hernández, J.: On the $(1/2,+)$-caloric capacity of Cantor sets. Annales Fennici Mathematici **49**(1), 211-–239 (2024).

[**Ho1**] Hofmann, S.: A characterization of commutators of parabolic singular integrals. In: García-Cuerva, J. (ed.) Fourier Analysis and Partial Differential Equations, pp. 195–210. CRC Press, Boca Raton (1995).

[**Ho2**] Hofmann, S.: Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials. Duke Mathematical Journal **90**(2), 209–259 (1997).

[**HoL**] Hofmann, S., Lewis, J. L.: L^2 solvability and representation by caloric layer potentials in time-varying domains. Annals of Mathematics **144**(2), 349–420 (1996).

[**Hy**] Hytönen, T.: Framework for non-homogeneous analysis on metric spaces, and the RMBO space of Tolsa. Publicacions Matemàtiques **54**(2), 485–504 (2010).

[**HyMar**] Hytönen, T., Martikainen, H.: Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. Journal of Geometric Analysis **22**(4), 1071–1107 (2012).

[**KMü**] Kaiser, W., Müller, B.: Removable sets for the heat equation. Vestnik Moskovskovo Universiteta. Seriya I. Matematika. Mekhanika **28**, 26–32 (1973).

[**Ka**] Kaufman, R.: Hausdorff measure, BMO, and analytic functions. Pacific Journal of Mathematics **102**(2), 369–371 (1982).

[**Ki**] Kishi, M.: Maximum Principles in the Potential Theory. Nagoya Mathematical Journal **23**, 165–187 (1963).

[**La**] Landkof, N. S.: Foundations of Modern Potential Theory. Springer-Verlag, Berlin (1972).

[**Lan**] Lanconelli, E.: Sul problema di Dirichlet per l'equazione del calore. Annali di Matematica Pura ed Applicata **97**(4), 83–114 (1973).

[**LeMu**] Lewis, J. L., Murray M. A. M.: The method of layer potentials for the heat equation in time-varying domains. Memoirs of the American Mathematical Society, Volume 114, Number 545. American Mathematical Society, Rhode Island (1995).

[**LeSi**] Lewis, J. L., Silver, J.: Parabolic measure and the Dirichlet problem for the heat equation in two dimensions. Indiana University Mathematics Journal **37**(4), 801–839 (1988).

[**Mae**] Maeda, F.-Y.: Capacities on harmonic spaces with adjoint structure. In: Král, J. et al. (ed.) Potential theory, pp. 231–236. Plenum Press, New York (1988).

[**MatP**] Mateu, J., Prat, L.: Removable singularities for solutions of the fractional heat equation in time varying domains. Potential Analysis **60**, 833–873 (2024).

[**MatPT**] Mateu, J., Prat, L., Tolsa, X.: Removable singularities for Lipschitz caloric functions in time varying domains. Revista Matemática Iberoamericana **38** (2), 547–588 (2022).

[**MatT**] Mateu, J., Tolsa, X.: Riesz Transforms and Harmonic Lip_1-Capacity in Cantor Sets. Proceedings of the London Mathematical Society **89**(3), 676–696 (2004).

[**Matti**] Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability. Cambridge University Press, Cambridge (1995).

[**MattiPar**] Mattila, P., Paramonov P.V.: On geometric properties of harmonic Lip_1-capacity. Pacific Journal of Mathematics **2**, 469–491 (1995).

[**MiSa**] Miller, K. S., Samko, S. G.: Completely monotonic functions. Integral Transforms and Special Functions **12**(4), 389-–402 (2001).

[**Me1**] Mel'nikov, M. S.: Metric properties of analytic α-capacity and approximation of analytic functions with a Hölder condition by rational functions. Mat. Sbornik **121**(8), 115–124 (1969).

[**Me2**] Mel'nikov, M. S.: Analytic capacity: discrete approach and curvature of measure. Sbornik: Mathematics **186**(6), 827–846 (1995).

[**MeVe**] Mel'nikov, M. S., Verdera, J.: A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs. International Mathematics Research Notices **7**, 325–331 (1995).

[**MoPu**] Mourgoglou, M., Puliatti, C.: Blow-ups of caloric measure in time varying domains and applications to two-phase problems. Journal de Mathématiques Pures et Appliquées **152**, 1–68 (2021).

[**MucWh**] Muckenhoupt, B., Wheeden, R.: Weighted bounded mean oscillation and the Hilbert transform. Studia Mathematica **54**(3), 221–237 (1976).

[**NTrVo1**] Nazarov, F., Treil, S., Volberg, A.: Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces. International Mathematics Reasearch Notices **9**, 463–487 (1998).

[**NTrVo2**] Nazarov, F., Treil, S., Volberg, A.: The Tb-theorem on non-homogeneous spaces that proves a conjecture of Vitushkin. CRM preprint **519**(2), 1–84 (2002).

[**NTrVo3**] Nazarov, F., Treil, S., Volberg, A.: The Tb-theorem on non-homogeneous spaces. Acta Mathematica **190**(2), 151–239 (2003).

[**Ni**] Ninomiya, N.: Sur le principe de continuité dans la théorie du potentiel. Jounal of the Institute of Polytechnics Osaka City University **8**, 51–56 (1957).

[**NySt**] Nyström, K., Strömqvist, M.: On the parabolic Lipschitz approximation of parabolic uniform rectifiable sets. Revista Matemática Iberoamericana **33**(4), 1397–1422 (2017).

[**O**] Ohtsuka, M.: On potential theory in locally compact spaces. Journal of Science of Hiroshima University **1A**, 135–352 (1961).

[**O'**] O'Farrell, A. G.: Rational Approximation and Weak Analyticity. II. Mathematische Annalen **281**, 169–176 (1988).

[**Pe**] Peetre, J.: On convolution operators leaving $L^{p,\lambda}$ spaces invariant. Annali di Matematica **72**, 295–304 (1966).

[**PrTa**] Pruitt, W. E., Taylor, S. J.: The potential kernel and hitting probabilities for the general stable process in $\mathbb{R}^n$. Transactions of the American Mathematical Society **146**, 299–321 (1969).

[**R**] Ransford, T.: Potential Theory in the Complex Plane (London Mathematical Society, Student Texts 28). Cambridge University Press, Cambridge (1995).

[**RoSe1**] Ros-Oton, X., Serra, J.: The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary. Journal de Mathématiques Pures et Appliquées **101**(3), 275–302 (2014).

[**RoSe2**] Ros-Oton, X., Serra, J.: Regularity theory for general stable operators. Journal of Differential Equations **260**(12), 8675–8715 (2016).

[**RoTo**] Ros-Oton, X., Torres-Latorre, C.: Optimal regularity for supercritical parabolic obstacle problems. Communications on Pure and Applied Mathematics **77**(3), 1724–1765 (2023).

[**Ru1**] Rudin, W.: Fourier Analysis on Groups. Interscience Publishers, New York (1962).

[**Ru2**] Rudin, W.: Real and Complex Analysis. McGraw-Hill Inc., Singapore (1987).

[**Sc**] Schoenberg, J.: Metric spaces and completely monotone functions. Annals of Mathematics **39**(4), 811–841 (1938).

[**Ste**] Stein, E. M.: Singular integrals and differentiability properties of functions. Princeton University Press, Princeton (1970).

[**SteW**] Stein, E. M., Weiss, G.: Fourier Analysis on Euclidean Spaces (PMS-32), Volume 32. Princeton University Press, Princeton (1971).

[**T1**] Tolsa, X.: BMO, H^1, and Calderón-Zygmund operators for non doubling measure. Mathematische Annalen, **319**(1), 89–149 (2001).

[**T2**] Tolsa, X.: Littlewood–Paley Theory and the $T(1)$ Theorem with Non-doubling Measures. Advances in Mathematics **164**(1), 57–116 (2001).

[**T3**] Tolsa, X.: Painlevé's problem and the semiadditivity of analytic capacity. Acta Mathematica **190**, 105–149 (2003).

[**T4**] Tolsa, X.: Calderón-Zygmund Capacities and Wolff Potentials on Cantor Sets. Journal of Geometric Analysis **21**, 195–223 (2011).

[**T5**] Tolsa, X.: Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón-Zygmund Theory. Birkhäuser Cham, Switzerland (2014).

[**U**] Ugaheri, T.: On the general potential and capacity. Japanese Journal of Mathematics **20**, 37–43 (1950).

[**Uy**] Uy, N. X.: Removable sets of analytic functions satisfying a Lipschitz condition. Arkiv för Matematik **17**(1–2), 19–27 (1979).

[**Va**] Vázquez, J. L.: Asymptotic behavior for the fractional heat equation in the Euclidean space. Complex Variables and Elliptic Equations **63**(7–8), 1216–1231 (2018).

[**Ve1**] Verdera, J.: BMO Rational Approximation and One-Dimensional Hausdorff Content. Transactions of the American Mathematical Society **297**(1), 283–304 (1986).

[**Ve2**] Verdera, J.: Classical potential theory and analytic capacity. Advanced courses of Mathematical Analysis II, World Scientific A, 174–192 (2007).

[**Vo**] Volberg, A.: Calderón-Zygmund Capacities and Operators on Nonhomogeneous Spaces. Volume 100, CBMS Regional Conference Series in Mathematics. American Mathematical Society, Rhode Island (2003).

[**Wa1**] Watson, N. A.: A theory of subtemperatures in several variables. Proceedings of the London Mathematical Society **26**(3), 385–417 (1973).

[**Wa2**] Watson, N. A.: Green functions, potentials, and the Dirichlet problem for the heat equation. Proceedings of the London Mathematical Society **33**(3), 251–298 (1976).

[**Wa3**] Watson, N. A.: Thermal capacity. Proceedings of the London Mathematical Society **37**(3), 342–362 (1978).

[**Wa4**] Watson, N. A.: Introduction to heat potential theory. Volume 18, Mathematical surveys and monographs. American Mathematical Society, Rhode Island (2012).