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Geometric properties of harmonic measure

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by

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Mathematics is, in my not-so-humble opinion, our most inexhaustible source of magic.

(math analogue of) Prof. Albus Dumbledore

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Abstract

This thesis focuses primarily on the study of the (harmonic and elliptic) Dirichlet boundary value problem. The final chapter treats an independent problem in geometric measure theory: the structure of the support of locally uniform and locally uniformly distributed measures.

Briefly speaking, given a (sufficiently regular) domain, the (harmonic) Dirichlet boundary value problem consists of finding a harmonic extension of a given function defined on the boundary. For continuous functions on the boundary, the solution is determined by the harmonic measure: the unique Radon probability measure that reconstructs the harmonic extension through integration of the boundary function. The elliptic Dirichlet boundary value problem and elliptic measure are defined similarly, with the Laplacian replaced by a second-order operator in divergence form with uniformly elliptic matrix.

The first three chapters investigate the Hausdorff dimension of elliptic measures in planar domains. Chapter I presents preliminary material and establishes fundamental results concerning elliptic measure. Chapter II focuses on two particular cases: symmetric matrices with determinant 1, and CDC domains. Through the application of quasiconformal mappings, we derive new bounds for the Hausdorff dimension that depend solely on the ellipticity constant. For matrices with additional regularity assumptions, we establish more refined dimensional estimates. These results extend the classical work of Makarov, Jones, and Wolff to the elliptic setting, while recovering their celebrated theorems in the harmonic case. In Chapter III, we extend Wolff's theorem to elliptic measures associated with Lipschitz matrices in Reifenberg flat domains, proving that these measures have σ-finite length provided small enough flatness depending on the ellipticity of the matrix.

Chapter IV addresses the L^2 solvability of the (harmonic) Dirichlet boundary value problem for domains with unbounded locally flat boundary. That is, we not only establish the existence of a harmonic extension for boundary data in L^2, but also prove that the L^2 norm of the nontangential maximal function of this extension is controlled by the L^2 norm of the boundary data. Following classical approaches, our analysis relies on layer potential techniques. We further extend these results to the L^p setting for p in a dimension-dependent range, and establish the $L^{p'}$ solvability for the corresponding Neumann problem.

The thesis concludes with Chapter V, on locally uniform and locally uniformly distributed measures. While the structure of (globally) uniform and (globally) uniformly distributed measures is well understood through work of Kirchheim, Kowalski, and Preiss (though some open problems remain), the local analogues are not yet fully characterized. We prove that for locally uniform measures, vanishing mean curvature (defined almost everywhere in the support in codimension 1) at a point forces flatness of the connected component that contains the point. Furthermore, we prove that for locally uniformly distributed measures, local flatness of a connected component implies its global flatness. These two results are presented in any codimension. These appear to be among the first results in a possible extension (if true) the Kirchheim and Preiss theorem on the real-analytic nature of globally uniformly distributed measures to this local setting. Moreover, we show that if a locally uniform measure satisfies an additional standard growth condition on balls centered in a neighborhood of a connected component of its support, then that connected component must be a linear variety.

Contents

Introduction

This introduction presents a general overview of the topics discussed in this thesis. It includes a survey of the results obtained, along with brief historical context for each area. The results are stated concisely to minimize technical preliminaries, with complete formulations referenced in the main text. More detailed introductions and specific historical background for each topic can be found in the corresponding chapter introductions.

This thesis focuses principally on the Dirichlet problem from two distinct perspectives: the Hausdorff dimension of harmonic/elliptic measures, and the L^p solvability of the Dirichlet problem in flat domains. The thesis concludes with an independent chapter examining the structure of locally uniform and locally uniformly distributed measures, a problem in geometric measure theory.

Dirichlet problem and elliptic measure

The Dirichlet boundary value problem is a central topic in harmonic analysis and the primary focus of this thesis. Given a domain $\Omega \subset \mathbb{R}^{n+1}$, the (harmonic) Dirichlet boundary value problem consists of finding, for any given function f defined on $\partial\Omega$, its harmonic extension $u = u_f : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega, \text{ in an appropriate sense.} \end{cases}$$

Perron's method yields a harmonic solution u_f for every given $f \in C(\partial\Omega)$. While this method applies to general domains, we emphasize that for sufficiently regular domains, the solution u_f satisfies

$$\lim_{\Omega \ni x \to \xi} u_f(x) = f(\xi) \text{ for all } \xi \in \partial\Omega.$$

For a fixed $p \in \Omega$, Perron's method defines a linear operator

$$\begin{array}{cccc} T: & C(\partial\Omega) & \to & \mathbb{R} \\ & f & \mapsto & u_f(p), \end{array}$$

which is bounded and positive by the maximum principle. The Riesz representation theorem then guarantees the existence and uniqueness of a Radon probability measure ω_Ω^p, the so-called harmonic measure, satisfying

$$u_f(p) = \int_{\partial\Omega} f(\xi)\, d\omega_\Omega^p(\xi) \text{ for every } f \in C(\partial\Omega).$$

This construction gives rise to the family of harmonic measures $\{\omega_\Omega^x\}_{x\in\Omega}$, yielding an explicit representation of u_f as

$$u_f(x) = \int_{\partial\Omega} f(\xi)\, d\omega_\Omega^x(\xi), \quad x \in \Omega.$$

Thus, the harmonic measure fully characterizes the solution to the Dirichlet problem as (roughly) formulated above.

The definition of the Dirichlet problem extends naturally to second-order operators in divergence form $L_A = -\operatorname{div}(A\nabla\cdot)$ with uniformly elliptic matrix A. Since Perron's method remains valid in this setting, the elliptic measure $\omega^p_{\Omega,A}$ can be defined analogously to the harmonic case. We refer to Chapter I.

Hausdorff dimension of elliptic measures

Given its connection to the Dirichlet problem, the geometric structure of harmonic and elliptic measures has been extensively studied. A particular focus of this work is the Hausdorff dimension of the elliptic measure $\omega^p_{\Omega,A}$ (for a fixed pole $p \in \Omega$), defined by

$$\dim_{\mathcal{H}} \omega^p_{\Omega,A} := \inf\{\dim_{\mathcal{H}} F : \omega^p_{\Omega,A}(F^c) = 0\}.$$

This topic has developed into an independent research direction.

In the harmonic case, $\omega := \omega^p_{\Omega,Id}$, Makarov proved in [Mak85] that for planar simply connected domains, the harmonic measure satisfies $\dim_{\mathcal{H}} \omega = 1$. For arbitrary planar domains, Jones and Wolff showed in [JW88] that $\dim_{\mathcal{H}} \omega \leq 1$. Wolff later refined this result in [Wol93], proving the existence of a set F with σ-finite $\mathcal{H}^1$ satisfying $\omega(F) = 1$. The higher-dimensional case remains almost completely open, as the value $\sup_{\Omega \subset \mathbb{R}^{n+1}} \dim_{\mathcal{H}} \omega$ is still unknown. Bourgain showed in [Bou87] that $\dim_{\mathcal{H}} \omega \leq n+1-b_n$ for some dimensional constant $b_n > 0$, and Wolff constructed in [Wol95] a domain $\Omega_n \subset \mathbb{R}^{n+1}$ such that $\dim_{\mathcal{H}} \omega_{\Omega_n} > n$.

Elliptic measures exhibit different behavior from harmonic measures. For every $\varepsilon > 0$, Sweezy constructed in [Swe92] (planar case) and [Swe94] (higher dimensional case) a domain $\Omega \subset \mathbb{R}^{n+1}$ and a uniformly elliptic matrix A such that $\dim_{\mathcal{H}} \omega_{\Omega,A} \geq n + 1 - \varepsilon$.

In order to understand geometric properties of elliptic measures $\omega_A = \omega^p_{\Omega,A}$ in planar domains, we establish elliptic analogues of the results by Makarov, Jones, and Wolff. This is achieved through two different approaches. First, in Chapter II, we employ quasiconformal mappings to derive dimensional properties for elliptic measures from classical harmonic measure results. Second, in Chapter III, we adapt Wolff's proof in [Wol93] to elliptic measures, when the domain is Reifenberg flat and the matrix has Lipschitz coefficients.

In Chapter II we consider two cases: either the matrix A is symmetric with determinant 1, or Ω is a CDC domain (see Definition I.5.1). Roughly speaking, a domain satisfies the capacity density condition, or it is CDC, if its complement is uniformly "fat" with respect to the scale.

Theorem I (See Theorems II.0.1 and II.0.5)**.** *Let $\Omega \subset \mathbb{R}^2$ be a domain, A be a uniformly elliptic matrix with constant $\lambda \geq 1$, and $K_\lambda \geq 1$ as defined in (II.0.2). If either*

- *A is symmetric with determinant 1, or*
- *Ω is CDC,*

then there is $F \subset \partial\Omega$ with σ-finite $\mathcal{H}^{\frac{2K_\lambda}{K_\lambda+1}}$ such that $\omega_{\Omega,A}(F) = 1$. If the domain is simply connected, then $\dim_{\mathcal{H}} \omega_{\Omega,A} \geq 2/(K_\lambda + 1)$.

For the cases described above, these results establish the analogue of Makarov's and Wolff's theorems for elliptic measures. We note that, as $K_1 = 1$ for the Laplacian, we recover the classical results for harmonic measure. Moreover, they reveal that in Sweezy's example [Swe92], the ellipticity of the matrix must necessarily degenerate as $\varepsilon \to 0$.

In the next two theorems we consider matrices with additional coefficient regularity.

Theorem II (See Theorem II.0.2). *Let $\Omega \subset \mathbb{R}^2$ be a domain and A be a uniformly elliptic matrix that is symmetric, has determinant* 1, *and has Hölder continuous coefficients*[(1)]. *Then there is $F \subset \partial\Omega$ with σ-finite $\mathcal{H}^1$ such that $\omega_{\Omega,A}(F) = 1$. If the domain is simply connected, then* $\dim_{\mathcal{H}} \omega_{\Omega,A} = 1$.

Theorem III (See Theorem II.0.6). *Let $\Omega \subset \mathbb{R}^2$ be a CDC domain and A be a uniformly elliptic matrix with continuous coefficients. Then* $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq 1$. *If the domain is simply connected, then* $\dim_{\mathcal{H}} \omega_{\Omega,A} = 1$.

In Chapter III we study elliptic measures arising from planar Reifenberg flat domains (see Definition III.1.4) and uniformly elliptic matrices with Lipschitz coefficients. Roughly speaking, a domain $\Omega \subset \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat if for every ball B centered at $\partial\Omega$ with radius $r < r_0$, the δr-neighborhood of a line through the center of B contains $B \cap \partial\Omega$. Under these assumptions, we obtain the σ-finite length of elliptic measures, even for nonsymmetric matrices.

Theorem IV (See Theorem III.0.1). *Let $\Omega \subset \mathbb{R}^2$ be a (δ, r_0)-Reifenberg flat domain and A be a uniformly elliptic matrix with Lipschitz coefficients. If δ is sufficiently small depending on the ellipticity of the matrix, then there is $F \subset \partial\Omega$ with σ-finite $\mathcal{H}^1$ such that $\omega_{\Omega,A}(F) = 1$.*

L^p Dirichlet and $L^{p'}$ Neumann problems

We return to the (harmonic) Dirichlet problem, in Ahlfors-David regular (ADR) domains (with surface measure $\sigma = \mathcal{H}^n|_{\partial\Omega}$) with the interior corkscrew condition. Informally, ADR regularity quantifies the n-dimensional nature in $\mathbb{R}^{n+1}$ of the boundary (see Section IV.1.2 for the definition), and the interior corkscrew condition provides a scale-invariant condition of openness (see Definition IV.1.2).

We now extend the setting from continuous functions on $\partial\Omega$ to boundary data in $L^p(\sigma)$ for some $1 < p < \infty$. In this case, the boundary condition $u_f = f$ on $\partial\Omega$ must be interpreted in the nontagential limit sense

$$\lim_{\substack{\Omega \ni z \to x \\ |z-x| < 2\operatorname{dist}(z,\partial\Omega)}} u_f(z) = f(x) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega,$$

see (IV.1.4). Furthermore, we study the L^p Dirichlet problem with the additional requirement (for the harmonic extension u_f with nontangential limit $f \in L^p(\sigma)$ σ-a.e.) that its nontangential maximal function (see (IV.1.2)), defined as

$$\mathcal{N}(u_f)(x) := \sup_{z \in \Omega : |z-x| < 2\operatorname{dist}(z,\partial\Omega)} |u_f(z)|, \quad x \in \partial\Omega,$$

satisfies

$$\|\mathcal{N}(u_f)\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}.$$

When this holds, we say that the Dirichlet problem is L^p solvable, or simply that (D_p) is solvable.

The L^p solvability of the Dirichlet problem is fundamentally connected to the properties of harmonic measure, and vice versa. Beyond the direct construction of harmonic extensions via harmonic measures, there exists the following precise relationship: (D_p) is solvable if and only if the harmonic measure satisfies a weak reverse Hölder inequality with exponent $p' = p/(p-1)$, the Hölder conjugate exponent. The solvability of the Dirichlet problem

[(1)]Thinner regularity cases are considered in Theorem II.0.3.

goes back to Dahlberg's 1977 work [Dah77], where he established the solvability of (D_2) for bounded Lipschitz domains using harmonic measure techniques. For further historical context on (D_p) solvability, see Chapter IV.

In Chapter IV we consider domains with unbounded boundary that fails to be sufficiently flat at only at finitely many dyadic boundary balls. The flatness conditions require control of both Jones' β_∞ numbers and the oscillation of the outward unit normal vector ν (see Remark IV.1.3). We refer to such domains as δ-$(s,S;R)$ domains, see Definition IV.0.1. Within this framework, we establish the solvability of (D_p).

Theorem V (See Theorem IV.0.2)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 2$, be a δ-$(s,S;R)$ domain. There exists $\varepsilon_n > 0$ such that for every $p \in (\frac{n}{n-1} - \varepsilon_n, \frac{2n}{n-1}]$, if δ is small enough depending on p and n, then (D_q) is solvable for all $q \geq p$.*

Our work extends (D_p) solvability results to the broader class of δ-$(s,S;R)$ domains, generalizing: bounded locally flat domains (known as regular SKT domains) by Hofmann, M. Mitrea, and Taylor [HMT10, Section 5], and unbounded domains with unbounded boundary that is flat at every scale, by Marín, Martell, D. Mitrea, I. Mitrea, and M. Mitrea [MMM+22a, Chapter 6]. Remark that both articles [HMT10, MMM+22a] deal with more general PDEs.

We employ the classical layer potential approach to boundary value problems. The interior double layer potential $\mathcal{D}$ is defined (see (IV.2.1)), for $g \in L^p(\sigma)$, as

$$\mathcal{D}g(x) \coloneqq \frac{1}{w_n} \int_{\partial\Omega} \frac{\langle \nu(y), y - x\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \Omega,$$

and the (boundary) double layer potential K is defined analogously via the principal value for $x \in \partial\Omega$ (see (IV.2.2)). For any $g \in L^p(\sigma)$, $\mathcal{D}g$ is harmonic in Ω, has nontangential limit $(\frac{1}{2}Id + K)g$ σ-a.e., and $\|\mathcal{N}(\mathcal{D}g)\|_{L^p(\sigma)} \lesssim \|g\|_{L^p(\sigma)}$. The method of layer potentials yields an explicit representation for u_f, provided the operator $\frac{1}{2}Id + K$ is invertible in $L^p(\sigma)$. In this setting, the solution is then obtained through the choice

$$g = \left(\frac{1}{2}Id + K\right)^{-1} f,$$

thus proving that (D_p) is solvable.

To establish the invertibility of $\frac{1}{2}Id + K$ in $L^p(\sigma)$ for δ-$(s,S;R)$ domains, we proceed in two steps. First, we prove that K is close to the set of compact operators (Definition IV.1.13), and second, applying the Fredholm alternative, we show its injectivity in $L^p(\sigma)$, for p in the range from our theorem. A key technical contribution of Chapter IV lies precisely in quantifying the closeness of K to the set of compact operators.

Theorem VI (See Theorem IV.0.4)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 2$, be a δ-$(s,S;R)$ domain and $1 < p < \infty$. For any given $\varepsilon > 0$, if δ is small enough depending on ε, p and n, then there is a compact operator T such that $\|K - T\|_{L^p(\sigma)} < \varepsilon$.*

The injectivity of $\frac{1}{2}Id + K$ in $L^p(\sigma)$ is established in Section IV.6. Notably, the proof requires only the 2-sided chord-arc condition (see Definition IV.1.7), rather than flatness assumptions. Unlike the divergence theorem approach for bounded domains, we instead use Moser type inequalities for harmonic functions with Dirichlet/Neumann vanishing data at the boundary.

We note that much of the framework developed for proving (D_p) solvability extends to establishing $L^{p'}$ solvability of the Neumann problem. Given $f \in L^{p'}(\sigma)$, it consists of finding a harmonic extension u_f such that its normal derivative converges nontangentially to f σ-a.e. (see (IV.1.5)) and $\|\mathcal{N}(\nabla u_f)\|_{L^{p'}(\sigma)} \lesssim \|f\|_{L^{p'}(\sigma)}$, with uniform constant. In this case, we say that $(N_{p'})$ is solvable.

The strategy for establishing that (N_p) is solvable is similar to the one for the Dirichlet problem. Arguing as above, we have that $-\frac{1}{2}Id + K^*$ is invertible in $L^{p'}(\sigma)$, and hence we can take

$$g = \left(-\frac{1}{2}Id + K^*\right)^{-1} f.$$

The choice $\mathcal{S}_{\text{mod}}g$ gives the solvability of (N_p), where $\mathcal{S}_{\text{mod}}$ is the modified single layer potential (see (IV.6.1)).

Theorem VII (See Theorem IV.0.3)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 2$, be a δ-$(s, S; R)$ domain. There exists $\varepsilon_n > 0$ such that for every $p \in (\frac{n}{n-1} - \varepsilon_n, \frac{2n}{n-1}]$, if δ is small enough depending on p and n, then $(N_{p'})$ is solvable.*

As in the solvability of the Dirichlet problem, the solution is explicitly constructed via the modified single layer potential. In contrast to (D_p), we cannot show extrapolation for the Neumann problem (though it likely holds, as in the bounded domain case).

Locally uniformly distributed measures

The thesis concludes with a purely geometric measure theory problem, focusing on geometric properties of the support

$$\operatorname{supp}\mu = \{x \in \mathbb{R}^{n+1} : \mu(B(x,r)) > 0 \text{ for every } r > 0\},$$

for a given locally k-uniform or locally uniformly distributed measure μ: A Radon measure μ is locally k-uniform (k integer) if

$$\mu(B(x,r)) = w_k r^k \text{ whenever } x \in \operatorname{supp}\mu \text{ and } 0 < r \leq 1,$$

where $w_k = \mathcal{H}^k(\{z \in \mathbb{R}^k : |z| < 1\})$, and a Radon measure μ is locally uniformly distributed if

$$0 < \mu(B(x,r)) = \mu(B(y,r)) < \infty \text{ whenever } x, y \in \operatorname{supp}\mu \text{ and } 0 < r \leq 1.$$

The structure of global versions of these measures (i.e., $0 < r \leq 1$ is replaced by $r > 0$) has been extensively studied. The most general results are due to Kirchheim, Kowalski and Preiss. Kowalski and Preiss showed in [KP87] that the support of globally k-uniform measures in $\mathbb{R}^{n+1}$ is contained in a quadratic variety, and in the codimensional 1 case $k = n$, they achieved a complete characterization: after a translation and rotation, either $\operatorname{supp}\mu = \{x_{n+1} = 0\}$ or (if $n \geq 3$) $\operatorname{supp}\mu = \{x_4^2 = x_1^2 + x_2^2 + x_3^2\}$. Other examples in higher codimension were recently constructed by Nimer [Nim22]. However, the characterization for arbitrary codimensions remains open. For globally uniformly distributed measures, it was proved by Kichheim and Preiss in [KP02] that their supports are analytic varieties.

It is natural to think that similar results may hold under the local condition. However, much less is known in this case. Kowalski and Preiss conjectured in [KP87, Conjecture 5.14] that each connected component of the support of a locally n-uniform measure in $\mathbb{R}^{n+1}$ is contained in the zero set of a quadratic polynomial. In the direction of proving this conjecture, and analogues of the results presented above, we show a unique continuation type property for locally k-uniform measures, in any codimension.

Theorem VIII (See Theorem V.0.2)**.** *If the mean curvature (whenever it is defined) of the support of a locally k-uniform measure vanishes at a point, then the connected component of* $\operatorname{supp}\mu$ *containing that point must be a k-plane.*

By the work of Preiss, Tolsa and Toro in [PTT09, Theorem 1.9], the best known result states that $\operatorname{supp}\mu$ is a k-dimensional $C^{1,\beta}$-manifold away from a closed set with zero Hausdorff measure $\mathcal{H}^k$. Combining this with [DKT01, Theorem 1.10] by David, Kenig and Toro, the mean curvature is well-defined μ-a.e. in the codimensional 1 case $k = n$.

As a consequence of the previous theorem, we derive a weak unique continuation type property for locally uniformly distributed measures.

Theorem IX (See Theorem V.0.3)**.** *Let μ be a locally uniformly distributed measure. If there is a ball $B(z_0, r_0)$ (with $z_0 \in \operatorname{supp}\mu$ and $r_0 > 0$) and a k-plane P such that $B(z_0, r_0) \cap \operatorname{supp}\mu = B(z_0, r_0) \cap P$, then the connected component of* $\operatorname{supp}\mu$ *containing z_0 is precisely the k-plane P.*

Our results provide evidence for local analogues of Kirchheim, Kowalski, and Preiss's theorems. Specifically, if such an analogue holds (i.e., if the support lies within the zero set of a quadratic polynomial or analytic function), then this function cannot have zero curvature anywhere unless it is the zero function, as in our theorems.

Furthermore, by adapting the arguments used in the strong unique continuation theorem above, we obtain the following flatness condition for locally k-uniform measures.

Theorem X (See Theorem V.0.4)**.** *Let μ be a locally k-uniform measure. If there is $z_0 \in \operatorname{supp}\mu$ with $\mu(B(x, r)) \leq w_k r^k$ for all $x \in B(z_0, 1)$ and $0 < r \leq 1$, then the connected component of* $\operatorname{supp}\mu$ *containing z_0 must be a k-plane.*

We remark that this was already known for globally k-uniform measures under the assumption that $\mu(B(x, r)) \leq w_k r^k$ for all $x \in \mathbb{R}^{n+1}$ and $0 < r < \infty$, see [Mat95, Theorem 17.3]. The proof in [Mat95, p. 235] relies on the known property that $\operatorname{supp}\mu \subset \{Q = 0\}$ for a quadratic polynomial Q. For locally k-uniform measures, however, we prove the theorem even though it remains open whether each connected component of $\operatorname{supp}\mu$ is contained in such a zero set, even in codimension 1 (see Conjecture V.0.1).

Structure of the thesis and chapter abstracts

The chapters of this thesis correspond to the articles contributed by the author, appearing in the following order: [GM24], [GMPT24] (joint work with Martí Prats and Xavier Tolsa), [GM25], and [EGM25] (joint work with Max Engelstein).

The article [EGM25] has been published in *The Journal of Geometric Analysis*, [GMPT24] has been accepted for publication in *Analysis and Mathematical Physics*, and [GM24, GM25] have been submitted for publication.

The results on the Hausdorff dimension of elliptic measures from [GMPT24, GM24] are organized as follows: Chapter I covers preliminary results, while Chapters II and III present the principal arguments from [GM24] for Theorems I to III, and from [GMPT24] for Theorem IV, respectively. The L^p solvability results in Theorems V to VII, established in [GM25], are treated in Chapter IV. Finally, Chapter V develops the theory on locally uniformly distributed measures from [EGM25], proving Theorems VIII to X.

In Chapter II, we obtain new bounds for the Hausdorff dimension of planar elliptic measure via the application of quasiconformal mappings, with these bounds depending solely on the ellipticity constant of the matrix. In fact, in our case studies, we find a quasiconformal mapping that relates the elliptic measure in a domain to the harmonic measure in its image domain, allowing us to deduce bounds for the dimension of the elliptic measure from the known results on the harmonic side. This extends previous works of Makarov, Jones and Wolff.

In Chapter III we show that, given a planar Reifenberg flat domain with small constant and a divergence form operator associated to a real (not necessarily symmetric) uniformly elliptic matrix with Lipschitz coefficients, the Hausdorff dimension of its elliptic measure is at most 1. More precisely, we prove that there exists a subset of the boundary with full elliptic measure and with σ-finite one-dimensional Hausdorff measure. For Reifenberg flat domains, this result extends a previous work of Wolff [Wol93] for the harmonic measure.

In Chapter IV we establish the solvability of the L^p-Dirichlet and $L^{p'}$-Neumann problems for the Laplacian for $p \in (\frac{n}{n-1} - \varepsilon, \frac{2n}{n-1}]$ for some $\varepsilon > 0$ in 2-sided chord-arc domains with unbounded boundary that is sufficiently flat at large scales and outward unit normal vector whose oscillation fails to be small only at finitely many dyadic boundary balls.

In Chapter V we show that the support of a locally k-uniform measure in $\mathbb{R}^{n+1}$ satisfies a kind of unique continuation property. As a consequence, we show that locally uniformly distributed measures satisfy a weaker unique continuation property. This continues work of Kirchheim and Preiss in [KP02] and David, Kenig and Toro in [DKT01] and lends additional evidence to the conjecture proposed by Kowalski and Preiss in [KP87] that each connected component of the support of a locally n-uniform measure in $\mathbb{R}^{n+1}$ is contained in the zero set of a quadratic polynomial.

Also in Chapter V, adapting the methods from [EGM25], we prove a flatness condition (Theorem X) which is not presented in the article(2): if a locally k-uniform measure is bounded by $w_k r^k$ when evaluated on balls of radius $0 < r \leq 1$ centered in a neighborhood of a connected component of its support, then that connected component must be a k-plane.

(2)After the journal publication of [EGM25], Max Engelstein and I realized that the methods developed therein could be adapted to prove this new result. I thank Max Engelstein for permitting the inclusion of this new result in this thesis.

General notation

Here, we present the general notation used throughout the thesis that is not otherwise introduced in the text.

$c, C \geq 1$	We use $c, C \geq 1$ to denote constants that may depend only on the dimension and the constants appearing in the hypotheses of the results, and whose values may change at each occurrence.
$a \lesssim b$, $a \approx b$	We write $a \lesssim b$ if there exists a constant $C \geq 1$ such that $a \leq Cb$, and $a \approx b$ if $C^{-1}b \leq a \leq Cb$.
$a \lesssim_\eta b$, $a \approx_\eta b$	If we want to stress the dependence of the constant on a parameter η, we write $a \lesssim_\eta b$ or $a \approx_\eta b$ meaning that $C = C(\eta) = C_\eta$.
$\mathbb{R}^{n+1}$	The ambient space is $\mathbb{R}^{n+1}$ with $n \geq 1$.
$\operatorname{diam} E$	The diameter of a set $E \subset \mathbb{R}^{n+1}$ is denoted by $\operatorname{diam} E \coloneqq \sup_{x,y\in E} \lvert x-y\rvert$. We allow $\operatorname{diam} E = \infty$ if E is unbounded.
$B_r(x)$, $B(x,r)$	We denote by $B_r(x)$ or $B(x,r)$ the open ball with center x and radius r, i.e., $B_r(x) = B(x,r) = \{y \in \mathbb{R}^{n+1} : \lvert y-x\rvert < r\}$. We denote $B_r \coloneqq B_r(0)$.
r_B, $r(B)$, c_B, $c(B)$	Given a ball B, we denote by r_B or $r(B)$ its radius, and by c_B or $c(B)$ its center.
tB	Given a ball B and $t > 1$, $tB \coloneqq B(c_B, tr_B)$.
$Q_r(x)$, $Q(x,r)$	We denote by $Q_r(x)$ or $Q(x,r)$ the open cube with center x and side length $2r$, i.e., $Q_r(x) = Q(x,r) = \{y \in \mathbb{R}^{n+1} : \lvert y_i - x_i\rvert < r \text{ for all } 1 \leq i \leq n+1\}$.
$\ell(Q)$, c_Q, $c(Q)$	Given a cube Q, we denote by $\ell(Q)$ its side length, and by c_Q or $c(Q)$ its center. That is, $Q = Q(c_Q, \ell(Q)/2)$.
tQ	Given a cube Q and $t > 1$, $tQ \coloneqq Q(c_Q, t\ell(Q)/2)$, that is, $c_{tQ} = c_Q$ and $\ell(tQ) = t\ell(Q)$.
$C^{0,\alpha}$, $C^{0,1}$	We say that a function f is Hölder continuous with exponent $\alpha \in (0,1]$ in a set U, or briefly $C^{0,\alpha}(U)$, if $$C_\alpha \coloneqq \sup_{x,y\in U, x\neq y} \frac{\lvert f(x)-f(y)\rvert}{\lvert x-y\rvert^\alpha} < \infty.$$ The constant C_α is called the Hölder seminorm of f in U. For shortness we write C^α instead of $C^{0,\alpha}$ if $\alpha \in (0,1)$, and when $\alpha = 1$ we say "Lipschitz continuous". In this case we write C_L instead of C_1, i.e., the Lipschitz seminorm is $$C_L \coloneqq \sup_{x,y\in U, x\neq y} \frac{\lvert f(x)-f(y)\rvert}{\lvert x-y\rvert} < \infty.$$
κ-Lipschitz	We say that a function f is κ-Lipschitz in U if $\lvert f(x)-f(y)\rvert \leq \kappa\lvert x-y\rvert$ for all $x, y \in U$.

$A \in X$	Given a space of function X, we say that a matrix $A = (a_{ij})_{i,j=1}^{n+1}$ belongs to X, $A \in X$ for short, if $a_{ij} \in X$ for all $1 \leq i, j \leq n+1$.
$A \in C^{0,\alpha}$, $A \in C^{0,1}$	The Hölder seminorm of a matrix $A \in C^{0,\alpha}(U)$ is $$C_\alpha \coloneqq \sup_{\substack{x,y\in U, x\neq y\\ 1\leq i,j\leq n+1}} \frac{\lvert a_{ij}(x) - a_{ij}(y)\rvert}{\lvert x-y\rvert^\alpha} < \infty.$$ Similarly, the Lipschitz seminorm of a matrix $A \in C^{0,1}(U)$ is $$C_\alpha \coloneqq \sup_{\substack{x,y\in U, x\neq y\\ 1\leq i,j\leq n+1}} \frac{\lvert a_{ij}(x) - a_{ij}(y)\rvert}{\lvert x-y\rvert} < \infty.$$
$\mathbf{1}_E$	We denote the characteristic function of a set E by $\mathbf{1}_E$.
$\mathcal{D}(\mathbb{R}^{n+1})$	Denote $\mathcal{D}(\mathbb{R}^{n+1})$ the standard dyadic grid. That is, $\mathcal{D}(\mathbb{R}^{n+1}) = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k(\mathbb{R}^{n+1})$ where $\mathcal{D}_k(\mathbb{R}^{n+1})$ is the collection of all cubes of the form $$\left\{x \in \mathbb{R}^{n+1} : m_i 2^{-k} \leq x_i < (m_i+1)2^{-k} \text{ for } i = 1,\dots,n+1\right\}$$ where $m_i \in \mathbb{Z}$.
$U_t(E)$	Given $t > 0$ and a set $E \subset \mathbb{R}^{n+1}$, we write $U_t(E) \coloneqq \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x,E) < t\}$ for the t-neighborhood E.
$\mathcal{H}^h$, $\mathcal{H}^h_\infty$, $\mathcal{H}^\alpha$, $\mathcal{H}^\alpha_\infty$	Given a continuous nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$, the h-Hausdorff measure (resp. content) is denoted by $\mathcal{H}^h$ (resp. $\mathcal{H}^h_\infty$), see [Mat95, Section 4.9] for instance. It is clear that $\mathcal{H}^h_\infty(E) \leq \mathcal{H}^h(E)$ for any set E. Despite the converse inequality does not hold, $\mathcal{H}^h_\infty(E) = 0$ implies $\mathcal{H}^h(E) = 0$. Although $\mathcal{H}^h_\infty$ is not measure, we write $\mathcal{H}^h \ll \mathcal{H}^h_\infty \ll \mathcal{H}^h$. If $h(r) = r^\alpha$, $0 \leq \alpha < \infty$, we simply write $\mathcal{H}^\alpha$ and $\mathcal{H}^\alpha_\infty$.
$\mu\lvert_S$	$\mu\lvert_S$ denotes the restriction of the measure μ to a set $S \subset \mathbb{R}^{n+1}$ defined as $\mu\lvert_S(E) \coloneqq \mu(S \cap E)$ for $E \subset \mathbb{R}^{n+1}$.
$m_E f$	Given a measure μ and a set E, if $\mu(E) \neq 0$ then we denote $m_E f \coloneqq ⨍_E f\, d\mu \coloneqq \frac{1}{\mu(E)} \int_E f\, d\mu$ for $f \in L^1_{\text{loc}}(\mu)$.

CHAPTER I

Preliminaries on elliptic measure

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set, and let $A \in \mathbb{R}^{(n+1)\times(n+1)}(\Omega)$ be a uniformly elliptic matrix. That is, a real and measurable matrix $A(\cdot) = (a_{ij}(\cdot))_{1\leq i,j\leq n+1}$ such that there exists $\lambda \geq 1$, the so-called ellipticity constant, with

$$\lambda^{-1}|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \text{ for all } \xi \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \Omega, \tag{I.0.1a}$$

$$\langle A(x)\xi, \eta\rangle \leq \lambda|\xi||\eta| \text{ for all } \xi, \eta \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \Omega. \tag{I.0.1b}$$

The uniform ellipticity condition implies that the matrix has bounded coefficients. We consider the second-order operator in divergence form $L_A u := -\operatorname{div}(A\nabla u)$ in Ω, understood in the weak sense, see Definition I.1.1. We say that a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is L_A-harmonic in Ω if $L_A u = 0$ in Ω.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set (bounded if $n \geq 2$) and fix a point $p \in \Omega$. Consider the operator

$$\begin{array}{rccc} T: & C(\partial\Omega) & \to & \mathbb{R} \\ & f & \mapsto & u_f^{L_A}(p), \end{array}$$

where $u_f^{L_A}$ is the L_A-harmonic extension of f in Ω via Perron's method. The operator T is linear, and by the maximum principle for L_A-harmonic functions with uniformly elliptic matrices (see [GT01, p. 46] for example), it is also bounded and positive.

The elliptic (L_A-harmonic) measure in Ω with pole $p \in \Omega$ is the unique Radon probability measure $\omega^p_{\Omega,A}$ (by the Riesz representation theorem) such that

$$u_f^{L_A}(p) = \int_{\partial\Omega} f(\xi)\, d\omega^p_{\Omega,A}(\xi) \text{ for every } f \in C(\partial\Omega).$$

Via the Perron method construction, although the characteristic function $\mathbf{1}_E$ of a set $E \subset \partial\Omega$ is not continuous in $\partial\Omega$, the function $x \mapsto \omega^x_{\Omega,A}(E)$ is L_A-harmonic in Ω. For a detailed construction of elliptic/harmonic measures, see [HKM06, Section 11]. If the set Ω, the matrix A, or the pole $p \in \Omega$ is clear from the context, we will omit it in $\omega^p_{\Omega,A}$.

Note that by the Harnack inequality of positive L_A-harmonic functions, given any two poles $x, y \in \Omega$, if Ω is connected, we have

$$\omega^x_{\Omega,A} \ll \omega^y_{\Omega,A} \ll \omega^x_{\Omega,A}.$$

Therefore, we can omit the pole from now on. Given two measures μ and ν on X, we recall that μ is absolutely continuous with respect to ν, we write $\mu \ll \nu$ in this case, if $\mu(S) = 0$ for all $S \subset X$ with $\nu(S) = 0$.

Harmonic and elliptic measure theory has seen significant development recently, particularly concerning the rectifiability of the boundary and dimensional bounds. For instance, when the boundary of the domain has codimension one, the relation between harmonic measure and rectifiability of the boundary has been studied by several authors. Roughly speaking, under some topological assumptions on the domain $\Omega \subset \mathbb{R}^{n+1}$, if $\partial\Omega$

is n-rectifiable, then $\omega_{\Omega,Id} \ll \mathcal{H}^n|_{\partial\Omega}$, see [ABHM19, AHM⁺20], and if $\partial\Omega$ is purely unrectifiable, then $\omega_{\Omega,Id}$ is singular with respect to $\mathcal{H}^n|_{\partial\Omega}$, see [AHM⁺16, AMT17]. Similar results hold for elliptic measure if the uniformly elliptic matrix is close to the identity, see [KP01, HMT17, HMM⁺21, AGMT23, HMM24]. The closeness of the matrix to the identity is usually measured in terms of a suitable Carleson measure condition on the gradient or the oscillation of the matrix. However, for general uniformly elliptic matrices, the elliptic measure can behave completely differently from the harmonic case, see examples [CFK81, MM81, Swe92, DM21] in the planar case.

In Chapters II and III, we study the Hausdorff dimension of elliptic measures arising from uniformly elliptic matrices. This is defined as

$$\dim_{\mathcal{H}} \omega_{\Omega,A} := \inf \{\dim_{\mathcal{H}} F : \omega_{\Omega,A}(F^c) = 0\}.$$

Naturally $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq \dim_{\mathcal{H}} \partial\Omega$. The Hausdorff dimension of the harmonic measure (i.e., with $\omega = \omega_{Id}$) has been studied by several authors, both in the plane and in higher dimensions, showing a different behavior in each case.

Let us introduce some notation to present the forthcoming results in Chapters I to III. Given a measure μ we will say that $\omega_{\Omega,A}$ has σ-finite with respect to μ if there is a set $F \subset \partial\Omega$ satisfying $\omega_{\Omega,A}(F) = 1$ and with σ-finite μ, that is, F is of the form $F = \bigcup_{i\geq 1} F_i$ with $\mu(F_i) < \infty$. For short we will write that $\omega_{\Omega,A}$ has σ-finite μ. The measure $\mathcal{H}^1$ will also be referred to as length.

In the plane, Carleson proved $\dim_{\mathcal{H}} \omega \leq 1$ for "snowflake type" sets and $\dim_{\mathcal{H}} \omega < 1$ for some self similar "Cantor type" sets, both results in [Car85]. Makarov showed in [Mak85, Theorem 1] that there is a constant $C_M > 0$ such that for any simply connected domain Ω, its harmonic measure ω_Ω satisfies $\omega_\Omega \ll \mathcal{H}^{r\exp(C_M\sqrt{\log r^{-1}\log\log\log r^{-1}})}$. This gives the lower bound $\dim_{\mathcal{H}} \omega \geq 1$. Actually, Makarov also proved in [Mak85, Theorem 3] that $\dim_{\mathcal{H}} \omega = 1$ for simply connected domains. The upper bound $\dim_{\mathcal{H}} \omega \leq 1$ with no assumptions on the domain was shown by Jones and Wolff in [JW88], and later it was improved by Wolff in [Wol93] by proving that there is a subset $F \subset \partial\Omega$ with full harmonic measure $\omega(F) = 1$ and σ-finite length.

For future reference, we write here the precise statements of Makarov[(I.1)] and Wolff mentioned above.

Theorem I.0.1 (Makarov)**.** *There exists a universal constant $C_M > 0$ such that*

$$\omega_\Omega \ll \mathcal{H}^{r\exp\left(C_M\sqrt{\log r^{-1}\log\log\log r^{-1}}\right)}$$

for every simply connected domain $\Omega \subset \mathbb{R}^2$.

Theorem I.0.2 (Wolff)**.** *For any domain $\Omega \subset \mathbb{R}^2$, ω_Ω has σ-finite length.*

In the same direction for higher dimensions, in $\mathbb{R}^{n+1}$ with $n \geq 2$, Bourgain proved in [Bou87] that there exists a dimensional constant $b_n > 0$ such that $\dim_{\mathcal{H}} \omega \leq n + 1 - b_n$. Contrary to the planar case, where we can take $b_1 = 1$ with optimal value by the work of Jones and Wolff [JW88], Wolff constructed a domain $\Omega_n \subset \mathbb{R}^{n+1}$ in [Wol95] satisfying $\dim \omega_{\Omega_n} > n$, that is, $b_n < 1$. In a recent work [BG24], Badger and Genshaw refined the proof in [Bou87] to find estimates on the Bourgain constant $b_n \in (0, 1)$. It remains a difficult open problem in the area to determine the optimal value $b_{n,\mathrm{opt}} := n + 1 - \sup\{\dim \omega_\Omega : \Omega \subset \mathbb{R}^{n+1}\} \in (0, 1)$.

(I.1) In Chapter II we will write $\omega_\Omega \ll \mathcal{H}^{\varphi_{1,C_M}}$, where $\varphi_{\cdot,\cdot}$ is the function defined in (II.0.1).

From the results in the previous paragraphs we have that $\dim_{\mathcal{H}} \omega < n+1$ (with some precise gap), but possibly $\dim_{\mathcal{H}} \omega = \dim_{\mathcal{H}} \partial\Omega$. In fact, the situation $\dim_{\mathcal{H}} \omega < \dim_{\mathcal{H}} \partial\Omega$ holds for many non-trivial domains. This phenomenon is frequently called the "dimension drop" for harmonic measure. The dimension drop is closely related to the results mentioned above. Indeed, similar techniques to the ones in [Bou87, JW88] are used in some of the following articles. The first work in this direction is due to Kaufman and Wu [KW85] for the planar $\log 4/\log 3$-dimensional Koch snowflake. Subsequently, Carleson [Car85] extended this result to self-similar Cantor sets in the plane. For this type of domains, see Makarov and Volberg [MV86], Batakis [Bat96], and also [Vol92, Vol93]. Jones and Wolff showed that the dimension drop happens for some uniform and disconnected planar domains, see Theorem 2.1 in [GM05, Section IX.2]. For IFS domains (iterated functions systems), see Urbański and Zdunik [UZ02], and Batakis and Zdunik [BZ15].

For general AD-regular domains of fractional dimension, one may ask if the dimension drop happens. It holds on uniformly "non-flat" AD-regular domains with codimension smaller than 1, as shown by Azzam in [Azz20]. Recently, for $n \geq 1$ and $s \in [n-\frac{1}{2}, n)$, the third author showed in [Tol24] that the dimension drop occurs for higher codimensional s-AD-regular subsets of C^1 n-dimensional manifolds in $\mathbb{R}^{n+1}$. In contrast, David, Jeznach and Julia proved in [DJJ23] that this last result may fail for s close enough to $n-1$.

The situation is different for uniformly elliptic matrices. In [Swe92] in the planar case and in [Swe94] in higher dimensions, for every $\varepsilon > 0$, Sweezy constructed a domain $\Omega \subset \mathbb{R}^{n+1}$ and an elliptic operator in divergence form L_A whose associated elliptic measure $\omega_{\Omega,A}$ satisfies $\dim_{\mathcal{H}} \omega_{\Omega,A} \geq n+1-\varepsilon$. However, as ε becomes smaller, the ellipticity constant of the resulting matrix increases. Such planar domains and elliptic measures are constructed by the push forward under quasiconformal mappings, and the higher dimensional analog is deduced from the planar case.

Using a new approach in the planar case, David and Mayboroda in [DM21] constructed an elliptic operator $L = -\operatorname{div} a\nabla$, where $a \approx 1$ is a continous scalar function on the complementary of the 1-dimensional four corner Cantor set $C \subset \mathbb{R}^2$, whose elliptic measure $\omega_{\mathbb{R}^2\setminus C, aId}$ is comparable to $\mathcal{H}^1|_C$, the one-dimensional Hausdorff measure on the Cantor set. Operators of this form are the so-called "good elliptic operators" in the terminology used in [DM21]. Following the same strategy, for $1 < d < \log 4/\log 3$, Perstneva in [Per25] constructed a "good elliptic operator" $L_{a_d Id}$ on the (unbounded component of the) complementary of a d-dimensional Koch-type snowflake $S_d \subset \mathbb{R}^2$ satisfying that $\omega_{\mathbb{R}^2\setminus S_d, a_d Id}$ is comparable to $\mathcal{H}^d|_{S_d}$.

After Sweezy's results, it is natural to ask which conditions on the matrix imply the analogous result of [JW88, Wol93] for the elliptic case. In Chapters II and III we study the metric properties of the elliptic measure when assuming regularity conditions on the matrix A. For other results in this line, see for example [PPT21], about the rectifiability of the elliptic measure for Hölder matrices.

I.1. Partial differential equations

We want to study the elliptic equation

$$L_A u(x) := -\operatorname{div}(A(\cdot)\nabla u(\cdot))(x) = 0, \tag{I.1.1}$$

which should be understood in the distributional sense. We simply write L instead of L_A when the matrix is clear from the context.

Definition I.1.1. We say that a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a solution of (I.1.1), or L_A-harmonic, in an open set $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, if

$$\int \langle A(y)\nabla u(y), \nabla\varphi(y)\rangle \, dy = 0, \text{ for all } \varphi \in C_c^\infty(\Omega).$$

By the De Giorgi-Nash-Moser theorem a solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of (I.1.1) is locally Hölder continuous. If the matrix has Hölder regularity then the solution is locally $C^{1,\beta}$ for some $\beta \in (0,1)$, see [HL11, Theorem 3.13]. Assuming Lipschitz regularity of the coefficients, L_A-harmonic functions enjoy more regularity. More precisely, weak solutions of $-\operatorname{div} A\nabla u = 0$ with A Lipschitz are twice weakly differentiable, and there is a "Caccioppoli type" inequality for second derivatives.

Theorem I.1.2 (See [GT01, Theorem 8.8])**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $L_A u = 0$, see (I.1.1), in $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, where A is uniformly elliptic and Lipschitz continuous in Ω. Then for any subdomain $\Omega' \subset\subset \Omega$, we have $u \in W^{2,2}(\Omega')$ and*

$$\|u\|_{W^{2,2}(\Omega')} \leq C\|u\|_{W^{1,2}(\Omega)}, \tag{I.1.2}$$

where C depends on the dimension, the ellipticity constant and Lipschitz constant of A, and the value $\operatorname{dist}(\Omega', \partial\Omega)$.

If, in addition, the domain has boundary of class C^2, then L_A-harmonic functions are globally in $W^{2,2}$.

Theorem I.1.3 (See [GT01, Theorem 8.12])**.** *Assume, in addition to the hypotheses of Theorem I.1.2, that Ω is bounded with $\partial\Omega$ of class C^2 and that there exists a function $\varphi \in W^{2,2}(\Omega)$ for which $u - \varphi \in W^{1,2}_0(\Omega)$. Then we have also $u \in W^{2,2}(\Omega)$ and*

$$\|u\|_{W^{2,2}(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\varphi\|_{W^{2,2}(\Omega)}\right),$$

where C depends on the dimension, the ellipticity constant and Lipschitz constant of A, and $\partial\Omega$.

Remark I.1.4. Let $U \subset \Omega$ an open subset. Assuming $A \in C^{0,1}(U)$, then any weak L_A-harmonic function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (in fact $u \in W^{2,2}_{\mathrm{loc}}(U)$ by Theorem I.1.2) satisfies $\operatorname{div} A\nabla u = 0$ a.e. in U. Indeed, since the matrix A is differentiable a.e. (by Rademacher's theorem) and $u \in W^{2,2}_{\mathrm{loc}}(U)$ (by Theorem I.1.2), we have $\operatorname{div} A\nabla u \in L^1_{\mathrm{loc}}(U)$. Moreover, since u is L_A-harmonic then $\int \operatorname{div} A\nabla u \cdot \psi = \int A\nabla u \nabla\psi = 0$ for any $\psi \in C_c^\infty(U) \subset C_c^\infty(\Omega)$. By the fundamental lemma of calculus of variations[(I.2)] we conclude $\operatorname{div} A\nabla u = 0$ a.e. in U.

The following theorem about the Hölder continuity of L_A-harmonic functions up to the boundary of regular enough domains will allow us (later in Lemma III.3.2) to bound the elliptic measure on a specific domain (an annulus) by means of studying the Green function near the boundary.

Theorem I.1.5 (See [GT01, Corollary 8.36])**.** *Let T be a (possibly empty) $C^{1,\alpha}$ boundary portion of a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and suppose $u \in W^{1,2}(\Omega)$ is a weak solution*

[(I.2)] The fundamental lemma of calculus of variations: If U open set, $f \in L^1_{\mathrm{loc}}(U)$ and $\int f\phi = 0$ for any $\phi \in C_c^\infty(U)$, then $f = 0$ a.e. in U.

of (I.1.1) in Ω, where $A \in C^\alpha$ $(0<\alpha<1)$, such that $u=0$ on T (in the sense of $W^{1,2}(\Omega)$). Then $u \in C^{1,\alpha}(\Omega\cup T)$, and for any $\Omega' \subset\subset \Omega \cup T$ we have

$$\|u\|_{C^{1,\alpha}(\Omega')} := \|u\|_{1;\Omega'} + [u]_{1,\alpha;\Omega'} \leq C \sup_{\Omega} |u|,$$

where

$$\|u\|_{1;\Omega'} := \sup_{\Omega'} |u| + \sup_{\Omega'} |\nabla u|, \quad [u]_{1,\alpha;\Omega'} := \sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\alpha},$$

for C depending on n, the value of $\operatorname{dist}(\Omega', \partial\Omega\setminus T)$, the $C^{1,\alpha}$ character of T, and the ellipticity constants and the Hölder norm of the matrix A.

Since bounded Lipschitz functions are also Hölder continuous for any exponent $\alpha \in (0,1)$, the previous theorem remains true for matrices with bounded Lipschitz coefficients.

I.2. Wiener regular domains

We recall the definition of Wiener regular domains.

Definition I.2.1 (Wiener regular)**.** Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a domain and A be a real and uniformly elliptic matrix. We say that $\xi \in \partial\Omega$ is a Wiener regular point for L_A if for every $f \in C(\partial\Omega)$, the L_A-harmonic extension $u_f^{L_A}$ of f in Ω via Perron's method satisfies $\lim_{\Omega\ni x\to\xi} u_f^{L_A}(x) = f(\xi)$. We say that Ω is Wiener regular for L_A if every point $\xi \in \partial\Omega$ is a Wiener regular point for L_A.

Remark I.2.2. In fact, the characterization of Wiener regular points does not depend on the operator L_A, see [LSW63]. Therefore, we say that a point $\xi \in \Omega$ is Wiener regular if and only if it is Wiener regular for some (and hence for all) L_A.

In the following lemma, we prove a Markov type property for elliptic measure in Wiener regular domains.

Lemma I.2.3. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a Wiener regular domain and A be a real uniformly elliptic (not necessarily symmetric) matrix. Let $\widetilde{\Omega} \subset \Omega$ be a Wiener regular domain. For any Borel set $E \subset \partial\Omega \cap \partial\widetilde{\Omega}$ and $p \in \widetilde{\Omega}$, there holds*

$$\omega^p_{\Omega,A}(E) - \omega^p_{\widetilde{\Omega},A}(E) = \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi_{\Omega,A}(E)\, d\omega^p_{\widetilde{\Omega},A}(\xi).$$

Proof. During the proof we write $\omega = \omega_{\Omega,A}$ and $\widetilde{\omega} = \omega_{\widetilde{\Omega},A}$.

Fixed $p \in \widetilde{\Omega}$, for $m \geq 1$, by the inner (just for Borel sets) and outer regularity of Radon measures, let $K_m \subset E$ be a compact set and $U_m \supset E$ be an open set such that

$$\begin{aligned} &\omega^p(U_m) - 1/m \leq \omega^p(E) \leq \omega^p(K_m) + 1/m, \text{ and} \\ &\widetilde{\omega}^p(U_m) - 1/m \leq \widetilde{\omega}^p(E) \leq \widetilde{\omega}^p(K_m) + 1/m. \end{aligned} \tag{I.2.1}$$

Moreover, we take $K_m \subset K_{m+1}$ and $U_m \supset U_{m+1}$ by redefining the sequences suitably. Finally, take $K := \bigcup_{m\geq 1} K_m$ and $U := \bigcap_{m\geq1} U_m$.

Let $\varphi = \varphi_m \in C_c(U_m)$ be such that $\mathbf{1}_{K_m} \leq \varphi \leq \mathbf{1}_{U_m}$, and let $u = u_m$ and $\widetilde{u} = \widetilde{u}_m$ denote the L_A-harmonic extension of φ in Ω and $\widetilde{\Omega}$ respectively. By the monotonicity of the integral,

$$\omega^\xi(K_m) \le u(\xi) \le \omega^\xi(U_m), \text{ for all } \xi \in \Omega, \text{ and} \tag{I.2.2a}$$

$$\widetilde{\omega}^\xi(K_m) \le \widetilde{u}(\xi) \le \widetilde{\omega}^\xi(U_m), \text{ for all } \xi \in \widetilde{\Omega}. \tag{I.2.2b}$$

Using (I.2.1), we get

$$|\omega^p(E) - \widetilde{\omega}^p(E) - (u(p) - \widetilde{u}(p))| \le 2/m. \tag{I.2.3}$$

On the other hand, note that $u(\xi) - \widetilde{u}(\xi) = (u(\xi) - \varphi(\xi)) \cdot \mathbf{1}_{\partial\widetilde{\Omega}\setminus\partial\Omega}(\xi)$ if $\xi \in \partial\widetilde{\Omega}$, and in particular, writing $u - \widetilde{u}$ as the L_A-harmonic extension in $\widetilde{\Omega}$ of its boundary values in $\partial\widetilde{\Omega}$ and evaluating at the point $p \in \Omega$ we get

$$u(p) - \widetilde{u}(p) = \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} (u(\xi) - \varphi(\xi))\, d\widetilde{\omega}^p(\xi).$$

From this, (I.2.2a) and $0 \le \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \varphi(\xi)\, d\widetilde{\omega}^p(\xi) \le \widetilde{\omega}^p(U_m \setminus K_m) \le 2/m$, we therefore obtain

$$\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi(K_m)\, d\widetilde{\omega}^p(\xi) - \frac{2}{m} \le u(p) - \widetilde{u}(p) \le \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi(U_m)\, d\widetilde{\omega}^p(\xi),$$

which together with (I.2.3) gives

$$\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi(K_m)\, d\widetilde{\omega}^p(\xi) - \frac{4}{m} \le \omega^p(E) - \widetilde{\omega}^p(E) \le \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi(U_m)\, d\widetilde{\omega}^p(\xi) + \frac{2}{m}.$$

Note that the set $K = \bigcup_{m\ge1} K_m \subset E$ satisfies $\omega^p(E) = \omega^p(K)$, and since K is measurable, then $\omega^p(E \setminus K) = 0$. Since elliptic measures are mutually absolutely continuous for any two different poles, this implies that $\omega^\xi(E \setminus K) = 0$ for any $\xi \in \Omega$. Again, since K is measurable we obtain that $\omega^\xi(E) = \omega^\xi(K)$ for any $\xi \in \Omega$. By the same argument, using now that E is Borel and so measurable for every ω^ξ with $\xi \in \Omega$, the set $U = \bigcap_{m\ge1} U_m$ satisfies $\omega^\xi(E) = \omega^\xi(U)$ for any $\xi \in \Omega$.

Taking $m \to \infty$, by the monotone convergence theorem, the equation above becomes

$$\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \lim_{m\to\infty} \omega^\xi(K_m)\, d\widetilde{\omega}^p(\xi) \le \omega^p(E) - \widetilde{\omega}^p(E) \le \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \lim_{m\to\infty} \omega^\xi(U_m)\, d\widetilde{\omega}^p(\xi),$$

and the lemma follows since $\lim_{m\to\infty} \omega^\xi(K_m) = \omega^\xi(K) = \omega^\xi(E) = \omega^\xi(U) = \lim_{m\to\infty} \omega^\xi(U_m)$. □

I.3. The fundamental solution and the Green function

We denote by $\mathcal{E}_x^A(y)$ the fundamental solution with pole at x for L_A in $\mathbb{R}^{n+1}$, $n \ge 1$, so that $L_A \mathcal{E}_x^A(\cdot) = \delta_x$ in the distributional sense, where δ_x is the Dirac mass at the point $x \in \mathbb{R}^{n+1}$. We write $\mathcal{E}_x(y)$ when the matrix A is clear from the context. For a construction of the fundamental solution for real and uniformly elliptic matrices we refer to [HK07] for higher dimensions, $\mathbb{R}^{n+1}$ with $n \ge 2$, and [KN85, Appendix] for the planar case.

In higher dimensions the fundamental solution behaves "in many senses" as in the harmonic case, see [HK07] for more details, but in the plane the situation is more delicate due to the change of sign of $\mathcal{E}_0^{Id}(x) = \log|x|$ in $|x| = 1$, the fundamental solution of the Laplacian in the plane.

Here we collect the formal definition and some properties of the fundamental solution in the plane.

Definition I.3.1 ([KR09, Definition 2.5])**.** A function $\mathcal{E}_x : \mathbb{R}^2 \to \mathbb{R}$ is called a fundamental solution for $L_A = \operatorname{div} A\nabla\cdot$ with pole at x if

(1) $\mathcal{E}_x \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x\}) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for all $p < 2$, and
$$\int_{\mathbb{R}^2} \langle A(z)\nabla\mathcal{E}_x(z), \nabla\varphi(z)\rangle\, dz = -\varphi(x), \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^2),$$

(2) $|\mathcal{E}_x(y)| = \mathcal{O}(\log|x-y|)$ as $|y| \to \infty$.

The following result controls the fundamental solution far from the pole similarly as the fundamental solution for the harmonic case.

Theorem I.3.2 ([KR09, Theorem 2.6])**.** *For each $x \in \mathbb{R}^2$ there exists a unique (modulo an additive constant) fundamental solution $\mathcal{E}_x$ for L_A with pole at x, and positive constants C_1, C_2, $R_1 < 1 < R_2$, which depend only on λ, such that*
$$\begin{aligned} C_1\log(1/|x-y|) \leq -\mathcal{E}_x(y) &\leq C_2\log(1/|x-y|) \text{ for } |x-y| < R_1, \text{ and}\\ C_1\log(|x-y|) \leq \mathcal{E}_x(y) &\leq C_2\log(|x-y|) \quad \text{ for } |x-y| > R_2.\end{aligned}$$

From the previous result and the maximum principle we obtain the following pointwise bound.

Corollary I.3.3. *$|\mathcal{E}_x(y)| \lesssim 1 + |\log|x-y||$ for all $x, y \in \mathbb{R}^2$, where the constant depends on λ.*

Proof. For $|x-y| < R_1$ and $|x-y| > R_2$, Theorem I.3.2 gives $|\mathcal{E}_x(y)| \lesssim |\log|x-y|| \leq 1 + |\log|x-y||$. In $\mathcal{A} = \{y \in \mathbb{R}^2 : R_1 < |x-y| < R_2\}$ we have $L\mathcal{E}_x(y) = 0$, and hence by the maximum principle we obtain $C_2\log R_1 \leq \mathcal{E}_x(\cdot) \leq C_2\log R_2$ in the annulus $\mathcal{A}$. So $|\mathcal{E}_x(y)| \lesssim 1 \leq 1 + |\log|x-y||$. ☐

We have the following relation between the fundamental solutions of the operators with matrices A and A^T. The same holds in higher dimensions, see [HK07, (3.43)].

Lemma I.3.4 ([KR09, Lemma 2.7])**.** *Fix $x, y \in \mathbb{R}^2$. Let $\mathcal{E}_x$ be the fundamental solution for an elliptic operator L_A with pole at x, and $\mathcal{E}_y^T$ be the fundamental solution to the adjoint operator L_{A^T} with pole at y. Then $\mathcal{E}_x(y) = \mathcal{E}_y^T(x)$.*

Now we focus on Green's function. Given a bounded Wiener regular domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and a uniformly elliptic matrix A, let $g_x^{\Omega,A}$ denote the Green function, in Ω with pole at $x \in \Omega$ with respect to the matrix A, constructed in [DK09, Theorem 2.12] in the planar case, and [GW82, Theorem 1.1] in higher dimensions.

Notation. If the set Ω, the matrix A, or the pole $x \in \Omega$ is clear from the context, we will omit it in $g_x^{\Omega,A}$. We will mainly skip the domain Ω in $g_x^{\Omega,A}$, that is, g_x^A. We denote $g_x^T = g_x^{\Omega,A^T}$ the Green function for the matrix A^T.

For all $x \in \Omega$, $g_x^{\Omega,A}$ satisfies[(I.3)]

(I.3.1) $g_x^{\Omega,A} \in W^{1,2}(\Omega\setminus B_r(x)) \cap W_0^{1,s}(\Omega)$, for all $s \in [1,(n+1)/n)$ and $0 < r < \operatorname{dist}(x,\partial\Omega)$,

and

$$\int_\Omega A(z)\nabla g_x^{\Omega,A}(z)\nabla\varphi(z)\, dz = \varphi(x), \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{I.3.2}$$

[(I.3)]If $\Omega \subset \mathbb{R}^2$ is bounded, then $Y_0^{1,2}(\Omega)$ in [DK09, Theorem 2.12] is precisely $W_0^{1,2}(\Omega)$, see between Lemma 3.1 and Section 3.1 of [DK09].

In particular, $g_x^{\Omega,A}$ is L_A-harmonic in $\Omega \setminus \{x\}$. By (I.3.1) and a density argument, (I.3.2) remains true for $\varphi \in W_c^{1,s'}(\Omega)$, with $n+1 < s' < \infty$. The Green function g_x also satisfies the following:

(1) $g_x(y) = g_y^T(x)$ for all $x, y \in \Omega$ and $x \neq y$. See [DK09, Theorem 2.12 (2.18)] in the plane, and [GW82, Theorem 1.3] in higher dimensions.
(2) For each $x \in \Omega$ and any $0 < r < \operatorname{dist}(x, \partial\Omega)$, $g_x \in W^{1,2}(\Omega \setminus B_r(x))$. See [DK09, Theorem 2.12 (2.15)] in the plane, and [GW82, Theorem 1.1 (1.3)] in higher dimensions.

If the domain Ω has boundary of class C^2 and the matrix is Lipschitz continuous in a neighborhood $U_{2s}(\partial\Omega) = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, \partial\Omega) < 2s\}$, then the Green function also satisfies:

(3) For each $x \in \Omega$ and any $0 < r < \min\{s, \operatorname{dist}(x, \partial\Omega)\}$, $g_x \in W^{2,2}(\Omega \cap U_r(\partial\Omega))$. This is a consequence of (2) by Theorems I.1.2 and I.1.3.

Next we show that $L_{A^T} g_x^T = \omega_{\Omega,A}^x - \delta_x$ in the distributional sense. With this identity we can move from integrating on the boundary to the interior of the set.

Lemma I.3.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded Wiener regular domain and $\varphi \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$. Then*

$$\int_{\partial\Omega} \varphi(\xi)\, d\omega_{\Omega,A}^x(\xi) - \varphi(x) = -\int_\Omega A^T(z) \nabla g_x^T(z) \nabla\varphi(z)\, dz, \text{ for a.e. } x \in \Omega. \tag{I.3.3}$$

Sketch of proof. In higher dimensions this is proved in [AGMT23, (2.6)]. Here we detail the differences in the planar case.

As Ω is bounded Wiener regular and $\varphi \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$, the L_A-harmonic function u solving the Dirichlet problem with boundary data $\varphi|_{\partial\Omega}$ can be taken to be in $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and $u - \varphi \in W_0^{1,2}(\Omega)$. Indeed, by the Lax-Milgram theorem in $W_0^{1,2}(\Omega)$ there is a unique function $v \in W_0^{1,2}(\Omega)$ with $\int_\Omega A\nabla v \nabla\vartheta = -\int_\Omega A\nabla\varphi\nabla\vartheta$ for any $\vartheta \in W_0^{1,2}(\Omega)$. Taking $u = v + \varphi$ it is clear that $L_A u = 0$, $u \in W^{1,2}(\Omega)$ and $u - \varphi \in W_0^{1,2}(\Omega)$. On the other hand, since Ω is bounded Wiener regular, the L_A-harmonic extension u of $\varphi|_{\partial\Omega}$ is continuous up to the boundary, see [HKM06, Theorem 6.27]. Moreover, by the definition of elliptic measure and the uniqueness of solutions we have

$$u(x) = \int_{\partial\Omega} \varphi(\xi)\, d\omega_{\Omega,A}^x(\xi).$$

Write

$$\int_\Omega A^T \nabla g_x^T \nabla\varphi = \int_\Omega A^T \nabla g_x^T \nabla u + \int_\Omega A^T \nabla g_x^T \nabla(\varphi - u) =: \mathrm{I} + \mathrm{II}.$$

The same proof of [AGMT23, (2.10) and (2.12)] applies also in the plane to have that the left-hand side integral is absolutely convergent and $\mathrm{II} = \varphi(x) - u(x)$ for a.e. $x \in \Omega$, replacing the use of (2.8) and (2.9) in [AGMT23] by the fact that the Green function satisfies $\nabla g_z \in L^p(\Omega)$ for all $p \in [1, 2)$ and $g_z \in W_{\mathrm{loc}}^{1,2}(\Omega \setminus \{z\})$, see [DK09, Remark 2.19 and (3.66)] and item 2 in p. 18 respectively. Using that $|g_x^T(z)| \lesssim |\log|z - x||$ when $|x - z| \leq 2\varepsilon$ is small enough, see [DK09, (2.17)], in the planar case the term I_ε^2 defined in [AGMT23, p. 10855] is controlled by

$$|\mathrm{I}_\varepsilon^2| \lesssim \frac{|\log\varepsilon|}{\varepsilon} \int_{B_{2\varepsilon}(x)} |\nabla u| \lesssim \varepsilon\, |\log\varepsilon|\, \mathcal{M}(\nabla u \mathbf{1}_\Omega)(x),$$

and as $\varepsilon\,|\log\varepsilon| \to 0$ as $\varepsilon \to 0$, the same proof there implies that $\mathrm{I} = 0$ for a.e. $x \in \Omega$. Therefore, (I.3.3) holds also in the planar case. $\square$

We will show below that from the equality (I.3.3) it follows that

$$g_y(x) = -\mathcal{E}_y(x) + \int_{\partial\Omega} \mathcal{E}_y(\xi)\, d\omega^x_{\Omega,A}(\xi), \text{ for all } x, y \in \Omega. \tag{I.3.4}$$

(Probably this is already known but we will show the full details in the plane for completeness). Recall that $x \mapsto \int_{\partial\Omega} \mathcal{E}_y(\xi)\, d\omega^x_{\Omega,A}(\xi)$ is the L_A-harmonic extension of $\mathcal{E}_y$ inside Ω. Assuming that $g_y(x) = 0$ if $y \notin \Omega$, we have that (I.3.4) also holds in this case, since Ω is Wiener regular and therefore the Green function is continuous through the boundary. Moreover, by (I.3.4) and since Ω is bounded, the Green function also satisfies:

(4) For each $x \in \Omega$, $g_x(y) \geq 0$ for any $y \in \Omega \setminus \{x\}$.

This was already proved in higher dimension in [GW82, Theorem 1.1]. However, since the situation is more delicate for unbounded planar domains due to the logarithmic behaviour of the fundamental solution, we only consider bounded planar Wiener regular domains.

Proof of (I.3.4) in the planar case. Let $0 < \varepsilon \ll \min\{|x-y|, \operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega)\}$, $\psi^y = \psi^y_\varepsilon \in C^\infty_c(B_{2\varepsilon}(y))$ such that $\psi^y_\varepsilon = 1$ in $B_\varepsilon(y)$ and $|\nabla\phi^y_\varepsilon| \lesssim 1/\varepsilon$, and $\psi^x = \psi^x_\varepsilon$ defined analogously.

Applying (I.3.3) to $(1-\psi^y)\mathcal{E}_y \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and using that $1 - \psi^y = 1$ in $\partial\Omega \cup \{x\}$, we have

$$\int_{\partial\Omega} \mathcal{E}_y(\xi)\, d\omega^x_{\Omega,A}(\xi) - \mathcal{E}_y(x) = -\int_\Omega A^T(z)\nabla g^T_x(z)\nabla((1-\psi^y)\mathcal{E}_y)(z)\, dz.$$

Write the right-hand side term as

$$\begin{aligned} -\int_\Omega A^T\nabla g^T_x\nabla((1-\psi^y)\mathcal{E}_y)\, dz &= \int_{B_{2\varepsilon}(y)\setminus B_\varepsilon(y)} \mathcal{E}_y A^T\nabla g^T_x\nabla\psi^y\, dz + \int_{B_{2\varepsilon}(y)} \psi^y A\nabla\mathcal{E}_y\nabla g^T_x\, dz \\ &\quad - \int_{B_{2\varepsilon}(x)\setminus B_\varepsilon(x)} g^T_x A\nabla\mathcal{E}_y\nabla\psi^x\, dz - \int_\Omega A\nabla\mathcal{E}_y\nabla((1-\psi^x)g^T_x)\, dz \\ &=: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon. \end{aligned}$$

Using that $|g^T_x(z)| \lesssim |\log|x-z||$ and $|\mathcal{E}_y(z)| \lesssim |\log|y-z||$ when $|x-z| \leq 2\varepsilon$ and $|y-z| \leq 2\varepsilon$ for small enough $\varepsilon > 0$, see [DK09, (2.17)] and Theorem I.3.2, the terms I_ε and III_ε are bounded by

$$|\mathrm{I}_\varepsilon| + |\mathrm{III}_\varepsilon| \lesssim \varepsilon\,|\log\varepsilon| \left\{ \left(\fint_{B_{2\varepsilon}(y)} |\nabla g^T_x|^2\, dz \right)^{1/2} + \left(\fint_{B_{2\varepsilon}(x)} |\nabla\mathcal{E}_y|^2\, dz \right)^{1/2} \right\}. \tag{I.3.5}$$

For the bound of II_ε, since g^T_x is L_{A^T}-harmonic in $\Omega\setminus\{x\} \supset B_{10\varepsilon}(y)$, there exists $p = p(\lambda) > 2$ such that

$$\left(\fint_{B_{2\varepsilon}(y)} |\nabla g^T_x|^p\, dz \right)^{1/p} \lesssim_\lambda \left(\fint_{B_{4\varepsilon}(y)} |\nabla g^T_x|^2\, dz \right)^{1/2},$$

see [Ken94, Lemma 1.1.12], and let $1 \leq q < 2$ be its Hölder exponent conjugate, i.e., $1/p + 1/q = 1$. By Hölder's inequality and the choice of $p > 2$, the term II_ε is controlled by

$$|\mathrm{II}_\varepsilon| = \left| \int_{B_{2\varepsilon}(y)} \psi^y A\nabla\mathcal{E}_y\nabla g^T_x\, dz \right| \lesssim \varepsilon^2 \left(\fint_{B_{2\varepsilon}(y)} |\nabla\mathcal{E}_y|^q\, dz \right)^{1/q} \left(\fint_{B_{4\varepsilon}(y)} |\nabla g^T_x|^2\, dz \right)^{1/2}.$$

By [CL92, Theorem 0.1] we have $\left(\fint_{B_{2\varepsilon}(y)} |\nabla \mathcal{E}_y|^q \, dz \right)^{1/q} \lesssim_\lambda 1/\varepsilon$, and so

$$|\mathrm{II}_\varepsilon| \lesssim \varepsilon \left(\fint_{B_{4\varepsilon}(x)} |\nabla g_x^T|^2 \, dz \right)^{1/2}. \tag{I.3.6}$$

Since $g_x^T \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{x\})$ and $\mathcal{E}_y \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{y\})$, in particular $|\nabla g_x^T|^2 \in L^1_{\mathrm{loc}}(\Omega \setminus \{x\})$ and $|\nabla \mathcal{E}_y|^2 \in L^1_{\mathrm{loc}}(\Omega \setminus \{y\})$, by the Lebesgue differentiation theorem we have that $\fint_{B_{4\varepsilon}(y)} |\nabla g_x^T|^2 \to |\nabla g_x^T(y)|^2$ for a.e. $y \in \Omega$ and $\fint_{B_{4\varepsilon}(x)} |\nabla \mathcal{E}_y|^2 \to |\nabla \mathcal{E}_y(x)|^2$ for a.e. $x \in \Omega$ respectively. That is, by (I.3.5) and (I.3.6) we have $|\mathrm{I}_\varepsilon| + |\mathrm{II}_\varepsilon| + |\mathrm{III}_\varepsilon| = 0$ a.e. $x, y \in \Omega$.

On the other hand, from the Dirac delta property of the fundamental solution, $(1 - \psi^x) g_x^T \in W^{1,2}_0(\Omega)$ and the density of $C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^{n+1})$ in $W^{1,2}_0(\Omega)$, we obtain $\mathrm{IV}_\varepsilon = (1 - \psi^x(y)) g_x^T(y) = g_y(x)$, and (I.3.4) is proved for a.e. $x, y \in \Omega$. By continuity, it also holds for all $x, y \in \Omega$. $\square$

To end this section, we see how the Green function is related to the density of the elliptic measure in smooth domains. Assume now A is Lipschitz continuous in an open neighborhood of $\partial\Omega$, say $U_s(\partial\Omega) = \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, \partial\Omega) < s\}$. Under this assumption A is differentiable a.e. by Rademacher's theorem, and L_A-harmonic functions are in $W^{2,2}$ by Theorem I.1.3.

Lemma I.3.6. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded domain with smooth boundary (and hence Wiener regular) and $A \in C^{0,1}(U_s(\partial\Omega))$. The elliptic measure $\omega^p_{\Omega,A}$ can be written as*

$$d\omega^p_{\Omega,A} = -\langle A^T \nabla g_p^T, \nu \rangle \, d\sigma, \ \text{for a.e. } p \in \Omega \setminus U_{2s}(\partial\Omega), \tag{I.3.7}$$

where ν is the unit outer normal to $\partial\Omega$ and σ is the surface measure on $\partial\Omega$.

Proof. Let $\varphi \in C_c^\infty(U_s(\partial\Omega))$ and set $\phi_p(z) = g_p^T(z) + \mathcal{E}_p^T(z)$ for $z \in \mathbb{R}^{n+1}$, see (I.3.4) when $z \in \Omega$. Notice that $\phi_p(z) = -\mathcal{E}_p^T(z)$ in Ω^c, and hence ϕ_p is L_{A^T}-harmonic in Ω and $\overline{\Omega}^c$.

The claim follows since the right-hand side of (I.3.3) is

$$-\int_{\partial\Omega} \varphi(\xi) \langle A^T(\xi) \nabla g_p^T(\xi), \nu(\xi) \rangle \, d\sigma(\xi) - \varphi(p).$$

Indeed, since A is differentiable a.e. in $\operatorname{supp}\varphi$ (Rademacher's theorem), $g_p^T \in W^{2,2}(\operatorname{supp}\varphi)$ (Theorem I.1.3) and $\varphi \in C_c^\infty(U_s(\partial\Omega))$, in particular $A^T \nabla g_p^T \cdot \varphi \in W^{1,2}(\operatorname{supp}\varphi)$. As the integration by parts formula holds for $W^{1,2}$ functions, then

$$\begin{aligned}
-\int_\Omega A^T \nabla g_p^T \nabla \varphi &= \int_\Omega \varphi \operatorname{div}\left(A^T \nabla g_p^T\right) - \int_\Omega \operatorname{div}\left(A^T \nabla g_p^T \cdot \varphi\right) \\
&= \int_\Omega \varphi \operatorname{div}\left(A^T \nabla g_p^T\right) - \int_{\partial\Omega} \varphi \langle A^T \nabla g_p^T, \nu \rangle \, d\sigma \\
&= \int_{\mathbb{R}^{n+1}} \varphi \operatorname{div}\left(A^T \nabla \phi_p\right) - \int_{\mathbb{R}^{n+1}} \varphi \operatorname{div}\left(A^T \nabla \mathcal{E}_p^T\right) - \int_{\partial\Omega} \varphi \langle A^T \nabla g_p^T, \nu \rangle \, d\sigma \\
&= \int_{\mathbb{R}^{n+1}} \varphi \operatorname{div}\left(A^T \nabla \phi_p\right) - \varphi(p) - \int_{\partial\Omega} \varphi \langle A^T \nabla g_p^T, \nu \rangle \, d\sigma,
\end{aligned}$$

where in the last equality we used that φ has compact support and $\int \varphi \operatorname{div}\left(A^T \nabla \mathcal{E}_p^T\right) = \varphi(p)$ by the definition of the fundamental solution. Since ϕ_p is L_{A^T}-harmonic in $\mathbb{R}^{n+1} \setminus \partial\Omega$, $\mathcal{H}^{n+1}(\partial\Omega) = 0$ and $A \in C^{0,1}(\operatorname{supp}\varphi)$, we have that $\operatorname{div}\left(A^T \nabla \phi_p\right) = 0$ a.e. in $\operatorname{supp}\varphi$ by Remark I.1.4, and the claim follows. $\square$

I.4. The capacity

In this section, we introduce capacity and present general properties of elliptic measure in Wiener regular domains.

Definition I.4.1. Let K be a compact subset of Ω. Its capacity is

$$\operatorname{Cap}(K,\Omega) = \inf\left\{\int_\Omega |\nabla u|^2\,dx : u \in C_0^\infty(\Omega) \text{ and } u \geq 1 \text{ on } K\right\}.$$

Just from the definition of capacity we obtain the following facts. For more properties see [HKM06, 2.2. Theorem].

Lemma I.4.2. *The set function $E \mapsto \operatorname{Cap}(E,\Omega)$, $E \subset \Omega$, enjoys the following properties:*

(1) If $E_1 \subset E_2$, then $\operatorname{Cap}(E_1,\Omega) \leq \operatorname{Cap}(E_2,\Omega)$.

(2) If $\Omega_1 \subset \Omega_2$ are open and $E \subset \Omega_1$, then $\operatorname{Cap}(E,\Omega_2) \leq \operatorname{Cap}(E,\Omega_1)$.

Let $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 1$, be a (bounded if $n \geq 2$) Wiener regular domain and let B be a ball centered at $\partial\Omega$. By the maximum principle [GT01, p. 46] we have

$$\omega^z_{\Omega\cap B,A}(E) \leq \omega^z_{\Omega,A}(E) \text{ for any } E \subset \partial\Omega \cap B \text{ and } z \in \Omega \cap B. \tag{I.4.1}$$

The converse inequality may fail. However, the following weaker relation holds.

Lemma I.4.3. *Let $\Omega \subset \mathbb{R}^2$ be a (possibly unbounded) Wiener regular domain, A be a real uniformly elliptic matrix and B be a ball centered at $\partial\Omega$ with* $\operatorname{Cap}(B \cap \partial\Omega, 4B) \neq 0$. *Let $E \subset \partial\Omega \cap B$ be a Borel set and $z_E \in \partial 1.5B$ such that* $\omega^{z_E}_{\Omega,A}(E) = \max_{z\in\partial 1.5B} \omega^z_{\Omega,A}(E)$. *Then*

$$\omega^{z_E}_{\Omega,A}(E) \lesssim \frac{\operatorname{Cap}(2B,4B)}{\operatorname{Cap}(B\cap\partial\Omega,4B)}\omega^{z_E}_{\Omega\cap 4B,A}(E),$$

where the constant involved depends on the ellipticity constant of the matrix A and the dimension. The same also holds for bounded Wiener regular domains $\Omega \subset \mathbb{R}^{n+1}$ when $n \geq 2$.

Proof. During this proof we write $\omega_\cdot$ instead of $\omega_{\cdot,A}$.

By Lemma I.2.3 we have

$$\omega^{z_E}_\Omega(E) - \omega^{z_E}_{\Omega\cap 4B}(E) = \int_{\partial 4B\cap\Omega} \omega^\xi_\Omega(E)\,d\omega^{z_E}_{\Omega\cap 4B}(\xi). \tag{I.4.2}$$

By the maximum principle[(I.4)] in $\Omega \setminus 1.5\overline{B}$, $\omega^\xi_\Omega(E) \leq \omega^{z_E}_\Omega(E)$ for $\xi \in 4\partial B \cap \Omega$. From this and (I.4.2), we get

$$\omega^{z_E}_\Omega(E) \leq \omega^{z_E}_{\Omega\cap 4B}(4\partial B\cap\Omega)\cdot\omega^{z_E}_\Omega(E) + \omega^{z_E}_{\Omega\cap 4B}(E). \tag{I.4.3}$$

It remains to bound $\omega^{z_E}_{\Omega\cap 4B}(4\partial B \cap \Omega)$. By the maximum principle we have

$$\omega^{z_E}_{\Omega\cap 4B}(4\partial B\cap\Omega) \leq \omega^{z_E}_{4B\setminus(B\cap\partial\Omega)}(4\partial B).$$

By [HKM06, Lemma 6.21], since $z_E \in 1.5\partial B$, we have

$$1 - \omega^{z_E}_{4B\setminus(B\cap\partial\Omega)}(4\partial B) \geq \widetilde{c}\cdot\operatorname{Cap}(B\cap\partial\Omega,4B)/\operatorname{Cap}(2B,4B) =: c \in (0,1). \tag{I.4.4}$$

In particular

$$\omega^{z_E}_{\Omega\cap 4B}(4\partial B\cap\Omega) \leq \omega^{z_E}_{4B\setminus(B\cap\partial\Omega)}(4\partial B) \leq 1-c. \tag{I.4.5}$$

[(I.4)] In the planar case the maximum principle on $\Omega \setminus 1.5\overline{B}$ holds even if Ω is not bounded, since $\omega^\cdot_\Omega(E) \in [0,1]$.

From (I.4.3) and (I.4.5) we get

$$\omega_\Omega^{z_E}(E) \leq \omega_{\Omega\cap 4B}^{z_E}(4\partial B\cap\Omega)\cdot\omega_\Omega^{z_E}(E) + \omega_{\Omega\cap 4B}^{z_E}(E) \leq (1-c)\cdot\omega_\Omega^{z_E}(E) + \omega_{\Omega\cap 4B}^{z_E}(E),$$

obtaining

$$c\cdot\omega_\Omega^{z_E}(E) \leq \omega_{\Omega\cap 4B}^{z_E}(E),$$

as claimed, with c as in step (I.4.4). □

We remark that $\mathrm{Cap}(B\cap\partial\Omega, 4B) = 0$ would imply $\omega_D(B\cap\partial\Omega) = 0$ for any domain D with $B\cap\partial\Omega\subset\partial D$, see [HKM06, Theorems 10.1 and 11.14].

In the following lemma, we construct a set with full elliptic measure by taking the union of local full-measure sets.

Lemma I.4.4. *Let $\Omega\subset\mathbb{R}^2$ be a (possibly unbounded) Wiener regular domain and A be a real uniformly elliptic matrix. Let $\{B_i\}_i$ be a countable family of balls centered at $\partial\Omega$ with $\omega_{\Omega,A}\left(\partial\Omega\setminus\bigcup_i B_i\right) = 0$, and $F_i\subset\partial(\Omega\cap 4B_i)$ with $\omega_{\Omega\cap 4B_i,A}(F_i) = 1$. Then $F := \partial\Omega\cap\bigcup_i F_i$ satisfies $\omega_{\Omega,A}(F) = 1$. The same also holds for bounded Wiener regular domains $\Omega\subset\mathbb{R}^{n+1}$ when $n\geq 2$.*

Proof. In this proof we denote $\omega := \omega_{\Omega,A}$ and $\omega_{\Omega\cap 4B_i} := \omega_{\Omega\cap 4B_i,A}$.

Let $p\in\Omega\setminus\{\infty\}$. Abusing notation, we write $F^c = \partial\Omega\setminus F$. Since $\omega^p(F^c\setminus\bigcup_i B_i)\leq\omega^p(\partial\Omega\setminus\bigcup_i B_i) = 0$, we have

$$\omega^p(F^c) = \omega^p\left(\bigcup_i F^c\cap B_i\right) \leq \sum_i \omega^p(F^c\cap B_i). \tag{I.4.6}$$

We claim that each term in the right-hand side is zero. Indeed, for each i fix a pole $p_i\in\Omega\cap 4B_i$, by the Borel regularity of $\omega_{\Omega\cap 4B_i}^{p_i}$ let $E_i\supset F^c\cap B_i$ be a Borel set with $\omega_{\Omega\cap 4B_i}^{p_i}(E_i) = \omega_{\Omega\cap 4B_i}^{p_i}(F^c\cap B_i)$, and finally let $z_i\in 1.5\partial B_i$ such that $\omega^{z_i}(E_i) = \max_{z\in 1.5\partial B_i}\omega^z(E_i)$. With this choice of z_i, by Lemma I.4.3 we have

$$\omega^{z_i}(E_i) \lesssim \omega_{\Omega\cap 4B_i}^{z_i}(E_i). \tag{I.4.7}$$

We have that $F^c\cap B_i\subset(F_i^c\cap B_i)\cap\partial\Omega\subset F_i^c$, so

$$\omega_{\Omega\cap 4B_i}^{p_i}(E_i) = \omega_{\Omega\cap 4B_i}^{p_i}(F^c\cap B_i) = 0.$$

Hence by the Harnack inequality (denoting its use with H), (I.4.7) and $\omega_{\Omega\cap 4B_i}^{p_i}(E_i) = 0$, for every index i there holds

$$\omega^p(F^c\cap B_i) \leq \omega^p(E_i) \overset{(H)}{\lesssim} \omega^{z_i}(E_i) \overset{\text{(I.4.7)}}{\lesssim} \omega_{\Omega\cap 4B_i}^{z_i}(E_i) \overset{(H)}{\lesssim} \omega_{\Omega\cap 4B_i}^{p_i}(E_i) = 0. \tag{I.4.8}$$

Notice that for each i the constants involved in the use of Harnack inequality and Lemma I.4.3 in (I.4.7) and (I.4.8) depend on i, but the right-hand side in (I.4.7) is zero.

By (I.4.8) we have that the sum in the right-hand side of (I.4.6) is zero as claimed. Therefore the set $F := \partial\Omega\cap\bigcup_i F_i$ satisfies $\omega^p(F) = 1$. □

We also want to note that from (I.4.1) and Lemma I.4.3, the dimension of elliptic measures only depends on how the matrix is around the boundary of the domain. We prove this in the following result.

Corollary I.4.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a (possibly unbounded if $n = 1$) Wiener regular domain, $\varepsilon > 0$ and let A, B be two real uniformly elliptic matrices with $A = B$ in $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$. Then $\omega_{\Omega,A}(F) = 1$ if and only if $\omega_{\Omega,B}(F) = 1$. That is, $\omega_{\Omega,A} \ll \omega_{\Omega,B} \ll \omega_{\Omega,A}$.*

Proof. Let $\omega_{\cdot} := \omega_{\cdot,A}$ and $\widetilde{\omega}_{\cdot} := \omega_{\cdot,B}$. We prove $\widetilde{\omega}_\Omega(F) = 1 \Rightarrow \omega_\Omega(F) = 1$, and the converse direction follows by symmetry. Let $F \subset \partial\Omega$ with $\widetilde{\omega}_\Omega(F) = 1$, we write $F^c = \partial\Omega \setminus F$, and let $\{B_i\}_i$ be a countable subfamily of $\{B_{\varepsilon/100}(\xi)\}_{\xi\in\partial\Omega}$ such that $\partial\Omega \subset \bigcup_i B_i$. We have

$$\omega_\Omega(F^c) \leq \sum_i \omega_\Omega(F^c \cap B_i).$$

We claim that each term on the right-hand side is zero, and in particular $\omega_\Omega(F) = 1$. Indeed, for each i fix a pole $p_i \in \Omega \cap 4B_i$. Since both matrices A and B coincide in $\Omega \cap 4B_i$ we have $\omega^{p_i}_{\Omega\cap 4B_i}(\cdot) = \widetilde{\omega}^{p_i}_{\Omega\cap 4B_i}(\cdot)$, and using (I.4.1) and $\widetilde{\omega}^{z_i}_\Omega(F) = 1$ we get

$$\omega^{p_i}_{\Omega\cap 4B_i}(F^c \cap B_i) = \widetilde{\omega}^{p_i}_{\Omega\cap 4B_i}(F^c \cap B_i) \leq \widetilde{\omega}^{p_i}_\Omega(F^c \cap B_i) \leq \widetilde{\omega}^{p_i}_\Omega(F^c) = 0.$$

Arguing as in the proof of Lemma I.4.4, where Lemma I.4.3 is used, $\omega^{p_i}_{\Omega\cap 4B_i}(F^c \cap B_i) = 0$ implies $\omega_\Omega(F^c \cap B_i) = 0$, as we claimed. $\square$

I.5. The capacity density condition

In this section, we present general properties of elliptic measure in CDC domains, which are defined as follows.

Definition I.5.1 (CDC domain. [HKM06, (11.20), (2.13)])**.** A domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, satisfies the capacity density condition, CDC for short, if there exist a constant $c_0 > 0$ and a radius $r_0 > 0$ such that

$$\operatorname{Cap}\left(\overline{B(x_0, r)} \cap \Omega^c, B(x_0, 2r)\right) \geq c_0 r^{n-1}$$

for all $x_0 \in \partial\Omega$, $x_0 \neq \infty$ and $r \leq r_0$. We write (c_0, r_0)-CDC in case we want to stress the parameters, and c_0-CDC if there is $r_0 > 0$ such that Ω is (c_0, r_0)-CDC.

All CDC domains belong to the class of Wiener regular domains, see [HKM06, Theorem 6.27, Section 9.5, and Theorem 9.20].

Annuli are trivial examples of CDC domains. For completeness, we compute here the CDC constants for a specific annulus; these will be used later in Chapter III to control the density in modified domains.

Lemma I.5.2. *Let $k > 3$. The annulus $\mathcal{A}_k = B(0, k^2 r) \setminus \overline{B(0, r)} \subset \mathbb{R}^{n+1}$, $n \geq 1$, satisfies the CDC with constant $c_0 = c_0(k)$ and radius $s_0 := \left(\frac{k+1}{2}\right)^2 r$, and moreover*

(1) $\overline{B(0, kr)} \subset B(x_0, s_0)$ and
(2) $\overline{B(x_0, 2s_0)} \subset B(0, k^2 r)$,

for any $x_0 \in \partial B(0, r)$, i.e., the inner circle.

Proof. We first determine s_0 for which the conditions (1) and (2) hold. From the first condition we get $s_0 > (k+1)r$, and from the second $2s_0 + r < k^2 r$, i.e., $2s_0 < (k^2 - 1)r$. In order to have existence in s_0 we need $2(k+1)r < (k^2-1)r$, whence we need $k > 3$. Let s_0 be the middle point in $\left((k+1)r, \frac{k^2-1}{2}r\right)$,

$$s_0 := \frac{(k+1)r + \frac{k^2-1}{2}r}{2} = \frac{k^2+2k+1}{4}r =: C(k)\cdot r.$$

Now we want to see that for $k > 3$, the annulus $\mathcal{A}_k$ satisfies the capacity density condition with $s_0 = \frac{k^2+2k+1}{4}r = C(k)\cdot r$. By definition of $C(k)$, given a point $x_0 \in \partial\mathcal{A}_k$, the ball $B(x_0, C(k)\cdot r)$ does not intersect the other component of $\partial\mathcal{A}_k$, by condition (2).

We want to see that there exists $c_0 = c_0(k)$ such that

$$\operatorname{Cap}\left(\overline{B(x_0,s)} \cap \mathcal{A}_k^c, B(x_0,2s)\right) \geq c_0(k)\cdot s^{n-1},$$

for all $x_0 \in \partial\mathcal{A}_k$ and $0 < s \leq s_0$.

Case 1. Suppose $x_0 \in \partial B(0,r) \subset \partial\mathcal{A}_k$. Let $0 < s \leq s_0 = C(k)\cdot r$. By the choice of $C(k)$ we have $B(x_0,s) \cap \mathcal{A}_k^c = B(x_0,s) \cap \overline{B(0,r)}$.

Set $\xi = x_0 - x_0\frac{s}{2s_0}$. So $|\xi - x_0| = \left|x_0\frac{s}{2s_0}\right| = \frac{rs}{2s_0} = \frac{s}{2C(k)}$, and note that $|\xi - x_0| \leq \frac{r}{2}$. In particular $B\left(\xi, \frac{s}{2C(k)}\right) \subset B(x_0,s)\cap\mathcal{A}_k^c$. Also, $B(x_0,2s) \subset B\left(\xi, 2s + |x_0-\xi|\right) = B\left(\xi, 2s + \frac{s}{2C(k)}\right)$. From these two inclusions, the monotonicity of the capacity and [HKM06, (2.13)], we have

$$\operatorname{Cap}\left(\overline{B(x_0,s)}\cap\mathcal{A}_k^c, B(x_0,2s)\right) \geq \operatorname{Cap}\left(\overline{B\left(\xi, \frac{s}{2C(k)}\right)}, B\left(\xi, 2s + \frac{s}{2C(k)}\right)\right) \approx_k s^{n-1}.$$

Case 2. Suppose $x_0 \in \partial B(0,k^2r) \subset \partial\mathcal{A}_k$. Let $0 < s \leq s_0 = C(k)\cdot r$. Define $\xi = x_0 + \frac{x_0}{|x_0|}\frac{s}{2}$. Hence $B\left(\xi, \frac{s}{2}\right) \subset B(x_0,s)\cap\mathcal{A}_k^c$ and $B(x_0,2s) \subset B\left(\xi, \frac{5}{2}s\right)$. Arguing as before we get

$$\operatorname{Cap}\left(\overline{B(x_0,s)}\cap\mathcal{A}_k^c, B(x_0,2s)\right) \gtrsim s^{n-1},$$

as claimed. □

A key property in CDC domains is the following uniform control of the elliptic measure, the so-called Bourgain's lemma, see [Bou87, Lemma 1] for the harmonic case.

Lemma I.5.3 (See [HKM06, Lemma 11.21])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded (c_0, r_0)-CDC domain, and let A be a uniformly elliptic matrix. Then there exists a constant $\tau \in (0,1)$, depending only on n, c_0 and the ellipticity constant of A, such that for every $E \subset \partial\Omega$, $x_0 \in \partial\Omega$ and $0 < r \leq r_0$, the following holds:*

(1) if $B(x_0,2r)\cap E = \emptyset$, then $\omega^p_{\Omega,A}(E) \leq 1-\tau < 1$, and
(2) if $B(x_0,2r)\cap\partial\Omega \subset E$, then $\omega^p_{\Omega,A}(E) \geq \tau > 0$,

for any point $p \in B(x_0,r)\cap\Omega$.

I.6. Elliptic measure and Green's function in CDC domains

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded Wiener regular domain and $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be a uniformly elliptic matrix. We denote by $g_x^{\Omega,A}(\cdot)$ the Green function in Ω with pole at $x \in \Omega$ of the operator $L_A = -\operatorname{div} A\nabla$ constructed in [DK09, Theorem 2.12] in the planar case and [GW82, Theorem 1.1] in higher dimensions, see Section I.3. We recall that $g_x^{\Omega,A}(y) \geq 0$ for each $y \in \Omega \setminus \{x\}$, see [GW82, Theorem 1.1] in higher dimensions and item 4 in p. 19 for the planar case.

Given a bounded Wiener regular domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, a uniformly elliptic matrix A, and a ball B centered at $\partial\Omega$, we claim

$$\omega^p_{\Omega,A}(B) \lesssim r_B^{n-1} \max_{x \in \overline{\Omega \cap 2B}} g_p^{A^T}(x) \text{ for all } p \in \Omega \setminus 4B, \tag{I.6.1}$$

with constant depending on the ellipticity of A. This follows from the identity (I.3.3). Indeed, taking $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi = 1$ in B, $\varphi = 0$ in $(1.5B)^c$ and $|\nabla\varphi| \lesssim 1/r_B$ in $1.5B \setminus B$, by (I.3.3), the ellipticity condition, and Hölder's and Cacciopolli's inequalities, for a.e. $p \in \Omega \setminus 4B$ we get

$$\omega^p_{\Omega,A}(B) \leq \int_{\partial\Omega} \varphi\, d\omega^p_{\Omega,A} = -\int_\Omega A^T \nabla g_p^{A^T} \nabla\varphi \lesssim \frac{1}{r_B} \int_{1.5B} |\nabla g_p^{A^T}| \lesssim r_B^n \left(\frac{1}{r_B^2} \fint_{2B} |g_p^{A^T}|^2 \right)^{1/2}.$$

The last element is trivially bounded by the maximum value of $r_B^{n-1} g_p^{A^T}$ in $\overline{\Omega \cap 2B}$, and (I.6.1) holds for all $p \in \Omega \setminus 4B$ by continuity.

For CDC domains, the following weaker reverse inequality holds.

Lemma I.6.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded (c_0, r_0)-CDC domain, A be a uniformly elliptic matrix and B be a ball centered at $\partial\Omega$. If $r_B \leq r_0$ then*

$$r_B^{n-1} g_p^{A^T}(x) \lesssim \omega^p_{\Omega,A}(4B) \textit{ for } x \in \Omega \cap B \textit{ and } p \in \Omega \setminus 2B,$$

with constant depending only on c_0 and the ellipticity of A.

In higher dimensions, $n \geq 2$, for bounded Wiener regular domains there holds the refined inequality[(I.5)]

$$r_B^{n-1} g_p^{A^T}(x) \inf_{z \in 2B} \omega^z_{\Omega,A}(4B) \lesssim \omega^p_{\Omega,A}(4B) \text{ for } x \in \Omega \cap B \text{ and } p \in \Omega \setminus 2B, \tag{I.6.2}$$

see [AHM+16, Lemma 3.3] for a proof in the harmonic case, which also works in the elliptic case, see [AM19, Lemma 3.6]. In particular, when the domain is CDC, by Lemma I.5.3 we have $\inf_{z \in 2B} \omega^z_{\Omega,A}(4B) \gtrsim 1$, and therefore the finer formulation (I.6.2) is equivalent to Lemma I.6.1.

We remark that the proof of (I.6.2) relies on the good behavior of the Green function (in fact, the fundamental solution) in higher dimensions, in contrast to what happens in the planar case. However, for planar CDC domains the Green function behaves well in a region around the pole.

Lemma I.6.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded (c_0, r_0)-CDC domain, and A be a uniformly elliptic matrix. Let $y \in \Omega$ be such that $\operatorname{dist}(y, \partial\Omega) < r_0$. Then*

$$g_y^A(x) \approx 1 \textit{ for } x \in \partial B(y, \operatorname{dist}(y, \partial\Omega)/2),$$

with constant depending only on c_0 and the ellipticity of A.

We leave its proof for the end of this section. Granted this, we obtain Lemma I.6.1.

Proof of Lemma I.6.1 when $n = 1$. Fix $x \in \Omega \cap B$. Since $r_B \leq r_0$, by Lemma I.6.2 we have $g_x^A(\cdot) \lesssim 1$ in $\partial B(x, \operatorname{dist}(x, \partial\Omega)/2)$. On the other hand, by Lemma I.5.3 there holds $\omega^{\cdot}_{\Omega,A}(4B) \gtrsim 1$ in $2B \supset \partial B(x, \operatorname{dist}(x, \partial\Omega)/2)$. In particular, $g_x^A(\cdot) \lesssim \omega^{\cdot}_{\Omega,A}(4B)$ in $\partial B(x, \operatorname{dist}(x, \partial\Omega)/2)$,

(I.5) In fact, one can replace $4B$ by aB with $a > 0$ and the involved constant does not depend on a.

and trivially also in $\partial\Omega$. By the maximum principle, the same inequality holds in $\Omega \setminus B(x, \operatorname{dist}(x, \partial\Omega)/2)$, and the lemma follows by taking $p \in \Omega \setminus 2B$ and $g_p^{A^T}(x) = g_x^A(p)$. $\square$

Although the good behavior in Lemma I.6.2 is not known for general planar domains, in the harmonic case, in [PT23, Lemma 7.21], there is a planar version of (I.6.2) with $⨍_B |g_x^{Id}(y) - m_B g_x^{Id}|\, dy$ instead of $g_p^{Id}(x)$. Moreover, in [PT23, Lemma 7.22], a pointwise estimate involving the logarithmic capacity is proved, which in particular gives Lemma I.6.1 in the harmonic case, with different dilatations of the ball.

We now turn to the proof of Lemma I.6.2. In the harmonic case, it is stated and proved in [AH08, Lemma 3.4 (1)][(I.6)]. Even though the same proof can be applied with minor technical changes to the elliptic case, for the sake of completeness all the details are provided here, as the author has found neither the statement nor any proof in the literature for the elliptic case. The rest of this section is dedicated to proving Lemma I.6.2.

Although in [AM18, Lemma 2.3] only the harmonic version of the following lemma is stated, exactly the same proof applies in this case.

Lemma I.6.3. *Let $\Omega \in \mathbb{R}^{n+1}$, $n \geq 1$, be a domain and A be a uniformly elliptic matrix, and $\xi \in \partial\Omega$. Suppose that there exists $\delta \in (0,1)$ such that $\omega^x_{\Omega\cap B(\xi,r),A}(\partial B(\xi, r) \cap \Omega) \leq \beta < 1$ for $x \in \partial B(\xi, \delta r) \cap \Omega$ and $r \in (0, R)$. Then there is $\alpha = \alpha(\beta, \delta)$ so that, for all $r \in (0, R)$,*

$$\omega^x_{\Omega,A}(\overline{B(\xi, r)}^c) \lesssim \left(\frac{|x - \xi|}{r}\right)^\alpha \text{ for } x \in \Omega \cap B(\xi, r),$$

with constant depending only on β and δ.

We note that if Ω is (c_0, r_0)-CDC, then for any ball B centered at $\partial\Omega$, the domain $\Omega \cap B \subset \Omega$ is also (c_0, r_0)-CDC, just because the complementary increases. Therefore, for any $\xi \in \partial\Omega$ and $r \leq 2r_0$, by Lemma I.5.3 we have $\omega^x_{\Omega\cap B(\xi,r),A}(\partial B(x, r) \cap \Omega) \leq 1 - \tau =: \beta$ for all $x \in B(\xi, r/2) \cap \Omega$, and in particular, we can apply the previous lemma, with $\delta = 1/4$ say. Consequently, for (c_0, r_0)-CDC domains Ω, Lemma I.6.3 holds with $R = 2r_0$ for all $\xi \in \partial\Omega$, where the constant now depends on c_0 and the ellipticity of A.

In particular, given a (c_0, r_0)-CDC domain Ω and $\xi \in \partial\Omega$, since $\Omega \cap B(\xi, r_0)$ is (c_0, r_0)-CDC as well, for all $r \in (0, 2r_0)$ there holds

$$\omega^x_{\Omega\cap B(\xi,r_0),A}(\overline{B(\xi, r)}^c) \lesssim \left(\frac{|x - \xi|}{r}\right)^\alpha \text{ for } x \in \Omega \cap B(\xi, r), \tag{I.6.3}$$

where the constant depends only on c_0 and the ellipticity of A.

We adapt the proof in [AH08, Lemma 3.4 (1)] with the technical changes for the elliptic case.

Proof of Lemma I.6.2. The ellipticity constant of A is denoted by $\lambda \geq 1$. To simplify the notation, we also write $\omega_{\cdot,A} = \omega_\cdot$, $g_y = g_y^A$ and $R := \operatorname{dist}(y, \partial\Omega) < r_0$.

For the lower bound, let $g_y^R := g_y^{B_R(y),A}$ and note that $g_y^R \leq g_y$ in $B_R(y)$ by the maximum principle, as $g_y - g_y^R$ is L_A-harmonic in $B_R(y)$ and positive in $\partial B_R(y)$. Finally, $g_y^R \approx_\lambda 1$ on $\partial B(y, R/2)$ and the lower bound is proved. This follows by writing g_y^R in terms of the Green function $g_0^{B_1(0),\widehat{A}}$ where $\widehat{A}(\cdot) = A(y + R\cdot)$ has the same ellipticity constant, and the pointwise estimates of the fundamental solution, see Section I.3 and the references therein. Note that the lower bound holds for general Wiener regular domains.

[(I.6)]The CDC is with respect the logarithmic capacity, see [AH08, Definition 3.1].

We now turn to the upper bound. Let $M_0 = \sup_{\partial B(y,R/2)} g_y(\cdot)$, and in particular, $g_y(\cdot) \leq M_0$ in $\Omega \setminus B(y, R/2)$ by the maximum principle. Let $y^* \in \partial\Omega$ be such that $|y - y^*| = \operatorname{dist}(y, \partial\Omega) = R$. Thus, by the maximum principle we have

$$g_y(z) \leq M_0 \omega^z_{\Omega \cap B(y^*, R/2)}(\Omega \cap \partial B(y^*, R/2)) \text{ for } z \in \Omega \cap B(y^*, R/2).$$

By (I.6.3) there exists $0 < \varepsilon_1 = \varepsilon_1(c_0, \lambda) < 1/4$ such that if $z \in \Omega \cap B(y^*, 2\varepsilon_1 R)$, then

$$\omega^z_{\Omega \cap B(y^*, R/2)}(\Omega \cap \partial B(y^*, R/2)) \leq \omega^z_{\Omega \cap B(y^*, R/2)}(\overline{B(y^*, R/4)}^c) \leq C\varepsilon_1^\alpha \leq \frac{1}{2}.$$

Therefore, combining the latter estimates we get

$$g_y(z) \leq M_0/2, \text{ for } z \in \Omega \cap B(y^*, 2\varepsilon_1 R).$$

We will compare the Green function g_y with the Green function $g_y^{\mathbf{B}} \coloneqq g_y^{\mathbf{B},A}$ in the ball $\mathbf{B} \coloneqq B(y, (1 - \varepsilon_1)R)$. Let us cover $\partial\mathbf{B}$ with a finite number $N = N(\varepsilon_1) = N(c_0, \lambda)$ of balls. Let y_1 be the point in the segment $\overline{yy^*}$ with $|y_1 - y^*| = \varepsilon_1 R$. Fix $N \coloneqq \left\lceil \frac{200\pi}{\varepsilon_1} \right\rceil$ and assume that we have already defined $y_1, \ldots y_j$. Let y_{j+1} be the point in $\partial\mathbf{B}$ with $\angle(y_{j+1} - y, y_j - y) = 2\pi/N > 0$.[I.7] This procedure gives N points as $y_{N+1} = y_1$. Let $r \coloneqq |y_2 - y_1|$, and for each $1 \leq j \leq N$, we define $B_j \coloneqq B(y_j, r)$. As all the points are equidistributed in $\partial\mathbf{B}$, in particular $y_{j-1}, y_{j+1} \in \partial B_j$ for all $1 \leq j \leq N$, with the correspondence $y_0 = y_N$ and $y_{N+1} = y_1$, and clearly $\partial\mathbf{B} \subset \bigcup_{j=1}^N B_j$. Moreover, $r = |y_2 - y_1| \leq \mathcal{H}^1(\operatorname{arc}(\widehat{y_1 y_2})) = \frac{2\pi(1-\varepsilon_1)R}{N} \leq \frac{2\pi R}{N} \leq \frac{\varepsilon_1 R}{100}$ by the choice of N. In particular, $g_y(\cdot) \leq M_0/2$ in B_1.

We claim that if $g_y \leq (1 - \alpha)M_0$ in $\overline{B_j}$ for some $\alpha > 0$, then $g_y \leq (1 - \tau\alpha)M_0$ in $\overline{B_{j+1}}$, with $\tau \in (0, 1)$ as in Lemma I.5.3. Indeed, since the annulus $\mathcal{A} \coloneqq 25B_j \setminus \overline{B_j}$ satisfies $\bar{\mathcal{A}} \subset B(y, R) \setminus \overline{B(y, R/2)}$, in particular $g_y \leq M_0$ in $\bar{\mathcal{A}}$. Since $\omega^z_{\mathcal{A}}(\partial 25B_j) = 1$ for $z \in \partial 25B_j$ and vanishes on ∂B_j, we can write

$$g_y(z) \leq \alpha M_0 \omega^z_{\mathcal{A}}(\partial 25B_j) + (1 - \alpha)M_0 \text{ for } z \in \partial\mathcal{A}.$$

By the maximum principle, the same holds for $z \in \bar{\mathcal{A}}$. The annulus $\mathcal{A}$ is a $(c_0, 9r)$-CDC domain with universal constant c_0 and radius $r_0 = 9r$, see Lemma I.5.2. In particular, for $z \in \overline{B_{j+1}} \subset 2B_{j+1} = B(y_{j+1}, 2r)$, since $4B_{j+1} \cap \partial 25B_j = \emptyset$, by Lemma I.5.3 we have $\omega^z_{\mathcal{A}}(\partial 25B_j) \leq 1 - \tau$. That is, $g_y \leq (1 - \tau\alpha)M_0$ in $\overline{B_{j+1}}$, as claimed.

Since $g_y \leq M_0/2$ in $\overline{B_1}$, from the previous paragraph we conclude that $g_y \leq (1 - c)M_0$ on $\partial\mathbf{B}$, with $c = 0.5\tau^N$, that is, $c = c(c_0, \lambda) \in (0, 1)$. Then,

$$g_y^{\mathbf{B}}(z) = g_y(z) - \int_{\partial\mathbf{B}} g_y(\xi)\, d\omega^z_{\mathbf{B},A}(\xi) \geq g_y(z) - (1 - c)M_0 \text{ for } z \in \mathbf{B},$$

and hence

$$\sup_{z \in \partial B(y, R/2)} g_y^{\mathbf{B}}(z) \geq M_0 - (1 - c)M_0 \geq cM_0.$$

As in the proof of the lower bound, we have $g_y^{\mathbf{B}}(z) \approx_\lambda 1$ on $\partial B(y, R/2)$. Therefore, $M_0 \leq C(c_0, \lambda)$ and the upper bound is also proved. $\square$

[I.7] This condition is just to ensure that we always choose the point going in the anticlockwise direction. We could do the same procedure in the clockwise direction as well.

CHAPTER II

The dimension of planar elliptic measures via quasiconformal mappings

In this chapter, we provide new bounds for the dimension of planar elliptic measures when the uniformly elliptic matrix is symmetric with constant determinant, or when the planar domain satisfies the capacity density condition (see Definition I.5.1). The bounds depend solely on the ellipticity constant of the matrix.

Let us quickly recall Makarov's Theorem I.0.1 and Wolff's Theorem I.0.2 regarding the dimension of the harmonic measure $\omega = \omega_{Id}$ in planar domains. Makarov [Mak85, Theorem 1] showed that there is a constant $C_M > 0$ such that for any simply connected domain Ω, its harmonic measure ω_Ω satisfies $\omega_\Omega \ll \mathcal{H}^{\varphi_{1,C_M}}$, where $\varphi_{\cdot,\cdot}$ is the function defined for $\rho \geq 1$ and $C > 0$ as

$$\varphi_{\rho,C}(r) = r^{\frac{2}{\rho+1}} \exp\left(C \frac{2\rho}{\rho+1} \sqrt{\log r^{-\rho} \log\log\log r^{-\rho}} \right). \tag{II.0.1}$$

We note that $\mathcal{H}^{\varphi_{\rho,C}} \ll \mathcal{H}^\alpha$ for any $\alpha < \frac{2}{\rho+1}$ because $\varphi_{\rho,C}(t)/t^\alpha \to 0$ as $t \to 0$. This in particular implies $\dim_{\mathcal{H}} \omega_\Omega \geq 1$. For the upper bound, Wolff [Wol93] proved that ω_Ω has σ-finite length for any domain $\Omega \subset \mathbb{R}^2$. For more geometric properties of the harmonic measure, see the introduction of Chapter I and the references therein.

Let us also recall the different behavior of the Hausdorff dimension of elliptic measures. For every $\varepsilon > 0$, using quasiconformal mappings, Sweezy constructed a domain $\Omega \subset \mathbb{R}^2$ and a uniformly elliptic matrix A such that $\dim_{\mathcal{H}} \omega_{\Omega,A} \geq 2 - \varepsilon$, see [Swe92]. Sweezy also provided the higher-dimensional analog in [Swe94]. Through a different technique, David and Mayboroda in [DM21] defined a continuous scalar function $a \approx 1$ on the complementary of the 1-dimensional four corner Cantor set $C \subset \mathbb{R}^2$ such that $\omega_{\mathbb{R}^2 \setminus C, aId}$ is comparable to $\mathcal{H}^1|_C$. Operators of the form $L_{aId} = -\operatorname{div} a\nabla$ that satisfy the above comparability are the so-called "good elliptic operators" in the terminology used in [DM21]. By the same strategy, for $1 < d < \log 4/\log 3$, Perstneva in [Per25] constructed a "good elliptic operator" $L_{a_d Id}$ on the (unbounded component of the) complementary of a d-dimensional Koch-type snowflake $S_d \subset \mathbb{R}^2$ satisfying that $\omega_{\mathbb{R}^2 \setminus S_d, a_d Id}$ is comparable to $\mathcal{H}^d|_{S_d}$.

When adding regularity to the matrix, one could expect that elliptic measures behave as the harmonic measure. In Chapter III below, in collaboration with Prats and Tolsa, for Lipschitz matrices we prove Theorem III.0.1, the analog of Wolff's Theorem I.0.2 for planar Reifenberg flat domains with small constant. More precisely, given a uniformly elliptic matrix A with Lipschitz coefficients, and a Reifenberg flat domain $\Omega \subset \mathbb{R}^2$ with small constant depending on the ellipticity of A, $\omega_{\Omega,A}$ has σ-finite length.

To the best of the author's knowledge, the above results are among the few concerning the dimension of the elliptic measure in the plane without any restriction on the dimension of the boundary. In the same spirit as Makarov's Theorem I.0.1 and Wolff's Theorem I.0.2, in this chapter we provide bounds for the dimension of the planar elliptic measure. Namely, we obtain σ-finiteness of the elliptic measure with respect to certain Hausdorff measures and we show absolute continuity of the elliptic measure on finitely connected domains with respect to a gauge Hausdorff measure $\mathcal{H}^{\varphi_{\rho,C}}$, where $\varphi_{\rho,C}$ is the gauge function defined in (II.0.1). Recall that, as discussed above, we have that

$$\omega_{\Omega,A} \ll \mathcal{H}^{\varphi_{\rho,C}} \implies \dim_{\mathcal{H}} \omega_{\Omega,A} \geq \frac{2}{\rho+1}.$$

The first bunch of results concerns the case when A is symmetric and has determinant 1.

Theorem II.0.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain, and $A \in \mathbb{R}^{2\times 2}(\mathbb{R}^2)$ be a uniformly elliptic (with ellipticity constant $\lambda \geq 1$) and symmetric matrix with $\det A = 1$. Then $\omega_{\Omega,A}$ has σ-finite $\mathcal{H}^{\frac{2\lambda}{\lambda+1}}$. If in addition Ω is finitely connected, then $\omega_{\Omega,A} \ll \mathcal{H}^{\varphi_{\lambda,C_M}}$, where $C_M > 0$ is the constant in Makarov's Theorem I.0.1.*

Theorem II.0.2. *Assume, in addition to the hypotheses of Theorem II.0.1, that $A \in C^{\alpha}_{\text{loc}}(\mathbb{R}^2)$, $0 < \alpha < 1$. Then $\omega_{\Omega,A}$ has σ-finite length. If in addition Ω is finitely connected, then $\omega_{\Omega,A} \ll \mathcal{H}^{\varphi_{1,C_M}}$, where $C_M > 0$ is the constant in Makarov's Theorem I.0.1.*

For the proof of Theorems II.0.1 and II.0.2, see pp. 46 and 49 respectively in Section II.3.

Note that the previous results are coherent with the results in the harmonic case of Makarov's Theorem I.0.1 and Wolff's Theorem I.0.2. If $A = Id$, we can apply Theorem II.0.2 using that the coefficients are Hölder, or we can just also use that the ellipticity constant is $\lambda = 1$ in Theorem II.0.1 and hence $2/(\lambda+1) = 2\lambda/(\lambda+1) = 1$. The consistency of using Theorem II.0.1 with $\lambda = 1$ is due to the sharpness of the distortion of the Hausdorff dimension and measures under quasiconformal mappings, see (II.2.5), Theorem II.2.7 and Corollary II.2.10.

In the case of symmetric matrices with determinant 1, the scheme of the proof only needs the uniform ellipticity condition on the matrix, and the key point is that the quasiconformal mapping which relates the elliptic measure of a domain with the harmonic measure in the image domain, is uniquely determined by the matrix. In particular, the more regular the coefficients of the matrix are, the more regular the quasiconformal mapping is. Arguing as in the proof of Theorem II.0.1 and using the additional regularity of the quasiconformal mapping when the coefficients enjoy extra regularity, we obtain Theorem II.0.2 and the following cases:

Theorem II.0.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain, and $A \in \mathbb{R}^{2\times 2}(\mathbb{R}^2)$ be a uniformly elliptic (with ellipticity constant $\lambda \geq 1$) and symmetric matrix with $\det A = 1$. Then the following holds:*

- *If $A(z) = A(\operatorname{Re}(z))$ (or $A(z) = A(\operatorname{Im}(z))$), $A \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ with $p > \frac{2\lambda^2}{\lambda^2+1}$ or $A \in \mathrm{DMO}_p$ (see Definition II.3.14) with $p > 1$, then $\omega_{\Omega,A}$ has σ-finite length.*
- *If $A \in \mathrm{VMO}_{\log}$ (see Definition II.3.18), then $\omega_{\Omega,A}$ has σ-finite $\mathcal{H}^{t/\log\frac{1}{t}}$.*
- *If $A \in \mathrm{VMO}$ (see Definition II.3.7), then $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq 1$.*

If in addition Ω is finitely connected, the following hold:

- *If $A(z) = A(\operatorname{Re}(z))$ (or $A(z) = A(\operatorname{Im}(z))$), or $A \in \mathrm{DMO}_1$, then $\omega_{\Omega,A} \ll \mathcal{H}^{\varphi_{1,C_M}}$.*
- *If $A \in \mathrm{VMO}_{\log}$, then $\omega_{\Omega,A} \ll \mathcal{H}^{(\varphi_{1,C_M})\circ(t\log\frac{1}{t})}$.*
- *If $A \in \mathrm{VMO}$, then $\dim_{\mathcal{H}} \omega_{\Omega,A} = 1$.*

Here $C_M > 0$ is the constant in Makarov's Theorem I.0.1.

We remark that Theorems II.0.2 and II.0.3 hold as long as the determinant is constant. The proofs of all cases can be found on p. 46 (Hölder case), p. 50 ($A(z) = A(\operatorname{Re}(z))$ case),

p. 50 (VMO case), p. 53 (Sobolev case), p. 55 (DMO case) and p. 60 ($\mathrm{VMO}_{\log}$ case). Also, as noted in Remarks II.3.15 and II.3.19, for $1 \le p < \infty$ there holds

$$\text{Hölder} \subset \text{Dini} \subset \mathrm{DMO}_p \subset \mathrm{DMO}_1 \subset \mathrm{VMO}_{\log} \subset \mathrm{VMO},$$

where Dini consists of functions satisfying the pointwise Dini condition (II.3.13).

In the same way, as a corollary of [Tol24, Theorems 5.1 and 5.3] and by the same approach of Theorem II.0.1, using the fact that the quasiconformal mapping and its inverse are C^1 when the matrix is DMO_p with $p > \lambda$, we derive the following result. The details are left to the reader.

Corollary II.0.4. *Let $A \in \mathbb{R}^{2\times 2}(\mathbb{R}^2)$ be a uniformly elliptic (with ellipticity constant $\lambda \ge 1$) and symmetric matrix with constant* $\det A$. *Let $E \subset \mathbb{R}^2$ be a compact subset contained in a C^1 curve which is (s, C_0)-AD regular for some $s > 0$, that is,*

$$C_0^{-1} r^s \le \mathcal{H}^s(E \cap B(x,r)) \le C_0 r^s \text{ for all } x \in E \text{ and } 0 < r \le \operatorname{diam}(E).$$

Let $\Omega = \mathbb{R}^2 \setminus E$, and assume also that $A \in \mathrm{DMO}_p$ with $p > \lambda$. We have that:

- *If $s \in \left[\frac{1}{2}, 1\right)$, then $\dim_{\mathcal{H}} \omega_{\Omega,A} < s$.*
- *There exists $\varepsilon = \varepsilon_{C_0} > 0$ small enough such that if $s \in \left[\frac{1}{2} - \varepsilon, 1\right)$, then $\dim_{\mathcal{H}} \omega_{\Omega,A} < s$.*

In the preceding result we concluded that $\dim_{\mathcal{H}} \omega_{\Omega,A}$ is strictly smaller than the dimension of the boundary. This phenomenon is known as dimension drop. For additional background in dimension drop, see for instance [Azz20, DJJ23, Tol24], the introduction of Chapter I, and the references therein.

The second group of results concerns general uniformly elliptic matrices but is restricted to CDC domains (Definition I.5.1), and they are proved in Section II.4.

Theorem II.0.5. *Let $\Omega \subset \mathbb{R}^2$ be a CDC domain, $A \in \mathbb{R}^{2\times 2}\left(\overline{\Omega}\right)$ be a uniformly elliptic matrix with ellipticity constant $\lambda \ge 1$, and*

$$K_\lambda = \begin{cases} \lambda & \text{if } A \text{ is symmetric}, \\ \lambda + \sqrt{\lambda^2 - 1} & \text{otherwise}. \end{cases} \tag{II.0.2}$$

Then $\omega_{\Omega,A}$ has σ-finite $\mathcal{H}^{2K_\lambda/(K_\lambda+1)}$. If in addition Ω is finitely connected[(II.1)], *then $\omega_{\Omega,A} \ll \mathcal{H}^{\varphi_{K_\lambda,C_M}}$, where $C_M > 0$ is the constant in Makarov's Theorem I.0.1.*

Contrary to what we had in the symmetric and determinant 1 case, for general uniformly elliptic matrices we cannot find a quasiconformal mapping such that the elliptic measure is the push-forward of the harmonic measure in some other domain. This will not allow us to relate, for any domain, its elliptic measure with the harmonic measure in some other domain. However, if the domain is CDC, we find a quasiconformal mapping that sends the elliptic Green function to the harmonic Green function in the image domain, which allows us to obtain the comparability between the elliptic measure and the harmonic measure in the image domain. Using this, the proof follows the same scheme as in the previous case in Theorem II.0.1.

We remark that, in the above situation, the quasiconformal mapping depends not only on the matrix but also on the Green function. Therefore, we cannot easily deduce better regularity of the quasiconformal mapping even if the matrix has additional regularity. However, if the matrix is continuous up to the boundary then we will deduce the following result.

[(II.1)]In particular it is CDC.

Theorem II.0.6. *Let $\Omega \subset \mathbb{R}^2$ be a CDC domain and $A \in \mathbb{R}^{2\times 2}(\overline{\Omega})$ be a uniformly elliptic matrix with continuous coefficients up to $\partial\Omega$, that is, $A \in C^0(\overline{\Omega})$. Then $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq 1$. If in addition Ω is finitely connected, then $\dim_{\mathcal{H}} \omega_{\Omega,A} = 1$.*

To prove this result, we will use a localization argument that relies on the fact that $K_\lambda \to 1$ as $\lambda \to 1$. We note that in Theorem III.0.1 below we obtain the σ-finite length of $\omega_{\Omega,A}$ under the assumptions that A is Lipschitz and $\Omega \subset \mathbb{R}^2$ is Reifenberg flat. In contrast, the result above shows that the upper bound $\dim \omega_{\Omega,A} \leq 1$ holds under much weaker assumptions: Ω is CDC and A is continuous up to the boundary.

We note that Theorem II.0.6 does not contradict the result of Perstneva in [Per25]. In fact, the scalar function $a_d \approx 1$ that she defined is continuous in the CDC domain Ω, the (unbounded component of the) complementary of a d-dimensional Koch-type snowflake, while in Theorem II.0.6 we require the matrix to be continuous up to the boundary.

An interesting question is the validity of Theorem II.0.6 in higher dimensions, in the sense that whether $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq n+1-b_{n,\mathrm{opt}}$ for any domain $\Omega \subset \mathbb{R}^{n+1}$ and any uniformly elliptic matrix $A \in C(\overline{\Omega})$, where $b_{n,\mathrm{opt}} \in (0,1)$ is the optimal (unknown) constant such that $\dim_{\mathcal{H}} \omega_{\Omega,Id} \leq n+1-b_{n,\mathrm{opt}}$ for any domain $\Omega \subset \mathbb{R}^{n+1}$.

II.1. Preliminaries

II.1.1. Notation. We retain the notation from p. 9 throughout this chapter. Specifically, the ambient space is $\mathbb{R}^2$, though auxiliary results will be stated in the general setting of $\mathbb{R}^{n+1}$ for $n \geq 1$.

II.1.2. Reduction to bounded Wiener regular domains. We present the lemmas to see that for the study of the dimension of planar elliptic measures, we can reduce to bounded Wiener regular domains. For the proof of the results in this chapter (Theorems II.0.1 to II.0.3, II.0.5 and II.0.6) for bounded Wiener regular domains, refer to Sections II.3 and II.4.

Using the Wiener regularity characterization for the Laplace operator, see Remark I.2.2, we directly note that finitely connected domains $\Omega \subset \mathbb{R}^2$ are Wiener regular, and in order to study the size of Wiener regular points (for any L_A) we can use the classical potential theory for the Laplacian operator. From this and the same proof of [PT23, Lemma 9.8], we obtain the following.

Lemma II.1.1. *Let A be a real and uniformly elliptic matrix. To prove that for any domain $\Omega \subset \mathbb{R}^2$, either $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq \beta$ (with $0 < \beta \leq 2$) or $\omega_{\Omega,A}$ has σ-finite (gauge) $\mathcal{H}^h$, it suffices to prove it for Wiener regular domains.*

Proof. Indeed, in the proof of [PT23, Lemma 9.8] one can replace with no modification harmonic measure by elliptic measure, the set G (given by assuming the result for Wiener regular domains) is constructed so that $\omega_{\Omega,A}(G) = 1$ and either $\dim_{\mathcal{H}} G \leq \beta$ or $\mathcal{H}^h(G)$ is σ-finite respectively, and the set F satisfies $\mathcal{H}^\gamma(F) = 0$ for all $0 < \gamma \leq 2$, see also [PT23, Lemma 6.19]. In particular, the set $G \cup F$ has full elliptic measure $\omega_{\Omega,A}$ and either $\dim_{\mathcal{H}}(G \cup F) \leq \beta$ or $\mathcal{H}^h(F \cup G)$ is σ-finite respectively, as claimed. $\square$

Lemma I.4.4 and the following lemma allow us to reduce to the case of bounded domains.

Lemma II.1.2. *Let $\Omega \subset \mathbb{R}^2$ be a finitely connected domain and A be a real and uniformly elliptic matrix. For any small enough ball B centered at $\partial\Omega$, each connected component of $\Omega \cap B$ is simply connected, and $(\omega_{\Omega\cap B,A})|_{\partial\Omega} \ll \omega_{\Omega,A}$. In particular $\dim \omega_{\Omega,A} \geq \dim \omega_{\Omega\cap B,A}$.*

Proof. Since Ω is finitely connected, each connected component of $\Omega \cap B$ is simply connected. By (I.4.1) we have $(\omega_{\Omega\cap B,A})|_{\partial\Omega} \ll \omega_{\Omega,A}$. $\square$

Remark II.1.3. By Lemmas I.4.4, II.1.1 and II.1.2 it suffices to prove our results for bounded Wiener regular domains. By translation, we can also assume $\Omega \subset B_{1/4}(0)$ if necessary.

Remark II.1.4. Assuming $\Omega \subset B_{1/4}(0)$, it may seem natural to assume that $A = Id$ in $B_1(0)^c$ by replacing A by the uniformly elliptic (with the same ellipticity constant) matrix

$$\widetilde{A} := \varphi A + (1-\varphi) Id,$$

for some function φ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B_{1/2}(0)$ and $\varphi = 0$ in $B_1(0)^c$. If φ is smooth, then $\widetilde{A}$ often has the same regularity as A. Note that $\widetilde{A}$ is symmetric wherever A is symmetric. However, $\widetilde{A}$ may no longer have determinant 1 even when A has determinant 1. If necessary, if we wanted to preserve also the determinant 1 property, we could replace the matrix A by the uniformly elliptic (also with the same ellipticity constant) matrix $\widehat{A}$ defined in Remark II.2.16.

Instead of modifying the matrix, in most of the cases we will modify the Beltrami coefficient, see the proof of the non-regularity case of Theorem II.0.1 in p. 46, the proof of the VMO case of Theorem II.0.3 in p. 50, and the proof of Theorem II.0.5 in p. 63, for example.

II.1.3. Small perturbation of the constant case. In this subsection, we study the dimension of elliptic measures arising from a uniformly elliptic matrix whose coefficients are close in the L^∞ norm to a uniformly elliptic constant matrix. The main result of this subsection and its proof will be used in the proof of the Dini mean oscillation case in Theorem II.0.3 and Theorem II.0.6.

For $\lambda \ge 1$ we denote

$$\overline{\dim}_{\mathcal{H}}(\lambda) = \sup\{\dim_{\mathcal{H}} \omega_{\Omega,A} : \Omega \subset \mathbb{R}^{n+1} \text{ and } A \text{ has ellipticity constant } \lambda\},$$
$$\underline{\dim}^{\mathrm{sc}}_{\mathcal{H}}(\lambda) = \inf\{\dim_{\mathcal{H}} \omega_{\Omega,A} : \Omega \subset \mathbb{R}^{2} \text{ is simply connected and } A \text{ has ellipticity constant } \lambda\}.$$

Using this notation, in the harmonic case, i.e., $\lambda = 1$, in the plane we have $\overline{\dim}_{\mathcal{H}}(1) \le 1$ and $\underline{\dim}^{\mathrm{sc}}_{\mathcal{H}}(1) \ge 1$ by Wolff's Theorem I.0.2 and Makarov's Theorem I.0.1 respectively.

In the result below, we estimate the dimension of the elliptic measure in terms of how close the matrix A is to a uniformly elliptic constant matrix.

Lemma II.1.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain and C be a constant uniformly elliptic matrix with ellipticity constant $\lambda \ge 1$. Let A be a (not necessarily constant) matrix and*

$$\varepsilon := \max_{i,j\in\{1,\dots,n+1\}} \|a_{ij}(\cdot) - c_{ij}\|_{L^\infty(\overline{\Omega})}.$$

If ε is small enough (depending on λ and n) then A is uniformly elliptic in $\overline{\Omega}$ and $\dim_{\mathcal{H}} \omega_{\Omega,A} \le \overline{\dim}_{\mathcal{H}}((1-(n+1)\varepsilon\lambda)^{-1})$. If $\Omega \subset \mathbb{R}^2$ is simply connected, then $\dim_{\mathcal{H}} \omega_{\Omega,A} \ge \underline{\dim}^{\mathrm{sc}}_{\mathcal{H}}((1-2\varepsilon\lambda)^{-1})$.

Note that when the matrix A is constant, the result agrees with the dimension of the harmonic measure. Before the proof, we need the following lemma.

Lemma II.1.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain and C be a uniformly elliptic and constant matrix. Then, in the weak form, for any $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ we have*

$$\operatorname{div} C\nabla u = \operatorname{div} C_s \nabla u,$$

where C_s denotes the symmetric part of the matrix C, that is $C_s = (C + C^T)/2$.

Proof. For $u \in C^\infty(\Omega)$, and as C is constant, we have

$$\operatorname{div} C\nabla u = \sum_{i,j} c_{ij}\partial_{ij}u = \sum_{ij} \frac{c_{ij}+c_{ji}}{2}\partial_{ij}u = \operatorname{div} C_s\nabla u.$$

Let us check the weak form of this identity for $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Given $\varphi \in C^\infty_c(\Omega)$, let U be an open set such that $\operatorname{supp}\varphi \subset U \subset \overline{U} \subset \Omega$, and we have $u \in W^{1,2}(U)$. Take $\{u_k\}_{k\geq 1} \subset C^\infty(U)$ so that $\lim_{k\to\infty} \|u_k - u\|_{W^{1,2}(U)} = 0$. As $\operatorname{div} C\nabla u_k = \operatorname{div} C_s\nabla u_k$ in the weak form for each $k \geq 1$, we get

$$\begin{aligned}\int_\Omega C\nabla u\nabla\varphi &= \int_\Omega C\nabla u_k\nabla\varphi + \int_\Omega C\nabla(u-u_k)\nabla\varphi \\ &= \int_\Omega C_s\nabla u_k\nabla\varphi + \int_\Omega C\nabla(u-u_k)\nabla\varphi \\ &= \int_\Omega C_s\nabla u\nabla\varphi + \int_\Omega C_s\nabla(u_k-u)\nabla\varphi + \int_\Omega C\nabla(u-u_k)\nabla\varphi.\end{aligned}$$

The lemma follows since $\int_\Omega |\nabla(u-u_k)||\nabla\varphi| \lesssim \|\nabla(u-u_k)\|_{L^2(U)} \to 0$ as $k \to \infty$. □

Using the previous lemma, we prove Lemma II.1.5.

Proof of Lemma II.1.5. If ε is small enough depending on λ and n then A is uniformly elliptic in $\overline{\Omega}$.

By Lemma II.1.6, in the weak form we have

$$\operatorname{div} A\nabla = \operatorname{div} C\nabla + \operatorname{div}(A-C)\nabla = \operatorname{div} C_s\nabla + \operatorname{div}(A-C)\nabla = \operatorname{div}(C_s + A - C)\nabla.$$

Note that the matrix $C_s + A - C$ is uniformly elliptic as it has bounded coefficients and $\langle B\xi, \xi\rangle = \langle B_s\xi, \xi\rangle$ for any (not necessarily constant) matrix B. Then, taking the linear change of variables given by S^T, where S is the square root of C_s (that is, $S = \sqrt{C_s}$, the matrix satisfying $S^TS = C_s$) and

$$\begin{aligned}M(z) &:= (S^T)^{-1}\left(C_s + A(S^Tz) - C\right)S^{-1} \\ &= Id + (S^T)^{-1}\left(A(S^Tz) - C\right)S^{-1} \text{ for } z \in (S^T)^{-1}(\overline{\Omega}),\end{aligned} \tag{II.1.1}$$

by Lemma II.2.6 below we have

$$\omega^p_{\Omega,A}(E) = \omega^{(S^T)^{-1}p}_{(S^T)^{-1}\Omega,M}((S^T)^{-1}E) \text{ for any } p \in \Omega \text{ and } E \subset \partial\Omega. \tag{II.1.2}$$

We claim that the ellipticity constant of M in $(S^T)^{-1}\overline{\Omega}$ is at most $(1-(n+1)\varepsilon\lambda)^{-1}$. Indeed, for $\xi, \eta \in \mathbb{R}^{n+1}$ and $x \in \overline{\Omega}$, by the Cauchy-Schwartz inequality and the well-known fact $|Bv| \leq (n+1)\delta|v|$ if $|b_{ij}| \leq \delta$ for all $1 \leq i,j \leq n+1$, we have

$$\langle\{(S^T)^{-1}\left(A(x) - C\right)S^{-1}\}\xi, \eta\rangle \leq |\left(A(x)-C\right)S^{-1}\xi||S^{-1}\eta| \leq (n+1)\varepsilon|S^{-1}\xi||S^{-1}\eta|.$$

Note that for any vector v there holds $|Sv|^2 = \langle Sv, Sv\rangle = \langle S^TSv, v\rangle = \langle C_sv, v\rangle \geq \lambda^{-1}|v|^2$. All in all,

$$\langle\{(S^T)^{-1}\left(A(x) - C\right)S^{-1}\}\xi, \eta\rangle \leq (n+1)\varepsilon\lambda|\xi||\eta|,$$

which implies

$$\langle M\xi, \eta\rangle \leq (1 + (n+1)\varepsilon\lambda)|\xi||\eta| \text{ and } \langle M\xi, \xi\rangle \geq (1-(n+1)\varepsilon\lambda)|\xi|^2.$$

The claim follows as $1/(1-(n+1)\varepsilon\lambda) \geq 1 + (n+1)\varepsilon\lambda$.

Since the linear mapping given by S^T is bi-Lipschitz and the ellipticity constant of M is at most $(1-(n+1)\varepsilon\lambda)^{-1}$, the lemma follows from (II.1.2) and the definition of $\overline{\dim}_{\mathcal{H}}$ and $\underline{\dim}_{\mathcal{H}}^{\mathrm{sc}}$. $\square$

Note that if the constant matrix C is symmetric, i.e., $C = C_s$, then (II.1.1) reads as

$$M(z) := (S^T)^{-1}A(S^Tz)S^{-1} \text{ for } z \in (S^T)^{-1}\overline{\Omega}. \tag{II.1.3}$$

As we mentioned in the introduction of this subsection, we will apply Lemma II.1.5 and adapt its proof in some proofs below. More precisely, we will compare the matrix A with the frozen coefficient matrix $C = A(\xi)$, for some point $\xi \in \partial\Omega$.

Remark II.1.7. Taking $C = A(\xi)$ (for some point $\xi \in \partial\Omega$) when the matrix A is symmetric with constant determinant, we have that the matrix M in (II.1.3) is symmetric and has determinant 1.

II.2. Quasiconformal mappings

In this section, we provide the definition and main properties of quasiconformal mappings. This is based on the monograph [AIM09].

Definition II.2.1 (Quasiconformal mapping)**.** An orientation-preserving homeomorphism $\phi : \Omega \to \Omega'$ between planar domains is called K-quasiconformal, $1 \le K < \infty$, if $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and the directional derivatives satisfy

$$\max_{\alpha\in[0,2\pi]} |\partial_\alpha f| \le K \min_{\alpha\in[0,2\pi]} |\partial_\alpha f| \text{ a.e. in } \Omega. \tag{II.2.1}$$

Remark II.2.2. In particular, a mapping is 1-quasiconformal if and only if it is conformal.

The condition in (II.2.1) is equivalent to

$$|\bar\partial f| \le k|\partial f| \text{ a.e. in } \Omega, \text{ for } k := \frac{K-1}{K+1},$$

where $\partial := \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar\partial := \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ are the Wirtinger derivatives.

A fundamental property of onto K-quasiconformal mappings $f : \Omega \to \Omega'$ is that its inverse function $f^{-1} : \Omega' \to \Omega$ is also K-quasiconformal, see [AIM09, Theorem 3.1.2]. Also, global K-quasiconformal mappings $f : \mathbb{C} \to \mathbb{C}$ are quasisymmetric, that is, there exists an increasing homeomorphism $\eta_K : [0,\infty) \to [0,\infty)$, only depending on K, such that for each triple $z_0, z_1, z_2 \in \mathbb{C}$ we have

$$\frac{|f(z_0)-f(z_1)|}{|f(z_0)-f(z_2)|} \le \eta_K\left(\frac{|z_0-z_1|}{|z_0-z_2|}\right), \tag{II.2.2}$$

see [AIM09, Theorem 3.5.3]. In this case, we say that f is η_K-quasisymmetric. For a local equivalence between quasiconformal and quasisymmetric mappings, see [AIM09, Theorem 3.6.2].

Another important property is that quasiconformal mappings are locally Hölder continuous, see [AIM09, Corollary 3.10.3].

Theorem II.2.3 (Local Hölder regularity)**.** *Every K-quasiconformal mapping $f : \Omega \to \Omega'$ is locally $\frac{1}{K}$-Hölder continuous. More precisely, if $B \subset 2B \subset \Omega$, then*

$$|f(z) - f(w)| \le C_K \mathrm{diam}(f(B)) \frac{|z-w|^{1/K}}{\mathrm{diam}(B)^{1/K}} \textit{ for all } z, w \in B,$$

where the constant C_K depends only on K.

Quasiconformal mappings are related to solutions of the Beltrami equation, see [AIM09, Theorem 2.5.4]. Given a homeomorphic $W^{1,2}_{\mathrm{loc}}(\Omega)$-mapping $f : \Omega \to \Omega'$, write $\mu(z) = \bar\partial f(z)/\partial f(z)$ when $\partial f(z) \neq 0$ and $\mu(z) = 0$ otherwise. With this, we have the equivalence

$$f \text{ is } K\text{-quasiconformal} \iff \bar\partial f = \mu \partial f \text{ a.e. in } \Omega,$$

where μ, called the Beltrami coefficient of f, is a bounded measurable function satisfying $\|\mu\|_\infty \leq (K-1)/(K+1) = k < 1$.

Notation. With this relation, given μ as above, we say that a function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is μ-quasiconformal if f solves the Beltrami equation $\bar\partial f = \mu \partial f$ a.e. in Ω.

The measurable Riemann mapping theorem ensures the existence (and uniqueness up to normalization) of solutions to a Beltrami equation, see [AIM09, Theorem 5.3.4].

Theorem II.2.4 (Measurable Riemann mapping theorem)**.** *Let $|\mu| \leq k < 1$ for almost every $z \in \mathbb{C}$. Then there is a solution $f : \mathbb{C} \to \mathbb{C}$ to the Beltrami equation*

$$\bar\partial f = \mu \partial f \ \text{ a.e. in } \mathbb{C},$$

which is a K-quasiconformal homeomorphism normalized by the three conditions $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Furthermore, the normalized solution f is unique.

For a compactly supported Beltrami coefficient μ, we call principal solution to the function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with $\bar\partial f = \mu \partial f$ normalized by the condition $f(z) = z + \mathcal{O}(1/z)$ near infinity.

II.2.1. Quasiconformal mappings and L_A-harmonic functions. Given a uniformly elliptic matrix $A \in \mathbb{R}^{2\times 2}$ and a global quasiconformal mapping f of $\mathbb{C}$, we define the uniformly elliptic matrix $\widetilde{A}_f$ as

$$\widetilde{A}_f(x) := (\det Df(x)) Df(x)^{-1} A(f(x)) (Df(x)^{-1})^T. \tag{II.2.3}$$

We remark that the matrix $\widetilde{A}_f$ is uniformly elliptic by the quasiconformal condition of f, see [HKM06, Lemma 14.38], and in particular, planar uniformly elliptic operators are invariant under quasiconformal homeomorphisms. Moreover, these matrices satisfy

$$\widetilde{(\widetilde{A}_f)}_{f^{-1}} = A = \widetilde{(\widetilde{A}_{f^{-1}})}_f.$$

With the notation in (II.2.3), we observe the behavior of L_A-harmonic functions and elliptic measures under quasiconformal changes of variables in the following two lemmas.

Lemma II.2.5 (See [HKM06, 14.39])**.** *Let $\Omega \subset \mathbb{R}^2$ be an open set, $A \in \mathbb{R}^{2\times 2}$ be a uniformly elliptic matrix, and f be a quasiconformal homeomorphism of $\mathbb{C}$. Then u is L_A-harmonic in Ω if and only if $u \circ f$ is $L_{\widetilde{A}_f}$-harmonic in $f^{-1}(\Omega)$.*

Lemma II.2.6 (See [HKM06, 14.51])**.** *Let $\Omega \subset \mathbb{R}^2$ be a domain, $A \in \mathbb{R}^{2\times 2}$ be a uniformly elliptic matrix and f be a quasiconformal homeomorphism of $\mathbb{C}$. For any $x \in \Omega$ and $E \subset \partial\Omega$, we have*

$$\omega^x_{\Omega,A}(E) = \omega^{f^{-1}(x)}_{f^{-1}(\Omega),\widetilde{A}_f}(f^{-1}(E)).$$

For a detailed proof for bi-Lipschitz mappings see [AM19, Lemmas 3.8 and 3.9]. For a more general version of these results see [HKM06, 14.39 and 14.51] respectively.

II.2.2. Distortion under quasiconformal mappings. Now we turn our attention to the distortion of the Hausdorff dimension and measure under K-quasiconformal mappings. Astala in [Ast94, Theorem 1.1] (see also [AIM09, Theorem 13.1.5]) proved the area distortion theorem, that is, there is a constant C_K depending only on $K \geq 1$, such that for any K-quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, for any disk $B \subset \mathbb{C}$ and for any subset $E \subset B$, we have

$$\frac{1}{C_K}\left(\frac{|E|}{|B|}\right)^K \leq \frac{|f(E)|}{|f(B)|} \leq C_K \left(\frac{|E|}{|B|}\right)^{1/K}. \tag{II.2.4}$$

As a corollary, see [Ast94, Corollary 1.3] and also [AIM09, Theorem 13.2.10], the area distortion theorem governs the distortion of Hausdorff dimension

$$\frac{1}{K}\left(\frac{1}{\dim_{\mathcal{H}}(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim_{\mathcal{H}}(f(E))} - \frac{1}{2} \leq K\left(\frac{1}{\dim_{\mathcal{H}}(E)} - \frac{1}{2}\right). \tag{II.2.5}$$

In the preprint [Tol09] (see also [ACT$^+$13, Theorem 1.2] for the published version), Tolsa generalized the previous two results by obtaining the control of the Hausdorff measure.

Theorem II.2.7 (Distortion of Hausdorff measure)**.** *Let $0 < t < 2$ and denote $t' = \frac{2Kt}{2+(K-1)t}$. Let $f : \mathbb{C} \to \mathbb{C}$ be K-quasiconformal. For any ball B and any compact set $E \subset B$, we have*

$$\frac{\mathcal{H}^{t'}(f(E))}{\operatorname{diam}(f(B))^{t'}} \leq C_{K,t}\left(\frac{\mathcal{H}^t(E)}{\operatorname{diam}(B)^t}\right)^{\frac{t'}{Kt}}, \tag{II.2.6}$$

In particular, if $\mathcal{H}^t(E)$ is finite (resp. σ-finite), then also $\mathcal{H}^{t'}(f(E))$ is finite (resp. σ-finite).

The case $t = t' = 2$ is precisely the area distortion theorem in (II.2.4). The σ-finiteness in the particular case $t' = 1$ had been solved in [ACM$^+$08]. The same control in (II.2.6) with Hausdorff content $\mathcal{H}_\infty$ instead of Hausdorff measures was proved in [LSUT10].

II.2.2.1. *Distortion of gauge Hausdorff measures.* Following Carathéodory's construction (see [Mat95, p. 54]), we define (gauge) spherical measures as in [ACT$^+$13, Section 2.3]. Let $\mathcal{B}$ denote the family of all closed balls contained in $\mathbb{C}$, and let $h : \mathcal{B} \to [0, \infty)$ be any function defined in $\mathcal{B}$ with $h(\overline{B(x,r)}) \to 0$ as $r \to 0$, for all $x \in \mathbb{C}$. Given $0 < \delta \leq \infty$ and $E \subset \mathbb{C}$, we set

$$\mathcal{S}^h_\delta(E) := \inf\left\{\sum_i h(\overline{B_i}) : E \subset \bigcup_i \overline{B_i},\ \text{with } \overline{B_i} \in \mathcal{B} \text{ and } r(B_i) \leq \delta\right\}.$$

The h-spherical measure $\mathcal{S}^h$ is defined as

$$\mathcal{S}^h(E) := \lim_{\delta\to 0} \mathcal{S}^h_\delta(E) = \sup_{\delta>0} \mathcal{S}^h_\delta(E),$$

as $\mathcal{S}^h_\delta(E)$ increases as δ decreases. The $\delta = \infty$ case $\mathcal{S}^h_\infty$ is called h-spherical content. It is clear that $\mathcal{S}^h_\infty(E) \leq \mathcal{S}^h(E)$. Despite the converse inequality does not hold, $\mathcal{S}^h_\infty(E) = 0$ implies $\mathcal{S}^h(E) = 0$ if the function $r \mapsto h(\overline{B(x,r)})$ is nondecreasing for all $x \in \mathbb{C}$. Although $\mathcal{S}^h_\infty$ is not measure, in this case we write $\mathcal{S}^h \ll \mathcal{S}^h_\infty \ll \mathcal{S}^h$.

With abuse of notation, we write $\mathcal{S}^h = \mathcal{S}^{\widetilde{h}}$ if $h(\overline{B}) = \widetilde{h}(r_B)$, $\overline{B} \in \mathcal{B}$, for some continuous nondecreasing function $\widetilde{h} : [0,\infty) \to [0,\infty)$ with $\widetilde{h}(0) = 0$. If in addition $C_h := \limsup_{s\to 0} \widetilde{h}(4r)/\widetilde{h}(r) < \infty$, a routinary computation gives

$$\mathcal{H}^{\widetilde{h}(\cdot/2)}(E) \leq \mathcal{S}^h(E) \leq C_h \mathcal{H}^{\widetilde{h}(\cdot/2)}(E) \text{ for any set } E \subset \mathbb{C}.$$

Let us introduce the notation used in Lemma II.2.8 below, see [ACT$^+$13, Section 2.3]. Let $\varepsilon : \mathcal{B} \to [0,\infty)$ be any function defined on $\mathcal{B}$, the family of all closed balls in $\mathbb{C}$. We set $\varepsilon(x,r) := \varepsilon(B(x,r))$. Given an arbitrary bounded set $S \subset \mathbb{C}$, let $\mathcal{B}(S)$ denote the family of balls with minimal diameter containing S, and set $\varepsilon(S) := \inf_{B\in\mathcal{B}(S)} \varepsilon(B)$.

As in [ACT$^+$13, Section 2.4], the families $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined as:

($\mathcal{G}_1$) We say that a function $\varepsilon : \mathcal{B} \to [0,\infty)$ belongs to $\mathcal{G}_1$ if there is a constant C_1 such that

$$C_1^{-1}\varepsilon(x,r) \le \varepsilon(y,s) \le C_1\varepsilon(x,r)$$

whenever $|x-y| \le 2r$ and $r/2 \le s \le 2r$.

($\mathcal{G}_2$) For each fixed $0 < t < 2$, a function $\varepsilon = \varepsilon(x,r)$ belongs to the family $\mathcal{G}_2 = \mathcal{G}_2(t)$ if there is a constant C_2 such that

$$\sum_{k\ge 0} 2^{-k(2-t)}\varepsilon(x,2^k r) \le C_2\varepsilon(x,r).$$

We present a technical lemma used in the proof in [ACT$^+$13] of Theorem II.2.7.

Lemma II.2.8 (See [ACT$^+$13, Lemma 4.4])**.** *Let $0 < t < 2$ and $t' = \frac{2Kt}{2+(K-1)t}$. Let $\varepsilon \in \mathcal{G}_1$ and set $h(x,r) = r^{t'}\varepsilon(x,r)$. Suppose that for any principal K-quasiconformal mapping $\psi : \mathbb{C} \to \mathbb{C}$ the function $(\varepsilon \circ \psi)^d$ belongs to $\mathcal{G}_2$ for any $\frac{t'}{Kt} \le d \le 1$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping, conformal on $\mathbb{C} \setminus \overline{\mathbb{D}}$, and set*

$$\widetilde{\varepsilon}(x,r) = \varepsilon(\phi(B(x,r)))^{\frac{Kt}{t'}}, \quad \widetilde{h}(x,r) = r^t\widetilde{\varepsilon}(x,r).$$

Then for every compact set $E \subset B(0,1/2)$ we have

$$\mathcal{S}^h_\infty(\phi(E)) \le C_K \mathcal{S}^{\widetilde{h}}_\infty(E)^{\frac{t'}{Kt}}.$$

To study the distortion under quasiconformal mappings of the gauge Hausdorff measure in Makarov's Theorem I.0.1, we aim to apply the previous lemma to $h = \varphi_{1,C}$, where $\varphi_{\rho,C}$ is defined in (II.0.1). We now recall some properties of $\varphi_{1,C}(r)/r$ for sufficiently small $r > 0$. For $s > e$, we set[(II.2)]

$$\mathbb{L}\mathrm{og}(s) := \log s \log\log\log s,$$

and for $s_0 > e^e$ and $s > 0$, we set

$$\mathbb{L}\mathrm{og}_{s_0}(s) := \begin{cases} \mathbb{L}\mathrm{og}(s_0), & \text{if } 0 < s \le s_0, \\ \mathbb{L}\mathrm{og}(s), & \text{if } s > s_0. \end{cases}$$

Note that this function is nondecreasing for $s > 0$.

Given $C > 0$ and $K \ge 1$, we define the nonincreasing function

$$\varepsilon(r) := \exp\left(C\sqrt{\mathbb{L}\mathrm{og}_{r_0^{-1}}(r^{-1})}\right),$$

and we fix $0 < r_0 = r_0(C,K) < e^{-e}$ (the existence of such r_0 is verified below) sufficiently small so that

$$r^{\frac{1}{2K}}\varepsilon(r) \text{ is nondecreasing for } r > 0, \text{ and} \tag{II.2.7a}$$

$$\varepsilon(r/2) \le 2\varepsilon(r) \text{ for all } 0 < r \le r_0. \tag{II.2.7b}$$

[(II.2)]Note that $\mathbb{L}\mathrm{og}(s)$ is not defined when $0 < s \le e$, and $\mathbb{L}\mathrm{og}(s) \ge 0$ whenever $s \ge e^e$.

The condition (II.2.7a) implies that φ and $\widetilde{\varphi}$ in Corollary II.2.9 below are nondecreasing for $r > 0$, and (II.2.7b) guarantees that

$$\varepsilon(r/2) \leq 2\varepsilon(r) \text{ for all } r > 0. \tag{II.2.8}$$

Indeed, $\varepsilon(r)$ is constant for $r > r_0$, and for $r_0 < r \leq 2r_0$, since $\varepsilon(r)$ is nonincreasing, we obtain $\varepsilon(r/2) \leq \varepsilon(r_0/2) \leq 2\varepsilon(r_0) \leq 2\varepsilon(r)$, as claimed.

The parameter $r_0 > 0$ in (II.2.7a) and (II.2.7b) exists because

$$\frac{\varepsilon(r/2)}{\varepsilon(r)} = \exp\left(C\left(\sqrt{Q_2(r)} - 1\right)\sqrt{\log\frac{1}{r}\log\log\log\frac{1}{r}}\right) \to 1 \text{ as } r \to 0,$$

where Q_2 is the particular case $T = 2$ of the function Q_T, defined for a fixed constant $T > 0$ and sufficiently small $r > 0$ as

$$Q_T(r) := \frac{\log(T\frac{1}{r})\log\log\log(T\frac{1}{r})}{\log\frac{1}{r}\log\log\log\frac{1}{r}} = \left(\frac{\log T}{\log\frac{1}{r}} + 1\right)\left(\frac{\log\left(\frac{\log\left(\frac{\log T}{\log\frac{1}{r}}+1\right)}{\log\log\frac{1}{r}} + 1\right)}{\log\log\log\frac{1}{r}} + 1\right), \tag{II.2.9}$$

which satisfies

$$\lim_{r\to 0}\left(\sqrt{Q_T(r)} - 1\right)\sqrt{\log\frac{1}{r}\log\log\log\frac{1}{r}} = 0. \tag{II.2.10}$$

Here we state the distortion of the gauge function in Theorem I.0.1 under quasiconformal mappings. The proof will be a routine computation to check the conditions of the previous lemma for this particular case. For this reason, its proof is located at the end of the subsection.

Corollary II.2.9. *Let $C > 0$, $K \geq 1$, $0 < r_0 = r_0(C, K) < e^{-e}$ as in (II.2.7a) and (II.2.7b), and $\phi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping and conformal on $\mathbb{C} \setminus \overline{\mathbb{D}}$. Then there exists a constant $C_{K,\phi}$ such that if for $r > 0$ we define*

$$\varphi(r) = r\exp\left(C\sqrt{\mathbb{L}\mathrm{og}_{r_0^{-1}}(r^{-1})}\right), \quad \widetilde{\varphi}(r) = r^{\frac{2}{K+1}}\exp\left(C\frac{2K}{K+1}\sqrt{\mathbb{L}\mathrm{og}_{r_0^{-1}}\left(C_{K,\phi}^{-1}r^{-K}\right)}\right),$$

then for every compact set $E \subset B(0, 1/2)$ we have

$$\mathcal{S}^{\varphi}_{\infty}(\phi(E)) \leq C_K \mathcal{S}^{\widetilde{\varphi}}_{\infty}(E)^{\frac{K+1}{2K}}.$$

Since for sets with zero Hausdorff or spherical measure and content we only need to consider the gauge function for small values, and the relations $\mathcal{S}^h \ll \mathcal{S}^h_\infty \ll \mathcal{S}^h$ and $\mathcal{H}^{\widetilde{h}(\cdot/2)} \ll \mathcal{S}^h \ll \mathcal{H}^{\widetilde{h}(\cdot/2)}$ hold when $h(E) = \widetilde{h}(\mathrm{diam}(E)/2)$ for some function nondecreasing function $\widetilde{h} : [0, \infty) \to [0, \infty)$ satisfying $\widetilde{h}(r) \to 0$ as $r \to 0$ and $\limsup_{r\to 0}\widetilde{h}(4r)/\widetilde{h}(r) < \infty$, we obtain the following immediate consequence of Corollary II.2.9.

Corollary II.2.10. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping and conformal on $\mathbb{C} \setminus \overline{\mathbb{D}}$, and $\varphi_{\cdot,\cdot}$ as in (II.0.1). Then $\mathcal{H}^{\varphi_{K,C}}(E) = 0$ implies $\mathcal{H}^{\varphi_{1,C}}(\phi(E)) = 0$ for all $E \subset B(0, 1/2)$.*

Proof. By the discussion before this lemma, it remains to see $\limsup_{r\to 0}\varphi_{\rho,C}(4r)/\varphi_{\rho,C}(r) < \infty$. For any constant $T > 0$, recall the function Q_T in (II.2.9). For any constant $S > 0$, by

(II.2.10) we have that if $r > 0$ is small enough, then

$$\varphi_{\rho,C}(Sr) = S^{\frac{2}{\rho+1}}\varphi_{\rho,C}(r)\exp\left(C\frac{2\rho}{\rho+1}\left(\sqrt{Q_{S^{-\rho}}(r^\rho)}-1\right)\sqrt{\log r^{-\rho}\log\log\log r^{-\rho}}\right)$$
$$\leq 1.1 S^{\frac{2}{\rho+1}}\varphi_{\rho,C}(r),$$

and the result follows from this and Corollary II.2.9. □

Proof of Corollary II.2.9. Take $t' = 1$, $t = \frac{2}{K+1}$, $\varepsilon(x,r) := \varphi(r)/r$ and $h(x,r) := r^{t'}\varepsilon(x,r)$ in Lemma II.2.8. Note that in this case the functions ε and h do not depend on x. So, we write $\varepsilon(r)$ and $h(r)$ instead of $\varepsilon(x,r)$ and $h(x,r)$ respectively. Recall that

$$\varepsilon(r) := \frac{\varphi(r)}{r} = \exp\left(C\sqrt{\mathbb{L}\mathrm{og}_{r_0^{-1}}(r^{-1})}\right)$$

is nonincreasing with $\lim_{r\to 0^+}\varepsilon(r) = +\infty$.

For now, assume we may invoke Lemma II.2.8 (we will verify its hypotheses later). Let us first show how this result implies the corollary. By the choice of t, t', $\varepsilon(r)$ and $h(r)$ at the beginning of this proof, setting

$$\widetilde{\varepsilon}(x,r) := \varepsilon(\phi(B(x,r)))^{\frac{2K}{K+1}}, \quad \widetilde{h}(x,r) := r^{\frac{2}{K+1}}\widetilde{\varepsilon}(x,r),$$

by Lemma II.2.8 we obtain

$$\mathcal{S}^{\varphi}_{\infty}(\phi(E)) \leq C_K \mathcal{S}^{\widetilde{h}}_{\infty}(E)^{\frac{K+1}{2K}}. \tag{II.2.11}$$

Now we want to remove the dependence of the quasiconformal mapping ϕ on the gauge function $\widetilde{h}$. Recall $\varepsilon(\phi(B(x,r))) = \inf_{B\in\mathcal{B}(\phi(B(x,r)))}\varepsilon(r_B)$ where $\mathcal{B}(\phi(B(x,r)))$ denotes the family of balls with minimal diameter containing $\phi(B(x,r))$. Since ε does not depend on the point x, we get

$$\varepsilon(\phi(B(x,r))) = \inf_{B\in\mathcal{B}(\phi(B(x,r)))}\varepsilon(r_B) = \varepsilon\left(\frac{\mathrm{diam}(\phi(B(x,r)))}{2}\right).$$

From the local Hölder continuity of K-quasiconformal mappings in Theorem II.2.3, there exists $C_{K,\phi}$ with $C_{K,\phi}^{-1}r^K \leq \mathrm{diam}(\phi(B(x,r)))/2 \leq C_{K,\phi}r^{1/K}$ for all $B(x,r)\subset B(0,100)$, and since ε is nonincreasing we have

$$\varepsilon(\phi(B(x,r))) \leq \varepsilon(C_{K,\phi}^{-1}r^K).$$

This implies

$$\widetilde{h}(x,r) = r^{\frac{2}{K+1}}\varepsilon(\phi(B(x,r)))^{\frac{2K}{K+1}} \leq r^{\frac{2}{K+1}}\varepsilon(C_{K,\phi}^{-1}r^K)^{\frac{2K}{K+1}} = \widetilde{\varphi}(r),$$

which together with (II.2.11) concludes the lemma.

It remain to check the assumptions in Lemma II.2.8, i.e., $\varepsilon \in \mathcal{G}_1$, and $(\varepsilon\circ\psi)^d \in \mathcal{G}_2$ for any principal K-quasiconformal mapping $\psi : \mathbb{C}\to\mathbb{C}$ and any $\frac{t'}{Kt} \leq d \leq 1$.

Since the function ε does not depend on the point, to verify $\varepsilon\in\mathcal{G}_1$ it suffices to check that $2^{-1}\varepsilon(r) \leq \varepsilon(s) \leq 2\varepsilon(r)$ whenever $r/2 \leq s \leq 2r$. It follows directly from (II.2.8) and the fact that ε is nonincreasing, because $\frac{1}{2}\varepsilon(r) \leq \varepsilon(2r) \leq \varepsilon(s) \leq \varepsilon(r/2) \leq 2\varepsilon(r)$, as claimed.

Let us see now that $(\varepsilon\circ\psi)^d \in \mathcal{G}_2$ for any principal K-quasiconformal mapping $\psi : \mathbb{C}\to\mathbb{C}$ and any $\frac{t'}{Kt} \leq d \leq 1$, that is, there exists a constant $C_2 > 0$ such that

$$\sum_{k\geq 0} 2^{-k(2-t)}\left((\varepsilon\circ\psi)(x,2^k r)\right)^d \leq C_2\left((\varepsilon\circ\psi)(x,r)\right)^d.$$

Recall that $\mathcal{B}(S)$ denotes the family of balls with minimal diameter containing the set S, $(\varepsilon\circ\psi)(x,s) = \varepsilon(\psi(B(x,s))) = \inf_{B\in\mathcal{B}(\psi(B(x,s)))}\varepsilon(r_B)$, and that the function ε does not depend on the point. Since ψ is an homeomorphism, we have $\psi(B(x,2^k r)) \subset \psi(B(x,2^{k+1}r))$ for any $k\geq 0$. In particular, a minimizer ball for $\psi(B(x,2^{k+1}r))$ contains the set $\psi(B(x,2^k r))$, and therefore the minimizer ball of $\psi(B(x,2^k r))$ has smaller diameter than the minimizer ball of $\psi(B(x,2^{k+1}r))$. This and the fact that the function ε is nonincreasing imply $(\varepsilon\circ\psi)(x,2^{k+1}r)\leq(\varepsilon\circ\psi)(x,2^k r)$ for any $k\geq 0$. Consequently, we have

$$\sum_{k\geq 0} 2^{-k(2-t)}\left((\varepsilon\circ\psi)(x,2^k r)\right)^d \leq ((\varepsilon\circ\psi)(x,r))^d\sum_{k\geq 0}2^{-k(2-t)}.$$

That is, the $\mathcal{G}_2$ condition is satisfied with $C_2 := \sum_{k\geq 0}2^{-k(2-t)} < \infty$. □

II.2.3. The L_A-harmonic conjugate in simply connected domains. This subsection is a brief summary of [AIM09, Section 16.1.3]. Let $\Omega\subset\mathbb{R}^2$ be a bounded simply connected domain. We remark that the simply connected condition on Ω used in this subsection is just an auxiliary assumption to be able to define the auxiliary L_A-harmonic conjugate function below.

Consider u a solution to

$$\operatorname{div} A(z)\nabla u(z) = 0, \quad u\in W^{1,2}_{\mathrm{loc}}(\Omega),$$

and consider the vector fields ∇u and $A(z)\nabla u$. The vector field ∇u has zero curl since $\operatorname{curl}\nabla u = \partial_y\partial_x u - \partial_x\partial_y u = 0$, and $A(z)\nabla u$ is divergence-free as a solution to the previous equation.

The Hodge star operator $*$ is defined as

$$* := \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} : \mathbb{R}^2\to\mathbb{R}^2,$$

which is simply the counterclockwise rotation by 90 degrees, and so $** = -Id$. The Hodge star operator $*$ transforms curl-free fields into divergence-free fields and vice versa. Therefore

$$\operatorname{div}(*\nabla u) = 0 \quad\text{and}\quad \operatorname{curl}(*A(z)\nabla u) = 0.$$

By the Poincaré lemma[(II.3)], see [AIM09, Lemma 16.1.3] for instance, we have that $*A(z)\nabla u$ is a gradient field. That is, there exists a real-valued function $v\in W^{1,2}_{\mathrm{loc}}(\Omega)$, called the L_A-harmonic conjugate of u, such that

$$\nabla v = *A(z)\nabla u. \tag{II.2.12}$$

Define now

$$A^* := *^t A^{-1} * = - * A^{-1} *,$$

which is also uniformly elliptic and

$$A^*(z)\nabla v = A^*(z)(*A(z)\nabla u) = (- * A^{-1} *)(*A(z)\nabla u) = *\nabla u.$$

Recall that $\operatorname{curl}\nabla u = 0$, and therefore $\operatorname{div}(*\nabla u) = 0$, i.e., the function v satisfies

$$\operatorname{div} A^*(z)\nabla v = 0, \quad v\in W^{1,2}_{\mathrm{loc}}(\Omega).$$

In the converse direction, the existence of $v\in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $\nabla v = *A\nabla u$ implies $\operatorname{div} A\nabla u = 0$, because $\operatorname{curl}(*A\nabla u) = \operatorname{curl}\nabla v = 0$, the Hodge star operator $*$ sends zero-curl to zero-divergence and $** = -Id$. As before, $\operatorname{div} A\nabla u = 0$ implies $\operatorname{div} A^*\nabla v = 0$.

[(II.3)] It is at this step that we need to work with simply connected domains $\Omega\subset\mathbb{R}^2$.

Remark II.2.11 (Symmetric matrices with determinant 1)**.** Wherever the matrix is symmetric with determinant 1 then $A^* = A$, and hence both u and its L_A-harmonic conjugate function v are L_A-harmonic.

The following theorem describes the strong connection between the Beltrami equation and PDEs.

Theorem II.2.12 (Weak [AIM09, Theorem 16.1.6])**.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a solution*

$$\operatorname{div} A(z)\nabla u = 0.$$

If $v \in W^{1,1}(\Omega)$ is a solution to $\nabla v = *A\nabla u$, then the function $f = u + iv$ satisfies the generalized Beltrami equation

$$\bar{\partial} f = \mu_A(z)\partial f + \nu_A(z)\overline{\partial f}, \tag{II.2.13}$$

where the coefficients μ_A, ν_A are given by

$$\begin{cases} \mu_A = & \frac{1}{\det(Id+A)}\left(a_{22} - a_{11} - i(a_{12} + a_{21})\right), \\ \nu_A = & \frac{1}{\det(Id+A)}\left(1 - \det A + i(a_{12} - a_{21})\right). \end{cases} \tag{II.2.14}$$

Conversely, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a mapping satisfying the generalized Beltrami equation (II.2.13), then $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ satisfy $\nabla v = *A\nabla u$ with A given by solving the complex equations in (II.2.14).

Notice also from (II.2.14) that when the matrix is symmetric and with $\det A = 1$, then $\nu_A = 0$ and (II.2.13) reads as the classical Beltrami equation

$$\bar{\partial} f = \mu_A(z)\partial f \text{ with } \mu_A = \frac{a_{22} - a_{11} - 2ia_{12}}{2 + a_{11} + a_{22}}.$$

In this case, we have the following relations:

Corollary II.2.13. *Let $A = (a_{ij})^2_{i,j=1}$ be a uniformly elliptic matrix. Wherever A is symmetric with determinant 1, then the Beltrami coefficients in (II.2.14) are*

$$\mu_A = \frac{a_{22} - a_{11} - 2ia_{12}}{2 + a_{11} + a_{22}}, \quad \nu_A = 0, \tag{II.2.15}$$

and given a domain $\Omega \subset \mathbb{R}^2$ and $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$, the following are equivalent:

(1) $\bar{\partial} f = \mu_A(z)\partial f$,
(2) $\nabla(\mathrm{Im} f) = *A\nabla(\mathrm{Re} f)$,
(3) $\bar{\partial}(f^{-1}) = -\mu_A(f^{-1}(z))\overline{\partial(f^{-1})}$,
(4) $\nabla\left(\mathrm{Im}(f^{-1})\right) = *B\nabla\left(\mathrm{Re}(f^{-1})\right)$,

where

$$B := \begin{pmatrix} 1/a_{11} & a_{12}/a_{11} \\ -a_{12}/a_{11} & 1/a_{11} \end{pmatrix} \circ f^{-1} \text{ and } B^* = \begin{pmatrix} 1/a_{22} & -a_{12}/a_{22} \\ a_{12}/a_{22} & 1/a_{22} \end{pmatrix} \circ f^{-1}. \tag{II.2.16}$$

are uniformly elliptic. In this case, we also have,

$$\operatorname{div} A\nabla(\mathrm{Re} f) = 0, \qquad \operatorname{div} A\nabla(\mathrm{Im} f) = 0,$$
$$\operatorname{div} B\nabla\left(\mathrm{Re}(f^{-1})\right) = 0, \qquad \operatorname{div} B^*\nabla\left(\mathrm{Im}(f^{-1})\right) = 0.$$

Proof. During the proof we write $\mu = \mu_A$. From Theorem II.2.12 we have $\bar{\partial} f = \mu \partial f$ if and only if $\nabla(\mathrm{Im} f) = *A\nabla(\mathrm{Re} f)$, with Beltrami coefficient μ as in (II.2.15), which gives (1) $\Leftrightarrow$ (2).

By [AIM09, Theorem 5.5.6 or Lemma 10.3.1], $\bar{\partial} f = \mu(z)\partial f$ if and only if

$$\bar{\partial}(f^{-1}) = -\mu(f^{-1}(z))\overline{\partial(f^{-1})}, \tag{II.2.17}$$

and so (1) $\Leftrightarrow$ (3). Hence, from Theorem II.2.12 there exists[II.4] a matrix B such that (II.2.17) is equivalent to having $\nabla\left(\mathrm{Im}(f^{-1})\right) = *B\nabla\left(\mathrm{Re}(f^{-1})\right)$. The matrix B defined in (II.2.16) satisfies (II.2.14) with $\nu = -\mu \circ f^{-1}$, i.e.,

$$-\frac{a_{22} - a_{11} - 2ia_{12}}{2 + a_{11} + a_{22}} = -\mu = \frac{1 - \det(B \circ f) + 2ib_{12}}{\det(Id + B \circ f)},$$

and hence we have (3) $\Leftrightarrow$ (4). Moreover, a routine computation $B^* = - * B^{-1} *$ gives the second matrix in (II.2.16).

It remains to see the last part. Since $\nabla\left(\mathrm{Im}(f^{-1})\right) = *B\nabla\left(\mathrm{Re}(f^{-1})\right)$ by (4), we have that

$$\operatorname{div} B\nabla\left(\mathrm{Re}(f^{-1})\right) = 0, \quad \operatorname{div} B^*\nabla\left(\mathrm{Im}(f^{-1})\right) = 0,$$

as we did in the paragraph before Remark II.2.11. The same holds for the real and imaginary parts of f with the matrix A and A^* respectively, but $A^* = A$ by Remark II.2.11. □

Notice that even though the matrix A is symmetric with determinant 1, the new matrix B in (II.2.16) is no longer symmetric with determinant 1. However, we still have that $\mathrm{Re}(f^{-1})$ (resp. $\mathrm{Im}(f^{-1})$) is L_B-harmonic (resp. $L_{B^\star}$-harmonic).

The following result gives the precise distortion of a quasiconformal mapping solving the generalized Beltrami equation (II.2.13).

Lemma II.2.14. *Let $A \in \mathbb{R}^{2\times 2}$ be a uniformly elliptic matrix with ellipticity constant $\lambda \geq 1$, μ_A and ν_A as in (II.2.14), and K_λ as in (II.0.2). Then*

$$\||\mu_A| + |\nu_A|\|_\infty \leq \frac{K_\lambda - 1}{K_\lambda + 1} < 1.$$

Proof. See [AN09, Proposition 1.8] for general matrices and [DKW17, Lemma 4.1] for the symmetric case. □

Remark II.2.15. Note that in both cases $K_\lambda \to 1^+$ as $\lambda \to 1^+$. In particular $\frac{K_\lambda - 1}{K_\lambda + 1} \to 0^+$ as $\lambda \to 1^+$.

Remark II.2.16. Let A be a uniformly elliptic and symmetric matrix with determinant 1. Recall from Remark II.1.4 we noted that the modification $\widetilde{A} = \varphi A + (1-\varphi)Id$ is symmetric but it has no longer determinant 1 in general. The right way to interpolate the matrices A and Id to preserve also that the modified matrix has determinant 1 is by replacing A by the uniformly elliptic (with the same ellipticity constant as A), symmetric and with determinant 1 matrix

$$\widehat{A} = \frac{\begin{pmatrix} a_{11}(1+\varphi)^2 + a_{22}(\varphi-1)^2 + 2(1-\varphi^2) & 4\varphi a_{12} \\ 4\varphi a_{12} & a_{11}(\varphi-1)^2 + (1+\varphi)(a_{22}(1+\varphi) + 2(1-\varphi)) \end{pmatrix}}{a_{11}(1-\varphi^2) + a_{22}(1-\varphi^2) + 2(1+\varphi^2)},$$

[II.4]Actually, Theorem II.2.12 gives us how to find the matrix B from A. Directly from (II.2.17) and (II.2.14) we deduce that the matrix B is antisymmetric with a common diagonal term.

which agrees with A when $\varphi = 1$ and with Id when $\varphi = 0$. This is precisely the matrix such that $\mu_{\widehat{A}} = \varphi\mu_A$ and $\nu_{\widehat{A}} = \varphi\nu_A = 0$, by the relation between uniformly elliptic matrices and Beltrami coefficients in Theorem II.2.12. As $\|\mu_{\widehat{A}}\|_\infty \leq \|\mu_A\|_\infty$, in particular we have that $\widehat{A}$ has the same ellipticity constant as A.

II.2.4. Harmonic factorization in general domains. In this subsection, we show that there exists a quasiconformal homeomorphism factorizing L_A-harmonic functions into harmonic functions in its image domains.

Let $\Omega \subset \mathbb{R}^2$ be an open set, A be a uniformly elliptic matrix, and μ_A and ν_A as in (II.2.14). Given an L_A-harmonic function $u \in W^{1,2}_{\text{loc}}(\Omega)$ and any simply connected domain $D \subset \Omega$, let $v \in W^{1,2}_{\text{loc}}(D)$ be the L_A-harmonic conjugate function of the L_A-harmonic function $u \in W^{1,2}_{\text{loc}}(D)$. By Theorem II.2.12, the function $f = u + iv \in W^{1,2}_{\text{loc}}(D)$ satisfies the Beltrami equation

$$\bar\partial f = \mu_A \partial f + \nu_A \overline{\partial f} \text{ a.e. in } D.$$

In particular, the function f satisfies the classical Beltrami equation

$$\bar\partial f = \widetilde{\mu}\partial f \text{ a.e. in } D, \text{ with } \widetilde{\mu} \coloneqq \begin{cases} \mu_A + \nu_A \frac{\overline{\partial f}}{\partial f} & \text{where } \overline{\partial f}/\partial f \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases} \tag{II.2.18}$$

Note that the Beltrami coefficient $\widetilde{\mu}$ seems to depend on $\overline{\partial f}/\partial f$, but in fact it only depends on (the derivative of) u. Indeed, by the relation $\nabla v = *A(z)\nabla u$, see (II.2.12), we can write in ∂f the terms $\partial_x v$ and $\partial_y v$ using $\partial_x u$ and $\partial_y u$. More precisely,

$$\partial f = (1 + a_{11})\partial_x u + a_{12}\partial_y u - i(a_{21}\partial_x u + (1 + a_{22})\partial_y u), \tag{II.2.19}$$

and in particular, $\widetilde{\mu}$ in (II.2.18) depends on μ_A, ν_A and the derivatives of the function u, but not on the auxiliar L_A-harmonic conjugate function v in D. By (II.2.18) and (II.2.19) we can redefine $\widetilde{\mu}$ in whole Ω as

$$\mu_{A,u} \coloneqq \mu_A + \nu_A \frac{(1 + a_{11})\partial_x u + a_{12}\partial_y u + i(a_{21}\partial_x u + (1 + a_{22})\partial_y u)}{(1 + a_{11})\partial_x u + a_{12}\partial_y u - i(a_{21}\partial_x u + (1 + a_{22})\partial_y u)} \text{ a.e. in } \Omega. \tag{II.2.20}$$

The main result of this subsection is the following.

Lemma II.2.17. *Let $\Omega \subset \mathbb{R}^2$ be an open set, A be a uniformly elliptic matrix with ellipticity constant $\lambda \geq 1$, and μ_A and ν_A as in (II.2.14). For any L_A-harmonic function $u \in W^{1,2}_{\text{loc}}(\Omega)$, if $\phi \in W^{1,2}_{\text{loc}}(\Omega)$ be a homeomorphic solution of the Beltrami equation*

$$\bar\partial\phi = \mu_{A,u}\partial\phi \textit{ a.e. in } \Omega, \textit{ where } \mu_{A,u} \textit{ as in (II.2.20),} \tag{II.2.21}$$

then $u \circ \phi^{-1} : \phi(\Omega) \to \mathbb{R}$ is harmonic.

The existence of ϕ is given by the Riemann mapping theorem, see Theorem II.2.4. Moreover, ϕ is K_λ-quasiconformal with K_λ as in (II.0.2), since $\|\mu_{A,u}\|_\infty \leq \||\mu_A| + |\nu_A|\|_\infty \leq \frac{K_\lambda - 1}{K_\lambda + 1} < 1$ by Lemma II.2.14.

Proof of Lemma II.2.17. We claim that it suffices to prove the lemma for simply connected domains. Indeed, for a general open set Ω, for each point $x \in \Omega$ take an open ball B with $x \in B \subset \Omega$, and in particular $u \in W^{1,2}_{\text{loc}}(B)$ and u is L_A-harmonic in B. Assuming the lemma for simply connected domains, we have that $u \circ \phi^{-1}_{u,A} : \phi_{u,A}(B) \to \mathbb{R}$ is harmonic. Since it is true for any $x \in \Omega$, we obtain that $u \circ \phi^{-1}_{u,A}$ is harmonic in $\phi_{u,A}(\Omega)$.

Let us see the proof assuming also that Ω is simply connected. Let $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be the L_A-harmonic conjugate function of the L_A-harmonic function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. By Theorem II.2.12, the function $f = u + iv \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies the Beltrami equation $\bar{\partial} f = \mu_A \partial f + \nu_A \overline{\partial f}$ a.e. in Ω, and in particular, also safisfies the classical Beltrami equation in (II.2.18). By the discussion before the statement of this lemma, we have $\widetilde{\mu} = \mu_{A,u}$ in Ω and therefore both $f = u + iv \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $\phi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ solve (II.2.21) in Ω. By Stoilow factorization (see [AIM09, Theorem 5.5.1]), the function $f \circ \phi^{-1} : \phi(\Omega) \to \mathbb{C}$ is holomorphic. Taking the real part on both sides we obtain

$$u(\phi^{-1}(z)) = \mathrm{Re}(f)(\phi^{-1}(z)) = \mathrm{Re}\left(f \circ \phi^{-1}\right)(z),$$

and so $u \circ \phi^{-1} : \phi(\Omega) \to \mathbb{R}$ is harmonic. □

II.2.4.1. *Symmetric matrices with determinant 1.* Let us move to the case where the matrix A is symmetric with $\det A = 1$ in Ω. Recall that we simply have $K_\lambda = \lambda$ in Ω since the matrix is symmetric there, see (II.0.2). Note that in this case, by (II.2.15) in Corollary II.2.13 we have that, for any L_A-harmonic function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, the Beltrami coefficient $\mu_{A,u}$ in (II.2.20) equals μ_A in Ω. In particular, we directly obtain the following result.

Corollary II.2.18. *Let $\Omega \subset \mathbb{R}^2$ be an open set and A be a uniformly elliptic matrix with ellipticity constant $\lambda \geq 1$. Assume also that A is symmetric with $\det A = 1$ in Ω. Let $\phi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a homeomorphic solution of the Beltrami equation*

$$\bar{\partial}\phi = \mu_A \partial \phi \text{ a.e. in } \Omega, \text{ where } \mu_A = \frac{a_{22} - a_{11} - 2ia_{12}}{2 + a_{11} + a_{22}}.$$

For any L_A-harmonic function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then $u \circ \phi^{-1} : \phi(\Omega) \to \mathbb{R}$ is harmonic.

As the function ϕ above factorizes every L_A-harmonic function in Ω into a harmonic function in $\phi(\Omega)$, the corollary suggests, and a straightforward computation confirms, that

$$\widetilde{A}_{\phi^{-1}} = Id \text{ a.e. in } \phi(\Omega), \tag{II.2.22}$$

equivalently $\widetilde{Id}_\phi = A$ in a.e. Ω; recall the definition of $\widetilde{A}_{\phi^{-1}}$ and $\widetilde{Id}_\phi$ in (II.2.3). That is, the factorization from L_A-harmonic functions to harmonic functions is not only given by the Stoilow factorization (in the proof of Lemma II.2.17) but by the previous pointwise relation given by just applying the change of variables ϕ in the weak form of L_A.

II.3. Symmetric matrices with determinant 1

Let $\Omega \subset \mathbb{R}^2$ be a domain and $A \in \mathbb{R}^{2\times 2}(\overline{\Omega})$ be a symmetric and uniformly elliptic (with ellipticity constant $\lambda \geq 1$) matrix with $\det A = 1$. Note that during all Section II.3 we will use $K_\lambda = \lambda$ since the matrix is symmetric, see (II.0.2).

II.3.1. Push-forward of the harmonic measure and proof of Theorem II.0.1. Given a pole $p \in \Omega$, the harmonic factorization of L_A-harmonic functions given by (II.2.22) leads us to obtain in the following lemma that the elliptic measure $\omega^p_{\Omega,A}$ is precisely the push-forward of the harmonic measure $\omega^{\phi(p)}_{\phi(\Omega),Id}$ under the map ϕ^{-1} given by Corollary II.2.18. The following result is a consequence of (II.2.22) and Lemma II.2.6.

Corollary II.3.1. *Under the assumptions in Corollary II.2.18, then*

$$\omega^p_{\Omega,A}(\phi^{-1}(E)) = \omega^{\phi(p)}_{\phi(\Omega),Id}(E) \textit{ for any } p \in \Omega \textit{ and } E \subset \partial(\phi(\Omega)). \tag{II.3.1}$$

By means of the distortion results of Hausdorff dimension and measure under quasiconformal mappings in Section II.2.2 we obtain the non-regularity case in Theorem II.0.1.

Proof of Theorem II.0.1. Recall we can assume that $\Omega \subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3, and we can replace finitely connected by simply connected by Lemma II.1.2.

Let $\phi_A \in W^{1,2}_{\text{loc}}(\mathbb{C})$ be a principal solution of the Beltrami equation

$$\bar\partial \phi_A = \mu_A \mathbf{1}_{\overline{B_1(0)}} \partial \phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. In particular, ϕ_A is a λ-quasiconformal homeomorphism, conformal in $\mathbb{C} \setminus \overline{\mathbb{D}}$, and ϕ_A^{-1} is also a λ-quasiconformal homeomorphism.

By Wolff's Theorem I.0.2 there is a set $F' \subset \partial(\phi_A(\Omega))$ such that $\omega^{\phi_A(p)}_{\phi_A(\Omega),Id}(F') = 1$ and with σ-finite length. Therefore, the set $F = \phi_A^{-1}(F') \subset \partial\Omega$ satisfies $\omega^p_{\Omega,A}(F) = 1$ by the relation (II.3.1) between elliptic and harmonic measures, and by the distortion of Hausdorff measures under quasiconformal mappings in Theorem II.2.7, the set F has σ-finite $\mathcal{H}^{\frac{2\lambda}{\lambda+1}}$.

Now consider the case where Ω is simply connected. Recall that the λ-quasiconformal homeomorphism ϕ_A is conformal in $\mathbb{C} \setminus \overline{\mathbb{D}}$. By the relation (II.3.1) between harmonic and elliptic measures, Makarov's Theorem I.0.1 and Corollary II.2.10 respectively, there exists $C_M > 0$ such that

$$\omega^p_{\Omega,A}(\cdot) = \omega^{\phi(p)}_{\phi_A(\Omega),Id}(\phi(\cdot)) \ll \mathcal{H}^{\varphi_{1,C_M}}(\phi(\cdot)) \ll \mathcal{H}^{\varphi_{\lambda,C_M}}(\cdot),$$

as claimed. $\square$

In the following sections, we will assume that the matrix A, and therefore also the Beltrami coefficient μ_A, enjoys more regularity. By regularity theory, we will have that the quasiconformal mapping ϕ_A given in Corollary II.2.18 has better estimates than the Hölder continuity in Theorem II.2.3 or the distortion of Hausdorff dimension (resp. measure) in (II.2.5) (resp. Theorem II.2.7 or Corollary II.2.10). The proof of the results in the following sections will have the same structure as the proof above, though we will use that the change of variables ϕ_A is not only λ-quasiconformal, but it is more regular.

II.3.2. Mean oscillation under quasiconformal mappings. In this subsection, we define the space BMO functions and collect some properties to control the regularity of the coefficients arising in the following sections.

Definition II.3.2. Let $1 \le p < \infty$, $f \in L^p_{\text{loc}}(\Omega)$ and a set $S \subset \Omega$. Denote the p-mean oscillation of f in S the quantity

$$[f]_{S,p} := \left(\fint_S |f(z) - m_S f|^p \, dz \right)^{1/p},$$

where $m_S f := \fint_S f(y)\,dy := \frac{1}{|S|}\int_S f(y)\,dy$.

Bounded functions satisfy $[f]_{S,p} \le 2\|f\|_{L^\infty(S)}$. In this lemma, we collect some fundamental bounds of the p-mean oscillation. We omit its proof.

Lemma II.3.3. *Let $1 \le p < \infty$. The following holds:*

(1) $[f+g]^p_{S,p} \lesssim [f]^p_{S,p} + [g]^p_{S,p}$.
(2) $[fg]^p_{S,p} \lesssim \|g\|^p_{L^\infty(S)}[f]^p_{S,p} + \|f\|^p_{L^\infty(S)}[g]^p_{S,p}$.
(3) $[1/f]^p_{S,p} \lesssim \|1/f\|^{2p}_{L^\infty(S)}\,[f]^p_{S,p}$.

(4) $[f/g]^p_{S,p} \lesssim \|1/g\|^p_{L^\infty(S)}[f]^p_{S,p} + \|f\|^p_{L^\infty(S)}\,\|1/g\|^{2p}_{L^\infty(S)}\,[g]^p_{S,p}$.

Let us now recall the definition of bounded mean oscillation functions.

Definition II.3.4. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is of bounded mean oscillation in Ω, $f \in \mathrm{BMO}(\Omega)$, if

$$\|f\|_{\Omega,*} := \sup_{B\subset\Omega \text{ ball}} \fint_B |f(z) - m_B f|\,dz < \infty.$$

If $\Omega = \mathbb{R}^{n+1}$ we write $\|f\|_*$.

With the notation in Definition II.3.2, $\|f\|_{\Omega,*} = \sup_{B\subset\Omega \text{ ball}}[f]_{B,1}$. Recall that $f \in \mathrm{BMO}$ implies $|f| \in \mathrm{BMO}$. Note also that it is equivalent considering cubes Q instead of balls B in the $\mathrm{BMO}(\mathbb{R}^{n+1})$ norm.

Given a function $f \in \mathrm{BMO}(\mathbb{R}^{n+1})$, the John-Nirenberg inequality, see [Duo01, Theorem 6.11] for example, states that there exist dimensional constants C_1 and C_2 such that for every cube $Q \subset \mathbb{R}^{n+1}$ and every $\lambda > 0$,

$$|\{z \in Q : |f(z) - m_Q f| > \lambda\}| \le C_1 e^{-C_2\lambda/\|f\|_*}|Q|.$$

Consequently, see [Duo01, Corollary 6.12] for a proof, for all $1 < p < \infty$,

$$\|f\|_* \approx_p \|f\|_{*,p} := \sup_{B\subset\Omega \text{ ball}} \left(\fint_B |f(z) - m_B f|^p\,dz\right)^{1/p}.$$

A close look at the proof of the John-Nirenberg inequality in [Duo01, Theorem 6.11] reveals that, in fact, for every cube $Q \subset \mathbb{R}^{n+1}$ and every $\lambda > 0$, if

$$\|f\|^{\text{cubes}}_{Q,*} := \sup_{R\subset Q \text{ cube}} \fint_R |f(z) - m_R f|\,dz$$

then

$$|\{z \in Q : |f(z) - m_Q f| > \lambda\}| \le C_1 e^{-C_2\lambda/\|f\|^{\text{cubes}}_{Q,*}}|Q|.$$

Following the steps in the proof of [Duo01, Corollary 6.12], given a cube $Q \subset \mathbb{R}^{n+1}$, we have

$$\sup_{R\subset Q \text{ cube}} \fint_R |f(z) - m_R f|\,dz \approx_p \sup_{R\subset Q \text{ cube}} \left(\fint_R |f(z) - m_R f|^p\,dz\right)^{1/p}. \tag{II.3.2}$$

Functions of bounded mean oscillation are invariant under quasiconformal mappings. This is a result of Reimann in [Rei74], which, in fact, also proved that if a $W^{1,1}_{\mathrm{loc}}$-homeomorphism $f : \mathbb{C} \to \mathbb{C}$ preserves the BMO space, then f is K-quasiconformal, with $K = K(\|f\|_*)$. In the following lemma, we control the mean oscillation under global quasiconformal mappings. As the statement of the lemma and its proof are slightly different from the one in [Rei74] and [AIM09, Theorem 13.4.1], for the sake of completeness we include the details.

Lemma II.3.5. *Let $K \ge 1$, $\phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be a K-quasiconformal homeomorphism of $\mathbb{C}$, and $\widetilde{B} \subset \mathbb{R}^2$ be an open ball. Given an open ball $B \subset \widetilde{B}$ (with center c_B and radius r_B), consider*

$$B_- := B\left(\phi^{-1}(c_B), \max_{|\xi - c_B| = r_B} |\phi^{-1}(\xi) - \phi^{-1}(c_B)|\right), \tag{II.3.3}$$

Then $r_{B_-} \le C_{K,\widetilde{B},\phi} \cdot r_B^{1/K}$, and for any $p \in (K,\infty)$,

$$\fint_B |f(\phi^{-1}(x)) - m_B(f\circ\phi^{-1})|\,dx \lesssim_{p,K} \left(\fint_{B_-} |f(x) - m_{B_-} f|^p\,dx\right)^{1/p}. \tag{II.3.4}$$

In particular $\|f\circ\phi^{-1}\|_* \lesssim_K \|f\|_*$.

Proof. The notation "$-$" in B_- is used to specify that the ball B_- "lives" in the image of ϕ^{-1}.

We control the radius of B_-. From the definition of B_- and the Hölder continuity with exponent $1/K$ of quasiconformal mappings in Theorem II.2.3 applied to ϕ^{-1}, we conclude

$$r_{B_-} = \max_{|\xi-c_B|=r_B} |\phi^{-1}(\xi)-\phi^{-1}(c_B)| \le C_{K,\widetilde{B},\phi} \max_{|\xi-c_B|=r_B} |\xi-c_B|^{1/K} = C_{K,\widetilde{B},\phi}\cdot r_B^{1/K}. \tag{II.3.5}$$

It remains to see (II.3.4). We can change from $m_B(f\circ\phi^{-1})$ to $m_{B_-}f$ because

$$\begin{aligned} \fint_B |f(\phi^{-1}(x)) - m_B(f\circ\phi^{-1})|\,dx &\le \fint_B\fint_B |f(\phi^{-1}(x)) - f(\phi^{-1}(y))|\,dy\,dx \\ &\le 2\fint_B |f(\phi^{-1}(x)) - m_{B_-}f|\,dx. \end{aligned} \tag{II.3.6}$$

Let q be the Hölder conjugate of p, i.e., $1/p+1/q=1$. In particular $p\in(K,\infty)$ if and only if $q\in(1,K/(K-1))$. By the change of variables $x=\phi(y)$, $\phi^{-1}(B)\subset B_-$ and Hölder's inequality, we bound the last term in (II.3.6) by

$$\begin{aligned} \fint_B |f(\phi^{-1}(x)) - m_{B_-}f|\,dx &= \frac{1}{|B|}\int_{\phi^{-1}(B)} |f(y)-m_{B_-}f||\det D\phi(y)|\,dy \\ &\le \frac{1}{|B|}\left(\int_{B_-}|f(x)-m_{B_-}f|^p\,dx\right)^{1/p}\left(\int_{B_-}|\det D\phi(x)|^q\,dx\right)^{1/q}. \end{aligned} \tag{II.3.7}$$

Since $q\in(1,K/(K-1))\subset[0,K/(K-1))$, the higher integrability of $|\det D\phi|$, see [AIM09, (13.24)], implies that the last term in (II.3.7) is controled by

$$\left(\int_{B_-}|\det D\phi(x)|^q\,dx\right)^{1/q} \lesssim_{p,K} |B_-|^{1/q}\frac{|\phi(B_-)|}{|B_-|}. \tag{II.3.8}$$

The inequality in (II.3.8) implies that (II.3.7) reduces to

$$\begin{aligned} \fint_B |f(\phi^{-1}(x)) - m_{B_-}f|\,dx &\lesssim_{p,K} \frac{|B_-|^{1/p}}{|B|}\left(\fint_{B_-}|f(x)-m_{B_-}f|^p\,dx\right)^{1/p}|B_-|^{1/q}\frac{|\phi(B_-)|}{|B_-|} \\ &= \frac{|\phi(B_-)|}{|B|}\left(\fint_{B_-}|f(x)-m_{B_-}f|^p\,dx\right)^{1/p}, \end{aligned} \tag{II.3.9}$$

where we used $1/p+1/q=1$ in the last inequality.

Now we want to bound $|\phi(B_-)|/|B|$. We have the inclusion

$$\phi(B_-)\subset B_+ := B\left(c_B, \max_{|\xi-\phi^{-1}(c_B)|=r_{B_-}}|\phi(\xi)-c_B|\right).$$

Here the notation B_+ is used to specify that the ball B_+ "lives" in the image of ϕ. By [AIM09, Lemma 3.4.5], there exists a constant $\delta=\delta(K)$ such that $\delta B_+\subset\phi(B_-)\subset B_+$. By definition $\delta B_+\subset B$, and hence

$$\frac{|\phi(B_-)|}{|B|} \le \frac{|B_+|}{|\delta B_+|} \approx_K 1. \tag{II.3.10}$$

Inequality in (II.3.4) follows from (II.3.6), and (II.3.10) applied to (II.3.9). Therefore, Lemma II.3.5 is proved. □

II.3.3. Proof of Theorems II.0.2 and II.0.3 via quasiconformal regularity. In this subsection, via quasiconformal theory, we will see that assuming regularity of the matrix, the global λ-quasiconformal homeomorphism ϕ_A in Corollary II.2.18 gains more regularity.

Let us present the sketch of this. If ϕ_A is a solution of $\bar{\partial}\phi_A = \mu_A \partial\phi_A$, then ϕ_A and ϕ_A^{-1} are λ-quasiconformal, in particular $\phi_A, \phi_A^{-1} \in C^{1/\lambda}_{\mathrm{loc}}$, and moreover ϕ_A^{-1} satisfies the Beltrami equation $\bar{\partial}(\phi_A^{-1}) = -(\mu_A \circ \phi_A^{-1})\overline{\partial(\phi_A^{-1})}$. From $\phi_A^{-1} \in C^{1/\lambda}_{\mathrm{loc}}$, assuming regularity on the matrix A will not only provide regularity on μ_A, but also on $-\mu_A \circ \phi_A^{-1}$. As a consequence, we will obtain extra regularity on the functions ϕ_A and ϕ_A^{-1}. For example, if $A \in C^\alpha$, then $\mu_A \in C^\alpha$ and $-\mu_A \circ \phi_A^{-1} \in C^{\alpha/\lambda}_{\mathrm{loc}}$, whence we obtain that $\phi_A \in C^{1,\alpha}_{\mathrm{loc}}$ and $\phi_A^{-1} \in C^{1,\alpha/\lambda}_{\mathrm{loc}}$, see [AIM09, Theorem 15.6.2].

II.3.3.1. *Hölder continuous coefficients.* In this subsection we prove Theorem II.0.2. The sketch is what is already explained above. For completeness, we provide the proof in full detail.

Although we saw in Corollary I.4.5 that the dimension of elliptic measures only depends on the nature of the matrix around the boundary, during the following proof we emphasize that the regularity of the matrix is only needed around the boundary. Recall $U_t(\partial\Omega)$ denotes the t-neighborhood of the boundary, i.e.,

$$U_t(\partial\Omega) := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \partial\Omega) < t\}.$$

Proof of Theorem II.0.2. Recall we can assume that $\Omega \subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3, and we can replace finitely connected by simply connected by Lemma II.1.2. Without loss of generality, we can also assume $\det A = 1$.

Assume that $A \in C^\alpha(U_\varepsilon(\partial\Omega))$ for $\varepsilon > 0$. Let $\phi_A \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar{\partial}\phi_A = \mu_A \partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. In particular ϕ is a λ-quasiconformal homeomorphism. Since A is uniformly elliptic, in particular $a_{11}, a_{22} \approx 1$ and hence $2 + a_{11} + a_{22} \geq 2$, and therefore $A \in C^\alpha(U_\varepsilon(\partial\Omega))$ implies $\mu_A \in C^\alpha(U_\varepsilon(\partial\Omega))$.

By the Hölder regularity of quasiconformal mappings in Theorem II.2.3 we have that $\phi_A^{-1} \in C^{1/\lambda}_{\mathrm{loc}}$, and hence $-\mu_A \circ \phi_A^{-1} \in C^{\alpha/\lambda}_{\mathrm{loc}}$ in an open neighborhood of $\partial(\phi_A(\Omega))$. Therefore, we get that $\phi_A \in C^{1,\alpha}_{\mathrm{loc}}$ (resp. $\phi_A^{-1} \in C^{1,\alpha/\lambda}_{\mathrm{loc}}$) in an open neighborhood of $\partial\Omega$ (resp. $\partial(\phi_A(\Omega))$), see [AIM09, Theorem 15.6.2]. In particular both ϕ_A and ϕ_A^{-1} are Lipschitz in an open neighborhood of $\partial\Omega$ and $\partial(\phi_A(\Omega))$ respectively.

The rest of the proof follows by the same proof of Theorem II.0.1 in p. 46 using that ϕ_A is bi-Lipschitz in an open neighborhood of $\partial\Omega$. □

II.3.3.2. *Only one variable dependence coefficients.* In this subsection, we consider matrices with no regularity, but satisfying $A(z) = A(\mathrm{Re}(z))$ (or $A(z) = A(\mathrm{Im}(z))$). Even with no regularity on the coefficients (only uniformly elliptic), in this case, the quasiconformal change of variables is bi-Lipschitz.

Lemma II.3.6 (See [BGMR13, Proposition 5.23])**.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be an arbitrary measurable function with $\|\mu\|_\infty \leq k < 1$ that depends on $x = \mathrm{Re}(z)$ only and let*

$$f(x) = \int_0^x \frac{1+\mu(t)}{1-\mu(t)}\, dt.$$

Then $\phi(z) = f(x) + iy$ represents a unique quasiconformal mapping on $\overline{\mathbb{C}}$ onto itself with complex dilatation μ and normalizations $\phi(0) = 0$, $\phi(i) = i$ and $\phi(\infty) = \infty$. Moreover, ϕ is bi-Lipschitz.

Proof of Theorem II.0.3: $A(z) = A(\mathrm{Re}(z))$ case. Recall we can assume that Ω is Wiener regular by Remark II.1.3, and we can replace finitely connected by simply connected by Lemma II.1.2. Without loss of generality, we can also assume $\det A = 1$.

We assume without loss of generality that $A(z) = A(\mathrm{Re}(z))$. In particular, μ_A as in (II.2.15) satisfies $\mu_A(z) = \mu_A(\mathrm{Re}(z))$ and hence ϕ_A, given by Lemma II.3.6, is bi-Lipschitz. The proof follows as in the non-regularity case of Theorem II.0.1 in p. 46 using that ϕ_A is bi-Lipschitz. □

II.3.3.3. *VMO coefficients.* Let us introduce the space of functions with vanishing mean oscillation.

Definition II.3.7. A functions f has vanishing mean oscillation, written $f \in \mathrm{VMO}$, if $f \in \mathrm{BMO}$ and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^2} \fint_{B_r(x)} |f(z) - m_{B_r(x)} f|\, dz = 0.$$

For the classical Beltrami equation with VMO_c coefficient, its quasiconformal solution preserves the dimension of sets.

Lemma II.3.8. *Let $\mu \in \mathrm{VMO}_c$ be a Beltrami coefficient and $\phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be a K-quasiconformal homeomorphism of $\mathbb{C}$ satisfying $\bar\partial\phi = \mu\partial\phi$. Then both ϕ and ϕ^{-1} are C^α_{loc} for every $\alpha \in (0, 1)$. In particular*

$$\dim_{\mathcal{H}}(\phi(S)) = \dim_{\mathcal{H}}(S) \text{ for any set } S.$$

This is well-known in the literature. For the sake of completeness, a proof of Lemma II.3.8 can be found below, as the author did not find any proof in the literature.

Using the result above, for VMO matrices we recover the Jones and Wolff theorem in [JW88] and the lower bound of the dimension of elliptic measure in simply connected domains.

Proof of Theorem II.0.3: VMO *case.* Recall we can assume that $\Omega \subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3, and we can replace finitely connected by simply connected by Lemma II.1.2. Without loss of generality, we can also assume $\det A = 1$.

Let us fix a function $\varphi \in C^\infty_c(\mathbb{R}^2)$ satisfying

$$0 \leq \varphi \leq 1,\ \varphi|_{B_{1/2}(0)} = 1,\ \varphi|_{B_1(0)^c} = 0 \text{ and } |\nabla\varphi| \lesssim 1, \tag{II.3.11}$$

and let $\phi_A \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar\partial\phi_A = \varphi\mu_A\partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. Since A is uniformly elliptic, in particular $a_{11}, a_{22} \approx 1$ and hence $2 + a_{11} + a_{22} \geq 2$. Since $A \in \mathrm{VMO}(\mathbb{R}^2)$, by item 4 in Lemma II.3.3 we have $\mu_A \in \mathrm{VMO}(\mathbb{R}^2)$ and so $\varphi\mu_A \in \mathrm{VMO}_c(\mathbb{R}^2)$. Therefore, by

Lemma II.3.8, the function ϕ_A preserves the dimension of sets. The theorem follows from this by the same proof of Theorem II.0.1, see p. 46. $\square$

We now turn to the proof of Lemma II.3.8. First, in the following result we see the regularity of a quasiconformal mapping with VMO_c coefficient.

Theorem II.3.9 (See [Iwa92, p. 42-43])**.** *Let $\mu \in \mathrm{VMO}_c$ be such that $\|\mu\|_\infty = k < 1$, and $p \in (0, \infty)$. Then for each $h \in L^p$ the equation*

$$\bar{\partial} f - \mu \partial f = h$$

admits a solution f with $\nabla f \in L^p$ which is unique up to an additive constant.

This result also holds for the generalized Beltrami equation, see [Kos11, Theorem 4.6] and [CC13, Theorem 8].

Remark II.3.10. Note that if $\bar{\partial}\phi = \mu \partial \phi$ with $\mu \in \mathrm{VMO}_c$, then $\phi \in L^p_{\mathrm{loc}}$ for all $p \in (1, \infty)$. Thus, $\phi \in C^\alpha_{\mathrm{loc}}$ for every $\alpha \in (0,1)$, and as a consequence, $\dim_{\mathcal{H}}(\phi(S)) \leq \dim_{\mathcal{H}}(S)$ for any set S.

Next we want to see the regularity of the Beltrami coefficient of ϕ^{-1}. Before that, as a consequence of the estimate (II.3.4) in Lemma II.3.5, we see that $\mathrm{VMO}_c \cap L^\infty$ is invariant under quasiconformal mappings.

Corollary II.3.11. $\mathrm{VMO}_c \cap L^\infty$ *is invariant under global homeomorphic quasiconformal mappings.*

Proof. Let $f \in \mathrm{VMO}_c \cap L^\infty$ be non-identically zero, otherwise we are done. By Definition II.3.7, for all $\varepsilon > 0$ there is $r_0 = r_0(\varepsilon)$ such that if $r < r_0$ then

$$\sup_{x \in \mathbb{R}^2} \fint_{B_r(x)} |f - m_{B_r(x)} f| < \varepsilon.$$

Let $\phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be a K-quasiconformal homomorphism of $\mathbb{C}$, with $K \geq 1$, $\widetilde{B}$ be a ball such that $\mathrm{supp}(f \circ \phi^{-1}) \subset \widetilde{B}/2$ and fix $\widetilde{C} \coloneqq C_{K, 2\widetilde{B}, \phi}$, the constant in Lemma II.3.5. We want to see that for all $\varepsilon' > 0$ there is $r_0' = r_0'(\varepsilon') \leq r(\widetilde{B})/100$ such that if $r < r_0'$ then

$$\sup_{x \in \mathbb{R}^2} \fint_{B_r(x)} |f \circ \phi^{-1} - m_{B_r(x)}(f \circ \phi^{-1})| < \varepsilon'.$$

Given $x \in \widetilde{B}$ and $r < r_0'$ (otherwise the integral above is zero), let $B = B_r(x)$ and B_- as in (II.3.3). By (II.3.4) (with $p = K+1$) we have

$$\begin{aligned} \fint_B |f \circ \phi^{-1} - m_B(f \circ \phi^{-1})| &\leq C_K \left(\fint_{B_-} |f - m_{B_-} f|^{K+1} \right)^{1/(K+1)} \\ &\leq C_K (2\|f\|_{L^\infty})^{K/(K+1)} \left(\fint_{B_-} |f - m_{B_-} f| \right)^{1/(K+1)}, \end{aligned}$$

where $r(B_-) \leq \widetilde{C} r(B)^{1/K}$. So, given ε', take $\varepsilon = (\varepsilon'/(C_K(2\|f\|_{L^\infty})^{K/(K+1)}))^{K+1}$ and $r_0' = \min\{(r_0(\varepsilon)/\widetilde{C})^K, r(\widetilde{B})/100\}$. With this choice we have $r(B_-) < \widetilde{C}(r_0')^{1/K} \leq r_0(\varepsilon)$ and therefore

$$C_K (2\|f\|_{L^\infty})^{K/(K+1)} \left(\fint_{B_-} |f - m_{B_-} f| \right)^{1/(K+1)} < \varepsilon',$$

as claimed. □

In the following lemma we prove that if ϕ solves the Beltrami equation $\bar{\partial}\phi = \mu\partial\phi$ with $\mu \in \mathrm{VMO}_c$, then ϕ^{-1} not only solves the generalized Beltrami equation $\bar{\partial}(\phi^{-1}) = -(\mu\circ\phi^{-1})\overline{\partial(\phi^{-1})}$ with $\mu\circ\phi^{-1} \in \mathrm{VMO}_c$, but it also satisfies $\bar{\partial}(\phi^{-1}) = \widetilde{\mu}\partial\phi$ with $\widetilde{\mu} \in \mathrm{VMO}_c$. In particular, we can also apply Theorem II.3.9 to ϕ^{-1}, see Remark II.3.10.

Lemma II.3.12. *Let $\mu \in \mathrm{VMO}_c$ be a Beltrami coefficient and $\phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be a K-quasiconformal homeomorphism of $\mathbb{C}$ satisfying $\bar{\partial}\phi = \mu\partial\phi$. Then ϕ^{-1} satisfies*

$$\bar{\partial}(\phi^{-1}) = -(\mu\circ\phi^{-1})\overline{\partial(\phi^{-1})} = -\left(\left(\mu\frac{\partial\phi}{\overline{\partial\phi}}\right)\circ\phi^{-1}\right)\partial(\phi^{-1}) \tag{II.3.12}$$

with $\mu\circ\phi^{-1}, (\mu\partial\phi/\overline{\partial\phi})\circ\phi^{-1} \in \mathrm{VMO}_c$.

Proof. The equalities in (II.3.12) are proved in Corollary II.2.13 and [CFM+09, Proposition 5].

From item 2 in Lemma II.3.3 and Corollary II.3.11 it suffices to see $\partial\phi/\overline{\partial\phi} \in \mathrm{VMO}$. The strategy is the same as in the proof of [CFM+09, Proposition 5]. Due to the work of Hamilton in [Ham89] we have $\lambda = \log\partial\phi \in \mathrm{VMO}$. Writing $\partial\phi = e^{\lambda}$, in particular we have

$$\frac{\partial\phi}{\overline{\partial\phi}} = \frac{e^{\lambda}}{\overline{e^{\lambda}}} = \frac{e^{\mathrm{Re}\lambda}e^{i\mathrm{Im}\lambda}}{e^{\mathrm{Re}\lambda}e^{-i\mathrm{Im}\lambda}} = e^{2i\mathrm{Im}\lambda}.$$

The function $\mathrm{Im}\lambda$ belongs in VMO as $\lambda \in \mathrm{VMO}$. Since $\mathbb{R} \ni t \mapsto e^{it}$ is 1-Lipschitz, for any ball B we have

$$\begin{aligned}\fint_B \left|e^{2i\mathrm{Im}\lambda(x)} - m_B e^{2i\mathrm{Im}\lambda}\right| dx &\le \fint_B\fint_B \left|e^{2i\mathrm{Im}\lambda(x)} - e^{2i\mathrm{Im}\lambda(y)}\right| dy\,dx \\ &\le 2\fint_B\fint_B |\mathrm{Im}\lambda(x) - \mathrm{Im}\lambda(y)|\,dy\,dx \le 4\fint_B |\mathrm{Im}\lambda(x) - m_B\mathrm{Im}\lambda|\,dx,\end{aligned}$$

implying $\partial\phi/\overline{\partial\phi} = e^{2i\mathrm{Im}\lambda} \in \mathrm{VMO}$ as $\mathrm{Im}\lambda \in \mathrm{VMO}$. □

We are now ready to prove Lemma II.3.8.

Proof of Lemma II.3.8. It follows from Theorem II.3.9, Remark II.3.10, and Lemma II.3.12. □

II.3.3.4. *$W^{1,p}_{\mathrm{loc}}$ coefficients with $p > \frac{2\lambda^2}{\lambda^2+1}$.* In Sections II.3.3.1 and II.3.3.2, we had that the quasiconformal mapping ϕ_A was bi-Lipschitz, which allowed us to prove the σ-finite length property of the elliptic measure. In this subsection we assume $A \in W^{1,p}_{\mathrm{loc}}$ with $p > \frac{2\lambda^2}{\lambda^2+1}$. Although this Sobolev regularity on A, and so also for the Beltrami coefficient μ_A, does not provide a bi-Lipschitz quasiconformal solution in general, it still preserves sets with σ-finite length, see Theorem II.3.13 below. In particular, this will give the Wolff's Theorem I.0.2 in this situation.

Theorem II.3.13 (See [CFM+09, Corollary 11 and Proposition 21(c)])**.** *Let $K \ge 1$ and $\frac{2K^2}{K^2+1} < p \le 2$. Let $\mu \in W^{1,p}_c(\mathbb{R}^2)$ be a compactly supported Beltrami coefficient with $\|\mu\|_\infty \le \frac{K-1}{K+1}$, and ϕ any μ-quasiconformal mapping. For every compact set E,*

$$\mathcal{H}^1(E) \text{ is } \sigma\text{-finite} \iff \mathcal{H}^1(\phi(E)) \text{ is } \sigma\text{-finite}.$$

For a non-trivial example of non-bi-Lipschitz μ-quasiconformal mapping with $\mu \in W^{1,2}$, see the function in [CFM$^+$09, (8)]. We remark that there are also bi-Lipschitz μ-quasiconformal mappings with $\mu \notin W^{1,2}$. In fact, for $L \geq 1$, L-bi-Lipschitz mappings are μ-quasiconformal with $\|\mu\|_\infty \leq (L^2-1)/(L^2+1)$, and in general $\mu \notin W^{1,2}$. Also, $\mu(z) = \frac{1}{2}\chi_{\mathbb{D}}(z) \notin W^{1,2}$ provides a bi-Lipschitz μ-quasiconformal mapping. The latter is a particular case in [MOV09].

The result above is precisely what we need to obtain the σ-finite length property when we invoke Wolff's Theorem I.0.2.

Proof of Theorem II.0.3: Sobolev case. Recall we can assume that $\Omega \subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3. Without loss of generality, we can also assume $\det A = 1$. When $p > 2$ we have the Sobolev embedding into Hölder continuous functions $W^{1,p} \subset C^{1-2/p}$ by Morrey's inequality, and hence the result follows from Theorem II.0.2. So, assume $p \in \left(\frac{2\lambda^2}{\lambda^2+1}, 2\right]$.

Let φ be as in (II.3.11) and $\phi_A \in W^{1,2}_{\text{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar{\partial}\phi_A = \varphi\mu_A\partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. Since A is uniformly elliptic, in particular $a_{11}, a_{22} \approx 1$ and hence $2 + a_{11} + a_{22} \geq 2$. Since $A \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)$, we have that $\mu_A \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ and therefore $\varphi\mu_A \in W^{1,p}_c(\mathbb{R}^2)$. The theorem follows by the same proof of Theorem II.0.1 in p. 46, using Wolff's Theorem I.0.2 and that the function ϕ_A preserve sets of σ-finite length, see Theorem II.3.13. That is, if $F \subset \partial(\phi_A(\Omega))$ is the set given by Wolff's Theorem I.0.2, then $\phi_A^{-1}(F) \subset \partial\Omega$ has full elliptic measure and σ-finite length. $\square$

II.3.4. Proof of Theorem II.0.3 via PDE regularity. Similarly as we did in the previous Section II.3.3, but using a PDE approach instead of applying quasiconformal theory, in this subsection we deduce better regularity of the quasiconformal change of variables assuming regularity on the matrix A, which allows proving the remaining cases of Theorem II.0.3.

Let us sketch this when the matrix has Hölder coefficients. As in the proof of Theorem II.0.2 in p. 49, we can assume that $A \in C^\alpha(U_\varepsilon(\partial\Omega))$ for $0 < \alpha < 1$ and $\varepsilon > 0$. Let $\phi_A \in W^{1,2}_{\text{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar{\partial}\phi_A = \mu_A\partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. By Corollary II.2.13, $u := \text{Re}\phi_A$, $v := \text{Im}\phi_A$, $R := \text{Re}\phi_A^{-1}$ and $I := \text{Im}\phi_A^{-1}$ satisfy $\operatorname{div} A\nabla u = 0$, $\operatorname{div} A\nabla v = 0$, $\operatorname{div} B\nabla u = 0$ and $\operatorname{div} B^*\nabla v = 0$ respectively, where B and B^* are the uniformly elliptic matrices in (II.2.16). By the Hölder regularity of quasiconformal mappings we have that $\phi_A^{-1} \in C^{1/\lambda}_{\text{loc}}$, and hence $A \in C^\alpha(U_\varepsilon(\partial\Omega))$ implies $B, B^* \in C^{\alpha/\lambda}_{\text{loc}}$ in an open neighborhood of $\partial(\phi_A(\Omega))$. All in all, u, v, I, R satisfy a PDE with Hölder coefficients and therefore ϕ_A is bi-Lipschitz in an open neighborhood of $\partial\Omega$. Theorem II.0.2 follows from the same proof of Theorem II.0.1 in p. 46 using that ϕ_A is bi-Lipschitz in an open neighborhood of $\partial\Omega$.

II.3.4.1. *Dini mean oscillation.* We consider matrices with p-Dini mean oscillation.

Definition II.3.14. For $1 \leq p < \infty$ and $f \in L^p_{\text{loc}}(\mathbb{R}^{n+1})$, denote

$$f_{\text{MO},p}(r) := \sup_{x\in\mathbb{R}^{n+1}} \left(\fint_{B(x,r)} \left|f(z) - m_{B(x,r)}f\right|^p \, dz \right)^{1/p},$$

where MO stands for "mean oscillation". We say that f is of p-Dini mean oscillation, $f \in \mathrm{DMO}_p$, if $f_{\mathrm{MO},p}$ satisfies the Dini condition

$$\int_0^1 f_{\mathrm{MO},p}(t)\,\frac{dt}{t} < \infty.$$

Remark II.3.15. From Hölder's inequality, we have the inclusion $\mathrm{DMO}_q \subset \mathrm{DMO}_p$ if $1 \leq p < q < \infty$. The Dini condition of the mean oscillation directly implies $\mathrm{DMO}_p \subset \mathrm{VMO}$ for every $1 \leq p < \infty$. Moreover, every DMO_p $(1 \leq p < \infty)$ contains the (pointwise) Dini space, that is, functions f satisfying the Dini condition

$$\int_0^1 \tau(t)\,\frac{dt}{t} < \infty, \text{ where } \tau(t) = \sup_{|x-y|\leq t} |f(x) - f(y)|. \tag{II.3.13}$$

All in all,

$$\text{Hölder} \subset \text{Dini} \subset \mathrm{DMO}_q \subset \mathrm{DMO}_p \subset \mathrm{VMO}, \text{ for any } 1 \leq p < q < \infty.$$

Under the 1-Dini mean oscillation regularity on the matrix coefficients, L_A-harmonic functions are C^1. More precisely, we have the following result.

Theorem II.3.16 (See [DK17, Theorem 1.5])**.** *Suppose the coefficients of $A(x) = (a_{ij}(x))_{i,j=1}^{n+1}$ satisfy*

$$\max_{1\leq i,j\leq n+1} \int_0^1 \eta_{a_{ij}}(t)\,\frac{dt}{t} < \infty, \textit{ where } \eta_{a_{ij}}(r) := \sup_{x\in B_3(0)} \fint_{B(x,r)} \left|a_{ij}(z) - m_{B(x,r)}a_{ij}\right| \, dz,$$

and let $u \in W^{1,2}(B_4(0))$ be a L_A-solution in $B_4(0)$. Then, we have $u \in C^1(\overline{B_1(0)})$.

In the following lemma, we see how the 1-Dini mean oscillation regularity changes under K-quasiconformal mappings.

Lemma II.3.17. *Let $K \geq 1$ and let $\phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be a K-quasiconformal homeomorphism of $\mathbb{C}$. For any function f and any $K < p < \infty$, there exists a constant $C_{K,\phi}$ such that*

$$\int_0^1 \eta_{(f\circ\phi^{-1})}(t)\frac{dt}{t} \lesssim_{p,K} \int_0^{C_{K,\phi}} f_{\mathrm{MO},p}(t)\frac{dt}{t}.$$

Proof. Fix $\widetilde{B} = B_{100}(0)$. Recall that ϕ^{-1} is η_K-quasisymmetric as it is global K-quasiconformal, see (II.2.2). Take $T = T_K \geq 2$ big enough so that $\eta_K(1/T) \leq 1/2$, and denote $r_m = T^{-m}$. Let $x \in B_3(0)$. First note that for $r_{m+1} \leq r \leq r_m$, and changing from $m_{B(x,r)}(f\circ\phi^{-1})$ to $m_{B(x,r_m)}(f\circ\phi^{-1})$ as we did in (II.3.6), we have

$$\eta_{(f\circ\phi^{-1})}(x,r) := \fint_{B(x,r)} \left|(f\circ\phi^{-1})(z) - m_{B(x,r)}(f\circ\phi^{-1})\right| \, dz \lesssim \eta_{(f\circ\phi^{-1})}(x,r_m).$$

Thus, $\int_0^1 \eta_{(f\circ\phi^{-1})}(x,t)\frac{dt}{t} \lesssim \sum_{m=0}^\infty \eta_{(f\circ\phi^{-1})}(x,r_m)$.

Denote now $\widetilde{r}_m(x) := \max_{|\xi-x|=r_m} |\phi^{-1}(\xi) - \phi^{-1}(x)|$, recall the ball defined in (II.3.3). We simply write $\widetilde{r}_m$ as $x \in B_3(0)$ is fixed in this proof and the constants below do not depend on x. By (II.3.4) in Lemma II.3.5 we have

$$\eta_{(f\circ\phi^{-1})}(x,r_m) \lesssim_{p,K} \left(\fint_{B(\phi^{-1}(x),\widetilde{r}_m)} \left|f(z) - m_{B(\phi^{-1}(x),\widetilde{r}_m(x))}f\right|^p\right)^{1/p} =: \eta_f(\phi^{-1}(x),\widetilde{r}_m,p).$$

By the quasisymmetry condition we have $1/\eta_K(T) \le \widetilde{r}_m/\widetilde{r}_{m-1} \le \eta_K(1/T) \le 1/2$, and hence $1 \le 2\int_{\widetilde{r}_m}^{\widetilde{r}_{m-1}} \frac{dt}{t}$. As before, for every $\widetilde{r}_m \le r \le \widetilde{r}_{m-1}$ we have $\eta_f(\phi^{-1}(x), \widetilde{r}_m, p) \lesssim \eta_f(\phi^{-1}(x), r, p)$. By Lemma II.3.5 we have $\widetilde{r}_m \le C_{K,\widetilde{B},\phi} r_m^{1/K}$, and in particular $\widetilde{r}_{-1} \le C_{K,\widetilde{B},\phi} T^{1/K}$. As a consequence of these observations respectively, we get

$$\begin{aligned}\sum_{m=0}^{\infty} \eta_{(f\circ\phi^{-1})}(x, r_m) &\lesssim \sum_{m=0}^{\infty} \eta_f(\phi^{-1}(x), \widetilde{r}_m, p) \lesssim \sum_{m=0}^{\infty} \int_{\widetilde{r}_m}^{\widetilde{r}_{m-1}} \eta_f(\phi^{-1}(x), \widetilde{r}_m, p)\frac{dt}{t} \\ &\lesssim \sum_{m=0}^{\infty} \int_{\widetilde{r}_m}^{\widetilde{r}_{m-1}} \eta_f(\phi^{-1}(x), t, p)\frac{dt}{t} = \int_0^{C_{K,\widetilde{B},\phi}T^{1/K}} \eta_f(\phi^{-1}(x), t, p)\frac{dt}{t}.\end{aligned}$$

The last element is bounded by $\int_0^{C_{K,\widetilde{B},\phi}T^{1/K}} f_{\mathrm{MO},p}(t)\frac{dt}{t}$. Taking the supremum over all the points $x \in B_3(0)$ we obtain the lemma with the constant $C_{K,\widetilde{B},\phi}T^{1/K}$. $\square$

Using the previous lemma we deduce the C^1 regularity ϕ_A^{-1} in the proof of the following result when $A \in \mathrm{DMO}_p$ with $p > \lambda$.

Proof of Theorem II.0.3: DMO *case.* Recall we can assume that $\Omega \subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3, and we can replace finitely connected by simply connected by Lemma II.1.2. Without loss of generality, we can also assume $\det A = 1$.

Let us see first the simply connected case. Assume $A \in \mathrm{DMO}_1$ and let $\phi_A \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar{\partial}\phi_A = \mu_A \partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. In particular $u := \mathrm{Re}\phi_A$ and $v := \mathrm{Im}\phi_A$ satisfy $\nabla v = *A\nabla u$, and $\operatorname{div} A\nabla u = 0$, see Corollary II.2.13. Now, since $A \in \mathrm{DMO}_1$ we have that $|\nabla u|$ is bounded in $B_{1/2}(0)$, see Theorem II.3.16, and by the relation $\nabla v = *A\nabla u$ we also have that $|\nabla v|$ is bounded in $B_{1/2}(0)$. In particular, u and v are Lipschitz and hence ϕ_A is Lipschitz in $B_{1/2}(0) \supset \partial\Omega$. This implies $\omega_{\Omega,A} \ll \mathcal{H}^{\varphi_{1,C}}$ by the same proof of Theorem II.0.1 in p. 46.

We now turn to the σ-finite length for general domains. Suppose first $A \in \mathrm{DMO}_p$ with $p > \lambda$, and later we will see we can always reduce to this case. Let ϕ_A be as above. Recall from Corollary II.2.13 that the functions $R := \mathrm{Re}(\phi_A^{-1})$ and $I := \mathrm{Im}(\phi_A^{-1})$ satisfy $\nabla I = *B\nabla R$, and $\operatorname{div} B\nabla R = 0$, where B is the uniformly elliptic matrix in (II.2.16). By Lemma II.3.17, item 4 in Lemma II.3.3, and $A \in \mathrm{DMO}_p$ with $p > \lambda$ we obtain that $B \in \mathrm{DMO}_1$. By Theorem II.3.16 we have that $R \in C^1$, i.e., $|\nabla R|$ is bounded around $\partial\phi_A(\Omega)$, and by the relation $\nabla I = *B\nabla R$ we also get that $|\nabla I|$ is bounded there. Hence, $R = \mathrm{Re}(\phi_A^{-1})$ and $I = \mathrm{Im}(\phi_A^{-1})$ are Lipschitz, and in particular ϕ_A^{-1} is Lipschitz. The same proof of Theorem II.0.1 in p. 46 implies the σ-finite length.

Let us see now the case $A \in \mathrm{DMO}_p$ with $p > 1$ follows from the previous reduction. As $A \in \mathrm{DMO}_p \subset \mathrm{DMO}_1$, by [HK20, Appendix] we have that it agrees a.e. with a matrix with (uniformly) continuous coefficients, which by the L_A weak form we can assume that we are in this latter case. In particular, given $\varepsilon > 0$, for each $\xi \in \overline{\Omega}$ there is $\delta_{\xi,\varepsilon} > 0$ such that

$$z \in \overline{6B_{\delta_{\xi,\varepsilon}}(\xi)} \implies |a_{ij}(z) - a_{ij}(\xi)| < \varepsilon \text{ for any } i, j \in \{1, 2\}.$$

Let $\{B_i\}_i \subset \{B_s(\xi)\}_{\xi\in\partial\Omega, 0<s\le\delta_{\xi,\varepsilon}}$ be a countably subfamily of disjoint balls such that $\omega^p(\partial\Omega \setminus \bigcup_i B_i) = 0$, by Vitali's covering theorem. The goal is to prove that for each index i there

exists a set $F_i \subset \partial(\Omega \cap 4B_i)$ satisfying $\omega_{\Omega\cap 4B_i,A}(F_i) = 1$ and with σ-finite $\mathcal{H}^1(F_i)$, because by Lemma I.4.4 we would have that the set $F = \bigcup_i F_i$ satisfies $\omega_{\Omega,A}(F) = 1$ and of course it has σ-finite length.

Let us now fix an index i. Let ξ_i denote the center of the ball B_i, and r_i its radius. Let $S = \sqrt{A(\xi_i)}$ be the matrix satisfying $S^T S = A(\xi_i)$, and M as in (II.1.1), i.e.,

$$M(z) = Id + (S^T)^{-1}(A(S^T) - A(\xi))S^{-1} \overset{\text{(II.1.3)}}{=} (S^T)^{-1}A(S^T)S^{-1},$$

which is symmetric and with determinant 1, see Remark II.1.7. Then, by Lemma II.2.6 we have

$$\omega^q_{\Omega\cap 4B_i,A}(E) = \omega^{(S^T)^{-1}q}_{(S^T)^{-1}(\Omega\cap 4B_i),M}((S^T)^{-1}E),$$

for any $q \in \Omega \cap 4B_i$ and $E \in \partial(\Omega \cap 4B_i)$. Given $\varphi \in C_c^\infty(6B_i)$ such that $0 \leq \varphi \leq 1$, $\varphi|_{5B_i} = 1$ and $|\nabla\varphi| \lesssim 1/r_i$, consider the function $\widetilde{\varphi} = \varphi \circ S^T$. Using this function $\widetilde{\varphi}$, we can directly replace the matrix M in the right-hand side above by simply the uniformly elliptic, symmetric and with determinant 1 matrix $\widehat{M}$ such that $\widetilde{\varphi}\mu_M = \mu_{\widehat{M}}$, see the precise definition in Remark II.2.16. That is,

$$\omega^q_{\Omega\cap 4B_i,A}(E) = \omega^{(S^T)^{-1}q}_{(S^T)^{-1}(\Omega\cap 4B_i),\widehat{M}}((S^T)^{-1}E)$$

for any $q \in \Omega \cap 4B_i$ and $E \in \partial(\Omega \cap 4B_i)$.

From the proof of Lemma II.1.5, M has ellipticity constant $\lambda_i = (1-2\varepsilon\lambda)^{-1}$ in $(S^T)^{-1}(6B_i)$. This implies that $|\mu_M| \leq (\lambda_i - 1)/(\lambda_i + 1)$ in $(S^T)^{-1}(6B_i)$. In particular, $|\mu_{\widehat{M}}| = |\widetilde{\varphi}\mu_M| \leq |\mathbf{1}_{(S^T)^{-1}(6B_i)}\mu_M| \leq (\lambda_i - 1)/(\lambda_i + 1)$ everywhere, implying that $\widehat{M}$ has ellipticity constant $\lambda_i = (1 - 2\varepsilon\lambda)^{-1}$ everywhere. If $\varepsilon > 0$ is small enough then $1 \leq \lambda_i = (1 - 2\varepsilon\lambda)^{-1} < p$; recall that $p > 1$ is the value so that $A \in \mathrm{DMO}_p$.

Let us see $\widehat{M} \in \mathrm{DMO}_p$. As $A \in \mathrm{DMO}_p$, we have that $(S^T)^{-1}A(\cdot)S^{-1} \in \mathrm{DMO}_p$, and as $z \mapsto S^T z$ is a linear mapping we get $(S^T)^{-1}A(S^T\cdot)S^{-1} \in \mathrm{DMO}_p$. Using now that $\widetilde{\varphi} = \varphi \circ S^T$ is smooth, bounded and with compact support, applying Lemma II.3.3 we conclude that $\widehat{M} \in \mathrm{DMO}_p$.

As the ellipticity constant of $\widehat{M}$ is $1 \leq \lambda_i < p$ and $\widehat{M} \in \mathrm{DMO}_p$, we are in the case of the second paragraph in this proof, and hence there is a set $F_i \subset \partial(\Omega \cap 4B_i)$ satisfying $\omega_{(S^T)^{-1}(\Omega\cap 4B_i),\widehat{M}}(F) = 1$ and with σ-finite $\mathcal{H}^1(F_i)$. In particular $\omega_{\Omega\cap 4B_i,A}(S^T(F)) = 1$ and $S^T(F)$ has σ-finite length, and the theorem follows. $\square$

II.3.4.2. *Mean oscillation with logarithm decay.* In this subsection, we consider matrices with the quantitative log decay of the mean oscillation in [Acq92, (1.3)].

Definition II.3.18. We say that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is of vanishing mean oscillation with logarithmic decay, $f \in \mathrm{VMO}_{\log}$ for short, if

$$\sup_{\substack{x\in\mathbb{R}^{n+1}\\ s\in(0,r]}} \left[(1 + |\log s|) \fint_{Q_s(x)} \left|f - m_{Q_s(x)}f\right|\right] \xrightarrow{r\to 0} 0, \tag{II.3.14}$$

recall $Q_s(x)$ denotes the cube with center x and side length $2s$.

Remark II.3.19. The logarithmic decay in (II.3.14) implies $\mathrm{VMO}_{\log} \subset \mathrm{VMO}$. On the other hand, for a function $f \in \mathrm{DMO}_1$, given $\varepsilon > 0$ we can take $s_0 = s_0(\varepsilon)$ such that $\int_0^{s_0} f_{\mathrm{MO},1}(t)\,\frac{dt}{t} < \varepsilon$, and hence $(1 + |\log t|)f_{\mathrm{MO},1}(t) < t(1 + |\log t|)\varepsilon/s_0$ for all $0 < t < s_0$. Thus, we can take $\delta = \delta(\varepsilon)$ small enough so that $0 < t < \delta$ implies $(1 + |\log t|)f_{\mathrm{MO},1}(t) < \varepsilon$.

Interchanging balls $B(x,t)$ by cubes $Q_t(x)$ in the definition of $f_{\mathrm{MO},1}(t)$ we obtain that f satisfies (II.3.14). That is, $\mathrm{DMO}_1 \subset \mathrm{VMO}_{\log}$. All in all,

$$\mathrm{DMO}_1 \subset \mathrm{VMO}_{\log} \subset \mathrm{VMO}.$$

The fundamental property (II.3.2) of BMO functions implies that, for $1 \leq p < \infty$, $f \in \mathrm{BMO}$ and $r \in (0,1)$, we have

(II.3.15)
$$\sup_{\substack{x\in\mathbb{R}^{n+1}\\ s\in(0,r]}} \left[(1+|\log s|) \fint_{Q_s(x)} |f - m_{Q_s(x)}f|\right] \approx_p \sup_{\substack{x\in\mathbb{R}^{n+1}\\ s\in(0,r]}} \left[(1+|\log s|) \left(\fint_{Q_s(x)} |f - m_{Q_s(x)}f|^p\right)^{1/p}\right],$$

and therefore $f \in \mathrm{VMO}_{\log}$ if and only if for some (and hence for all) $1 \leq p < \infty$ the right-hand side term above goes to 0 as $r \to 0$. Indeed, for $x \in \mathbb{R}^2$ and $s \in (0,r]$, using (II.3.2) we have

(II.3.16)
$$\left(\fint_{Q_s(x)} |f - m_{Q_s(x)}f|^p\right)^{1/p} \leq \sup_{Q\subset Q_s(x)} \left(\fint_Q |f - m_Q f|^p\right)^{1/p} \approx_p \sup_{Q\subset Q_s(x)} \fint_Q |f - m_Q f|.$$

Since $|\log t|$ is decreasing in $(0,1)$ and every $Q \subset Q_s(x)$ satisfies $\ell(Q) \leq 2s$, we obtain

$$\begin{aligned}\sup_{Q\subset Q_s(x)} (1+|\log s|) \fint_Q |f - m_Q f| &\leq \sup_{Q\subset Q_s(x)} \left(1 + \left|\log \frac{\ell(Q)}{2}\right|\right) \fint_Q |f - m_Q f| \\ &\leq \sup_{\substack{x\in\mathbb{R}^2\\ s\in(0,r]}} \left[(1+|\log s|) \fint_{Q_s(x)} |f - m_{Q_s(x)}f|\right],\end{aligned} \tag{II.3.17}$$

and inequality $\gtrsim_p$ in (II.3.15) follows from (II.3.16) and (II.3.17) by taking the supremum over $x \in \mathbb{R}^2$ and $s \in (0,r]$. The other inequality in (II.3.15) is just Hölder's inequality.

Assuming that the matrix has $\mathrm{VMO}_{\log}$ coefficients, the following result states that L_A-harmonic solutions have the gradient in BMO.

Theorem II.3.20 (See [Acq92, Theorem 4.1])**.** *Let $A \in L^\infty(Q_3(0)) \cap \mathrm{VMO}_{\log}$ and $u \in W^{1,2}(Q_3(0))$ be a L_A-solution in $Q_3(0)$. Then $\nabla u \in \mathrm{BMO}(Q_1(0))$.*

In the following lemma, we see that functions with derivatives in the space BMO satisfy the so-called log-Lip regularity.

Lemma II.3.21. *Let $f : B_1(0) \subset \mathbb{R}^{n+1} \to \mathbb{R}$ be a function with $\nabla f \in L^1(B_1(0)) \cap \mathrm{BMO}(B_1(0))$, and let $x, y \in B_{1/2}(0)$. If $|x-y|$ is small enough, only depending on the L^1 and* BMO *norms, then*

$$|f(x) - f(y)| \lesssim \|\nabla f\|_* |x-y| \log \frac{1}{|x-y|}.$$

Proof. Consider $B := B(\frac{x+y}{2}, |x-y|) \subset B_1(0)$, and write $r_B = |x-y|$. By triangle inequality $|f(x) - f(y)| \leq |f(x) - m_B f| + |f(y) - m_B f|$, and by symmetry, it suffices to control one of them. By [GT01, Lemma 7.16],

$$|f(x) - m_B f| \lesssim \int_B \frac{|\nabla f(z)|}{|x-z|}\, dz.$$

Splitting the domain of integration B of the right-hand side integral in annulus, we obtain

$$\int_B \frac{|\nabla f(z)|}{|x-z|}\,dz = \sum_{k\geq 0}\int_{2^{-k}B\setminus 2^{-(k+1)}B}\frac{|\nabla f(z)|}{|x-z|}\,dz \approx \frac{2^k}{r_B}\sum_{k\geq 0}\int_{2^{-k}B\setminus 2^{-(k+1)}B}|\nabla f(z)|\,dz$$
$$\lesssim r_B\sum_{k\geq 0}2^{-k}\fint_{2^{-k}B}|\nabla f(z)|\,dz.$$

Let us bound each element on the right-hand side in terms of $k\geq 0$. For each $k\geq 0$, let $N_k\in\mathbb{N}$ the largest index such that $2^{N_k}r_{2^{-k}B}=2^{N_k}2^{-k}r_B\leq 1/2$, i.e., $N_k := \lfloor \log_2\frac{2^k}{2|x-y|}\rfloor$. With this choice,

$$\fint_{2^{-k}B}|\nabla f(z)|\,dz = \sum_{m=0}^{N_k-1}\left(\fint_{2^m2^{-k}B}|\nabla f(z)|\,dz - \fint_{2^{m+1}2^{-k}B}|\nabla f(z)|\,dz\right) + \fint_{2^{N_k}B}|\nabla f(z)|\,dz.$$

Each element in the sum is controlled in absolute value by $\lesssim \|\nabla f\|_*$. The latter term $\fint_{2^{N_k}B}|\nabla f(z)|\,dz$ is controlled by $\lesssim \|\nabla f\|_{L^1(B_1(0))}$ as $2^{N_k}B\subset B_1(0)$ and $2^{N_k}r_B\approx 1$ by the choice of N_k. In particular

$$\fint_{2^{-k}B}|\nabla f(z)|\,dz \lesssim N_k\|\nabla f\|_* + \|\nabla f\|_{L^1(B_1(0))}.$$

If $|x-y|$ is small enough depending on L^1 and BMO norm, more precisely, take small enough $|x-y|$ to satisfy $\|\nabla f\|_{L^1(B_1(0))}\leq \lfloor\log_2\frac{1}{2|x-y|}\rfloor\|\nabla f\|_*$, then $\|\nabla f\|_{L^1(B_1(0))}\leq N_k\|\nabla f\|_*$, and therefore

$$\fint_{2^{-k}B}|\nabla f(z)|\,dz \lesssim N_k\|\nabla f\|_*.$$

Summing over $k\geq 0$,

$$|f(x)-m_Bf| \lesssim \int_B\frac{|\nabla f(z)|}{|x-z|}\,dz \lesssim r_B\sum_{k\geq 0}2^{-k}\fint_{2^{-k}B}|\nabla f(z)|\,dz \lesssim r_B\|\nabla f\|_*\sum_{k\geq 0}2^{-k}N_k.$$

Recall $N_k=\lfloor\log_2\frac{2^k}{2|x-y|}\rfloor\approx\log\frac{2^k}{|x-y|}$. Hence, the sum in the right-hand side is controlled by

$$\sum_{k\geq 0}2^{-k}\log\frac{2^k}{|x-y|} = \sum_{k\geq 0}2^{-k}\log 2^k + \log\frac{1}{|x-y|}\sum_{k\geq 0}2^{-k}.$$

As $\sum_{k\geq 0}2^{-k}\log 2^k<\infty$, for small enough $|x-y|$ we have $\sum_{k\geq 0}2^{-k}\log 2^k\leq\log\frac{1}{|x-y|}$, and hence $\sum_{k\geq 0}2^{-k}\log\frac{2^k}{|x-y|}\lesssim\log\frac{1}{|x-y|}$. $\square$

Here we see the distortion of Hausdorff measures under log-Lip functions.

Lemma II.3.22. *If $f:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ satisfies $|f(x)-f(y)|\leq C_{\mathrm{logLip}}|x-y|\log\frac{1}{|x-y|}$ for $|x-y|$ small enough, $g(t)=t/\log\frac{1}{t}$, $h(t)=t\log\frac{1}{t}$, and $A\subset\mathbb{R}^{n+1}$, then*

(1) $\mathcal{H}^g(f(A))\leq 2C_{\mathrm{logLip}}\mathcal{H}^1(A)$, and

(2) $\mathcal{H}^{\varphi_{\rho,C}}(f(A))\leq 2C_{\mathrm{logLip}}^{\frac{2}{\rho+1}}\mathcal{H}^{(\varphi_{\rho,C})\circ h}(A)$.

Proof. For small enough $\delta>0$, the function $t\mapsto t\log\frac{1}{t}$ is increasing in $t\in(0,\delta)$. Let $\eta=C_{\mathrm{logLip}}\delta\log\frac{1}{\delta}>0$, and note that $\eta\to 0$ as $\delta\to 0$.

For $\varepsilon > 0$, let $\{E_i\}_i$ such that $A \subset \bigcup_i E_i$, $\operatorname{diam} E_i \leq \delta$ and $\sum_i \operatorname{diam} E_i \leq \mathcal{H}^1_\delta(A) + \varepsilon$. Then $f(A) \subset \bigcup_i f(E_i)$ and $\operatorname{diam}(f(E_i)) \leq C_{\mathrm{logLip}} \operatorname{diam} E_i \log \frac{1}{\operatorname{diam} E_i} \leq C_{\mathrm{logLip}} \delta \log \frac{1}{\delta} = \eta$. As $g(t) = t / \log \frac{1}{t}$ is increasing in $(0, \eta)$ if η is small enough, we have

$$g(\operatorname{diam} f(E_i)) \leq g\left(C_{\mathrm{logLip}} \operatorname{diam} E_i \log \frac{1}{\operatorname{diam} E_i} \right) = \frac{C_{\mathrm{logLip}} \operatorname{diam} E_i \log \frac{1}{\operatorname{diam} E_i}}{\log \left(\frac{1}{C_{\mathrm{logLip}} \operatorname{diam} E_i \log \frac{1}{\operatorname{diam} E_i}} \right)}$$
$$= \frac{C_{\mathrm{logLip}} \operatorname{diam} E_i \log \frac{1}{\operatorname{diam} E_i}}{\log \frac{1}{C_{\mathrm{logLip}}} + \log \frac{1}{\operatorname{diam} E_i} + \log \log \frac{1}{\operatorname{diam} E_i}} \leq 2 C_{\mathrm{logLip}} \operatorname{diam} E_i,$$

where we used that $\operatorname{diam} E_i \leq \delta$ is small enough. All in all,

$$\begin{aligned} \mathcal{H}^g_\eta(f(A)) &\leq \sum_i g(\operatorname{diam} f(E_i)) \leq 2 C_{\mathrm{logLip}} \sum_i \operatorname{diam} E_i \\ &\leq 2 C_{\mathrm{logLip}} (\mathcal{H}^1_\delta(A) + \varepsilon) \leq 2 C_{\mathrm{logLip}} (\mathcal{H}^1(A) + \varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$ and $\delta \to 0$, i.e., $\eta \to 0$, we get $\mathcal{H}^g(f(A)) \leq 2 C_{\mathrm{logLip}} \mathcal{H}^1(A)$ as claimed.

Similarly, for $\varepsilon > 0$, let $\{E_i\}_i$ such that $A \subset \bigcup_i E_i$, $\operatorname{diam} E_i \leq \delta$ and $\sum_i ((\varphi_{\rho,C}) \circ h)(\operatorname{diam} E_i) \leq \mathcal{H}^{(\varphi_{\rho,C}) \circ h}_\delta(A) + \varepsilon$. As before,

$$\begin{aligned} \mathcal{H}^{\varphi_{\rho,C}}_\eta(f(A)) &\leq \sum_i \varphi_{\rho,C}(\operatorname{diam} f(E_i)) \leq \sum_i \varphi_{\rho,C}(C_{\mathrm{logLip}} h(\operatorname{diam} E_i)) \\ &\leq 2 C_{\mathrm{logLip}}^{\frac{2}{\rho+1}} \sum_i ((\varphi_{\rho,C}) \circ h)(\operatorname{diam} E_i) \leq 2 C_{\mathrm{logLip}}^{\frac{2}{\rho+1}} (\mathcal{H}^{(\varphi_{\rho,C}) \circ h}(A) + \varepsilon), \end{aligned}$$

where we used $\varphi_{\rho,C}(C_{\mathrm{logLip}} h(\operatorname{diam} E_i)) \leq 2 C_{\mathrm{logLip}}^{\frac{2}{\rho+1}} \varphi_{\rho,C}(h(\operatorname{diam} E_i))$, see the proof of Corollary II.2.10. Letting $\varepsilon, \delta \to 0$, and hence also $\eta \to 0$, we obtain the bound $\mathcal{H}^{\varphi_{\rho,C}}(f(A)) \leq 2 C_{\mathrm{logLip}} \mathcal{H}^{(\varphi_{\rho,C}) \circ h}(A)$. □

Lemma II.3.23. $\mathrm{VMO}_{\log,c}$ *is invariant under global homeomorphic quasiconformal mappings.*

Proof. Let $f \in \mathrm{VMO}_{\log}$ be non-identically zero (otherwise we are done) with compact support and $\phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be a K-quasiconformal homeomormism of $\mathbb{C}$, with $K \geq 1$. Let $\widetilde{B}$ be a ball such that $\operatorname{supp}(f \circ \phi^{-1}) \subset \widetilde{B}/2$, $r < r(\widetilde{B})/100$ and fix $\widetilde{C} := C_{K, 2\widetilde{B}, \phi}$, the constant in Lemma II.3.5 so that $r_{B_-} \leq \widetilde{C} r_B^{1/K}$ for every ball $B \subset 2\widetilde{B}$. For $x \in \widetilde{B}$ and $s \in (0, r]$ (otherwise the first integral below is zero), by Lemma II.3.5 and (II.3.2)[(II.5)] we have

$$\begin{aligned} \fint_{Q_s(x)} \left| f \circ \phi^{-1} - m_{Q_s(x)}(f \circ \phi^{-1}) \right| &\leq \sup_{Q \subset Q_s(x)} \fint_Q \left| f \circ \phi^{-1} - m_Q(f \circ \phi^{-1}) \right| \\ &\lesssim \sup_{Q \subset Q(\phi(x), C s^{1/K})} \left(\fint_Q |f - m_Q f|^{2K} \right)^{\frac{1}{2K}} \\ &\approx \sup_{Q \subset Q(\phi(x), C s^{1/K})} \fint_Q |f - m_Q f| . \end{aligned} \tag{II.3.18}$$

[(II.5)] The last step using (II.3.2) is unnecessary since the rest of the proof works with the exponent $2K$ by (II.3.15). However, to simplify the notation, we remove the exponent $2K$.

As we did in (II.3.17), we obtain
(II.3.19)

$$(1+|\log(Cs^{1/K})|)\sup_{Q\subset Q(\phi(x),Cs^{1/K})}\fint_Q |f-m_Qf| \leq \sup_{\substack{x\in\mathbb{R}^2\\ s\in(0,Cr^{1/K}]}}\Big[(1+|\log s|)\fint_{Q_s(x)}\big|f-m_{Q_s(x)}f\big|\Big].$$

The lemma follows from (II.3.18), (II.3.19), the fact that $(1+|\log s|)\leq 2K(1+|\log(Cs^{1/K})|)$ for small enough s, and finally taking $r\to 0$. □

Using the previous results, we conclude this subsection with the following result about the dimension of the elliptic measure for $\mathrm{VMO}_{\log}$ matrices.

Proof of Theorem II.0.3: $\mathrm{VMO}_{\log}$ *case.* Recall we can assume that $\Omega\subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3, and we can replace finitely connected by simply connected by Lemma II.1.2. In particular $\Omega\subset Q_{1/3}(0)$ (cube centered at 0 with side length 2/3). Without loss of generality, we can also assume $\det A=1$.

Let $\widetilde{\varphi}(\cdot)=\varphi(\cdot/100)$, with φ as in (II.3.11), and let $\phi_A\in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar\partial\phi_A=\widetilde{\varphi}\mu_A\partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ where } \mu_A \text{ as in (II.2.15)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. Let $\widehat{A}$ be the uniformly elliptic, symmetric and with determinant 1 matrix such that $\widetilde{\varphi}\mu_A=\mu_{\widehat{A}}$, see Remark II.2.16. Therefore, the Beltrami equation above reads as

$$\bar\partial\phi_A=\mu_{\widehat{A}}\partial\phi_A \text{ a.e. in } \mathbb{C}, \text{ with } \mu_{\widehat{A}}=\mu_A \text{ in } B_{50}(0).$$

In particular $u:=\mathrm{Re}\phi_A$ and $v:=\mathrm{Im}\phi_A$ satisfy $\mathrm{div}\,A\nabla u=0$ and $\mathrm{div}\,A\nabla v=0$ in $B_{50}(0)$, see Corollary II.2.13. Now, since $A\in\mathrm{VMO}_{\log}$ we have that $\nabla u,\nabla v\in\mathrm{BMO}$, see Theorem II.3.20.

Now we want to see the same property for ϕ_A^{-1}. By Corollary II.2.13, the functions $R:=\mathrm{Re}(\phi_A^{-1})$ and $I:=\mathrm{Im}(\phi_A^{-1})$ satisfy $\mathrm{div}\,B\nabla R=0$ and $\mathrm{div}\,B^*\nabla I=0$ in $\phi_A(B_{50}(0))$, where B and B^* are the uniformly elliptic matrices in (II.2.16). Since $A\in\mathrm{VMO}_{\log}$, by Lemma II.3.3 we have that $\mu_A\in\mathrm{VMO}_{\log}$ and hence $\mu_{\widehat{A}}\in\mathrm{VMO}_{\log,c}$. By the invariance of $\mathrm{VMO}_{\log,c}$ under quasiconformal mappings in Lemma II.3.23, we have that $-\mu_{\widehat{A}}\circ\phi_A^{-1}\in\mathrm{VMO}_{\log,c}$, the coefficient of the conjugated Beltrami equation in item 3 of Corollary II.2.13. Therefore, by Lemma II.3.3 the uniformly elliptic matrices $\widehat{B}$ and $\widehat{B}^*$ in (II.2.16) (in terms of $\widehat{A}$) belong to $\mathrm{VMO}_{\log}$. By Theorem II.3.20 and since $\widehat{B}=B$ and $\widehat{B}^*=B^*$ in $\phi_A(B_{50}(0))$, we conclude $\nabla R,\nabla I\in\mathrm{BMO}$.

Since both $\nabla\phi_A,\nabla\phi_A^{-1}\in\mathrm{BMO}$, by the log-Lip regularity in Lemma II.3.21 and the distortion under these type of functions in Lemma II.3.22, continuing as in the end of the proof of Theorem II.0.1 in p. 46 we have that there is a set $F\subset\partial\Omega$ satisfying $\omega_{\Omega,A}(F)=1$ and with σ-finite $\mathcal{H}^{t/\log\frac{1}{t}}$ measure; and if the domain is simply connected, by Makarov's Theorem I.0.1 there exists $C_M>0$ such that $\omega_{\Omega,A}\ll\mathcal{H}^{(\varphi_{1,C_M})\circ(t\log\frac{1}{t})}$. □

II.4. Uniformly elliptic matrices in CDC domains

This part is dedicated to the proof of Theorems II.0.5 and II.0.6. That is, we work with general uniformly elliptic matrices $A\in\mathbb{R}^{2\times2}$ (not necessarily symmetric nor constant determinant) in CDC domains.

II.4.1. Proof of Theorem II.0.5. Let us start by the harmonic factorization of the Green function for the operator L_A, which turns out to be the harmonic Green function in the image domain.

Lemma II.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Wiener regular domain, $p \in \Omega$, A be a uniformly elliptic matrix with $A = Id$ in an open neighborhood of p. If μ_{A,g_p^A} and ϕ are as in Lemma II.2.17, then $g_p^{\Omega,A} \circ \phi^{-1} = g_{\phi(p)}^{\phi(\Omega),Id}$ is the Green function in $\phi(\Omega)$ with pole $\phi(p)$ for the Laplacian operator $L_{Id} = -\Delta$.*

Remark II.4.2. Note that the Green function g_p^A has a singularity at $p \in \Omega$. However, the Beltrami coefficient μ_{A,g_p^A} should be understood as $\mu_{A,g_p^A} = \mu_A$ in the open neighborhood of p where the matrix $A = Id$, as $\nu_A = 0$ in this case. In particular, g_p^A is L_A-harmonic in $\Omega \setminus \{x : A(x) \neq Id\} \subset \Omega \setminus \{p\}$.

Proof of Lemma II.4.1. We check $\int_{\phi(\Omega)} \nabla(g_p^A \circ \phi^{-1})\nabla\varphi = \varphi(\phi(p))$ for any $\varphi \in C_c^\infty(\phi(\Omega))$. Let $B \subset 10B \subset \Omega$ be a ball centered at $p \in \Omega$ with small enough radius such that $A = Id$ in $2B$, and let $\psi \in C_c^\infty(\mathbb{R}^2)$ be so that $\psi|_{B/2} = 1$ and $\psi|_{B^c} = 0$. Then

$$\int_{\phi(\Omega)} \nabla(g_p^A \circ\phi^{-1})\nabla\varphi = \int_{\phi(\Omega)} \nabla(g_p^A\circ\phi^{-1})\nabla(\varphi\cdot(\psi\circ\phi^{-1})) + \int_{\phi(\Omega)} \nabla(g_p^A\circ\phi^{-1})\nabla(\varphi\cdot(1-\psi\circ\phi^{-1})).$$

Since $\varphi(1-\psi\circ\phi^{-1}) \in W_c^{1,2}(\phi(\Omega))$ with $\varphi(1-\psi\circ\phi^{-1}) = 0$ in $\phi(B/2) \ni \phi(p)$, and $g_p^A \circ \phi^{-1}$ is harmonic in $\phi(\Omega \setminus B/2)$ by Lemma II.2.17, we have that the second term on the right-hand side is zero, and therefore

$$\int_{\phi(\Omega)} \nabla(g_p^A \circ \phi^{-1})\nabla\varphi = \int_{\phi(\Omega)} \nabla(g_p^A \circ \phi^{-1})\nabla(\varphi \cdot (\psi \circ \phi^{-1})).$$

As the matrix $A = Id$ in $2B$ and ϕ is conformal in $2B$, we have that the matrix after the change of variables is $\widetilde{A}_{\phi^{-1}} = Id$ in $\phi(2B)$, recall its definition in (II.2.3). In particular, since $\operatorname{supp} \nabla(\varphi \cdot (\psi \circ \phi^{-1})) \subset \overline{\phi(B)}$, we can just write

$$\int_{\phi(\Omega)} \nabla(g_p^A \circ \phi^{-1})\nabla\varphi = \int_{\phi(\Omega)} \widetilde{A}_{\phi^{-1}}\nabla(g_p^A \circ \phi^{-1})\nabla(\varphi \cdot (\psi \circ \phi^{-1})).$$

By [AIM09, Theorem 3.8.2] (applied to every $\Omega' \subset \Omega \setminus \{p\}$) we have

$$\nabla(g_p^A \circ \phi^{-1})(z) = (D\phi^{-1}(z))^T \nabla g_p^A(\phi^{-1}(z)) \text{ for a.e. } z \in \phi(\Omega),$$

and applying the change of variables given by ϕ, see [AIM09, Theorem 3.8.1], we have

$$\int_{\phi(\Omega)} \widetilde{A}_{\phi^{-1}}\nabla(g_p^A \circ \phi^{-1})\nabla(\varphi \cdot (\psi \circ \phi^{-1})) = \int_\Omega A\nabla g_p^A \nabla\left((\varphi\circ\phi)\cdot\psi\right).$$

By the higher integrability of quasiconformal mappings in [AIM09, (13.24)], see also (II.3.8), there exists $\varepsilon > 0$ such that $(\varphi\circ\phi)\cdot\psi \in W_c^{1,2+\varepsilon}(\Omega)$, see [OP17, Theorem 2.7] for instance[(II.6)], and as noted right below (I.3.2), the integral above equals $\varphi(\phi(p))\psi(p) = \varphi(\phi(p))$, as claimed.

As we have that both $g_p^A \circ \phi^{-1}$ and $g_{\phi(p)}^{Id}$ are the dirac delta in the weak sense, we obtain

$$\int_{\phi(\Omega)} \nabla(g_{\phi(p)}^{Id} - g_p^A \circ \phi^{-1})\nabla\varphi = 0 \text{ for any } \varphi \in C_c^\infty(\phi(\Omega)).$$

[(II.6)] Cover $\operatorname{supp}\varphi$ by a finite number of balls B_i with $2B_i \subset \phi(\Omega)$.

Since $g_p^A \in W^{1,s}(\Omega)$ and $g_{\phi(p)}^{Id} \in W^{1,s}(\phi(\Omega))$ for all $1 \le s < 2$ by (I.3.1), in particular $g_p^A \circ \phi^{-1}, g_{\phi(p)}^{Id} \in W^{1,1}(\phi(\Omega))$, see [OP17, Theorem 2.7] for instance. The existence of weak derivatives in $L^1(\phi(\Omega))$ implies that the left-hand side is $-\int_{\phi(\Omega)} \varphi\Delta(g_{\phi(p)}^{Id} - g_p^A \circ \phi^{-1})$, and therefore

$$\int_{\phi(\Omega)} \varphi\Delta(g_{\phi(p)}^{Id} - g_p^A \circ \phi^{-1}) = 0 \text{ for any } \varphi \in C_c^\infty(\phi(\Omega)).$$

By Weyl's lemma, $g_{\phi(p)}^{Id} - g_p^A \circ \phi^{-1} \in C^\infty(\phi(\Omega))$ and is harmonic in $\phi(\Omega)$. Since both functions $g_{\phi(p)}^{Id}$ and $g_p^A \circ \phi^{-1}$ vanish on $\partial(\phi(\Omega))$, in particular $g_{\phi(p)}^{Id} - g_p^A \circ \phi^{-1}$ vanishes on $\partial(\phi(\Omega))$. By the maximum principle for harmonic functions, the function $g_{\phi(p)}^{Id} - g_p^A \circ \phi^{-1}$ must be zero in $\phi(\Omega)$, and therefore $g_{\phi(p)}^{Id} = g_p^A \circ \phi^{-1}$ in $\phi(\Omega)$, as claimed. □

Given a function ϕ as in Lemma II.4.1 for the matrix A^T, using the notation in (II.2.3) and Lemma II.2.5, recall that $\widetilde{Id}_\phi$ denotes the matrix such that

$$u \text{ is harmonic in } \phi(\Omega) \iff u \circ \phi \text{ is } L_{\widetilde{Id}_\phi}\text{-harmonic in } \Omega.$$

With this definition, in the previous lemma we saw that $g_p^{A^T}$ is the Green function in Ω for both operators L_{A^T} and $L_{\widetilde{Id}_\phi}$. Consequently, we obtain the following relation between their elliptic measures. When applying (I.6.1) and Lemma I.6.1 in the following proof, note that the matrix $\widetilde{Id}_\phi$ is symmetric.

Lemma II.4.3. *Under the same hypothesis in Lemma II.4.1, assume also that Ω is (c_0, r_0)-CDC. If $\mu_{A^T, g_p^{A^T}}$ and ϕ are as in Lemma II.2.17, then for any ball B centered at $\partial\Omega$ with $r_B \le r_0/2$ and satisfying that $p \in \Omega \setminus 4B$, there holds*

$$\omega_{\Omega,A}^p(B) \lesssim \omega_{\Omega,\widetilde{Id}_\phi}^p(8B) \text{ and } \omega_{\Omega,\widetilde{Id}_\phi}^p(B) \lesssim \omega_{\Omega,A}^p(8B),$$

where the constants depend only on c_0 and the ellipticity of A.

Proof. Let $\omega = \omega_{\Omega,A}^p$ and $\widetilde{\omega} = \omega_{\Omega,\widetilde{Id}_\phi}^p$. By (I.6.1) we have $\omega^p(B) \lesssim \max_{x \in \overline{\Omega \cap 2B}} g_p^{A^T}(x)$. By Lemma II.4.1, $g_p^{A^T}$ is also the Green function for the operator with matrix $\widetilde{Id}_\phi$, that is $g_p^{A^T} = g_p^{\widetilde{Id}_\phi}$ in Ω. Applying Lemma I.6.1, we get $g_p^{\widetilde{Id}}(x) \lesssim \widetilde{\omega}(8B)$ for any $x \in \Omega \cap 2B$. All in all, $\omega(B) \lesssim \widetilde{\omega}(8B)$. We can do the same argument to obtain $\widetilde{\omega}(B) \lesssim \omega(8B)$. □

Corollary II.4.4. *Under the same hypothesis in Lemma II.4.1, assume also that Ω is c_0-CDC. If $\mu_{A^T, g_p^{A^T}}$ and ϕ are as in Lemma II.2.17, then*

$$\omega_{\Omega,A}^p(E) \approx \omega_{\Omega,\widetilde{Id}_\phi}^p(E) \text{ for any } E \subset \partial\Omega,$$

where the constant depends only on c_0 and the ellipcity of A. By Lemma II.2.6, this is equivalent to

$$\omega_{\Omega,A}^p(E) \approx \omega_{\phi(\Omega),Id}^{\phi(p)}(\phi(E)) \text{ for any } E \subset \partial\Omega.$$

Proof. Let $\omega = \omega_{\Omega,A}^p$ and $\widetilde{\omega} = \omega_{\Omega,\widetilde{Id}_\phi}^p$. For every $\varepsilon > 0$ let $V_\varepsilon \supset E$ be an open set such that $\widetilde{\omega}(V_\varepsilon) \le \widetilde{\omega}(E) + \varepsilon$.

Split $E = E_{\mathsf{d}} \cup E_{\mathsf{nd}}$, where "d" stands for doubling and "nd" for non-doubling, as

$$E_{\mathsf{d}} := \{x \in E : \exists\{r_k\}_{k\ge1} \text{ with } r_k \xrightarrow{k\to\infty} 0 \text{ such that } \omega(8B_{r_k}(x)) \le 100\omega(B_{r_k}(x))\}, \text{ and}$$
$$E_{\mathsf{nd}} := E \setminus E_{\mathsf{d}}.$$

With this choice we have $\omega(E_{\mathsf{nd}})=0$, see [Tol14, Lemma 2.8], and hence $\omega(E)=\omega(E_{\mathsf{d}})$.

For each $x\in E_{\mathsf{d}}$ take an small enough radius $r_x>0$ such that

(1) $8B_{r_x}(x)\subset V_\varepsilon$,
(2) $\omega(8B_{r_x}(x))\lesssim\omega(B_{r_x}(x))$, since $x\in E_{\mathsf{d}}$; and
(3) $\omega(B_{r_x}(x))\lesssim\widetilde{\omega}(8B_{r_x}(x))$, by Lemma II.4.3,

and let $\{8B_i\}_i$ be the subfamily of $\{8B_{r_x}(x)\}_{x\in E_{\mathsf{d}}}$ covering E_{d} and with finite overlapping, given by Besicovitch's covering theorem. Therefore,

$$\omega(E)=\omega(E_{\mathsf{d}})\leq\omega\Big(\bigcup_i 8B_i\Big)\leq\sum_i\omega(8B_i)\overset{(2),(3)}{\lesssim}\sum_i\widetilde{\omega}(8B_i)\approx\widetilde{\omega}\Big(\bigcup_i 8B_i\Big)\overset{(1)}{\leq}\widetilde{\omega}(V_\varepsilon)\leq\widetilde{\omega}(E)+\varepsilon.$$

Letting $\varepsilon\to 0$ we conclude $\omega(E)\lesssim\widetilde{\omega}(E)$. The lemma follows since the exactly same argument holds if we interchange ω and $\widetilde{\omega}$. $\square$

Let us see the proof of Theorem II.0.5.

Proof of Theorem II.0.5. Recall we can assume that $\Omega\subset B_{1/4}(0)$ is bounded Wiener regular by Remark II.1.3 and we can replace finitely connected by simply connected by Lemma II.1.2. Furthermore, if $\varepsilon>0$ is small enough so that $B_{2\varepsilon}(p)\subset\Omega$, by Corollary I.4.5 we can replace A by $\mathbf{1}_{B_1(0)\setminus\overline{B_\varepsilon(p)}}A+(\mathbf{1}_{B_1^c(0)\cup\overline{B_\varepsilon(p)}})Id$.

Let $\phi_{A^T,g_p^{A^T}}\in W^{1,2}_{\text{loc}}(\mathbb{C})$ be the normalized solution of the Beltrami equation

$$\bar\partial\phi_{A^T,g_p^{A^T}}=\mu_{A^T,g_p^{A^T}}\partial\phi_{A^T,g_p^{A^T}}\ \text{ a.e. in }\mathbb{C},\ \text{ where }\mu_{A^T,g_p^{A^T}}\text{ as in (II.2.20)},$$

given by the measurable Riemann mapping theorem, see Theorem II.2.4. In particular $\phi_{A^T,g_p^{A^T}}$ is a K_λ-quasiconformal homeomorphism with K_λ as in (II.0.2) by Lemma II.2.14, and conformal in $\mathbb{C}\setminus\overline{\mathbb{D}}$. We remark that the dependence of the function $\phi_{A^T,g_p^{A^T}}$ in terms of $g_p^{A^T}$ and A^T is due to the fact that $\mu_{A^T,g_p^{A^T}}$ depends on (the derivatives of) $g_p^{A^T}$ and the coefficients of A^T.

The theorem follows from the exactly same proof of the non-regularity case of Theorem II.0.1 in p. 46, using $\omega_{\Omega,A}\ll\omega_{\phi(\Omega),Id}(\phi(\cdot))\ll\omega_{\Omega,A}$ by Corollary II.4.4 (although we do not have the equality in (II.3.1)) and that $\phi_{A^T,g_p^{A^T}}$ is K_λ-quasiconformal instead of λ-quasiconformal. $\square$

II.4.2. Continuous matrices. Proof of Theorem II.0.6. In this subsection, we prove Theorem II.0.6 via a localization argument. More precisely, we use the continuity of the matrix to see that locally, it is as close as we want to a constant matrix. Once in this case, we use the following remark.

Remark II.4.5. A quick look of the proof of Lemma II.1.5 reveals that the same result remains true for CDC domains if we replace $\overline{\dim}_{\mathcal{H}}(\lambda)$ by

$$\overline{\dim}_{\mathcal{H}}^{\text{CDC}}(\lambda)\coloneqq\sup\{\dim_{\mathcal{H}}\omega_{\Omega,A}:\Omega\subset\mathbb{R}^{n+1}\text{ is CDC and }A\text{ has ellipticity constant }\lambda\}.$$

That is, given a CDC domain $\Omega\subset\mathbb{R}^{n+1}$ and a constant uniformly elliptic matrix C with ellipticity constant $\lambda\geq 1$, if A is a (not necessarily constant) matrix with small enough

$$\varepsilon\coloneqq\max_{i,j\in\{1,\dots,n+1\}}\|a_{ij}(\cdot)-c_{ij}\|_{L^\infty(\overline{\Omega})}$$

depending on λ and n, then A is uniformly elliptic in $\overline{\Omega}$ and $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq \overline{\dim}_{\mathcal{H}}^{\mathrm{CDC}}((1-(n+1)\varepsilon\lambda)^{-1})$.

Proof of Theorem II.0.6. First, we prove the upper bound for CDC domains, and then the lower bound for finitely connected domains.

"Upper bound". It suffices to see that for every $\tau > 0$ there exists $F_\tau \subset \partial\Omega$ with $\dim_{\mathcal{H}} F_\tau \leq 1+\tau$ and $\omega_{\Omega,A}(F_\tau) = 1$. Indeed, $F := \bigcap_{n\geq 1} F_{1/n}$ satisfies $\dim_{\mathcal{H}} F \leq 1$ and $\omega_{\Omega,A}(F) = 1$.

Since A has continuous coefficients in $\overline{\Omega}$, for each $\xi \in \partial\Omega$ and every $\varepsilon > 0$ there is $\delta_{\xi,\varepsilon} > 0$ such that

$$z \in \overline{4B_{\delta_{\xi,\varepsilon}}(\xi) \cap \Omega} \implies |a_{ij}(z) - a_{ij}(\xi)| < \varepsilon \text{ for any } i,j \in \{1,2\}. \tag{II.4.1}$$

Fix $\varepsilon > 0$. Let $\{B_i\}_i \subset \{B_s(\xi)\}_{\xi\in\partial\Omega, 0<s\leq\delta_{\xi,\varepsilon}}$ be a countable subfamily of disjoint balls such that $\omega^p(\partial\Omega \setminus \bigcup_i B_i) = 0$, by Vitali's covering theorem.

We claim that it suffices to see that for each index i there exists a set $F_i \subset \partial(\Omega \cap 4B_i)$ with $\omega_{\Omega\cap 4B_i,A}(F_i) = 1$ and $\dim_{\mathcal{H}} F_i \leq 1+\tau$ provided $\varepsilon > 0$ is small enough. Indeed, this implies that the set $F_\tau := \partial\Omega \cap \bigcup_i F_i$ satisfies $\dim_{\mathcal{H}} F_\tau \leq 1+\tau$ by definition, and $\omega_{\Omega,A}(F_\tau) = 1$ by Lemma I.4.4.

Applying Remark II.4.5 for each index i, by (II.4.1) and as $\Omega \cap 4B_i$ is CDC we have that there exists a set $F_i \subset \partial(\Omega \cap 4B_i)$ with $\omega_{\Omega\cap 4B_i,A}(F_i) = 1$ and $\dim_{\mathcal{H}} F_i \leq \overline{\dim}_{\mathcal{H}}^{\mathrm{CDC}}((1-2\varepsilon\lambda)^{-1})$. By Theorem II.0.5 and taking $\varepsilon > 0$ small enough respectively we have

$$\overline{\dim}_{\mathcal{H}}^{\mathrm{CDC}}((1-2\varepsilon\lambda)^{-1}) \leq \frac{2K_{(1-2\varepsilon\lambda)^{-1}}}{K_{(1-2\varepsilon\lambda)^{-1}}+1} \leq 1+\tau,$$

and the upper bound is proved.

"Lower bound". We turn now to the proof of the lower bound for finitely connected domains. Let $F \subset \partial\Omega$ with $\omega_{\Omega,A}(F) = 1$ and fix $\tau > 0$. Given $\varepsilon > 0$, for each $\xi \in \partial\Omega$ let $\delta_{\xi,\varepsilon} > 0$ be small enough to satisfy (II.4.1) and that $\Omega \cap B(\xi, \delta_{\xi,\varepsilon})$ is simply connected, see Lemma II.1.2. Let $\{B_i\}_i \subset \{B_s(\xi)\}_{\xi\in\partial\Omega, 0<s\leq\delta_{\xi,\varepsilon}}$ be a countable subfamily with finite overlapping by Besicovitch covering theorem. For each ball B_i, by Lemma II.1.2 we have $\omega_{\Omega\cap B_i}(\partial\Omega \setminus F) = 0$, which implies

$$\omega_{\Omega\cap B_i}((\partial B_i \cap \Omega) \cup (F \cap B_i)) = 1.$$

From this, $\dim_{\mathcal{H}} \partial B = 1$, (II.4.1) and Lemma II.1.5 we have

$$\dim_{\mathcal{H}} F \geq \dim_{\mathcal{H}}(F \cap B_i) \geq \underline{\dim}_{\mathcal{H}}^{\mathrm{sc}}((1-2\varepsilon\lambda)^{-1}).$$

By Theorem II.0.5 and taking $\varepsilon > 0$ small enough respectively we have

$$\underline{\dim}_{\mathcal{H}}^{\mathrm{sc}}((1-2\varepsilon\lambda)^{-1}) \geq \frac{2}{K_{(1-2\varepsilon\lambda)^{-1}}+1} \geq 1-\tau.$$

As this can be done for every $\tau > 0$ we have that $\dim_{\mathcal{H}} F \geq 1$ and the lower bound of $\dim_{\mathcal{H}} \omega_{\Omega,A}$ for finitely connected domains is proved. $\square$

CHAPTER III

The dimension of planar elliptic measures arising from Lipschitz matrices in Reifenberg flat domains

In this chapter, we study the dimension of planar elliptic measures in Reifenberg flat domains with small constant, assuming also Lipschitz continuity of the coefficients of the matrix. In fact, that regularity is only needed near the boundary.

Suppose from now on that the domain is (δ, r_0)-Reifenberg flat and the coefficients of the matrix are also Lipschitz. Roughly speaking, a domain $\Omega \subset \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat if for every ball B centered at $\partial\Omega$ and with radius $r \leq r_0$, the δr-neighborhood of a line through the center of B contains $B \cap \partial\Omega$. See Definition III.1.4 for the details. In this situation we show that the dimension of the elliptic measure in Reifenberg flat domains with small constant is at most 1. More precisely, for this type of domains we obtain the analogous result of Wolff's Theorem I.0.2 in the following theorem.

Theorem III.0.1. *Let $\Omega \subset \mathbb{R}^2$ be a (δ, r_0)-Reifenberg flat domain, $p \in \Omega$, and A be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ. Suppose also that A is Lipschitz. Then there exists $\delta_0 = \delta_0(\lambda) > 0$ such that if $0 < \delta \leq \delta_0$ then $\omega_{\Omega,A}$ has σ-finite length. In particular $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq 1$.*

Despite we are requiring to work with δ-Reifenberg flat domains with small enough constant δ, such sets can be constructed with Hausdorff dimension strictly larger than 1. For example, a suitable variant of the Koch snowflake can be constructed to be δ-Reifenberg flat.

It is well-known that the Hausdorff dimension of elliptic measures only depends on how the matrix is near the boundary (see Corollary I.4.5) and hence it is only necessary to assume the Lipschitz regularity around the boundary. In fact, in our proof we use the regularity only around the boundary, and so the theorem is still true assuming Lipschitz continuity in a small neighbourhood of $\partial\Omega$.

Similarly as in [Wol93, Proof of Theorem 1], Theorem III.0.1 follows from a more quantitative result involving a good covering of a subset of the boundary with big elliptic measure, see Main Lemma III.2.1 for the precise statement.

One might think that this result could be obtained by the application of quasiconformal mappings. In Chapter II, for symmetric matrices with determinant 1, the principal solution of the associated Beltrami equation, which depends only on the matrix, can act as a change of variables which inherits the extra regularity of the coefficients. Then, we obtain that the elliptic measure is the pushforward of the harmonic measure in the image domain. In the general case, we can obtain the elliptic measure as a pushforward of a harmonic measure using a quasiconformal change of variables which depends also on the Green function of the domain, as long as it satisfies the capacity density condition. In this case, the extra dependence of the Green function does not allow us to obtain extra regularity estimates for the change of variables, and the σ-finiteness of length can not be attained.

A key point, and the main difficulty in this chapter is to obtain the lower bound

$$\int_{\partial\widetilde{\Omega}} \log \frac{d\widetilde{\omega}^p}{d\sigma}(\xi)\, d\widetilde{\omega}^p(\xi) \geq -\text{const} > -\infty,$$

with a bound independent of the smoothness, where $\widetilde{\Omega}$ is the modified domain appearing in the proof of Lemma III.2.3 and $\widetilde{\omega}^p$ the elliptic measure in $\widetilde{\Omega}$ with respect to the matrix A. This lower bound is known in the harmonic case for general smooth domains in the plane. This fact is the key point in the study of the dimension in the planar case, see for example [Car85, JW88, Wol93]. Actually, the behavior of this integral is also crucial in the study of the dimension of harmonic measures in higher dimensions in [Wol95].

The first occurrence (as far as we know) of the use of this lower bound in the study of the dimension of the harmonic measure is in [Car85]. For the proof see [JW88], and for further details see [CTV19].

The previous lower bound is used in the proof of Lemma III.2.3 to obtain a subset with big elliptic measure as it is done in [JW88] and [Wol93], and it can be deduced from

$$\int_{\partial\widetilde{\Omega}} \log |S\nabla g_p^T(\xi)|^2\, d\widetilde{\omega}^p(\xi) \geq -\text{const} > -\infty, \tag{III.0.1}$$

where g_p^T is the Green function in $\widetilde{\Omega}$ with respect to the matrix A^T, and S is the square root matrix of the symmetric part $A_0 = (A + A^T)/2$, i.e., $S^T S = A_0$. In Section III.6 we obtain also an upper bound for the integral above, see Lemma III.6.1.

A fundamental tool to obtain the estimate (III.0.1) is the relation

$$|\nabla g_p^T(y)| \gtrsim \frac{g_p^T(y)}{\operatorname{dist}(y, \partial\widetilde{\Omega})}$$

near the boundary. In Reifenberg flat domains this estimate was obtained by Lewis, Lundström and Nyström in [LLN08], see Lemma III.1.5 below. The converse inequality is well-known for general domains and Hölder matrices and follows by Schauder estimates. Obtaining an inequality of this type for other domains may allow to apply the techniques exposed in this chapter.

Very similar results to the one in Theorem III.0.1 about the dimension are true for p-harmonic measures, i.e., when the associated operator is the p-Laplacian $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for Reifenberg flat sets with small constant (see [LNV13]), and simply connected sets in the plane (see [LNPC11]).

III.1. Preliminaries and definitions

III.1.1. Notation. We retain the notation from p. 9 throughout this chapter. Specifically, the ambient space is $\mathbb{R}^2$, though auxiliary results will be stated in the general setting of $\mathbb{R}^{n+1}$ for $n \geq 1$.

III.1.2. NTA and Reifenberg flat domains. In this subsection we introduce non-tangentially accessible domains (NTA) and Reifenberg flat domains, the main object of our study.

In order to define NTA domains we need to introduce some concepts. Given a domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and a fixed constant C, we define:

- *C-Whitney ball*: A ball $B(x,r) \subset \Omega$ is C-Whitney if $C^{-1}r < \operatorname{dist}(B(x,r), \partial\Omega) < Cr$.

- *C-Harnack chain*: For $p_1, p_2 \in \Omega$, a C-Harnack chain from p_1 to p_2 in Ω is a sequence of C-Whitney balls such that the first ball contains p_1, the last contains p_2, and such that consecutive balls intersect. The number of balls is called the length of the C-Harnack chain.

Consecutive balls in a C-Harnack chain must have comparable radius. Given a positive L_A-harmonic function in Ω, a C-Harnack chain between two points $p_1, p_2 \in \Omega$ allows us (via Harnack's inequality) to obtain $u(p_1) \approx u(p_2)$, where the constant involved depends on C and the length of the C-Harnack chain.

Definition III.1.1 (NTA domain. [JK82, Section 3])**.** A domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is called nontangentially accessible, NTA for short, if there exists constants $C > 1$ and $r_0 > 0$ such that:

(1) *Interior Corkscrew condition*: For any $\xi \in \partial\Omega$, $0 < r < r_0$, there exists a point $A_r(\xi) \in \Omega$ such that $|A_r(\xi) - \xi| < r$ and $\operatorname{dist}(A_r(\xi), \partial\Omega) > C^{-1}r$. The point $A_r(\xi) = A(\xi, r)$ is called the Corkscrew point of the point x at radius r.
(2) *Exterior Corkscrew condition*: $\overline{\Omega}^c$ satisfies the interior Corkscrew condition.
(3) *Harnack chain condition*: If $\varepsilon > 0$ and $p_1, p_2 \in \Omega$ satisfy that $p_1, p_2 \in \Omega \cap B(\xi, r_0/4)$ for some $\xi \in \partial\Omega$, $\operatorname{dist}(p_j, \partial\Omega) > \varepsilon$, and $|p_1 - p_2| < 2^k\varepsilon$, then there exists a Harnack chain from p_1 to p_2 of length Ck and such that the diameter of each ball is bounded below by $C^{-1} \min_{j=1,2} \operatorname{dist}(p_j, \partial\Omega)$.

Remark III.1.2. A domain with the exterior Corkscrew condition satisfies the capacity density condition, see [HKM06, Theorem 6.31]. In particular, NTA domains satisfy the capacity density condition.

Definition III.1.3 (Hausdorff distance)**.** Given nonempty sets A and B, we denote their Hausdorff distance $\operatorname{dist}_{\mathcal{H}}(A, B)$ by

$$\operatorname{dist}_{\mathcal{H}}(A, B) = \max \left\{ \sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A) \right\}.$$

Definition III.1.4 (Reifenberg flat domain)**.** Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set, and let $0 < \delta < 1/2$ and $r_0 > 0$. We say that Ω is a (δ, r_0)-Reifenberg flat domain if it satisfies the following conditions:

(1) For every $x \in \partial\Omega$ and every $0 < r \leq r_0$, there exists a hyperplane $\mathcal{P}(x, r)$ containing x such that

$$\operatorname{dist}_{\mathcal{H}} \left(\partial\Omega \cap B(x, r), \mathcal{P}(x, r) \cap B(x, r) \right) \leq \delta r.$$

(2) For every $x \in \partial\Omega$ and every $0 < r \leq r_0$, one of the connected components of

$$B(x, r) \cap \left\{ y \in \mathbb{R}^{n+1} : \operatorname{dist}(y, \mathcal{P}(x, r)) \geq 2\delta r \right\}$$

is contained in Ω and the other is contained in $\mathbb{R}^{n+1} \setminus \Omega$.

For small enough $\delta > 0$ we have that a (δ, r_0)-Reifenberg flat domain is also an NTA domain (see [KT97, Section 3]) and it satisfies the capacity density condition, see Remark III.1.2.

III.1.3. Non-degeneracy of $|\nabla u|$ in Reifenberg flat domains with small constant. Despite the following result applies to more general matrices (see [LLN08, Definition 1.1 and Lemma 3.35]), it is only stated in our setting for our purposes, i.e., for uniformly elliptic (I.0.1a)-(I.0.1b) and Hölder continuous matrices. In fact, the Hölder continuity is only used near the boundary.

Lemma III.1.5 ([LLN08, Lemma 3.35]). *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a (δ, r_0)-Reifenberg flat domain, $\xi \in \partial\Omega$, and $0 < r < \min\{r_0, 1\}$. Let $0 < \alpha < 1$ and $A \in C^\alpha(\{x : \operatorname{dist}(x, \partial\Omega) < 10r_0\})$ be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ and Hölder seminorm C_α. Suppose that u is a positive L_A-harmonic function in $\Omega \cap B(\xi, 4r)$, that is continuous in $\overline{\Omega} \cap B(\xi, 4r)$, and that $u = 0$ on $\partial\Omega \cap B(\xi, 4r)$. There exist $\hat{\delta} = \hat{\delta}(n, \lambda, C_\alpha, \alpha)$, $\gamma = \gamma(n, \lambda, C_\alpha, \alpha)$ and $\hat{c} = \hat{c}(n, \lambda, C_\alpha, \alpha)$ such that if $0 < \delta \leq \hat{\delta}$, then*

$$\gamma^{-1} \frac{u(y)}{\operatorname{dist}(y, \partial\Omega)} \leq |\nabla u(y)| \leq \gamma \frac{u(y)}{\operatorname{dist}(y, \partial\Omega)} \text{ whenever } y \in \Omega \cap B(\xi, r/\hat{c}).$$

Remark III.1.6. Note that if the matrix has bounded Lipschitz coefficients, i.e., $\alpha = 1$, then the same result holds with constants depending on n, the ellipticity constant λ and the value $C_L \|A\|_{L^\infty(\mathbb{R}^{n+1})}$, where C_L is the Lipschitz seminorm of A. The value $C_L \|A\|_{L^\infty(\mathbb{R}^{n+1})}$ comes from the fact that bounded Lipschitz functions are Hölder continuous for any exponent. Indeed, a quick computation shows that the matrix A is Hölder continuous with exponent $1/2$ with Hölder seminorm $C_{1/2} := \big(2C_L \|A\|_{L^\infty(\mathbb{R}^{n+1})}\big)^{1/2}$.

The comparability in Lemma III.1.5 will allow to bound the error terms in the study of the key term in (III.6.1).

III.2. Main Lemma and preliminary reductions

As in [Wol93], Theorem III.0.1 will follow from the following more quantitative result.

Main Lemma III.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded (δ, r_0)-Reifenberg flat domain, a point $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$, and A be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ, and suppose also that A is κ-Lipschitz in $U_{r_0}(\partial\Omega) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial\Omega) < r_0\}$. For a given $0 < r \leq 1$ satisfying $r\kappa\|A\|_{L^\infty(\mathbb{R}^2)} \leq 1$, there exists $\delta_0 = \delta_0(\lambda) > 0$ such that for every $0 < \delta \leq \delta_0$ we have the following:*

For any $0 < \tau < 1$, sufficiently large M, and $\rho \in (0, r/M)$ there is a set $F \subset \partial\Omega$ such that $\omega^p_{\Omega,A}(F) \geq C^{-1}\tau$ and a countable covering $F \subset \bigcup_i B(z_i, r_i)$ where

(1) $\sum_i r_i \leq CM^\tau$,
(2) $\sum_{\{i : r_i > \rho\}} r_i \leq CM^{-1}$,

with universal constant C.

Remark III.2.2. Given M sufficiently large to satisfy the conclusions of the lemma, the particular choice $\rho = r/(2M)$ yields that the number of balls $B(z_i, r_i)$ with $r_i > r/(2M)$ is universally bounded, that is, $\sum_{\{i : r_i > r/(2M)\}} 1 \leq 2C/r$.

By means of a linear deformation of the plane (see Section III.2.1 below) and a rescaling, we see that it suffices to prove the following weaker lemma to obtain Main Lemma III.2.1.

Lemma III.2.3 (Weak form of Main Lemma III.2.1). *Let $\Omega \subset \mathbb{R}^2$, $p \in \Omega$ and A as in Main Lemma III.2.1. Suppose also that $A_0 = \frac{A+A^T}{2}$ is of the form $A_0 = R^T BR$ with*

$R \in C^{0,1}(U_{r_0}(\partial\Omega))$ a rotation, and $B \in C^{0,1}(U_{r_0}(\partial\Omega))$ diagonal. Then there exists $\delta_0 = \delta_0(\lambda, \kappa\|A\|_{L^\infty(\mathbb{R}^2)}) > 0$ such that for every $0 < \delta \leq \delta_0$ we have the following:

For any $0 < \tau < 1$, sufficiently large M (how large depends on τ and on the constants in the hypothesis), and $\rho \in (0, 1/M)$ there is a set $F \subset \partial\Omega$ such that $\omega^p_{\Omega,A}(F) \geq C^{-1}\tau$ and a countable covering $F \subset \bigcup_i B(z_i, r_i)$ with

(1) $\sum_i r_i \leq CM^\tau$,*

(2) $\sum_{\{i : r_i > \rho\}} r_i \leq CM^{-1}$,*

with universal constant C.

We remark that this weaker form replaces the assumption on the parameter r by the additional assumption $A_0 = R^T BR$, and allows δ_0 to depend also on $\kappa\|A\|_{L^\infty}$, and ρ to be in $(0, 1/M)$. For the proof of Lemma III.2.3, see Section III.4.

III.2.1. The change of variables in Reduction 1. In this subsection we first collect some auxiliary results about changes of variables that will be useful to prove some technical lemmas, secondly we construct the precise linear deformation in the plane that allows the reduction from Main Lemma III.2.1 to Lemma III.2.3, and finally we see how it distorts planar Reifenberg flat domains.

III.2.1.1. *Linear changes of variables.* We will see how L_A-harmonic functions behave under linear changes of variables. See [AM19, Lemmas 3.8 and 3.9] for a detailed proof of the following two results.

Lemma III.2.4. *Let $D \in \mathbb{R}^{(n+1)\times(n+1)}$ be a constant matrix with $\det D \neq 0$, $n \geq 0$. A function f is L_A-harmonic in Ω if and only if $\widetilde{f} = f \circ D$ is $L_{\widetilde{A}}$-harmonic in $D^{-1}(\Omega)$, where $\widetilde{A}(\cdot) = D^{-1}A(D\cdot)(D^{-1})^T$ and $D^{-1}(\Omega) = \{D^{-1}x : x \in \Omega\}$.*

By the definition of elliptic measure, the previous lemma implies the following relation of elliptic measures under a linear change of variables.

Corollary III.2.5. *Let $D \in \mathbb{R}^{(n+1)\times(n+1)}$ be a constant matrix such that $\det D \neq 0$, $n \geq 0$, and let Ω be a Wiener regular domain. Let $\omega = \omega_{\Omega,A}$ be the elliptic measure in Ω with matrix A, and $\widetilde{\omega} = \omega_{D^{-1}(\Omega),\widetilde{A}}$ where $\widetilde{A}(\cdot) = D^{-1}A(D\cdot)\,(D^{-1})^T$. Then $\omega^x(E) = \widetilde{\omega}^{D^{-1}x}(D^{-1}(E))$ for every $x \in \Omega$ and $E \subset \partial\Omega$.*

These two results are particular cases of Lemmas II.2.5 and II.2.6.

III.2.1.2. *Lipschitz diagonalization of symmetric matrices in the plane.* In the study of the integral (III.6.1) in Section III.6 we will use that after a suitable linear change of variables D, the symmetric part of the matrix $\widetilde{A}$ in Corollary III.2.5 diagonalizes in the form $R^T BR$, where R is a Lipschitz rotation and B is Lipschitz diagonal. In this subsection we see that we can always reduce to this case.

We need to follow this strategy because in general it is not true that Lipschitz elliptic symmetric matrices diagonalize in the aforementioned form, as we can see in the following example.

Example III.2.6. Let A_1, A_2 be two constant symmetric matrices diagonalizing with different eigenvectors, and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a Lipschitz function with $f\mathbf{1}_{\{|x|<1\}} < 0$, $f\mathbf{1}_{\{|x|>1\}} > 0$ and $\|f\|_\infty \leq \varepsilon$ for small enough fixed constant $\varepsilon > 0$.

Set $A(x) := Id + f(x)\mathbf{1}_{\{|x|<1\}}(x)A_1 + f(x)\mathbf{1}_{\{|x|>1\}}(x)A_2$ and take $\varepsilon > 0$ small enough to ensure the ellipticity condition on the matrix A. Moreover, with this choice of the function f we have that the matrix A has Lipschitz coefficients.

Let v_1 be an eigenvector of A_1 with eigenvalue μ_1, i.e., $A_1 v_1 = \mu_1 u_1$, and let v_2 be an eigenvector of A_2 with eigenvalue μ_2, i.e., $A_2 v_2 = \mu_2 v_2$. Then, for $|x| < 1$ the vector u is an eigenvector of A,

$$A(x)v_1 = (Id + f(x)A_1)v_1 = (1 + f(x)\mu_1)v_1,$$

and for $|x| > 1$ the vector v is an eigenvector of A,

$$A(x)v_2 = (Id + f(x)A_2)v_2 = (1 + f(x)\mu_2)v_2.$$

From this we get that the matrix A diagonalizes with the same basis as A_1 if $|x| < 1$, and with the same basis as A_2 if $|x| > 1$, whence we obtain that the basis is not continuous.

In the following lemma we see that we can avoid the situation seen in the previous example by using a linear change of variables.

Lemma III.2.7. *Let $U \subset \mathbb{R}^2$ be a set. Let $A \in C^{0,1}(U)$ be a uniformly elliptic and symmetric 2×2 matrix with ellipticity constant λ, and let $D = \begin{pmatrix} 1/K & 0 \\ 0 & 1 \end{pmatrix}$ with $K^2 \geq \lambda^2 + \lambda$. Then the matrix $\widetilde{A}(\cdot) = D^{-1}A(D\cdot)D^{-1}$ is of the form $\widetilde{A} = R^T B R \in C^{0,1}(D^{-1}(U))$, with $B \in C^{0,1}(D^{-1}(U))$ diagonal and $R \in C^{0,1}(D^{-1}(U))$ a rotation.*

Proof. Denote the matrix $A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix}$, and let

$$\widetilde{A} = D^{-1}(A \circ D)D^{-1} = \begin{pmatrix} K^2\widetilde{a} & K\widetilde{b} \\ K\widetilde{b} & \widetilde{d} \end{pmatrix},$$

where we write $\widetilde{a}(x) = a(Dx)$ and the analogous expressions for the other elements of the matrix. We want to see that when $K^2 \geq \lambda^2 + \lambda$ we can write $\widetilde{A} = R^T B R$ where B is Lipschitz and diagonal, and R is a Lipschitz rotation matrix.

The eigenvalues $\lambda_\pm = \lambda_\pm(\cdot)$ of $\widetilde{A}$ are

$$\lambda_\pm = \frac{K^2\widetilde{a} + \widetilde{d} \pm \sqrt{\left(K^2\widetilde{a} - \widetilde{d}\right)^2 + 4K^2\widetilde{b}^2}}{2}, \tag{III.2.1}$$

and we want to see that they are Lipschitz if $K^2 \geq \lambda^2 + \lambda$. Note that

$$\lambda_+ + \lambda_- = K^2\widetilde{a} + \widetilde{d}. \tag{III.2.2}$$

For shortness, let $f = \left(K^2\widetilde{a} - \widetilde{d}\right)^2 + 4K^2\widetilde{b}^2$ be an auxiliar function. Note that since $a, b, d \in C^{0,1}(U) \cap L^\infty(U)$, and so $\widetilde{a}, \widetilde{b}, \widetilde{d} \in C^{0,1}(D^{-1}U) \cap L^\infty(D^{-1}U)$, we have $f \in C^{0,1}(D^{-1}U) \cap L^\infty(D^{-1}U)$.

For $x, y \in D^{-1}U$, i.e., $Dx, Dy \in U$,

(III.2.3)

$$\begin{aligned} 2|\lambda_\pm(x) - \lambda_\pm(y)| &\leq K^2|\widetilde{a}(x) - \widetilde{a}(y)| + |\widetilde{d}(x) - \widetilde{d}(y)| + |\sqrt{f(x)} - \sqrt{f(y)}| \\ &\leq C|x - y| + |\sqrt{f(x)} - \sqrt{f(y)}| = C|x - y| + \frac{|f(x) - f(y)|}{|\sqrt{f(x)} + \sqrt{f(y)}|}. \end{aligned}$$

Since $a \geq \lambda^{-1}$ and $d \leq \lambda$ by ellipticity (indeed $a, d \approx_\lambda 1$), and so $\widetilde{a} \geq \lambda^{-1}$ and $\widetilde{d} \leq \lambda$, in particular $K^2\widetilde{a} - \widetilde{d} \geq K^2\lambda^{-1} - \lambda$. Since $K^2 \geq \lambda^2 + \lambda$, we obtain that

$$K^2\widetilde{a} - \widetilde{d} \geq K^2\lambda^{-1} - \lambda \geq 1, \tag{III.2.4}$$

and with this we have

$$\sqrt{f} = \sqrt{\left(K^2\widetilde{a} - \widetilde{d}\right)^2 + 4K^2\widetilde{b}^2} \geq K^2\widetilde{a} - \widetilde{d} \geq 1. \tag{III.2.5}$$

Combining the estimates (III.2.3) and (III.2.5) we get

$$|\lambda_\pm(x) - \lambda_\pm(y)| \lesssim |x - y| + |f(x) - f(y)| \lesssim |x - y|,$$

i.e., $\lambda_\pm$ are Lipschitz.

It remains to see that the matrix diagonalizes in the form $R^T B R$ and that the eigenvectors are also Lipschitz. Let $u^\pm = (u_1^\pm, u_2^\pm)$ be the eigenvectors of the eigenvalues $\lambda_\pm$. Hence, $\left(\widetilde{A} - \lambda_\pm Id\right) u^\pm = 0$, i.e.,

$$\begin{cases} (K^2\widetilde{a} - \lambda_\pm)u_1^\pm + K\widetilde{b}u_2^\pm = 0, \\ K\widetilde{b}u_1^\pm + (\widetilde{d} - \lambda_\pm)u_2^\pm = 0. \end{cases} \tag{III.2.6}$$

Consider the vectors $v^+ = \left(\lambda_+ - \widetilde{d}, K\widetilde{b}\right)$ and $v^- = \left(K\widetilde{b}, \lambda_- - K^2\widetilde{a}\right)$, which are clearly Lipschitz by the preceding discussion. We claim that v^+ and v^- satisfy (III.2.6). Indeed, v^+ satisfy the second equality in (III.2.6) by the definition of v^+, and the first equality follows from the definition of the eigenvalues $\lambda_\pm$ and the equality $\lambda_+ - \widetilde{d} = K^2\widetilde{a} - \lambda_-$, see (III.2.2). The vector v^- satisfy (III.2.6) by the same reason.

By (III.2.2), we can write $v^+ = \left(K^2\widetilde{a} - \lambda_-, K\widetilde{b}\right)$ (and so v^- and v^+ are orthogonal), and hence $\|v^+\| = \|v^-\|$. Moreover,

$$\|v^\pm\|^2 = \left(K^2\widetilde{a} - \lambda_-\right)^2 + K^2\widetilde{b}^2 \geq \left(K^2\widetilde{a} - \lambda_-\right)^2 = \left(\frac{K^2\widetilde{a} - \widetilde{d} + \sqrt{f}}{2}\right)^2 \geq 1,$$

by (III.2.4) and (III.2.5). To conclude, set the unitary vectors $u^\pm := \frac{v^\pm}{\|v^\pm\|}$. They are orthonormal and hence we conclude that $\widetilde{A} = R^T B R$ with

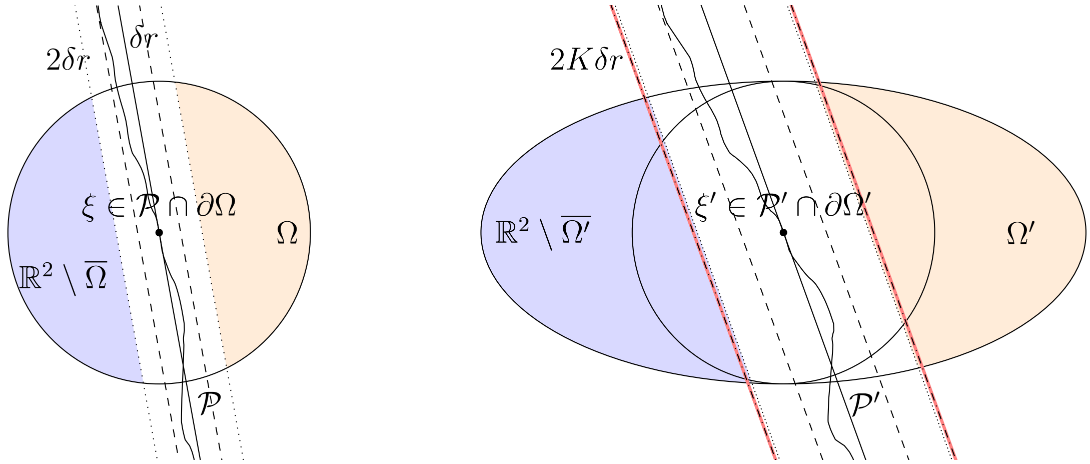
$$B = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \quad R = \begin{pmatrix} u_1^- & u_1^+ \\ u_2^- & u_2^+ \end{pmatrix} = \begin{pmatrix} u_2^+ & u_1^+ \\ -u_1^+ & u_2^+ \end{pmatrix},$$

and $u^\pm$ are Lipschitz since $v^\pm$ are Lipschitz and $\|v^\pm\| \geq 1$. □

We also need to control how Reifenberg flat sets change under the linear planar deformation in the previous lemma.

Lemma III.2.8. *Let $K \geq 1$, $D = \begin{pmatrix} 1/K & 0 \\ 0 & 1 \end{pmatrix}$ and Ω be a (δ, r_0)-Reifenberg flat domain. If $\delta < \sqrt{15}/(16K)$ then $D^{-1}(\Omega)$ is a $\left(\frac{8\sqrt{15}}{15}K\delta, r_0\right)$-Reifenberg flat domain.*

Proof. Let $0 < r \leq r_0$, $\xi \in \partial\Omega$ and $\mathcal{P} := \mathcal{P}(\xi, r) \ni \xi$. Denote $\Omega' = D^{-1}(\Omega)$, $\xi' = D^{-1}\xi$ and $\mathcal{P}' = D^{-1}(\mathcal{P})$. We want to check the conditions in Definition III.1.4 with the point ξ', radius r and the hyperplane $\mathcal{P}'$. See Figure 1.



(A) Ω, $0 < r \leq r_0$ and $\xi \in \mathcal{P} \cap \partial\Omega$. (B) D^{-1} of fig. 1a and $\{x : \mathrm{dist}(x, \mathcal{P}') = 2K\delta r\}$ in red.

FIGURE 1. Almost the worst situation: $\mathcal{P} \perp \{y = 0\}$.

Claim III.2.9. *For any $x, y \in \mathbb{R}^2$, $K^{-1}\mathrm{dist}(x, y) \leq \mathrm{dist}(Dx, Dy) \leq \mathrm{dist}(x, y)$.*

Proof. Indeed, since the minimum (resp. maximum) eigenvalue of D is K^{-1} (resp. 1), we have

$$\frac{\mathrm{dist}(Dx, Dy)}{\mathrm{dist}(x, y)} = \frac{\|Dx - Dy\|}{\|x - y\|} \in [K^{-1}, 1].$$

□

Claim III.2.10. *One component of*

$$B_r(\xi') \cap \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathcal{P}') \geq 2K\delta r\} \tag{III.2.7}$$

is contained in Ω' and the other is contained in $\mathbb{R}^2 \setminus \Omega'$.

Proof. By the previous claim we get

$$D^{-1}\left(\{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathcal{P}) \geq 2\delta r\}\right) \supset \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathcal{P}') \geq 2K\delta r\}, \tag{III.2.8}$$

and from the definition of D^{-1} we have $B_r(\xi') \subset D^{-1}(B_r(\xi))$. By (III.2.8), and since $2K\delta < 2\sqrt{15}/16 < 1$, in particular $B_r(\xi') \cap D^{-1}(\{x : \mathrm{dist}(x, \mathcal{P}) \geq 2\delta r\}) \neq \emptyset$, and the claim follows from the Reifenberg flat condition of Ω. □

First we check

$$\sup_{y \in \partial\Omega' \cap B_r(\xi')} \mathrm{dist}(y, \mathcal{P}' \cap B_r(\xi')) \leq K\delta r. \tag{III.2.9}$$

From the Reifenberg flatness of Ω, see Definition III.1.4(1), we have

$$\sup_{x \in \partial\Omega \cap B_r(\xi)} \mathrm{dist}(x, \mathcal{P} \cap B(\xi, r)) \leq \delta r.$$

Given $y \in \partial\Omega' \cap B_r(\xi')$, take $z \in \mathcal{P} \cap B_r(\xi)$ with $\mathrm{dist}(Dy, z) \leq \delta r$. Since $\mathcal{P}'$ is a line through ξ, the distance $\mathrm{dist}(y, \mathcal{P}')$ is attained at $B_r(\xi')$, that is,

$$\mathrm{dist}(y, \mathcal{P}' \cap B_r(\xi')) = \mathrm{dist}(y, \mathcal{P}') \leq \mathrm{dist}(y, D^{-1}z).$$

By Claim III.2.9,

$$\mathrm{dist}(y, \mathcal{P}' \cap B_r(\xi')) \leq K\mathrm{dist}(Dy, z) \leq K\delta r,$$

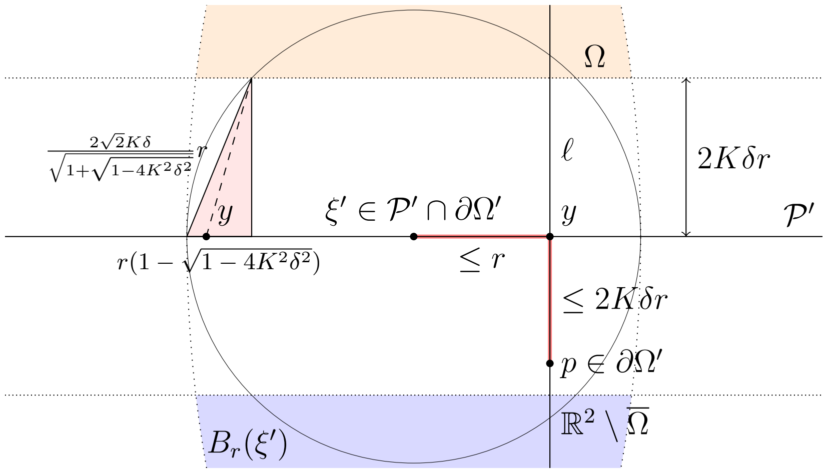


FIGURE 2. Setting of (III.2.10).

and (III.2.9) follows.

Now we turn to prove

$$\sup_{y\in\mathcal{P}'\cap B_r(\xi')} \operatorname{dist}(y, \partial\Omega' \cap B_r(\xi')) \leq \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r. \tag{III.2.10}$$

By Claim III.2.10 we have that for each line ℓ orthogonal to $\mathcal{P}'$ such that

$$\ell \cap B_r(\xi') \cap \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\} \neq \emptyset,$$

there is a point $p \in \ell \cap \partial\Omega' \cap \{x \in B_r(\xi') : \operatorname{dist}(x, \mathcal{P}') < 2K\delta r\}$, since $\partial\Omega'$ must separate each component in (III.2.7). Using this fact and $\xi' \in \mathcal{P}' \cap \partial\Omega'$ (the center of the ball), for every $y \in \mathcal{P}' \cap B_r(\xi')$ we obtain

$$\operatorname{dist}(y, \partial\Omega' \cap B_r(\xi')) \leq \min\left\{r, \operatorname{dist}\left(y, B_r(\xi') \cap \{x : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\}\right)\right\}.$$

Using basic trigonometric computations, see Figure 2, we have that for every $y \in \mathcal{P}' \cap B_r(\xi')$,

$$\operatorname{dist}\left(y, B_r(\xi') \cap \{x : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\}\right) \leq \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r.$$

Since $K\delta < \sqrt{15}/16 < \sqrt{3}/4$, the previous value is less than r, and hence (III.2.10) is proved.

Notice that $K\delta r < 2K\delta r < \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r$. From this, (III.2.9) and (III.2.10) we get

$$\operatorname{dist}_{\mathcal{H}}(\partial\Omega' \cap B_r(\xi'), \mathcal{P}' \cap B_r(\xi')) \leq \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r,$$

and Definition III.1.4(1) is verified.

Since the last term is strictly larger than $2K\delta r$, and $2\frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}} < 1$ when $\delta < \frac{\sqrt{15}}{16K}$, we get

$$B_r(\xi') \cap \{x : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\} \supset B_r(\xi') \cap \left\{x : \operatorname{dist}(x, \mathcal{P}') \geq 2\frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r\right\} \neq \emptyset,$$

and the second condition of Reifenberg flat, Definition III.1.4(2), is achieved by Claim III.2.10.

In conclusion, if $\delta < \sqrt{15}/(16K)$ then $\Omega' = D^{-1}(\Omega)$ is $\left(\frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}, r_0\right)$-Reifenberg flat. In particular $\Omega' = D^{-1}(\Omega)$ is $\left(\frac{8\sqrt{15}}{15}K\delta, r_0\right)$-Reifenberg flat. □

III.2.2. Reduction 1: Lipschitz diagonalization of the symmetric part. Let us see how Lemma III.2.3 implies Main Lemma III.2.1. We will do this in two steps. First we show how to get rid of the assumption on the decomposition of A_0, using the linear transformation in Lemma III.2.7.

Claim III.2.11. *In Lemma III.2.3, the hypothesis $A_0 = R^T BR$ is unnecessary.*

Proof. For every square matrix X we define its symmetric part $X_0 := \frac{X+X^T}{2}$.

By Lemma III.2.7 there exists a constant diagonal matrix $D = \left(\begin{smallmatrix} 1/K & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $K = K(\lambda) \geq 1$ such that $\widetilde{(A_0)}(\cdot) = D^{-1}A_0(D\cdot)D^{-1}$ can be written as $\widetilde{(A_0)} = R^T BR \in C^{0,1}(D^{-1}(U_{r_0}(\partial\Omega)))$ with $R \in C^{0,1}(D^{-1}(U_{r_0}(\partial\Omega)))$ a rotation, and $B \in C^{0,1}(D^{-1}(U_{r_0}(\partial\Omega)))$ diagonal.

Setting $\widetilde{A}(\cdot) := D^{-1}A(D\cdot)D^{-1}$, we have that the symmetric part of the matrix $\widetilde{A}$ is

$$\widetilde{A}_0 := \frac{D^{-1}A(D\cdot)D^{-1} + (D^{-1}A(D\cdot)D^{-1})^T}{2} = \frac{D^{-1}A(D\cdot)D^{-1} + D^{-1}A^T(D\cdot)D^{-1}}{2} = \widetilde{(A_0)},$$

and hence $\widetilde{A}_0 = R^T BR$ as before.

Note that

$$U_{r_0}(\partial D^{-1}(\Omega)) := \{x : \operatorname{dist}(x, \partial D^{-1}\Omega) < r_0\} \subset D^{-1}(U_{r_0}(\partial\Omega)),$$

and so these matrices are Lipschitz in $U_{r_0}(\partial D^{-1}(\Omega))$.

Denoting $\widetilde{\omega} := \omega_{D^{-1}\Omega,\widetilde{A}}$ the elliptic measure in $D^{-1}\Omega$ with matrix $\widetilde{A}$, by Corollary III.2.5 we have $\omega^x(\cdot) = \widetilde{\omega}^{D^{-1}x}(D^{-1}\cdot)$ for any $x \in \Omega$. By Lemma III.2.8 we have that $D^{-1}\Omega$ is $(8\sqrt{15}K\delta/15, r_0)$-Reifenberg flat. Set $\widetilde{p} := D^{-1}p \in D^{-1}(\Omega)$. Since $\operatorname{dist}(p, \partial\Omega) > r_0$, $\operatorname{dist}(\widetilde{p}, \partial D^{-1}\Omega) > r_0$ and we are in position to apply Lemma III.2.3 with this pole $\widetilde{p}$.

First, we need to compute the ellipticity constant, the Lipschitz seminorm, and the L^∞ norm of the matrix $\widetilde{A}$. Recall $\lambda \geq 1$ is the ellipticity constant of A. For $\xi, \eta \in \mathbb{R}^2$,

$$\begin{aligned}
\langle \widetilde{A}\xi, \eta\rangle &= \langle A(D\cdot)D^{-1}\xi, D^{-1}\eta\rangle \leq \lambda|D^{-1}\xi||D^{-1}\eta| \leq \lambda K^2|\xi||\eta|,\\
\langle \widetilde{A}\xi, \xi\rangle &= \langle A(D\cdot)D^{-1}\xi, D^{-1}\xi\rangle \geq \lambda^{-1}|D^{-1}\xi|^2 \geq \lambda^{-1}|\xi|^2 \geq \left(\lambda K^2\right)^{-1}|\xi|^2,
\end{aligned}$$

i.e., the ellipticity constant of $\widetilde{A}$ is λK^2. If one seeks optimal constants, choosing $\widetilde{A}/K$ the ellipticity constant becomes λK. The Lipschitz seminorm of $\widetilde{A}$ in $D^{-1}(U_{r_0}(\partial\Omega))$ is at most $K^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))}$. Indeed, for any two points $x, y \in D^{-1}(U_{r_0}(\partial\Omega))$ with $x \neq y$,

$$\begin{aligned}
\frac{|D^{-1}A(Dx)D^{-1} - D^{-1}A(Dy)D^{-1}|}{|x-y|} &\leq K^2\frac{|A(Dx) - A(Dy)|}{|x-y|}\\
&\leq K^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))}\frac{|Dx - Dy|}{|x-y|}\\
&\leq K^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))}.
\end{aligned}$$

The L^∞ norm is $\|\widetilde{A}\|_{L^\infty(\mathbb{R}^2)} \leq K^2\|A\|_{L^\infty(\mathbb{R}^2)}$.

By Lemma III.2.3 there exists

$$\delta_0 = \delta_0(K(\lambda)^2\lambda, K(\lambda)^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))} \cdot K(\lambda)^2\|A\|_{L^\infty(\mathbb{R}^2)}) > 0$$

such that if $0 < 8\sqrt{15}K(\lambda)\delta/15 \le \delta_0$, i.e., $0 < \delta \le 15\delta_0/(8\sqrt{15}K(\lambda))$, and taking M big enough such that Lemma III.2.3 holds, then there is a set $\widetilde{F} \subset \partial D^{-1}(\Omega)$ such that $\widetilde{\omega}^{\widetilde{p}}(\widetilde{F}) \ge C^{-1}\tau$ and with a covering $\widetilde{F} \subset \bigcup_i B(\widetilde{z}_i, r_i)$ where

(1*) $\sum_i r_i \le CM^\tau$,
(2*) $\sum_{\{i:\, r_i > \rho\}} r_i \le CM^{-1}$,

with universal constant C.

Defining $F := D(\widetilde{F})$ we have $\omega^p(F) = \widetilde{\omega}^{\widetilde{p}}(\widetilde{F}) \ge C^{-1}\tau$, and $F \subset \bigcup_i D(B(\widetilde{z}_i, r_i))$. Finally, as $D(B(\widetilde{z}_i, r_i)) \subset B(D\widetilde{z}_i, r_i)$, then $\{B(D\widetilde{z}_i, r_i)\}_i$ is a covering of F satisfying the same properties. □

III.2.3. Reduction 2: The dependence on the Lipschitz seminorm. This is the second step to show that Lemma III.2.3 implies Main Lemma III.2.1. Note that the flatness constant of the Reifenberg flat domain on Lemma III.2.3 (hence also on Claim III.2.11) depends also on the Lipschitz seminorm κ of the matrix. Below, we see that in fact these results imply Main Lemma III.2.1 by a rescaling argument. Here the flatness constant is determined solely by the ellipticity of the matrix.

Proof of Main Lemma III.2.1 assuming Lemma III.2.3. Fix $r \in (0,1]$ with $r\kappa\|A\|_{L^\infty} \le 1$, and let $\widetilde{A}(\cdot) = A(r\cdot)$, $\widetilde{\Omega} = \Omega/r$ and $\widetilde{\omega} := \omega_{\widetilde{\Omega},\widetilde{A}}$ be the elliptic measure with respect to the matrix $\widetilde{A}$ in $\widetilde{\Omega}$.

The matrix $\widetilde{A}$ is Lipschitz in $\{x/r : \operatorname{dist}(x, \partial\Omega) < r_0\} = \{x : \operatorname{dist}(x, \partial\widetilde{\Omega}) < r_0/r\}$, and $\widetilde{\Omega}$ is $(\delta, r_0/r)$-Reifenberg flat since Ω is (δ, r_0)-Reifenberg flat. By the uniqueness of the elliptic measure we have $\widetilde{\omega}^{z/r}(\cdot/r) = \omega^z(\cdot)$ for any $z \in \Omega$.

With this "zoom" the ellipticity constant of $\widetilde{A}$ becomes the same as the one of A, $\|\widetilde{A}\|_{L^\infty} = \|A\|_{L^\infty}$ and $[\widetilde{A}]_{C^{0,1}} = r[A]_{C^{0,1}}$, which implies $[\widetilde{A}]_{C^{0,1}} \cdot \|\widetilde{A}\|_{L^\infty} = r\kappa\|A\|_{L^\infty} \le 1$.

This allows us to invoke Claim III.2.11 for the elliptic measure $\widetilde{\omega}^{p/r}$ since $\operatorname{dist}(p/r, \Omega/r) > r_0/r$, as $\operatorname{dist}(p, \Omega) > r_0$. Note that now we don't have the dependence on the Lipschitz seminorm and L^∞ norm of the matrix since we are in the case $[\widetilde{A}]_{C^{0,1}}\|\widetilde{A}\|_{L^\infty} \le 1$. Hence, there exists $\delta_0 = \delta_0(\lambda) > 0$ such that if $0 < \delta \le \delta_0$, then for M big enough (to satisfy Claim III.2.11) and setting ρ such that $0 < \rho/r < 1/M$, we can find a set $F' \subset \partial\Omega/r$ such that $\widetilde{\omega}^{p/r}(F') \ge C^{-1}\tau$ and with a covering $F' \subset \bigcup_i B(z_i', r_i)$ such that

(1*) $\sum_i r_i \le CM^\tau$,
(2*) $\sum_{\{i:\, r_i > \rho/r\}} r_i \le CM^{-1}$.

Set $F = rF'$. Then $\omega^p(F) = \widetilde{\omega}^{p/r}(F') \ge C^{-1}\tau$, and $F = rF' \subset \bigcup_i B(z_i, rr_i)$, which implies

(1) $\sum_i rr_i \le rCM^\tau \le CM^\tau$,
(2) $\sum_{\{i:\, rr_i > \rho\}} rr_i = \sum_{\{i:\, r_i > \rho/r\}} rr_i \le rCM^{-1} \le CM^{-1}$,

as claimed. □

III.3. Elliptic measures in CDC domains

In this section we collect the key properties of elliptic measures in CDC domains for the proof of Main Lemma III.2.1. The proof of Main Lemma III.2.1 is based on a modification of the domain, without losing the initial information. In the following lemma we obtain the first step in that modification. This is the analogue of [Wol93, Lemma 1.1].

Lemma III.3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded CDC domain with constants c_0 and radius s_0, let $x \in \partial\Omega$ and $0 < r \leq s_0$, and let A be a real uniformly elliptic (not necessarily symmetric) matrix. Then for any $k > 2$, there exists a constant C depending only on n, c_0 and the ellipticity constant of the matrix A such that*

$$\omega^p_{\Omega\setminus\overline{B(x,r)},A}\left(\overline{B(x,r)}\right) \leq C\omega^p_{\Omega,A}(B(x,kr)), \text{ for all } p \in \Omega \setminus \overline{B(x,r)}.$$

Proof. The proof follows the same argument as [Wol93, Lemma 1.1].

Since $k > 2$, by Lemma I.5.3(2) there exists $C > 1$, depending on n, c_0 and the ellipticity constant of the matrix, such that $\omega^p_{\Omega,A}(B(x,kr)) \geq \omega^p_{\Omega,A}(\overline{B(x,2r)}) \geq C^{-1}$ for any $p \in \overline{B(x,r)} \cap \Omega$. The lemma follows by the maximum principle in $\Omega \setminus \overline{B(x,r)}$ by standard techniques. □

Later on we will need to have some control on the Radon-Nikodym derivative of the elliptic measure of the modified domain with respect to its surface measure, see Lemma III.3.3 below. First we study the density of the elliptic measure in an annulus. For a Hölder matrix A, here we use that L_A-harmonic functions are Hölder continuous up to the boundary, see Theorem I.1.5.

Lemma III.3.2. *Let $k > 3$, $0 < r \leq 1$ and $\mathcal{A}_k = B(x,k^2r) \setminus \overline{B(x,r)} \subset \mathbb{R}^{n+1}$, $n \geq 1$. Let A be a real uniformly elliptic (not necessarily symmetric) matrix. Suppose also that $A \in C^\alpha(B(x,2k^2r))$ with $0 < \alpha < 1$. Then the elliptic measure in the annulus $\mathcal{A}_k$ (arising from the matrix A) satisfies*

$$\omega^z_{\mathcal{A}_k,A}(Y) \lesssim \frac{\sigma(Y)}{r^n}, \text{ for any } z \in \partial B(x,kr) \text{ and any } Y \subset \partial B(x,r),$$

with constant depending only on k, $[A]_{C^\alpha}$ and the ellipticity of A.

Proof. Suppose without loss of generality that the annulus is centered at the origin and denote $B_t := B(0,t)$. We can also assume that $r = 1$. Indeed, denote $\omega := \omega_{\mathcal{A}_k,A}$ and $\widetilde{\omega} := \omega_{\widetilde{\mathcal{A}_k},\widetilde{A}}$ the elliptic measure associated to the matrix $\widetilde{A}(\cdot) := A(r\cdot)$, where the rescaled annulus is $\widetilde{\mathcal{A}_k} = B_{k^2} \setminus \overline{B_1}$. After rescaling and by the uniqueness of the elliptic measure,

$$\omega^z(Y) = \widetilde{\omega}^{z'}(Y') \text{ where } z' = z/r \text{ and } Y' = Y/r,$$

see Corollary III.2.5. The matrix $\widetilde{A}$ has the same ellipticity constant as A, and the Hölder seminorm is improved because $[\widetilde{A}]_{C^\alpha} = [A]_{C^\alpha} \cdot r^\alpha$ whenever $0 < r < 1$. If the lemma were true with $r = 1$ then writing $p = z/r$ we would get

$$\omega^z(Y) = \widetilde{\omega}^p(Y') \leq C_k\sigma(Y') = C_k\frac{\sigma(Y)}{r^n},$$

as claimed.

Let $p \in \partial B_k$ and let g^T_p the Green function of the annulus $\mathcal{A}_k$ with pole at p. Then, using (I.3.7),

$$\omega^p(Y) = -\int_Y \langle A^T(\xi)\nabla g^T_p(\xi), \nu(\xi)\rangle\, d\sigma(\xi) \lesssim \int_Y |\nabla g^T_p(\xi)|\, d\sigma(\xi). \tag{III.3.1}$$

We would be done if we can bound $|\nabla g^T_p| \leq C_k$. To obtain this we apply Theorem I.1.5. In the next paragraphs we check its hypothesis.

The function g_p^T is L_{A^T}-harmonic in $B_2 \setminus B_1$ since we are in the case $k > 3$. Moreover $g_p^T \equiv 0$ in ∂B_1. We need to verify that $g_p^T \in W^{1,2}(B_2 \setminus B_1)$. Recall that the Green function is constructed in (I.3.4) as $g_p^T = -\mathcal{E}_p^T + h$ where h is a $L_{\widetilde{A}^T}$-harmonic function with $h = \mathcal{E}_p^T$ on $\partial \mathcal{A}_k$, and $\mathcal{E}_p^T$ is the fundamental solution with pole at p. Hence,

$$\begin{aligned}\|g_p^T\|_{L^2(B_2\setminus B_1)} \leq &\|g_p^T\|_{L^2\left(B_{5/2}\setminus B_1\right)} \leq \max_{y\in\partial B_{5/2}} g_p^T(y) \leq \max_{y\in\partial B_{5/2}} |\mathcal{E}_p^T(y)| + \max_{y\in\partial B_{5/2}} |h(y)| \\ &\leq \max_{y\in\partial B_{5/2}} |\mathcal{E}_p^T(y)| + \max_{y\in\partial B_{k^2}\cup\partial B_1} |\mathcal{E}_p^T(y)|.\end{aligned} \tag{III.3.2}$$

In the planar case, we have $|\mathcal{E}_p^T(y)| \lesssim 1 + |\log|y-p||$ by Corollary I.3.3, and in particular, from (III.3.2) we obtain

$$\|g_p^T\|_{L^2(B_2\setminus B_1)} \lesssim \max_{y\in\partial B_{5/2}} [1 + |\log|y-p||] + \max_{y\in\partial B_{k^2}\cup\partial B_1} [1 + |\log|y-p||] \leq C_k.$$

In higher dimensions, $n \geq 2$, the fundamental solution is bounded by $|\mathcal{E}_p^T(y)| \lesssim |p-y|^{1-n}$ (see [HK07, Theorem 3.1 (3.55)]). From this bound and (III.3.2) we get

$$\|g_p^T\|_{L^2(B_2\setminus B_1)} \lesssim \max_{y\in\partial B_{5/2}} |y-p|^{n-1} + \max_{y\in\partial B_{k^2}\cup\partial B_1} |y-p|^{n-1} \leq C_k.$$

In fact, we have obtained $\|g_p^T\|_{L^2\left(B_{5/2}\setminus B_1\right)} \lesssim C_k$, and hence by Caccioppoli's inequality in the annulus $B_2 \setminus B_1$ we also obtain the upper bound for the gradient,

$$\|\nabla g_p^T\|_{L^2(B_2\setminus B_1)} \lesssim \|g_p^T\|_{L^2\left(B_{5/2}\setminus B_1\right)} \lesssim C_k,$$

implying $g_p^T \in W^{1,2}(B(0,2) \setminus B(0,1))$ depending only on k and the ellipticity constant.

Consider $\Omega = B_2 \setminus \overline{B}_1$, $T = \partial B_1$ and $\Omega' = B_{3/2} \setminus \overline{B}_1$ in Theorem I.1.5. Note that $\Omega' \subset \Omega$, $T \subset \partial\Omega'$ and $\operatorname{dist}\left((B_{3/2} \setminus \overline{B}_1) \cup \partial B_1, \partial B_2\right) = 1/2$. Then we get $g_p^T \in C^{1,\alpha}((B_{3/2} \setminus \overline{B}_1) \cup \partial B_1)$ with

$$\max_{y\in\partial B_1} |\nabla g_p^T(y)| \leq \|g_p^T\|_{1;\Omega'} \overset{\text{Thm I.1.5}}{\lesssim} \max_{y\in B_2\setminus\overline{B}_1} g_p^T(y) \leq \max_{y\in\partial B_{5/2}} g_p^T(y) \leq C_k,$$

and the lemma follows. $\square$

Lemma III.3.1 relates the elliptic measure on the initial domain with the elliptic measure on the domain minus a fixed ball. Next in Lemma III.3.3, which is the analogue of [Wol93, Lemma 1.2], we study the elliptic measure on this last setting. Combining Lemmas III.3.1 and III.3.3 we will obtain density properties of the elliptic measure on a modified domain.

Lemma III.3.3. *Set $k > 3$ and $0 < r \leq 1$. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded Wiener regular domain, $x \in \partial\Omega$ and let A be a real uniformly elliptic (not necessarily symmetric) matrix. Suppose also that $A \in C^\alpha(\{y \in \mathbb{R}^{n+1} : \operatorname{dist}(y, \partial\Omega) < 2k^2 r\})$ with $0 < \alpha < 1$.*

Set $\widetilde{\Omega} = \Omega \setminus \overline{B}$ where $B = B(x,r)$. Then $\omega^p_{\widetilde{\Omega},A}\big|_{\partial B}$ is absolutely continuous with respect to σ for any $p \in \Omega \setminus k\overline{B}$, and for $z \in \partial B$,

$$\frac{d\omega^p_{\widetilde{\Omega},A}}{d\sigma}(z) \leq \frac{C}{r^n} \omega^p_{\Omega\setminus k\overline{B},A}\left(k\overline{B}\right), \text{ for any } p \in \Omega \setminus k\overline{B},$$

with constant C depending only on n, k, $[A]_{C^\alpha}$ and the ellipticity of A.

Following the scheme of the proof of [Wol93, Lemma 1.2], to obtain Lemma III.3.3 we study the elliptic measure of the annulus $\mathcal{A}_k := B(x, k^2 r) \setminus \overline{B(x,r)}$ when $k > 3$ (in order to apply Lemma I.5.2) and $0 < r \leq 1$ (to have a control on the Hölder seminorm of the matrix $A \in C^\alpha$). However, some technicalities are needed due to the variability of the coefficients of the matrix.

Proof of Lemma III.3.3. During the proof we write $\omega_\cdot$ instead of $\omega_{\cdot,A}$.

To obtain the result it suffices to prove $\omega^p_{\widetilde{\Omega}}(Y) \lesssim \frac{\sigma(Y)}{r^n} \omega^p_{\Omega\setminus k\overline{B}}(k\overline{B})$ for all $p \in \Omega \setminus k\overline{B}$ and every $Y \subset \partial B$, and in fact, it is enough to assume that Y is open. Indeed, fixed $p \in \Omega \setminus k\overline{B}$, for $\varepsilon > 0$ let $U \supset Y$ be an open set (relative to ∂B) such that $\sigma(U) \leq \sigma(Y) + \varepsilon$. If the lemma were true for open sets, then $\omega^p_{\widetilde{\Omega}}(Y) \leq \omega^p_{\widetilde{\Omega}}(U) \lesssim \frac{\sigma(U)}{r^n} \omega^p_{\Omega\setminus k\overline{B}}(k\overline{B}) = \frac{\sigma(Y)+\varepsilon}{r^n} \omega^p_{\Omega\setminus k\overline{B}}(k\overline{B})$ and the general case would follow taking $\varepsilon \to 0$.

Let us assume Y is an open set relative to ∂B, and fix $p \in \Omega \setminus k\overline{B}$. Again, for $\varepsilon > 0$ let $V \supset k\overline{B}$ be an open set such that $\omega^p_{\Omega\setminus k\overline{B}}(V) \leq \omega^p_{\Omega\setminus k\overline{B}}(k\overline{B}) + \varepsilon$, $\psi \in C_c(V)$ such that $\mathbf{1}_{k\overline{B}} \leq \psi \leq \mathbf{1}_V$, and let $v_{\Omega\setminus k\overline{B}}$ denote the L_A-harmonic extension of ψ in $\Omega \setminus k\overline{B}$.

Let $N := \max_{x\in\partial kB} \omega^x_{\widetilde{\Omega}}(Y)$ and let $x_0 \in \partial kB$ such that $\omega^{x_0}_{\widetilde{\Omega}}(Y) = N$. Define the annulus $\mathcal{A}_k := k^2 B \setminus \overline{B}$. By Lemma I.2.3 we obtain

$$N - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) = \omega^{x_0}_{\widetilde{\Omega}}(Y) - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) = \int_{\Omega\cap\partial k^2 B} \omega^{\xi}_{\widetilde{\Omega}}(Y)\, d\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\xi). \tag{III.3.3}$$

By the maximum principle in $\widetilde{\Omega} \setminus k\overline{B}$ we have that $\omega^{\xi}_{\widetilde{\Omega}}(Y) \leq \omega^{x_0}_{\widetilde{\Omega}}(Y) = N$ for $\xi \in \partial k^2 B$, and hence

$$\int_{\Omega\cap\partial k^2 B} \omega^{\xi}_{\widetilde{\Omega}}(Y)\, d\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\xi) \leq N \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\Omega \cap \partial k^2 B). \tag{III.3.4}$$

All in all, from (III.3.3) and (III.3.4) then

$$N - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \leq N \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\Omega \cap \partial k^2 B). \tag{III.3.5}$$

Also, by the maximum principle and the fact that the annulus $\mathcal{A}_k = k^2 B \setminus \overline{B}$ when $k > 3$ satisfies the CDC with the precise conditions in Lemma I.5.2, we have that the right-hand side of (III.3.5) is controlled by

$$\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\Omega \cap \partial k^2 B) \leq \omega^{x_0}_{\mathcal{A}_k}(\partial k^2 B) \leq 1 - \tau < 1, \tag{III.3.6}$$

for $\tau \in (0,1)$, depending also on k. Indeed, this last step follows by applying Lemma I.5.3(1) to any $y \in \partial B$ with the choice of s_0 in Lemma I.5.2, because in that case $x_0 \in \partial kB \subset B(y, s_0)$ and $B(y, 2s_0) \cap \partial k^2 B = \emptyset$.

From (III.3.5) and (III.3.6) we obtain $N - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \leq N \omega^{p_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\partial k^2 B) \leq (1-\tau)N$, equivalently $\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \geq \tau N = \tau \omega^{x_0}_{\widetilde{\Omega}}(Y)$. From this and the maximum principle,

$$\tau \omega^{x_0}_{\widetilde{\Omega}}(Y) \leq \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \leq \omega^{x_0}_{\mathcal{A}_k}(Y).$$

We have reduced to the elliptic measure in the annulus $\mathcal{A}_k$. By Lemma III.3.2 we have $\omega^{x_0}_{\mathcal{A}_k}(Y) \lesssim \sigma(Y)/r^n$, and so $\omega^{x_0}_{\widetilde{\Omega}}(Y) \lesssim \omega^{x_0}_{\mathcal{A}_k}(Y) \lesssim \sigma(Y)/r^n$. Since $x_0 \in \partial kB$ was chosen to achieve the maximum of $\omega^{\cdot}_{\widetilde{\Omega}}(Y)$ in ∂kB, we obtain that for any $x \in \widetilde{\Omega} \cap \partial kB$,

$$\omega^x_{\widetilde{\Omega}}(Y) \leq \omega^{x_0}_{\widetilde{\Omega}}(Y) \lesssim \frac{\sigma(Y)}{r^n} = \frac{\sigma(Y)}{r^n} \cdot v_{\Omega\setminus k\overline{B}}(x),$$

where the last equality is just because $v_{\Omega\setminus k\overline{B}}(\xi) = \psi = 1$ for $\xi \in \widetilde{\Omega} \cap \partial kB$. Moreover, $\omega^x_{\widetilde{\Omega}}(Y) = 0$ if $x \in \partial\widetilde{\Omega} \setminus kB$. The same inequality follows for $x \in \widetilde{\Omega} \setminus k\overline{B}$ by the maximum principle in $\widetilde{\Omega} \setminus k\overline{B}$. Evaluating at the fixed pole $p \in \Omega \setminus k\overline{B}$, by the choice of V we have

$$\omega^p_{\widetilde{\Omega}}(Y) \lesssim \frac{\sigma(Y)}{r^n} \cdot v_{\Omega\setminus k\overline{B}}(p) \leq \frac{\sigma(Y)}{r^n} \cdot \omega^p_{\Omega\setminus k\overline{B}}(V) \leq \frac{\sigma(Y)}{r^n} \cdot (\omega^p_{\Omega\setminus k\overline{B}}(k\overline{B}) + \varepsilon),$$

and the lemma follows by taking $\varepsilon \to 0$. □

III.4. Proof of the weak version of the Main Lemma

According to the previous reductions in Section III.2, to obtain the Main Lemma III.2.1 it suffices to prove Lemma III.2.3, which we intend to do in this section modulo the proof of (III.0.1) which is deferred to Section III.6.

In this section we work with bounded (δ, r_0)-Reifenberg flat domains $\Omega \subset \mathbb{R}^2$. Recall that for $\delta > 0$ small enough we have that Ω is an NTA domain (see [KT97, Section 3]), and hence it satisfies the capacity density condition. See Remark III.1.2.

Proof of Lemma III.2.3. Let $M > 0$ be big enough and $0 < \rho < 1/M$. Denote $\omega \coloneqq \omega^p_{\Omega,A}$.

For $x \in \partial\Omega$ define the 'high density value' as

$$h(x) \coloneqq \sup\{r \geq \rho : \omega(B(x,r)) \geq Mr\}, \tag{III.4.1}$$

and $h(x) = \rho$ if the supremum runs over an empty set. Note that $\rho \leq h(x) \leq 1/M$ for every $x \in \partial\Omega$, because ω is a probability measure.

Definition III.4.1 (Good balls)**.** For $x \in \partial\Omega$, we say that the ball $B(x,r)$ is good, $B(x,r) \in \mathsf{Good}$, if $r > h(x)$, i.e., for any $s \geq r$ we have $\omega(B(x,s)) < Ms$.

For $x \in \mathbb{R}^2$ define

$$d(x) \coloneqq \inf_{B\in\mathsf{Good}} \left[|x - c(B)| + r(B)\right],$$

where $c(B)$ is the center of the ball and $r(B)$ its radius. For any $B \in \mathsf{Good}$ we have that $r(B) \geq h(c(B)) \geq \rho$, which implies $d(x) \geq \rho$.

Note that for every ball $B(\xi, r) \in \mathsf{Good}$ we have $r \geq h(\xi)$. Therefore

$$d(x) = \inf_{\xi\in\partial\Omega} \left[|x - \xi| + h(\xi)\right].$$

This function is 1-Lipschitz as it is the infimum of 1-Lipschitz functions.

Remark III.4.2. For $x \in \partial\Omega$ it follows $\rho \leq d(x) \leq h(x)$, and hence $h(x) = \rho$ implies $d(x) = h(x)$. Moreover, if $d(x) = \rho$ then $x \in \partial\Omega$.

Let $0 < \varepsilon < 1$ be small enough (to be fixed in (III.4.8) below) depending on the ellipticity constant λ and the product $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$. Let $\mathcal{I} = \mathcal{I}_{\varepsilon^{-2}}$ be the family of maximal dyadic cubes $Q \in \mathcal{D}(\mathbb{R}^2)$ such that $Q \cap \partial\Omega \neq \emptyset$ and $\ell(Q) \leq \varepsilon^2 d(x)$ for all $x \in Q$.

Lemma III.4.3. *Let $\mathcal{I}$ be the family defined above. Then:*

(1) *Every $Q \in \mathcal{I}$ satisfy $\frac{\varepsilon^2 d(x)}{3} < \ell(Q) \leq 2\varepsilon^2 d(x)$ for all $x \in \frac{\varepsilon^{-2}}{4}Q$.*
(2) *If $Q_1, Q_2 \in \mathcal{I}$ and $\frac{\varepsilon^{-2}}{4}Q_1 \cap \frac{\varepsilon^{-2}}{4}Q_2 \neq \emptyset$, then $\frac{\ell(Q_1)}{6} < \ell(Q_2) < 6\ell(Q_1)$.*
(3) *$\{\frac{\varepsilon^{-2}}{4}Q\}_{Q\in\mathcal{I}}$ has finite superposition, with superposition number $N = N_\varepsilon$ depending on ε only.*

Proof. Let $x \in \frac{\varepsilon^{-2}}{4}Q$. We start by proving $\ell(Q) \leq 2\varepsilon^2 d(x)$ in (1). Take any $y \in Q$. By the election of y and since d is 1-Lipschitz,

$$d(x) = d(y) + d(x) - d(y) \geq \varepsilon^{-2}\ell(Q) - |x-y| \geq \varepsilon^{-2}\ell(Q) - \operatorname{diam}\frac{\varepsilon^{-2}}{4}Q \geq \frac{\ell(Q)}{2\varepsilon^2}.$$

For the other inequality in (1), let $\widehat{Q}$ be the dyadic father of Q, i.e., the unique $\widehat{Q} \in \mathcal{D}(\mathbb{R}^2)$ such that $Q \subset \widehat{Q}$ and $\ell(\widehat{Q}) = 2\ell(Q)$. Since Q is maximal, there exists $y \in \widehat{Q}$ such that $2\ell(Q) = \ell(\widehat{Q}) > \varepsilon^2 d(y)$, and hence

$$d(x) = d(x) - d(y) + d(y) < |x-y| + \frac{2\ell(Q)}{\varepsilon^2} \leq \operatorname{diam}\frac{\varepsilon^{-2}}{4}Q + \frac{2\ell(Q)}{\varepsilon^2} \leq \frac{3\ell(Q)}{\varepsilon^2},$$

and with this we conclude the proof of (1).

Let $Q_1, Q_2 \in \mathcal{I}$ such that $\frac{\varepsilon^{-2}}{4}Q_1 \cap \frac{\varepsilon^{-2}}{4}Q_2 \neq \emptyset$. Take $x \in \frac{\varepsilon^{-2}}{4}Q_1 \cap \frac{\varepsilon^{-2}}{4}Q_2$, and then (2) follows from (1) by

$$\ell(Q_1) \leq 2\varepsilon^2 d(x) < 6\ell(Q_2).$$

Given $Q \in \mathcal{I}$, there is only a finite number $N = N_\varepsilon$ of cubes $P \in \mathcal{D}(\mathbb{R}^2)$ such that $\ell(Q)/6 < \ell(P) < 6\ell(Q)$ and $\frac{\varepsilon^{-2}}{4}Q \cap \frac{\varepsilon^{-2}}{4}P \neq \emptyset$, which gives (3). □

Lemma III.4.4. *There exists $\gamma \geq 1$ such that for every $Q \in \mathcal{I}$ there exists a ball $G_Q \in \mathsf{Good}$ with $r(G_Q) \approx \varepsilon^{-2}\ell(Q)$, satisfying the inclusions $\varepsilon^{-2}Q \subset \gamma G_Q$ and $G_Q \subset \gamma\varepsilon^{-2}Q$.*

Proof. Given $Q \in \mathcal{I}$, fix any $x \in Q \cap \partial\Omega$, and take $\xi \in \partial\Omega$ such that

$$d(x) \leq |x-\xi| + h(\xi) \leq 1.1 d(x). \tag{III.4.2}$$

Define the ball $G_Q \coloneqq B\left(\xi, 2(|x-\xi| + h(\xi))\right)$, and hence $G_Q \in \mathsf{Good}$, since $r(G_Q) \geq 2h(\xi) \geq h(\xi) + \rho > h(\xi)$.

We claim that $r(G_Q) \approx \varepsilon^{-2}\ell(Q)$. Indeed, from (III.4.2) and (1) in Lemma III.4.3,

$$\frac{1}{2}r(G_Q) = |x-\xi| + h(\xi) \overset{\text{(III.4.2)}}{\approx} d(x) \overset{(1)}{\approx} \varepsilon^{-2}\ell(Q).$$

Also, using the previous comparability, the distance between x and ξ is controlled above by

$$|x-\xi| \leq |x-\xi| + h(\xi) = \frac{1}{2}r(G_Q) \approx \varepsilon^{-2}\ell(Q),$$

which implies that there exists a universal constant $\gamma \geq 1$ such that $\varepsilon^{-2}Q \subset \gamma G_Q$ and $G_Q \subset \gamma\varepsilon^{-2}Q$. □

For each cube $Q \in \mathcal{I}$, fix a point $z_Q \in Q \cap \partial\Omega$ and define $B_Q \coloneqq B(z_Q, r_Q)$ with $r_Q \coloneqq \varepsilon d(z_Q)$. Next, we modify the domain as in [Wol93], but using the family $\{B_Q\}_{Q\in\mathcal{I}}$. To do so, define

$$\widetilde{\Omega} \coloneqq \Omega \setminus \bigcup_{Q\in\mathcal{I}} B_Q,$$

and denote $\widetilde{\omega} \coloneqq \omega^p_{\widetilde{\Omega},A}$ its elliptic measure with pole p. From (1) in Lemma III.4.3 and $d(\cdot) \geq \rho$ on $\partial\Omega$ we have that $\mathcal{I}$ is finite. In particular, the family $\{B_Q\}_{Q\in\mathcal{I}}$ is finite, and $\partial\widetilde{\Omega}$ is smooth except at finitely many points.

Recall $\mathcal{P}$ is the approximating hyperplane in Definition III.1.4. Since the function $d(\cdot)$ is 1-Lipschitz, and so $\varepsilon d(\cdot)$ is ε-Lipschitz, the same proof as in [AMT17, Lemma 2.2] applies to obtain the following lemma.

Lemma III.4.5. *Let $r_0 \in (0, \infty]$ and let $\varepsilon > 0$ be small enough. Then:*

(1) *(Analogue of* [AMT17, Lemma 2.2]*) There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\Omega \subset \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat with $0 < \delta < \delta_0$, then the modified domain $\widetilde{\Omega}$ as above is $(c\varepsilon^{1/2}, r_0/2)$-Reifenberg flat.*

(2) *(See* [AMT17, Lemma 2.3(c)]*) For every $Q \in \mathcal{I}$, there exists a Lipschitz function $f_Q : \mathcal{P}(z_Q, 30r(B_Q)) \cap 10B_Q \to \mathcal{P}(z_Q, 30r(B_Q))^\perp$ with Lipschitz constant at most $c\varepsilon^{1/2}$.*

For any $Q \in \mathcal{I}$, by the maximum principle and Lemma III.3.1 (with $k = 10$) respectively,

$$\widetilde{\omega}\left(\overline{B_Q}\right) \le \omega_{\Omega \setminus \overline{B_Q}}\left(\overline{B_Q}\right) \lesssim \omega(10B_Q). \tag{III.4.3}$$

Moreover, for all $z \in \partial\widetilde{\Omega} \cap \partial B_Q$, by Lemma III.3.3 (with $k = \sqrt{10}$), the maximum principle and Lemma III.3.1 (with $k = \sqrt{10}$) respectively,

$$\frac{d\widetilde{\omega}}{d\sigma}(z) \lesssim \frac{\omega_{\widetilde{\Omega} \setminus \overline{\sqrt{10}B_Q}}\left(\overline{\sqrt{10}B_Q}\right)}{r(B_Q)} \le \frac{\omega_{\Omega \setminus \overline{\sqrt{10}B_Q}}\left(\overline{\sqrt{10}B_Q}\right)}{r(B_Q)} \lesssim \frac{\omega\left(10B_Q\right)}{r(B_Q)}. \tag{III.4.4}$$

By the existence of a good ball $G_Q \in \mathsf{Good}$ with $r(G_Q) \approx \varepsilon^{-2}\ell(Q)$ and $\varepsilon^{-2}Q \subset \gamma G_Q$ (see Lemma III.4.4), the ball $10B_Q$ has bounded density with respect to the initial elliptic measure:

$$\begin{aligned}\omega\left(10B_Q\right) = \omega\left(B(z_Q, 10\varepsilon d(z_Q))\right) &\overset{\text{L.III.4.3(1)}}{\le} \omega(3 \cdot 3 \cdot 10\varepsilon^{-1}Q) \overset{\varepsilon \ll 1}{\le} \omega(\varepsilon^{-2}Q) \le \omega(\gamma G_Q) \\ &\overset{\gamma G_Q \in \mathsf{Good}}{\le} \gamma r(G_Q) M \approx \gamma\varepsilon^{-2}\ell(Q)M \overset{\text{L.III.4.3(1)}}{\approx} \gamma d(z_Q) M = \gamma\varepsilon^{-1} r(B_Q) M.\end{aligned}$$

In particular, combined with (III.4.4) this implies

$$\frac{d\widetilde{\omega}}{d\sigma}(z) \lesssim M \text{ for all } z \in \partial\widetilde{\Omega}, \tag{III.4.5}$$

where the involved constant depends on γ and ε.

Let us smooth out the domain $\widetilde{\Omega}$. Recall that $\widetilde{\Omega}$ is smooth except at finitely many points $\{\xi_j\}_{j \in J}$, with $\#J < \infty$ depending on M and ρ. Let $0 < s < \min_{Q \in \mathcal{I}} r(B_Q)/1000$ be a small enough parameter to be fixed later. For each point ξ_j, $j \in J$, let $B_1, B_2 \in \{B_Q : Q \in \mathcal{I}\}$ be the two intersecting balls such that $\xi_j \in \partial B_1 \cap \partial B_2$, and let c_1, c_2 be their centers and r_1, r_2 be their radii respectively. Take the unique point $q_j \in \widetilde{\Omega} \cap \partial B(c_1, r_1 + s) \cap \partial B(c_2, r_2 + s)$, and denote $B_j := B(q_j, s)$. The ball B_j is tangent to B_1 and B_2, and define T_j to be the bounded open region enclosed between B_1, B_2 and B_j. We define the new smooth domain

$$\widetilde{\widetilde{\Omega}} = \widetilde{\widetilde{\Omega}}_s := \widetilde{\Omega} \setminus \bigcup_{j \in J} \overline{T_j},$$

taking small enough s such that

(1) for each $Q \in \mathcal{I}$, $\sigma(\partial B_Q \cap \partial\widetilde{\widetilde{\Omega}}) \ge 0.9\sigma(\partial B_Q \cap \partial\widetilde{\Omega})$, and
(2) $\widetilde{\widetilde{\Omega}}$ is a Lipschitz domain with the same Lipschitz character as $\widetilde{\Omega}$.

Note that $\sigma(\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}) \leq \#J \cdot 2\pi s$, and so we can take this value to be as small as needed.

Denote $\widetilde{\widetilde{\omega}} := \omega^p_{\widetilde{\widetilde{\Omega}},A}$. By the maximum principle, $\widetilde{\widetilde{\omega}}(E) \leq \widetilde{\omega}(E)$ for any $E \subset \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$. Consequently,

$$\widetilde{\widetilde{\omega}}(\overline{B_Q}) \leq \widetilde{\omega}(\overline{B_Q}) \overset{\text{(III.4.3)}}{\lesssim} \omega(10B_Q) \text{ for any } Q \in \mathcal{I}, \tag{III.4.6}$$

and

$$\frac{d\widetilde{\widetilde{\omega}}}{d\sigma}(z) \leq \frac{d\widetilde{\omega}}{d\sigma}(z) \overset{\text{(III.4.5)}}{\lesssim} M \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}. \tag{III.4.7}$$

Let $K(\cdot) := \frac{d\widetilde{\widetilde{\omega}}}{d\sigma}(\cdot)$ be the Radon-Nikodym derivative. By (I.3.7) we have

$$K(\xi) = -\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle \text{ for } \xi \in \partial\widetilde{\widetilde{\Omega}},$$

where g^T is the Green function in $\widetilde{\widetilde{\Omega}}$ with respect to the matrix A^T. By (III.0.1) (proved in Section III.6), if $\varepsilon > 0$ is small enough depending on the ellipticity constant λ and the product $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$, then there is a constant $\text{const}_{\Omega,A} > 0$ such that

$$-\infty < -\text{const}_{\Omega,A} \leq \int_{\partial\widetilde{\widetilde{\Omega}}} \log|S(\xi)\nabla g^T(\xi)|\, d\widetilde{\widetilde{\omega}}(\xi) = \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log|S(\xi)\nabla g^T(\xi)|\, d\sigma(\xi), \tag{III.4.8}$$

where $A_0 = \frac{A+A^T}{2}$ and $S = A_0^{1/2}$, i.e., $S^TS = A_0$.

For every $\xi \in \partial\widetilde{\widetilde{\Omega}}$ we can write

$$\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle = \langle \nabla g^T(\xi), A(\xi)\nu(\xi)\rangle = \langle \nabla g^T(\xi), c_1(\xi)\nu(\xi)\rangle + \langle \nabla g^T(\xi), c_2(\xi)t(\xi)\rangle,$$

where ν (resp. t) is the outward normal (resp. tangential) vector of $\partial\widetilde{\widetilde{\Omega}}$, and $c_1(\xi)$ (resp. $c_2(\xi)$) is the projection of $A(\xi)\nu(\xi)$ into $\nu(\xi)$ (resp. $t(\xi)$). In particular $c_1(\xi) = \langle A(\xi)\nu(\xi), \nu(\xi)\rangle \approx 1$, and since $\partial\widetilde{\widetilde{\Omega}}$ is smooth, we have that $\langle \nabla g^T(\xi), c_2(\xi)t(\xi)\rangle = c_2(\xi)\partial_t g^T(\xi) = 0$ in $\partial\widetilde{\widetilde{\Omega}}$ as $g^T|_{\partial\Omega} = 0$. Hence $-\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle \approx |\nabla g^T(\xi)|$, and since $|\nabla g^T(\xi)|^2 \approx \langle A_0\nabla g^T(\xi), \nabla g^T(x)\rangle = |S(\xi)\nabla g^T(\xi)|^2$, we obtain

$$|S(\xi)\nabla g^T(\xi)| \leq -C\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle = CK(\xi). \tag{III.4.9}$$

From (III.4.8) and (III.4.9),

$$\begin{aligned}
-\infty < -\text{const}_{\Omega,A} &\leq \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log|S(\xi)\nabla g^T(\xi)|\, d\sigma(\xi) \leq \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log CK(\xi)\, d\sigma(\xi) \\
&= \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log C\, d\sigma(\xi) + \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log K(\xi)\, d\sigma(\xi) \\
&= \log C + \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log K(\xi)\, d\sigma(\xi),
\end{aligned}$$

whence we obtain

$$-\infty < -C_{\Omega,A} \leq \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log K(\xi)\, d\sigma(\xi). \tag{III.4.10}$$

In view of (III.4.10) and the fact that $K(\cdot) \lesssim M$ on $\partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$ by (III.4.7), if M is big enough (provided s is small enough depending on M and ρ), then the set of points $\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$

with density $K(\xi) < M^{-\tau}$ indeed has elliptic measure at most $1-\tau/4$, uniformly in ρ and M. More precisely, we will obtain

$$\widetilde{\widetilde{\omega}}\left(\left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}\right) = \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}:K(\xi)\geq M^{-\tau}\right\}} K(\xi)\, d\sigma(\xi) \geq \frac{\tau}{4} \tag{III.4.11}$$

whenever M is big enough depending on τ and the constant $\mathrm{const}_{\Omega,A}$, taking small enough s depending on M and ρ. Note that if we had $K(\cdot) \lesssim M$ in the whole boundary $\partial\widetilde{\widetilde{\Omega}}$, using Tchebyshoff's inequality as in [Wol93, p. 170] we would directly get (III.4.11) from (III.4.10). However, we only have the bound of K on $\partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$ and we will need to estimate the elliptic measure (and the $K \log^+ K\, d\sigma$ measure) of the "bad" set $\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}$.

Proof of (III.4.11). Let $\log^+(\cdot) := \max\{0, \log(\cdot)\}$ and $\log^-(\cdot) := -\min\{0, \log(\cdot)\}$, and recall that $\widetilde{\widetilde{\Omega}} = \widetilde{\widetilde{\Omega}}_s$. We first establish that

$$\int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K \log^+ K\, d\sigma \to 0 \text{ as } s \to 0, \tag{III.4.12}$$

which in particular implies

$$\widetilde{\widetilde{\omega}}(\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}) = \int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K\, d\sigma \leq e\sigma(\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}) + \int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K \log^+ K\, d\sigma \to 0 \text{ as } s \to 0. \tag{III.4.13}$$

As previously noted, for sufficiently small $s > 0$, the Lipschitz character of $\widetilde{\widetilde{\Omega}}$ is controlled by the one of $\widetilde{\Omega}$, and therefore we have that it is NTA with uniform constants. Thus, the involved constants in the subsequent computations will be independent of s.

We claim that there exists $\alpha > 0$ such that

$$\widetilde{\widetilde{\omega}}(B(x,r)) \lesssim \left(\frac{r}{r_0}\right)^\alpha \text{ for all } x \in \partial\Omega \text{ and } 0 < r \leq r_0/1000, \tag{III.4.14}$$

where r_0 is the scale where Reifenberg flatness of Ω is granted. To prove this, note that there exists $\beta \in (0,1)$ such that $\widetilde{\widetilde{\omega}}(B \setminus B/2) \geq \beta\widetilde{\widetilde{\omega}}(B/2)$ for any ball B centered at $\partial\widetilde{\widetilde{\Omega}}$ with radius $r_B \leq r_0/100$, as a consequence of the doubling property of elliptic measure in NTA domains (see [Ken94, (1.3.7)]) and the fact that $B \setminus B/2 \neq \emptyset$ by Reifenberg flatness. It follows that $\widetilde{\widetilde{\omega}}(B/2) \leq \widetilde{\widetilde{\omega}}(B)/(1+\beta)$. Iterating this inequality for a fixed $x \in \partial\Omega$ and $B_0 = B(x, r_0/200)$, we have $\widetilde{\widetilde{\omega}}(2^{-k}B_0) \leq \widetilde{\widetilde{\omega}}(B_0)/(1+\beta)^k \leq 1/(1+\beta)^k$ for all $k \geq 0$ integer. This readily implies the existence of $\alpha = \alpha(\beta) > 0$ such that (III.4.14) holds.

Let us fix $j \in J$ momentarily. For B_j, let $q_j = c(B_j)$ denote its center. By the change of pole formula in NTA domains (see [Ken94, Corollary 1.3.8]), we have

$$K(z) \approx \frac{d\widetilde{\widetilde{\omega}}^{q_j}}{d\sigma}(z)\widetilde{\widetilde{\omega}}(\overline{B_j}) \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j. \tag{III.4.15}$$

Let $\widetilde{\widetilde{g}}_{q_j}^T$ be the Green function of $\widetilde{\widetilde{\Omega}}$ with pole at q_j, and $u(\cdot) = \widetilde{\widetilde{g}}_{q_j}^T(s \cdot +q_j)$, which is the Green function of $\{(z-q_j)/s : z \in \widetilde{\widetilde{\Omega}}\}$ with pole at 0 for the matrix $A(s \cdot +q_j)^T$. Arguing as in Lemma III.3.2 (using Theorem I.1.5 with $T = \{(z-q_j)/s : z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j\}$) we obtain $s|\nabla\widetilde{\widetilde{g}}_{q_j}^T(z)| = |\nabla u((z-q_j)/s))| \lesssim \max_{\partial B_{1/2}(0)} u \approx 1$ for all $z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j$, where we used

Lemma I.6.2 in the last step[III.1]. Consequently,

$$\frac{d\widetilde{\widetilde{\omega}}^{q_j}}{d\sigma}(z) \lesssim \frac{1}{s} \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j.$$

Using both this and (III.4.14) in (III.4.15), since $j \in J$ was arbitrary, we get

$$K(z) \lesssim \frac{s^{\alpha-1}}{r_0^\alpha} \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}.$$

This bound suffices to establish (III.4.12), as we can now estimate

$$\int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K\log^+ K\,d\sigma \lesssim \frac{1}{r_0^\alpha}\int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} s^{\alpha-1}\log^+(s^{\alpha-1})\,d\sigma \lesssim \frac{\#Js^\alpha\log^+(s^{\alpha-1})}{r_0^\alpha},$$

which goes to zero as $s \to 0$.

Taking $s > 0$ small enough in (III.4.13), to prove (III.4.11) it suffices to see

$$\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K\,d\sigma \geq \frac{\tau}{2}. \tag{III.4.16}$$

We now prove (III.4.16). First we bound

$$\int_{\partial\widetilde{\widetilde{\Omega}}} K\log^+ K\,d\sigma = \int_{\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}} K\log^+ K\,d\sigma + \int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K\log^+ K\,d\sigma \overset{\text{(III.4.12)}}{\leq} \int_{\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}} K\log^+ K\,d\sigma + 1, \tag{III.4.17}$$

where in the last step we took $s > 0$ small enough in (III.4.12). Writing $K(\cdot) \leq e^C M$ on $\partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$, see (III.4.7),

$$\int_{\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}} K\log^+ K\,d\sigma \leq (\log M + C)\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq 1\right\}} K\,d\sigma \leq \log M\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq 1\right\}} K\,d\sigma + C. \tag{III.4.18}$$

By (III.4.17) and (III.4.18) we can control the term with $K\log^+ K$ as

$$\int_{\partial\widetilde{\widetilde{\Omega}}} K\log^+ K\,d\sigma \leq \log M\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq 1\right\}} K\,d\sigma + C + 1.$$

From this and (III.4.10),

$$\begin{aligned}\int_{\partial\widetilde{\widetilde{\Omega}}} K\log^- K\,d\sigma &= \int_{\partial\widetilde{\widetilde{\Omega}}} K\log^+ K\,d\sigma - \int_{\partial\widetilde{\widetilde{\Omega}}} K\log K\,d\sigma \leq \log M\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq 1\right\}} K\,d\sigma + C_{\Omega,A}\\ &\leq \log M\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K\,d\sigma + C_{\Omega,A}.\end{aligned}$$

[III.1] In fact, it would suffice to show that $|\nabla\widetilde{\widetilde{g}}^T_{q_j}(z)| \lesssim \frac{|\log s|}{s} \leq s^{-\alpha/2-1}$ for small enough $s = s(\alpha, \operatorname{diam}\widetilde{\widetilde{\Omega}})$, which follows from the pointwise estimate in Corollary I.3.3.

Using this in the last inequality of the following computations, we obtain

$$\begin{aligned}
\tau \log M - \tau \log M \int_{\{\xi\in\partial\widetilde{\Omega}: K(\xi)\geq M^{-\tau}\}} K\,d\sigma &= \log M^{\tau} \int_{\partial\widetilde{\Omega}} K\,d\sigma - \log M^{\tau} \int_{\{\xi\in\partial\widetilde{\Omega}: K(\xi)\geq M^{-\tau}\}} K\,d\sigma \\
&= \log M^{\tau} \int_{\{\xi\in\partial\widetilde{\Omega}: K(\xi)< M^{-\tau}\}} K\,d\sigma \leq \int_{\{\xi\in\partial\widetilde{\Omega}: K(\xi)< M^{-\tau}\}} K \log^{-} K\,d\sigma \leq \int_{\partial\widetilde{\Omega}} K\log^{-} K\,d\sigma \\
&\leq \log M \int_{\{\xi\in\partial\widetilde{\Omega}: K(\xi)\geq M^{-\tau}\}} K\,d\sigma + C_{\Omega,A},
\end{aligned}$$

which gives

$$\int_{\{\xi\in\partial\widetilde{\Omega}: K(\xi)\geq M^{-\tau}\}} K\,d\sigma \geq \frac{\tau \log M - C_{\Omega,A}}{(1+\tau)\log M} = \frac{\tau}{1+\tau} - \frac{C_{\Omega,A}}{(1+\tau)\log M} \geq \frac{\tau}{2},$$

as claimed in (III.4.16), by letting $M > 0$ be big enough depending on τ and $C_{\Omega,A}$. □

We are now in position to find the final set F with the claimed properties. Let $\{B_{Q_n}\}_n \subset \{B_Q\}_{Q\in\mathcal{I}}$ be the subfamily satisfying either (HD) or (TS):

(HD) $h(z_{Q_n}) > \rho$, where $z_{Q_n} = c(B_{Q_n})$ is the center of the ball B_{Q_n}.

(TS) $h(z_{Q_n}) = \rho$ and $\partial B_{Q_n} \cap \left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\} \neq \emptyset$.

Here HD stands for 'high density' and TS for 'touching set'.

Notation. We write $B_Q \in \mathsf{HD}$ if B_Q satisfies (HD), and $B_Q \in \mathsf{TS}$ if B_Q satisfies (TS).

With this choice,

$$\frac{\tau}{4} \overset{\text{(III.4.11)}}{\leq} \widetilde{\widetilde{\omega}}\left(\left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}\right) \leq \widetilde{\widetilde{\omega}}\left(\overline{\bigcup_n B_{Q_n}}\right) \leq \sum_n \widetilde{\widetilde{\omega}}\left(\overline{B_{Q_n}}\right),$$

and by (III.4.6),

$$\frac{\tau}{4} \leq \sum_n \widetilde{\widetilde{\omega}}\left(\overline{B_{Q_n}}\right) \lesssim \sum_n \omega(10B_{Q_n}).$$

If ε is small enough, then $10B_Q = B(z_Q, 10\varepsilon d(z_Q)) \subset \frac{\varepsilon^{-2}}{4}Q$ for each $Q \in \mathcal{I}$. Moreover, since $\{\frac{\varepsilon^{-2}}{4}Q\}_{Q\in\mathcal{I}}$ has finite overlapping by Lemma III.4.3(3), $\{10B_Q\}_{Q\in\mathcal{I}}$ has also finite overlapping with constant depending on ε only. From this we obtain,

$$\tau \lesssim \sum_n \omega(10B_{Q_n}) \lesssim \omega\left(\bigcup_n 10B_{Q_n}\right). \tag{III.4.19}$$

At this point we have found a subset of $\partial\Omega$ (covered by balls) with elliptic measure bounded uniformly from below. Moreover, the radii $\varepsilon d(z_{Q_n})$ of these balls B_{Q_n} are smaller than the 'high density value' $h(z_{Q_n})$, which will allow us to have a control on the sum of the radii.

First we need to define the set $F \subset \partial\Omega$ and its covering. For each $B_{Q_n} \in \mathsf{HD}$, we have $10B_{Q_n} = B(z_{Q_n}, 10\varepsilon d(z_{Q_n}))$, and since $d(\cdot) \leq h(\cdot)$ on $\partial\Omega$, $10B_{Q_n} \subset B(z_{Q_n}, 10\varepsilon h(z_{Q_n})) \subset B(z_{Q_n}, 10h(z_{Q_n}))$. Since the family $\{B(z_{Q_n}, 10h(z_{Q_n}))\}_{B_{Q_n}\in\mathsf{HD}}$ is finite, by means of the $3R$-covering theorem consider a disjoint subfamily

$$\{B_m\}_m \subset \{B(z_{Q_n}, 10h(z_{Q_n}))\}_{B_{Q_n}\in\mathsf{HD}}$$

such that

$$\bigcup_{B_{Q_n}\in \mathsf{HD}} B\left(z_{Q_n}, 10h(z_{Q_n})\right) \subset \bigcup_m 3B_m.$$

Let us define

$$F := \left(\bigcup_m 3B_m \cup \bigcup_{B_{Q_n}\in \mathsf{TS}} 10B_{Q_n}\right) \cap \partial\Omega.$$

Note that $\bigcup_n 10B_{Q_n} \subset \bigcup_m 3B_m \cup \bigcup_{B_{Q_n}\in\mathsf{TS}} 10B_{Q_n}$ implies $\omega(F) \gtrsim \tau$ by (III.4.19). Next we show that the covering

$$\{3B_m\}_m \cup \{10B_{Q_n}\}_{B_{Q_n}\in\mathsf{TS}}$$

satisfies the properties (1*) and (2*) in Lemma III.2.3.

We can control the radius of the balls with high density

$$\sum_m r(3B_m) = 30\sum_m h\left(c(B_m)\right) \overset{\text{(III.4.1)}}{\leq} \frac{30}{M}\sum_m \omega\left(B_m\right) = \frac{30}{M}\omega\left(\bigcup_m B_m\right) \lesssim \frac{1}{M},$$

by the definition of $h(\cdot)$ in (III.4.1), and the fact that the balls $\{B_m\}_m$ are pairwise disjoint. Also, $r\left(3B_m\right) = 30h\left(c(B_m)\right) > 30\rho$. We have shown the second property of the covering.

Recall that the balls in TS intersect the set $\left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}$. For each $B_{Q_n} \in \mathsf{TS}$ consider a point $x_{Q_n} \in \partial B_{Q_n} \cap \left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}$. Then, by (III.4.7) and (III.4.4) we have

$$M^{-\tau} \leq \frac{d\widetilde{\widetilde{\omega}}}{d\sigma}(x_{Q_n}) \overset{\text{(III.4.7)}}{\leq} \frac{d\widetilde{\omega}}{d\sigma}(x_{Q_n}) \overset{\text{(III.4.4)}}{\lesssim} \frac{\omega\left(10B_{Q_n}\right)}{r(B_{Q_n})},$$

which implies

$$\sum_{B_{Q_n}\in\mathsf{TS}} r\left(10B_{Q_n}\right) \lesssim M^\tau \sum_{B_{Q_n}\in\mathsf{TS}} \omega\left(10B_{Q_n}\right) \lesssim M^\tau \omega\left(\bigcup_{Q_n\in\mathsf{TS}} 10B_{Q_n}\right) \leq M^\tau,$$

obtaining the first property of the covering. Note that for $B_{Q_n} \in \mathsf{TS}$ we have $h(z_{Q_n}) = \rho$, and in particular $d(z_{Q_n}) = \rho$, which implies $r(10B_{Q_n}) = 10\varepsilon d(z_Q) = 10\varepsilon\rho$. $\square$

III.5. Proof of Theorem III.0.1

In this section we will prove Theorem III.0.1. First we make the reduction to the case of bounded domains (Claim III.5.2) and then we prove the theorem using Main Lemma III.2.1.

III.5.1. Reduction to bounded domains.

First, we state the following lemma.

Lemma III.5.1. *Let $r_0 \in (0,\infty]$ and let $\varepsilon > 0$ be small enough. There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\Omega \in \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat with $\delta \in (0,\delta_0)$ and B is a ball centered at $\partial\Omega$ with radius $r_B \leq r_0/100$, then there exists a bounded (ε, r)-Reifenberg flat domain $D \subset \{z : \operatorname{dist}(z, \partial\Omega) < r_0/2\}$ (for some $r \in (0, r_0/2)$) with $\Omega \cap 4B = D \cap 4B$.*

Note that we are not interested in the precise dependence of r with respect to r_0, because we seek for a qualitative result in Theorem III.0.1. It is quite likely that with some care the previous result could be made quantitative.

Proof of Lemma III.5.1. The proof uses the construction in [AMT17, Definition 2.1 and Lemma 2.2]. Set $0 < \varepsilon < 1/100$ and $E := \partial\Omega \cap 5\overline{B}$. Let $\mathcal{W}_{\varepsilon^{-2}}(E^c)$ be the set of maximal dyadic cubes $Q \subset E^c$ such that $\mathrm{diam}(\varepsilon^{-2}Q) \leq r_0$ and $\varepsilon^{-2}Q \cap E = \emptyset$.

Denote $\mathcal{I}$ the family of cubes $Q \in \mathcal{W}_{\varepsilon^{-2}}(E^c)$ such that $Q \cap \partial\Omega \neq \emptyset$. For each cube $Q \in \mathcal{I}$ fix a point $z_Q \in Q \cap \partial\Omega$, and set $r_Q := \varepsilon \min\{r_0, \mathrm{dist}(z_Q, E)\}$ and $B_Q := B(z_Q, r_Q)$. Consider the enlarged domain

$$\Omega_\varepsilon^+ := \Omega \cup \bigcup_{Q \in \mathcal{I}} B_Q \supset \Omega.$$

By [AMT17, Lemma 2.2], this new domain is $(c\varepsilon^{1/2}, r_0/2)$-Reifenberg flat, where the constant c depends only on the dimension, provided the initial domain is (δ, r_0)-Reifenberg flat with $\delta \in (0, \delta_0)$ and δ_0 is small enough depending on ε.

Consider the domain $D_0 := \Omega_\varepsilon^+ \cap 10B$. Clearly $D_0 \subset \{z : \mathrm{dist}(z, \partial\Omega) < r_0/2\}$ as $10r_B \leq r_0/10$. Let us smooth the corners of D_0 out, where the $(c\varepsilon^{1/2}, s)$-Reifenberg flat condition fails for all $s > 0$. Note that this may only happen in a finite number of points $\{\xi_j\}_{j\in J} \in \partial D_0 \cap \partial 10B$ because Ω_ε^+ is contructed as a countable union of balls. Fix a small parameter τ. For each ξ_j of these points in $\partial D_0 \cap \partial 10B$, let $B_j \in \{B_Q : Q \in \mathcal{I}\}$ such that $\xi_j \in \partial 10B \cap \partial B_j$, and let c_j and r_j denote its centers and radii respectively. Consider now the unique point $p_j \in D_0 \cap (\partial B(c_B, 10r_B - \tau) \cap \partial B(c_j, r_j - \tau))$. In particular, the ball $B(p_j, \tau)$ is tangent to $\partial 10B$ and ∂B_j. Let $T_j = 10B \cap B_j \cap B(p_j, \tau)^c$ the bounded open region enclosed between the previous balls. Taking τ to be small enough, the final domain

$$D := D_0 \setminus \bigcup_{j\in J} T_j \subset D_0$$

satisfies $D \cap 4B = \Omega_\varepsilon^+ \cap 4B = \Omega \cap 4B$ and is $(c\varepsilon^{1/2}, r)$-Reifenberg flat for some $r > 0$ depending on τ. □

We now show that the proof of Theorem III.0.1 reduces to the case of bounded domains.

Claim III.5.2. *If Theorem III.0.1 holds for bounded (δ_0, r_0)-Reifenberg flat domains, then there exists $\delta_1 = \delta_1(\delta_0) > 0$ such that Theorem III.0.1 holds for unbounded (δ_1, r_0)-Reifenberg flat domains.*

Proof. First we want to remark that if Theorem III.0.1 holds for (δ, r_0)-Reifenberg flat domains for a fixed $r_0 > 0$, then by means of a dilation it holds for (δ, r)-Reifenberg flat domains for any $r > 0$.

Let $\varepsilon > 0$ given by Lemma III.5.1 and let $\varepsilon' := \min\{\varepsilon, \delta_0/2\}$ be small enough. Let $\delta_1 = \delta_1(\varepsilon')$ given by Lemma III.5.1. Let $\Omega \subset \mathbb{R}^2$ be an unbounded (δ_1, r_0)-Reifenberg flat domain. Let $\{B_i\}_i \subset \{B(\xi, r)\}_{\xi\in\partial\Omega, 0<r<r_0/100}$ be a disjoint family with $\omega_{\Omega,A}(\partial\Omega \setminus \bigcup_i B_i) = 0$, by Vitali's covering theorem. For each ball B_i let D_i be the bounded (ε, r_i)-Reifenberg flat domain from Lemma III.5.1, for some $r_i \in (0, r_0/2)$. As $\varepsilon' < \delta_0$, in particular each D_i is a bounded (δ_0, r_i)-Reifenberg flat domain.

As we are assuming that Theorem III.0.1 holds for bounded (δ_0, r)-Reifenberg flat domains for any $r > 0$, for each i take $F_i \subset \partial D_i$ with $\omega_{D_i,A}(F_i) = 1$ and σ-finite one-dimensional

Hausdorff measure. From $\Omega \cap 4B_i = D_i \cap 4B_i$, the maximum principle and $\omega_{D_i}(F_i) = 1$ we get

$$\omega_{\Omega\cap 4B_i,A}((\partial\Omega \setminus F_i) \cap 4B_i) = \omega_{D_i\cap 4B_i,A}((\partial D_i \setminus F_i) \cap 4B_i) \leq \omega_{D_i,A}((\partial D_i \setminus F_i) \cap 4B_i) = 0.$$

In particular $\omega_{\Omega\cap 4B_i,A}((F_i \cap 4B_i) \cup \partial 4B_i) = 1$. As F_i has σ-finite length, so does $\widetilde{F}_i := (F_i \cap 4B_i) \cup \partial 4B_i$. By Lemma I.4.4, the set $F = \partial\Omega \cap \bigcup_i \widetilde{F}_i$ satisfies $\omega_\Omega(F) = 1$, and clearly has σ-finite length. $\square$

III.5.2. Proof for bounded domains. Theorem III.0.1 follows from Main Lemma III.2.1 as it is done in [Wol93, Proof of Theorem 1], with some small modifications. For the sake of completeness we give the detailed proof.

Proof of Theorem III.0.1. By Claim III.5.2 we can assume without loss of generality that Ω is bounded. We denote $\omega := \omega_{\Omega,A}$.

Let $\phi : [0,\infty) \to [0,\infty)$ be any increasing function with $\lim_{t\to 0} \phi(t)/t = 0$, and consider the ϕ-Hausdorff content

$$h_\phi(E) = \inf \left\{ \sum_i \phi(r_i) : E \subset \bigcup_i B(z_i, r_i) \right\}.$$

Now we claim that there exists $F_\phi \subset \partial\Omega$ with $\omega(F_\phi) = 1$ and $h_\phi(F_\phi) = 0$. Indeed, suppose that the pole $z \in \Omega$ is such that $\operatorname{dist}(z, \partial\Omega) > r_0$. Set $\tau = 1/2$ and fix $0 < r \leq 1$ to be small enough as in Main Lemma III.2.1. Let $\varepsilon > 0$ and fix $M > \varepsilon^{-1}$ satisfying the hypothesis in Main Lemma III.2.1. Take ρ small enough such that $0 < \rho/r < 1/M$ and $\phi(\gamma) < \varepsilon M^{-1/2}\gamma$ for all $\gamma \leq \rho$. Then, by Main Lemma III.2.1 with these parameters, we obtain a set $F_\varepsilon \subset \partial\Omega$ such that $\omega^z(F_\varepsilon) \geq C^{-1}$ and with a covering $F_\varepsilon \subset \bigcup_i B(z_i, r_i)$ with $\sum_i r_i \leq CM^{1/2}$ and $\sum_{\{i : r_i > \rho\}} r_i \leq CM^{-1}$. This covering satisfies

$$h_\phi(F_\varepsilon) \leq \sum_{r_i \leq \rho} \phi(r_i) + \sum_{r_i > \rho} \phi(r_i) \leq \varepsilon M^{-1/2} \sum_{r_i \leq \rho} r_i + \sum_{r_i > \rho} r_i \leq \varepsilon M^{-1/2} C M^{1/2} + CM^{-1} \leq C\varepsilon.$$

Define now $F_\infty \subset \partial\Omega$ as

$$F_\infty := \limsup_{j\to\infty} F_{1/j^2} = \bigcap_{j\geq 1} \bigcup_{k\geq j} F_{1/k^2}.$$

With this choice we have

$$\omega^z(F_\infty) = \lim_{j\to\infty} \omega^z \left(\bigcup_{k\geq j} F_{1/k^2} \right) \geq \limsup_{j\to\infty} \omega^z(F_{1/j^2}) \geq C^{-1},$$

and as $F_\infty \subseteq \bigcup_{k\geq j} F_{1/k^2}$ for any $j \geq 1$, then

$$h_\phi(F_\infty) \leq h_\phi \left(\bigcup_{k\geq j} F_{1/k^2} \right) \leq \sum_{k\geq j} h_\phi(F_{1/k^2}) \leq C \sum_{k\geq j} \frac{1}{k^2},$$

which gives $h_\phi(F_\infty) = 0$ letting $j \to \infty$.

Let $\{z_k\}_{k=1}^\infty$ be a countable dense subset of Ω. Fix z_k and set $d_k = \operatorname{dist}(z_k, \partial\Omega)$. As Ω is (δ, r_0)-Reifenberg flat and the matrix A is Lipschitz in $\{x : \operatorname{dist}(x, \partial\Omega) < r_0\}$, then Ω is (δ, r_k)-Reifenberg flat and A is Lipschitz in $\{x : \operatorname{dist}(x, \partial\Omega) < r_k\}$ with $r_k = \min\{r_0, d_k/2\}$.

By the choice of r_k we are in the situation $\operatorname{dist}(z_k, \partial\Omega) > r_k$. By the same argument done in the previous paragraphs we get a set $F_k \subset \partial\Omega$ (relative to z_k) such that

$$\omega^{z_k}(F_k) \geq C^{-1} \text{ and } h_\phi(F_k) = 0.$$

Define $F_\phi := \bigcup_{k=1}^\infty F_k$. The condition $h_\phi(F_k) = 0$ for every $k \geq 1$ means that for every $\epsilon > 0$ there exists a covering $F_k \subset \bigcup_i B(z_i^{k,\epsilon}, r_i^{k,\epsilon})$ such that $\sum_i \phi(r_i^{k,\epsilon}) \leq \epsilon/2^k$. Hence, $F_\phi = \bigcup_{k=1}^\infty F_k \subset \bigcup_{k=1}^\infty \bigcup_i B(z_i^{k,\epsilon}, r_i^{k,\epsilon})$ which gives $h_\phi(F_\phi) = 0$ because

$$h_\phi(F_\phi) \leq \sum_{k=1}^\infty \sum_i \phi(r_i^{k,\epsilon}) \leq \sum_{k=1}^\infty \frac{\epsilon}{2^k} = \epsilon.$$

Moreover $\omega^{z_k}(F_\phi) \geq \omega^{z_k}(F_k) \geq C^{-1}$ for any z_k. As $\{z_k\}_{k=1}^\infty \subset \Omega$ is dense and $\omega^z(F_\phi)$ is L_A-harmonic with respect to z (in particular continuous), then $\omega^p(F_\phi) \geq C^{-1}$ for any $p \in \Omega$. By [HKM06, Lemma 11.16] we conclude $\omega^p(F_\phi) = 1$ for any $p \in \Omega$.

Finally let

$$F = \left\{\xi \in \partial\Omega : \limsup_{r\to 0} \frac{\omega^p(B(\xi, r))}{r} > 0\right\}.$$

We claim that this set has σ-finite length and $\omega(F) = 1$. We start by proving $\omega(F) = 1$, and later we will show that it has σ-finite length.

Suppose to get a contradiction that $\omega^p(F) \neq 1$, that is,

$$\omega^p(F^c) = \omega^p\left(\left\{\xi \in \partial\Omega : \lim_{r\to 0} \frac{\omega^p(B(\xi, r))}{r} = 0\right\}\right) > 0.$$

Egorov's theorem ensures that for every $s > 0$ there exists a measurable set $V = V_s \subset F^c$ such that $\omega^p(V) < s$ and $\omega^p(B(\xi, r))/r \to 0$ as $r \to 0$ uniformly on $F^c \setminus V$. Since $0 < \omega^p(F^c) = \omega^p(V) + \omega^p(F^c \setminus V) < s + \omega^p(F^c \setminus V)$, if we take $s > 0$ small enough, say $s = \omega^p(F^c)/2$, then $\omega^p(F^c \setminus V) > 0$. The set $Y := F^c \setminus V$ has non zero elliptic measure and the limit

$$\lim_{r\to 0} \left\| \frac{\omega^p(B(\cdot, r))}{r} \right\|_{L^\infty(Y)} = 0.$$

Note that the function $\phi(r) := \sup_{\xi\in Y} \omega^p(B(\xi, r))$ satisfies the conditions of the rate function in the beginning of this proof, and by definition it satisfies $\omega^p(B(\xi, r)) \leq \phi(r)$ for all $\xi \in Y$ and all $r > 0$. All in all,

(1) $\omega^p(B(\xi, r)) \leq \phi(r)$ for all $\xi \in Y$ and all $r > 0$,
(2) ϕ is increasing, and
(3) $\phi(r)/r \to 0$ as $r \to 0$.

For this particular function ϕ, let F_ϕ be the set constructed in the beginning of this proof. That is, a set F_ϕ with $h_\phi(F_\phi) = 0$ and $\omega^p(F_\phi) = 1$ for every $p \in \Omega$. Hence $\omega^p(Y \cap F_\phi) = \omega^p(Y) > 0$, and moreover $h_\phi(Y \cap F_\phi) \leq h_\phi(F_\phi) = 0$. Consequently, we can cover $Y \cap F_\phi$ with balls $B(\xi_i, r_i)$ centered at $Y \cap F_\phi$ such that $\sum_i \phi(r_i) < \omega^p(Y \cap F_\phi)/2$. With this we get

$$\omega^p(Y \cap F_\phi) \leq \sum_i \omega^p(B(\xi_i, r_i)) \overset{(1)}{\leq} \sum_i \phi(r_i) < \frac{\omega^p(Y \cap F_\phi)}{2}.$$

This is a contradiction, and hence $\omega^p(F) = 1$.

It remains to prove that F has σ-finite one-dimensional Hausdorff measure. The set F can be written as

$$F = \bigcup_{j\geq 1} F^{1/j} \text{ with } F^{1/j} := \left\{ \xi \in \partial\Omega : \limsup_{r\to 0} \frac{\omega^p(B(\xi,r))}{r} > 1/j \right\}.$$

Therefore, it suffices to see that every $F^{1/j}$ has finite one-dimensional Hausdorff measure. Each point $\xi \in F^{1/j}$ has arbitrarily small neighborhoods $B(\xi, r)$ such that $\omega^p(B(\xi,r)) \geq r/j$. Given $\varepsilon > 0$ small, consider the family of these balls centered at $F^{1/j}$ with radius at most ε, i.e.,

$$\mathcal{B}_\varepsilon := \{B(\xi,r) : \xi \in F^{1/j}, r < \varepsilon, \text{ and } \omega^p(B(\xi,r)) \geq r/j\}.$$

By the Besicovitch covering theorem there is a countable subfamily $\{B(\xi_i, r_i)\}_i \subset \mathcal{B}_\varepsilon$ such that no point belongs to more than a fixed finite number C (it depends on the dimension only) of these balls. Hence,

$$\sum_i r_i \leq j \sum_i \omega^p(B(\xi_i, r_i)) \leq Cj,$$

and letting $\varepsilon \to 0$ we obtain $\mathcal{H}^1(F^{1/j}) \leq Cj$, as claimed. □

Hence Theorem III.0.1 is proved under the assumption that (III.0.1) holds.

III.6. $L\log L(d\sigma)$ type estimate for small densities: Proof of (III.0.1)

The purpose of this section is to prove (III.0.1), under the hypothesis of Lemma III.2.3. More specifically, we prove the following result.

Lemma III.6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded (δ, r_0)-Reifenberg flat domain, $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$, and A be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ. Suppose also that A is κ-Lipschitz in $U_{r_0}(\partial\Omega) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x,\partial\Omega) < r_0\}$ and that its symmetric part $A_0 = \frac{A+A^T}{2}$ is of the form $A_0 = R^T B R$, with $R \in C^{0,1}(U_{r_0}(\partial\Omega))$ a rotation and $B \in C^{0,1}(U_{r_0}(\partial\Omega))$ diagonal.*

Then there exists $\delta_0 = \delta_0(\lambda, \kappa\|A\|_{L^\infty(\mathbb{R}^2)}) > 0$ and $C = C(\lambda, \kappa, r_0, \operatorname{diam}\partial\Omega) \in (0,\infty)$ such that if $\delta \leq \delta_0$, then for any (δ, r_0)-Reifenberg flat domain $\widetilde{\Omega} \subset \Omega$ with smooth boundary $\partial\widetilde{\Omega}$ and small enough $\operatorname{dist}_{\mathcal{H}}(\partial\Omega, \partial\widetilde{\Omega})$, we have

$$\left| \int_{\partial\widetilde{\Omega}} \log |S\nabla g_p^T(\xi)|^2 \, d\widetilde{\omega}^p(\xi) \right| \leq C < +\infty, \tag{III.6.1}$$

where $\widetilde{\omega}^p = \omega^p_{\widetilde{\Omega},A}$ is the elliptic measure in $\widetilde{\Omega}$ with respect to the matrix A, g_p^T is the Green function in $\widetilde{\Omega}$ with respect to the matrix A^T, and S is the square root matrix of the symmetric part $A_0 = (A + A^T)/2$, i.e., $S^T S = A_0$.

Remark III.6.2. As argued from (III.4.8) to (III.4.10), this implies that the Radon-Nykodym derivative $\frac{d\widetilde{\omega}^p}{d\sigma}$ satisfies the following $L\log L(d\sigma)$ type estimate

$$-\infty < C'(\lambda, \kappa, \operatorname{diam}\partial\Omega) \leq \int_{\partial\widetilde{\Omega}} \frac{d\widetilde{\omega}^p}{d\sigma}(\xi) \log \frac{d\widetilde{\omega}^p}{d\sigma}(\xi)\, d\sigma(\xi).$$

By symmetry, estimate (III.6.1) is equivalent to the existence of a constant $C = C(\lambda, \kappa, r_0, \operatorname{diam}\partial\Omega)$ depending only on the ellipticity constant, the Lipschitz seminorm of the matrix A, r_0, and the diameter $\operatorname{diam}\partial\Omega$ (but not on $\widetilde{\Omega}$) such that

$$\left|\int_{\partial\widetilde{\Omega}} \log|S\nabla g(\xi)|^2\, d\widetilde{\omega}_T^p(\xi)\right| \leq C < \infty, \tag{III.6.2}$$

where $S = A_0^{1/2}$, i.e., $S^T S = A_0$, $g = g_p$ is the Green function in $\widetilde{\Omega}$ with respect to the matrix A with pole p, and $\widetilde{\omega}_T = \omega^p_{\widetilde{\Omega},A^T}$ denotes the elliptic measure in $\widetilde{\Omega}$ with respect to the matrix A^T and a pole $p \in \widetilde{\Omega}$ such that $\operatorname{dist}(p, \partial\Omega) > r_0$. The existence of the matrix S is granted by the fact that the symmetric matrix A_0 is uniformly elliptic with the same ellipticity constant as A, and hence positive definite. In fact, $S = R^T\sqrt{B}R$, where $\sqrt{B} = (\delta_{i=j}\sqrt{b_{ij}})_{1\leq i,j\leq 2}$.

Throughout all this section, when dealing with terms within solid integrals we will use the following notation:

- We write $h(x) = \mathcal{O}(f(x))$ to denote "$|h(x)| \leq Cf(x)$ almost everywhere with respect to the Lebesgue measure".
- Given a Lipschitz function h, we write $|\nabla h(x)| \leq C$ instead of "$|\nabla h(x)| \leq C$ almost everywhere with respect to the Lebesgue measure". Recall that Lipschitz functions are differentiable almost everywhere by Rademacher's theorem, see [Mat95, Theorem 7.3].

III.6.1. Directional derivatives and the dual space. We introduce some extra notation regarding the directional derivatives.

Notation. Definition of directional derivatives.

- The (vertical) vector $e_i \coloneqq (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)^T$ has 1 in position i and 0's otherwise. In $\mathbb{R}^2$ there are only two such vectors, namely $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$.
- $R_i \coloneqq e_i^T \cdot R$ corresponds to the i-th row of R.
- The ∂^i-directional derivative $\partial^i \coloneqq \partial_{R_i^T}$ (superscript) is defined as
$$\partial^i f \coloneqq \partial_{R_i^T} f = \langle \nabla f, R_i^T\rangle = R_i \cdot \nabla f.$$
Here $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. The derivative $\partial_i = \partial_{e_i}$ (subscript) is the usual one in the direction e_i.
- The R-directional gradient ∇_R is defined as
$$\nabla_R f \coloneqq R \cdot \nabla f = \left(\partial^i f\right)_{i=1}^{n+1}.$$

Remark III.6.3. The directional derivative ∂^i preserves the usual properties in sums, products, and logarithms, i.e., $\partial^i(f+g) = \partial^i f + \partial^i g$, $\partial^i(fg) = g\partial^i f + f\partial^i g$, and $\partial^i \log f = \frac{\partial^i f}{f}$.

Note that

$$\left|\partial^{ij} f\right| = \left|\partial^i(R_j \cdot \nabla f)\right| = \left|\partial^i R_j \cdot \nabla f + R_j \partial^i \nabla f\right| \lesssim |\nabla f| + \left|\nabla^2 f\right|, \tag{III.6.3}$$

where we denote $|\nabla^2 f|^2 \coloneqq \sum_{i,j=1}^{n+1}(\partial_{i,j} f)^2$.

Since the directions R_i change depending on the point, the usual integration by parts formula does not apply. Instead we have the following formula.

Claim III.6.4 (Integration by parts formula)**.** *Let $U \subset \mathbb{R}^{n+1}$ be an open set. For $f, h \in W^{1,2}(U)$, if $fh \in W_0^{1,2}(U)$ then*

$$\int_U \partial^i f(x)h(x)\,dx = -\int_U f(x)h(x)\operatorname{div} R_i^T(x)\,dx - \int_U f(x)\partial^i h(x)\,dx. \tag{III.6.4}$$

Proof. The product rule gives $\partial^i f(x)h(x) = \partial^i(f(x)h(x)) - f(x)\partial^i h(x)$. By definition of the directional derivative ∂^i we can write

$$\begin{aligned}\partial^i(f(x)h(x)) = R_i(x)\cdot\nabla(f(x)h(x)) &= \sum_{j=1}^{n+1}\partial_j(f(x)h(x))R_{i,j}(x)\\ &= \sum_{j=1}^{n+1}\partial_j\left(f(x)h(x)R_{i,j}(x)\right) - f(x)h(x)\sum_{j=1}^{n+1}\partial_j R_{i,j}(x)\\ &= \operatorname{div}(fhR_i^T)(x) - f(x)h(x)\operatorname{div} R_i^T(x),\end{aligned}$$

which imply

$$\int \partial^i f(x)h(x)\,dx = \int \operatorname{div}(fhR_i^T)(x)\,dx - \int f(x)h(x)\operatorname{div} R_i^T(x)\,dx - \int f(x)\partial^i h(x)\,dx.$$

Using that $C_c^\infty(U)$ is dense in $W_0^{1,2}(U)$ and the divergence theorem, the first element in the right-hand side is $\int \operatorname{div}(fhR_i^T)(x)\,dx = 0$. □

Again, since the directions R depend on the point, the directional derivatives do not commute. Instead, the following formula relating $\partial^{\alpha,\beta}$ and $\partial^{\beta,\alpha}$ is available, where $\partial^{\alpha,\beta} := \partial^\alpha\partial^\beta$. We refer to this as the 'almost'-commutative property.

Claim III.6.5. *Let $U \subset \mathbb{R}^{n+1}$ be an open set and $f \in W^{2,2}(U)$. For $\alpha, \beta \in \{1, \dots, n+1\}$,*

$$\partial^{\alpha,\beta} f - \partial^{\beta,\alpha} f = (\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot\nabla f.$$

Proof. By definition we have $\partial^\beta f = R_\beta\cdot\nabla f = \sum_{i=1}^{n+1}\partial_i f R_{\beta,i}$. Applying ∂^α gives

$$\partial^{\alpha,\beta} f = R_\alpha\cdot\nabla\left(\sum_{i=1}^{n+1}\partial_i f R_{\beta,i}\right) = \sum_{j=1}^{n+1}\partial_j\left(\sum_{i=1}^{n+1}\partial_i f R_{\beta,i}\right)R_{\alpha,j}.$$

Expanding the derivative of this last expression and arranging we obtain

$$\partial^{\alpha,\beta} f = \sum_{i,j=1}^{n+1}\partial_{j,i} f R_{\beta,i}R_{\alpha,j} + \sum_{i=1}^{n+1}\partial_i f\sum_{j=1}^{n+1}\partial_j R_{\beta,i}R_{\alpha,j}.$$

The sum inside the second term in the right-hand side is precisely $\sum_{j=1}^{n+1}\partial_j R_{\beta,i}R_{\alpha,j} = R_\alpha\cdot\nabla R_{\beta,i} = \partial^\alpha R_{\beta,i}$. Hence

$$\partial^{\alpha,\beta} f = \sum_{i,j=1}^{n+1}\partial_{j,i} f R_{\beta,i}R_{\alpha,j} + \sum_{i=1}^{n+1}\partial_i f\,\partial^\alpha R_{\beta,i} = \sum_{i,j=1}^{n+1}\partial_{j,i} f R_{\beta,i}R_{\alpha,j} + \partial^\alpha R_\beta\cdot\nabla f.$$

By symmetry we obtain $\partial^{\beta,\alpha} f = \sum_{i,j=1}^{n+1}\partial_{j,i} f R_{\alpha,i}R_{\beta,j} + \partial^\beta R_\alpha\cdot\nabla f$, and subtracting $\partial^{\alpha,\beta} f - \partial^{\beta,\alpha} f$ we get Claim III.6.5. □

In particular, setting $\alpha = 1$ and $\beta = 2$ in the planar case we have

$$\partial^{1,2} f - \partial^{2,1} f = (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla f,$$

and applying it to the Green function g, we get

$$\partial^{1,2} g - \partial^{2,1} g = (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla g = \mathcal{O}(|\nabla g|). \tag{III.6.5}$$

III.6.1.1. *Definition of the dual space.* Let $U \subset \widetilde{\Omega}$ and $u \in L^1_{\mathrm{loc}}(U)$. Define the linear functional

$$T_u(\psi) \coloneqq \int_U u\psi,$$

whenever it makes sense. In particular, $T_u \in L^\infty_c(U)'$. To simplify the notation, we will also write $u(\psi)$ to denote $T_u(\psi)$.

The ∂_i-derivative functional is defined as

$$T_{\partial_i u}(\psi) \coloneqq -\int_U u\partial_i \psi,$$

whenever it makes sense. Note that $T_{\partial_i u} \in W^{1,\infty}_c(U)'$ for $u \in L^1_{\mathrm{loc}}(U)$. Moreover, we also have $T_{\partial_i u} \in W^{1,2}_c(U)'$ whenever $u \in L^2_{\mathrm{loc}}(U)$. As before, we will also write $Du(\psi)$ to denote $T_{Du}(\psi)$ for any differential operator D. In general, the functional

$$T_{fD(u)}(\psi) \coloneqq T_{D(u)}(f\psi),$$

is in $W^{1,\infty}_c(U)'$ as long as either $f \in W^{1,2}_{\mathrm{loc}}(U)$ and $u \in L^2_{\mathrm{loc}}(U)$, or $f \in W^{1,\infty}_{\mathrm{loc}}(U)$ and $u \in L^1_{\mathrm{loc}}(U)$.

We claim that whenever $u \in L^2_{\mathrm{loc}}(U)$ and $f \in W^{1,2}_{\mathrm{loc}}(U)$, in the dual space $W^{1,\infty}_c(U)'$, the ∂_i-derivative functional satisfies the product rule

$$T_{\partial_i (fu)} = T_{\partial_i f u} + T_{f\partial_i u}. \tag{III.6.6}$$

Indeed, for any $\psi \in W^{1,\infty}_c(U)$, the previous equality reads as

$$-\int_U fu\partial_i\psi = \int_U \partial_i f u\psi - \int_U u\partial_i(f\psi),$$

which holds by the Leibniz's rule a.e. for $W^{1,2}$-functions.

Using the definitions of T_{∂_i} and the directional derivative $\partial^i(\cdot) = R_i\nabla(\cdot)$, see the beginning of Section III.6.1, the directional ∂^i-derivative functional $T_{\partial^i u} : W^{1,\infty}_c(U) \to \mathbb{R}$, for $u \in L^1_{\mathrm{loc}}(U)$, is defined as

$$T_{\partial^i u} \coloneqq T_{R_i \nabla u} = \sum_j T_{R_{i,j}\partial_j u}. \tag{III.6.7}$$

Equivalently, for $\psi \in W^{1,\infty}_c(U)$,

$$\begin{aligned} T_{\partial^i u}(\psi) &= \sum_j T_{\partial_j u}(R_{i,j}\psi) = -\sum_j \int_U u\partial_j(R_{i,j}\psi) = -\sum_j \int_U u\psi\partial_j R_{i,j} - \sum_j \int_U uR_{i,j}\partial_j\psi \\ &= -\int_U u\psi \operatorname{div} R_i^T - \int_U u\partial^i\psi. \end{aligned} \tag{III.6.8}$$

This agrees with the integration by parts formula in (III.6.4) when $u \in W^{1,2}(U)$.

Next we claim that, in the dual space $W_c^{1,\infty}(U)'$, for $u \in L^2_{\mathrm{loc}}(U)$ and $f \in W^{1,2}_{\mathrm{loc}}(U)$, the directional ∂^i-derivative functional satisfies the product rule

$$T_{\partial^i(fu)} = T_{\partial^i f u} + T_{f\partial^i u}. \tag{III.6.9}$$

Indeed, $R_{i,j}\psi \in W_c^{1,\infty}(U)$ when $\psi \in W_c^{1,\infty}(U)$, and hence, for $f \in W^{1,2}_{\mathrm{loc}}(U)$,

$$\begin{aligned} T_{\partial^i(fu)}(\psi) &\overset{(\text{III.6.7})}{=} \sum_j T_{\partial_j(fu)}(R_{i,j}\psi) \overset{(\text{III.6.6})}{=} \sum_j T_{\partial_j f u}(R_{i,j}\psi) + T_{f\partial_j u}(R_{i,j}\psi) \\ &\overset{(\text{III.6.7})}{=} T_{\partial^i f u}(\psi) + T_{f\partial^i u}(\psi), \end{aligned}$$

as claimed.

Another important property is the following 'almost'-commutative property of the directional derivatives (compare to Claim III.6.5). For $u \in W^{1,1}_{\mathrm{loc}}(U)$, in the dual space $W_c^{1,\infty}(U)'$, we claim that

$$T_{\partial^1(\partial^2 u)} - T_{\partial^2(\partial^1 u)} = T_{(\partial^1 R_2 - \partial^2 R_1)\cdot\nabla u}. \tag{III.6.10}$$

Indeed, for $\psi \in W_c^{1,\infty}(U)$ and $\alpha, \beta \in \{1,2\}$,

$$T_{\partial^\alpha(\partial^\beta u)}(\psi) \overset{(\text{III.6.8})}{=} -\int_U \partial^\beta u \psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta u \partial^\alpha \psi.$$

Given $\{u_k\}_{k\geq 1} \subset C_c^\infty(U)$ with $\lim_{k\to\infty} \|u_k - u\|_{W^{1,1}(\operatorname{supp}\psi)} = 0$, this equals

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &-\int_U \partial^\beta u_k \psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta u_k \partial^\alpha \psi \\ &-\int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta(u-u_k)\partial^\alpha\psi. \end{aligned} \tag{III.6.11}$$

By the integration by parts in (III.6.4) applied to $\partial^\beta u_k$, we have that the first row in the right-hand side is precisely $\int_U \partial^\alpha(\partial^\beta u_k)\psi$. Applying now the 'almost'-commutative property (Claim III.6.5) here we obtain

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &\int_U \partial^\beta(\partial^\alpha u_k)\psi + \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla u_k)\psi \\ &-\int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta(u-u_k)\partial^\alpha\psi. \end{aligned}$$

Again, by the integration by parts in (III.6.4), we can replace the first term in the right-hand side to obtain

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &-\int_U \partial^\alpha u_k \psi \operatorname{div} R_\beta^T - \int_U \partial^\alpha u_k \partial^\beta \psi \\ &+\int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla u_k)\psi \\ &-\int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta(u-u_k)\partial^\alpha\psi. \end{aligned}$$

Now, adding and subtracting u in the second row we get

$$\begin{aligned}T_{\partial^\alpha(\partial^\beta u)}(\psi) = &-\int_U \partial^\alpha u_k \psi \operatorname{div} R_\beta^T - \int_U \partial^\alpha u_k \partial^\beta \psi \\ &+ T_{(\partial^\alpha R_\beta - \partial^\beta R_\alpha)\nabla u}(\psi) + \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla(u_k - u))\psi, \\ &- \int_U \partial^\beta(u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta(u-u_k)\partial^\alpha \psi.\end{aligned}$$

By symmetry in (III.6.11) and subtracting we obtain

$$\begin{aligned}T_{\partial^\alpha(\partial^\beta u)}(\psi) - T_{\partial^\beta(\partial^\alpha u)}(\psi) = T_{(\partial^\alpha R_\beta - \partial^\beta R_\alpha)\nabla u}(\psi) &+ \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla(u_k - u))\psi, \\ &- \int_U \partial^\beta(u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta(u-u_k)\partial^\alpha \psi \\ &+ \int_U \partial^\alpha(u-u_k)\psi \operatorname{div} R_\beta^T + \int_U \partial^\alpha(u-u_k)\partial^\beta \psi.\end{aligned}$$

By the convergence $\lim_{k\to\infty} \|u_k - u\|_{W^{1,1}(\operatorname{supp}\psi)} = 0$ we get (III.6.10).

III.6.1.2. *Properties of directional derivatives and the dual space.* The following claim allows us to move from the initial matrix A to its symmetric part A_0, which we will relate to the directional derivatives in Claim III.6.7.

Claim III.6.6. *Let $U \subset \widetilde{\Omega}$ be an open set and $u \in W^{1,2}_{\mathrm{loc}}(U)$. Every $\psi \in W^{1,\infty}_c(U)$ satisfies*

$$(\operatorname{div} A\nabla u)(\psi) = (\operatorname{div} A_0 \nabla u)(\psi) + \left(\sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j} - a_{j,i}}{2}\right) \partial_j u\right)(\psi).$$

Proof. Let $u \in C^\infty(U)$. By definition of the divergence and differentiating we have

$$\operatorname{div} A\nabla u = \sum_{i,j=1}^{n+1} \partial_i a_{i,j} \partial_j u + \sum_{i,j=1}^{n+1} a_{i,j}\partial_{i,j} u = \sum_{i,j=1}^{n+1} \partial_i a_{i,j}\partial_j u + \sum_{i,j=1}^{n+1} \frac{a_{i,j} + a_{j,i}}{2}\partial_{i,j} u.$$

Note that $a^0_{i,j} := \frac{a_{i,j}+a_{j,i}}{2}$ are precisely the coefficients of the symmetric matrix A_0. The product derivative rule gives

$$a^0_{i,j}\partial_{i,j} u = \partial_i \left(a^0_{i,j}\partial_j u\right) - \partial_i a^0_{i,j}\partial_j u,$$

and applying this relation to each pair of indexes $i, j \in \{1, \dots, n+1\}$ we get

$$\sum_{i,j=1}^{n+1} a^0_{i,j}\partial_{i,j} u = \operatorname{div} A_0 \nabla u - \sum_{i,j=1}^{n+1} \partial_i a^0_{i,j} \partial_j u.$$

Summing up,

$$\begin{aligned}\operatorname{div} A\nabla u &= \operatorname{div} A_0 \nabla u + \sum_{i,j=1}^{n+1} \partial_i a_{i,j}\partial_j u - \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j} + a_{j,i}}{2}\right)\partial_j u \\ &= \operatorname{div} A_0\nabla u + \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j} - a_{j,i}}{2}\right)\partial_j u,\end{aligned}$$

as claimed, for functions in $C^\infty(U)$.

Let us check this in the dual sense for a function $u \in W^{1,2}_{\mathrm{loc}}(U)$. Let $\psi \in W^{1,\infty}_c(U)$, and hence $u \in W^{1,2}(\operatorname{supp}\psi)$. Take $\{u_k\}_{k\geq 1} \subset C^\infty_c(U)$ such that $\lim_{k\to\infty} \|u_k - u\|_{W^{1,2}(\operatorname{supp}\psi)} = 0$. As we have the claim for each function u_k, in particular

$$\begin{aligned}\int_U A\nabla u\nabla\psi &= \int_U A\nabla(u-u_k)\nabla\psi + \int_U A\nabla u_k\nabla\psi\\ &= \int_U A\nabla(u-u_k)\nabla\psi + \int_U A_0\nabla u_k\nabla\psi - \int_U \sum_{i,j=1}^{n+1} \partial_i\left(\frac{a_{i,j}-a_{j,i}}{2}\right)\partial_j u_k\cdot\psi\\ &= \int_U A_0\nabla u\nabla\psi - \int_U \sum_{i,j=1}^{n+1} \partial_i\left(\frac{a_{i,j}-a_{j,i}}{2}\right)\partial_j u\cdot\psi + \int_U A\nabla(u-u_k)\nabla\psi\\ &\quad + \int_U A_0\nabla(u_k-u)\nabla\psi - \int_U \sum_{i,j=1}^{n+1} \partial_i\left(\frac{a_{i,j}-a_{j,i}}{2}\right)\partial_j (u_k-u)\psi.\end{aligned}$$

Now, since the matrix is Lipschitz and $\lim_{k\to\infty} \|u_k - u\|_{W^{1,2}(\operatorname{supp}\psi)} = 0$, we conclude

$$\int_U A\nabla u\nabla\psi = \int_U A_0\nabla u\nabla\psi - \int_U \sum_{i,j=1}^{n+1} \partial_i\left(\frac{a_{i,j}-a_{j,i}}{2}\right)\partial_j u\cdot\psi,$$

as claimed. ∎

We treat one of the main terms in (III.6.1) by means of a perturbation argument. To do that, we note that we can write $\operatorname{div}(A_0\nabla u)$ as in the diagonal case using the R-directional derivatives, plus an error term. More precisely, we have the following claim.

Claim III.6.7. *Let $A_0 = R^T BR$ satisfy the conditions in Lemma III.6.1, let $U \subset \widetilde{\Omega}$ be an open set and $u \in W^{1,2}_{\mathrm{loc}}(U)$. Every $\psi \in W^{1,\infty}_c(U)$ satisfies*

$$\left(\operatorname{div}(A_0\nabla u)\right)(\psi) = \left(\sum_{i=1}^{n+1} \partial^i(b_i\partial^i u)\right)(\psi) + \left(\sum_{i=1}^{n+1} b_i\partial^i u\cdot \operatorname{div} R_i^T\right)(\psi).$$

Proof. Assume first $u \in C^\infty(U)$. Let us write $R\nabla u = (R_j\cdot\nabla u)_{j=1}^{n+1} = (\partial^j u)_{j=1}^{n+1}$, and so $BR\nabla u = (b_j\partial^j u)_{j=1}^{n+1}$. Hence,

$$R^T BR\nabla u = R^T(b_j\partial^j u)_{j=1}^{n+1} = \left(\sum_{j=1}^{n+1} R^T_{i,j} b_j \partial^j u\right)_{i=1}^{n+1}.$$

Therefore,

$$\operatorname{div}(A_0\nabla u) = \sum_{i=1}^{n+1}\partial_i\left(\sum_{j=1}^{n+1} R^T_{i,j} b_j\partial^j u\right) = \sum_{j=1}^{n+1} b_j\partial^j u \sum_{i=1}^{n+1}\partial_i(R_{j,i}) + \sum_{j=1}^{n+1}\sum_{i=1}^{n+1} R_{j,i}\partial_i(b_j\partial^j u).$$

Note that the sum inside the first term in the right-hand side is precisely $\sum_{i=1}^{n+1}\partial_i(R_{j,i}) = \operatorname{div} R_j^T$, and the sum inside the second term in the right-hand side is $\sum_{i=1}^{n+1} R_{j,i}\partial_i(b_j\partial^j u) = R_j\cdot\nabla(b_j\partial^j u) = \partial^j(b_j\partial^j u)$. Thus, the claim follows for $u \in C^\infty(U)$.

From the definition of the directional ∂^i-derivative in (III.6.7) and (III.6.8), the claim follows by a density argument as in Claim III.6.6. ∎

III.6.2. Sketch of the proof. Throughout this section the pole $p \in \Omega$ so that $\operatorname{dist}(p, \partial\Omega) > r_0$ is fixed unless it is otherwise stated (see Lemma III.6.10 below). In any case, it will be far from $\partial\Omega$ and so from $\partial\widetilde{\Omega}$. Recall also that $S := A_0^{1/2}$, i.e., $S^T S = A_0$.

Fix $R := \min\{1, r_0/2\}/2$, and so we have $0 < R < \min\{1, r_0/2\}$. By [LLN08, Lemma 3.35], see Lemma III.1.5 and Remark III.1.6 above, if $\widetilde{\Omega}$ is Reifenberg flat enough, depending on the ellipticity constant λ and the value $\kappa\|A\|_{L^\infty(\mathbb{R}^{n+1})}$, then there exists a constant $c = c(\lambda, \kappa\|A\|_{L^\infty(\mathbb{R}^{n+1})}) \geq 1$ such that

$$c^{-1}|\nabla g(y)| \leq \frac{g(y)}{\operatorname{dist}\left(y, \partial\widetilde{\Omega}\right)} \leq c|\nabla g(y)| \text{ for every } y \in B(\xi, R/c) \text{ with } \xi \in \partial\widetilde{\Omega}.$$

Here we need to work with (δ, r_0)-Reifenberg flat domains with flatness parameter δ smaller than some constant depending on the ellipticity constant λ and the value $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$.

Let us now fix the support function φ with $\varphi(p) = 0$:

Remark III.6.8. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ with

- $\varphi = 1$ in $U_{\frac{R}{2000c}}(\partial\Omega) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial\Omega) \leq \frac{R}{2000c}\}$,
- $\varphi = 0$ in $U_{\frac{R}{1500c}}(\partial\Omega)^c$, and
- $|\nabla\varphi| \lesssim 1$.

So $\operatorname{supp}\varphi \subseteq U_{\frac{R}{1500c}}(\partial\Omega)$. Note that $\partial\widetilde{\Omega} \subset \Omega \cap \operatorname{supp}\varphi$ if we assume that $\operatorname{dist}_{\mathcal{H}}(\partial\widetilde{\Omega}, \partial\Omega)$ is small enough. With this choice of φ we have the comparability

$$|\nabla g(y)| \approx \frac{g(y)}{\operatorname{dist}(y, \partial\widetilde{\Omega})} \text{ for } y \in \widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega) \supset \widetilde{\Omega} \cap \operatorname{supp}\varphi. \tag{III.6.12}$$

We claim

$$\log|S\nabla g|^2 \in W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega)). \tag{III.6.13}$$

Indeed, this follows as $|\nabla g| \approx g/\operatorname{dist}(\cdot, \partial\widetilde{\Omega}) > 0$ in $\widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega)$ and $g \in W^{2,2}(\widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega))$, see (III.6.20).

Notation. From now on the variables and the region of integration will not be written unless they are not clear from the context.

Estimate (III.6.2) will follow from the following partial results.

For $a \geq 10$, let $\log_{(a)} x := \max\{\log x, -a\}$. Note that in the weak sense $(\log_{(a)}(x))' = \mathbf{1}_{\{x \geq e^{-a}\}}(x) \log x$.

Lemma III.6.9 (Step 1)**.** *For $a \geq 10$ big enough,*

$$\int_{\partial\widetilde{\Omega}} \log|S\nabla g|^2\, d\widetilde{\omega}_T^p = \int_{\partial\widetilde{\Omega}} \log_{(a)}|S\nabla g|^2\, d\widetilde{\omega}_T^p + \sigma(\partial\widetilde{\Omega})\mathcal{O}(ae^{-a/2}).$$

Since $\sigma(\partial\widetilde{\Omega}) < \infty$, the second term in the right-hand side tends to zero as $a \to \infty$.

We prove the preceding lemma in Section III.6.5.1.

Lemma III.6.10 (Step 2)**.** *For any $a \geq 10$ so that $e^{-a} < \min_{z \in \operatorname{supp}\nabla\varphi} |S\nabla g(z)|^2$, and for a.e. $p \in \Omega$ with* $\operatorname{dist}(p, \partial\Omega) > r_0$*,*

$$\int_{\partial\widetilde{\Omega}} \log_{(a)}|S\nabla g|^2\, d\widetilde{\omega}_T^p = -\int_{\widetilde{\Omega}} \langle A^T \nabla \log_{(a)}|S\nabla g|^2, \nabla(\varphi g)\rangle + \mathcal{O}\left(1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g\right).$$

We prove the preceding lemma in Section III.6.5.2. Note that the left-hand side integral is supported on the boundary, while the one on the right-hand side is a solid integral.

Next, fix $\widetilde{\Omega}_\varphi \subset \widetilde{\Omega}$ a smooth domain satisfying

$$\widetilde{\Omega} \cap \operatorname{supp} \varphi \subset \widetilde{\Omega}_\varphi \subset U_{\frac{R}{1000c}}(\partial\Omega) \cap \widetilde{\Omega}.$$

With this choice we have the comparability $|\nabla g| \approx g/\operatorname{dist}(\cdot, \partial\widetilde{\Omega})$ in $\widetilde{\Omega}_\varphi$, see (III.6.12). We prove the following two lemmas in Section III.6.5.1.

Lemma III.6.11 (Step 3)**.** *For $a \geq 10$ big enough,*

$$-\int_{\widetilde{\Omega}} \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle = -\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle + \mathcal{O}\left(\mathcal{H}^2(\Omega) e^{-a/2} + \int_{\widetilde{\Omega} \cap \operatorname{supp}\varphi \cap \{|S\nabla g|^2 \leq e^{-a}\}} |\nabla^2 g|\right).$$

Note that the last term in the right-hand side also tends to zero as $a \to \infty$ because $g \in W^{2,2}(\widetilde{\Omega}_\varphi)$ and $\bigcap_{a\geq 10}\{|S\nabla g|^2 \leq e^{-a}\} \cap \widetilde{\Omega}_\varphi = \emptyset$.

For $\varepsilon > 0$ as small as desired, we consider a given function $\psi_\varepsilon \in C^\infty(\mathbb{R}^2)$ satisfying

(1) $0 \leq \psi_\varepsilon \leq 1$ everywhere,
(2) $\psi_\varepsilon = 0$ in $U_\varepsilon(\partial\widetilde{\Omega})$,
(3) $\psi_\varepsilon = 1$ in $U_{3\varepsilon}(\partial\widetilde{\Omega})^c$, and
(4) $|\nabla \psi_\varepsilon| \lesssim \frac{1}{\varepsilon}$.

Lemma III.6.12 (Step 4)**.** *For ψ_ε as above we have*

$$\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle = \lim_{\varepsilon\to 0}\left\{\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\psi_\varepsilon \varphi g)\rangle + \mathcal{O}\left(\frac{1}{\varepsilon}\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g\right)\right\}.$$

In the weak sense, we can write the first term in the right-hand side in Lemma III.6.12 as $\operatorname{div}(A^T \nabla \log |S\nabla g|^2)$ acting on $\psi_\varepsilon \varphi g \in W^{1,\infty}_c(\widetilde{\Omega}_\varphi)$, which can be understood as test functions. When studying the harmonic measure in the plane, i.e., $A = Id$, this term is 0 by the harmonicity of $\log |\nabla g|$ far from the critical points, a property that does not hold in general in higher dimensions. This was a key point to establish (III.0.1) for the Laplacian in [JW88, Lemma 3.1].

In the following remark we discuss this argument in the constant coefficient case.

Remark III.6.13 (Constant matrix and L_A-harmonic functions)**.** Suppose now that the matrix A is constant. Given any function u, by Claim III.6.6,

$$\operatorname{div}(A^T \nabla u) = \operatorname{div}(A_0 \nabla u),$$

i.e., we can reduce the study to the symmetric part. Moreover, by means of a linear change of variables, see Lemma III.2.4, we have that $\operatorname{div}(A^T \nabla u) = 0$ if and only if $\widetilde{u} = u \circ D$ satisfies $\operatorname{div}\left(\widetilde{A_0}\nabla\widetilde{u}\right) = 0$ where $\widetilde{A_0} = D^{-1}A_0(D^{-1})^T$, for any constant matrix D with $\det D \neq 0$. Since A is constant now, if we choose $D = S^T$ where $S = A_0^{1/2}$, i.e., $S^T S = A_0$, then $\widetilde{A_0} = (S^T)^{-1}A_0 S^{-1} = (S^T)^{-1}S^T S S^{-1} = Id$, and so $\Delta\widetilde{u} = 0$. It is known that if $\Delta\widetilde{u} = 0$, then $\Delta \log |\nabla\widetilde{u}|^2 = 0$ (only) in the plane, whenever $\nabla\widetilde{u} \neq 0$. If we undo the previous change of variables then we obtain

$$\operatorname{div}\left(A_0 \nabla\left(\log |S\nabla u|^2\right)\right) = 0,$$

off the critical points.

Back to the general case, when the matrix A is non-constant, instead of moving from our matrix to the Laplacian by means of a change of variables as it is done in the constant case (meaning that we would need a non-constant change of variables), we will work with its symmetric part. Using the particular form $A_0 = R^T BR$, the rotation matrices will play the role of the directional derivatives, so we will be left with the diagonal matrix B, and this allows us to apply the previous strategy.

Now we study the first term appearing in the right-hand side in Lemma III.6.12,

$$-\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\psi_\varepsilon \varphi g)\rangle,$$

which is the most delicate, and the key point in the different behavior between the planar case and higher dimensions. Recall that, in the dual space, this term is

$$\big(\mathrm{div}(A^T \nabla \log |S\nabla g|^2)\big)\,(\psi_\varepsilon \varphi g) \text{ where } \psi_\varepsilon \varphi g \in W^{1,\infty}_c(\widetilde{\Omega}_\varphi).$$

Notation. From now on, a boxed term $\boxed{X}$ (representing the functional T_X) can be understood as a pointwise function, while a functional written as T_X needs to be understood strictly in the dual sense.

In the following lemma we decompose this last term.

Lemma III.6.14 (Step 5)**.** *In the dual space $W^{1,\infty}_c(\widetilde{\Omega}_\varphi)'$, we have*

$$\mathrm{div}(A^T \nabla \log |S\nabla g|^2) = \boxed{M1} + \boxed{M2} + T_{M3} + T_E,$$

where

$$\boxed{M1} := -\sum_{i=1}^{2} \frac{4b_i}{|S\nabla g|^4} \langle \partial^i \nabla_R g, B\nabla_R g\rangle^2,$$

$$\boxed{M2} := \sum_{i=1}^{2} \frac{2b_i}{|S\nabla g|^2} \langle \partial^i \nabla_R g, B\partial^i \nabla_R g\rangle,$$

$$T_{M3} := \frac{2}{|S\nabla g|^2} \left\langle \sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g\right), B\nabla_R g \right\rangle,$$

and T_E is an error term (involving derivatives of the matrix A) of the form

$$T_E = \sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i \left(\mathcal{O}(|\nabla g|^2)\right) + \mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right). \tag{III.6.14}$$

We prove the preceding lemma in Section III.6.5.3.

Note that, using (III.6.3), we have that the two main terms satisfy

$$\left|\boxed{M1}\right| + \left|\boxed{M2}\right| \lesssim 1 + \left(\frac{|\nabla^2 g|}{|\nabla g|}\right)^2, \tag{III.6.15}$$

which is bad for our purposes. A similar behavior occurs with the term T_{M3}. Instead, we need to exploit their cancellation, as illustrated in the constant matrix case:

Remark III.6.15 (Constant matrix and main terms)**.** If the matrix A were constant, and hence R and B were constant as well, then we would have $T_E = 0$. That is,

$$\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) = \boxed{M1} + \boxed{M2} + T_{M3}.$$

This suggests that these 3 terms are the main terms, and the others must be bounded error terms. Moreover, we have the following points:

- The key point in the plane is that $\sum_{i=1}^2 \partial^i(b_i\partial^i g) = 0$ would imply $b_1\partial^{1,1}g = -b_2\partial^{2,2}g$ (off the critical points). One can show that, as a consequence, the main terms $\boxed{M1}$ and $\boxed{M2}$ would cancel each other in the sense $\boxed{M1}+\boxed{M2} = 0$ off the critical points of g.
- Since the derivatives would commute, i.e., $\partial^{1,2} = \partial^{2,1}$, we would have

$$\sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g\right) = \nabla_R \left(\sum_{i=1}^{2} \partial^i \left(b_i \partial^i g\right)\right) = \nabla_R(0) = 0, \tag{III.6.16}$$

  where we used that the Green function g satisfies $\sum_{i=1}^2 \partial^i(b_i\partial^i g) = 0$. This is why we call T_{M3} the "zero" term. In contrast to the previous point, here it is not needed that we are in the plane. However, we use that the Green function is L_A-harmonic, that is, this term is zero even in higher dimensions.

All in all, in the constant matrix case we would have

$$\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) = \boxed{M1} + \boxed{M2} + T_{M3} = 0$$

off the critical points of g.

The strategy explained in the previous remark is, morally, what we will do to prove the following two lemmas.

Lemma III.6.16 (Step 6)**.** *Pointwise* $\left|\boxed{M1} + \boxed{M2}\right| \lesssim 1 + |\nabla^2 g|/|\nabla g|$ *in* $\widetilde{\Omega}_\varphi$.

We prove the preceding lemma in Section III.6.5.4.

Lemma III.6.17 (Step 7)**.** *In the dual space* $W_c^{1,\infty}(\widetilde{\Omega}_\varphi)'$, *the functional* T_{M3} *is of the form*

$$T_{M3} = \sum_{j=1}^{2} \frac{2b_j\partial^j g}{|S\nabla g|^2}\left(\mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla^2 g|) + \partial^1\left(\mathcal{O}(|\nabla g|)\right) + \partial^2\left(\mathcal{O}(|\nabla g|)\right)\right).$$

Recall that this equality must be understood at a functional level, see (III.6.36) below.

We prove the preceding lemma in Section III.6.5.5.

In the following lemma, we bound the error terms that have appeared in the previous computations.

Lemma III.6.18 (Step 8)**.** *For small enough* $\varepsilon > 0$,

$$|T_{M3}(\psi_\varepsilon \varphi g)| + |T_E(\psi_\varepsilon\varphi g)| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g.$$

We prove the preceding lemma in Section III.6.5.6.

Lemma III.6.19 (Step 9)**.** *We have* $\int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g \leq C < \infty$, *and for small enough* $\varepsilon > 0$, $\int_{U_\varepsilon(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g \lesssim \varepsilon$.

We prove the preceding lemma at the end of Section III.6.4.

Using the previous lemmas we obtain (III.6.2).

Proof of (III.6.2). Let $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$ so that Lemma III.6.10 holds for all $a > 10$ integer. By Lemmas III.6.9 to III.6.11 we have

$$\left|\int_{\partial\widetilde{\Omega}} \log |S\nabla g|^2 \, d\widetilde{\omega}_T^p\right| \leq \left|-\int \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle\right| + \mathcal{O}\left(1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g\right).$$

By Lemmas III.6.12, III.6.14, III.6.16 and III.6.18, we get

$$\left|-\int \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle\right| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g + \mathcal{O}\left(\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g\right).$$

In particular,

$$\left|\int_{\partial\widetilde{\Omega}} \log |S\nabla g|^2 \, d\widetilde{\omega}_T^p\right| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g + \mathcal{O}\left(\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g\right).$$

Finally, (III.6.2) follows from this and Lemma III.6.19. Note that by Harnack inequality this is true for every $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$. □

III.6.3. Second derivatives of the Green function. In this subsection, we collect a fundamental property of the Green function in our situation, and some useful equalities and bounds that we will use later on involving the second order derivatives of the Green function.

The following claim points out the main difference between the planar and the higher dimensional case. This is a key point in the proof of Lemmas III.6.16 and III.6.17.

Claim III.6.20. *The Green function g satisfies*

$$\sum_{i=1}^{2} \partial^i (b_i \partial^i g) = -\sum_{i=1}^{2} b_i \partial^i g \operatorname{div} R_i^T - \sum_{i,j=1}^{2} \partial_i \left(\frac{a_{ij} - a_{ji}}{2}\right) \partial_j g = \mathcal{O}(|\nabla g|).$$

a.e. in $U_{r_0}(\partial\Omega) \cap \widetilde{\Omega}$. In particular,

$$b_1 \partial^{1,1} g = -b_2 \partial^{2,2} g + \mathcal{O}(|\nabla g|). \tag{III.6.17}$$

Proof. By Claims III.6.6 and III.6.7,

$$\begin{aligned}
\operatorname{div} A\nabla g &= \sum_{i=1}^{2} \partial^i (b_i \partial^i g) + \sum_{i=1}^{2} b_i \partial^i g \operatorname{div} R_i^T + \sum_{i,j=1}^{2} \partial_i \left(\frac{a_{ij} - a_{ji}}{2}\right) \partial_j g \\
&= \sum_{i=1}^{2} \partial^i (b_i \partial^i g) + \mathcal{O}(|\nabla g|),
\end{aligned}$$

where we used that the matrices A, B and R are Lipschitz. The claim follows from the fact that the Green function satisfies $\operatorname{div} A\nabla g = 0$ a.e. in $U_{r_0}(\partial\Omega) \cap \widetilde{\Omega}$, see Remark I.1.4.

As a consequence of $\partial^i (b_i \partial^i g) = \partial^i b_i \partial^i g + b_i \partial^{i,i} g$ and $\partial^i b_i \partial^i g \in \mathcal{O}(|\nabla g|)$, we have (III.6.17). □

A close look to the proof reveals that if the matrix A were constant, then we would have $b_1 \partial^{1,1} g = -b_2 \partial^{2,2} g$ in the planar case, as it was explained in Remark III.6.15. Instead, we get the analog with some error terms in (III.6.17).

In the following claim we collect some basic equalities and estimates. These equalities will be useful in the decomposition procedure in Lemma III.6.14.

Claim III.6.21. *Using the notation $\nabla_R f = R\cdot\nabla f$, see Section III.6.1, we have*

$$\partial^i|S\nabla g|^2 = 2\langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle. \tag{III.6.18}$$

Consequently,

$$\partial^i\left(\frac{1}{|S\nabla g|^2}\right) = \frac{-\partial^i|S\nabla g|^2}{|S\nabla g|^4} \overset{(III.6.18)}{=} -\frac{2\langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle}{|S\nabla g|^4}, \tag{III.6.19}$$

and

$$|\nabla\log|S\nabla g|^2| = \frac{|\nabla|S\nabla g|^2|}{|S\nabla g|^2} \lesssim 1 + \frac{|\nabla^2 g|}{|\nabla g|} \text{ in } \widetilde{\Omega}_\varphi. \tag{III.6.20}$$

Proof. Since $S^TS = A_0$ and $A_0 = R^TBR$, we can write

$$|S\nabla g|^2 = \langle S\nabla g, S\nabla g\rangle = \langle\nabla g, A_0\nabla g\rangle = \langle R\nabla g, BR\nabla g\rangle = \langle\nabla_R g, B\nabla_R g\rangle.$$

Recall that R and B are Lipschitz regular, thus we can apply the usual derivative rules a.e. to obtain

$$\partial^i|S\nabla g|^2 = \langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle + \langle\nabla_R g, B\partial^i\nabla_R g\rangle,$$

and (III.6.18) follows from the symmetry of B.

The inequality in (III.6.20) follows from

$$\begin{aligned}\big|\nabla|S\nabla g|^2\big| &\overset{(III.6.18)}{\approx} \sum_{i=1}^2 \big|2\langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle\big| \\ &\le \sum_{i=1}^2 2|\langle\partial^i\nabla_R g, B\nabla_R g\rangle| + \sum_{i=1}^2 |\langle\nabla_R g, \partial^i B\nabla_R g\rangle| \\ &\lesssim |\nabla g|^2 + \sum_{i=1}^2 |\partial^i\nabla_R g||\nabla g| \overset{(III.6.3)}{\lesssim} |\nabla g|^2 + |\nabla g||\nabla^2 g|,\end{aligned}$$

and this concludes the proof of the claim. ☐

Remark III.6.22. $1/|S\nabla g|^2 \in W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}_\varphi)$ because of (III.6.19), $|\nabla g| \approx g/\mathrm{dist}(\cdot, \partial\widetilde{\Omega}) > 0$ in $\widetilde{\Omega}_\varphi$ and $g \in W^{2,2}(\widetilde{\Omega}_\varphi)$. By the same reason, $\log|S\nabla g|^2 \in W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}_\varphi)$.

III.6.4. Whitney cubes and proof of Step 9. A classical way to compute an integral over a given set is to discretize it. We will do so using Whitney cubes, that is, we divide the domain into regions (cubes) which have diameter comparable to their distance to the boundary, so that Harnack's and Caccioppoli's inequalities can be used locally.

Let us start by defining the Whitney covering of $\widetilde{\Omega}$, and then we will move to study the properties of those cubes touching $\operatorname{supp}\varphi$.

Definition III.6.23. There exists a collection $W_{\widetilde{\Omega}}$ of dyadic cubes satisfying the following properties:

(W1) $\sqrt{2}\ell(Q) \le \mathrm{dist}(Q, \partial\widetilde{\Omega}) \le 4\sqrt{2}\ell(Q)$, equivalently, $\mathrm{diam}Q \le \mathrm{dist}(Q, \partial\widetilde{\Omega}) \le 4\mathrm{diam}Q$.

(W2) $1.5Q \cap \partial\widetilde{\Omega} = \emptyset$, since $\mathrm{dist}(1.5Q, \partial\widetilde{\Omega}) \ge \mathrm{dist}(Q, \partial\widetilde{\Omega}) - \sqrt{2}\cdot 0.25\ell(Q) \ge 0.75\sqrt{2}\ell(Q) > 0$,

which we call *Whitney cubes*, see [Gra08, Appendix J]. We define $W_\varphi := \{Q \in W_{\widetilde{\Omega}} : Q \cap \operatorname{supp}\varphi \ne \emptyset\}$.

Lemma III.6.24. *Every $Q \in W_\varphi$ satisfies also:*

(W3) $\operatorname{dist}(x, \partial\widetilde{\Omega}) \leq 7\frac{R}{2000c} \leq R/c$ *for* $x \in 1.5Q$.
(W4) $\ell(Q) \lesssim 1$.
(W5) If $Q \cap \operatorname{supp} \nabla\varphi \neq \emptyset$, *then* $\ell(Q) \gtrsim 1$.
(W6) For $x \in 1.5Q$ *we have*

$$g(x) \approx g(A(\xi, 10\ell(Q))) \lesssim \widetilde{\omega}_T(B(\xi, 10\ell(Q))) \leq \widetilde{\omega}_T(50Q) \leq 1,$$

where $\xi \in \partial\widetilde{\Omega}$ *is such that* $\operatorname{dist}(\xi, Q) = \operatorname{dist}(\partial\widetilde{\Omega}, Q)$, *and* $A(\xi, 10\ell(Q))$ *is the Corkscrew point at* ξ *with radius* $10\ell(Q)$.

Proof. Note that for $x \in 1.5Q$, Harnack's inequality gives $g(x) \approx g(A(\xi, 10\ell(Q)))$. Using the relation between the elliptic measure and Green's function in NTA domains (see [Ken94, Lemma 1.3.3]), we get

$$g(A(\xi, 10\ell(Q))) \lesssim \widetilde{\omega}_T(B(\xi, 10\ell(Q))).$$

Here we need $p \notin B(\xi, 3 \cdot 10\ell(Q))$, which is granted as long as $\partial\widetilde{\Omega}$ and $\partial\Omega$ are close enough.

The remaining estimates in the lemma follow from the definition by standard arguments. The details are left to the reader. □

In order to bound the error terms arising in the proofs of Section III.6.2, we will use (without mention) the following remark.

Remark III.6.25. If $f \in \mathcal{O}(|\nabla g|^k)$ then $\left|\int \frac{f}{|\nabla g|^k}\varphi g\right| \leq C < \infty$.

Proof. Since $g \lesssim 1$ in $\operatorname{supp}\varphi$ (see Lemma III.6.24(W6)), we obtain

$$\left|\int \frac{f}{|\nabla g|^k}\varphi g\right| \leq \int \frac{|f|}{|\nabla g|^k}|\varphi g| \lesssim \int |\varphi g| \lesssim \mathcal{H}^2(\Omega) \leq (\operatorname{diam}\partial\Omega)^2 < \infty,$$

where we used that Ω is bounded. □

Next we continue by controlling the Green function over Whitney cubes:

Lemma III.6.26. *We have:*

(1) $\sum_{Q\in W_\varphi} \ell(Q) \cdot \widetilde{\omega}_T(50Q) \lesssim 1$.
(2) For $\varepsilon > 0$, $\sum_{\{Q\in W_{\widetilde{\Omega}}:Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset\}} \ell(Q) \cdot \widetilde{\omega}_T(50Q) \lesssim \varepsilon$.
(3) For $\varepsilon > 0$, $\int_{U_\varepsilon(\partial\widetilde{\Omega})} g \lesssim \varepsilon^2$.
(4) For each $Q \in W_\varphi$, $\left(\int_{1.5Q} g^2\right)^{1/2} \lesssim \ell(Q) \cdot \widetilde{\omega}_T(50Q)$.
(5) For each $Q \in W_\varphi$, $\int_Q |\nabla g| \lesssim \ell(Q) \cdot \widetilde{\omega}_T(50Q)$.
(6) For each $Q \in W_\varphi$, $\int_Q |\nabla^2 g| \lesssim \widetilde{\omega}_T(50Q)$.

Proof. Item 1: As $\ell(Q) \lesssim 1$ for every $Q \in W_\varphi$ (see Lemma III.6.24(W4)), there is k_0 such that every $Q \in W_\varphi$ satisfies $\ell(Q) = 2^{-k}$ with $k \geq k_0$. Note also that, for each scale $k \geq k_0$, the family $\{50Q\}_{\{Q\in W_\varphi\,:\,\ell(Q)=2^{-k}\}}$ has finite overlapping. Therefore,

$$\sum_{Q\in W_\varphi} \ell(Q) \cdot \widetilde{\omega}_T(50Q) = \sum_{k\geq k_0} \sum_{\substack{Q\in W_\varphi \\ \ell(Q)=2^{-k}}} \ell(Q) \cdot \widetilde{\omega}_T(50Q) = \sum_{k\geq k_0} 2^{-k} \sum_{\substack{Q\in W_\varphi \\ \ell(Q)=2^{-k}}} \widetilde{\omega}_T(50Q)$$

$$\lesssim \sum_{k\geq k_0} 2^{-k}\widetilde{\omega}_T\Bigg(\bigcup_{\substack{Q\in W_\varphi \\ \ell(Q)=2^{-k}}} 50Q\Bigg) \leq \sum_{k\geq k_0} 2^{-k} < \infty.$$

Item 2: Recall that the cubes $Q \in W_{\widetilde{\Omega}}$ with $Q \cap U_\varepsilon(\partial\widetilde{\Omega}) \neq \emptyset$ satisfy $\ell(Q) \lesssim \varepsilon$, and $\ell(Q) \approx \varepsilon$ if $Q \cap (\partial U_\varepsilon(\partial\widetilde{\Omega}) \setminus \partial\widetilde{\Omega}) \neq \emptyset$, see (W1). Hence, as in the proof of item 1, taking $k_0(\varepsilon) \approx -\log_2 \varepsilon$ we have

$$\sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q)\widetilde{\omega}_T(50Q) \leq \sum_{k\geq k_0(\varepsilon)} 2^{-k} \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ \ell(Q)=2^{-k}}} \widetilde{\omega}_T(50Q) \lesssim \sum_{k\geq k_0(\varepsilon)} 2^{-k} \approx 2^{\log_2 \varepsilon} = \varepsilon.$$

Item 3: Using that $g \lesssim \widetilde{\omega}_T(50Q)$ in Q, see (W6), that the cubes $Q \in W_{\widetilde{\Omega}}$ with $Q \cap U_\varepsilon(\partial\widetilde{\Omega}) \neq \emptyset$ satisfy $\ell(Q) \lesssim \varepsilon$, see (W1), and item 2, we have

$$\int_{U_\varepsilon(\partial\widetilde{\Omega})} g \leq \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \int_Q g \overset{(W6),(W1)}{\lesssim} \varepsilon \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q)\widetilde{\omega}_T(50Q) \overset{(2)}{\lesssim} \varepsilon^2.$$

Item 4 follows since $g(x) \lesssim \widetilde{\omega}_T(50Q)$ for $x \in 1.5Q$ (see Lemma III.6.24(W6)). Item 5 follows from Cauchy-Schwarz and Caccioppoli's inequalities and item 4.

Item 6: Since the Green function g of $\widetilde{\Omega}$ is solution of $\operatorname{div}(A\nabla\cdot) = 0$ in $1.1Q$, the function $\widehat{g}(\cdot) = g(\ell(Q)\cdot)$ is solution of $\operatorname{div}(\widehat{A}\nabla\cdot)$ in $1.1\widehat{Q}$, where $\widehat{Q} \coloneqq \{x/\ell(Q) : x \in Q\}$ has side-length one, and $\widehat{A}(\cdot) = A(\ell(Q)\cdot)$. Moreover, since $\partial_i\partial_j\widehat{g}(\cdot) = \ell(Q)^2\partial_i\partial_j g(\ell(Q)\cdot)$, we get

$$\int_Q |\nabla^2 g(x)|^2\,dx = \ell(Q)^2 \int_{\widehat{Q}} |\nabla^2 g(\ell(Q)x)|^2\,dx = \frac{1}{\ell(Q)^2}\int_{\widehat{Q}} |\nabla^2 \widehat{g}(x)|^2\,dx.$$

The matrix $\widehat{A}$ has the same ellipticity constant as A, but as $\ell(Q) \lesssim 1$ for every $Q \in W_\varphi$, see Lemma III.6.24(W4), the Lipschitz norm cannot grow too much:

$$\|\widehat{A}\|_{C^{0,1}\left(1.1\widehat{Q}\right)} \coloneqq \|\widehat{A}\|_{L^\infty\left(1.1\widehat{Q}\right)} + [\widehat{A}]_{C^{0,1}\left(1.1\widehat{Q}\right)} = \|A\|_{L^\infty(1.1Q)} + \ell(Q)[A]_{C^{0,1}(1.1Q)} \leq C\|A\|_{C^{0,1}(1.1Q)}.$$

Moreover, $\operatorname{dist}\left(\widehat{Q}, (1.1\widehat{Q})^c\right) \approx 1$. Therefore we can apply (I.1.2) in Theorem I.1.2 to obtain

$$\int_Q |\nabla^2 g|^2\,dx = \frac{1}{\ell(Q)^2}\int_{\widehat{Q}} |\nabla^2\widehat{g}|^2\,dx \overset{(I.1.2)}{\lesssim} \frac{1}{\ell(Q)^2}\left[\left(\int_{1.1\widehat{Q}} |\nabla\widehat{g}|^2\right)^{1/2} + \left(\int_{1.1\widehat{Q}} \widehat{g}^2\right)^{1/2}\right]^2.$$

Applying Caccioppoli's inequality and item 4, this implies

$$\int_Q |\nabla^2 g|^2\,dx \lesssim \frac{1}{\ell(Q)^2}\int_{1.5\widehat{Q}} \widehat{g}^2 = \frac{1}{\ell(Q)^4}\int_{1.5Q} g^2 \overset{(4)}{\lesssim} \frac{\widetilde{\omega}_T(50Q)^2}{\ell(Q)^2}.$$

Using the Cauchy-Schwarz inequality and this, we get

$$\int_Q |\nabla^2 g| \leq \ell(Q)\left(\int_Q |\nabla^2 g|^2\right)^{1/2} \lesssim \widetilde{\omega}_T(50Q).$$

□

Using this lemma we obtain Lemma III.6.19.

Proof of Lemma III.6.19. Recall that, by (III.6.12) we have

$$|\nabla g(x)| \gtrsim \frac{g(x)}{\operatorname{dist}\left(x, \partial\widetilde{\Omega}\right)} \text{ for } x \in \operatorname{supp}\varphi,$$

provided the domain $\widetilde{\Omega}$ is $(\delta, r_0/2)$-Reifenberg flat with δ small enough depending only on λ and $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$. With this it suffices to show

$$\int_{\operatorname{supp}\varphi} |\nabla^2 g(x)| \operatorname{dist}\left(x, \partial\widetilde{\Omega}\right) dx \leq C.$$

Summing in W_φ, by (6) and (1) in Lemma III.6.26 we obtain

$$\int_{\operatorname{supp}\varphi} |\nabla^2 g(x)| \operatorname{dist}(x, \partial\Omega)\, dx \lesssim \sum_{Q\in W_\varphi} \ell(Q) \int_Q |\nabla^2 g| \overset{(6)}{\lesssim} \sum_{Q\in W_\varphi} \ell(Q)\cdot \widetilde{\omega}_T(50Q) \overset{(1)}{\lesssim} 1,$$

as claimed.

Let us see the localized version. As above, using (III.6.12), and items 2 and 6 in Lemma III.6.26,

$$\int_{U_\varepsilon(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g \overset{\text{(III.6.12)}}{\lesssim} \int_{U_\varepsilon(\partial\widetilde{\Omega})} |\nabla^2 g| \operatorname{dist}(x, \partial\widetilde{\Omega}) \leq \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q) \int_Q |\nabla^2 g|$$

$$\overset{(6)}{\lesssim} \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q) \widetilde{\omega}_T(50Q) \overset{(2)}{\lesssim} \varepsilon.$$

□

III.6.5. Proof of Steps 1 to 8. In this subsection we prove remaining Steps 1 to 8 in Section III.6.2.

III.6.5.1. *Proof of Lemmas III.6.9, III.6.11 and III.6.12.*

Proof of Lemma III.6.9. We have

$$\left| \int_{\partial\widetilde{\Omega}} \log |S(\xi)\nabla g(\xi)|^2\, d\widetilde{\omega}_T^p(\xi) - \int_{\partial\widetilde{\Omega}} \log_{(a)} |S(\xi)\nabla g(\xi)|^2\, d\widetilde{\omega}_T^p(\xi) \right|$$
$$= \left| \int_{\{\xi\in\partial\widetilde{\Omega}: |S(\xi)\nabla g(\xi)|^2 \leq e^{-a}\}} (\log |S(\xi)\nabla g(\xi)|^2 + a)\, d\widetilde{\omega}_T^p(\xi) \right|$$

Since the domain is smooth, we can use (I.3.7) to write and bound the right-hand side term as

$$\left| \int_{\{\xi\in\partial\widetilde{\Omega}: |S(\xi)\nabla g(\xi)|^2 \leq e^{-a}\}} - \langle A(\xi)\nabla g(\xi), \nu\rangle (\log |S(\xi)\nabla g(\xi)|^2 + a)\, d\sigma(\xi) \right|$$
$$\lesssim \int_{\{\xi\in\partial\widetilde{\Omega}: |S(\xi)\nabla g(\xi)|^2 \leq e^{-a}\}} |S(\xi)\nabla g(\xi)| (|\log |S(\xi)\nabla g(\xi)||)\, d\sigma(\xi)$$
$$\leq \sigma(\partial\widetilde{\Omega}) \left(e^{-a/2} |\log e^{-a/2}|\right).$$

□

Proof of Lemma III.6.11. We have

$$-\int \langle A^T\nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle + \int \langle A^T\nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle$$
$$= \int \langle A^T\nabla \min\{0, a + \log |S\nabla g|^2\}, \nabla(\varphi g)\rangle.$$

On the other hand, by (III.6.12) we have that if $a \geq 10$ is big enough then $\varphi = 1$ in $\widetilde{\Omega}_\varphi \cap \{|S\nabla g|^2 \leq e^{-a}\}$, and hence

$$\int \langle A^T\nabla \min\{0, a + \log |S\nabla g|^2\}, \nabla(\varphi g)\rangle = \int \langle A^T\nabla \log |S\nabla g|^2, \nabla g\rangle \mathbf{1}_{\{|S\nabla g|^2 \leq e^{-a}\}}.$$

By (III.6.20), this is controlled in absolute value by

$$\int_{\{|S\nabla g|^2 \leq e^{-a}\}} \left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right) |\nabla g| \leq \int_{\{|S\nabla g|^2 \leq e^{-a}\}} |\nabla g| + \int_{\{|S\nabla g|^2 \leq e^{-a}\}} |\nabla^2 g|.$$

The lemma follows because the first term in the right-hand side is bounded by a constant times $\mathcal{H}^2(\Omega)e^{-a/2}$. □

Proof of Lemma III.6.12. For $\varepsilon > 0$ small we have $(1 - \psi_\varepsilon)\varphi = (1 - \psi_\varepsilon)$. Hence

$$-\int \langle A^T\nabla \log |S\nabla g|^2, \nabla((1 - \psi_\varepsilon)\varphi g)\rangle = -\int \langle A^T\nabla \log |S\nabla g|^2, \nabla((1 - \psi_\varepsilon)g)\rangle$$
$$= \int \langle A^T\nabla \log |S\nabla g|^2, \nabla\psi_\varepsilon\rangle g - \int \langle A^T\nabla \log |S\nabla g|^2, \nabla g\rangle(1 - \psi_\varepsilon).$$

Using (III.6.20), the second term in the right-hand side is controlled in absolute value by

$$\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right) |\nabla g| = \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} |\nabla g| + \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} |\nabla^2 g| \xrightarrow{\varepsilon\to 0} 0,$$

and the first term is controlled in absolute value by

$$\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right) g|\nabla\psi_\varepsilon| = \frac{1}{\varepsilon}\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} g + \frac{1}{\varepsilon}\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g.$$

The lemma follows by applying (3) in Lemma III.6.26 in the last line. □

III.6.5.2. *Proof of Lemma III.6.10.* Recall that the Green function g in $\widetilde{\Omega}$ for the operator $L_A u = \operatorname{div} A\nabla u$ satisfies, for every $\phi \in C^0(\overline{\widetilde{\Omega}}) \cap W^{1,2}(\widetilde{\Omega})$,

(III.6.21) $$\int_{\partial\widetilde{\Omega}} \phi(\xi)\, d\widetilde{\omega}_T^x(\xi) - \phi(x) = -\int_{\widetilde{\Omega}} \langle A^T(y)\nabla\phi(y), \nabla g_x(y)\rangle\, dy, \text{ for a.e. } x \in \widetilde{\Omega},$$

see (I.3.3).

We claim that

$$\varphi \log_{(a)} |S\nabla g|^2 \in C^0(\overline{\widetilde{\Omega}}) \cap W^{1,2}(\widetilde{\Omega}).$$

Indeed, $\varphi \log_{(a)} |S\nabla g|^2 \in C^0(\overline{\widetilde{\Omega}})$ because the function φ avoids the pole and ∇g is continuous up to the boundary, see Theorem I.1.5. Let us see that $\varphi \log_{(a)} |S\nabla g|^2 \in W^{1,2}(\widetilde{\Omega})$. Using

Jensen's inequality as the function $x \mapsto |\log x|^2$ is concave for $x > e$, the L^2-norm is controlled (depending on a) by

(III.6.22)
$$\begin{aligned}\int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} \left|\log_{(a)} |S\nabla g|^2\right|^2 &\le \int_{\widetilde{\Omega}\cap\{|S\nabla g|^2\le e\}} \left|\log_{(a)} |S\nabla g|^2\right|^2 + \int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi\cap\{|S\nabla g|^2>e\}} \left|\log |S\nabla g|^2\right|^2 \\ &\lesssim \mathcal{H}^2(\widetilde{\Omega}) \left(a^2 + \left|\log\left(\fint_{\widetilde{\Omega}\cap \operatorname{supp}\varphi\cap\{|S\nabla g|^2>e\}} |S\nabla g|^2\right)\right|^2\right) < \infty,\end{aligned}$$

which is bounded since $g \in W^{1,2}(\widetilde{\Omega} \cap \operatorname{supp}\varphi)$. The L^2 norm of $\nabla(\varphi \log_{(a)} |S\nabla g|^2)$ is

$$\int_{\widetilde{\Omega}} |\nabla(\varphi \log_{(a)} |S\nabla g|^2)|^2 \le \int_{\widetilde{\Omega}} |\nabla\varphi|^2 \left|\log_{(a)} |S\nabla g|^2\right|^2 + \int_{\widetilde{\Omega}} \varphi^2 |\nabla \log_{(a)} |S\nabla g|^2|^2.$$

The first term is finite by (III.6.22) because $\varphi \in C_c^\infty(\mathbb{R}^2)$. From the definition of $\log_{(a)} x := \max\{\log x, -a\}$ and by (III.6.20), we get

$$|\nabla \log_{(a)} |S\nabla g|^2| \lesssim \mathbf{1}_{\{x\ge e^{-a}\}}(|S\nabla g|^2)\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right),$$

and hence the second term is controlled by

$$\int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} |\nabla \log_{(a)} |S\nabla g|^2|^2 \lesssim \int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi\cap\{|S\nabla g|^2\ge e^{-a}\}} 1 + \frac{|\nabla^2 g|^2}{|\nabla g|^2} \lesssim \mathcal{H}^2(\widetilde{\Omega}) + e^a \int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} |\nabla^2 g|^2,$$

which is bounded as $g \in W^{2,2}(\widetilde{\Omega} \cap \operatorname{supp}\varphi)$.

By the choice of φ with $\varphi(p) = 0$ and $\varphi|_{\partial\widetilde{\Omega}} = 1$, and plugging $\phi = \varphi \log_{(a)} |S\nabla g|^2 \in C^0(\overline{\widetilde{\Omega}}) \cap W^{1,2}(\widetilde{\Omega})$ in (III.6.21), for a.e. $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$ we have

(III.6.23)
$$\begin{aligned}\int_{\partial\widetilde{\Omega}} \log_{(a)} |S\nabla g|^2 \, d\widetilde{\omega}_T^p &= -\int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} \langle A^T \nabla\left(\varphi \log_{(a)} |S\nabla g|^2\right), \nabla g\rangle \\ &= -\int \langle A^T \nabla\varphi, \nabla g\rangle \cdot \log_{(a)} |S\nabla g|^2 - \int \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \varphi\nabla g\rangle \\ &= \boxed{1} + \boxed{2} + \boxed{3},\end{aligned}$$

with

$$\begin{aligned}\boxed{1} &:= -\int \langle A^T \nabla\varphi, \nabla g\rangle \cdot \log_{(a)} |S\nabla g|^2, \\ \boxed{2} &:= \int \langle A^T \; \nabla \log_{(a)} |S\nabla g|^2, \nabla\varphi\rangle \cdot g, \text{ and} \\ \boxed{3} &:= -\int \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle.\end{aligned}$$

We claim that $\left|\boxed{1}\right| \lesssim 1$ and $\left|\boxed{2}\right| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g$, and hence Lemma III.6.10 follows.

We start by controlling the term $\boxed{1}$. Recall that we have fixed φ so that $|\nabla\varphi| \lesssim 1$, and assumed that a is big enough so that $e^{-a} < \min_{z\in \operatorname{supp}\nabla\varphi} |S\nabla g(z)|^2$, that is, $\log_{(a)} |S\nabla g(z)|^2 =$

$\log|S\nabla g(z)|^2$ for $z\in\operatorname{supp}\nabla\varphi$. Thus,

$$\left|\boxed{1}\right| \lesssim \int_{\operatorname{supp}\nabla\varphi} |\nabla g|\,|\log|S\nabla g|| \approx \int_{\operatorname{supp}\nabla\varphi} |S\nabla g|\,|\log|S\nabla g||\,.$$

Since $\operatorname{supp}\nabla\varphi$ is far from $\partial\widetilde{\Omega}$, in particular $\operatorname{dist}(x,\partial\widetilde{\Omega})\approx 1$ for $x\in\operatorname{supp}\nabla\varphi$. Consequently, from this and (III.6.12) we have $|\nabla g|\approx g/\operatorname{dist}(\cdot,\partial\widetilde{\Omega})\approx g$ in the support of $\nabla\varphi$, and hence

$$|S\nabla g(x)|\approx|\nabla g(x)|\approx g(x)\overset{(W6)}{\lesssim} 1 \text{ for } x\in\operatorname{supp}\nabla\varphi.$$

Note that for $0<t<t_0$ we have $t\,|\log t|\le\max\{1,t_0\,|\log t_0|\}$. Therefore,

$$\boxed{1}\lesssim\int_{\operatorname{supp}\nabla\varphi}|S\nabla g|\,|\log|S\nabla g||\lesssim\mathcal{H}^2(\operatorname{supp}\nabla\varphi)\le C<\infty.$$

Now we study the term $\boxed{2}$. Since φ is fixed, we have $|\nabla\varphi|\lesssim 1$ and so

$$\left|\boxed{2}\right|\lesssim\int_{\operatorname{supp}\nabla\varphi} g\cdot|\nabla\log_{(a)}|S\nabla g|^2|\le\int_{\operatorname{supp}\varphi} g\cdot|\nabla\log_{(a)}|S\nabla g|^2|.$$

Plugging (III.6.20) in here we get

$$\left|\boxed{2}\right|\lesssim\int_{\operatorname{supp}\varphi} g+\int_{\operatorname{supp}\varphi}\frac{|\nabla^2 g|}{|\nabla g|}g\overset{(W6)}{\lesssim}1+\int_{\operatorname{supp}\varphi}\frac{|\nabla^2 g|}{|\nabla g|}g.$$

□

III.6.5.3. *Proof of Lemma III.6.14.* In this subsection we study, via a perturbation argument, the functional term

$$\operatorname{div}\left(A^T\nabla\log|S\nabla g|^2\right)\in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)'.$$

Note that the action of this functional on φg gives rise to a modified version of the term $\boxed{3}$ in (III.6.23).

First, we move from the matrix A to its symmetric part A_0 by Claim III.6.6, and we write its divergence in terms of the directional derivatives using Claim III.6.7 (see Remark III.6.22):

$$\begin{aligned}\operatorname{div}\left(A^T\nabla\log|S\nabla g|^2\right)&=\operatorname{div}\left(A_0\nabla\log|S\nabla g|^2\right)+\sum_{i,j=1}^2\partial_i\left(\frac{a_{ij}^T-a_{ji}^T}{2}\right)\partial_j\log|S\nabla g|^2\\&=\sum_{i=1}^2\partial^i(b_i\partial^i(\log|S\nabla g|^2))+\sum_{i=1}^2 b_i\partial^i(\log|S\nabla g|^2)\operatorname{div}R_i^T\\&\quad+\sum_{i,j=1}^2\partial_i\left(\frac{a_{ji}-a_{ij}}{2}\right)\partial_j(\log|S\nabla g|^2)=:T_{3.A}+\boxed{3.B}+\boxed{3.C}.\end{aligned}$$

Note that the terms $\boxed{3.B}$ and $\boxed{3.C}$ contain derivatives of the matrix A. In particular, if the matrix were constant then these two terms would be zero, which suggests that the terms $\boxed{3.B}$ and $\boxed{3.C}$ must be bounded error terms.

Next we deal with $T_{3.A}$. As $1/|S\nabla g|^2 \in W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}_\varphi)$ (see Remark III.6.22), by (III.6.9) we have

$$
\begin{aligned}
T_{3.A} &= \sum_{i=1}^{2} \partial^i \left(b_i \frac{\partial^i |S\nabla g|^2}{|S\nabla g|^2} \right) \\
&= \sum_{i=1}^{2} \partial^i \left(\frac{1}{|S\nabla g|^2} \right) b_i \partial^i |S\nabla g|^2 + \sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i (b_i \partial^i |S\nabla g|^2).
\end{aligned} \tag{III.6.24}
$$

We infer that each element in the sum in the first term in the right-hand side in (III.6.24) is

$$
\partial^i \left(\frac{1}{|S\nabla g|^2} \right) b_i \partial^i |S\nabla g|^2 \overset{\text{(III.6.19)}}{=} \frac{-b_i \left(\partial^i |S\nabla g|^2\right)^2}{|S\nabla g|^4}. \tag{III.6.25}
$$

Therefore, using identities (III.6.18) and (III.6.25) and expanding, the first term in the right-hand side of (III.6.24) can be written as

$$
\sum_{i=1}^{2} \partial^i \left(\frac{1}{|S\nabla g|^2} \right) b_i \partial^i |S\nabla g|^2 = \boxed{M1} + \boxed{E1} + \boxed{E2},
$$

where

$$
\begin{aligned}
\boxed{M1} &:= -\sum_{i=1}^{2} \frac{4b_i}{|S\nabla g|^4} \langle \partial^i \nabla_R g, B\nabla_R g \rangle^2, \\
\boxed{E1} &:= -\sum_{i=1}^{2} \frac{4b_i}{|S\nabla g|^4} \langle \partial^i \nabla_R g, B\nabla_R g \rangle \langle \nabla_R g, \partial^i B \nabla_R g \rangle, \text{ and} \\
\boxed{E2} &:= -\sum_{i=1}^{2} \frac{b_i}{|S\nabla g|^4} \langle \nabla_R g, \partial^i B \nabla_R g \rangle^2.
\end{aligned}
$$

Note that $\boxed{E1}$ and $\boxed{E2}$ can be considered "error terms" from a perturbation point of view, since in case B were constant we would get $\boxed{E1} = \boxed{E2} = 0$, while $\boxed{M1}$ can be considered a "main term".

Next we perform a similar decomposition for $\partial^i (b_i \partial^i |S\nabla g|^2)$, in the second term in the right-hand side in (III.6.24). In this case, we need to pay attention to the first term in the right-hand side of (III.6.18):

$$
\partial^i \left(b_i \langle \partial^i \nabla_R g, B\nabla_R g \rangle \right) = \left\langle \partial^i \left(b_i \partial^i \nabla_R g \right), B\nabla_R g \right\rangle + b_i \langle \partial^i \nabla_R g, \partial^i B \nabla_R g \rangle + b_i \langle \partial^i \nabla_R g, B \partial^i \nabla_R g \rangle. \tag{III.6.26}
$$

Then using (III.6.18) and (III.6.26) we get that the second term in the right-hand side of (III.6.24) can be written as

$$
\sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i (b_i \partial^i |S\nabla g|^2) = \boxed{M2} + T_{M3} + \boxed{E3} + T_{E4},
$$

where

$$\boxed{M2} := \sum_{i=1}^{2} \frac{2b_i}{|S\nabla g|^2} \langle \partial^i \nabla_R g, B\partial^i \nabla_R g \rangle,$$
$$T_{M3} := \frac{2}{|S\nabla g|^2} \left\langle \sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g \right), B\nabla_R g \right\rangle,$$
$$\boxed{E3} := \sum_{i=1}^{2} \frac{2b_i}{|S\nabla g|^2} \langle \partial^i \nabla_R g, \partial^i B \nabla_R g \rangle, \text{ and}$$
$$T_{E4} := \sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i \left(b_i \langle \nabla_R g, \partial^i B \nabla_R g \rangle \right).$$

All together we have

$$\begin{aligned} \operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) &= T_{3.A} + \boxed{3.B} + \boxed{3.C} \\ &= \boxed{M1} + \boxed{M2} + T_{M3} && \text{(Main terms)} \\ &\quad + \boxed{3.B} + \boxed{3.C} + \boxed{E1} + \boxed{E2} + \boxed{E3} + T_{E4}, && \text{(Error terms)} \end{aligned}$$

and the first part of the lemma follows taking

$$T_E := \boxed{3.B} + \boxed{3.C} + \boxed{E1} + \boxed{E2} + \boxed{E3} + T_{E4}.$$

Let us see now the second part of the lemma. Indeed, $\boxed{E2}$ is $\mathcal{O}(1)$ since $|\langle \nabla_R g, \partial^i B \nabla_R g \rangle| \lesssim |\nabla g|^2$. By (III.6.20) we get that $\boxed{3.B}$ and $\boxed{3.C}$ are in $\mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right)$, and (III.6.3) implies that $\boxed{E1}$ and $\boxed{E3}$ are of the form $\mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right)$. All in all,

$$T_E = T_{E4} + \mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right),$$

as claimed. □

III.6.5.4. *Proof of Lemma III.6.16.* In this subsection we prove that the two main terms cancel out in the sense

$$\left| \boxed{M1} + \boxed{M2} \right| \lesssim 1 + \frac{|\nabla^2 g|}{|\nabla g|}.$$

This cancellation is related to what happens in the constant case, see Remark III.6.15.

We will see that they both have a common term in opposite sign, and exploiting this cancellation we will obtain a sum of error terms. The key identity is (III.6.17), which only applies in the plane.

The key idea to prove the lemma is that using (III.6.5), which relates $\partial^{1,2} g$ and $\partial^{2,1} g$, and (III.6.17), which relates $b_1 \partial^{1,1} g$ and $b_2 \partial^{2,2} g$, we will be able to write $\partial^{1,1}$, $\partial^{2,2}$, $\partial^{1,2}$ and $\partial^{2,1}$ in terms of $\partial^{1,1}$ and $\partial^{2,1}$ only.

We start by studying the term $\boxed{M1} := -\sum_i \frac{4b_i}{|S\nabla g|^4}\langle\partial^i\nabla_R g, B\nabla_R g\rangle^2$ in $\widetilde{\Omega}_\varphi$. Expanding the numerator,

$$\sum_{i=1}^{2} b_i\langle\partial^i\nabla_R g, B\nabla_R g\rangle^2 = b_1\langle\partial^1\nabla_R g, B\nabla_R g\rangle^2 + b_2\langle\partial^2\nabla_R g, B\nabla_R g\rangle^2$$
$$= b_1(b_1\partial^1 g\partial^{1,1}g + b_2\partial^2 g\ \partial^{1,2}g)^2 + b_2(b_1\partial^1 g\partial^{2,1}g + b_2\partial^2 g\ \partial^{2,2}g)^2.$$

By (III.6.5) and (III.6.17), which read as $\partial^{1,2}g = \partial^{2,1}g + \mathcal{O}(|\nabla g|)$ and $b_2\partial^{2,2}g = -b_1\partial^{1,1}g + \mathcal{O}(|\nabla g|)$ respectively, we have

$$\sum_{i=1}^{2} b_i\langle\partial^i\nabla_R g, B\nabla_R g\rangle^2 = b_1(b_1\partial^1 g\partial^{1,1}g + b_2\partial^2 g\partial^{2,1}g + \mathcal{O}(|\nabla g|^2))^2$$
$$+ b_2(b_1\partial^1 g\partial^{2,1}g - b_1\partial^2 g\partial^{1,1}g + \mathcal{O}(|\nabla g|^2))^2.$$

Expanding, and using $b_1(\partial^1 g)^2 + b_2(\partial^2 g)^2 = \langle\nabla_R g, B\nabla_R g\rangle = |S\nabla g|^2$ and the cancellation of cross terms, we get

$$\sum_{i=1}^{2} b_i\langle\partial^i\nabla_R g, B\nabla_R g\rangle^2 = |S\nabla g|^2\left(b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2\right) + (\partial^{1,1}g + \partial^{2,1}g)\mathcal{O}(|\nabla g|^3) + \mathcal{O}(|\nabla g|^4).$$

In conclusion, $\boxed{M1}$ is of the form

$$\boxed{M1} = -4\frac{b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2}{|S\nabla g|^2} + \frac{\mathcal{O}(\partial^{1,1}g + \partial^{2,1}g)}{|S\nabla g|} + \mathcal{O}(1). \tag{III.6.27}$$

With the same strategy we study the term $\boxed{M2} := \sum_i \frac{2b_i}{|S\nabla g|^2}\langle\partial^i\nabla_R g, B\partial^i\nabla_R g\rangle$ in $\widetilde{\Omega}_\varphi$. Using (III.6.5) and (III.6.17) as before and expanding, the numerator can be written as

$$\sum_{i=1}^{2} b_i\langle\partial^i\nabla_R g, B\partial^i\nabla_R g\rangle = b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{1,2}g)^2 + b_1b_2(\partial^{2,1}g)^2 + b_2^2(\partial^{2,2}g)^2$$
$$= b_1^2(\partial^{1,1}g)^2 + b_1b_2\left(\partial^{2,1}g + \mathcal{O}(|\nabla g|)\right)^2 + b_1b_2(\partial^{2,1}g)^2 + \left(-b_1\partial^{1,1}g + \mathcal{O}(|\nabla g|)\right)^2$$
$$= 2\left(b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2\right) + \left(\partial^{1,1}g + \partial^{2,1}g\right)\mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla g|^2).$$

Hence, $\boxed{M2}$ is of the form

$$\boxed{M2} = 4\frac{b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2}{|S\nabla g|^2} + \frac{\mathcal{O}(\partial^{1,1}g + \partial^{2,1}g)}{|S\nabla g|} + \mathcal{O}(1). \tag{III.6.28}$$

Note that both $\boxed{M1}$ in (III.6.27) and $\boxed{M2}$ in (III.6.28) have the term

$$4\frac{b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2}{|S\nabla g|^2}$$

in common with opposite sign, which allows us to remove the square in (III.6.15). That is, adding up both terms,

$$\boxed{M1} + \boxed{M2} = \frac{\mathcal{O}(\partial^{1,1}g + \partial^{2,1}g)}{|S\nabla g|} + \mathcal{O}(1),$$

whence we obtain

$$\left|\boxed{M1}+\boxed{M2}\right| \overset{\text{(III.6.3)}}{\lesssim} 1+\frac{|\nabla^2 g|}{|\nabla g|},$$

as claimed. □

III.6.5.5. *Proof of Lemma III.6.17.* We study, for $\psi \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)$, the term

$$T_{M3}(\psi) := \left(\frac{2}{|S\nabla g|^2}\left\langle \sum_{i=1}^{2} \partial^i\left(b_i\partial^i\nabla_R g\right), B\nabla_R g\right\rangle\right)(\psi).$$

In this subsection we don't use the fact that we are in the plane. Instead, the key ingredient is the L_A-harmonicity of the Green function far from the pole, and so the computations in this subsection could be done also in higher dimensions.

The term T_{M3} must be read as

$$\begin{aligned} T_{M3} &= \frac{2}{|S\nabla g|^2}\left\langle \sum_{i=1}^{2}\begin{pmatrix}\partial^i(b_i\partial^{i,1}g)\\ \partial^i(b_i\partial^{i,2}g)\end{pmatrix}, \begin{pmatrix} b_1\partial^1 g\\ b_2\partial^2 g\end{pmatrix}\right\rangle \\ &= \frac{2}{|S\nabla g|^2}\sum_{i=1}^{2}\left\{\partial^i(b_i\partial^{i,1}g)b_1\partial^1 g+\partial^i(b_i\partial^{i,2}g)b_2\partial^2 g\right\} \qquad \text{(III.6.29)} \\ &= \frac{2}{|S\nabla g|^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\partial^i(b_i\partial^{i,j}g)b_j\partial^j g =: \sum_{i=1}^{2}\sum_{j=1}^{2} T_{M3}^{(i,j)}. \end{aligned}$$

Here, for each $i,j\in\{1,2\}$,

$$T_{M3}^{(i,j)}(\psi) = T_{\partial^i(b_i\partial^{i,j}g)}\left(\frac{2}{|S\nabla g|^2}b_j\partial^j g\psi\right). \qquad \text{(III.6.30)}$$

We want to study the functionals $T_{M3}^{(i,j)}$ for $i,j\in\{1,2\}$. First note that

$$\frac{2}{|S\nabla g|^2}b_j\partial^j g\psi \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi),$$

while $b_i\partial^{i,j}g\in L^2(\widetilde{\Omega}_\varphi)$. Hence, (III.6.30) makes sense and it suffices to study the functionals $T_{\partial^i(b_i\partial^{i,j}g)}\in W_c^{1,\infty}(\widetilde{\Omega})'$. In fact, fixed $j\in\{1,2\}$, we will exploit the cancellation of the functional $\sum_{i=1}^2 T_{\partial^i(b_i\partial^{i,j}g)}$. For simplicity we will write $\partial^i(b_i\partial^{i,j}g)$ instead of $T_{\partial^i(b_i\partial^{i,j}g)}$.

Compared to the strategy seen in Remark III.6.15, now the matrix is not constant and hence the directional derivatives do not commute. For this reason, some error terms will appear in the procedure of extracting the gradient ∇_R outside from

$$\sum_{i=1}^{2}\partial^i\left(b_i\partial^i\nabla_R g\right) = \begin{pmatrix}\partial^1(b_1\partial^{1,1}g)+\partial^2(b_2\partial^{2,1}g)\\ \partial^1(b_1\partial^{1,2}g)+\partial^2(b_2\partial^{2,2}g)\end{pmatrix}, \qquad \text{(III.6.31)}$$

as it is done in the constant matrix case in (III.6.16). The idea is the same as in the proof of Lemma III.6.16, see Section III.6.5.4. That is, using the 'almost'-commutative property (III.6.5) (relating $\partial^{1,2}g$ and $\partial^{2,1}g$) of the directional derivatives, and its functional version in (III.6.10), we manage to extract the gradient ∇_R outside.

Let us fix $j\in\{1,2\}$. If $i=j$ (clearly $\partial^{i,j}=\partial^{j,i}$ in this case), then

$$\partial^i(b_i\partial^{i,j}g) = \partial^j\left(\partial^i(b_i\partial^i g)\right) - \partial^i(\partial^j b_i\partial^i g), \qquad \text{(III.6.32)}$$

and when $i \neq j$, using (III.6.5) and (III.6.10) we get

$$
\begin{aligned}
\partial^i(b_i\partial^{i,j}g) &\overset{\text{(III.6.5)}}{=} \partial^i\left(b_i\partial^{j,i}g\right) + \partial^i\left(b_i(\partial^i R_j - \partial^j R_i)\nabla g\right) \\
&= \partial^i\left(\partial^j(b_i\partial^i g)\right) - \partial^i\left(\partial^j b_i\partial^i g\right) + \partial^i\left(b_i(\partial^i R_j - \partial^j R_i)\nabla g\right) \\
&\overset{\text{(III.6.10)}}{=} \partial^j\left(\partial^i(b_i\partial^i g)\right) - (\partial^j R_i - \partial^i R_j)\nabla(b_i\partial^i g) \\
&\qquad - \partial^i\left(\partial^j b_i\partial^i g\right) + \partial^i\left(b_i(\partial^i R_j - \partial^j R_i)\nabla g\right).
\end{aligned}
\tag{III.6.33}
$$

Plugging (III.6.32) and (III.6.33) in (III.6.31), we get, for $j=1$,

$$
\begin{aligned}
\sum_{i=1}^{2}\partial^i(b_i\partial^{i,1}g) = \partial^1\left(\sum_{i=1}^{2}\partial^i\left(b_i\partial^i g\right)\right) - \sum_{i=1}^{2}\partial^i(\partial^1 b_i\partial^i g) - (\partial^1 R_2 - \partial^2 R_1)\cdot\nabla(b_2\partial^2 g) \\
+ \partial^2\left(b_2(\partial^2 R_1 - \partial^1 R_2)\cdot\nabla g\right),
\end{aligned}
\tag{III.6.34}
$$

and for $j=2$,

$$
\begin{aligned}
\sum_{i=1}^{2}\partial^i(b_i\partial^{i,2}g) = \partial^2\left(\sum_{i=1}^{2}\partial^i\left(b_i\partial^i g\right)\right) - \sum_{i=1}^{2}\partial^i(\partial^2 b_i\partial^i g) - (\partial^2 R_1 - \partial^1 R_2)\cdot\nabla(b_1\partial^1 g) \\
+ \partial^1\left(b_1(\partial^1 R_2 - \partial^2 R_1)\cdot\nabla g\right).
\end{aligned}
\tag{III.6.35}
$$

Note that in both cases there is the term $\sum_{i=1}^{2}\partial^i\left(b_i\partial^i g\right)$, which is studied in Claim III.6.20. By (III.6.29) and using Claim III.6.20 in both (III.6.34) and (III.6.35), in the dual space we get

$$
\begin{aligned}
T_{M3} &= \frac{2}{|S\nabla g|^2}\sum_{j=1}^{2} b_j\partial^j g\left(\sum_{i=1}^{2}\partial^i(b_i\partial^{i,j}g)\right) \\
&= \sum_{j=1}^{2}\frac{2b_j\partial^j g}{|S\nabla g|^2}\left\{\partial^j\left(-\sum_{i=1}^{2}b_i\partial^i g\operatorname{div}R_i^T - \sum_{i,k=1}^{2}\partial_i\left(\frac{a_{ik}-a_{ki}}{2}\right)\partial_k g\right) - \sum_{i=1}^{2}\partial^i(\partial^j b_i\partial^i g)\right\} \\
&\quad + \frac{2b_1\partial^1 g}{|S\nabla g|^2}\left\{-(\partial^1 R_2 - \partial^2 R_1)\cdot\nabla(b_2\partial^2 g) + \partial^2\left(b_2(\partial^2 R_1 - \partial^1 R_2)\cdot\nabla g\right)\right\} \\
&\quad + \frac{2b_2\partial^2 g}{|S\nabla g|^2}\left\{-(\partial^2 R_1 - \partial^1 R_2)\cdot\nabla(b_1\partial^1 g) + \partial^1\left(b_1(\partial^1 R_2 - \partial^2 R_1)\cdot\nabla g\right)\right\}.
\end{aligned}
\tag{III.6.36}
$$

As a consequence of this, the right-hand equality in Claim III.6.20 and $|\nabla\partial^i g| = \mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla^2 g|)$, see (III.6.3), we conclude that the functional T_{M3} is of the form

$$
T_{M3} = \sum_{j=1}^{2}\frac{2b_j\partial^j g}{|S\nabla g|^2}\left(\mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla^2 g|) + \partial^1\left(\mathcal{O}(|\nabla g|)\right) + \partial^2\left(\mathcal{O}(|\nabla g|)\right)\right),
$$

as claimed. □

III.6.5.6. *Proof of Lemma III.6.18.* First, we need the following claim, which we will prove later.

Claim III.6.27. *Let $\psi \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)$. Then, for $i,j\in\{1,2\}$, both*

$$T_{\frac{1}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|^2))}(\psi) \text{ and } T_{\frac{b_j\partial^j g}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|))}(\psi)$$

are of the form

$$\int \frac{\mathcal{O}(|\nabla^2 g|)}{|\nabla g|}\psi + \int \mathcal{O}(1)\psi + \int \mathcal{O}(1)\partial^i\psi.$$

Granted this, by Lemma III.6.17 and (III.6.14) we have

$$|T_{M3}(\psi_\varepsilon\varphi g)| + |T_E(\psi_\varepsilon\varphi g)| \lesssim \int \frac{|\nabla^2 g|}{|\nabla g|}\psi_\varepsilon\varphi g + \int \psi_\varepsilon\varphi g + \int |\partial^i(\psi_\varepsilon\varphi g)|.$$

The second term in the right-hand side is bounded by a constant times $\mathcal{H}^2(\Omega)<\infty$ as Ω is bounded and $g\lesssim 1$ in $\operatorname{supp}\varphi$, see (W6). Hence, it suffices to prove

$$\int \left|\partial^i(\psi_\varepsilon\varphi g)\right| \lesssim 1, \text{ for } i\in\{1,2\}.$$

By the product derivative rule and the notation $\partial^i f = R_i\cdot\nabla f$, we can use

$$\int \left|\partial^i\left(\psi_\varepsilon\varphi g\right)\right| \lesssim \int g|\nabla\varphi| + \int_{\operatorname{supp}\varphi}|\nabla g| + \frac{1}{\varepsilon}\int_{\operatorname{supp}\nabla\psi_\varepsilon} g.$$

Since $|\nabla\varphi|\lesssim 1$ and $g(x)\lesssim 1$ in $\operatorname{supp}\varphi$ (see (W6)), we get $\int g|\nabla\varphi|\lesssim 1$. For the second integral on the right-hand side we sum over Whitney cubes in W_φ and apply items 1 and 5 in Lemma III.6.26

$$\int_{\operatorname{supp}\varphi}|\nabla g| \le \sum_{Q\in W_\varphi}\int_Q|\nabla g| \overset{(5)}{\lesssim} \sum_{Q\in W_\varphi}\ell(Q)\cdot\widetilde{\omega}_T(50Q) \overset{(1)}{\lesssim} 1.$$

On the other hand, using $\operatorname{supp}\nabla\psi_\varepsilon\subset U_{3\varepsilon}(\partial\widetilde{\Omega})$ and Lemma III.6.26(3) respectively, we have

$$\frac{1}{\varepsilon}\int_{\operatorname{supp}\nabla\psi_\varepsilon} g \le \frac{1}{\varepsilon}\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} g \lesssim \varepsilon \le 1,$$

and Lemma III.6.18 follows. □

We now turn to the proof of Claim III.6.27.

Proof of Claim III.6.27. First we study $T_{\frac{1}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|^2))}(\psi)$. We have

$$\begin{aligned}T_{\frac{1}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|^2))}(\psi) &= T_{\partial^i(\mathcal{O}(|\nabla g|^2))}\left(\frac{1}{|S\nabla g|^2}\psi\right)\\ &\overset{(III.6.8)}{=} -\int \mathcal{O}(|\nabla g|^2)\frac{1}{|S\nabla g|^2}\psi\operatorname{div}R_i^T - \int \mathcal{O}(|\nabla g|^2)\partial^i\left(\frac{1}{|S\nabla g|^2}\psi\right).\end{aligned}$$

The first term in the right-hand side is of the form $\int\mathcal{O}(1)\psi$. Let us study the second term in the right-hand side. By (III.6.19),

$$\partial^i\left(\frac{1}{|S\nabla g|^2}\psi\right) = -\frac{2\langle\partial^i\nabla_R g, B\nabla_R g\rangle}{|S\nabla g|^4}\psi - \frac{\langle\nabla_R g, \partial^i B\nabla_R g\rangle}{|S\nabla g|^4}\psi + \frac{\partial^i\psi}{|S\nabla g|^2},$$

and hence we get

$$-\int \mathcal{O}(|\nabla g|^2)\partial^i\left(\frac{1}{|S\nabla g|^2}\psi\right) = \int \frac{\mathcal{O}(\partial^i\nabla_R g)}{|S\nabla g|}\psi + \int \mathcal{O}(1)\psi + \int \mathcal{O}(1)\partial^i\psi$$

$$\overset{\text{(III.6.3)}}{=} \int \frac{\mathcal{O}(|\nabla^2 g|)}{|\nabla g|}\psi + \int \mathcal{O}(1)\psi + \int \mathcal{O}(1)\partial^i\psi.$$

Next we study $T_{\frac{b_j\partial^j g}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|))}(\psi)$, similarly as we did with the previous term. Now we have

$$T_{\frac{b_j\partial^j g}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|))}(\psi) = T_{\partial^i(\mathcal{O}(|\nabla g|))}\left(\frac{b_j\partial^j g}{|S\nabla g|^2}\psi\right)$$

$$\overset{\text{(III.6.8)}}{=} -\int \frac{\mathcal{O}(|\nabla g|)b_j\partial^j g}{|S\nabla g|^2}\psi \operatorname{div} R_i - \int \mathcal{O}(|\nabla g|)\partial^i\left(\frac{b_j\partial^j g}{|S\nabla g|^2}\psi\right).$$

As before, the first term in the right-hand side is of the form $\int \mathcal{O}(1)\psi$ and the second term is

$$-\int \mathcal{O}(|\nabla g|)\partial^i\left(\frac{b_j\partial^j g}{|S\nabla g|^2}\psi\right) = \int \frac{\mathcal{O}(1)\psi}{|S\nabla g|}\partial^i\left(b_j\partial^j g\right) + \int \mathcal{O}(|\nabla g|^2)\partial^i\left(\frac{1}{|S\nabla g|^2}\right)\psi + \int \mathcal{O}(1)\partial^i\psi.$$

The claim follows by using (III.6.3) and (III.6.19) in the previous line. $\square$

CHAPTER IV

L^p-solvability of boundary value problems for the Laplacian in locally flat unbounded domains

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an ADR domain (see Section IV.1.2), and denote its surface measure by

$$\sigma = \sigma_\Omega \coloneqq \mathcal{H}^n|_{\partial\Omega}.$$

For $1 < p < \infty$, we say that the Dirichlet problem is solvable in L^p (denoted as (D_p) is solvable) in Ω if there is $C_{D_p} \geq 1$ such that for any given $f \in L^p(\sigma)$, there exists $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}u \in L^p(\sigma), \\ u|_{\partial\Omega}^{\mathrm{nt}} = f, \ \sigma\text{-a.e.}, \end{cases} \tag{IV.0.1}$$

and

$$\|\mathcal{N}u\|_{L^p(\sigma)} \leq C_{D_p} \|f\|_{L^p(\sigma)}. \tag{IV.0.2}$$

Here $\mathcal{N}$ is the nontangential operator (see (IV.1.2)) and $u|_{\partial\Omega}^{\mathrm{nt}}$ is the nontangential limit (see (IV.1.4)). We say that the Neumann problem is solvable in L^p (denoted as (N_p) is solvable) in Ω if there is $C_{N_p} \geq 1$ such that for any given $f \in L^p(\sigma)$, there exists $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\sigma), \\ \partial_{\nu_\Omega} u = f, \ \sigma\text{-a.e.}, \end{cases} \tag{IV.0.3}$$

and

$$\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \leq C_{N_p} \|f\|_{L^p(\sigma)}. \tag{IV.0.4}$$

Here ∂_{ν_Ω} is the interior nontangential derivative (see (IV.1.5)).

We note that the definition of solvability for the Dirichlet and Neumann problems may vary slightly across the literature. We will not distinguish between these variations in the subsequent articles mentioned in the introduction. Furthermore, while several of the results below were originally established for more general divergence-form operators in their respective papers, we restrict our historical overview to the harmonic case.

In ADR domains, it is well-known that (D_p) is solvable for some $1 < p < \infty$ if and only if the harmonic measure ω is locally in weak-$A_\infty(\sigma)$. Specifically, there exist constants $C, s > 0$ such that for every ball $B = B(x, r)$ with $x \in \partial\Omega$ and $r < \operatorname{diam}(\partial\Omega)/4$, there holds $\omega^p(E) \leq C(\sigma(E)/\sigma(B))^s \omega^p(2B)$ for any $p \in \Omega \setminus 4B$ and any Borel set $E \subset B$. We refer the reader to [HL18, Hof19]. For ADR domains satisfying the corkscrew condition (see Definition IV.1.2), (D_p) is solvable if and only if the harmonic measure satisfies a weak reverse Hölder inequality with exponent $p' \coloneqq p/(p-1)$ (the Hölder conjugate of p), see [MPT23, Proposition 2.20] for instance. Consequently, if (D_p) is solvable for some $1 < p < \infty$, then Gehring's lemma guarantees the existence of $\varepsilon > 0$ such that (D_q) is solvable for all $p - \varepsilon \leq q < \infty$.

The solvability of the Dirichlet and Neumann problems is a long-standing and active area of research. In 1963, Lavrent'ev [Lav63] showed that in bounded planar simply connected chord-arc domains (see Definition IV.1.7), the harmonic measure is locally in weak-$A_\infty(\sigma)$, implying the solvability of (D_p) for some $1 < p < \infty$. However, Jerison [Jer83] later proved in 1983 that for every $1 < p < \infty$, there exists a planar chord-arc domain where (D_p) fails to be solvable. A significant breakthrough came in 1977 with Dahlberg [Dah77], who proved that in any bounded Lipschitz domain Ω, its harmonic measure satisfies a reverse Hölder inequality with exponent 2. Consequently, there exists $\varepsilon_\Omega >$ such that (D_p) is solvable for all $2 - \varepsilon_\Omega \leq p < \infty$. This result is sharp: for every $\varepsilon > 0$ there is a Lipschitz domain Ω_ε where $(D_{2-\varepsilon})$ is not solvable, see [Ken86, pp. 153-54]. In 1978, Fabes, Jodeit and Rivière [FJR78] employed double and single layer potentials (see Sections IV.2 and IV.6.1) to establish the solvability of both (D_p) and (N_p) for all $1 < p < \infty$ in bounded C^1 domains. Notably, Dahlberg [Dah79] had already proven the solvability of (D_p) for all $1 < p < \infty$ in bounded C^1 domains without using layer potentials, though this was published later in 1979.[(IV.1)] Subsequent work by Jerison and Kenig [JK80, JK81a] simplified the proofs of Dahlberg's results through the application of the so-called Rellich identity [Rel40]. Furthermore, in [JK81b], they resolved (N_2) in bounded Lipschitz domains.

Briefly speaking, the layer potential approach relies on the invertibility of the operators $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ (here K is the double layer potential and K^* is its adjoint) and gives an "explicit" solution to the Dirichlet and Neumann problems, respectively. In $C^{1,\alpha}$ domains, $\alpha > 0$, it is not difficult to see that the double layer potential is compact in $L^p(\sigma)$ and (by Fredholm theory) that those operators are invertible, for all $1 < p < \infty$. For a proof of this, see the lecture notes [DK85, Ken86]. While this argument does not directly extend to C^1 domains, Fabes, Jodeit and Rivière [FJR78] succeeded to show that in bounded C^1 domains, the double layer potential K is compact in $L^p(\sigma)$, and both $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ are invertible in $L^p(\sigma)$ for all $1 < p < \infty$. For Lipschitz domains, however, the compactness of the double layer potential generally fails, as shown by Fabes, Jodeit and Lewis in [FJL77]. Nevertheless, using the Rellich identity mentioned above, Verchota [Ver84] established in 1984 the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ in $L^2(\sigma)$, thereby recovering the solvability of (D_2) and (N_2) for bounded Lipschitz domains, as originally shown in [Dah77, JK81b] respectively.

In 1987, Dahlberg and Kenig [DK87] established that for every bounded Lipschitz domain Ω, there exists $\varepsilon_\Omega > 0$ such that (N_p) is solvable for all $1 < p \leq 2 + \varepsilon_\Omega$, and in fact, they showed that its solution, as well as the solution of $(D_{p'})$ in Dahlberg's result [Dah77], can be obtained using the method of layer potentials. As in the Dirichlet problem, this range of solvability is sharp: for every $\varepsilon > 0$, there exists a Lipschitz domain Ω_ε for which $(N_{2+\varepsilon})$ fails to be solvable, see [Ken86, pp. 153-54].

Dahlberg's result [Dah77] was extended to chord-arc domains (see Definition IV.1.7) independently by David and Jerison [DJ90], and Semmes [Sem90] in 1990. They proved that for any bounded chord-arc domain, there exists $1 < p < \infty$ such that the harmonic measure satisfies a reverse Hölder inequality with exponent p. A complete geometric characterization came in 2020 when Azzam, Hofmann, Martell, Mourgoglou, and Tolsa [AHM$^+$20] identified the class of ADR domains with interior corkscrews where (D_p) is solvable for some $1 < p < \infty$.

[(IV.1)] Fabes, Jodeit and Rivière cite a technical report of [Dah79] in [FJR78].

These are precisely domains having interior big pieces of chord-arc domains (IBPCAD), as defined in [AHM+20, Definition 2.12].

Let us roughly introduce the regularity boundary value problem for an ADR domain Ω. When well-defined, we say that the regularity problem is solvable in L^p (denoted as (R_p) is solvable) in Ω if for any given $f \in C^{0,1}(\partial\Omega)$, there is a harmonic function $u : \Omega \to \mathbb{R}$ such that $u|_{\partial\Omega}^{\text{nt}} = f$ holds σ-a.e. on $\partial\Omega$ and $\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \lesssim \|\nabla f\|_{L^p(\sigma)}$, where the implicit constant does not depend on f. The exact definition may vary slightly across literature; see [HS25, Definition 5.35] and [MPT23, Definition 1.4] for technical formulations.

We observe that the regularity problem is closely connected to the Dirichlet problem: for bounded domains with the corkscrew condition and having UR boundary (see Definition IV.1.1), Mourgoglou, Poggi and Tolsa proved in [MPT23, Theorem 1.33] that $(R_p) \iff (D_{p'})$ for all $p \in (1, \infty)$, see also [MT24a, Theorems 1.2 and 1.6]. One might expect similar results for 2-sided chord-arc domains with unbounded boundaries. However, to the best of our knowledge, such results have not been established in the literature. It is quite likely that the arguments in [MPT23, MT24a] could be extended to domains with unbounded boundaries, but this extension has not yet been documented.

The question of whether (N_p) or (R_p) is solvable for some $1 < p < \infty$ in chord-arc domains was first posed by Kenig [Ken94, Problem 3.2.2] in 1994 and later reintroduced by Toro [Tor10, Question 2.5] at the ICM 2010. While the regularity part of this question was recently resolved in 2023 by Mourgoglou, Poggi, and Tolsa in [MPT23, Corollary 1.36], the Neumann part remains an open problem.

Although the solvability of the Neumann problem remains open for chord-arc domains, significant progress has been made in understanding its extrapolation properties under the assumption that $(D_{p'})$ or/and (R_p) is solvable for some $1 < p < \infty$. As noted earlier, in bounded chord-arc domains (which have UR boundary by Theorem IV.1.8), the equivalence $(R_p) \iff (D_{p'})$ holds for any $1 < p < \infty$. In 1993, Kenig and Pipher [KP93, Theorem 6.3] showed the extrapolation $(D_{p'}) + (N_p) \implies (N_q)$ for all $1 < q < p + \varepsilon$, for some $\varepsilon > 0$, for bounded Lipschitz domains. In 2024, Feneuil and Li [FL24, Corollary 1.22] extended this result to bounded chord-arc domains. In the same year, Mourgoglou and Tolsa showed in [MT24b, Theorem 1.1] that (N_p) is solvable for a fixed $p \in (1, 2)$, whenever Ω is a bounded chord-arc domain such that (R_q) is solvable for some $q > p$, $\partial\Omega$ supports a weak p-Poincaré inequality, and Ω has very big pieces of chord-arc superdomains for which (N_q) is solvable, Most recently in 2025, Hofmann and Sparrius proved in [HS25, Theorem 5.54] that $(N_p) + (R_p) + (D_{p'}) \implies (N_q)$ for all $1 < q < p$ in 2-sided chord-arc domains with unbounded boundary.

Given the strong connection to the Neumann problem, we briefly address the extrapolation properties of the regularity problem. The equivalence $(R_p) \iff (D_{p'})$ mentioned above, combined with the well-known extrapolation of solvability for the Dirichlet problem, immediately yields extrapolation of solvability of the regularity problem. A more subtle endpoint case was resolved in 2025 by Gallegos, Mourgoglou, and Tolsa, who proved solvability extrapolation for (R_1) (even for $(R_{1-\varepsilon})$) in ADR domains satisfying the interior corkscrew condition, see [GMT25, Theorems 1.3 and 1.6].

Let us now return to our main discussion of the solvability of Dirichlet and Neumann problems. In 2010, Hofmann, M. Mitrea and Taylor studied the solvability of the Dirichlet and Neumann problems (among other) in bounded δ-regular SKT (Semmes-Kenig-Toro) domains. Roughly speaking, a domain is δ-regular SKT if it is Reifenberg flat (with small enough constant) and its geometric measure theoretic outward unit normal vector ν (see Remark IV.1.3) satisfies $\mathrm{dist}_{\mathrm{BMO}(\sigma)}(\nu, \mathrm{VMO}(\sigma)) \leq \delta$. The precise definition of δ-regular SKT domains can be found in [HMT10, Definition 4.9]. In [HMT10, Section 5], the authors proved that for any $1 < p < \infty$, both (D_p) and $(N_{p'})$ are solvable in δ-regular SKT domains when δ is small enough. Consequently, (D_p) and $(N_{p'})$ are solvable for all $1 < p < \infty$ in regular SKT domains, that is, domains that are δ-regular SKT for all $\delta > 0$, see [HMT10, Definition 4.8].

The approach in [HMT10] employs layer potentials. For any $1 < p < \infty$ and a bounded regular SKT domain, the authors show in [HMT10, Theorem 4.36] that the double layer potential K is compact. More precisely, for bounded δ-regular SKT domains with sufficiently small $\delta = \delta(p) > 0$, [HMT10, Theorem 4.36] establishes that K is close enough in L^p norm to the set of compact operators (see Definition IV.1.13). This property nevertheless allows the application of Fredholm theory to prove the invertibility of both $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ from the injectivity of $\frac{1}{2}Id + K^*$ in $L^2(\sigma)$, see [HMT10, Proposition 5.11].

Recently in 2022, Marín, Martell, D. Mitrea, I. Mitrea and M. Mitrea [MMM$^+$22a, Chapter 6] showed the L^p solvability of the Dirichlet and Neumann problems (among others) for 2-sided chord-arc domains with unbounded boundary, under the BMO smallness condition $\|\nu\|_* < \delta$ where $\delta > 0$ is sufficiently small depending on $1 < p < \infty$. They proved that the L^p norm of K tends to zero as $\delta \to 0$. This enabled them to see that both $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ are invertible via Neumann series, thereby solving the Dirichlet and Neumann problems for any $1 < p < \infty$. More specifically, the smallness of $\delta > 0$ depends on the dimension, the chord-arc parameters of the domain and p. We emphasize that the results in [MMM$^+$22a] hold for weighted L^p spaces and systems of divergence form operators with constant coefficient matrices.

Throughout this work, we will work in $\mathbb{R}^{n+1}$ with $n \geq 2$, although similar results may hold in the planar case $n = 1$. We focus on domains defined as follows:

Definition IV.0.1 (δ-$(s, S; R)$ domain)**.** A domain $\Omega \subset \mathbb{R}^{n+1}$ is called δ-$(s, S; R)$ domain if it is a 2-sided chord-arc domain (see Definition IV.1.7) with unbounded boundary and there exist $\delta > 0$, scales $S > s > 0$ and a radius $R > 0$ such that the outward unit normal vector ν (see Remark IV.1.3) satisfies

$$\fint_{B(x,r)} |\nu(z) - m_{B(x,r)}\nu| \, d\sigma(z) \leq \delta, \text{ provided } \begin{cases} x \in \partial\Omega \setminus B_R(0) \text{ and } r \in (0, \infty), \text{ or} \\ x \in B_R(0) \cap \partial\Omega \text{ and } r \notin (s, S), \end{cases}$$

(here $m_{B(x,r)}\nu = \frac{1}{\sigma(B(x,r))} \int_{B(x,r)} \nu \, d\sigma$) and for all $x \in \partial\Omega$ and $r \geq S$, there holds

$$\beta_{\infty,\partial\Omega}(B(x, r)) := \inf_{n\text{-plane } L \ni x} \sup_{y \in \partial\Omega \cap B(x,r)} \frac{\mathrm{dist}(y, L)}{r} \leq \delta,$$

where the infimum is taken over all n-planes $L \subset \mathbb{R}^{n+1}$ through x.

That is, these are domains whose failure of sufficient flatness is limited to finitely many dyadic boundary balls. We note that the domains studied in [MMM$^+$22a] satisfy the first condition in Definition IV.0.1 for every $x \in \partial\Omega$ and every scale $r \in (0, \infty)$, implying $\beta_{\infty,\partial\Omega}(B(x, r)) \lesssim \delta^{\frac{1}{2n}}$ for all $x \in \partial\Omega$ and all $r \in (0, \infty)$, see [MMM$^+$22a, Theorem 2.2].

In this chapter, we study the L^p-solvability of the Dirichlet and Neumann problems in δ-$(s, S; R)$-domains, and we also provide uniqueness results for (IV.0.1) and (IV.0.3) respectively. In the subsequent results, $\mathcal{D}$ denotes the interior double layer potential (see (IV.2.1)), K the (boundary) double layer potential (see (IV.2.2)), K^* the adjoint of K, and $\mathcal{S}_{\rm mod}$ the modified interior single layer potential (see (IV.6.1)).

Theorem IV.0.2 (L^p-Dirichlet problem). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s, S; R)$ domain. There is $\varepsilon_D = \varepsilon_D(n, CAD) \in (0, \frac{1}{n-1})$ such that for every $p_0 \in (\frac{n}{n-1} - \varepsilon_D, \frac{2n}{n-1}]$ there exists $\delta_0 = \delta_0(n, p_0, CAD) > 0$ such that if $\delta \leq \delta_0$, then $\frac{1}{2}Id + K$ is invertible in $L^{p_0}(\sigma)$ and given $f \in L^{p_0}(\sigma)$, the function*

$$u := \mathcal{D}\left(\left(\frac{1}{2}Id + K\right)^{-1} f\right), \tag{IV.0.5}$$

is the solution of (D_{p_0}). Furthermore, there exists $\varepsilon > 0$ such that (D_p) is solvable for all $p \in (p_0 - \varepsilon, \infty)$, and the solution of (D_p) is the unique solution of the Dirichlet problem (IV.0.1).

Theorem IV.0.3 (L^p-Neumann problem). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s, S; R)$ domain. There is $\varepsilon_N = \varepsilon_N(n, CAD) > 0$ such that for every $p \in [\frac{2n}{n+1}, n+\varepsilon_N)$ there exists $\delta_0 = \delta_0(n, p, CAD) > 0$ such that if $\delta \leq \delta_0$, then $-\frac{1}{2}Id+K^*$ is invertible in $L^p(\sigma)$, (N_p) is solvable and given $f \in L^p(\sigma)$, the function*

$$u := \mathcal{S}_{\rm mod}\left(\left(-\frac{1}{2}Id + K^*\right)^{-1} f\right), \tag{IV.0.6}$$

is the unique (modulo constants) solution of (N_p). Furthermore, it is the unique (modulo constants) solution of the Neumann problem (IV.0.3).

Theorems IV.0.2 and IV.0.3 mainly follow from the invertibility of $\frac{1}{2}Id+K$ and $-\frac{1}{2}Id+K^*$, respectively. We emphasize that the existence of $\varepsilon_D > 0$ in Theorem IV.0.2 and $\varepsilon_N > 0$ in Theorem IV.0.3 guarantees the solvability of both (D_2) and (N_2).

Let us briefly address the lack of extrapolation in Theorem IV.0.3. While for 2-sided chord-arc domains with unbounded boundary there holds $(N_p) + (R_p) + (D_{p'}) \implies (N_q)$ for all $1 < q < p$, our results for δ-$(s, S; R)$ domains in Theorems IV.0.2 and IV.0.3 establish $(D_{p'})$ and (N_p) solvability for a fixed $p \in [2n/(n+1), n+\varepsilon)$ (δ is sufficiently enough), but not the solvability of (R_p). Since we have not yet established the solvability of (R_p), we cannot consequently derive that (N_q) is solvable for all $1 < q \leq p$.

Let us outline the proof strategy for Theorems IV.0.2 and IV.0.3. Unlike the approach in [MMM+22a] discussed above, our setting with δ-$(s, S; R)$-domains presents a key difference: the double layer potential K does not a priori have small norm, and thus we cannot directly derive the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ via Neumann series. As in [HMT10, Theorem 4.36], in the following result (one of the main points in this chapter), we show that K and its adjoint K^* are close to the set of compact operators (see Definition IV.1.13). This enables us to employ Fredholm theory to characterize their invertibility.

Theorem IV.0.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s, S; R)$ domain, $1 < p < \infty$, and let $p' = p/(p-1)$ be its Hölder conjugate exponent. For all $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon, p, CAD, n) > 0$ such that if $\delta \leq \delta_0$, then there is a compact operator $T : L^p(\sigma) \to L^p(\sigma)$ such that $\|K - T\|_{L^p(\sigma)} = \|K^* - T^*\|_{L^{p'}(\sigma)} < \varepsilon$.*

Here, $T^* : L^{p'}(\sigma) \to L^{p'}(\sigma)$ is the adjoint of T, which is compact by Schauder's theorem. To prove this theorem we truncate the double layer potential K at small, intermediate ("close" and “far” from $B_R(0)$) and large scales, see (IV.2.10). The operator on the “close” intermediate scales turns out to be compact (Lemma IV.2.2). The operators at small and “far” intermediate scales have small norm (Lemmas IV.2.3 and IV.2.4), by using the already known behavior of the double layer potential at these scales (Theorem IV.3.4) via Semmes' decomposition Theorem IV.3.3. One of the main difficulties in this chapter is establishing the small norm of the operator at large scales (Theorem IV.2.5), which we address in Section IV.4.

Combining Theorem IV.0.4 with the Fredholm alternative Theorem IV.1.14, in Corollary IV.2.6, for $\lambda \in \mathbb{R} \setminus \{0\}$ we establish several equivalent conditions in order to see that $\lambda Id + K$ (respectively $\lambda Id + K^*$) is invertible in $L^p(\sigma)$ (respectively $L^{p'}(\sigma)$). Furthermore, the invertibility of $\lambda Id + K$ in $L^p(\sigma)$ and $\lambda Id + K^*$ in $L^{p'}(\sigma)$ is shown to be equivalent to the injectivity of $\lambda Id + K^*$ in $L^{p'}(\sigma)$.

As previously noted, the solvability of the Dirichlet and Neumann problems mainly relies on the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$. The equivalences discussed in the previous paragraph allow us to reduce this invertibility problem to showing the injectivity of $\pm\frac{1}{2}Id+K^*$ in $L^p(\sigma)$ for some range of p. Using the modified single layer potential (see Section IV.6.1) and Moser estimates for harmonic functions with vanishing Dirichlet or Neumann boundary conditions, in Section IV.6 we prove that $\pm\frac{1}{2}Id + K^*$ are indeed injective in $L^p(\sigma)$ when $p \in [2n/(n+1), n+\varepsilon)$, for some $\varepsilon > 0$.

In Section IV.8, we establish the L^p solvability of the Dirichlet and Neumann problems in δ-$(s, S; R)$ domains in Theorems IV.0.2 and IV.0.3. This mainly follows from the invertibility explained above, and also from some already-known properties of the single and double layer potentials. The uniqueness results for (IV.0.1) and (IV.0.3) are proved in Sections IV.5 and IV.7.

A natural question is whether Theorems IV.0.2 to IV.0.4 generalize to second-order divergence form operators $\operatorname{div}(A\nabla\cdot)$ for uniformly elliptic matrices A with Dini mean oscillation coefficients, in the spirit of recent work [MMPT23] by Molero, Mourgoglou, Puliatti, and Tolsa.

IV.1. Preliminaries and definitions

IV.1.1. Notation. We retain the notation from p. 9 throughout this chapter. In this chapter, the ambient space is $\mathbb{R}^{n+1}$ for $n \geq 2$, and we introduce the following additional notation:

- Given a domain Ω, we denote the boundary ball centered at $x \in \partial\Omega$ with $r > 0$ by $\Delta(x,r) = \Delta_r(x) := B(x,r) \cap \partial\Omega$.
- Given a boundary ball Δ, we denote by r_Δ or $r(\Delta)$ its radius, and by c_Δ or $c(\Delta)$ its center.
- Given a boundary ball Δ and $t > 1$, $t\Delta = \Delta(c_\Delta, tr_\Delta)$.
- Given a domain $\Omega \subset \mathbb{R}^{n+1}$, for each $x \in \partial\Omega$ we define the nontangentially approach cone with aperture $\alpha > 0$ as

$$\Gamma(x) = \Gamma_\alpha^\Omega(x) := \{y \in \Omega : |y - x| < (1+\alpha)\operatorname{dist}(y, \partial\Omega)\}. \tag{IV.1.1}$$

For a function $u : \Omega \to \mathbb{R}$ we define the nontangentially maximal function

$$\mathcal{N}u(x) = \mathcal{N}_\alpha u(x) := \sup_{y \in \Gamma_\alpha^\Omega(x)} |u(y)|, \quad x \in \partial\Omega, \tag{IV.1.2}$$

and the δ-nontangential maximal function $\mathcal{N}^\delta u$ of u by

$$\mathcal{N}^\delta u(x) = \mathcal{N}^\delta_\alpha u(x) := \sup_{y \in \Gamma^\Omega_\alpha(x) \cap \overline{B_{2\delta}(x)}} |u(y)|, \quad x \in \partial\Omega. \tag{IV.1.3}$$

For a fixed $\alpha > 0$, we introduce the following definitions whenever well-defined. The nontangential limit is

$$u|_{\partial\Omega}(x) = u|^{\mathrm{nt}}_{\partial\Omega}(x) := \lim_{\Gamma^\Omega(x) \ni z \to x} u(z), \quad x \in \partial\Omega. \tag{IV.1.4}$$

If in addition the outward unit normal ν_Ω exists (see Remark IV.1.3) and $u \in C^1(\Omega)$, the interior normal nontangential derivative is

$$\partial^{\mathrm{int}}_{\nu_\Omega} u(x) := \lim_{\Gamma^\Omega(x) \ni z \to x} \langle \nu_\Omega(x), \nabla u(z) \rangle, \quad x \in \partial\Omega. \tag{IV.1.5}$$

For shortness, we will also write ∂_{ν_Ω} or ∂_ν instead $\partial^{\mathrm{int}}_{\nu_\Omega}$. If $u \in C^1(\overline{\Omega}^c)$, the exterior normal nontangential derivative is

$$\partial^{\mathrm{ext}}_{\nu_\Omega} u(x) := \lim_{\Gamma^{\overline{\Omega}^c}(x) \ni z \to x} \langle \nu_\Omega(x), \nabla u(z) \rangle, \quad x \in \partial\Omega. \tag{IV.1.6}$$

- Given an $\mathcal{H}^n$-measurable set $E \subset \mathbb{R}^{n+1}$ and $1 < q < \infty$, we define the uncentered Hardy-Littlewood maximal operator $\mathcal{M}_{q,E}$, for $f \in L^q_{\mathrm{loc}}(\mathcal{H}^n|_E)$, as

$$\mathcal{M}_{q,E} f(x) := \sup_{\substack{r>0 \\ y \in E \\ B(y,r) \ni x}} \left(\fint_{B(y,r)} |f(z)|^q \, d\mathcal{H}^n|_E(z) \right)^{\frac{1}{q}}, \quad x \in E, \tag{IV.1.7}$$

  It is well known that it is bounded from $L^p(\mathcal{H}^n|_E)$ to $L^p(\mathcal{H}^n|_E)$ if $q < p \leq \infty$, with norm $C_{p,q,n} > 0$. If the set E is clear from the context we write $\mathcal{M}_q = \mathcal{M}_{q,E}$. We will use the uncentered Hardy-Littlewood maximal operator with $E = \partial\Omega$, where Ω is an ADR domain (see Section IV.1.2).
- Let $E \subset \mathbb{R}^{n+1}$ be an ADR closed set (see Section IV.1.2) and let $\mu := \mathcal{H}^n|_E$. Given $f \in L^2_{\mathrm{loc}}(\mu)$, $x \in E$, $R > 0$, we set

$$\|f\|_*(B(x,R)) := \sup_{B \subset B(x,R)} \left(\fint_B |f(z) - m_B f|^2 \, d\mu(z) \right)^{1/2},$$

  where the supremum is taken over all balls B centered at E included in $B(x,R)$, and $m_B f := \fint_B f \, d\mu$.
- Given an ADR domain Ω satisfying the 2-sided corkscrew condition (see Section IV.1.2) with outward unit normal ν_Ω (see Remark IV.1.3), the tangential gradient of a Lipschitz function f in $\partial\Omega$ is

$$\nabla_t f(y) := \nabla \widetilde{f}(y) - \langle \nabla \widetilde{f}(y), \nu_\Omega(y) \rangle \nu_\Omega(y) \text{ for } \sigma\text{-a.e. } y \in \partial\Omega,$$

  where $\widetilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}$ is any Lipschitz extension of f to $\mathbb{R}^{n+1}$.

IV.1.2. ADR, UR, Uniform, NTA and CAD. We say that a Radon measure μ in $\mathbb{R}^{n+1}$ is (n-dimensional) Ahlfors-David regular if there exists $C \geq 1$ (called the ADR constant) such that

$$C^{-1}r^n \leq \mu(B(x,r)) \leq Cr^n \text{ for all } x \in \operatorname{supp}\mu \text{ and } 0 < r < \operatorname{diam}(\operatorname{supp}\mu), \tag{IV.1.8}$$

where $\operatorname{diam}(\operatorname{supp}\mu)$ may be infinite. A closed set $E \subset \mathbb{R}^{n+1}$ is said to be Ahlfors-David regular if $\mathcal{H}^n|_E$ is Ahlfors-David regular. A domain $\Omega \subset \mathbb{R}^{n+1}$ is said to be an Ahlfors-David regular domain if $\partial\Omega$ is Ahlfors-David regular.

Notation. From now on, the term Ahlfors-David regular may be shortened to AD regular or ADR. Moreover, given an ADR domain Ω we will denote its surface measure by

$$\sigma = \sigma_\Omega := \mathcal{H}^n|_{\partial\Omega}.$$

Definition IV.1.1 (UR set)**.** A set $E \subset \mathbb{R}^{n+1}$ is called (n-dimensional) uniformly rectifiable, UR for short, if it is ADR and there exist $\varepsilon, M \in (0,\infty)$ (called the UR constants of E) such that for every $x \in E$ and $r \in (0, \operatorname{diam} E)$, there is a Lipschitz map $\varphi = \varphi_{x,r} : \{y \in \mathbb{R}^n : |y| < r\} \to \mathbb{R}^{n+1}$ with Lipschitz constant $\leq M$, such that

$$\mathcal{H}^n(E \cap B(x,r) \cap \varphi(\{y \in \mathbb{R}^n : |y| < r\})) \geq \varepsilon r^n.$$

This is a quantitative version of rectifiability introduced by David and Semmes in [DS91, DS93]. It is well known that any UR set is rectifiable, for a detailed proof see [HMT10, p. 2629].

Definition IV.1.2 (Corkscrew ball conditions)**.** We say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies

- the (interior) corkscrew condition if there is a constant $M > 1$ such that for every $x \in \partial\Omega$ and $r \in (0, \operatorname{diam}(\partial\Omega))$ there exists a point $A_r(x) \in \Omega$ such that $B(A_r(x), M^{-1}r) \subset B(x,r) \cap \Omega$. $A_r(x)$ is called the corkscrew point of the point x at radius r.
- the exterior corkscrew condition if $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the corkscrew condition.
- the 2-sided corkscrew condition if it satisfies the interior and exterior corkscrew condition.

Remark IV.1.3 (The geometric measure theoretic outward unit normal vector ν)**.** If $\Omega \subset \mathbb{R}^{n+1}$ is an ADR domain satisfying the 2-sided corkscrew condition, then for σ-a.e. $x \in \partial\Omega$ there exists a unique unit vector $\nu(x)$ (called the geometric measure theoretic outward unit vector) satisfying for $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{R}^{n+1} \setminus \Omega$, both

$$\lim_{r\to 0} \frac{m(\Omega^\pm \cap \{y \in B(x,r) : \pm\langle \nu(x), y - x\rangle \geq 0\})}{r^{n+1}} = 0.$$

We remark that the vector ν exists under more general conditions, see for instance [HMT10, Section 2.2] and the references therein.

Definition IV.1.4 (Harnack chain condition)**.** We say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the Harnack chain condition if there is a constant $M > 1$ such that for every $\varepsilon > 0$ and $x_1, x_2 \in \Omega$ with $\operatorname{dist}(x_i, \partial\Omega) \geq \varepsilon$ $(i = 1, 2)$ and $|x_1 - x_2| \leq 2^j\varepsilon$ for some integer $j \geq 1$, there exists a chain of open balls $\{B_k\}_{1\leq k\leq N}$ inside Ω with $N \leq Mj$ satisfying that $x_1 \in B_1$, $x_2 \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ (for $1 \leq k \leq N-1$) and $M^{-1}r(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq Mr(B_k)$ (for $1 \leq k \leq N$).

Definition IV.1.5 (Uniform domain)**.** A domain $\Omega \subset \mathbb{R}^{n+1}$ is called uniform domain if it satisfies the Harnack chain and interior corkscrew conditions.

Definition IV.1.6 (NTA domain)**.** A domain $\Omega \subset \mathbb{R}^{n+1}$ is called nontangentially accessible (NTA for short) domain if it is a uniform domain and it satisfies the exterior corkscrew condition.

Definition IV.1.7 (CAD)**.** A domain $\Omega \subset \mathbb{R}^{n+1}$ is called chord-arc domain (1-sided CAD or CAD for shortness) if it is an NTA domain and $\partial\Omega$ is ADR. We say that Ω is a 2-sided chord arc domain (2-sided CAD) if Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ are CAD.

Notation. Given a (2-sided) CAD Ω, we will write $C = C(\text{CAD})$ if the constant C depends on the CAD constants of Ω.

The following is from [DJ90, Sem90], see also [HMT10, Corollary 3.9].

Theorem IV.1.8. *If $\Omega \subset \mathbb{R}^{n+1}$ is a domain satisfying the 2-sided corkscrew condition and whose boundary is ADR, then $\partial\Omega$ is UR.*

Remark IV.1.9. Most of the results in [HMT10] are presented for 2-sided local John domains with ADR boundary. However, it was shown in [TT24] that this apparently weaker condition is, in fact, equivalent to being 2-sided CAD. This allows us to apply the known results in the literature for 2-sided local John domains with ADR boundary when working with 2-sided CAD. For instance, the Semmes decomposition in [HMT10, Theorem 4.16], restated in Theorem IV.3.3 below.

IV.1.3. The nontangential maximal operator and boundary dyadic cubes in ADR domains. For ADR domains $\Omega \subset \mathbb{R}^{n+1}$, the L^p norm (with $1 < p < \infty$) of the nontangential maximal function $\mathcal{N}$ in (IV.1.2) does "not" depend on the aperture in the sense that, for every $\alpha, \beta > 0$ and any $u : \Omega \to \mathbb{R}$ there holds

$$\|\mathcal{N}_\alpha u\|_{L^p(\sigma)} \approx_{\alpha,\beta} \|\mathcal{N}_\beta u\|_{L^p(\sigma)}, \tag{IV.1.9}$$

see [HMT10, Proposition 2.2]. For this fact, we will omit the aperture $\alpha > 0$ in $\mathcal{N}_\alpha$ and Γ_α from now on. We may fix $\alpha = 1$ for instance.

The following two lemmas provide the control of interior integrals by the nontangential maximal function. The first is for solid interior integrals (see [HMT10, (2.3.25) in Proposition 2.12]) and the second[IV.2] is for interior sets with n-growth (see [MT24c, Lemma 5.1]).

Lemma IV.1.10. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain, and fix $\alpha > 0$. Then there exists $C = C(n, \alpha, ADR) > 0$ such that for any measurable function $u : \Omega \to \mathbb{R}$ there holds*

$$\frac{1}{\delta}\int_{U_\delta(\partial\Omega)} |u(z)|\, dm(z) \le C\|\mathcal{N}_\alpha^\delta u\|_{L^1(\sigma)},\ 0 < \delta \le \operatorname{diam}(\Omega).$$

Lemma IV.1.11. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain, B_0 a ball centered at $\partial\Omega$, and $E \subset B_0 \cap \Omega$ such that*

$$\mathcal{H}^n(B(x,r) \cap E) \le C_0 r^n \text{ for all } x \in E \text{ and } r > 0.$$

Then, for any Borel function $u : \Omega \to \mathbb{R}$ such that $u \in L^1_{\text{loc}}(\mathcal{H}^n|_E)$,

$$\int_E |u(z)|\, d\mathcal{H}^n(z) \lesssim \int_{2B_0} \mathcal{N}_\beta^{2r(B_0)} u(z)\, d\sigma(z),$$

[IV.2] This statement and its proof is written in [MT24c, Lemma 5.1], the first version of its more general version (under more general assumptions) in [MT24b, Lemma 5.1].

assuming the aperture $\beta > 0$ to be large enough (depending only on n). The implicit constant above depends only on n, C_0, and the ADR constants of $\partial\Omega$.

For the construction of the Lipschitz graph in Section IV.4.5 we will follow [DS91, Section 8]. So, given an ADR domain Ω with surface measure σ of $\partial\Omega$, we consider dyadic lattice $\mathcal{D}_\sigma$ of "cubes" built by David and Semmes, see [DS93, Chapter 3 of Part I] with codimension 1.

Lemma IV.1.12 (Boundary dyadic cubes)**.** *Given an ADR domain Ω with surface measure σ, for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_{\sigma,j}$ of Borel subsets of* $\operatorname{supp}\sigma = \partial\Omega$*, called the dyadic cubes of the j-th generation, with the following properties:*

(1) each $\mathcal{D}_{\sigma,j}$ is a partition of $\partial\Omega$, i.e., $\partial\Omega = \bigcup_{Q\in\mathcal{D}_{\sigma,j}} Q$ with $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{D}_{\sigma,j}$ with $Q \neq Q'$,
(2) if $Q \in \mathcal{D}_{\sigma,i}$ and $Q' \in \mathcal{D}_{\sigma,j}$ for some $i \leq j$, then either $Q \cap Q' = \emptyset$ or $Q \subset Q'$,
(3) for all $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}_{\sigma,j}$, we have that $2^j \leq \operatorname{diam} Q \leq C_{\mathcal{D}} 2^j$ and $C^{-1}2^{jn} \leq \sigma(Q) \leq C2^{jn}$, and
(4) for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_{\sigma,j}$ and $0 < \tau < 1$, we have the so-called "thin boundary condition":

$$\sigma(\{x \in Q : \operatorname{dist}(x, \partial\Omega \setminus Q) \leq \tau 2^j\}) + \sigma(\{x \in \partial\Omega \setminus Q : \operatorname{dist}(x, Q) \leq \tau 2^j\}) \leq C_{\mathcal{D}} \tau^{1/C_{\mathcal{D}}} 2^{jn}.$$

We set $\mathcal{D}_\sigma = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_{\sigma,j}$. The constants $C_{\mathcal{D}}, C \geq 1$ in (3) and (4) above do not depend on j, Q or τ.

IV.1.4. Compact operators. Let us briefly recall the definition of compact operators and the Fredholm alternative.

Definition IV.1.13. Given Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, a bounded linear operator $T : X \to Y$ is called compact if for every bounded sequence $\{x_k\}_{k\geq 1} \subset X$, the sequence $\{Tx_k\}_{k\geq 1} \subset Y$ has a convergent subsequence.

Theorem IV.1.14 (Fredholm alternative)**.** *Let $(X, \|\cdot\|_X)$ be a Banach space, $T : X \to X$ be a compact operator, and $\lambda \in \mathbb{C} \setminus \{0\}$. Then exactly one of the following holds:*

- *the equation $Tv - \lambda v = 0$ has a non-zero solution $v \in X$, or*
- *for every $u \in X$, the equation $Tv - \lambda v = u$ has a unique solution $v \in X$. In this case, the solution v depends continuously on u.*

Remark IV.1.15. Since the composition $T \circ B$ of a compact operator T with a bounded operator B is again compact, the same holds when replacing λv by $\mathcal{I}v$ for an invertible bounded operator $\mathcal{I} : X \to X$.

IV.1.5. Calderón-Zygmund operators and the Riesz transform. We say that $k : \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : x \neq y\} \to \mathbb{C}$ is a Calderón-Zygmund kernel if there exist constants $C \geq 1$ and $0 < \tau \leq 1$ such that for all $x, x', y \in \mathbb{R}^{n+1}$ with $x \neq y$, $x' \neq y$, there holds

$$|k(x,y)| \leq C\frac{1}{|x-y|^n}, \text{ and}$$
$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq C\frac{|x-x'|^\tau}{|x-y|^{n+\tau}}, \text{ if } |x-x'| \leq |x-y|/2.$$

Given a Radon measure μ and a Calderón-Zygmund kernel k, we define

$$T^k\mu(x) := \int k(x,y)\, d\mu(y), \quad x \in \mathbb{R}^{n+1} \setminus \operatorname{supp}\mu,$$

and as this may not converge for $x \in \operatorname{supp}\mu$, for $\varepsilon > 0$ we define the truncated operator

$$T^k_\varepsilon \mu(x) \coloneqq \int_{|y-x|>\varepsilon} k(x,y)\, d\mu(y), \quad x \in \mathbb{R}^{n+1}.$$

Given a Radon measure μ and $f \in L^1_{\mathrm{loc}}(\mu)$, we define

$$T^k_\mu f(x) \coloneqq T^k(f\mu)(x), \text{ for } x \in \mathbb{R}^{n+1} \setminus \operatorname{supp}\mu,$$
$$T^k_{\mu,\varepsilon} f(x) \coloneqq T^k_\varepsilon(f\mu)(x), \text{ for } \varepsilon > 0 \text{ and } x \in \mathbb{R}^{n+1},$$

and the maximal operator

$$\text{(IV.1.10)} \qquad T^k_{\mu,*} f(x) \coloneqq \sup_{\varepsilon>0} |T^k_{\mu,\varepsilon} f(x)|, \quad x \in \operatorname{supp}\mu.$$

We say that T^k_μ is bounded in $L^p(\mu)$, $1 < p < \infty$, if the trucanted operators $T^k_{\mu,\varepsilon}$ are bounded in $L^p(\mu)$ uniformly on $\varepsilon > 0$. In this case, we write $T^k_\mu : L^p(\mu) \to L^p(\mu)$ is bounded. We remark that, if μ has growth of degree n (i.e., μ satisfies the upper bound in (IV.1.8)), then the boundedness of T^k_μ in $L^p(\mu)$ is equivalent to the boundedness of the maximal operator $T^k_{\mu,*}$ in $L^p(\mu)$, by [Tol14, Theorem 2.16](IV.3) and Cotlar's inequality (take $s = 1$ in [Tol14, (2.26)] for instance).

Definition IV.1.16 (Riesz transform)**.** The (n-dimensional) Riesz kernel is the Calderón-Zygmund vector-valued kernel (with $\tau = 1$)

$$\widetilde{k}_{\mathcal{R}}(x) \coloneqq \frac{x}{|x|^{n+1}} \text{ for } x \in \mathbb{R}^{n+1} \setminus \{0\}.$$

The (n-dimensional) Riesz transform $\mathcal{R}$ is defined as

$$\mathcal{R} \coloneqq T^k, \text{ with } k(x,y) = \widetilde{k}_{\mathcal{R}}(x-y) \text{ for } x \neq y.$$

By [Dav88, Proposition 4 bis], if $\Omega \subset \mathbb{R}^{n+1}$ is an ADR domain with the 2-sided corkscrew condition (in particular $\partial\Omega$ is UR by Theorem IV.1.8), then

$$\|\mathcal{R}_{\sigma,*} f\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR}} \|f\|_{L^p(\sigma)} \text{ for all } f \in L^p(\sigma),$$

see also [HMT10, Proposition 3.18] for instance.

IV.2. The double layer potential

Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition (in particular $\partial\Omega$ is UR by Theorem IV.1.8), let $\nu \coloneqq \nu_\Omega$ be the geometric measure theoretic outward unit vector of Ω (see Remark IV.1.3), and let $f \in L^1\left(\frac{d\sigma(x)}{1+|x|^n}\right)$. The interior double layer potential operator associated with Ω is

$$\text{(IV.2.1)} \qquad \mathcal{D}f(x) = \mathcal{D}_\Omega f(x) \coloneqq \frac{1}{w_n} \int_{\partial\Omega} \frac{\langle \nu(y), y-x \rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \mathbb{R}^{n+1} \setminus \partial\Omega.$$

Here w_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$. The double layer potential satisfies $\Delta(\mathcal{D}f) = 0$ in $\mathbb{R}^{n+1} \setminus \partial\Omega$.

Remark IV.2.1. Note that $L^p(\sigma) \subset L^1\left(\frac{d\sigma(x)}{1+|x|^n}\right)$ for all $p > 1$, as $\partial\Omega$ is ADR.

(IV.3)A quick inspection of its proof reveals that the same holds if $L^2(\mu)$ is replaced by $L^p(\mu)$ for any $1 < p < \infty$.

The boundary double layer potential, that is, the principal value version of the interior double layer potential, is defined as

$$Kf(x) = K_\Omega f(x) := \lim_{\varepsilon \to 0^+} K_\varepsilon f(x), \quad x \in \partial\Omega, \tag{IV.2.2}$$

where

$$K_\varepsilon f(x) := \frac{1}{w_n} \int_{\{y \in \partial\Omega : |y-x| > \varepsilon\}} \frac{\langle \nu(y), y - x \rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{IV.2.3}$$

Also, the maximal operator of the boundary double layer potential is defined as

$$K_* f(x) := \sup_{\varepsilon > 0} |K_\varepsilon f(x)|, \quad x \in \partial\Omega.$$

We also define

$$K_\varepsilon^* f(x) := \frac{1}{w_n} \int_{\{y \in \partial\Omega : |y-x| > \varepsilon\}} \frac{\langle \nu(x), x - y \rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

and the maximal operator

$$K_*^* f(x) := \sup_{\varepsilon > 0} |K_\varepsilon^* f(x)|, \quad x \in \partial\Omega.$$

A quick computation shows that the operator K^* defined as

$$K^* f(x) := \lim_{\varepsilon \to 0^+} K_\varepsilon^* f(x), \quad x \in \partial\Omega, \tag{IV.2.4}$$

is the adjoint operator of K. That is, K^* satisfies $\int Kf(y) g(y)\, d\sigma(y) = \int f(y) K^* g(y)\, d\sigma(y)$ for every $f \in L^p(\sigma)$ and every $g \in L^{p'}(\sigma)$, where $1 < p < \infty$ and p' is the conjugate exponent p.

The interior double layer potential satisfies the jump relation

$$(\mathcal{D}f)|_{\partial\Omega}^{\mathrm{nt}}(x) = \left(\frac{1}{2} Id + K\right) f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{IV.2.5}$$

see [MMM+22a, (3.31)]. If in addition $f \in L^p(\sigma)$ with $1 < p < \infty$, then the boundary and interior double layer potentials satisfy

$$\|K_* f\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR}} \|f\|_{L^p(\sigma)}, \tag{IV.2.6a}$$

$$\|\mathcal{N}(\mathcal{D}f)\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR}} \|f\|_{L^p(\sigma)}, \tag{IV.2.6b}$$

see [HMT10, (3.3.5) and (3.3.6)] respectively. Here UR denotes that the constant depends on the UR constants of $\partial\Omega$. The second estimate also depends on the aperture of $\mathcal{N}$.

IV.2.1. Proof of Theorem IV.0.4 and consequences. In order to study the boundary double layer potential K in Theorem IV.0.4, for $0 < t < T < \infty$ we define its truncations by

$$K_s f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{w_n} \int_{\{y \in \partial\Omega : \varepsilon < |y-x| \le t\}} \frac{\langle \nu(y), y - x \rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{IV.2.7}$$

$$K_i f(x) := \frac{1}{w_n} \int_{\{y \in \partial\Omega : t < |y-x| \le T\}} \frac{\langle \nu(y), y - x \rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{IV.2.8}$$

$$K_l f(x) := \frac{1}{w_n} \int_{\{y \in \partial\Omega : |y-x| > T\}} \frac{\langle \nu(y), y - x \rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{IV.2.9}$$

where s, m and l stand for small, intermediate and large scales respectively. If we want to stress the scales we will write $K_{s(t)}$, $K_{i(t,T)}$ and $K_{l(T)}$ respectively. We define $K^*_{s(t)}$, $K^*_{i(t,T)}$ and $K^*_{l(T)}$ analogously from the definition of K^*.

For $\widetilde{R} > 0$ we also break the intermediate scales operator as $K_i = \mathbf{1}_{B_{\widetilde{R}}(0)} K_i + \mathbf{1}_{B_{\widetilde{R}}(0)^c} K_i$. So, for $f \in L^p(\sigma)$ and $x \in \partial\Omega$ we will decompose the double layer potential as

$$Kf(x) = K_s f(x) + K_l f(x) + \mathbf{1}_{B_{\widetilde{R}}(0)}(x) K_i f(x) + \mathbf{1}_{B_{\widetilde{R}}(0)^c}(x) K_i f(x). \tag{IV.2.10}$$

Next, we present a series of results to summarize the relevant properties of the operators in the decomposition (IV.2.10).

Lemma IV.2.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition. For any $0 < t < T < \infty$ and $\widetilde{R} > 0$, the operator $\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)} : L^p(\sigma) \to L^p(\sigma)$ is compact for all $p \in (1, \infty)$.*

We write the proof in Section IV.2.2. The compact operator in Theorem IV.0.4 will be in fact $T = \mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$, for some choice of parameters.

The following two results control the L^p norm of the boundary double layer potential on small scales and scales far from the "bad" balls in Definition IV.0.1 respectively.

Lemma IV.2.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD. Assume also that there is $\delta > 0$ and $s > 0$ such that*

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r \le s}} \fint_{B(x,r)} |\nu - m_{B(x,r)}\nu| \, d\sigma \le \delta.$$

Given $p \in (1, \infty)$, there exists $t_0 = t_0(s, \delta, p, CAD, n) > 0$ such that for every $0 < t \le t_0$ there holds

$$\|K_{s(t)}\|_{L^p(\sigma)} \lesssim \delta^{1/4},$$

where the involved constant depends on n, the CAD constants of Ω and p.

Lemma IV.2.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s, S; R)$ domain (see Definition IV.0.1). Given $p \in (1, \infty)$ and $t > 0$, there exists $\widetilde{R} = \widetilde{R}(R, \delta, t, p, CAD, n)$ such that there holds*

$$\|\mathbf{1}_{B_{\widetilde{R}}(0)^c} K_{s(t)}\|_{L^p(\sigma)} \lesssim \delta^{1/4},$$

where the involved constant depends on n, the CAD constants of Ω and p.

We prove the preceding two lemmas in Section IV.3. Note that, to obtain the claimed bound, the first lemma gives a sufficiently small parameter $t > 0$, while the second one provides the truncation parameter $\widetilde{R}$ given a scale parameter $t > 0$, which is not assumed to be small in this case.

The next result is the main work in this chapter, and provides the small norm of the large scales double layer potential maximal operator, defined for $f \in L^p(\sigma)$ as

$$K_{l,*} f(x) \coloneqq \sup_{\varepsilon \ge T} |K_\varepsilon f(x)|, \quad x \in \partial\Omega,$$

assuming flatness conditions on large scales. The proof is deferred to Section IV.4.

Theorem IV.2.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD with unbounded boundary. Assume also that there are $\delta_\beta, \delta_* > 0$ and $S > 0$ such that the following two conditions hold:*

- *Small Jones' $\beta_{\infty,\partial\Omega}$ coefficient on large scales: for $x \in \partial\Omega$ and $r \ge S$,*

$$\beta_{\infty,\partial\Omega}(B(x,r)) \le \delta_\beta. \tag{IV.2.11}$$

• *Small* BMO *norm of* ν *on large scales: for* $x \in \partial\Omega$ *and* $r \geq S$,

$$\fint_{B(x,r)} |\nu(z) - m_{B(x,r)}\nu| \, d\sigma(z) \leq \delta_*. \tag{IV.2.12}$$

Denote $\delta = \max\{\delta_\beta, \delta_*\}$. *Given* $1 < p < \infty$, *there exists* $\theta = \theta(n,p) > 0$ *(see (IV.4.4)) and* $T = T(p, \delta, S, CAD, n) \gg 100S$ *such that*

$$\|K_{l(T),*}\|_{L^p(\sigma)} \lesssim \delta^\theta, \tag{IV.2.13}$$

where the involved constant depends on n, *the CAD constants of* Ω *and* p.

Despite the truncated operators appearing in (IV.2.10) and their properties in Lemmas IV.2.2 to IV.2.4 and Theorem IV.2.5 have not yet been studied and proved, we now turn to the proof of Theorem IV.0.4.

Proof of Theorem IV.0.4. Given $\varepsilon > 0$, by Lemmas IV.2.3 and IV.2.4 and Theorem IV.2.5 there exists $\delta_0 = \delta_0(\varepsilon, p, \mathrm{CAD}, n)$ and $0 < t \ll 1 \ll T \ll \widetilde{R}$ such that if $\delta \leq \delta_0$, then for all $f \in L^p(\sigma)$ there holds

$$\|Kf - \mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)} f\|_{L^p(\sigma)} \overset{\text{(IV.2.10)}}{=} \|K_{s(t)} f + K_{l(T)} f + \mathbf{1}_{B_{\widetilde{R}}(0)^c} K_{i(t,T)} f\|_{L^p(\sigma)} < \varepsilon,$$

where we used that $|K_{i(t,T)} f| \leq |K_{s(t)} f| + |K_{s(T)} f|$ σ-a.e. on $\partial\Omega$. By Lemma IV.2.2, the operator $T = \mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$ is compact (with abuse of notation using T for both the compact operator and the scale). Finally, since T is compact, its adjoint T^* is also compact by Schauder's theorem and moreover

$$\|K^* - T^*\|_{L^{p'}(\sigma)} = \|K - T\|_{L^p(\sigma)} < \varepsilon,$$

as claimed. $\square$

As a consequence of Theorem IV.0.4, in the following result we obtain that injectivity implies invertibility for $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$, under the assumption of enough flatness.

Corollary IV.2.6. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a* δ-$(s, S; R)$ *domain (see Definition IV.0.1),* $\lambda_0 > 0$, $1 < p < \infty$ *and* $p' = p/(p-1)$ *its Hölder conjugate exponent. There exists* $\delta_0 = \delta_0(\lambda_0, p, CAD, n)$ *such that if* $\delta \leq \delta_0$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| \geq \lambda_0$, *then the following are equivalent:*

(1) $\lambda Id + K$ *is invertible in* $L^p(\sigma)$,
(2) $\lambda Id + K$ *is injective in* $L^p(\sigma)$,
(3) $\lambda Id + K$ *is surjective in* $L^p(\sigma)$,
(4) $\lambda Id + K^*$ *is invertible in* $L^{p'}(\sigma)$,
(5) $\lambda Id + K^*$ *is injective in* $L^{p'}(\sigma)$,
(6) $\lambda Id + K^*$ *is surjective in* $L^{p'}(\sigma)$.

Proof. By Theorem IV.0.4 there is $\delta_0 = \delta_0(\lambda_0, p, \mathrm{CAD}, n)$ such that if $\delta \leq \delta_0$ then there is a compact operator T such that $\|K - T\|_{L^p(\sigma)} = \|K^* - T^*\|_{L^{p'}(\sigma)} < \lambda_0 \leq |\lambda|$.

By Neumann series, the operator $\mathcal{I} \coloneqq \lambda Id + K - T$ is invertible. By the Fredholm alternative Theorem IV.1.14 and Remark IV.1.15, if either $\lambda Id + K$ is injective or surjective in $L^p(\sigma)$ we get that it is bijective, and by the bounded inverse theorem we conclude that $(\lambda Id + K)^{-1} : L^p(\sigma) \to L^p(\sigma)$ is a linear bounded operator. This concludes the equivalence between items (1), (2) and (3).

The same argument holds mutatis mutandis with $\lambda Id + K^*$ in $L^{p'}(\sigma)$, whence we get the equivalences between (4), (5) and (6).

Now, since (any arbitrary) an operator U is injective if its adjoint U^* is surjective, in particular (6) implies (2), and (3) implies (5). $\square$

Remark IV.2.7. Recall that "U is injective implies that U^* is surjective" is not true in general, therefore the six conditions above are not equivalent in general.

IV.2.2. The compact operator. Proof of Lemma IV.2.2. We conclude this section by seeing that $\mathbf{1}_{B_{\widetilde{R}}(0)}K_i$ is compact.

Proof of Lemma IV.2.2. First note that the kernels of $\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)})^*$ are

$$\mathbf{1}_{B_{\widetilde{R}}(0)}(x)\mathbf{1}_{\{y\in\partial\Omega:t<|y-x|\leq T\}}(y)\frac{\langle\nu(y),y-x\rangle}{|x-y|^{n+1}},\text{ and}$$
$$\mathbf{1}_{\{y\in B_{\widetilde{R}}(0)\cap\partial\Omega:t<|y-x|\leq T\}}(y)\frac{\langle\nu(x),x-y\rangle}{|x-y|^{n+1}},$$

respectively. In particular, both satisfy the so-called Hilbert-Schmidt condition, see the line before [Fol95, Theorem 0.45] for instance. Hence, by [Fol95, Theorem 0.45] we have that both $\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)})^*$ are bounded and compact from $L^2(\sigma)$ to $L^2(\sigma)$. This concludes the proof for the case $p=2$.

For $p\in(1,2)$, fix any $p_0\in(1,p)$. Since the boundary is UR, we have that both $\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)})^*$ are bounded from $L^{p_0}(\sigma)$ to $L^{p_0}(\sigma)$. By the interpolation theorem in [Kra60] (see also [KZPS76, Theorem 3.10]) between compact operators and bounded operators, we conclude that both $\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)}K_{i(t,T)})^*$ are compact from $L^p(\sigma)$ to $L^p(\sigma)$. This gives the case $p\in(1,2)$.

The case $p\in(2,\infty)$ follows from the result obtained in the previous paragraph and Schauder's theorem, which states that a bounded linear operator between Banach spaces is compact if and only if its adjoint is compact. This concludes the proof of the lemma. $\square$

IV.3. The double layer potential in small scales. Proof of Lemmas IV.2.3 and IV.2.4

In this section we study the domain and the double layer potential for a fixed scale. We conclude the section by proving Lemmas IV.2.3 and IV.2.4.

Theorem IV.3.1 ([TT24, Theorem 1.3])**.** *Let $\Omega\subset\mathbb{R}^{n+1}$ be a 2-sided CAD. Then the following weak 1-Poincaré inequality for Lipschitz functions on $\partial\Omega$ holds: there exist constants*[(IV.4)] *$C_P\geq 1$ and $\Lambda\geq 1$ such that for every Lipschitz function f on $\partial\Omega$, every $x\in\partial\Omega$ and every $r>0$, for $\Delta\coloneqq\Delta(x,r)$ we have*

$$\fint_\Delta |f(z)-m_\Delta f|\,d\sigma(z)\leq C_P r\fint_{\Lambda\Delta}|\nabla_t f(z)|\,d\sigma(z),$$

where ∇_t is the tangential gradient of f.

This is a no-tail version of [HMT10, Proposition 4.13] for Lipschitz functions, which is enough for our applications.

By the same proof in [HMT10, Theorem 4.14], but using the refined Poincaré inequality above, we obtain the following estimate for the theoretical unit normal vector ν. For the sake of completeness, we provide the proof below.

(IV.4)The notation C_P is to specify the constant appearing in the Poincaré inequality.

Theorem IV.3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD. Then there exist $C'_P = C(C_P) > 0$ such that for every $\alpha \in (0,1)$, every $x \in \partial\Omega$ and every $r > 0$, for $\Delta := \Delta(x,r)$ there holds*

$$\sup_{y\in 2\Delta} r^{-1}|\langle x-y, m_\Delta \nu\rangle| \leq C'_P \left(\fint_{\Lambda\Delta} |\nu(z) - m_\Delta \nu|^{\frac{n}{1-\alpha}}\, d\sigma(z)\right)^{\frac{1-\alpha}{n}},$$

with $\Lambda \geq 1$ as in the weak 1-Poincaré inequality in Theorem IV.3.1.

Note that the term on the right-hand side can be controlled by

$$\left(\fint_{\Lambda\Delta} |\nu(z) - m_\Delta \nu|^{\frac{n}{1-\alpha}}\, d\sigma(z)\right)^{\frac{1-\alpha}{n}} \lesssim_\Lambda \left(\fint_{\Lambda\Delta} |\nu(z) - m_{\Lambda\Delta} \nu|^{\frac{n}{1-\alpha}}\, d\sigma(z)\right)^{\frac{1-\alpha}{n}} + \fint_{\Lambda\Delta} |\nu(z) - m_{\Lambda\Delta}\nu|\, d\sigma(z).$$

From this we conclude two things. First, by the John-Nirenberg inequality, we have

$$\sup_{y\in\Delta(x,2r)} r^{-1}|\langle x-y, m_{\Delta(x,r)}\nu\rangle| \lesssim_{\alpha,\Lambda,C_P} \|\nu\|_*(B(x,2\Lambda r)), \text{ for any } r>0, \tag{IV.3.1}$$

and second, under the assumption (IV.2.12), if $\Lambda r \geq S$ then

$$\sup_{y\in\Delta(x,2r)} r^{-1}|\langle x-y, m_{\Delta(x,r)}\nu\rangle| \lesssim_{\Lambda,C_P} \delta_*^{\frac{1-\alpha}{n}}. \tag{IV.3.2}$$

Proof of Theorem IV.3.2. Define the Lipschitz function $g_x(z) := \langle x - z, m_\Delta\nu\rangle$ for $z \in \mathbb{R}^{n+1}$. As in [HMT10, (4.2.25)], the claim follows from the particular case $y' = x$ of

$$|g_x(y) - g_x(y')| \leq C'_P r^{1-\alpha}|y-y'|^\alpha \left(\fint_{\Lambda\Delta} |\nu(z) - m_\Delta \nu|^{\frac{n}{1-\alpha}}\, d\sigma(z)\right)^{\frac{1-\alpha}{n}}, \tag{IV.3.3}$$

for all $y, y' \in 2\Delta$ and each $\alpha \in (0,1)$.

Let us see this. Fixed $\alpha \in (0,1)$, let $p = n/(1-\alpha) > 1$, equivalently $\alpha = 1 - n/p$. For σ-a.e. $z \in \partial\Omega$ we have

$$|\nabla_t g_x(z)| = |m_\Delta\nu - \langle m_\Delta\nu, \nu(z)\rangle\nu(z)| = |m_\Delta\nu - \nu(z) - \langle m_\Delta\nu - \nu(z), \nu(z)\rangle\nu(z)| \leq 2|\nu(z) - m_\Delta\nu|.$$

Now, for any arbitrary boundary ball $\Delta_s \subset \Delta$ (centered at $\partial\Omega$) of radius s, by the 1-Poincaré inequality in Theorem IV.3.1 we have

$$\begin{aligned}\frac{1}{s}\fint_{\Delta_s} |g_x(z) - m_{\Delta_s} g_x|\, d\sigma(z) &\leq C_P \fint_{\Lambda\Delta_s} |\nabla_t g_x(z)|\, d\sigma(z) \leq 2C_P \fint_{\Lambda\Delta_s} |\nu(z) - m_\Delta\nu|\, d\sigma(z)\\ &\leq 2C_P \left(\fint_{\Lambda\Delta_s} |\nu(z) - m_\Delta\nu|^p\, d\sigma(z)\right)^{1/p} \lesssim C_P \frac{r^{\frac{n}{p}}}{s^{\frac{n}{p}}} \left(\fint_{\Lambda\Delta} |\nu(z) - m_\Delta\nu|^p\, d\sigma(z)\right)^{1/p}.\end{aligned}$$

By the choice of p in terms of α, for all $\Delta_s \subset \Delta$ we get

$$\frac{1}{s^\alpha}\fint_{\Delta_s} |g_x(z) - m_{\Delta_s} g_x|\, d\sigma(z) \leq C'_P r^{1-\alpha} \left(\fint_{\Lambda\Delta} |\nu(z) - m_\Delta \nu|^{\frac{n}{1-\alpha}}\, d\sigma(z)\right)^{\frac{1-\alpha}{n}}.$$

This implies the Hölder regularity in (IV.3.3) by Meyer's criterion in [Mey64]. $\square$

For completeness, we state the Semmes decomposition, as in [HMT10, Theorem 4.16].

Theorem IV.3.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD. Then there exist $C_* \geq 1$ and C_1, C_2, C_3, $C_4 > 0$ (depending on the CAD constants of Ω and n) with the property that if for every compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$, there exists $R_{\mathcal{K}} > 0$ for which*

$$\sup_{x \in \mathcal{K} \cap \partial\Omega} \|\nu\|_* \left(\Delta\left(x, R_{\mathcal{K}}\right)\right) \leq \delta \leq 1/C_*,$$

then for every compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ and $T \geq 1$, setting

$$\widetilde{\mathcal{K}}_T \coloneqq \{y \in \mathbb{R}^{n+1} : \operatorname{dist}(y, \mathcal{K}) \leq T\},$$

and

$$R_{*,T,\mathcal{K}} = \min\{\delta R_{\widetilde{\mathcal{K}}_T}/C_*, \operatorname{diam}(\partial\Omega)/C_*, T\}, \tag{IV.3.4}$$

for $x \in \mathcal{K} \cap \partial\Omega$ and $0 < r \leq R_{*,T,\mathcal{K}}$ the following holds:

(1) There exists a unit vector $\vec{n}_{x,r}$ and a Lipschitz function

$$h : H(x,r) \coloneqq \langle \vec{n}_{x,r}\rangle^{\perp} \to \mathbb{R} \text{ with } \|\nabla h\|_{L^\infty} \leq C_3\sqrt{\delta},$$

and whose graph

$$\mathcal{G} \coloneqq \{y = x + \zeta + t\vec{n}_{x,r} : \zeta \in H(x,r), t = h(\zeta)\}$$

(in the coordinate system $y = (\zeta, t) \iff y = x + \zeta + t\vec{n}_{x,r}$, $\zeta \in H(x,r)$, $t \in \mathbb{R}$) is a good approximation of $\partial\Omega$ in the cylinder

$$\mathcal{C}(x,r) \coloneqq \{x + \zeta + t\vec{n}_{x,r} : \zeta \in H(x,r), |\zeta| \leq r, |t| \leq r\}$$

in the sense

$$\sigma\left(\mathcal{C}(x,r) \cap ((\partial\Omega \setminus \mathcal{G}) \cup (\mathcal{G} \setminus \partial\Omega))\right) \leq C_1 w_n r^n \exp\left(-C_2/\sqrt{\delta}\right).$$

(2) There exist two disjoint sets $G(x,r)$ ("good") and $E(x,r)$ ("evil") such that

$$\begin{aligned}
&\mathcal{C}(x,r) \cap \partial\Omega = G(x,r) \cup E(x,r) \text{ with } G(x,r) \subset \mathcal{G},\\
&\sigma(E(x,r)) \leq C_1 w_n r^n \exp\left(-C_2/\sqrt{\delta}\right),
\end{aligned}$$

and moreover, if $\Pi : \mathbb{R}^{n+1} \to H(x,r)$ is defined by $\Pi(y) = \zeta$ if $y = x + \zeta + t\vec{n}_{x,r} \in \mathbb{R}^{n+1}$ with $\zeta \in H(x,r)$ and $t \in \mathbb{R}$, then

$$|y - (x + \Pi(y) + h(\Pi(y))\vec{n}_{x,r})| \leq C_3\sqrt{\delta}\operatorname{dist}(\Pi(y), \Pi(G(x,r))) \text{ for all } y \in E(x,r),$$

and

$$\begin{aligned}
&\mathcal{C}(x,r) \cap \partial\Omega \subset \left\{x + \zeta + t\vec{n}_{x,r} : |t| \leq C_3\sqrt{\delta}r, \zeta \in H(x,r)\right\}\\
&\Pi(\mathcal{C}(x,r) \cap \partial\Omega) = \{\zeta \in H(x,r) : |\zeta| < r\}.
\end{aligned}$$

(3) $\left(1 - C_4\sqrt{\delta}\right) w_n r^n \leq \sigma(\Delta(x,r)) \leq \left(1 + C_4\sqrt{\delta}\right) w_n r^n.$

A few comments are in order. The case $T \geq 1$ follows from the case $T = 1$ by scaling. For the case $T = 1$, a careful inspection of [HMT10, Proof of Theorem 4.16] reveals the following:

(1) The constants $C_* \geq 1$ and $C_1, C_2, C_3, C_4 > 0$ in [HMT10, Theorem 4.16] do not depend on the compact set $\mathcal{K}$.

(2) In [HMT10, p. 2703, l. 6] the authors define

$$R_* = \min\{\delta R_{\widetilde{\mathcal{K}}_1}/(8C), R_{\widetilde{\mathcal{K}}_1}/8, R_0/100, 1\},$$

where R_0 is the constant used in the statement of [HMT10, Theorem 4.14] and $C > 0$ is the geometrical constant appearing in [HMT10, (4.2.20)], i.e., $C = C(\text{CAD})$. It turns out that $R_0 \approx_{\text{CAD}} \operatorname{diam}(\partial\Omega)$ since Ω is a 2-sided CAD, by Definition IV.1.7 and the fact that being a 2-sided local John domain (see [HMT10, Definition 3.12] with $R = \operatorname{diam}(\partial\Omega)$) with ADR boundary is equivalent to being 2-sided CAD, see [TT24, Theorem 1.2 and Corollary 1.5].

All in all, reusing the same notation for the constants, there exists constant $C_* \geq 1$ such that the conclusions of the theorem hold with the choice of $R_{*,T,\mathcal{K}}$ in (IV.3.4).

Given a 2-sided CAD $\Omega \subset \mathbb{R}^{n+1}$, for a boundary ball $\Delta_0 := \Delta(\xi, r_0)$, i.e., $\xi \in \partial\Omega$ and $r_0 > 0$, let $k_0 \in \mathbb{Z}$ the minimal index satisfying $r_0 \leq 2^{k_0}$ and we define

$$I_0(\Delta_0) := \bigcup_{Q \in \mathcal{Q}_0(\Delta_0)} Q, \ \text{ where } \mathcal{Q}_0(\Delta_0) := \{Q \in \mathcal{D}_{\sigma,k_0} : Q \cap 2\Delta_0 \neq \emptyset\}, \tag{IV.3.5}$$

recall $\mathcal{D}_{\sigma,k_0}$ is the family of dyadic cubes of $\partial\Omega$ in Lemma IV.1.12.

Here we state the localized L^p norm of the double layer potential. This is proved in [HMT10, Theorem 4.36], though it is not explicitly stated as a separate theorem. For completeness, at the end of this section in p. 136 we provide the detailed proof following the argument in [HMT10, Theorem 4.36], omitting computations that follow analogously to those in the original argument.

Theorem IV.3.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD, $p \in (1, \infty)$, $\xi_0 \in \partial\Omega$ and $r_0 > 0$. There exists $C_0 = C_0(ADR, p, n) \geq 1$ such that if the conclusions in the Semmes decomposition Theorem IV.3.3 are valid with $\delta > 0$ for all $x \in I_0(\Delta(\xi_0, r_0))$ and all $0 < r \leq C_0 r_0$, and*

$$\sup_{x \in I_0(\Delta(\xi_0, r_0))} \|\nu\|_*(\Delta(x, C_0 r_0)) \leq \delta, \tag{IV.3.6}$$

then

$$\int_{I_0(\Delta(\xi_0,r_0))} |K_* f|^p \, d\sigma \lesssim \delta^{p/4} \int_{\Delta(\xi_0,r_0)} |f|^p \, d\sigma \ \text{for any } f \in L^p(\Delta(\xi_0, r_0)), \tag{IV.3.7}$$

where the involved constant depends on the CAD constants of Ω, p and n.

Remark IV.3.5. When invoking Theorem IV.3.4, we may suppose that $C_0 = C_0(\text{ADR}, p, n)$ is sufficiently large so that $I_0(\Delta(x, r)) \subset \Delta(x, C_0 r)$ for all $x \in \partial\Omega$ and all $r > 0$.

Using Theorem IV.3.4, we are now ready to prove Lemmas IV.2.3 and IV.2.4.

Proof of Lemma IV.2.3. By the John-Nirenberg inequality, let $C_1 \geq 1$ be the constant satisfying

$$\sup_{x \in \partial\Omega} \|\nu\|_*(\Delta(x, s/2)) \leq C_1 \delta.$$

Let C_* and C_0 be the constants in Theorems IV.3.3 and IV.3.4 respectively. We can assume $\delta \leq 1/(C_1 C_*)$, since otherwise, for any $t > 0$, we have

$$\|K_{s(t)}\|_{L^p(\sigma)} \leq 2\|K_*\|_{L^p(\sigma)} \overset{\text{(IV.2.6a)}}{\lesssim} 1 \lesssim \delta^{1/4}.$$

We fix

$$t_0 = \frac{1}{2C_0} \min\{s/2, \delta s/(2C_*), \operatorname{diam}(\partial\Omega)/C_*, 1\},$$

(recall $R_{*,1,\cdot}$ from (IV.3.4)) so that for all $0 < t \le t_0$ we have

$$\sup_{x\in\partial\Omega} \|\nu\|_*(B(x, 2C_0 t)) \le C_1\delta,$$

and the conclusions in the Semmes decomposition Theorem IV.3.3 hold (with $C_1\delta$) for all $x \in \partial\Omega$ and all $0 < r \le 2C_0 t$. For a fixed $0 < t \le t_0$ and any $x \in \partial\Omega$, applying Theorem IV.3.4 we obtain

(IV.3.8)

$$\|\mathbf{1}_{I_0(\Delta(x,2t))} K_{s(t)}(f\mathbf{1}_{\Delta(x,2t)})\|_{L^p(\sigma)} \le 2\|\mathbf{1}_{I_0(\Delta(x,2t))} K_*(f\mathbf{1}_{\Delta(x,2t)})\|_{L^p(\sigma)} \lesssim \delta^{1/4}\|f\mathbf{1}_{\Delta(x,2t)}\|_{L^p(\sigma)},$$

with $I_0(\cdot)$ as in (IV.3.5).

By the $5R$-covering theorem, let $\{\Delta_i\}_{i\in\mathbb{N}}$ be a subfamily of $\{\Delta(x,t)\}_{x\in\partial\Omega}$ such that $\partial\Omega \subset \bigcup_{i\in\mathbb{N}} \Delta_i$ and $\{\Delta_i/5\}_{i\in\mathbb{N}}$ is pairwise disjoint. Since $\{\Delta_i/5\}_{i\in\mathbb{N}}$ is pairwise disjoint and all balls have the same radius, we have that the family $\{2\Delta_i\}_{i\in\mathbb{N}}$ has finite overlapping, with constant depending only on the dimension. For any $f \in L^p(\sigma)$ we have

$$\|K_{s(t)}f\|^p_{L^p(\sigma)} \le \sum_{i\in\mathbb{N}} \|\mathbf{1}_{\Delta_i} K_{s(t)} f\|^p_{L^p(\sigma)} = \sum_{i\in\mathbb{N}} \|\mathbf{1}_{\Delta_i} K_{s(t)}(f\mathbf{1}_{2\Delta_i})\|^p_{L^p(\sigma)}.$$

For each $i \in \mathbb{N}$, let $I_i := I_0(2\Delta_i) \supset 4\Delta_i$. Applying (IV.3.8), we obtain

$$\|\mathbf{1}_{\Delta_i} K_{s(t)}(f\mathbf{1}_{2\Delta_i})\|^p_{L^p(\sigma)} \le \|\mathbf{1}_{I_i} K_{s(t)}(f\mathbf{1}_{2\Delta_i})\|^p_{L^p(\sigma)} \lesssim \delta^{p/4}\|f\mathbf{1}_{2\Delta_i}\|^p_{L^p(\sigma)}.$$

By the finite overlapping of the family $\{2\Delta_i\}_{i\in\mathbb{N}}$, we conclude

$$\|K_{s(t)}f\|_{L^p(\sigma)} \lesssim \delta^{1/4}\|f\|_{L^p(\sigma)}$$

as claimed. □

The proof of Lemma IV.2.4 follows a similar approach to the proof of Lemma IV.2.3. However, in the previous proof, we chose the small truncation parameter to ensure that Theorem IV.3.4 is satisfied. In contrast, here, given a truncation parameter for the "small" scales, we must choose a sufficiently large radius $\widetilde{R}$ so that Theorem IV.3.4 holds in the complementary of the ball $B_{\widetilde{R}}(0)$.

Proof of Lemma IV.2.4. By the John-Nirenberg inequality, let $C_1 \ge 1$ be the constant such that for any $x \in \partial\Omega$ and any $s > 0$ there holds

$$\|\nu\|_*(B(x, s/2)) \le C_1 \sup_{\substack{B\subset B(x,s)\\ c_B\in\partial\Omega}} ⨍_B |\nu(z) - m_B\nu|\, d\sigma(z),$$

where the supremum is taken over all balls B centered on $\partial\Omega$ and contained in $B(x,s)$.

Let C_* and C_0 be the constants in Theorems IV.3.3 and IV.3.4 respectively. As in the proof of Lemma IV.2.3, we may assume $\delta \le 1/(C_1C_*)$ and the lemma will follow by finding $\widetilde{R} = \widetilde{R}(t,R)$ such that for all $\xi_0 \in \partial\Omega \setminus B_{\widetilde{R}}(0)$ there holds

$$\|\mathbf{1}_{I_0(\Delta(\xi_0,2t))} K_*(f\mathbf{1}_{\Delta(\xi_0,2t)})\|_{L^p(\sigma)} \lesssim \delta^{1/4}\|f\mathbf{1}_{\Delta(\xi_0,2t)}\|_{L^p(\sigma)},$$

with $I_0(\cdot)$ as in (IV.3.5). By Theorem IV.3.4, to see this it suffices to find $\widetilde{R}$ such that for all $\xi_0 \in \partial\Omega \setminus B_{\widetilde{R}}(0)$, we have

$$\sup_{x \in I_0(\Delta(\xi_0, 2t))} \|\nu\|_*(\Delta(x, 2C_0 t)) \leq C_1 \delta,$$

and the conclusions of the Semmes decomposition Theorem IV.3.3 hold (with $C_1\delta$) for all $x \in I_0(\Delta(\xi_0, 2t))$ and all $0 < r \leq 2C_0 t$.

We now determine $\widetilde{R}$ to ensure these two conditions hold. By the John-Nirenberg inequality, the δ-$(s, S; R)$ domain Ω satisfies the assumption of the Semmes decomposition Theorem IV.3.3. Recall that for every $x \in \partial\Omega \setminus B_R(0)$ we have

$$\fint_{B(x,r)} |\nu(z) - m_{B(x,r)}\nu| \, d\sigma(z) \leq \delta, \text{ for all } r \in (0, \infty).$$

In particular, by the John-Nirenberg inequality, we have

$$\sup_{x \in \partial\Omega \setminus B_{R+T}(0)} \|\nu\|_*(\Delta(x, T)) \leq C_1 \delta, \tag{IV.3.9}$$

This implies that for any compact set $\mathcal{K} \subset \partial\Omega \setminus B_{R+T}(0)$, we can take $R_{\widetilde{\mathcal{K}}_T} = T$ with $C_1\delta \leq 1/C_*$ in Theorem IV.3.3. Consequently, the value in (IV.3.4) is $R_{*,T,\mathcal{K}} = \delta T / C_*$, since $\operatorname{diam}(\partial\Omega) = \infty$.

We fix $T = \max\{2C_* C_0 t/\delta, 2C_0 t\}$ and we take any

$$\widetilde{R} > R + T + 2C_0 t.$$

With this choice, for any $\xi_0 \in \partial\Omega \setminus B_{\widetilde{R}}(0)$, we have

$$I_0(\Delta(\xi_0, 2t)) \subset \Delta(\xi_0, 2C_0 t) \subset \partial\Omega \setminus B_{R+T}(0),$$

and therefore,

$$\sup_{x \in I_0(\Delta(\xi_0, 2t))} \|\nu\|_*(\Delta(x, 2C_0 t)) \leq \sup_{x \in \partial\Omega \setminus B_{R+T}(0)} \|\nu\|_*(\Delta(x, T)) \overset{\text{(IV.3.9)}}{\leq} C_1 \delta,$$

and the conclusions in the Semmes decomposition Theorem IV.3.3 hold (with $C_1\delta$) for all $x \in \partial\Omega \setminus B_{R+T}(0) \supset I_0(\Delta(\xi_0, 2t))$ and all $0 < r \leq 2C_0 t$. This concludes the proof. $\square$

We now proceed with the proof of Theorem IV.3.4.

Proof of Theorem IV.3.4. Let $\Lambda \geq 1$ be the constant in Theorem IV.3.1. By (IV.3.1) (having fixed $\alpha = 1/2$, for instance) and (IV.3.6), for $0 < r \leq C_0 r_0/(2\Lambda)$, we obtain

$$\sup_{x \in I_0(\Delta(\xi_0, r_0))} \sup_{y \in \Delta(x, 2r)} r^{-1} |\langle x - y, m_{\Delta(x,r)}\nu \rangle| \lesssim \sup_{x \in I_0(\Delta(\xi_0, r_0))} \|\nu\|_*(B(x, 2\Lambda r)) \leq \delta, \tag{IV.3.10}$$

Setting $\widetilde{C}_0 = C_0/(2\Lambda)$, by (IV.3.6) and (IV.3.10), we may now assume

$$\sup_{x \in I_0(\Delta(\xi_0, r_0))} \|\nu\|_*(\Delta(x, \widetilde{C}_0 r_0)) \leq \delta, \text{ and} \tag{IV.3.11a}$$

$$\sup_{x \in I_0(\Delta(\xi_0, r_0))} \sup_{y \in \Delta(x, 2r)} r^{-1} |\langle x - y, m_{\Delta(x,r)}\nu \rangle| \lesssim \delta \text{ for all } r \in (0, \widetilde{C}_0 r_0). \tag{IV.3.11b}$$

A careful inspection of the proof [HMT10, Theorem 4.36] shows that (IV.3.11a) and (IV.3.11b) imply (IV.3.7). For completeness, we provide its proof following that of [HMT10, Theorem 4.36], omitting computations that follow mutatis mutandis.

We will use constants $C_j = C_j(\mathrm{ADR}, p, n) \geq 1$ for $j \geq 0$. The constant C_0 is the one in the statement, C_j with $j = 1, 2, 3, 4$ are the ones in the Semmes decomposition Theorem IV.3.3, and C_j with $j \geq 5$ will be fixed during the proof.

By density arguments, we may assume that $f \in C_c^{0,1}(\Delta(\xi_0, r_0))$. Throughout the proof, we denote $\Delta_0 = \Delta(\xi_0, r_0)$, $\mathcal{Q}_0 = \mathcal{Q}_0(\Delta_0)$ and $I_0 = I_0(\Delta_0)$ (see (IV.3.5)). We fix $\gamma := (p-1)/2$, and recall the definition of $\mathcal{M}_{1+\gamma}$ in (IV.1.7). We allow the implicit constants in the following arguments to depend on p and the UR character of the domain, recall Theorem IV.1.8.

We may assume that δ is sufficiently small, depending on p and the ADR constants; otherwise, (IV.3.7) follows from (IV.2.6a). During all the proof we fix

$$A = \delta^{-1/4}.$$

A standard and routine computation shows that it suffices to prove the good lambda inequality

$$\sigma(\{x \in I_0 : K_*f(x) > 3\lambda \text{ and } \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\}) \leq c_\delta \sigma(\{x \in I_0 : K_*f(x) > \lambda\}), \tag{IV.3.12}$$

where $c_\delta \to 0$ as $\delta \to 0$. We define

$$\mathcal{F}_\lambda := \{x \in I_0 : K_*f(x) > 3\lambda \text{ and } \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\}.$$

Clearly, we may assume that $\mathcal{F}_\lambda \neq \emptyset$. We decompose $I_0 = \mathcal{P}_\lambda \cup \mathcal{S}_\lambda$ where

$$\mathcal{P}_\lambda := \{x \in I_0 : K_*f(x) \leq \lambda\} \text{ and } \mathcal{S}_\lambda := \{x \in I_0 : K_*f(x) > \lambda\}.$$

We consider two cases: either there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \emptyset$, or $\mathcal{P}_\lambda \cap Q \neq \emptyset$ for all $Q \in \mathcal{Q}_0$, as in [MMM$^+$22a, (4.176)].

First, assume that there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \emptyset$. Equivalently, $Q_0 \subset \mathcal{S}_\lambda$. By the Semmes decomposition assumption, we decompose I_0 as the disjoint union $I_0 = G \cup E$, where $\sigma(E) \leq C_1 \exp(-C_2/\sqrt{\delta})\sigma(I_0)$ and G coincides with the graph of a Lipschitz function h with $\|\nabla h\|_{L^\infty} \leq C_3\sqrt{\delta}$. By [HMT10, (4.5.23) to (4.5.25)] and the fact that $\sigma(I_0) \approx \sigma(Q_0) \leq \sigma(\mathcal{S}_\lambda)$ by AD regularity, we obtain that (IV.3.12) holds with

$$c_\delta \lesssim \exp(-C_2/\sqrt{\delta}) + A^{1+\gamma}\sqrt{\delta}^{1+\gamma} + A^{\sqrt{1+\gamma}} \exp\left(-\frac{C_2}{\sqrt{\delta}} \frac{\sqrt{1+\gamma}-1}{\sqrt{1+\gamma}}\right), \tag{IV.3.13}$$

that is, $c_\delta \to 0$ as $\delta \to 0$.

Now, we consider the case $\mathcal{P}_\lambda \cap Q \neq \emptyset$ for all $Q \in \mathcal{Q}_0$. We write

$$\mathcal{S}_\lambda = \bigcup_{k \in \mathcal{K}} Q_k,$$

where $Q_k \in \mathcal{D}_\sigma$ is the maximal cube with $Q_k \subset I_0$, $Q_k \in \bigcup_{j \leq k_0} \mathcal{D}_{\sigma,j}$ and $Q_k \subset \mathcal{S}_\lambda$. That is, if $\widetilde{Q}_k \subset I_0$ denotes the dyadic parent of Q_k with $k \in \mathcal{K}$, then

$$\text{there exists } x_k^* \in \widetilde{Q}_k \cap (I_0 \setminus \mathcal{S}_\lambda) = \widetilde{Q}_k \cap \mathcal{P}_\lambda.$$

For each $k \in \mathcal{K}$, we fix (any) $x_k \in Q_k$ and define $\Delta_k := \Delta(x_k, \operatorname{diam} Q_k)$ and $\widetilde{\Delta}_k := \Delta(x_k, 2\operatorname{diam} \widetilde{Q}_k)$. We split $\mathcal{K}$ as

$$\mathcal{K}_1 := \{k \in \mathcal{K} : \text{there exists } x_k^{**} \in \Delta_k \text{ such that } \mathcal{M}_{1+\gamma}f(x_k^{**}) \leq A\lambda\}, \text{ and}$$
$$\mathcal{K}_2 := \mathcal{K} \setminus \mathcal{K}_1.$$

Clearly, $\mathcal{F}_\lambda \cap \Delta_k = \emptyset$ for all $k \in \mathcal{K}_2$, and since $\mathcal{F}_\lambda \subset \mathcal{S}_\lambda$, we have

$$\sigma(\mathcal{F}_\lambda) = \sum_{k \in \mathcal{K}_1} \sigma(\mathcal{F}_\lambda \cap \Delta_k).$$

We now denote

$$F_k := \{x \in \Delta_k : K_* f(x) > 3\lambda\}, \quad k \in \mathcal{K}_1.$$

Note that $\mathcal{F}_\lambda \cap \Delta_k \subset F_k$. We claim

$$\sigma(F_k) \le c_\delta \sigma(\Delta_k) \text{ for all } k \in \mathcal{K}_1. \tag{IV.3.14}$$

By the computations in [HMT10, (4.5.33)], this implies (IV.3.12).

Let us see (IV.3.14) for a fixed $k \in \mathcal{K}_1$. To shorten the notation, from now on we write $\Delta = \Delta_k$, $\widetilde{\Delta} = \widetilde{\Delta}_k$, $F = F_k$, $x^* = x_k^*$ and $x^{**} = x_k^{**}$. Moreover, we define $\Delta^* := \Delta(x^*, C_5 \operatorname{diam} \Delta)$ with C_5 chosen so that $2\widetilde{\Delta} \subset \Delta^*$. We note that $\Delta^* \subset C_6 \Delta_0$ for a suitable large constant C_6. We decompose

$$f = g_1 + g_2, \text{ where } g_1 := f\mathbf{1}_{2\Delta^*} \text{ and } g_2 := f\mathbf{1}_{\partial\Omega \setminus 2\Delta^*},$$

so that

$$\sigma(F) \le \sigma(\{x \in \Delta : K_* g_1(x) > \frac{3}{2}\lambda\}) + \sigma(\{x \in \Delta : K_* g_2(x) > \frac{3}{2}\lambda\}). \tag{IV.3.15}$$

Arguing as in the previous case (there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \emptyset$), using now that $\mathcal{M}_{1+\gamma} f(x^{**}) \le A\lambda$, the first term in the right-hand side is controlled by

$$\sigma(\{x \in \Delta : K_* g_1(x) > \frac{3}{2}\lambda\}) \le c_\delta \sigma(\Delta), \tag{IV.3.16}$$

with c_δ as in (IV.3.13).

The rest of the proof is dedicated to the other term. As $x^* \in \widetilde{\Delta} \cap \mathcal{P}_\lambda$, we note that

$$|K_\varepsilon g_2(x^*)| = |K_{\max\{\varepsilon, 2C_5 \operatorname{diam} \Delta\}} f(x^*)| \le K_* f(x^*) \le \lambda \text{ for all } \varepsilon > 0. \tag{IV.3.17}$$

Let us a fix $\varepsilon > 0$ and an arbitrary point $x \in \Delta$. As in [HMT10, (4.5.38) and (4.5.39)], we bound

$$|K_\varepsilon g_2(x) - K_\varepsilon g_2(x^*)| \le \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I} := \frac{1}{w_n} \int_{\partial\Omega \setminus 2\Delta^*} \left| \frac{\langle \nu(y), x - y \rangle}{|x - y|^{n+1}} - \frac{\langle \nu(y), x^* - y \rangle}{|x^* - y|^{n+1}} \right| |f(y)| \, d\sigma(y),$$
$$\mathrm{II} := \frac{1}{w_n} \int_{\substack{y \in \partial\Omega \setminus 2\Delta^* \\ |x-y| > \varepsilon, |x^*-y| < \varepsilon}} \frac{|\langle \nu(y), x - y \rangle|}{|x - y|^{n+1}} |f(y)| \, d\sigma(y), \text{ and}$$
$$\mathrm{III} := \frac{1}{w_n} \int_{\substack{y \in \partial\Omega \setminus 2\Delta^* \\ |x^*-y| > \varepsilon, |x-y| < \varepsilon}} \frac{|\langle \nu(y), x^* - y \rangle|}{|x^* - y|^{n+1}} |f(y)| \, d\sigma(y).$$

As in [HMT10, (4.5.40) and (4.5.41)], we bound

$$\mathrm{I} \lesssim \mathrm{I}_0 + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,$$

where

$$\mathrm{I}_0 := \int_{\Delta_0\setminus 2\Delta^*} \left| \frac{x-y}{|x-y|^{n+1}} - \frac{x^*-y}{|x^*-y|^{n+1}} \right| |\nu(y) - m_{\Delta^*}\nu| |f(y)|\, d\sigma(y),$$

$$\mathrm{I}_1 := \sum_{\substack{j\in\mathbb{N} \\ 2^{j+1}\Delta^*\subset C_6\Delta_0}} \int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*} \frac{|\langle x-x^*, m_{\Delta^*}\nu\rangle|}{|x^*-y|^{n+1}} |f(y)|\, d\sigma(y),$$

$$\mathrm{I}_2 := r(\Delta^*) \sum_{\substack{j\in\mathbb{N} \\ 2^{j+1}\Delta^*\subset C_6\Delta_0}} \int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*} \frac{|\langle x-y, m_{\Delta^*}\nu - m_{2^{j+1}\Delta^*}\nu\rangle|}{|x^*-y|^{n+2}} |f(y)|\, d\sigma(y), \text{ and}$$

$$\mathrm{I}_3 := r(\Delta^*) \sum_{\substack{j\in\mathbb{N} \\ 2^{j+1}\Delta^*\subset C_6\Delta_0}} \int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*} \frac{|\langle x-y, m_{2^{j+1}\Delta^*}\nu\rangle|}{|x^*-y|^{n+2}} |f(y)|\, d\sigma(y).$$

As in [HMT10, (4.5.40) and (4.5.43)] we have

$$\mathrm{I}_0 + \mathrm{I}_2 \lesssim A\lambda \|\nu\|_*(C_6\Delta_0) \overset{\text{(IV.3.11a)}}{\leq} A\delta\lambda,$$

provided C_0 is big enough. If $C_0 \gg C_6$, by (IV.3.11b) we have $|\langle x-x^*, m_{\Delta^*}\nu\rangle| \lesssim \delta \operatorname{diam}\Delta^* \approx \delta \operatorname{diam}\Delta$ and $|\langle x-y, m_{2^{j+1}\Delta^*}\nu\rangle| \lesssim 2^j\delta \operatorname{diam}\Delta^* \approx 2^j\delta\operatorname{diam}\Delta$ for all $y \in 2^{j+1}\Delta^*$, and therefore, as in [HMT10, (4.5.52)] we have

$$\mathrm{I}_1 + \mathrm{I}_3 \lesssim A\delta\lambda.$$

Let us bound II and III. Since $x, x^{**} \in \Delta$ and $x^* \in \widetilde{\Delta}$, we have $|x-y| \approx |x^*-y| \approx |x^{**}-y|$ for all $y \in \partial\Omega \setminus 2\Delta^*$. In particular, $|x-y| \approx |x^*-y| \approx \varepsilon$ in the domain of integration in II and III. Since $\operatorname{supp} f \subset \Delta_0$, we may assume

$$0 < \varepsilon \leq C_7 r_0,$$

for a suitable large $C_7 = C_7(\text{ADR})$, otherwise $\mathrm{II} = \mathrm{III} = 0$. As in [HMT10, (4.5.47)], we bound $\mathrm{II} \lesssim \mathrm{II}_1 + \mathrm{II}_2$ where

$$\mathrm{II}_1 := \varepsilon^{-n} \int_{\substack{|x^{**}-y|<C_8\varepsilon \\ |x-y|<C_9\varepsilon}} \frac{|\langle x-y, \nu(y) - m_{\Delta(x,C_9\varepsilon)}\nu\rangle|}{|x-y|} |f(y)|\, d\sigma(y), \text{ and}$$

$$\mathrm{II}_2 := \varepsilon^{-n} \int_{\substack{|x^{**}-y|<C_8\varepsilon \\ |x-y|<C_9\varepsilon}} \frac{|\langle x-y, m_{\Delta(x,C_9\varepsilon)}\nu\rangle|}{|x-y|} |f(y)|\, d\sigma(y)$$

As in [HMT10, (4.5.48)], since $\varepsilon \leq C_7 r_0$, for $C_0 \gg C_7$ we have

$$\mathrm{II}_1 \lesssim A\lambda \|\nu\|_*(C_0\Delta_0) \overset{\text{(IV.3.11a)}}{\leq} A\delta\lambda,$$

and since $|\langle x-y, m_{\nu(x,C_9\varepsilon)}\nu\rangle| \lesssim \varepsilon\delta$ for all $y \in \Delta(x, 2C_9\varepsilon)$ by (IV.3.11b), as in [HMT10, (4.5.49)] we have

$$\mathrm{II}_2 \lesssim A\delta\lambda.$$

Hence, $\mathrm{II} \lesssim A\delta\lambda$, and in a similar manner, $\mathrm{III} \lesssim A\delta\lambda$.

All in all, we have shown

$$|K_\varepsilon g_2(x) - K_\varepsilon g_2(x^*)| \leq C_{10} A\delta\lambda \text{ for all } x \in \Delta \text{ and all } \varepsilon > 0.$$

By (IV.3.17), taking $0 < \delta < \frac{1}{4C_{10}^2}$ here (recall $A = \delta^{-1/4}$ is fixed) we have

$$K_* g_2(x) \leq \frac{3}{2}\lambda \text{ for all } x \in \Delta.$$

This implies that $\{x \in \Delta : K_* g_2(x) > \frac{3}{2}\lambda\} = \emptyset$, and therefore

$$\sigma(F) \overset{\text{(IV.3.15)}}{\leq} \sigma(\{x \in \Delta : K_* g_1(x) > \frac{3}{2}\lambda\}) \overset{\text{(IV.3.16)}}{\leq} c_\delta \sigma(\Delta),$$

as claimed in (IV.3.14). This concludes the proof. □

IV.4. The double layer potential in large scales. Proof of Theorem IV.2.5: $\|K_{l,*}\|_{L^p(\sigma)}$ is small

This section is dedicated to the proof of Theorem IV.2.5. During this section we assume $\Omega \subset \mathbb{R}^{n+1}$ as in the statement of Theorem IV.2.5. Briefly, Ω is a 2-sided CAD with unbounded boundary satisfying the small Jones' $\beta_{\infty,\partial\Omega}$ coefficient and small BMO norm of ν conditions (IV.2.11) and (IV.2.12) on large scales.

We allow the constants to depend on n, $1 < p < \infty$ and the CAD constants of Ω, and this will not be specified in the computations from now in this section.

IV.4.1. Relation between unit normal vectors from $\beta_{\infty,\partial\Omega}$ and BMO oscillation. Given by (IV.2.11), for each $x \in \partial\Omega$ and $r \geq S$, we fix an n-plane $L_{B(x,r)} \ni x$ such that

$$\sup_{y \in \partial\Omega \cap B(x,r)} \frac{\text{dist}(y, L_{B(x,r)})}{r} \leq 2\delta_\beta, \tag{IV.4.1}$$

and we denote its orthogonal unit vectors as $\pm N_{B(x,r)}$. The orientation of $N_{B(x,r)}$ is fixed below in (IV.4.3).

Lemma IV.4.1. *Let $x \in \partial\Omega$ and $r \geq S$. For all $y \in B(x,r) \cap \partial\Omega$,*

(1) $|\langle x - y, N_{B(x,r)}\rangle| \leq 2\delta_\beta r$, and

(2) $|\langle x - y, m_{B(x,r)}\nu\rangle| \lesssim \delta_*^{\frac{1}{2n}} r$.

Proof. The second item is proved in (IV.3.2). For the first item, write $B = B(x,r)$ and let N_B and $L_B \ni x$ as in (IV.4.1). Hence $\text{dist}(y, L_B) \leq 2\delta_\beta r$ for all $y \in B(x,r) \cap \partial\Omega$. For each $y \in B(x,r)$ let $\alpha_y = \angle(N, y - x)$ denote the angle between N and $y - x$. Therefore, $|\cos\alpha_y| = |\sin(\pi/2 - \alpha_y)| = \text{dist}(y, L_B)/|x - y| \leq 2\delta_\beta r/|x - y|$, and so $|\langle x - y, N_B\rangle| = |x - y||\cos\alpha_y| \leq 2\delta_\beta r$. □

Under the conditions of the lemma above, as in [DS91, Lemma 5.8], if A is big enough only depending only on the dimension and the ADR constant, then there exists $\{y_j\}_{j=0}^n \subset B(x,r) \cap \partial\Omega$ with $y_0 = x$ and $\text{dist}(y_j, L_{j-1}) \geq A^{-1}r$, where L_k is the k-plane passing through the points $y_0, \ldots, y_k$. In particular $|x - y_j| \approx_A r$ and by Lemma IV.4.1 we have

$$\left|\left\langle \frac{x - y_j}{|x - y_j|}, N_{B(x,r)}\right\rangle\right| \leq 2A\delta_\beta, \text{ and}$$
$$\left|\left\langle \frac{x - y_j}{|x - y_j|}, m_{B(x,r)}\nu\right\rangle\right| \lesssim A\delta_*^{\frac{1}{2n}}.$$

Therefore, (recall $\delta = \max\{\delta_\beta, \delta_*\}$)

$$|\langle m_{B(x,r)}\nu, N_{B(x,r)}\rangle| \geq 1 - C_A \max\{\delta_\beta, \delta_*^{\frac{1}{2n}}\} \geq 1 - C_A \delta^{\frac{1}{2n}}, \tag{IV.4.2}$$

meaning that $N_{B(x,r)}$ and $\nu_{B(x,r)}$ are almost parallel provided both δ's are small enough. By (IV.4.2), choosing appropriately the orientation of $N_{B(x,r)}$, we assume that

$$|m_{B(x,r)}\nu - N_{B(x,r)}|^2 \lesssim \delta^{\frac{1}{2n}}. \tag{IV.4.3}$$

IV.4.2. Notation and parameters for the proof. Here we write the notation we use in the following sections.

- $S > 0$: the first (large) scale where we have the smallness condition on the Jones' $\beta_{\infty,\partial\Omega}$ coefficient (bound given by δ_β) in (IV.2.11) and the BMO norm of the unitari normal vector ν (bound given by δ_*) in (IV.2.12).
- $T \gg 100S$: scale where we truncate the kernel on the large scales.
- L, L_B, L_j, etc: planes, depending on the situation.
- N_B or similar: unitari normal vector in the Jones' $\beta_{\infty,\partial\Omega}$ coefficient in (IV.4.1).
- N: big parameter in the proof of (IV.4.8a).
- α: the stopping condition in the construction of the Lipschitz graph, to obtain the Lipschitz graph with norm $\lesssim \alpha$.
- We fix
$$\gamma := \frac{1}{2}\min\left\{1, p-1, \frac{1}{2n-1}\right\} > 0.$$
In particular, $\gamma/(1+\gamma) < 1/2n$. We will use $\tau^{\frac{1}{2n}} < \tau^{\frac{\gamma}{1+\gamma}}$ when $\tau \in (0,1)$ repeatedly.
- Let $\mathcal{M}_{1+\gamma} = \mathcal{M}_{1+\gamma,\partial\Omega}$ denote the uncentered Hardy-Littlewood maximal function defined in (IV.1.7). Recall it is bounded from $L^p(\sigma)$ to $L^p(\sigma)$ with norm C_p, since $1+\gamma < p \leq \infty$ and $\gamma > 0$ is already fixed depending on p.

Given $S > 0$ and $\delta = \max\{\delta_\beta, \delta_*\}$ we take

$$A = \delta^{-\theta} \text{ with } \theta = \frac{\gamma}{2(n+3)(1+\gamma)}, \tag{IV.4.4}$$

$N = A^{1+\frac{1}{n+1}}$, $\alpha = \delta^{\frac{\gamma}{3(1+\gamma)}}$ and $T = SA^2 \gg 100S$. With this choice, we claim that, as $\delta \to 0$, there holds:

(C1) $A \to \infty$,
(C2) $A/N \to 0$,
(C3) $\delta^{\frac{\gamma}{1+\gamma}} A \leq \delta^{\frac{1}{4n}(1-\frac{1}{\sqrt{1+\gamma}})} A^2 \leq \delta^{\frac{\gamma}{1+\gamma}} N^{n+1} A \to 0$,
(C4) $\delta^{\frac{\gamma}{1+\gamma}} \leq \alpha^3 \ll \alpha \to 0$,
(C5) $\alpha A^2 \to 0$, and
(C6) $AS/T \to 0$.

Indeed, (C1), (C2), (C4) and (C6) are clear, for (C3) we have

$$\delta^{\frac{\gamma}{1+\gamma}} N^{n+1} A = \delta^{\frac{\gamma}{1+\gamma}} A^{(1+\frac{1}{n+1})(n+1)+1} = \delta^{\frac{\gamma}{1+\gamma}-\theta(n+3)} = \delta^{\frac{1}{2}\frac{\gamma}{1+\gamma}},$$

and for (C5),

$$\alpha A^2 = \delta^{\frac{\gamma}{3(1+\gamma)}-2\theta} = \delta^{(\frac{1}{3}-\frac{1}{n+3})\frac{\gamma}{1+\gamma}}.$$

IV.4.3. Reduction to a good lambda inequality. In order to estimate the L^p norm of $K_{l,*} = K_{l(T),*}$ in (IV.2.13), it suffices to prove the following good lambda inquality

$$\sigma(\{x \in \partial\Omega : K_{l,*}f(x) > 101\lambda, \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\}) \leq c_\delta \sigma(\{x \in \partial\Omega : K_{l,*}f(x) > \lambda\}), \tag{IV.4.5}$$

for the parameters fixed above in (IV.4.4) with δ small enough, and $c_\delta \to 0$ as $\delta \to 0$. A standard and routinary computation using Fubini's theorem (see [Mat95, Theorem 1.15] for instance), the L^p bound of the maximal operator $\mathcal{M}_{1+\gamma}$ (see (IV.1.7)) and the good lambda inequality (IV.4.5) provides that there exists $\delta_p > 0$ such that if $\delta \leq \delta_p$ then

$$\|K_{l,*}f\|_{L^p(\sigma)} \lesssim_{p,n} \frac{1}{A}\|f\|_{L^p(\sigma)}. \tag{IV.4.6}$$

By the choice of $A = \delta^{-\theta}$ in (IV.4.4), we obtain (IV.2.13) with constants depending only on the dimension and p. In the other case, i.e., if $\delta > \delta_p$, then (IV.2.13) follows directly from (IV.2.6a), with constants depending also on the CAD constants of Ω.

IV.4.4. Localization of the good lambda inequality (IV.4.5). First, let us see the classical Whitney decomposition.

Lemma IV.4.2 (Whitney decomposition). *If $U \subset \mathbb{R}^{n+1}$ is open, $U \neq \mathbb{R}^{n+1}$, then U can be covered as*

$$U = \bigcup_{i \in I} Q_i,$$

where Q_i, $i \in I$, are (classical) dyadic cubes in $\mathcal{D}(\mathbb{R}^{n+1})$ with disjoint interiors such that the following holds:

1. *$5Q_i \subset U$ for each $i \in I$.*
2. *$11Q_i \cap U^c \neq \emptyset$ for each $i \in I$.*
3. *If $3Q_i \cap 3Q_j \neq \emptyset$, $i, j \in I$, then $1/(7\sqrt{n+1}) \leq \ell(Q_i)/\ell(Q_j) \leq 7\sqrt{n+1}$.*
4. *The family $\{3Q_i\}_{i\in I}$ has finite overlapping with constant depending only on the dimension.*

Proof. The family $\mathcal{F} = \{Q_i\}_{i\in I}$ of maximal dyadic cubes $Q \in \mathcal{D}(\mathbb{R}^{n+1})$ such that $5Q \subset U$ satisfies the above properties by standard arguments.

Let us check them for completeness. By definition $5Q \subset U$, and clearly every $x \in U$ belongs to one $Q \in \mathcal{F}$. This gives the disjoint decomposition of U.

As $Q \in \mathcal{F}$ is maximal, its parent $\widehat{Q} \in \mathcal{D}(\mathbb{R}^{n+1})$ satisfies that $5\widehat{Q} \cap U^c \neq \emptyset$, and (2) follows as $5\widehat{Q} \subset 11Q$.

From $5Q \subset U$ we directly have that $\operatorname{dist}(3Q, U^c) \geq \ell(Q)$ for each $Q \in \mathcal{F}$. From $11Q \cap U^c \neq \emptyset$, see (2), we get $\operatorname{dist}(3Q, U^c) \leq 4\sqrt{n+1}\ell(Q)$. From these two estimates, if $x \in 3Q$ then $\ell(Q) \leq \operatorname{dist}(x, U^c) \leq \operatorname{diam}(3Q) + 4\sqrt{n+1}\ell(Q) = 7\sqrt{n+1}\ell(Q)$, and (3) is proved.

Finally, given $Q \in \mathcal{F}$, (4) follows as there is only a finite number (depending on the dimension) of cubes $Q' \in \mathcal{D}(\mathbb{R}^{n+1})$ such that $1/(7\sqrt{n+1}) \leq \ell(Q_i)/\ell(Q_j) \leq 7\sqrt{n+1}$ and $3Q \cap 3Q' \neq \emptyset$. □

For $\lambda > 0$, let W_λ be the family of dyadic cubes given by Lemma IV.4.2 of the open set $\Omega_\lambda \coloneqq \mathbb{R}^{n+1} \setminus \{x \in \partial\Omega : K_{l,*}f(x) \leq \lambda\}$. We will refer to W_λ as the Whitney decomposition of Ω_λ. So, we can write

$$V_\lambda \coloneqq \partial\Omega \cap \Omega_\lambda = \{x \in \partial\Omega : K_{l,*}f(x) > \lambda\} = \bigcup_{Q \in W_\lambda} Q \cap \partial\Omega.$$

Note that $11Q \cap (\partial\Omega \setminus V_\lambda) \neq \emptyset$ for all $Q \in W_\lambda$ and $\{3Q\}_{Q\in W_\lambda}$ has finite overlapping.

Notation. We will sometimes use Q instead of $Q \cap \partial\Omega$ when it is clear from the context.

Lemma IV.4.3. *For $Q \in W_\lambda$, there holds*

(IV.4.7)
$$\{x \in Q\cap\partial\Omega : K_{l,*}f(x) > 101\lambda, \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\} \subset \{x \in Q\cap\partial\Omega : K_{l,*}(f\mathbf{1}_{3Q})(x) > 50\lambda\}.$$

We note that if A were small, the proof of (IV.4.7) would be standard. However, its dependence on δ makes the proof more involved, relying on the flatness assumptions (IV.2.11) and (IV.2.12).

Proof of Lemma IV.4.3. Let $x \in Q\cap\partial\Omega$ satisfy $K_{l,*}f(x) > 101\lambda$ and $\mathcal{M}_{1+\gamma}f(x) \leq A\lambda$, and let $z \in 11Q\cap(\partial\Omega\setminus V_\lambda) \neq \emptyset$ so that $K_{l,*}f(z) \leq \lambda$. We have $x, z \in 11Q \subset B(z, 11(n+1)^{1/2}\ell(Q)) =:$ B, and so $r(B) \approx \ell(Q)$.

The lemma will follow from

$$K_{l,*}(f\mathbf{1}_{B^c})(x) \leq 41\lambda, \text{ and} \tag{IV.4.8a}$$
$$K_{l,*}(f\mathbf{1}_{B\setminus 3Q})(x) \leq 10\lambda, \tag{IV.4.8b}$$

because

$$K_{l,*}(f\mathbf{1}_{3Q})(x) \geq K_{l,*}f(x) - K_{l,*}(f\mathbf{1}_{B^c})(x) - K_{l,*}(f\mathbf{1}_{B\setminus 3Q})(x) \geq 50\lambda.$$

We first prove (IV.4.8a). For $N > 0$ big enough (auxiliary parameter only used in this proof, already defined depending on A, see (C2) and (C3)) write

$$\begin{aligned} K_{l,*}(f\mathbf{1}_{B^c})(x) &= K_{l,*}(f\mathbf{1}_{B^c})(z) + K_{l,*}(f\mathbf{1}_{B^c})(x) - K_{l,*}(f\mathbf{1}_{B^c})(z) \\ &\leq K_{l,*}(f\mathbf{1}_{B^c})(z) + \left|K_{l,*}(f\mathbf{1}_{NQ\setminus B})(x) - K_{l,*}(f\mathbf{1}_{NQ\setminus B})(z)\right| \\ &\quad + \left|K_{l,*}(f\mathbf{1}_{(NQ)^c})(x) - K_{l,*}(f\mathbf{1}_{(NQ)^c})(z)\right| \\ &=: \boxed{i} + \boxed{ii} + \boxed{iii}. \end{aligned}$$

Since B is centered at z and by the choice of z we have

$$\boxed{i} \leq K_{l,*}f(z) \leq \lambda.$$

Let us bound the term $\boxed{ii}$. We simply have

$$\boxed{ii} \lesssim \frac{1}{r_B^{n+1}} \left(\int_{NQ\setminus B} |\langle \nu(y), y - x\rangle||f(y)|\, d\sigma(y) + \int_{NQ\setminus B} |\langle \nu(y), y - z\rangle||f(y)|\, d\sigma(y) \right).$$

For $\xi = x$ or $\xi = z$, we have

$$\int_{NQ\setminus B} |\langle \nu(y), y - \xi\rangle||f(y)|\, d\sigma(y) \leq \int_{B(\xi,\sqrt{n+1}N\ell(Q))} |\langle \nu(y), y - \xi\rangle||f(y)|\, d\sigma(y) =: \boxed{ii_\xi}.$$

Set $B_\xi := B(\xi, \sqrt{n+1}N\ell(Q))$. By the triangle inequality (after adding $\pm m_{B_\xi}\nu$), Hölder's inequality, (IV.3.2) and $\mathcal{M}_{1+\gamma}f(x) \leq A\lambda$ we have

$$\begin{aligned}
\boxed{ii_\xi} &\leq \int_{B_\xi} |\langle \nu(y) - m_{B_\xi}\nu, y-\xi\rangle||f(y)|\,d\sigma(y) + \int_{B_\xi} |\langle m_{B_\xi}\nu, y-\xi\rangle||f(y)|\,d\sigma(y) \\
&\lesssim N^{n+1}\ell(Q)^{n+1}\left(\fint_{B_\xi} |\nu(y) - m_{B_\xi}\nu|^{\frac{1+\gamma}{\gamma}}\,d\sigma(y)\right)^{\frac{\gamma}{1+\gamma}}\left(\fint_{B_\xi} |f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}} \\
&\quad + \int_{B_\xi} |\langle m_{B_\xi}\nu, y-\xi\rangle||f(y)|\,d\sigma(y) \\
&\overset{\text{(IV.3.2)}}{\lesssim} N^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}}\mathcal{M}_{1+\gamma}f(x) + N^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{1}{2n}}\mathcal{M}_1 f(x) \\
&\lesssim N^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}}A\lambda.
\end{aligned}$$

Note that even though we are studying the term $\boxed{ii_\xi}$, in any case we controlled $\fint_{B_\xi}|f|\,d\sigma$ and $(\fint_{B_\xi}|f|^{1+\gamma}\,d\sigma)^{1/(1+\gamma)}$ by $\mathcal{M}_1 f(x)$ and $\mathcal{M}_{1+\gamma}f(x)$ respectively, and so we finally used that $\mathcal{M}_{1+\gamma}f(x) \leq A\lambda$. Hence,

$$\boxed{ii} \lesssim \frac{1}{r_B^{n+1}}N^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}}A\lambda \approx N^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}}A\lambda.$$

Let us bound $\boxed{iii}$. For $\varepsilon \geq T$, we have

$$|K_\varepsilon(f\mathbf{1}_{(NQ)^c})(x) - K_\varepsilon(f\mathbf{1}_{(NQ)^c})(z)| \leq \boxed{A} + \boxed{B} + \boxed{C},$$

where

$$\begin{aligned}
\boxed{A} &:= \left|\int_{\substack{|y-x|>\varepsilon \\ |y-z|>\varepsilon}} \left\langle \nu(y)f(y)\mathbf{1}_{(NQ)^c}(y), \frac{y-x}{|x-y|^{n+1}} - \frac{y-z}{|z-y|^{n+1}}\right\rangle d\sigma(y)\right|, \\
\boxed{B} &:= \left|\int_{\substack{|y-x|>\varepsilon \\ |y-z|\leq\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}} f(y)\mathbf{1}_{(NQ)^c}(y)\,d\sigma(y)\right|, \text{ and} \\
\boxed{C} &:= \left|\int_{\substack{|y-z|>\varepsilon \\ |y-x|\leq\varepsilon}} \frac{\langle \nu(y), y-z\rangle}{|z-y|^{n+1}} f(y)\mathbf{1}_{(NQ)^c}(y)\,d\sigma(y)\right|.
\end{aligned}$$

We first study the term $\boxed{A}$. Using the cancellation property of the Calderon-Zygmund kernel first (this is just the mean value theorem), and $B(x, N\ell(Q)/2) \subset NQ$ and $|x-z| \lesssim \ell(Q)$ second, we have

$$\boxed{A} \lesssim \int_{y\notin NQ} |f(y)|\frac{|x-z|}{|x-y|^{n+1}}\,d\sigma(y) \lesssim \ell(Q)\int_{y\notin B(x,\frac{N}{2}\ell(Q))} \frac{|f(y)|}{|x-y|^{n+1}}\,d\sigma(y).$$

Note that the latter integral does not depend on z. Breaking the latter integral in annulus and by the AD regularity of σ, using standard estimates we get

$$\begin{aligned}\boxed{A} &\lesssim \ell(Q)\sum_{k\geq 0}\int_{2^{k+1}B(x,\frac{N}{2}\ell(Q))\setminus 2^k B(x,\frac{N}{2}\ell(Q))}\frac{|f(y)|}{|x-y|^{n+1}}\,d\sigma(y)\\ &\lesssim \frac{\ell(Q)}{\frac{N}{2}\ell(Q)}\sum_{k\geq 0}\frac{1}{2^{k+1}}\fint_{2^{k+1}B(x,\frac{N}{2}\ell(Q))}|f(y)|\,d\sigma(y)\lesssim \frac{1}{N}\mathcal{M}_{1+\gamma}f(x)\leq \frac{A}{N}\lambda.\end{aligned}$$

We now turn to the terms $\boxed{B}$ and $\boxed{C}$. First we note that we can assume that $\varepsilon \geq \frac{N-10}{2}\ell(Q)\geq \frac{N}{4}\ell(Q)$, otherwise $B(z,\varepsilon)\subset NQ$ and so $\boxed{B}=0$. For $\varepsilon > \frac{N}{4}\ell(Q)$ we have $B(z,\varepsilon)\subset B(x,2\varepsilon)$, and we get

$$\begin{aligned}\boxed{B} &\leq \int_{B(x,2\varepsilon)\setminus\overline{B(x,\varepsilon)}}\frac{|\langle \nu(y),y-x\rangle|}{|x-y|^{n+1}}|f(y)|\,d\sigma(y)\lesssim \frac{1}{\varepsilon}\fint_{B(x,2\varepsilon)}|\langle \nu(y),y-x\rangle||f(y)|\,d\sigma(y)\\ &\leq \frac{1}{\varepsilon}\fint_{B(x,2\varepsilon)}|\langle \nu(y)-m_{B(x,2\varepsilon)}\nu,y-x\rangle||f(y)|\,d\sigma(y)\\ &\quad+\frac{1}{\varepsilon}\fint_{B(x,2\varepsilon)}|\langle m_{B(x,2\varepsilon)}\nu,y-x\rangle||f(y)|\,d\sigma(y)\\ &\lesssim \left(\fint_{B(x,2\varepsilon)}|\nu(y)-m_{B(x,2\varepsilon)}\nu|^{\frac{1+\gamma}{\gamma}}\,d\sigma(y)\right)^{\frac{\gamma}{1+\gamma}}\left(\fint_{B(x,2\varepsilon)}|f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}}\\ &\quad+\left(\sup_{y\in\partial\Omega\cap B(x,2\varepsilon)}\frac{|\langle x-y,m_{B(x,2\varepsilon)}\nu\rangle|}{\varepsilon}\right)\left(\fint_{B(x,2\varepsilon)}|f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}}\\ &\overset{\text{(IV.3.2)}}{\lesssim}(\delta_*^{\frac{\gamma}{1+\gamma}}+\delta_*^{\frac{1}{2n}})\mathcal{M}_{1+\gamma}f(x)\lesssim \delta_*^{\frac{\gamma}{1+\gamma}}\mathcal{M}_{1+\gamma}f(x)\leq \delta_*^{\frac{\gamma}{1+\gamma}}A\lambda.\end{aligned}$$

By symmetry, but using that $B(z,\varepsilon)\subset B(x,2\varepsilon)$ (recall we can assume $\varepsilon>\frac{N}{4}\ell(Q)$) before applying the bound of the maximal operator $\mathcal{M}_{1+\gamma}$, we have the same bound for $\boxed{C}$, that is, $\boxed{C}\lesssim \delta_*^{\frac{\gamma}{1+\gamma}}\mathcal{M}_{1+\gamma}f(x)\leq \delta_*^{\frac{\gamma}{1+\gamma}}A\lambda$. Therefore, the term $\boxed{iii}$ is controlled by

$$\boxed{iii}\leq \boxed{A}+\boxed{B}+\boxed{C}\lesssim \left(\frac{A}{N}+\delta_*^{\frac{\gamma}{1+\gamma}}A\right)\lambda$$

From the bound of $\boxed{i}$, $\boxed{ii}$ and $\boxed{iii}$, and conditions (C2) and (C3) we conclude

$$K_{l,*}(f\mathbf{1}_{B^c})(x)\leq \lambda + C\left(N^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}}A+\frac{A}{N}+\delta_*^{\frac{\gamma}{1+\gamma}}A\right)\lambda\leq 41\lambda,$$

as claimed in (IV.4.8a).

It remains to prove (IV.4.8b). For $\varepsilon\geq T$, we may assume $\{y : |y-x|>\varepsilon\}\cap B\neq\emptyset$, otherwise $K_\varepsilon(f\mathbf{1}_{B\setminus 3Q})(x)=0$. Thus, $r(B)\gtrsim\varepsilon\geq T$, allowing us to apply (IV.3.2). Using that $\ell(Q)\approx r(B)$ and that $|y-x|\gtrsim \ell(Q)$ if $y\notin 3Q$, and arguing in the last inequality as in the bound for $\boxed{ii_\xi}$, we get

$$\begin{aligned}K_\varepsilon(f\mathbf{1}_{B\setminus 3Q})(x) &= \left|\int_{|y-x|>\varepsilon} \frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}} f(y)\mathbf{1}_{B\setminus 3Q}(y)\,d\sigma(y)\right| \\ &\lesssim \frac{1}{\ell(Q)}\bigg(\fint_B |\langle \nu(y)-m_B\nu, y-x\rangle||f(y)|\,d\sigma(y) \\ &\qquad\qquad + \fint_B |\langle m_B\nu, y-x\rangle||f(y)|\,d\sigma(y)\bigg) \\ &\lesssim \delta_*^{\frac{\gamma}{1+\gamma}} A\lambda,\end{aligned}$$

uniformly on $\varepsilon \geq T$. Then (IV.4.8b) follows by condition (C3). $\square$

By (IV.4.7), in order to prove (IV.4.5) we claim that it suffices to see

$$\sigma(\{x\in Q : K_{l,*}(f\mathbf{1}_{3Q}) > 50\lambda, \mathcal{M}_{1+\gamma}f \leq A\lambda\}) \leq c_\delta \sigma(3Q), \tag{IV.4.9}$$

for all $Q\in W_\lambda$. Indeed,

$$\begin{aligned}\sigma(\{K_{l,*}f > 101\lambda, \mathcal{M}_{1+\gamma}f \leq A\lambda\}) &\leq \sum_{Q\in W_\lambda} \sigma(\{x\in Q : K_{l,*}f > 101\lambda, \mathcal{M}_{1+\gamma}f\leq A\lambda\}) \\ &\overset{\text{(IV.4.7)}}{\leq} \sum_{Q\in W_\lambda} \sigma(\{x\in Q : K_{l,*}(f\mathbf{1}_{3Q}) > 50\lambda, \mathcal{M}_{1+\gamma}f \leq A\lambda\}) \\ &\overset{\text{(IV.4.9)}}{\leq} c_\delta \sum_{Q\in W_\lambda} \sigma(3Q) \lesssim c_\delta\sigma(\Omega_\lambda) = c_\delta \sigma(V_\lambda),\end{aligned}$$

where in the last step we used the finite overlapping of the family $\{3Q\}_{Q\in W_\lambda}$.

Remark IV.4.4. From now on we can assume without loss of generality that $\ell(Q) \geq \frac{T}{10(n+1)^{1/2}}$, otherwise, if $x\in Q$ then $\{y\in 3Q : |y-x| > T\} = \emptyset$ and hence

$$K_{l,*}(f\mathbf{1}_{3Q})(x) = \sup_{\varepsilon\geq T}\left|\int_{y\in 3Q\setminus\overline{B(x,\varepsilon)}} \frac{\langle\nu(y), y-x\rangle}{|x-y|^{n+1}} f(y)\mathbf{1}_{3Q}(y)\,d\sigma(y)\right| = 0.$$

IV.4.5. Construction of a Lipschitz graph. During this subsection, let $\Omega\subset\mathbb{R}^{n+1}$ be an ADR domain with unbounded boundary, and let $\mathcal{D}_\sigma$ denote its dyadic lattice in Lemma IV.1.12. For $t>1$ and $Q\in\mathcal{D}_\sigma$ we set

$$tQ \coloneqq \{x\in\partial\Omega : \operatorname{dist}(x,Q) \leq (t-1)\operatorname{diam} Q\}. \tag{IV.4.10}$$

Notation (Dependence of parameters)**.** Fixed $m>1$, we choose $k_0 = k_0(m) \gg m$ and $k = k(k_0) \gg k_0$. We consider $\varepsilon,\alpha \ll 1$ such that $\varepsilon \ll \alpha$ and $\varepsilon \ll 1/k_0$. Throughout this subsection, we allow the constant C to depend on the ADR constant, but not on the parameters mentioned above.

Following [DS91, Section 8], in this subsection we construct a Lipschitz approximating graph on large scales (see Proposition IV.4.9), which only uses the following flatness assumption of large scales: We assume that for each cube $Q\in\mathcal{D}_\sigma$ with $\operatorname{diam} Q \geq S$, there exists an n-plane L_Q such that

$$\operatorname{dist}(x, L_Q) \leq \varepsilon\operatorname{diam} Q \text{ for all } x\in kQ. \tag{IV.4.11}$$

We will see in Remark IV.4.16 in Section IV.4.6 below that this is guaranteed under the assumption (IV.2.11).

We recall that the n-plane in (IV.4.11) is almost unique in the following sense.

Lemma IV.4.5 (See [DS91, Lemma 5.13])**.** *Let $Q \in \mathcal{D}_\sigma$ and assume that L_1 and L_2 are two n-planes such that* $\operatorname{dist}(x, L_i) \leq \varepsilon' \operatorname{diam} Q$ *for all $x \in Q$, $i = 1, 2$. Then $\angle(L_1, L_2) \leq C\varepsilon'$.*

Let us fix $R \in \mathcal{D}_{\sigma,j_0}$ with $j_0 \in \mathbb{Z}$ so that $2^{j_0} \geq S$; in particular $\operatorname{diam} Q \geq S$ for all $Q \in \mathcal{D}_{\sigma,j_0}$. Fixed $m > 1$, we define the "m-neighborhood cubes" of R as the collection of cubes in $\mathcal{D}_{\sigma,j_0}$ touching mR. That is,

$$U_m(R) \coloneqq \{Q \in \mathcal{D}_{\sigma,j_0} : Q \cap mR \neq \emptyset\}.$$

So, for every $Q \in U_m(R)$ there holds $\operatorname{dist}(Q, R) \leq (m-1)\operatorname{diam} R$. For any two cubes $Q_1, Q_2 \in U_m(R)$, if $x \in Q_1$ then $\operatorname{dist}(x, Q_2) \leq \operatorname{diam} Q_1 + \operatorname{dist}(Q_1, R) + \operatorname{diam} R + \operatorname{dist}(R, Q_2) \leq C_{\mathcal{D}} 2m \operatorname{diam} Q_2$ and therefore

$$Q_1 \subset (C_{\mathcal{D}} 2m + 1)Q_2.$$

Fix a constant $k \gg m$ (at least $k \geq C_{\mathcal{D}} 2m + 1$), so that the inclusion above implies

$$Q_1 \subset kQ_2 \text{ for any } Q_1, Q_2 \in U_m(R).$$

As a consequence of this, (IV.4.11) and Lemma IV.4.5, we have

$$\angle(L_{Q_1}, L_{Q_2}) \lesssim \varepsilon \text{ for any } Q_1, Q_2 \in U_m(R). \tag{IV.4.12}$$

Definition IV.4.6. We say that $Q \in \bigcup_{j \leq j_0} \mathcal{D}_{\sigma,j}$ is a stopping cube with respect to R, and we write $Q \in \mathsf{Stop}(R)$, if $Q \subset \bigcup_{Q' \in U_m(R)} Q'$ and Q is maximal such that either

(1) $\operatorname{diam} Q < S$, or
(2) $\operatorname{diam} Q \geq S$ and $\angle(L_R, L_Q) > \alpha$, recall L_R and L_Q as in (IV.4.11).

Definition IV.4.7. We define the set $\operatorname{Tree}(R)$ as the collection of cubes $P \in \bigcup_{j \leq j_0} \mathcal{D}_{\sigma,j}$ such that

(1) $P \subset \bigcup_{Q' \in U_m(R)} Q'$, and
(2) $P \not\subset Q$ for any $Q \in \mathsf{Stop}(R)$.

In particular, every $Q \in \operatorname{Tree}(R)$ satisfies $\operatorname{diam} Q \geq S$ and $\angle(L_R, L_Q) \leq \alpha$.

Remark IV.4.8. In contrast to the construction in [DS91, Section 8], where only dyadic subcubes of R are considered, here we take dyadic subcubes of $U_m(R)$ in the construction of $\operatorname{Tree}(R)$. In our situation, since $\varepsilon \ll \alpha$ and $\operatorname{diam} Q \geq S$ for all $Q \in U_m(R)$, we note that (IV.4.12) ensures that $Q \notin \mathsf{Stop}(R)$ for all $Q \in U_m(R)$. Consequently, given a cube in $\mathsf{Stop}(R)$, its parent is in $\operatorname{Tree}(R)$.

Consider

$$d(x) = d_R(x) \coloneqq \inf_{Q \in \operatorname{Tree}(R)} \{\operatorname{dist}(x, Q) + \operatorname{diam} Q\}, \quad x \in \mathbb{R}^{n+1}. \tag{IV.4.13}$$

Since $\#\operatorname{Tree}(R) < \infty$, the infimum is in fact a minimum. Note that $d(\cdot) \geq S$ and is 1-Lipschitz, as it is the infimum of 1-Lipschitz functions. Clearly, if $x \in Q \in \operatorname{Tree}(R)$ then $d(x) \leq \operatorname{diam} Q$.

In this subsection, we prove the following result, which is the analog of [DS91, Proposition 8.2]. In this lemma and throughout this section, $L_R^\perp$ denotes the orthogonal line to the n-plane L_R (see (IV.4.11)), $\Pi = \Pi_R$ denotes the orthogonal projection onto L_R, and $\Pi^\perp = \Pi_R^\perp$ denotes the orthogonal projection onto $L_R^\perp$.

Proposition IV.4.9. *For $R \in \mathcal{D}_{\sigma,j_0}$ with $j_0 \in \mathbb{Z}$ so that $2^{j_0} \geq S$ (in particular $\operatorname{diam} R \geq S$), there exists $A : L_R \to L_R^\perp$ Lipschitz graph with norm $\leq C\alpha$ such that*

$$\operatorname{dist}(x, (\Pi_R(x), A(\Pi_R(x)))) \leq C\varepsilon d_R(x) \text{ for all } x \in k_0 R. \tag{IV.4.14}$$

We begin the proof of Proposition IV.4.9. As in the rest of this subsection, the argument follows [DS91, Section 8], adapted to our stopping conditions in Definition IV.4.6. Most of the results below are proved mutatis mutandis as in [DS91, Section 8]. These are Lemmas IV.4.10, IV.4.11, IV.4.13 and IV.4.14, as well as the existence of $Q(i)$ satisfying (IV.4.19a) and (IV.4.19b). For completeness and to keep track of some constants, we provide detailed proofs.

We define in L_R the 1-Lipschitz function

$$D(p) = D_R(p) := \inf_{x \in \Pi^{-1}(p)} d_R(x), \quad p \in L_R, \tag{IV.4.15}$$

which can be rewritten as

$$D(p) = \inf_{x \in \Pi^{-1}(p)} \inf_{Q \in \operatorname{Tree}(R)} \{\operatorname{dist}(x, Q) + \operatorname{diam} Q\} = \inf_{Q \in \operatorname{Tree}(R)} \{\operatorname{dist}(p, \Pi_R(Q)) + \operatorname{diam} Q\}.$$

As $d \geq S$ in $\mathbb{R}^{n+1}$, in particular $D \geq S$ in L_R. As before, since $\#\operatorname{Tree}(R) < \infty$, the latter infimum is in fact a minimum.

Arguing as in the proof of [DS91, Lemma 8.4], we obtain the following, as the proof relies only on the fact that every $Q \in \operatorname{Tree}(R)$ satisfies $\angle(L_R, L_Q) \leq \alpha$.

Lemma IV.4.10 ([DS91, Lemma 8.4])**.** *If $x, y \in 10k_0R$ satisfy $|x-y| \geq 10^{-3} \min\{d(x), d(y)\}$, then*

$$|\Pi^\perp(x) - \Pi^\perp(y)| \leq 2\alpha |\Pi(x) - \Pi(y)|.$$

Proof. Without loss of generality assume that $|x - y| \geq 10^{-3} d(x)$. Let $Q_0 \in \operatorname{Tree}(R)$ such that

$$\operatorname{dist}(x, Q_0) + \operatorname{diam} Q_0 \leq 2d(x) \leq 2 \cdot 10^3 |x - y|.$$

Let $Q \in \operatorname{Tree}(R)$ be the maximal cube with $Q_0 \subset Q$ and $\operatorname{diam} Q \leq 2 \cdot 10^3 |x-y|$. In particular we claim that $|x-y|/k_0 \lesssim \operatorname{diam} Q \leq 2 \cdot 10^3 |x-y|$. Indeed, as $|x-y| \lesssim k_0 \operatorname{diam} R$, if $Q \in \mathcal{D}_{\sigma,j_0}$ then $|x - y| \lesssim k_0 \operatorname{diam} R \approx k_0 \operatorname{diam} Q$, and on the other hand, if $Q \in \mathcal{D}_{\sigma,j}$ with $j < j_0$ then its parent Q' is in $\operatorname{Tree}(R)$ but $\operatorname{diam} Q' > 2 \cdot 10^3 |x - y|$, and hence $\operatorname{diam} Q \gtrsim |x - y|$.

Using that $Q \supset Q_0$, we have

$$\begin{aligned}
\operatorname{dist}(x, Q) + \operatorname{diam} Q &\leq \operatorname{dist}(x, Q_0) + \operatorname{diam} Q \\
&\leq \operatorname{dist}(x, Q_0) + \operatorname{diam} Q_0 + \operatorname{diam} Q \\
&\leq 2 \cdot 10^3 |x - y| + \operatorname{diam} Q \leq 4 \cdot 10^3 |x - y|.
\end{aligned}$$

From this we have $\operatorname{dist}(x, Q) \lesssim |x - y| \lesssim k_0 \operatorname{diam} Q$, and also $\operatorname{dist}(y, Q) \leq |x - y| + \operatorname{dist}(x, Q) \lesssim k_0 \operatorname{diam} Q$. In particular, $x, y \in kQ$ provided $k \gg k_0$. Therefore, by (IV.4.11) and $\varepsilon \ll \alpha$ we have

$$\operatorname{dist}(x, L_Q) + \operatorname{dist}(y, L_Q) \overset{\text{(IV.4.11)}}{\leq} 2\varepsilon \operatorname{diam} Q \leq C\varepsilon |x - y| \ll \alpha |x - y|.$$

The lemma now follows from the stopping rule $\angle(L_R, L_Q) \leq \alpha$. $\square$

Now we decompose L_R in classical dyadic cubes using the function D_R. First, we shall identify L_R with $\mathbb{R}^n$, and in particular we equip L_R with classical dyadic cubes $\mathcal{D}_{L_R} := \mathcal{D}_{\mathbb{R}^n}$.

For each $x \in L_R$ (recall $D_R \geq S > 0$ in L_R), let $R_x \in \mathcal{D}_{L_R}$ be the largest dyadic cube containing x and satisfying

$$\operatorname{diam} R_x \leq 20^{-1} \inf_{u \in R_x} D(u). \tag{IV.4.16}$$

We relabel them without repetition as $\{R_i\}_{i\in I}$. Thus, the family $\{R_i\}_{i\in I}$ is pairwise disjoint and covers L_R. (As in [DS91, Section 8], we use the convention that dyadic cubes are closed but are called disjoint if their interiors are disjoint.)

Recall that the definition of tQ is slightly different depending on whether $Q \in \mathcal{D}_\sigma$ or $Q \in \mathcal{D}_{L_R} = \mathcal{D}_{\mathbb{R}^n}$, see (IV.4.10) and p. 9 respectively.

From the definition of the family $\{R_i\}_{i\in I}$ using (IV.4.16) and that D is 1-Lipschitz in L_R, by the same proof of [DS91, Lemma 8.7] we have:

Lemma IV.4.11 ([DS91, Lemma 8.7])**.** *For all $y \in 10R_i$, $i \in I$,*

$$10\operatorname{diam} R_i \leq D(y) \leq 60\operatorname{diam} R_i, \tag{IV.4.17}$$

and in particular, if $10R_i \cap 10R_j \neq \emptyset$, $i, j \in I$, then

$$6^{-1}\operatorname{diam} R_i \leq \operatorname{diam} R_j \leq 6\operatorname{diam} R_i. \tag{IV.4.18}$$

Proof. Since D is 1-Lipschitz, we have $D(y) \geq \min_{u\in R_i} D(u) - 10\operatorname{diam} R_i \geq 10\operatorname{diam} R_i$. For the other inequality, the parent R' of R_i satisfies $\operatorname{diam} R' > 20^{-1} \inf_{u\in R'} D(u)$. Let $z \in R'$ such that $D(z) < 20\operatorname{diam} R' = 40\operatorname{diam} R_i$. Therefore, since D is 1-Lipschitz, we get $D(y) \leq D(z) + 20\operatorname{diam} R_i \leq 60\operatorname{diam} R_i$, as claimed. □

Let $U_0 := L_R \cap B(\Pi(x_R), 2k_0\operatorname{diam} R)$, where x_R is any fixed point in R, and $I_0 := \{i \in I : R_i \cap U_0 \neq \emptyset\}$. We fix $k_0 \gg m$ so that $\bigcup_{Q'\in U_m(R)} \Pi(Q') \subset U_0$. For each $i \in I_0$, let $Q(i) \in \operatorname{Tree}(R)$ such that[IV.5]

$$C^{-1}\frac{1}{k_0}\operatorname{diam} R_i \leq \operatorname{diam} Q(i) \leq C\operatorname{diam} R_i, \text{ and} \tag{IV.4.19a}$$

$$\operatorname{dist}(\Pi(Q(i)), R_i) \leq C\operatorname{diam} R_i. \tag{IV.4.19b}$$

Such cubes $Q(i)$ exist, because if $p \in R_i$, then there exist $Q \in \operatorname{Tree}(R)$ such that

$$\operatorname{dist}(p, \Pi(Q)) + \operatorname{diam} Q \leq 2D(p) \overset{(IV.4.17)}{\leq} 120\operatorname{diam} R_i, \tag{IV.4.20}$$

and we can take $Q(i)$ to be the maximal cube in $\operatorname{Tree}(R)$ with $Q \subset Q(i)$ and $\operatorname{diam} Q(i) \leq 120\operatorname{diam} R_i$. Indeed, just from $Q \subset Q(i)$ we have

$$\operatorname{dist}(\Pi(Q(i)), R_i) \leq \operatorname{dist}(\Pi(Q), R_i) \leq \operatorname{dist}(p, \Pi(Q)) \overset{(IV.4.20)}{\leq} 120\operatorname{diam} R_i,$$

and it remains to see that $k_0\operatorname{diam} Q(i) \gtrsim \operatorname{diam} R_i$. If $Q(i) \in \mathcal{D}_{\sigma,j}$ with $j < j_0$, then its parent Q' is in $\operatorname{Tree}(R)$ but $\operatorname{diam} Q' > 120\operatorname{diam} R_i$, whence we get $\operatorname{diam} Q(i) \gtrsim \operatorname{diam} R_i$. When $Q(i) \in \mathcal{D}_{\sigma,j_0}$ (in particular $\operatorname{diam} Q(i) \approx \operatorname{diam} R$), we want to see that $\operatorname{diam} R_i \lesssim k_0\operatorname{diam} Q(i)$. Let us assume that $\operatorname{diam} R_i \geq \widetilde{C}k_0\operatorname{diam} Q(i) \approx \widetilde{C}k_0\operatorname{diam} R$ for a sufficiently large fixed constant $\widetilde{C} = \widetilde{C}(\mathrm{ADR})$, since otherwise we are done. In this case, we have $10R_i \supset U_0 \supset \Pi(Q(i))$ (since $R_i \cap U_0$ as $i \in I_0$), and therefore, taking any $x_i \in Q(i)$, since $\Pi(x_i) \in \Pi(Q(i)) \subset 10R_i$, we have

$$10\operatorname{diam} R_i \overset{(IV.4.17)}{\leq} D(\Pi(x_i)) \leq \operatorname{dist}(\Pi(x_i), \Pi(Q(i))) + \operatorname{diam} Q(i) = \operatorname{diam} Q(i).$$

[IV.5]The correspondence $I_0 \ni i \mapsto Q(i)$ may not be injective.

This contradicts $\operatorname{diam} R_i \geq \widetilde{C} k_0 \operatorname{diam} Q(i)$ if $10\widetilde{C}k_0 > 1$, whence we get that $\operatorname{diam} R_i \leq Ck_0 \operatorname{diam} Q(i)$, as claimed.

Definition IV.4.12. For $i \in I_0$, let $A_i : L_R \to L_R^\perp$ denote the afine function whose graph is the n-plane $L_{Q(i)}$. By the stopping condition, the Lispchitz norm is $\leq 2\alpha$.

We consider a partition of the unity for $V = \bigcup_{i\in I_0} 2R_i$. That is, a family $\{\phi_i\}_{i\in I_0}$ satisfying for all $i \in I_0$ that $\phi_i \geq 0$, $\phi_i \in C_c^1(3R_i)$ (hence $\{\operatorname{supp}\phi_i\}_{i\in I_0}$ has finite overlapping, by (IV.4.18)) and

$$|\nabla \phi_i| \lesssim \frac{1}{\operatorname{diam} R_i}. \tag{IV.4.21}$$

Finally we define A on V by

$$A(p) := \sum_{i\in I_0} \phi_i(p) A_i(p) \text{ for } p \in V. \tag{IV.4.22}$$

By the same proof in [DS91, Lemma 8.17] we have:

Lemma IV.4.13 ([DS91, Lemma 8.17])**.** *If* $10R_i \cap 10R_j \neq \emptyset$*, then* $\operatorname{dist}(Q(i), Q(j)) \leq C\operatorname{diam} R_j$ *and*

$$|A_i(q) - A_j(q)| \leq C\varepsilon \operatorname{diam} R_j \textit{ for all } q \in 100R_j. \tag{IV.4.23}$$

Proof. Let $x \in Q(i)$ and $y \in Q(j)$. If $|x-y| \leq \frac{1}{3}\operatorname{diam} Q(j)$, then

$$\operatorname{dist}(Q(i), Q(j)) \leq |x-y| \leq \frac{1}{3}\operatorname{diam} Q(j) \overset{\text{(IV.4.19a)}}{\leq} \frac{1}{3} C\operatorname{diam} R_j,$$

and the first claim is proved in this case.

Assume $|x-y| \geq \frac{1}{3}\operatorname{diam} Q(j)$. Since $y \in Q(j) \in \operatorname{Tree}(R)$ we have

$$d(y) \leq \operatorname{diam} Q(j) \leq 3|x-y| \leq 10^3|x-y|.$$

So, we can invoke Lemma IV.4.10 (and simply $\alpha \leq 1/2$) to obtain

$$|\Pi^\perp(x) - \Pi^\perp(y)| \leq |\Pi(x) - \Pi(y)|.$$

By the triangle inequality, for every $z, \xi \in L_R$ we have $|\Pi(x) - \Pi(y)| \leq |\Pi(x) - z| + |z - \xi| + |\xi - \Pi(y)|$, and so, taking $z \in R_i$ and $\xi \in R_j$ minimizing the first and third term respectively, we have

$$|\Pi(x) - \Pi(y)| \leq \operatorname{dist}(\Pi(x), R_i) + \max_{z\in R_i, \xi \in R_j} |z - \xi| + \operatorname{dist}(\Pi(y), R_j).$$

The third term in the right-hand side is bounded by

$$\begin{aligned}
\operatorname{dist}(\Pi(y), R_j) &\leq \operatorname{diam} \Pi(Q(j)) + \operatorname{dist}(\Pi(Q(j)), R_j) \\
&\leq \operatorname{diam} Q(j) + \operatorname{dist}(\Pi(Q(j)), R_j) \\
&\leq C\operatorname{diam} R_j
\end{aligned}$$

where the last inequality is by the choice of $Q(j)$ in (IV.4.19a) and (IV.4.19b). By symmetry, the first term in the right-hand side is bounded by $\operatorname{dist}(\Pi(x), R_i) \leq C\operatorname{diam} R_i$, which is controlled by $\leq 6C\operatorname{diam} R_j$ by (IV.4.18). For the second term in the right-hand side, for any $z \in R_i$, since $10R_i \cap 10R_j \neq \emptyset$ implies that $\operatorname{diam} R_i \approx \operatorname{diam} R_j$ by (IV.4.18), in particular $z \in CR_j$, we also have $\max_{z\in R_i, \xi\in R_j} |z - \xi| \lesssim \operatorname{diam} R_j$. All in all,

$$|\Pi^\perp(x) - \Pi^\perp(y)| \leq |\Pi(x) - \Pi(y)| \leq C\operatorname{diam} R_j.$$

Therefore, using simply that $x \in Q(i)$ and $y \in Q(j)$ in the first inequality, we conclude

$$\operatorname{dist}(Q(i), Q(j)) \leq |x-y| \leq |\Pi^{\perp}(x) - \Pi^{\perp}(y)| + |\Pi(x) - \Pi(y)| \leq 2|\Pi(x) - \Pi(y)| \leq C \operatorname{diam} R_j,$$

which gives the first claim in the lemma.

Directly from the (already proved) inequality $\operatorname{dist}(Q(i), Q(j)) \leq C \operatorname{diam} R_j$, since by (IV.4.18) and (IV.4.19a) we have $\operatorname{diam} R_i \approx \operatorname{diam} R_j$, $\operatorname{diam} Q(i) \lesssim \operatorname{diam} R_i$ and $\operatorname{diam} Q(j) \lesssim \operatorname{diam} R_j$, for large enough k there holds $Q(j) \subset kQ(i)$. By (IV.4.11), taking $Q = Q(j)$, $L_1 = L_{Q(i)}$ and $L_2 = L_{Q(j)}$ in Lemma IV.4.5, we have $\angle(L_{Q(i)}, L_{Q(j)}) \leq C\varepsilon$, and thus $|A_i(q) - A_j(q)| \leq C\varepsilon \operatorname{diam} R_j$, which concludes the second claim in the lemma. □

Using the previous lemma, the same proof in [DS91, Section 8, p. 46] applies to see that the restriction of A in $2R_j$, $j \in I_0$, is 3α-Lipschitz.

Lemma IV.4.14 ([DS91, (8.19)]). *If $p, q \in 2R_j$, $j \in I_0$, then*

$$|A(p) - A(q)| \leq 3\alpha |p-q|. \tag{IV.4.24}$$

Proof. From the definition of A in (IV.4.22), $\sum_{i \in I_0} \phi_i(\cdot) = 1$ in V, and since A_i is 2α-Lipschitz,

$$\begin{aligned} |A(p) - A(q)| &\leq \left| \sum_{i \in I_0} \phi_i(p)(A_i(p) - A_i(q)) \right| + \left| \sum_{i \in I_0} (\phi_i(p) - \phi_i(q)) A_i(q) \right| \\ &\leq \sum_{i \in I_0} \phi_i(p) |A_i(p) - A_i(q)| + \left| \sum_{i \in I_0} (\phi_i(p) - \phi_i(q)) A_i(q) \right| \\ &\leq 2\alpha |p-q| + \left| \sum_{i \in I_0} (\phi_i(p) - \phi_i(q))(A_i(q) - A_j(q)) \right|. \end{aligned}$$

If $\phi_i(p) \neq 0$ or $\phi_i(q) \neq 0$, without loss of generality $\phi_i(p) \neq 0$ say, then $p \in 3R_i$ (since $\operatorname{supp} \widetilde{\phi}_i \subset 3R_i$). On the other hand, since $p, q \in 2R_j$ by assumption, in particular we have $p \in 3R_i \cap 2R_j \neq \emptyset$, which imply $\operatorname{diam} R_i \approx \operatorname{diam} R_j$ by (IV.4.18). Also from the fact $3R_i \cap 2R_j \neq \emptyset$ and $q \in 2R_j$ (by assumption), we can invoke (IV.4.23) in Lemma IV.4.13 and the last term is bounded by

$$\begin{aligned} \sum_{\substack{i \in I_0 \\ \phi_i(p) + \phi_i(q) > 0}} |\phi_i(p) - \phi_i(q)| |A_i(q) - A_j(q)| &\overset{\text{(IV.4.21)}}{\lesssim} \sum_{\substack{i \in I_0 \\ \phi_i(p) + \phi_i(q) > 0}} \frac{|p-q|}{\operatorname{diam} R_i} |A_i(q) - A_j(q)| \\ &\overset{\text{(IV.4.23)}}{\leq} C\varepsilon |p-q| \sum_{\substack{i \in I_0 \\ \phi_i(p) + \phi_i(q) > 0}} \frac{\operatorname{diam} R_j}{\operatorname{diam} R_i} \\ &\overset{\text{(IV.4.18)}}{\leq} C\varepsilon |p-q| \sum_{\substack{i \in I_0 \\ \operatorname{supp} \phi_i \cap \operatorname{supp} \phi_j \neq 0}} 1 \\ &\leq C\varepsilon |p-q|, \end{aligned}$$

where the last inequality is by the finite overlapping of $\{\operatorname{supp} \phi_i\}_{i \in I_0}$. For $C\varepsilon \leq \alpha$, this concludes the proof of the lemma. □

We now aim to show that A is $C\alpha$-Lipschitz in U_0. We remark that this is the analog of [DS91, (8.20)], with the difference that in our case, we have $D(\cdot) \geq S > 0$ in L_R. For the sake of completeness, we provide the full details of the proof.

Lemma IV.4.15. *If $p, q \in U_0$, then $|A(p) - A(q)| \leq C\alpha|p - q|$.*

Proof. Since $D \geq S > 0$, all points in U_0 belong to some (unique) R_k, where $k \in I_0$. Let $i, j \in I_0$ be such that $p \in R_i$ and $q \in R_j$. If $p, q \in 2R_i$ or $p, q \in 2R_j$, then $|A(p) - A(q)| \leq 3\alpha|p - q|$ by Lemma IV.4.14, and we are done. Hence, from now on, we may assume that $q \notin 2R_i$ and $p \notin 2R_j$. In particular, this implies

$$|p - q| \geq \max\{\operatorname{diam} R_i, \operatorname{diam} R_j\}. \tag{IV.4.25}$$

Let $y \in Q(i)$ and $z \in Q(j)$ be such that $|y - z| = \sup_{a \in Q(i), b \in Q(j)} |a - b|$, which in particular implies[(IV.6)]

$$|y - z| \geq \frac{1}{2} \min\{\operatorname{diam} Q(i), \operatorname{diam} Q(j)\}. \tag{IV.4.26}$$

We have

$$\begin{aligned} |A(p) - A(q)| \leq\, & |A(p) - A_i(p)| + |A_i(p) - A_i(\Pi(y))| + |A_i(\Pi(y)) - \Pi^\perp(y)| \\ & + |\Pi^\perp(y) - \Pi^\perp(z)| \\ & + |A(q) - A_j(q)| + |A_j(q) - A_j(\Pi(z))| + |A_j(\Pi(z)) - \Pi^\perp(z)|. \end{aligned}$$

Let us see that all the terms are bounded by $\lesssim \alpha$. We bound the first four terms, and the last three follow by symmetry.

Term $|A(p) - A_i(p)|$: Using the partition of unity in the second equality, and by (IV.4.23) in Lemma IV.4.13 (since $p \in R_i$, the fact that $\phi_j(p) \neq 0$ implies that $\operatorname{supp} \phi_j \cap R_i \neq \emptyset$, and in particular $3R_j \cap R_i \neq \emptyset$) and since the last sum runs over the index $j \in I_0$ around R_i (in particular a finite number of candidates), we have

$$\begin{aligned} |A(p) - A_i(p)| &= \left| \left(\sum_{j \in I_0} \phi_j(p) A_j(p) \right) - A_i(p) \right| = \left| \sum_{j \in I_0} \phi_j(p)(A_j(p) - A_i(p)) \right| \\ &\overset{(IV.4.23)}{\leq} C\varepsilon \operatorname{diam} R_i \overset{(IV.4.25)}{\leq} C\varepsilon|p - q| \leq \alpha|p - q|. \end{aligned} \tag{IV.4.27}$$

Term $|A_i(p) - A_i(\Pi(y))|$: First, $|A_i(p) - A_i(\Pi(y))| \leq 2\alpha|p - \Pi(y)|$ since A_i is 2α-Lipschitz, and second $|p - \Pi(y)| \leq \operatorname{diam} R_i + \operatorname{dist}(R_i, \Pi(Q(i))) + \operatorname{diam} Q(i)$, which by the choice of $Q(i)$ in (IV.4.19a) and (IV.4.19b) the last two term are controlled by $\lesssim \operatorname{diam} R_i$. We conclude this term by using (IV.4.25).

Term $|A_i(\Pi(y)) - \Pi^\perp(y)|$: Since $\angle(L_{Q(i)}, L_R) \leq \alpha$ is small and by (IV.4.11), we have $|A_i(\Pi(y)) - \Pi^\perp(y)| \leq 2\operatorname{dist}(y, L_{Q(i)}) \leq 2\varepsilon \operatorname{diam} Q(i)$. By the choice of $Q(i)$ in (IV.4.19a) this is controlled by $\leq C\varepsilon \operatorname{diam} R_i$, and by (IV.4.25) the last term is controlled by $\leq C\varepsilon|p - q| \leq \alpha|p - q|$.

Term $|\Pi^\perp(y) - \Pi^\perp(z)|$: First, $y \in Q(i)$ implies that $d(y) \leq \operatorname{diam} Q(i)$, and $z \in Q(j)$ implies $d(z) \leq \operatorname{diam} Q(j)$. Thus, from (IV.4.26) and this we have that $|y - z| \geq$

[(IV.6)]This holds for any two sets A and B, because if we assume that $|a - b| < \frac{1}{2}\operatorname{diam} B$ for all $a \in A$ and $b \in B$, then for any $b_1, b_2 \in B$ and $a_1 \in A$ we have $|b_1 - b_2| \leq |b_1 - a| + |a - b_2| < \operatorname{diam} B$ and hence $\operatorname{diam} B < \operatorname{diam} B$, contradiction.

$\frac{1}{2}\min\{\operatorname{diam} Q(i), \operatorname{diam} Q(j)\} \geq \frac{1}{2}\min\{d(y), d(z)\}$. Using Lemma IV.4.10 in the first inequality we get

$$|\Pi^\perp(y) - \Pi^\perp(z)| \leq 2\alpha|\Pi(y) - \Pi(z)| \leq 2\alpha\left(|\Pi(y) - p| + |p - q| + |q - \Pi(z)|\right).$$

As in the bound of the second term (that is, $|A_i(p) - A_i(\Pi(y))|$), the first term inside the brackets is bounded by $\leq C\operatorname{diam} R_i$. By symmetry, the third term is controlled by $\leq C\operatorname{diam} R_j$. This term follows by (IV.4.25) as well. ☐

Now we can use the Whitney extension theorem to extend A from U_0 to a $C\alpha$-Lipschitz function on all L_R. This gives the existence of the $C\alpha$-Lipschitz graph in Proposition IV.4.9. It remains to see now that this $C\alpha$-Lipschitz graph approximates k_0R in the (IV.4.14) sense.

Proof of (IV.4.14). Let $x \in k_0R$ and set $p = \Pi(x)$. Recall $D(p) \geq S > 0$, and let R_i, $i \in I_0$, so that $p \in R_i$. We first break

$$\operatorname{dist}(x, (\Pi(x), A(\Pi(x)))) \leq |\Pi^\perp(x) - A_i(\Pi(x))| + |A(\Pi(x)) - A_i(\Pi(x))|.$$

Applying [DS91, Lemma 8.21] with $r = D(p)$ and $Q = Q(i)$, we get that $x \in C_{k_0}Q(i)$ for some C_{k_0} depending on k_0, so taking $k \gg C_{k_0}$ we have $x \in kQ(i)$. Using that $\angle(L_R, L_{Q(i)}) \leq \alpha$ is small, the flatness condition in (IV.4.11) and the choice of $Q(i)$ in (IV.4.19a) respectively, we have

$$|\Pi^\perp(x) - A_i(\Pi(x))| \leq 2\operatorname{dist}(x, L_{Q(i)}) \leq 2\varepsilon\operatorname{diam} Q(i) \lesssim \varepsilon\operatorname{diam} R_i.$$

The term $|A(\Pi(x)) - A_i(\Pi(x))|$ is simply the first term in the proof of Lemma IV.4.15, whence we obtain directly from (IV.4.27) that

$$|A(\Pi(x)) - A_i(\Pi(x))| \leq C\varepsilon\operatorname{diam} R_i.$$

All in all, $\operatorname{dist}(x, (\Pi(x), A(\Pi(x)))) \leq C\varepsilon\operatorname{diam} R_i$. From the definition of the family $\{R_i\}_{i\in I}$ in (IV.4.16), we have $\operatorname{diam} R_i \leq 20^{-1}D(p)$. Just from the definition D in (IV.4.15), $D(p) \leq d(x)$. This concludes the proof of (IV.4.14). ☐

IV.4.6. Proof of the localized good lambda inequality (IV.4.9). Let Ω be as in the statement of Theorem IV.2.5. To apply the notation of the section of the Lipschitz graph construction, for $\lambda > 0$ we write $R_W \in W_\lambda$ for the cube from the Whitney covering of Ω_λ (the letters Q, R are reserved for boundary dyadic cubes $\mathcal{D}_\sigma$), and we rewrite the good lambda inequality (IV.4.9) as

(IV.4.28) $$\sigma(\{x \in R_W : K_{l,*}(f\mathbf{1}_{3R_W})(x) > 50\lambda, \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\}) \leq c_\delta\sigma(3R_W).$$

We will use with no mention that we assume that there exists $x_0 \in R_W$ with $\mathcal{M}_{1+\gamma}f(x_0) \leq A\lambda$, otherwise the left-hand side in (IV.4.28) is zero and we are done. Recall also from Remark IV.4.4 that we have $\ell(R_W) \gtrsim T \gg S$.

The thin boundary condition (4) in Lemma IV.1.12 implies, for each $Q \in \mathcal{D}_\sigma$, the existence of a point $c_Q \in Q$, called the center of Q, such that $\operatorname{dist}(c_Q, \partial\Omega \setminus Q) \gtrsim_{\mathrm{ADR},n} \operatorname{diam} Q$, see [DS93, Lemma 3.5 of Part I]. We denote

(IV.4.29) $$B(Q) := B(c_Q, c_1\operatorname{diam} Q), \quad B_Q := B(c_Q, \operatorname{diam} Q) \supset Q,$$

where $c_1 \in (0, 1]$ is chosen so that $B(Q') \cap \partial\Omega \subset Q'$ for all $Q' \in \mathcal{D}_\sigma$, satisfying also that $B(Q_1) \cap B(Q_2) = \emptyset$ for all disjoint $Q_1, Q_2 \in \mathcal{D}_\sigma$. Indeed, assuming without loss of generality that $\operatorname{diam} Q_2 \leq \operatorname{diam} Q_1$, if $B(Q_1) \cap B(Q_2) \neq \emptyset$, then

$$\operatorname{diam} Q_1 \lesssim_{\mathrm{ADR},n} \operatorname{dist}(c_{Q_1}, \partial\Omega \setminus Q_1) \leq |c_{Q_1} - c_{Q_2}| \leq c_1(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \leq 2c_1\operatorname{diam} Q_1,$$

which is not possible if c_1 is chosen to be sufficiently small, depending only on the ADR character of $\partial\Omega$.

We begin the proof of (IV.4.28). Given $R_W \in W_\lambda$, let $j \in \mathbb{Z}$ be so that $\ell(R_W) = 2^j$ (recall $R_W \in \mathcal{D}(\mathbb{R}^{n+1})$), fix any $x \in R_W \cap \partial\Omega$ (if such point does not exist then the left-hand side of (IV.4.28) is zero), and let $R \in \mathcal{D}_{\sigma,j}$ be the cube with $x \in R$. We fix $m = 3$ so that $\partial\Omega \cap 3R_W \subset mR$. Indeed, since $\operatorname{diam} R \geq 2^j$, for all $y \in R_W$ we have $|y - x| \leq 2\operatorname{diam} R_W = 2 \cdot 2^j \leq 2\operatorname{diam} R$, thus $\operatorname{dist}(y, R) \leq |y - x| \leq 2\operatorname{diam} R$, see (IV.4.10).

For every $Q \in U_m(R)$ and every $x \in Q$,

$$|x - c_R| \leq \operatorname{diam} Q + (m-1)\operatorname{diam} R + \operatorname{diam} R \leq C_{\mathcal{D}}(m+1)\operatorname{diam} R,$$

where $C_{\mathcal{D}}$ is the constant in (3) in the boundary dyadic lattice Lemma IV.1.12. Also, if $x \in B_Q \supset B(Q)$ (not necessarily in Q), then

$$|x - c_R| \leq |x - c_Q| + |c_Q - c_R| \leq \operatorname{diam} Q + C_{\mathcal{D}}(m+1)\operatorname{diam} R \leq C_{\mathcal{D}}(m+2)\operatorname{diam} R.$$

This implies that $\bigcup_{Q\in U_m(R)} B_Q \supset \bigcup_{Q\in U_m(R)} B(Q)$ and $\bigcup_{Q\in U_m(R)} Q$ are subsets of

$$\mathfrak{B}_R \coloneqq C_{\mathcal{D}}(m+2)B_R, \tag{IV.4.30}$$

with B_R as in (IV.4.29).

Since σ is ADR, we will use that

$$\sigma(3R_W) \approx \sigma(R) \approx_m \sigma(\mathfrak{B}_R) \approx_m \sigma\left(\bigcup_{Q\in U_m(R)} Q\right).$$

Remark IV.4.16. Under the choice in (IV.4.1), the estimate (IV.4.11) holds with $\varepsilon = 2k\delta_\beta$ and $L_Q = L_{kB_Q}$. Indeed, as $kQ \subset kB_Q$ we have

$$\sup_{x\in kQ} \operatorname{dist}(x, L_Q) \leq \sup_{x\in\partial\Omega\cap kB_Q} \operatorname{dist}(y, L_Q) \overset{\text{(IV.4.1)}}{\leq} 2\delta_\beta k \operatorname{diam} Q.$$

Recall in Section IV.4.5 we needed $\varepsilon = 2k\delta_\beta \ll \alpha$, which is granted by (C4).

We define the following subfamilies of $\mathsf{Stop}(R)$ (see Definition IV.4.6) of big and small angle respectively as

$$\begin{aligned}\mathsf{BA}(R) &\coloneqq \{Q \in \mathsf{Stop}(R) : \angle(L_R, L_Q) > \alpha\}, \text{ and}\\ \mathsf{SA}(R) &\coloneqq \{Q \in \mathsf{Stop}(R) : \angle(L_R, L_Q) \leq \alpha\},\end{aligned}$$

so that $\mathsf{Stop}(R) = \mathsf{BA}(R) \cup \mathsf{SA}(R)$ and the union is disjoint. By the stopping conditions in Definition IV.4.6 in the construction of the Lipschitz graph, we have

$$\begin{aligned}\mathsf{BA}(R) &\supset \{Q \in \mathsf{Stop}(R) : \operatorname{diam} Q \geq S\}, \text{ and}\\ \mathsf{SA}(R) &\subset \{Q \in \mathsf{Stop}(R) : \operatorname{diam} Q < S\}.\end{aligned}$$

Remark IV.4.17. If $Q \in \mathsf{SA}(R)$, then $\operatorname{diam} Q < S$ but its parent $\widehat{Q} \in \operatorname{Tree}(R)$ satisfies $\operatorname{diam} \widehat{Q} \geq S$ as $\widehat{Q} \notin \mathsf{Stop}(R)$, and therefore $\operatorname{diam} Q \approx S$.

Let us define

$$G_R \coloneqq \bigcup_{Q\in\mathsf{SA}(R)} Q.$$

The notation G_R stands for "good" in the following sense:

Lemma IV.4.18. *We have* $\sigma\left(\bigcup_{Q\in U_m(R)} Q\setminus G_R\right) = \sigma(\bigcup_{Q\in\mathsf{BA}(R)} Q) \lesssim \delta^{1/3}\sigma(3R_W)$.

Proof. For each $Q\in\mathsf{BA}(R)$, let L_Q as in (IV.4.11) (with $\varepsilon = 2k\delta_\beta$) and L_{B_Q} as in (IV.4.1). By the same computations in Remark IV.4.16, both L_Q and L_{B_Q} satisfy the assumptions of Lemma IV.4.5 with $\varepsilon = 2k\delta_\beta$, and hence we conclude that $\angle(L_Q, L_{B_Q}) \lesssim k\delta_\beta$. By the same argument, $\angle(L_R, L_{\mathfrak{B}_R}) \lesssim k\delta_\beta$, where $\mathfrak{B}_R$ is the ball in (IV.4.30) so that $\bigcup_{Q\in U_m(R)} Q\subset \mathfrak{B}_R$.

From $\angle(L_Q, L_{B_Q}) \lesssim k\delta_\beta$, $\angle(L_R, L_{\mathfrak{B}_R}) \lesssim k\delta_\beta$, and the stopping condition $\angle(L_R, L_Q) > \alpha$, we get $\angle(L_{\mathfrak{B}_R}, L_{B_Q}) > \alpha/2$ provided $\delta_\beta \ll \alpha/k$, see (C4). From this and (IV.4.2) we have $\angle(m_{\mathfrak{B}_R}\nu, m_{B_Q}\nu) > \alpha/3$ provided $\delta^{\frac{1}{2n}} \ll \alpha$, which is granted by (C4). In particular $|m_{\mathfrak{B}_R}\nu - m_{B_Q}\nu| \gtrsim 1-\cos\alpha$ by the law of cosines.

Therefore, by Chebysheff's inequality we have

$$\begin{aligned}\sigma\left(\bigcup_{Q\in\mathsf{BA}(R)} Q\right) &= \sum_{Q\in\mathsf{BA}(R)} \sigma(Q) \lesssim \frac{1}{1-\cos\alpha}\sum_{Q\in\mathsf{BA}(R)}\int_Q |m_{B_Q}\nu - m_{\mathfrak{B}_R}\nu|\,d\sigma\\ &\le \frac{1}{1-\cos\alpha}\left(\sum_{Q\in\mathsf{BA}(R)}\int_Q |\nu - m_{B_Q}\nu|\,d\sigma + \sum_{Q\in\mathsf{BA}(R)}\int_Q |\nu - m_{\mathfrak{B}_R}\nu|\,d\sigma\right)\\ &\le \frac{1}{1-\cos\alpha}\left(\sum_{Q\in\mathsf{BA}(R)}\int_{B_Q} |\nu - m_{B_Q}\nu|\,d\sigma + \int_{\mathfrak{B}_R} |\nu - m_{\mathfrak{B}_R}\nu|\,d\sigma\right),\end{aligned}$$

where we simply used in the last step that $Q\subset B_Q$ and $\bigcup_{Q\in\mathsf{BA}(R)} Q\subset\bigcup_{Q\in U_m(R)} Q\subset\mathfrak{B}_R$. The first term is controlled by

$$\begin{aligned}\sum_{Q\in\mathsf{BA}(R)}\int_{B_Q}|\nu - m_{B_Q}\nu|\,d\sigma &\overset{\text{(IV.2.12)}}{\le} \delta_*\sum_{Q\in\mathsf{BA}(R)}\sigma(B_Q) \approx \delta_*\sum_{Q\in\mathsf{BA}(R)}\sigma(Q)\\ &= \delta_*\sigma\left(\bigcup_{Q\in\mathsf{BA}(R)} Q\right) \le \delta_*\sigma\left(\bigcup_{Q\in U_m(R)} Q\right) \lesssim \delta_*\sigma(\mathfrak{B}_R).\end{aligned}$$

The other term is controlled by

$$\int_{\mathfrak{B}_R}|\nu - m_{\mathfrak{B}_R}\nu|\,d\sigma \overset{\text{(IV.2.12)}}{\le} \delta_*\sigma(\mathfrak{B}_R).$$

All in all, from the last three inline equations, and using $\delta_*\le\delta$, $\delta\le\alpha^3$ (see (C4)) and $\frac{\delta}{1-\cos(\delta^{1/3})}\le 3\delta^{1/3}$ for small enough δ, we have

$$\sigma\left(\bigcup_{Q\in\mathsf{BA}(R)} Q\right) \lesssim \frac{\delta_*}{1-\cos\alpha}\sigma(\mathfrak{B}_R) \lesssim \delta^{1/3}\sigma(\mathfrak{B}_R).$$

The lemma follows as $\sigma(3R_W)\approx_m \sigma(\mathfrak{B}_R)$. □

Using the set G_R defined above, we decompose

$$\sigma(\{x\in R_W : K_{l,*}(f\mathbf{1}_{3R_W})(x) > 40\lambda, \mathcal{M}_{1+\gamma}f(x)\le A\lambda\}) \le \boxed{1} + \boxed{2} + \boxed{3},$$

where

$$\boxed{1} := \sigma(R_W \setminus G_R),$$
$$\boxed{2} := \sigma(\{x \in G_R : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 20\lambda, \mathcal{M}_{1+\gamma}f(x) \le A\lambda\}), \text{ and}$$
$$\boxed{3} := \sigma(\{x \in \partial\Omega : K_{l,*}(f\mathbf{1}_{3R_W\setminus G_R})(x) > 20\lambda\}).$$

Using the control of the measure of $\bigcup_{Q\in U_m(R)} Q \setminus G_R$ in Lemma IV.4.18, we estimate $\boxed{1}$ and $\boxed{3}$:

Lemma IV.4.19. *We have* $\boxed{1} \le C\delta^{1/3}\sigma(3R_W)$.

Proof. This is just by the inclusion $\partial\Omega \cap 3R_W \subset mR \subset \bigcup_{Q\in U_m(R)} Q$ and Lemma IV.4.18. □

Lemma IV.4.20. *We have* $\boxed{3} \lesssim A^{\sqrt{1+\gamma}}\delta^{\frac{1}{3}(1-\frac{1}{\sqrt{1+\gamma}})}\sigma(3R_W)$. *In particular* $\boxed{3} \le c_\delta\sigma(3R_W)$ *with* $c_\delta \to 0$ *as* $\delta \to 0$, *by (C3).*

Proof. By Chebysheff's inequality and the fact that $K_{l,*}g \le K_*g$ for any $g \in L^p(\sigma)$, we have

$$\boxed{3} \lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}}\int_{\partial\Omega}|K_{l,*}(f\mathbf{1}_{3R_W\setminus G_R})|^{\sqrt{1+\gamma}}\,d\sigma \le \frac{1}{\lambda^{\sqrt{1+\gamma}}}\int_{\partial\Omega}|K_*(f\mathbf{1}_{3R_W\setminus G_R})|^{\sqrt{1+\gamma}}\,d\sigma.$$

Since $L^p(\sigma) \subset L^{\sqrt{1+\gamma}}_{\mathrm{loc}}(\sigma)$ (as we are assuming that $0 < \gamma < p-1$), by (IV.2.6a) (since $\partial\Omega$ is uniformly rectifiable) and Hölder's inequality respectively, the right-hand side term is bounded by

$$\begin{aligned}\frac{1}{\lambda^{\sqrt{1+\gamma}}}\int_{\partial\Omega}|K_*(f\mathbf{1}_{3R_W\setminus G_R})|^{\sqrt{1+\gamma}}\,d\sigma &\lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}}\int_{\partial\Omega}|f\mathbf{1}_{3R_W\setminus G_R}|^{\sqrt{1+\gamma}}\,d\sigma \\ &\lesssim \frac{\sigma(3R_W\setminus G_R)^{1-\frac{1}{\sqrt{1+\gamma}}}}{\lambda^{\sqrt{1+\gamma}}}\left(\int_{3R_W}|f|^{1+\gamma}\,d\sigma\right)^{\frac{1}{\sqrt{1+\gamma}}} \\ &= \frac{\sigma(3R_W\setminus G_R)^{1-\frac{1}{\sqrt{1+\gamma}}}\sigma(3R_W)^{\frac{1}{\sqrt{1+\gamma}}}}{\lambda^{\sqrt{1+\gamma}}}\left(\fint_{3R_W}|f|^{1+\gamma}\,d\sigma\right)^{\frac{1}{\sqrt{1+\gamma}}}.\end{aligned}$$

Using that $\left(\fint_{3R_W}|f|^{1+\gamma}\,d\sigma\right)^{\frac{1}{\sqrt{1+\gamma}}} \lesssim \mathcal{M}_{1+\gamma}f(x_0)^{\sqrt{1+\gamma}}$, $\mathcal{M}_{1+\gamma}f(x_0) \le A\lambda$ and $3R_W \subset U_m(R)$ in the latter term, we conclude

$$\boxed{3} \lesssim A^{\sqrt{1+\gamma}}\left(\frac{\sigma\left(\bigcup_{Q\in U_m(R)} Q\setminus G_R\right)}{\sigma(3R_W)}\right)^{1-\frac{1}{\sqrt{1+\gamma}}}\sigma(3R_W),$$

and the lemma follows by applying Lemma IV.4.18 in the last term. □

It remains to study the term $\boxed{2}$. As in the terms $\boxed{1}$ and $\boxed{3}$, we want to see:

Lemma IV.4.21. *We have* $\boxed{2} \le c_\delta\sigma(3R_W)$, *with* $c_\delta \to 0$ *as* $\delta \to 0$.

The rest of the section is devoted to the proof of Lemma IV.4.21. By Proposition IV.4.9 there exists a Lipschitz graph Γ_R given by $A : L_R \to L_R^\perp$ with norm $\le C\alpha$ such that

$$\mathrm{dist}(x, (\Pi_R(x), A(\Pi_R(x)))) \le Ck\delta_\beta d_R(x) \text{ for all } x \in k_0R \supset mR \supset 3R_W.$$

In particular, for $x \in Q \in \mathsf{SA}(R)$, if $\widehat{Q} \supset Q$ is the parent of Q then $\widehat{Q} \in \mathrm{Tree}(R)$ and hence $d_R(x) = \inf_{Q\in\mathrm{Tree}(R)}\{\mathrm{dist}(x,Q) + \mathrm{diam}\, Q\} \leq \mathrm{diam}\, \widehat{Q} \approx \mathrm{diam}\, Q \approx S$ and

$$\mathrm{dist}(x, (\Pi_R(x), A(\Pi_R(x)))) \leq Ck\delta_\beta d_R(x) \lesssim k\delta_\beta S.$$

So, if $\delta_\beta \ll 1/k$ is small enough we have

$$\Gamma_R \cap \frac{1}{2}B(Q) \neq \emptyset;$$

see (IV.4.29) for the definition of $B(Q)$. This in particular implies

$$\mathcal{H}^n|_{\Gamma_R}(B(Q)) \approx \sigma(Q).$$

Given $f\mathbf{1}_{3R_W\cap G_R} \in L^p(\sigma)$, let us define an auxiliar function f_{Γ_R} in Γ_R. First, for each $Q \in \mathsf{SA}(R)$, let

(IV.4.31)

$$f_{\Gamma_R}(Q) := \frac{1}{\mathcal{H}^n(\Gamma_R \cap B(Q))}\int_Q f(y)\mathbf{1}_{3R_W\cap G_R}(y)\, d\sigma(y) \approx \fint_Q f(y)\mathbf{1}_{3R_W\cap G_R}(y)\, d\sigma(y),$$

and we define the function f_{Γ_R} in Γ_R as

$$f_{\Gamma_R}(x) := \sum_{Q\in\mathsf{SA}(R)} f_{\Gamma_R}(Q)\mathbf{1}_{B(Q)}(x), \quad x \in \Gamma_R. \tag{IV.4.32}$$

Note that f_{Γ_R} is a piecewise constant function, since $\{B(Q)\}_{Q\in\mathsf{SA}(R)}$ is pairwise disjoint family. This follows from the fact that $\{Q\}_{Q\in\mathsf{SA}(R)}$ is a pairwise disjoint family and the choice of c_1 in (IV.4.29). By definition of f_{Γ_R} and (IV.4.30), we have

$$\mathrm{supp}\, f_{\Gamma_R} \subset \bigcup_{Q\in\mathsf{SA}(R)} \overline{B(Q)} \subset \mathfrak{B}_R. \tag{IV.4.33}$$

From the definition of f_{Γ_R} from $f\mathbf{1}_{3R_W\cap G_R}$ in (IV.4.32) we have that both $f\mathbf{1}_{3R_W\cap Q}\, d\sigma$ and $f_{\Gamma_R}\mathbf{1}_{B(Q)}\, d\mathcal{H}^n|_{\Gamma_R}$ have the same mass for each $Q \in \mathsf{SA}(R)$. That is,

$$\int_{\Gamma_R} f_{\Gamma_R}\mathbf{1}_{B(Q)}\, d\mathcal{H}^n = \mathcal{H}^n(\Gamma_R \cap B(Q))f_{\Gamma_R}(Q) = \int_Q f\mathbf{1}_{3R_W\cap Q}\, d\sigma. \tag{IV.4.34}$$

Before seeing some properties of f_{Γ_R}, let us define the subfamily I_R of $\mathsf{SA}(R)$ as

$$I_R := \{Q \in \mathsf{SA}(R) : \exists x \in Q \text{ with } K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda \text{ and } \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\}.$$

For each $Q \in I_R$, we fix $z_Q \in Q$ (do not confuse with c_Q, the "center" of Q) such that

$$\mathcal{M}_{1+\gamma}f(z_Q) \leq A\lambda. \tag{IV.4.35}$$

The following lemma shows that f_{Γ_R} inherits a localized maximal bound from f, restricted to the cubes $Q \in I_R$.

Lemma IV.4.22. *For each $Q \in I_R$ we have*

$$\left(\fint_{\Gamma_R\cap B(Q)} |f_{\Gamma_R}|^{1+\gamma}\, d\mathcal{H}^n\right)^{\frac{1}{1+\gamma}} \lesssim A\lambda \quad \text{and} \quad \left(\fint_{\Gamma_R\cap\mathfrak{B}_R} |f_{\Gamma_R}|^{1+\gamma}\, d\mathcal{H}^n\right)^{\frac{1}{1+\gamma}} \lesssim A\lambda.$$

Proof. Let z_Q denote the point in Q such that $\mathcal{M}_{1+\gamma}f(z_Q) \leq A\lambda$, see (IV.4.35). The first inequality follows from the definition of f_{Γ_R}, (IV.4.31) and $Q \in I_R$. That is,

$$\left(\fint_{\Gamma_R\cap B(Q)} |f_{\Gamma_R}(y)|^{1+\gamma}\, d\mathcal{H}^n(y)\right)^{\frac{1}{1+\gamma}} = |f_{\Gamma_R}(Q)| \lesssim \fint_Q |f(y)|\, d\sigma(y) \lesssim \mathcal{M}_{1+\gamma}f(z_Q) \leq A\lambda.$$

Similarly, the second inequality is obtained as

$$
\begin{aligned}
\fint_{\Gamma_R\cap\mathfrak{B}_R} |f_{\Gamma_R}(y)|^{1+\gamma}\,d\mathcal{H}^n(y) &\lesssim \frac{1}{\mathcal{H}^n(\Gamma_R\cap\mathfrak{B}_R)}\int_{\Gamma_R\cap\mathfrak{B}_R}\left(\sum_{Q\in\mathsf{SA}(R)}|f_{\Gamma_R}(Q)|\mathbf{1}_{B(Q)}(y)\right)^{1+\gamma} d\mathcal{H}^n(y)\\
&\approx \frac{1}{\ell(R)^n}\sum_{Q\in\mathsf{SA}(R)}\int_{\Gamma_R\cap B(Q)}|f_{\Gamma_R}(Q)|^{1+\gamma}\,d\mathcal{H}^n(y)\\
&\overset{\text{(IV.4.31)}}{\lesssim} \frac{S^n}{\ell(R)^n}\sum_{Q\in\mathsf{SA}(R)}\left(\fint_Q |f(y)|\,d\sigma(y)\right)^{1+\gamma}\\
&\overset{\text{(Jensen)}}{\lesssim} \frac{1}{\ell(R)^n}\sum_{Q\in\mathsf{SA}(R)}\int_Q |f(y)|^{1+\gamma}\,d\sigma(y) \overset{\text{(IV.4.30)}}{\lesssim} \fint_{\mathfrak{B}_R}|f(y)|^{1+\gamma}\,d\sigma(y)\\
&\lesssim \mathcal{M}_{1+\gamma}f(z_Q)^{1+\gamma} \le (A\lambda)^{1+\gamma},
\end{aligned}
$$

where we took any z_Q with $Q\in I_R$ in the last inequality. $\square$

For shortness on the notation, for $\varepsilon>0$ we denote

$$\Phi_\varepsilon(z) := \frac{z}{|z|^{n+1}}\mathbf{1}_{\{|z|>\varepsilon\}} \text{ for } z\in\mathbb{R}^{n+1}\setminus\{0\}.$$

We present two lemmas that we will need in the proof of Lemma IV.4.21.

Lemma IV.4.23. *For all $x\in Q\in I_R$, all $x'\in B(Q)\cap\Gamma_R$ and every $\varepsilon\ge T$,*

$$\left|\int_{\partial\Omega}\Phi_\varepsilon(x-y)f(y)\mathbf{1}_{3R_W\cap G_R}(y)\,d\sigma(y) - \int_{\Gamma_R}\Phi_\varepsilon(x'-y)f_{\Gamma_R}(y)\,d\mathcal{H}^n(y)\right| \le \frac{1}{2}\lambda.$$

Proof. We write

$$
\begin{aligned}
\boxed{M} &= \left|\int(\Phi_\varepsilon(x-y)f\mathbf{1}_{3R_W\cap G_R}\,d\sigma(y) - \Phi_\varepsilon(x'-y)f_{\Gamma_R}\,d\mathcal{H}^n|_{\Gamma_R}(y))\right|\\
&\le \left|\int_{\partial\Omega}(\Phi_\varepsilon(x-y)-\Phi_\varepsilon(x'-y))f\mathbf{1}_{3R_W\cap G_R}\,d\sigma(y)\right|\\
&\quad + \left|\int\Phi_\varepsilon(x'-y)f\mathbf{1}_{3R_W\cap G_R}\,d\sigma(y) - \int\Phi_\varepsilon(x'-y)f_{\Gamma_R}\,d\mathcal{H}^n|_{\Gamma_R}(y)\right|\\
&=: \boxed{M1}+\boxed{M2}.
\end{aligned}
$$

Let us bound the term $\boxed{M1}$. Since $x\in Q$ and $x'\in B(Q)$ satisfy $|x-x'|\le 2\operatorname{diam}Q<2S$, and Φ_ε is the truncated kernel of big scales $\varepsilon\ge T\gg 100S$, in particular $|x-y|\ge T\gg 100S$, we have $|x-x'|/|x-y|\le S/T\le 1/2$. Thus, we can use the cancellation property of the Calderón-Zygmund kernel (or just the mean value theorem) to obtain

$$|\Phi_\varepsilon(x-y)-\Phi_\varepsilon(x'-y)| \lesssim \frac{|x-x'|}{|x-y|^{n+1}}\mathbf{1}_{B(x,\varepsilon/2)^c}(y) \lesssim \frac{S}{|x-y|^{n+1}}\mathbf{1}_{B(x,T/2)^c}(y),$$

where we used in the last step that $\varepsilon \geq T$ and that $x \in Q$ and $x' \in B(Q)$ satisfy $|x-x'| \leq 2S$. From this we get

$$\boxed{M1} \leq \int_{\partial\Omega} |\Phi_\varepsilon(x-y) - \Phi_\varepsilon(x'-y)||f(y)|\mathbf{1}_{G_R}(y)\,d\sigma(y) \lesssim S \int_{B(x,T/2)^c} \frac{|f(y)|}{|x-y|^{n+1}}\,d\sigma(y)$$
$$= S \sum_{k=0}^{\infty} \int_{B(x,2^kT)\setminus B(x,2^{k-1}T)} \frac{|f(y)|}{|x-y|^{n+1}}\,d\sigma(y) \lesssim \frac{S}{T} \sum_{k=0}^{\infty} \frac{1}{2^k} \fint_{B(x,2^kT)} |f|\,d\sigma(y) \overset{\text{(IV.4.35)}}{\lesssim} \frac{S}{T} A\lambda,$$

where we used in the last step that $Q \in I_R$.

It remains to bound the term $\boxed{M2}$. We have

$$\boxed{M2} = \left| \int \Phi_\varepsilon(x'-y) f \mathbf{1}_{3R_W \cap G_R}\,d\sigma(y) - \int \Phi_\varepsilon(x'-y) f_{\Gamma_R}\,d\mathcal{H}^n|_{\Gamma_R}(y) \right|$$
$$= \left| \sum_{\widetilde{Q} \in \mathsf{SA}(R)} \int_{\widetilde{Q}} \Phi_\varepsilon(x'-y) f(y) \mathbf{1}_{3R_W}(y)\,d\sigma(y) - \sum_{\widetilde{Q} \in \mathsf{SA}(R)} \int_{B(\widetilde{Q})} \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\,d\mathcal{H}^n|_{\Gamma_R}(y) \right|$$
$$\leq \sum_{\widetilde{Q} \in \mathsf{SA}(R)} \left| \int \Phi_\varepsilon(x'-y) f(y) \mathbf{1}_{3R_W \cap \widetilde{Q}}(y)\,d\sigma(y) - \int \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y) \mathbf{1}_{B(\widetilde{Q})}(y)\,d\mathcal{H}^n|_{\Gamma_R}(y) \right|.$$

To shorten the notation, we write $\boxed{M2} \leq \sum_{\widetilde{Q} \in \mathsf{SA}(R)} \boxed{M2(\widetilde{Q})}$ where

$$\boxed{M2(\widetilde{Q})} := \left| \int \Phi_\varepsilon(x'-y) \Big(f(y) \mathbf{1}_{3R_W \cap \widetilde{Q}}(y)\,d\sigma(y) - f_{\Gamma_R}(y) \mathbf{1}_{B(\widetilde{Q})}(y)\,d\mathcal{H}^n|_{\Gamma_R}(y) \Big) \right|.$$

Let us study $\boxed{M2(\widetilde{Q})}$ for each $\widetilde{Q} \in \mathsf{SA}(R)$. Since we have that $f(y)\mathbf{1}_{3R_W \cap \widetilde{Q}}(y)\,d\sigma(y) - f_{\Gamma_R}(y)\mathbf{1}_{B(\widetilde{Q})}(y)\,d\mathcal{H}^n|_{\Gamma_R}(y)$ has zero mass for each $\widetilde{Q} \in \mathsf{SA}(R)$, see (IV.4.34), we can add $\Phi_\varepsilon(x'-c_{\widetilde{Q}})$ inside the integral sign in the last term, where $c_{\widetilde{Q}}$ is the "center" of $\widetilde{Q}$,[IV.7] and therefore

$$\boxed{M2(\widetilde{Q})} = \left| \int (\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}}))(f(y)\mathbf{1}_{3R_W \cap \widetilde{Q}}(y)\,d\sigma(y) - f_{\Gamma_R}(y)\mathbf{1}_{B(\widetilde{Q})}(y)\,d\mathcal{H}^n|_{\Gamma_R}(y)) \right|.$$

With this we have

$$\boxed{M2(\widetilde{Q})} \leq \int |\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}})||f(y)|\mathbf{1}_{3R_W \cap \widetilde{Q}}(y)\,d\sigma(y)$$
$$+ \int |\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}})||f_{\Gamma_R}(y)|\mathbf{1}_{B(\widetilde{Q})}(y)\,d\mathcal{H}^n|_{\Gamma_R}(y).$$

For $y \in \widetilde{Q}$ or $y \in B(\widetilde{Q})$, note first that $|y - c_{\widetilde{Q}}| \leq \operatorname{diam} \widetilde{Q} < S$, and second, $|x'-y| > \varepsilon \geq T \gg 100S$ by the truncation of the kernel Φ_ε. Hence, $|y-c_{\widetilde{Q}}|/|y-x'| \leq 1/2$. Therefore, by the cancellation of the Calderón-Zygmund kernel Φ_ε we have

$$|\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}})| \lesssim \frac{|y-c_{\widetilde{Q}}|}{|x'-y|^{n+1}} \leq \frac{S}{|x'-y|^{n+1}}.$$

[IV.7] Any point in $\widetilde{Q}$ would do the job.

Thus, defining

$$\boxed{M2(\widetilde{Q})_\sigma} := \int_{\widetilde{Q}\cap\{|y-x'|\geq T/2\}} \frac{S}{|x'-y|^{n+1}}|f(y)|\,d\sigma(y), \text{ and}$$
$$\boxed{M2(\widetilde{Q})_{\mathcal{H}^n}} := \int_{B(\widetilde{Q})\cap\Gamma_R\cap\{|y-x'|\geq T/2\}} \frac{S}{|x'-y|^{n+1}}|f_{\Gamma_R}(y)|\,d\mathcal{H}^n(y).$$

we have

$$\boxed{M2(\widetilde{Q})} \lesssim \boxed{M2(\widetilde{Q})_\sigma} + \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}}.$$

Summing $\boxed{M2(\widetilde{Q})_\sigma}$ over $\widetilde{Q}\in \mathsf{SA}(R)$, as we did in the bound of $\boxed{M1}$ we have

$$\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_\sigma} \leq S\int_{B(x',T/2)^c} \frac{|f(y)|}{|x'-y|^{n+1}}\,d\sigma(y) \lesssim \frac{S}{T}\sum_{k=0}^\infty \frac{1}{2^k} \fint_{B(x',2^kT)} |f|\,d\sigma \overset{\text{(IV.4.35)}}{\lesssim} \frac{S}{T}A\lambda,$$

where we used in the last step that $Q\in I_R$.

We obtain the same bound for $\sum_{\widetilde{Q}\in\mathsf{SA}(R)}\boxed{M2(\widetilde{Q})_{\mathcal{H}^n}}$. Indeed, by the same computations above we have

(IV.4.36)
$$\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}} \lesssim S\sum_{k=0}^\infty \int_{B(x',T/2)^c} \frac{|f_{\Gamma_R}(y)|}{|x'-y|^{n+1}}\,d\mathcal{H}^n|_{\Gamma_R}(y) \lesssim \frac{S}{T}\sum_{k=0}^\infty \frac{1}{2^k}\fint_{B(x',2^kT)} |f_{\Gamma_R}|\,d\mathcal{H}^n|_{\Gamma_R}.$$

For each $k\geq 0$, we define

$$\mathsf{SA}_k(R) := \{\widetilde{Q}\in\mathsf{SA}(R) : \widetilde{Q}\cap B(x',2^kT)\neq\emptyset\}.$$

Since $|f_{\Gamma_R}(\widetilde{Q})|\mathcal{H}^n|_{\Gamma_R}(B(\widetilde{Q})) \leq \int_{\widetilde{Q}}|f|\,d\sigma$ for all $\widetilde{Q}\in\mathsf{SA}(R)$ (see (IV.4.31)), and $\widetilde{Q}\subset B(x',2^{k+1}T)$ if $\widetilde{Q}\in\mathsf{SA}_k(R)$ (because $\operatorname{diam}\widetilde{Q}<S\ll T$ for all $\widetilde{Q}\in\mathsf{SA}(R)$), we have

(IV.4.37)
$$\begin{aligned}\int_{B(x',2^kT)} |f_{\Gamma_R}|\,d\mathcal{H}^n|_{\Gamma_R} &\leq \sum_{\widetilde{Q}\in\mathsf{SA}_k(R)} \int_{B(\widetilde{Q})} |f_{\Gamma_R}|\,d\mathcal{H}^n|_{\Gamma_R} = \sum_{\widetilde{Q}\in\mathsf{SA}_k(R)} |f_{\Gamma_R}(\widetilde{Q})|\mathcal{H}^n|_{\Gamma_R}(B(\widetilde{Q}))\\ &\lesssim \sum_{\widetilde{Q}\in\mathsf{SA}_k(R)} \int_{\widetilde{Q}} |f|\,d\sigma \leq \int_{B(x',2^{k+1}T)} |f|\,d\sigma.\end{aligned}$$

By (IV.4.36) and (IV.4.37), using now that $Q\in I_R$ (i.e., (IV.4.35)), we conclude

$$\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}} \lesssim \frac{S}{T}\sum_{k=0}^\infty \frac{1}{2^k}\fint_{B(x',2^{k+1}T)} |f|\,d\sigma \lesssim \frac{S}{T}A\lambda, \tag{IV.4.38}$$

as claimed.

All in all, we conclude

$$\boxed{M} \leq \boxed{M1} + \boxed{M2} \leq C\frac{S}{T}A\lambda \leq \frac{1}{2}\lambda,$$

where we use (C6) in the last step. □

From now on, we write the double layer potential in Γ_R as K^{Γ_R}, replacing $\partial\Omega$ and its surface measure σ by Γ_R and $\mathcal{H}^n|_{\Gamma_R}$ in (IV.2.3).

The second lemma for the proof of Lemma IV.4.21 is about a continuity-type result for the large scale double layer potential in Γ_R.

Lemma IV.4.24. *If there is $x \in Q \in I_R$ with $K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(x) > 2\lambda$ then $K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z) > \lambda$ for all $z \in B(Q) \cap \Gamma_R$.*

Proof. It suffices to see that $|K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(x) - K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z)| \leq \lambda$ for all $z \in B(Q) \cap \Gamma_R$. For $\varepsilon \geq T$, since $x, z \in B(Q) \cap \Gamma_R$, $\operatorname{diam} Q < S$ and $\varepsilon \geq T \gg S$, we have $|x-y| \approx |z-y|$ if $|y-x| > \varepsilon$. So, using the cancellation of the Calderón-Zygmund kernel, we have

$$\begin{aligned}|K_\varepsilon^{\Gamma_R}(f_{\Gamma_R})(x) - K_\varepsilon^{\Gamma_R}(f_{\Gamma_R})(z)| &= \left| \int \langle \Phi_\varepsilon(x-y) - \Phi_\varepsilon(z-y), \nu_{\Gamma_R}(y)\rangle f_{\Gamma_R}\, d\mathcal{H}^n|_{\Gamma_R}(y) \right| \\ &\lesssim \int_{|y-x|\geq T/2} \frac{|x-z|}{|x-y|^{n+1}} |f_{\Gamma_R}(y)|\, d\mathcal{H}^n|_{\Gamma_R}(y) \\ &\leq S \int_{|y-x|\geq T/2} \frac{1}{|x-y|^{n+1}} |f_{\Gamma_R}(y)|\, d\mathcal{H}^n|_{\Gamma_R}(y).\end{aligned}$$

Note that the last element is in fact $\sum_{\widetilde{Q}\in \mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}}$ in the proof of Lemma IV.4.23, replacing x' by $x \in Q$. Thus, by the same computations in (IV.4.36), (IV.4.37) and (IV.4.38), we have $|K_\varepsilon^{\Gamma_R}(f_{\Gamma_R})(x) - K_\varepsilon^{\Gamma_R}(f_{\Gamma_R})(z)| \lesssim \frac{S}{T} A\lambda$. By the condition (C6), we conclude that this is bounded by $\leq \lambda$. $\square$

For the proof of Lemma IV.4.21, we will compare the double layer potential in $\partial\Omega$ with the double layer potential in the Lipschitz graph Γ_R, so that we can use Lemmas IV.4.23 and IV.4.24, and known properties of K^{Γ_R}. The following is a simplified version of [HMT10, Theorem 4.34], see also [HMT10, (4.4.9)].

Theorem IV.4.25. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, $\Gamma = \{(x, \varphi(x)) : x \in \mathbb{R}^n\}$ its graph, and $1 < p < \infty$. Then*

$$\|K_*^\Gamma\|_{L^p(\Gamma)} \leq C_{n,p} \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} (1 + \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)})^N, \tag{IV.4.39}$$

for some $N = N(n)$.

Let us see the proof of Lemma IV.4.21. During the proof, $\mathcal{R} = \mathcal{R}^{\partial\Omega}$ and $\mathcal{R}^{\Gamma_R}$ denote the Riesz transform in $\partial\Omega$ and Γ_R respectively, see Definition IV.1.16, and $\mathcal{R}_*$ and $\mathcal{R}_*^{\Gamma_R}$ their respective maximal operators as in (IV.1.10).

Proof of Lemma IV.4.21. For each $Q \in I_R$, let us study first

$$\sigma(\{x \in Q : K_{l,*}(f\mathbf{1}_{3R_W \cap G_R})(x) > 20\lambda, \mathcal{M}_{1+\gamma} f(x) \leq A\lambda\}).$$

For any $x \in Q \in I_R$ and any $x' \in B(Q) \cap \Gamma_R$, we have

$$\begin{aligned}K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) \leq{}& K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(x') + \mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(x) \\ &+ \mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(x') \\ &+ \sup_{\varepsilon\geq T} \left| \int_{\partial\Omega} \Phi_\varepsilon(x-\cdot) f\mathbf{1}_{3R_W\cap G_R}\, d\sigma - \int_{\Gamma_R} \Phi_\varepsilon(x'-\cdot) f_{\Gamma_R}\, d\mathcal{H}^n \right|,\end{aligned}$$

where $N_{\mathfrak{B}_R}$ is the orthogonal unit vector to $L_{\mathfrak{B}_R}$ in (IV.4.1) for the ball $\mathfrak{B}_R$ defined in (IV.4.30). Therefore, if we take $x' \in B(Q) \cap \Gamma_R$ satisfying

$$\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(x') \leq \inf_{z \in B(Q)\cap\Gamma_R} \mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) + \frac{\lambda}{2},$$

then we have that

$$\begin{aligned}
\sigma(\{x \in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda\}) \leq\ & \sigma(\{x \in Q : K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(x') > 2\lambda\}) \\
& + \sigma(\{x \in Q : \mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(x) > \lambda\}) \\
& + \sigma(\{x \in Q : \inf_{z\in B(Q)\cap\Gamma_R} \mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) > \lambda\}) \\
& + \sigma(\{x \in Q : \sup_{\varepsilon\geq T} \Big| \int_{\partial\Omega} \Phi_\varepsilon(x-y) f(y)\mathbf{1}_{3R_W\cap G_R}(y)\, d\sigma(y) \\
& \qquad - \int_{\Gamma_R} \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n(y) \Big| > \lambda\}).
\end{aligned}$$

By Lemma IV.4.23, the last term drops out since the corresponding set is empty, and Lemma IV.4.24 implies that the first is controlled by $\sigma(\{x \in Q : \inf_{z\in B(Q)\cap\Gamma_R} K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z) > \lambda\})$. That is,

$$\begin{aligned}
\sigma(\{x \in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda\}) &\leq \sigma(\{x \in Q : \inf_{z\in B(Q)\cap\Gamma_R} K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z) > \lambda\}) \\
& \quad + \sigma(\{x \in Q : \mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(x) > \lambda\}) \\
& \quad + \sigma(\{x \in Q : \inf_{z\in B(Q)\cap\Gamma_R} \mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) > \lambda\}) \\
&=: \boxed{2_{Q,i}} + \boxed{2_{Q,ii}} + \boxed{2_{Q,iii}}.
\end{aligned}$$

Using the infimum property, $\boxed{2_{Q,i}}$ is treated as

$$\begin{aligned}
\boxed{2_{Q,i}} &\leq \frac{\sigma(Q)}{\lambda^{1+\gamma}} \left(\inf_{z\in B(Q)\cap\Gamma_R} K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z) \right)^{1+\gamma} \approx \frac{\mathcal{H}^n|_{\Gamma_R}(B(Q))}{\lambda^{1+\gamma}} \left(\inf_{z\in B(Q)\cap\Gamma_R} K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z) \right)^{1+\gamma} \\
&\leq \frac{1}{\lambda^{1+\gamma}} \int_{B(Q)\cap\Gamma_R} |K_*^{\Gamma_R}(f_{\Gamma_R})(y)|^{1+\gamma}\, d\mathcal{H}^n(y),
\end{aligned}$$

in the last step we removed the restriction on large scales. By the same computations above,

$$\boxed{2_{Q,iii}} \lesssim \frac{1}{\lambda^{1+\gamma}} \int_{B(Q)\cap\Gamma_R} |\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(y)|^{1+\gamma}\, d\mathcal{H}^n(y).$$

Using Chebysheff's inequality, the term $\boxed{2_{Q,ii}}$ is controlled directly by

$$\boxed{2_{Q,ii}} \leq \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_Q |\mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(y)|^{\sqrt{1+\gamma}}\, d\sigma(y).$$

All in all,

$$
\begin{aligned}
\sigma(\{x \in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda\}) \lesssim \; & \frac{1}{\lambda^{1+\gamma}} \int_{B(Q)\cap\Gamma_R} |K_*^{\Gamma_R}(f_{\Gamma_R})(y)|^{1+\gamma}\, d\mathcal{H}^n(y) \\
& + \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_Q |\mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(y)|^{\sqrt{1+\gamma}}\, d\sigma(y) \\
& + \frac{1}{\lambda^{1+\gamma}} \int_{B(Q)\cap\Gamma_R} |\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(y)|^{1+\gamma}\, d\mathcal{H}^n(y).
\end{aligned}
$$

Summing over all $Q \in I_R \subset \mathsf{SA}(R)$ and using the above inequality in the last step, we have

$$
\begin{aligned}
\boxed{2} &= \sigma(\{x \in G_R : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 20\lambda, \mathcal{M}_{1+\gamma}f(x) \le A\lambda\}) \\
&= \sum_{Q\in I_R} \sigma(\{x \in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 20\lambda, \mathcal{M}_{1+\gamma}f(x) \le A\lambda\}) \\
&\le \sum_{Q\in I_R} \sigma(\{x \in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda\}) \\
&\lesssim \frac{1}{\lambda^{1+\gamma}} \int_{\Gamma_R} |K_*^{\Gamma_R}(f_{\Gamma_R})(y)|^{1+\gamma}\, d\mathcal{H}^n(y) \\
&\quad + \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_{\partial\Omega} |\mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(y) - N_{\mathfrak{B}_R}))(y)|^{\sqrt{1+\gamma}}\, d\sigma(y) \\
&\quad + \frac{1}{\lambda^{1+\gamma}} \int_{\Gamma_R} |\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}))(y)|^{1+\gamma}\, d\mathcal{H}^n(y) \\
&=: \boxed{2_i} + \boxed{2_{ii}} + \boxed{2_{iii}}.
\end{aligned}
$$

The smallness of the term $\boxed{2_i}$ will come from the L^p norm of $K_*^{\Gamma_R}$ in the Lipschitz graph with small constant $\lesssim \alpha$, while we will bound the terms $\boxed{2_{ii}}$ and $\boxed{2_{iii}}$ using the smallness assumption on the unit normal vectors, after using that the Riesz transform is bounded in L^p (with constant depending in p). We do this in the rest of the proof.

Using that the Lipschitz norm of Γ_R is $\lesssim \alpha$ small, by Theorem IV.4.25 we have that the L^q norm (in every $1 < q < \infty$) of $K_*^{\Gamma_R}$ is $\lesssim_q \alpha$, and hence we get

$$
\boxed{2_i} \overset{\text{(IV.4.39)}}{\lesssim} \frac{\alpha^{1+\gamma}}{\lambda^{1+\gamma}} \int_{\Gamma_R} |f_{\Gamma_R}(y)|^{1+\gamma}\, d\mathcal{H}^n(y) \approx \frac{\alpha^{1+\gamma}}{\lambda^{1+\gamma}} \sigma(\mathfrak{B}_R) \fint_{\Gamma_R\cap\mathfrak{B}_R} |f_{\Gamma_R}(y)|^{1+\gamma}\, d\mathcal{H}^n(y),
$$

where we used in the last step that $\operatorname{supp} f_{\Gamma_R} \subset \mathfrak{B}_R$, see (IV.4.33), and we allow the implicit constant to depend on γ. By Lemma IV.4.22, the last averaged integral is controlled by $\lesssim (A\lambda)^{1+\gamma}$ and hence

$$
\boxed{2_i} \lesssim \alpha^{1+\gamma} A^{1+\gamma} \sigma(\mathfrak{B}_R) \le \alpha A^2 \sigma(\mathfrak{B}_R) \approx \alpha A^2 \sigma(3R_W),
$$

where the last second inequality is simply because $0 < \gamma < 1$.

Using that the Riesz transform is bounded in UR sets, Hölder's inequality, and that there exists some point $z \in 3R_W \subset \mathfrak{B}_R$ such that $\mathcal{M}_{1+\gamma} f(z) \leq A\lambda$, we have

$$\begin{aligned}\boxed{2_{ii}} &\lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_{3R_W} (|f(y)||\nu_{\partial\Omega}(y) - N_{\mathfrak{B}_R}|)^{\sqrt{1+\gamma}} \, d\sigma(y) \\ &\lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}} \sigma(3R_W) \left(\fint_{\mathfrak{B}_R} |f(y)|^{1+\gamma} \, d\sigma(y) \right)^{1/\sqrt{1+\gamma}} \left(\fint_{\mathfrak{B}_R} |\nu_{\partial\Omega}(y) - N_{\mathfrak{B}_R}| \, d\sigma(y) \right)^{1-1/\sqrt{1+\gamma}} \\ &\lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}} \sigma(3R_W)(A\lambda)^{\sqrt{1+\gamma}} \left(\fint_{\mathfrak{B}_R} |\nu_{\partial\Omega}(y) - N_{\mathfrak{B}_R}| \, d\sigma(y) \right)^{1-1/\sqrt{1+\gamma}}.\end{aligned}$$

Using the smallness assumption of the oscillation of the unit normal vector in (IV.2.12) and that $|m_{\sigma,\mathfrak{B}_R}\nu_{\partial\Omega} - N_{\mathfrak{B}_R}| \lesssim \delta^{\frac{1}{4n}}$ by (IV.4.3), we have

$$\fint_{\mathfrak{B}_R} |\nu_{\partial\Omega}(y) - N_{\mathfrak{B}_R}| \, d\sigma(y) \leq \fint_{\mathfrak{B}_R} |\nu_{\partial\Omega}(y) - m_{\sigma,\mathfrak{B}_R}\nu_{\partial\Omega}| \, d\sigma(y) + |m_{\sigma,\mathfrak{B}_R}\nu_{\partial\Omega} - N_{\mathfrak{B}_R}| \lesssim \delta^{\frac{1}{4n}}.$$

We conclude then that

$$\boxed{2_{ii}} \leq C\delta_*^{\frac{1-1/\sqrt{1+\gamma}}{4n}} A^2 \sigma(3R_W).$$

The term $\boxed{2_{iii}}$ is treated similarly, using the L^∞ bound of the oscillation of the unit vector as Γ_R is a Lipschitz graph of norm $\lesssim \alpha$. That is, since Γ_R is the Lipschitz of $A : L_R \to L_R^\perp$ in Proposition IV.4.9 with norm $\lesssim \alpha$, then $\angle(\nu_{\Gamma_R}(y), N_R) \lesssim \alpha$ for all $y \in \Gamma_R$, where $N_R \perp L_R$, and on the other hand, since $\operatorname{dist}(x, L_R) \leq 2k\delta_\beta \operatorname{diam} R$ for all $x \in kR \supset R$ by (IV.4.11) and Remark IV.4.16, and $\operatorname{dist}(x, L_{\mathfrak{B}_R}) \leq 2\delta_\beta r_{\mathfrak{B}_R}$ for all $x \in \partial\Omega \cap \mathfrak{B}_R \supset R$ by (IV.4.1), by Lemma IV.4.5 we have $\angle(N_{\mathfrak{B}_R}, N_R) = \angle(L_{\mathfrak{B}_R}, L_R) \lesssim k\delta_\beta \leq \alpha$, provided $\delta_\beta \leq \alpha/k$. So, $\angle(\nu_{\Gamma_R}(y), N_{\mathfrak{B}_R}) \lesssim \alpha$ for all $y \in \Gamma_R$, and by the law of cosines we conclude

$$|\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}|^2 = 2 - 2|\cos(\angle(\nu_{\Gamma_R}(y), N_{\mathfrak{B}_R}))| \leq 2 - 2\cos(C\alpha) \approx \alpha^2.$$

Using that the Riesz transform is bounded in Lipschitz sets, $\operatorname{supp} f_{\Gamma_R} \subset \mathfrak{B}_R$ and $|\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}| \lesssim \alpha$, and Lemma IV.4.22, we have

$$\begin{aligned}\boxed{2_{iii}} &\lesssim \frac{1}{\lambda^{1+\gamma}} \int_{\Gamma_R} (|f_{\Gamma_R}(y)||\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}|)^{1+\gamma} \, d\mathcal{H}^n(y) \\ &\lesssim \alpha^{1+\gamma} \mathcal{H}^n|_{\Gamma_R}(\mathfrak{B}_R) \frac{1}{\lambda^{1+\gamma}} \fint_{\Gamma_R \cap \mathfrak{B}_R} |f_{\Gamma_R}(y)|^{1+\gamma} \, d\mathcal{H}^n(y) \\ &\lesssim \alpha^{1+\gamma} A^{1+\gamma} \sigma(3R_W) \leq \alpha A^2 \sigma(3R_W).\end{aligned}$$

All in all,

$$\boxed{2} \leq C\left(\boxed{2_i} + \boxed{2_{ii}} + \boxed{2_{iii}}\right) \leq C(\alpha A^2 + \delta_*^{\frac{1-1/\sqrt{1+\gamma}}{4n}} A^2)\sigma(3R_W),$$

and we conclude the proof by applying the conditions (C3) and (C5). ☐

With this we have that Lemmas IV.4.19 to IV.4.21 are proved. Recall that this implies the good lambda (IV.4.28), a rewrite of the good lambda (IV.4.9). As we already saw below (IV.4.9), it implies the good lambda (IV.4.5), and in particular (IV.4.6) and (IV.2.13). This concludes the proof of Theorem IV.2.5. ☐

IV.5. Uniqueness of the solution of the Neumann problem

In this section we prove the uniqueness (modulo constant) for the Neumann problem in terms of weak derivatives in unbounded 1-sided CAD with unbounded boundary for the range of $2n/(n+1) \leq p < n + \varepsilon_1$.

Proposition IV.5.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded 1-sided CAD with unbounded boundary. Then there exists $\varepsilon_1 = \varepsilon_1(n, CAD) > 0$ such that if $u : \Omega \to \mathbb{R}$ satisfies $\mathcal{N}(\nabla u) \in L^p(\sigma)$ with $p \in [2n/(n+1), n + \varepsilon_1)$ and*

$$\int_\Omega \nabla u(z) \nabla \varphi(z)\, dm(z) = 0, \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}),$$

then u is constant.

Before its proof, let us state some fundamental properties that will be useful in this section. By [HMT10, Proposition 3.24], if Ω is an ADR domain which is either bounded or has an unbounded boundary and $p \in (0, \infty)$, then

$$\|u\|_{L^{p(n+1)/n}(\Omega)} \lesssim \|\mathcal{N} u\|_{L^p(\sigma)}, \tag{IV.5.1}$$

where the involved constant depends on the ADR parameter, independent of the function $u : \Omega \to \mathbb{R}$. If Ω is a 1-sided CAD, by [MMM+22a, (2.609)], there exists a sufficiently large $C > 1$ such that if $\mathcal{N}_\alpha^\varepsilon(\nabla u) \in L^p_{\mathrm{loc}}(\sigma)$ for some $p \in (0, \infty]$ and some $\varepsilon > 0$, then

$$\mathcal{N}_\alpha^{\varepsilon/C} u \in L^p_{\mathrm{loc}}(\sigma). \tag{IV.5.2}$$

In particular, for $p \in (0, \infty)$ and a 1-sided CAD Ω that is either bounded or has unbounded boundary, we claim

$$\mathcal{N}(\nabla u) \in L^p(\sigma) \implies u \in W^{1,p(n+1)/n}(\Omega \cap B) \text{ for any ball } B \text{ centered at } \partial\Omega. \tag{IV.5.3}$$

Indeed, directly from (IV.5.1) we obtain $\nabla u \in L^{p(n+1)/n}(\Omega)$. For a ball B centered at $\partial\Omega$, again by (IV.5.1) we have $\|u\|_{L^{p(n+1)/n}(\Omega\cap B)} \lesssim \|\mathcal{N}^{r_B}(u\mathbf{1}_B)\|_{L^p(\sigma)}$. We note that $\mathrm{supp}(\mathcal{N}^{r_B}(u\mathbf{1}_B)) \subset (2+\alpha)B$, whence we get $\|\mathcal{N}^{r_B}(u\mathbf{1}_B)\|_{L^p(\sigma)} \leq \|\mathcal{N}^{r_B} u\|_{L^p(\sigma|_{(2+\alpha)B})}$, which is bounded using that we can take any $\varepsilon > 0$ in (IV.5.2).

For the type of domains in Proposition IV.5.1, the following lemma relates the nontangential and the weak derivatives.

Lemma IV.5.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition, and let $u : \Omega \to \mathbb{R}$ be harmonic. If $\mathcal{N}(\nabla u) \in L^p(\sigma)$ for some $p \in (1, \infty)$ and the pointwise nontangential limit $(\nabla u)|_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$, then*

$$\int_\Omega \nabla u(z) \nabla \varphi(z)\, dm(z) = \int_{\partial\Omega} \varphi(z) \partial_\nu u(z)\, d\sigma(z), \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}).$$

Proof. If Ω is bounded or has unbounded boundary, we define $D = \Omega$, and if Ω is unbounded and has bounded boundary, then we take a ball B so that $\partial\Omega \subset B/2$ and we define $D = \Omega \cap B$. Note that ∂D is ADR, and we denote $\sigma_D = \mathcal{H}^n|_{\partial D}$. During the proof $\mathcal{N}_\Omega$ and $\mathcal{N}_D$ denote the nontangentially maximal operators in Ω and D respectively. We claim

$$\mathcal{N}_D(\nabla u) \in L^p(\sigma_D). \tag{IV.5.4}$$

This is by assumption when $D = \Omega$. If Ω is unbounded with bounded boundary, we have $\mathcal{N}_D(\nabla u) \leq \mathcal{N}_\Omega(\nabla u)$ on $\partial\Omega$, and since u is harmonic in Ω (in particular $u \in C^\infty(\Omega)$) and for

$x \in \partial B$ we have that $\Gamma_{D,\alpha}(x)$ is uniformly far from $\partial\Omega$ (depending on r_B and $\alpha > 0$), we therefore get $\mathcal{N}_D(\nabla u) < +\infty$ on ∂B. This gives (IV.5.4).

Fixing $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and assuming also $B \supset \operatorname{supp} \varphi$, this is an almost direct consequence of the divergence theorem in [HMT10, Theorem 2.8] for the domain D and the vector field $\varphi\nabla u$.[(IV.8)] Let us check its hypothesis. First, ∂D is ADR, and [HMT10, (2.3.1)] for D is satisfied by the 2-sided corkscrew condition of Ω. Next, $\varphi\nabla u \in C^0(D)$ holds because u is harmonic (and hence smooth) in Ω. For the nontangential maximal function, $\mathcal{N}_D(\varphi\nabla u) \in L^p(\sigma)$ follows from the bound $\mathcal{N}_D(\varphi\nabla u) \le C_\varphi \mathcal{N}_D(\nabla u)$ and the fact that $\mathcal{N}_D(\nabla u) \in L^p(\sigma_D)$ by (IV.5.4). The pointwise nontangential limit $(\varphi\nabla u)|_{\partial D}$ exists σ_D-a.e., since $(\nabla u)|_{\partial\Omega}$ exists σ-a.e. by assumption. Finally, $\operatorname{div}(\varphi\nabla u) \in L^1(D)$ because

$$\int_D |\operatorname{div}(\varphi\nabla u)|\, dm = \int_D |\nabla u \nabla\varphi + \varphi\Delta u|\, dm \le C_\varphi \int_{D\cap B} |\nabla u|\, dm < \infty,$$

where we used $\Delta u = 0$ in D and that $\nabla u \in L^{p(n+1)/n}(D)$ by (IV.5.4) and (IV.5.1).

Having checked these conditions, as $\Delta u = 0$ in Ω and by the divergence theorem in [HMT10, Theorem 2.8] for the domain D and the vector field $\varphi\nabla u$ (with bounded support in D), we have

$$\int_D \nabla u \nabla\varphi\, dm = \int_D \operatorname{div}(\varphi\nabla u)\, dm = \int_{\partial\Omega} \varphi\partial_\nu u\, d\sigma,$$

as claimed. □

As a consequence of Proposition IV.5.1 and Lemma IV.5.2, we get the uniqueness of the Neumann problem with zero nontangential derivative.

Corollary IV.5.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ and $\varepsilon_1 > 0$ be as in Proposition IV.5.1. If $p \in [2n/(n+1), n+\varepsilon_1)$, then constant functions in Ω are the unique solutions of the Neumann boundary value problem*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\sigma), \\ \partial_\nu u = 0, \ \sigma\text{-a.e. on } \partial\Omega. \end{cases} \tag{IV.5.5}$$

The rest of the section is devoted to the proof of Proposition IV.5.1. The proof will use the extra regularity in (IV.5.3) for functions satisfying $\mathcal{N}(\nabla u) \in L^p(\sigma)$, the Hölder regularity (up to the boundary) of $W^{1,2}$-solutions of the Neumann problem in terms of weak derivatives (see Theorem IV.5.4 below), and the well-known Poincaré inequality for uniform domains (see Lemma IV.5.5 below).

Theorem IV.5.4 ([HS25, Corollary 4.49])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be as in Proposition IV.5.1 and let $B := B(\xi, R)$ be a ball of radius $R > 0$ centered at $\xi \in \partial\Omega$. There exist $K = K(CAD, n) \ge 1$, $C = C(CAD, n) < \infty$ and $\alpha = \alpha(CAD, n) > 0$ such that if $u \in W^{1,2}(\Omega \cap B_{KR}(\xi))$ satisfies*

$$\int_\Omega \nabla u(z)\nabla\varphi(z)\, dm(z) = 0, \ \text{for all } \varphi \in W_c^{1,2}(B_{KR}(\xi)), \tag{IV.5.6}$$

(IV.8) As noted in [HMT10, lines 1–6 in proof of Theorem 3.25], the divergence theorem in [HMT10, Theorem 2.8] continues to hold for unbounded domains with unbounded boundary and vector fields with bounded support in the domain.

then for all $0 < r < R/2$ *there holds*

$$|u(x) - u(y)| \leq C \left(\frac{|x-y|}{r}\right)^{\alpha} \left(\fint_{\Omega \cap B_{2r}(\xi)} u(z)^2 \, dm(z)\right)^{1/2} \quad \textit{for all } x, y \in \Omega \cap B_r(\xi).$$

Lemma IV.5.5 (Poincaré inequality)**.** *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a uniform domain and let* B *be a ball centered at* $\partial\Omega$*. There exists* $K > 2$*, depending only on the uniformity constants of* Ω*, such that if* $u \in W^{1,p}(KB \cap \Omega)$*, for* $1 < p < n+1$*, then with* $p + \varepsilon_p = p(n+1)/(n+1-p)$ *there holds*

$$\left(\fint_{\Omega \cap B} |u(z) - m_{\Omega \cap B} u|^{p+\varepsilon_p} \, dm(z)\right)^{1/(p+\varepsilon_p)} \lesssim R \left(\fint_{\Omega \cap KB} |\nabla u(z)|^p \, dm(z)\right)^{1/p}. \tag{IV.5.7}$$

Proof. See [MT24b, Theorem 4.1] for instance. A quick inspection of [MT24b, p. 25, l. 5] reveals that $p + \varepsilon_p = p(n+1)/(n+1-p)$. □

We now turn to the proof of Proposition IV.5.1.

Proof of Proposition IV.5.1. Let us fix a constant $K > 2$ for which both Theorem IV.5.4 and the Poincaré inequality in Lemma IV.5.5 hold, and we allow all subsequent involved constants to depend on K. Fixed $\xi \in \partial\Omega$, we write $B_R := B_R(\xi)$ for $R > 0$. By (IV.5.3) we have $u \in W^{1,p(n+1)/n}(\Omega \cap KB_{10R}) \subset W^{1,p}(\Omega \cap KB_{10R})$, in particular $u \in W^{1,2}(\Omega \cap KB_{10R})$ since we are assuming $p \geq 2n/(n+1)$. By assumption (and a density argument since $u \in W^{1,2}(\Omega \cap KB_{10R})$), it follows that we can apply Theorem IV.5.4 and therefore, for any $x, y \in B(\xi, R/2)$ we have

$$|u(x) - u(y)| \lesssim \left(\frac{|x-y|}{R}\right)^{\alpha} \left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^2 \, dm\right)^{1/2}, \tag{IV.5.8}$$

with $\alpha = \alpha(n, \mathrm{CAD}) > 0$.

If $p \geq 2$, then by the Poincaré inequality (IV.5.7) and Hölder's inequality we have

$$\left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^2 \, dm\right)^{1/2} \lesssim R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^2 \, dm\right)^{1/2} \leq R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^p \, dm\right)^{1/p}.$$

If $2n/(n+1) \leq p < 2$, then by Hölder's inequality and the Poincaré inequality (IV.5.7) (with $p + \varepsilon_p := (n+1)p/(n+1-p)$ which is ≥ 2 since $p \geq 2n/(n+1) \geq 2(n+1)/(n+3)$),

$$\begin{aligned}\left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R(\xi)} u|^2 \, dm\right)^{1/2} &\leq \left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^{p+\varepsilon_p} \, dm\right)^{1/(p+\varepsilon_p)} \\ &\lesssim R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^p \, dm\right)^{1/p}.\end{aligned}$$

That is, in any case we obtain the same bound. From this, Hölder's inequality and (IV.5.1) we have

(IV.5.9)

$$\begin{aligned}\left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^2 \, dm\right)^{1/2} &\lesssim R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^p \, dm\right)^{1/p} \\ &\lesssim \frac{R}{R^{(n+1)\frac{n}{p(n+1)}}} \|\nabla u\|_{L^{p(n+1)/n}(\Omega)} \lesssim R^{1-n/p} \|\mathcal{N}(\nabla u)\|_{L^p(\sigma)}.\end{aligned}$$

From (IV.5.8) and (IV.5.9) we get

$$|u(x) - u(y)| \lesssim |x-y|^\alpha R^{1-n/p-\alpha} \|\mathcal{N}(\nabla u)\|_{L^p(\sigma)}.$$

Having fixed $x, y \in \Omega$ and $\xi \in \partial\Omega$, this bound holds as long as $R > 0$ is big enough so that $x, y \in B(\xi, R/2)$. Taking $R \to \infty$, we obtain that u is constant in Ω provided $p < n/(1-\alpha)$, i.e., $p < n + \varepsilon_1$ with $\varepsilon_1 = \varepsilon_1(\alpha, n) = \varepsilon_1(\mathrm{CAD}, n) > 0$. $\square$

IV.6. Injectivity of $\pm\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ for $p \in [2n/(n+1), n+\varepsilon)$

Recall that for Theorems IV.0.2 and IV.0.3, we are interested in the invertibility of the operators $\frac{1}{2}Id + K$ in $L^{p'}(\sigma)$ and $-\frac{1}{2}Id + K^*$ in $L^p(\sigma)$. By Corollary IV.2.6, it suffices to show that $\frac{1}{2}Id + K^*$ and $-\frac{1}{2}Id + K^*$ are injective in $L^p(\sigma)$.

More precisely, in this section we see the injectivity of $\pm\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ with $2n/(n+1) \leq p < n + \varepsilon_3$, for 2-sided CAD's with unbounded boundary. This is stated in Corollary IV.6.2, and follows from the injectivity of $-\frac{1}{2}Id + K^*$ for unbounded 1-sided CAD with unbounded boundary.

Proposition IV.6.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ and $\varepsilon_1 > 0$ be as in Proposition IV.5.1. Then there exists $0 < \varepsilon_2 = \varepsilon_2(n, CAD) \leq \varepsilon_1$ such that $-\frac{1}{2}Id + K^*$ is injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_2)$.*

Proposition IV.6.1 is proved in Section IV.6.2. Applying this result to $\overline{\Omega}^c$ we deduce the following corollary.

Corollary IV.6.2. *Assume that $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ is an unbounded 1-sided CAD with unbounded boundary, and let $\varepsilon_2 > 0$ be as in Proposition IV.6.1 for the domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Then $\frac{1}{2}Id + K^*$ is injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_2)$.*

Proof. A quick computation reveals $K^*_\Omega = -K^*_{\overline{\Omega}^c}$. Hence, $\frac{1}{2}Id + K^*_\Omega = \frac{1}{2}Id - K^*_{\overline{\Omega}^c}$, which is injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_2)$ by Proposition IV.6.1. $\square$

Corollary IV.6.3. *If $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded 2-sided CAD with unbounded boundary, then there exists $\varepsilon_3 = \varepsilon_3(n, CAD) > 0$ such that both $\pm\frac{1}{2}Id + K^*$ are injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_3)$.*

Proof. Take $\varepsilon_3 := \min\{\varepsilon_2, \widetilde{\varepsilon}_2\}$, with ε_2 and $\widetilde{\varepsilon}_2$ as in Proposition IV.6.1 and Corollary IV.6.2 for the domains Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ respectively. $\square$

By Corollary IV.2.6, this immediately implies that for $p \in [2n/(n+1), n+\varepsilon_3)$, the operators $\pm\frac{1}{2}Id + K$ and $\pm\frac{1}{2}Id + K^*$ are invertible in $L^{p'}(\sigma)$ and $L^p(\sigma)$ respectively, under small enough flatness assumption of the δ-$(s, S; R)$ domain Ω, see Definition IV.0.1.

Remark IV.6.4. For bounded 2-sided corkscrew ADR domains, the injectivity of $\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ for $p \in [2n/(n+1), \infty)$ is proved in [MMM23b, Theorem 1.7.2 (4)]. Since the domain is bounded, the injectivity for the full range $[2n/(n+1), \infty)$ follows from the particular case $p = 2n/(n+1)$ and Hölder's inequality. The case $p = 2$ was already proved in [HMT10, Proposition 5.11]. Both proofs are based on the divergence theorem. In fact, for the so-called regular SKT domains (see [HMT10, Definition 4.8]), the injectivity in $L^2(\sigma)$ of $\frac{1}{2}Id + K^*$ is enough to show that all four operator $(\pm)\frac{1}{2}Id + K^{(*)}$ are invertible in $L^p(\sigma)$, for all $1 < p < \infty$, see [HMT10, Proposition 5.12].

IV.6.1. The single layer potential. Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition (in particular $\partial\Omega$ is UR by Theorem IV.1.8) and $f \in L^1\left(\frac{d\sigma(x)}{1+|x|^n}\right)$. The modified single layer potential operator associated with Ω is

(IV.6.1)
$$\mathcal{S}_{\text{mod}} f(x) = \mathcal{S}_{\text{mod},\Omega} f(x) := \frac{1}{w_n(1-n)} \int_{\partial\Omega} \left(\frac{1}{|x-y|^{n-1}} - \frac{\mathbf{1}_{B_1(0)^c}(y)}{|y|^{n-1}} \right) f(y)\, d\sigma(y), \; x \in \mathbb{R}^{n+1} \setminus \partial\Omega,$$

and its boundary version is defined as

$$S_{\text{mod}} f(x) = S_{\text{mod},\Omega} f(x) := \frac{1}{w_n(1-n)} \int_{\partial\Omega} \left(\frac{1}{|x-y|^{n-1}} - \frac{\mathbf{1}_{B_1(0)^c}(y)}{|y|^{n-1}} \right) f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

Remark IV.6.5. The classical single layer potential $\mathcal{S}$ as well as its boundary version S are defined as $\mathcal{S}_{\text{mod}}$ and S_{mod} but replacing their kernel by $\frac{1}{|x-y|^{n-1}}$. However, this only makes sense for domains with bounded boundary, and in this case, the difference between the classical and the modified single layer potential is constant.

The single layer potential satisfies

$$\nabla \mathcal{S}_{\text{mod}} f(x) = \frac{1}{w_n} \int_{\partial\Omega} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \mathbb{R}^{n+1} \setminus \partial\Omega,$$

see [MMM+22a, (3.39)]. Note that this is (modulo a multiplicative constant) the Riesz transform $\mathcal{R}_\sigma f$. Moreover, it satisfies $\Delta(\mathcal{S}_{\text{mod}} f) = 0$ in $\mathbb{R}^{n+1} \setminus \partial\Omega$, see [MMM+22a, (3.40)], and $\mathcal{S}_{\text{mod}} f$ is continuous through the boundary $\partial\Omega$ in the nontangential sense

$$(\mathcal{S}_{\text{mod}} f)\big|^{\text{nt}}_{\partial\Omega}(z) = S_{\text{mod}} f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{IV.6.2}$$

see [MMM+22a, (3.47)].

Recall the interior and exterior normal nontangential derivatives in (IV.1.5) and (IV.1.6). There is a jump formula for the interior derivative, see [MMM+22a, (3.67)], which is

$$\partial^{\text{int}}_{\nu_\Omega} \mathcal{S}_{\text{mod}} f(x) = \left(-\frac{1}{2} Id + K^* \right) f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{IV.6.3}$$

A quick computation reveals that $K^*_{\overline{\Omega}^c} = -K^*_\Omega$, and using this in the last equality, for σ-a.e. $x \in \partial\Omega$ we have the jump formula for the exterior derivative:

$$\begin{aligned} \partial^{\text{ext}}_{\nu_\Omega} \mathcal{S}_{\text{mod}} f(x) &= \lim_{\substack{z \to x \\ z \in \Gamma^{\overline{\Omega}^c}_\alpha(x)}} \langle \nu(x), \nabla \mathcal{S}_{\text{mod}} f(z) \rangle = - \lim_{\substack{z \to x \\ z \in \Gamma^{\overline{\Omega}^c}_\alpha(x)}} \langle \nu_{\overline{\Omega}^c}(x), \nabla \mathcal{S}_{\text{mod},\overline{\Omega}^c} f(z) \rangle \\ &\overset{\text{(IV.6.3)}}{=} - \left(-\frac{1}{2} Id + K^*_{\overline{\Omega}^c} \right) f(x) = \left(\frac{1}{2} Id + K^* \right) f(x). \end{aligned} \tag{IV.6.4}$$

Given any $\varepsilon > 0$, by [MMM+22a, (3.41)] we have

$$\mathcal{N}^\varepsilon(\mathcal{S}_{\text{mod}} f) \in \bigcap_{0<p<\frac{n}{n-1}} L^p_{\text{loc}}(\sigma). \tag{IV.6.5}$$

Arguing as in the proof of (IV.5.3) (using (IV.6.5) with $p = 1$ instead of (IV.5.2)) we obtain $\mathcal{S}_{\text{mod}} f \in L^{(n+1)/n}(\Omega \cap B)$ for any ball B centered at $\partial\Omega$. Note that the same holds for $\overline{\Omega}^c$ (as it satisfies the same assumptions as Ω), and therefore we have

$$\mathcal{S}_{\text{mod}} f \in L^{(n+1)/n}_{\text{loc}}(\mathbb{R}^{n+1}). \tag{IV.6.6}$$

For $f \in L^p(\sigma)$ with $1 < p < \infty$, the single layer potential satisfies

$$\|\mathcal{N}(\nabla \mathcal{S}_{\mathrm{mod}} f)\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR},\alpha} \|f\|_{L^p(\sigma)}, \tag{IV.6.7}$$

see [MMM+22a, (3.127)]. By (IV.6.7) (and arguing as in (IV.5.4) if Ω is unbounded with bounded boundary) and (IV.5.1), the single layer potential satisfies

$$\nabla(\mathcal{S}_{\mathrm{mod}} f) \in L^{p(n+1)/n}(\Omega \cap B) \text{ for any ball } B \text{ centered at } \partial\Omega. \tag{IV.6.8}$$

In fact, if Ω is bounded or has unbounded boundary, a direct application of (IV.6.7) and (IV.5.1) gives $\nabla(\mathcal{S}_{\mathrm{mod}} f) \in L^{p(n+1)/n}(\Omega)$. As before, the same holds for $\overline{\Omega}^c$. Arguing as in the proof of Lemma IV.5.2[(IV.9)], weak derivatives of the single layer potential in $\mathbb{R}^{n+1}$ exist and its gradient is $\mathbf{1}_{\mathbb{R}^{n+1}\setminus\partial\Omega} \nabla \mathcal{S}_{\mathrm{mod}} f$, whence we get from (IV.6.8) (with both domains Ω and $\overline{\Omega}^c$) that in the weak sense there holds

$$\nabla(\mathcal{S}_{\mathrm{mod}} f) \in L^{p(n+1)/n}_{\mathrm{loc}}(\mathbb{R}^{n+1}), \tag{IV.6.9}$$

and in fact $\nabla(\mathcal{S}_{\mathrm{mod}} f) \in L^{p(n+1)/n}(\mathbb{R}^{n+1})$ if $\partial\Omega$ is unbounded, i.e., both Ω and $\overline{\Omega}^c$ are unbounded. For the sake of completeness, we justify the existence of weak derivatives of $u = \mathcal{S}_{\mathrm{mod}} f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Indeed, for any $1 \leq i \leq n+1$ and $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, we want to see

$$-\int u \partial_{x_i} \varphi \, dm = \int \varphi \mathbf{1}_{\mathbb{R}^{n+1}\setminus\partial\Omega} \partial_{x_i} u \, dm. \tag{IV.6.10}$$

As $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, writing $\Omega^+ = \Omega$, $\Omega^- = \overline{\Omega}^c$ we have

$$\int u \partial_{x_i} \varphi \, dm = \int_{\Omega^+} u \partial_{x_i} \varphi \, dm + \int_{\Omega^-} u \partial_{x_i} \varphi \, dm.$$

We have

$$\int_{\Omega^\pm} u \partial_{x_i} \varphi \, dm = \int_{\Omega^\pm} \partial_{x_i}(u\varphi) \, dm - \int_{\Omega^\pm} \varphi \partial_{x_i} u \, dm$$

For $\Omega^\pm$, we define $D^\pm$ and $\sigma_{D^\pm}$ as in the proof of (IV.5.4), after fixing a ball $B \supset \operatorname{supp} \varphi$ satisfying also $\partial\Omega^\pm \subset B/2$ if $\partial\Omega^\pm$ is bounded. Let us check the assumptions on the divergence theorem in [HMT10, Theorem 2.8] for the ADR domain $D^\pm$ and the vector field V defined as $V_j = u\varphi$ if $j = i$ and $V_j = 0$ otherwise. Clearly $u\varphi \in C^0(D^\pm)$, by (IV.6.6) and (IV.6.8) we have $\operatorname{div} V = \partial_{x_i}(u\varphi) \in L^1(D^\pm)$, by (IV.6.2) we have that $(u\varphi)|_{\partial D^\pm}$ exists $\sigma_{D^\pm}$-a.e. on ∂D, and by (IV.6.5) with $p = 1 + 1/n$ (arguing as in the proof of (IV.5.4)) we get that $\mathcal{N}_{D^\pm}(u\varphi) \leq C_\varphi \mathcal{N}_{D^\pm} u \in L^{1+1/n}(\sigma_{D^\pm})$. Hence, by the divergence theorem [HMT10, Theorem 2.8] we get

$$\int_{D^\pm} \partial_{x_i}(u\varphi) \, dm = \int_{\partial\Omega} u|_{\partial\Omega} \varphi \nu_{\Omega^\pm, i} \, d\sigma,$$

where $\nu_{\Omega^\pm,i}$ is the i-th coordinate of the outward unit vector $\nu_{\Omega^\pm}$. Since $\nu_\Omega = -\nu_{\Omega^-}$ σ-a.e. (see [MMM22b, Lemma 5.10.9(6)]) and by the σ-a.e. continuity of $\mathcal{S}_{\mathrm{mod}} f$ through the boundary (IV.6.2), all in all we obtain (IV.6.10).

(IV.9) By the divergence theorem in [HMT10, Theorem 2.8] using now (IV.6.2), (IV.6.5), (IV.6.6) and (IV.6.8).

IV.6.2. Proof of Proposition IV.6.1. The proof of Proposition IV.6.1 is presented in several steps. In Step 0 we define the notation during the entire proof, in Step 1 we show that the single layer potential is constant in Ω and σ-a.e. on $\partial\Omega$, in Step 2 we find a submean value property for modulo of the single layer potential minus the constant found in Step 1, and from this we deduce in Step 3 that the single layer potential is also constant in the complementary of Ω, with the same value. From this, in Step 4 we conclude the proof of the injectivity.

Step 0: Let $\varepsilon_2 \in (0, \varepsilon_1]$, with $\varepsilon_1 > 0$ as in Corollary IV.5.3. During the proof we write $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{R}^{n+1} \setminus \overline{\Omega}$. Let $f \in L^p(\sigma)$ satisfy $(-\frac{1}{2}Id + K^*)f = 0$ in $L^p(\sigma)$, equivalently,

$$\left(-\frac{1}{2}Id + K^*\right) f = 0 \ \sigma\text{-a.e. on } \partial\Omega. \tag{IV.6.11}$$

We want to show that $f = 0$ σ-a.e. on $\partial\Omega$. To this end, we consider the modified single layer potential

$$u := \mathcal{S}_{\mathrm{mod}} f,$$

which is harmonic in $\mathbb{R}^{n+1} \setminus \partial\Omega$.

Step 1: In this step we prove

$$u \equiv c_0 \text{ constant in } \Omega \text{ and } \sigma\text{-a.e. on } \partial\Omega. \tag{IV.6.12}$$

By (IV.6.3) and (IV.6.11) we have $\partial_{\nu_\Omega} u = (-\frac{1}{2}Id + K^*)f = 0$ σ-a.e. on $\partial\Omega$. Recall that $u : \Omega \to \mathbb{R}$ is harmonic and $\mathcal{N}(\nabla u) \in L^p(\sigma)$ by (IV.6.7). All in all, the assumptions of Corollary IV.5.3 are satisfied, and therefore we conclude that u is constant in Ω. By (IV.6.2), we conclude the proof of (IV.6.12).

Step 2: In this step we prove the following submean value property

$$|u(z) - c_0| \leq \fint_{B_r(z)} |u(x) - c_0| \, dm(x) \text{ for all } z \in \mathbb{R}^{n+1} \setminus \partial\Omega \text{ and } r > 0, \tag{IV.6.13}$$

where c_0 is the constant in (IV.6.12). Note that if u were a continuous function in $\mathbb{R}^{n+1}$, this would immediately follow from the fact that h is harmonic in $\overline{\Omega}^c$ and $h = 0$ in $\overline{\Omega}$. However, due to the lack of continuity, a more careful argument is required.

We define $h(\cdot) := u(\cdot) - c_0$ to shorten the notation. To prove (IV.6.13), we first see that it satisfies

$$\int_{\mathbb{R}^{n+1}} \nabla|h| \nabla\varphi \, dm \leq 0 \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}) \text{ with } \varphi \geq 0, \tag{IV.6.14}$$

called weak subharmonic condition.

Note that, by (IV.6.9) and since $|\nabla|h|| = |\nabla u|$ m-a.e. in $\mathbb{R}^{n+1}$, we have

$$\nabla|h| \in L^{p(n+1)/n}_{\mathrm{loc}}(\mathbb{R}^{n+1}). \tag{IV.6.15}$$

Let us fix $B \supset \operatorname{supp}\varphi$ (not necessarily centered at $\partial\Omega$), and write $\Omega_B^- := \Omega^- \cap B$ from now on. By $\operatorname{supp}\varphi \subset B$ and $h = u - c_0 \equiv 0$ in Ω and σ-a.e. in $\partial\Omega$, we have

$$\int_{\mathbb{R}^{n+1}} \nabla|h| \nabla\varphi \, dm = \int_{\Omega_B^-} \nabla|h| \nabla\varphi \, dm. \tag{IV.6.16}$$

As in [HMT10, (2.3.37)] for the domain Ω_B^-, for $\delta > 0$ we define

$$\chi_\delta(x) := \begin{cases} 1 & \text{if } x \in \Omega_B^- \setminus U_\delta(\partial\Omega_B^-), \\ 2\delta^{-1}\operatorname{dist}(x, \partial U_{\delta/2}(\partial\Omega_B^-)) & \text{if } x \in \Omega_B^- \cap (U_\delta(\partial\Omega_B^-) \setminus U_{\delta/2}(\partial\Omega_B^-)), \\ 0 & \text{if } x \in U_{\delta/2}(\partial\Omega_B^-) \cup (\mathbb{R}^{n+1} \setminus \Omega_B^-). \end{cases}$$

Moreover, we fix $\psi \in C_c^\infty(B_1(0))$ satisfying $\psi \geq 0$ and $\int_{B_1(0)} \psi \, dm = 1$, and define

$$\psi_\varepsilon(\cdot) := \frac{1}{\varepsilon^{n+1}}\psi\left(\frac{\cdot}{\varepsilon}\right), \text{ for } \varepsilon > 0. \tag{IV.6.17}$$

Since by Hölder's inequality and (IV.6.15) we have

$$\int_{\Omega_B^-} \big|(1-\chi_\delta)\nabla|h|\nabla\varphi\big| \, dm \lesssim \|\nabla|h|\|_{L^p(m|_B)} m(U_\delta(\partial\Omega_B^-))^{1-\frac{1}{p}} \lesssim \|\nabla|h|\|_{L^p(m|_B)} \delta^{1-\frac{1}{p}} \xrightarrow{\delta\to 0} 0,$$

we get

$$\int_{\Omega_B^-} \nabla|h|\nabla\varphi \, dm = \lim_{\delta\to 0} \int_{\Omega_B^-} \chi_\delta \nabla|h|\nabla\varphi \, dm. \tag{IV.6.18}$$

Also, since $\nabla|h| \in L^1(m|_B)$ by (IV.6.15), for every $\delta > 0$ we have

$$\int_{\Omega_B^-} \chi_\delta \nabla|h|\nabla\varphi \, dm = \lim_{\varepsilon\to 0} \int_{\Omega_B^-} \chi_\delta \nabla(|h| * \psi_\varepsilon)\nabla\varphi \, dm \tag{IV.6.19}$$

For every $\delta > 0$ and every $0 < \varepsilon < \delta/2$, by $\nabla(\chi_\delta\varphi) = \varphi\nabla\chi_\delta + \chi_\delta\nabla\varphi$ and $\operatorname{div}(\chi_\delta\varphi\nabla(|h|*\psi_\varepsilon)) = \nabla(\chi_\delta\varphi)\nabla(|h| * \psi_\varepsilon) + \chi_\delta\varphi\Delta(|h| * \psi_\varepsilon)$, we have

$$\begin{aligned} \int_{\Omega_B^-} \chi_\delta \nabla(|h| * \psi_\varepsilon)\nabla\varphi \, dm &= \int_{\Omega_B^-} \operatorname{div}(\chi_\delta\varphi\nabla(|h| * \psi_\varepsilon)) \, dm \\ &\quad - \int_{\Omega_B^-} \chi_\delta\varphi\Delta(|h| * \psi_\varepsilon) \, dm - \int_{\Omega_B^-} \varphi\nabla(|h| * \psi_\varepsilon)\nabla\chi_\delta \, dm \end{aligned}$$

The first term on the right-hand side is zero by the classical divergence theorem. Also, we have that $-\Delta(|h| * \psi_\varepsilon) \leq 0$ in $\Omega^- \setminus U_\delta(\partial\Omega)$ for $\varepsilon < \delta/2$, because $|h| * \psi_\varepsilon$ is smooth and $(|h| * \psi_\varepsilon)(z) \leq m_{B(z,r)}(|h| * \psi_\varepsilon)$ for all $z \in \Omega^- \setminus U_\delta(\partial\Omega)$ and $0 < r < \delta/2$, which follows from the fact that $|h|(z) \leq m_{B(z,r)}|h|$ for all $z \in \Omega^-$ and $0 < r < \operatorname{dist}(z, \partial\Omega)$ (since $\Delta h = 0$ in Ω^-). Hence, we get

$$\int_{\Omega_B^-} \chi_\delta \nabla(|h| * \psi_\varepsilon)\nabla\varphi \, dm \leq -\int_{\Omega_B^-} \varphi\nabla(|h| * \psi_\varepsilon)\nabla\chi_\delta \, dm. \tag{IV.6.20}$$

From (IV.6.18), (IV.6.19), (IV.6.20), and that the latter integral in (IV.6.20) converges to $-\int_{\Omega_B^-} \varphi\nabla|h|\nabla\chi_\delta \, dm$ as $\varepsilon \to 0$ because $\nabla|h| \in L^1(m|_B)$, we get

$$\int_{\Omega_B^-} \nabla|h|\nabla\varphi \, dm \leq \lim_{\delta\to 0} -\int_{\Omega_B^-} \nabla\chi_\delta(\varphi\nabla|h|) \, dm. \tag{IV.6.21}$$

Rewriting the latter integral using the notation $\nu_\delta(x) := -\nabla(\operatorname{dist}(x, \partial U_{\delta/2}(\partial\Omega_B^-)))$ and $\widetilde{U}_\delta(\partial\Omega_B^-) := \Omega_B^- \cap (U_\delta(\partial\Omega_B^-) \setminus U_{\delta/2}(\partial\Omega_B^-))$, we now claim (and prove below) that

$$\lim_{\delta\to 0} \frac{2}{\delta} \int_{\widetilde{U}_\delta(\partial\Omega_B^-)} \langle \nu_\delta, \varphi\nabla|h| \rangle \, dm = \int_{\partial\Omega_B^-} \langle \nu_{\Omega_B^-}, (\varphi\nabla|h|)|_{\partial\Omega_B^-} \rangle \, d\mathcal{H}^n. \tag{IV.6.22}$$

Before its proof, let us conclude the proof of (IV.6.14). Using that $\varphi \equiv 0$ on ∂B and the fact that $\partial_{\nu_{\Omega^-}}|h| \leq 0$ because $|h| \geq 0$ converges to zero nontangentially σ-a.e. at $\partial\Omega$ (by (IV.6.12) in Step 1), we have

$$\int_{\partial\Omega_B^-} \langle \nu_{\Omega_B^-}, (\varphi\nabla|h|)|_{\partial\Omega_B^-}\rangle \, d\mathcal{H}^n = \int_{B\cap\partial\Omega} \varphi\partial_{\nu_{\Omega^-}}|h| \, d\sigma \leq 0.$$

This, together with (IV.6.16), (IV.6.21) and (IV.6.22), concludes the proof of (IV.6.14), in the absence of the justification of (IV.6.22).

Let us see the claim in (IV.6.22). We first need to check that

$$\text{(IV.6.23)} \qquad \mathcal{N}(\nabla|h|) \in L^p(\sigma) \text{ and } (\nabla|h|)|_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega.$$

The first condition is satisfied by $|\nabla|h|| = |\nabla u|$ m-a.e. in Ω^- and (IV.6.7). Regarding the second condition, we want to see that for σ-a.e. $x \in \partial\Omega$ there exists a vector v_x such that, for all $\varepsilon > 0$ there is $\delta > 0$ satisfying that

$$z \in \Gamma_\alpha^{\Omega^-}(x) \cap B_\delta(x) \implies |\nabla|h| - v_x| < \varepsilon.$$

By the jump formula (IV.6.4) and the assumption $(-\frac{1}{2}Id + K^*)f = 0$, see (IV.6.11), we have $\partial_{\nu_{\Omega^-}} h = f$ for σ-a.e., in particular the pointwise nontangential limit $(\nabla h)|_{\partial\Omega}$ exists σ-a.e., which we denote by w_x for such $x \in \partial\Omega$. If $w_x = 0$, then $v_x = w_x$ does the job as $|\nabla|h|| = |\nabla h|$. Since $h \equiv 0$ σ-a.e., if $w_x \neq 0$, then for $\delta > 0$ small enough we have that either $h > 0$ or $h < 0$ in $\Gamma_\alpha^{\Omega^-}(x) \cap B_\delta(x)$. If $\pm h > 0$, then we take $v_x = \pm w_x$, and hence we have

$$|\nabla|h| - \pm w_x| = |\pm\nabla u \mp w_x| = |\nabla u - w_x| < \varepsilon,$$

where we used that $\nabla|h| = \frac{h}{|h|}\nabla u = \pm\nabla u$ in $\Gamma_\alpha^{\Omega^-}(x) \cap B_\delta(x)$ (since we are in the case $\pm h > 0$ there) and the fact that the pointwise nontangential limit exists σ-a.e. and is $(\nabla u)|_{\partial\Omega} = w_x$.

As argued in [HMT10, p. 2596, lines 6–8], the equality (IV.6.22) directly holds if one replaces $\varphi\nabla|h|$ by $v \in C^{0,1}(\overline{\Omega_B^-})$. We adapt the end of Step I in the proof of [MMM22b, Theorem 1.3.1] to our situation. By the density of $C_c^\infty(\mathbb{R}^{n+1})$ functions in $L^1(\sigma|_{B\cap\partial\Omega})$, for any $\eta > 0$ there exists $w \in C_c^\infty(\mathbb{R}^{n+1})$ such that

$$\|(\nabla|h|)|_{\partial\Omega} - w\|_{L^1(\sigma|_{B\cap\partial\Omega})} < \eta.$$

First,

$$\int_{\partial\Omega_B^-} |\langle \nu_{\Omega_B^-}, \varphi w - (\varphi\nabla|h|)|_{\partial\Omega}\rangle| \, d\mathcal{H}^n \lesssim \|(\nabla|h|)|_{\partial\Omega} - w\|_{L^1(\sigma|_{B\cap\partial\Omega})} < \eta,$$

second, as we already noted, since φw is in particular Lipschitz in $\overline{\Omega_B^-}$, the equality in (IV.6.22) holds in this case and so we have

$$\lim_{\delta\to 0} \frac{2}{\delta} \int_{\widetilde{U}_\delta(\partial\Omega_B^-)} \langle \nu_\delta, \varphi w\rangle \, dm = \int_{\partial\Omega_B^-} \langle \nu_{\Omega_B^-}, \varphi w\rangle \, d\mathcal{H}^n,$$

and third, by Lemma IV.1.10, for small enough $\delta > 0$ so that $\varphi = 0$ in $B \setminus (1-\delta)B$ we have

$$\frac{2}{\delta} \int_{\widetilde{U}_\delta(\partial\Omega_B^-)} |\langle \nu_\delta, \varphi\nabla|h| - \varphi w\rangle| \, dm \lesssim \|\mathcal{N}^\delta(\nabla|h| - w)\|_{L^1(\sigma|_{B\cap\partial\Omega})},$$

recall the definition of $\mathcal{N}^\delta$ in (IV.1.3). Note that the second condition in (IV.6.23) implies that $\mathcal{N}^\delta(\nabla|h| - w)(x) \to (\nabla|h|)|_{\partial\Omega}(x) - w(x)$ as $\delta \to 0$ for σ-a.e. $x \in \partial\Omega$. From this and

$\mathcal{N}(\nabla|h|) \in L^1(\sigma|_{B\cap\partial\Omega})$, see (IV.6.23), by the dominated convergence theorem we have

$$\lim_{\delta\to 0} \|\mathcal{N}^\delta(\nabla|h| - w)\|_{L^1(\sigma|_{B\cap\partial\Omega})} = \|(\nabla|h|)|_{\partial\Omega} - w\|_{L^1(\sigma|_{B\cap\partial\Omega})} < \eta.$$

All in all, we have

$$\left|\lim_{\delta\to 0}\frac{2}{\delta}\int_{\widetilde{U}_\delta(\partial\Omega_B^-)} \langle \nu_\delta, \varphi\nabla|h|\rangle\, dm - \int_{\partial\Omega_B^-}\langle \nu_{\Omega_B^-}, (\varphi\nabla|h|)|_{\partial\Omega}\rangle\, d\mathcal{H}^n\right| \lesssim \eta,$$

with uniform constant, and as $\eta > 0$ is arbitrary, we conclude the claim in (IV.6.22), and therefore the proof of (IV.6.14) is now complete.

Let us see now how (IV.6.14) implies the submean value property (IV.6.13). With ψ_ε as in (IV.6.17), the function $|h| * \psi_\varepsilon$ is smooth and for $z \in \mathbb{R}^{n+1}$ there holds

$$\text{(IV.6.24)}\quad \begin{aligned} -\Delta(|h| * \psi_\varepsilon)(z) &= \sum_{i=1}^{n+1} -(\partial_{x_i}|h| * \partial_{x_i}\psi_\varepsilon)(z) = -\int \nabla|h|(y)\nabla\psi_\varepsilon(z-y)\, dm(y) \\ &= \int \nabla|h|(y)\nabla(\psi_\varepsilon(z-y))\, dm(y) \overset{\text{(IV.6.14)}}{\leq} 0, \end{aligned}$$

whence we get that $|h| * \psi_\varepsilon$ is subharmonic in $\mathbb{R}^{n+1}$ and in particular we get the submean value property

$$(|h| * \psi_\varepsilon)(z) \leq \fint_{B_r(z)} (|h| * \psi_\varepsilon)(x)\, dm(x) \text{ for all } z \in \mathbb{R}^{n+1} \text{ and } r > 0.$$

For $z \in \mathbb{R}^{n+1} \setminus \partial\Omega$, the left-hand side converges to $|h(z)|$ because $h \in C^0(\mathbb{R}^{n+1}\setminus\partial\Omega)$. On the other hand, since $\||h| * \psi_\varepsilon - |h|\|_{L^1(m|_{B_r(z)})} \to 0$ as $\varepsilon \to 0$ by (IV.6.6), the right-hand side term converges to $\fint_{B_r(z)} |h(x)|\, dm(x)$ as $\varepsilon \to 0$. This concludes the proof of the submean value property (IV.6.13). □

Step 3: In this step we prove

$$\text{(IV.6.25)}\qquad u \equiv c_0 \text{ constant in } \Omega^-.$$

For a fixed $z \in \Omega^-$, taking $r > 2\operatorname{dist}(z, \partial\Omega)$ and $\xi \in \partial\Omega$ so that $|z - \xi| = \operatorname{dist}(z, \partial\Omega)$, we have $B(z, r) \subset B(\xi, 2r)$ and therefore

$$|u(z) - c_0| \overset{\text{(IV.6.13)}}{\leq} \fint_{B(z,r)} |u(x) - c_0|\, dm(x) \lesssim \fint_{B(\xi,2r)} |u(x) - c_0|\, dm(x).$$

Note that c_0 is the mean of u over any ball inside Ω. So, taking $\widetilde{B}$ the interior corkscrew ball of $B(\xi, 2r)$, we have that $\widetilde{B} \subset B(\xi, 2r) \cap \Omega$ with $r_{\widetilde{B}} \approx r$, and therefore we get that the last term above is

$$\fint_{B(\xi,2r)} |u(x) - c_0|\, dm(x) = \fint_{B(\xi,2r)} |u(x) - m_{\widetilde{B}}u|\, dm(x).$$

Adding $\pm m_{B(\xi,2r)}u$, by the classical Poincaré inequality since $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^{n+1})$ (see (IV.6.6) and (IV.6.15)), Hölder's inequality and the estimates (IV.5.1) and (IV.6.7), the latter term is controlled by

$$\fint_{B(\xi,2r)} |u(x) - m_{\widetilde{B}}u|\, dm(x) \lesssim \fint_{B(\xi,2r)} |u(x) - m_{B(\xi,2r)}u|\, dm(x) \lesssim r \fint_{B(\xi,2r)} |\nabla u(x)|\, dm(x).$$

$$\lesssim r^{1-n/p}\|\nabla u\|_{L^{p(n+1)/n}(\Omega)} \overset{\text{(IV.5.1),(IV.6.7)}}{\lesssim} r^{1-n/p}\|f\|_{L^p(\sigma)}$$

All in all, for any $z \in \Omega^-$ and any $r > 2\text{dist}(z, \partial\Omega)$ we get

$$|u(z) - c_0| \lesssim r^{1-n/p}\|f\|_{L^p(\sigma)}. \tag{IV.6.26}$$

In the case $2n/(n+1) \le p < n$, letting $r \to \infty$ we deduce that $u(z) = c_0$, as claimed. For $n \le p < n + \varepsilon_2$, we will use the following lemma to see that (IV.6.26) implies a slightly better bound, as in (IV.5.8).

Lemma IV.6.6. *Let $D \subset \mathbb{R}^{n+1}$ be a domain satisfying the exterior corkscrew condition with constant $M > 1$. There are $\alpha = \alpha(M, n) > 0$ and $C = C(M, n) \ge 1$ such that for every nonnegative smooth subharmonic function u in $\mathbb{R}^{n+1}$, all $\xi \in \partial\Omega$ and all $r > 0$, if $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus U_{r/(4M)^{k_0}}(D)$ for some integer $k_0 \ge 1$, then*

$$u(x) \le C \left(\frac{|x-\xi|}{r}\right)^{\alpha} \sup_{B_r(\xi)} u, \quad \textit{for all } x \in B(\xi, r) \setminus \overline{B(\xi, r/(4M)^{k_0+1})}. \tag{IV.6.27}$$

This is well-known for harmonic functions vanishing continuously on $\partial\Omega$, even for more general domains. However, to keep track of the parameters, the proof is written in full detail at the end of this subsection. Assume the lemma to be true for the moment.

Let us fix $z \in \Omega^-$, and take $r > 2\text{dist}(z, \partial\Omega)$ and $\xi \in \partial\Omega$ so that $\text{dist}(z, \partial\Omega) = |z - \xi|$. With ψ_ε as in (IV.6.17) for $\varepsilon > 0$, as we already saw in (IV.6.24), $|h| * \psi_\varepsilon$ is nonnegative, smooth and subharmonic in $\mathbb{R}^{n+1}$, and identically zero in $\mathbb{R}^{n+1} \setminus U_\varepsilon(\Omega^-)$. So, by Lemma IV.6.6 there is $\alpha_2 > 0$ such that if $0 < \varepsilon \ll r$, then

$$(|h| * \psi_\varepsilon)(z) \overset{\text{(IV.6.27)}}{\lesssim} \left(\frac{|z-\xi|}{r}\right)^{\alpha_2} \sup_{B(\xi,r)} (|h| * \psi_\varepsilon).$$

Also, for $x \in B(\xi, r)$ we have

$$(|h| * \psi_\varepsilon)(x) = \int_{B(x,\varepsilon)} |h|(y)\psi_\varepsilon(x-y)\, dm(y) \overset{\text{(IV.6.26)}}{\lesssim} r^{1-n/p}\|f\|_{L^p(\sigma)},$$

as long as $\varepsilon < r$. All in all, and as $(|h| * \psi_\varepsilon)(z) \to |h(z)| = |u(z) - c_0|$ as $\varepsilon \to 0$ because $h \in C^0(\Omega^-)$, we conclude

$$|u(z) - c_0| \lesssim |z - \xi|^{\alpha_2} r^{1-\frac{n}{p}-\alpha_2}\|f\|_{L^p(\sigma)}.$$

This goes to zero as $r \to \infty$ if $1 < p < n/(1-\alpha_2)$. The parameter $\varepsilon_2 > 0$ is taken so that $n + \varepsilon_2 = \min\{n + \varepsilon_1, n/(1-\alpha_2)\}$. We conclude that $u \equiv c_0$ in Ω^-. □

Step 4: Let us finish the proof by seeing that $f \equiv 0$ in $L^p(\sigma)$. By (IV.6.4) and since $u \equiv c_0$ in Ω^- (see (IV.6.25) in Step 3), we have $(\frac{1}{2}Id + K^*)f = \partial_{\nu_{\overline{\Omega}^c}} u = 0$. On the other hand, recall that $(-\frac{1}{2}Id + K^*)f = 0$ by assumption, see (IV.6.11). We conclude that

$$f = \left(\frac{1}{2}Id + K^*\right) f - \left(-\frac{1}{2}Id + K^*\right) f = 0 \quad \sigma\text{-a.e. on } \partial\Omega,$$

as claimed. □

We conclude this subsection with the proof of Lemma IV.6.6. We adapt the standard argument to show the Hölder continuity of harmonic functions that vanish continuously on the boundary of a domain satisfying the exterior corkscrew condition.

Proof of Lemma IV.6.6. We first claim that there is $c_1 = c_1(M,n) \in (0,1)$ such that for a fixed $\xi \in \partial\Omega$ and $s > 0$, there holds

$$\sup_{B(\xi,s/(4M))} u \le (1-c_1) \sup_{B(\xi,s)} u, \text{ as long as } u \equiv 0 \text{ in } \mathbb{R}^{n+1} \setminus U_{s/(2M)}(D). \tag{IV.6.28}$$

Indeed, by the exterior corkscrew condition there is $B(A_s(\xi), s/M) \subset B(\xi,s) \setminus \overline{D}$, and therefore there exists $\rho \in (s/(2M), s)$ such that

$$\mathcal{H}^n(\partial B(\xi,\rho) \setminus D) \ge \mathcal{H}^n(\partial B(\xi,\rho) \cap B(A_s(\xi), s/(2M))) \gtrsim_{M,n} \rho^n. \tag{IV.6.29}$$

We are using $2M > 2$ to ensure that $u \equiv 0$ in $B(A_s(\xi), s/(2M)) \subset \mathbb{R}^{n+1} \setminus U_{s/(2M)}(D)$. Let H_u be harmonic extension in $B(\xi,\rho)$ of $u|_{\partial B(\xi,\rho)}$. For $x \in B(\xi,\rho)$, using the Poisson kernel and that $u \equiv 0$ in $B(A_s(\xi), s/(2M)) \subset \mathbb{R}^{n+1} \setminus U_{s/(2M)}(D)$, we write H_u as

$$\begin{aligned} H_u(x) &= \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho)} \frac{u(z)}{|x-z|^{n+1}} \, d\mathcal{H}^n(z) \\ &= \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho) \setminus B(A_s(\xi), s/(2M))} \frac{u(z)}{|x-z|^{n+1}} \, d\mathcal{H}^n(z), \end{aligned}$$

where $w_n = \mathcal{H}^n(\partial B(0,1))$. We bound $H_u(x)$ for $x \in B(\xi, \rho/2)$. Bounding u by its supremum in $B(\xi,s) \supset B(\xi,\rho)$, we have

$$\begin{aligned} H_u(x) &\le \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho) \setminus B(A_s(\xi), s/(2M))} \frac{\sup_{B(\xi,s)} u}{|x-z|^{n+1}} \, d\mathcal{H}^n(z) \\ &= \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho)} \frac{\sup_{B(\xi,s)} u}{|x-z|^{n+1}} \, d\mathcal{H}^n(z) \\ &\quad - \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho) \cap B(A_s(\xi), s/(2M))} \frac{\sup_{B(\xi,s)} u}{|x-z|^{n+1}} \, d\mathcal{H}^n(z) \\ &= \left(1 - \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho) \cap B(A_s(\xi), s/(2M))} \frac{1}{|x-z|^{n+1}} \, d\mathcal{H}^n(z)\right) \sup_{B(\xi,s)} u, \end{aligned}$$

where we used in the last equality that the harmonic extension (using the Poisson kernel) of a constant function is the constant itself. Now, using that $|x-\xi| \le \rho/2$ and $|x-z| \le 3\rho/2$ if $x \in B(\xi,\rho/2)$ and $z \in \partial B(\xi,\rho)$, we get

$$\begin{aligned} &\frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int_{\partial B(\xi,\rho) \cap B(A_s(\xi), s/(2M))} \frac{1}{|x-z|^{n+1}} \, d\mathcal{H}^n(z) \\ &\qquad \ge \frac{2^{n-1}}{3^n w_n \rho^n} \mathcal{H}^n(\partial B(\xi,\rho) \cap B(A_s(\xi), s/(2M))) \overset{\text{(IV.6.29)}}{\ge} c_1 \in (0,1), \end{aligned}$$

for some $c_1 = c_1(M,n) \in (0,1)$. All in all, by the inclusion, the maximum principle, and the estimates above, we get

$$\sup_{B(\xi,s/(4M))} u \le \sup_{B(\xi,\rho/2)} u \le \sup_{B(\xi,\rho/2)} H_u \le (1-c_1) \sup_{B(\xi,s)} u,$$

provided that $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus U_{s/(2M)}(D)$, as claimed in (IV.6.28).

As we are assuming $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus U_{r/(4M)^{k_0}}(D)$, iterating (IV.6.28) we get

$$\sup_{B(\xi,r/(4M)^k)} u \leq (1-c_1)^k \sup_{B(\xi,r)} u \text{ for all } 0 \leq k \leq k_0. \tag{IV.6.30}$$

This readily proves the lemma. □

IV.7. Uniqueness of the solution of the Dirichlet problem

Given a Wiener regular[(IV.10)] domain Ω (not necessarily bounded) and $x \in \Omega$, we denote by $g_x = g_x^\Omega$ ($\omega^x = \omega_\Omega^x$ respectively) the harmonic Green function (harmonic measure respectively) of Ω with pole at x. If the domain Ω is ADR and ω^x is absolutely continuous with respect to σ ($\omega^x \ll \sigma$ for shortness) for some (and hence for all) $x \in \Omega$, then there exists a function, which we denote by $d\omega^x/d\sigma$ and it is referred to as the Radon-Nikodym derivative (also known as the Poisson kernel), such that

$$\omega^x(E) = \int_E \frac{d\omega^x}{d\sigma}\, d\sigma, \text{ for any Borel set } E \subset \mathbb{R}^{n+1},$$

see [Mat95, Theorem 2.17] for instance.

This section is dedicated to the uniqueness of the solution of the L^p Dirichlet problem in (IV.0.1) under the assumption that the harmonic measure satisfies the p'-reverse Hölder inequality.

Proposition IV.7.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD, $x \in \Omega$, $p \in (1,\infty)$, and let p' so that $1/p + 1/p' = 1$. Assume $\omega_\Omega^x \ll \sigma$ and that there exists $\gamma > 0$ such that*

$$\left(\fint_{B(\xi,r)} \left| \frac{d\omega_\Omega^x}{d\sigma} \right|^{p'} d\sigma \right)^{1/p'} \lesssim \frac{\omega^x(B(\xi,r))}{\sigma(B(\xi,r))}. \tag{IV.7.1}$$

for every $\xi \in \partial\Omega$ and $0 < r \leq \gamma$. Then the zero function is the unique solution of the homogeneous Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ \mathcal{N} u \in L^p(\sigma) \\ u|_{\partial\Omega}^{\text{nt}} = 0 \ \sigma\text{-a.e. on } \partial\Omega. \end{cases} \tag{IV.7.2}$$

This is well-known when the domain is assumed to be bounded, see for instance [MMM23a, Theorem 5.7.7]. For completeness, we provide a detailed proof in the general case, where Ω is not necessarily bounded. Its proof is a direct combination of Lemmas IV.7.3 and IV.7.4 below. A quick inspection of the proof of Lemma IV.7.3 below reveals that the NTA condition is only used for balls with sufficiently small radii.

We state the extrapolation of the solvability of (D_p) with uniqueness of (IV.0.1).

Corollary IV.7.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD. If (D_{p_0}) is solvable for some $1 < p_0 < \infty$, then there exists $\varepsilon > 0$ such that (D_p) is solvable for all $p \in (p_0 - \varepsilon, \infty)$ and the solution of (D_p) is the unique solution of (IV.0.1).*

[(IV.10)] This ensures that the Green function is well-defined. For instance, domains with the exterior corkscrew condition or whose boundary is ADR are Wiener regular. In fact, satisfying one of these conditions at small scales is sufficient.

Proof. Using that the harmonic measure is doubling in NTA domains (see [JK82, Lemma 4.9][(IV.11)]), this follows by the well-known equivalence between the solvability of the Dirichlet problem and the reverse Hölder for the harmonic measure (see [MPT23, Proposition 2.20][(IV.12)] for instance), Gehring's lemma (see [GM12, Theorem 6.38] for instance), and Proposition IV.7.1. □

We now turn to the first step in the proof of Proposition IV.7.1.

Lemma IV.7.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD domain, $x \in \Omega$, $p \in (1,\infty)$, and let p' so that $1/p + 1/p' = 1$. Assume that there exists $\gamma > 0$ such that (IV.7.1) holds for every $\xi \in \partial\Omega$ and $0 < r \leq \gamma$. Then for any $\delta < \min\{\operatorname{diam}(\partial\Omega), \operatorname{dist}(x, \partial\Omega)\}/100$ there holds*

$$\|\mathcal{N}^\delta g_x^\Omega\|_{L^{p'}(\sigma)} \lesssim \gamma^{-n/p}\delta,$$
$$\|\mathcal{N}^\delta(\nabla g_x^\Omega)\|_{L^{p'}(\sigma)} \lesssim \gamma^{-n/p},$$

where the involved constant depends on p, the CAD character of Ω, the constant in (IV.7.1) (and the aperture on the definition of $\mathcal{N}^\delta$ in (IV.1.3)).

Proof. We define the Hardy-Littlewood maximal function

$$M_\sigma\omega^x(\xi) \coloneqq \sup_{r>0} \frac{\omega^x(B_r(\xi))}{\sigma(B_r(\xi))} = \sup_{r>0} \fint_{B_r(\xi)} \frac{d\omega^x}{d\sigma}\, d\sigma, \quad \xi \in \partial\Omega.$$

Let us fix $\xi \in \partial\Omega$. For any $y \in \Gamma(\xi) \cap \overline{B_{2\delta}(\xi)}$, in particular, $|y-\xi| < (1+\alpha)\operatorname{dist}(y, \partial\Omega)$, we have that y is essentially a corkscrew point at scale $2|y-\xi|$ in the sense that both $y, A_{2|y-\xi|}(\xi) \in B_{2|y-\xi|}(\xi)$ and are uniformly far from the boundary with distance $\gtrsim 2|y-\xi|$. Thus, by Harnack inequality and the relation in [JK82, Lemma 4.8][(IV.11)] between the Green function and the harmonic measure in NTA domains, we have

$$g_x(y) \approx g_x(A_{2|y-\xi|}(\xi)) \lesssim \frac{\omega^x(B(\xi, 2|y-\xi|))}{|y-\xi|^{n-1}}.$$

Therefore, from this and the definition of $M_\sigma\omega^x$, we have

$$\mathcal{N}^\delta g_x(\xi) = \sup_{y \in \Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} g_x(y) \lesssim \sup_{y \in \Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} |y-\xi| \frac{\omega^x(B(\xi, 2|y-\xi|))}{|y-\xi|^n} \lesssim \delta M_\sigma\omega^x(\xi).$$

So, from this and the $L^{p'}$-bound of the Hardy-Littlewood maximal function $M_\sigma\omega^x$ we have

$$\int_{\partial\Omega} |\mathcal{N}^\delta g_x|^{p'}\, d\sigma \lesssim \delta^{p'} \int_{\partial\Omega} |M_\sigma\omega^x|^{p'}\, d\sigma \lesssim_{p'} \delta^{p'} \int_{\partial\Omega} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma.$$

Given by the $5R$-covering theorem, let $\{B_k\}_{k\geq1}$ be a subfamily of $\{B(\xi,\gamma)\}_{\xi\in\partial\Omega}$ such that $\partial\Omega \subset \bigcup_{k\geq1} B_k$ and $\{B_k/5\}_{k\geq1}$ is pairwise disjoint. So, (IV.7.1) applies for each B_k since

(IV.11) This is originally for bounded domains, but the same continues to hold for unbounded domains.

(IV.12) The solvability of the Dirichlet problem in [MPT23, Definition 1.4] is stated for $C_c(\partial\Omega)$ functions, instead of $L^p(\sigma)$. However, it is well-known that this is equivalent to the L^p solvability definition of (D_p) in (IV.0.1) and (IV.0.2), by a density argument.

$r(B_k) = \gamma$. Also, since $\{B_k/5\}_{k\geq 1}$ is a pairwise disjoint family and each ball has the same radius, we also have that the family $\{B_k\}_{k\geq 1}$ has finite overlapping. Thus, we have

$$\int_{\partial\Omega} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma \leq \sum_{k\geq 1} \int_{B_k} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma \approx \gamma^n \sum_{k\geq 1} \fint_{B_k} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma \lesssim \gamma^n \sum_{k\geq 1} \left(\frac{\omega^x(B_k)}{\sigma(B_k)}\right)^{p'}$$
$$\approx \gamma^{n(1-p')} \sum_{k\geq 1} \omega^x(B_k)^{p'} \leq \gamma^{n(1-p')} \sum_{k\geq 1} \omega^x(B_k) \lesssim \gamma^{n(1-p')}.$$

All in all, $\|\mathcal{N}^\delta g_x\|_{L^{p'}(\sigma)} \lesssim \gamma^{-n/p}\delta$.

The $L^{p'}$-norm of the nontangential maximal operator of ∇g_x follows from the same computations above, using that $|\nabla g_x(\cdot)| \lesssim g_x(\cdot)/\mathrm{dist}(\cdot, \partial\Omega)$ far from the pole. Indeed, we now have

$$\mathcal{N}^\delta(\nabla g_x)(\xi) = \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} |\nabla g_x(y)| \lesssim \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} \frac{g_x(y)}{\mathrm{dist}(y, \partial\Omega)} \lesssim \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} \frac{g_x(y)}{|y-\xi|},$$

and the other computations that are needed have already been done. $\square$

We now proceed to the second step in the proof of Proposition IV.7.1.

Lemma IV.7.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ an ADR domain and let u be a solution of (IV.7.2) with $1 < p < \infty$. Take p' so that $1/p + 1/p' = 1$, and fix $z \in \Omega$ and $g_z = g_z^\Omega$. If $\|\mathcal{N}^\delta g_z\|_{L^{p'}(\sigma)} \lesssim \delta$ for small enough $\delta > 0$, then $u \equiv 0$ in Ω.*

Proof. First note that $\|\mathcal{N}^\delta g_x\|_{L^{p'}(\sigma)} \lesssim \delta$ holds for every $x \in \Omega$ with constant depending on $x \in \Omega$ and provided δ is small enough also depending on $x \in \Omega$, because by the symmetry of the harmonic Green function we have

$$g_x(y) = g_y(x) \approx_{x,z} g_y(z) = g_z(y).$$

Let us fix $x \in \Omega$ and $R \geq 2\mathrm{dist}(x, \partial\Omega)$, and define $\Omega_R := \Omega \cap B_R(x)$. We remark that the auxiliary domain Ω_R and its Green function are only necessary if the domain is unbounded, either with bounded or unbounded boundary. If the domain Ω is bounded, one can directly remove the dependence on R by taking $R = 2\mathrm{diam}(\Omega)$ on the computations below.

We will make use of three Green's functions: g_x^Ω, $g_x^{\Omega_R}$ and $g_x^{B_R(x)}$ are the Green functions with pole at x of Ω, Ω_R and $B_R(x)$ respectively. Moreover, it holds that $g_x^{\Omega_R} \leq g_x^\Omega$ and $g_x^{\Omega_R} \leq g_x^{B_R(x)}$ in $\Omega_R \setminus \{x\}$.

Let us fix a dimensional constant $C_1 \geq 1$. For $\delta < \mathrm{dist}(x, \partial\Omega_R)/(4C_1)$, as in [HMT10, (7.1.10) and (7.1.11)], let $\psi_\delta^\Omega \in C^\infty(\Omega)$ such that $0 \leq \psi_\delta^\Omega \leq 1$, $\psi_\delta^\Omega = 1$ in $\Omega \setminus U_{C_1\delta}(\partial\Omega)$, $\psi_\delta^\Omega = 0$ in $U_{C_1\delta/2}(\partial\Omega) \cap \Omega$, and $|\partial^\alpha \psi_\delta^\Omega| \lesssim_\alpha (C_1\delta)^{-|\alpha|}$ for all multi-indices α. Similarly for $B_R(x)$ (with δ instead of $C_1\delta$), define $\psi_\delta^{B_R(x)} \in C_c^\infty(B_R(x))$ such that $0 \leq \psi_\delta^{B_R(x)} \leq 1$, $\psi_\delta^{B_R(x)} = 1$ in $B_R(x) \setminus U_\delta(\partial B_R(x))$, $\psi_\delta^{B_R(x)} = 0$ in $U_{\delta/2}(\partial B_R(x)) \cap B_R(x)$, and $|\partial^\alpha \psi_\delta^{B_R(x)}| \lesssim_\alpha \delta^{-|\alpha|}$ for all multi-indices α. Finally, we define

$$\psi_\delta := \psi_\delta^\Omega \psi_\delta^{B_R(x)} \in C_c^\infty(\Omega_R),$$

which in particular satisfies $0 \leq \psi_\delta \leq 1$, $|\partial^\alpha \psi_\delta| \lesssim_{\alpha, C_1} \delta^{-|\alpha|}$ for all multi-indices α, and moreover

$$\mathrm{dist}(y, \partial\Omega_R) \geq \delta/2 \text{ for all } y \in \operatorname{supp} \nabla\psi_\delta.$$

As in [HMT10, (7.1.12)], we have

$$u(x) = u(x)\psi_\delta(x) = \int_{\Omega_R} \langle \nabla g_x^{\Omega_R}, \nabla(u\psi_\delta)\rangle \, dm = 2\int_{\Omega_R} \langle \nabla g_x^{\Omega_R}, \nabla \psi_\delta\rangle u \, dm + \int_{\Omega_R} g_x^{\Omega_R} u \Delta\psi_\delta \, dm.$$

Using that $|\nabla g_x^{\Omega_R}(\cdot)| \lesssim g_x^{\Omega_R}(\cdot)/\mathrm{dist}(\cdot, \partial\Omega_R) \lesssim g_x^{\Omega_R}(\cdot)/\delta$ in $\operatorname{supp} \nabla\psi_\delta$, and

$$\operatorname{supp} \nabla\psi_\delta \subset (U_{C_1\delta}(\partial\Omega) \cap B_R(x)) \cup ((B_R(x) \setminus \overline{B_{R-\delta}(x)}) \setminus U_{C_1\delta}(\partial\Omega)),$$

we have

$$\begin{aligned} |u(x)| &\lesssim \frac{1}{\delta^2} \int_{\operatorname{supp} \nabla\psi_\delta} g_x^{\Omega_R} |u| \, dm|_\Omega \\ &\le \frac{1}{\delta^2} \int_{U_{C_1\delta}(\partial\Omega) \cap B_R(x)} g_x^{\Omega_R} |u| \, dm|_\Omega + \frac{1}{\delta^2} \int_{(B_R(x) \setminus \overline{B_{R-\delta}(x)}) \setminus U_{C_1\delta}(\partial\Omega)} g_x^{\Omega_R} |u| \, dm|_\Omega \\ &=: \mathrm{I}_\delta + \mathrm{II}_\delta. \end{aligned} \tag{IV.7.3}$$

Let us study the term I_δ. Using that $g^{\Omega_R} \le g^\Omega$ in $\Omega_R \setminus \{x\}$, Lemma IV.1.10 and Hölder's inequality respectively, we have

$$\mathrm{I}_\delta \le \frac{1}{\delta^2} \int_{U_{C_1\delta}(\partial\Omega) \cap B_R(x)} g^\Omega |u| \, dm|_\Omega \lesssim \frac{1}{\delta} \int_{\partial\Omega} \mathcal{N}^{C_1\delta}(g^\Omega |u|) \, d\sigma \le \frac{1}{\delta} \|\mathcal{N}^{C_1\delta} g^\Omega\|_{L^{p'}(\sigma)} \|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)}. \tag{IV.7.4}$$

We now turn to the term II_δ, which is directly zero if the domain is bounded and $R \ge 2\mathrm{diam}(\Omega)$. Before, recall that

$$g_x^{B_R(x)}(y) = \frac{1}{\kappa_n(n-1)} \left(\frac{1}{|y-x|^{n-1}} - \frac{1}{R^{n-1}} \right),$$

where κ_n is the surface area of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. So, for $y \in B_R(x) \setminus \overline{B_{R-\delta}(x)}$ we get

$$g_x^{B_R(x)}(y) \lesssim \frac{1}{|R-\delta|^{n-1}} - \frac{1}{R^{n-1}} \lesssim \frac{\delta}{R^n}.$$

Hence, using that $g^{\Omega_R} \le g^{B_R}$ in $\Omega_R \setminus \{x\}$ and this, we have

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n} \frac{1}{\delta} \int_{(B_R(x) \setminus \overline{B_{R-\delta}(x)}) \setminus U_{C_1\delta}(\partial\Omega)} |u| \, dm|_\Omega.$$

For shortness we write $F_\delta := (B_R(x) \setminus \overline{B_{R-\delta}(x)}) \setminus U_{C_1\delta}(\partial\Omega)$. Let W_Ω denote the family of Whitney cubes of Ω, as in Lemma IV.4.2, and

$$W_\Omega^\delta := \{Q \in W_\Omega : Q \cap F_\delta \neq \emptyset\}.$$

Then we have

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n} \frac{1}{\delta} \int_{F_\delta} |u| \, dm|_\Omega = \frac{1}{R^n} \frac{1}{\delta} \sum_{Q \in W_\Omega^\delta} \int_{Q \cap F_\delta} |u| \, dm.$$

Since $m(Q \cap F_\delta) \lesssim \frac{\delta}{\ell(Q)} m(Q) = \delta \ell(Q)^n$ for each $Q \in W_\Omega^\delta$, we have

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n} \frac{1}{\delta} \sum_{Q \in W_\Omega^\delta} \int_{Q \cap F_\delta} |u| \, dm \lesssim \frac{1}{R^n} \sum_{Q \in W_\Omega^\delta} \ell(Q)^n \sup_Q |u|.$$

Note that having fixed $C_1 \geq 1$ big enough depending on the dimension, then every $Q \in W_\Omega^\delta$ satisfies $\ell(Q) \geq 100\delta$, and therefore $3Q \cap \partial B_R(x) \neq \emptyset$ with $\mathcal{H}^n(3Q \cap \partial B_R(x)) \approx \ell(Q)^n$. So, we get

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n} \sum_{Q \in W_\Omega^\delta} \ell(Q)^n \sup_Q |u| \lesssim \frac{1}{R^n} \sum_{Q \in W_\Omega} \mathcal{H}^n(3Q \cap \partial B_R(x)) \sup_Q |u|, \tag{IV.7.5}$$

uniformly on $\delta > 0$, where we also used in the last step that $W_\Omega^\delta \subset W_\Omega$.

Let us bound the sum on the right-hand side. Defining

$$f(y) \coloneqq \sum_{Q \in W_\Omega} \mathbf{1}_{3Q \cap \partial B_R(x)}(y) \sup_Q |u|, \quad y \in \Omega,$$

we have

$$\begin{aligned} \sum_{Q \in W_\Omega} \mathcal{H}^n(3Q \cap \partial B_R(x)) \sup_Q |u| &= \sum_{Q \in W_\Omega} \int_{3Q \cap \partial B_R(x)} \sup_Q |u| \, d\mathcal{H}^n \\ &\leq \sum_{Q \in W_\Omega} \int_{3Q \cap \partial B_R(x)} f \, d\mathcal{H}^n \lesssim \int_{\partial B_R(x)} f \, d\mathcal{H}^n, \end{aligned} \tag{IV.7.6}$$

where we used the finite overlapping of $\{3Q\}_{Q \in W_\Omega}$ in the last step. Applying Lemma IV.1.11 to f (with $\xi_x \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - \xi_x|$, $C_2 \geq 1$ is a fixed big enough dimensional constant so that $\operatorname{supp} f \subset B(\xi, C_2 R)$, and with aperture β_1), the latter term is bounded by

$$\int_{\partial B_R(x)} f \, d\mathcal{H}^n \lesssim \int_{2B(\xi_x, C_2 R) \cap \partial\Omega} \mathcal{N}_{\beta_1} f \, d\sigma. \tag{IV.7.7}$$

Given $\xi \in \partial\Omega$, let

$$W_\Omega(\xi) \coloneqq \{Q \in W_\Omega : 3Q \cap \Gamma_{\beta_1}(\xi) \neq \emptyset\}.$$

Using this definition, for any $y \in \Gamma_{\beta_1}(\xi)$, the sum in the definition of $f(y)$ runs over the cubes in $W_\Omega(\xi)$ (instead of W_Ω), which in particular implies

$$\mathcal{N}_{\beta_1} f(\xi) = \sup_{y \in \Gamma_{\beta_1}(\xi)} f(y) = \sup_{y \in \Gamma_{\beta_1}(\xi)} \sum_{Q \in W_\Omega(\xi)} \mathbf{1}_{3Q \cap \partial B_R(x)}(y) \sup_Q |u|. \tag{IV.7.8}$$

By the relation $\ell(Q) \approx \operatorname{dist}(Q, \partial\Omega)$ for any $Q \in W_\Omega$, it is clear that there exists an aperture $\beta_2 \geq \beta_1$ (depending on β_1 and the dimension) such that

$$\bigcup_{Q \in W_\Omega(\xi)} 3Q \subset \Gamma_{\beta_2}(\xi).$$

From (IV.7.8), this and the finite overlapping of $\{3Q\}_{Q \in W_\Omega}$, we get

$$\mathcal{N}_{\beta_1} f(\xi) \lesssim \mathcal{N}_{\beta_2} u,$$

Therefore, from (IV.7.6), (IV.7.7) and this, we conclude

$$\sum_{Q \in W_\Omega} \mathcal{H}^n(3Q \cap \partial B_R(x)) \sup_Q |u| \lesssim \int_{2B(\xi_x, C_2 R) \cap \partial\Omega} \mathcal{N}_{\beta_2} u \, d\sigma. \tag{IV.7.9}$$

All in all, we have

$$\mathrm{II}_\delta \overset{\text{(IV.7.5)}}{\lesssim} \frac{1}{R^n} \sum_{Q \in W_\Omega} \mathcal{H}^n(3Q \cap \partial B_R(x)) \sup_Q |u| \overset{\text{(IV.7.9)}}{\lesssim} \frac{1}{R^n} \int_{2B(\xi_x, C_2 R) \cap \partial\Omega} \mathcal{N}_{\beta_2} u \, d\sigma,$$

and by Hölder's inequality (and (IV.1.9)) we get

$$\mathrm{II}_\delta \lesssim R^{-n/p}\|\mathcal{N}u\|_{L^p(\sigma)}, \tag{IV.7.10}$$

and we conclude the control of the term II_δ.

Collecting estimates (IV.7.4) and (IV.7.10) in (IV.7.3), we obtain, for all $R \geq 2\mathrm{dist}(x, \partial\Omega)$ and all $\delta < \mathrm{dist}(x, \partial\Omega_R)/(4C_1)$,

$$|u(x)| \lesssim \frac{1}{\delta}\|\mathcal{N}^{C_1\delta} g^\Omega\|_{L^{p'}(\sigma)}\|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)} + R^{-n/p}\|\mathcal{N}u\|_{L^p(\sigma)}.$$

First, for $\delta > 0$ small enough (depending on x), by Lemma IV.7.3 we have

$$\frac{1}{\delta}\|\mathcal{N}^{C_1\delta} g^\Omega\|_{L^{p'}(\sigma)}\|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)} \lesssim \|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)}.$$

Since $u|^{\mathrm{nt}}_{\partial\Omega} = 0$ σ-a.e. on $\partial\Omega$ implies $\mathcal{N}^{C_1\delta}u(x) \to 0$ as $\delta \to 0$ for σ-a.e. $x \in \partial\Omega$, and $\mathcal{N}u \in L^p(\sigma)$, by the dominated convergence theorem we have $\|\mathcal{N}^{C_1\delta}u\|_{L^p(\sigma)} \to 0$ as $\delta \to 0$. Second, as we are assuming $\|\mathcal{N}u\|_{L^p(\sigma)} < \infty$ we have $R^{-n/p}\|\mathcal{N}u\|_{L^p(\sigma)} \to 0$ as $R \to \infty$. That is, $u(x) = 0$. Since the point $x \in \Omega$ is arbitrary, we conclude that u is identically zero in Ω. □

IV.8. The Dirichlet and Neumann problems

In this section we solve the Dirichlet and Neumann problems, stated in Theorems IV.0.2 and IV.0.3 respectively.

Proof of Theorem IV.0.2. Let $\varepsilon_D > 0$ be so that $n/(n-1) - \varepsilon_D$ is the Hölder conjugate exponent of $n + \varepsilon_2$, with $\varepsilon_2 > 0$ as in Corollary IV.6.2. Fix $n/(n-1) - \varepsilon_D < p_0 \leq 2n/(n+1)$, and let $\delta_0 = \delta_0(p_0, n, \mathrm{CAD}) > 0$ be given by Corollary IV.2.6.

By Corollaries IV.2.6 and IV.6.2 we have that $(\frac{1}{2}Id + K)^{-1} : L^{p_0}(\sigma) \to L^{p_0}(\sigma)$ is a bounded linear operator, as claimed in the theorem. Hence, $(\frac{1}{2}Id + K)^{-1}f \in L^{p_0}(\sigma)$ with $\|(\frac{1}{2}Id + K)^{-1}f\|_{L^{p_0}(\sigma)} \lesssim \|f\|_{L^{p_0}(\sigma)}$, where the involved constant does not depend on $f \in L^{p_0}(\sigma)$. It is clear then that the function $u = \mathcal{D}\left((\frac{1}{2}Id + K)^{-1}f\right)$ in (IV.0.5) is harmonic in Ω. Moreover, by the jump formula (IV.2.5), we have

$$u|^{\mathrm{nt}}_{\partial\Omega}(x) = f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega,$$

whence we conclude that it solves the Dirichlet problem in (IV.0.1) with p_0 instead of p. Finally, by (IV.2.6b) and as $\left(\frac{1}{2}Id + K\right)^{-1}$ is bounded in $L^{p_0}(\sigma)$, we conclude

$$\|\mathcal{N}u\|_{L^{p_0}(\sigma)} \lesssim \left\|\left(\frac{1}{2}Id + K\right)^{-1} f\right\|_{L^{p_0}(\sigma)} \lesssim \|f\|_{L^{p_0}(\sigma)},$$

as claimed in (IV.0.2). That is, (D_{p_0}) is solvable with solution u.

By Corollary IV.7.2, there is $\varepsilon > 0$ such that (D_p) is solvable for all $p \in (p_0 - \varepsilon, \infty)$, and the solution of (D_p) is the unique solution of (IV.0.1). □

Proof of Theorem IV.0.3. Let $\varepsilon_N = \varepsilon_2$ with $\varepsilon_2 > 0$ as in Proposition IV.6.1. We fix $2n/(n+1) \leq p < n + \varepsilon_N$, and let $\delta_0 = \delta_0(p, n, \mathrm{CAD})$ be given by Corollary IV.2.6.

By Corollary IV.2.6 and Proposition IV.6.1 we have that $(-\frac{1}{2}Id + K^*)^{-1} : L^p(\sigma) \to L^p(\sigma)$ is a bounded linear operator. Hence, $(-\frac{1}{2}Id + K^*)^{-1}f \in L^p(\sigma)$ with $\|(-\frac{1}{2}Id + K^*)^{-1}f\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$, where the involved constant does not depend on $f \in L^p(\sigma)$. It is clear then that the function $u = \mathcal{S}_{\mathrm{mod}}\left((-\frac{1}{2}Id + K^*)^{-1} f\right)$ in (IV.0.6) is harmonic in Ω. Moreover, by the jump formula (IV.6.3) we have

$$\partial_{\nu_\Omega} u(x) = f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega,$$

whence we conclude that it solves the Neumann problem in (IV.0.3). Finally, by (IV.6.7) and as $\left(-\frac{1}{2}Id + K^*\right)^{-1}$ is bounded in $L^p(\sigma)$, we conclude

$$\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \lesssim \left\| \left(-\frac{1}{2}Id + K^*\right)^{-1} f \right\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)},$$

as claimed in (IV.0.4). That is, (N_p) is solvable with solution u, and by Corollary IV.5.3, it is the unique (modulo constant) solution of (IV.0.3). $\square$

It is straightforward to verify that any solution u of the Dirichlet problem (IV.0.1) (respectively, Neumann problem (IV.0.3)) satisfies $\|\mathcal{N}u\|_{L^p(\sigma)} \geq \|f\|_{L^p(\sigma)}$ (respectively, $\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \geq \|f\|_{L^p(\sigma)}$). Indeed, for $x \in \partial\Omega$ we have $\mathcal{N}u(x) = \sup_{z\in\Gamma^\Omega(x)} |u(z)| \geq \sup_{\substack{z\in\Gamma^\Omega(x)\\ |z-x|<\varepsilon}} |u(z)|$ for all $\varepsilon > 0$, and therefore

$$\mathcal{N}u(x) \geq \lim_{\substack{z\to x\\ z\in\Gamma^\Omega(x)}} |u(z)| \geq \left| \lim_{\substack{z\to x\\ z\in\Gamma^\Omega(x)}} u(z) \right| = |f(x)|, \text{ for } \sigma\text{-a.e. } x \in \partial\Omega.$$

implying $\|\mathcal{N}u\|_{L^p(\sigma)} \geq \|f\|_{L^p(\sigma)}$. By the same argument, using that $|\nu| = 1$ σ-a.e., we have

$$\begin{aligned} \mathcal{N}(\nabla u)(x) &\geq \lim_{\substack{z\to x\\ z\in\Gamma^\Omega(x)}} |\nabla u(z)| \geq \lim_{\substack{z\to x\\ z\in\Gamma^\Omega(x)}} |\langle \nu(x), \nabla u(z)\rangle| \\ &\geq \left| \lim_{\substack{z\to x\\ z\in\Gamma^\Omega(x)}} \langle \nu(x), \nabla u(z)\rangle \right| = |f(x)|, \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \end{aligned}$$

which gives $\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \geq \|f\|_{L^p(\sigma)}$.

CHAPTER V

Unique continuation for locally uniformly distributed measures

In this chapter we consider locally uniform measures, first introduced by Kowalski and Preiss in [KP87] and later studied by David, Kenig and Toro in [DKT01]. For $0 \leq k \leq n+1$ integer, a Radon measure μ with $\operatorname{supp}\mu \subset \mathbb{R}^{n+1}$ is locally k-uniform if

$$\mu(B(x,r)) = w_k r^k \text{ whenever } x \in \operatorname{supp}\mu \text{ and } 0 < r \leq 1, \tag{V.0.1}$$

where $w_k = \mathcal{H}^k(B^k(0,1))$ is the measure of the unit ball $B^k(0,1)$ in $\mathbb{R}^k$. When $\operatorname{supp}\mu$ is a smooth manifold and $\mu = \mathcal{H}^k|_{\operatorname{supp}\mu}$ it is a natural question in differential geometry to ask what the condition (V.0.1) tells us about the manifold $\operatorname{supp}\mu$, see e.g., [KP89, Oss75].

For general μ, the condition (V.0.1) and its global cousin (i.e., (V.0.1) without the restriction that $r \leq 1$) are extremely well studied, as they represent "end point cases" in the study of measure densities, see e.g., [Pre87, Mar64]. These measure density theorems have also been applied to understand rectifiability in measure spaces (see [DL08, Mer22]) and higher notions of smoothness, see [DKT01, PTT09, Tol15].

The condition (V.0.1), is extremely rigid and constrains not just the measure, but also its support. For example any two locally uniform measures (in fact locally uniformly distributed, see (V.0.2)) with the same support must be scalar multiples of one another, see for instance [Mat95, Theorem 3.4][(V.1)]. Furthermore, by Marstrand's theorem, [Mar64], there are no locally k-uniform measures for non-integer k.

For globally uniform measures ((V.0.1) without the restriction $r \leq 1$), even more is known. It was proven by Kowalski and Preiss in [KP87] that globally k-uniform measures in $\mathbb{R}^{n+1}$ must be of the form $\mathcal{H}^k|_\Sigma$ where Σ is a quadratic variety. To be more precise, in [KP87, Theorem 4.1], it is only proven that in codimension 1 the support is a subset of a quadratic variety. The proof for arbitrary codimension works mutatis mutandis, see [Mat95, Remark 17.4], and can be found in [Mat95, Theorem 17.3]. That the measure is equal to $\mathcal{H}^k|_\Sigma$ then follows from [KP02, Theorem 1.4]. When $k = n$ (the codimension 1 case) the admissible quadratic varieties were fully characterized by Kowalski and Preiss in [KP87], with Σ given by (after a translation and rotation)

$$\begin{aligned}
&\Sigma = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}, \text{ or} \\
&\Sigma = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\} \text{ if } n \geq 3.
\end{aligned}$$

In higher codimension classifying globally 0-uniform and globally 1-uniform measures is straightforward and it follows from work in [KP87] that any connected component of the

[(V.1)] The theorem is written for uniformly distributed measures but the proof only uses the local structure of the measures.

support of a globally 2-uniform measure must be contained in a sphere or a plane[(V.2)]. However, the characterization of which quadratic varieties support a globally k-uniform measures when $2 < k < n$ remains an open problem and even constructing interesting examples is extremely difficult (see [Nim22]).

However, given their success in classifying globally n-uniform measures in $\mathbb{R}^{n+1}$ it was natural for Kowalski and Preiss, [KP87, Conjecture 5.14], to conjecture that each connected smooth hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ satisfying $\mathcal{H}^n(\mathcal{M} \cap B(x,r)) = w_n r^n$ for each $x \in \mathcal{M}$ and each sufficiently small $r > 0$ is a subset of an n-dimensional quadric. We reformulate this conjecture in terms of locally n-uniform measures in $\mathbb{R}^{n+1}$:

Conjecture V.0.1 (See [KP87, Conjecture 5.14])**.** *Let μ be a locally n-uniform measure in $\mathbb{R}^{n+1}$. Then each connected component of the support of μ is contained in the zero set of a quadratic polynomial.*

This conjecture remains completely open for $n \geq 3$ with little progress (apart from work of [DKT01] which we will discuss below). In this note, provide some evidence towards Conjecture V.0.1, *in any codimension*. In particular, we prove that the support of a locally k-uniform measure in $\mathbb{R}^{n+1}$ must satisfy the following "strong unique continuation property" in terms of the mean curvature vector:

Theorem V.0.2 (Strong Unique Continuation)**.** *Let μ be a locally k-uniform measure in $\mathbb{R}^{n+1}$. Assume there is a point $z_0 \in \operatorname{supp}\mu$ such that the mean curvature vector of $\operatorname{supp}\mu$ vanishes at z_0. Then the connected component of $\operatorname{supp}\mu$ containing z_0 must be a k-plane.*

We prove this theorem in Section V.2, where we also recall the definition of the mean curvature vector and establish a seemingly novel and key identity, Lemma V.2.1. The assumption that the mean curvature vector vanishes (and is, in particular, well-defined) requires $\operatorname{supp}\mu$ be of class C^2 in a neighborhood of $z_0 \in \operatorname{supp}\mu$. For $k < n$, all that is known (to the best of the authors' knowledge) is that $\operatorname{supp}\mu$ is a k-dimensional $C^{1,\beta}$-manifold $(0 < \beta < 1)$ away from a closed set $\mathcal{S}$ such that $\mathcal{H}^k(\mathcal{S}) = 0$, see [PTT09, Theorem 1.10]. For $k = n$, work of [DKT01, Theorem 1.9] by David, Kenig and Toro, combined with [PTT09, Corollary 1.11] by Preiss, Tolsa and Toro, shows $\operatorname{supp}\mu = \mathcal{R} \cup \mathcal{S}$, where $\dim \mathcal{S} \leq n-3$ (discrete if $n = 3$) and around every $z_0 \in \mathcal{R}$ there exists a $r_0 > 0$ such that $B(z_0, r_0) \cap \operatorname{supp}\mu$ is the graph of a C^∞ function. Thus in codimension 1 asking that the mean curvature vector exists is not particularly restrictive, while in higher codimensions it may in fact place additional constraints on μ (though it seems more likely that a similar regularity theory is true in the higher codimension case).

Observe that a standard density argument[(V.3)] shows that, if μ is locally k-uniform, then in any ball, B, such that $\operatorname{supp}\mu \cap B$ is of class C^1, we have $\mu|_B = \mathcal{H}^k|_{\operatorname{supp}\mu\cap B}$. Therefore, (V.0.1) (and the work of [PTT09]) immediately implies that any (locally) k-uniform measure μ can be written $\mu = \mathcal{H}^k|_{\operatorname{supp}\mu}$. As a consequence, it is not more restrictive to study manifolds, $\mathcal{M}$, such that $\mathcal{H}^k(\mathcal{M} \cap B(x,r)) = w_k r^k$ for all $x \in \mathcal{M}$ and $r > 0$ sufficiently small. Additionally,

[(V.2)]According to [DKT01], in unpublished work Kowalski showed that the smooth support of any locally 2-uniform measure must be a union of spheres and planes in $\mathbb{R}^3$, see also [KP87, Theorem 3.2(b)] and [KP89, Proposition 3.1]. Also note, that while a single sphere is not the support of a globally 2-uniform measure, it is hard to rule out the possibility that countably many disjoint spheres, carefully placed, support a globally 2-uniform measure, see [Nim22] for interesting computations in this direction.

[(V.3)]See for instance the proof of [KP87, Theorem 4.5], or the proof of Theorem V.0.3, more precisely (V.1.2), in Section V.1.

in the setting of Theorem V.0.2 above, we can conclude that $\mu|_P = \mathcal{H}^k|_P$, where P is the k-plane coinciding with the connected component of $\operatorname{supp}\mu$ containing z_0.

By the curvature restrictions on the support of (locally smooth) locally n-uniform measures in $\mathbb{R}^{n+1}$, see [KP87, Corollary 3.1], Theorem V.0.2 still holds if the mean curvature is replaced by scalar curvature in codimension 1.

Theorem V.0.2 provides some evidence towards Conjecture V.0.1 (in any codimension), since if $\Sigma = \{Q = 0\}$ for some quadratic polynomial Q, and the second fundamental form of Σ vanishes at a point, then Σ must be a plane. We should mention that we know of only one other result regarding the mean curvature of the support of a locally n-uniform measure in $\mathbb{R}^{n+1}$: [KP87, Theorem 3.3] shows, under the much stronger assumption that $\operatorname{supp}\mu$ is minimal (i.e., the mean curvature vanishes everywhere) that $\operatorname{supp}\mu$ must be a union of planes.

As a simple corollary, Theorem V.0.2 implies a "weak" unique continuation type property for the support of a broader class of measures, which we call locally uniformly distributed. A Radon measure μ with $\operatorname{supp}\mu \subset \mathbb{R}^{n+1}$ is locally uniformly distributed if

$$0 < \mu(B(x,r)) = \mu(B(y,r)) < \infty \text{ whenever } x, y \in \operatorname{supp}\mu \text{ and } 0 < r \leq 1. \tag{V.0.2}$$

Every locally uniform measure is locally uniformly distributed but the opposite is not always the case. Indeed, $\mathcal{H}^3|_{\mathbb{S}^3}$ is locally uniformly distributed but a straightforward computation shows that it is not locally 3-uniform in $\mathbb{R}^4$ (one can notice that $\mathbb{S}^3$ does not satisfy the curvature conditions of [KP87, Corollary 3.1]). In contrast $\mathcal{H}^2|_{\mathbb{S}^2}$ is locally 2-uniform in $\mathbb{R}^3$ (and locally uniformly distributed). Locally uniformly distributed measures naturally arise as the conical part of globally uniform measures ((V.0.1) without the restriction $r \leq 1$), and so understanding their structure is an important step towards producing more examples of globally uniform measures, see e.g., [Nim22].

Very little is known even about globally uniformly distributed measures ((V.0.2) without the restriction that $r \leq 1$). Kirchheim and Preiss showed in [KP02] that globally uniformly distributed measures are supported on analytic varieties. Given the few known examples, it is natural to conjecture that the support of a locally uniformly distributed measure must also be an analytic variety. Our next theorem supports this conjecture.

Theorem V.0.3 (Weak Unique Continuation)**.** *Let μ be a locally uniformly distributed measure in $\mathbb{R}^{n+1}$. Assume there is an open ball $B(z_0, r_0)$ (with $z_0 \in \operatorname{supp}\mu$ and $r_0 > 0$) and a k-plane P such that $B(z_0, r_0) \cap \operatorname{supp}\mu = B(z_0, r_0) \cap P$. Then there is a unique connected component of* $\operatorname{supp}\mu$ *which intersects non-trivially with $B(z_0, r_0)$ and this connected component is the k-plane P.*

This follows from the strong unique continuation Theorem V.0.2 for locally k-uniform measures in $\mathbb{R}^{n+1}$, and its proof is given in Section V.1.

Using arguments similar to those used in the proof of Theorem V.0.2, in Section V.3 we prove the following flatness condition for locally k-uniform measures.

Theorem V.0.4. *Let μ be a locally k-uniform measure in $\mathbb{R}^{n+1}$. Assume there is a point $z_0 \in \operatorname{supp}\mu$ such that*

$$\mu(B(x,r)) \leq w_k r^k \text{ for all } x \in B(z_0, 1) \text{ and } 0 < r \leq 1.$$

Then the connected component of $\operatorname{supp}\mu$ *containing z_0 must be a k-plane.*

This is the local counterpart of [Mat95, Theorem 17.3]: If a (globally) k-uniform measure μ satisfies $\mu(B(x,r)) \leq w_k r^k$ for all $x \in \mathbb{R}^{n+1}$ and $0 < r < \infty$, then $\mu = \mathcal{H}^k|_P$ for a k-plane

P. We remark that the proof in [Mat95, p. 235] relies on the fact that $\operatorname{supp}\mu \subset \{Q = 0\}$ for a quadratic polynomial Q, a property known for (globally) k-uniform measures but still open for locally k-uniform measures, where it remains conjectured (even in codimension 1, see Conjecture V.0.1) that each connected component of $\operatorname{supp}\mu$ is contained in the zero set of a quadratic polynomial. Instead, we prove Theorem V.0.4 using the particular case $b_{0,r} = 0$ in the key equality Lemma V.2.1.

V.1. Proof of Theorem V.0.3: Weak unique continuation property

Before proving Theorem V.0.2, we show how the weak unique continuation property in Theorem V.0.3 follows from our Theorem V.0.2.

Proof of Theorem V.0.3. We mainly follow the first part of [KP02, Proof of Theorem 1.4(ii)]. The Lebesgue differentiation theorem implies that the limit

$$K(y) \coloneqq \lim_{r\to 0} \frac{\mu(B(y,r))}{(\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})(B(y,r))}$$

exists at $(\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})$-a.e. $y \in \operatorname{supp}\mu$, and that

$$\mu(E) = \int_E K(y)\, d(\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})(y) \text{ for any } E \subset B(z_0, r_0), \tag{V.1.1}$$

since μ is (clearly) absolutely continuous with respect to $\mu + \mathcal{H}^k|_{\operatorname{supp}\mu}$, i.e., $\mu \ll (\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})$. Note also that the lower bound in (V.0.2) implies $K(y) > 0$ for $(\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})$-a.e. $y \in \operatorname{supp}\mu$.

Fixed a (any) point $p \in \operatorname{supp}\mu$, we define

$$g_\mu(r) \coloneqq \mu(B(p,r)) \text{ for } 0 < r \le 1,$$

see (V.0.2). By the flatness assumption $B(z_0, r_0) \cap \operatorname{supp}\mu = B(z_0, r_0) \cap P$ with k-plane P, for $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$ we have

$$K(y) = \lim_{r\to 0} \frac{g_\mu(r)}{g_\mu(r) + w_k r^k} \le 1,$$

which is independent of $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$. Since $K(y)$ exists $(\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})$-a.e., we conclude that the limit $K(y)$ exists for all $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$. Furthermore, since $K(y) > 0$ for $(\mu + \mathcal{H}^k|_{\operatorname{supp}\mu})$-a.e. $y \in \operatorname{supp}\mu$, it follows that $0 < K(y) \le 1$ for all $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$.

We define (before checking it makes sense)

$$c \coloneqq \lim_{r\to 0} \frac{g_\mu(r)}{w_k r^k}.$$

We have that this limit exists by the existence of the limit $K(y)$ for all $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$ and since

$$\frac{1}{c} = \lim_{r\to 0} \frac{w_k r^k}{g_\mu(r)} = \lim_{r\to 0} \frac{w_k r^k + g_\mu(r)}{g_\mu(r)} - 1 = \frac{1}{K(y)} - 1.$$

Moreover, the fact that $K(y) > 0$ implies that $c > 0$. Furthermore if $c = \infty$ then $K(y) = 1$ for $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$ and therefore (V.1.1) would give $\mathcal{H}^k|_{\operatorname{supp}\mu}(E) = 0$ for any $E \subset B(z_0, r_0) \cap \operatorname{supp}\mu$, but this cannot happen since $B(z_0, r_0) \cap \operatorname{supp}\mu = B(z_0, r_0) \cap P$.

Since $c \in (0, \infty)$, we can write $K(y) = \frac{c}{c+1} \in (0,1)$ for all $y \in B(z_0, r_0) \cap \operatorname{supp}\mu$, and therefore by (V.1.1) we conclude that

$$\mu(E) = c\mathcal{H}^k|_{\operatorname{supp}\mu}(E) \text{ for any } E \subset B(z_0, r_0). \tag{V.1.2}$$

The equality (V.1.2) in particular implies $g_\mu(r) = \mu(B(x,r)) = cw_k r^k$ for all $x \in B(z_0, r_0/2) \cap \operatorname{supp}\mu$ and $0 < r \le r_0/4$. However, by the locally uniformly distributed condition (V.0.2), the same holds for any $x \in \operatorname{supp}\mu$. That is,

$$g_\mu(r) = \mu(B(x,r)) = cw_k r^k \text{ for all } x \in \operatorname{supp}\mu \text{ and } 0 < r \le r_0/4.$$

Thus, the measure

$$\widetilde{\mu}(\cdot) := \frac{1}{c}\frac{1}{\left(\frac{r_0}{4}\right)^k} g\left(\frac{r_0}{4}\cdot\right)$$

satisfies the local uniform condition (V.0.1), is flat in the open ball $B(\frac{4z_0}{r_0}, 4) = \frac{B(z_0,r_0)}{(r_0/4)}$, and $\operatorname{supp}\widetilde{\mu}$ is a dilation of $\operatorname{supp}\mu$. The result follows by Theorem V.0.2 applied to the locally k-uniform measure $\widetilde{\mu}$. □

V.2. Proof of Theorem V.0.2: Strong unique continuation property

In this section we prove the strong continuation-type property in Theorem V.0.2 for locally k-uniform measures in $\mathbb{R}^{n+1}$ that is, if the mean curvature vector of the support vanishes at a point, then the whole connected component containing the point must be a k-plane.

Let us recall that the mean curvature vector of a k-dimensional C^2-manifold in $\mathbb{R}^{n+1}$ is the trace of the (vector-valued) second fundamental form, divided by k. Let $\mathcal{M} \cap B(z_0, r_0)$, with $z_0 \in \mathcal{M}$ and $r_0 > 0$, be a C^2-manifold of dimension k. If $\{\hat{n}_j(z)\}_{j=k+1}^{n+1}$ is an orthonormal basis of the normal space $N_{\mathcal{M}}(z)$ at $z \in \mathcal{M} \cap B(z_0, r_0)$ then, intuitively, the j-component of the mean curvature vector is a measure of "flatness" in that it infinitesimally captures how much the measure of $\mathcal{M}$ changes under perturbations in the direction of $\hat{n}_j(z)$.

For our purposes it is enough to work in coordinates, where the definition becomes more concrete. Assume that $\mathcal{M} \cap B(z_0, r_0)$ can be described by a C^2 function $\varphi = (\varphi_{k+1}, \ldots, \varphi_{n+1}) : \mathbb{R}^k \to \mathbb{R}^{n+1-k}$ such that $z = (z', \varphi(z'))$ for $z \in \mathcal{M} \cap B(z_0, r_0)$, with $z' \in \mathbb{R}^k$. Note that, in this case, the tangent space $T_{\mathcal{M}}(z)$ is

$$T_{\mathcal{M}}(z) := \operatorname{span}\left(\{t_j\}_{j=1}^k\right), \text{ with } t_j(z) := \frac{e_j^k + \partial_{x_j}\varphi(z')}{\sqrt{1 + |\partial_{x_j}\varphi(z')|^2}},$$

and $\{e_j^k\}_{j=1}^k$ is the standard orthonormal basis of $\mathbb{R}^k$. As above, $\{\hat{n}_j(z)\}_{j=k+1}^{n+1}$ is an orthonormal basis of the normal space $N_{\mathcal{M}}(z)$. Then the vector-valued second fundamental form $\mathrm{II}_{\mathcal{M}}(\cdot,\cdot)(z) : T_{\mathcal{M}}(z) \times T_{\mathcal{M}}(z) \to N_{\mathcal{M}}(z)$ is given by

$$\mathrm{II}_{\mathcal{M}}(u,v)(z) := \Pi_{N_{\mathcal{M}}(z)}(u \cdot \nabla v) = \sum_{j=k+1}^{n+1} \langle u \cdot \nabla v, \hat{n}_j(z)\rangle\, \hat{n}_j(z).$$

(Where the inner product is the Euclidean inner product).

The mean curvature vector is then the trace so we have that

$$H_{\mathcal{M}}(z) = \frac{1}{k}\sum_{i,j=1}^k g^{ij}(z) \sum_{\ell=k+1}^{n+1} \langle t_j(z) \cdot \nabla t_i(z), \hat{n}_\ell(z)\rangle\, \hat{n}_\ell(z),$$

where $g^{ij}(z)$ is the inverse of the first fundamental form, given by $g_{ij}(z) = \langle t_i(z), t_j(z)\rangle$.

In generality, this expression is extremely complicated, but it simplifies greatly if we assume that $T_{\mathcal{M}}(0) = \mathbb{R}^k \times \{0\}^{n+1-k}$ (that is, $\partial_{x_i}\varphi(0) = 0$, for every $i = 1, \ldots, k$). In this setting we can take $t_j(0) = e_j$, $j = 1, \ldots, k$, and $\hat{n}_j(0) = e_j$, $j = k+1, \ldots, n+1$ (here $\{e_i\}_{i=1}^{n+1}$

is the standard orthonormal basis of $\mathbb{R}^{n+1}$). It follows that $g^{ij}(0) = \delta_{ij}$ (i.e., is 1 when $i = j$ and 0 otherwise). Thus our formula simplifies to

$$H_{\mathcal{M}}(0) = \frac{1}{k} \sum_{i=1}^{k} \sum_{\ell=k+1}^{n+1} \langle e_i \cdot \nabla t_i(0), e_\ell \rangle \, e_\ell .$$

We recall that $\nabla t_i(0)$ is a matrix and we need only compute its (ℓ, i)-entry. This is a straightforward computation that gives

$$(\nabla t_i(0))_{\ell,i} = \partial_{x_i} \left(\frac{\partial_{x_i} \varphi_\ell}{\sqrt{1 + |\partial_{x_i} \varphi|^2}} \right) (0) = \partial^2_{x_i} \varphi_\ell(0),$$

where we have used in the second equality that $\partial_{x_i}\varphi(0) = 0$ for all $i = 1 \dots, k$. Thus we have the final form

$$H_{\mathcal{M}}(0) = \frac{1}{k} \sum_{\ell=k+1}^{n+1} \Delta\varphi_\ell(0) e_\ell = \frac{1}{k}(0, \Delta\varphi(0)), \text{ if } T_{\mathcal{M}}(0) = \mathbb{R}^k \times \{0\}^{n+1-k}. \tag{V.2.1}$$

Before we begin our proofs, let us introduce some preliminary notation. We define the vector $b_{z,r}$ and its nonnormalized version $\widetilde{b}_{z,r}$ as

$$b_{z,r} \coloneqq \frac{k+2}{2w_k r^{k+2}} \int_{B(z,r)} \left(r^2 - |y-z|^2\right)(y-z)\, d\mu(y), \tag{V.2.2a}$$

$$\widetilde{b}_{z,r} \coloneqq \frac{2w_k r^{k+2}}{k+2} b_{z,r} = \int_{B(z,r)} \left(r^2 - |y-z|^2\right)(y-z)\ d\mu(y), \tag{V.2.2b}$$

and the quadratic form $Q_{z,r}$ and its nonnormalized version $\widetilde{Q}_{z,r}$ on $\mathbb{R}^{n+1}$ as

$$Q_{z,r}(x) \coloneqq \frac{k+2}{w_k r^{k+2}} \int_{B(z,r)} \langle x, y-z \rangle^2 \, d\mu(y), \tag{V.2.3a}$$

$$\widetilde{Q}_{z,r}(x) \coloneqq \frac{w_k r^{k+2}}{k+2} Q_{z,r} = \int_{B(z,r)} \langle x, y-z \rangle^2 \, d\mu(y), \tag{V.2.3b}$$

all first introduced in [KP87, (4.3) and (4.4)], see also [DKT01, (7.2) and (7.3)]. Throughout this section, $b^j_{z,r}$ and $\widetilde{b}^j_{z,r}$ denote the j-component ($1 \le j \le n+1$) of the vectors $b_{z,r}$ and $\widetilde{b}_{z,r}$ respectively.

Here we present the key equality in the proof of our main result Theorem V.0.2. This relates the orthogonal (see (V.2.6)) vector $\widetilde{b}_{0,r}$ and the mean curvature (given in local coordinates by $\Delta\varphi(0)$, see (V.2.1)), to the integral of "height" of the support squared. This identity is the main novel part of our work.

Lemma V.2.1 (Key equality). *Let μ be a locally k-uniform measure in $\mathbb{R}^{n+1}$ and $0 < r < 1/2$. Assume $0 \in \operatorname{supp}\mu$ and that $\operatorname{supp}\mu \cap B(0, r_0)$ (for some $r_0 > 0$) is given by the graph of a $C^{1,\beta}$, $\beta > 0$, function $\varphi = (\varphi_{k+1}, \dots, \varphi_{n+1}) : \mathbb{R}^k \to \mathbb{R}^{n+1-k}$ with $|\nabla\varphi_j(0)| = 0$ for all $j = k+1, \dots, n+1$. Then there holds*

$$\sum_{j=k+1}^{n+1} \int_{B(0,r)} y_j^2 \, d\mu(y) = \begin{cases} 0 & \text{if } \widetilde{b}_{0,r} = 0, \text{ and} \\ \frac{1}{2}\sum_{j=k+1}^{n+1} \widetilde{b}^j_{0,r} \Delta\varphi_j(0) & \text{if } \varphi \text{ is of class } C^2. \end{cases}$$

We prove this in Section V.2.1. We are now ready to turn to the proof of Theorem V.0.2.

Proof of Theorem V.0.2. Since we are C^2-smooth in a neighborhood of z_0, we can work in coordinates: after a harmless rotation and translation, let $\operatorname{supp}\mu$ in a neighborhood of $0 \in \operatorname{supp}\mu$ be given by $(x', \varphi(x'))$, $x' \in \mathbb{R}^n$, with $\varphi(x) = (\varphi_{k+1}(x'), \ldots, \varphi_{n+1}(x'))$ satisfying $\varphi(0) = 0$ and $|\nabla\varphi_j(0)| = 0$ for all $j = k+1, \ldots, n+1$. As we computed in (V.2.1) above, this implies that

$$H_{\operatorname{supp}\mu}(0) = \frac{1}{k}(0, \Delta\varphi(0)).$$

Fixed $0 < r < 1/2$, by Lemma V.2.1 we have

$$\sum_{j=k+1}^{n+1} \int_{B(0,r)} y_j^2 \, d\mu(y) = \frac{1}{2} \sum_{j=k+1}^{n+1} \widetilde{b}_{0,r}^j \Delta\varphi_j(0).$$

Since $(0, \Delta\varphi(0)) = 0$ (in particular $\Delta\varphi_j(0) = 0$ for all $j = k+1, \ldots, n+1$), then the integral in the left-hand side is zero, implying that $y_j = 0$ for μ-a.e. $y \in B(0,r)$ for all $j = k+1, \ldots, n+1$. But this is only possible if $B(0,r) \cap \operatorname{supp}\mu = B(0,r) \cap P$ for the k-plane $P := \mathbb{R}^k \times \{0\}^{n+1-k}$. Since this holds for every $0 < r < 1/2$, we conclude that $\operatorname{supp}\mu \cap B(0,1/2) = P \cap B(0,1/2)$.

To show that $B(z_0,1/2) \cap \operatorname{supp}\mu = B(z_0,1/2) \cap P$ for any z_0 in the same connected component of $\operatorname{supp}\mu$, consider a chain of balls of radius $1/2$ centered at points in $\operatorname{supp}\mu$ such that the center of each new ball is in the previous ball with 0 as the center of the first ball and z_0 as the center of the last ball. The result follows by applying this argument to each ball in the chain in sequence. □

V.2.1. Proof of Key equality: Lemma V.2.1. Recall the definition of $b_{z,r}$ and $Q_{z,r}$ from (V.2.2a) and (V.2.3a). Let us state two fundamental properties satisfied by a locally k-uniform measure μ in $\mathbb{R}^{n+1}$. The first is the elementary identity

$$\int_{B(z,r)} |y-z|^2 \, d\mu(y) = \frac{k w_k}{k+2} r^{k+2} \text{ for all } z \in \operatorname{supp}\mu \text{ and } 0 < r < 1, \tag{V.2.4}$$

for the detailed straightforward computation (for $k = n$) see [KP87, (4.5)] for instance. Secondly, proved in [DKT01, (10.3)][(V.4)], for $z \in \operatorname{supp}\mu$ and $0 < r < 1/2$ there holds

$$\big|\langle 2b_{z,r}, x-z\rangle + Q_{z,r}(x-z) - |x-z|^2\big| \le C\frac{|x-z|^3}{r} \text{ for all } x \in \operatorname{supp}\mu \cap B(z,r/2). \tag{V.2.5}$$

Since $Q_{z,r}(x-z) \le \frac{k+2}{w_k}|x-z|^2$, this in particular implies that for $|x-z|$ small enough there holds

$$|\langle 2b_{z,r}, x-z\rangle| \le C'|x-z|^2.$$

From this latter inequality we conclude that, if $\operatorname{supp}\mu \cap B(z,r_0)$ (for some $r_0 > 0$) is at least of class C^1, then either $b_{z,r} = 0$ or

$$b_{z,r} \text{ is orthogonal to the tangent } k\text{-plane of } \operatorname{supp}\mu \text{ at } z \in \operatorname{supp}\mu. \tag{V.2.6}$$

Having recalled these facts we can now establish the following algebraic relation, which gets us most of the way to Lemma V.2.1. We thank the reviewer for pointing out that similar estimates (in spirit) actually date back at least to the work of [Mar64], see Lemma 4 there. We should also state explicitly that the general idea of working in local coordinates

[(V.4)] The codimension 1 version is proved in [DKT01, (10.3)], but its proof does not depend on the codimension. For a statement in any codimension, see [PTT09, Proposition 2.3 (2.12)].

and estimating errors using the smoothness of the support is ubiquitous in the literature, appearing in [KP87, KP89, Mat95, DKT01, PTT09] and surely elsewhere.

Lemma V.2.2. *Under the assumptions of Lemma V.2.1, for all $e \in \mathbb{S}^{k-1} \times \{0\}^{n+1-k}$ there holds*

$$1 - Q_{0,r}(e) = \begin{cases} 0 & \text{if } b_{0,r} = 0, \text{ and} \\ \sum_{j=k+1}^{n+1} b_{0,r}^j \langle \nabla^2 \varphi_j(0) e, e\rangle & \text{if } \varphi \text{ is of class } C^2. \end{cases} \tag{V.2.7}$$

Proof. During the proof, by the $C^{1,\beta}$ local representation of $\operatorname{supp}\mu$ we write $x = (x', \varphi(x'))$ with $x' \in \mathbb{R}^k$ and $\varphi(x') = (\varphi_{k+1}(x'), \ldots, \varphi_{n+1}(x')) \in \mathbb{R}^{n+1-k}$. As we are assuming that $0 \in \operatorname{supp}\mu$ and $|\nabla\varphi_j(0)| = 0$ for all $j = k+1, \ldots, n+1$, by the $C^{1,\beta}$ regularity in the first case and Taylor's theorem in the second case, for all $j = k+1, \ldots, n+1$ we then have

$$\varphi_j(x') = \begin{cases} o(|x'|) & \text{if } b_{0,r} = 0, \text{ and} \\ \frac{\langle \nabla^2\varphi_j(0)x', x'\rangle}{2} + o(|x'|^2) & \text{if } \varphi \text{ is of class } C^2. \end{cases}$$

(We use little o notation, so that $o(s)$ is a quantity that when divided by s goes to zero as $s \downarrow 0$).

With $z = 0$, (V.2.5) reads as

$$\left|\langle 2b_{0,r}, x\rangle + Q_{0,r}(x) - |x|^2\right| \leq C\frac{|x|^3}{r} \text{ for all } x \in \operatorname{supp}\mu \cap B(0, r/2). \tag{V.2.8}$$

In both cases, we have

$$\begin{aligned} Q_{0,r}(x) &= \frac{k+2}{w_k r^{k+2}} \int_{B(0,r)} \langle x, y\rangle^2 \, d\mu(y) \\ &= \frac{k+2}{w_k r^{k+2}} \int_{B(0,r)} \left(\langle x', y'\rangle + \sum_{j=k+1}^{n+1} \varphi_j(x') y_j\right)^2 d\mu(y) \\ &= \frac{k+2}{w_k r^{k+2}} \int_{B(0,r)} \langle x', y'\rangle^2 \, d\mu(y) + o(|x'|^2), \end{aligned}$$

and simply $|x|^2 = |x'|^2 + o(|x'|^2)$. Also from this, we have that $|x| \approx |x'|$ for $x \in \operatorname{supp}\mu \cap B(0, r/2)$ with small enough modulus.

For the case $b_{0,r} = 0$, from (V.2.8) and these estimates, there exists a $\theta \in (0,1)$ such that for all $x \in \operatorname{supp}\mu \cap B(0, r/2)$ with $|x| \leq \theta r$ we get

$$\left|\frac{k+2}{w_k r^{k+2}} \int_{B(0,r)} \langle x', y'\rangle^2 \, d\mu(y) - |x'|^2\right| \leq \frac{1}{r} o(|x'|^2). \tag{V.2.9}$$

Note that we can simply write the first term as $Q_{0,r}((x', 0))$.

We now consider the case where φ is of class C^2. From (V.2.6) we have that $b_{0,r}$ is orthogonal to the tangent k-plane of $\operatorname{supp}\mu$ at 0, or it may be zero. That is, in local coordinates, this means $b_{0,r} \in (\mathbb{R}^k \times \{0\}^{n+1-k})^\perp$, since $|\nabla\varphi_j(0)| = 0$ for all $j = k+1, \ldots, n+1$. This implies $b_{0,r}^j = 0$ for all $j = 1, \ldots, k$, and therefore,

$$\langle 2b_{0,r}, x\rangle = 2\sum_{j=k+1}^{n+1} b_{0,r}^j \varphi_j(x') = \sum_{j=k+1}^{n+1} b_{0,r}^j \langle \nabla^2\varphi_j(0)x', x'\rangle + o(|x'|^2).$$

As before, from (V.2.8) and the previous estimates, there exists a $\theta \in (0,1)$ (depending on $\|\varphi\|_{C^2}$) such that for all $x \in \operatorname{supp}\mu \cap B(0, r/2)$ with $|x| \le \theta r$ we get

$$\left| \sum_{j=k+1}^{n+1} b_{0,r}^j \langle \nabla^2 \varphi_j(0) x', x' \rangle + \frac{k+2}{w_k r^{k+2}} \int_{B(0,r)} \langle x', y' \rangle^2 \, d\mu(y) - |x'|^2 \right| \le \frac{1}{r} o(|x'|^2). \tag{V.2.10}$$

Note that we can simply write the middle term as $Q_{0,r}((x', 0))$.

Given $e \in \mathbb{S}^{k-1} \times \{0\}^{n+1-k}$, let $\theta r/2 > \varepsilon > 0$ and $x' = \varepsilon e$ in (V.2.10). Dividing both sides of (V.2.9) and (V.2.10) by $|x'|^2 = \varepsilon^2$ and letting $\varepsilon \downarrow 0$, we get (V.2.7). □

We now turn to the proof of Lemma V.2.1.

Proof of Lemma V.2.1. Multiplying the equality (V.2.7) in Lemma V.2.2 by $w_k r^{k+2}/(k+2)$, for all $e \in \mathbb{S}^{k-1} \times \{0\}^{n+1-k}$ we have

$$\frac{w_k}{k+2} r^{k+2} - \widetilde{Q}_{0,r}(e) = \begin{cases} 0 & \text{if } b_{0,r} = 0, \text{ and} \\ \frac{1}{2} \sum_{j=k+1}^{n+1} \widetilde{b}_{0,r}^j \langle \nabla^2 \varphi_j(0) e, e \rangle & \text{if } \varphi \text{ is of class } C^2. \end{cases} \tag{V.2.11}$$

For $1 \le i \le k$, taking $e = e_i \in \mathbb{S}^{k-1} \times \{0\}^{n+1-k}$ in (V.2.11), recall $\{e_i\}_{i=1}^{n+1}$ is the standard orthonormal basis of $\mathbb{R}^{n+1}$, we have

$$\frac{w_k}{k+2} r^{k+2} - \int_{B(0,r)} y_i^2 \, d\mu(y) = \begin{cases} 0 & \text{if } b_{0,r} = 0, \text{ and} \\ \frac{1}{2} \sum_{j=k+1}^{n+1} \widetilde{b}_{0,r}^j \partial_{x_i, x_i} \varphi_j(0) & \text{if } \varphi \text{ is of class } C^2. \end{cases} \tag{V.2.12}$$

Adding up (V.2.12) for $1 \le i \le k$ we get

$$\frac{k w_k}{k+2} r^{k+2} - \sum_{i=1}^{k} \int_{B(0,r)} y_i^2 \, d\mu(y) = \begin{cases} 0 & \text{if } b_{0,r} = 0, \text{ and} \\ \frac{1}{2} \sum_{j=k+1}^{n+1} \widetilde{b}_{0,r}^j \Delta \varphi_j(0) & \text{if } \varphi \text{ is of class } C^2. \end{cases}$$

The lemma follows from the above after applying the identity (V.2.4). □

V.3. Proof of Theorem V.0.4

Let μ be a locally k-uniform measure, and recall the definition of $b_{z,r}$ and $Q_{z,r}$ from (V.2.2a) and (V.2.3a). A quick inspection of the proof of [Mat95, (10) on p. 235, proof of Theorem 17.3] reveals that, if $z_0 \in \operatorname{supp}\mu$ satisfies

$$\mu(B(x,r)) \le w_k r^k \text{ for all } x \in B(z_0, 1) \text{ and } 0 < r \le 1,$$

then for $0 < r < 1/2$ there holds

$$\langle 2 b_{z_0,r}, x - z_0 \rangle + Q_{z_0,r}(x - z_0) - |x - z_0|^2 \le C \frac{|x - z_0|^3}{r} \text{ for all } x \in B(z_0, r/2). \tag{V.3.1}$$

Note that the left-hand side lacks an absolute value, and this inequality holds for all $x \in B(z_0, r/2)$, not just for $x \in \operatorname{supp}\mu \cap B(z_0, r/2)$, in contrast to (V.2.5).

By (V.3.1) and the case $b_{0,r} = 0$ in the key equality Lemma V.2.1, we are now ready to prove Theorem V.0.4.

Proof of Theorem V.0.4. We first observe that it suffices to prove the theorem under the assumption that $\operatorname{supp}\mu$ is $C^{1,\beta}$-smooth in a neighborhood of z_0. Let us suppose instead that $\operatorname{supp}\mu$ is not $C^{1,\beta}$-smooth in any neighborhood of z_0, and we will show that this leads to a contradiction. By [PTT09, Theorem 1.10], $\operatorname{supp}\mu$ is a k-dimensional $C^{1,\beta}$-manifold ($0 < \beta < 1$) away from a closed set $\mathcal{S}$ such that $\mathcal{H}^k(\mathcal{S}) = 0$. As explained above in the introduction in this Chapter V, we have $\mu = \mathcal{H}^k|_{\operatorname{supp}\mu}$. Since $\mu(B(z_0, \frac{1}{2}) \setminus \mathcal{S}) = \mu(B(z_0, \frac{1}{2})) = w_k(1/2)^k > 0$, there exists a point $z_0' \in B(z_0, \frac{1}{2}) \setminus \mathcal{S}$, and we now note that

$$\mu(B(x,r)) \leq w_k r^k \text{ for all } x \in B(z_0', 1/2) \text{ and } 0 < r \leq 1/2.$$

Rescaling μ and applying the result for z_0' (where $\operatorname{supp}\mu$ is $C^{1,\beta}$-smooth nearby), we conclude that the connected component containing z_0' is a k-plane P. The condition (V.0.1) implies that any other connected component $\mathcal{C}$ of $\operatorname{supp}\mu$ satisfies $\inf_{x\in P, y\in\mathcal{C}} |x-y| \geq 1$. However, since $|z_0 - z_0'| < 1/2$, it follows that $z_0' \in P$, contradicting the non-smoothness assumption at z_0.

Let us assume from now on that $\operatorname{supp}\mu$ is $C^{1,\beta}$-smooth in a neighborhood of z_0. In this case, we can work in coordinates: after a harmless rotation and translation, let $\operatorname{supp}\mu$ in a neighborhood of $0 \in \operatorname{supp}\mu$ be given by $(x', \varphi(x'))$, $x' \in \mathbb{R}^n$, with $\varphi(x) = (\varphi_{k+1}(x'), \ldots, \varphi_{n+1}(x'))$ satisfying $\varphi(0) = 0$ and $|\nabla\varphi_j(0)| = 0$ for all $j = k+1, \ldots, n+1$.

Fixed $0 < r < 1/2$, we aim to see that

$$b_{0,r} = 0.$$

Granted this, by Lemma V.2.1 we get

$$\sum_{j=k+1}^{n+1} \int_{B(0,r)} y_j^2 \, d\mu(y) = 0.$$

That is, $y_j = 0$ for μ-a.e. $y \in B(0,r)$ for all $j = k+1, \ldots, n+1$, and therefore $B(0,r) \cap \operatorname{supp}\mu = B(0,r) \cap P$ for the k-plane $P := \mathbb{R}^k \times \{0\}^{n+1-k}$. The result follows by arguing as in the end of the proof of Theorem V.0.2, or simply applying it directly.

The rest of the proof is dedicated to showing that $b_{0,r} = 0$. With $z_0 = 0$, (V.3.1) reads as

$$\langle 2b_{0,r}, x\rangle + Q_{0,r}(x) - |x|^2 \leq C\frac{|x|^3}{r} \text{ for all } x \in B(0, r/2).$$

Combining this with the bound $Q_{0,r}(x) \leq \frac{k+2}{w_k}|x|^2$, we obtain

$$\langle 2b_{0,r}, x\rangle \leq C\frac{|x|^3}{r} + |Q_{0,r}(x)| \leq C'|x|^2 \text{ for all } x \in B(0, r/2).$$

For any sufficiently small $\varepsilon > 0$, taking $x = \varepsilon b_{0,r}$ above yields $2\varepsilon|b_{0,r}|^2 \leq C'\varepsilon^2|b_{0,r}|^2$, which can only hold if $b_{0,r} = 0$, as claimed. $\square$

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