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A metric approach to the study of manifolds of positive scalar curvature
Approches métriques pour l'étude des variétés à courbure scalaire strictement positive

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Abstract

This thesis is dedicated to the topology and the geometry of Riemannian manifolds of positive scalar curvature. To explore the structure of such manifolds, we adopt a metric perspective, specifically through two metric generalisations of the notion of positive scalar curvature.

First, we focus on the topology of 3-manifolds of positive scalar curvature, and we provide a new obstruction to the existence of complete Riemannian metrics of positive scalar curvature on non-compact 3-manifolds. More precisely, we prove that if an orientable 3-manifold M admits a complete Riemannian metric whose scalar curvature is positive and has a subquadratic decay at infinity, then it decomposes as a (possibly infinite) connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands. As a consequence, the manifold M admits a complete metric of uniformly positive scalar curvature. This result constitutes a generalisation of a theorem by Gromov and Wang, and its proof builds upon a different approach of metric and topological nature. More generally, the topological decomposition holds without any assumption on the scalar curvature, relying instead on a metric estimate on the filling discs of closed curves in the universal cover, based in the notion of fill radius. Moreover, the decay rate in the decomposition theorem is optimal, since the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ admits a complete metric of positive scalar curvature decaying exactly quadratically at infinity, yet it does not decompose as a connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands.

Then, we explore the notion of macroscopic scalar curvature and its relation to the systolic geometry of a manifold. More precisely, we derive a macroscopic version of a celebrated systolic inequality by Bray–Brendle–Neves on the area of non-contractible 2-spheres in a manifold of positive scalar curvature. We show that if a complete Riemannian n-manifold with non-trivial codimension 1 homology with $\mathbb{Z}_2$-coefficients or $\mathbb{Z}$-coefficients has positive macroscopic scalar curvature, then it admits a non-nullhomologous hypersurface of small Urysohn $(n-2)$-widht. The proof of this result is based on an adaptation of Guth's macroscopic version of the Schoen–Yau descent argument.

Keywords: scalar curvature, 3-manifold topology, quadratic decay, fill radius, macroscopic scalar curvature, Urysohn width, systole.

Résumé

Cette thèse est consacrée à la topologie et à la géométrie des variétés riemanniennes de courbure scalaire strictement positive. Afin d'explorer la structure de telles variétés, nous adoptons un point de vue métrique, en particulier à travers deux généralisations métriques de la notion de courbure scalaire strictement positive.

Tout d'abord, nous nous concentrons sur la topologie des 3-variétés à courbure scalaire strictement positive, et nous fournissons une nouvelle obstruction à l'existence de métriques riemanniennes complètes à courbure scalaire strictement positive sur des 3-variétés non compactes. Plus précisément, nous prouvons que si une 3-variété orientable M admet une métrique riemannienne complète à courbure scalaire strictement positive à décroissance sous-quadratique, alors elle se décompose en une somme connexe (possiblement infinie) de variétés sphériques et de termes $\mathbb{S}^2 \times \mathbb{S}^1$. En conséquence, la variété M admet une métrique complète à courbure scalaire uniformément strictement positive. Ce résultat constitue une généralisation d'un théorème de Gromov et Wang, et sa preuve repose sur une approche différente, de nature métrique et topologique. Plus généralement, nous dérivons la décomposition topologique à partir d'une estimation métrique des disques de remplissage des courbes fermées dans le revêtement universel, basée sur la notion de rayon de remplissage, et sans hypothèse supplémentaire sur la courbure scalaire. De plus, le taux de décroissance dans le théorème de décomposition topologique est optimal, puisque la variété $\mathbb{R}^2 \times \mathbb{S}^1$ admet une métrique complète à courbure scalaire strictement positive à décroissance exactement quadratique, mais elle ne se décompose pas comme une somme connexe des variétés sphériques et de termes $\mathbb{S}^2 \times \mathbb{S}^1$.

Ensuite, nous explorons la notion de courbure scalaire macroscopique et sa relation avec la géométrie systolique des variétés. Plus concrètement, nous établissons une version macroscopique de la célèbre inégalité systolique de Bray–Brendle–Neves sur l'aire des 2-sphères non contractiles dans une variété à courbure scalaire strictement positive. Nous montrons que si une n-variété riemannienne complète avec une $(n-1)$-homologie à coefficients dans $\mathbb{Z}_2$ ou dans $\mathbb{Z}$ non triviale a une courbure scalaire macroscopique strictement positive, alors elle contient une hypersurface non nulle en homologie de petite $(n-2)$-largeur d'Urysohn. La preuve de ce résultat s'appuie sur une adaptation d'une version macroscopique, due à Guth, de l'argument de descente de Schoen–Yau.

Mots-clés: courbure scalaire, topologie des 3-variétés, décroissance quadratique, rayon de remplissage, courbure scalaire macroscopique, largeur d'Urysohn, systole.

Resum

Aquesta tesi està dedicada a la topologia i la geometria de les varietats riemannianes de curvatura escalar estrictament positiva. Per abordar el seu estudi hem adoptat un punt de vista mètric, concretament a través de dues generalitzacions mètriques de la noció de curvatura escalar estrictament positiva.

En primer lloc, ens centrem en la topologia de les 3-varietats riemannianes de curvatura escalar estrictament positiva, tot proporcionant una nova obstrucció a l'existència de mètriques riemannianes completes de curvatura escalar estrictament positiva per les 3-varietats no compactes. Concretament, demostrem que si una 3-varietat orientable M admet una mètrica riemanniana completa la curvatura escalar de la qual és estrictament positiva i decreix subquadràticament a l'infinit, aleshores M es descompon com una suma connexa (possiblement infinita) de varietats esfèriques i de sumands $\mathbb{S}^2 \times \mathbb{S}^1$. Com a conseqüència, la varietat M admet una mètrica Riemanniana completa de curvatura escalar uniformement estrictament positiva, resolent parcialment una conjectura de Gromov. Aquest resultat constitueix una generalització d'un teorema de Gromov i Wang, tot utilitzant una aproximació al problema de natura diferent, basada en tècniques mètriques i topològiques. Més generalment, derivem la descomposició topològica sota una condició en termes dels discs d'emplenament de corbes tancades en el recobriment universal, basada en la noció de radi d'emplenament, sense cap hipòtesi addicional sobre la curvatura de la varietat. Així mateix, la taxa de decreixement de la curvatura escalar en el teorema de descomposició és òptima. En efecte, la varietat $\mathbb{R}^2 \times \mathbb{S}^1$ admet una mètrica riemanniana completa de curvatura escalar estrictament positiva amb un decreixement exactament quadràtic, però no es descompon com una suma connexa de varietats esfèriques i de productes $\mathbb{S}^2 \times \mathbb{S}^1$.

Tot seguit, ens dediquem a explorar la noció de curvatura escalar macroscòpica i la seva relació amb la geometria sistòlica de les varietats. Més precisament, establim una versió macroscòpica d'una cèlebre desigualtat sistòlica deguda a Bray–Brendle–Neves sobre l'àrea de les 2-esferes no contràctils dins una varietat de curvatura escalar estrictament positiva. Demostrem que si una n-varietat riemanniana completa amb una $(n-1)$-homologia amb coefficients a $\mathbb{Z}_2$ o a $\mathbb{Z}$ no trivial té curvatura escalar macroscòpica estrictament positiva, aleshores la varietat conté una hipersuperfície no nul·la en homologia amb una $(n-2)$-amplada d'Urysohn petita. La prova d'aquest resultat es fonamenta en una adaptació d'una versió macroscòpica, deguda a Guth, de l'argument de descens de Schoen–Yau.

Mots clau: curvatura escalar, topologia de 3-varietats, decreixement quadràtic, radi d'emplenament, curvatura escalar macrsocòpica, amplada d'Urysohn, sístole.

Acknowledgements

Contents

Introduction

This thesis is about the topology and the geometry of Riemannian manifolds of positive scalar curvature. The scalar curvature of a Riemannian n-manifold M is a central invariant in Riemannian geometry. The *scalar curvature* $\mathrm{scal}(x)$ at a point $x \in M$ is defined as

$$\mathrm{scal}(x) := \sum_{i \neq j} \mathrm{sect}_x(e_i \wedge e_j),$$

where sect_x denotes the sectional curvature of the manifold M at the point x and (e_i) is an orthonormal basis of the tangent space T_xM. The scalar curvature can be equivalently defined through the volumetric deviation of geodesic balls of infinitesimal radius with respect to Euclidean balls of the same radius. More precisely, the volume of the geodesic ball $B(x,r)$ centered at a point $x \in M$ satisfies

$$|B(x,r)| = b_n r^n \left(1 - \frac{\mathrm{scal}(x)}{6(n+2)} r^2 + O(r^3)\right), \tag{1}$$

for radii $r > 0$ small enough, where b_n denotes the volume of the unit ball in the Euclidean n-dimensional space.

The scalar curvature constitutes the weakest curvature notion among all the classical curvatures which can be defined from the Riemann curvature tensor. Hence, a central problem in Riemannian geometry consists in understanding the relation between scalar curvature and the global topology and geometry of a manifold.

Throughout this introduction, we shall provide a brief overview of the different developments in the study of scalar curvature, and we will present our results within the context of this field. In Section I, we will focus on the problem of determining which manifolds admit Riemannian metrics of positive scalar curvature, a question that remains one of the central challenges in scalar curvature geometry to this day. Next, in Section II, we will address this question in the specific case of 3-dimensional manifolds, starting with closed 3-manifolds and then turning to the case of open 3-manifolds. Finally, in Section III, we will address the interaction between scalar curvature and systolic geometry. The results obtained during this thesis will be presented in Sections II and III.

I Manifolds of positive scalar curvature

A fundamental question in the study of scalar curvature is to determine when a manifold admits a complete metric of positive scalar curvature. Among the numerous motivations for studying manifolds admitting metrics of positive scalar curvature, there is the following theorem.

Trichotomy Theorem ([KW75a, KW75b, KW75c]). *Let M be a closed connected n-manifold. Then M belongs to exactly one of the following three classes:*

- ***Type 1**: Closed manifolds admitting a Riemannian metric of positive scalar curvature.*
- ***Type 2**: Closed manifolds not admitting a Riemannian metric of positive scalar curvature, but admitting a Riemannian metric of scalar curvature identically zero. In this case, such a metric is Ricci-flat.*
- ***Type 3**: Closed manifolds not admitting a Riemannian metric of non-negative scalar curvature.*

Moreover, if $n \geq 3$, then:

1. *If M is of type 1, then any function $f \in \mathcal{C}^\infty(M)$ can be realised as the scalar curvature of a Riemannian metric on M.*
2. *If M is of type 2, then a function $f \in \mathcal{C}^\infty(M)$ can be realised as the scalar curvature of a Riemannian metric on M if and only if either $f(x) < 0$ for some $x \in M$ or $f \equiv 0$.*
3. *If M is of type 3, then a function $f \in \mathcal{C}^\infty(M)$ can be realised as the scalar curvature of a Riemannian metric on M if and only if $f(x) < 0$ for some $x \in M$.*

The Trichotomy Theorem has deep implications. First, any manifold of dimension at least three can be endowed with a Riemannian metric of negative scalar curvature, which can be assumed to be constant. This result was first proven by Aubin [Aub70], and it was later extended to the non-compact case for complete Riemannian metrics in [BK89]. On the contrary, we will see that there exist topological obstructions to admitting metrics of positive scalar curvature. Moreover, the Trichotomy Theorem implies that deciding whether the scalar curvature of a manifold can be prescribed to be any function corresponds to determining if the manifolds admits a metric of positive scalar curvature. Secondly, it follows from the Trichotomy Theorem that, on non-compact manifolds, there is no restriction to admitting metrics of positive scalar curvature if one does not additionally suppose the metric to be complete. Finally, it also implies that if a manifold of non-negative scalar curvature does not admit a metric of positive scalar curvature, then the manifold is Ricci-flat. This is known as Kazdan's Deformation Theorem [Kaz82]. Recall that in dimension 3, Ricci-flatness implies Riemannian-flatness, that is, the Riemann curvature tensor vanishes. In higher dimensions, by the Cheeger-Gromoll Splitting Theorem [CG71, FW75], any smooth closed Ricci-flat manifold is finitely covered by the product of a torus and a closed simply connected Ricci-flat manifold. Hence, Kazdan's Deformation Theorem implies rigidity results for manifolds of non-negative scalar curvature which do not admit metrics of positive scalar curvature.

The classification of manifolds admitting complete Riemannian metrics of positive scalar curvature can be divided into two separated questions: the determination of topological obstructions to admit metrics of positive scalar curvature, and the elaboration of techniques to construct manifolds with positive scalar curvature.

I.I Topological obstructions to complete metrics of positive scalar curvature

The Dirac operator method

The first obstructions to admit metrics of positive scalar curvature were derived from index theoretical considerations on the Dirac operator on spin manifolds. This approach has the advantage to hold in any dimensions, but only for spin manifolds, that is, manifolds whose second Stiefel–Whitney class vanishes.

Let M be a closed spin Riemannian n-manifold. The Dirac operator $\mathcal{D}$ is a self-adjoint elliptic first order differential operator which may be defined on the sections of a spinor bundle $\mathcal{S} \to M$ over M. We refer the reader to [LM89] for the precise construction of spinor bundles and the Dirac operator on spin manifolds. The Dirac operator satisfies the Lichnerowicz Formula [Lic63]

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4},$$

where ∇ denotes the covariant derivative on the spinor bundle induced by the Levi–Civita connection and ∇^* is the adjoint of ∇. A spinor $\varphi \in \mathcal{C}^\infty(M, \mathcal{S})$ is *harmonic* if it satisfies the Dirac equation $\mathcal{D}\varphi = 0$. The Lichnerowicz Formula implies that, if the Riemannian manifold M has positive scalar curvature, then any harmonic spinor must be trivial.

On the other hand, the Atiyah–Singer Index Theorem [AS71a, AS71b] for the Dirac operator $\mathcal{D}$ relates the existence of non-trivial harmonic spinors to a topological invariant, called the α-genus $\alpha(M)$ of the manifold M. More precisely, the Atiyah–Singer Index Theorem implies that, if $\alpha(M) \neq 0$, then M admits non-trivial spinors. Therefore, it follows from the Lichnerowicz Formula that manifolds admitting metrics of positive scalar curvature have a vanishing α-genus.

A considerable collection of obstruction results for spin manifolds was derived using the Dirac operator method. First, Lichnerowicz [Lic63] showed that there exist smooth closed manifolds of dimension divisible by 4 with non-trivial α-genus, and therefore which do not admit a metric of positive scalar curvature. An example of such manifolds is the Kummer surface, which is the 4-manifold given by the equation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ in $\mathbb{C}\mathrm{P}^3$. Later, Hitchin [Hit74] proved the existence of exotic spheres (that is, n-manifolds homeomorphic but not diffeomorphic to the standard sphere of dimension n) of dimension $n \equiv 1, 2 \pmod 8$ not admitting metrics of positive scalar curvature. The Dirac operator method also allowed to give a full description of which simply connected closed manifolds of dimension at least 5 admit metrics of positive scalar curvature. More precisely, Gromov–Lawson [GL80b] and Stolz [Sto92] showed that a simply connected closed manifold M of dimension $n \geq 5$ admits a metric of positive scalar curvature if and only if either M is not spinable, or M is spinable and $\alpha(M) = 0$. Finally, Gromov–Lawson [GL80a] derived from a twisted version of the Dirac operator and of the Lichnerowicz formula that enlargeable manifolds do not admit metrics of positive scalar curvature. The class of enlargeable manifolds includes the n-torus, closed solvmanifolds, closed hyperbolic manifolds and, more generally, any closed manifold of non-negative sectional curvature. In particular, Gromov–Lawson proved that the n-torus does not admit a metric of positive scalar curvature. This statement is known as the Geroch Conjecture, and was one of the leading problems in the study of scalar curvature. The twisted Dirac operator method also allowed Gromov–Lawson [GL83] to prove that closed aspherical 3-manifolds do not admit Riemannian metrics with positive scalar curvature.

The minimal stable hypersurface descent method

Let M be a closed Riemannian n-manifold and let $\Sigma \subset M$ be a two-sided stable minimal hypersurface. The stability of the hypersurface Σ together with the second variation formula implies that any function $f \in \mathcal{C}^\infty(\Sigma)$ satisfies the Stability Inequality

$$\int_\Sigma \left(|\nabla f|^2 - \left(\mathrm{Ric}(\nu, \nu) + \|\mathrm{II}\|^2 \right) f^2 \right) dV \geq 0, \tag{2}$$

where Ric denotes the Ricci curvature tensor of M, ν is the unit vector field normal to Σ and II is the second fundamental form of Σ. Schoen–Yau [SY79b] noticed that one can cleverly rearrange the Gauss equations to obtain the following identity

$$\mathrm{scal} - \mathrm{scal}_\Sigma + \|\mathrm{II}\|^2 = 2\left(\mathrm{Ric}(\nu,\nu) + \|\mathrm{II}\|^2\right), \tag{3}$$

where scal and scal_Σ are the scalar curvatures of M and Σ, respectively. Therefore, after using Schoen–Yau's rearrangement (3), the Stability Inequality (2) implies that, for any function $f \in \mathcal{C}^\infty(\Sigma)$,

$$\int_\Sigma \left(|\nabla f|^2 + \frac{1}{2}\mathrm{scal}_\Sigma f^2\right) dV \geq \frac{1}{2}\int_\Sigma \mathrm{scal}\, f^2 dV. \tag{4}$$

It follows from the inequality (4) that if the manifold M has positive scalar curvature, then Σ also admits a metric of positive scalar curvature. Indeed, for $n = 3$, it suffices to take $f \equiv 1$ and to use the Gauss–Bonnet formula, to obtain from the inequality (4) that

$$\int_\Sigma \mathrm{scal} \leq 2\int_\Sigma \kappa_\Sigma = 8\pi\chi(\Sigma),$$

where κ_Σ and $\chi(\Sigma)$ denote the Gauss curvature and the Euler characteristic of the surface Σ, respectively. Therefore, the surface Σ must be either homeomorphic to a 2-sphere or to a projective plane. The case $n \geq 4$ needs to be treated with more care. One obtains a metric of positive scalar curvature on Σ by conformally modifying the metric on Σ, and using the inequality (4) to ensure that the new metric has positive scalar curvature. Notice that, in both cases, the induced metric on Σ from the ambient manifold M does not necessarily have positive scalar curvature.

Hence, we conclude that if a Riemannian n-manifold M has positive scalar curvature, then any stable minimal hypersurface in M also has positive scalar curvature. This is known as Schoen–Yau's descent argument ([SY79a] for $n = 3$, and [SY79b] for $n \geq 4$). More generally, if the manifold M has positive scalar curvature and one is able to construct a descending sequence

$$M \supset \Sigma^{n-1} \supset \cdots \supset \Sigma^2$$

of closed oriented stable minimal k-submanifolds Σ^k, then Σ^2 must be a disjoint union of 2-spheres or projective planes. Hence, the descent method consists in showing that a manifold does not admit metrics of positive scalar curvature by constructing a descending sequence $M \supset \Sigma^{n-1} \supset \cdots \supset \Sigma^2$ of stable minimal hypersurfaces ending at a surface Σ^2 of non-positive Euler characteristic.

The precise topological conditions that allow one to define such a descending sequence of stable minimal surfaces were captured by Schick [Sch98] in the notion of SYS manifold. A closed orientable n-manifold M is SYS if there exist cohomology classes $\alpha_1, \ldots, \alpha_{n-2} \in H^1(M;\mathbb{Z})$ such that the homology class

$$[M] \frown (\alpha_1 \smile \cdots \smile \alpha_{n-2}) \in H_2(M;\mathbb{Z})$$

does not lie in the image of the Hurewicz map $\pi_2(M) \to H_2(M)$. Indeed, if M is a SYS n-manifold, then one can consider the non-trivial homology class $[M] \frown \alpha_1 \in H_{n-1}(M;\mathbb{Z})$. By a classical result in Geometric Measure Theory [FF60, Fed70], if the dimension of the manifold M is $n \leq 7$, then the class $[M] \frown \alpha_1$ can be represented by a stable minimal hypersurface Σ^{n-1} which minimises the volume among all representatives. The restriction in the dimension is due to the well-known fact that, in higher dimensions, the volume-minimising hypersurface may present singularities [BDGG69]. Since the minimal stable hypersurface Σ^{n-1} inherits the SYS property, one can apply the construction inductively to construct a descending sequence $M \supset \Sigma^{n-1} \supset \cdots \supset \Sigma^2$ with Σ^2 of non-positive Euler characteristic. Thus, we conclude that SYS manifolds of dimension $n \leq 7$ do not admit metrics of positive scalar curvature. This result was extended to the case $n = 8$

by Joachim–Schick [JS00] using a result of Smale [Sma93]. Recently, Schoen–Yau [SY22] were able to circumvent the restriction on the dimension, at least in certain situations.

The main advantage of the descent method is that, as opposed to the Dirac operator method, it does not require the manifold M to be spin. It was conjectured by Gromov–Lawson–Rosenberg [GL83, Ros91] that all the obstructions to admitting metrics of positive scalar curvature can be determined by the Dirac operator method, at least for spin manifolds. However, Schick [Sch98] disproved the Gromov–Lawson–Rosenberg conjecture by constructing a closed spin 5-manifold for which all obstructions coming from the Dirac operator method vanish, but which does not admit a metric of positive scalar curvature by the descent method. Therefore, the descent method provides new obsructions to admitting metrics of positive scalar curvature. On the other hand, it requires $H^1(M;\mathbb{Z})$ to be non-zero, and the regularity of stable minimal hypersurfaces imposes restrictions on the dimension of M.

Obstructions in the non-compact case

The determination of which non-compact manifolds admit a complete metric of positive scalar curvature or not is more delicate than in the compact case. In fact, when dealing with non-compact manifolds, one must also consider complete metrics with uniformly positive scalar curvature, since there exist complete Riemannian manifolds with positive scalar curvature which cannot be endowed with a complete metric of uniformly positive scalar curvature.

Let M be a closed n-manifold. Rosenberg–Stolz [RS94] proved that $M \times \mathbb{R}^2$ can be endowed with a complete metric of positive scalar curvature, and $M \times \mathbb{R}^k$ admits a complete metric of uniformly positive scalar curvature when $k \geq 3$. Consequently, in [RS94] the authors also conjectured that if M does not admit a metric of positive scalar curvature, then $M \times \mathbb{R}$ admits no complete metric of positive scalar curvature, and $M \times \mathbb{R}^2$ does not admit a metric of uniformly positive scalar curvature. Rosenberg–Stolz's conjecture was established when the dimension of M is $n \leq 2$ by Gromov–Lawson [GL83].

Recent methods on the study of scalar curvature

A new tool for the study of scalar curvature that has led to interesting results are the μ-bubbles introduced by Gromov in [Gro23]. The μ-bubble method may be understood as an extension of the minimal hypersurface approach which allows more flexibility when adapting the construction to the topology and geometry of the manifold.

Given a Riemannian n-manifold M and a smooth function h on M, a *μ-bubble (with respect to h)* is a subset $\Omega \subset \operatorname{int}(M)$ which minimises a certain functional involving the $(n-1)$-volume of the boundary $\partial\Omega$ and a term depending on the values of h on Ω. The existence of μ-bubbles in dimensions $3 \leq n \leq 7$ and for suitable choices of the function h was established in [Gro23, Zhu21]. The variation formulae associated with such functional imply that the boundary $\partial\Omega$ of a μ-bubble Ω with respect to h has mean curvature h, and that it satisfies a stability inequality analogous to equation (2), involving the scalar curvatures of M and of Σ. In particular, the existence of μ-bubbles can be seen as a mean-curvature prescription problem, and therefore the functional minimised by μ-bubbles is often called the prescribed-mean-curvature functional. The stability inequality for μ-bubbles may be used to derive significant geometric estimates to study the geometry of M from lower bounds on the scalar curvature.

The μ-bubble method has led to a number of important results. In [Gro86], Gromov conjectured that any closed aspherical n-manifold does not support a Riemannian metric of positive scalar curvature, for any $n \geq 2$. As mentioned above, Gromov's conjecture was positively solved in dimension 3 by Gromov–Lawson [GL83] using the twisted Dirac operator method. Using μ-bubbles, Chodosh–Li [CL24] and independently Gromov [Gro20] recently established the conjecture in dimensions 4 and 5. However, Gromov's conjecture remains wide open in higher dimensions. In [Gro17, Gro23], Gromov stated the stronger conjecture that $\mathbb{Q}$-essential manifolds do not admit metrics of positive scalar curvature, which will be discussed in Section 1.3.4.

As we will see later, μ-bubbles have also played a central role in recent advances on the topological classification of 3-manifolds of uniformly positive scalar curvature [Gro23, Wan23a].

In [Ste22], Stern developed an innovative approach to scalar curvature for dimension 3, which can be understood as a dual version to the minimal surface method. Recall that if M is a closed oriented 3-manifold then, by Poincaré's Duality, 2-homology classes are in bijective correspondence with homotopy classes of $\mathbb{S}^1$-valued maps on M. Instead of working with stable minimal hypersurfaces, which are constructed by minimising the area functional in a non-trivial 2-homology class, Stern considered harmonic maps $M \to \mathbb{S}^1$, that is, maps minimising the Dirichlet energy in their homotopy class. Using Bochner's Identity, Stern derived an inequality relating the topology of the level sets of a harmonic map $u : M \to \mathbb{S}^1$ with the scalar curvature of the ambient manifold. Among other applications, the harmonic map method can be used to obtain an alternative proof of the Geroch Conjecture for the 3-torus or for Bray–Brendle–Neves' Systolic Inequality [BBN10] that will be presented below.

The Hamilton–Ricci flow has also been employed to unravell the structure of Riemannian 3-manifolds of bounded geometry with uniformly positive scalar curvature [BBM11] and to understand the topology of the moduli spaces of such metrics [Cod12, BBMC21].

There are other approaches to the study of scalar curvature specially suited to dimension 4, based on the Seiberg–Witten Theory.

I.II Existence of manifolds of positive scalar curvature

The fundamental example of manifolds which admit metrics of positive scalar curvature are compact symmetric spaces, since they are non-flat and have non-negative sectional curvature. Examples of compact symmetric spaces include the n-sphere $\mathbb{S}^n$, the projective n-spaces $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ (and, more generally, Grassmannian manifolds over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$), the Cayley plane $\mathbb{C}\mathrm{aP}^2$, and their Riemannian products. More generally, apart from the flat torus, compact homogeneous spaces admit metrics of positive scalar curvature. Also, manifolds obtained as quotients of non-flat compact homogeneous spaces by a free isometric action of a compact group will also admit a metric of positive scalar curvature. Furthermore, strictly convex hypersurfaces have positive sectional curvature, and thus positive scalar curvature. In the realm of complex geometry, complex hypersurfaces of $\mathbb{C}\mathrm{P}^n$ of degree at most n, and more generally K-stable Fano varieties, admit metrics of positive scalar curvature.

There are simple procedures for producing manifolds of positive scalar curvature metrics from previously known examples. For instance, if M is a closed Riemannian manifold of positive scalar curvature and N is any closed manifold, then $M \times N$ admits a metric of positive scalar curvature. This follows from the additivity of scalar curvature under Riemannian products, that is, if M and N are two Riemannian manifolds, then

$$\mathrm{scal}_{M\times N} = \mathrm{scal}_M \circ \pi_M + \mathrm{scal}_N \circ \pi_N,$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the corresponding projections.

The additivity of scalar curvature can be used in the more general setting of fibred manifolds to produce examples of manifolds of positive scalar curvature. Given $\pi : M \to B$ a Riemannian submersion with totally geodesic fibres, one can always deform the Riemannian metric g of M by rescaling the metric along the fibres of π by a factor of $\varepsilon > 0$. This procedure gives a family of metrics $(g_\varepsilon)_{\varepsilon>0}$ on M known as the canonical variation of g, for which $\pi : M \to B$ is still a Riemannian submersion. By O'Neill's formulae [O'N66] (see also [Bes87, Proposition 9.70]), the scalar curvature $\mathrm{scal}_\varepsilon$ of (M, g_ε) for a fixed $\varepsilon > 0$ is given by

$$\mathrm{scal}_\varepsilon = \frac{1}{\varepsilon^2}\,\mathrm{scal}_F + \mathrm{scal}_B \circ \pi - \varepsilon^2\,|A|^2\,,$$

with scal_B and scal_F the scalar curvatures of the base B and of the fibre F through the corresponding point, respectively, and where $|A|^2$ is the squared norm of O'Neill's integrability tensor A. Therefore, if $\pi : M \to B$ is a submersion whose fibres are totally geodesic and admit metrics of positive scalar curvature, then M also can be endowed with a metric of positive scalar curvature, just by sufficiently shrinking the fibres through the canonical variation. However, this techniques provide only a limited number of examples.

A breakthrough in the study of scalar curvature is the Surgery Theorem [GL80a, SY79b].

Surgery Theorem ([GL80a, SY79b]). *Let M be a (non-necessarily connected) closed n-manifold of positive scalar curvature. If N is a manifold obtained from M by a surgery of codimension at least 3, then N also admits a metric of positive scalar curvature.*

In particular, the connected sum of two manifolds of positive scalar curvature can also be endowed with a metric of positive scalar curvature. The Surgery Theorem is a powerful tool for constructing new manifolds admitting metrics of positive scalar curvature.

II The topology of 3-manifolds of positive scalar curvature

Now, let us turn our attention to the specific case of 3-manifolds. As discussed previously, specific examples of 3-manifolds admitting metrics of positive scalar curvature are the 3-sphere, spherical manifolds, the product $\mathbb{S}^2 \times \mathbb{S}^1$, and their connected sums. Recall that a *spherical 3-manifold* is a manifold $\mathbb{S}^3/\Gamma$ obtained as the quotient of the 3-sphere by a subgroup $\Gamma < O(4)$ of isometries acting freely on $\mathbb{S}^3$. In his Problem Section, Yau asked for a classification of 3-manifolds admitting complete metrics of positive scalar curvature [Yau82, Problem 27].

Let us first discuss the case of closed 3-manifolds. Using the twisted Dirac operator method, Gromov–Lawson [GL83] proved that, if a closed (not necessarily orientable) 3-manifold admits a metric of positive scalar curvature, then it cannot contain an aspherical summand in its prime decomposition (see Section 1.1.1). Hence, from the Kneser–Milnor Prime Decomposition Theorem [Kne29, Mil62], the Surgery Theorem [GL80a, SY79b] and Perelman's resolution of the Elliptisation Conjecture [Per02, Per03a, Per03b] it follows that a closed orientable 3-manifold admits a metric of positive scalar curvature if and only if it decomposes as a finite connected sum

$$\mathbb{S}^3/\Gamma_1 \# \ldots \# \mathbb{S}^3/\Gamma_p \# \mathbb{S}^2 \times \mathbb{S}^1 \# \ldots \# \mathbb{S}^2 \times \mathbb{S}^1$$

of spherical 3-manifolds $\mathbb{S}^3/\Gamma_i$ and $\mathbb{S}^2 \times \mathbb{S}^1$ summands. Recall that, unlike in the orientable case, non-orientable closed prime 3-manifolds are not classified. As a consequence, the structure of non-orientable closed 3-manifolds admitting metrics of positive scalar curvature is less understood.

The first problem one encounters when considering orientable open 3-manifolds is that the Kneser–Milnor Prime Decomposition Theorem does not hold in general. Indeed, Scott [Sco77] showed that not every open manifold decomposes as a connected sum of prime manifolds, even if one considers infinite connected sums. Other examples of open non-prime 3-manifolds which are indecomposable as an infinite connected sum can be found in [ST89, Mai08].

However, a similar decomposition theorem has recently been proved for open 3-manifolds admitting complete Riemannian metrics of uniformly positive scalar curvature. Gromov [Gro23] and Wang [Wan23a] used μ-bubble theory to show that if a complete orientable Riemannian 3-manifold admits a metric of uniformly positive scalar curvature, then it decomposes as a possibly infinite connected sum of spherical 3-manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$. This decomposition result had already been proven previously under additional hypothesis: more specifically, for manifolds with finitely generated fundamental group using K-theory methods [CWY10], and for manifolds with bounded geometry using Ricci flow techniques [BBM11].

In [BGS24], we generalised the decomposition theorem of Gromov and Wang to complete orientable Riemannian 3-manifolds whose scalar curvature is positive and exhibits a certain decay at infinity.

Theorem A ([BGS24, Theorem 1.3]). *Let M be a complete orientable Riemannian 3-manifold. Let $x \in M$ be a point. Suppose that M has positive scalar curvature, and that there exists a constant $C > 64\pi^2$ such that, for every point $y \in M$ with $d(x,y) \geq 1$,*

$$\mathrm{scal}(y) > \frac{C}{d(x,y)^2}. \tag{5}$$

Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands.

One may wonder whether the conclusion of Theorem A holds under a weaker decay rate with respect to the distance to the point x. The example of the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ [Gro23, Section 3.10.2] shows this is impossible. Indeed, the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ admits a complete metric of positive scalar curvature decaying quadratically with respect to the distance to x with a constant $C = \frac{1}{2}$, but does not decompose as an infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$, see Section 1.5.2.

The proof of Theorem A relies on an estimate on the filling discs of closed curves, based on the notion of fill radius introduced in [GL83, SY79a, SY83], which generalises the notion of positive scalar curvature with the quadratic decay condition in equation (5). Let M be a Riemannian n-manifold with empty boundary. The fill radius $\mathrm{fillrad}\,(\gamma)$ of a contractible closed curve γ in M is the supremal positive real number $R > 0$ such that the curve does not bound a disc in its closed R-neighbourhood. Gromov–Lawson [GL83] and Schoen–Yau [SY83] proved that if a complete Riemannian 3-manifold M with bounded geometry has uniformly positive scalar curvature $\mathrm{scal} \geq s > 0$, then any contractible closed curve γ in M satisfies

$$\mathrm{fillrad}\,(\gamma) \leq \frac{2\pi}{\sqrt{s}}.$$

If a complete orientable Riemannian 3-manifold M has positive scalar curvature decaying at infinity, then the fill radius of contractible closed curves in M is not uniformly bounded in general. Still, if the decay is not too pronounced, one can control the growth of the fill radius of contractible closed curves in M, or rather, of their lifts to the Riemannain universal cover of M. We prove the topological decomposition of Theorem A by replacing the scalar curvature assumption with this weaker condition on the growth of the fill radius of lifts of contractible closed curves to the Riemannian universal cover of M.

The following rigidity result is a direct consequence of Theorem A and an adaptation of the Surgery Theorem, see Section 1.6.

Corollary B ([BGS24, Corollary 1.5])**.** *Let M be a complete orientable Riemannian 3-manifold. Let $x \in M$ be a point. Suppose that M has positive scalar curvature, and that there exists a constant $C > 64\pi^2$ such that, for every point $y \in M$ with $d(x,y) \geq 1$,*

$$\operatorname{scal}(y) > \frac{C}{d(x,y)^2}.$$

Then M admits a complete Riemannian metric with uniformly positive scalar curvature.

III Systolic geometry of manifolds of positive scalar curvature

Regarding the relation of scalar curvature to the geometry of a Riemannian manifold, we will mainly consider its effect to systolic quantities.

Let M be a closed Riemannian 3-manifold with $\pi_2(M) \neq 0$. The *homotopical* 2*-systole* $\operatorname{sys}\pi_2(M)$ of M is defined to be the least area among non-contractible 2-spheres immersed in M, see Section 2.2. Bray–Brendle–Neves [BBN10] proved that if the scalar curvature of M satisfies $\operatorname{scal} \geq s > 0$, then

$$\operatorname{sys}\pi_2(M) \leq \frac{8\pi}{s}. \tag{6}$$

Moreover, equality holds if and only if the Riemannian universal cover M is isometric to the standard Riemannian cylinder $\mathbb{S}^2(1) \times \mathbb{R}$ up to scaling. The proof of Bray–Brendle–Neves' Systolic Inequality (6) relies on the Stability Inequality (2) applied on a non-contractible 2-sphere of least area in its homotopy class.

Bray–Brendle–Neves' Systolic Inequality (6) has been generalised in multiple directions. For example, Bray–Brendle–Eichmair–Neves proved an analogous inequality for embedded projective planes, see [BBEN10]. In higher dimensions, Riemannian products of round spheres show that one cannot expect in general a control of the 2-systole solely from a lower bound on the scalar curvature. However, some generalisations have been derived under further topological assumptions on the manifold M. For instance, Zhu proved that Bray–Brendle–Neves' Systolic Inequality (6) holds up to dimension 7 if the manifold admits a non-zero degree map to $\mathbb{S}^2 \times \mathbb{T}^{n-2}$, see [Zhu20]. The author also generalised the Systolic Inequality (6) to the non-compact case for manifolds admitting a non-zero degree map to $\mathbb{S}^2 \times \mathbb{T}^{n-3} \times \mathbb{R}$, again up to dimension 7, see [Zhu23]. In another direction, Richard obtained an estimate for the homotopical 2-systole of $\mathbb{S}^2 \times \mathbb{S}^2$ endowed with a metric of positive scalar curvature satisfying a certain stretching condition, see [Ric20].

Bray–Brendle–Neves' Systolic Inequality (6) has also motivated analogous results for hypersurfaces which are minimising within their homology class. In [Ste22], Stern gave a direct proof of the homological analogue of the Systolic Inequality (6). A generalisation to dimensions from 4 to 7 was addressed by Chu–Lee–Zhu in [CLZ24], where they proved an upper bound on the codimension 1 homological systole (see Section 2.2) under a stronger curvature positivity condition, namely positive bi-Ricci curvature, and obtained a rigidity statement for the equality case.

III.I Systolic geometry and positive macroscopic scalar curvature

Because of equation (1), scalar curvature can be understood as a measure of the volumetric deviation of geodesic balls of infinitessimal radii with respect to the Euclidean balls of the same radius. In [Gut10a], Guth introduced a macroscopic analogue of scalar curvature, which quantifies the volumetric deviation of geodesic balls of a fixed finite radius.

Let M be a Riemannian n-manifold and denote by $\tilde{M}$ the universal Riemannian cover of M. The *macroscopic scalar curvature* $\operatorname{mscal}(x,R)$ of M at a point $x \in M$ and scale $R > 0$ satisfies

$\mathrm{mscal}(x,R) \geq s > 0$ if and only if the volume of the geodesic ball $B_{\tilde{M}}(\tilde{x},R)$ in $\tilde{M}$ centered at a lift $\tilde{x}$ of x and of radius R verifies

$$\left|B_{\tilde{M}}(\tilde{x},R)\right| \leq V_s^n(R),$$

where $V_s^n(R)$ is the volume of any ball of radius R in the n-sphere of constant scalar curvature s. The macroscopic scalar curvature is defined through the volumes of balls in the universal cover $\tilde{M}$ of M in order to ensure that flat manifolds have macroscopic scalar curvature equal to zero at any scale, see Section 2.1.

One may wonder whether there is a macroscopic analogue of Bray–Brendle–Neves' Systolic Inequality (6). The following proposition shows that one cannot hope for a control of the homotopical systoles of a closed Riemannian manifold solely from a lower bound on its macroscopic scalar curvature. The *homotopical k-systole* $\mathrm{sys}\,\pi_k(M)$ of a Riemannian manifold M is defined to be the k-volume of the smallest non-contractible k-dimensional sphere immersed in M. If instead of dealing with non-contractible k-spheres, one considers non-nullhomologous k-dimensional submanifolds, one obtains analogously the notion of *homological k-systole* $\mathrm{sys}\,H_k(M)$ of M.

Proposition C ([Gil25, Proposition 1.5])**.** *Let $n \geq 3$ and $k \in \{2,\dots,n-1\}$. For every $s > 0$, there is a family of product Riemannian metrics $(g_\varepsilon)_{\varepsilon\in(0,1)}$ on $\mathbb{S}^k \times \mathbb{S}^{n-k}$ such that the following holds.*

1. *For any point $x \in \mathbb{S}^k \times \mathbb{S}^{n-k}$ and any scale $R > 0$, one has that for every $\varepsilon \in (0,1)$*

$$\mathrm{mscal}_{(\mathbb{S}^k\times\mathbb{S}^{n-k},g_\varepsilon)}(x,R) \geq s.$$

2. *The homotopical k-systole and the homological k-systole verify*

$$\lim_{\varepsilon\to 0} \mathrm{sys}\,\pi_k(\mathbb{S}^k \times \mathbb{S}^{n-k}, g_\varepsilon) = \lim_{\varepsilon\to 0} \mathrm{sys}\,H_k(\mathbb{S}^k \times \mathbb{S}^{n-k}, g_\varepsilon) = +\infty.$$

However, one could hope to have an analogue of Bray–Brendle–Neves' Systolic Inequality (6) holding for a weaker metric invariant describing the size of topologically non-trivial hypersurfaces, as for instance their codimension 1 Urysohn width.

Given a metric space X, the *k-dimensional Urysohn width* $\mathrm{UW}_k(X)$ of X is a measure of how far X is from being k-dimensional. For a precise definition of the notion of Urysohn width, see Section 2.3. When dealing with Riemannian manifolds, the codimension 1 Urysohn width is particularly relevant, since it is the first non-trivial Urysohn width. In [Gut17], Guth showed that the codimension 1 Urysohn width of a n-dimensional Riemannian manifold M is related to its volume $|M|$ by

$$\mathrm{UW}_{n-1}(M) \leq C_n \, |M|^n \, ,$$

where $C_n > 0$ is a constant depending only on the dimension of M. As a consequence, the infimum of the Urysohn $(n-2)$-width among all non nullhomologous hypersurfaces immersed in M is a weaker invariant than its homological $(n-1)$-systole $\mathrm{sys}\,H_{n-1}(M)$.

The main result in [Gil25] is the following macroscopic version of Bray–Brendle–Neves' Systolic Inequality (6). Let $G = \mathbb{Z}_2$ or $\mathbb{Z}$. Consider a non-simply connected complete Riemannian n-manifold M such that $H_{n-1}(M;G) \neq 0$. Notice that when the manifold M is compact and G-orientable, having non-trivial codimension 1 G-homology already implies that M is not simply connected, by Poincaré Duality and the Universal Coefficient Theorem. However, it is no longer true when one considers non-compact manifolds. Consider the homotopical 1-systole $\mathrm{sys}\,\pi_1(M)$ of M, that is, the length of the shortest non-contractible closed curve on M. Notice that if M is non-compact, one may have $\mathrm{sys}\,\pi_1(M) = 0$.

Theorem D ([Gil25, Theorem 1.9])**.** *There is a dimensional constant $\kappa_n > 0$ such that the following holds. Let $G = \mathbb{Z}_2$ or $\mathbb{Z}$. Let M be a non-simply connected complete Riemannian n-manifold such that $H_{n-1}(M; G) \neq 0$ and $\operatorname{sys}\pi_1(M) > 0$. Fix $R > 0$ and $s > 0$ such that $\kappa_n/\sqrt{s} < R < \frac{1}{2}\operatorname{sys}\pi_1(M)$. Suppose that $\operatorname{mscal}(x, R) \geq s$ for every point $x \in M$. Then there exists a closed embedded hypersurface Σ such that $[\Sigma] \neq 0 \in H_{n-1}(M; G)$ and*

$$\mathrm{UW}_{n-2}(\Sigma) \leq \frac{n-1}{n} R.$$

In fact, one cannot expect Theorem D to hold for arbitrarily large values of the scale R, as the volume growth of the metric balls centered at a fixed point is significantly affected beyond the injectivity radius at that point, as shown by the following proposition.

Proposition E. *Fix $\kappa > 0$. There is a family of Riemannian metrics $(\bar{g}_s)_{s>0}$ on the real projective space $\mathbb{R}\mathrm{P}^3$ satisfying the following properties.*

1. *For every $s > 0$ large enough, there is a scale $R_s > \max\left\{\frac{1}{2}\operatorname{sys}\pi_1(\mathbb{R}\mathrm{P}^3, \bar{g}_s), \frac{\kappa}{\sqrt{s}}\right\}$ such that for every $x \in M$*
$$\operatorname{mscal}_{\bar{g}_s}(x, R_s) \geq s.$$
2. *For every $s > 0$ large enough, every closed embedded surface Σ in $(\mathbb{R}\mathrm{P}^3, \bar{g}_s)$ such that $[\Sigma] \neq 0 \in H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2)$ has*
$$\mathrm{UW}_1(\Sigma) > w,$$
for some constant $w > 0$ (which does not depend on s).

Structure of the thesis

This thesis is divided into two chapters. Chapter 1 is dedicated to the topology of 3-dimensional manifolds of positive scalar curvature, with the principal aim of providing the proof of Theorem A. In Chapter 2 we will present the different notions introduced in Section III, detail Propositions C and E, and finally the proof of Theorem D.

Notation

Throughout this thesis, we will generally assume that all manifolds are smooth and connected, unless explicitly otherwise. We will consider both manifolds without and with boundary. Accordingly, we will explicitly state in each case whether or not the manifold has a boundary. We will also specify whether the manifold is orientable or not necessarily in each case.

Metric balls will be considered closed. The *closed metric ball* centered at a point x and of radius R will be denoted by $B(x, R)$, and its boundary will be denoted by $S(x, R) := \partial B(x, R)$. To avoid confusion, we will indicate the corresponding metric space when necessary. For instance, $B_{\tilde{M}}(\tilde{x}, R)$ denotes a metric ball in the universal Riemannian cover $\tilde{M}$ of a manifold M.

We shall also consider closed metric neighbourhoods. Given a metric space X and a subset $Z \subset X$, we will denote the *closed R-neighbourhood* of Z in M by

$$U(Z, R) := \{x \in X \mid d(x, Z) \leq R\}.$$

Introduction

Cette thèse porte sur la topologie et la géométrie des variétés riemanniennes à courbure scalaire strictement positive. La courbure scalaire constitue un invariant fondamental en géométrie riemannienne. La *courbure scalaire* $\mathrm{scal}(x)$ d'une variété riemannienne de dimension n en un point $x \in M$ est définie par

$$\mathrm{scal}(x) := \sum_{i \neq j} \mathrm{sect}_x(e_i \wedge e_j),$$

où sect_x désigne la courbure sectionnelle de la variété M au point x, et (e_i) est une base orthonormée de l'espace tangent T_xM. La courbure scalaire peut également être définie de manière équivalente à partir de la déviation volumique des boules géodésiques de rayon infinitéssimal par rapport aux boules euclidiennes de même rayon. Plus précisément, le volume de la boule géodésique $B(x, r)$ centrée au point $x \in M$ vérifie

$$|B(x,r)| = b_n r^n \left(1 - \frac{\mathrm{scal}(x)}{6(n+2)} r^2 + O(r^3)\right), \tag{1}$$

pour des rayons $r > 0$ suffisamment petits, où b_n représente le volume de la boule unité dans l'espace euclidien de dimension n.

La courbure scalaire constitue la notion de courbure la plus faible parmi toutes les notions de courbure que l'on peut définir à partir du tenseur de courbure de Riemann. Ainsi, un problème majeur en géométrie riemannienne est de comprendre comment la courbure scalaire est liée à la topologie et à la géométrie globales d'une variété.

Au fil de cette introduction, nous présenterons brièvement les principaux développements dans l'étude de la courbure scalaire, puis nous énoncerons nos résultats dans le cadre de ce domaine. Dans la Section I, nous nous intéresserons au problème de déterminer quelles variétés admettent des métriques riemanniennes à courbure scalaire strictement positive, une question qui reste l'un des principaux défis du domaine à ce jour. Ensuite, dans la Section II, nous aborderons cette question dans le cadre des variétés de dimension 3, en commençant par les variétés fermées, puis en passant aux variétés ouvertes. Enfin, dans la Section III, nous examinerons l'interaction entre la courbure scalaire et la géométrie systolique. Les résultats obtenus au cours de cette thèse seront présentés dans les Sections II et III.

I Variétés à courbure scalaire strictement positive

Une question fondamentale dans l'étude de la courbure scalaire consiste à déterminer quand une variété admet une métrique complète à courbure scalaire strictement positive. Parmi les nombreuses motivations pour l'étude des variétés admettant des métriques à courbure scalaire strictement positive, on peut citer le théorème suivant.

Théorème de Trichotomie ([KW75a, KW75b, KW75c])**.** *Soit M une variété fermée et connexe de dimension n. Alors M appartient exactement à l'une des trois classes suivantes :*

- ***Type 1**: Variétés fermées admettant une métrique riemannienne à courbure scalaire strictement positive.*
- ***Type 2**: Variétés fermées n'admettant pas de métrique riemannienne à courbure scalaire strictement positive, mais admettant une métrique riemannienne à courbure scalaire identiquement nulle. Dans ce cas, une telle métrique est Ricci-plate.*
- ***Type 3**: Variétés fermées n'admettant pas de métrique riemannienne à courbure scalaire positive.*

De plus, si $n \geq 3$, alors :

1. *Si M est de type 1, toute fonction $f \in \mathcal{C}^\infty(M)$ peut être réalisée comme la courbure scalaire d'une métrique riemannienne sur M.*
2. *Si M est de type 2, une fonction $f \in \mathcal{C}^\infty(M)$ peut être réalisée comme la courbure scalaire d'une métrique riemannienne sur M si et seulement si $f(x) < 0$ pour un certain $x \in M$ ou $f \equiv 0$.*
3. *Si M est de type 3, une fonction $f \in \mathcal{C}^\infty(M)$ peut être réalisée comme la courbure scalaire d'une métrique riemannienne sur M si et seulement si $f(x) < 0$ pour un certain $x \in M$.*

Le Théorème de Trichotomie a des implications profondes. Premièrement, toute variété de dimension au moins trois peut être munie d'une métrique riemannienne à courbure scalaire strictement négative, qui peut être supposée constante. Ce résultat a d'abord été démontré par Aubin [Aub70], et a ensuite été étendu au cas non compact pour les métriques riemanniennes complètes dans [BK89]. En revanche, il existe des obstructions topologiques à l'admission de métriques à courbure scalaire strictement positive. De plus, le Théorème de Trichotomie implique que décider si la courbure scalaire d'une variété peut être prescrite par une fonction quelconque revient à déterminer si la variété admet une métrique à courbure scalaire strictement positive. Deuxièmement, il découle du Théorème de Trichotomie que, sur les variétés non compactes, il n'existe aucune restriction à l'admission de métriques à courbure scalaire strictement positive si l'on ne suppose pas en plus que la métrique est complète. Enfin, le Théorème de Trichotomie implique en outre que si une variété à courbure scalaire positive n'admet pas de métrique à courbure scalaire strictement positive, alors elle est Ricci-plate. Ce résultat est connu sous le nom de Théorème de Déformation de Kazdan [Kaz82]. Rappelons qu'en dimension 3, être Ricci-plat implique être Riemann-plat, c'est-à-dire que le tenseur de courbure de Riemann s'annule identiquement. En dimensions supérieures, d'après le Théorème de Séparation de Cheeger–Gromoll [CG71, FW75], toute variété fermée Ricci-plate admet un revêtement fini par le produit d'un tore et d'une variété fermée simplement connexe et Ricci-plate. Ainsi, le Théorème de Déformation de Kazdan entraîne des résultats de rigidité pour les variétés à courbure scalaire positive n'admettant pas de métriques à courbure scalaire strictement positive.

La classification des variétés admettant des métriques riemanniennes complètes à courbure scalaire strictement positive se divise en deux questions distinctes : la détermination des obstructions topologiques à l'admission de telles métriques, et l'élaboration de techniques pour construire des variétés à courbure scalaire strictement positive.

I.I Obstructions topologiques à la courbure scalaire strictement positive

La méthode de l'opérateur de Dirac

Les premières obstructions à l'existence de métriques à courbure scalaire strictement positive furent établies à partir de considérations issues de la théorie de l'indice appliquée à l'opérateur de Dirac sur les variétés spin. Cette approche a l'avantage de pouvoir s'appliquer en toute dimension, mais uniquement aux variétés spin, c'est-à-dire aux variétés dont la seconde classe de Stiefel–Whitney est nulle.

Soit M une variété riemannienne fermée spin de dimension n. L'opérateur de Dirac $\mathcal{D}$ est un opérateur différentiel elliptique auto-adjoint d'ordre 1 défini sur les sections d'un fibré spinoriel $\mathcal{S} \to M$ sur M. Pour une construction précise des fibrés spinoriels et de l'opérateur de Dirac sur les variétés spin, on pourra consulter [LM89]. L'opérateur de Dirac satisfait la Formule de Lichnerowicz [Lic63]

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{\text{scal}}{4},$$

où ∇ désigne la dérivée covariante sur le fibré spinoriel induite par la connexion de Levi–Civita, et ∇^* est l'adjoint de ∇. Un spineur $\varphi \in \mathcal{C}^\infty(M, \mathcal{S})$ est dit *harmonique* s'il satisfait l'équation de Dirac $\mathcal{D}\varphi = 0$. La Formule de Lichnerowicz implique que, si la variété riemannienne M est à courbure scalaire strictement positive, alors tout spineur harmonique est nécessairement trivial.

D'autre part, le Théorème de l'Indice d'Atiyah–Singer [AS71a, AS71b] appliqué à l'opérateur de Dirac $\mathcal{D}$ relie l'existence de spineurs harmoniques non triviaux à un invariant topologique, appelé le α-genre $\alpha(M)$ de la variété M. Plus précisement, le Théorème de l'Indice d'Atiyah–Singer implique que, si $\alpha(M) \neq 0$, alors M admet des spineurs non triviaux. En conséquence, la Formule de Lichnerowicz implique que les variétés admettant des métriques à courbure scalaire strictement positive ont un α-genre nul.

Un grand nombre de résultats d'obstruction pour les variétés spin ont été dérivés grâce à l'application de la méthode de l'opérateur de Dirac. Tout d'abord, Lichnerowicz [Lic63] montra qu'il existe des variétés fermées et lisses de dimension divisible par 4, ayant un α-genre non trivial, et qui n'admettent donc pas de métrique à courbure scalaire strictement positive. Un exemple de telles variétés est la surface de Kummer, qui est la variété de dimension 4 donnée par l'équation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ dans $\mathbb{C}P^3$. Plus tard, Hitchin [Hit74] démontra l'existence de sphères exotiques (c'est-à-dire des variétés n-dimensionnelles qui sont homéomorphes, mais non difféomorphes, à la sphere standard de dimension n) de dimension $n \equiv 1, 2 \pmod 8$ qui ne peuvent être équipées de métriques à courbure scalaire strictement positive. La méthode de l'opérateur de Dirac a également permis de fournir une description complète des variétés fermées simplement connexes de dimension au moins 5 qui peuvent admettre des métriques à courbure scalaire strictement positive. Plus précisement, Gromov–Lawson [GL80b] et Stolz [Sto92] ont montré qu'une variété fermée simplement connexe M de dimension $n \geq 5$ admet une métrique à courbure scalaire strictement positive si et seulement si soit M n'est pas spinable, soit M est spinable et $\alpha(M) = 0$. Enfin, Gromov–Lawson [GL80a] ont obtenu, à partir d'une version tordue de l'opérateur de Dirac et de la Formule de Lichnerowicz, que les variétés agrandissables n'admettent pas de métriques à courbure scalaire strictement positive. La classe des variétés agrandissables comprend le tore n-dimensionnel, les solvariétés fermées, les variétés hyperboliques fermées et, plus généralement, toute variété fermée à courbure sectionelle positive. En particulier, Gromov–Lawson ont démontré que le tore n-dimensionnel n'admet pas de métrique à courbure scalaire strictement positive. Cette assertion est connue sous le nom de Conjecture de Geroch et a constitué l'un des principaux enjeux dans l'étude de la courbure scalaire. Par ailleurs, grâce à la méthode de l'opérateur de Dirac tordu, Gromov–Lawson [GL83] ont prouvé que les 3-variétés asphériques fermées n'admettent pas de métriques riemanniennes à courbure scalaire strictement positive.

La méthode de descente des hypersurfaces minimales stables

Soit M une variété riemannienne fermée de dimension n, et $\Sigma \subset M$ une hypersurface minimale stable à deux faces. La stabilité de l'hypersurface Σ et la formule de la variation seconde entraînent que toute fonction $f \in \mathcal{C}^\infty(\Sigma)$ satisfait l'Inégalité de Stabilité

$$\int_\Sigma \left(|\nabla f|^2 - \left(\mathrm{Ric}(\nu,\nu) + \|\mathrm{II}\|^2 \right) f^2 \right) dV \geq 0, \tag{2}$$

où Ric désigne le tenseur de courbure de Ricci de M, ν est le champ de vecteurs unitaires normal à Σ, et II est la seconde forme fondamentale de Σ. Schoen–Yau [SY79b] ont observé qu'il est possible, par un réarrangement ingénieux des équations de Gauss, d'obtenir l'identité suivante

$$\mathrm{scal} - \mathrm{scal}_\Sigma + \|\mathrm{II}\|^2 = 2\left(\mathrm{Ric}(\nu,\nu) + \|\mathrm{II}\|^2 \right), \tag{3}$$

où scal et scal_Σ désignent les courbures scalaires de M et de Σ, respectivement. Par conséquent, en appliquant le réarrangement de Schoen–Yau (3), l'Inégalité de Stabilité (2) implique que, pour toute fonction $f \in \mathcal{C}^\infty(\Sigma)$,

$$\int_\Sigma \left(|\nabla f|^2 + \frac{1}{2}\mathrm{scal}_\Sigma f^2 \right) dV \geq \frac{1}{2}\int_\Sigma \mathrm{scal}\, f^2 dV. \tag{4}$$

Il découle de l'inégalité (4) que si la variété M est à courbure scalaire strictement positive, alors Σ admet également une métrique à courbure scalaire strictement positive. En effet, pour $n = 3$, il suffit de prendre $f \equiv 1$ et d'utiliser la formule de Gauss–Bonnet, ce qui permet d'obtenir à partir de l'inégalité (4) que

$$\int_\Sigma \mathrm{scal} \leq 2\int_\Sigma \kappa_\Sigma = 8\pi\chi(\Sigma),$$

où κ_Σ et $\chi(\Sigma)$ désignent respectivement la courbure de Gauss et la caractéristique d'Euler de la surface Σ. Ainsi, la surface Σ doit être soit homéomorphe à une 2-sphere, soit au plan projectif. Le cas où $n \geq 4$ nécessite un traitement plus attentif. On peut obtenir une métrique à courbure scalaire strictement positive sur Σ en modifiant de manière conforme la métrique induite sur Σ, et en utilisant l'inégalité (4) pour garantir que la nouvelle métrique est bien à courbure scalaire strictement positive.

Nous concluons donc que si une variété riemannienne n-dimensionnelle M est à courbure scalaire strictement positive, alors toute hypersurface minimale stable dans M possède également une courbure scalaire strictement positive. Cela constitue l'argument de descente de Schoen–Yau ([SY79a] pour $n = 3$, et [SY79b] pour $n \geq 4$). Plus généralement, si la variété M est à courbure scalaire strictement positive et qu'il est possible de construire une suite descendante

$$M \supset \Sigma^{n-1} \supset \cdots \supset \Sigma^2$$

de sous-variétés minimales stables, fermées et orientées, de dimension k, notées Σ^k, alors Σ^2 doit être une union disjointe de 2-sphères ou de plans projectifs. Ainsi, la méthode de descente consiste à démontrer qu'une variété n'admet pas de métriques à courbure scalaire strictement positive en construisant une suite descendante $M \supset \Sigma^{n-1} \supset \cdots \supset \Sigma^2$ d'hypersurfaces minimales stables, se terminant par une surface Σ^2 dont la caractéristique d'Euler est négative.

Les conditions topologiques précises permettant de définir une telle suite descendante de surfaces minimales stables ont été formulées par Schick [Sch98] à travers la notion de variété SYS. Une variété M fermée, orientable et de dimension n est dite SYS s'il existe des classes de cohomologie $\alpha_1, \ldots, \alpha_{n-2} \in H^1(M;\mathbb{Z})$ telles que la classe d'homologie

$$[M] \frown (\alpha_1 \smile \cdots \smile \alpha_{n-2}) \in H_2(M;\mathbb{Z})$$

n'appartient pas à l'image de l'application de Hurewicz $\pi_2(M) \to H_2(M)$. En effet, si M est une variété SYS de dimension n, alors on peut considérer la classe d'homologie non triviale donnée par $[M] \frown \alpha_1 \in H_{n-1}(M;\mathbb{Z})$. D'après un résultat classique de la Théorie Géométrique de la Mesure [FF60, Fed70], si la dimension de la variété M est $n \leq 7$, alors la classe $[M] \frown \alpha_1$ peut être représentée par une hypersurface minimale stable Σ^{n-1} qui minimise le volume parmi tous les représentants de la classe. Cette restriction sur la dimension provient du fait bien connu qu'en dimension supérieure, les hypersurfaces minimisantes peuvent présenter des singularités [BDGG69]. Puisque l'hypersurface minimale stable Σ^{n-1} hérite de la propriété SYS, on peut appliquer la construction de manière inductive pour obtenir une suite descendante $M \supset \Sigma^{n-1} \supset \cdots \supset \Sigma^2$, où Σ^2 est une surface de caractéristique d'Euler négative. Il en résulte que les variétés SYS de dimension $n \leq 7$ n'admettent pas de métrique à courbure scalaire strictement positive. Ce résultat a été étendu au cas $n = 8$ par Joachim–Schick [JS00], à l'aide d'un résultat de Smale [Sma93]. Plus récemment, Schoen–Yau [SY22] ont réussi à lever la restriction dimensionnelle, au moins dans certaines situations.

Le principal avantage de la méthode de descente est que, contrairement à la méthode de l'opérateur de Dirac, elle ne nécessite pas que la variété M soit spin. Gromov–Lawson–Rosenberg [GL83, Ros91] ont conjecturé que toutes les obstructions à l'existence de métriques à courbure scalaire strictement positive pouvaient être détectées par la méthode de l'opérateur de Dirac, au moins pour les variétés spin. Cependant, Schick [Sch98] a réfuté la conjecture de Gromov–Lawson–Rosenberg en construisant une variété spin fermée de dimension 5 pour laquelle toutes les obstructions provenant de l'opérateur de Dirac s'annulent, mais dont on peut montrer, par la méthode de descente, qu'elle n'admet pas de métrique à courbure scalaire strictement positive. Ainsi, la méthode de descente fournit de nouvelles obstructions à l'admission de métriques à courbure scalaire strictement positive. D'autre part, elle nécessite que $H^1(M;\mathbb{Z})$ soit non nul, et la régularité des hypersurfaces minimales stables impose des restrictions sur la dimension de M.

Obstructions dans le cas non compact

La détermination des variétés non compactes qui admettent ou non une métrique complète à courbure scalaire strictement positive se révèle plus subtile que dans le cas compact. En effet, lorsqu'il s'agit de variétés non compactes, il faut également prendre en considération les métriques complètes à courbure scalaire uniformément positive, car il existe des variétés riemanniennes complètes à courbure scalaire strictement positive qui ne peuvent pas être munies d'une métrique complète à courbure scalaire uniformément positive.

Soit M une variété fermée de dimension n. Rosenberg–Stolz [RS94] ont démontré que $M \times \mathbb{R}^2$ peut être munie d'une métrique complète à courbure scalaire strictement positive, et que $M \times \mathbb{R}^k$ admet une métrique complète à courbure scalaire uniformément positive lorsque $k \geq 3$. En conséquence, Rosenberg–Stolz [RS94] ont également conjecturé que si M n'admet pas de métrique complète à courbure scalaire strictement positive, alors $M \times \mathbb{R}$ n'admet pas de métrique complète à courbure scalaire strictement positive, et $M \times \mathbb{R}^2$ n'admet pas de métrique à courbure scalaire uniformément positive. La conjecture de Rosenberg–Stolz fut établie par Gromov–Lawson [GL83] lorsque la dimension de M est $n \leq 2$.

Méthodes récentes dans l'étude de la courbure scalaire

Un nouvel outil dans l'étude de la courbure scalaire, ayant conduit à des résultats remarquables, sont les μ-bulles introduites par Gromov dans [Gro23]. La méthode des μ-bulles peut être comprise comme une extension de l'approche des hypersurfaces minimales, offrant une plus grande flexibilité pour adapter la construction à la topologie et à la géométrie de la variété.

Soit M une variété riemannienne de dimension n et h une fonction lisse sur M. Une *μ-bulle (associée à h)* est une partie $\Omega \subset \operatorname{int}(M)$ qui minimise une certaine fonctionnelle faisant intervenir le volume $(n-1)$-dimensionnel de la frontière $\partial\Omega$ ainsi qu'un terme dépendant des valeurs de h sur Ω. L'existence de μ-bulles en dimensions $3 \leq n \leq 7$, pour des choix appropriés de la fonction h, a été établie dans [Gro23, Zhu21]. Les formules de variation associées à cette fonctionnelle impliquent que la frontière $\partial\Omega$ d'une μ-bulle Ω associée à h a pour courbure moyenne la fonction h, et qu'elle satisfait une inégalité de stabilité analogue à l'équation (2), mettant en jeu les courbures scalaires de M et de Σ. En particulier, l'existence de μ-bulles peut être interprétée comme un problème de prescription de la courbure moyenne, et ainsi la fonctionnelle minimisée par les μ-bulles est fréquemment appelée la fonctionnelle de courbure moyenne prescrite. L'inégalité de stabilité pour les μ-bulles peut être utilisée pour dériver des estimations géométriques significatives, permettant d'étudier la géométrie de M à partir des bornes inférieures de la courbure scalaire.

La méthode des μ-bulles a conduit à un certain nombre de résultats importants. Dans [Gro86], Gromov a conjecturé que toute variété asphérique fermée de dimension n n'admet pas de métrique riemannienne à courbure scalaire strictement positive quand $n \geq 2$. Comme mentionné précédemment, la conjecture de Gromov a été résolue positivement en dimension 3 par Gromov–Lawson [GL83] à l'aide de la méthode de l'opérateur de Dirac tordu. En utilisant les μ-bulles, Chodosh–Li [CL24] et, indépendamment, Gromov [Gro20] ont récemment établi la conjecture en dimensions 4 et 5. Cependant, la conjecture de Gromov reste largement ouverte en dimensions supérieures. Dans [Gro17, Gro23], Gromov a énoncé une conjecture plus forte, selon laquelle les variétés $\mathbb{Q}$-essentielles n'admettent pas de métriques à courbure scalaire strictement positive, sujet qui sera discuté dans la Section 1.3.4.

Comme nous le verrons ultérieurement, les μ-bulles ont joué un rôle central dans les récentes avancées concernant la classification topologique des 3-variétés à courbure scalaire uniformément positive [Gro23, Wan23a].

Dans [Ste22], Stern a développé une nouvelle approche pour aborder la courbure scalaire en dimension 3, qu'il est possible de concevoir comme une version duale de la méthode des surfaces minimales. Rappelons que si M est une variété orientée fermée de dimension 3, alors, d'après la Dualité de Poincaré, les classes de 2-homologie correspondent aux classes d'homotopie des applications de M à valeurs dans $\mathbb{S}^1$. Au lieu de travailler avec des hypersurfaces minimales stables, construites en minimisant la fonctionnelle d'aire dans une classe de 2-homologie non triviale, Stern a opté pour travailler avec des applications harmoniques $M \to \mathbb{S}^1$, c'est-à-dire des applications minimisant l'énergie de Dirichlet dans leur classe d'homotopie. En utilisant l'Identité de Bochner, il a obtenu une inégalité reliant la topologie des ensembles de niveau d'une application harmonique $u : M \to \mathbb{S}^1$ à la courbure scalaire de la variété ambiante M. Parmi ses nombreuses applications, la méthode des applications harmoniques permet de fournir une démonstration alternative de la Conjecture de Geroch pour le tore tridimensionnel ou de l'Inégalité Systolique de Bray–Brendle–Neves [BBN10], qui sera présentée ci-dessous.

Le flot de Hamilton–Ricci a également été utilisé pour élucider la structure des variétés riemanniennes de dimension 3 à géométrie bornée et à courbure scalaire uniformément positive [BBM11], ainsi que pour décrire la topologie des espaces de modules de telles métriques [Cod12, BBMC21].

Il existe d'autres approches pour l'étude de la courbure scalaire, spécialement adaptées à la dimension 4, basées sur la théorie de Seiberg–Witten.

I.II Existence de variétés à courbure scalaire strictement positive

L'exemple fondamental de variétés qui admettent des métriques à courbure scalaire strictement positive sont les espaces symétriques compacts, car ils sont non plats et possèdent une courbure sectionnelle positive. Parmi les examples d'espaces symétriques compacts, on trouve la n-sphère $\mathbb{S}^n$, les espaces projectifs $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ (et, de manière plus générale, les variétés grassmanniennes sur $\mathbb{R}$, $\mathbb{C}$ et $\mathbb{H}$), le plan de Cayley $\mathbb{C}\mathrm{aP}^2$, ainsi que leurs produits riemanniens. Plus généralement, à part le tore plat, les espaces homogènes compacts admettent des métriques à courbure scalaire strictement positive. De plus, les variétés obtenues comme quotients d'espaces homogènes compacts et non plats par l'action isométrique libre d'un groupe compact admettent également des métriques à courbure scalaire positive. Par ailleurs, les hypersurfaces strictement convexes possèdent une courbure sectionnelle strictement positive, et donc leur courbure scalaire est strictement positive. Dans le cadre de la géométrie complexe, les hypersurfaces complexes de $\mathbb{C}\mathrm{P}^n$ de degré inférieur ou égal à n, et plus généralement les variétés de Fano K-stables, admettent des métriques à courbure scalaire strictement positive.

Il existe des procédés simples permettant de construire de nouvelles variétés admettant des métriques à courbure scalaire strictement positive à partir d'exemples déjà connus. Par exemple, si M est une variété riemannienne fermée à courbure scalaire strictement positive et N est une variété fermée quelconque, alors le produit $M \times N$ admet une métrique à courbure scalaire strictement positive. Ce fait découle de l'additivité de la courbure scalaire dans les produits riemanniens : si M et N sont deux variétés riemanniennes, alors

$$\mathrm{scal}_{M\times N} = \mathrm{scal}_M \circ \pi_M + \mathrm{scal}_N \circ \pi_N,$$

où $\pi_M : M \times N \to M$ et $\pi_N : M \times N \to N$ sont les projections correspondantes.

L'additivité de la courbure scalaire peut être utilisée dans le cadre plus général des variétés fibrées pour produire des exemples de variétés à courbure scalaire strictement positive. Étant donnée $\pi : M \to B$ une submersion riemannienne dont les fibres sont totalement géodésiques, il est toujours possible de déformer la métrique riemannienne g de M en la redimensionnant le long des fibres de π par un facteur $\varepsilon > 0$. Cette procédure engendre une famille de métriques $(g_\varepsilon)_{\varepsilon>0}$ sur M, connue sous le nom de variation canonique de g, pour laquelle $\pi : M \to B$ reste une submersion riemannienne. Par la Formule d'O'Neill [O'N66] (voir aussi [Bes87, Proposition 9.70]), la courbure scalaire $\mathrm{scal}_\varepsilon$ de (M, g_ε) pour un $\varepsilon > 0$ fixé est donnée par

$$\mathrm{scal}_\varepsilon = \frac{1}{\varepsilon^2}\,\mathrm{scal}_F + \mathrm{scal}_B \circ \pi - \varepsilon^2\,|A|^2\,,$$

où scal_B et scal_F désignent respectivement les courbures scalaires de la base B et de la fibre F au point considéré, et où $|A|^2$ est la norme au carré du tenseur d'intégrabilité d'O'Neill, noté par A. Cela implique que, si $\pi : M \to B$ est une submersion dont les fibres sont totalement géodésiques et admettent des métriques à courbure scalaire strictement positive, alors M peut également être munie d'une métrique à courbure scalaire strictement positive, simplement en contractant suffisamment les fibres via la variation canonique. Toutefois, ces techniques ne permettent de produire qu'un nombre limité d'exemples.

Une avancée majeure dans l'étude de la courbure scalaire est le Théorème de Chirurgie [GL80a, SY79b].

Théorème de Chirurgie ([GL80a, SY79b]). *Soit M une variété fermée de dimension n à courbure scalaire strictement positive (non nécessairement connexe). Si N est la variété obtenue en effectuant une chirurgie de codimension au moins 3 sur M, alors N admet aussi une métrique à courbure scalaire strictement positive.*

En particulier, la somme connexe de deux variétés à courbure scalaire strictement positive peut être munie d'une métrique à courbure scalaire strictement positive. Le Théorème de Chirurgie est un outil puissant pour construire de nouvelles variétés admettant des métriques à courbure scalaire strictement positive.

II La topologie des 3-variétés à courbure scalaire strictement positive

Passons maintenant au cas spécifique des variétés de dimension 3. Comme discuté précédemment, des exemples spécifiques de variétés 3-dimensionnelles admettant des métriques à courbure scalaire strictement positive sont la 3-sphère, les variétés sphériques, le produit $\mathbb{S}^2 \times \mathbb{S}^1$ et leurs sommes connexes. Rappelons qu'une 3-*variété sphérique* est une variété $\mathbb{S}^3/\Gamma$ obtenue comme quotient de la 3-sphère par un sous-groupe $\Gamma < O(4)$ d'isométries agissant librement sur $\mathbb{S}^3$. Dans sa Section de Problèmes, Yau posa la question d'une classification des 3-variétés admettant des métriques à courbure scalaire strictement positive [Yau82, Problème 27].

Commençons par discuter le cas des 3-variétés fermées. En utilisant la méthode de l'opérateur de Dirac tordu, Gromov–Lawson [GL83] ont démontré que si une variété fermée (non nécessairement orientable) de dimension 3 admet une métrique à courbure scalaire strictement positive, alors elle ne peut pas contenir un terme asphérique dans sa décomposition en facteurs premiers (voir la Section 1.1.1). Ainsi, à partir du Théorème de Decomposition de Kneser–Milnor [Kne29, Mil62], du Théorème de Chirurgie [GL80a, SY79b] et de la résolution de la Conjecture d'Elliptisation par Perelman [Per02, Per03a, Per03b], il s'ensuit qu'une 3-variété orientable fermée admet une métrique à courbure scalaire strictement positive si et seulement si elle se décompose comme une somme connexe finie

$$\mathbb{S}^3/\Gamma_1\# \dots \#\mathbb{S}^3/\Gamma_p\#\mathbb{S}^2 \times \mathbb{S}^1\# \dots \#\mathbb{S}^2 \times \mathbb{S}^1$$

de 3-variétés sphériques $\mathbb{S}^3/\Gamma_i$ et de termes $\mathbb{S}^2 \times \mathbb{S}^1$. Rappelons que, contrairement au cas orientable, les 3-variétés fermées non orientables premières ne sont pas classifiées. Par conséquent, la structure des variétés fermées non orientables admettant des métriques à courbure scalaire strictement positive reste moins bien comprise.

Le premier problème que l'on rencontre lorsqu'on considère des 3-variétés orientables ouvertes est que le Théorème de Decomposition de Kneser–Milnor ne s'applique pas en général. En effet, Scott [Sco77] a montré qu'il existe des variétés ouvertes qui ne se décomposent pas en somme connexe de variétés premières, même si l'on considère des sommes connexes infinies. D'autres exemples de variétés ouvertes non premières qui sont indécomposables en somme connexe infinie peuvent être trouvés dans [ST89, Mai08].

Cependant, un théorème de décomposition similaire a récemment été démontré pour les variétés ouvertes de dimension 3 admettant des métriques riemanniennes complètes à courbure scalaire uniformément positive. Gromov [Gro23] et Wang [Wan23a] ont utilisé la méthode des μ-bulles pour montrer que si une variété riemannienne orientable complète admet une métrique à courbure scalaire uniformément positive, alors elle se décompose en une somme connexe, possiblement infinie, de variétés sphériques et de $\mathbb{S}^2 \times \mathbb{S}^1$. Ce résultat de décomposition avait déjà été prouvé auparavant sous des hypothèses supplémentaires : plus précisément, pour les variétés à groupe fondamental finiment engendré en utilisant des méthodes de K-théorie [CWY10], et pour les variétés à géométrie bornée en utilisant le flot de Ricci [BBM11].

Dans [BGS24], nous avons généralisé le théorème de décomposition de Gromov et Wang aux variétés riemanniennes orientables complètes de dimension 3 à courbure scalaire strictement positive et présentant une certaine décroissance à l'infini.

Théorème A ([BGS24, Théorème 1.3])**.** *Soit M une variété riemannienne orientable complète de dimension 3. Soit $x \in M$ un point. Supposons que M est à courbure scalaire strictement positive, et qu'il existe une constante $C > 64\pi^2$ telle que, pour chaque point $y \in M$ avec $d(x,y) \geq 1$, on ait l'inégalité*

$$\mathrm{scal}(y) > \frac{C}{d(x,y)^2}. \tag{5}$$

Alors M se décompose en somme connexe, possiblement infinie, de 3-variétés sphériques et de termes $\mathbb{S}^2 \times \mathbb{S}^1$.

On pourrait se demander si la conclusion du Théorème A est vraie pour un taux de décroissance en terme de la distance au point x plus faible. L'exemple de la variété $\mathbb{R}^2 \times \mathbb{S}^1$ montre que cela est impossible. En effet, la variété $\mathbb{R}^2 \times \mathbb{S}^1$ admet une métrique complète à courbure scalaire strictement positive à décroissance quadratique en la distance au point x avec une constante $C = \frac{1}{2}$, mais elle ne se décompose pas en somme connexe infinie de variétés sphériques et de $\mathbb{S}^2 \times \mathbb{S}^1$, voir la Section 1.5.2.

La démonstration du Théorème A repose sur une estimation des disques de remplissage des courbes fermées dans la variété, basée sur la notion de rayon de remplissage introduite dans [GL83, SY79a, SY83], qui généralise la notion de courbure scalaire strictement positive avec la condition de décroissance quadratique dans l'équation (5). Soit M une variété riemannienne sans bord de dimension n. Le rayon de remplissage $\mathrm{fillrad}(\gamma)$ d'une courbe fermée contractile γ dans M est le plus grand nombre réel $R > 0$ tel que la courbe γ ne borde pas un disque dans son R-voisinage fermé. Gromov–Lawson [GL83] et Schoen–Yau [SY83] ont prouvé que si une variété riemannienne 3-dimensionnelle complète à géométrie bornée a une courbure scalaire uniformément positive $\mathrm{scal} \geq s > 0$, alors toute courbe fermée contractile γ dans M satisfait

$$\mathrm{fillrad}(\gamma) \leq \frac{2\pi}{\sqrt{s}}.$$

Si une variété riemannienne orientable 3-dimensionnelle complète M possède une courbure scalaire strictement positive qui décroît à l'infini, alors le rayon de remplissage des courbes fermées contractiles dans M n'est pas nécessairement uniformément borné. Toutefois, si la décroissance n'est pas trop prononcée, il est possible de contrôler la croissance du rayon de remplissage des courbes fermées contractiles dans M, ou plus précisément, de leurs relevés dans le revêtement universel riemannien de M. Nous prouvons la décomposition topologique du Théorème A en remplaçant l'hypothèse sur la courbure scalaire par cette condition plus faible sur la croissance du rayon de remplissage des relevés des courbes fermées contractiles dans le revêtement universel riemannien de M.

Le résultat de rigidité suivant découle directement du Théorème A, ainsi que d'une adaptation du Théorème de Chirurgie, voir la Section 1.6.

Corollaire B ([BGS24, Corollary 1.5])**.** *Soit M une variété riemannienne orientable complète de dimension 3, et soit $x \in M$ un point. Supposons que M est à courbure scalaire strictement positive,* *et qu'il existe une constante $C > 64\pi^2$ telle que, pour tout point $y \in M$ vérifiant $d(x,y) \geq 1$, on ait l'inégalité*

$$\mathrm{scal}(y) > \frac{C}{d(x,y)^2}.$$

Alors M admet une métrique riemannienne complète à courbure scalaire uniformément positive.

III Géométrie systolique des variétés à courbure scalaire strictement positive

En ce qui concerne la relation entre la courbure scalaire et la géométrie d'une variété riemannienne, nous nous intéresserons avant tout à son effet sur les quantités de nature systolique.

Soit M une variété riemannienne fermée de dimension 3 avec $\pi_2(M) \neq 0$. La 2-*systole homotopique* $\mathrm{sys}\,\pi_2(M)$ de M désigne l'infimum des aires des 2-spheres immergées dans M non contractiles, voir la Section 2.2. Bray–Brendle–Neves [BBN10] ont démontré que si la courbure scalaire de M satisfait $\mathrm{scal} \geq s > 0$, alors

$$\mathrm{sys}\,\pi_2(M) \leq \frac{8\pi}{s}. \tag{6}$$

De plus, on a égalité si et seulement si le revêtement universel riemannien de M est isométrique au cylindre riemannien standard $\mathbb{S}^2(1) \times \mathbb{R}$, à une homothétie près. La démonstration de l'Inégalité Systolique de Bray–Brendle–Neves (6) repose sur l'Inégalité de Stabilité (2) appliquée à une 2-sphère non-contractile d'aire minimale dans sa classe d'homotopie.

L'Inégalité Systolique de Bray–Brendle–Neves (6) a été généralisée dans plusieurs directions. Par exemple, Bray–Brendle–Eichmair–Neves [BBEN10] ont démontré une inégalité analogue pour les plans projectifs plongés. En dimension supérieure, les produits riemanniens de sphères rondes montrent qu'on ne peut pas en général s'attendre à un contrôle de la 2-systole uniquement à partir d'une borne inférieure sur la courbure scalaire. Néanmoins, certaines généralisations ont été dérivées sous des hypothèses topologiques supplémentaires sur la variété M. Par exemple, Zhu [Zhu20] a prouvé que l'Inégalité Systolique de Bray–Brendle–Neves (6) est valable jusqu'à la dimension 7 si la variété admet une application vers $\mathbb{S}^2 \times \mathbb{T}^{n-2}$ de degré non nul. Il a également généralisé l'Inégalité Systolique (6) au cas non compact pour les variétés admettant une application de degré non nul vers $\mathbb{S}^2 \times \mathbb{T}^{n-3} \times \mathbb{R}$, toujours jusqu'à la dimension 7, voir [Zhu23]. Dans une autre direction, Richard [Ric20] a obtenu une estimation pour la 2-systole homotopique de $\mathbb{S}^2 \times \mathbb{S}^2$ équipée d'une métrique à courbure scalaire strictement positive satisfaisant une certaine condition d'étirement.

L'Inégalité Systolique de Bray–Brendle–Neves (6) a également motivé des résultats analogues pour des hypersurfaces minimisantes dans leur classe d'homologie. Dans [Ste22], Stern a fourni une preuve directe de l'analogue homologique de l'Inégalité Systolique (6). Une généralisation aux dimensions de 4 à 7 a été abordée par Chu–Lee–Zhu dans [CLZ24], où ils ont donné une majoration de la systole homologique en codimension 1 (voir Section 2.2) sous une condition de positivité de courbure plus forte, portant sur la courbure bi-Ricci positive, et ont obtenu un résultat de rigidité pour le cas de l'égalité.

III.I Géométrie systolique et courbure scalaire macroscopique strictement positive

Comme conséquence de l'équation (1), la courbure scalaire peut être interprétée comme une mesure de la déviation du volume des boules géodésiques de rayons infinitésimaux par rapport aux boules euclidiennes de même rayon. Dans [Gut10a], Guth a introduit un analogue macroscopique de la courbure scalaire, qui quantifie la déviation volumique des boules géodésiques d'un rayon fini fixé.

Soit M une variété riemannienne de dimension n, et soit $\tilde{M}$ le revêtement universel riemannien de M. La *courbure scalaire macroscopique* $\mathrm{mscal}(x, R)$ de M au point $x \in M$ et à l'échelle $R > 0$

satisfait $\mathrm{mscal}(x,R) \geq s > 0$ si et seulement si le volume de la boule géodésique $B_{\tilde{M}}(\tilde{x},R)$ dans $\tilde{M}$ centrée en un relevé $\tilde{x}$ de x et de rayon R vérifie

$$\left|B_{\tilde{M}}(\tilde{x},R)\right| \leq V_s^n(R),$$

où $V_s^n(R)$ dénote le volume d'une boule de rayon R dans la n-sphère de courbure scalaire constante s. La courbure scalaire macroscopique est définie à travers les volumes des boules dans le revêtement universel $\tilde{M}$ de M, afin de garantir que les variétés plates ont une courbure scalaire macroscopique égale à zero à toute échelle, voir la Section 2.1.

On pourrait se demander s'il existe un analogue macroscopique de l'Inégalité Systolique de Bray–Brendle–Neves (6). La proposition suivante montre qu'on ne peut pas espérer un contrôle des systoles homotopiques d'une variété riemannienne fermée uniquement à partir d'une borne inférieure de la courbure scalaire macroscopique. La *k-systole homotopique* $\mathrm{sys}\,\pi_k(M)$ d'une variété riemannienne M est définie comme le k-volume de la plus petite k-sphère immergée dans M non contractile. Si, au lieu des k-sphères non contractiles, on considérait des sous-variétés k-dimensionnelles non nulles en homologie, on obtiendrait de manière analogue la notion de *k-systole homologique* $\mathrm{sys}\,H_k(M)$ de M.

Proposition C ([Gil25, Proposition 1.5])**.** *Soit $n \geq 3$ et $k \in \{2,\dots,n-1\}$. Pour chaque $s > 0$, il existe une famille de métriques riemanniennes produit $(g_\varepsilon)_{\varepsilon\in(0,1)}$ sur $\mathbb{S}^k \times \mathbb{S}^{n-k}$ telle que les conditions suivantes sont vérifiées.*

1. *Pour tout point $x \in \mathbb{S}^k \times \mathbb{S}^{n-k}$ et toute échelle $R > 0$, on a, pour chaque $\varepsilon \in (0,1)$,*

$$\mathrm{mscal}_{(\mathbb{S}^k\times\mathbb{S}^{n-k},g_\varepsilon)}(x,R) \geq s.$$

2. *La k-systole homotopique et la k-systole homologique vérifient*

$$\lim_{\varepsilon\to 0} \mathrm{sys}\,\pi_k(\mathbb{S}^k \times \mathbb{S}^{n-k}, g_\varepsilon) = \lim_{\varepsilon\to 0} \mathrm{sys}\,H_k(\mathbb{S}^k \times \mathbb{S}^{n-k}, g_\varepsilon) = +\infty.$$

Cependant, on pourrait espérer avoir un analogue de l'Inégalité Systolique de Bray–Brendle–Neves (6) pour un invariant métrique plus faible décrivant la taille des hypersurfaces topologiquement non triviales, comme par exemple leur largeur d'Urysohn de codimension 1.

La *largeur d'Urysohn de dimension k* d'un espace métrique X mesure à quel point X diffère d'être k-dimensionnel. Pour une définition précise de la notion de largeur d'Urysohn, nous renvoyons à la Section 2.3. Lorsqu'on considère des variétés riemanniennes, la largeur d'Urysohn de codimension 1 est particulièrement pertinente, car elle est la première largeur d'Urysohn non triviale. Dans [Gut17], Guth a montré que la largeur d'Urysohn de codimension 1 d'une variété riemannienne M de dimension n est liée à son volume $|M|$ par :

$$\mathrm{UW}_{n-1}(M) \leq C_n\,|M|^n\,,$$

où $C_n > 0$ est une constante qui dépend uniquement de la dimension de M. Par conséquent, l'infimum de la $(n-2)$-largeur d'Urysohn parmi toutes les hypersurfaces immergées dans M homologiquement non nulles est un invariant plus faible que la $(n-1)$-systole homologique $\mathrm{sys}\,H_{n-1}(M)$ de M.

Le résultat principal de [Gil25] est la version macroscopique suivante de l'Inégalité Systolique de Bray–Brendle–Neves (6). Soit $G = \mathbb{Z}_2$ ou $\mathbb{Z}$. Considérons une n-variété riemannienne complète non simplement connexe M telle que $H_{n-1}(M;G) \neq 0$. Si la variété M est compacte et G-orientable, le fait d'avoir une G-homologie de codimension 1 non triviale implique déjà, par la Dualité de Poincaré et le Théorème des Coefficients Universels, que M n'est pas simplement connexe. Cependant, cela n'est plus vrai lorsqu'on considère des variétés non compactes. Considérons la 1-systole homotopique $\operatorname{sys}\pi_1(M)$ de M, c'est-à-dire la longueur de la plus courte courbe fermée non contractile sur M. Remarquons que si M est non compacte, on peut avoir $\operatorname{sys}\pi_1(M) = 0$.

Théorème D ([Gil25, Theorem 1.9])**.** *Il existe une constante dimensionnelle $\kappa_n > 0$ telle que ce qui suit a lieu. Soit $G = \mathbb{Z}_2$ ou $\mathbb{Z}$. Soit M une n-variété riemannienne complète non simplement connexe telle que $H_{n-1}(M;G) \neq 0$ et $\operatorname{sys}\pi_1(M) > 0$. Fixons $R > 0$ et $s > 0$ tels que $\kappa_n/\sqrt{s} < R < \frac{1}{2}\operatorname{sys}\pi_1(M)$. Supposons que $\operatorname{mscal}(x,R) \geq s$ pour tout point $x \in M$. Alors il existe une hypersurface fermée plongée Σ telle que $[\Sigma] \neq 0 \in H_{n-1}(M;G)$ et*

$$\mathrm{UW}_{n-2}(\Sigma) \leq \frac{n-1}{n} R.$$

Il ne faut pas s'attendre à ce que le théorème D reste valable pour des échelles R arbitrairement grandes, car la croissance du volume des boules métriques centrées en un point donné est fortement perturbée au-delà du rayon d'injectivité en ce point, comme le montre la proposition suivante.

Proposition E. *Fixons $\kappa > 0$. Il existe une famille de métriques riemanniennes $(\bar{g}_s)_{s>0}$ sur l'espace projectif réel $\mathbb{R}\mathrm{P}^3$ vérifiant les propriétés suivantes.*

1. *Pour tout $s > 0$ suffisamment grand, il existe une échelle $R_s > \max\left\{\frac{1}{2}\operatorname{sys}\pi_1(\mathbb{R}\mathrm{P}^3, \bar{g}_s), \frac{\kappa}{\sqrt{s}}\right\}$ telle que pour tout $x \in M$*
$$\operatorname{mscal}_{\bar{g}_s}(x, R_s) \geq s.$$
2. *Pour tout $s > 0$ suffisamment grand, toute surface fermée plongée Σ dans $(\mathbb{R}\mathrm{P}^3, \bar{g}_s)$ telle que $[\Sigma] \neq 0 \in H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2)$ vérifie*
$$\mathrm{UW}_1(\Sigma) > w,$$
pour une constante $w > 0$ indépendante de s.

Structure de la thèse

Cette thèse se compose de deux chapitres. Le Chapitre 1 est consacré à la topologie des variétés 3-dimensionnelles à courbure scalaire strictement positive, avec l'objectif principal de fournir la démonstration du Théorème A. Dans le Chapitre 2, nous exposons les différentes notions introduites dans la Section III, puis nous détaillons les Propositions C et E, et enfin nous présentons la preuve du Théorème D.

Notation

Tout au long de cette thèse, nous supposerons généralement que toutes les variétés sont connexes, sauf mention explicite du contraire. Nous considérerons des variétés avec ou sans bord. Ainsi, nous préciserons explicitement, dans chaque cas, si la variété possède un bord ou non. Nous indiquerons également si la variété est supposée orientable.

Nous travaillerons avec des boules métriques fermées. On notera la *boule métrique fermée* centrée en un point x et de rayon R par $B(x,R)$, et son bord sera noté $S(x,R) := \partial B(x,R)$. Afin d'éviter toute confusion, nous indiquerons l'espace métrique correspondant lorsque cela sera nécessaire. Par exemple, $B_{\tilde{M}}(\tilde{x}, R)$ désigne une boule métrique dans le revêtement riemannien universel $\tilde{M}$ d'une variété M.

Nous considérerons de même des voisinages métriques fermés. Étant donné un espace métrique X et un sous-ensemble $Z \subset X$, nous noterons le *voisinage fermé* de rayon R de Z dans X par

$$U(Z, R) := \{x \in X \mid d(x, Z) \leq R\} .$$

Chapter 1

The topology of 3-manifolds of positive scalar curvature

The objective of this first chapter is to prove Theorem A, that is, we will show that if an orientable 3-manifold M admits a complete Riemannian metric whose scalar curvature is positive and has a subquadratic decay at infinity, then it decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$.

To this end, in Sections 1.1 and 1.2, we will begin by presenting several classical results in 3-manifold topology, ultimately leading us to the definition of infinite connected sum, and discuss the theory of ends, respectively. In Section 1.3 we will present the notion of fill radius, and we will unravel the relation between the decay of scalar curvature and the growth of the fill radius. Finally, in Sections 1.4 and 1.5 we will conclude the proof of Theorem A. In Section 1.6 we will prove Corollary B using the Surgery Theorem.

1.1 The topology of 3-manifolds

In this section, we will present several classical results about the topology of 3-dimensional manifolds that will be useful in the proof of Theorem A.

Recall that, since every topological 3-manifold admits a unique smooth structure up to diffeomorphism [Moi52b, HM74], there is little difference between working in the category of topological 3-manifolds or the category of smooth 3-manifolds. Hence, throughout this section we will consider topological 3-manifolds, and all maps will always be assumed to be continuous unless explicitely stated otherwise.

Let us introduce some notation. Let M be a 3-manifold. Let $\mathbb{B}^n$ denote the Euclidean unit n-dimensional ball. A *disc* D in M is the image of $\mathbb{B}^2$ by a continuous map $i : \mathbb{B}^2 \to M$. We will specify in each situation whether a given disc $D \subset M$ is embedded, that is, whether the map $i : \mathbb{B}^2 \to M$ is an embedding. A *ball* B in M is the image of $\mathbb{B}^3$ by an embedding $i : \mathbb{B}^3 \to M$. That is, we will only consider embedded balls.

We will say that a closed curve $\gamma \subset M$ *bounds a disc* in a subset $B \subset M$ if there exists a map $i : \mathbb{B}^2 \to B \subset M$ whose restriction to $\partial\mathbb{B}^2$ coincides with γ. This is equivalent to γ being contractible in B.

A surface Σ embedded in M is *two-sided* if its normal bundle in M is trivial. A two-sided surface Σ embedded in M is *separating* if the complement $M - \Sigma$ of Σ in M consists of two connected components. If Σ is a two-sided surface embedded in M, we define the *splitting of M along* Σ as the manifold resulting from removing an open tubular neighbourhood of Σ in M. Notice that the splitting of M along Σ has two new boundary components, each of them homeomorphic to Σ.

If M is a 3-manifold with non-empty boundary and Σ is a boundary component of M homeomorphic to the 2-sphere, the *capping off* of Σ is the operation consisting in gluing a copy of the 3-ball $\mathbb{B}^3$ to M along a homeomorphism $\varphi : \partial\mathbb{B}^3 \to \Sigma$. This construction does not depend on the choice of the homeomorphism φ. A 3-manifold M has *spherical boundary* if its boundary ∂M is non-empty and homeomorphic to a disjoint union of 2-spheres. If M is a 3-manifold with spherical boundary, the result of capping off all its boundary components is denoted by $\hat{M}$.

1.1.1 The Kneser–Milnor Prime Decomposition Theorem

Let us start by introducing the notion of connected sum of two 3-manifolds.

Definition 1.1.1. Let M_1 and M_2 be two connected oriented 3-manifolds. Fix two embedded balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and an orientation-reversing homeomorphism $\varphi : \partial B_1 \to \partial B_2$. The *connected sum* of the manifolds M_1 and M_2 is the manifold defined as

$$M_1 \# M_2 := (M_1 - \operatorname{int}(B_1)) \sqcup (M_2 - \operatorname{int}(B_2))/x \sim \varphi(x),\ x \in \partial B_1.$$

The resulting manifold depends neither on the choice of the balls B_1 and B_2 nor on the choice of the homeomorphism $\varphi : \partial B_1 \to \partial B_2$, hence the connected sum of oriented 3-manifolds is well defined. However, it may depend on the orientation of the summands M_1 and M_2. The connected sum is associative and commutative, and the 3-sphere $\mathbb{S}^3$ verifies that for any oriented 3-manifold M, the connected sum $M\#\mathbb{S}^3$ is homeomorphic to the original manifold M.

Conversely, 3-manifolds can be decomposed as a connected sum of simpler manifolds. Let $\Sigma \subset M$ be a separating embedded 2-sphere. Then the splitting of M along Σ is composed of two connected components N_1 and N_2, and the boundary of each of them is homeomorphic to the 2-sphere. Denote by $M_1 = \hat{N}_1$ and $M_2 = \hat{N}_2$ the result of capping off the manifolds N_1 and N_2 respectively. Then M is homeomorphic to the connected sum $M_1 \# M_2$. This fact motivates the following definition.

Definition 1.1.2. A connected 3-manifold P is *prime* if whenever P decomposes as a connected sum $P_1 \# P_2$, then either P_1 or P_2 is homeomorphic to the 3-sphere $\mathbb{S}^3$.

By Alexander's Theorem [Ale24, Moi52a], a connected 3-manifold M is prime if and only if any separating embedded 2-sphere in M bounds a ball, that is, there is an embedded ball $B \subset M$ such that $\partial B = \Sigma$. Observe that, by definition, the 3-sphere $\mathbb{S}^3$ is a prime 3-manifold. Other examples of prime 3-manifolds include $\mathbb{R}^3$, spherical 3-manifolds, flat 3-manifolds, hyperbolic 3-manifolds and the product $\mathbb{S}^2 \times \mathbb{S}^1$. Spherical manifolds, that is, manifolds obtained as a quotient $\mathbb{S}^3/\Gamma$ of the 3-sphere by a subgroup $\Gamma < O(4)$ of isometries acting freely on $\mathbb{S}^3$, will be discussed in more detail in Section 1.1.2.

Orientable closed prime 3-manifolds are classified by their fundamental group, as a consequence of Perelman's resolution of the Elliptisation Conjecture [Per02, Per03a, Per03b]. Recall that a manifold M is *aspherical* if all its higher homotopy groups $\pi_k(M)$, $k \geq 2$, are trivial. Equivalently, by Whitehead's Theorem [Hat02, Theorem 4.5], a manifold is aspherical if its universal cover is contractible.

Theorem 1.1.3 ([Per02, Per03a, Per03b])**.** *Let P be an orientable closed prime 3-manifold. Then P is either spherical, aspherical or homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.*

Proof. If $\pi_1(P)$ is finite, then the Elliptisation Theorem [Per02, Per03a, Per03b] implies that P is a spherical manifold.

Now, suppose that $\pi_1(P)$ is infinite. Then the universal cover $\tilde{P}$ of P is non-compact. By the long exact homotopy sequence of a fibration [Hat02, Theorem 4.41], we have $\pi_2(\tilde{P}) \simeq \pi_2(P)$.

Suppose first that $\pi_2(P) = 0$. Then $\pi_2(\tilde{P}) = 0$ and $\tilde{P}$ is 2-homotopically connected. By Hurewicz's Theorem [Hat02, Theorem 4.32], there is an isomorphism $\pi_3(\tilde{P}) \simeq H_3(\tilde{P})$. Since $\tilde{P}$ is non-compact, we have by Poincaré Duality [Hat02, Theorem 3.35] that $H_3(\tilde{P};\mathbb{Z})$ vanishes, so the group $\pi_3(\tilde{P})$ vanishes as well. Since the homology groups $H_k(\tilde{P};\mathbb{Z})$ vanish for $k \geq 4$, we can apply Hurewicz's Theorem inductively to conclude that $\pi_k(\tilde{P}) = 0$ for $k \geq 0$. Therefore, the manifold P is aspherical.

Finally, suppose that $\pi_2(P)$ is nontrivial. Then, by Papakyriakopoulos' Sphere Theorem 1.1.14, which will be presented in Section 1.1.4, the manifold P contains an embedded 2-sphere S not bounding a ball. Since P is prime, the 2-sphere S is non-separating. Then there exists a simple closed curve γ in P intersecting S at a single point. Let T be a tubular neighbourhood of $S \cup \gamma$, see Figure 1.1.

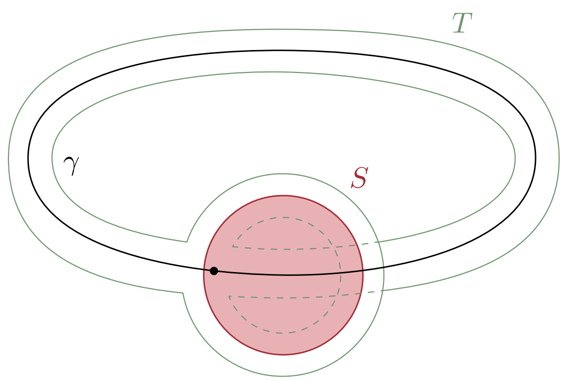

Figure 1.1: Tubular neighbourhood T of $S \cup \gamma$.

It is easy to see that T is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ with a ball removed, and that its boundary ∂T is a separating 2-sphere in P. Again, since P is a prime manifold, the 2-sphere ∂T must bound a ball in P. Therefore, P is obtained from T by capping off its spherical boundary with a ball. We conclude that P is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. □

The following fundamental theorem states that every closed oriented 3-manifold decomposes uniquely as the connected sum of closed prime 3-manifolds. The existence of such a decomposition was proven by Kneser [Kne29], and its uniqueness was shown by Milnor [Mil62].

Theorem 1.1.4 (Kneser–Milnor Prime Decomposition Theorem [Kne29, Mil62])**.** *Let M be a closed oriented 3-manifold. Then M is homeomorphic to a finite connected sum*

$$P_1 \# \cdots \# P_k,$$

where each P_i is a closed prime 3-manifold. This decomposition is unique up to permutation of the summands and addition or deletion of $\mathbb{S}^3$ summands.

Let us finish this section with the following lemma, which is a consequence of Van Kampen's Theorem [Hat02, Theorem 1.20].

Lemma 1.1.5. *Let M_1 and M_2 be two oriented 3-manifolds. Then there is an isomorphism*

$$\pi_1(M_1 \# M_2) \simeq \pi_1(M_1) * \pi_1(M_2).$$

It is also true that every decomposition as a free product of the fundamental group of a closed 3-manifold M is realised as a connected sum decomposition. This is known as the Kneser Conjecture [Hem76, Theorem 7.1], and it was proven by Stallings [Sta59].

1.1.2 Spherical 3-manifolds

Let us recall the definition of spherical 3-manifold.

Definition 1.1.6. A *spherical 3-manifold* is a quotient $\mathbb{S}^3/\Gamma$ of the 3-sphere by a subgroup $\Gamma < O(4)$ acting on $\mathbb{S}^3$ freely and by isometries.

Equivalently, by the Killing–Hopf Theorem [GHL04, Theorem 3.82], a 3-manifold is spherical if and only if it admits a metric of constant positive sectional curvature. By Synge's Theorem [dC92, Corollary 3.10], spherical 3-manifolds are orientable. Moreover, the fundamental group Γ of a spherical 3-manifold is necessarily a subgroup $\Gamma < SO(4)$.

Therefore, the classification of spherical 3-manifolds can be reduced to describing which subgroups of $SO(4)$ act on $\mathbb{S}^3$ freely and by isometries. Spherical 3-manifolds were completely classified by Seifert–Threlfall [TS31, TS33]. We will not describe the classification of spherical 3-manifolds here, and we refer the reader to [Thu97, Section 4.4] instead.

A particularly well-known class of spherical 3-manifolds are the lens spaces, which correspond to abelian subgroups of $SO(4)$. Recall that, given two coprime integers $p \geq 1$ and q, the *lens space* $L_p(q)$ is the quotient of $\mathbb{S}^3$ under the free action of $\mathbb{Z}_p$ generated by

$$(z, w) \mapsto \left(e^{\frac{2\pi}{p}i} z, e^{\frac{2\pi q}{p}i} w\right),$$

where we identified the 3-sphere $\mathbb{S}^3$ with $\{(z, w) \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$. For instance, the trivial case of $L_1(1)$ corresponds to the 3-sphere $\mathbb{S}^3$, and $L_2(q)$ is homeomorphic to $\mathbb{RP}^3$, for any $q \geq 3$ odd.

Lens spaces constitute the simplest example of a closed manifold whose topology cannot be determined solely from homotopical considerations.

Proposition 1.1.7 ([Hem76, Exercise 2.11, Lemma 3.23])**.** *Let $L_p(q)$ and $L_{p'}(q')$ two lens spaces, where p, q and p', q' are two pairs of coprime numbers, with $p, p' \geq 1$.*

1. *$L_p(q)$ and $L_{p'}(q')$ are homeomorphic if and only if $p = p'$, and either $q \equiv \pm q' \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$.*

2. *$L_p(q)$ and $L_{p'}(q')$ are homotopy equivalent if and only if $p = p'$ and $qq' \equiv \pm k^2 \pmod{p}$, for some $k \in \mathbb{N}$.*

For instance, the lens spaces $L_7(1)$ and $L_7(2)$ have the same exact homotopy type, but they are not homeomorphic.

Apart from the lens spaces, another example of spherical 3-manifold, which was of great historical importance, is the Poincaré homology sphere. It corresponds to the binary icosahedral group I^*, which admits the presentation $\langle a, b \mid a^2 = b^3 = (b^{-1}a)^5 \rangle$. The binary icosahedral I^* has order 120, and it is the only perfect subgroup of $SO(4)$. Hence, every homology 3-sphere is homeomorphic to the Poincaré homology sphere.

1.1.3 Infinite connected sums

Let us now turn our attention to open 3-manifolds, that is, non-compact manifolds without boundary. To study the topology of open 3-manifolds, we must generalise the notion of connected sum in order to describe connected sums of infinitely many prime manifolds.

Let us present the notion of connected sum along a graph, which was introduced in [Sco77]. We shall use the formalism from [BBM20]. Recall that a *colouring* of a graph $\mathcal{G}$ is a map $f : V(\mathcal{G}) \to I$ from the vertex set $V(\mathcal{G})$ of $\mathcal{G}$ to a subset $I \subset \mathbb{N}$, and the pair $(\mathcal{G}, f)$ is called a *coloured graph*.

Definition 1.1.8. Let $I \subset \mathbb{N}$ be a subset. Let $\mathcal{F} = \{M_i\}_{i\in I}$ be a family of connected 3-manifolds indexed by I, and $(\mathcal{G}, f)$ a locally finite coloured graph. By convention $0 \in I$ and $M_0 = \mathbb{S}^3$. The *connected sum over $\mathcal{F}$ along the locally finite coloured graph* $(\mathcal{G}, f)$ is the manifold obtained as follows:

1. For each vertex $v \in V(\mathcal{G})$, consider the manifold Y_v obtained from $M_{f(v)}$ by removing a number $\deg(v)$ of disjoint balls from its interior,
2. For each edge e joining two vertices v and v', glue two spherical boundary components from ∂Y_v and $\partial Y_{v'}$ along an orientation-reversing homeomorphism.

See Figure 1.2.

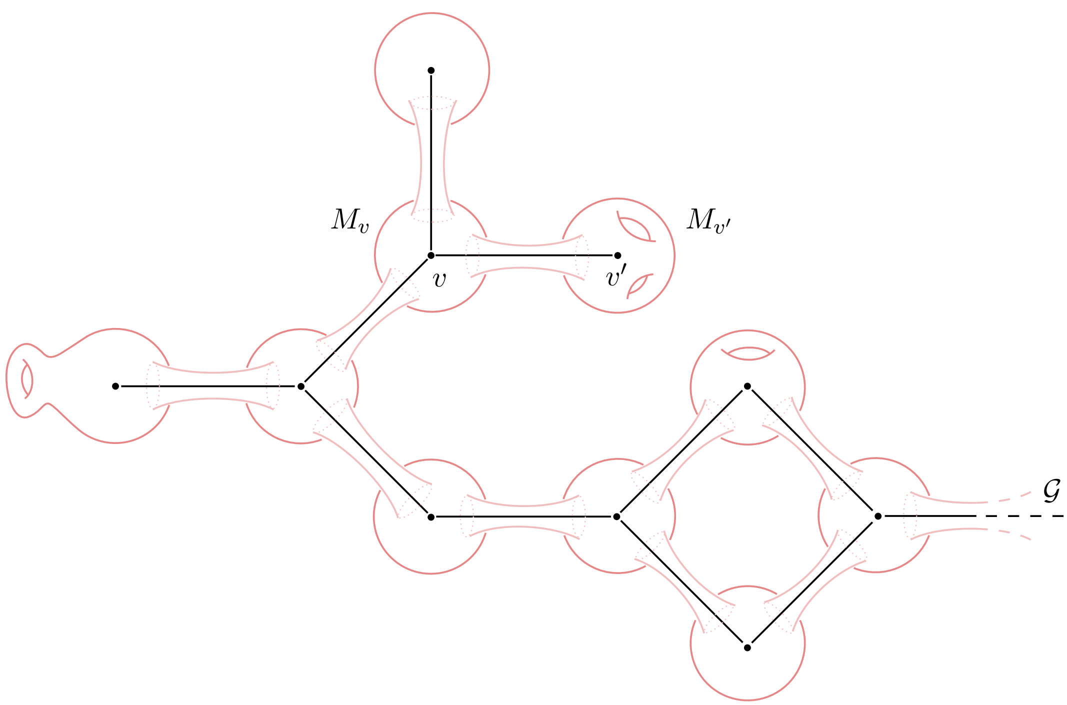

Figure 1.2: Connected sum modelled on a graph.

Equivalently, a 3-manifold M decomposes as a connected sum over $\mathcal{F}$ (along a certain locally finite coloured graph) if there exists a locally finite collection of pairwise disjoint 2-spheres embedded in M such that cutting M along the collection of 2-spheres and then capping off each new spherical boundary component by a 3-ball results in a disjoint collection of manifolds belonging to $\mathcal{F}$.

Clearly, the resulting manifold depends on the coloured graph on which it is modelled. If we assume the graph $\mathcal{G}$ to be a finite tree, then we recover the usual notion of connected sum. In any case, one can turn any infinite connected sum modelled on a graph into one modelled on a tree by adding some additional $\mathbb{S}^2 \times \mathbb{S}^1$ summands.

Proposition 1.1.9 ([BBM20, Theorem 2.3])**.** *Let M be a 3-manifold. Suppose that M decomposes as a connected sum over a family $\mathcal{F}$ of connected 3-manifolds along a locally finite coloured graph. Then M is isomorphic to a connected sum over $\mathcal{F} \cup \{\mathbb{S}^2 \times \mathbb{S}^1\}$ along a locally finite coloured tree.*

It would be convenient to have a generalisation of the Kneser–Milnor Prime Decomposition Theorem 1.1.4 to the case of open 3-manifolds to study their topology. Unfortunately, not every open 3-manifold is homeomorphic to a connected sum of prime manifolds along a locally finite graph. Some examples of such manifolds can be found in [Sco77, Mai08]. Even when an open 3-manifold is homeomorphic to a connected sum of closed prime 3-manifolds, this decomposition may fail to be unique.

In [BBM20], the authors gave a classification of open 3-manifolds which decompose over a finite family of closed prime oriented 3-manifolds. In order to introduce the topological invariants of their classification, we shall need the following definition. We will denote by $\mathcal{P}$ the family of all closed oriented prime 3-manifolds.

Definition 1.1.10. Let M be an oriented 3-manifold, and fix a prime 3-manifold $P \in \mathcal{P}$. Given a (possibly disconnected) compact oriented 3-manifold K with spherical boundary, we denote by $n_P(K)$ the sum of the number of summands homeomorphic to P in the Kneser–Milnor prime decomposition of each connected component of the capped-off manifold $\hat{K}$. We define

$$n_P(M) := \sup \{n_P(K) \mid K \subset M \text{ compact submanifold with spherical boundary}\} \in \mathbb{N} \cup \{\infty\}.$$

Also, define

$$E_P(M) := \{e \in E(M) \mid n_P(U) = \infty, \text{ for every open neighbourhood } U \text{ of } e\} \subset E(M),$$

where $E(M)$ denotes the space of ends of M, see Section 1.2.1.

Intuitively, given a prime 3-manifold $P \in \mathcal{P}$, the number $n_P(M)$ counts how many times the summand P appears in an infinite connected sum decomposition of M, and $E_P(M)$ is the subset of ends for which any of their neighbourhood contains infinitely many P summands.

Theorem 1.1.11 ([BBM20, Theorem 1.2])**.** *Let M and M' be two open oriented 3-manifolds. Suppose that M and M' both decompose as a connected sum over a finite subfamily $\mathcal{F}$ of $\mathcal{P}$. Then M is homeomorphic to M' if and only if the following conditions are satisfied.*

1. *For every prime 3-manifold $P \in \mathcal{P}$, $n_P(M) = n_P(M')$.*
2. *There is an homeomorphism $\phi : E(M) \to E(M')$ such that $\phi(E_P(M)) = E_P(M')$, for every prime 3-manifold $P \in \mathcal{P}$.*

Theorem 1.1.11 has some counterintuitive consequences. Let $(P_i)_{i\in\mathbb{N}}$ be a sequence of manifolds in $\mathcal{P}$ which are pairwise non-homeomorphic. By Theorem 1.1.11, the manifolds

$$P_1 \# P_2 \# P_1 \# P_2 \# P_1 \# P_2 \# P_1 \# \dots$$

and

$$P_1 \# P_2 \# P_2 \# P_1 \# P_2 \# P_2 \# P_1 \# \dots,$$

which are infinite connected sums modelled on the halfline, are homeomorphic. In [BBM20], the authors also showed that the manifolds

$$\dots\#\mathbb{S}^3\#\mathbb{S}^3\#\mathbb{S}^3\#P_1\#P_2\#P_3\#\dots$$

and

$$\dots\#P_5\#P_3\#P_1\#P_2\#P_4\#P_6\#\dots$$

are not homeomorphic.

1.1.4 The Loop Theorem and the Sphere Theorem

We will now present the Loop Theorem and the Sphere Theorem, two fundamental results in 3-manifold topology. The Loop Theorem will be crucial for the proof of Theorem 1.5.1, see Section 1.4.1. In essence, both the Loop Theorem and the Sphere Theorem state that a feature of the algebraic topology of a given 3-manifold can be realised by an object of geometric nature.

Historically, the origin of the Loop and Sphere Theorems can be traced back to 1910, when Dehn [Deh10] published what today is known as Dehn's Lemma. Apart from the fact that Dehn's Lemma is less general than the Loop and Sphere Theorems, Dehn's proof contained a significant gap. It was not until later that Papakyriakopoulos [Pap57b, Pap57a] proved definitively Dehn's Lemma and established the Loop Theorem and the Sphere Theorem. The modern formulation of the Loop Theorem is due to Stallings [Sta60].

Theorem 1.1.12 (Loop Theorem)**.** *Let Σ be a closed (not necessarily connected) surface embedded into a 3-manifold M. If, for a certain basepoint $x \in \Sigma$, the homomorphism induced by the inclusion $\pi_1(\Sigma, x) \to \pi_1(M, x)$ is not injective, then there is a simple closed curve γ of Σ representing a nontrivial element of* $\ker(\pi_1(\Sigma, x) \to \pi_1(M, x))$ *and an embedded disc $D \subset M$ such that $\gamma = \partial D = D \cap \Sigma$.*

The Loop Theorem is the basic tool for performing a topological operation on surfaces embedded in a 3-manifold, known as compression.

Definition 1.1.13. Let Σ be a closed (not necessarily connected) surface embedded into an orientable 3-manifold M. The surface Σ is *incompressible* if the morphism $\pi_1(\Sigma, x) \to \pi_1(M, x)$ induced by inclusion is injective for any basepoint $x \in \Sigma$. Otherwise, the surface Σ is *compressible*.

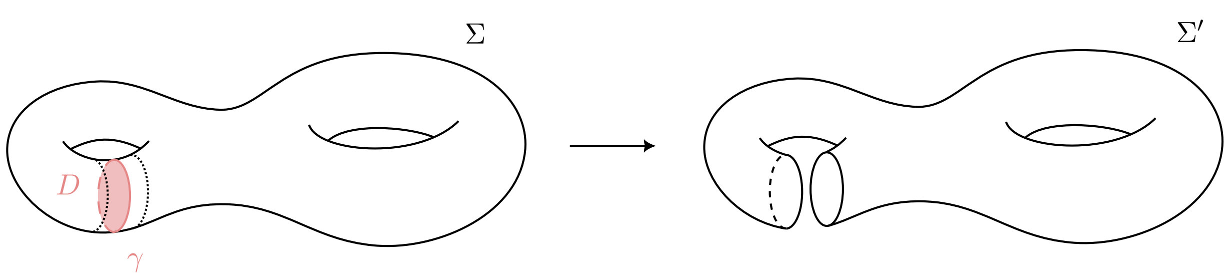

Figure 1.3: Compressing a surface.

A compressible surface Σ can be compressed into an (embedded) incompressible surface homologous to Σ as follows. By the Loop Theorem 1.1.12, take a simple closed curve γ in Σ representing a nontrivial element of $\ker(\pi_1(\Sigma, x) \to \pi_1(M, x))$ which bounds an embedded disc $D \subset M$ intersecting Σ only along its boundary. There is a diffeomorphism onto its image $\varphi : D \times [-1, 1] \to M$ such that $\varphi(\cdot, 0) = id_D$ and $\varphi(D \times [-1,1]) \cap \Sigma = \varphi(\partial D \times [-1,1])$. Then compress Σ along the disc D as follows. Remove the band $\varphi(\partial D \times [-1,1])$ and glue two discs $D^{\pm} := \varphi(D \times \{\pm 1\})$ to Σ along the corresponding curve $\varphi(\partial D \times \{\pm 1\})$, see Figure 1.3. Notice that $\varphi(D \times [-1,1])$ is a 1-handle in M, whose boundary corresponds to $\varphi(\partial D \times [-1,1]) \cup D^{\pm}$. Therefore, the compression, which consists in substituting $\varphi(\partial D \times [-1,1])$ by $D^{\pm}$, yields a new surface Σ' homologous to Σ in M.

The compression of Σ along a disc reduces its complexity. Namely, if γ is not separating in Σ, then the compression simply reduces the genus of the compressed connected component of Σ by one. On the other hand, if γ is separating in Σ, then the compression splits the corresponding connected component into two new connected components. Each of them has strictly lower genus than the compressed connected component of Σ. Hence, by iterating the procedure finitely many times, we finally obtain an incompressible surface Σ'.

For the sake of completeness, we will finish this section by stating the Sphere Theorem.

Theorem 1.1.14 (Sphere Theorem [Pap57a])**.** *Let M be an orientable 3-manifold with $\pi_2(M) \neq 0$. Then there is an embedded 2-sphere S representing a non-trivial element of $\pi_2(M)$.*

1.2 The theory of ends

The study of the topology of non-compact manifolds (or, more generally, non-compact topological spaces) requires the use of topological invariants describing their topological structure at infinity. An object which appears naturally in the study of the topology at infinity of a topological space is the notion of end. Intuitively, an end of a non-compact topological space X is a connected component of the complementary $X - K$, for an arbitrarily large compact subset $K \subset X$. The space of ends was first defined by Freudenthal [Fre31], in his search for a definition of a notion of compactification with some desirable properties, known nowadays as Freudenthal's end compactification.

1.2.1 The space of ends

Although Freudenthal's definition of space of ends holds for a larger class of topological spaces, for our purposes it will suffice to assume that X is a connected countable locally finite CW-complex. We say that a subset of X is *bounded* if it is contained in a compact subset of X, otherwise we say it is *unbounded*.

Definition 1.2.1. An *neighbourhood system of infinity* in X is a descending sequence

$$U_1 \supset U_2 \supset \dots$$

of connected open unbounded subsets $U_i \subset X$ whose boundary ∂U_i is compact and satisfying $\bigcap_i U_i = \emptyset$. An *end* e of X is an equivalence class of neighbourhood systems of infinity in X by the equivalence relation

$$(U_1 \supset U_2 \supset \dots) \sim (V_1 \supset V_2 \supset \dots)$$

if and only if for every U_i there is a j such that $U_i \subset V_j$, and for every V_j there is an i such that $V_j \subset U_i$. Each open subset U_i is said to be an *open neighbourhood* of the end e. The set of ends of X is denoted by $E(X)$.

For every open subset $U \subset X$ with compact boundary we define

$$U^* := \{e \in E(X) \mid e \text{ is represented by } U_1 \supset U_2 \supset \dots \text{ such that } U_i \subset U \text{ for large enough } i\}.$$

The collection $\{U^* \mid U \subset X \text{ open subset}\}$ defines a basis for a topology on $E(X)$. The set $E(X)$ endowed with this topology is the *space of ends* of X.

The *number of ends* of X is defined as $e(X) := |E(X)|$.

Notice that the space of ends of a compact CW-complex X is $E(X) = \emptyset$, and therefore $e(X) = 0$. There is an equivalent definition of the notion of end that uses equivalence classes of proper rays under proper homotopy equivalence, see [Geo08, Section 13.4], but we shall not make use of it.

Freudenthal [Fre31] proved the following structure theorem.

Theorem 1.2.2 ([Fre31])**.** *Let X be a connected countable locally finite CW-complex. The space of ends $E(X)$ of X is compact, metrisable and totally disconnected. In particular, $E(X)$ is homeomorphic to a subset of the Cantor set.*

The number of ends of a CW-complex X can be expressed in terms of the cohomology of ends of X. Let $\mathcal{C}^*(X)$ be the simplicial cochain complex corresponding to X. A cochain $\varphi \in \mathcal{C}^*(X)$ is *compactly supported* if there is a compact subset $K \subset X$ such that $\varphi(\sigma) = 0$ for any chain $\sigma \in \mathcal{C}_*(X - K)$. Denote by $\mathcal{C}_c^*(X)$ the cochain subcomplex of $\mathcal{C}^*(X)$ consisting of compactly supported cochains, and define

$$\mathcal{C}_e^*(X) := \mathcal{C}^*(X)/\mathcal{C}_c^*(X).$$

The *cohomology of ends* $H_e^*(X;G)$ of X with coefficients in an abelian group G is the cohomology associated to the complex $\mathcal{C}_e^*(X)$ and the abelian group G. It can be shown that

$$H_e^*(X;G) = \varinjlim\nolimits_K H^*(X-K;G),$$

where the direct limit runs over all compact subsets $K \subset X$.

Lemma 1.2.3 ([Eps61, Theorem 1])**.** *Let F be a field. Then $e(X) = \dim_F(H_e^0(X;F))$.*

The following result is an application of Lemma 1.2.3 in the computation of the number of ends of contractible manifolds, see for instance [Geo08, Proposition 16.4.1].

Proposition 1.2.4. *Let M be a contractible n-manifold with $n \geq 2$. Then $e(M) = 1$.*

Proof. Fix a compact subset $K \subset M$. The long exact sequence corresponding to the pair $(M, M-K)$ for reduced cohomology (with coefficients in a field) may be written

$$\cdots \leftarrow \tilde{H}^1(M) \leftarrow H^1(M, M-K) \leftarrow \tilde{H}^0(M-K) \leftarrow \tilde{H}^0(M) \leftarrow H^0(M, M-K) \leftarrow 0,$$

and the contractibility of M gives an isomorphism $H^1(M, M-K) \simeq \tilde{H}^0(M-K)$. Now, Poincaré Duality [Hat02, Theorem 3.35] gives an isomorphism $H_{n-1}(M) \simeq H_c^1(M)$, where $H_c^1(M)$ denotes the 1-cohomology group with compact support. Recall that $H_c^1(M) \simeq \varinjlim_K H^1(M, M-K)$ (see [Hat02, Section 3.3] for instance). Therefore,

$$H_{n-1}(M) \simeq \varinjlim\nolimits_K \tilde{H}^0(M, M-K).$$

Since M is contractible, the former space must vanish. Therefore

$$e(M) = \dim\left(\varinjlim\nolimits_K H^0(M-K)\right) = \dim\left(\varinjlim\nolimits_K \tilde{H}^0(M-K)\right) + 1 = 1.$$

∎

1.2.2 The number of ends of a group

In Section 1.2.1 we defined the number of ends of a connected topological space with a countable locally finite CW-structure. The following result motivates the definition of the notion of number of ends of a finitely generated group.

Theorem 1.2.5 ([Eps61, Theorem 3])**.** *Let $\bar{X}$ be a simplicial complex. Let G be a finitely generated group acting cocompactly by covering transformations on $\bar{X}$. Then the number of ends $e(\bar{X})$ depends uniquely on the group G.*

Therefore, the number of ends of such a simplicial cover is a group invariant.

Definition 1.2.6. The *number of ends* of a finitely generated group G is defined as $e(G) := e(\bar{X})$, where $\bar{X} \to X$ is any regular covering of a finite simplicial complex X with covering transformation group G.

In particular, the number of ends of a finitely generated group G coincides with the number of topological ends of its Cayley graph, and does not depend on the set of generators chosen to define it. Using the theory of covering spaces one can show that the number of groups is invariant when passing to a subgroup of finite index.

Theorem 1.2.7 ([Eps61, Theorem 11])**.** *Let G be a finitely generated group and H a subgroup of G of finite index. Then $e(H) = e(G)$.*

Unlike the number of topological ends of a CW-complex, the number of ends that a group may have is limited. Recall that a group is *virtually infinite cyclic* if and only if it contains a finite index group isomorphic to $\mathbb{Z}$.

Theorem 1.2.8 ([Hop44; Sta71, Theorem 5.A.9])**.** *The number of ends of a finitely generated group G is either 0, 1, 2 or infinite. More precisely, we have the following.*

1. *$e(G) = 0$ if and only if G is finite.*
2. *$e(G) = 2$ if and only if G is virtually infinite cyclic.*
3. *$e(G) = \infty$ if and only if G is isomorphic to*
    - *an amalgamated free product $A *_C B$ with A, B subgroups of G and C a proper finite subgroup of both A and B, or*
    - *an HNN extension $\langle H, t \mid tct^{-1} = \varphi(c) \text{ for } c \in C_1\rangle$ over a finite subgroup H of G relative to an isomorphism $\varphi : C_1 \to C_2$ between two subgroups C_1 and C_2 of H.*
4. *$e(G) = 1$ if and only if G is not of type 1, 2 or 3.*

It follows from Definition 1.2.6 that the number of ends of the fundamental group of a closed manifold coincides with the number of ends of its universal cover.

Lemma 1.2.9. *Let M be a closed n-manifold, and denote by $\tilde{M}$ the universal cover of M. Then $e(\pi_1(M)) = e(\tilde{M})$.*

As a consequence of Proposition 1.2.4 and Lemma 1.2.9 we obtain the following.

Corollary 1.2.10. *If M is a closed aspherical n-manifold with $n \geq 2$, then $e(\pi_1(M)) = 1$.*

1.3 The fill radius

This section is devoted to the notion of fill radius and its interaction with the geometry of manifolds of positive scalar curvature. The fill radius of contractible closed curves was first introduced in [GL83, SY79a, SY83], and it is related to the more general notion of filling radius of a Riemannian manifold presented in [Gro83], see Section 2.3.1.

Let M be a Riemannian n-manifold with possibly nonempty boundary. Recall that, given a subset $Z \subset M$, the closed R-neighbourhood of Z in M is denoted by

$$U(Z,R) := \{x \in M \mid d(x,Z) \leq R\}. \tag{1.1}$$

Definition 1.3.1. Let M be a Riemannian n-manifold with possibly nonempty boundary. The *fill radius* of a contractible closed curve γ in M is defined as

$$\operatorname{fillrad}(\gamma) := \sup\{R \geq 0 \mid d(\gamma, \partial M) > R \text{ and } [\gamma] \neq 0 \in \pi_1(U(\gamma,R))\}.$$

Define also

$$\operatorname{fillrad}(M) := \sup\{\operatorname{fillrad}(\gamma) \mid \gamma \text{ contractible closed curve of } M\}.$$

Notice that if a manifold M has bounded diameter, then $\operatorname{fillrad}(M) \leq \operatorname{diam}(M)$.

Example 1.3.2 ([SY83, Remark 1])**.** A ball $B^3(R)$ of radius $R > 0$ in the Euclidean space $\mathbb{R}^3$ has $\operatorname{fillrad}(\mathbb{B}^3(R)) = \frac{R}{2}$. The standard Riemannian cylinder $\mathbb{S}^2(1) \times \mathbb{R}$ has $\operatorname{fillrad}(\mathbb{S}^2(1) \times \mathbb{R}) = \frac{\pi}{2}$.

1.3.1 Positive scalar curvature and fill radius

The fill radius of a Riemannian manifold is affected by its curvature. Intuitively, the more positively curved the manifold is, the smaller the fill radius. For instance, by the Bonnet–Myers Theorem [GHL04, Theorem 3.85], a manifold of uniformly positive Ricci curvature must have finite diameter, and hence bounded fill radius. In [RW10], the authors conjectured that some weaker conditions of positive curvature (namely, uniformly 2-positive Ricci curvature and uniformly positive isotropic curvature) also imply a bound on the fill radius.

For 3-manifolds of uniformly positive scalar curvature, the following bound on the fill radius was established in [GL83, SY83], see also [Wol12, Theorem 9.3.1] for another exposition. A Riemannian manifold M has *uniformly positive scalar curvature* if there exists a positive constant $s > 0$ such that $\operatorname{scal} \geq s > 0$.

Theorem 1.3.3 ([GL83, Theorem 10.7], [SY83])**.** *Let M be a complete Riemannian 3-manifold with bounded geometry, possibly with nonempty boundary. If $\operatorname{scal} \geq s > 0$, then*

$$\operatorname{fillrad}(M) \leq \frac{2\pi}{\sqrt{s}}. \tag{1.2}$$

The value of the constant 2π was obtained by Gromov–Lawson [GL83]. Using a different argument, Schoen–Yau [SY83] derived Theorem 1.3.3 for the sharper value of the constant $\sqrt{\frac{8}{3}}\pi$, and recently Hu–Xu–Zhang in [HXZ24] showed that the latter is optimal. However, we will use Gromov–Lawson's constant, since their proof is more suitable to our case, see Proposition 1.3.9.

Theorem 1.3.3 implies that when M is a complete orientable 3-manifold with bounded geometry and uniformly positive scalar curvature $\operatorname{scal} \geq s > 0$, then the universal Riemannian cover $\tilde{M}$ of M satisfies $\operatorname{fillrad}(\tilde{M}) \leq 2\pi/\sqrt{s}$. Therefore, an upper bound on the fill radius of the universal cover provides a generalisation of the notion of uniformly positive scalar curvature.

More generally, we will consider 3-manifolds admitting a complete Riemannian metric of positive scalar curvature with at most a quadratic decay at infinity.

Definition 1.3.4. Let M be a complete Riemannian n-dimensional manifold. Fix a basepoint $x \in M$, and denote by $r_x(y) = d(x, y)$ the distance function to x.

1. The scalar curvature of M has a *decay at infinity of rate* $\alpha \geq 0$ *and constant* $C > 0$ if there exists a constant $R_0 > 0$ such that for every $y \in M$ with $r_x(y) \geq R_0$,

$$\operatorname{scal}(y) > \frac{C}{r_x(y)^\alpha}.$$

   Notice that a change of basepoint would only modify the constant R_0 and would leave the constant C unaltered. Hence, the basepoint $x \in M$ can be taken arbitrarily.

2. The scalar curvature of M has a *subquadratic decay at infinity* if it decays at infinity at rate α and constant C, for some $\alpha < 2$ and $C > 0$.

3. The scalar curvature of M has *at most C-quadratic decay at infinity* if it has a decay at infinity of rate $\alpha = 2$ and constant C, for some $C > 0$.

With these definitions, if the scalar curvature of M is uniformly positive with $\operatorname{scal} \geq s > 0$, it decays at infinity at rate α and constant C, for any $\alpha \geq 0$ and $0 < C \leq s$. Similarly, if the scalar curvature of M has a subquadratic decay at infinity, it has at most C-quadratic decay at infinity for any $C > 0$.

If a complete orientable 3-manifold M has positive scalar curvature decaying at infinity, then the conclusion of Theorem 1.3.3 does not longer hold, since the fill radius of M is not necessarily bounded in general. Still, if the decay is not too pronounced, one can control the growth of the fill radius of the lifts to the universal cover of the closed curves contractible in M. This property will serve as a generalisation of the notion of positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$, see Theorem 1.3.7. Notice that if M is a compact manifold, the fill radius of contractible curves of M is uniformly bounded by $\operatorname{diam}(M)$, which is why we consider the lifts of such curves to the universal cover.

Definition 1.3.5. Let M be a complete Riemannian manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Fix a basepoint $x \in M$. Denote by $B(x, R)$ the closed metric ball of radius R centered at x.

1. The fill radius of $\tilde{M}$ *grows at infinity at rate* $\beta \geq 0$ *and with constant* $c > 0$ if there is a constant $R_0' \geq 0$ such that if $R \geq R_0'$, then for every closed curve γ lying in $B(x, R)$ and contractible in M, any of its lifts $\tilde{\gamma}$ to $\tilde{M}$ satisfies

$$\operatorname{fillrad}(\tilde{\gamma}) < cR^\beta.$$

   Again, notice that the basepoint $x \in M$ may be taken arbitrarily since a change of basepoint does not affect the rate β, nor the value of the constant c.

2. The fill radius of $\tilde{M}$ has *sublinear growth at infinity* if it grows at infinity at rate β and constant c, for some $\beta < 1$ and $c > 0$.

3. The fill radius of $\tilde{M}$ has *at most c-linear growth at infinity* if it grows a rate $\beta = 1$ and constant c, for some $c > 0$.

Example 1.3.6. The fill radius of the Euclidean 3-dimensional space $\mathbb{R}^3$ grows at infinity at rate $\beta = 1$ and constant $c = 1$.

The decay at infinity of the scalar curvature of a manifold and the growth at infinity of the fill radius of its universal Riemannian cover are related as follows.

Theorem 1.3.7. *Let M be an orientable complete Riemannian 3-manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Suppose that M has positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$. Then, the fill radius of $\tilde{M}$ has at most c-linear growth at infinity for some $c < \frac{1}{3}$.*

We will prove the topological decomposition of Theorem A by replacing the scalar curvature assumption with this weaker condition about the filling discs of the lifts of contractible closed curves, namely that the fill radius of $\tilde{M}$ has at most c-linear growth at infinity with $c < \frac{1}{3}$, see Theorem 1.5.1 and Section 1.5.

The rest of Section 1.3.1 will be devoted to the proof of Theorem 1.3.7. In coherence with the definition of the fill radius, see Definition 1.3.1, we introduce the following notation.

Definition 1.3.8. Let K be a subset of a complete Riemannian 3-manifold M. Let γ be a closed curve lying in K and contractible in M. Define the *fill radius of γ relative to K* as

$$\operatorname{fillrad}(\gamma \subset K) := \sup\left\{R \geq 0 \mid d(\gamma, \partial K) > R \text{ and } [\gamma] \neq 0 \in \pi_1(U(\gamma, R))\right\}.$$

Recall that $U(\gamma, R)$ is the closed R-neighbourhood of γ in M.

With this notation, one can adapt the proof of Theorem 1.3.3 in [GL83, Proof of Theorem 10.7] to prove the following result.

Proposition 1.3.9. *Let M be a complete Riemannian 3-manifold with* $\operatorname{scal} > 0$. *Let $K \subset M$ be a compact subset. Let $s_K > 0$ be a constant such that* $\operatorname{scal}(x) \geq s_K > 0$ *for every $x \in K$. Then, any closed curve γ lying in K and contractible in M satisfies:*

$$\operatorname{fillrad}(\gamma \subset K) \leq \frac{2\pi}{\sqrt{s_K}}.$$

Proof. We argue by contradiction following [GL83, Proof of Theorem 10.7]. The only difference here is that we will need to carefully apply the Stability Inequality [GL83, Inequality 10.17] to functions with support in K. We also give more details of some parts of the argument.

Let γ be a closed curve lying in K and contractible in M. Let $\rho > \pi/\sqrt{s_K}$. Suppose that $d(\gamma, \partial K) > 2\rho$ and that γ does not bound a disc in $U(\gamma, 2\rho)$ (in the proof, all the discs are non-necessarily embedded topological discs). Consider $B \subset M$ a closed metric ball whose interior contains the domain K as well as a disc $D \subset M$ bounded by γ. After slightly deforming B, we may assume that the boundary ∂B of B is a smooth surface. Now, we modify the Riemannian metric in a tubular neighbourhood of ∂B away from K to make it isometric to the Riemannian product $\partial B \times [0, 1]$. With the modified metric, the subset B is a manifold with (mean) convex boundary. By [MY82], there exists a disc $\Sigma \subset B$ bounded by γ of least area. Observe that Σ is a stable minimal disc which is not contained in $U(\gamma, 2\rho)$. Using Schoen-Yau's rearrangement [SY79b], the stability of the minimal surface Σ implies

$$\int_\Sigma \left(|\nabla f|^2 + \kappa_\Sigma f^2 - \frac{1}{2}\left(\operatorname{scal} + \|\mathrm{II}\|^2\right) f^2\right) dA \geq 0 \tag{1.3}$$

for any function $f \in \mathcal{C}_c^\infty(\Sigma)$, where κ_Σ and II denote the Gauss curvature and the second fundamental form of Σ.

Now, consider the level set $\gamma_\rho := \{x \in \Sigma \mid d_\Sigma(\gamma, x) = \rho\}$ which, after considering a smooth aproximation of the function $d_\Sigma(\gamma, \cdot)$, consists of a disjoint collection of closed curves. Since γ does not bound a disc in $U(\gamma, 2\rho)$, there is at least one connected component σ of γ_ρ which does not bound a disc in $U(\gamma, 2\rho)$. Let $\Omega \subset \Sigma$ be a small neighbourhood of σ. For every $s \in [0, \rho]$, denote by

$$\Omega(s) := \{x \in \Sigma \mid d_\Sigma(x, \Omega) \leq s\}$$

the closed s-neighbourhood of Ω in Σ, see Figure 1.4.

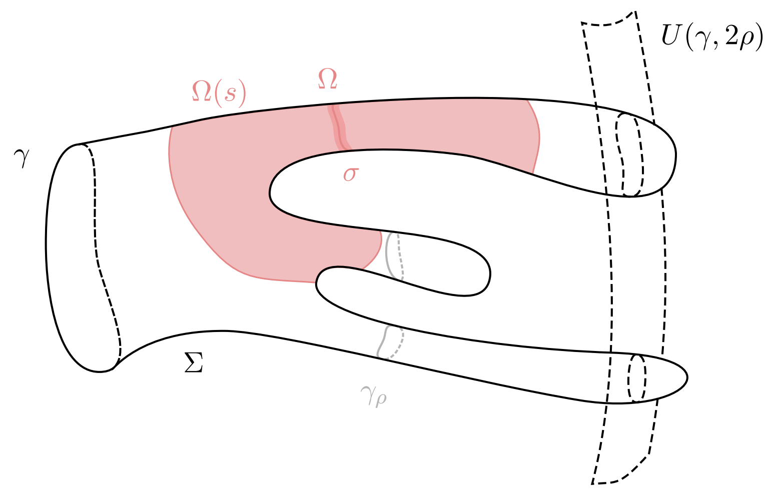

Figure 1.4: The neighbourhood $\Omega(s)$ in the stable minimal disc Σ.

Following [GL83, Proof of Theorem 10.2], by analytic approximation of the domain Ω and the distance function $d_\Sigma(\cdot, \Omega)$, we may assume that the level set $\partial\Omega(s)$ is piecewise smooth, for every $s \in [0, \rho]$. Notice that, as $d(\gamma, \partial K) > 2\rho$, the set $\Omega(\rho)$ is contained in the compact K and its interior does not intersect the curve γ. Therefore, if $f \in \mathcal{C}_c^\infty(\Sigma)$ is a function supported in $\Omega(\rho)$, the Stability Inequality (1.3) implies

$$\int_{\Omega(\rho)} \left(|\nabla f|^2 + \kappa_\Sigma f^2 - \frac{s_K}{2} f^2 \right) dA \geq 0. \tag{1.4}$$

For every $s \in [0, \rho]$, we denote by

- $\gamma(s) := \partial\Omega(s)$ the boundary of $\Omega(s)$,
- $\ell(s) := \ell(\gamma(s))$ the length of the curve $\gamma(s)$,
- $\Gamma(s) := \int_{\gamma(s)} \kappa_g(s) + \sum_i \theta_i(s)$, where $\kappa_g(s)$ is the geodesic curvature of the curve $\gamma(s)$ and $\theta_i(s)$ are the external angles defined by the curve $\gamma(s)$ at its vertices,
- $\kappa(s) := \int_{\gamma(s)} \kappa_\Sigma$ the integral of the Gauss curvature κ_Σ of Σ over the curve $\gamma(s)$,
- $A(s) := |\Omega(s)|$ the area of $\Omega(s)$, and
- $\chi(s) := \chi(\Omega(s))$ the Euler characteristic of the surface $\Omega(s)$.

The Gauss–Bonnet Theorem [dC76, Section 4.5] applied to the surface $\Omega(s_1) - \operatorname{int}(\Omega(s_0))$ gives

$$\int_{s_0}^{s_1} \kappa(s) ds = 2\pi \left(\chi(s_1) - \chi(s_0)\right) - \left(\Gamma(s_1) - \Gamma(s_0)\right),$$

which implies that, in the distributional sense,

$$\kappa(s) = (2\pi\chi(s) - \Gamma(s))'. \tag{1.5}$$

Now consider the function $f : \Sigma \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \cos\left(\frac{\pi}{2\rho} d_\Sigma(x, \Omega)\right) & \text{if } x \in \Omega(\rho) \\ 0 & \text{if } x \notin \Omega(\rho) \end{cases}.$$

Notice that the function f is supported in $\Omega(\rho)$. For this choice of function f, and using Coarea Formula [BZ88, Theorem 13.4.2], the inequality (1.4) may be written as follows

$$\begin{aligned} \frac{\pi^2}{4\rho^2}\int_0^\rho \ell(s)\sin^2\left(\tfrac{\pi}{2\rho}s\right)ds + \int_0^\rho \kappa(s)\cos^2\left(\tfrac{\pi}{2\rho}s\right)ds + 2\pi\chi(0) - \Gamma(0) \geq \\ \geq \frac{s_K}{2}\left(\int_0^\rho \ell(s)\cos^2\left(\tfrac{\pi}{2\rho}s\right)ds + |\Omega|\right). \end{aligned} \tag{1.6}$$

Using equation (1.5) and integrating by parts, we have

$$\int_0^\rho \kappa(s)\cos^2\left(\tfrac{\pi}{2\rho}s\right)ds + 2\pi\chi(0) - \Gamma(0) = \frac{\pi}{2\rho}\int_0^\rho (2\pi\chi(s) - \Gamma(s))\sin\left(\tfrac{\pi}{\rho}s\right)ds. \tag{1.7}$$

Now we use the fact that $\ell'(s) \leq \Gamma(s)$ for every $s \in (0, \rho)$ [Fia41] (see also [GL83, Proof of Theorem 10.2]), and we integrate by parts to obtain

$$\begin{aligned} -\frac{\pi}{2\rho}\int_0^\rho \Gamma(s)\sin\left(\tfrac{\pi}{\rho}s\right)ds \leq -\frac{\pi}{2\rho}\int_0^\rho \ell'(s)\sin\left(\tfrac{\pi}{\rho}s\right)ds = \\ = \frac{\pi^2}{2\rho^2}\int_0^\rho \ell(s)\left(\cos^2\left(\tfrac{\pi}{2\rho}s\right) - \sin^2\left(\tfrac{\pi}{2\rho}s\right)\right)ds. \end{aligned} \tag{1.8}$$

Hence, by equations (1.7) and (1.8), the inequality (1.6) implies

$$\begin{aligned} \left(\frac{\pi^2}{\rho^2} - s_K\right)\int_0^\rho \ell(s)\cos^2\left(\tfrac{\pi}{2\rho}s\right)ds + \frac{4\pi^2}{\rho}\int_0^\rho \chi(s)\sin\left(\tfrac{\pi}{2\rho}s\right)\cos\left(\tfrac{\pi}{2\rho}s\right)ds \geq \\ \geq s_K|\Omega| + \frac{\pi^2}{2\rho^2}\int_0^\rho \ell(s)\sin^2\left(\tfrac{\pi}{2\rho}s\right)ds. \end{aligned} \tag{1.9}$$

Since the curve γ does not bound a disc in $U(\gamma, 2\rho)$, we have $\chi(s) \leq 0$ for every $s \in [0, \rho]$. Therefore, we obtain

$$\left(\frac{\pi^2}{\rho^2} - s_K\right)\int_0^\rho \ell(s)\cos^2\left(\tfrac{\pi}{2\rho}s\right)ds \geq s_K|\Omega| > 0.$$

We conclude that $\rho < \pi/\sqrt{s_K}$, which is a contradiction. □

From Proposition 1.3.9, one can derive Theorem 1.3.7. More generally, we have the following result.

Proposition 1.3.10. *Let M be a complete Riemannian 3-manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Suppose that M has positive scalar curvature with a decay at infinity of rate $\alpha \geq 0$ and constant $C > 0$.*

1. *If the decay is subquadratic (that is, $\alpha \in [0, 2)$), then the fill radius of $\tilde{M}$ has sublinear growth at infinity of rate $\beta = \alpha/2 \in [0, 1)$.*

2. *If the decay is at most C-quadratic with $C > 4\pi^2$, then the fill radius of $\tilde{M}$ has at most linear growth at infinity with constant*
$$c = \frac{2\pi}{\sqrt{C} - 2\pi}.$$

In particular, if the scalar curvature has at most C-quadratic decay at infinity with $C > 64\pi^2$, then the fill radius of $\tilde{M}$ has at most c-linear growth at infinity with $c < \frac{1}{3}$.

Proof. Fix a point $x \in M$. Consider $R_0 > 0$ such that every point $y \in M$ with $r_x(y) \geq R_0$ satisfies
$$\operatorname{scal}(y) > \frac{C}{r_x(y)^\alpha},$$
and define $s_0 := \min_{B(x,R_0)} \operatorname{scal}$. Fix $\mu > 1$. Let $R \geq R_0'$, where
$$R_0' = \begin{cases} \max\left\{R_0, \left(\frac{C}{s_0}\right)^{\frac{1}{\alpha}}, \left(\frac{2\pi}{\sqrt{C}}\right)^{\frac{2}{2-\alpha}} \left(\frac{\mu^\alpha}{(\mu-1)^2}\right)^{\frac{1}{2-\alpha}}\right\} & \text{if } \alpha \in [0,2) \\ \max\left\{R_0, \sqrt{\frac{C}{s_0}}\right\} & \text{if } \alpha = 2 \end{cases}.$$
Note that $R \geq \max\left\{R_0, (\frac{C}{s_0})^{\frac{1}{\alpha}}\right\}$ in both cases.

Let γ be a closed curve lying in the closed metric ball $B(x,R)$ and contractible in M. Take any lift $\tilde{x}$ of x in the universal cover $\tilde{M}$, and lift γ to a closed curve $\tilde{\gamma}$ lying in the set $p^{-1}(B(x,R)) = U(\pi_1(M)\cdot\tilde{x}, R)$ formed by the points of $\tilde{M}$ at distance at most R from a point in the orbit of $\tilde{x}$ by the action of the fundamental group of M, see equation (1.1).

Consider the μR-neighbourhood $K = U(\pi_1(M)\cdot\tilde{x}, \mu R)$ of the orbit $\pi_1(M)\cdot\tilde{x}$. Since $R \geq (\frac{C}{s_0})^{\frac{1}{\alpha}}$, we have
$$\min_K \operatorname{scal} > \frac{C}{(\mu R)^\alpha}.$$
By Proposition 1.3.9, the closed curve $\tilde{\gamma}$ has
$$\operatorname{fillrad}(\tilde{\gamma} \subset K) < \frac{2\pi}{\sqrt{C}}(\mu R)^{\frac{\alpha}{2}}.$$
Notice that
$$d(\tilde{\gamma}, \partial K) \geq \mu R - R.$$
Hence, the curve $\tilde{\gamma}$ bounds a (non-necessarily embedded) disc in its $\frac{2\pi}{\sqrt{C}}(\mu R)^{\frac{\alpha}{2}}$-neighbourhood if
$$\mu R - R \geq \frac{2\pi}{\sqrt{C}}(\mu R)^{\frac{\alpha}{2}}. \tag{1.10}$$

1. Let us consider first the subquadratic case, that is, suppose $\alpha \in [0,2)$. The inequality
$$R \geq \left(\frac{2\pi}{\sqrt{C}}\right)^{\frac{2}{2-\alpha}} \left(\frac{\mu^\alpha}{(\mu-1)^2}\right)^{\frac{1}{2-\alpha}}$$
is equivalent to the inequality (1.10). Therefore,
$$\operatorname{fillrad}(\tilde{\gamma}) < \frac{2\pi}{\sqrt{C}}\mu^{\frac{\alpha}{2}} R^{\frac{\alpha}{2}}.$$
That is, the fill radius of $\tilde{M}$ has sublinear growth of rate $\beta = \alpha/2$ and constant $c = 2\pi\mu^{\frac{\alpha}{2}}/\sqrt{C}$. Notice that this estimate holds for any $\mu > 1$. In particular, one can let μ go to 1 in order to make c as close to $2\pi/\sqrt{C}$ as desired, in which case R_0' diverges to infinity.

2. Now, suppose that $\alpha = 2$ and $C > 4\pi^2$. In this case, the inequality (1.10) is equivalent to

$$\mu \geq \frac{\sqrt{C}}{\sqrt{C} - 2\pi}.$$

Hence, for any $\mu \geq \frac{\sqrt{C}}{\sqrt{C}-2\pi}$, we have

$$\operatorname{fillrad}(\tilde{\gamma}) < \frac{2\pi}{\sqrt{C}} \mu R.$$

In particular,

$$\operatorname{fillrad}(\tilde{\gamma}) < \frac{2\pi}{\sqrt{C} - 2\pi} R.$$

That is, the fill radius of $\tilde{M}$ has at most c-linear growth, with $c = \frac{2\pi}{\sqrt{C}-2\pi}$. Notice that $C > 64\pi^2$ if and only if $c < \frac{1}{3}$.

□

1.3.2 Fill radius and simply connectedness at infinity

A slow growth at infinity of the fill radius of the universal Riemannian cover of a manifold may constrain its topology at infinity. In this section, we will show that complete Riemannian manifolds whose universal Riemannian cover has fill radius with at most c-linear growth at infinity with $c < 1$ are simply connected at infinity on contractible curves.

Let us begin by recalling the well-known notion of manifold simply connected at infinity (see [Geo08, Section 16] for a detailed discussion on simply connectedness at infinity and related notions).

Definition 1.3.11. A manifold M is *simply connected at infinity* if for any compact subset $B \subset M$, there is a compact subset $K \subset M$ containing B such that the morphism

$$\pi_1(M - K) \to \pi_1(M - B)$$

induced by the inclusion is trivial.

Intuitively, a manifold is simply connected at infinity if for any compact subset B, closed curves sufficiently far from B can be contracted to a point avoiding B. Notice that simply connected manifolds are not necessarily simply connected at infinity. For instance, the plane in dimension two and the Whitehead manifold are simply connected (even contractible) but they are not simply connected at infinity. If instead we require this condition to hold only for curves that are contractible in M, we say that M is simply connected at infinity on contractible curves. More precisely:

Definition 1.3.12. A manifold M is *simply connected at infinity on contractible curves* if for any compact subset $B \subset M$, there is a compact $K \subset M$ containing B such that

$$\ker(\pi_1(M - K) \to \pi_1(M)) = \ker(\pi_1(M - K) \to \pi_1(M - B)),$$

that is, every closed curve $\gamma \subset M - K$ contractible in M is already contractible in $M - B$.

Example 1.3.13. Clearly, simply connectedness at infinity implies simply connectedness at infinity on contractible curves. The converse is not true, as shown by the following example. Consider the 3-manifold M obtained as an infinite connected sum (see Section 1.1.3) of manifolds $\mathbb{S}^2 \times \mathbb{S}^1$ modelled on the halfline $[0, +\infty)$ with vertices at the integer points. The manifold M is simply connected at infinity on contractible curves. However, M is not simply connected at infinity, since the complementary of any compact contains (infinitely many) non-contractible curves.

Let us now prove that if the fill radius of the universal Riemannian cover of a complete Riemannian manifold M grows at most c-linearly at infinity with $c < 1$, then M is simply connected at infinity on contractible curves. Notice that this result is valid for manifolds of any dimension.

Proposition 1.3.14. *Let M be a complete Riemannian manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Suppose the fill radius of $\tilde{M}$ has at most c-linear growth at infinity for some $c < 1$. Then the manifold M is simply connected at infinity on contractible curves.*

Proof. Fix a point $x \in M$. Consider $c < 1$ and $R_0' > 0$ such that if γ is a closed contractible curve lying in the closed metric ball $B(x, R)$ for $R \geq R_0'$, then any of its lifts $\tilde{\gamma}$ to the universal cover $\tilde{M}$ satisfies

$$\operatorname{fillrad}(\tilde{\gamma}) < cR.$$

Notice that, since the projection from the universal cover to M is distance non-increasing, the curve γ also satisfies $\operatorname{fillrad}(\gamma) < cR$.

For any compact $B \subset M$, choose an $r \geq R_0'$ such that $B \subset B(x, r)$. Let $R := \frac{r}{1-c} > r$ and consider the closed metric ball $K = B(x, R)$. Suppose that $\eta \subset M - K$ is a closed curve that is contractible in M. If η bounds a (non-necessarily embedded) disc in $M - K$, there is nothing to prove, so suppose that η bounds a disc D which intersects K. That is, there exists a continuous map $i : \mathbb{B}^2 \to M$ such that $i(\mathbb{B}^2) = D$ and whose restriction to $\partial\mathbb{B}^2$ coincides with η. By an approximation argument [Hir76, Theorem 2.12, Theorem 3.4], one can assume that $i : \mathbb{B}^2 \to M$ is an immersion, that is, that D is an immersed disc in M. Approximate the distance function $d(x, \cdot)$ on the manifold M by a smooth function, and denote by f its restriction to the disc D. By Sard's Theorem [Hir76, Theorem 1.3], we can slightly adjust R to a regular value of $f \circ i$ so that the preimage $(f \circ i)^{-1}(R)$ consists of a finite collection of disjoint simple closed curves γ_j in $\mathbb{B}^2$, each of which bounds a topological disc $\mathcal{D}_j \subset \mathbb{B}^2$. Observe that two such discs are either disjoint or one is contained within the other. It follows that the maximal discs $\mathcal{D}_j'$ in this family are disjoint. Note also that $i(\mathbb{B}^2 - \sqcup_j \mathcal{D}_j') \subset M - K$. The images γ_j' of $\partial\mathcal{D}_j'$ under the immersion i are contractible closed curves in D, and hence in M. Since each closed curve γ_j' lies in the metric ball $B(x, R)$, we have

$$\operatorname{fillrad}(\gamma_j') < cR = \frac{c}{1-c} r.$$

The distance of each curve γ_j' to the boundary $\partial B(x, r)$ satisfies

$$d(\gamma_j', \partial B(x, r)) \geq R - r \geq \frac{c}{1-c} r.$$

Therefore, each curve γ_j' bounds a disc inside the complement $M - B(x, r)$. That is, for each closed curve γ_j', there exists a continuous map $i_j : \mathcal{D}_j' \to M - B(x, r)$, which can be assumed to be an immersion, and whose restriction to $\partial\mathcal{D}_j'$ coincides with γ_j'. By combining the maps $i : \mathbb{B}^2 - \sqcup_j \mathcal{D}_j' \to M - B(x, r)$ and $i_j : \mathcal{D}_j' \to M - B(x, r)$, we obtain a map $\mathbb{B}^2 \to M - B(x, r)$ whose restriction to $\partial\mathbb{B}^2$ coincides with η. Thus, the curve η bounds a disc inside $M - B(x, r) \subset M - B$. □

Remark 1.3.15. A result closely related to Proposition 1.3.14 is [GL83, Corollary 10.9], where the authors proved that a complete 3-manifold of uniformly positive scalar curvature and finitely generated fundamental group is simply connected at infinity. Note that this result fails without the assumption that the fundamental group is finitely generated; see Example 1.3.13 for a counterexample. The hypothesis in Proposition 1.3.14 are more general (in particular, we do not make any assumption on the dimension of M, nor on the fundamental group of M), but we derive a weaker result, namely simply connectedness at infinity on contractible curves.

Remark 1.3.16. By Proposition 1.3.10, Proposition 1.3.14 applies to complete Riemannian 3-manifolds of positive scalar curvature with at most C-quadratic decay at infinity with $C > 16\pi^2$. Although we shall not make use of this fact, one can show directly this consequence by constructing a minimising annulus in a compact domain $K \subset M$ where the scalar curvature is bounded from below by some $s_K > 0$, without relying on the fill radius, as in the proof of Proposition 1.3.9. This alternative approach applies when $C > \pi^2$ and was first used in [GL83, Proof of Corollary 10.9]. Notice that for such manifolds, the fill radius of their universal cover has at most c-linear growth at infinity, for any $c > 0$.

The following result is a direct consequence of the application of Proposition 1.3.14 to contractible manifolds.

Corollary 1.3.17. *Let M be a complete contractible Riemannian manifold. Suppose the fill radius of M has at most c-linear growth at infinity for some $c < 1$. Then M is homeomorphic to $\mathbb{R}^n$.*

Proof. Proposition 1.3.14 together with the fact that the manifold M is contractible imply that M is simply connected at infinity. And the only contractible manifold which is simply connected at infinity is $\mathbb{R}^n$. This result was proven first for manifolds of dimension $n \geq 5$ [Sta62, Theorem 4], then for 3-dimensional manifolds [Edw63], and finally for 4-dimensional manifolds [Fre82, Corollary 1.2]. □

In particular, any contractible complete 3-manifold with positive scalar curvature of subquadratic decay at infinity, and more generally of C-quadratic decay at infinity for some $C > 16\pi^2$, is homeomorphic to $\mathbb{R}^3$. This result was already obtained by Wang [Wan19]. It is an open question whether any contractible complete 3-manifold with positive scalar curvature is homeomorphic to $\mathbb{R}^3$, despite some recent progress in this direction [Wan23b, Wan24a, Wan24b, CLX25].

1.3.3 Fill radius and number of ends

In this section, we will examine how the growth of the fill radius of the universal Riemannian cover of a manifold M influences the fundamental group $\pi_1(M)$ of M.

In [RW10], Ramachandran–Wolfson showed that the fundamental group of a closed manifold of any dimension, whose universal Riemannian cover has bounded fill radius cannot contain finitely generated subgroups with one end, and as a consequence, the fundamental group of such a manifold is virtually free. Recall that a group G is *virtually free* if it contains a finite index free subgroup. Recently, the same strategy was used in [CLL23] to prove a classification of closed manifolds admitting a metric of positive scalar curvature in dimensions 4 and 5, under some additional topological assumptions.

In this section, we derive the main result of [RW10] under a weaker hypothesis (the fill radius of $\tilde{M}$ is not necessarily bounded, but has at most c-linear growth at infinity with $c < \frac{1}{3}$), adapting ideas present in [RW10, CLL23, GL83]. Note that the results in this section are valid for manifolds of any dimension.

Theorem 1.3.18. *Let M be a complete Riemannian manifold, and consider its Riemannian universal cover $\tilde{M}$. Suppose the fill radius of $\tilde{M}$ has at most c-linear growth at infinity, with $c < \frac{1}{3}$. Then, the fundamental group $\pi_1(M)$ does not contain any finitely generated subgroup with exactly one end.*

We start by proving the following version of [GL83, Corollary 10.11] for manifolds whose Riemannian universal cover has fill radius with a certain growth at infinity.

Lemma 1.3.19. *Let M be a complete Riemannian manifold, and denote by $p : \tilde{M} \to M$ its universal Riemannian cover. Let $x \in M$ be a point. Let $\beta \geq 0$ and $c > 0$ be constants, and suppose that either $\beta < 1$, or $\beta = 1$ and $c < \frac{1}{2}$. Suppose there is a constant $R_0' \geq 0$ such that if $R \geq R_0'$ then, for every closed curve γ lying in $B(x, R)$ and contractible in M, any of its lifts $\tilde{\gamma}$ to $\tilde{M}$ verifies*

$$\operatorname{fillrad}(\tilde{\gamma}) < cR^{\beta}.$$

Let $Z \subset M$ be a path-connected compact subset and consider a path-connected component $\bar{Z}$ of $p^{-1}(Z) \subset \tilde{M}$. Then, there is a constant $R_0'' \geq R_0'$ such that if $R \geq R_0''$, every path-connected component C_R of the level set $\partial U(\bar{Z}, R)$ satisfies

$$\operatorname{diam}(C_R) < 6c\,(d(x, Z) + \operatorname{diam}(Z) + R)^{\beta}.$$

Proof. Let $R_0'' \geq R_0'$ be a constant to be determined later. Let $R \geq R_0'$ and denote $L := d(x, Z) + \operatorname{diam}(Z) + R$. Fix a connected component C_R of the level set $\partial U(\bar{Z}, R)$ and consider two points $z_1, z_2 \in C_R$. We shall prove that z_1 and z_2 lie within a distance at most $6cL^{\beta}$. Let $\tilde{\eta}$ be a curve in C_R joining z_1 to z_2. Join each point z_i to $\bar{Z}$ by a minimal geodesic $\tilde{\eta}_i$, and join the endpoints of $\tilde{\eta}_1$ and $\tilde{\eta}_2$ lying in $\bar{Z}$ by a curve $\tilde{\eta}'$ lying in $\bar{Z}$; see Figure 1.5. The concatenation $\tilde{\gamma} = \tilde{\eta}_1 * \tilde{\eta} * \tilde{\eta}_2 * \tilde{\eta}'$ is a closed curve contained in $U(\bar{Z}, R)$. Now, consider the projection γ of $\tilde{\gamma}$ to M. Then the closed curve γ lies in the closed metric ball $B(x, d(x, Z) + \operatorname{diam}(Z) + R)$. Therefore,

$$\operatorname{fillrad}(\tilde{\gamma}) < c\,(d(x, Z) + \operatorname{diam}(Z) + R)^{\beta} = cL^{\beta}.$$

That is, there exists a disc $D \subset \tilde{M}$, which can be assumed immersed, with $\partial D = \tilde{\gamma}$ such that any point $a \in D$ satisfies $d(a, \tilde{\gamma}) < cL^{\beta}$.

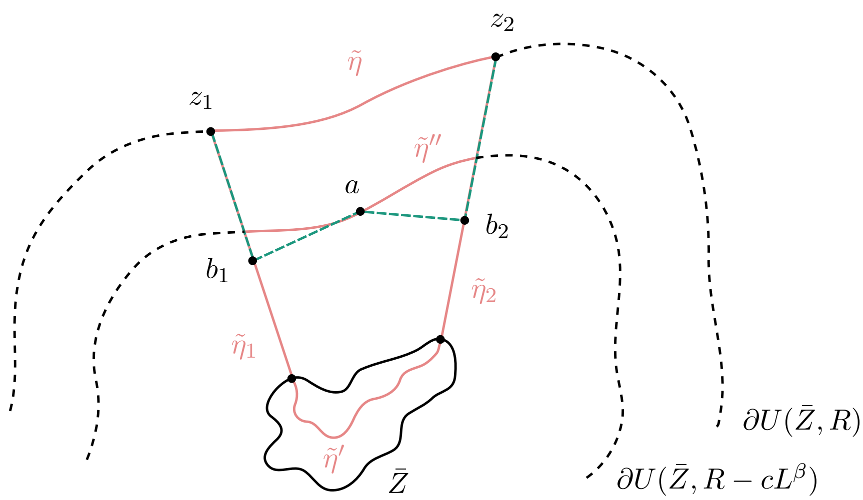

Figure 1.5: Scheme of the proof of Lemma 1.3.19.

Now, consider the curve $\tilde{\eta}'' = D \cap \partial U(\bar{Z}, R - cL^{\beta})$. Since either $\beta < 1$ or $\beta = 1$ and $c < \frac{1}{2}$, for R_0'' large enough, if $R \geq R_0''$ then every point $a \in \tilde{\eta}''$ satisfies

$$d(a, \tilde{\eta}') \geq d(a, \bar{Z}) = R - cL^{\beta} \geq cL^{\beta},$$

and clearly $d(a, \tilde{\eta}) \geq cL^{\beta}$. Therefore, there is a point $a \in D$ lying within a distance less than cL^{β} from both curves $\tilde{\eta}_1$ and $\tilde{\eta}_2$. That is, there are points $b_i \in \tilde{\eta}_i$ such that $d(a, b_i) < cL^{\beta}$, for $i = 1, 2$. Since $d(b_i, z_i) < 2cL^{\beta}$ (otherwise the inequality $d(a, b_i) < cL^{\beta}$ would not hold), we obtain

$$d(z_1, z_2) \leq d(z_1, b_1) + d(b_1, a) + d(a, b_2) + d(b_2, z_2) < 6cL^{\beta}.$$

That is,

$$d(z_1, z_2) < 6c\,(d(x, Z) + \operatorname{diam}(Z) + R)^{\beta}.$$

□

Now, we prove Theorem 1.3.18.

Proof of Theorem 1.3.18. Let $x \in M$ be a point and $R_0' \geq 0$ such that if γ is a contractible closed curve lying in the closed metric ball $B(x, R)$ for $R \geq R_0'$, then any of its lifts $\tilde{\gamma}$ to the universal cover $\tilde{M}$ satisfies

$$\operatorname{fillrad}(\tilde{\gamma}) < cR,$$

with $c < \frac{1}{3}$. Suppose that G is a finitely generated subgroup of $\pi_1(M, x)$ with exactly one end. In particular, the subgroup G is infinite. Consider a collection of closed curves $\eta_1, \ldots, \eta_k$ based at x representing the generators of G. Now, consider the lift $\bar{X}$ of $X = \cup_i \eta_i$ to the universal cover $\tilde{M}$, and fix a lift $\tilde{x} \in \bar{X}$ of the point x. Notice that $\bar{X}$ is homeomorphic to the Cayley graph of G associated to the generating set represented by the homotopy classes of the curves $\{\eta_i\}$.

We shall need the following result.

Lemma 1.3.20. *There exists a constant $H > 0$ such that for any minimising geodesic $\tilde{\gamma}$ joining two points y_1, y_2 lying in $\bar{X}$, we have*

$$\max_{y \in \tilde{\gamma}} d(y, \bar{X}) \leq H.$$

Proof. We argue by contradiction. Suppose that for any $H > 0$, there is a minimal geodesic $\tilde{\gamma}$ joining two points y_1 and y_2 lying in $\bar{X}$ and a point $y_0 \in \tilde{\gamma}$ such that $d(y_0, \bar{X}) = H$; see Figure 1.6. In particular, $d(y_0, y_j) \geq H$ for $j = 1, 2$, and $d(y_1, y_2) \geq 2H$.

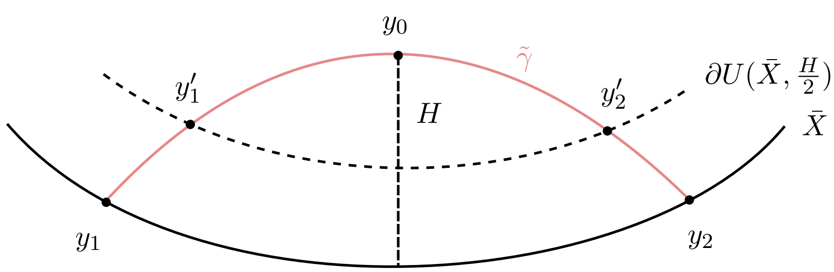

Figure 1.6: Minimising curve $\tilde{\gamma}$ joining two points y_1 and y_2 lying in $\bar{X}$.

Let y_j' be the intersection point of $\tilde{\gamma}$ with the level set $\partial U(\bar{X}, \frac{H}{2})$ which is closest to y_j, for$j = 1, 2$. The points y_1' and y_2' must lie in the same connected component of $\partial U(\bar{X}, \frac{H}{2})$, otherwise the concatenation of $\tilde{\gamma}$ with a curve lying in $\bar{X}$ joining y_1 with y_2, which is contractible, would have nontrivial intersection with the cycle $\partial U(\bar{X}, \frac{H}{2})$. Applying Lemma 1.3.19 to $\bar{Z} = \bar{X}$, $\beta = 1$ and $c < \frac{1}{3}$, we have that for H large enough,

$$d(y_1, y_2) \leq d(y_1, y_1') + d(y_1', y_2') + d(y_2', y_2) < H + 6c\,(\operatorname{diam}(X) + \tfrac{H}{2}).$$

Since $c < \frac{1}{3}$, this contradicts the fact that $d(y_1, y_2) \geq 2H$, for H sufficiently large. □

Now, let $R \geq R_0'$. Fix a fundamental domain Δ of $\bar{X}$ for the action of G. Since $e(\bar{X}) = e(G) = 1$, there is a compact $K \subset \bar{X}$ containing $U(\Delta, R+H) \cap \bar{X}$ such that $\bar{X} - K$ has a unique (unbounded) connected component.

Take a minimising curve $\tilde{\gamma}$ of $\tilde{M}$ joining two points y_1 and y_2 in $\bar{X}$ at distance at least $2T$ in $\tilde{M}$, where $T > \mathrm{diam}\,(K) + H$. Choose a point $y_0 \in \tilde{\gamma}$ with $d(y_0, y_j) \geq T$ for $j = 1, 2$ and a point $\bar{y}_0 \in \bar{X}$ at minimal distance from y_0. By Lemma 1.3.20, the minimising curve $\tilde{\gamma}$ is at distance at most H from $\bar{X}$. Thus, $d(y_0, \bar{y}_0) \leq H$. Translating the curve $\tilde{\gamma}$ by an element of G if necessary, we can assume that $\bar{y}_0$ lies in Δ. It follows that

$$B(y_0, R) \cap \bar{X} \subset B(\bar{y}_0, R+H) \cap \bar{X} \subset K.$$

Now, by the triangle inequality,

$$d(\bar{y}_0, y_j) \geq d(y_0, y_j) - d(y_0, \bar{y}_0) \geq T - H > \mathrm{diam}\,(K).$$

Thus, both y_1 and y_2 lie in $\bar{X} - K$ and can be joined by a curve τ lying in $\bar{X} - K$ by construction. Since $B(y_0, R) \cap \bar{X} \subset K$, the curve τ lies outside $B(y_0, R)$.

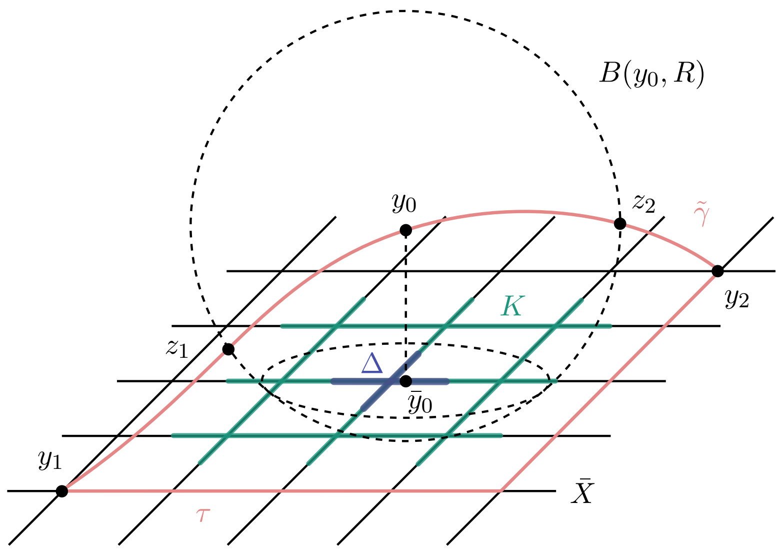

Figure 1.7: Sketch of the proof of Theorem 1.3.18.

Consider the intersection point z_j of $\tilde{\gamma}$ with the sphere $S(y_0, R) = \partial B(y_0, R)$ which is the closest to the point y_j for $j = 1, 2$. The points z_1 and z_2 must lie in the same connected component of $S(y_0, R)$, otherwise the concatenation $\tilde{\gamma} * \tau$ would have a nontrivial intersection number with the cycle $S(y_0, R)$, which is absurd since $\tilde{\gamma} * \tau$ is contractible. By Lemma 1.3.19 applied to $\bar{Z} = \{y_0\}$, $\beta = 1$ and $c < \frac{1}{3}$, we have

$$d(z_1, z_2) < 6c\,(d(x, p(y_0)) + R) \tag{1.11}$$

where $p : \tilde{M} \to M$ is the universal covering. Now, recall that $p(y_0)$ is at distance at most H from X by Lemma 1.3.20. Thus, the distance between $p(y_0)$ and the basepoint x of X is bounded uniformly in R. More precisely,

$$d(x, p(y_0)) \leq \mathrm{diam}\,(X) + H.$$

Since the curve $\tilde{\gamma}$ is minimising, we have $d(z_1, z_2) = 2R$. By taking R large enough in the inequality (1.11), we obtain a contradiction with $c < \frac{1}{3}$. □

Finally, we recover the main result of [RW10] under more general assumptions (that is, not just for closed manifolds whose universal cover has bounded fill radius).

Corollary 1.3.21. *Let M be a complete Riemannian manifold with finitely presented fundamental group. Denote by $\tilde{M}$ its Riemannian universal cover. Suppose the fill radius of $\tilde{M}$ has at most c-linear growth at infinity, with $c < \frac{1}{3}$. Then $\pi_1(M)$ is virtually free.*

Proof. We follow the argument of [RW10]. By Theorem 1.3.18, $\pi_1(M)$ does not contain finitely generated subgroups of exactly one end. Since $\pi_1(M)$ is finitely presented, then it is accessible by [Dun85]. Recall that a group is accessible if it is the fundamental group of a graph of groups such that every edge group is finite and every vertex group is at most one-ended. Finally, an accessible group without one-ended subgroups is virtually free, by a result of Serre [Ser80, Chapter 2, Section 2.6, Proposition 11]. □

Remark 1.3.22. In particular, if the fill radius of the universal cover of a complete Riemannian manifold with finitely presented fundamental group is uniformly bounded above, then its fundamental group is virtually free.

1.3.4 Fill radius and aspherical summands

Another consequence of Theorem 1.3.18, combined with Corollary 1.2.10, is that manifolds which decompose as the connected sum of a manifold with an aspherical summand do not admit complete metrics such that the Riemannian universal cover has at most c-linear growth at infinity with $c < \frac{1}{3}$.

Corollary 1.3.23. *Let P be a closed aspherical n-manifold and N be an arbitrary n-manifold with $n \geq 2$. Then the connected sum $M = P \# N$ does not admit any complete Riemannian metric such that the fill radius of $\tilde{M}$ has at most c-linear growth at infinity, with $c < \frac{1}{3}$.*

By [GL83], closed aspherical 3-manifolds do not support Riemannian metrics of positive scalar curvature. The same statement for any dimension was conjectured by Gromov [Gro86].

Conjecture 1.3.24. *A closed aspherical n-manifold does not admit any Riemannian metric with positive scalar curvature.*

Conjecture 1.3.24 was solved for $n \in \{4, 5\}$ by Chodosh–Li [CL24], and independently by Gromov [Gro20]. Gromov [Gro17, Section 4, Conjecture 12], [Gro23, Section 3.2] also conjectured a stronger version of Conjecture 1.3.24, involving the notion of $\mathbb{Q}$-essential manifold.

Definition 1.3.25. Let G be an abelian group. A closed n-manifold M is G-*essential* if the classifying map $f : M \to K(\pi_1(M), 1)$ induces a nontrivial homomorphism $f_* : H_n(M; G) \to H_n(K; G)$ in top dimensional G-homology, that is,

$$f_*[M] \neq 0 \in H_n(K; G).$$

A closed n-manifold M is *essential* if it is orientable and $\mathbb{Z}$-essential, or if it is non-orientable and $\mathbb{Z}_2$-essential.

Notice that any closed manifold M admitting a non-zero degree map to an essential manifold M' is also essential. In particular, the connected sum $M \# M'$ of any closed manifold M with an essential manifold M' is also essential.

Clearly, any closed aspherical manifold is $\mathbb{Q}$-essential, and any closed $\mathbb{Q}$-essential manifold is essential. However, it is not true that every essential manifold is $\mathbb{Q}$-essential.

Example 1.3.26. The real projective spaces $\mathbb{RP}^n$ are essential manifolds. Indeed, the inclusion map $i : \mathbb{RP}^n \to \mathbb{RP}^\infty$ into the $K(\mathbb{Z}_2, 1)$-space $\mathbb{RP}^\infty$ induces a non-trivial homomorphism in G-homology for $G = \mathbb{Z}$ when n is odd and $G = \mathbb{Z}_2$ when n is even. However, $\mathbb{RP}^n$ is not $\mathbb{Q}$-essential, since $H_n(\mathbb{RP}^\infty; \mathbb{Q}) = 0$ for every $n \in \mathbb{N}$.

More generally, the lens spaces $L_p(q_1, \ldots, q_n)$ are essential manifolds which are not $\mathbb{Q}$-essential. Recall that, given $p \in \mathbb{N}$ and a sequence $(q_j)_{j\in\mathbb{N}}$ of integers such that $(p, q_i) = 1$ for every $i \in \{1, \ldots, n\}$, the *lens space* $L_p(q_1, \ldots, q_n)$ is the $(2n-1)$-manifold obtained as the quotient of the $(2n-1)$-sphere $\mathbb{S}^{2n-1} = \{|z_1|^2 + \cdots + |z_n|^2 = 1\} \subset \mathbb{C}^n$ by the free action of $\mathbb{Z}_p$ generated by the rotation

$$(z_1, \ldots, z_n) \mapsto \left(e^{\frac{2\pi q_1}{p} i} z_1, \ldots, e^{\frac{2\pi q_n}{p} i} z_n\right).$$

Similarly, the *infinite dimensional lens space* $L_p^\infty(q_j)$ is defined as the quotient of the infinite dimensional sphere $\mathbb{S}^\infty = \{\sum_{j=1}^\infty |z_j|^2 = 1\} \subset \mathbb{C}^\infty$ by the free action of $\mathbb{Z}_p$ generated by

$$(z_j)_{j\in\mathbb{N}} \mapsto \left(e^{\frac{2\pi q_j}{p} i} z_j\right).$$

Notice that for any sequence of integers $(q_j)_{j\in\mathbb{N}}$, $L_2(q_1, \ldots, q_n) = \mathbb{RP}^{2n-1}$ and $L_2^\infty(q_j) = \mathbb{RP}^\infty$. Moreover, $L_p(1, q)$ corresponds to the 3-dimensional lens space $L_p(q)$ introduced in Section 1.1.2. The infinite dimensional lens space $L_p(q_j)$ has a CW-complex structure, and contains $L_p(q_1, \ldots, q_n)$ as its $(2n-1)$-skeleton, for every $n \in \mathbb{N}$. As in the case of projective spaces, the inclusion

$$i : L_p(q_1, \ldots, q_n) \to L_p^\infty(q_j)$$

induces a non-trivial homomorphism in $\mathbb{Z}$-homology. Since the universal cover of $L_p^\infty(q_j)$ is $\mathbb{S}^\infty$, which is a contractible space, we conclude that $L_p^\infty(q_j)$ is a $K(\mathbb{Z}_p, 1)$-space. Therefore, $L_p(q_1, \ldots, q_n)$ is essential, and it is not $\mathbb{Q}$-essential since $H_n(L_p^\infty(q_j); \mathbb{Q}) = 0$ for every $n \in \mathbb{N}$.

Gromov conjectured that $\mathbb{Q}$-essential closed manifolds do not support metrics of positive scalar curvature.

Conjecture 1.3.27 ([Gro17, Gro23])**.** *A closed $\mathbb{Q}$-essential n-manifold does not admit any Riemannian metric with positive scalar curvature.*

The results in Section 1.3.3 allow us to prove a weaker version of Conjecture 1.3.27 involving the fill radius.

Proposition 1.3.28. *Let M be a closed $\mathbb{Q}$-essential n-manifold with $n \geq 2$. Then M does not admit any Riemannian metric such that the fill radius of its universal cover $\tilde{M}$ is bounded.*

Proof. Suppose that $\tilde{M}$ has bounded fill radius. Then Corollary 1.3.21 implies that the fundamental group G of M is virtually free. This means that there is a finite covering $p : N \to M$ of nonzero degree k such that the fundamental group $F = \pi_1(N)$ is free. Consider the classifying map $f : M \to K(G, 1)$. We can lift the map f to obtain the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\bar{f}} & K(F,1) \\ {\scriptstyle p}\downarrow & & \downarrow \\ M & \xrightarrow{f} & K(G,1). \end{array}$$

The corresponding commutative diagram induced in n-dimensional rational homology is

$$\begin{array}{ccc} H_n(N;\mathbb{Q}) & \xrightarrow{\bar{f}_*} & H_n(F;\mathbb{Q}) \\ {\scriptstyle p_*}\downarrow & & \downarrow \\ H_n(M;\mathbb{Q}) & \xrightarrow{f_*} & H_n(G;\mathbb{Q}). \end{array}$$

Since F is a free group, its classifying space $K(F,1)$ is homotopy equivalent to a graph, and therefore $H_n(F;\mathbb{Q}) = 0$. In particular $\bar{f}_*[N] = 0$. But, on the other hand, we have

$$(f \circ p)_*[N] = k f_*[M],$$

which implies $f_*[M] = 0$. Therefore, M is not $\mathbb{Q}$-essential. □

It follows from Proposition 1.3.28 and Theorem 1.3.3 that $\mathbb{Q}$-essential closed 3-manifolds, and more generally 3-manifolds containing a closed $\mathbb{Q}$-essential term in their prime decomposition, do not admit Riemannian metrics with positive scalar curvature, result that was already proved in [GL83].

Remark 1.3.29. It is not true that a closed manifold whose Riemannian universal cover has bounded fill radius is not essential. Indeed, $\mathbb{R}\mathrm{P}^n$ is essential (but not $\mathbb{Q}$-essential), and the fill radius of its universal cover is clearly bounded (for any metric). More generally, the latter holds for all lens spaces.

1.4 Simply connectedness at infinity on contractible curves and exhaustion by compact domains with incompressible boundaries

In Section 1.3.2 we proved that Riemannian manifolds whose universal cover has at most c-linear growth at infinity, for some $c < 1$, must be simply connected at infinity on contractible curves. In this section, we derive two topological consequences of the simply connectedness at infinity on contractible curves.

1.4.1 Localised compression of surfaces

A consequence of the Loop Theorem 1.1.12 is that a compressible surface Σ can be compressed into an incompressible surface, see Section 1.1.4. When the ambient manifold is simply connected at infinity on contractible curves, such a compression may be performed in a localised manner.

Proposition 1.4.1. *Let M be an orientable 3-manifold. Suppose M is simply connected at infinity on contractible curves. Let $B \subset M$ be a compact subset, and consider a compact subset $K \subset M$ containing B such that*

$$\ker(\pi_1(M-K) \to \pi_1(M)) = \ker(\pi_1(M-K) \to \pi_1(M-B)). \tag{1.12}$$

Let $\Sigma \subset M-K$ be a compressible embedded orientable surface. Then Σ can be compressed into an incompressible surface $\Sigma' \subset M-B$ so that the surface obtained at each step of the compression is contained in $M-B$.

Proof. Suppose that $\Sigma' \subset M-B$ is the result of compressing Σ a number k of times. One can write Σ' as the union of $\Sigma_0 = \Sigma' \cap \Sigma$, the part corresponding to the original surface Σ, which is contained in $M-K$, and some embedded discs $D_1^{\pm}, \dots, D_k^{\pm} \subset M-B$ glued to Σ_0 during the former compressions.

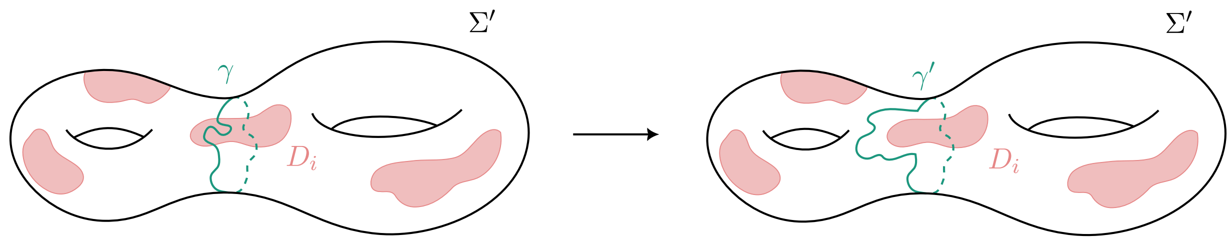

Figure 1.8: Deforming a loop.

Now, take a closed curve $\gamma \subset \Sigma$ representing a nontrivial element of $\ker(\pi_1(\Sigma') \to \pi_1(M))$. If γ intersects some disc $D_i^{\pm}$, one can homotope every arc of $\gamma \cap D_i^{\pm}$ to the boundary $\partial D_i^{\pm}$, and push them slightly beyond, so that the resulting curve avoids $D_i^{\pm}$; see Figure 1.8. Repeating the procedure for each disc, the closed curve γ can be homotoped to a closed curve γ' lying in Σ_0. Now, since γ' is a contractible loop contained in $\Sigma_0 \subset M - K$, the loop γ' is already contractible in $M - B$ by the relation (1.12). We obtain

$$\ker(\pi_1(\Sigma') \to \pi_1(M)) = \ker(\pi_1(\Sigma') \to \pi_1(M - B)).$$

Now, we conclude by applying the Loop Theorem to the surface Σ' contained in $M - B$, which gives an embedded disc $D \subset M - B$ such that $\Sigma' \cap D = \partial D$ is a closed curve representing a nontrivial element in $\ker(\pi_1(\Sigma') \to \pi_1(M - B))$. Thus, the result of the compression of Σ' along D is again contained in $M - B$. □

1.4.2 Exhaustion by compact domains with incompressible boundaries

Using Proposition 1.4.1, we will prove that Riemannian manifolds which are simply connected at infinity may be decomposed into compact domains along incompressible surfaces.

Proposition 1.4.2. *Let M be a complete orientable Riemannian 3-manifold. Suppose that M is simply connected at infinity on contractible curves. Then M admits an exhaustion by compact connected domains*

$$K_1 \subset K_2 \subset \cdots \subset M$$

with $K_i \subset \mathring{K}_{i+1}$ and $M = \cup_i K_i$, such that $\cup_i \partial K_i$ is a locally finite disjoint collection of embedded orientable closed incompressible surfaces.

Proof. Let $x \in M$ and consider the singleton $K_0 = \{x\}$. Suppose we have constructed $K_0 \subset \cdots \subset K_i$, with $K_j \subset U(K_j, 1) \subset K_{j+1}$, where $U(K_j, 1)$ is the closed 1-neighbourhood of K_j (see equation (1.1)), such that ∂K_j is a closed incompressible surface. Take a closed metric ball B containing $U(K_i, 1)$. Since M is simply connected at infinity on contractible curves, there is a compact subset K containing B such that

$$\ker(\pi_1(M - K) \to \pi_1(M)) = \ker(\pi_1(M - K) \to \pi_1(M - B)).$$

Now, take a metric ball B' containing K. After approximating the distance function $d(x, \cdot)$ by a smooth function and slightly perturbing the radius of B', we can suppose that the boundary of B' is an embedded orientable surface Σ. Since the surface Σ lies in $M - K$, we can use Proposition 1.4.1 to compress it inside $M - B$ into an incompressible surface Σ' (if Σ is already incompressible, we have $\Sigma' = \Sigma$). Since $B \subset B'$ and Σ' is homologous to Σ, the surface Σ' encloses a compact connected component K_{i+1} containing B. The boundary ∂K_{i+1} of K_{i+1} lies in Σ', and thus, is incompressible.

Finally, since $K_i \subset U(K_i, 1) \subset B \subset K_{i+1}$, we conclude that the compact subset K_{i+1} contains the closed metric ball $B(x, i+1)$. Hence, $\cup_i K_i = M$. □

1.5 The Decomposition Theorem

As discussed in Section 1.3.1, using Theorem 1.3.7, Theorem A can be proven under a weaker hypothesis on the growth at infinity of the fill radius of the universal cover, which generalises the scalar curvature decay assumption.

Theorem 1.5.1. *Let M be an orientable complete Riemannian 3-manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Suppose that the fill radius of $\tilde{M}$ has at most c-linear growth at infinity for some $c < \frac{1}{3}$. Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$.*

1.5.1 The proof of the Decomposition Theorem

We finally prove Theorem 1.5.1.

Proof of Theorem 1.5.1. By Proposition 1.4.2, consider an exhaustion of M by compact domains $K_1 \subset K_2 \subset \cdots \subset M$, whose boundaries form a locally finite collection of orientable closed connected incompressible surfaces, denoted by $\{\Sigma_\alpha\}$. Since each surface is incompressible, $\pi_1(\Sigma_\alpha)$ is a finitely generated subgroup of $\pi_1(M)$, which cannot have exactly one end by Theorem 1.3.18. But since $\pi_1(\Sigma_\alpha)$ is a surface group associated to a closed orientable surface, the only possibility is that Σ_α is a 2-sphere.

The result of cutting M along the collection of 2-spheres $\{\Sigma_\alpha\}$ consists of the connected components Y_{ij} of the pieces $Y_i = \overline{K_i - K_{i-1}}$. Consider $\hat{Y}_{ij}$ the result of capping the spherical boundary components of Y_{ij} off by 3-balls. By the Kneser–Milnor Decomposition Theorem 1.1.4, each $\hat{Y}_{ij}$ decomposes as a connected sum of prime closed 3-manifolds, see Section 1.1.1. Namely, there is a finite collection of disjoint spheres $\left\{\Sigma_{\beta_{ij}}\right\}$, which can be taken inside $\operatorname{int}(Y_{ij})$, such that $M - (\cup_\alpha \Sigma_\alpha \cup_{\beta_{ij}} \Sigma_{\beta_{ij}})$ consists of the disjoint union of prime manifolds with some punctures, see Figure 1.9. Denote each of these punctured prime manifolds together with their boundary spheres by P_{ijk}, and the result of capping their spherical boundary components off by $\hat{P}_{ijk}$.

Now, construct a locally finite coloured graph $(\mathcal{G}, f)$ as follows. Consider the countable family of manifolds $\mathcal{F} = \{M_l\}_{l \in I}$ satisfying that each M_l is homeomorphic to exactly one $\hat{P}_{ijk}$. Clearly, $\mathcal{F}$ is a subfamily of the family $\mathcal{P}$ of closed oriented prime 3-manifolds. The graph $\mathcal{G}$ has a vertex v for each prime manifold P_{ijk}, for which we set $f(v) = l$ if $\hat{P}_{ijk}$ is homeomorphic to M_l. The graph $\mathcal{G}$ has an edge e joining a pair of vertices v, v' for each sphere in the common boundary $\partial P_v \cap \partial P_{v'}$, where P_v and $P_{v'}$ are the prime manifolds corresponding to v and v'.

By construction, the manifold M decomposes as a (possibly infinite) connected sum over $\mathcal{F}$ along the locally finite coloured graph $(\mathcal{G}, f)$, see Section 1.1.3. By Theorem 1.1.3, each closed prime 3-manifold in $\mathcal{F}$ is either spherical, aspherical or homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. By Corollary 1.3.23, the family $\mathcal{F}$ cannot contain aspherical 3-manifolds. Therefore, M is homeomorphic to a (possibly infinite) connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ modelled on the coloured graph $(\mathcal{G}, f)$. □

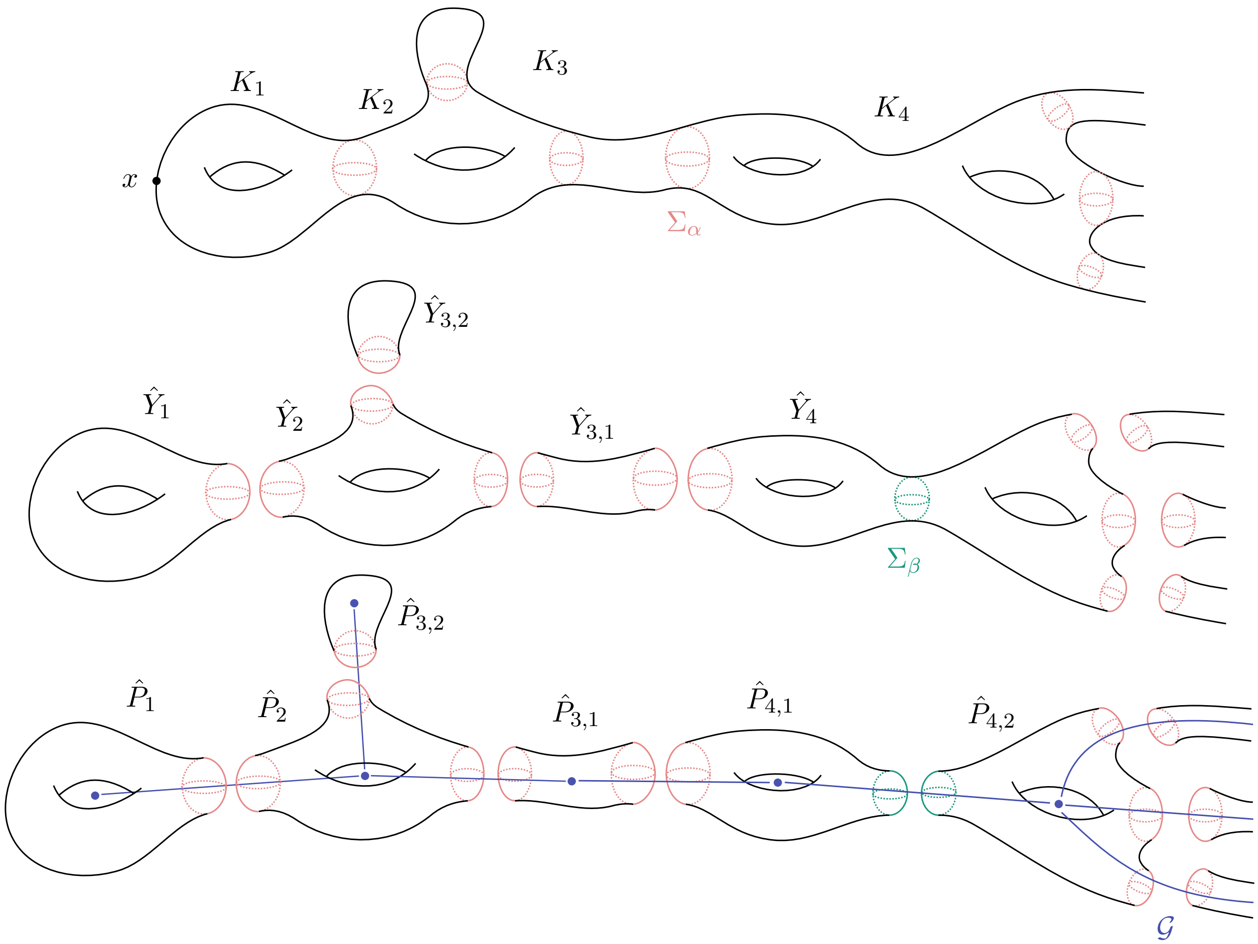

Figure 1.9: Prime decomposition of M along the locally finite graph $\mathcal{G}$.

1.5.2 An indecomposable 3-manifold of positive scalar curvature with quadratic decay

As observed in [Gro23, Section 3.10.2], the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ admits a complete Riemannian metric of positive scalar curvature with a C-quadratic decay for a constant $C < 64\pi^2$. Since it does not decompose as a connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$, this shows the optimality of the decay rate in Theorem A.

Proposition 1.5.2 ([Gro23, Section 3.10.2])**.** *The manifold $\mathbb{R}^2 \times \mathbb{S}^1$ admits a complete Riemannian metric of positive scalar curvature with $\frac{1}{2}$-quadratic decay at infinity.*

Proof. Take polar coordinates (r, θ) on the first factor $\mathbb{R}^2$, and consider the rotationally invariant product metric

$$g = dr^2 + f(r)^2 d\theta^2 + dt^2.$$

By definition of the scalar curvature, the scalar curvature of g is twice the Gauss curvature of $(\mathbb{R}^2, dr^2 + f(r)^2 d\theta^2)$. Since the metric is rotationally invariant, its scalar curvature is given by (see [GHL04, 3.50, p.147], for instance)

$$\mathrm{scal} = -2\,\frac{f''}{f}.$$

Now, consider the function

$$f(r) = \begin{cases} \sin(r) & \text{if } r \in [0, \frac{\pi}{4}] \\ \varphi(r) & \text{if } r \in (\frac{\pi}{4}, 2) \\ \sqrt{r} & \text{if } r \in [2, +\infty) \end{cases}$$

where φ is a concave smooth interpolation between $\sin(r)$ and $\sqrt{r}$. For this particular function, the scalar curvature is

$$\mathrm{scal} = \begin{cases} 2 & \text{if } r \in [0, \frac{\pi}{4}] \\ -2\,\frac{\varphi''}{\varphi} & \text{if } r \in (\frac{\pi}{4}, 2) \\ \frac{1}{2}\frac{1}{r^2} & \text{if } r \in [2, +\infty) \end{cases}$$

In particular, the scalar curvature is positive with exactly quadratic decay at infinity. □

Proposition 1.5.3. *The manifold $\mathbb{R}^2 \times \mathbb{S}^1$ does not decompose as a connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$.*

Proof. Suppose that $\mathbb{R}^2 \times \mathbb{S}^1$ is homeomorphic to a connected sum M modelled on a locally finite graph $\mathcal{G}$ of closed prime manifolds P_i, each of them homeomorphic to some spherical manifold or to $\mathbb{S}^2 \times \mathbb{S}^1$. Then, Lemma 1.1.5 implies

$$\pi_1(M) \simeq *_i\, \pi_1(P_i) * \pi_1(\mathcal{G}).$$

We can always add $\mathbb{S}^2 \times \mathbb{S}^1$ summands to the connected sum to turn $\mathcal{G}$ into a locally finite tree. Hence, $\pi_1(M) \simeq *_i\pi_1(P_i)$. However, $\pi_1(\mathbb{R}^2 \times \mathbb{S}^1) \simeq \mathbb{Z}$ is torsion free, so, after permuting the summands in the connected sum, one can assume that $P_1 \simeq \mathbb{S}^2 \times \mathbb{S}^1$ and $P_i \simeq \mathbb{S}^3$ for $i \geq 2$. Notice that, since $\mathbb{R}^2 \times \mathbb{S}^1$ has one end, the tree $\mathcal{G}$ must have one end as well. So, after removing some $\mathbb{S}^3$ terms, the tree $\mathcal{G}$ can be assumed to be homeomorphic to a half-line. Since an infinite connected sum of 3-spheres modelled on a half-line graph is homeomorphic to $\mathbb{R}^3$, the manifold M is homeomorphic to $(\mathbb{S}^2 \times \mathbb{S}^1)\#\mathbb{R}^3$, which is homotopically equivalent to the wedge sum $\mathbb{S}^2 \vee \mathbb{S}^1$. We obtain a contradiction with $\pi_2(\mathbb{R}^2 \times \mathbb{S}^1) = 0$. □

Remark 1.5.4. In fact, the product $\mathbb{R}^2 \times \mathbb{S}^1$ is a prime (non-compact) 3-manifold, since it satisfies $\pi_2(\mathbb{R}^2 \times \mathbb{S}^1) = 0$.

As for the optimal value of the decay constant C under which the conclusion of Theorem A holds, Propositions 1.5.2 and 1.5.3 show that we cannot hope for more than $C > \frac{1}{2}$ (while Theorem A holds for $C > 64\pi^2$). More generally, Gromov conjectured the following [Gro23, Section 3.6.1].

Conjecture 1.5.5 (Critical Rate of Decay Conjecture [Gro23])**.** *There exists a dimensional constant $C_n > 0$ such that the following holds. Let M be an orientable n-manifold that admits a complete Riemannian metric of positive scalar curvature.*

1. *For every $C < C_n$, there exists a complete Riemannian metric on M of positive scalar curvature with at most C-quadratic decay at infinity.*

2. *If M admits a complete Riemannian metric with positive scalar curvature with C-quadratic decay at infinity for $C > C_n$, then M admits a complete Riemannian metric with uniformly positive scalar curvature.*

1.6 The Surgery Theorem

In this section, we prove that a complete Riemannian manifold of positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$ admits a complete Riemannian metric with uniformly positive scalar curvature, see Corollary B. This rigidity result addresses the case (2) of Conjecture 1.5.5.

By Theorem A, Corollary B follows from the following result.

Theorem 1.6.1. *Let M be an orientable complete Riemannian 3-manifold which decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$. Then M admits a complete Riemannian metric of uniformly positive scalar curvature.*

The proof of Theorem 1.6.1 is based on the following local construction, which provides a control on the lower bound of the scalar curvature. This construction is key to the proof of Gromov–Lawson's Surgery Theorem [GL80b, Theorem A], which states that the connected sum of two closed 3-manifolds of positive scalar curvature admits a metric of positive scalar curvature, compare [Gro23, Section 1.3].

Theorem 1.6.2 ([GL80b])**.** *Let (M, g) be a complete Riemannian 3-manifold with uniformly positive scalar curvature* $\mathrm{scal} \geq s > 0$. *Fix $x \in M$. Let $R \leq \min\left\{\frac{1}{2}\,\mathrm{inj}_M(x), 1\right\}$, where $\mathrm{inj}_M(x)$ denotes the injectivity radius of M at x. Let $\alpha \in (0, 1)$ be a constant. Then there exist a radius $r \in (0, R)$ and a Riemannian metric g' on the punctured ball $B(x, R) - \{x\}$ satisfying the following properties:*

1. *the new Riemannian metric g' coincides with g on $\partial B(x, R)$;*
2. *the punctured ball $(B(x, r) - \{x\}, g')$ is isometric to the standard Riemannian cylinder $\mathbb{S}^2(r) \times \mathbb{R}$;*
3. *the scalar curvature of g' satisfies* $\mathrm{scal}_{g'} \geq \alpha s > 0$.

Let us prove Theorem 1.6.1.

Proof of Theorem 1.6.1. Let $(\mathcal{G}, f)$ be a locally finite coloured graph such that M decomposes as a connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ along $(\mathcal{G}, f)$. For each vertex v of $\mathcal{G}$, endow the corresponding manifold $M_{f(v)}$ with a metric of $\mathrm{scal} \geq s > 0$. For every edge e joining two vertices v^+ and v^- of $\mathcal{G}$, let (x_e^+, x_e^-) be a pair of points such that $x_e^\pm \in M_{f(v^\pm)}$. For every edge e of $\mathcal{G}$, let $R_e > 0$ be a radius satisfying the following two conditions:

1. $R_e \leq \min\left\{\frac{1}{2}\,\mathrm{inj}_{M_{f(v^+)}}(x_e^+), \frac{1}{2}\,\mathrm{inj}_{M_{f(v^-)}}(x_e^-), 1\right\}$,
2. the balls $B(x_e^\pm, R_e)$ are pairwise disjoint.

Now, fix $\alpha \in (0, 1)$. For each edge e of $\mathcal{G}$, let $r_e \in (0, R_e)$ and $(g_e^\pm)'$ be the radius and the Riemannian metric on $B(x_e^\pm, R_e)$ given by Theorem 1.6.2. For each edge e, glue $(B(x_e^+, R_e) - B(x_e^+, r_e), (g_e^+)')$ with $(B(x_e^-, R_e) - B(x_e^-, r_e), (g_e^-)')$ by identifying their inner boundaries, both of which are isometric to the round sphere $\mathbb{S}^2(r_e)$ of radius r_e. The resulting Riemannian manifold (M', g') is homeomorphic to M by construction, and by Theorem 1.6.2, the Riemannian metric g' has uniformly positive scalar curvature $\mathrm{scal}_{g'} \geq \alpha s > 0$. □

Remark 1.6.3. The same strategy was used in [BBMC21], where the authors proved that an orientable 3-manifold admits a complete Riemannian metric of positive scalar curvature and bounded geometry if and only if the manifold decomposes as a possibly infinite connected sum of spherical 3-manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ with finitely many summands up to homeomorphism.

Another consequence of Theorems 1.6.1 and 1.6.2 is the following.

Corollary 1.6.4. *Let M be a complete Riemannian 3-manifold with (uniformly) positive scalar curvature and let $x \in M$ be a point. Then the punctured manifold $M - \{x\}$ admits a complete Riemannian metric with (uniformly) positive scalar curvature.*

Proof. The punctured manifold $M - \{x\}$ is homeomorphic to the connected sum $M \# \mathbb{R}^3$. Notice that $\mathbb{R}^3$ decomposes as an infinite connected sum of 3-spheres $\mathbb{S}^3$ along the halfline $[0, \infty)$ with vertices at the integer points. Hence, by Theorem 1.6.1, the space $\mathbb{R}^3$ can be endowed with a metric of uniformly positive scalar curvature. If M has (uniformly) positive scalar curvature, using Theorem 1.6.2 as in the proof of Theorem 1.6.1, one can construct a metric of (uniformly) positive scalar curvature on $M \# \mathbb{R}^3 \simeq M - \{x\}$. □

Chapter 2

Systolic geometry and positive macroscopic scalar curvature

In this chapter we will explore the systolic geometry of manifolds subject to lower bounds on their macroscopic scalar curvature, with the aim of generalising Bray–Brendle–Neves' Systolic Inequality 2.2.3 to the macroscopic setting. The main result proven in this chapter is Theorem D, which states that a complete Riemannian n-manifold with non-trivial codimension 1 homology (with $\mathbb{Z}_2$-coefficients or $\mathbb{Z}$-coefficients) and with positive macroscopic scalar curvature large enough must contain a non-nullhomologous hypersurface of small Urysohn $(n-2)$-width. Besides, we will present the details of the proofs of Propositions C and E.

We will begin this second chapter by introducing the notion of macroscopic scalar curvature Section 2.1 and discussing its main properties. In Section 2.2, we provide a brief overview of systolic geometry and prove Proposition C. The rigorous definition of the Urysohn width will be presented in Section 2.3, where we will also discuss its relation to other metric invariants and scalar curvature. Section 2.4 is dedicated to the Coarea Formula, while in Section 2.5 we present a macroscopic analogue of Schoen–Yau's Stability Inequality (4), both of which are central ingredients to the proof of Theorem D. In Section 2.6 we prove Theorem D. Finally, in Section 2.7 we give an explicit construction showing Proposition E.

2.1 Macroscopic scalar curvature

It is a classical fact that the scalar curvature of a Riemannian n-manifold M can be defined in terms of volumes of small geodesic balls [GHL04, Theorem 3.98]. More precisely, the volume of a geodesic ball $B(x,r)$ centered at a point $x \in M$ obeys the expansion:

$$|B(x,r)| = b_n r^n \left(1 - \frac{\operatorname{scal}(x)}{6(n+2)} r^2 + O(r^3)\right), \tag{2.1}$$

for $r > 0$ small enough. Recall that b_n denotes the volume of the Euclidean n-dimensional unit ball.

As Gromov pointed out in his Four Lectures [Gro23], despite its more geometrical flavour, the definition of the scalar curvature in terms of the volume of infinitesimal balls is of little practical use. This is due to the fact that the estimate in equation (2.1) is of infinitesimal nature, and does not provide any estimate on the volume of a ball of fixed radius $R > 0$.

In order to illustrate this phenomenon, it is enlightening to compare estimates arising from a lower bound on the scalar curvature with analogue estimates coming from lower bounds on the Ricci curvature. Let M be a Riemannian n-manifold. Denote by $V^n_s(R)$ the volume of any ball of radius R in the simply connected n-dimensional space form of constant scalar curvature s. By equation (2.1), if the scalar curvature of M satisfies $\mathrm{scal} > s$ for some $s \in \mathbb{R}$, then for every $x \in M$ there is a small radius $r > 0$ (depending on x) such that

$$|B(x,r)| \leq V^n_s(r). \tag{2.2}$$

On the other hand, it is well-known that a lower bound $\mathrm{Ric} > s/n$, with $s \in \mathbb{R}$, implies by the Bishop-Gromov Inequality [Bis63, Gro07] (see also [Ber03, Theorem 107]) that for any point $x \in M$ and any radius $R > 0$ one has

$$|B(x,R)| \leq V^n_s(R).$$

Hence, a lower bound on the Ricci curvature gives a global estimate on the volume of balls, holding at any scale, which can be effectively used to study the geometry and the topology of M. In comparison, the volume estimate arising from a lower bound on the scalar curvature is infinitesimal, and therefore it is extremely difficult, if not impossible, to get global information from it.

Motivated by the estimate in equation (2.2), Guth [Gut10a] introduced a macroscopic version of scalar curvature, known as the macroscopic scalar curvature, which is defined through the volume of geodesic balls of a fixed radius.

Before giving the definition of the notion of macroscopic scalar curvature, let us fix some notation. Let $\mathbb{M}^n_\sigma$ denote the simply connected n-dimensional space form of constant sectional curvature σ. By the Killing–Hopf Theorem [GHL04, Theorem 3.82], if $\sigma > 0$, $\sigma = 0$ or $\sigma < 0$ then $\mathbb{M}^n_\sigma$ is, up to rescaling of the metric, isometric to the round n-sphere, the Euclidean n-space or the hyperbolic n-space, respectively. The space form $\mathbb{M}^n_\sigma$ has constant scalar curvature equal to $s = n(n-1)\sigma$. Define the radius ρ of $\mathbb{M}^n_\sigma$ to be $\rho = 1/\sqrt{|\sigma|}$. Recall that $V^n_s(R)$ denotes the volume of any ball of radius R in the space form $\mathbb{M}^n_\sigma$ of constant sectional curvature $\sigma = s/n(n-1)$. We will denote by

$$b_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

the n-dimensional volume of the unit ball of $\mathbb{R}^n$, where Γ denotes the Gamma function, and by $w_n = (n+1)b_{n+1}$ the n-dimensional volume of the unit n-sphere.

The quantity $V^n_s(R)$ can be expressed explicitely in terms of $s \in \mathbb{R}$, the radius $R > 0$ and the dimension n.

Lemma 2.1.1 ([GHL04, Section 3.H.3])**.** *Let $s \in \mathbb{R}$ and $R > 0$. Then,*

$$V^n_s(R) = \begin{cases} w_{n-1}\displaystyle\int_0^R \left(\frac{\sin(\sqrt{\sigma}t)}{\sqrt{\sigma}}\right)^{n-1} dt, & \text{if } s > 0 \text{ and } R < \pi\rho \\ w_n\rho^n, & \text{if } s > 0 \text{ and } R \geq \pi\rho \\ b_n R^n, & \text{if } s = 0 \\ w_{n-1}\displaystyle\int_0^R \left(\frac{\sinh(\sqrt{-\sigma}t)}{\sqrt{-\sigma}}\right)^{n-1} dt, & \text{if } s < 0 \end{cases}.$$

From the explicit form of Lemma 2.1.1, we obtain the following corollary.

Corollary 2.1.2. *The function $V^n_s(R)$ satisfies the following properties.*

1. *Let $R > 0$ and $s \in \mathbb{R}$. Then, for any $\lambda > 0$,*

$$V^n_s\left(\frac{R}{\lambda}\right) = \frac{1}{\lambda^n} V^n_{s/\lambda^2}(R).$$

In particular, $V^n_s(R) = V^n_{sR^2}(1)R^n$.

2. *At every fixed scale $R > 0$, $s \mapsto V_s^n(R)$ is a strictly decreasing function (see Figure 2.1), which verifies*

$$\lim_{s\to-\infty} V_s^n(R) = +\infty \text{ and } \lim_{s\to+\infty} V_s^n(R) = 0.$$

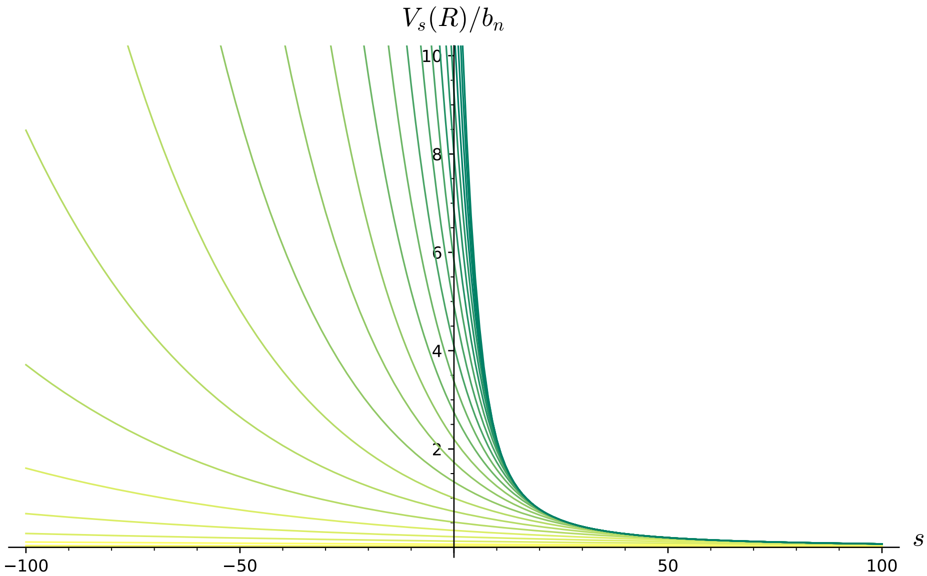

Figure 2.1: The function $s \mapsto V_s^n(R)/b_n$ for $n = 3$ and $R = 0.1, 0.2, \ldots, 2.5$.

Let us now define the notion of macroscopic scalar curvature, following [Gut10a].

Definition 2.1.3. The *macroscopic scalar curvature* $\operatorname{mscal}(x, R)$ of a Riemannian n-manifold M at a point $x \in M$ and scale $R > 0$ is the unique $s \in \mathbb{R}$ such that

$$\left|B_{\tilde{M}}(\tilde{x}, R)\right| = V_s^n(R),$$

where $\tilde{x}$ is a lift of x to the Riemannian universal cover $\tilde{M}$ of M. Equivalently, the macroscopic scalar curvature at a point $x \in M$ satisfies $\operatorname{mscal}(x, R) \geq s$ if and only if

$$\left|B_{\tilde{M}}(\tilde{x}, R)\right| \leq V_s^n(R). \tag{2.3}$$

By Corollary 2.1.2 (2), the macroscopic scalar curvature is well defined. The macroscopic scalar curvature satisfies the following properties.

Proposition 2.1.4. *Let (M, g) be a Riemannian n-manifold. Fix a point $x \in M$ and a scale $R > 0$.*

1. *For any $\lambda > 0$, $\operatorname{mscal}_{\lambda^2 g}(x, R) = \dfrac{1}{\lambda^2} \operatorname{mscal}_g(x, \frac{R}{\lambda})$.*

2. *If M is a Riemannian-flat manifold, then $\operatorname{mscal}(x, R) = 0$ at any scale $R > 0$.*

3. $\lim_{R\to 0} \operatorname{mscal}(x, R) = \operatorname{scal}(x)$.

Notice that, taking the infinitesimal limit, one recovers the classical scalar curvature. In the asymptotic limit, the macroscopic scalar curvature is related with another Riemannian invariant, the volume entropy [Man79].

In [Gut10a], Guth conjectured macroscopic versions of some deep results and open questions involving scalar curvature. For instance, Guth conjectured the Macroscopic Geroch Conjecture, stating that the n-torus $\mathbb{T}^n$ does not admit any Riemannian metric such that $\operatorname{mscal}(x, R) > 0$ at any point $x \in \mathbb{T}^n$ and any scale $R > 0$. Guth also asked about a macroscopic version of Schoen's Conjecture [Sch89], which was later addressed in [Gut11, Kar15, BK19, Sab22].

Guth's macroscopic approach to the study of scalar curvature consequently gave rise to a series of works involving the macroscopic scalar curvature [Gut10b, AF17, BS21, Alp22, Sab22, ABG24]. In fact, some of these papers explore variations of the notion of macroscopic scalar curvature originally introduced by Guth and presented in Definition 2.1.3. For instance, in [ABG24] the authors work with a condition of positive macroscopic scalar curvature defined through condition 2.3 along with an additional acyclicity hypothesis, resembling the bounded fill radius condition of Theorem 1.3.3 presented in Chapter 1.

2.2 Systolic geometry

Let M be a non-simply connected Riemannian manifold. The systole of M is the length of the shortest non-contractible closed curve on M. Since it is defined in terms of closed curves representing non-trivial elements of $\pi_1(M)$, we will denote the systole of M by $\operatorname{sys}\pi_1(M)$. The origins of systolic geometry date back to 1949, when Loewner (unpublished, see [Pu52]) proved that, for any Riemannian metric g on the 2-torus $\mathbb{T}^2$, the systole of $(\mathbb{T}^2, g)$ satisfies

$$\operatorname{sys}\pi_1(\mathbb{T}^2, g)^2 \leq \frac{2}{\sqrt{3}} \operatorname{area}(\mathbb{T}^2, g), \tag{2.4}$$

where $\operatorname{area}(\mathbb{T}^2, g)$ denotes the area of $(\mathbb{T}^2, g)$. Inequality (2.4) was the first example of a systolic inequality. A closed n-manifold M satisfies a *systolic inequality* if there exists a constant $C > 0$ such that for any Riemannian metric g on M, the systole of (M, g) satisfies

$$\operatorname{sys}\pi_1(M, g)^n \leq C \operatorname{vol}(M, g).$$

Notice that the constant C may depend on the manifold M, but should be independent of the Riemannian metric g. If for a given manifold M, such a constant does not exist, we say that M exhibits *systolic freedom*.

Loewner's Systolic Inequality (2.4) was followed by the corresponding systolic inequalities for other surfaces [Pu52, Bla61, Bav86, Heb82, BZ88, Gro83]. Gromov [Gro83] was the first to address the case of manifolds of higher dimension, and he proved the existence of non-trivial systolic inequalities for essential manifolds. Later, Babenko [Bab00] showed that essential manifolds are precisely those manifolds who admit non-trivial systolic inequalities.

Another possible generalisation of Loewner's Systolic Inequality (2.4) consists in considering higher dimensional analogues of the systole $\operatorname{sys}\pi_1(M)$. The higher dimensional systoles were first considered by Berger [Ber72].

Definition 2.2.1. Let M be a Riemannian n-manifold with $\pi_k(M) \neq 0$ for some $k \in \{1, \ldots, n-1\}$. The *homotopical k-systole* of M is defined as

$$\operatorname{sys}\pi_k(M) := \inf\left\{|\Sigma| \mid \Sigma \subset M \text{ immersed } k\text{-sphere such that } [\Sigma] \neq 0 \in \pi_k(M)\right\},$$

where $|\Sigma|$ denotes the k-dimensional volume of the k-sphere Σ.

One can also consider the homological analogue of the notion of systole.

Definition 2.2.2. Let M be a Riemannian n-manifold with $H_k(M;\mathbb{Z}) \neq 0$ for some $k \in \{1, \dots, n-1\}$. The *homological k-systole* of M is defined as

$$\operatorname{sys} H_k(M) := \inf\left\{|\Sigma| \mid \Sigma \subset M \text{ immersed } k\text{-submanifold such that } [\Sigma] \neq 0 \in H_k(M;\mathbb{Z})\right\},$$

where $|\Sigma|$ denotes the k-dimensional volume of the submanifold Σ in M.

In higher dimensions, one would be interested in determining whether a given n-manifold M satisfies an *intersystolic inequality*, that is, if a certain product of k-systoles, for $k \in \{1, \dots, n-1\}$, admits an upper bound in terms of a power of the volume, for any Riemannian metric g on M. Unfortunately, a large number of examples of manifolds have been proven to exhibit intersystolic freedom, see [Ber93, Gro96] for a discussion on the different results on intersystolic inequalities.

One way of overcoming intersystolic freedom is to consider smaller classes of Riemannian metrics. For instance, one could consider Riemannian metrics with a lower bound on some curvature. Regarding lower bounds on the scalar curvature, Bray–Brendle–Neves proved the following result.

Theorem 2.2.3 (Bray–Brendle–Neves' Isosystolic Inequality [BBN10])**.** *Let M be a closed Riemannian 3-manifold with $\pi_2(M) \neq 0$. Suppose that* $\operatorname{scal} \geq s > 0$. *Then*

$$\operatorname{sys} \pi_2(M) \leq \frac{8\pi}{s}.$$

Moreoever, equality holds if and only if the Riemannian universal cover of M is isometric to the standard Riemannian cylinder $\mathbb{S}^2(1) \times \mathbb{R}$.

We refer the reader to the Introduction for an overview on recent results and generalisations related to Theorem 2.2.3.

We are interested in obtaining analogous estimates on systolic quantities arising from a positive lower bound on the macroscopic scalar curvature. As stated in Proposition C, lower bounds on the macroscopic scalar curvature do not necessarily imply an upper bound on the intermediate systoles. Proposition C follows from the following proposition after appropriately rescaling the Riemannian metric.

Proposition 2.2.4. *Let $n \geq 3$ and $k \in \{2, \dots, n-1\}$. For every $s > 0$, there is a family of product metrics $(g_\varepsilon)_{\varepsilon \in (0,1)}$ on $\mathbb{S}^k \times \mathbb{S}^{n-k}$ such that the following holds.*

1. *For any point $x \in \mathbb{S}^k \times \mathbb{S}^{n-k}$ and any scale $R > 0$, one has*

$$\lim_{\varepsilon \to 0} \operatorname{mscal}_{g_\varepsilon}(x, R) = \infty.$$

2. *The homotopical and the homological k-systoles verify*

$$\operatorname{sys} \pi_k(\mathbb{S}^k \times \mathbb{S}^{n-k}, g_\varepsilon) = \operatorname{sys} H_k(\mathbb{S}^k \times \mathbb{S}^{n-k}, g_\varepsilon) = w_k$$

for every $\varepsilon \in (0,1)$, where w_k is the k-dimensional volume of the round k-sphere.

Proof. Given $0 < \varepsilon \leq a$, consider the prolate k-dimensional hyperellipsoid given by

$$E^k(\varepsilon, a) = \left\{ \frac{x_1^2}{\varepsilon^2} + \cdots + \frac{x_k^2}{\varepsilon^2} + \frac{x_{k+1}^2}{a^2} = 1 \right\} \subset \mathbb{R}^{k+1}.$$

For every $0 < \varepsilon \leq 1$, let $a(\varepsilon) \geq 1$ be the unique real number such that

$$\left| E^k(\varepsilon, a(\varepsilon)) \right| = w_k.$$

Consider the product Riemannian manifold $(M, g_\varepsilon) = E^k(\varepsilon, a(\varepsilon)) \times \mathbb{S}^{n-k}(1)$, which is diffeomorphic to $\mathbb{S}^k \times \mathbb{S}^{n-k}$. The Riemannian universal cover $(\tilde{M}, \tilde{g}_\varepsilon)$ of (M, g_ε) is given by the Riemannian product of $E^k(\varepsilon, a(\varepsilon))$ with the Riemannian universal cover $\tilde{\mathbb{S}}^{n-k}(1)$ of $\mathbb{S}^{n-k}(1)$. Notice that $\tilde{\mathbb{S}}^{n-k}(1)$ is isometric to the round $(n-k)$-sphere if $1 \leq k \leq n-2$, and the standard real line for $k = n-1$.

Now fix a point $x \in \mathbb{S}^k \times \mathbb{S}^{n-k}$ and a scale $R > 0$. Consider the metric ball $B_{(\tilde{M}, \tilde{g}_\varepsilon)}(\tilde{x}, R)$ of radius R centered at a lift $\tilde{x}$ of x to $\tilde{M}$. Since $(\tilde{M}, \tilde{g}_\varepsilon)$ is a Riemannian product, the ball $B_{(\tilde{M}, \tilde{g}_\varepsilon)}(\tilde{x}, R)$ is contained in the product of metric balls

$$B_{E^k(\varepsilon, a(\varepsilon))}(\tilde{x}_1, R) \times B_{\tilde{\mathbb{S}}^{n-k}(1)}(\tilde{x}_2, R),$$

where $\tilde{x}_1$ and $\tilde{x}_2$ denote the projections of $\tilde{x}$ to the corresponding factors. It is easy to show that

$$\left| B_{E^k(\varepsilon, a(\varepsilon))}(\tilde{x}_1, R) \right| \leq 2 w_{k-1} R \varepsilon^{k-1}.$$

Besides, the quantity $\left| B_{\tilde{\mathbb{S}}^{n-k}(1)}(\tilde{x}_2, R) \right|$ coincides with the volume $V^{n-k}_{(n-k)(n-k-1)}(R)$ of any ball of radius R in the unit round $(n-k)$-sphere, which has scalar curvature $(n-k)(n-k-1)$. Therefore,

$$\left| B_{(\tilde{M}, \tilde{g}_\varepsilon)}(\tilde{x}, R) \right| \leq 2 w_{k-1} V^{n-k}_{(n-k)(n-k-1)}(R) R \varepsilon^{k-1}.$$

Hence, if one takes $\varepsilon \to 0$ for a fixed scale $R > 0$, then $\mathrm{mscal}_{(M, g_\varepsilon)}(x, R) \to \infty$ uniformly in $x \in M$. Nonetheless, for any $0 < \varepsilon \leq 1$, the k-systoles of (M, g_ε) are given by

$$\mathrm{sys}\, \pi_k(M, g_\varepsilon) = \mathrm{sys}\, H_k(M, g_\varepsilon) = \left| E^k(\varepsilon, a(\varepsilon)) \right| = w_k.$$

□

2.3 Urysohn width

The notion of Urysohn width was introduced by Urysohn in the 1920s in the context of dimension theory [Ury25]. Yet, it was not until 1983 that Gromov started using Urysohn width in a geometric setting in his works on systolic geometry [Gro83, Gro86, Gro88]. Intuitively, the k-dimensional Urysohn width of a metric space X quantifies how well X can be approximated by a k-dimensional simplicial complex.

Definition 2.3.1. Let X be a metric space and $k \in \mathbb{N}$. The *k-dimensional Urysohn width* $\mathrm{UW}_k(X)$ of X is defined as the infimal positive real number $w > 0$ such that there exists a continuous map $p : X \to Y$ into a k-dimensional simplicial complex Y whose fibres satisfy

$$\mathrm{diam}_X(p^{-1}(y)) \leq w$$

for every $y \in Y$.

Notice that, after modifying the projection map $p : X \to Y$, one can always assume that the fibre $p^{-1}(y)$ is connected, for any $y \in Y$. From Definition 2.3.1 it follows directly that $\mathrm{UW}_0(X) = \mathrm{diam}(X)$. The Urysohn width satisfies the following monotonicity properties.

Proposition 2.3.2. *Let X be a metric space. The following properties hold.*

1. *Let $f : X \to X'$ be a distance non-decreasing map. Then* $\mathrm{UW}_k(X) \leq \mathrm{UW}_k(X')$ *for all* $k \in \mathbb{N}$.

2. $\mathrm{UW}_0(X) \geq \mathrm{UW}_1(X) \geq \mathrm{UW}_2(X) \geq \dots$

The original definition of Urysohn width was formulated in terms of open covers. Let $\{U_i\}_i$ be an open cover of a metric space X. The open cover $\{U_i\}_i$ has *multiplicity at most* m if every point $x \in X$ lies in at most m different open sets U_i of the cover.

Proposition 2.3.3. *Let X be a metric space and $k \in \mathbb{N}$. Then $\mathrm{UW}_k(X) \leq w$ if and only if there is an open cover $\{U_i\}$ of X of multiplicity at most $k+1$ such that $\mathrm{diam}_X(U_i) \leq w$ for every i.*

See [Gut17, Lemma 0.8] for a proof of Proposition 2.3.3. The following classical theorem from dimension theory has deep implications, since it can be used together with Proposition 2.3.3 in order to derive Urysohn width estimates.

Theorem 2.3.4 (Lebesgue Covering Theorem [Leb11])**.** *Let $[0,1]^n$ be the Euclidean unit cube. Suppose that $\{U_i\}$ is an open cover of multiplicity at most n. Then there is an open set U_i of the cover which contains points of two opposite faces.*

The Lebesgue Covering Theorem 2.3.4 was first stated by Lebesgue [Leb11] and proven later by Brouwer [Bro13]. For a proof of 2.3.4, see [HW41, Theorem IV.2]. The Lebesgue Covering Theorem 2.3.4, together with other similar theorems from dimension theory, may be used to derive precise estimates for the codimension 1 Urysohn width of some metric spaces.

Proposition 2.3.5. *The following properties hold true.*

1. *Let $[0,1]^n$ denote the Euclidean unit n-cube. Then $\mathrm{UW}_{n-1}([0,1]^n) = 1$.*

2. *Let Δ^n denote the Euclidean regular unit n-simplex. Then $\mathrm{UW}_{n-1}(\Delta^n) = \frac{1}{n}$.*

3. *Let $\mathbb{B}^n$ denote the Euclidean unit n-ball. Then $\mathrm{UW}_{n-1}(\mathbb{B}^n) = \sqrt{\frac{2n+2}{n}}$.*

More precisely, the Lebesgue Covering Theorem 2.3.4 implies Proposition 2.3.5 (1), and Proposition 2.3.5 (2) and (3) follow from the Knaster–Kuratowski—Mazurkiewicz Theorem [KKM29].

Suppose now that the metric space X arises from the distance structure induced by a Riemannian metric on a n-manifold M. Then the decreasing sequence in Proposition 2.3.2 (2) ends at the dimension of the manifold. Moreover, by the Lebesgue Covering Theorem and Proposition 2.3.2 (1), the first non-zero Urysohn width of a manifold occurs at codimension 1.

Corollary 2.3.6. *Let M be a Riemannian n-manifold. Then $\mathrm{UW}_n(M) = 0$ and $\mathrm{UW}_{n-1}(M) > 0$.*

The explicit values of the Urysohn widths of the round n-sphere up to half its dimension are also known, and they coincide with its diameter.

Theorem 2.3.7 ([Šče74])**.** *Let $\mathbb{S}^n(1)$ denote the unit round n-sphere. Then*

- $\mathrm{UW}_k(\mathbb{S}^n(1)) = \pi$, *if* $k \leq \frac{n}{2}$, *and*

- $\mathrm{UW}_k(\mathbb{S}^n(1)) < \pi$, *if* $k > \frac{n}{2}$.

In fact, Theorem 2.3.7 holds also for the n-sphere $\mathbb{S}^n$ endowed with the extrinsic metric inherited from its inclusion in the Euclidean space $\mathbb{R}^{n+1}$.

2.3.1 Urysohn width and other metric invariants

The Urysohn widths of a manifold, and specially the codimension 1 Urysohn width, are related to other important metric invariants. For instance, Guth [Gut17] derived an estimate for the codimension 1 Urysohn width of a Riemannian manifold from a control on the volume of geodesic balls in M.

Theorem 2.3.8 ([Gut17, Theorem 0.1])**.** *There is a dimensional constant $c_n > 0$ such that the following holds. Let M be a complete Riemannian n-manifold. Suppose that there is a radius $R > 0$ such that, for every $x \in M$, the closed geodesic ball $B(x, R)$ centered a x has volume $|B(x, R)| \leq c_n R^n$. Then*

$$\mathrm{UW}_{n-1}(M) \leq R.$$

From Theorem 2.3.8 it follows that the codimension 1 Urysohn width of a Riemannian manifold can be estimated in terms of its volume.

Corollary 2.3.9 ([Gut17, Corollary 0.3])**.** *Let M be a closed Riemannian n-manifold. Then*

$$\mathrm{UW}_{n-1}(M) \leq c_n^{-1/n} |M|^{1/n},$$

where $c_n > 0$ is the dimensional constant in Theorem 2.3.8.

A fundamental metric invariant is the filling radius, introduced by Gromov [Gro83] in its proof of the systolic inequality. Let M be a closed Riemannian n-manifold. The *Kuratowski embedding* of M is the map

$$\begin{array}{rccc} f: & M & \to & L^\infty(M) \\ & x & \mapsto & d(x, \cdot) \end{array}$$

from the manifold M to the space $L^\infty(M)$ of bounded functions on M endowed with the supremum norm. Notice that the Kuratoswki embedding is an isometry of metric spaces between M and $L^\infty(M)$.

Definition 2.3.10. Let M be a closed Riemannian n-manifold and denote by $f : M \to L^\infty(M)$ the Kuratowski embedding of M. Let $G = \mathbb{Z}$ if M is orientable and $G = \mathbb{Z}_2$ otherwise. The *filling radius* $\mathrm{FillRad}\,(M)$ of M is the infimal positive real number $R > 0$ such that

$$f_*[M] = 0 \in H_n(U(f(M), R); G).$$

In other words, the filling radius of M is the smaller radius $R > 0$ such that the image $f(M)$ of M by the Kuratowski embedding bounds a $(n+1)$-chain in its closed R-neighbourhood $U(f(M), R)$ in $L^\infty(M)$. Notice that the fill radius $\mathrm{fillrad}\,(\gamma)$ of a closed curve γ in a manifold M introduced in Section 1.3 is a homotopical and extrinsic version of Gromov's filling radius for closed curves in a manifold with boundary. Gromov proved the following estimate.

Theorem 2.3.11 ([Gro83, Appendix 1])**.** *Let M be a closed Riemannian n-manifold. Then*

$$\mathrm{FillRad}\,(M) \leq \frac{1}{2}\,\mathrm{UW}_{n-1}(M).$$

Another metric invariant which quantifying the size of manifold is the hyperspherical radius, which was also defined by Gromov [GL83, Gro86].

Definition 2.3.12. Let M be a closed Riemannian n-manifold. The *hyperspherical radius* $\mathrm{HS}(M)$ of M is the supremal positive real number $R > 0$ such that there exists a 1-Lipschitz map of non-zero degree

$$f : M \to \mathbb{S}^n(R)$$

from M to the round n-sphere of radius R.

The following theorem of Gromov can be used to estimate the hyperspherical radius of a manifold.

Theorem 2.3.13 ([Gro88, Proposition F_1]). *Let X be a metric space. Suppose that X admits a map $\varphi : X \to \mathbb{S}^k(\rho)$ to the k-dimensional round sphere of radius ρ which is L-Lipschitz and not null-homotopic. Then*

$$\mathrm{UW}_{k-1}(X) > \frac{\pi}{2} \cdot \frac{\rho}{L}.$$

Corollary 2.3.14. *Let M be a closed Riemannian n-manifold. Then*

$$\mathrm{HS}(M) \leq \frac{\pi}{2} \mathrm{UW}_{n-1}(M).$$

2.3.2 Urysohn width and scalar curvature

In [Gro86], Gromov conjectured that the scalar curvature and the codimension 2 Urysohn width of a Riemannian manifold are related as follows.

Conjecture 2.3.15 ([Gro86, 2.A]). *There is a dimensional constant $\lambda_n > 0$ such that the following holds. Let M be a complete Riemannian n-manifold with uniformly positive scalar curvature* $\mathrm{scal} \geq s > 0$. *Then* $\mathrm{UW}_{n-2}(M) \leq \frac{\lambda_n}{\sqrt{s}}$.

The conjecture has been proven true for 3-manifolds [CL24, LM23], and some progress has been made in the case of dimension 4 [DD22].

One could ask whether the macroscopic version of Conjecture 2.3.15 holds true. Notice that Theorem 2.3.8 may be understood as a macroscopic generalisation of Conjecture 2.3.15 for the codimension 1 Urysohn width. Indeed, given a point $x \in M$ in a complete n-Riemannian manifold M and a scale $R > 0$, the condition $|B(x, R)| \leq c_n R^n$ is equivalent to a certain lower bound on the macroscopic scalar curvature $\mathrm{mscal}(x, R)$. Regarding the codimension 2 Urysohn width, Alpert–Balitskiy–Guth [ABG24] showed one cannot control $\mathrm{UW}_{n-2}(M)$ only from a lower bound of the macroscopic scalar curvature at a certain scale. However, they showed that the macroscopic version of Conjecture 2.3.15 holds true in dimension 3 for manifolds with finitely generated 1-homology under an additional acyclicity condition on the geodesic balls. Alpert–Balitskiy–Guth [ABG24] also conjectured that a lower bound on the macroscopic scalar curvature together with the acyclicity condition imply an upper bound on the codimension 2 Urysohn width in general dimension.

2.4 The Coarea Formula

The Coarea Formula is a fundamental result in Geometric Measure Theory, established in its general form by Federer [Fed69, Theorem 3.2.22]. It generalises Fubini's Theorem in the context of manifolds, and allows to decompose an integral over a manifold as an iterate integral over the level sets of a given function. We will use the formulation given in [BZ88, Theorem 13.4.2].

Let M and N be Riemannian manifolds of dimensions m and n respectively, with $m > n$. If $f : M \to N$ is a Lipschitz map between M and N, then Rademacher's Theorem [Rad19] implies the differentiability of f almost everywhere in M. That is, for almost every point $x \in M$, one can consider the differential map $d_x f : T_x M \to T_{f(x)} N$. Fix a point $x \in M$ of differentiability of f, and denote by $f^* T_x N$ the orthogonal complement of $\ker(d_x f)$ in $T_x M$. The *Jacobian* $Jf(x)$ of the map f at a point $x \in M$ of differentiability of f is defined as follows. If $\dim(f^* T_x N) = \operatorname{rank}(d_x f) = n$, then $Jf(x)$ is defined to be the Jacobian of the restriction of the differential $d_x f$ to $f^* T_x N$. If $\dim(f^* T_x N) = \operatorname{rank}(d_x f) < n$, then $Jf(x) = 0$.

Given a Riemannian manifold M, we will denote the *k-dimensional Hausdorff measure* on M by $\mathcal{H}^k_M$.

Theorem 2.4.1 (Coarea Formula [BZ88, Theorem 13.4.2])**.** *Let M and N be Riemannian manifolds of dimensions m and n respectively, with $m > n$, and let $f : M \to N$ be a Lipschitz map. Given any Lebesgue measurable subset $A \subset M$, we have*

$$\int_A Jf(x)\, d\mathcal{H}^m_M(x) = \int_N \mathcal{H}^{m-n}_M \left(A \cap f^{-1}(y)\right)\, d\mathcal{H}^n_N(y).$$

Now let M be a Riemannian manifold and fix a point $x \in M$. The application of the Coarea Formula 2.4.1 to the distance function $d(x, \cdot) : M \to \mathbb{R}$ gives the following corollary.

Corollary 2.4.2. *Let M be a Riemannian manifold. The volume of the geodesic ball $B(x, R)$ centered at a point $x \in M$ and of radius $R > 0$ satisfies*

$$|B(x, R)| = \int_0^R |S(x, \tau)|\, d\tau.$$

2.5 The Macroscopic Stability Inequality

As discussed in the Introduction, Schoen–Yau [SY79a, SY79b] used the Stability Inequality (2) to show that given a closed orientable Riemannian n-manifold M with positive scalar curvature, then any stable minimal hypersurface Σ embedded in M inherits a metric of positive scalar curvature, after a conformal change on the induced metric from M. In other words, by equation (2.1), if Σ is a stable minimal hypersurface in M, then an upper bound on the volume of geodesic balls in M of infinitesimal radius descends (up to a conformal change of the Riemannian metric of Σ) to an upper bound on the volume of infinitesimal balls in Σ.

In [Gut10b], Guth developed a macroscopic analogue of Schoen–Yau's descent based on the Coarea Formula 2.4.1, called the Macroscopic Stability Inequality, relating the volume of geodesic balls of a fixed radius in a stable minimal hypersurface to the volume of geodesic balls of a larger radius in the ambient manifold. A crucial step in the proof of Theorem D consists in the application of Guth's Macroscopic Stability Inequality. Originally, the Macroscopic Stability Inequality was developed for $\mathbb{Z}_2$-coefficients by Guth in [Gut10b] in order to give a shorter proof of Gromov's Isosystolic Inequality [Gro83] for the n-torus. Recently, the Macroscopic Stability Inequality has been extended to $\mathbb{Z}$-coefficients by Alpert in [Alp22].

In fact, the applicability of Guth's Macroscopic Stability Inequality is not restricted to stable minimal hypersurfaces, but also holds for hypersurfaces which are almost area-minimising in their homology class. We start by rigorously defining the notion of almost minimising hypersurface. Hereafter G will denote a coefficient group for homology, that will always be either $\mathbb{Z}_2$ or $\mathbb{Z}$.

Definition 2.5.1. Let M be a complete Riemannian n-manifold with $H_{n-1}(M; G) \neq 0$. Let Σ be a closed hypersurface embedded in M such that $[\Sigma] \neq 0 \in H_{n-1}(M; G)$. The hypersurface Σ is *δ-almost minimising in its G-homology class* if any embedded hypersurface Σ' homologous to Σ in $H_{n-1}(M; G)$ satisfies

$$|\Sigma| \leq \left|\Sigma'\right| + \delta.$$

Let us now state the Macroscopic Stability Inequality

Theorem 2.5.2 (Macroscopic Stability Inequality [Gut10b, Alp22])**.** *Let $G = \mathbb{Z}_2$ or $\mathbb{Z}$. Let M be a non-simply connected complete Riemannian n-manifold such that $H_{n-1}(M; G) \neq 0$ and $\operatorname{sys}\pi_1(M) > 0$. Let Σ be a hypersurface embedded in M which is δ-almost minimising in its G-homology class. Fix $r > 0$ and $R > 0$ such that $0 < r < R < \frac{1}{2}\operatorname{sys}\pi_1(M)$. Then, for every $x \in \Sigma$,*

$$|B_\Sigma(x, r)| \leq \frac{1}{R - r}\,|B(x, R)| + \delta.$$

The following lemma due to Gromov constitutes an intermediate step in the proof of the Macroscopic Stability Inequality 2.5.2.

Lemma 2.5.3 (Gromov's Curve Factoring Lemma [Gro07, Proposition 5.28])**.** *Let M be a Riemannian n-manifold. Fix a point $x \in M$. Let γ be a closed curve contained in a closed geodesic ball $B(x, R)$ and let $\varepsilon > 0$. Then γ is $\mathbb{Z}$-homologous to a 1-cycle $\sum_i \gamma_i$, where each γ_i is a closed curve of length $\ell(\gamma_i) < 2t + \varepsilon$.*

Proof of Gromov's Curve Factoring Lemma 2.5.3. Fix $\varepsilon > 0$. Let $y_1, \ldots, y_k$ be a finite collection of cyclically ordered points lying in γ such that every arc σ_i in γ joining two consecutive points y_i and y_{i+1} has length $\ell(\sigma_i) < \varepsilon$. For each point y_i, let τ_i be a curve joining x with y_i of length $\ell(\tau_i) \leq R$, see Figure 2.2.

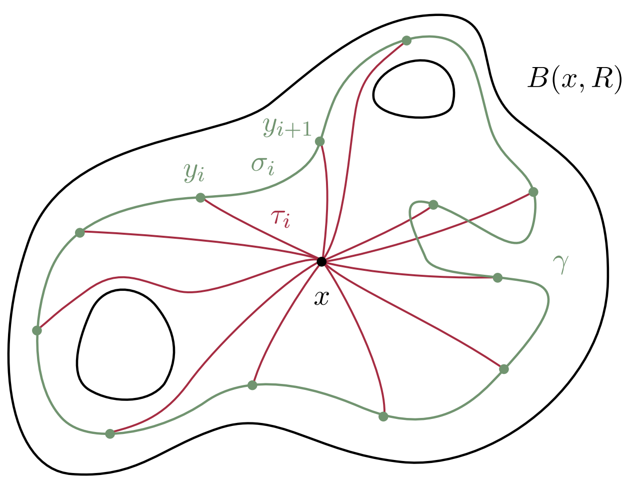

Figure 2.2: Scheme of the proof of Gromov's Curve Factoring Lemma 2.5.3.

Define γ_i to be the 1-cycle defined by $-\tau_{i+1} + \sigma_i + \tau_i$, which has length $\ell(\gamma_i) < 2R + \varepsilon$. Then $\gamma = \sum_{i=1}^k \gamma_i$ as 1-chains. In particular, the closed curve γ and the 1-cycle $\sum_{i=1}^k \gamma_i$ define the same homology class in $H_1(M; \mathbb{Z})$. □

We reproduce the proofs of the Macroscopic Stability Inequality 2.5.2 by [Gut10b] and [Alp22], giving more detail.

Proof of the Macroscopic Stability Inequality 2.5.2. Let $x \in \Sigma$ be a point. Recall that as a consequence of the Coarea Formula 2.4.1, for any radius $R > 0$ we have

$$|B(x, R)| = \int_0^R |S(x, \tau)|\, d\tau,$$

see Corollary 2.4.2. Hence there exists a radius $t \in (r, R)$ for which

$$|B(x,R)| \geq \int_r^R |S(x,\tau)|\, d\tau = (R-r)\, |S(x,t)|\,. \tag{2.5}$$

Consider the closed ball $B(x,t)$ of radius t. We shall now prove that every closed curve lying in $B(x,t)$ intersects Σ trivially.

Suppose that γ is a closed curve lying in $B(x,t)$ with non-trivial intersection with Σ. Let $\varepsilon = \mathrm{sys}\,(M) - 2R$. By Gromov's Curve Factoring Lemma 2.5.3, the closed curve γ is $\mathbb{Z}$-homologous to a sum $\sum_i \gamma_i$ of closed curves γ_i of length $\ell(\gamma_i) < 2t + \varepsilon$. Since γ intersects non-trivially the hypersurface Σ, one curve γ_j in the sum $\sum_i \gamma_i$ has non-trivial intersection with Σ. In particular, the curve γ_j has to be non contractible. However,

$$\ell(\gamma_j) < 2t + \varepsilon \leq \mathrm{sys}\,\pi_1(M),$$

which is a contradiction.

Consider the cycle $\Sigma \cap B(x,t)$ in $B(x,t)$ relative to the boundary $S(x,t)$. Notice that, since $t < \frac{1}{2}\,\mathrm{sys}\,\pi_1(M)$, the ball $B(x,t)$ is orientable. Otherwise the ball $B(x,t)$ would contain a closed curved along which the orientation of $B(x,t)$ (and of the manifold M) is reversed. Such a curve is non-contractible in M. Then, by Gromov's Curve Factoring Lemma 2.5.3, there would exist a non-contractible closed curve γ_i of length $\ell(\gamma_i) < \mathrm{sys}\,\pi_1(M)$, which is a contradiction. Hence, by Lefschetz's Duality [Hat02, Theorem 3.43] and the Universal Coefficient Theorem [Hat02, Theorem 3.2], there are isomorphisms

$$H_{n-1}(B(x,t), S(x,t); \mathbb{Z}) \simeq H^1(B(x,t); \mathbb{Z}) \simeq H_1(B(x,t); \mathbb{Z}).$$

Since every 1-cycle in $B(x,t)$ intersects Σ trivially, we have

$$[\Sigma \cap B(x,t)] = 0 \in H_{n-1}(B(x,t), S(x,t); \mathbb{Z}),$$

which implies that the chain $\Sigma \cap B(x,t)$ is $\mathbb{Z}$-homologous to a chain $\sum_i m_i Z_i$ in $S(x,t)$, where $Z_i \subset S(x,t)$ are connected components of $S(x,t) \setminus \Sigma$ and $m_i \in \mathbb{Z}$. We will discuss the cases $G = \mathbb{Z}_2$ and $G = \mathbb{Z}$ separately hereafter.

Let us first discuss the case $G = \mathbb{Z}_2$. Projecting to the chain complex with $\mathbb{Z}_2$-coefficients, we obtain that the chain $\Sigma \cap B(x,t)$ is $\mathbb{Z}_2$-homologous to a chain $\sum_i Z_i$. Consider the embedded hypersurface Σ' obtained from Σ by replacing $\Sigma \cap B(x,t)$ with $\cup_i Z_i \subset S(x,t)$ and smoothing out the resulting cycle. Since the hypersurface Σ is $\mathbb{Z}_2$-homologous to Σ', the δ-almost minimality of Σ implies

$$|\Sigma \cap B(x,t)| \leq |\cup_i Z_i| + \delta \leq |S(x,t)| + \delta.$$

We conclude by noting that $B_\Sigma(x,r) \subset \Sigma \cap B(x,t)$ and using the inequality (2.5).

Finally, we address the case $G = \mathbb{Z}$. For $\mathbb{Z}$-coefficients, the chain $\Sigma \cap B(x,t)$ may fail to be $\mathbb{Z}$-homologous to a chain $\sum_i Z_i$ in $S(x,t)$ consisting of a disjoint union of connected components Z_i of $S(x,t) \setminus \Sigma$. Still, the different connected components of $\Sigma \cap B(x,t)$ may be grouped into a collection $D_1, \ldots, D_{N-1}$ such that $|D_i| \leq |S(x,t)| + \delta$ for every $i \in \{1, \ldots, N-1\}$.

We proceed as follows. Since every closed curve lying in $B(x,t)$ has trivial intersection with Σ, one can group the connected components of $B(x,t) \setminus \Sigma$ into levels $L_1, \ldots, L_N$ in a way such that every path starting at L_i and ending at L_j has signed intersection number with Σ equal to $j - i$. For each $i \in \{1, \ldots, N\}$, define $S_i := L_i \cap S(x,t)$. Finally, group the connected components of $\Sigma \cap B(x,t)$

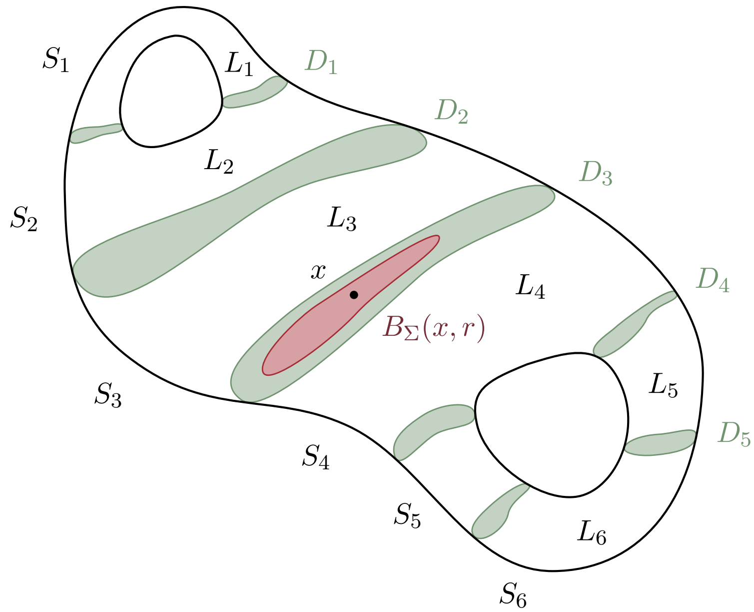

Figure 2.3: Subdivision of the ball $B(x,t)$ into levels $L_1, \ldots, L_N$ separated by the dividers $D_1, \ldots, D_N$.

into dividers $D_1, \ldots, D_{N-1}$ so that the divider D_i is the common boundary between L_i and L_{i+1} for $i \in \{1, \ldots, N-1\}$, see Figure 2.3. For convenience, we set $D_0 = \emptyset$ and $D_N = \emptyset$.

For every $i \in \{1, \ldots, N-1\}$ and every $k \in \{0, \ldots, N\}$, consider the chain $D_{i,k}$ defined by

$$D_{i,k} := \begin{cases} D_k + \sum_{j=k+1}^{i} S_j, & \text{if } k < i \\ D_i, & \text{if } k = i \ . \\ D_k + \sum_{j=i+1}^{k} S_j, & \text{if } k > i \end{cases}$$

In particular, we have $D_{i,0} = \sum_{j=0}^{i} S_j$ and $D_{i,N} = \sum_{j=i+1}^{N} S_j$ the two connected components of $S(x,t) \setminus D_i$. Notice that, for every $i \in \{1, \ldots, N-1\}$ and every $k \in \{0, \ldots, N\}$, the chain D_i is $\mathbb{Z}$-homologous to $D_{i,k}$. For each $i \in \{1, \ldots, N-1\}$, let $k_i \in \{0, \ldots, N\}$ be such that

$$|D_{i,k_i}| = \min_{k \in \{0,\ldots,N\}} |D_{i,k}| \,,$$

and define $D_i' := D_{k_i}$. That is, for each $i \in \{1, \ldots, N-1\}$, the chain D_i' denotes the combination $D_{i,k}$ of least area. By the minimality of the D_{k_i} with respect to the combinations $D_{i,k}$, one can always assume that $0 \leq k_1 \leq \cdots \leq k_{N-1} \leq N$.

Now, modify the hypersurface Σ by replacing each D_i by the corresponding D_i' and perturb the resulting hypersurface to make it embedded, see Figure 2.4.

We obtain an embedded hypersurface Σ' which is $\mathbb{Z}$-homologous to the original hypersurface Σ. By the δ-almost minimality of Σ, we have

$$\sum_{i=1}^{N-1} |D_i| \leq \sum_{i=1}^{N-1} |D_i'| + \delta. \tag{2.6}$$

From the inequality (2.6) and the minimality of the D_i', it follows that, for every $i \in \{1, \ldots, N-1\}$,

$$|D_i| \leq |D_i'| + \delta.$$

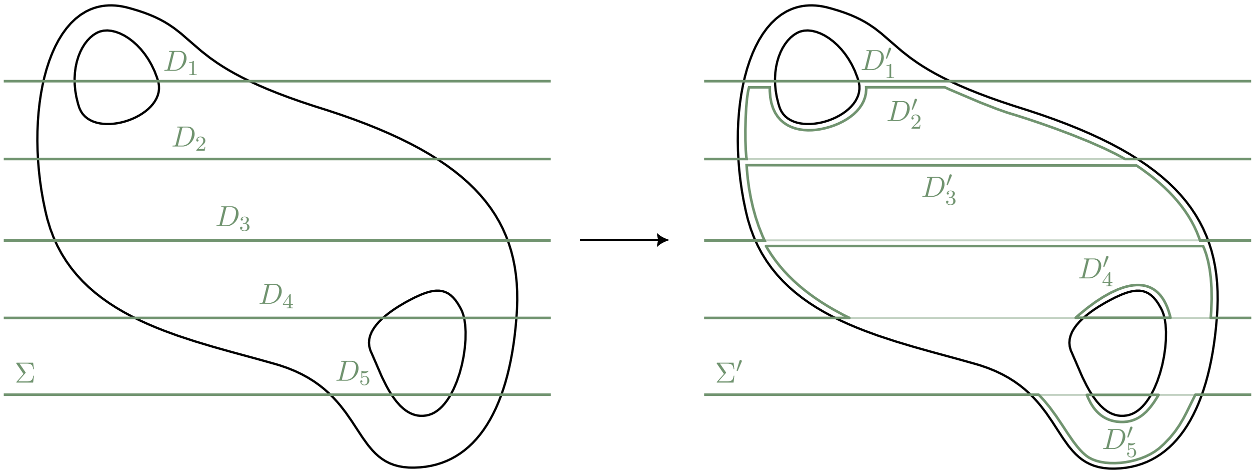

Figure 2.4: Modification of the surface Σ by replacing each D_i by the corresponding D'_i.

The minimality of D'_i implies that $|D'_i| \leq \sum_{j=0}^{i} |S_j|$ and $|D'_i| \leq \sum_{j=i+1}^{N} |S_j|$. We derive that for every $i \in \{1, \ldots, N-1\}$,

$$|D_i| \leq |S(x,t)| + \delta.$$

We conclude by observing that there is an $i \in \{1, \ldots, N-1\}$ such that $B_\Sigma(x,r) \subset D_i$ and using the inequality (2.5). □

2.6 The proof of Theorem D

The aim of this section is to present the proof of Theorem D.

Proof of Theorem D. Let $G = \mathbb{Z}_2$ or $\mathbb{Z}$. Recall that M is a non-simply connected complete Riemannian n-manifold with $H_{n-1}(M;G) \neq 0$ and $\operatorname{sys}\pi_1(M) > 0$. Fix any non-trivial homology class $h \in H_{n-1}(M;G)$. Recall that every codimension 1 homology class with coefficients either in $\mathbb{Z}_2$ or in $\mathbb{Z}$ can be represented by a smooth closed embedded hypersurface [Tho54]. Let Σ be a δ-almost minimising closed embedded hypersurface representing h. Fix a point $x \in \Sigma$ and consider two radii $0 < r < R < \frac{1}{2}\operatorname{sys}\pi_1(M)$ to be determined later. Let $B_\Sigma(x,r)$ be the metric ball centered at the point x of radius r with respect to the induced metric on Σ. The Macroscopic Stability Inequality 2.5.2 together with the lower bound on the macroscopic scalar curvature of M at point x and scale R imply

$$|B_\Sigma(x,r)| \leq \frac{1}{R-r} V_s^n(R) + \delta.$$

If, given $0 < r < R$ and $s > 0$, the inequality

$$\frac{1}{R-r} V_s^n(R) < c_{n-1} r^{n-1} \tag{2.7}$$

holds, then Theorem 2.3.8 applied to the δ-almost minimising hypersurface Σ for $\delta > 0$ small enough implies that $\mathrm{UW}_{n-2}(\Sigma) \leq r$. By Corollary 2.1.2.(1), the inequality (2.7) is equivalent to

$$\frac{1}{(r/R)^{n-1}} \cdot \frac{1}{1 - r/R} V_{sR^2}^n(1) < c_{n-1}. \tag{2.8}$$

The value of the inner radius r that makes the left-hand term in the inequality (2.8) as small as possible is $r = \frac{n-1}{n}R$. In this case, the inequality (2.8) becomes

$$V^n_{sR^2}(1) < \frac{(n-1)^{n-1}}{n^n} c_{n-1},$$

which is equivalent to $sR^2 > \kappa_n := f_n\left(\frac{(n-1)^{n-1}}{n^n} c_{n-1}\right)$, where $f_n : (0,\infty) \to \mathbb{R}$ is the inverse function of the map $s \mapsto V^n_s(1)$, see Corollary 2.1.2.(2). □

2.7 Berger metrics on $\mathbb{R}\mathrm{P}^3$

This section is dedicated to the construction of the family of Riemannian metrics $(\bar{g}_s)_{s>0}$ on the real projective space $\mathbb{R}\mathrm{P}^3$ presented in Proposition E.

Throughout this section, we will identify the 3-sphere $\mathbb{S}^3$ with $\{(z,w) \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$, and the 2-sphere $\mathbb{S}^2$ with $\{(z,t) \mid |z|^2 + t^2 = 1\} \subset \mathbb{C} \times \mathbb{R}$. Consider the *Hopf action* on $\mathbb{S}^3$, that is, the free action of $\mathbb{S}^1$ on $\mathbb{S}^3$ given by

$$\theta \cdot (z,w) = (e^{i\theta} z, e^{i\theta} w),$$

for every $\theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and every $(z,w) \in \mathbb{S}^3$. The quotient space $\mathbb{S}^3/\mathbb{S}^1$ corresponding to the Hopf action is homeomorphic to $\mathbb{S}^2$, and the projection $\mathbb{S}^3 \to \mathbb{S}^2$ defines a circle bundle structure on $\mathbb{S}^3$. Let $V_{(z,w)} = (iz, iw) \in \mathbb{C}^2$ denote the *Hopf vector field*, which is a unit vector field on $\mathbb{S}^3$ (with respect to the round metric) tangent to the orbits of the Hopf action.

Definition 2.7.1. The *Berger metric* of parameter $\varepsilon > 0$ on $\mathbb{S}^3$ is the metric defined by

$$g_\varepsilon(X,Y) = g(X,Y) + (\varepsilon^2 - 1) g(X,V) g(V,Y),$$

for any pair of vectors X, Y tangent to $\mathbb{S}^3$, where g denotes the standard round Riemannian metric on $\mathbb{S}^3$.

Intuitively, the Berger metric g_ε is obtained from the round metric g by shrinking the metric in the direction of the Hopf fibres by a factor ε (so that they have length $2\pi\varepsilon$ with respect to the metric g_ε). The Berger metrics $(g_\varepsilon)_{\varepsilon>0}$ define a 1-parameter family of Riemannian metrics on $\mathbb{S}^3$, and the Berger metric g_ε corresponding to $\varepsilon = 1$ coincides with the standard round metric g on $\mathbb{S}^3$.

Lemma 2.7.2. *The quotient map*

$$\begin{array}{rccc} \mathcal{H}\colon & (\mathbb{S}^3, g_\varepsilon) & \longrightarrow & \mathbb{S}^2(\frac{1}{2}) \\ & (z,w) & \longmapsto & \left(z\bar{w}, \frac{1}{2}(|z|^2 - |w|^2)\right) \end{array},$$

known as the Hopf map, *is a Riemannian submersion.*

In fact, in more technical terms, the family of Berger metrics $(g_\varepsilon)_{\varepsilon>0}$ corresponds to the canonical variation of the round metric on $\mathbb{S}^3$ associated to the Riemannian submersion $\mathcal{H} : \mathbb{S}^3(1) \to \mathbb{S}^2(\frac{1}{2})$, see [Bes87, Section 9.G].

The antipodal action of $\mathbb{Z}_2$ on the Berger sphere $(\mathbb{S}^3, g_\varepsilon)$ is an isometric action. Hence, the real projective space $\mathbb{R}\mathrm{P}^3$ inherits a Riemannian metric from $(\mathbb{S}^3, g_\varepsilon)$, that we denote by $\bar{g}_\varepsilon$. The map $\mathcal{H}$ induces a Riemannian submersion $\bar{\mathcal{H}} : (\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon) \to \mathbb{S}^2(\frac{1}{2})$, which defines a circle bundle on $\mathbb{R}\mathrm{P}^3$. In particular, the map $\bar{\mathcal{H}}$ is 1-Lipschitz.

Proposition E follows from Proposition 2.7.3 and Proposition 2.7.4.

Proposition 2.7.3. *Fix $\kappa > 0$. For every $\varepsilon \in (0,1)$, there is a scale $R_\varepsilon > 0$ satisfying $R_\varepsilon \geq \frac{1}{2}\operatorname{sys}\pi_1(\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon)$ and $R_\varepsilon > \kappa/\sqrt{s_\varepsilon}$, with $s_\varepsilon := 6/\varepsilon^{2/3}$, such that for any point $x \in \mathbb{R}\mathrm{P}^3$,*

$$\operatorname{mscal}_{(\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon)}(x, R_\varepsilon) \geq s_\varepsilon.$$

Proof. Fix $\varepsilon \in (0,1)$. Let $\kappa' > \max\left\{\kappa/\sqrt{6}, \pi\right\}$ be a constant, and consider the scale $R_\varepsilon = \kappa' \sqrt[3]{\varepsilon}$. Notice that $R_\varepsilon > \kappa/\sqrt{s_\varepsilon}$ and $R_\varepsilon > \operatorname{sys}\pi_1(\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon) = \pi\varepsilon$. By the Coarea Formula 2.4.1 applied to the fibration $\mathcal{H} : (\mathbb{S}^3, g_\varepsilon) \to \mathbb{S}^2(\frac{1}{2})$ we have

$$\left|B_{(\mathbb{S}^3, g_\varepsilon)}(\tilde{x}, R_\varepsilon)\right| \leq \left|(\mathbb{S}^3, g_\varepsilon)\right| \leq 2\pi\varepsilon \left|\mathbb{S}^2(\tfrac{1}{2})\right| = 2\pi^2\varepsilon.$$

Notice that the volume of the unit 3-sphere is $2\pi^2$, that is, $w_3 = 2\pi^2$. Therefore

$$\left|B_{(\mathbb{S}^3, g_\varepsilon)}(\tilde{x}, R_\varepsilon)\right| \leq w_3\varepsilon = \left|\mathbb{S}^3(\sqrt[3]{\varepsilon})\right| = V^3_{6/\varepsilon^{2/3}}(R_\varepsilon).$$

The last equality holds since $R_\varepsilon \geq \pi\sqrt[3]{\varepsilon}$. Therefore, the macroscopic scalar curvature of $(\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon)$ at a scale R_ε satisfies $\operatorname{mscal}_{(\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon)}(x, R_\varepsilon) \geq 6/\varepsilon^{2/3}$. □

Finally we prove Proposition E (2).

Proposition 2.7.4. *Let Σ be any closed immersed surface in $(\mathbb{R}\mathrm{P}^3, \bar{g}_\varepsilon)$ representing the non-trivial homology class in $H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Then*

$$\mathrm{UW}_1(\Sigma) > \frac{\pi}{4}.$$

Proof of Proposition 2.7.4. Suppose that the inclusion map $i : \Sigma \to \mathbb{R}\mathrm{P}^3$ satisfies $i_*[\Sigma] = [\mathbb{R}\mathrm{P}^2]$, where $[\Sigma] \in H_2(\Sigma; \mathbb{Z}_2)$ denotes the fundamental class of the surface Σ and $[\mathbb{R}\mathrm{P}^2]$ is the generator of $H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Consider the map

$$\varphi = \bar{\mathcal{H}} \circ i : \Sigma \to \mathbb{S}^2(\tfrac{1}{2}),$$

given by the restriction of the map $\bar{\mathcal{H}}$ to Σ. The map φ is 1-Lipschitz, since it is the restriction of the 1-Lipschitz map $\bar{\mathcal{H}}$ to Σ.

Let us show that φ is not null-homotopic. The Gysin sequence [Hat02, Section 4.D] applied to the circle bundle $\bar{\mathcal{H}} : \mathbb{R}\mathrm{P}^3 \to \mathbb{S}^2$ yields the exact sequence

$$\cdots \to H^0(\mathbb{S}^2; \mathbb{Z}_2) \to H^2(\mathbb{S}^2; \mathbb{Z}_2) \xrightarrow{\bar{\mathcal{H}}^*} H^2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \to H^1(\mathbb{S}^2; \mathbb{Z}_2) \to \cdots.$$

Since $H^1(\mathbb{S}^2; \mathbb{Z}_2)$ is trivial, the map

$$\bar{\mathcal{H}}^* : H^2(\mathbb{S}^2; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \to H^2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \simeq \mathbb{Z}_2$$

is an epimorphism, and therefore an isomorphism. By the Universal Coefficient Theorem [Hat02, Theorem 3.2], the corresponding induced map in homology

$$\bar{\mathcal{H}}_* : H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \to H_2(\mathbb{S}^2; \mathbb{Z}_2)$$

is an isomorphism, and it sends the generator $[\mathbb{R}\mathrm{P}^2]$ to the fundamental class $[\mathbb{S}^2]$. Therefore

$$\varphi_*[\Sigma] = \bar{\mathcal{H}}_*[\mathbb{R}\mathrm{P}^2] = [\mathbb{S}^2],$$

which implies that $\varphi_* : H_2(\Sigma; \mathbb{Z}_2) \to H_2(\mathbb{S}^2; \mathbb{Z}_2)$ is an isomorphism.

Hence the 1-Lipschitz map $\varphi : \Sigma \to \mathbb{S}^2(\frac{1}{2})$ is not null-homotopic. By Theorem 2.3.13, we conclude that $\mathrm{UW}_1(\Sigma) > \frac{\pi}{4}$. □

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Notation Index

scal	scalar curvature
Ric	Ricci curvature
sect	sectional curvature
$\mathrm{inj}_M(x)$	injectivity radius of the manifold M at x
κ_Σ	Gauss curvature of a surface Σ
II	second fundamental form of a surface Σ in M
$\chi(\Sigma)$	Euler characteristic of a surface Σ
$M_1 \# M_2$	connected sum of M_1 and M_2
$\mathcal{P}$	family of closed oriented prime 3-manifolds
$\hat{M}$	result of capping off the boundary components of a 3-manifold M with spherical boundary
$\tilde{M}$	universal (Riemannian) cover of a (Riemannian) manifold M
$\mathbb{S}^n$	n-dimensional sphere
$\mathbb{S}^\infty$	infinite dimensional sphere
$\mathbb{T}^n$	n-dimensional torus
$\mathbb{R}\mathrm{P}^n$	real projective n-dimensional space
$\mathbb{R}\mathrm{P}^\infty$	infinite dimensional real projective space
$L_p(q_1, \ldots, q_n)$	$(2n-1)$-dimensional lens space of parameters p and $q_1, \ldots, q_n$
$L_p(q_j)$	infinite dimensional lens space of parameters p and $(q_j)_{j\in\mathbb{N}}$
$\mathbb{B}^n$	n-dimensional ball
$\mathbb{S}^n(r)$	n-dimensional round sphere of radius r
$E^n(\varepsilon, a)$	n-dimensional sphere with the prolate ellipsoid metric of axes $\varepsilon \leq a$
$\mathbb{M}^n_\sigma$	simply connected n-dimensional space form of constant scalar curvature
$\|\Sigma\|$	k-volume of a k-dimensional submanifold Σ
b_n	n-dimensional volume of the unit Euclidean n-dimensional ball
w_n	n-dimensional volume of the unit round n-dimensional sphere.
$V^n_s(R)$	volume of a ball of radius R in the simply connected n-dimensional space form of constant scalar curvature
$U(Z, R)$	closed R-neighbourhood of the subset Z
$B(x, R)$	closed metric ball of radius R centered at x

$S(x, R)$	closed metric sphere of radius R centered at x
$\mathcal{H}^k_X$	k-dimensional Hausdorff measure on a metric space X
$\mathrm{mscal}(x, R)$	macroscopic scalar curvature at the point x and scale $R > 0$
$\mathrm{sys}\, \pi_k(M)$	k-dimensional homotopical systole of the manifold M
$\mathrm{sys}\, H_k(M)$	k-dimensional homological systole of the manifold M
$\mathrm{diam}\,(X)$	diameter of a metric space X
$\mathrm{diam}_X\,(Z)$	diameter of a subset Z in a metric space X
$\mathrm{UW}_k(X)$	k-dimensional Urysohn width of the metric space X
$\mathrm{HS}(M)$	hypersphericity of the manifold M
$\mathrm{FillRad}(M)$	filling radius of the manifold M
$\mathrm{fillrad}\,(\gamma)$	fill radius of the contractible closed curve γ
$\mathrm{fillrad}\,(M)$	fill radius of the manifold M
$E(X)$	space of ends of the connected locally finite CW-complex X
$e(X)$	number of ends of connected locally finite CW-complex X
$e(G)$	number of ends of the group G
$H_k(X; G)$	k-dimensional homology group of the topological space X with coefficients in G
$H^k(X; G)$	k-dimensional cohomology group of the topological space X with coefficients in G
$\tilde{H}^k(X; G)$	k-dimensional reduced cohomology group of the topological space X with coefficients in G
$H^k_c(X; G)$	k-dimensional cohomology group with compact support of the topological space X with coefficients in G
$H^k_e(X; G)$	k-dimensional cohomology of ends group with compact support of the topological space X with coefficients in G
$\mathcal{C}^\infty(M)$	space of smooth functions on the manifold M
$\mathcal{C}^\infty_c(M)$	space of smooth functions with compact support on the manifold M
$(\mathcal{G}, f)$	coloured graph