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Relatively Big Projective Modules, Direct Sum Decompositions of Modules, and Representations of Finite Groups

Román Álvarez Arias

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Facultat de Ciències

TESI DOCTORAL

Relatively Big Projective Modules, Direct Sum Decompositions of Modules, and Representations of Finite Groups

2025 Román Álvarez Arias

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Relatively Big Projective Modules, Direct Sum Decompositions of Modules, and Representations of Finite Groups

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Certifico que la present Memòria ha
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Dedication.

Este trabajo es el resultado de muchos años de esfuerzo, aprendizaje y, sobre todo, del apoyo de todas las personas que han estado a mi lado en diferentes momentos de este camino.

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Title: Relatively Big Projective Modules, Direct Sum Decompositions of Modules, and Representations of Finite Groups

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Abstract: In this dissertation we study relatively big projective modules and their applications to direct sum decompositions. The first part addresses the class of direct sums of finitely generated torsion-free modules over a commutative domain, focusing on its closure under direct summands. For local domains of Krull dimension 1, we characterize this closure in terms of endomorphism rings of indecomposable modules, and extend to h-local domains by establishing necessary and sufficient conditions involving two-generated ideals and integral closure. In the second part we show that in broad classes of rings —including right noetherian PI-rings, semiperfect rings, and suitable locally semiperfect torsion-free algebras over h-local domains— every countably generated projective module is relatively big. Going back to applications we include structural results for endomorphism rings of torsion-free modules and conditions ensuring decompositions into finitely generated summands. Finally, we extend the analysis to locally semiperfect algebras with semisimple artinian rings of quotients. This leads to precise criteria for the existence and classification of projective modules via dimension sequences and decomposition matrices, obtaining representation-theoretic descriptions of projectives. The main examples of this part are group algebras of finite groups over a suitable Dedekind domain.

Keywords: projective modules, torsion-free modules, direct sum decompositions, commutative domains, h-local domains, Krull dimension 1, relatively big projective modules, locally semiperfect algebras, decomposition matrices, group algebras

Frequently used notation

Λ	associative ring with 1 or algebra over a commutative domain R
R	commutative associative ring with 1
M, N	right Λ-module
P	projective right Λ-module
I, J	ideal of a ring Λ
$\mathfrak{m}, \mathfrak{n}$	maximal ideal of a commutative domain R
$\mathrm{J}(\Lambda)$	Jacobson radical of Λ
$\mathbb{N}_0$	monoid of non-negative integers
$\mathbb{N}_0^*$	monoid of non-negative integers together with ∞
$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rationals
ω	first infinite ordinal number
$\mathrm{M}_n(\Lambda)$	ring of $n \times n$ matrices with entries in the ring Λ
$M^{(A)}$	direct sum of a family of copies of a right Λ-module M indexed by the set A
$\mathrm{Hom}_\Lambda(M, N)$	abelian group of right Λ-module homomorphisms from M to N
$\mathrm{End}_\Lambda(M)$	endomorphism ring of a right module M
$\mathrm{Tr}_\Lambda(M)$	trace ideal of the right Λ-module M
$\mathrm{Add}(M)$	category of direct summands of a direct sum of an arbitrary number of copies of M
$\mathrm{add}(M)$	category of direct summands of a direct sum of a finite number of copies of M
$\mathrm{Add}_{\aleph_0}(M)$	category of direct summands of a direct sum of a countable number of copies of M
$\mathrm{V}(\Lambda)$	monoid of isomorphism classes of finitely generated projective right Λ-modules
$\mathrm{V}^*(\Lambda)$	monoid of isomorphism classes of countably generated projective right Λ-modules
$\mathrm{B}(\Lambda)$	semigroup of isomorphism classes of relatively big projective right Λ-modules which are not finitely generated
$R_\mathfrak{m}$	localization of a commutative domain R at the maximal ideal $\mathfrak{m}$
Q	field of fractions of a commutative domain R
$\Lambda_\mathfrak{m}$	localization of an algebra Λ over a commutative domain R at the maximal ideal $\mathfrak{m}$ of R
$I_\mathfrak{m}$	localization of an ideal I of an algebra Λ over a commutative domain R at the maximal ideal $\mathfrak{m}$ of R
$M_\mathfrak{m}$	localization of a right module M over an algebra Λ over a commutative domain R at the maximal ideal $\mathfrak{m}$ of R
Λ_Q	ring of quotients of Λ, that is, the tensor product of an algebra Λ over a commutative domain R and the field of fractions Q of R
I_Q	tensor product of an ideal I of an algebra Λ over a commutative domain R and the field of fractions Q of R
M_Q	tensor product of a module M over an algebra Λ over a commutative domain R and the field of fractions Q of R

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Introduction

This dissertation studies relatively big projective modules and their applications to direct sum decompositions. Relatively big projective modules are countably generated projective modules which are liftings of finitely generated projective modules modulo a trace ideal. They are a particular class of countably generated projective modules that can be well understood. In this dissertation, the focus is to study the cases where every countably generated projective module is relatively big.

A powerful tool is the equivalence between the category of direct summands of countably many copies of a finitely generated module M and the category of projective modules over the endomorphism ring of M. This allows us to exchange information from projective modules to direct summands of direct sums of finitely generated modules. Because of this, one of the main examples of rings we are studying in this dissertation is the endomorphism ring of a finitely generated torsion-free module over a commutative domain.

The first part of this dissertation explains preliminaries on projective modules, decompositions of projective modules over suitable rings, and the trace ideal of a projective module. We recall that every projective module is the direct sum of countably generated projective modules, a result proved by Kaplansky. This justifies why, from the beginning, we restrict to countably generated projective modules. The main tool used in this dissertation is that projective modules lift modulo a trace ideal.

The second part of this dissertation establishes precise criteria —both local and global— for when the class of direct sums of finitely generated torsion-free modules is closed under direct summands. Our global setting is that of h-local domains —commutative domains satisfying strong noetherianity conditions on the maximal spectrum; typical examples are noetherian domains of Krull dimension one— and we develop methods to pass from the local case to the global case. The results in this part connect directly to the work on generalized lattices by Butler, Rump, and Přihoda. This part corresponds to the first original paper in this dissertation [1].

The third part of this dissertation shifts focus to relatively big projective modules. By introducing the monoid notation, we develop a framework for determining when every countably generated projective module is relatively big. Two examples where this happens are locally semiperfect algebras over an h-local domain with simple artinian ring of quotients, and right noetherian rings satisfying a polynomial identity. The endomorphism ring of a suitable finitely generated torsion-free module over an h-local domain is an example of the first kind (linking with the previous part of the dissertation). The group algebra of a finite group over a commutative noetherian domain is an example of the latter (linking with integral representation theory of finite groups). Understanding these two examples was the initial motivation for this dissertation. This part corresponds to the second original paper in this dissertation [2].

In the following paragraphs, we unravel the core of this dissertation.

Torsion-free modules over commutative domains of Krull dimension 1

Let R be a commutative domain, and let Λ be an R-algebra. Let $\mathcal{F}_\Lambda$ denote the class of modules over Λ that are direct sums of finitely generated modules which are torsion-free over R (we should write $\mathcal{F}$ if the ring is clear). The class $\mathcal{F}_\Lambda$ is closed under arbitrary direct sums, and we inquire whether $\mathcal{F}_\Lambda$ is closed under direct summands.

If R is a semihereditary domain, then the modules in $\mathcal{F}_R$ are projective modules. It is well known that, in this case, all projective modules are direct sums of finitely generated ideals of R. In particular, $\mathcal{F}_R$ is the class of all projective modules, and it is closed under direct summands with no extra assumption on the Krull dimension of R.

The interest in these kinds of questions was awakened by the study of the so-called *generalized lattices*. Let R be a Dedekind domain with field of fractions Q, and let Λ be an R-order in a separable Q-algebra (hence, Λ is finitely generated and projective as an R-module). Classically, a right Λ-module is a lattice if it is finitely generated and projective as an R-module. Dropping the finitely generated condition from the definition of a lattice, we encounter the generalized Λ-lattices. In this context, the class $\mathcal{F}_\Lambda$ coincides with the (arbitrary) direct sums of Λ-lattices. If Λ is locally lattice-finite, the generalized lattices are precisely the direct summands of modules in $\mathcal{F}_\Lambda$. See Theorem 4 in [10] for the proof in the lattice-finite case and Corollary on p. 112 of [60] for its extension to the locally lattice-finite case.

In Theorem 4 of [60], W. Rump proved that if generalized Λ-lattices are direct sums of lattices, then Λ is locally lattice-finite. Moreover, he answered the question of when $\mathcal{F}_\Lambda$ is closed under direct summands, finding that the answer depended on representation-theoretical data of Λ.

Příhoda, in the paper [48], translated the question to $\Lambda = R$ being a noetherian domain of Krull dimension 1 with module-finite normalization. In the local case, he proved that $\mathcal{F}_R$ is closed under direct summands if and only if its normalization (which is assumed to be finitely generated) is local. In the global case, and under the assumption that R is, in addition, lattice-finite (i.e., there are only finitely many indecomposable finitely generated torsion-free R-modules up to isomorphism) he characterized when $\mathcal{F}_R$ is closed under direct summands by certain ring-theoretical properties of R that will also appear in Theorem B.

In this dissertation, we consider the case $\Lambda = R$ being a commutative domain of Krull dimension 1, and we also follow the strategy of first studying the case of a local domain and then extending it to a suitable global situation. For the case of local domains, we obtain a satisfactory characterization that is summarized in the following theorem.

Theorem A (Corollary 2.2.3, Corollary 2.2.9)**.** *Let R be a local domain of Krull dimension* 1*. The class $\mathcal{F}_R$ is closed under direct summands if and only if any finitely generated indecomposable torsion-free module has local endomorphism ring. If, in addition, R is noetherian, this is equivalent to having local integral closure.*

Therefore, in this situation, any module M in $\mathcal{F}_R$ can be written as $M = \bigoplus_{i\in I} N_i$, where each N_i is a finitely generated module with local endomorphism

ring, and such a decomposition is unique up to isomorphism. In addition, any direct summand of M is isomorphic to a direct sum of modules from the family $\{N_i\}_{i\in I}$.

Our global setting will be that of h-local domains. Matlis introduced h-local domains in the 1960s in [38] as commutative domains satisfying the condition that they have finite character, that is, every non-zero ideal is contained only in finitely many maximal ideals; and every non-zero prime ideal is contained only in one maximal ideal.

However, Jaffard appears to be the first author to give an equivalent definition of h-local domains in [28], but under the name of *domains of Dedekind type*. For the story and characterizations of h-local domains, the reader is referred to Fuchs and Salce's monograph (see Section IV.3 of [18]) and to the paper by Olberding (see Theorem 2.1 in [43]).

A domain of Krull dimension 1 is h-local if and only if it has finite character, and noetherian domains of Krull dimension 1 are always h-local (equivalently, always have finite character).

If R is a domain such that $\mathcal{F}_R$ is closed under direct summands, then, in particular, all projective modules are direct sums of finitely generated projective modules. Not much is known about projective modules over arbitrary domains. However, Hinohara [27] proved that, over an h-local domain, an infinitely generated projective module, which is not finitely generated, is always free.

The following theorem summarizes our results for this class of domains, which translates the property of being closed under direct summands to arithmetical properties between the ranks of the indecomposables at different maximal ideals and the minimal number of generators of the ideals.

Theorem B (Theorem 4.1.6, Corollary 4.2.3, Corollary 4.2.4, Proposition 4.4.6, Theorem 4.4.8)**.** *Let R be an h-local domain of Krull dimension 1. If the class $\mathcal{F}_R$ is closed under direct summands, then R satisfies that*

- **(1)** *$\mathcal{F}_{R_\mathfrak{m}}$ is closed under direct summands for any maximal ideal $\mathfrak{m}$ of R;*
- **(2)** *for any maximal ideal $\mathfrak{m}$ of R, except may be one, all finitely generated ideals of $R_\mathfrak{m}$ are at most two-generated;*
- **(3)** *the integral closure $\overline{R}$ of R in its field of fractions satisfies that $\mathcal{F}_{\overline{R}}$ is closed under direct summands.*

Conversely, if R is an h-local domain of Krull dimension 1 satisfying **(1)** *and* **(2)**, *then $\mathcal{F}_R$ is closed under direct summands in the following situations:*

- **(a)** *R is integrally closed, or*
- **(b)** *R is semilocal noetherian, or*
- **(c)** *R is noetherian and has module-finite normalization.*

That the closure of $\mathcal{F}_R$ under direct summands implies conditions **(1)** and **(2)** was already proven by Příhoda for noetherian domains of Krull dimension 1 with module-finite normalization [48]. The fact that a condition like **(2)** still holds in

the generality with which we are working has been an interesting surprise, and its proof is one of the more involved aspects of this part of the dissertation.

A good description for torsion-free modules over a local ring with the 2-generated property for finitely generated ideals is only available in the noetherian case: if R has module-finite integral closure, then it is a Bass domain, and the theory goes back to Bass' fundamental paper [6]. The general case of noetherian domains with two-generated ideals remained open for a long time and was finally settled by Rush in [61]. In both cases, it is proven that finitely generated torsion-free modules are isomorphic to a direct sum of (finitely generated) ideals. As far as we know, it is an open question whether such a result could still hold outside the noetherian case.

We also study the isomorphism classes of modules in $\mathcal{F}$. Recall that two modules M and N are in the same genus if and only if $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. For finitely generated modules, the isomorphism class of a module M is usually determined by its genus and some extra information about the *class group*. This is the case, for example, with Bass domains, as shown in Theorem 4.2 of [35], and for general noetherian domains with 2-generated ideals [61].

For infinite countable direct sums in $\mathcal{F}$, we show that the genus is enough to determine the isomorphism class. Now we state the precise result we prove; notice that it is for finitely generated R-algebras and that there is no assumption on the Krull dimension of the domain R.

Theorem C (Theorem 4.3.4)**.** *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that $\Lambda \otimes_R Q$ is a simple artinian ring. Let $M = \bigoplus_{i\in\mathbb{N}_0} A_i$ and $N = \bigoplus_{i\in\mathbb{N}_0} B_i$ be direct sums of non-zero finitely generated right Λ-modules which are torsion-free as R-modules. If M and N are in the same genus, then there are decompositions*

$$M = \bigoplus_{i\in\mathbb{N}_0} M_i \qquad \text{and} \qquad N = \bigoplus_{i\in\mathbb{N}_0} N_i$$

such that both M_i and N_i are finitely generated, and $M_i \cong N_i$ for every $i \in \mathbb{N}_0$. In particular, M and N are isomorphic.

Theorem C is related to a result proven by Rump in [60] for Λ a locally lattice-finite order in a separable algebra over a Dedekind domain R (cf. Theorem 3 in [60]).

Relatively big projective modules and their applciations to direct sum decompositions

For a long time, the study of projective modules has focused on the finitely generated ones. Outside the finitely generated setting, the classical theorem of Kaplansky [29] stating that every projective module is a direct sum of countably generated projective modules shows that one can consider only the countably generated case. Interesting monoids associated with infinitely generated projective modules have recently been studied in several different contexts: the monoid $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ of relatively big projectives, formally defined in [25], which is the main object of interest in this paper; the analogous Cuntz monoid for general rings defined in [4]; and the κ-monoids defined in [41].

Let Λ be a ring, and let $\mathrm{V}(\Lambda)$ (resp. $\mathrm{V}^*(\Lambda)$) denote the set of representatives of the isomorphism classes of finitely generated (resp. countably generated) projective right Λ-modules. These sets form commutative monoids with addition induced by the direct sum of modules, with $\mathrm{V}(\Lambda)$ naturally embedded in $\mathrm{V}^*(\Lambda)$.

The concept of relatively big projective modules was introduced by Příhoda in [46], extending Bass' uniformly big projectives [6].

Let J be the trace of a countably generated projective module. A countably generated projective right Λ-module P is said to be J-big if every countably generated projective Λ-module whose trace is contained in J is a homomorphic image of P. A countably generated projective module P is said to be *relatively J-big* if P is J-big and P/PJ is finitely generated. Here, J is maximal with respect to the property that P is J-big and minimal with respect to the property that P/PJ is finitely generated (cf. Section 5.2). A crucial property is that a relatively J-big projective module P is uniquely determined, up to isomorphism, by the trace ideal J and the isomorphism class of P/PJ (cf. Lemma 5.1.7).

The relatively big projectives are the countably generated projective modules that are relatively J-big for some J. The important fact that boils down to the concept of relatively big projective is that projective modules modulo a trace ideal can be lifted to projective modules over the ring. If the projective module over the quotient is finitely generated, then there is a unique lifting to a relatively big projective module over the ring.

The monoid $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ of isomorphism classes of relatively big projective modules was introduced in [25] to capture the properties of such projectives. This monoid embeds in $\mathrm{V}^*(\Lambda)$ and can be thought of as the *disjoint union* of $\mathrm{V}(\Lambda/J)$, where J runs through the trace ideals of the countably generated projective right Λ-modules, see Corollary 5.3.5.

In [46], it was proved that over a noetherian ring Λ which satisfies a certain descending condition on two-sided ideals, which we call the $(*)$-condition, $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ (cf. Lemma 2.8 in [46] and Theorem 5.2.4). Commutative noetherian rings, finitely generated algebras over commutative noetherian rings of Krull dimension one, and noetherian semilocal rings satisfy the $(*)$-condition. Therefore, all countably generated projective modules over these rings are relatively big. These results were the main ingredients used to characterize $\mathrm{V}^*(\Lambda)$ for Λ a noetherian semilocal ring in [22], or to study some infinite direct sum decompositions in [25].

The main results of this part of the thesis provide new classes of rings for which $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$. The following result summarizes all these results.

Theorem D (Corollaries 6.2.9 and 6.2.10, Theorem 6.3.17)**.** *Let Λ be a ring. Then $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ in the following situations:*

- **(a)** *Λ is a right noetherian PI-ring, in particular, if it is a module-finite algebra over a commutative noetherian ring,*
- **(b)** *Λ is a locally semiperfect torsion-free algebra over an h-local domain with field of fractions Q and such that Λ_Q is simple artinian.*

In particular, statement **(a)** of Theorem D applies to group algebras of finite groups over commutative noetherian rings or to endomorphism rings of finitely

generated modules over commutative noetherian rings. It is important to note that the restriction on Krull dimension imposed in previous works was superfluous.

Beyond the noetherian setting, it is easy to show that for semiperfect rings, all countably generated projective right (or left) modules are relatively big (cf. Proposition 6.3.4). Recall from Part II (cf. Lemma 2.1.11) that if Λ is the endomorphism ring of a finitely generated torsion-free module M over an h-local domain R, then Λ_Q is simple artinian. Then statement **(b)** of Theorem D applies to the resulting algebras of combining Theorems A and B.

When Λ is locally semiperfect and Λ_Q is simple artinian, we can give an equational description of the genus of every countably generated projective module. In particular, for every maximal ideal $\mathfrak{m}$ of R, there exists a finite set $\{U_{\mathfrak{m},1}, \ldots, U_{\mathfrak{m},\ell_\mathfrak{m}}\}$ of indecomposable projective modules over $\Lambda_\mathfrak{m}$ such that every countably generated projective module over $\Lambda_\mathfrak{m}$ can be written uniquely as a direct sum of copies of them. As Λ_Q is simple artinian, there exists a finitely generated simple module S over Λ_Q such that every countably generated projective module over Λ_Q can be written uniquely as a direct sum of copies of S.

For every maximal ideal $\mathfrak{m}$ of R, we can tensor each indecomposable projective $\Lambda_\mathfrak{m}$-module with Q and write it uniquely as a direct sum of copies of S. That is, for each maximal ideal $\mathfrak{m}$ of R and each $j = 1, \ldots, \ell_\mathfrak{m}$, there exists a unique $d_{\mathfrak{m},j} \in \mathbb{N}_0$ such that $U_{\mathfrak{m},j} \otimes_{R_\mathfrak{m}} Q \cong S_i^{d_{\mathfrak{m},j}}$. We say that $d_{\mathfrak{m},i}$ is the *length* of $U_{\mathfrak{m},i}$.

Consider a package $\{\boldsymbol{r}_\mathfrak{m} = (r_{\mathfrak{m},1}, \ldots, r_{\mathfrak{m},\ell_\mathfrak{m}}) \in \mathbb{N}_0^{\ell_\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{mSpec}(R)\}$. Our goal is to determine when a projective Λ-module P exists such that $P_\mathfrak{m} \cong \bigoplus_{i=1}^{\ell_\mathfrak{m}} U_{\mathfrak{m},i}^{r_{\mathfrak{m},i}}$. A set of such sequences $\{\boldsymbol{r}_\mathfrak{m}\}_{\mathfrak{m}\in\operatorname{mSpec}(R)}$ is called the *genus sequence* of P.

Theorem E (Lemma 6.3.9, Propositions 7.1.7 and 7.1.9)**.** *Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian, so that $\Lambda_Q \cong \mathrm{M}_k(D)$ for a suitable division ring D. Then*

- **(i)** *There exists a finite set of maximal ideals $\mathcal{S}(\Lambda)$ such that $\Lambda_\mathfrak{m} \cong \mathrm{M}_k(T_\mathfrak{m})$ for some local ring $T_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$. In particular, for every maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $\Lambda_\mathfrak{m}$ has exactly one indecomposable projective module up to isomorphism.*

- **(ii)** *For each maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, if U and V are projective $\Lambda_\mathfrak{m}$-modules, then $U \cong V$ are isomorphic if and only if $U_Q \cong V_Q$*

- **(iii)** *For each maximal ideal $\mathfrak{m} \in \mathcal{S}(\Lambda)$, let $d_{\mathfrak{m},1}, \ldots, d_{\mathfrak{m},\ell_\mathfrak{m}}$ be the lengths of the indecomposable projective right $\Lambda_\mathfrak{m}$-modules, and fix $\boldsymbol{r}_\mathfrak{m} \in (\mathbb{N}_0^*)^{\ell_\mathfrak{m}}$.*

  Then, there exists a countably generated projective right Λ-module P with genus sequence $\{\boldsymbol{r}_\mathfrak{m}\}_{\mathfrak{m}\in\operatorname{mSpec}(R)}$ if and only if there exists $r \in \mathbb{N}_0^*$ such that for every maximal ideal $\mathfrak{m}$ of R

$$\sum_{j=1}^{\ell_\mathfrak{m}} d_{\mathfrak{m},j} r_{\mathfrak{m},j} = \begin{bmatrix} d_{\mathfrak{m},1} & \cdots & d_{\mathfrak{m},\ell_\mathfrak{m}} \end{bmatrix} \begin{bmatrix} r_{\mathfrak{m},1} \\ \vdots \\ r_{\mathfrak{m},\ell_\mathfrak{m}} \end{bmatrix} = r.$$

As an application of the theory of relatively big projective modules developed in this part, we go back to consider Λ as the endomorphism ring of a finitely

generated torsion-free module M over an h-local domain R from the previous part. Recall that, in this situation, Λ_Q is simple artinian (cf. Lemma 2.1.22). If, in addition, R has Krull dimension 1 and all the direct summands of $\mathrm{Add}(R^{(\omega)} \oplus M)$ are again a direct sum of finitely generated modules, by Proposition 4.1.1, Λ is locally semiperfect. The converse of this result is not true in general, as additional conditions on the ranks of indecomposable modules over the localizations of R at its maximal ideals must be satisfied, see Theorem F.

Theorem F (Theorem 7.3.2)**.** *Let R be an h-local domain and let M be a non-zero finitely generated torsion-free R-module with endomorphism ring $\Lambda = \mathrm{End}_R(M)$. Consider the following statements,*

- **(i)** *Λ is locally semiperfect and if X and Y are indecomposable direct summands of $M_{\mathfrak{m}}$ and $M_{\mathfrak{n}}$, respectively, for some $\mathfrak{m} \neq \mathfrak{n} \in \mathrm{mSpec}(R)$. Then $\mathrm{rank}(X)$ and $\mathrm{rank}(Y)$ are coprime.*

- **(ii)** *Every module in $\mathrm{Add}(M)$ is a direct sum of finitely generated modules.*

Then **(i)** *always implies* **(ii)**, *and the converse is true if M is a generator and R has Krull dimension* 1.

We finish this dissertation by considering locally semiperfect algebras over a commutative domain R such that Λ_Q is semisimple artinian. Such algebras arise naturally in the representation theory of finite groups and, more generally, in the theory of maximal orders (see [11, 55] for a detailed discussion).

Let Λ be a locally semiperfect algebra such that Λ_Q is semisimple artinian. As before, since it is locally semiperfect, for every maximal ideal $\mathfrak{m}$ of R, there exists a finite set $\{U_{\mathfrak{m},1}, \dots, U_{\mathfrak{m},\ell_{\mathfrak{m}}}\}$ of indecomposable projective modules over $\Lambda_{\mathfrak{m}}$ such that every countably generated projective module over $\Lambda_{\mathfrak{m}}$ can be written uniquely as a direct sum of copies of them. As Λ_Q is semisimple artinian, there exists a finite set $\{S_1, \dots, S_k\}$ of finitely generated simple modules over Λ_Q such that every countably generated projective module over Λ_Q can be written uniquely as a direct sum of copies of them.

For every maximal ideal $\mathfrak{m}$ of R, we can tensor each indecomposable projective $\Lambda_{\mathfrak{m}}$-module with Q and write it uniquely as a direct sum of copies of the simple modules over Λ_Q. That is, for each maximal ideal $\mathfrak{m}$ of R and each $j = 1, \dots, \ell_{\mathfrak{m}}$, there exists a unique sequence $(d_{\mathfrak{m},1j}, \dots, d_{\mathfrak{m},kj}) \in \mathbb{N}_0^k$ such that $U_{\mathfrak{m},j} \otimes_{R_{\mathfrak{m}}} Q \cong \bigoplus_{i=1}^{k} S_i^{d_{\mathfrak{m},ij}}$. The set of matrices $\{\mathbf{D}_{\mathfrak{m}} = (d_{\mathfrak{m},ij}) \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$ is called the set of *decomposition matrices* associated with Λ. Their name comes from the integral representation theory (cf. Definition 16.20 in [11]).

The natural question is the same as for Theorem E, and the answer is precisely related to the decomposition matrices of Λ.

Theorem G (Corollary 8.2.5)**.** *Let R be a noetherian domain of Krull dimension* 1 *with field of fractions Q. Let Λ be a locally semiperfect module-finite torsion-free R-algebra such that Λ_Q is semisimple artinian. Let $\{\mathbf{D}_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$ denote the decomposition matrices associated with Λ. For each maximal ideal $\mathfrak{m}$ of R, let $U_{\mathfrak{m},1}, \dots, U_{\mathfrak{m},\ell_{\mathfrak{m}}}$ be a set of representatives of the indecomposable projective right $\Lambda_{\mathfrak{m}}$-modules, and fix $\boldsymbol{r}_{\mathfrak{m}} \in (\mathbb{N}_0^*)^{\ell_{\mathfrak{m}}}$.*

Then, there exists a countably generated projective right Λ-module P with genus sequence $\{\boldsymbol{r}_{\mathfrak{m}}\}_{\mathfrak{m}\in \mathrm{mSpec}(R)}$ if and only if for any pair of maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R

$$\mathbf{D}_{\mathfrak{m}}\boldsymbol{r}_{\mathfrak{m}}^{t} = \mathbf{D}_{\mathfrak{n}}\boldsymbol{r}_{\mathfrak{n}}^{t}.$$

Thesis structure

Now, we briefly describe the content of the different chapters of this dissertation, taking the opportunity to highlight some of the main ideas. This dissertation is divided into three clearly distinguished parts.

Part I. We present a collection of established results that will serve as a foundation for the later parts of the dissertation. We begin in Chapter 1 with the definition of a projective module and a review of their fundamental properties. In the following sections, we introduce the notion of a trace ideal and we exploit the fact that projective modules can be lifted modulo the trace ideal of a projective module (cf. Theorems 1.5.4). We apply it to construct possibly infinitely generated direct summands by lifting modules over the endomorphism ring of a module modulo a trace ideal (cf. Theorems 1.5.5 and Corollary 1.5.12). It is also used implicitly for relatively big projective modules when we see them as liftings of finitely generated projective modules modulo a trace ideal. This machinery for projective modules was developed in the noetherian setting in [46], extended to general rings in [24], and the connection with the study of the category $\mathrm{Add}(M)$, of direct summands of direct sums of copies of a finitely generated module M, was developed in [25].

Part II. This second part is divided into three chapters, each corresponding to the paper [1].

To approximate the problem of whether the class $\mathcal{F}$ is closed under direct summands, we begin by considering the local case in Chapter 2.

Section 2.1 is quite technical but essential for the rest of this part of the dissertation. In it, we survey the results required on finitely generated (torsion-free) and/or finite rank modules over a domain R, as well as the properties of their endomorphism rings. Given a domain R, we also need to treat the case of finitely generated modules over an R-algebra Λ that is torsion-free as R-modules. We also introduce the concepts of a domain of finite character and of an h-local domain, developing the first basic properties of endomorphism rings of finitely generated modules over these classes of rings. These are domains that satisfy strong noetherianity conditions on the maximal spectrum, more precisely, any non-zero ideal of R is contained in only a finite number of maximal ideals, and any non-zero prime ideal of R is contained in a unique maximal ideal. In particular, noetherian domains of Krull dimension one are always h-local. The definition of h-local domain is attributed to Matlis. However, Jaffard in [28] introduced Dedekind-type domains as those domains R such that any non-zero ideal is a finite product of ideals that are contained in a unique maximal ideal of R. Jaffard proved that Dedekind-type domains coincide with h-local domains (cf. Theorem 2.1.5 in [17]).

In Section 2.2, we characterize local domains of Krull dimension 1 that satisfy the condition that $\mathcal{F}$ is closed under direct summands. The results in this section prove Theorem A.

In Section 2.3, we develop tools to establish a relationship between the property of being locally a direct summand and being a direct summand in the context of finitely generated torsion-free modules. If R is a domain of finite character, and N and M are finitely generated torsion-free modules such that $N \otimes_R Q$ is a direct summand of $M \otimes_R Q$, then $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R. This implies that we only have a finite number of "problematic maximal ideals." In the remainder of the section, we prove several results on how to proceed for these problematic maximal ideals. Analogous results were proven in the one-dimensional noetherian setting in [21] and were inspired by the representation theory of orders.

In Chapter 3, we turn to the study of finitely generated modules over an algebra Λ over an h-local domain R. We establish the foundational results that serve as a bridge between the local and global cases. Subsequently, we address the following question: Given a family of right $\Lambda_{\mathfrak{m}}$-modules $\{X(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec} R\}$, when does there exist a right Λ-module N such that $N_{\mathfrak{m}} \cong X(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R? This question has been previously explored by Levy and Odenthal in [34] for the specific case of R being a noetherian domain of Krull dimension 1. Their findings indicate that the existence of N is contingent upon the fulfillment of mild, compatible conditions between the localizations. In this work, we generalize Levy-Odenthal's Package Deal Theorem to localizations over h-local domains (cf. Theorem 3.2.3). Additionally, we present a generalized version for localizations of projective modules (cf. Theorem 3.3.4) and another for localizations of trace ideals of countably generated projective modules (cf. Theorem 3.4.3).

The case $\Lambda = R$ presents a particularly elegant solution. The localization of a finitely generated torsion-free module at a maximal ideal is free for almost all maximal ideals (cf. Corollary 3.1.3). Conversely, a family $\{X(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec} R\}$ of $R_{\mathfrak{m}}$-modules can be "glued together" to form a finitely generated torsion-free R-module N provided that all the modules $X_{\mathfrak{m}}$ possess the same rank and are free for almost all maximal ideals $\mathfrak{m}$ (cf. Corollary 3.2.4).

Finally, we conclude this section with Chapter 4. In Section 4.1, we utilize the theory developed in the preceding sections to establish the first part of Theorem B. An important intermediate result is Theorem 4.1.2, which demonstrates that if $\mathcal{F}_R$ is closed under direct summands, then finitely generated indecomposable torsion-free modules over distinct localizations at maximal ideals must have coprime ranks. This type of result was initially identified by Příhoda in [48], and our proof adheres to the same strategy as the original result.

In Section 4.2, the remaining portion of the characterization of integrally closed domains with $\mathcal{F}$ closed under direct summands, as contained in Theorem B, is proven.

In Section 4.3, we provide the proof of Theorem C. This section also includes examples demonstrating that, over a semilocal domain, being a direct summand of an infinite direct sum of finitely generated torsion-free modules could be satisfied locally but not globally. This contrasts with the case of finitely generated modules (cf. Corollary 2.3.5).

The noetherian part of Theorem B is proven in Section 4.4. We emphasize

that Theorem 4.4.8 provides the converse of one of the main results in [48].

We conclude this chapter in Section 4.5, with a family of examples of h-local domains of Krull dimension 1 that satisfy Theorem B. These examples are of the form $R = K + xL[x]$ where $K \subseteq L$ is a field extension. If such an extension is finitely generated, then R is noetherian. If the extension is transcendental, then R is integrally closed in its field of fractions (and it is not a Prüfer domain!). Moreover, under mild restrictions over the dimension of the field extension, R has indecomposable finitely generated torsion-free modules of all ranks.

Part III. This part is divided into four chapters. The first three chapters correspond to [2], while the fourth chapter contains unpublished results.

In Chapter 5.1, we provide the basic definitions of monoids of infinitely generated projective modules and their notation. In Sections 5.1 and 5.2, we introduce the necessary background, presenting the concept of a relatively big projective module, the $(*)$-condition, and the primary results relating both notions. In Section 5.3, we examine the monoid of isomorphism classes of relatively big projective modules, its algebraic properties, and investigate the right/left (non)symmetry of the monoids, see Example 5.3.7 and Corollary 5.3.9. We also recall here that $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ if Λ is a left and right noetherian ring satisfying the $(*)$-condition (cf. Theorem 5.3.10).

In Chapter 6, we aim to expand the classes of rings Λ for which $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$. We start in Section 6.1 by proving that countably generated projective right modules lift modulo a right T-nilpotent ideal, as shown in Lemma 6.1.2. We establish that for a right T-nilpotent ideal N of Λ, there exist monoid isomorphisms between $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ and $\mathrm{V}(\Lambda/N) \sqcup \mathrm{B}(\Lambda/N)$ (cf. Corollary 6.1.4).

In Section 6.2, these results are crucial in proving that right noetherian PI-rings satisfy the $(*)$-condition (cf. Theorem 6.2.8) and concluding that $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ for right noetherian PI-rings (cf. Corollary 6.2.9). In particular, this holds for module-finite algebras over commutative noetherian rings (cf. Corollary 6.2.10).

In Section 6.3, we turn to semiperfect rings, and locally semiperfect algebras over h-local domains. In Proposition 6.3.4, we observe that $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ for semiperfect rings. Subsequently, we investigate locally semiperfect algebras Λ over an h-local domain R with field of fractions Q such that Λ_Q is a simple artinian module. We show that for such algebras, every trace of a projective module is the trace of a finitely generated projective module (cf. Lemma 6.3.7) and that every countably generated projective module is relatively big. Equivalently, that $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ (cf. Theorem 6.3.17).

Chapter 7 deals with the applications of the previous results. In Section 7.1, we study in detail the genus of countably generated projective modules over locally semiperfect algebras over h-local domains with such algebras. In Proposition 7.1.1, we show that two countably generated projective modules that are not finitely generated are isomorphic if and only if they are in the same genus. This is also true for finitely generated projective modules, provided Λ is a finitely generated algebra over a semilocal domain R, cf. Proposition 7.1.2.

These tools allow us to study in Section 7.2 when every projective module is a direct sum of finitely generated ones. Finally, in Section 7.3, we apply our results to Λ the endomorphism ring of a finitely generated torsion-free module M over a h-local domain R of Krull dimension 1. In Theorem 7.3.2, we give necessary and

sufficient conditions that imply that when any direct summand of an arbitrary direct sum of copies of M is a sum of finitely generated modules. These results complement and clarify the ones obtained in Part II.

Finally, in Chapter 8, we extend the results of the previous sections to locally semiperfect algebras when Λ_Q is semisimple artinian. These algebras naturally arise in the integral representation theory of finite groups, specifically in the group algebras of finite groups over a particular commutative noetherian domain.

In Section 8.1, we establish analogous results to those presented in Section 6.3, changing Λ_Q to be semisimple artinian. The conclusions drawn in this section align with the classical concepts of maximal orders in Number Theory. Consequently, we start by revisiting the foundational knowledge pertaining to maximal orders over Dedekind domains (cf. Lemma 8.1.6). Subsequently, we study the structure of a locally semiperfect algebra modulo a trace ideal of a countably generated projective right Λ-module (as shown in Lemmas 8.1.8 and 8.1.10). Concluding this section, we present a Package Deal Theorem for projectives (cf. Theorem 8.1.13) and for idempotent ideals (cf. Proposition 8.1.14) in this context.

In Section 8.2, we introduce the concept of decomposition matrices associated with a locally semiperfect algebra Λ over a commutative domain R with semisimple artinian ring of quotients Λ_Q.

In Section 8.3, we review previously known results about idempotent ideals of group algebras of finite groups over a noetherian domain (see Theorem 8.3.1). By a result of Whitehead [72], over a noetherian ring, trace ideals are precisely the idempotent ideals of the ring. Therefore, to classify every relatively large projective module over a group algebra, we need to know how to compute every idempotent ideal of $\Lambda = RG$. We distinguish between the idempotent ideals such that extended to QG are QG, and those that extended to QG are the augmentation ideal of QG, which is an idempotent ideal whenever G is perfect (that is, $[G, G] = G$, cf. Proposition 8.3.4). We provide an algorithm that allows us to compute every full idempotent ideal in a group algebra with respect to either $\mathrm{Aug}(\Lambda)$ or Λ (see Algorithm 8.3.6).

In Section 8.4, we apply the results presented in this chapter to provide classifications of infinitely generated projective modules over group algebras of finite groups. Our focus is on the first two simple alternating groups, namely A_5 and A_6. The second group serves as an illustrative example in which the quotient Λ/I possesses both artinian and non-artinian indecomposable projective modules for an idempotent ideal I of Λ.

Throughout this dissertation, rings are associative with 1, and morphisms are unital. Our main topic is torsion-free modules over commutative domains, so most of the time R is a commutative domain with field of fractions Q. We reserve Λ to denote a non-necessarily commutative ring or an algebra over a domain R. If M is a finitely generated Λ-module which is torsion-free as an R-module, we denote by S its endomorphism ring.

Part I

Preliminaries

1 Projective modules and trace ideals

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In this introductory chapter, we give the basic concepts of projective modules, their decomposition properties, and their lifting properties modulo a trace ideal. This will put the basis for what is going to be developed in this dissertation.

In Section 1.1, we introduce the definition of a projective module, give some characterizations and examples, and recall an important result: projective modules are completely determined modulo the radical (cf. Proposition 1.1.8).

In Section 1.2, we study several classes of rings for which projective modules admit particularly well-behaved decompositions. We recall rings in which every projective module is free, as well as on those where projective modules decompose as direct sums of rank-one projectives. In addition, we introduce semisimple artinian and semiperfect rings, which play a central role in the study of direct sum decompositions presented in Part III. Of special significance is a theorem due to Kaplansky, which states that every projective module can be written as a direct sum of countably generated projective modules (cf. Theorem 1.2.1).

In Section 1.3, we briefly introduce the concept of a pure-projective module, which constitutes a weaker notion than that of a projective module. Although pure-projective modules do not play a central role in this dissertation, much of the analysis in Part II originated from the question of when every pure-projective module can be expressed as a direct sum of finitely presented modules.

In Section 1.4, we define what a trace ideal is, an ideal closely related to projective modules, and prove some basic properties of it.

Finally, in Section 1.5, we exploit that projective modules can be lifted modulo a trace ideal (cf. Theorem 1.5.5). We characterize the trace ideals of projective S-modules (cf. Lemma 1.5.10), where S is the endomorphism ring of a finitely generated module M over a ring Λ. We apply this fact to lift modules —and direct sums of finitely generated modules— over S modulo a trace ideal to construct *interesting and possibly infinitely generated* direct summands of $\Lambda^{(\omega)} \oplus M$ (cf. Corollary 1.5.12). We close this section by considering Λ to be semiperfect and how modules over S modulo a trace ideal lift under assumptions on the direct summands of $\Lambda^{(\omega)} \oplus M$ (cf. Proposition 1.5.16).

1.1 Definition and characterizations

Definition 1.1.1. Let Λ be a ring. A right Λ-module P is said to be *projective* if it satisfies the following *lifting property*: if $\varphi\colon M \longrightarrow\!\!\!\!\!\rightarrow N$ is an epimorphism of right Λ-modules, then every Λ-module homomorphism from P into N lifts to a Λ-module homomorphism into M, i.e., given $f \in \mathrm{Hom}_\Lambda(P, N)$ there is a lift $F \in \mathrm{Hom}_\Lambda(P, M)$ making the following diagram commute:

$$\begin{array}{ccccc} & & P & & \\ & \overset{F}{\swarrow} & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0. \end{array}$$

The following proposition presents several equivalent characterizations of projective modules.

Proposition 1.1.2. *Let Λ be a ring, and let P be a right Λ-module. Then the following are equivalent:*

(i) *P is projective.*

(ii) *The functor $\mathrm{Hom}_\Lambda(P, -)$ from the category of right Λ-modules to the category of abelian groups is exact, i.e., for any right Λ-modules L, M, and N, if*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \mathrm{Hom}_\Lambda(P, L) \longrightarrow \mathrm{Hom}_\Lambda(P, M) \longrightarrow \mathrm{Hom}_\Lambda(P, N) \longrightarrow 0 \tag{1.1}$$

is also a short exact sequence.

(iii) *Every short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$$

splits, i.e., if P is a quotient of the right Λ-module M then P is isomorphic to a direct summand of M.

(iv) *P is a direct summand of a free Λ-module.*

(v) **(Dual Basis Lemma)** *P has a dual basis, i.e., for some index set I, there is a family $\{x_i\}_{i\in I}$ in P and a family $\{\varphi_i\}_{i\in I}$ in $\mathrm{Hom}_\Lambda(P, \Lambda)$ such that for each $x \in P$, $\varphi_i(x) = 0$ for almost all $i \in I$, and $x = \sum_{i\in I} x_i\varphi_i(x)$.*

Proof. One can find a proof of this result in [13, 30, 59]. □

Remark 1.1.3. If P is a finitely generated projective module, then P is a direct summand of a finitely generated free module. Indeed, if $P = \sum_{i=1}^n x_i\Lambda$, then P is a direct summand of Λ^n.

Example 1.1.4. Let Λ be a ring, and let e be an idempotent in Λ. Then

$$\Lambda = e\Lambda \oplus (1-e)\Lambda,$$

so $P = e\Lambda$ is a projective right Λ-module. One can check that (e, ι) is a dual basis of P, where $\iota\colon P \longrightarrow \Lambda$ denotes the inclusion map. Indeed, if $a = e\lambda$, then

$$e\iota(a) = ea = e^2\lambda = e\lambda = a.$$

Another important property of projective modules is the following.

Proposition 1.1.5. *A direct sum of modules is projective if and only if every direct summand is projective.*

In general, the direct product of projective modules need not be projective. The following example is attributed to Baer. For a proof of this fact, the reader can check Lemma 2.8' in [30].

Example 1.1.6. The direct product $\mathbb{Z}^\omega = \mathbb{Z}\times\mathbb{Z}\times\cdots$ is not a projective $\mathbb{Z}$-module.

For a ring Λ, the intersection of all the maximal left ideals is equal to the intersection of all the maximal right ideals. This common intersection is a two-sided ideal of Λ called the *Jacobson radical* of Λ and denoted $\mathrm{J}(\Lambda)$. The Jacobson radical is also one of each of the following sets:

(a) $\{x \in \Lambda \mid 1 - rx$ has a left inverse for all $r \in \Lambda\}$.

(b) $\{x \in \Lambda \mid 1 - xs$ has a right inverse for all $s \in \Lambda\}$.

(c) $\{x \in \Lambda \mid 1 - rxs$ has a two-sided inverse for all $r, s \in \Lambda\}$.

(cf. Theorem 15.3 in [3]). The Jacobson radical $\mathrm{J}(\Lambda)$ is also the annihilator of the class of all simple right Λ-modules (cf. Corollary 15.5 in [3]). A useful result related to the Jacobson radical is the following.

Lemma 1.1.7 (Nakayama)**.** *Let Λ be a ring, and let I be an ideal of Λ. The following are equivalent:*

(i) $I \subseteq \mathrm{J}(\Lambda)$

(ii) *For every finitely generated right Λ-module M, if $MI = M$, then $M = \{0\}$.*

(iii) *For every finitely generated right Λ-module M, if N is a submodule of M and $N + M\,\mathrm{J}(\Lambda) = M$. Then $N = M$.*

Proof. For a proof of this result, see Corollary 15.13 in [3]. □

It is a well-known result that finitely generated projective modules are isomorphic if and only if they are isomorphic modulo the Jacobson radical (cf. Proposition 17.18 in [3]). In [45], Příhoda proved that this is also valid for general projective modules.

Proposition 1.1.8 (Theorem 2.3 in [45])**.** *Let Λ be a ring, and let P, Q be projective right Λ-modules. Then $P \cong Q$ if and only if $P/P\,\mathrm{J}(\Lambda) \cong Q/Q\,\mathrm{J}(\Lambda)$.*

1.2 Direct summand decomposition of projective modules

We now shift our focus to the structure of projective modules. It is well known that projectives over local rings or over principal ideal domains are free (cf. [29] and Corollary 2.27 in [30]), while projectives over Dedekind domains can be decomposed into direct sums of projectives of rank 1. Moreover, finitely generated projective modules over Dedekind domains are completely determined by the class group of the ring (cf. Theorem 1.4.12 in [58]).

There is another important instance of a ring where every *finitely generated* projective module is free, which is the polynomial ring over the field K in $n \geq 1$ variables, $K[x_1, \ldots, x_n]$. This is a famous theorem of Quillen [54] and Suslin [65], also known as Serre's conjecture. This result is not valid if we assume K to be a non-commutative division ring instead of a field (cf. Proposition 17.6 in [44]).

The following result from Kaplansky simplifies the study of projective modules to countably generated projective modules.

Theorem 1.2.1 (Kaplansky, [29])**.** *Every projective module is the direct sum of countably generated modules.*

This theorem follows directly from the subsequent lemma, which we present without proof.

Lemma 1.2.2 (Theorem 1 in [29])**.** *If a module M is the direct sum of modules that are countably generated, then every direct summand of M is a direct sum of submodules which are countably generated.*

In this dissertation, we will make repeated use of the following remark.

Remark 1.2.3 (Eilenberg's Trick). Let Λ be a ring. Let P be a countably generated projective right Λ-module, and let M be a direct summand of P. Then $M \oplus P^{(\omega)} \cong P^{(\omega)}$. Indeed, since M is a direct summand of P, there exists a right projective Λ-module N such that $P = N \oplus M$. Then

$$\begin{aligned} M \oplus P^{(\omega)} &\cong M \oplus (N \oplus M) \oplus (N \oplus M) \oplus \cdots \\ &\cong (M \oplus N) \oplus (M \oplus N) \oplus \cdots = P^{(\omega)}. \end{aligned}$$

In order to understand direct sum decompositions of modules into indecomposable direct summands, one wants to have a necessary condition to ensure that the decomposition is unique up to isomorphism. The following result will be useful in Part II, when we consider the question of when the class of direct sums of finitely generated torsion-free modules is closed under direct summands.

Theorem 1.2.4 (Krull-Schmidt-Remak-Azumaya)**.** *Let Λ be a ring. Let M be a right Λ-module that is a direct sum of modules with local endomorphism rings. Let $\{M_i\}_{i \in I}$ and $\{N_j\}_{j \in J}$ be two families of indecomposable submodules of M such that*

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j.$$

Then there is a bijective map $\varphi \colon I \longrightarrow J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$.

Proof. For a proof, see Theorem 2.12 in [14]. □

There is still an open classical question called Matlis' problem (cf. page 517 in [36]). Let Λ be a ring and $M = \bigoplus_{i\in I} E_i$ be a right Λ-module that is a direct sum of indecomposable injective modules. Is every direct summand of M also a direct sum of indecomposable injective modules? Equivalently, is the class of direct sums of indecomposable injective modules closed under direct summands? If Λ is right noetherian this is true (cf. Theorem 2.5 in [36]), because in this case the module M is injective (cf. Proposition 18.13 in [3]), so every direct summand of M is a direct sum of indecomposable injective modules (cf. Theorem 25.6 in [3]). There are other partial answers in the literature.

On the other hand, (semi)simple artinian rings and semiperfect rings are interesting classes of rings where projective modules have a unique decomposition into finitely generated indecomposable projective modules. These will appear in Part III. Recall that a non-zero module M is *simple* if 0 and M are the only submodules of M. A ring Λ is *simple* if 0 and Λ are the only two-sided ideals of Λ. A module M over a ring Λ is said to be a *semisimple module* if it is the direct sum of a family of simple submodules.

Remark 1.2.5. Let $\Lambda = \mathrm{M}_n(D)$ for some $n \in \mathbb{N}$ and some division ring D. We will show that Λ_Λ is a direct sum of n pairwise isomorphic simple right Λ-modules, and ${}_\Lambda\Lambda$ is a direct sum of n pairwise isomorphic simple left Λ-modules.

For each $i, j = 1, \ldots, n$, let E_{ij} denote the $n \times n$ matrix with a 1 in the (i, j)-entry and zeros elsewhere. For each $i = 1, \ldots, n$, consider the right ideals $I_i = E_{ii}\mathrm{M}_n(D)$ and $J_i = \mathrm{M}_n(D)E_{ii}$. Note that

$$I_1 = \begin{bmatrix} D & \cdots & D \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \qquad \ldots, \qquad I_n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ D & \cdots & D \end{bmatrix}$$

and

$$J_1 = \begin{bmatrix} D & \cdots & 0 \\ \vdots & \ddots & \vdots \\ D & \cdots & 0 \end{bmatrix}, \qquad \ldots, \qquad J_n = \begin{bmatrix} 0 & \cdots & D \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D \end{bmatrix},$$

so $\Lambda_\Lambda = I_1 \oplus \cdots \oplus I_n$ as right Λ-modules and ${}_\Lambda\Lambda = J_1 \oplus \cdots \oplus J_n$ as left Λ-modules.

Since $I_i = E_{ij}I_j$ and $J_i = J_jE_{ji}$ for every $i, j = 1, \ldots, n$, the map given by $x \mapsto E_{ij}x$ is an isomorphism of *right* Λ-modules between I_j and I_i, with the inverse given by $y \mapsto E_{ji}y$ (similarly, the map given by $x \mapsto xE_{ji}$ is an isomorphism of *left* Λ-modules between J_j and J_i, with the inverse given by $y \mapsto yE_{ij}$).

Fix some $i \in \{1, \ldots, n\}$. Let U be a non-zero submodule of $I_i = E_{ii}\mathrm{M}_n(D)$, and let $x \in U \setminus \{0\}$. Then, there exist $a_1, \ldots, a_n \in D$ such that $x = \sum_{j=1}^n E_{ij}a_j$. Fix some $k \in \{1, \ldots, n\}$ such that $a_k \neq 0$, and let $y = E_{ki}a_k^{-1} \in U$. Then

$$xy = \Big(\sum_{j=1}^n E_{ij}a_j\Big)E_{ki}a_k^{-1} = \sum_{j=1}^n E_{ij}E_{ki}a_ja_k^{-1} = E_{ik}E_{ki}a_ka_k^{-1} = E_{ii}.$$

Therefore, $E_{ii} \in U$, so $I_i \subseteq U$, and hence I_i is a simple *right* Λ-module for every $i = 1, \ldots, n$. Similarly, J_i is a simple *left* Λ-module for every $i = 1, \ldots, n$.

Therefore, Λ_Λ is a direct sum of n pairwise isomorphic simple right Λ-modules, and ${}_\Lambda\Lambda$ is a direct sum of n pairwise isomorphic simple left Λ-modules.

Theorem 1.2.6. *Let Λ be a ring. The following are equivalent:*

(i) *Λ is a semisimple right Λ-module.*

(ii) *Λ is a semisimple left Λ-module.*

(iii) *Every right (resp. left) Λ-module is semisimple.*

(iv) *Every right (resp. left) Λ-module is projective.*

(v) *The ring Λ is right artinian and $\mathrm{J}(\Lambda) = \{0\}$.*

(vi) *The ring Λ is left artinian and $\mathrm{J}(\Lambda) = \{0\}$.*

Proof. For a proof, see Theorem 1.2 in [14] or Theorem 4.4 in [20]. □

Definition 1.2.7. A ring Λ that satisfies any of the equivalent conditions of Theorem 1.2.6 is said to be a *semisimple artinian ring*.

Theorem 1.2.8 (Wedderburn-Artin)**.** *Let Λ be a ring. Then Λ is semisimple artinian if and only if either Λ is the zero ring or $\Lambda \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for some positive integers $n_1, \ldots, n_k$ and some division rings $D_1, \ldots, D_k$.*

Moreover, there are exactly k isomorphism classes of simple right Λ-modules, say $S_1, \ldots, S_k$, such that for every $i = 1, \ldots, k$, $\mathrm{End}_\Lambda(S_i) \cong D_i$, and $\dim_{D_i} S_i = n_i$.

Proof. For a proof, one can check Theorem 3.5 in [31]. □

Definition 1.2.9. A ring Λ is called *semiperfect* if $\Lambda/\mathrm{J}(\Lambda)$ is semisimple artinian, and idempotents of $\Lambda/\mathrm{J}(\Lambda)$ can be lifted to Λ.

If Λ is a semiperfect ring, then every finitely generated right Λ-module M has a *projective cover*, that is, an epimorphism $\theta\colon P \longrightarrow M$, where P is a projective right Λ-module and $\ker\theta$ is a small submodule of P (cf. Theorem 3.6 in [14]).

Lemma 1.2.10 (Theorem 3.10 in [14])**.** *Let Λ be a semiperfect ring, and let $\{U_1, \ldots, U_\ell\}$ be a set of representatives of the isomorphism classes of indecomposable projective right Λ-modules. Then, for every projective right Λ-module P, there exist sets $A_1, \ldots, A_\ell$, uniquely determined up to cardinality, such that $P \cong U_1^{(A_1)} \oplus \cdots \oplus U_r^{(A_\ell)}$.*

Before proceeding, we fix notation for the categories of (finitely generated) right Λ-modules, for the subcategory of (finitely generated) projective right Λ-modules, and for the additive subcategories $\mathrm{Add}(M)$ and $\mathrm{add}(M)$ generated by a right Λ-module M.

Notation 1.2.11. Given a ring Λ, let $\mathbf{Mod}_\Lambda$ denote the category of all right Λ-modules, and let $\mathbf{mod}_\Lambda$ denote the category of all finitely generated right Λ-modules.

If M is a right Λ-module, let $\mathrm{Add}(M)$ denote the full subcategory of $\mathbf{Mod}_\Lambda$ whose objects are all the modules isomorphic to direct summands of arbitrary direct sums of copies of M, and let $\mathrm{add}(M)$ denote the full subcategory of $\mathbf{mod}_\Lambda$ whose objects are all the modules isomorphic to direct summands of direct sums

of a finite number of copies of M. It is also convenient to consider the full subcategory $\mathrm{Add}_{\aleph_0}(M)$ of $\mathbf{Mod}_\Lambda$ whose objects are all the modules isomorphic to direct summands of $M^{(\omega)}$.

Note that $\mathrm{Add}(\Lambda)$ ($\mathrm{add}(\Lambda)$) is the category of (finitely generated) projective right Λ-modules. While $\mathrm{Add}_{\aleph_0}(\Lambda)$ is the category of countably generated projective right Λ-modules.

It is possible to transfer the information about projective modules over the endomorphism ring of a finitely generated module M to $\mathrm{Add}(M)$, via the following well-known result from Dress (cf. [12]).

Proposition 1.2.12 (Theorem 4.7 in [14])**.** *Let Λ be a ring, and let M be a right Λ-module. Let $S = \mathrm{End}_\Lambda(M)$. Then the functors*

$$\mathrm{Hom}_\Lambda(M, -)\colon \mathbf{Mod}_\Lambda \longrightarrow \mathbf{Mod}_S \quad \textit{and} \quad - \otimes_S M\colon \mathbf{Mod}_S \longrightarrow \mathbf{Mod}_\Lambda$$

induce an equivalence between $\mathrm{add}(M)$ *and* $\mathrm{add}(S)$*. Moreover, if M is finitely generated, then they induce an equivalence between* $\mathrm{Add}(M)$ *and* $\mathrm{Add}(S)$*. This equivalence restricts to an equivalence between* $\mathrm{Add}_{\aleph_0}(M)$ *and* $\mathrm{Add}_{\aleph_0}(S)$*.*

1.3 Pure-projective modules

In this section, we introduce a weaker notion of projective modules, namely, pure-projective modules. They were introduced by Warfield in [71], and they are projective modules with respect to pure short exact sequences. We recall the definition of pure-exact sequences.

Definition 1.3.1. Let Λ be a ring. A short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{1.2}$$

of right Λ-modules is *pure exact* if the induced sequence of abelian groups

$$0 \longrightarrow L \otimes_\Lambda D \longrightarrow M \otimes_\Lambda D \longrightarrow N \otimes_\Lambda D \longrightarrow 0$$

is exact for every left Λ-module D. A submodule L of a right Λ-module M is a *pure submodule* if the canonical short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0$ is pure. A right ideal I of Λ is a *pure ideal* if I is a pure submodule of Λ as a right Λ-module.

Recall that a right Λ-module M is *flat* if and only if $M \otimes_\Lambda -$ is an exact functor from the category of right Λ-modules to the category of abelian groups.

Lemma 1.3.2 (Theorem 4.85 in [30])**.** *A right Λ-module N is flat if and only if any short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ of right Λ-modules is pure.*

Corollary 1.3.3. *Let Λ be a ring. A right ideal I of Λ is pure if and only if the quotient ring Λ/I is a flat right Λ-module.*

Proof. Assume that I is a pure ideal. That is, the short exact sequence $0 \longrightarrow I \longrightarrow \Lambda \longrightarrow \Lambda/I \longrightarrow 0$ is pure. Then the induced sequence of abelian groups $0 \longrightarrow I \otimes_\Lambda D \longrightarrow \Lambda \otimes_\Lambda D \longrightarrow \Lambda/I \otimes_\Lambda D \longrightarrow 0$ is exact for every left Λ-module D. Therefore, $\mathrm{Tor}_1^\Lambda(\Lambda/I, D) = 0$ for every left Λ-module D, so Λ/I is flat.

Conversely, assume that Λ/I is flat. Then, by Lemma 1.3.2, the short exact sequence $0 \longrightarrow I \longrightarrow \Lambda \longrightarrow \Lambda/I \longrightarrow 0$ is pure, so I is a pure ideal. □

Lemma 1.3.4 (Theorem 4.89 in [30]). *Let Λ be a ring. The following are equivalent:*

(i) *The sequence* (1.2) *is pure.*

(ii) *The induced sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}_\Lambda(D, L) \longrightarrow \mathrm{Hom}_\Lambda(D, M) \longrightarrow \mathrm{Hom}_\Lambda(D, N) \longrightarrow 0$$

is exact for every finitely presented right Λ-module D.

(iii) *The sequence* (1.2) *is the direct limit of a direct system of split short exact sequences $\{0 \longrightarrow L \longrightarrow M_i \longrightarrow N_i \longrightarrow 0\}_{i \in I}$, where for each $i \in I$, N_i is a finitely presented right Λ-module.*

Following Lemma 1.3.4 **(ii)**, one can define the concept of pure-projective module with respect to pure short exact sequences.

Definition 1.3.5. Let Λ be a ring. A right Λ-module P is said to be *pure-projective* if, for every pure exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

the induced sequence $0 \longrightarrow \mathrm{Hom}_\Lambda(P, L) \longrightarrow \mathrm{Hom}_\Lambda(P, M) \longrightarrow \mathrm{Hom}_\Lambda(P, N) \longrightarrow 0$ is exact.

We have the following characterization of pure-projective modules due to Warfield [71].

Proposition 1.3.6 (Proposition 1.24 in [14]). *Let Λ be a ring, and let P be a right Λ-module. Then the following are equivalent:*

(i) *P is pure-projective.*

(ii) *Every pure short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$ splits.*

(iii) *P is a direct summand of a direct sum of finitely presented modules.*

As with projective modules, the following result follows from Lemma 1.2.2.

Corollary 1.3.7 (Kaplansky, [29]). *Every pure-projective module is a direct sum of countably generated modules.*

In [49], Příhoda, Puninski and Toffalori showed that, over a commutative noetherian ring, every pure-projective torsion-free module is a direct summand of a direct sum of finitely presented torsion-free modules (cf. Proposition 2.1 in [50]).

The problem of classifying pure-projective modules can be difficult, even if we know the structure of finitely presented modules. For instance, over a suitable uniserial ring Λ, there exists a pure-projective Λ-module without indecomposable decompositions (see [52]).

In [71], Warfield proved that over a commutative complete noetherian local ring R, every pure-projective R-module is a direct sum of finitely presented R-modules. In [53], Puninski and Rothmaler asked the general question of *when every pure-projective module is a direct sum of finitely presented modules.* They proved that over a hereditary noetherian ring, every pure-projective right module is a direct sum of finitely presented modules (cf. Corollary 6.5 in [53]). We will extend this result in Part II to noetherian domains of Krull dimension 1 with module-finite normalization (cf. Theorem 4.4.8).

1.4 Trace ideals

An important object related to projective modules is the set of their trace ideals. We recall the definition and some basic properties.

Definition 1.4.1. Let Λ be a ring, and let M be a right Λ-module. Then the *trace* of M in Λ is the two-sided ideal of Λ defined as

$$\mathrm{Tr}_\Lambda(M) = \sum_{f \in \mathrm{Hom}_\Lambda(M,\Lambda)} f(M).$$

To simplify the notation, we will just use $\mathrm{Tr}(M)$ to denote the trace in Λ of the module M when the ring Λ is clear.

Remark 1.4.2. Let Λ be a ring, and let M be a right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Then M is also a left S-module. Since $\mathrm{Hom}_\Lambda(M, \Lambda)$ is both a right S-module and a left Λ-module, the tensor product $\mathrm{Hom}_\Lambda(M, \Lambda) \otimes_S M$ is well-defined. Then $\mathrm{Tr}_\Lambda(M)$ is the image of the *trace map*:

$$\begin{aligned} \tau\colon\ \mathrm{Hom}_\Lambda(M, \Lambda) \otimes_S M &\longrightarrow \Lambda \\ f \otimes x &\longmapsto f(x). \end{aligned}$$

In the following results, we state some basic properties of trace ideals. Some of them are well-known.

Proposition 1.4.3. *Let Λ be a ring.*

(i) *Let M, N be right Λ-modules. If M is a homomorphic image of N, then $\mathrm{Tr}_\Lambda(M) \subseteq \mathrm{Tr}_\Lambda(N)$.*

(ii) *Let $\{M_i\}_{i \in A}$ be a family of right Λ-modules. Then*

$$\mathrm{Tr}_\Lambda\Big(\bigoplus_{i \in I} M_i\Big) = \sum_{i \in I} \mathrm{Tr}_\Lambda(M_i).$$

(iii) (Proposition 2.40 in [30]) *Let J be an ideal of Λ, and let P be a projective right Λ-module. Then $\mathrm{Tr}_\Lambda(P) \subseteq J$ if and only if $PJ = P$. In addition, the trace of a projective module is always an idempotent ideal.*

Notation 1.4.4. We will denote by $\mathcal{T}_r(\Lambda)$ (resp. $\mathcal{T}_\ell(\Lambda)$) the set of ideals of Λ that are traces of countably generated projective right (resp. left) Λ-modules. When we simply write $\mathcal{T}(\Lambda)$ we mean $\mathcal{T}_r(\Lambda)$.

Lemma 1.4.5. *Let Λ be a ring, and let P be a right Λ-module. Then, for any two-sided ideal I of Λ,*

(i) $(\mathrm{Tr}_\Lambda(P) + I)/I \subseteq \mathrm{Tr}_{\Lambda/I}(P/PI)$;

(ii) *If, in addition, P is projective,* $\mathrm{Tr}_{\Lambda/I}(P/PI) = (\mathrm{Tr}_\Lambda(P) + I)/I$.

Proof. **(i)** Let $x \in \mathrm{Tr}_\Lambda(P)$, then there exist

$$f_1, \ldots, f_n \in \mathrm{Hom}_\Lambda(P, \Lambda) \quad \text{and} \quad p_1, \ldots, p_n \in P$$

such that $x = \sum_{i=1}^n f_i(p_i)$. Then, for each $i = 1, \ldots, n$, there is an induced homomorphism $\overline{f}_i \colon P/PI \longrightarrow \Lambda/I$ given by $\overline{f}_i(p + PI) = f(p) + I$. Since $\overline{f}_i(P/PI) \subseteq \mathrm{Tr}_{\Lambda/I}(P/PI)$, in particular, $f_i(p_i) + I \in \mathrm{Tr}_{\Lambda/I}(P/PI)$. Since $x = \sum_{i=1}^n f_i(p_i)$, it follows that $x + I \in \mathrm{Tr}_{\Lambda/I}(P/PI)$.

Statement **(ii)** follows from the lifting property of projective modules, since every Λ/I-module homomorphism $\overline{f} \colon P/PI \longrightarrow \Lambda/I$ lifts to an Λ-module homomorphism $f \colon P \longrightarrow \Lambda$. □

Remark 1.4.6. For a commutative ring R, Vasconcelos showed that trace ideals of projective modules are exactly the pure ideals (see page 430 in [69] or Proposition 2.12 in [24]).

In [72], Whitehead gave a characterization of ideals that are traces of projective modules for general rings. In particular, he showed that an idempotent ideal of Λ that is finitely generated as a left ideal is always the trace of a countably generated projective right Λ-module. We shall use this result throughout this dissertation, sometimes without prior acknowledgment.

An alternative approach to the characterization of the traces of projective modules can be found in the paper [24].

However, if the ring is not commutative, trace ideals are not necessarily pure.

Example 1.4.7 (Remark 2.10 (3) in [24])**.** Let K be a field, and let

$$\Lambda = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}.$$

Then the artinian ring Λ has two idempotents,

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The indecomposable projective right Λ-modules are

$$P_1 = e_1\Lambda = \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = e_2\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix},$$

and the indecomposable projective left Λ-modules are

$$Q_1 = \Lambda e_1 = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_2 = \Lambda e_2 = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}.$$

Therefore $P_1 = \mathrm{Tr}(P_1) = \Lambda e_1 \Lambda = \mathrm{Tr}(Q_1)$ is pure as a right ideal but not as a left ideal, while $Q_2 = \mathrm{Tr}(Q_2) = \Lambda e_2 \Lambda = \mathrm{Tr}(P_2)$ is pure as a left ideal but not as a right ideal. Then, the two-sided ideal generated by the idempotent (e_1, e_2) in $\Lambda \times \Lambda$ is a trace ideal of a projective module that is neither right nor left pure.

1.5 Lifting projective modules modulo a trace ideal

Lemma 1.5.1. *Let Λ be a ring with Jacobson radical $\mathrm{J}(\Lambda)$, and let I and I' be elements in $\mathcal{T}(\Lambda)$. Then*

(i) *$I = I'$ if and only if $I + \mathrm{J}(\Lambda) = I' + \mathrm{J}(\Lambda)$.*

(ii) *$(I + \mathrm{J}(\Lambda))/\mathrm{J}(\Lambda) \in \mathcal{T}(\Lambda/\mathrm{J}(\Lambda))$, so the assignment defined by $I \mapsto (I + \mathrm{J}(\Lambda))/\mathrm{J}(\Lambda)$ defines an injective map $\varphi\colon \mathcal{T}(\Lambda) \longrightarrow \mathcal{T}(\Lambda/\mathrm{J}(\Lambda))$.*

Proof. **(i)** is included in Corollary 2.9 of [24].

To prove **(ii)**, note that if I is the trace of the projective right Λ-module P, then by Lemma 1.4.5, $(I + \mathrm{J}(\Lambda))/\mathrm{J}(\Lambda)$ is the trace of $P/P\,\mathrm{J}(\Lambda)$. This shows that φ is well defined. The injectivity of φ follows from **(i)**. □

We recall the definition of the class of modules generated by a projective module.

Definition 1.5.2. Let Λ be a ring. Let M be a right Λ-module. We define the *class of modules generated by M*, denoted by $\mathrm{Gen}(M)$, as the class of right Λ-modules that are a homomorphic image of a direct sum of copies of M.

For further quoting, we recall the following characterization of the class $\mathrm{Gen}(P)$, when P is a projective module.

Lemma 1.5.3 (Lemma 2.10 in [25])**.** *Let Λ be a ring, and let P be a projective right Λ-module with trace ideal I. Then*

$$\mathrm{Gen}(P) = \{M \in \mathbf{Mod}_\Lambda : MI = M\}.$$

The following result states that projective modules can be lifted modulo a trace ideal. This is a crucial result in the development of the theory of relatively big projective modules.

Theorem 1.5.4 (Theorem 3.1, Corollary 3.2 in [24])**.** *Let Λ be a ring, and let I be the trace of a projective right Λ-module. Let P' be a projective right Λ/I-module. Then there is a projective right Λ-module X such that $I \subseteq \mathrm{Tr}_\Lambda(X)$, $X/XI \cong P'$, and $\mathrm{Tr}_\Lambda(X)/I = \mathrm{Tr}_{\Lambda/I}(P')$.*

Moreover, if I is the trace of a countably generated projective right Λ-module and P' is countably generated, then X can be taken to be countably generated.

The following result shows some properties of this lifting.

Theorem 1.5.5. *Let Λ be a ring, and let I be an ideal of Λ that is the trace of a countably generated projective right Λ-module Q. Then*

(i) *If P_1 and P_2 are countably generated projective modules such that $P_1/P_1I \cong P_2/P_2I$, then $Q^{(\omega)} \oplus P_1 \cong Q^{(\omega)} \oplus P_2$.*

(ii) *If Q' is a countably generated projective Λ-module such that $\mathrm{Tr}_\Lambda(Q') = I$, then $Q^{(\omega)} \cong (Q')^{(\omega)}$.*

Proof. Statement **(i)** follows from the proof of Lemma 2.5 in [46]. We provide a self-contained proof for the reader's convenience.

For each $i = 1, 2$, let $\pi_i \colon P_i \longrightarrow P_i/P_iI$ denote the canonical projection. Let $f \colon P_1/P_1I \longrightarrow P_2/P_2I$ be an isomorphism. Then there exists $g_1 \colon P_1 \longrightarrow P_2$ such that the diagram

$$\begin{array}{ccc} & & P_1 \\ & \overset{g_1}{\swarrow} & \big\downarrow {\scriptstyle f\circ\pi_1} \\ P_2 & \xrightarrow{\pi_2} & P_2/P_2I \end{array}$$

is commutative. Hence, $P_2 = g_1(P_1) + P_2I$.

Since I is the trace ideal of the countably generated projective module Q, it is idempotent and, by Lemma 1.5.3, there exists an onto module homomorphism $g_2 \colon Q^{(\kappa)} \longrightarrow P_2I$, where κ is an infinite cardinal. Let $h \colon Q^{(\kappa)} \oplus P_1 \longrightarrow P_2/P_2I$ be the module homomorphism defined by $h(q, p) = f \circ \pi_1(p)$ for any $(q, p) \in Q^{(\kappa)} \oplus P_1$. Then, there is a commutative diagram

$$\begin{array}{ccc} & & Q^{(\kappa)} \oplus P_1 \\ & \overset{g}{\swarrow} & \big\downarrow {\scriptstyle h} \\ P_2 & \xrightarrow{\pi_2} & P_2/P_2I \end{array}$$

where g is the onto module homomorphism defined by $g(q, p) = g_2(q) + g_1(p)$ for any $(q, p) \in Q^{(\kappa)} \oplus P_1$. Since P_2 is countably generated, there exists $C \subseteq \kappa$ countably infinite such that $P_2 = g(Q^{(C)} \oplus P_1)$. Let $g' \colon Q^{(C)} \oplus P_1 \longrightarrow P_2$ be the restriction of g to $Q^{(C)} \oplus P_1$. Since P_2 is projective, $Q^{(C)} \oplus P_1 = P_2' \oplus \ker g'$, where P_2' is a submodule of $Q^{(C)} \oplus P_1$ isomorphic to P_2.

As $\ker g' \subseteq \ker h = Q^{(\kappa)} \oplus P_1I$, we deduce that

$$Q^{(C)} \oplus P_1I = \ker g' \oplus (P_2' \cap (Q^{(C)} \oplus P_1I)).$$

By Lemma 1.5.3, $P_1I \in \mathrm{Gen}(Q)$. Hence, the countably generated module $\ker g'$ is a homomorphic image of $Q^{(\omega)}$. Since it is also projective, it is a direct summand of $Q^{(\omega)}$. By the Eilenberg trick (cf. Remark 1.2.3), $Q^{(\omega)} \oplus \ker g' \cong Q^{(\omega)}$. Therefore,

$$Q^{(\omega)} \oplus P_1 \cong Q^{(\omega)} \oplus P_2 \oplus \ker g' \cong Q^{(\omega)} \oplus P_2,$$

as we wanted to prove.

Statement **(ii)** is a consequence of **(i)**. Taking $P_1 = Q^{(\omega)}$ and $P_2 = (Q')^{(\omega)}$, we deduce from **(ii)**,

$$Q^{(\omega)} \cong Q^{(\omega)} \oplus Q^{(\omega)} \cong Q^{(\omega)} \oplus (Q')^{(\omega)}.$$

Exchanging the roles of Q and Q', we deduce from **(ii)** that $(Q')^{(\omega)} \cong (Q')^{(\omega)} \oplus Q^{(\omega)}$, which yields the claimed isomorphism. □

As a corollary of Theorem 1.5.5, we show that direct sums of projective modules can be lifted modulo a trace ideal. This result is implicit in [25]. But first, we note the following property of direct sums of finitely generated modules.

Lemma 1.5.6. *Let Λ be a ring, and assume there is a direct sum decomposition of right Λ-modules $M' \oplus M = X \oplus Y$ with M finitely generated and X, Y direct sums of finitely generated modules. Then there exist A and B direct summands of X and Y, respectively, such that $M \oplus Z = A \oplus B$ with Z a finitely generated direct summand of M'.*

Proof. Since M is finitely generated, by the hypothesis, $M \subseteq A \oplus B$ with A and B finitely generated direct summands of X and Y, respectively. Therefore, by the modular law, $A \oplus B = M \oplus Z$ where $Z = M' \cap (A \oplus B)$. □

Corollary 1.5.7. *Let Λ be a ring, and let P be a countably generated projective right Λ-module. Let I be the trace of a countably generated projective right Λ-module Q. Assume that $P/PI = X' \oplus Y'$. Then there exist countably generated projective right Λ-modules X and Y such that $Q^{(\omega)} \oplus P \cong X \oplus Y$ and $X/XI \cong X'$, $Y/YI \cong Y'$ as Λ/I-modules.*

If, in addition, P is finitely generated and X, Y are direct sums of finitely generated modules, then there exist A and B isomorphic to direct summands of X and Y, respectively, such that $P \oplus Z = A \oplus B$, with Z a finitely generated direct summand of $Q^{(\omega)}$, and satisfying that $A/AI \cong X'$, $B/BI \cong Y'$.

Proof. By Theorem 1.5.5, there exist countably generated projective right Λ-modules X_1 and Y_1, such that $X_1/X_1I \cong X'$, $Y_1/Y_1I \cong Y'$. Hence, $P/PI \cong X_1/X_1I \oplus Y_1/Y_1I$.

Set $X = Q^{(\omega)} \oplus X_1$ and $Y = Q^{(\omega)} \oplus Y_1$. By Theorem 1.5.5, $Q^{(\omega)} \oplus P \cong X \oplus Y$.

To prove the final part of the statement, assume for simplicity that $Q^{(\omega)} \oplus P = X \oplus Y$. Notice that the existence of A, B, and Z follows from Lemma 1.5.6. Therefore, there are direct sum decompositions $X = A \oplus A'$ and $Y = B \oplus B'$. Now

$$Q^{(\omega)} \oplus P = A \oplus A' \oplus B \oplus B' = P \oplus (Z \oplus A' \oplus B').$$

so that the canonical projection $\pi\colon Q^{(\omega)} \oplus P \longrightarrow Z \oplus A' \oplus B'$ induces an isomorphism $Q^{(\omega)} \longrightarrow Z \oplus A' \oplus B'$. Therefore Z, A', and B' are modules in $\mathrm{Gen}(Q)$. By Lemma 1.5.3, we deduce that $X' \cong X/XI \cong A/AI$ and $Y' \cong Y/YI \cong B/BI$. This concludes the proof of the statement. □

The following fact will be used in the next part of this dissertation.

Remark 1.5.8. Let Λ be a ring and M a right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Let L be a two-sided ideal of Λ. Then $I = \{f \in S \mid f(M) \subseteq ML\}$ is a two-sided ideal of S.

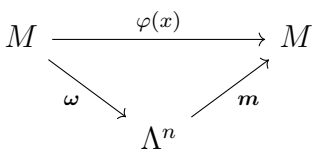
In addition, the Λ-Λ bimodule structure of Λ gives a structure of left Λ-module to $\mathrm{Hom}_\Lambda(M, \Lambda)$ where $r \cdot \omega$, for $r \in \Lambda$ and $\omega \in \mathrm{Hom}_\Lambda(M, \Lambda)$, is defined as the module homomorphism $r \cdot \omega\colon M \longrightarrow \Lambda$ such that $r \cdot \omega(x) = r\omega(x)$ for any $x \in M$. This gives a canonical morphism

$$\varphi\colon M \otimes_\Lambda \mathrm{Hom}_\Lambda(M, \Lambda) \longrightarrow \mathrm{End}_\Lambda(M)$$

where, for $m \in M$ and $\omega \in \mathrm{Hom}_\Lambda(M, \Lambda)$, $\varphi(m \otimes \omega)$ is the endomorphism of M defined by $\varphi(m \otimes \omega)(x) = m\omega(x)$, for any $x \in M$. It is not difficult to show that

$$\varphi(M \otimes_\Lambda \mathrm{Hom}_\Lambda(M, \Lambda)) = \{f \in S \mid f \text{ factors through } \Lambda^n \text{ for some } n \geq 1\}.$$

Indeed, let $x = \sum_{i=1}^n m_i \otimes \omega_i \in M \otimes_\Lambda \mathrm{Hom}_\Lambda(M, \Lambda)$. Since $\varphi(x) = \sum_{i=1}^n m_i\omega_i$, taking $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)^t \in \mathrm{M}_{n\times 1}(\Lambda) \cong \mathrm{Hom}_\Lambda(M, \Lambda^n)$ and $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathrm{M}_{1\times n}(\Lambda) \cong \mathrm{Hom}_\Lambda(\Lambda^n, M)$, the following diagram commutes:



Therefore, every element in $\varphi(M \otimes_\Lambda \operatorname{Hom}_\Lambda(M,\Lambda))$ is precisely an endomorphism of M which factors through Λ^n, where n is the number of simple tensors appearing in the decomposition of the preimage of the element in $\varphi(M \otimes_\Lambda \operatorname{Hom}_\Lambda(M,\Lambda))$.

Throughout the following sections, we will use the notation

$$M \operatorname{Hom}_\Lambda(M,\Lambda) = \varphi(M \otimes_\Lambda \operatorname{Hom}_\Lambda(M,\Lambda)).$$

Lemma 1.5.9. *Let Λ be a ring. Let M be a right Λ-module with endomorphism ring $S = \operatorname{End}_\Lambda(M)$. Let $I = \operatorname{Tr}_\Lambda(M)$ and $J = \{f \in S \mid f(M) \subseteq MI\}$ (which is a two-sided ideal of S by Remark 1.5.8). Then $M \operatorname{Hom}_\Lambda(M,\Lambda) \subseteq J$.*

Proof. In the notation of Remark 1.5.8 and because of the definition of φ, it follows that $\varphi(m \otimes \omega)(M) \subseteq MI$. So that, also $\varphi(\sum_{i=1}^n m_i \otimes \omega_i)(M) \subseteq MI$ for $m_i \in M$ and $\omega_i \in \operatorname{Hom}_\Lambda(M,\Lambda)$. □

Lemma 1.5.10 (Theorem 2.11**(i)** in [25])**.** *Let Λ be a ring. Let M be a finitely generated right Λ-module. Let $S = \operatorname{End}_\Lambda(M)$. Let X be an object of $\operatorname{Add}(M)$, and let $P_S = \operatorname{Hom}_\Lambda(M,X)$. Set $I = \operatorname{Tr}_S(P)$. Then,*

(i) *$I = \{f \in S \mid f$ factors through X^n for some $n \in \mathbb{N}\}$, that is*

$$I = \operatorname{Hom}_\Lambda(X,M) \operatorname{Hom}_\Lambda(M,X).$$

(ii) *For any $Y \in \operatorname{Add}(M)$,*

$$\operatorname{Hom}_\Lambda(M,Y)I = \{f \in \operatorname{Hom}_\Lambda(M,Y) \mid f \text{ factors through } X^n \text{ for some } n \geq 1\}.$$

Remark 1.5.11. Let M and M' be two right modules over a ring Λ, with endomorphism rings S and S', respectively. Let

$$T := \operatorname{End}_\Lambda(M' \oplus M) = \begin{bmatrix} S' & \operatorname{Hom}_\Lambda(M,M') \\ \operatorname{Hom}_\Lambda(M',M) & S \end{bmatrix}$$

Consider the finitely generated projective right T-module

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T \cong \operatorname{Hom}_\Lambda(M' \oplus M, M').$$

Its trace ideal is

$$I = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} S' & \operatorname{Hom}_\Lambda(M,M') \\ \operatorname{Hom}_\Lambda(M',M) & \operatorname{Hom}_\Lambda(M',M) \operatorname{Hom}_\Lambda(M,M') \end{bmatrix}.$$

Then $T/I \cong S/\operatorname{Hom}_\Lambda(M',M) \operatorname{Hom}_\Lambda(M,M')$.

We will use the case $M' = \Lambda$, so that $T/I \cong S/M \operatorname{Hom}_\Lambda(M,\Lambda)$. That is, it is the endomorphism ring of M_Λ in the stable category.

Corollary 1.5.12. *Let Λ be a ring. Let M be a finitely generated right Λ-module with endomorphism ring S. Let*

$$J = \{f \in S \mid f \text{ factors through } \Lambda^n \text{ for some } n \in \mathbb{N}\} = M \operatorname{Hom}_\Lambda(M, \Lambda).$$

Let e be an idempotent of S/J. Then, there is a direct sum decomposition $\Lambda^{(\omega)} \oplus M = X \oplus Y$ such that

$$\begin{aligned}\operatorname{Hom}_\Lambda(M, X)/X \operatorname{Hom}_\Lambda(M, \Lambda) &\cong \operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/X \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) \\ &\cong e(S/J)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_\Lambda(M, Y)/Y \operatorname{Hom}_\Lambda(M, \Lambda) &\cong \operatorname{Hom}_\Lambda(\Lambda \oplus M, Y)/Y \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) \\ &\cong (1-e)(S/J).\end{aligned}$$

If, in addition, X and Y are direct sums of finitely generated modules, then there exist A and B direct summands of X and Y, respectively, such that $M \oplus P = A \oplus B$ with P a finitely generated projective right Λ-module and satisfying that

$$\begin{aligned}\operatorname{Hom}_\Lambda(M, A)/A \operatorname{Hom}_\Lambda(M, \Lambda) &\cong \operatorname{Hom}_\Lambda(\Lambda \oplus M, A)/A \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) \\ &\cong e(S/J)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_\Lambda(M, B)/B \operatorname{Hom}_\Lambda(M, \Lambda) &\cong \operatorname{Hom}_\Lambda(\Lambda \oplus M, B)/B \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) \\ &\cong (1-e)(S/J).\end{aligned}$$

Proof. Let $T = \operatorname{End}_\Lambda(\Lambda \oplus M)$. Taking into account Remark 1.5.11, $S/J \cong T/I$ with I the trace ideal of the projective right T-module $Q = \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda)$. That is, following the notation in Remark 1.5.11, $I = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T$.

By Corollary 1.5.7 and its proof, there is a direct sum decomposition $Q^{(\omega)} \oplus \operatorname{Hom}(\Lambda \oplus M, M) = X' \oplus Y'$ with $X'/X'I \cong e(S/J)$ and $Y'/Y'I \cong (1-e)(S/J)$ such that $X' \oplus Q^{(\omega)} \cong X'$ and $Y' \oplus Q^{(\omega)} \cong Y'$. Notice that, by Theorem 1.5.5, these properties determine X' and Y' up to endomorphism.

Set $X = X' \otimes_T (\Lambda \oplus M)$ and $Y = Y' \otimes_T (\Lambda \oplus M)$. By Proposition 1.2.12, $\Lambda^{(\omega)} \oplus M \cong X \oplus Y$, $\Lambda^{(\omega)} \oplus X \cong X$, $\Lambda^{(\omega)} \oplus Y \cong Y$. By Lemma 1.5.10 and using the notation explained in Remark 1.5.8, X and Y also satisfy

$$\operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/X \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) \cong e(S/J)$$

and

$$\operatorname{Hom}_\Lambda(\Lambda \oplus M, Y)/Y \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) \cong (1-e)(S/J).$$

These properties determine X and Y up to isomorphism.

Let $\pi\colon \Lambda \oplus M \longrightarrow M$ denote the canonical projection. The assignment $g \mapsto g \circ \pi$, for any $g \in \operatorname{Hom}_\Lambda(M, X)$, defines a homomorphism of abelian groups $\operatorname{Hom}_\Lambda(M, X) \to \operatorname{Hom}_\Lambda(\Lambda \oplus M, X)$. Composed with the projection $\operatorname{Hom}_\Lambda(\Lambda \oplus M, X) \to \operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/X \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda)$ gives an onto homomorphism of right S-modules

$$\Phi\colon \operatorname{Hom}_\Lambda(M, X) \to \operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/X \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda).$$

Recall that, since $X \in \operatorname{Add}(\Lambda \oplus M)$,

$$\operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/X \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda) = \operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/\operatorname{Hom}_\Lambda(\Lambda \oplus M, X)I.$$

Clearly,

$$\ker \Phi = \{g \in \operatorname{Hom}_\Lambda(M, X) \mid g \text{ factors through } \Lambda^n \text{ for some } n \in \mathbb{N}\}.$$

That is, $\ker \Phi = X \operatorname{Hom}_\Lambda(M, \Lambda)$. This finishes the proof of the isomorphism $\operatorname{Hom}_\Lambda(M, X)/X \operatorname{Hom}_\Lambda(M, \Lambda) \cong \operatorname{Hom}_\Lambda(\Lambda \oplus M, X)/X \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda)$ as right S-modules.

In a similar way,

$$\operatorname{Hom}_\Lambda(M, Y)/Y \operatorname{Hom}_\Lambda(M, \Lambda) \cong \operatorname{Hom}_\Lambda(\Lambda \oplus M, Y)/Y \operatorname{Hom}_\Lambda(\Lambda \oplus M, \Lambda)$$

as right S-modules.

If X and Y are sums of finitely generated modules, then Lemma 1.5.6 allows us to proceed as above to prove the final claim of the statement. □

Lemma 1.5.13. *Let Λ be a ring. Let M be a right Λ-module with semiperfect endomorphism ring. If X is a right Λ-module such that $M \oplus X = Y_1 \oplus Y_2$ for suitable right Λ-modules Y_1 and Y_2, then for $i = 1, 2$, $Y_i \cong N_i \oplus Y_i'$ where $N_1 \oplus N_2 \cong M$ and $Y_1' \oplus Y_2' \cong X$.*

Proof. This is an easy consequence of the fact that modules with a semiperfect endomorphism ring are a finite direct sum of modules with local endomorphism rings, and that modules with local endomorphism rings satisfy the exchange property (cf. Lemma 2.7 and Theorem 2.8 in [14]). Namely, if $M \oplus X = Y_1 \oplus Y_2$ and M has local endomorphism ring, then we may assume that M is isomorphic to a direct summand of Y_1 so that $Y_1 \cong M \oplus Y_1'$ and, moreover, $X \cong Y_1' \oplus Y_2$. Now, if $M = M_1 \oplus \cdots \oplus M_n$ with each M_i having local endomorphism ring, the statement follows by induction on n. □

Lemma 1.5.14. *Let Λ be a semiperfect ring. Let M be a right Λ-module. Then $\operatorname{Tr}_\Lambda(M) \subseteq \mathrm{J}(\Lambda)$ if and only if M has no non-zero projective direct summands.*

Proof. Assume that $\operatorname{Tr}_\Lambda(M) \not\subseteq \mathrm{J}(\Lambda)$. Hence, there exist $f\colon M \longrightarrow \Lambda$ and $e^2 = e \in \Lambda$, with $e\Lambda/e\,\mathrm{J}(\Lambda)$ being simple, such that $ef(M) \subsetneq e\,\mathrm{J}(\Lambda)$. Therefore $e\Lambda = ef(M) + e\,\mathrm{J}(\Lambda)$. By Nakayama's Lemma, $e\Lambda = ef(M)$ so $e\Lambda$ is a projective direct summand of M.

The converse statement is clear because the trace of a non-zero projective module is never contained in $\mathrm{J}(\Lambda)$. □

Lemma 1.5.15. *Let Λ be a semiperfect ring. Then:*

- **(i)** *If M is a finitely generated module, $M = P \oplus M'$ with P a finitely generated projective module and M' with no non-zero projective direct summands. In addition, this decomposition is unique up to isomorphism.*
- **(ii)** *If M_1 and M_2 are Λ-modules with no non-zero projective direct summand then $M_1 \oplus M_2$ has the same property.*

Proof. Statement **(i)** is Theorem 3.15 in [14].

To prove statement **(ii)**, notice that if P is an indecomposable projective Λ-module, then its endomorphism ring is local because Λ is semiperfect. Therefore, if such P is a direct summand of $M_1 \oplus M_2$, then it is a direct summand either of M_1 or of M_2, which is not possible under our assumptions. □

Proposition 1.5.16. *Let Λ be a semiperfect ring. Let M be a finitely generated right Λ-module with endomorphism ring S. Set $J = M\operatorname{Hom}_\Lambda(M,\Lambda)$. If every direct summand of $\Lambda^{(\omega)} \oplus M$ is a direct sum of finitely generated modules, then the following statements hold:*

- **(i)** *Let $e \in S/J$ be an idempotent, then there exists $f^2 = f \in S$ such that $(f+J)S/J \cong eS/J$ and $(1-f+J)S/J \cong (1-e)S/J$;*
- **(ii)** *if M is indecomposable, then so is the right S-module S/J;*
- **(iii)** *every direct summand of $\Lambda^{(\omega)} \oplus M$ is of the form $P \oplus X$, where X is a direct summand of M and P is a direct summand of $\Lambda^{(\omega)}$ (that is, a countably generated projective module).*

Proof. **(i)** We consider the setting of Remark 1.5.11 with $M' = \Lambda$, and we follow the notation from that remark.

By Corollary 1.5.12, there is a decomposition $P \oplus M = A \oplus B$ with P finitely generated projective and A and B finitely generated such that

$$\operatorname{Hom}_\Lambda(M,A)/A\operatorname{Hom}_\Lambda(M,\Lambda) \cong e(S/J)$$

and

$$\operatorname{Hom}_\Lambda(M,B)/B\operatorname{Hom}_\Lambda(M,\Lambda) \cong (1-e)(S/J).$$

By Lemma 1.5.13, $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$ with $M \cong A_1 \oplus B_1$, and A_2 and B_2 projective Λ-modules. Therefore, $\operatorname{Hom}_\Lambda(M,A_1) \cong fS$ and $\operatorname{Hom}_\Lambda(M,B_1) \cong (1-f)S$ for $f^2 = f \in S$.

By Lemma 1.5.10,

$$\begin{aligned} e(S/J) &\cong \operatorname{Hom}_\Lambda(M,A)/A\operatorname{Hom}_\Lambda(M,\Lambda) \cong \operatorname{Hom}_\Lambda(M,A_1)/A_1\operatorname{Hom}_\Lambda(M,\Lambda) \\ &= \operatorname{Hom}_\Lambda(M,A_1)/\operatorname{Hom}_\Lambda(M,A_1)J \cong (f+J)S/J. \end{aligned}$$

Similarly, $(1-e)S/J \cong (1-f+J)S/J$.

Statement **(ii)** is a consequence of (i).

(iii) Since Λ is semiperfect, $M = P \oplus M'$ where P is finitely generated projective and M' has no non-zero projective direct summands (cf. Lemma 1.5.15). Since $\Lambda^{(\omega)} \oplus P \cong \Lambda^{(\omega)}$, to prove our statement, we may assume that $M = M'$ has no non-zero projective direct summands.

Let N be a finitely generated direct summand of $\Lambda^{(\omega)} \oplus M$. Then, there exists $n \geq 0$ such that N is a direct summand of $\Lambda^n \oplus M$. By Lemma 1.5.13, we can deduce that N is of the required form.

Assume now that N is a direct summand of $\Lambda^{(\omega)} \oplus M$ that is not finitely generated. By hypothesis, $N = \bigoplus_{i\in\mathcal{A}} N_i$ where N_i are finitely generated modules. By the previous step, for each $i \in \mathcal{A}$, $N_i = P_i \oplus M_i$ with P_i finitely generated

projective, and M_i isomorphic to a direct summand of M. Notice that each M_i has no non-zero projective direct summands, by our assumptions.

Since the ring Λ is semiperfect, $M/M\,\mathrm{J}(\Lambda)$ is a semisimple module of length ℓ. Then the maximal number of non-zero direct summands in a direct sum decomposition of M is ℓ. We claim that this implies that only ℓ of the direct summands M_i are different from zero.

Indeed, take $i_1, \dots, i_{\ell+1}$ to be $\ell+1$ different elements of the index set $\mathcal{A}$. Then $N' = \bigoplus_{j=1}^{\ell+1} M_{i_j}$ is a direct summand of $\Lambda^{(\omega)} \oplus M$. By Lemma 1.5.15, N' has no non-zero projective direct summands. By the first step, N' is a direct summand of M. Hence, there is some $M_{i_j} = \{0\}$.

Since only finitely many M_i's are different from zero, as before, we can deduce that its direct sum is a direct summand of M. Since $\bigoplus_{i\in\mathcal{A}} P_i$ is a countably generated projective module, it is a direct summand of $\Lambda^{(\omega)}$. This concludes the proof of the claim. □

Now we want to interpret these results in terms of lifting of idempotents modulo an ideal. The following fact characterizing when two idempotents are conjugated by a unit will be useful.

Lemma 1.5.17 (Exercise 21.15 in [31])**.** *Let Λ be a ring, and let e and f be idempotents of Λ. Then there exists a unit $u \in \Lambda$ such that $ueu^{-1} = f$ if and only if $e\Lambda \cong f\Lambda$ and $(1-e)\Lambda \cong (1-f)\Lambda$.*

Proof. Assume that $f = ueu^{-1}$. Then $ue = fu$, so $ue\Lambda = fu\Lambda = f\Lambda$. Therefore, left multiplication by u defines a homomorphism of right Λ-modules $e\Lambda \longrightarrow f\Lambda$, showing that $e\Lambda \cong f\Lambda$. Similarly, since $1-f = 1-ueu^{-1} = u(1-e)u^{-1}$, applying the same argument, we deduce that $(1-e)\Lambda \cong (1-f)\Lambda$.

On the other hand, let $\varphi\colon e\Lambda \longrightarrow f\Lambda$ and $\psi\colon (1-e)\Lambda \longrightarrow (1-f)\Lambda$ denote isomorphisms of right Λ-modules. Then $\varphi \oplus \psi$ is an automorphism of Λ as a right Λ-module, given by left multiplication by some unit $u \in U(\Lambda)$. Since $ueu^{-1} \in f\Lambda u^{-1} = f\Lambda$ and $ueu^{-1} + u(1-e)u^{-1} = u[e+(1-e)]u^{-1} = 1$, the direct sum decomposition $\Lambda = f\Lambda \oplus (1-f)\Lambda$ shows that $ueu^{-1} = f$. □

Lemma 1.5.18. *Let Λ be a ring, and let J be an ideal of Λ contained in $\mathrm{J}(\Lambda)$. Let $e = e^2 \in \Lambda/J$. Assume that $\Lambda_\Lambda \cong X_1 \oplus X_2$, and $X_1/X_1J \cong e(\Lambda/J)$ and $X_2/X_2J \cong (1-e)(\Lambda/J)$. Then there exists $f^2 = f \in \Lambda$ such that $f + J = e$ and $f\Lambda \cong X_1$ and $(1-f)\Lambda \cong X_2$.*

Proof. Let $g^2 = g \in \Lambda$ be such that there exist two isomorphisms $f_1\colon g\Lambda \longrightarrow X_1$ and $f_2\colon (1-g)\Lambda \longrightarrow X_2$. Then the restriction of f_1 and f_2 gives isomorphisms $gJ \cong X_1J$ and $(1-g)J \cong X_2J$, so that

$$(g+J)\Lambda/J \cong g\Lambda/gJ \cong X_1/X_1J,$$
$$(1-g+J)\Lambda/J \cong (1-g)\Lambda/(1-g)J \cong X_2/X_2J.$$

By Lemma 1.5.17, there exists v a unit of Λ/J such that $v(g+J)v^{-1} = e$. Since J is contained in $\mathrm{J}(\Lambda)$, any $u \in \Lambda$ such that $u + J = v$ is also a unit of Λ. Pick an invertible element u, and set $f = ugu^{-1}$. Then f is an idempotent element of Λ with $f + J = v(g+J)v^{-1} = e$. Moreover, as f is conjugate to g, we can apply Lemma 1.5.17 to deduce that $f\Lambda \cong X_1$ and $(1-f)\Lambda \cong X_2$. □

Corollary 1.5.19. *Let Λ be a semiperfect ring. Let M be a finitely generated right Λ-module with endomorphism ring S. Assume that onto endomorphisms of M are bijective and that M has no non-zero projective direct summands. Then:*

(i) *S is semilocal and the ideal $J = M \operatorname{Hom}_\Lambda(M, \Lambda)$ is contained in $\mathrm{J}(S)$.*

(ii) *If the direct summands of $\Lambda^{(\omega)} \oplus M$ are direct sum of finitely generated modules, then idempotents of S/J can be lifted to idempotents of S. If, in addition, $\mathrm{J}(S)/J$ is a nil-ideal then S is semiperfect.*

Proof. **(i)** By hypothesis and Lemma 1.5.14, $f \in J$ implies that $f(M) \subseteq M\,\mathrm{J}(\Lambda)$, so $f(M)$ is superfluous in M. Then $1 - f$ is onto, hence bijective. This implies that $J \subseteq \mathrm{J}(S)$. The ring S is semilocal because $M/M\,\mathrm{J}(\Lambda)$ is a semisimple module of finite length, so we can apply the results in [26].

(ii) By Proposition 1.5.16, any direct summand of S/J can be lifted to a direct summand of S. By **(i)**, $J \subseteq \mathrm{J}(S)$, we deduce that idempotents can be lifted modulo J from Lemma 1.5.18. By Theorem 21.28 in [31], if $\mathrm{J}(S)/J$ is nil, then idempotents of $S/\mathrm{J}(S)$ can be lifted to S/J. Hence, to S. Since, by **(i)**, S is semilocal, we deduce that S is semiperfect. □

Part II

Torsion-free modules over commutative domains of Krull dimension one

Let R be a commutative domain of Krull dimension 1, and let $\mathcal{F}$ denote the class of R-modules which are direct sums of finitely generated torsion-free modules. In this part, we study when the class $\mathcal{F}$ of modules over R that are arbitrary direct sums of finitely generated torsion-free modules is closed under direct summands.

In Chapter 2, we show that $\mathcal{F}$ is closed under direct summands over a local domain of Krull dimension 1 if and only if any indecomposable, finitely generated, torsion-free module has local endomorphism ring. If, in addition, R is noetherian this is equivalent to saying that the normalization of R is a local ring.

In Chapter 3 we study the following question: given $\{X(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$ a family of right $\Lambda_{\mathfrak{m}}$-modules, when is there a right Λ-module N such that $N_{\mathfrak{m}} \cong X(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. We provide an answer for h-local domains, projective modules, and trace ideals of projective modules.

Then, in Chapter 4, we prove that if $\mathcal{F}$ is closed under direct summands over an h-local domain of Krull dimension 1, then the property is inherited by the localizations of R at maximal ideals. Moreover, any localization of R at a maximal ideal, except for one, satisfies that any finitely generated ideal is 2-generated. The converse is true when the domain R is, in addition, integrally closed, noetherian semilocal, or noetherian with module-finite normalization.

Finally, over a commutative domain of finite character and with no restrictions on the Krull dimension, we show that the isomorphism classes of countably generated modules in $\mathcal{F}$ are determined by their genus.

2 Direct sums of finitely generated torsion-free modules: the local case

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In this chapter, we will prove the following Theorem A already mentioned in the introduction. We recall it:

Theorem A (Corollary 2.2.3, Corollary 2.2.9)**.** *Let R be a local domain of Krull dimension* 1*. The class $\mathcal{F}$ is closed under direct summands if and only if any finitely generated indecomposable torsion-free module has local endomorphism ring. If, in addition, R is noetherian, this is equivalent to having local integral closure.*

Therefore, in this situation, any module M in $\mathcal{F}_R$ can be written as $M = \bigoplus_{i\in I} N_i$, where each N_i is a finitely generated module with local endomorphism ring, and such a decomposition is unique up to isomorphism. In addition, any direct summand of M is isomorphic to a direct sum of modules from the family $\{N_i\}_{i\in I}$.

Section 2.1 provides a crucial technical overview of the necessary results for the subsequent sections of this dissertation. It discusses the properties of finitely generated (torsion-free) and/or finite rank modules over a domain R, their endomorphism rings, and finitely generated modules over an R-algebra Λ that are torsion-free as R-modules. It also introduces the concepts of a domain of finite character and an h-local domain and develops the foundational properties of endomorphism rings for finitely generated modules over these classes of rings.

In Section 2.2, we characterize local domains of Krull dimension 1 such that $\mathcal{F}$ is closed under direct summands. These results prove Theorem A.

In Section 2.3, we develop tools to relate the property of being locally a direct summand to being a direct summand in finitely generated torsion-free modules. If R is a domain of finite character, and N and M are finitely generated torsion-free modules such that N_Q is a direct summand of M_Q, then $N_\mathfrak{m}$ is a direct summand of $M_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R. This means we only have a finite number of problematic maximal ideals. We prove several results on how to proceed with these problematic maximal ideals. Analogous results were proven in the one-dimensional noetherian setting in [21] and were inspired by the representation theory of orders.

2.1 The endomorphism ring of a finitely generated module

2.1.1 Finitely generated modules

As implicit in Corollary 1.5.19, the condition that onto endomorphisms of a finitely generated module are bijective is going to play an important role in our study. In this section, we recall some well-known facts about the endomorphism ring of a finitely generated module over a commutative ring.

Lemma 2.1.1. *Let R be a commutative ring. Let M be a finitely generated R-module with endomorphism ring S. Then:*

(i) (Vasconcelos, Proposition 1.2 in [68]) *Any onto endomorphism of M is bijective.*

(ii) *The central ring extension $R/\mathrm{ann}_R(M) \hookrightarrow S$ is integral (that is, any element of S satisfies a monic polynomial with coefficients in $R/\mathrm{ann}_R(M)$).*

(iii) *Let L be a two-sided ideal of R containing $\mathrm{ann}_R(M)$ and let*

$$I = \{f \in S \mid f(M) \subseteq ML\}.$$

If M can be generated by ℓ elements, then for every $f \in I$, there exist $b_1, \ldots, b_\ell \in L$ such that $f^\ell + f^{\ell-1}\overline{b}_\ell + \cdots + f\overline{b}_2 + \overline{b}_1 = 0$, where for each $i = 1, \ldots, \ell$, $\overline{b}_i$ denotes the class of the element b_i in $R/\mathrm{ann}_R(M)$ viewed inside S.

Proof. Statements **(ii)** and **(iii)** are applications of the determinant trick. To prove **(ii)**, let $M = m_1R + \cdots + m_\ell R$ and let $f \in S$. Then there exists a matrix $A \in \mathrm{M}_\ell(R)$ such that

$$(f\mathrm{Id}_\ell - A)\begin{bmatrix} m_1 \\ \vdots \\ m_\ell \end{bmatrix} = 0.$$

Then we deduce that $\det(f\mathrm{Id}_\ell - A)m_i = 0$ for any $i = 1, \ldots, \ell$. Hence,

$$\overline{\det(f\mathrm{Id}_\ell - A)} \in (R/\mathrm{ann}_R(M))[f]$$

is zero. Hence, f is an integral element over $R/\mathrm{ann}_R(M)$.

To prove **(iii)**, just observe that the coefficients of the polynomial $\overline{\det(f\mathrm{Id}_\ell - A)}$ satisfy the claimed properties if $f \in I$. □

Remark 2.1.2. Let R be a commutative ring, and let Λ be a (non-necessarily commutative) module-finite R-algebra. Let M be a finitely generated right Λ-module. Then the conclusions of Lemma 2.1.1 also hold for $S = \mathrm{End}_\Lambda(M)$. M_Λ being finitely generated and Λ being module-finite imply that M_R is also finitely generated as an R-module. Moreover, $\mathrm{End}_\Lambda(M)$ is a subring of $\mathrm{End}_R(M)$.

Goodearl in [19] characterized which rings satisfy that any onto endomorphism of a finitely generated module is bijective. The following result follows easily from such characterization, and it includes an extension of Lemma 2.1.1 and Remark 2.1.2.

Proposition 2.1.3. *Let R be a commutative ring, and let Λ be a (non-necessarily commutative) module-finite R-algebra. Let M be a finitely generated right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Let X be a finitely generated right S-module. Then*

(i) *X satisfies that any onto endomorphism of X is bijective.*

(ii) *If S is semilocal, then so is the endomorphism ring of any finitely generated right (or left) S-module. In particular, if Λ is semilocal, then so is S.*

(iii) *$I = \{f \in \mathrm{End}_S(X) \mid f(X) \subseteq X\,\mathrm{J}(S)\}$ is a two-sided ideal of $\mathrm{End}_S(X)$ contained in $\mathrm{J}(\mathrm{End}_S(X))$. In particular, $\{f \in S \mid f(M) \subseteq M\,\mathrm{J}(\Lambda)\}$ is a two-sided ideal of S contained in $\mathrm{J}(S)$.*

Proof. **(i)** Since $\mathrm{M}_n(S) \cong \mathrm{End}_\Lambda(M^n)$ for any $n \geq 1$, by Lemma 2.1.1**(ii)** and Remark 2.1.2, we deduce that the extension $R/\mathrm{ann}_R(M) \longrightarrow \mathrm{M}_n(S)$ is integral for any $n \geq 1$. Now, the conclusion follows from Corollary 6 in [19].

(ii) The first part of the result is a particular case of Proposition 3.2 in [15]. Taking $M = \Lambda$ gives the second part of the statement.

(iii) By Remark 1.5.8, I is a two-sided ideal of S, so it is enough to show that $1 - f$ is bijective for every $f \in I$.

Let $f \in I$. Then $X = f(X) + (1-f)(X) = X\,\mathrm{J}(S) + (1-f)(X)$. By Nakayama's Lemma, $X = (1-f)(X)$. By **(i)**, $1 - f$ is bijective. □

Definition 2.1.4. A ring homomorphism $f\colon R \longrightarrow S$ is said to be *local* if, for any $r \in R$, $f(r)$ is invertible in S if and only if r is invertible in R.

Lemma 2.1.5. *Let R be a commutative ring. Let $R \subseteq S$ be a central integral extension of rings, where S is not necessarily commutative. Then,*

(i) *For any $s \in S$, the inclusion $R[s] \hookrightarrow S$ is a local ring homomorphism. Therefore $\mathrm{J}(S) \cap R[s] \subseteq \mathrm{J}(R[s])$.*

(ii) *If R is a ring of Krull dimension 0, then $\mathrm{J}(S)$ is a nil-ideal.*

Proof. **(i)** Let $s \in S$. To show that the inclusion $R[s] \subseteq S$ is a local homomorphism, we need to prove that if $x \in R[s]$ is invertible in S, then the inverse is in $R[s]$. This is a well-known fact for integral extensions of commutative rings. We recall the argument to see that it also works in our situation.

Since $R \subseteq S$ is an integral extension, the element x^{-1} satisfies

$$x^{-n} - r_1 x^{-n+1} - \cdots - r_n = 0$$

for suitable $r_1, \ldots, r_n \in R$. Since x^{-1} is invertible, we may assume $r_n \neq 0$. Therefore, multiplying by x^n we obtain the equality $1 = (r_1 + r_2 x + \cdots + r_n x^{n-1})x$. Therefore, $x^{-1} = r_1 + r_2 x + \cdots + r_n x^{n-1} \in R[s]$.

The second part of the statement follows immediately from the first part.

(ii) Assume now that R has Krull dimension 0, and let $s \in \mathrm{J}(S)$. By **(i)**, $s \in \mathrm{J}(R[s])$. Since $R \subseteq R[s]$ is an integral extension of commutative rings, also $R[s]$ is a ring of Krull dimension 0. Hence, $\mathrm{J}(R[s])$ coincides with the nilradical of $R[s]$. So we deduce that s is nilpotent. This shows that $\mathrm{J}(S)$ is a nil-ideal. □

Lemma 2.1.6. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. If M is a finitely generated right Λ-module, then*

$$\mathrm{End}_\Lambda(M)\,\mathrm{J}(R) \subseteq \mathrm{J}(\mathrm{End}_\Lambda(M)).$$

Proof. Let $f_1, \ldots, f_k \in \mathrm{End}_\Lambda(M)$ and let $r_1, \ldots, r_k \in \mathrm{J}(R)$. Since M_R is finitely generated, by Lemma 2.1.1, to show that $1 - \sum_{i=1}^k f_i r_i$ is invertible it suffices to show that $1 - \sum_{i=1}^k f_i r_i$ is an onto Λ-endomorphism of M. To this aim, notice that

$$M = \Big(1 - \sum_{i=1}^k f_i r_i\Big)(M) + \Big(\sum_{i=1}^k f_i r_i\Big)(M) = \Big(1 - \sum_{i=1}^k f_i r_i\Big)(M) + M\,\mathrm{J}(R).$$

Since $M\,\mathrm{J}(R)$ is a small R-submodule of M, we deduce that

$$M = \Big(1 - \sum_{i=1}^k f_i r_i\Big)(M). \qquad \square$$

Lemma 2.1.7. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let M be a non-zero finitely generated right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Let $\varphi\colon R \longrightarrow S$ denote the canonical homomorphism. Let $\mathfrak{n}$ be a two-sided maximal ideal of S. Then*

(i) *$\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$ is a maximal ideal of R.*

(ii) *$I = \{f \in S \mid f(M) \subseteq M\mathfrak{m}\}$ is a two-sided ideal of S contained in $\mathfrak{n}$ and satisfying $\mathfrak{m} = \varphi^{-1}(I)$.*

Proof. **(i)** By Lemma 2.1.1 and Remark 2.1.2, the morphism $\varphi\colon R \longrightarrow S$ induces a central integral extension $R/\mathrm{ann}_R(M) \longrightarrow S$. Then $S/\mathfrak{n}$ is a simple ring and $R/\varphi^{-1}(\mathfrak{n}) \longrightarrow S/\mathfrak{n}$ is also a central integral extension. The center of the simple ring $S/\mathfrak{n}$ is always a field K (the two-sided ideal generated by a non-zero element x in the center must be the total and, being the element central, this ideal is just the set of multiples of x; this is to say that x is invertible in $S/\mathfrak{n}$ because x is central, so is its inverse). We deduce that $R/\varphi^{-1}(\mathfrak{n}) \longrightarrow K$ is an integral extension of commutative rings. Hence, $R/\varphi^{-1}(\mathfrak{n})$ is a field (cf. Lemma 2.1.5) and $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$ is a maximal ideal of R.

(ii) By Remark 1.5.8, I is a two-sided ideal of S. Clearly, $\mathfrak{m} \subseteq \varphi^{-1}(I)$. Since $\mathrm{ann}_R(M)$ is contained in the maximal ideal $\mathfrak{m}$ of R and M_R is finitely generated, $M_\mathfrak{m} \neq \{0\}$. By Nakayama's Lemma, $M \neq M\mathfrak{m}$, so we conclude that $1 \notin \varphi^{-1}(I)$. Since $\mathfrak{m}$ is a maximal ideal contained in $\varphi^{-1}(I)$, we deduce that $\mathfrak{m} = \varphi^{-1}(I)$.

By Lemma 2.1.1**(iii)**, $(I + \mathfrak{n})/\mathfrak{n}$ is a two-sided nil-ideal of $S/\mathfrak{n}$. Since $S/\mathfrak{n}$ is simple, we deduce that $(I + \mathfrak{n})/\mathfrak{n}$ is the zero ideal, so $I \subseteq \mathfrak{n}$, as claimed. $\square$

2.1.2 Finitely generated torsion-free modules

We start this subsection by recalling the following well-known result.

Proposition 2.1.8 (Theorems 2.37 and 2.38 in [55]). *Let R be a commutative ring, let Λ and Γ be R-algebras, and let M, N be right Λ-modules. Let*

$$\varphi\colon \operatorname{Hom}_\Lambda(M,N)\otimes_R\Gamma \longrightarrow \operatorname{Hom}_{\Lambda\otimes_R\Gamma}(M\otimes_R\Gamma, N\otimes_R\Gamma)$$

be the two-sided homomorphism of Γ-modules defined by

$$\varphi(f\otimes\gamma):(m\otimes\gamma')\mapsto(f\otimes\gamma)(m\otimes\gamma')=f(m)\otimes\gamma\gamma'.$$

Assume that Γ is flat as an R-module.

(i) *If M is finitely generated as a right Λ-module, then φ is a monomorphism.*

(ii) *If M is finitely presented as a right Λ-module, then φ is an isomorphism.*

Proposition 2.1.9 (Lemma 10.1 in [25]). *Let R be a commutative ring, let Λ and Γ be R-algebras and let M be a right Λ-module.*

(i) *Let P be a countably generated projective right Λ-module. Suppose that Γ is flat as an R-module. Then $P\otimes_R\Gamma$ is a countably generated projective right $\Lambda\otimes_R\Gamma$-module and*

$$\operatorname{Tr}_{\Lambda\otimes_R\Gamma}(P\otimes_R\Gamma)=\operatorname{Tr}_\Lambda(P)\otimes_R\Gamma.$$

(ii) *Suppose that Γ is commutative and flat as an R-module, and that M is a finitely presented Λ-module. Then*

$$\operatorname{Tr}_{\Lambda\otimes_R\Gamma}(M\otimes_R\Gamma)=\operatorname{Tr}_\Lambda(M)\otimes_R\Gamma.$$

Definition 2.1.10. Let R be a commutative ring with total ring of fractions Q. We say that an R-module M is *torsion-free* if the non-zero divisors of R act as non-zero divisors on M, that is, if the natural map $M\longrightarrow M\otimes_R Q$ is injective. Sometimes we will use the notation M_Q to denote the tensor product $M\otimes_R Q$.

The following proposition shows that Proposition 2.1.8**(ii)** is true for finitely generated torsion-free modules if $\Gamma=R_\Sigma$, where Σ is a multiplicative subset of R consisting of non-zero divisors. For a similar result (and proof), the reader can check Warfield's paper (Lemma 3.5 in [70]).

Proposition 2.1.11. *Let R be a commutative domain and let Λ be an R-algebra. Let Σ be a multiplicative subset of R. If M is a finitely generated right Λ-module which is torsion-free as an R-module, then the canonical injective homomorphism*

$$\varphi\colon \operatorname{Hom}_\Lambda(M,N)\otimes_R R_\Sigma \longrightarrow \operatorname{Hom}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma, N\otimes_R R_\Sigma).$$

is an isomorphism for every Λ-module N which is torsion-free as an R-module.

Proof. By Theorem 2.1.8**(i)**, it suffices to prove surjectivity. Note that Λ-modules which are torsion-free as R-modules can be seen as submodules of their localization at Σ.

Let $g\in\operatorname{Hom}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma, N\otimes_R R_\Sigma)$. Let $m_1,\dots,m_n$ be a set of generators for M. Then there is an element $s\in\Sigma$ such that for every $i\in\{1,\dots,n\}$, we can write $f(m_i)=\frac{n_i}{s}$ for some $n_i\in N$. Taking $f=sg|_M\in\operatorname{Hom}_\Lambda(M,N)$, we have that $\varphi(f\otimes\frac{1}{s})=g$. This shows that φ is surjective, as claimed in the statement. □

Proposition 2.1.12. *Let R be a commutative ring, let Λ be an R-algebra, and let M, N be right Λ-modules. Let Σ be a multiplicative subset of R consisting of non-zero divisors. If M is a finitely generated right Λ-module which is torsion-free as an R-module, then*

$$\mathrm{Tr}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma) = \mathrm{Tr}_\Lambda(M)\otimes_R R_\Sigma.$$

Proof. Let $x \in \mathrm{Tr}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma)$. Then there is an element $t\in\Sigma$ and elements $f_1,\dots,f_n \in \mathrm{Hom}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma, \Lambda\otimes_R R_\Sigma)$, $m_1,\dots,m_n\in M$ such that $x = \sum_{i=1}^n f_i\left(m_i\otimes\frac{1}{t}\right)$. Let

$$\varphi\colon \mathrm{Hom}_\Lambda(M,\Lambda)\otimes_R R_\Sigma \longrightarrow \mathrm{Hom}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma, \Lambda\otimes_R R_\Sigma)$$

denote the canonical homomorphism. By Lemma 2.1.11, φ is an isomorphism, so there is an element $r\in\Sigma$ and elements $g_1,\dots,g_n\in\mathrm{Hom}_\Lambda(M,\Lambda)$ such that $\varphi(g_i\otimes\frac{1}{r}) = f_i$ for each $i=1,\dots,n$. Then

$$x = \sum_{i=1}^n f_i\left(m_i\otimes\frac{1}{t}\right) = \sum_{i=1}^n \varphi\left(g_i\otimes\frac{1}{r}\right)\left(m_i\otimes\frac{1}{t}\right) = \sum_{i=1}^n g_i(m_i)\otimes\frac{1}{rt}.$$

Therefore, $x\in\mathrm{Tr}_\Lambda(M)\otimes_R R_\Sigma$.

To prove the reverse containment, let $x\in\mathrm{Tr}_\Lambda(M)$ and $t\in R_\Sigma$. Then there are $f_1,\dots,f_n\in\mathrm{Hom}_\Lambda(M,\Lambda)$, $m_1,\dots,m_n\in M$ such that $x=\sum_{i=1}^n f_i(m_i)$. Therefore, $x\otimes t = \sum_{i=1}^n f_i(m_i)\otimes t = \sum_{i=1}^n \varphi(f_i\otimes 1)(m_i\otimes t)$. Hence, $x\otimes t\in\mathrm{Tr}_{\Lambda\otimes_R R_\Sigma}(M\otimes_R R_\Sigma)$. This concludes the proof of the statement. □

Let Λ be an algebra over a commutative domain R. We say that a short exact sequence

$$0\longrightarrow L\longrightarrow M\longrightarrow N\longrightarrow 0 \tag{2.1}$$

of right Λ-modules is *locally split* if it is split when we apply the functor $R_\mathfrak{m}\otimes_R -$ for any maximal ideal $\mathfrak{m}$ of R. Therefore, if X is a left Λ-module, we obtain an exact sequence of R-modules

$$0\longrightarrow K\longrightarrow L\otimes_\Lambda X\longrightarrow M\otimes_\Lambda X\longrightarrow N\otimes_\Lambda X\longrightarrow 0.$$

As (2.1) is locally split, $R_\mathfrak{m}\otimes_R K = 0$ for any maximal ideal $\mathfrak{m}$ of R. Therefore $K=0$. This proves that a locally split exact sequence of right Λ-modules is pure.

Now we show that the finitely generated Λ-modules that are torsion-free as R-modules are projective with respect to the class of locally split exact sequences of right Λ-modules that are torsion-free as R-modules.

Lemma 2.1.13. *Let R be a commutative domain, and let Λ be an R-algebra. Let L, M, N, and X be right Λ-modules which are torsion-free as R-modules. Assume that there is a short exact sequence,*

$$0\longrightarrow L\xrightarrow{f} M\xrightarrow{g} N\longrightarrow 0. \tag{2.2}$$

Then:

- **(i)** *If X_Λ is finitely generated, and (2.2) is a locally split exact sequence, then the map $\mathrm{Hom}_\Lambda(X,g)\colon \mathrm{Hom}_\Lambda(X,M)\longrightarrow\mathrm{Hom}_\Lambda(X,N)$ is onto. Equivalently, finitely generated torsion-free modules are projective with respect to the class of locally split exact sequences of torsion-free modules.*

(ii) *If N is a direct summand of a direct sum of finitely generated Λ-modules that are torsion-free as R-modules, then the exact sequence* (2.2) *splits if and only if it is locally split.*

Proof. **(i)** If $g_{\mathfrak{m}}\colon M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$ is a splitting epimorphism for every maximal ideal $\mathfrak{m}$ of R. Then, $\mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, g_{\mathfrak{m}})\colon \mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, M_{\mathfrak{m}}) \longrightarrow \mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, N_{\mathfrak{m}})$ is onto for every maximal ideal $\mathfrak{m}$ of R. By Lemma 2.1.11 we may identify $\mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, g_{\mathfrak{m}})$ with $R_{\mathfrak{m}} \otimes_R \mathrm{Hom}_{\Lambda}(X, g)$. Therefore, $R_{\mathfrak{m}} \otimes_R \mathrm{Hom}_{\Lambda}(X, g)$ is onto for every maximal ideal $\mathfrak{m}$ of R, so $\mathrm{Hom}_{\Lambda}(X, g)$ is onto, as desired.

(ii) As the class of modules that are projective with respect to the class of locally split exact sequences is closed under direct sums and direct summands, we deduce the statement **(ii)** from **(i)**. □

Corollary 2.1.14. *Let R be a commutative domain, and let Λ be an R-algebra. Let N be a direct summand of a direct sum of finitely generated right Λ-modules which are torsion-free as R-modules. Then N is a projective Λ-module if and only if $N_{\mathfrak{m}}$ is a projective $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let $f\colon F \longrightarrow N$ be an epimorphism of Λ-modules for some free Λ-module F. By Lemma 2.1.13**(ii)**, f is a splitting epimorphism if and only if $f_{\mathfrak{m}}$ is a splitting epimorphism for every maximal ideal $\mathfrak{m}$ of R. This is equivalent to the statement. □

Lemma 2.1.15. *Let R be a commutative domain, and let Λ be an R-algebra. Let Σ be a multiplicative subset of R. Let N be a right Λ-module which is torsion-free as an R-module. If M is a finitely generated Λ-submodule of N_Σ, then dM is isomorphic to a Λ-submodule of N for some $d \in \Sigma$.*

Proof. Note that, since N is torsion-free as an R-module, it can be seen as a Λ-submodule of N_Σ.

Let $m \in M$ and let $\{n_1/1, \ldots, n_t/1\} \subseteq N_\Sigma$ be a finite set of Λ-generators of M. Then,

$$m = \frac{n_1}{1}\frac{a_1}{s_1} + \cdots + \frac{n_t}{1}\frac{a_t}{s_t} \in N_\Sigma$$

for some $a_1, \ldots, a_t \in \Lambda$, and $s_1, \ldots, s_t \in \Sigma$. Multiplying by $d = s_1 \cdots s_t \in \Sigma$,

$$dm = \frac{n_1 a_1' + \cdots + n_t a_t'}{1} \in \lambda(N),$$

where $\lambda\colon N \longrightarrow N_\Sigma$ denotes the localization map. Therefore, $dM \subseteq \lambda(N) \cong N$. □

Remark 2.1.16. We note that Lemmas 2.1.11 and 2.1.15 could be stated for general commutative rings, provided that the multiplicative set Σ consists of non-zero divisors of R. In this general context, we mean that a module M is torsion-free if $md = 0$, for $m \in M$ and d a non-zero divisor of R, implies that $m = 0$.

Let R be a commutative domain, and let Q denote the field of fractions of R. The *rank* of an R-module M is the dimension of M_Q as a Q-vector space.

Remark 2.1.17. *Let R be a commutative domain with field of fractions Q. Let N be a torsion-free R-module of rank n. Then N contains a free R-module of rank n.*

Note that, since N is torsion-free of rank n, we can see N as an essential submodule of its injective hull $E(N) \cong Q^n$. On the other hand, R^n has the same injective hull Q^n. Therefore, by Lemma 2.1.15, since R^n is a finitely generated R-submodule of $N_Q = Q^n$, $dR^n \subseteq N$ for some non-zero element $d \in R$.

Lemma 2.1.18. *Let R be a commutative domain with field of fractions Q. Let M and N be non-zero torsion-free R-modules of the same finite rank n. Let $f\colon M \longrightarrow N$ be an R-module homomorphism. Then,*

(i) *f is onto if and only if it is bijective;*

(ii) *f is injective if and only if* $\operatorname{im} f$ *is an essential submodule of N.*

Proof. Applying the functor $- \otimes_R Q$ to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow \ker f \otimes_R Q \longrightarrow M \otimes_R Q \xrightarrow{f\otimes_R \mathrm{Id}_Q} N \otimes_R Q \longrightarrow 0$$

in which the Q-module homomorphism $f \otimes_R \mathrm{Id}_Q$ is an isomorphism because it is an onto morphism between Q-vector spaces of the same finite dimension. Therefore, $\ker f \otimes_R Q = 0$ and $\ker f$ being torsion-free, it must be zero. Hence, f is bijective.

The statement **(ii)** is easy to prove. □

Proposition 2.1.19. *Let R be a commutative domain with field of fractions Q, and let Λ be a torsion-free R-algebra of finite rank. Let M be a non-zero finitely generated right Λ-module which is torsion-free as an R-module. Then,*

(i) *any onto endomorphism of M is bijective;*

(ii) *if* $\mathrm{J}(R) \subseteq \mathrm{J}(\Lambda)$ *then* $\mathrm{End}_\Lambda(M)\,\mathrm{J}(R) \subseteq \mathrm{J}(\mathrm{End}_\Lambda(M))$*;*

(iii) *if Λ is semilocal then so is* $\mathrm{End}_\Lambda(M)$*.*

Proof. **(i)** Since M is a homomorphic image of a finite number of copies of Λ, M is a torsion-free R-module of finite rank. Since a Λ-module homomorphism is also an R-module homomorphism, the statement follows from Lemma 2.1.18**(i)**.

(ii) Our hypothesis allows us to repeat the proof of Lemma 2.1.6 to conclude the statement.

(iii) If Λ is semilocal, then $M\,\mathrm{J}(\Lambda)$ is a small submodule of M and $M/M\,\mathrm{J}(\Lambda)$ is semisimple artinian of finite length. By **(i)**, any onto endomorphism ring of M is bijective. As an application of [26], we deduce that $\mathrm{End}_\Lambda(M)$ is semilocal. □

2.1.3 Domains of finite character and h-local domains

Definition 2.1.20. A commutative domain R is said to have *finite character* if any non-zero element is contained in only a finite number of maximal ideals. A commutative domain of finite character is said to be *h-local* if, in addition, any non-zero prime ideal of R is contained in a unique maximal ideal.

The domain R has finite character if R/I is a semilocal ring for every non-zero ideal I of R. If R is h-local, then it also satisfies that $R/\mathfrak{p}$ is a local domain for every non-zero prime ideal $\mathfrak{p}$ of R.

If R is a domain of Krull dimension 1, then the notions of h-locality and being of finite character coincide. They are equivalent to saying that, for every non-zero ideal I of R, R/I is a semiperfect ring whose Jacobson radical is a nil-ideal. Because R/I is a ring of Krull dimension 0 with only a finite number of maximal ideals, $\mathrm{J}(R/I)$ coincides with the nilradical of R/I.

Matlis introduced the notion of h-local domains in [38] as a generalization of local domains and noetherian domains of Krull dimension 1. In the next result, we recall the key property of h-local domains that we will use throughout this dissertation.

Lemma 2.1.21. *A commutative domain R is h-local if and only if R/I is semiperfect for every non-zero proper ideal I of R. In particular, if R is h-local, I is a non-zero proper ideal of R, and $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$ is the finite set of maximal ideals of R containing I, then*

- **(i)** *$I = I_1 \cdots I_\ell$, where $I_i = I_{\mathfrak{m}_i} \cap R$, so $I_1, \ldots, I_\ell$ are pairwise comaximal ideals of R such that each I_i is contained in exactly one maximal ideal of R.*
- **(ii)** *the canonical map $R/I \longrightarrow (R/I)_{\mathfrak{m}_1} \times \cdots \times (R/I)_{\mathfrak{m}_\ell}$, given by the localization at each component, is an isomorphism, and $(R/I)_{\mathfrak{m}_i} = (R/I_i)_{\mathfrak{m}_i}$ for $i = 1, \ldots, \ell$.*

Proof. The first part is Theorem 4.9 in [7]. Statement **(i)** is proven in Lemma 5.1 of [42]. To prove **(ii)**, notice that, by the Chinese Remainder Theorem, there is an isomorphism

$$\varphi\colon R/I \longrightarrow R/I_1 \times \cdots \times R/I_\ell.$$

Since each I_i is contained only in the maximal ideal $\mathfrak{m}_i$, we deduce that R/I_i is a local ring and that $R/I_i = (R/I_i)_{\mathfrak{m}_i}$. So that we have the isomorphisms claimed. □

Lemma 2.1.22. *Let R be a commutative domain with field of fractions Q. Let M be a non-zero finitely generated torsion-free module with endomorphism ring S. Then*

- **(i)** *S is a torsion-free R-module containing R, and S is integral over R. The restriction of the embedding $R \subseteq S$ gives an embedding $\mathrm{J}(R) \subseteq \mathrm{J}(S)$.*
- **(ii)** *For any $f \in S$, there is some non-zero $d \in R$ such that $df \in M \operatorname{Hom}_R(M, R)$. In particular, $M \operatorname{Hom}_R(M, R) \neq \{0\}$.*

(iii) *$S_Q \cong \mathrm{M}_n(Q)$ where n is the rank of M. So S is a subring of the simple artinian algebra $\mathrm{M}_n(Q)$.*

(iv) *For any non-zero two-sided ideal I of S, $I \cap R \neq \{0\}$.*

(v) *$S_{\mathfrak{m}} \cong \mathrm{End}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is a semilocal ring for all maximal ideals $\mathfrak{m}$ of R.*

If, in addition, R is h-local, then

(vi) *Let I be any non-zero two-sided ideal of S. Then S/I is a semilocal ring. Moreover, $I_{\mathfrak{m}} = S_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R. If R has Krull dimension 1 then $\mathrm{J}(S/I)$ is a nil-ideal, so S/I is semiperfect.*

(vii) *If I is the trace ideal of a non-zero projective right S-module, then I is contained only in a finite number of trace ideals of projective right S-modules.*

Proof. Recall that, since R is commutative, there is a ring morphism $\varphi\colon R \longrightarrow S$ where, for any $r \in S$, $\varphi(r)$ is the morphism given by multiplication of r. Since M is a faithful R-module, φ is injective, and we may identify R as a subring of S. We will use this identification throughout the proof.

(i) Let $f \in S$ be such that there is some non-zero $r \in R$ with $rf = 0$. Equivalently, $rf(M) = \{0\}$ so that $f(M) = \{0\}$ because M is torsion-free.

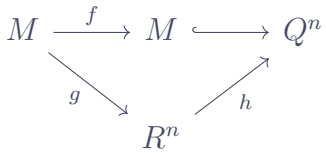
The extension $R \subseteq S$ is integral because of Lemma 2.1.1. Repeating the arguments in Lemma 2.1.6, we deduce that $\mathrm{J}(R) \subseteq \mathrm{J}(S)$.

(ii) Since M is finitely generated and torsion-free, we can view M as an essential submodule of its injective hull $E(M) \cong Q^n$ for some $n \geq 1$. On the other hand, R^n has the same injective hull Q^n. Therefore, since M is finitely generated, by Lemma 2.1.15, $gM \subseteq R^n$ for some non-zero $g \in R \subseteq S$.

Now, we proceed with the proof of the statement. The claim is clear regarding the zero endomorphism. Pick $f \in S \setminus \{0\}$. Since Q^n is an injective R-module, there exists

$$h \in \mathrm{Hom}_R(R^n, Q^n) \cong \mathrm{M}_n(Q)$$

such that the diagram



commutes. Note that, since R^n is finitely generated and M is essential in Q^n, by Lemma 2.1.15, $dh(R^n) \subseteq M$ for some non-zero $d \in R$, that is, $dh \in \mathrm{Hom}_R(R^n, M)$. Therefore, $df = (dh) \circ g \in M\,\mathrm{Hom}_R(M, R)$.

(iii) By Lemma 2.1.11, $S_Q \cong \mathrm{End}_Q(M_Q) \cong \mathrm{M}_n(Q)$, where n is the rank of M. By **(i)**, S is torsion-free as an R-module, so the localization map $S \longrightarrow S_Q$ is injective.

(iv) By **(iii)**, we can identify S with a subring of its central localization at the non-zero divisors of R, which is the simple ring $\mathrm{M}_n(Q)$. If I is a non-zero ideal of S, then $I_Q = I\mathrm{M}_n(Q)$ is a non-zero ideal of $\mathrm{M}_n(Q)$. Therefore $I\mathrm{M}_n(Q) = \mathrm{M}_n(Q)$. Therefore, there is some $f \in I$ and some non-zero $d \in R$ such that $f/d = \mathrm{Id}_n$. This implies that f is the endomorphism of M given by multiplication by d. So $f \in R \cap I$.

(v) By Lemma 2.1.11, $S_{\mathfrak{m}} \cong \mathrm{End}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$. By Lemma 2.1.1, $S_{\mathfrak{m}}$ is semilocal.

From now on, we assume that R is h-local.

(vi) Let I be a non-zero two-sided ideal of S. The inclusion $\varphi\colon R \longrightarrow S$ induces an injective ring homomorphism $R/(I \cap R) \longrightarrow S/I$. By **(iv)**, $I \cap R \neq \{0\}$, and since R is h-local, there are only finitely many maximal ideals of R, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$, containing $I \cap R$. By Lemma 2.1.21, the canonical homomorphisms

$$R/(I \cap R) \longrightarrow (R/(I \cap R))_{\mathfrak{m}_1} \times \cdots \times (R/(I \cap R))_{\mathfrak{m}_t}$$

and

$$S/I \longrightarrow (S/I)_{\mathfrak{m}_1} \times \cdots \times (S/I)_{\mathfrak{m}_t}$$

are isomorphisms. Therefore, we have a commutative diagram

$$\begin{array}{ccc} R/(I \cap R) & \lhook\joinrel\longrightarrow & S/I \\ \downarrow \cong & & \downarrow \cong \\ (R/(I \cap R))_{\mathfrak{m}_1} \times \cdots \times (R/(I \cap R))_{\mathfrak{m}_\ell} & \lhook\joinrel\longrightarrow & (S/I)_{\mathfrak{m}_1} \times \cdots \times (S/I)_{\mathfrak{m}_\ell} \end{array}$$

By **(v)**, S/I is a finite product of semilocal rings. Thus, it is a semilocal ring.

Since $I \cap R$ is contained in only a finite number of maximal ideals of R, it follows that $(I \cap R)_{\mathfrak{m}} = R_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R. Therefore, $I_{\mathfrak{m}} = S_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R.

Assume now that R has Krull dimension 1. As $R/(I \cap R) \subseteq S/I$ is an integral extension, and $R/(I \cap R)$ is a ring of Krull dimension 0, Lemma 2.1.5 implies that $\mathrm{J}(S/I)$ is a nil-ideal. As idempotents can be lifted modulo a nil-ideal, we deduce that S/I is a semiperfect ring.

The statement **(vii)** follows from **(vi)** because semilocal rings have only a finite number of trace ideals of projective modules. □

2.2 Local domains of Krull dimension one

Now we are ready to start our study of direct summands of direct sums of torsion-free modules. In this section, we will specialize in the case of local domains of Krull dimension one. The next lemma points out the key properties of endomorphism rings of torsion-free modules over such domains that we will need.

Lemma 2.2.1. *Let R be a commutative local domain of Krull dimension 1, and with field of fractions Q. Let M be a non-zero finitely generated torsion-free R-module with endomorphism ring S. If M does not contain a non-zero free direct summand, then:*

- **(i)** *$J = M\,\mathrm{Hom}_R(M, R) \subseteq \mathrm{J}(S)$ and $\mathrm{J}(S)/M\,\mathrm{Hom}_R(M, R)$ is a nil-ideal of S/J.*
- **(ii)** *If every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules, then S is semiperfect.*

Proof. By Lemma 2.1.1, M satisfies that any onto endomorphism ring of M is bijective.

(i) By Corollary 1.5.19,

$$J = M \operatorname{Hom}_R(M, R) \subseteq \mathrm{J}(S).$$

The rest of the statement is included in Lemma 2.1.22.

(ii) Follows from **(i)** and Corollary 1.5.19. □

Proposition 2.2.2. *Let R be a commutative local domain of Krull dimension 1, and with field of fractions Q. Let M be a finitely generated torsion-free R-module. The following statements are equivalent:*

- **(i)** *every module in $\operatorname{Add}(R \oplus M)$ is a direct sum of finitely generated modules;*
- **(ii)** *every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules;*
- **(iii)** *$\operatorname{End}_R(M)$ is semiperfect; and*
- **(iv)** *every finitely generated indecomposable module in $\operatorname{Add}(R \oplus M)$ has local endomorphism ring.*

Proof. **(i)** $\Rightarrow$ **(ii)** is clear.

(ii) $\Rightarrow$ **(iii)** Write $M = F \oplus M'$, with F finitely generated free, and M' having no projective direct summands. By Lemma 2.2.1**(ii)**, $\operatorname{End}_R(M')$ is semiperfect. As R is local, so is $\operatorname{End}_R(M)$.

Statement **(iii)** is equivalent to saying that $M = Y_1 \oplus \cdots \oplus Y_m$, where each Y_i has local endomorphism ring for $i \in \{1, \ldots, n\}$ (cf. Theorem 3.14 in [14]).

In general, over a commutative ring, direct summands of direct sums of finitely generated modules with local endomorphism rings are also direct sums of finitely generated modules with local endomorphism rings, and such decomposition is unique (cf. Corollary 2.55 in [14]). Therefore, every element in $\operatorname{Add}(R \oplus M)$ is isomorphic to a direct sum of copies of $R, Y_1, \ldots, Y_m$. This proves **(i)**.

In particular, since every indecomposable in $\operatorname{Add}(R \oplus M)$ is isomorphic to one in $\{R, Y_1, \ldots, Y_m\}$, we deduce **(iv)**.

Note that **(iv)** also implies that M is a direct sum of indecomposable modules with local endomorphism rings. Thus, this implies **(iii)**. □

Corollary 2.2.3. *Let R be a commutative local domain of Krull dimension 1, and with field of fractions Q. The following statements are equivalent:*

- **(i)** *the class of direct sums of finitely generated torsion-free R-modules is closed under direct summands;*
- **(ii)** *for any finitely generated torsion-free R-module M, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules;*
- **(iii)** *for any finitely generated torsion-free R-module M, every module in $\operatorname{Add}(M)$ is a direct sum of finitely generated modules; and*
- **(iv)** *every finitely generated, indecomposable, torsion-free R-module has local endomorphism ring.*

In addition, the above equivalent conditions are fulfilled by any intermediate ring S between R and its integral closure.

Proof. The equivalence of the four statements is an immediate consequence of Proposition 2.2.2.

To finish the proof of the statement, let S be a ring such that $R \subseteq S \subseteq \overline{R}$. Notice that if X_S is a finitely generated torsion-free S-module, then it has finite rank n. Identifying X with a submodule of Q^n,

$$\mathrm{End}_S(X) = \{A \in \mathrm{End}_Q(Q^n) \mid AX \subseteq X\} = \mathrm{End}_R(X).$$

In particular, X is indecomposable as an S-module if and only if it is indecomposable as an R-module.

If S is finitely generated over R, then X is also finitely generated over R. So, by **(iv)**, X_S is indecomposable if and only if $\mathrm{End}_S(X)$ is local.

Assume S is not finitely generated over R, and fix $x_1, \ldots, x_r$ a family of generators of X_S. Then $\mathrm{End}_S(X)$ is the directed union of $\mathrm{End}_T(\sum_{i=1}^r x_i T)$ where T varies among all subrings of S that are finite extensions of S. If X_S is indecomposable, then so is the T-module $\sum_{i=1}^r x_i T$. By the previous step, $\mathrm{End}_T(\sum_{i=1}^r x_i T)$ is a local ring for any T. Therefore, $\mathrm{End}_S(X)$ is also a local ring. □

We do not know of a characterization of local domains (of Krull dimension 1) such that their indecomposable finitely generated torsion-free modules have local endomorphism ring. If we restrict the condition to finitely generated, torsion-free, rank one modules, then it is equivalent to having a local integral closure, as we show in the next result.

Proposition 2.2.4. *Let R be a commutative local domain of Krull dimension 1, and with field of fractions Q. The following statements are equivalent:*

- **(i)** *the class of direct sums of finitely generated, rank-one, torsion-free R-modules is closed under direct summands;*
- **(ii)** *for any finitely generated, rank-one, torsion-free R-module M, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules;*
- **(iii)** *every finitely generated, rank-one, torsion-free module has local endomorphism ring; and*
- **(iv)** *the integral closure of R (into its field of fractions) is a local ring.*

Proof. The equivalence of **(i)**, **(ii)**, and **(iii)** follows from Proposition 2.2.2.

(iii) $\Rightarrow$ **(iv)**. By **(iii)** any finitely generated integral extension of R inside Q is local. Let $S \subseteq Q$ be any integral extension of R inside Q. If S has two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, then for any $i = 1, 2$, choose $s_i \in \mathfrak{m}_i$ such that $s_1 + s_2 = 1$. Then $R \subseteq R[s_1, s_2]$ is a finitely generated integral extension of R. By the first part of the proof, $R[s_1, s_2]$ is a local ring, but its maximal ideal should contain s_1 and s_2, which contradicts our assumptions. Therefore, S is a local ring.

(iv) $\Rightarrow$ **(iii)**. If M is a finitely generated, rank-one, torsion-free module, then its endomorphism ring S is a subring of Q, integral over S. Hence, it is contained in the integral closure $\overline{R}$ of R. Since $S \subseteq \overline{R}$ is integral and $\overline{R}$ is local, so is S (cf. Theorem 9.3 in [39]). □

In the case of noetherian domains of Krull dimension 1 with finitely generated integral closure, Příhoda in [48] proved that having local integral closure (so the integral closure is a discrete valuation ring) already implies that all indecomposable finitely generated torsion-free modules have local endomorphism ring.

The following criteria to ensure local endomorphism ring encodes Příhoda's ideas in Proposition 1 of [48]. It will allow us to prove that for any noetherian local domain of Krull dimension 1, having local integral closure is equivalent to the fact that indecomposable, finitely generated, torsion-free modules have local endomorphism ring.

Lemma 2.2.5. *Let R be a commutative local domain of Krull dimension* 1*, and with field of fractions Q. Let M_R be a finitely generated, indecomposable, torsion-free module. Assume that:*

- **(i)** *there exists a local over ring T of R, of Krull dimension* 1*, contained in Q, such that MT_T is a direct sum of T-modules with local endomorphism ring;*
- **(ii)** *the conductor ideal $\mathfrak{c} = \{r \in R \mid rT \subseteq R\}$ is non-zero.*

Then M_R has local endomorphism ring.

Proof. Since MT_T is a (finite) direct sum of modules with local endomorphism ring, then $\mathrm{End}_T(MT)$ is a semiperfect ring. Since $\mathfrak{c}$ is a non-zero proper ideal of both the rings R and T, it is contained in their corresponding maximal ideals. Since R and T have Krull dimension 1, $R/\mathfrak{c}$ and $T/\mathfrak{c}$ are rings of Krull dimension zero.

Let $S = \mathrm{End}_R(M)$ and $J = \{f \in S \mid f(M) \subseteq M\mathfrak{c}\}$. By Proposition 2.1.3, J is a two-sided ideal of S contained in $\mathrm{J}(S)$. So it is enough to show that S/J is a local ring.

Since $R \subseteq T \subseteq Q = E(R)$, M can be identified with an essential submodule of Q^n for some n. Via this identification, $S = \{A \in \mathrm{M}_n(Q) \mid AM \subseteq M\}$ and $\mathrm{End}_T(MT) = \{A \in \mathrm{M}_n(Q) \mid AMT \subseteq MT\}$, so that S is a subring of the semiperfect ring $\mathrm{End}_T(MT_T)$. Moreover, $J = \{A \in \mathrm{M}_n(Q) \mid AM \subseteq M\mathfrak{c}\} = \{A \in \mathrm{M}_n(Q) \mid AMT \subseteq M\mathfrak{c}\}$, so it is also a two-sided ideal of $\mathrm{End}_T(MT_T)$ contained in its Jacobson radical.

By Lemma 2.1.1, $R/(J \cap R) \subseteq S/J$ is an integral extension. As $\mathfrak{c} \subseteq J \cap R$, $R/(J \cap R)$ has Krull dimension 0, and by Lemma 2.1.5, we deduce that $\mathrm{J}(S)/J$ is a nil-ideal. So the idempotents of the semisimple artinian ring $S/\mathrm{J}(S)$ can be lifted to S/J. Let $e \in S$ be such that $e + J$ is an idempotent of $S/J \subseteq \mathrm{End}_T(MT_T)/J$. Since $\mathrm{End}_T(MT_T)$ is a semiperfect ring, by Proposition 27.4 and Theorem 27.6 in [3], there exists an idempotent $e' \in \mathrm{End}_T(MT_T)$ such that $e - e' \in J$. But then $e'M \subseteq (e'-e)M + eM \subseteq M\mathfrak{c} + M = M$. Hence $e' \in S$. Since M is indecomposable, e' is a trivial idempotent. Therefore, $e + J$ is also a trivial idempotent of S/J and we deduce that S/J is a local ring. □

Lemma 2.2.6. *Let R be a commutative local domain of Krull dimension* 1*, and with local integral closure. Let M be a finitely generated indecomposable torsion-free R-module. Assume that there is a finitely generated integral extension T of R such that MT_T is a direct sum of T-modules with local endomorphism ring. Then M_R has local endomorphism ring.*

Proof. Since the integral closure of R is local, T is also local. Moreover, T is also a ring of Krull dimension 1, and since T_R is finitely generated, the conductor ideal $\mathfrak{c} = \{r \in R \mid rT \subseteq R\}$ is different from zero. Therefore, the statement follows from Lemma 2.2.5. □

Lemma 2.2.7. *Let R be a commutative local domain of Krull dimension 1 with field of fractions Q. Assume that the integral closure $\overline{R}$ of R is a valuation domain. For any non-zero finitely generated torsion-free module M, there exists a finite integral extension T of R inside Q such that $MT \cong T \oplus M'$ for a suitable T-module M'.*

Proof. Let $n > 0$ be the rank of M. We may assume that M is an essential submodule of Q^n that contains R^n as an essential submodule. Since $\overline{R}$ is a valuation domain, the module $M\overline{R}_{\overline{R}}$ is free (see, for example, Corollary V.2.8 in [18]), so that $\mathrm{Tr}_{\overline{R}}(M\overline{R}) = \overline{R}$. Therefore, there exist $f_1, \dots, f_\ell \in \mathrm{Hom}_{\overline{R}}(M\overline{R}, \overline{R})$ and $m_1, \dots, m_\ell \in M$ such that $1 = f_1(m_1) + \cdots + f_\ell(m_\ell)$.

Since $\overline{R}^n$ is a submodule of $M\overline{R}$,

$$\mathrm{Hom}_{\overline{R}}(M\overline{R}, \overline{R}) = \{(s_1, \dots, s_n) \in \overline{R}^n \mid (s_1, \dots, s_n)M \subseteq \overline{R}\}.$$

So that, for $i = 1, \dots, \ell$, $f_i = (s_1^i, \dots, s_n^i)$. Let T be the finite integral extension of R generated by the elements

$$\{s_j^i\}_{\substack{i=1,\dots,\ell \\ j=1,\dots,n}} \quad \text{and} \quad \bigcup_{i=1}^{\ell} f_i(G),$$

where G is a finite set of generators of M.

By construction, $\mathrm{Tr}_T(MT) = T$, and then, by Lemma 1.5.14, we deduce that there exists M' such that $MT \cong T \oplus M'$. This finishes the proof of the lemma. □

Proposition 2.2.8. *Let R be a commutative local domain of Krull dimension 1 with field of fractions Q. Assume that the integral closure $\overline{R}$ of R is a valuation domain. Let M_R be a finitely generated torsion-free module. Then M is a direct sum of modules with local endomorphism ring.*

Proof. We are going to prove the statement by induction on the rank n of M. If $n = 1$, Lemma 2.2.7 implies that $MT \cong T$. Then, since T must also be a local ring, we may conclude by Lemma 2.2.6. Now assume that $n > 1$ and that the statement is proven for modules of smaller rank. By the previous claim, there exists a finite integral extension of T such that $MT \cong T \oplus M'$ for a suitable T-module M' of smaller rank. By the inductive hypothesis, M' is a T-module that is a direct sum of modules with local endomorphism ring. By Lemma 2.2.6, we conclude that M is a direct sum of modules with local endomorphism ring. □

Corollary 2.2.9. *Let R be a commutative local noetherian domain of Krull dimension 1, and with field of fractions Q. Then the following statements are equivalent:*

- **(i)** *every direct summand of a direct sum of finitely generated, torsion-free modules of rank one is a direct sum of finitely generated modules;*

(ii) *for any finitely generated, torsion-free R-module M of rank one, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules;*

(iii) *every finitely generated and torsion-free module of rank one has local endomorphism ring;*

(iv) *the integral closure of R is local;*

(v) *every direct summand of a direct sum of finitely generated, torsion-free modules is a direct sum of finitely generated modules;*

(vi) *for any finitely generated, torsion-free R-module M, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules;*

(vii) *every finitely generated, indecomposable, and torsion-free module has local endomorphism ring.*

Proof. The first four statements are equivalent because of Proposition 2.2.4. Since R is a commutative local noetherian ring of Krull dimension 1, its integral closure is a discrete valuation ring. Hence, by Proposition 2.2.8, all finitely generated torsion-free modules are direct sums of modules with local endomorphism rings. Now, the equivalence of the remaining statements follows from Corollary 2.2.3. This concludes the proof of the equivalence of the statements. □

The reader is referred to Lemma 4.3.6 for examples that satisfy the conclusion of Corollary 2.2.9.

2.3 Local versus global direct summands

The following result, for finitely generated torsion-free modules over a domain R, gives a general relation between the property of being locally a direct summand and being a direct summand. Afterwards, we will see how the result can be refined when we also assume that the domain has finite character.

Proposition 2.3.1. *Let R be a commutative domain, and Λ be an R-algebra. Let M and N be finitely generated right Λ-modules which are torsion-free as R-modules. If $M_{\mathfrak{m}}$ is a direct summand of $N_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, then there exists $k > 0$ such that M is a direct summand of N^k.*

Proof. Let $\{\mathfrak{m}_\alpha\}_{\alpha\in\Omega}$ denote the set of all maximal ideals of R and fix $\alpha \in \Omega$. Since $M_{\mathfrak{m}_\alpha}$ is a direct summand of $N_{\mathfrak{m}_\alpha}$, there are $\Lambda_{\mathfrak{m}_\alpha}$-module homomorphisms $\tilde{f}_\alpha \colon N_{\mathfrak{m}_\alpha} \to M_{\mathfrak{m}_\alpha}$ and $\tilde{g}_\alpha \colon M_{\mathfrak{m}_\alpha} \to N_{\mathfrak{m}_\alpha}$ such that $\tilde{f}_\alpha \circ \tilde{g}_\alpha = \mathrm{Id}_{M_{\mathfrak{m}_\alpha}}$. By Lemma 2.1.11, there are Λ-homomorphisms $f_\alpha \colon N \to M$ and $g_\alpha \colon M \to N$ such that $\tilde{f}_\alpha = f_\alpha/s_\alpha$, $\tilde{g}_\alpha = g_\alpha/s_\alpha$, and $f_\alpha \circ g_\alpha = s_\alpha^2 \mathrm{Id}_M$ for some $s_\alpha \in R \setminus \mathfrak{m}_\alpha$.

Note that by the definition of the elements s_α,

$$I = \sum_{\alpha\in\Omega} s_\alpha^2 R = R,$$

because no maximal ideal of R contains I. Therefore, there exist $k > 0$, $\alpha_1, \ldots, \alpha_k \in \Omega$, and $r_1, \ldots, r_k \in R$ such that $1 = \sum_{i=1}^k s_{\alpha_i}^2 r_i$. Hence

$$\sum_{i=1}^k r_i f_{\alpha_i} \circ g_{\alpha_i} = \Big(\sum_{i=1}^k s_{\alpha_i}^2 r_i \Big) \mathrm{Id}_M = \mathrm{Id}_M,$$

and we conclude that M is a direct summand of N^k. □

The following lemma is an extension of Lemma 2.1 in [21] to the case of finitely generated torsion-free modules over commutative domains of finite character.

Lemma 2.3.2. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be an R-algebra, and let M and N be right Λ-modules which are torsion-free as R-modules. Assume that there exists a homomorphism of Λ_Q-modules $F\colon M_Q \to N_Q$. If M is finitely generated, then*

(i) *If F is a monomorphism (respectively, an epimorphism), then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (respectively, an epimorphism) for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, if N is also finitely generated, then

(ii) *If F is a splitting monomorphism (respectively, a splitting epimorphism), then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting monomorphism (respectively, a splitting epimorphism) for almost all maximal ideals $\mathfrak{m}$ of R.*

(iii) *If F is an isomorphism, then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is an isomorphism for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. **(i)** By Lemma 2.1.11, there is a Λ-module homomorphism $f\colon M \to N$ such that $F = f/s$ for some non-zero $s \in R$. Since R is of finite character, s is contained only in finitely many maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $s/1$ is a unit in $R_{\mathfrak{m}}$. Therefore, $f_{\mathfrak{m}}$ is a monomorphism (respectively, an epimorphism) for every maximal ideal not containing s.

(ii) If F is a splitting monomorphism, then there is a Λ_Q-module homomorphism $G\colon N_Q \to M_Q$ such that $G \circ F = \mathrm{Id}_{M_Q}$. In particular, G is a splitting epimorphism. By Lemma 2.1.11, there are Λ-module homomorphisms $f\colon M \to N$ and $g\colon N \to M$ such that $F = f/s$, $G = g/s$, and $g \circ f = s^2\mathrm{Id}_M$ for some non-zero $s \in R$. For any maximal ideal $\mathfrak{m}$ not containing s, $s/1$ is a unit in $R_{\mathfrak{m}}$. Therefore, $f_{\mathfrak{m}}$ is a splitting monomorphism and $g_{\mathfrak{m}}$ is a splitting epimorphism for every maximal ideal not containing s. The proof in the case where F is a splitting epimorphism follows by changing the roles of F and G.

The proof of **(iii)** follows from **(i)** and **(ii)** since F is an isomorphism if and only if it is a monomorphism and a splitting epimorphism. □

Corollary 2.3.3. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be an R-algebra, and let N be a finitely generated right Λ-module which is torsion-free as an R-module. If N_Q is projective, then $N_{\mathfrak{m}}$ is projective for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. Suppose N_Q is projective and let $\tilde{f}\colon F_Q \to N_Q$ be a splitting epimorphism for some finitely generated free Λ-module F. By Lemma 2.3.2, there is a Λ-module homomorphism $f\colon F \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon F_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting epimorphism for almost all maximal ideals $\mathfrak{m}$ of R. Thus, $N_{\mathfrak{m}}$ is projective for almost all maximal ideals $\mathfrak{m}$ of R. □

Lemma 2.3.4. *Let R be a commutative domain, and let Λ be a module-finite R-algebra. Let M and N be finitely generated right Λ-modules which are torsion-free as R-modules. If $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for each $\mathfrak{m}$ in some finite subset $\mathcal{M} \subseteq \operatorname{mSpec} R$, then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting epimorphism for every $\mathfrak{m} \in \mathcal{M}$.*

Proof. Let $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ and fix $i \in \{1, \dots, k\}$. Since $N_{\mathfrak{m}_i}$ is a direct summand of $M_{\mathfrak{m}_i}$, there are $\Lambda_{\mathfrak{m}_i}$-module homomorphisms $\tilde{f}_i\colon M_{\mathfrak{m}_i} \to N_{\mathfrak{m}_i}$ and $\tilde{g}_i\colon N_{\mathfrak{m}_i} \to M_{\mathfrak{m}_i}$ such that $\tilde{f}_i \circ \tilde{g}_i = \mathrm{Id}_{N_{\mathfrak{m}_i}}$. By Lemma 2.1.11, there are Λ-module homomorphisms $f_i\colon M \to N$ and $g_i\colon N \to M$ such that $\tilde{f}_i = f_i/s_i$, $\tilde{g}_i = g_i/s_i$, and $f_i \circ g_i = s_i^2 \mathrm{Id}_N$ for some $s_i \notin \mathfrak{m}_i$. Note that, for any $r \notin \mathfrak{m}_i$, $(rf_i)_{\mathfrak{m}_i}$ is a splitting epimorphism.

For any $i \in \{1, \dots, k\}$, because $R = \mathfrak{m}_i + \bigcap_{j \neq i} \mathfrak{m}_j$, $1 = s_i + r_i$ with $s_i \in \mathfrak{m}_i$ and $r_i \in \bigcap_{j \neq i} \mathfrak{m}_j$. Let $f = \sum_{i=1}^{k} r_i f_i$. We claim that $f_{\mathfrak{m}_i}$ is a splitting epimorphism. Since $(r_i f_i)_{\mathfrak{m}_i}$ is a splitting epimorphism, there is a $\Lambda_{\mathfrak{m}_i}$-module homomorphism $h\colon N_{\mathfrak{m}_i} \to M_{\mathfrak{m}_i}$ such that $(r_i f_i)_{\mathfrak{m}_i} \circ h = \mathrm{Id}_{N_{\mathfrak{m}_i}}$. Hence

$$f_{\mathfrak{m}_i} \circ h = \mathrm{Id}_{N_{\mathfrak{m}_i}} + h', \quad \text{where } h' \in \operatorname{End}_{\Lambda_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i})\mathfrak{m}_i R_{\mathfrak{m}_i}.$$

By Lemma 2.1.6, $h' \in \mathrm{J}(\operatorname{End}_{\Lambda_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}))$. Therefore, $f_{\mathfrak{m}_i} \circ h$ is invertible, and $f_{\mathfrak{m}_i}$ is a splitting epimorphism, as claimed. □

Corollary 2.3.5. *Let R be a commutative semilocal domain, and let Λ be a module-finite R-algebra. Let M and N be finitely generated right Λ-modules which are torsion-free as R-modules. Then*

- **(i)** *N is a direct summand of M if and only if $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*
- **(ii)** *N is isomorphic to M if and only if $N_{\mathfrak{m}}$ is isomorphic to $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Statement **(i)** follows immediately from Lemma 2.3.4. To prove **(ii)**, observe that by **(i)** there exist K and K' finitely generated Λ-modules such that $M \cong N \oplus K$ and $N \cong M \oplus K'$. Since N and M are torsion-free R-modules of the same rank, we deduce that $K = K' = 0$. □

The following is an extension of Lemma 6.1 in [21] to the case of torsion-free modules over domains of finite character.

Proposition 2.3.6. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra, and let M, N, and K be finitely generated right Λ-modules which are torsion-free as R-modules. Assume that:*

- **(i)** *$M_{\mathfrak{m}}$ is a direct summand of $N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R, and*
- **(ii)** *M_Q is a direct summand of K_Q.*

Then M is a direct summand of $N \oplus K$.

Proof. It suffices to find Λ-module homomorphisms $f\colon N \to M$, $g\colon K \to M$ such that, for any $\mathfrak{m} \in \operatorname{mSpec} R$, either $f_\mathfrak{m}$ or $g_\mathfrak{m}$ is a splitting epimorphism. For then $(f, g)\colon N \oplus K \to M$ is locally a splitting epimorphism and, by Lemma 2.1.13, it is a splitting epimorphism.

By condition **(ii)**, since M_Q is a direct summand of K_Q, there is a splitting epimorphism $\tilde{g}\colon K_Q \to M_Q$. By Lemma 2.3.2, there is a Λ-module homomorphism $g\colon K \to M$ such that the induced homomorphism $g_\mathfrak{m}\colon K_\mathfrak{m} \to M_\mathfrak{m}$ is a splitting epimorphism for almost all maximal ideals $\mathfrak{m}$ of R.

By condition **(i)**, since $M_\mathfrak{m}$ is a direct summand of $N_\mathfrak{m}$, there is a splitting epimorphism $\tilde{f}_\mathfrak{m}\colon N_\mathfrak{m} \to M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{M} \subseteq \operatorname{mSpec} R$ be the finite set of maximal ideals such that $g_\mathfrak{m}$ is not a splitting epimorphism. By Lemma 2.3.4, there is a Λ-module homomorphism $f\colon N \to M$ such that the induced homomorphism $f_\mathfrak{m} : N_\mathfrak{m} \to M_\mathfrak{m}$ is a splitting epimorphism for every $\mathfrak{m} \in \mathcal{M}$. This concludes the proof of the proposition. □

3 Package deal theorems for localizations over h-local domains

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In this chapter, we will develop tools to deal with modules over an algebra Λ over an h-local domain R. A particular instance of this situation is when Λ is the endomorphism ring of a finitely generated torsion-free R-module (recall Lemma 2.1.22).

In Section 3.1, we prove basic properties of the localization at maximal ideals of R of finitely generated modules over an algebra Λ over a domain of finite character.

In Section 3.2, we enter the study of constructing Λ-modules having prescribed localization at the maximal ideals of R and proving the so-called *Package Deal Theorems*. To do that, we extend the methods of Levy and Odenthal from algebras over noetherian rings of Krull dimension 1 to algebras over h-local domains.

In Section 3.3, we discuss Package Deal Theorems for projective modules over Λ, whenever Λ is an algebra over an h-local domain. Our goal is to prove some extension needed in Part III.

Finally, in Section 3.4, we prove a package deal theorem for traces of countably generated projective modules. Trace ideals are not, in general, finitely generated. However, as we will see, they satisfy enough finiteness conditions to extend our methods to this case.

3.1 Finitely generated modules over domains of finite character

Lemma 3.1.1. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be an R-algebra, and let M be a finitely generated right Λ-module which is torsion-free as an R-module. Let N be a Λ-submodule of M such that $M_Q = N_Q$. Then $M_\mathfrak{m} = N_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

In particular, $N_\mathfrak{m}$ is finitely generated as a $\Lambda_\mathfrak{m}$-module for almost all maximal ideals $\mathfrak{m}$ of R.

Proof. Since M is torsion-free as an R-module, it can be seen as a Λ-submodule of $M_Q = N_Q$. Therefore, $M \subseteq N_Q$, and since M is finitely generated, by Lemma 2.1.15, $dM \subseteq N$ for some non-zero $d \in R$. Since R is of finite character and d is non-zero, d is contained only in finitely many maximal ideals of R. For

any other maximal ideal $\mathfrak{m}$, $d/1$ is a unit in $R_\mathfrak{m}$. Hence, $M_\mathfrak{m} = N_\mathfrak{m}$, and $N_\mathfrak{m}$ is finitely generated. □

Lemma 3.1.2. *Let R be a commutative domain of finite character with field of fractions Q. Let Y be a submodule of a finitely generated torsion-free R-module M of rank r. Then there exist $y_1, \dots, y_r \in Y$ such that $Y_\mathfrak{m}$ is a finitely generated free $R_\mathfrak{m}$-module with basis $y_1/1, \dots, y_r/1$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. By Remark 2.1.17, we may assume that $M \leq R^s$ for $s = \operatorname{rank} M$, and then Y is a submodule of R^s.

Let $r = \operatorname{rank} Y$, and notice that $r \leq s$. Let $y_1, \dots, y_r \in Y$ be such that they are R-linearly independent (that is, they form a basis of Y_Q). Then, there is an embedding $f\colon R^r \to R^s$ given by $f(a_1, \dots a_r) = \sum_{i=1}^r y_i a_i$. Since f_Q is a splitting monomorphism, by Lemma 2.3.2, $f_\mathfrak{m}$ is a splitting monomorphism for almost all maximal ideals of R, which implies that $Z_\mathfrak{m} = \operatorname{Im} f_\mathfrak{m} = \sum_{i=1}^r \frac{y_i}{1} R_\mathfrak{m}$ is a direct summand of $R^s_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R. Since $Z_\mathfrak{m} \subseteq Y_\mathfrak{m}$ and they have the same rank, we deduce, from the modular law, that $Z_\mathfrak{m} = Y_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$. Therefore, $Y_\mathfrak{m}$ has $y_1/1, \dots, y_r/1$ as a basis for almost all maximal ideals $\mathfrak{m}$ of R. □

Corollary 3.1.3. *Let R be a commutative domain of finite character with field of fractions Q. Let M be a finitely generated R-module. Then,*

- **(i)** *If M is torsion-free then $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for almost all maximal ideals $\mathfrak{m}$ of R;*
- **(ii)** *$M_\mathfrak{m}$ is a finitely presented $R_\mathfrak{m}$-module of projective dimension at most one for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. **(i)** Since M is torsion-free as an R-module, it can be regarded as an essential submodule of its injective hull Q^n. By Remark 2.1.17, $dR^n \subseteq M$ for some non-zero $d \in R$. By Lemma 3.1.1 with $\Lambda = R$ and $N = dR^n$, $M_\mathfrak{m} = R^n_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R, as claimed.

(ii) Assume now that M is a finitely generated R-module. Consider a presentation of M

$$0 \longrightarrow N \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

By Lemma 3.1.2, $N_\mathfrak{m}$ is a finitely generated free module for almost all maximal ideals $\mathfrak{m}$ of R. This proves the claim. □

Corollary 3.1.4. *Let R be a commutative domain of finite character. Let M be a countable direct sum of finitely generated torsion-free R-modules. Then $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all but countably many maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $M = \bigoplus_{i \in \mathbb{N}} M_i$, where each M_i is a finitely generated torsion-free module. By Corollary 3.1.3, $(M_i)_\mathfrak{m}$ is free for almost all maximal ideals $\mathfrak{m}$ of R. Therefore, $M_\mathfrak{m}$ is free for all but countably many maximal ideals $\mathfrak{m}$ of R. □

3.2 Gluing localizations over h-local domains

It is very important to keep in mind the following structural result of torsion modules over h-local domains. It was already proved by Matlis, see Theorem 22 in [37], but since the result and its proof are quite important to our work, we also include a proof of it.

Lemma 3.2.1. *Let R be an h-local domain, and let Λ be an R-algebra. Let M be a Λ-module that is torsion as an R-module. Then $M \cong \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}(R)} M_{\mathfrak{m}}$, each $M_{\mathfrak{m}}$ is a homomorphic image of M, and its structure as a $\Lambda_{\mathfrak{m}}$-module is the same as its structure as a Λ-module.*

Moreover, M is finitely generated as a Λ-module if and only if there exists a finite set $\mathcal{S}$ of maximal ideals of R such that $M_{\mathfrak{m}} = \{0\}$ for any $\mathfrak{m} \notin \mathcal{S}$ and $M_{\mathfrak{m}}$ is a finitely generated $\Lambda_{\mathfrak{m}}$-module for any $\mathfrak{m} \in \mathcal{S}$.

Proof. For any maximal ideal $\mathfrak{m}$ of R, let $\lambda_{\mathfrak{m}} \colon M \to M_{\mathfrak{m}}$ denote the localization map. Then there is an inclusion $\lambda \colon M \hookrightarrow \prod_{\mathfrak{m}\in \mathrm{mSpec}(R)} M_{\mathfrak{m}}$ defined by $m \mapsto (\lambda_{\mathfrak{m}}(m))_{\mathfrak{m}\in \mathrm{mSpec}(R)}$ for any $m \in M$.

For any $m \in M$, $m\Lambda$ is an $R/\mathrm{ann}_R(m)$-module. Being M torsion as an R-module, $\mathrm{ann}_R(m) \neq \{0\}$. Since R is h-local, $\mathrm{ann}_R(m)$ is contained only in finitely many maximal ideals of R. Therefore, $m\Lambda_{\mathfrak{m}} = \{0\}$ for almost all maximal ideals $\mathfrak{m}$ of R. Hence, λ has its image in $\bigoplus_{\mathfrak{m}\in \mathrm{mSpec}(R)} M_{\mathfrak{m}}$.

Let $\mathcal{S}$ denote the finite set of maximal ideals containing $\mathrm{ann}_R(m)$. By Lemma 2.1.21,

$$m\Lambda \cong \bigoplus_{\mathfrak{m}\in\mathcal{S}} (m\Lambda)_{\mathfrak{m}},$$

and also

$$\Lambda/\mathrm{ann}_R(m) \cong \prod_{\mathfrak{m}\in\mathcal{S}} (\Lambda/\mathrm{ann}_R(m))_{\mathfrak{m}}.$$

This allows us to conclude that, for any $m \in M$ and $\mathfrak{m} \in \mathrm{mSpec}(R)$, there exists $m' \in M$ such that $\lambda_{\mathfrak{n}}(m') = 0$ for any $\mathfrak{n} \neq \mathfrak{m}$ and such that $\lambda_{\mathfrak{m}}(m') = \lambda_{\mathfrak{m}}(m)$. This implies that λ induces an isomorphism $M \cong \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}(R)} M_{\mathfrak{m}}$.

In particular, for any maximal ideal $\mathfrak{m}$ of R, $M_{\mathfrak{m}}$ is a homomorphic image of M and the structure as Λ-module is the same as the structure as $\Lambda_{\mathfrak{m}}$-module. So the statement about the finite generation of M easily follows from our previous discussion. □

The nect lemma will be used to determine when we obtain finitely generated modules in our package deal results.

Lemma 3.2.2. *Let R be an h-local domain with field of fractions Q. Let Λ be an R-algebra, and let M be a finitely generated right Λ-module, which is torsion-free as an R-module. Let N be a Λ-submodule of M such that $M_Q = N_Q$. Then N is finitely generated as a Λ-module if and only if $N_{\mathfrak{m}}$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. It is clear that if N is finitely generated as a Λ-module, then $N_{\mathfrak{m}}$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.

To prove the converse implication, assume that $N_{\mathfrak{m}}$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R. Since M is torsion-free as an

R-module, it can be seen as a Λ-submodule of $M_Q = N_Q$. Therefore, $M \subseteq N_Q$, and since M is finitely generated as a Λ-module, by Lemma 2.1.15, $dM \subseteq N$ for some non-zero $d \in R$. Note that dM is also finitely generated. So we only need to prove that N/dM is a finitely generated $\Lambda/d\Lambda$-module.

Since R is h-local and d is non-zero, d is contained only in finitely many maximal ideals of R, say $\mathcal{M} = \{\mathfrak{n}_1, \ldots, \mathfrak{n}_t\}$. By Lemma 2.1.21, there is an isomorphism

$$\varphi\colon R/dR \to (R/dR)_{\mathfrak{n}_1} \times \cdots \times (R/dR)_{\mathfrak{n}_t}.$$

Therefore, $N/dM \cong N_{\mathfrak{n}_1}/dM_{\mathfrak{n}_1} \oplus \cdots \oplus N_{\mathfrak{n}_t}/dM_{\mathfrak{n}_t}$. By Lemma 3.2.1, N/dM is finitely generated as a $\Lambda/d\Lambda$-module. Therefore, N is finitely generated as a Λ-module. □

Package Deal Theorem 3.2.3 (Localization of submodules)**.** *Let R be an h-local domain with field of fractions Q. Let Λ be an R-algebra, and let M be a finitely generated right Λ-module, which is torsion-free as an R-module. For each maximal ideal $\mathfrak{m}$ of R, let $X(\mathfrak{m})$ be a $\Lambda_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$, which is torsion-free as an $R_{\mathfrak{m}}$-module, and such that $X(\mathfrak{m})_Q = (M_{\mathfrak{m}})_Q$. Then the following statements are equivalent*

- **(i)** *There is a Λ-submodule $N \subseteq M$, which is torsion-free as an R-module, and such that $N_{\mathfrak{m}} = X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.*
- **(ii)** *$X(\mathfrak{m}) = M_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, if each of the $X(\mathfrak{m})$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module, then N is also finitely generated as a Λ-module.

Proof. **(ii) ⇒ (i)** Suppose that $X(\mathfrak{m}) = M_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{M} = \{\mathfrak{n}_1, \ldots, \mathfrak{n}_t\}$ denote the finite set of maximal ideals of R such that $X(\mathfrak{m}) \neq M_{\mathfrak{m}}$.

Fix $i \in \{1, \ldots, t\}$. Since $M_{\mathfrak{n}_i}$ is torsion-free as an $R_{\mathfrak{n}_i}$-module, it can be seen as a $\Lambda_{\mathfrak{n}_i}$-submodule of $(M_{\mathfrak{n}_i})_Q = X(\mathfrak{n}_i)_Q$. Therefore, $M_{\mathfrak{n}_i} \subseteq X(\mathfrak{n}_i)_Q$, and since $M_{\mathfrak{n}_i}$ is finitely generated, by Lemma 2.1.15, $d_i M_{\mathfrak{n}_i} \subseteq X(\mathfrak{n}_i)$ for some non-zero $d_i \in R$. Since $X(\mathfrak{n}_i) \neq M_{\mathfrak{n}_i}$, $d_i \in \mathfrak{n}_i$ (otherwise $d_i/1$ would be a unit in $R_{\mathfrak{n}_i}$ and, hence $X(\mathfrak{n}_i) = M_{\mathfrak{n}_i}$, a contradiction).

Let $d = d_1 \cdots d_t \in \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_t$. Since R is h-local and d is non-zero, d is contained only in finitely many maximal ideals of R, say $\{\mathfrak{m}_1 = \mathfrak{n}_1, \ldots, \mathfrak{m}_t = \mathfrak{n}_t, \ldots, \mathfrak{m}_k\}$. By Lemma 2.1.21, the canonical homomorphism

$$\varphi : R/dR \to (R/dR)_{\mathfrak{m}_1} \times \cdots \times (R/dR)_{\mathfrak{m}_k}$$

is an isomorphism. Therefore, there are non-zero $b_1, \ldots, b_t, b \in R$ such that $\varphi(\bar{b}_i) = (\bar{0}, \ldots, \bar{1}^{[(\mathbf{i})]}, \ldots, \bar{0})$ and $\varphi(\bar{b}) = (\bar{0}, \overset{(t}{\ldots}, \bar{0}, \bar{1}, \ldots, \bar{1})$. Note that $b_i/1$ is a unit in $R_{\mathfrak{n}_i}$ for every $i = 1, \ldots, t$, $b/1, b_j/1 \in dR_{\mathfrak{n}_i}$ for every $j \neq i$, and $b/1$ is a unit in $R_{\mathfrak{m}_i}$ for every $i = t+1, \ldots, k$.

For each $i \in \{1, \ldots, t\}$, let X_i' be the Λ-submodule of M generated by the numerators of some set of $\Lambda_{\mathfrak{n}_i}$-generators of $X(\mathfrak{n}_i)$. Let $N = b_1X_1' + \cdots + b_tX_t' + bM + dM$, which is a Λ-submodule of M. Since $dM_{\mathfrak{n}_i} \subseteq X(\mathfrak{n}_i)$, $N_{\mathfrak{n}_i} = X(\mathfrak{n}_i) + dM_{\mathfrak{n}_i} = X(\mathfrak{n}_i)$ for every $i = 1, \ldots, t$. On the other hand, $N_{\mathfrak{m}_i} = M_{\mathfrak{m}_i} + dM_{\mathfrak{m}_i} = M_{\mathfrak{m}_i}$ for

every $i = t+1, \ldots, k$. For any other maximal ideal $\mathfrak{m}$, $d/1$ is a unit in $R_\mathfrak{m}$, hence $N_\mathfrak{m} = M_\mathfrak{m}$, as claimed.

(i) $\Rightarrow$ (ii) Conversely, let N be a Λ-submodule of $M \subseteq M_Q$, which is torsion-free as an R-module, and such that $N_\mathfrak{m} = X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R. Since $M_Q = X(\mathfrak{m})_Q = (N_\mathfrak{m})_Q$, by Lemma 3.1.1, $M_\mathfrak{m} = N_\mathfrak{m} = X(\mathfrak{m})$ for almost all maximal ideals $\mathfrak{m}$ of R.

Finally, the last part of the statement follows from Lemma 3.2.2. This finishes the proof of the theorem. □

Corollary 3.2.4. *Let R be an h-local domain with field of fractions Q. For each maximal ideal $\mathfrak{m}$ of R, let $X(\mathfrak{m})$ be a finitely generated torsion-free $R_\mathfrak{m}$-module of rank n. Then the following statements are equivalent:*

- **(i)** *There is a finitely generated torsion-free R-module N of rank n, such that $N_\mathfrak{m} \cong X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.*
- **(ii)** *$X(\mathfrak{m})$ is a free $R_\mathfrak{m}$-module of rank n for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. **(ii) $\Rightarrow$ (i)** Suppose that $X(\mathfrak{m})$ is a free $R_\mathfrak{m}$-module of rank n for almost all maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{M} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ be the finite set of maximal ideals of R such that $X(\mathfrak{m})$ is not free.

Fix $i \in \{1, \ldots, t\}$. Since $X(\mathfrak{m}_i)$ is finitely generated and torsion-free, we can see $X(\mathfrak{m}_i)$ as an essential submodule of its injective hull Q^n. On the other hand, $R^n_{\mathfrak{m}_i}$ has the same injective hull Q^n. Since $X(\mathfrak{m}_i)$ is finitely generated, by Lemma 2.1.15, $d_i X(\mathfrak{m}_i) \subseteq R^n_{\mathfrak{m}_i}$ for some non-zero $d_i \in R$.

Let $d = d_1 \ldots d_t \in R$ and let $M = d^{-1}R^n$. Note that $X(\mathfrak{m})$ is an $R_\mathfrak{m}$-submodule of $M_\mathfrak{m}$. Since R is h-local and d is non-zero, d is contained only in finitely many maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $d/1$ is a unit in $R_\mathfrak{m}$. Therefore, $M_\mathfrak{m} = R^n_\mathfrak{m} = X(\mathfrak{m})$ for almost all maximal ideals $\mathfrak{m}$ of R. By the Package Deal Theorem 3.2.3, there is a finitely generated torsion-free R-module N, such that $N_\mathfrak{m} = X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.

(i) $\Rightarrow$ (ii) The converse is just Corollary 3.1.3. □

In Section 2.3 we proved some results that allow us to make the following considerations about the (non)uniqueness of the modules constructed in Theorem 3.2.3 and Corollary 3.2.4.

Remark 3.2.5. It is well known that two finitely generated modules M and M' over a commutative domain R that are in the same genus (that is, with isomorphic localizations at the maximal ideals of R) need not be isomorphic.

Moreover, if M and M' are finitely generated torsion-free modules over a commutative domain R that are in the same genus, by Proposition 2.3.1, $\mathrm{add}(M) = \mathrm{add}(M')$. If R has finite character, by Proposition 2.3.6, M is a direct summand of $M' \oplus M'$ and M' is a direct summand of $M \oplus M$. If R is semilocal then $M \cong M'$ by Corollary 2.3.5.

3.3 Projective versus locally projective modules

In general, one has the following easy property for modules over torsion-free algebras over a domain.

Lemma 3.3.1. *Let R be a commutative domain, and let Λ be an R-algebra. Let Σ be a multiplicative subset of R. Let U be a right Λ-module which is torsion-free as an R-module. If V is a finitely generated Λ-submodule that is torsion-free and such that V_Σ is isomorphic to a submodule of U_Σ, then U has a submodule isomorphic to V.*

Proof. By hypothesis, and because V is torsion-free, U_Σ has a submodule V' that is isomorphic to V. Since V' is finitely generated, there exists $d \in \Sigma$ such that $V' \subseteq \frac{1}{d}U \subseteq U_\Sigma$. Then $dV' \leq U$ is the module we are looking for. □

In the following result, we specialize in two-sided ideals of Λ and assume that Λ_Q is a simple artinian ring.

Lemma 3.3.2. *Let R be an h-local domain with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is simple artinian, and let I be a non-zero two-sided ideal of Λ. Then:*

(i) *$I \cap R \neq \{0\}$;*

(ii) *$I_\mathfrak{m} = \Lambda_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R;*

(iii) *There exist $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$ maximal ideals of R such that $\Lambda/I = \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}$.*

Proof. **(i)** Since Λ_Q is simple artinian and Λ is torsion-free, I_Q is a non-zero two-sided ideal of Λ_Q. So $I_Q = \Lambda_Q$. Therefore, there exist $x_1, \ldots, x_n \in I$ and $d \in R \setminus \{0\}$ such that $1 = \sum_{i=1}^{n} \frac{x_i}{d}$. Thus $d \in R \cap I$ and this finishes the proof of **(i)**. Note that d is contained only in a finite number of maximal ideals, so this yields **(ii)**.

Statement **(iii)** is a direct application of Lemma 2.3.2. □

Corollary 3.3.3. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is simple artinian. Let S denote a simple right Λ_Q-module, and assume that $\Lambda_Q \cong S^k$ as right Λ_Q-modules. Then any finitely generated projective right Λ-module P satisfies that $P_\mathfrak{m}^k$ is free for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. By assumption $P_Q \cong S^\ell$ for suitable ℓ. Therefore $P_Q^k \cong \Lambda_Q^\ell$. Now the conclusion follows from Lemma 2.3.2 using the fact that finitely generated projective modules over Λ are torsion-free because Λ is torsion-free. □

Package Deal Theorem 3.3.4 (Localization of Projectives)**.** *Let R be an h-local domain with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is simple artinian. For each maximal ideal $\mathfrak{m}$ of R, let $P(\mathfrak{m})$ be a finitely generated projective right $\Lambda_\mathfrak{m}$-module with trace ideal $I(\mathfrak{m})$, and such that $P(\mathfrak{m})_Q \cong \Lambda_Q^r$ for some positive integer r. Then the following statements are equivalent:*

(i) *There is a finitely generated projective right Λ-module P such that $P_\mathfrak{m} \cong P(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R.*

(ii) *$P(\mathfrak{m}) \cong \Lambda_\mathfrak{m}^r$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, if P is a finitely generated projective satisfying **(i)** *and $I = \mathrm{Tr}_\Lambda(P)$, then $I_\mathfrak{m} = I(\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{mSpec}\ R$.*

Proof. (**ii**) $\Rightarrow$ (**i**). Let $\mathcal{M}$ be the finite set of maximal ideals of R such that $P(\mathfrak{m}) \ncong \Lambda_\mathfrak{m}^r$. Consider a module $F = \Lambda^r$. Since Λ is torsion-free as an R-module, for every $\mathfrak{m} \in \mathcal{M}$ there exists a submodule $F(\mathfrak{m})$ of $F_\mathfrak{m}$ such that $F(\mathfrak{m}) \cong P(\mathfrak{m})$. For $\mathfrak{n} \in \mathrm{mSpec}\, R \setminus \mathcal{M}$ set $F(\mathfrak{n}) = F_\mathfrak{n}$. By Package Deal Theorem 3.2.3, there is a finitely generated right Λ-module P contained in F such that $P_\mathfrak{m} = F(\mathfrak{m}) \cong P(\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{mSpec}R$. By Corollary 2.1.14, P is a projective module. Also note that $\mathrm{Tr}_\Lambda(P)_\mathfrak{m} = \mathrm{Tr}_{\Lambda_\mathfrak{m}}(P_\mathfrak{m})$ by Lemma 2.1.9.

(**i**) $\Rightarrow$ (**ii**) Apply Lemma 2.3.2. □

Corollary 3.3.5. *Let R be an h-local domain with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is simple artinian. Let S denote a simple right Λ_Q-module, and assume that $\Lambda_Q \cong S^k$ as right Λ_Q-modules. Let r be a positive integer, and for each maximal ideal $\mathfrak{m}$ of R, let $P(\mathfrak{m})$ be a finitely generated projective right $\Lambda_\mathfrak{m}$-module with trace ideal $I(\mathfrak{m})$ such that $P(\mathfrak{m})_Q \cong S^r$. Then the following statements are equivalent:*

- **(i)** *There is a finitely generated projective right Λ-module P with trace ideal I such that $P_\mathfrak{m} \cong P(\mathfrak{m})^k$ and $I_\mathfrak{m} = I(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R.*
- **(ii)** *$P(\mathfrak{m})^k$ is free for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. Apply Theorem 3.3.4 to a package $P(\mathfrak{m})' := P(\mathfrak{m})^k$, for each maximal ideal $\mathfrak{m}$ of R. □

3.4 Package deal for traces of projective modules

Now we want to focus on localizations of trace ideals of countably generated projective modules over suitable algebras over h-local domains. Trace ideals of countably generated projective right modules were characterized by Whitehead in [72] and with more detail by Herbera and Příhoda in [24]. We recall here this characterization.

Proposition 3.4.1 (Proposition 2.4 in [24])**.** *Let Λ be a ring, and let I be a two-sided ideal of Λ. Then I is the trace ideal of a countably generated projective right Λ-module if and only if there exists an ascending chain of finitely generated left ideals $(J_n)_{n\geq 1}$ such that $J_{n+1}J_n = J_n$ and $I = \bigcup_{n\geq 1} J_n\Lambda$.*

It is useful to keep in mind the following lemma, as it explains some modifications that can be made in the ascending chain in Proposition 3.4.1.

Lemma 3.4.2 (Lemma 2.2 in [24])**.** *Let Λ be a ring. Let $J_1 \subseteq J_2$ be finitely generated left ideals of Λ satisfying that $J_2J_1 = J_1$. For $i = 1, 2$, fix A_i a finite set of generators of J_i. Let X be a finite subset of λ such that $1 \in X$. For $i = 1, 2$, set*

$$J_i' = \sum_{\substack{x\in X \\ a\in A_i}} \Lambda ax.$$

Then $J_1' \subseteq J_2'$ and $J_2'J_1' = J_1'$. Moreover, for $i = 1, 2$, $J_i \subseteq J_iR = J_i'R$.

Package Deal Theorem 3.4.3 (Localization of trace ideals)**.** *Let R be an h-local domain with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is a simple artinian ring. For each maximal ideal $\mathfrak{m}$ of R, let $I(\mathfrak{m})$ be a non-zero two-sided ideal of $\Lambda_\mathfrak{m}$ which is the trace ideal of a countably generated projective right $\Lambda_\mathfrak{m}$-module. Then the following statements are equivalent*

- **(i)** *There is a two-sided ideal I of Λ, which is the trace ideal of a countably generated projective right Λ-module, and such that $I_\mathfrak{m} = I(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.*
- **(ii)** *$I(\mathfrak{m}) = \Lambda_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, the ideal I that satisfies the equivalent conditions **(i)** *and* **(ii)** *is unique.*

Proof. By Lemmas 10.3 and 10.4 in [25] two trace ideals with isomorphic localizations at maximal ideals of R are equal. This implies that the ideal I in statement **(i)** is unique. Now we proceed to prove the equivalence of the two statements.

(ii) $\Rightarrow$ **(i)** Let $\mathcal{M} = \{\mathfrak{n}_1, \dots, \mathfrak{n}_t\}$ be the set of maximal ideals of R such that $I(\mathfrak{m}) \neq \Lambda_\mathfrak{m}$, and fix some $i \in \{1, \dots, t\}$. We may assume that $\mathcal{M}$ is not an empty set, otherwise by Lemma 10.3 and 10.4 in [25], $I = \Lambda$ and Λ is a trace of a free module. Now, since $I(\mathfrak{n}_i)$ is the trace ideal of a countably generated projective right $\Lambda_{\mathfrak{n}_i}$-module, by Proposition 3.4.1, there is an ascending chain of non-zero finitely generated left ideals $(J_{i,n})_{n\geq 1}$ of $\Lambda_{\mathfrak{n}_i}$ such that

$$J_{i,n+1}J_{i,n} = J_{i,n} \quad \text{and} \quad I(\mathfrak{n}_i) = \bigcup_{n\geq 1} J_{i,n}\Lambda_{\mathfrak{n}_i}.$$

Fix $n \geq 1$. Let $A_{i,n} \subseteq \Lambda$ be a finite set of $\Lambda_{\mathfrak{n}_i}$-generators of $J_{i,n}$. Since $J_{i,n}$ is non-zero and Λ_Q is simple artinian, $(J_{i,n})_Q\Lambda_Q = \Lambda_Q$. Since Λ is torsion-free, it can be seen as a Λ-submodule of $\Lambda_Q = (J_{i,n})_Q\Lambda_Q$. Therefore, $\Lambda \subseteq (J_{i,n})_Q\Lambda_Q$, and by Lemma 2.1.15, $d_{i,n}\Lambda_{\mathfrak{n}_i} \subseteq J_{i,n}\Lambda_{\mathfrak{n}_i}$ for some non-zero $d_{i,n} \in \mathfrak{n}_i$. Then,

$$\frac{1}{d_{i,n}} \sum_{j=1}^{m_{i,n}} \sum_{a_{i,n}\in A_{i,n}} \frac{r_{j,i,n}}{1}\frac{a_{i,n}}{1}\frac{s_{j,i,n}}{1} = 1_{\Lambda_Q} \in \Lambda_Q,$$

for some $r_{j,i,n}, s_{j,i,n} \in \Lambda$ for every $j = 1, \dots, m_{i,n}$.

Let

$$X_n = \{1_\Lambda\} \cup \bigcup_{i=1}^{t} \{s_{1,i,n}, s_{2,i,n}, \dots, s_{m_{i,n},i,n}\} \subseteq \Lambda$$

and define

$$J'_{i,n} = \sum_{\substack{x_n\in X_n \\ a_{i,n}\in A_{i,n}}} \Lambda_{\mathfrak{n}_i} a_{i,n} x_n.$$

By Lemma 3.4.2, $J'_{i,n} \subseteq J'_{i,n+1}$, $J'_{i,n+1}J'_{i,n} = J'_{i,n}$, and $J_{i,n} \subseteq J_{i,n}\Lambda_{\mathfrak{n}_i} = J'_{i,n}\Lambda_{\mathfrak{n}_i}$. Let $d_n = d_{1,n} \dots d_{t,n} \in \mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_t$. Note that $d_n\Lambda_{\mathfrak{n}_i} \subseteq J'_{i,n}$ by construction. Define

$$L_{i,n} = \sum_{\substack{x_n\in X_n \\ a_{i,n}\in A_{i,n}}} \Lambda a_{i,n} x_n + \Lambda d_n,$$

which are left ideals of Λ. Note that $(L_{i,n})_{\mathfrak{n}_i} = J'_{i,n}$ because $d_n\Lambda_{\mathfrak{n}_i} \subseteq J'_{i,n}$. Since R is of finite character and d_n is non-zero, d_n is contained only in finitely many

maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $d_n/1$ is a unit in $R_{\mathfrak{m}}$, hence $(L_{i,n})_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$. Let $\mathcal{M}_n = \{\mathfrak{m}_1 = \mathfrak{n}_1, \ldots, \mathfrak{m}_t = \mathfrak{n}_t, \mathfrak{m}_{t+1}, \ldots, \mathfrak{m}_{k_n}\}$ be the finite set of maximal ideals containing d_n. By Lemma 2.1.21, the canonical homomorphism

$$\varphi_n : R/d_nR \longrightarrow (R/d_nR)_{\mathfrak{m}_1} \times \cdots \times (R/d_nR)_{\mathfrak{m}_{k_n}}$$

is an isomorphism. Therefore, there are non-zero elements $b_{1,n}, \ldots, b_{t,n}, b_n \in R$ such that $\varphi_n(\bar{b}_{i,n}) = (\bar{0}, \ldots, \bar{1}^{[(\mathbf{i})}, \ldots, \bar{0})$ and $\varphi_n(\bar{b}_n) = (\bar{0}, \overset{t)}{\ldots}, \bar{0}, \bar{1}, \ldots, \bar{1})$.

Let $J_n = L_{1,n}b_{1,n} + \cdots + L_{t,n}b_{t,n} + \Lambda b_n + \Lambda d_n$. Note that $(J_n)_{\mathfrak{n}_i} = J'_{i,n}$, and $(J_n)_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for every $\mathfrak{m} \neq \mathfrak{n}_i$. Consider the short exact sequence of left Λ-modules

$$0 \longrightarrow J_{n+1} \longrightarrow J_{n+1} + J_n \longrightarrow (J_{n+1} + J_n)/J_{n+1} \longrightarrow 0.$$

Since localization is an exact functor,

$$0 \longrightarrow (J_{n+1})_{\mathfrak{m}} \longrightarrow (J_{n+1} + J_n)_{\mathfrak{m}} \longrightarrow (J_{n+1} + J_n)_{\mathfrak{m}}/(J_{n+1})_{\mathfrak{m}} \longrightarrow 0$$

is also a short exact sequence of left $\Lambda_{\mathfrak{m}}$-modules. Note that, since $J'_{i,n} \subseteq J'_{i,n+1}$, $(J_{n+1} + J_n)_{\mathfrak{n}_i}/(J_{n+1})_{\mathfrak{n}_i} = (J'_{i,n+1} + J'_{i,n})/J'_{i,n+1} = 0$ for every $i = 1, \ldots, t$. On the other hand, $(J_{n+1} + J_n)_{\mathfrak{m}}/(J_{n+1})_{\mathfrak{m}} = (\Lambda_{\mathfrak{m}} + \Lambda_{\mathfrak{m}})/\Lambda_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \neq \mathfrak{n}_i$. Therefore, $(J_{n+1} + J_n)/J_{n+1} = 0$ and we deduce that $(J_n)_{n\geq 1}$ is an increasing sequence of left ideals of Λ.

Note that $(J_{n+1})_{\mathfrak{n}_i}(J_n)_{\mathfrak{n}_i} = J'_{i,n+1}J'_{i,n} = J'_{i,n} = (J_n)_{\mathfrak{n}_i}$ for every $i = 1, \ldots, t$; and also $(J_{n+1})_{\mathfrak{m}}(J_n)_{\mathfrak{m}} = \Lambda^2_{\mathfrak{m}} = \Lambda_{\mathfrak{m}} = (J_n)_{\mathfrak{m}}$ for every $\mathfrak{m} \neq \mathfrak{n}_i$. Therefore, $J_{n+1}J_n = J_n$ for every $n \geq 1$, and by Proposition 3.4.1, $I = \bigcup_{n\geq 1} J_n\Lambda$ is a two-sided ideal of Λ such that it is the trace ideal of a countably generated projective right Λ-module, $I_{\mathfrak{n}_i} = I(\mathfrak{n}_i)$ and $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ otherwise.

(i) $\Rightarrow$ (ii) The converse is just Lemma 3.3.2. □

4 Direct sums of finitely generated torsion-free modules: from local to global

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We finish this part with a study over commutative domains of finite character or h-local domains. Through this chapter, we prove Theorem B stated in the introduction:

Theorem B (Theorem 4.1.6, Corollary 4.2.3, Corollary 4.2.4, Proposition 4.4.6, Theorem 4.4.8)**.** *Let R be an h-local domain of Krull dimension* 1*. If the class $\mathcal{F}_R$ is closed under direct summands, then R satisfies that*

(1) *$\mathcal{F}_{R_\mathfrak{m}}$ is closed under direct summands for any maximal ideal $\mathfrak{m}$ of R;*

(2) *for any maximal ideal $\mathfrak{m}$ of R, except maybe one, all finitely generated ideals of $R_\mathfrak{m}$ are at most two-generated;*

(3) *the integral closure $\overline{R}$ of R in its field of fractions satisfies that $\mathcal{F}_{\overline{R}}$ is closed under direct summands.*

Conversely, if R is an h-local domain of Krull dimension 1 *satisfying* **(1)** *and* **(2)***, then $\mathcal{F}_R$ is closed under direct summands in the following situations:*

(a) *R is integrally closed, or*

(b) *R is semilocal noetherian, or*

(c) *R is noetherian and has module-finite normalization.*

In Section 4.1 we exploit the theory developed in the previous chapters, proving the first part of Theorem B. An important intermediate result is Theorem 4.1.2, which shows that if $\mathcal{F}_R$ is closed under direct summands, then finitely generated indecomposable torsion-free modules over different localizations at maximal ideals must have coprime rank. This type of result was first spotted by Příhoda in [48], and our proof follows the same strategy as the original result.

In Section 4.2, the remaining part of the characterization of integrally closed domains with $\mathcal{F}$ closed under direct summands, contained in Theorem B, is proven. The noetherian part of Theorem B is proven in Section 4.4. We stress that Theorem 4.4.8 gives the converse of one of the main results in [48].

In Section 4.3, we prove Theorem C stated in the introduction, which says that the isomorphism class of infinite countable direct sums in $\mathcal{F}$ is determined by the genus over suitable rings (cf. Theorem 4.3.4). This section also includes examples showing that over a semilocal domain, being a direct summand of an infinite direct sum of finitely generated torsion-free modules could be satisfied locally but not globally. This contrasts with the case of finitely generated modules (cf. Corollary 2.3.5).

We close this chapter in Section 4.5, with a family of examples of h-local domains of Krull dimension 1 satisfying Theorem B. These examples are of the form $R = K + xL[x]$ where $K \subseteq L$ is a field extension. If such an extension is finitely generated, then R is noetherian. If the extension is transcendental, then R is integrally closed in its field of fractions (and it is not a Prüfer domain!). Moreover, under mild restrictions over the dimension of the field extension, R has indecomposable finitely generated torsion-free modules of all ranks.

4.1 The global case

In this section we study when the class $\mathcal{F}$, of direct sums of finitely generated torsion-free modules over an h-local domain R, is closed under direct summands. In the next Proposition we prove that this property localizes at maximal ideals of R, so the results from Section 2.2 hold for $R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R.

It is worth mentioning that we do not know how to prove directly that if $\mathcal{F}$ is closed under direct summands then the same holds for $R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R. The property that passes to the localization of R is that, for any finitely generated torsion-free R-module, every direct summand of $R^{(\omega)} \oplus X$ is a direct sum of finitely generated modules.

Proposition 4.1.1. *Let R be an h-local domain of Krull dimension 1. Let X be a finitely generated torsion-free R-module, and assume that every direct summand of $R^{(\omega)} \oplus X$ is a direct sum of finitely generated modules. Then, for any maximal ideal $\mathfrak{m}$ of R, $\operatorname{End}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}})$ is a semiperfect ring, so that $X_{\mathfrak{m}}$ satisfies the equivalent conditions of Proposition 2.2.2.*

Proof. Fix a maximal ideal $\mathfrak{m}$ of R, and set $M' = X_{\mathfrak{m}}$. By Lemma 2.1.22**(v)**, $\operatorname{End}_{R_{\mathfrak{m}}}(M')$ is a semilocal ring. We want to prove that it is semiperfect. We decompose $M' = F \oplus M$, where F is a finitely generated free $R_{\mathfrak{m}}$-module and M is a finitely generated module with no projective direct summands. Notice that $\operatorname{End}_{R_{\mathfrak{m}}}(M')$ is semiperfect if and only if so is $\operatorname{End}_{R_{\mathfrak{m}}}(M)$.

By Lemma 2.2.1**(i)** (see also Corollary 1.5.19), to prove that $\operatorname{End}_{R_{\mathfrak{m}}}(M)$ is semiperfect, it suffices to show that the idempotents of

$$\operatorname{End}_{R_{\mathfrak{m}}}(M)/M \operatorname{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}})$$

can be lifted to idempotents of $\operatorname{End}_{R_{\mathfrak{m}}}(M)$. Therefore, we may assume that M is non-zero and that $J = M \operatorname{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}})$ is a proper ideal of $\operatorname{End}_{R_{\mathfrak{m}}}(M)$ contained in $\operatorname{J}(\operatorname{End}_{R_{\mathfrak{m}}}(M))$.

Let $e \in \operatorname{End}_{R_{\mathfrak{m}}}(M)/J$ be a non-trivial idempotent. By Lemma 1.5.18 and Proposition 1.2.12, to show that e can be lifted to an idempotent of $\operatorname{End}_{R_{\mathfrak{m}}}(M)$ we only need to find a decomposition $M \cong A_1 \oplus A_2$ such that, if we set $P_i =$

$\mathrm{Hom}_{R_\mathfrak{m}}(M, A_i)$ for $i = 1, 2$, then $P_1/P_1J \cong e(\mathrm{End}_{R_\mathfrak{m}}(M)/J)$ and $P_2/P_2J \cong (1-e)(\mathrm{End}_{R_\mathfrak{m}}(M)/J)$.

As $M \cong A_1 \oplus A_2$, it is easy to see that, for $i = 1, 2$,

$$\begin{aligned} P_iJ &= \{f \in \mathrm{Hom}_{R_\mathfrak{m}}(M, A_i) \mid f \text{ factors throught a free module}\} \\ &= A_i\mathrm{Hom}_{R_\mathfrak{m}}(M, R_\mathfrak{m}). \end{aligned}$$

Therefore, we need to prove that $M \cong A_1 \oplus A_2$ with

$$\mathrm{Hom}_{R_\mathfrak{m}}(M, A_1)/A_1\mathrm{Hom}_{R_\mathfrak{m}}(M, R_\mathfrak{m}) \cong e(\mathrm{End}_{R_\mathfrak{m}}(M)/J)$$

and

$$\mathrm{Hom}_{R_\mathfrak{m}}(M, A_2)/A_2\mathrm{Hom}_{R_\mathfrak{m}}(M, R_\mathfrak{m}) \cong (1-e)(\mathrm{End}_{R_\mathfrak{m}}(M)/J).$$

By Lemma 2.1.22 there is an extension of rings

$$R/I \subseteq \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R),$$

where $0 \neq I = R \cap X\,\mathrm{Hom}_R(X, R)$. Since R is h-local, I is contained only in finitely many maximal ideals $\{\mathfrak{m} = \mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$ of R. So we have an isomorphism

$$\begin{aligned} \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R) &\to \prod_{i=1}^{\ell} (\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R))_{\mathfrak{m}_i} \\ &\cong \prod_{i=1}^{\ell} \mathrm{End}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i})/X_{\mathfrak{m}_i}\,\mathrm{Hom}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i}, R_{\mathfrak{m}_i}) \end{aligned}$$

given by taking localization at $\mathfrak{m}_i$ at each component (cf. Lemma 3.2.1).

Hence, there exists $\widetilde{e}^2 = \widetilde{e} \in \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R)$ such that $\lambda_\mathfrak{n}(\widetilde{e}) = 0$ for any maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ and $\lambda_\mathfrak{m}(\widetilde{e}) = e$, where

$$\begin{aligned} \lambda_\mathfrak{n}\colon \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R) &\to (\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R))_\mathfrak{n} \cong \\ &\cong \mathrm{End}_{R_\mathfrak{n}}(X_\mathfrak{n})/X_\mathfrak{n}\,\mathrm{Hom}_{R_\mathfrak{n}}(X_\mathfrak{n}, R_\mathfrak{n}) \end{aligned}$$

denotes the localization morphism.

By Corollary 1.5.12, $R^{(\omega)} \oplus X = X_1 \oplus X_2$ with

$$\begin{aligned} \mathrm{Hom}_R(R \oplus X, X_1)/X_1\,\mathrm{Hom}_R(R \oplus X, R) &\cong \widetilde{e}(\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R)) \\ &\cong e(\mathrm{End}_{R_\mathfrak{m}}(M)/J) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_R(R \oplus X, X_2)/X_2\,\mathrm{Hom}_R(R \oplus X, R) &\cong (1-\widetilde{e})(\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X, R)) \\ &\cong (1-e)(\mathrm{End}_{R_\mathfrak{m}}(M)/J) \times \Big(\prod_{i=2}^{\ell} \mathrm{End}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i})/X_{\mathfrak{m}_i}\,\mathrm{Hom}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i}, R_{\mathfrak{m}_i})\Big). \end{aligned}$$

Therefore,

$$R_\mathfrak{m}^{(\omega)} \oplus M \cong R_\mathfrak{m}^{(\omega)} \oplus X_\mathfrak{m} = (X_1)_\mathfrak{m} \oplus (X_2)_\mathfrak{m},$$

$$\mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m} \oplus X_\mathfrak{m}, (X_1)_\mathfrak{m})/(X_1)_\mathfrak{m}\,\mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m} \oplus X_\mathfrak{m}, R_\mathfrak{m}) \cong e(\mathrm{End}_{R_\mathfrak{m}}(M)/J)$$

and

$$\mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m} \oplus X_\mathfrak{m}, (X_2)_\mathfrak{m})/(X_2)_\mathfrak{m}\,\mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m} \oplus X_\mathfrak{m}, R_\mathfrak{m}) \cong (1-e)(\mathrm{End}_{R_\mathfrak{m}}(M)/J).$$

By hypothesis, for $i = 1, 2$, X_i is a direct sum of finitely generated R-modules; hence, $N_i = (X_i)_{\mathfrak{m}}$ is also a direct sum of finitely generated $R_{\mathfrak{m}}$-modules. By Lemma 1.5.13, we obtain a decomposition $M = A_1 \oplus A_2$ such that, for each $i \in \{1, 2\}$,

$$\begin{aligned}\operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus M, N_i)/N_i \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus M, R_{\mathfrak{m}}) \\ \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(M, A_i)/A_i \operatorname{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}).\end{aligned}$$

Therefore $\operatorname{Hom}_{R_{\mathfrak{m}}}(M, A_1)/A_1 \operatorname{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}) \cong e(\operatorname{End}_{R_{\mathfrak{m}}}(M)/J)$ and

$$\operatorname{Hom}_{R_{\mathfrak{m}}}(M, A_2)/A_2 \operatorname{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}) \cong (1-e)(\operatorname{End}_{R_{\mathfrak{m}}}(M)/J),$$

as we wanted. So we can conclude that e can be lifted to an idempotent of $\operatorname{End}_{R_{\mathfrak{m}}}(M)$ by an application of Lemma 1.5.18. □

Next result shows that, over an h-local domain R, to have the class $\mathcal{F}$ closed under direct summands we need that the ranks of indecomposable modules over different localizations at maximal ideals are coprime.

This somewhat surprising Theorem 4.1.2 is the extension of Lemma 4 in [48] to our setting. For its proof, we need our versions of the Package Deal Theorems 3.2.3 and 3.4.3, as well as the results from Section 1.5.

Theorem 4.1.2. *Let R be an h-local domain with at least two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. For each $i = 1, 2$, let M_i be a finitely generated, indecomposable, torsion-free $R_{\mathfrak{m}_i}$-module with local endomorphism ring and with rank r_i. Then,*

- **(i)** *for $i = 1, 2$, there exists a finitely generated, indecomposable, torsion-free R-module X_i such that $(X_i)_{\mathfrak{m}_i} \cong M_i$ and $X_{\mathfrak{m}} \cong R_{\mathfrak{m}}^{r_i}$ for every maximal ideal $\mathfrak{m}$ different from $\mathfrak{m}_i$;*
- **(ii)** *if r_1 and r_2 are not coprime then $\operatorname{Add}(X_1 \oplus X_2)$ has elements that are not a direct sum of finitely generated modules.*

Proof. Corollary 3.2.4 ensures the existence of finitely generated, indecomposable, torsion-free R-modules X_1 and X_2 with the claimed properties in statement **(i)**.

To prove **(ii)**, let $d = \gcd(r_1, r_2)$. We show that if $d > 1$, there exists a module in $\operatorname{Add}(X_1 \oplus X_2)$ which is not a direct sum of finitely generated torsion-free R-modules.

Let $\Lambda = \operatorname{End}_R(X_1 \oplus X_2)$. By Lemma 2.1.22**(i)** and **(iii)**, Λ is a torsion-free R-module and Λ_Q a simple artinian ring. Recall from Remark 1.5.11 that Λ can be identified with the matrix ring:

$$\Lambda = \begin{bmatrix} \operatorname{End}_R(X_1) & \operatorname{Hom}_R(X_2, X_1) \\ \operatorname{Hom}_R(X_1, X_2) & \operatorname{End}_R(X_2) \end{bmatrix}$$

By Lemma 2.1.11, $\Lambda_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}((X_1)_{\mathfrak{m}} \oplus (X_2)_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of R. Let

$$I_1 = \Lambda_{\mathfrak{m}_1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{\mathfrak{m}_1} \qquad \text{and} \qquad I_2 = \Lambda_{\mathfrak{m}_2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Lambda_{\mathfrak{m}_2}.$$

By the Package Deal Theorem 3.4.3, there is a non-zero two-sided ideal I of Λ, which is the trace of a countably generated projective right Λ-module, such that $I_{\mathfrak{m}_i} = I_i$ $(i = 1, 2)$ and $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R different from $\mathfrak{m}_i$.

By Lemma 2.1.22**(iv)**, $I \cap R \neq \{0\}$, so Λ/I is a torsion R-module. Note that the only maximal ideals containing $I \cap R$ are $\mathfrak{m}_1$ and $\mathfrak{m}_2$ (otherwise, if there exists another maximal ideal $\mathfrak{n}$ containing $I \cap R$, $I_{\mathfrak{n}} \cap R_{\mathfrak{n}} \neq R_{\mathfrak{n}}$, and then $I_{\mathfrak{n}} \neq \Lambda_{\mathfrak{n}}$, a contradiction). By Lemma 2.1.22[**(vi)**, the canonical homomorphism

$$\Lambda/I \to \bigoplus_{\mathfrak{m}\in \operatorname{mSpec} R}(\Lambda/I)_{\mathfrak{m}} = (\Lambda/I)_{\mathfrak{m}_1} \times (\Lambda/I)_{\mathfrak{m}_2}$$

is an isomorphism.

From Remark 1.5.11,

$$I_1 = \begin{bmatrix} \operatorname{End}_{R_{\mathfrak{m}_1}}(M_1) & \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1, R_{\mathfrak{m}_1}^{r_2}) \\ \operatorname{Hom}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2}, M_1) & \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1, R_{\mathfrak{m}_1}^{r_2}) \operatorname{Hom}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2}, M_1) \end{bmatrix}.$$

Therefore, $(\Lambda/I)_{\mathfrak{m}_1} \cong \Lambda_{\mathfrak{m}_1}/I_1 \cong \mathrm{M}_{r_2}(R_{\mathfrak{m}_1})/J$, where

$$J = \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1, R_{\mathfrak{m}_1}^{r_2}) \operatorname{Hom}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2}, M_1).$$

Notice that $J \neq \operatorname{End}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2})$ because this would imply that the identity map of $R_{\mathfrak{m}_1}^{r_2}$ is in J, so that $R_{\mathfrak{m}_1}^{r_2}$ is a direct summand of M_1^n for some $n \in \mathbb{N}$. Since M_1 has local endomorphism ring, the Krull-Schmidt-Remak-Azumaya Theorem implies that $M_1 \cong R_{\mathfrak{m}_1}$, but we are assuming that $r_1 > 1$, a contradiction.

Therefore, there is an isomorphism $\varphi\colon \Lambda/I \to \mathrm{M}_{r_2}(R_{\mathfrak{m}_1})/J \times (\Lambda/I)_{\mathfrak{m}_2}$. Then there is an idempotent element $e \in \Lambda/I$ such that $\varphi(e) = (E_{11} + J, 0)$, where E_{11} is the idempotent matrix with 1 in the 1,1-entry and zeros elsewhere. Since $\mathrm{M}_{r_2}(R_{\mathfrak{m}_1})E_{11}\mathrm{M}_{r_2}(R_{\mathfrak{m}_1}) = \mathrm{M}_{r_2}(R_{\mathfrak{m}_1})$, we deduce that $E_{11} \notin J$.

By Theorem 1.5.5**(i)**, there is a countably generated projective right Λ-module Q such that $Q/QI \cong e(\Lambda/I)$. We claim that such Q is neither finitely generated nor a direct sum of finitely generated modules.

Recall that $\Lambda_{\mathfrak{m}_1} \cong \operatorname{End}_{R_{\mathfrak{m}_1}}(M_1 \oplus R_{\mathfrak{m}_1}^{r_2})$. Since M_1 has local endomorphism ring, there are only two finitely generated indecomposable projective $\Lambda_{\mathfrak{m}_1}$-modules up to isomorphism, namely

$$P_{1a} = \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1 \oplus R_{\mathfrak{m}_1}^{r_2}, M_1) \quad \text{and} \quad P_{1b} = \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1 \oplus R_{\mathfrak{m}_1}^{r_2}, R_{\mathfrak{m}_1}).$$

Notice that P_{1a} and P_{1b} are not isomorphic because M_1 has rank $r_1 > 1$. In addition, by the Krull-Schmidt-Remak-Azumaya Theorem, every projective $\Lambda_{\mathfrak{m}_1}$-module can be written uniquely as a direct sum of copies of P_{1a} and P_{1b}.

By the definition of P_{1a}, $\operatorname{Tr}(P_{1a}) = I_1$. On the other hand,

$$(Q/QI)_{\mathfrak{m}_1} \cong (e(\Lambda/I))_{\mathfrak{m}_1} \cong (E_{11} + J)\mathrm{M}_{r_2}(R_{\mathfrak{m}_1}),$$

so $Q_{\mathfrak{m}_1} \cong P_{1a}^{(\kappa_1)} \oplus P_{1b}$, where κ_1 is, at most, a countable cardinal.

Similarly, there are two finitely generated indecomposable projective $\Lambda_{\mathfrak{m}_2}$-modules up to isomorphism

$$P_{2a} = \operatorname{Hom}_{R_{\mathfrak{m}_2}}(R_{\mathfrak{m}_2}^{r_1} \oplus M_2, M_2) \quad \text{and} \quad P_{2b} = \operatorname{Hom}_{R_{\mathfrak{m}_2}}(R_{\mathfrak{m}_2}^{r_1} \oplus M_2, R_{\mathfrak{m}_2}).$$

By the definition of P_{2a}, $\operatorname{Tr}(P_{2a}) = I_2$. Thus, since $(Q/QI)_{\mathfrak{m}_2} \cong (e(\Lambda/I))_{\mathfrak{m}_2} = \{0\}$, we have $Q_{\mathfrak{m}_2} \cong P_{2a}^{(\kappa_2)}$, where κ_2 is, at most, a countable cardinal.

Let $L = Q \otimes_\Lambda (X_1 \oplus X_2)$, which is in $\operatorname{Add}(X_1 \oplus X_2)$. By Proposition 1.2.12, L is a direct summand of $(X_1 \oplus X_2)^{(\omega)}$ and $\operatorname{Hom}_R(X_1 \oplus X_2, L) \cong Q$. Moreover, by Proposition 1.2.12,

$$(*) \quad L_{\mathfrak{m}_1} \cong M_1^{(\kappa_1)} \oplus R_{\mathfrak{m}_1} \qquad \text{and} \qquad (**) \quad L_{\mathfrak{m}_2} \cong M_2^{(\kappa_2)}.$$

If Q is finitely generated, κ_1 and κ_2 are finite. But then, by the isomorphism $(*)$, the rank of L is congruent to 1 modulo d and, by the isomorphism $(**)$, the rank of L is divisible by d, a contradiction.

To prove the second statement in the claim, assume that $Q = \bigoplus_{i\in\mathbb{N}} Q_i$, where the Q_i's are finitely generated. Since $Q/QI \cong e(\Lambda/I)$ is finitely generated, $Q_i = Q_iI$ for almost all $i \in \mathbb{N}$. If $I_0 := \{i \in \mathbb{N} \mid Q_i \neq Q_iI\}$, then $Q_f = \bigoplus_{i\in I_0} Q_i$ is a finitely generated projective right Λ-module such that $Q_f/Q_fI \cong e(\Lambda/I)$. By the previous part of the proof, such Q_f does not exist, hence Q is not a direct sum of finitely generated modules. By Proposition 1.2.12, $L \in \mathrm{Add}(X_1 \oplus X_2)$ is not a direct sum of finitely generated modules. This finishes the proof of (2). □

Next proposition reviews a construction, that goes back to an idea of Bass [6], of an indecomposable finitely generated torsion-free module of rank two over any local domain that has a finitely generated ideal that cannot be generated by two elements.

This construction is particularly relevant to us in view of Theorem 4.1.2. It is going to imply that if $\mathcal{F}$ is closed under direct summands, then finitely generated ideals of $R_{\mathfrak{m}}$ are two-generated for all maximal ideals of R except maybe one, see Theorem 4.1.6.

Proposition 4.1.3. *Let R be a commutative local domain, with field of fractions Q, and maximal ideal $\mathfrak{m}$. Let a, b, c be elements in R such that the ideal $K = aR + bR + cR$ cannot be generated by two elements. Set $\alpha = (a, b, c) \in R^3$ and let $H = R^3 \cap Q\alpha$. Then:*

- **(i)** *$M = R^3/H$ is an indecomposable, finitely generated torsion-free module of rank 2.*
- **(ii)** *If $\overline{R}$ is local, then so is $\mathrm{End}_R(M)$.*

Proof. **(i)** Let $k = R/\mathfrak{m}$ be the residue field of R. Let $M = A_1/H \oplus A_2/H$ be a nontrivial decomposition with A_i R-submodules of R^3 containing H. Consider $M \otimes_R k \cong (A_1/H \otimes_R k) \oplus (A_2/H \otimes_R k)$. Then, since $M \otimes_R k$ is at most a three-dimensional vector space, one of the direct summands is one-dimensional. We may assume that it is $A_2/H \otimes_R k$, which in turn implies that $A_2/H \cong R^3/A_1$ is isomorphic to R. Therefore, the exact sequence

$$0 \longrightarrow A_1 \longrightarrow R^3 \longrightarrow R^3/A_1 \longrightarrow 0$$

splits, and we deduce that A_1 is a 2-generated free module. So that if v_1 and v_2 is a basis of A_1, there exists $r_1, r_2 \in R$ such that $(a, b, c) = v_1r_1 + v_2r_2$. This implies that $K \subseteq r_1R + r_2R$. Since v_1 and v_2 can be completed to a basis of R^3, it follows that $K = r_1R + r_2R$, which contradicts the assumption that K cannot be generated by two elements. This finishes the proof that M is indecomposable.

(ii) The module M fits into the exact sequence

$$0 \longrightarrow H \longrightarrow R^3 \longrightarrow M \longrightarrow 0.$$

Let $T = \{f \in \mathrm{End}_R(R^3) \mid f(H) \subseteq H\}$, which is a subring of $\mathrm{End}_R(R^3)$. Then there is an onto ring endomorphism $\varphi\colon T \to \mathrm{End}_R(M)$. We will prove that T is a local ring, and then so is $\mathrm{End}_R(M)$.

Restriction to H induces a ring morphism $\psi\colon T \to \mathrm{End}_R(H)$ with kernel $I = \{f \in T \mid f(H) = 0\}$.

We claim that the embedding $T \hookrightarrow \mathrm{End}_R(R^3)$ is local. Indeed, let $f \in T$ be such that f is invertible in $\mathrm{End}_R(R^3)$. Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & R^3 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi(f)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi(f)} & & \\ 0 & \longrightarrow & H & \longrightarrow & R^3 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since f is invertible, it is onto and then so is $\varphi(f)$. Since M is a finitely generated module over a commutative ring, we deduce that $\varphi(f)$ is bijective. Now the Snake Lemma implies that $\psi(f)$ is also bijective, and then we can deduce that $f^{-1} \in T$.

Now we prove that $I \subseteq \mathrm{J}(T)$. Notice that if $f \in I$, then it induces a module homomorphism $\tilde{f}\colon M \to R^3$ such that $\operatorname{im} f = \operatorname{im} \tilde{f}$. Since M is indecomposable, and R is local $\operatorname{im} \tilde{f} \subseteq R^3\mathfrak{m}$. Hence, $1 - f$ is invertible in $\mathrm{End}_R(R^3)$, and since the embedding $T \hookrightarrow \mathrm{End}_R(R^3)$ is local, we deduce that $1 - f$ is invertible in T. This proves that the two-sided ideal $I \subseteq \mathrm{J}(T)$.

The image of the ring morphism ψ is the ring

$$S = \{g \in \mathrm{End}_R(H) \mid g \text{ can be extended to an endomorphism of } R^3\}.$$

Hence $T/I \cong S$, and T is local provided S is local. Notice that for any $g \in S$ its extension to R^3 satisfies a monic polynomial of degree 3 with coefficients in R, so g also satisfies that polynomial. This is to say that the extension $R \hookrightarrow S$ is integral.

Since H is a torsion-free module of rank 1, $\mathrm{End}_R(H)$ can be identified with a subring of Q. Then S is a subring of $\overline{R}$. Since $\overline{R}$ is local, so is S. This finishes the proof of the result. □

The next corollary shows that the conclusions of Proposition 4.1.3 can be extended to any ring S between R and its integral closure.

Corollary 4.1.4. *Let R be a commutative local domain, with field of fractions Q and maximal ideal $\mathfrak{m}$ such that $\overline{R}$ is local. Let S be an intermediate ring $R \subseteq S \subseteq \overline{R}$. Let a, b, c be elements in S such that the ideal $K = aS + bS + cS$ cannot be generated by two elements. Set $\alpha = (a, b, c) \in S^3$ and let $H = S^3 \cap Q\alpha$. Then:*

- **(i)** *$M = S^3/H$ is a torsion-free R-module of rank 2 and $\mathrm{End}_R(M) = \mathrm{End}_S(M)$ is a local ring. In particular, M is indecomposable.*
- **(ii)** *Consider the ring $T = R[a, b, c]$ which is a finite integral extension of R. Let $H' = H \cap T^3$. Then $M' = T^3/H'$ is an indecomposable, finitely generated R-module of rank 2 with local endomorphism ring.*

Proof. Since $\overline{R}$ is local, so is any intermediate ring $R \subseteq S \subseteq \overline{R}$.

Notice also that an S-module M has finite rank n as S-module if and only if it has finite rank n as R-module because $M \otimes_S Q = M \otimes_S S \otimes_R Q = M \otimes_R Q$.

This implies, in addition, that $\mathrm{End}_S(M) \cong \{A \in \mathrm{M}_n(Q) \mid AM \subseteq M\}$ coincides with $\mathrm{End}_R(M)$.

Therefore, the result is a consequence of Proposition 4.1.3 applied to S in **(i)** and to T in **(ii)**. □

The following lemma is a variation for domains of finite character of a well-known fact about bounds on the number of generators of finitely generated ideals.

Lemma 4.1.5 (Proposition 1.4 in [5])**.** *Let R be a commutative domain of finite character. Let $k \geq 2$. If I is a non-zero finitely generated ideal of R such that $IR_{\mathfrak{m}}$ is k-generated for every maximal ideal $\mathfrak{m}$ of R, then I is also k-generated.*

Proof. Let I be a non-zero ideal of R. Since R is of finite character, I is contained only in finitely many maximal ideals $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ of R. First, observe that there exists $\alpha \in I$ such that $\alpha/1 \notin \mathfrak{m}_i I_{\mathfrak{m}_i}$ for every $i = 1, \ldots, n$. Indeed, let $\alpha_i \in I$ be such that $\alpha_i/1 \notin \mathfrak{m}_i I_{\mathfrak{m}_i}$ and let $e_1, \ldots, e_n$ be such that $e_i - 1 \in \mathfrak{m}_i$ for every $i = 1, \ldots, n$ and $e_i \in \mathfrak{m}_j$ whenever $1 \leq i \neq j \leq n$. Then let $\alpha = \sum_{i=1}^{n} e_i \alpha_i$.

Since R is of finite character, α is contained only in finitely many maximal ideals of R, say $\mathcal{M} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n, \mathfrak{m}_{n+1}, \ldots, \mathfrak{m}_{n+\ell}\}$.

For each $i = 1, \ldots, n$, there are $\alpha_{i,1}, \ldots, \alpha_{i,k-1} \in I$ such that

$$I_{\mathfrak{m}_i} = \frac{\alpha}{1} R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \frac{\alpha_{i,j}}{1} R_{\mathfrak{m}_i}.$$

For each $i = n+1, \ldots, n+\ell, j = 1, \ldots, k-1$ let $\alpha_{i,1} \in I \setminus \mathfrak{m}_i$ and $\alpha_{i,j} = 0$ if $j > 1$. Notice that $I_{\mathfrak{m}_i} = \frac{\alpha}{1} R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \frac{\alpha_{i,j}}{1} R_{\mathfrak{m}_i}$ for every $i = 1, \ldots, n+\ell$. As before, consider $f_1, \ldots, f_{n+\ell} \in R$ such that $f_i - 1 \in \mathfrak{m}_i$ and $f_i \in \mathfrak{m}_j$ if $1 \leq j \neq i \leq n+\ell$. Set $\beta_{i,j} = f_i \alpha_{i,j}$ for every $i = 1, \ldots, n+\ell$ and $j = 1, \ldots, k-1$. For $j = 1, \ldots, k-1$ set $\alpha_j := \sum_{i=1}^{n+\ell} \beta_{i,j}$. We claim that $I = \alpha R + \sum_{j=1}^{k-1} \alpha_j R$. It is sufficient to verify this equality locally. If $\mathfrak{m} \notin \mathcal{M}$ then both sides localize to $R_{\mathfrak{m}}$. Also,

$$\alpha R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \alpha_j R_{\mathfrak{m}_i} + \mathfrak{m}_i I_{\mathfrak{m}_i} = \alpha R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \alpha_{i,j} R_{\mathfrak{m}_i} + \mathfrak{m}_i I_{\mathfrak{m}_i}.$$

Since I is finitely generated, we can conclude by Nakayama's lemma. □

Now we are ready to prove the main result of the section.

Theorem 4.1.6. *Let R be a commutative domain of finite character, and of Krull dimension 1. Assume that for any finitely generated torsion-free R-module X, every element of $\mathrm{Add}(X)$ is a direct sum of finitely generated modules. Then R satisfies the following properties:*

- **(1)** *For any maximal ideal $\mathfrak{m}$ of R and any ring S such that $R_{\mathfrak{m}} \subseteq S \subseteq \overline{R_{\mathfrak{m}}}$ we have that finitely generated indecomposable torsion-free S-modules have local endomorphism ring;*

- **(2)** *for any maximal ideal $\mathfrak{m}$ of $R_{\mathfrak{m}}$, except maybe one maximal ideal $\mathfrak{m}_0$, all finitely generated ideals of $R_{\mathfrak{m}}$ are at most two-generated. Then $\overline{R_{\mathfrak{m}}}$ is a valuation ring for any maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$.*

(3) *for each maximal ideal* $\mathfrak{m}$ *of* R*, there is a unique maximal ideal of* $\overline{R}$ *lying over* $\mathfrak{m}$*. In particular,* $\overline{R}$ *has also finite character.*

If $\overline{R_{\mathfrak{m}}}$ *satisfies the two-generated property for any maximal ideal* $\mathfrak{m}$*, then* $\overline{R}$ *is a Prüfer domain of Krull dimension* 1*, so any finitely generated ideal of* R *is, at most, two-generated.*

Proof. By Corollary 2.2.3 and Proposition 4.1.1, we deduce that for any maximal ideal $\mathfrak{m}$ of R, any ring S such that $R_{\mathfrak{m}} \subseteq S \subseteq \overline{R_{\mathfrak{m}}}$ satisfies that every finitely generated indecomposable torsion-free S-module has local endomorphism ring. This shows (1).

Notice that **(1)** implies that $\overline{R}$ is also a ring of finite character because there is just one maximal ideal of $\overline{R}$ lying over each maximal ideal of R. This shows (3).

By Theorem 4.1.2, the ranks of two finitely generated, indecomposable, torsion-free modules over different localizations of R at maximal ideals must be coprime. In view of Corollary 4.1.4, we deduce that finitely generated ideals of $R_{\mathfrak{m}}$ are 2-generated for any maximal ideal $\mathfrak{m}$ of R, except maybe for one that we will denote by $\mathfrak{m}_0$. By Proposition III.1.11 in [18], $\overline{R_{\mathfrak{m}}}$ is a valuation ring for any maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$. This proves statement (2).

If the maximal ideal $\mathfrak{m}_0$ does not exist, then $\overline{R}$ is a Prüfer domain of Krull dimension 1. Moreover, by Lemma 4.1.5, any finitely generated ideal of R is, at most, 2-generated. □

4.2 The integrally closed case

In this section, we will use the results developed until now to show that the converse of Theorem 4.1.6 is true for integrally closed h-local domains of Krull dimension 1. This will be done in Corollary 4.2.3.

We recall the following definition, which will be needed for this section.

Definition 4.2.1. Let R be a ring. Let M be a right R-module. A submodule N of M is called relatively divisible or an RD-*submodule* if $Nr = N \cap Mr$ for each $r \in R$.

In particular, this definition tells us that if for some $r \in R$, $a \in N$, the equation $xr = a$ has a solution in M, then it has a solution in N as well. In this sense, pure submodules are a generalization of RD-submodules to the case where, instead of an equation, we have a system of linear equations which has a solution in N^n whenever it has a solution in M^n, for some positive integer $n \geq 1$.

Proposition 4.2.2. *Let* R *be an* h*-local domain of Krull dimension* 1 *with field of fractions* Q*. Assume that* $R_{\mathfrak{m}}$ *is a valuation domain for all maximal ideals* $\mathfrak{m}$ *of* R *except for maybe one maximal ideal* $\mathfrak{m}_0$*.*

Then the class of R*-modules that are direct sums of finitely generated torsion-free* R*-modules is closed under direct summands if and only if the class of direct sums of finitely generated torsion-free* $R_{\mathfrak{m}_0}$*-modules is closed under direct summands.*

Proof. By Proposition 4.1.1, if the class of R-modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands, then so is the corresponding class for $R_{\mathfrak{m}_0}$. We need to prove the converse result.

Let $\{N_i \mid i \in \mathbb{N}\}$ be a countable family of non-zero finitely generated torsion-free R-modules and set $N = \bigoplus_{i\in\mathbb{N}} N_i$. Let A be a direct summand of N. Then $A_{\mathfrak{m}_0}$ is a direct summand of $N_{\mathfrak{m}_0}$. By hypothesis, $A_{\mathfrak{m}_0}$ is a direct sum of finitely generated $R_{\mathfrak{m}_0}$-modules, say $A_{\mathfrak{m}_0} = \bigoplus_{i\in\mathbb{N}} X_i$. For $i \in \mathbb{N}$, let $A_i := \{a \in A \mid \frac{a}{1} \in \bigoplus_{1\leq j\leq i} X_j\}$. We claim that A_i is an RD-submodule of A. Let $a \in A$ and $r \in R \setminus \{0\}$ be such that $\frac{a}{1}\frac{r}{1} \in \bigoplus_{1\leq j\leq i} X_j$, since $\frac{r}{1}$ is not a zero divisor of X_ℓ for any ℓ, we deduce that $\frac{a}{1} \in A_i$. This finishes the proof of the claim.

Note also that there is an $\ell \geq 1$ such that $\bigoplus_{1\leq j\leq i} X_j$ is a submodule of $\bigoplus_{i=1}^{\ell}(N_i)_{\mathfrak{m}_0}$. Hence, since all modules involved are torsion-free, $A_i \leq \bigoplus_{i=1}^{\ell} N_i$. We claim that this is also an RD-embedding. Assume that $n \in \bigoplus_{i=1}^{\ell} N_i$ is such that there exists $r \in R$ such that $nr \in A_i \leq A$. As A is an RD-submodule of N, we deduce that $n \in A$, and since A_i is an RD-submodule of A we deduce that $n \in A_i$.

Now we shall prove that A_i is finitely generated. Since A_i is a submodule of a free module R^r, by Lemma 3.1.2 there exist $a_1, \ldots, a_s \in A_i$ and a finite set of maximal ideals of the form $\mathcal{S} = \{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ such that $(A_i)_{\mathfrak{m}} = \sum_{j=1}^{s} \frac{a_j}{1} R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m} \notin \mathcal{S}$. In particular, $(A_i)_Q = \sum_{j=1}^{s} \frac{a_j}{1} Q$ and in the exact sequence,

$$0 \longrightarrow \sum_{j=1}^{s} a_j R \longrightarrow A_i \longrightarrow A_i/\Big(\sum_{j=1}^{s} a_j R\Big) \longrightarrow 0$$

the module $A_i/(\sum_{j=1}^{s} a_j R)$ is torsion. Since R is an h-local domain of Krull dimension 1,

$$A_i/\Big(\sum_{j=1}^{s} a_j R\Big) = \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}(R)} \Big(A_i/\Big(\sum_{j=1}^{s} a_j R\Big)\Big)_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}\in\mathcal{S}} \Big(A_i/\Big(\sum_{j=1}^{s} a_j R\Big)\Big)_{\mathfrak{m}}.$$

Hence, to prove the claim, it is enough to show that $(A_i)_{\mathfrak{m}_j}$ is finitely generated as a $\Lambda_{\mathfrak{m}_j}$-module for any $j = 0, \ldots, t$, cf. Lemma 3.2.1.

Note that $(A_i)_{\mathfrak{m}_0} = \bigoplus_{1\leq j\leq i} X_j$ is finitely generated. Assume now that $j \geq 1$. Since A_i is an RD-submodule of $\bigoplus_{k=1}^{\ell} N_k$, we deduce that $(A_i)_{\mathfrak{m}_j}$ is an RD-submodule of $\bigoplus_{k=1}^{\ell}(N_k)_{\mathfrak{m}_j}$. Since $R_{\mathfrak{m}_j}$ is a valuation domain, and the module $\bigoplus_{k=1}^{\ell}(N_k)_{\mathfrak{m}_j}/(A_i)_{\mathfrak{m}_j}$ is finitely generated and torsion-free, it is projective. Therefore, $(A_i)_{\mathfrak{m}_j}$ is a direct summand of $\bigoplus_{k=1}^{\ell}(N_k)_{\mathfrak{m}_j}$, so it is finitely generated, as claimed.

Now, A is a union of a chain of finitely generated modules $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. For any $i \in \mathbb{N}$, the exact sequence $0 \to A_i \to A_{i+1} \to A_{i+1}/A_i \to 0$ splits upon localization at $\mathfrak{m}_0$ by construction. It also splits when localized at other maximal ideals since it is RD-exact, $R_{\mathfrak{m}}$ is a valuation domain, and all the involved modules are finitely generated and torsion-free, and therefore projective.

By Lemma 2.1.13, the sequence $0 \to A_i \to A_{i+1} \to A_{i+1}/A_i \to 0$ splits in R for any $i \in \mathbb{N}$. Therefore, $A \cong A_1 \oplus (\bigoplus_{i\in\mathbb{N}} A_{i+1}/A_i)$ is a direct sum of finitely generated modules. □

Corollary 4.2.3. *Let R be an h-local domain of Krull dimension 1 that is integrally closed in its field of fractions Q. Then the following statements are equivalent:*

(i) *The class of R-modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands.*

(ii) *For any finitely generated torsion-free R-module X, every element in* $\mathrm{Add}(X)$ *is a direct sum of finitely generated modules.*

(iii) *The ring R satisfies one of the following conditions:*

- **(1)** *R is a Prüfer domain; or*
- **(2)** *there is a maximal ideal $\mathfrak{m}_0$ such that for every maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$, $R_{\mathfrak{m}}$ is a valuation domain, $R_{\mathfrak{m}_0}$ is an integrally closed domain that is not a valuation domain, and every indecomposable finitely generated torsion-free $R_{\mathfrak{m}_0}$-module has local endomorphism ring.*

Proof. It is clear that **(i)** implies **(ii)**. Theorem 4.1.6 shows that **(ii)** implies **(iii)**.

Assume **(iii)** holds. If R is a Prüfer domain, then it is semihereditary, so finitely generated torsion-free modules are projective, and all projective modules are the direct sum of finitely generated ideals. So **(i)** is trivially satisfied.

Now assume that R satisfies the condition (2). By Corollary 2.2.3, $R_{\mathfrak{m}_0}$ satisfies that the class of finitely generated torsion-free modules is closed under direct summands. Then **(i)** follows from Proposition 4.2.2. □

Corollary 4.2.4. *Let R be an h-local domain of Krull dimension* 1*. Assume that the class of R-modules that are direct sums of finitely generated torsion-free modules is closed under direct summands. Then its integral closure also satisfies this property.*

Proof. By Theorem 4.1.6, $\overline{R}$ satisfies conditions **(1)** and **(2)** of part **(iii)** of Corollary 4.2.3 so we can deduce that the class of finitely generated torsion-free $\overline{R}$-modules is closed by direct summands. □

4.3 Infinite direct sums are determined by the genus

Definition 4.3.1. Let R be a commutative ring, and let Λ be an R-algebra. Let M and N be right Λ-modules. We say that M and N are *in the same genus* if $M_{\mathfrak{m}}$ is isomorphic to $N_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.

Lemma 4.3.2. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let M, N, and X be right Λ-modules. If X is finitely generated over R and $M \oplus X$ and $N \oplus X$ are in the same genus, then M and N are in the same genus.*

Proof. Recall that if a module Y has semilocal endomorphism ring, then it has the cancellation property, that is, $Y \oplus A \cong Y \oplus B$ implies $A \cong B$ (see, for example, Theorem 4.5 in [14]).

By Proposition 2.1.3, $X_{\mathfrak{m}}$ has semilocal endomorphism ring. Therefore, if $M_{\mathfrak{m}} \oplus X_{\mathfrak{m}} \cong N_{\mathfrak{m}} \oplus X_{\mathfrak{m}}$ then $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. □

Lemma 4.3.3. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that Λ_Q is a simple artinian ring. Let $M = \bigoplus_{i\in I} M_i$ be an infinite direct sum of non-zero finitely generated right Λ-modules which are torsion-free as R-modules, and let N be a finitely generated right Λ-module which is torsion-free as an R-module. If $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, then N is a direct summand of M.*

Proof. Suppose that $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. Since M_Q is a free Λ_Q-module and N is finitely generated, there is some finite subset $I_1 \subseteq I$ such that N_Q is a direct summand of $\bigoplus_{i\in I_1}(M_i)_Q$. By Lemma 2.3.2, there is a Λ-module homomorphism $f\colon \bigoplus_{i\in I_1} M_i \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon \bigoplus_{i\in I_1}(M_i)_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting epimorphism for almost all maximal ideals $\mathfrak{m}$ of R.

Let $\mathcal{M}$ be the finite set of maximal ideals $\mathfrak{m}$ such that $f_{\mathfrak{m}}$ is not a splitting epimorphism. Since N is finitely generated, there is some finite subset $I_2 \subseteq I$ containing I_1 such that $N_{\mathfrak{m}}$ is a direct summand of $\bigoplus_{i\in I_2}(M_i)_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathcal{M}$. Then $N_{\mathfrak{m}}$ is a direct summand of $\bigoplus_{i\in I_2}(M_i)_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{mSpec} R$.

Since Λ_Q is simple artinian and N is finitely generated, there exists $I_3 \subseteq I \setminus I_2$ finite such that N_Q is a direct summand of $\bigoplus_{i\in I_3}(M_i)_Q$. Then, by Proposition 2.3.6, N is a direct summand of $\bigoplus_{i\in I_0} M_i$, where $I_0 = I_2 \cup I_3$. Hence, N is a direct summand of M. □

Theorem 4.3.4. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that Λ_Q is a simple artinian ring. Let $M = \bigoplus_{i\in\mathbb{N}_0} A_i$ and $N = \bigoplus_{i\in\mathbb{N}_0} B_i$ be direct sums of non-zero finitely generated right Λ-modules which are torsion-free as R-modules. If M and N are in the same genus, then there are decompositions*

$$M = \bigoplus_{i\in\mathbb{N}_0} M_i \qquad \text{and} \qquad N = \bigoplus_{i\in\mathbb{N}_0} N_i$$

such that both M_i and N_i are finitely generated, and $M_i \cong N_i$ for every $i \in \mathbb{N}_0$. In particular, M and N are isomorphic.

Proof. Let $\{\mathfrak{m}_\alpha\}_{\alpha\in\Omega}$ denote the set of maximal ideals of R. Let $\{a_i\}_{i\in\mathbb{N}_0}$ and $\{b_j\}_{j\in\mathbb{N}_0}$ be countable sets of generators for M and N, respectively.

By Lemma 2.1.3, $(A_i)_{\mathfrak{m}_\alpha}$ and $(B_i)_{\mathfrak{m}_\alpha}$ have semilocal endomorphism rings for every $\alpha \in \Omega$, and $i \in \mathbb{N}$. Hence, both A_i and B_i satisfy the cancellation property locally for every $i \in \mathbb{N}$.

We claim that there exist $m_0, n_0 \in \mathbb{N}$ and decompositions

$$M = M_0 \oplus Z_0 \oplus \textstyle\bigoplus_{i=m_0+1}^{\infty} A_i \quad \text{and} \quad N = N_0 \oplus \bigoplus_{j=n_0+1}^{\infty} B_j$$

for some finitely generated modules M_0, N_0 and Z_0 such that M_0 is isomorphic to N_0, $a_0 \in M_0$, $b_0 \in N_0$, and $Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i$ and $\bigoplus_{j=n_0+1}^{\infty} B_j$ are in the same genus.

Let $m_0' = \min\{m \in \mathbb{N}_0 \mid a_0 \in \bigoplus_{i=0}^{m} A_i\}$, and consider $X_0 = \bigoplus_{i=0}^{m_0'} A_i$. By Lemma 4.3.3 applied to X_0, there is $n_0' \in \mathbb{N}_0$ such that X_0 is isomorphic to a direct summand of $\bigoplus_{j=0}^{n_0'} B_j$. Therefore, we can write

$$M = X_0 \oplus \textstyle\bigoplus_{i=m_0'+1}^{\infty} A_i \qquad \text{and} \qquad N = Y_0 \oplus Z_0' \oplus \bigoplus_{j=n_0'+1}^{\infty} B_j$$

with $X_0 \cong Y_0$ and $Y_0 \oplus Z_0' = \bigoplus_{j=0}^{n_0'} B_j$. Thanks to Lemma 4.3.2, $\bigoplus_{i=m_0+1}^{\infty} A_i$ and $Z_0' \oplus \bigoplus_{j=n_0'+1}^{\infty} B_j$ are in the same genus.

Let $n_0 = \max\{n_0' + 1, \min\{n \in \mathbb{N}_0 \mid b_0 \in \bigoplus_{j=0}^{n} B_j\}\}$, and consider $Y_1 = Z_0' \oplus \bigoplus_{j=n_0'+1}^{n_0} B_j$. By Lemma 4.3.3 applied to Y_1, there is $m_0 > m_0'$ such that Y_1 is isomorphic to a direct summand of $\bigoplus_{i=m_0'+1}^{m_0} A_i$. Then

$$M = X_0 \oplus X_1 \oplus Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i \qquad \text{and} \qquad N = Y_0 \oplus Y_1 \oplus \bigoplus_{j=n_0+1}^{\infty} B_j$$

with $X_0 \cong Y_0$ and $X_1 \cong Y_1$. By Lemma 4.3.2, $Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i$ and $\bigoplus_{j=n_0+1}^{\infty} B_j$ are in the same genus. Take $M_0 = X_0 \oplus X_1$ and $N_0 = Y_0 \oplus Y_1$. We can repeat this argument ω times to obtain ascending chains of positive integers $m_0 < \cdots < m_\ell < \cdots$ and $n_0 < \cdots < n_\ell < \cdots$, and decompositions

$$M = (\bigoplus_{i=0}^{\ell} M_i) \oplus Z_\ell \oplus (\bigoplus_{i=m_\ell+1}^{\infty} A_i) \quad \text{and} \quad N = (\bigoplus_{i=0}^{\ell} N_i) \oplus (\bigoplus_{j=n_\ell+1}^{\infty} B_j)$$

such that M_i is finitely generated and isomorphic to N_i for every $i = 0, \ldots, \ell$, $a_0, \ldots, a_\ell \in \bigoplus_{i=0}^{\ell} M_i$, $b_0, \ldots, b_\ell \in \bigoplus_{i=0}^{\ell} N_i$, and $Z_\ell \oplus (\bigoplus_{i=m_\ell+1}^{\infty} A_i)$ and $\bigoplus_{j=n_\ell+1}^{\infty} B_j$ are in the same genus. Therefore, we obtain two families of finitely generated Λ-modules which are torsion-free as R-modules $\{M_i\}_{i\in\mathbb{N}_0} \subseteq M$ and $\{N_i\}_{i\in\mathbb{N}_0} \subseteq N$ such that M_i is finitely generated and isomorphic to N_i, $\{a_i\}_{i\in\mathbb{N}_0} \subseteq \bigoplus_{i\in\mathbb{N}_0} M_i$, and $\{b_i\}_{i\in\mathbb{N}_0} \subseteq \bigoplus_{i\in\mathbb{N}_0} N_i$. We deduce that $M = \bigoplus_{i\in\mathbb{N}_0} M_i$ and $N = \bigoplus_{i\in\mathbb{N}_0} N_i$. Therefore, M is isomorphic to N. This finishes the proof of the proposition. □

The following proposition generalizes Lemma 4.3.3.

Proposition 4.3.5. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that Λ_Q is a simple artinian ring. Let $M = \bigoplus_{i\in I} M_i$ be an infinite direct sum of non-zero finitely generated right Λ-modules which are torsion-free as R-modules, and let A be a direct summand of M of infinite rank. Let N be a finitely generated right Λ-module which is torsion-free as an R-module. If $N_{\mathfrak{m}}$ is isomorphic to a direct summand of $A_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R, then N is isomorphic to a direct summand of A.*

Proof. Since A is a direct summand of M, there are Λ-homomorphisms $\iota\colon A \hookrightarrow M$ and $\pi\colon M \to A$ such that $\pi\iota = 1_A$. For every subset $J \subseteq I$, let $\pi_J\colon M \to \bigoplus_{j\in J} M_j$ denote the canonical projection, and let $\iota_J\colon \bigoplus_{j\in J} M_j \hookrightarrow M$ denote the canonical embedding.

Since N_Q is a direct summand of A_Q, there are Λ_Q-homomorphisms $f\colon N_Q \to A_Q$ and $g\colon A_Q \to N_Q$ such that $gf = 1_{N_Q}$. By Lemma 2.1.11, there is a Λ-module homomorphism $f_0\colon N \to A$ such that $f = f_0/s$ for some non-zero $s \in R$. Let J_0 be a finite subset of I such that $\operatorname{im} f_0 \subseteq \bigoplus_{j\in J_0} M_j$. Hence $g(\iota_{J_0}\pi_{J_0}\iota)_Q f = 1_{N_Q}$.

Again, by Lemma 2.1.11, there is a Λ-homomorphism $g'\colon \bigoplus_{j\in J_0} M_j \to N$ such that $g(\iota_{J_0})_Q = g'/t$ for some non-zero $t \in R$. Let $g_0 = g'\pi_{J_0}\iota$, and note that $g_0/t\colon A_Q \to N_Q$ is a Λ_Q-homomorphism satisfying that $(g_0/t)f = 1_{N_Q}$. Hence, we may assume that there exists a Λ-homomorphism $g_0\colon A \to N$ that factors through $\pi_{J_0}\iota$, and such that g is of the form g_0/t for some non-zero $t \in R$. In particular, $g_0 f_0 = st1_N$. Also note that $g_0(A \cap \bigoplus_{i\in I\setminus J_0} M_i) = 0$ and $\operatorname{im} f_0 \subseteq \bigoplus_{j\in J_0} M_j$.

Let $r_0 = st \in R$. Since R is of finite character, r_0 is contained only in finitely many maximal ideals of R, say $\mathcal{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k\}$. A similar argument as

above shows that there are Λ-homomorphisms $f_1, \dots, f_k \colon N \to A$, $g_1, \dots, g_k \colon A \to N$, and non-zero elements $r_1, \dots, r_k \in R$ such that $g_i f_i = r_i 1_N$ and $r_i \notin \mathfrak{m}_i$, for each $i = 1, \dots, k$. These morphisms can be chosen such that there are finite subsets $J_1, J_2, \dots, J_k \subseteq I$ such that $g_j(A \cap \bigoplus_{i \in I \setminus J_j} M_i) = 0$ and $\operatorname{im} f_j \subseteq \bigoplus_{i \in J_j} M_i$.

Let $J = \bigcup_{i=1}^{k} J_i$, and let $J' := I \setminus J$. Consider an exact sequence

$$0 \longrightarrow X \xrightarrow{\nu} A \xrightarrow{\pi_J|_A} \bigoplus_{j \in J} M_j,$$

where $X = A \cap \bigoplus_{j \in J'} M_j$. Since A is of infinite rank, X has infinite rank as well, and hence there is a Λ_Q-monomorphism $f' \colon N_Q \to X_Q$. Then $\nu_Q f'$ splits because Λ_Q is simple artinian. Let $g' \colon A_Q \to N_Q$ be a Λ_Q-homomorphism such that $g' \nu_Q f' = 1_{N_Q}$. As in the first part of the proof, we may assume that g' factors through $(\pi_K \iota)_Q$ for some finite subset $K \subseteq J'$. Therefore, there are Λ-homomorphisms $f_\infty \colon N \to A$ and $g_\infty \colon A \to N$ such that $g_\infty f_\infty = r_\infty 1_N$ for some non-zero $r_\infty \in R$. Moreover, $\operatorname{im} f_\infty \subseteq X$ and $g_\infty(A \cap \bigoplus_{j \in J} M_j) = 0$.

Since R is of finite character, r_∞ is contained only in finitely many maximal ideals of R, say $\mathcal{N} = \{\mathfrak{n}_1, \mathfrak{n}_2, \dots, \mathfrak{n}_\ell\}$. By the Chinese Remainder Theorem, we can find $e_1, \dots, e_\ell \in R$ such that $e_j \equiv 0 \mod \mathfrak{n}_i$ whenever $j \neq i$ and $e_j \equiv 1 \mod \mathfrak{n}_j$, for every $j = 1, \dots, \ell$. Note that any maximal ideal of R either does not contain r_0 or is in $\mathcal{M}$. Hence, by the definition of the $r_1, \dots, r_k$, no maximal ideal of R contains the whole set $\{r_0, \dots, r_k\}$.

For every $j = 1, \dots, \ell$, let $i(j) \in \{0, \dots, k\}$ be such that $r_{i(j)} \notin \mathfrak{n}_j$. Let $g \colon A \to N$ be the Λ-homomorphism given by $g = g_\infty + \sum_{j=1}^{\ell} e_j g_{i(j)}$.

We claim that g is a locally split epimorphism, so by Lemma 2.1.13 it is a split epimorphism. Note that $g f_\infty = g_\infty f_\infty = r_\infty 1_N$, so $g_{\mathfrak{n}}$ splits if $\mathfrak{n}$ does not contain r_∞. On the other hand,

$$g f_{i(t)} = \sum_{j=1}^{\ell} e_j g_{i(j)} f_{i(t)} = e_t r_{i(t)} 1_N + \sum_{\substack{1 \leq j \leq \ell \\ j \neq t}} e_j g_{i(j)} f_{i(t)}.$$

When we localize at $\mathfrak{n}_t$, the first summand is an invertible element, while the second summand is in $\mathrm{J}(\operatorname{End}_{\Lambda_{\mathfrak{n}_t}}(N_{\mathfrak{n}_t}))$ by Lemma 2.1.6. Hence, $g_{\mathfrak{n}}$ splits for any maximal ideal of R, and we conclude with the proof of the statement. □

Now we are going to give an example showing that Lemma 4.3.3 is not true for direct summands that are not finitely generated. We start with the following well-known lemma that will provide us a source of noetherian local domains of Krull dimension 1 with local integral closure. So that indecomposable finitely generated torsion-free modules have local endomorphism ring (see, for example, Corollary 2.2.9).

Lemma 4.3.6. *Let $\alpha_1, \dots, \alpha_n$ be (non-zero) coprime elements of $\mathbb{N}$. Let K be a field, $R = K[t^{\alpha_1}, \dots, t^{\alpha_n}]$ and let $\mathfrak{m} = t^{\alpha_1} R + \cdots + t^{\alpha_n} R$. Set $\Sigma = R \setminus \mathfrak{m}$. Then:*

- **(i)** *R is a noetherian domain of Krull dimension 1 and field of quotients $K(t)$. The integral closure of R into its field of fractions is $K[t]$.*
- **(ii)** *The integral closure of $R_{\mathfrak{m}}$ into its field of fractions is $K[t]_\Sigma = K[t]_{(t)}$. In particular, it is a local ring.*

(iii) *Every finitely generated, indecomposable, torsion-free $R_{\mathfrak{m}}$-module has local endomorphism ring.*

Proof. **(i)** The field of fractions of R is a subfield of $K(t)$ that contains R and also t (because $\alpha_1, \ldots, \alpha_n$ are coprime) so it coincides with $K(t)$. Since $K[t]$ is a PID, it is integrally closed, and being an integral extension of R, it is the integral closure.

(ii) By **(i)** the extension $R_{\mathfrak{m}} \subseteq K[t]_\Sigma$ is integral. If $\mathfrak{n}$ is a maximal ideal of $K[t]_\Sigma$ then $\mathfrak{n} \cap R = \mathfrak{m}$. Therefore $t \in \mathfrak{n}$, so that $\mathfrak{n} = tK[t]_\Sigma$, and $K[t]_\Sigma$ is a local ring.

The statement **(iii)** follows from **(ii)** and Corollary 2.2.9. □

Example 4.3.7. *There are a semilocal noetherian domain R of Krull dimension 1, and R-modules M and N that can be written as infinite direct sums of non-zero finitely generated torsion-free R-modules such that $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R but N is not a direct summand of M.*

Proof. Let K be an infinite field. Let $R_1 = K[t^2, t^3]_{(t^2,t^3)}$ and $R_2 = K[t^3, t^7]_{(t^3,t^7)}$. So Lemma 4.3.6 applies to R_1 and R_2.

The local domain R_1 is a Bass domain with just two indecomposable finitely generated torsion-free modules (or rank one) up to isomorphism. Namely, $X = R_1$ and $Y = \mathfrak{m}_1$ where $\mathfrak{m}_1$ denotes the maximal ideal of R_1. Note that, by Lemma 4.3.6, X and Y have local endomorphism ring.

The domain R_2 has infinitely many indecomposable finitely generated torsion-free modules of all ranks ≥ 2, because it fails to satisfy the Drozd-Roiter conditions (cf. Theorem 4.2 in [33]). More precisely, if we denote by $\mathfrak{m}_2$ the maximal ideal of R_2, the R_2-module $(\mathfrak{m}_2 K[t]_{(t)} + R_2)/R_2$ is not cyclic because a minimal set of generators is t^4 and t^5.

We single out an infinite family $Z_1, Z_2, \ldots$ of indecomposable finitely generated torsion-free R_2-modules of rank 2. Note that, by Lemma 4.3.6, such modules have local endomorphism ring.

Let $\varphi\colon K(t) \to K(t)$ be the automorphism that fixes K and such that $\varphi(t) = t + 1$. Let R be the ring that fits in the pull-back diagram

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{\varphi'} & K(t) \end{array}$$

where $\varphi'\colon R_2 \hookrightarrow K(t) \xrightarrow{\varphi} K(t)$.

By Theorem 4.4 in [73], R is a noetherian domain of Krull dimension 1 with exactly two maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ and satisfying that $R_{\mathfrak{m}} \cong R_1$ and $R_{\mathfrak{n}} \cong R_2$.

Apply the results on [73] (or just Corollary 3.2.4) to construct two sequences of finitely generated torsion-free R-modules $M_1, M_2, \ldots, N_1, N_2, \ldots$ such that

$$(M_i)_{\mathfrak{n}} = (N_i)_{\mathfrak{n}} = Z_i, \qquad (M_i)_{\mathfrak{m}} = X \oplus Y \qquad \text{and} \qquad (N_i)_{\mathfrak{m}} = X \oplus X.$$

Call $M = \bigoplus_{i\in\mathbb{N}} M_i$ and $N = \bigoplus_{i\in\mathbb{N}} N_i$. Then N is locally a direct summand of M, but N is not isomorphic to a direct summand of M. Indeed, if $N \oplus N' \cong M$ then

$$(\bigoplus_{i\in\mathbb{N}} Z_i) \oplus N'_{\mathfrak{n}} \cong \bigoplus_{i\in\mathbb{N}} Z_i.$$

Since $\{Z_i\}_{i\in\mathbb{N}}$ are non-isomorphic and have local endomorphism rings, $N'_{\mathfrak{n}} = \{0\}$. Hence, as N' is torsion-free, $N' = \{0\}$. But $N_{\mathfrak{m}} \ncong M_{\mathfrak{m}}$, a contradiction. □

Remark 4.3.8. By Proposition 4.4.6, the ring R in Example 4.3.7 satisfies that the class of direct sums of finitely generated torsion-free modules is closed under direct summands.

Proposition 4.3.9. *Let R be a commutative domain of finite character. Let M be a torsion-free R-module of countable rank. Let $\mathfrak{m}_0$ be a maximal ideal of R. Assume that*

(a) $M_{\mathfrak{m}}$ is a direct sum of finitely generated torsion-free modules of rank one with local endomorphism ring for every maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$, and

(b) $M_{\mathfrak{m}_0}$ is a direct sum of finitely generated torsion-free modules.

Then M is in the same genus as a direct sum of finitely generated torsion-free modules if and only if

(i) *$M_{\mathfrak{m}}$ is a free module for all but countably many maximal ideals $\mathfrak{m}$ of R.*

(ii) *If $\mathcal{M} = \{\mathfrak{m} \in \operatorname{mSpec} R \mid M_{\mathfrak{m}}$ is not free$\}$, and if $r_{\mathfrak{m}}$ denotes the number of free direct summands in the decomposition of $M_{\mathfrak{m}}$, then for every $b \in \mathbb{N}$ the set $\{\mathfrak{m} \in \mathcal{M} \mid r_{\mathfrak{m}} \leq b\}$ is finite.*

Proof. Let $N = \bigoplus_{i\in\mathbb{N}} N_i$, where each N_i is a finitely generated torsion-free module, and assume that M and N are in the same genus. Statement **(i)** follows from Corollary 3.1.4. If $\mathcal{M}$ is finite, **(ii)** is clear. Assume $\mathcal{M}$ is infinite and let $b \in \mathbb{N}$. Then there are only finitely many maximal ideals $\mathfrak{m} \in \mathcal{M}$ such that at least one of $(N_1)_{\mathfrak{m}}, \ldots, (N_b)_{\mathfrak{m}}$ is not free, so $r_{\mathfrak{m}} \geq b$ for almost all $\mathfrak{m} \in \mathcal{M}$. This proves **(ii)**.

Conversely, assume that **(i)** and **(ii)** are satisfied and let $M_{\mathfrak{m}} = \bigoplus_{i\in\mathbb{N}} M_{\mathfrak{m},i}$, where each $M_{\mathfrak{m},i}$ is a finitely generated torsion-free module and such that $M_{\mathfrak{m},i}$ has rank one for every maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$. For each $i \in \mathbb{N}$, let d_i denote the rank of the module $M_{\mathfrak{m}_0,i}$.

First, assume that $\mathcal{M}$ is finite. For each $i \in \mathbb{N}$, apply the Package Deal Theorem 3.2.3 to

$$X_i(\mathfrak{m}_0) = M_{\mathfrak{m}_0,i} \quad \text{and} \quad X_i(\mathfrak{m}) = \bigoplus_{i=d_0+\cdots+d_{i-1}+1}^{d_0+\cdots+d_i} M_{\mathfrak{m},i} \quad \text{for each } \mathfrak{m} \neq \mathfrak{m}_0$$

(note that, since $\mathcal{M}$ is finite, $X_i(\mathfrak{m})$ is free for almost all maximal ideals $\mathfrak{m}$ of R). Then, for each $j \in \mathbb{N}$, there is an R-module N_j such that $(N_j)_{\mathfrak{m}} \cong X_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Therefore, $\bigoplus_{j\in\mathbb{N}} N_j$ is in the same genus as M.

Now, assume that $\mathcal{M}$ is infinite and consider the case where $r_{\mathfrak{m}}$ is finite for every maximal ideal $\mathfrak{m}$ of R. First, write the elements of $\mathcal{M}$ in a sequence $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ in such a way that $r_{\mathfrak{m}_1} \leq r_{\mathfrak{m}_2} \leq \cdots$ is non-decreasing, and assume that the direct summands in the decompositions of $M_{\mathfrak{m}}$ are indexed such that $M_{\mathfrak{m},1}, \ldots, M_{\mathfrak{m},r_{\mathfrak{m}}}$ are

the free direct summands. For each $i \in \mathbb{N}$, apply the Package Deal Theorem 3.2.3 to

$$X_i(\mathfrak{m}_0) = M_{\mathfrak{m}_0,i} \qquad \text{and} \quad X_i(\mathfrak{m}) = \bigoplus_{j=d_0+\cdots+d_{i-1}+1}^{d_0+\cdots+d_i} M_{\mathfrak{m},j} \quad \text{for each } \mathfrak{m} \neq \mathfrak{m}_0$$

(note that, with this reordering and considering **(i)** and **(ii)** with $b = d_0 + \cdots + d_i$, $M_{\mathfrak{m},i}$ is free for almost all maximal ideals). Then, for each $j \in \mathbb{N}$, there is an R-module N_j such that $(N_j)_{\mathfrak{m}} \cong X_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Therefore, $\bigoplus_{j\in\mathbb{N}} N_j$ is in the same genus as M.

Finally, assume that $\mathcal{M}$ is infinite and there is at least one maximal ideal $\mathfrak{m}$ with $r_{\mathfrak{m}}$ infinite. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ be a (finite or infinite) list of elements in $\{\mathfrak{m} \in \mathcal{M} \mid r_{\mathfrak{m}} \text{ is finite}\}$ and let $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ be a (finite or infinite) list of elements in $\{\mathfrak{n} \in \mathcal{M} \mid r_{\mathfrak{n}} \text{ is infinite}\}$. The first sequence is chosen such that $r_{\mathfrak{m}_1} \leq r_{\mathfrak{m}_2} \leq \cdots$ is non-decreasing, and again we assume the direct summands in the decompositions of $M_{\mathfrak{m}_i}$ are indexed such that $M_{\mathfrak{m},1}, \ldots, M_{\mathfrak{m},r_{\mathfrak{m}_i}}$ are the free direct summands. Moreover, we assume that the direct summands in the decompositions of $M_{\mathfrak{n}_i}$ are indexed such that $M_{\mathfrak{n}_i,1}, \ldots, M_{\mathfrak{n}_i,i}$ are free.

Let us check that each $M_{\mathfrak{m},j}$ is free for almost all $\mathfrak{m} \in \mathcal{M}$. If $M_{\mathfrak{n}_i,j}$ is not free, then $i < j$. If $M_{\mathfrak{m},j}$ is not free for infinitely many $\mathfrak{m}$'s with finite $r_{\mathfrak{m}}$, then $b = j - 1$ would contradict **(ii)**. Hence, as before, we find finitely generated torsion-free R-modules N_j such that $(N_j)_{\mathfrak{m}} \cong X_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, $\bigoplus_{j\in\mathbb{N}} N_j$ is in the same genus as M. □

4.4 The noetherian case

The results of Section 4.3 allow us to show in this section that the converse of Theorem 4.1.6 is true for semilocal noetherian domains of Krull dimension 1 (cf. Proposition 4.4.6), and this result will allow us also to prove the converse of Theorem 4.1.6 for noetherian domains of Krull dimension 1 and with finitely generated integral closure in Theorem 4.4.8.

Let $\mathcal{C}$ be a pre-additive category, let M be an object of $\mathcal{C}$, and let I be a two-sided ideal of the ring $\mathrm{End}_{\mathcal{C}}(M)$. Recall that the ideal of $\mathcal{C}$ associated to I is the ideal $\mathcal{A}_I$ of the category $\mathcal{C}$ defined as follows. A morphism $f\colon X \to Y$ belongs to $\mathcal{A}_I(X,Y)$ if and only if $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon M \to X$ and $\beta\colon Y \to M$ in the category $\mathcal{C}$.

Notice that if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. Then the restriction of $\mathcal{A}_I$ to $\mathcal{C}'$ gives an ideal of the category $\mathcal{C}'$.

Remark 4.4.1. Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let M be a non-zero finitely generated right Λ-module with endomorphism ring $S = \mathrm{End}_{\Lambda}(M)$. Let $\varphi\colon R \to S$ denote the canonical homomorphism. Let $\mathfrak{n}$ be a two-sided maximal ideal of S, and let $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$. By Lemma 2.1.7**(i)**, $\mathfrak{m}$ is a maximal ideal of R.

Let X and Y be two objects in $\mathbf{Mod}\Lambda$. If $f \in \mathrm{Hom}_{\Lambda}(X,Y)$ is such that $f(X) \subseteq Y\mathfrak{m}$, then $f \in \mathcal{A}_{\mathfrak{n}}$. Indeed, $\beta f \alpha(M) \subseteq M\mathfrak{m}$ for every pair of morphisms $\alpha\colon M \to X$ and $\beta\colon Y \to M$. Hence, by Lemma 2.1.7**(ii)**, $\beta f \alpha \in \mathfrak{n}$.

If $\mathcal{C}$ is a full subcategory of $\mathbf{Mod}\Lambda$, we will still denote by $\mathcal{A}_{\mathfrak{n}}$ the restriction to $\mathcal{C}$ of the ideal defined in the whole module category.

Lemma 4.4.2. *Let R be a commutative ring, and let $\mathfrak{m}$ be a maximal ideal of the ring R. Let $f\colon X \to Y$ be a homomorphism of R-modules such that $f_{\mathfrak{m}}(X_{\mathfrak{m}}) \subseteq Y_{\mathfrak{m}}\mathfrak{m}R_{\mathfrak{m}}$. Then $f(X) \subseteq Y\mathfrak{m}$.*

Proof. Let $x \in X$. Then there exist $r_i \in \mathfrak{m}, y_i \in Y$ and $s \in R \setminus \mathfrak{m}$ such that

$$\frac{f(x)}{1} = \sum_{i=1}^{n} \frac{y_i r_i}{1}\frac{1}{s} = \frac{y}{s}, \qquad y \in Y\mathfrak{m}.$$

Therefore, there exists $t \notin \mathfrak{m}$ such that $f(x)st = yt$. Then there exists $u \notin \mathfrak{m}$ such that $1 - stu \in \mathfrak{m}$ and $f(x) = f(x)(1 - stu) + f(x)stu = f(x)(1 - stu) + ytu \in Y\mathfrak{m}$. □

Proposition 4.4.3. *Under the assumptions of Remark 4.4.1, let $N = \bigoplus_{i\in\mathbb{N}} M_i$ be a direct sum of finitely generated right Λ-modules which are torsion-free as R-modules, and consider two direct summands A and A' of N. Assume that $\mathcal{C}$ contains M, A, A' and every M_i, for each $i \in \mathbb{N}$. If $A_{\mathfrak{m}} \cong A'_{\mathfrak{m}}$, then A is isomorphic to A' in the factor category $\mathcal{C}/\mathcal{A}_{\mathfrak{n}}$.*

Proof. We have to check that there are homomorphisms $f\colon A \to A'$ and $g\colon A' \to A$ such that $\mathrm{Id}_A - gf, \mathrm{Id}_{A'} - fg \in \mathcal{A}_{\mathfrak{n}}$. By Remark 4.4.1 and Lemma 4.4.2, it is sufficient to find $f\colon A \to A'$ and $g\colon A' \to A$ such that $(\mathrm{Id}_A - gf)_{\mathfrak{m}}(A_{\mathfrak{m}}) \subseteq A_{\mathfrak{m}}\mathfrak{m}R_{\mathfrak{m}}$ and $(\mathrm{Id}_{A'} - fg)_{\mathfrak{m}}(A'_{\mathfrak{m}}) \subseteq A'_{\mathfrak{m}}\mathfrak{m}R_{\mathfrak{m}}$.

Assume now that $A_{\mathfrak{m}} \cong A'_{\mathfrak{m}}$. Let $\alpha\colon A_{\mathfrak{m}} \to A'_{\mathfrak{m}}$ and $\beta\colon A'_{\mathfrak{m}} \to A_{\mathfrak{m}}$ denote mutually inverse isomorphisms. Then $\gamma = \iota_{A'_{\mathfrak{m}}}\alpha\pi_{A_{\mathfrak{m}}}$ and $\delta = \iota_{A_{\mathfrak{m}}}\beta\pi_{A'_{\mathfrak{m}}}$ are elements that belong to $\mathrm{End}_{R_{\mathfrak{m}}}(\bigoplus_{i\in\mathbb{N}}(M_i)_{\mathfrak{m}})$ such that the following diagram commutes

$$\begin{array}{ccc}
\bigoplus_{i\in\mathbb{N}}(M_i)_{\mathfrak{m}} & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & \bigoplus_{i\in\mathbb{N}}(M_i)_{\mathfrak{m}} \\
\iota_{A_{\mathfrak{m}}} \uparrow \quad \downarrow \pi_{A_{\mathfrak{m}}} & & \iota_{A'_{\mathfrak{m}}} \uparrow \quad \downarrow \pi_{A'_{\mathfrak{m}}} \\
A_{\mathfrak{m}} & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & A'_{\mathfrak{m}}
\end{array}$$

By Lemma 2.1.11, we can consider γ as a column-finite matrix such that the i-th column represents

$$\mathrm{Hom}_{\Lambda_{\mathfrak{m}}}((M_i)_{\mathfrak{m}}, \bigoplus_{j\in I}(M_j)_{\mathfrak{m}}) \cong \mathrm{Hom}_{\Lambda}(M_i, \bigoplus_{j\in I} M_j) \otimes_R R_{\mathfrak{m}}.$$

For each $i \in \mathbb{N}$, let $s_i \notin \mathfrak{m}$ be the product of the denominators in the i-th column, and define $\tau = \bigoplus_{i\in\mathbb{N}} s_i \mathrm{Id}_{(M_i)_{\mathfrak{m}}} \in \mathrm{Aut}_{\Lambda_{\mathfrak{m}}}(N_{\mathfrak{m}})$. By Lemma 2.1.11, there exists $h \in \mathrm{End}_{\Lambda}(N)$ such that $\gamma \circ \tau = h_{\mathfrak{m}}$. For each $i \in \mathbb{N}$, take $t_i \notin \mathfrak{m}$ such that $1 - t_i s_i \in \mathfrak{m}$ and define $\theta = \bigoplus_{i\in\mathbb{N}} t_i \mathrm{Id}_{M_i} \in \mathrm{End}_{\Lambda}(N)$. Similarly, define τ' and θ' starting with the endomorphism δ, i.e., $\delta \circ \tau' = h'_{\mathfrak{m}}$ for some $h' \in \mathrm{End}_{\Lambda}(N)$. Note that $\tau\theta_{\mathfrak{m}} = \mathrm{Id}_{N_{\mathfrak{m}}} - r$ for some $r \in \mathrm{End}_{\Lambda_{\mathfrak{m}}}(N_{\mathfrak{m}})$ with $\operatorname{im} r \subseteq (N_{\mathfrak{m}})\mathfrak{m}R_{\mathfrak{m}}$. Similarly, there exists $r' \in \mathrm{End}_{\Lambda_{\mathfrak{m}}}(N_{\mathfrak{m}})$ with $\operatorname{im} r' \subseteq (N_{\mathfrak{m}})\mathfrak{m}R_{\mathfrak{m}}$ such that $\tau'\theta'_{\mathfrak{m}} = \mathrm{Id}_{N_{\mathfrak{m}}} - r'$.

Set

$$f = \pi_{A'} h \theta \iota_A \quad \text{and} \quad g = \pi_A h' \theta' \iota_{A'}.$$

Then

$$\begin{aligned}(\mathrm{Id}_A - gf)_{\mathfrak{m}} &= (\mathrm{Id}_A - \pi_A h'\theta'\iota_{A'}\pi_{A'}h\theta\iota_A)_{\mathfrak{m}} \\ &= \mathrm{Id}_{A_{\mathfrak{m}}} - \pi_{A_{\mathfrak{m}}}\delta\tau'\theta'_{\mathfrak{m}}\iota_{A'_{\mathfrak{m}}}\pi_{A'_{\mathfrak{m}}}\gamma\tau\theta_{\mathfrak{m}}\iota_{A_{\mathfrak{m}}} \\ &= \mathrm{Id}_{A_{\mathfrak{m}}} - \pi_{A_{\mathfrak{m}}}\iota_{A_{\mathfrak{m}}}\beta\pi_{A'_{\mathfrak{m}}}\tau'\theta'_{\mathfrak{m}}\iota_{A'_{\mathfrak{m}}}\pi_{A'_{\mathfrak{m}}}\iota_{A'_{\mathfrak{m}}}\alpha\pi_{A_{\mathfrak{m}}}\tau\theta_{\mathfrak{m}}\iota_{A_{\mathfrak{m}}} \\ &= \mathrm{Id}_{A_{\mathfrak{m}}} - \pi_{A_{\mathfrak{m}}}\iota_{A_{\mathfrak{m}}}\beta\pi_{A'_{\mathfrak{m}}}(\mathrm{Id}_{N_{\mathfrak{m}}} - r')\iota_{A'_{\mathfrak{m}}}\pi_{A'_{\mathfrak{m}}}\iota_{A'_{\mathfrak{m}}}\alpha\pi_{A_{\mathfrak{m}}}(\mathrm{Id}_{N_{\mathfrak{m}}} - r)\iota_{A_{\mathfrak{m}}} \\ &= \mathrm{Id}_{A_{\mathfrak{m}}} - \pi_{A_{\mathfrak{m}}}\iota_{A_{\mathfrak{m}}}\beta\pi_{A'_{\mathfrak{m}}}1\iota_{A'_{\mathfrak{m}}}\pi_{A'_{\mathfrak{m}}}\iota_{A'_{\mathfrak{m}}}\alpha\pi_{A_{\mathfrak{m}}}1\iota_{A_{\mathfrak{m}}} + s,\end{aligned}$$

where $s \in \mathrm{End}_{\Lambda_{\mathfrak{m}}}(A_{\mathfrak{m}})$ satisfies that $\operatorname{im} s \subseteq A_{\mathfrak{m}}\mathfrak{m}R_{\mathfrak{m}}$. A symmetric computation shows that $\operatorname{im}(\mathrm{Id}_{A'} - fg)_{\mathfrak{m}} \subseteq A'_{\mathfrak{m}}\mathfrak{m}R_{\mathfrak{m}}$. □

Remark 4.4.4. The statement in Proposition 4.4.3 is also true if we assume that N is a direct sum of finitely presented right Λ-modules instead of a direct sum of finitely generated right Λ-modules which are torsion-free as R-modules using Theorem 3.18 in [55] instead of Lemma 2.1.11.

Corollary 4.4.5. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let $M = \bigoplus_{i\in\mathbb{N}} M_i$ be a direct sum of non-zero finitely presented right Λ-modules with semilocal endomorphism rings. If A and A' are direct summands of M, then $A \cong A'$ if and only if $A_{\mathfrak{m}} \cong A'_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Consider $\mathcal{C} = \mathrm{Add}(\bigoplus_{i\in\mathbb{N}} M_i)$. We aim to apply Proposition 3.6 in [47]. We have to check if $\mathfrak{n}$ is a maximal ideal of $\mathrm{End}_\Lambda(M_i)$ for some $i \in \mathbb{N}$, then A and A' are isomorphic objects of the factor category $\mathcal{C}/\mathcal{A}_{\mathfrak{n}}$. Note that the preimage of $\mathfrak{n}$ in the canonical homomorphism $R \to \mathrm{End}_\Lambda(N_i)$ is a maximal ideal of R by Lemma 2.1.7**(i)**, so we can apply Proposition 4.4.3 and Remark 4.4.4. □

From now on, we only consider the case when R is a noetherian domain of Krull dimension 1. We can try to find appropriate generalizations reflecting the results in the previous sections of this chapter. For the semilocal case below, we need Krull dimension 1 because of Corollary 2.2.9.

Proposition 4.4.6. *Let R be a semilocal noetherian domain of Krull dimension 1. Let $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be the list of maximal ideals of R. Assume that*

- **(i)** *$R_{\mathfrak{m}_i}$ has the two-generator property for $i = 1, \ldots, k$.*
- **(ii)** *The normalization of $R_{\mathfrak{m}_i}$ is a discrete valuation domain for every $i = 0, 1, \ldots, k$.*

Then every pure projective torsion-free R-module is a direct sum of finitely presented modules.

Proof. By Kaplansky's Theorem, see Theorem 1 in [29], every pure projective module is a direct sum of countably generated modules. Therefore, we may consider A to be a direct summand of $\bigoplus_{i\in\mathbb{N}} M_i$, where each M_i is a non-zero finitely presented torsion-free R-module, say $A \oplus A' = \bigoplus_{i\in\mathbb{N}} M_i$. Also, we may assume that A is not finitely generated.

If $\mathfrak{m}$ is a maximal ideal of R, then $A_{\mathfrak{m}} \oplus A'_{\mathfrak{m}} = \bigoplus_{i\in\mathbb{N}}(M_i)_{\mathfrak{m}}$, and by **(ii)** and Corollary 2.2.9, $A_{\mathfrak{m}}$ is a direct sum of finitely generated $R_{\mathfrak{m}}$-modules with local endomorphism ring.

For each $i = 0, \dots, k$, let $A_{\mathfrak{m}_i} = \bigoplus_{j\in\mathbb{N}} N_{i,j}$, where $N_{i,j}$ is a finitely generated indecomposable $R_{\mathfrak{m}_i}$-module. In Theorem 4.3 of [61], Rush proved that over a noetherian ring with the two-generator property any finitely generated indecomposable torsion-free module has rank 1, thus **(i)** implies that $N_{i,j}$ is a module of rank 1 if $i \geq 1$.

Therefore, the module A fulfills the hypothesis of Proposition 4.3.9, and since the number of maximal ideals of R is finite, we can deduce that A is in the same genus as a module B that is a direct sum of finitely generated torsion-free modules. Since A and B are direct summands of $B \oplus (\bigoplus_{i\in\mathbb{N}} M_i)$, we deduce from Corollary 4.4.5 that $A \cong B$. □

The following lemma is a variation of Proposition 4.2.2 adapted to the noetherian setting.

Lemma 4.4.7. *Let R be a noetherian domain of Krull dimension 1, and with module-finite normalization $\overline{R}$. Let $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ be the list of maximal ideals of R such that $R_{\mathfrak{m}}$ is a principal ideal domain for any maximal ideal $\mathfrak{m} \notin \mathcal{M}$, and let $\Sigma = R \setminus \bigcup_{i=1}^{k} \mathfrak{m}_i$. Further, let $M \subseteq \overline{R}^{(\omega)}$ be such that M_Σ is a direct sum of finitely generated R_Σ-modules. Then M is a direct sum of finitely generated modules.*

Proof. We may assume that the rank of M is infinite. Let $M_\Sigma = \bigoplus_{i=1}^{\infty} M_i$, where each M_i is a non-zero finitely generated R_Σ-module. For each $i = 1, 2, \dots$ let

$$N_i := \{m \in M \mid m/1 \in \textstyle\bigoplus_{j=1}^{i} M_j\}.$$

Note that $N_i \subseteq \overline{R}^{\ell}$ for some ℓ, hence N_i is a finitely generated submodule of M. Further, we claim that N_i is an RD-submodule of M. Consider $m \in M, n \in N_i$ and $0 \neq r \in R$ such that $mr = n$. Then $(m/1)(r/1) = n/1$ in M_Σ. Since $r/1$ is not a zero-divisor on each M_t, $m1/1$ is an element of $\bigoplus_{j=1}^{i} M_j$.

We claim that for every $i \in \mathbb{N}$ the inclusion $\iota_i \colon N_i \to N_{i+1}$ splits. By Lemma 2.1.13, it is enough to check that $\iota_i \otimes_R R_{\mathfrak{m}}$ splits for every maximal ideal $\mathfrak{m}$ of R. Since $\iota_i \otimes_R R_{\mathfrak{m}} \cong \iota_i \otimes_R R_\Sigma \otimes_{R_\Sigma} R_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathcal{M}$ and $\iota_i \otimes_R R_\Sigma$ splits, $\iota_i \otimes_R R_{\mathfrak{m}}$ splits for any $\mathfrak{m} \in \mathcal{M}$.

Now suppose that $\mathfrak{m} \notin \mathcal{M}$. Then it is easy to check that $(N_i)_{\mathfrak{m}}$ is an RD-submodule of $(N_{i+1})_{\mathfrak{m}}$ and since $R_{\mathfrak{m}}$ is a valuation domain, also a pure submodule of $(N_i)_{\mathfrak{m}}$. The pure exact sequence

$$0 \longrightarrow (N_i)_{\mathfrak{m}} \xrightarrow{\iota\otimes_R R_{\mathfrak{m}}} (N_{i+1})_{\mathfrak{m}} \longrightarrow (N_{i+1}/N_i)_{\mathfrak{m}} \longrightarrow 0$$

splits, since the right-hand term is finitely presented. From this, we conclude that ι_i splits.

Overall, M is a union of the chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ where each N_i splits in N_{i+1}, i.e., there exists a submodule $D_{i+1} \subseteq N_{i+1}$ such that $N_{i+1} = N_i \oplus D_{i+1}$. Further let $D_1 := N_1$. Then $M = \bigoplus_{i=1}^{\infty} D_i$ is a direct sum of finitely generated R-modules. □

Theorem 4.4.8. *Let R be a noetherian domain of Krull dimension 1, and with module-finite normalization $\overline{R}$. Then the following statements are equivalent:*

(1) *Every pure projective torsion-free R-module is a direct sum of finitely presented modules.*

(2) *For any finitely generated, torsion-free R-module X, every element in* $\mathrm{Add}(X)$ *is a direct sum of finitely generated modules.*

(3) *R satisfies the following two conditions:*

- **(i)** *there exists at most one maximal ideal $\mathfrak{m}_0$ of R such that $R_{\mathfrak{m}_0}$ is not a Bass domain, and*
- **(ii)** *the normalization of $R_{\mathfrak{m}}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R.*

Proof. It is clear that **(1)** implies **(2)**. Statement **(2)** implies **(3)** by Theorem 4.1.6. We only need to prove that **(3)** implies **(1)**.

Note that, if R is integrally closed, then R is a Dedekind domain, and pure projective torsion-free modules are projective. Assume that $R \neq \overline{R}$, and let $\mathcal{M} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$ be a finite set of maximal ideals of R such that $R_{\mathfrak{m}}$ is integrally closed whenever $\mathfrak{m} \notin \mathcal{M}$. Let $\Sigma := R \setminus \bigcup_{i=1}^{k} \mathfrak{m}_i$. Therefore, R_Σ is a semilocal ring satisfying the assumptions of Proposition 4.4.6. Hence, any pure projective torsion-free R_Σ-module is a direct sum of finitely presented modules.

Let A be a countably generated pure projective torsion-free R-module, i.e., a direct summand of $\bigoplus_{i\in\mathbb{N}} M_i$, where each M_i is a finitely generated torsion-free module. Note that each M_i can be considered as a submodule of $M_i\overline{R}$ (the $\overline{R}$-submodule of $(M_i)_0$ generated by M_i) which is a projective $\overline{R}$-module. Hence, we may consider A as a submodule of $\overline{R}^{(\omega)}$.

Since A_Σ is a pure projective torsion-free R_Σ-module, it has to be a direct sum of finitely generated modules. By Lemma 4.4.7, A is a direct sum of finitely generated modules. □

4.5 A family of examples

In this section, we provide a family of examples of h-local domains of Krull dimension 1, not necessarily noetherian, that exemplifies well the situations described in Theorem 4.1.6 and also in Corollary 4.2.3 as we show that they satisfy that direct summands of finitely generated torsion-free modules are direct sums of finitely generated modules. This family of examples was suggested to us by Carmelo Antonio Finocchiaro and Paolo Zanardo.

Let $K \subseteq L$ be a field extension, and consider the ring $R = K + xL[x]$ and its localization $R_{\mathfrak{m}}$ at the maximal ideal $\mathfrak{m} = xL[x]$. The field of fractions of R is $L(x)$, and R fits in the pullback diagram of rings

$$\begin{array}{ccc} R = K + xL[x] & \xhookrightarrow{\text{incl.}} & L[x] \\ \Big\downarrow{\scriptstyle \mathrm{ev}_0} & & \Big\downarrow{\scriptstyle \mathrm{ev}_0} \\ K & \xhookrightarrow{\text{incl.}} & L \end{array} \tag{4.1}$$

where ev_0 denotes the evaluation at 0. In fact, this diagram is a conductor square, with conductor ideal $\mathfrak{c} = \mathfrak{m}$ since $R/\mathfrak{m} \cong K$ and $L[x]/\mathfrak{m} \cong L$.

Note that $R_\mathfrak{m} = K + xL[x]_\mathfrak{m}$, since every element in $R_\mathfrak{m}$ can be written as

$$\frac{a + xp(x)}{1 + xq(x)} = a + \frac{x(p(x) - aq(x))}{1 + xq(x)},$$

with $a \in K$, $p(x), q(x) \in L[x]$.

The localization of (4.1) at $\mathfrak{m}$ gives also a conductor square for $R_\mathfrak{m}$

$$\begin{array}{ccc} R_\mathfrak{m} = K + xL[x]_\mathfrak{m} & \xhookrightarrow{\text{incl.}} & L[x]_\mathfrak{m} \\ \Big\downarrow{\scriptstyle \mathrm{ev}_0} & & \Big\downarrow{\scriptstyle \mathrm{ev}_0} \\ K & \xhookrightarrow{\text{incl.}} & L \end{array} \tag{4.2}$$

with conductor $\mathfrak{c}_\mathfrak{m} = xL[x]_\mathfrak{m}$.

Lemma 4.5.1. *Let $K \subseteq L$ be a field extension, let R denote the ring $K + xL[x]$, and $\mathfrak{m} = xL[x]$. Then*

- **(i)** *If I is an ideal of R, then $IL = L[x]$ if and only if $I = R$.*
- **(ii)** *Every maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ of R is generated by a polynomial $q(x) \in R$ that is irreducible in $L[x]$ with $q(0) = 1$.*
- **(iii)** *$R_\mathfrak{n} = L[x]_\mathfrak{n}$, that is, $R_\mathfrak{n}$ is a discrete valuation ring for every maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ in R.*
- **(iv)** *R is an h-local domain of Krull dimension 1.*

Proof. **(i)** Suppose that $IL = L[x]$, so $y_0 + y_1a_1 + \cdots + y_na_n = 1$ for some elements $y_i \in I$ and $a_i \in L \setminus K$. Then, for every polynomial $q(x) \in L[x]$, $y_0xq(x) + y_1a_1xq(x) + \cdots + y_na_nxq(x) = xq(x) \in I$. Therefore $xL[x] \subseteq I$. Since $xL[x]$ is a maximal ideal of R and $I \not\subseteq xL[x]$, $I = K + xL[x]$.

(ii) Let $\mathfrak{n} \neq \mathfrak{m}$ be a maximal ideal of R. By **(i)** and because $\mathfrak{n}$ is maximal, $\mathfrak{n}L \neq L[x]$ and $\mathfrak{n}L \cap R = \mathfrak{n}$. There exists $p(x) \in L[x]$ with $p(0) = 1$ such that $p(x)L[x] = \mathfrak{n}L$. Since $p(x) \in \mathfrak{n}L \cap R$, $p(x) \in \mathfrak{n}$. Notice that if $q(x) \in L[x]$ and $p(x)q(x) \in R$ then also $q(x) \in R$. Therefore $p(x)R = \mathfrak{n}$. If $p(x) = q_1(x)q_2(x)$ in $L[x]$, and since $p(0) = 1$, we can assume that $q_1(0) = q_2(0) = 1$ so that it is also a decomposition in R. Since $\mathfrak{n}$ is maximal, either $q_1(x)$ or $q_2(x) \in \mathfrak{n}$ which implies that either $q_1(x)$ or $q_2(x)$ is equal 1. This shows that $p(x)$ is irreducible in $L[x]$.

(iii) Since the element $x \in R \setminus \mathfrak{n}$ is invertible in $R_\mathfrak{n}$, we have $a = ax/x \in R_\mathfrak{n}$ for every element $a \in L$, so we deduce that $L \subseteq R_\mathfrak{n}$, that is, $L[x]_\mathfrak{n} \subseteq R_\mathfrak{n}$.

(iv) Let $p(x)$ be a non-zero polynomial in R. If $p(0) \neq 0$, $p(x)$ can be written as $p(x) = kq_1(x) \cdots q_n(x)$, where $k = p(0) \in K^*$ and $q_i(x)$ are irreducible polynomials in $L[x]$ with $q_i(0) = 1$ for every $i = 1, \ldots, n$ so, in particular, $q_i(x) \in R$. This decomposition is unique up to a unit (an element in K) because $L[x]$ is a UFD, and in this case, $p(x)$ is only contained in the maximal ideals generated by the irreducible polynomials $q_i(x)$ for $i = 1, \ldots, n$.

If $p(0) = 0$, then $p(x)$ can be written as $p(x) = x^m q(x)$, where $m \geq 1$ and $q(x) \in L[x]$ with $q(0) \neq 0$. Then $q(x)$ can be written as $q(x) = lq_1(x) \cdots q_n(x)$, where $l = q(0) \in L^*$ and $q_i(x)$ are irreducible polynomials in $L[x]$ with $q_i(0) = 1$ for every $i = 1, \ldots, n$, so again, $q_i(x) \in R$. Therefore, $p(x)$ is only contained in the maximal ideals generated by the irreducible polynomials $q_i(x)$ for $i = 1, \ldots, n$ and it is also contained in $\mathfrak{m}$. Therefore, R has finite character.

Now we prove that $R_\mathfrak{m}$ has Krull dimension 1. Let $\mathfrak{p}$ be a non-zero prime ideal of $R_\mathfrak{m}$ and let $0 \neq g \in \mathfrak{p}$. We can assume, up to a unit of $R_\mathfrak{m}$, that g is of the form $x^n a$ with $a \in L \setminus \{0\}$ and $n \geq 1$. If $n > 1$, then $x^n a = x \cdot (x^{n-1}a)$ which implies that either x or $x^{n-1}a$ are in $\mathfrak{p}$. By induction on n we can deduce that $\mathfrak{p}$ contains an element of the form xa with $a \in L \setminus \{0\}$ so it also contains $x^2 b = xa \cdot xa^{-1}b$ for any $b \in L$. As $(xb)^2 \in \mathfrak{p}$ for any $b \in L$, we deduce that $xb \in \mathfrak{p}$. Hence $\mathfrak{p} = \mathfrak{m}$.

Since the Krull dimension of R localized at any maximal ideal is 1, we deduce that R has Krull dimension 1 and, being of finite character, it is h-local. □

From **(iii)** it follows that finitely generated torsion-free $R_\mathfrak{n}$-modules are free for every maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ in R.

Remark 4.5.2. Assume $K \subsetneq L$.

(1) *R is integrally closed if and only if the extension $K \subseteq L$ is purely transcendental.* If $\alpha \in L \setminus K$ is algebraic over K then α is integral over R. So if R is integrally closed, $K \subseteq L$ is a purely transcendental extension.

To prove the converse, let $f \in L(x)$ satisfy a polynomial equation $f^n + a_{n-1}f^{n-1} + \cdots + a_0 = 0$ with $a_i \in R$ for $i = 0, \ldots, n-1$. Since $L[x]$ is already integrally closed, we can assume that $f \in L[x]$. Let f_0 denote the evaluation at 0 of f. Then f_0 satisfies a polynomial equation in K, so f_0 is algebraic over K. Since L is purely transcendental, we deduce that $f_0 \in K$, so $f \in R$.

(2) *$R_\mathfrak{m}$ is noetherian if and only if the extension $K \subseteq L$ has finite degree.* Indeed, let $(a_i)_{i \in A}$ be a basis of L as K-vector space. The maximal ideal $\mathfrak{m} = xL[x]$ is generated by $\mathcal{A} = \{xa_i\}_{i \in A}$ and no proper subset of $\mathcal{A}$ generates $\mathfrak{m}$. Then $R_\mathfrak{m}$ is noetherian if and only if A is finite.

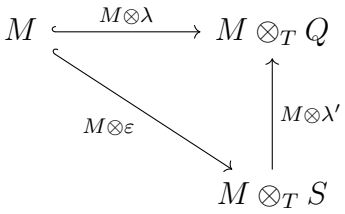
(3) *$R_\mathfrak{m}$ is not a valuation ring.* Note that every element $a \in L \setminus K$ is in its field of fractions $L(x)$, but neither a nor a^{-1} belong to $R_\mathfrak{m}$.

Now we will study torsion-free modules over the ring R and over the ring $R_\mathfrak{m}$. The inclusion of R into the principal ideal domain $L[x]$ allows us to use the techniques of the conductor square and artinian pairs (see [33]). First, we fix some notation.

We shall describe a class of torsion-free modules over the ring T that can be either R of $R_\mathfrak{m}$. The ring T is included in a principal ideal domain S, where $S = L[x]$ if $T = R$ and $S = L[x]_\mathfrak{m}$ if $T = R_\mathfrak{m}$. The field of fractions of T coincides with the one of S and it is $Q = L(x)$. We will denote by $\mathfrak{c}$ the conductor of both conductor squares (4.1) and (4.2).

We denote by $\lambda\colon T \to Q$ and $\lambda'\colon S \to Q$ the corresponding localization maps, and by $\varepsilon\colon T \to S$ the ring inclusion. Then if M_T is a torsion-free T-module, there

is a commutative diagram



Therefore, we can identify M_T with an essential submodule of $Q^{(A)}$ for a suitable set A, and then, we set $(M\otimes\lambda')(M\otimes_T S) = MS$.

In general, $M\otimes\lambda'$ is not an injective map; it is when M_T is projective or, more generally, when M_T is flat.

Lemma 4.5.3. *Let M_T be a torsion-free module over T. Then:*

(i) *$MS = ML$;*

(ii) *$ML\cap M \supseteq M\mathfrak{c}$;*

(iii) *As a K-vector space, $M = V\oplus M\mathfrak{c}$, where V is any complement of $M\mathfrak{c}$ in M. Moreover, as L-vector space, $ML = VL\oplus M\mathfrak{c}$. Setting $W = VL$, it follows that $\dim_K(V)\geq\dim_L(W)$.*

(iv) *As an L-vector space, $ML = W\oplus M\mathfrak{c}$, where W is any complement of $M\mathfrak{c}$ in M. Moreover, $V = W\cap M$ is a K-vector space such that $M = V\oplus M\mathfrak{c}$ and $W = VL$.*

(v) *For any pair V, W chosen as in* **(iii)** *or* **(iv)**, *$V\cong M/M\mathfrak{c}$ and $W\cong ML/M\mathfrak{c}$. In addition, there is a pullback diagram*

$$\begin{array}{ccc} M & \xhookrightarrow{\text{incl.}} & ML \\ \downarrow & & \downarrow \\ V & \xhookrightarrow{\text{incl.}} & W \end{array} \tag{4.3}$$

Proof. Statement **(i)** follows because $TS = TL$, cf. Lemma 4.5.1**(i)**. Statement **(ii)** is clear.

The ring T fits in an exact sequence of T-modules

$$0\longrightarrow\mathfrak{c}\longrightarrow T\longrightarrow K\longrightarrow 0$$

shows that (4.3) is a pull-back diagram. □

Definition 4.5.4. The category $\mathcal{B}$ of modules over the artinian pair $K\hookrightarrow L$ has as objects the inclusions $V\hookrightarrow W$ where V is a K-subspace of the L-vector space W satisfying that $VL = W$.

If $V \hookrightarrow W$ and $V' \hookrightarrow W'$ are two objects of $\mathcal{B}$, a morphism between them consists of one K-linear map $f\colon V \to V'$ and one L-linear map $g\colon W \to W'$ making the diagram

$$\begin{array}{ccc} V & \xhookrightarrow{\text{incl.}} & W \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V' & \xhookrightarrow{\text{incl.}} & W' \end{array}$$

commutative.

Corollary 4.5.5. *As usual, T denotes the ring $R = K + xL[x]$ or $R_{\mathfrak{m}}$ where $\mathfrak{m} = xL[x]$. Let $\mathcal{A}_T$ be the category of torsion-free modules over T. Let $\mathcal{B}$ be the category of modules over the artinian pair $K \hookrightarrow L$. Then there is a functor F_T that assigns to each object M_T in $\mathcal{A}_T$ the object of $\mathcal{B}$, $M/M\mathfrak{c} \hookrightarrow ML/M\mathfrak{c}$.*

Moreover, $F_R(M) = F_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ for any object M of $\mathcal{A}_R$.

Proof. Lemma 4.5.3(**v**) and the fact that a morphism between torsion-free modules over T induces a morphism between the modules over the artinian pair, imply the existence of such a functor.

If M is an object in $\mathcal{A}_R$ then $M/M\mathfrak{c}$ is an R-module and also an $R_{\mathfrak{m}}$-module. Also, $ML/M\mathfrak{c}$ is an $L[X]$ module as well as an $L[x]_{\mathfrak{m}}$-module. So, the second part of the statement is clear. □

For a general torsion-free module M the pull-back diagram of Lemma 4.5.3(**v**) can be trivial. Take, for example, $M = Q$. Then, $M = ML = M\mathfrak{c}$ and $V = W = 0$. So, in the notation of Corollary 4.5.5, $F_T(Q) = 0$.

To get a better correspondence, we need to restrict the class of torsion-free modules we are interested in. We shall consider the class

$$\mathcal{C}_T = \{M \mid M \text{ is a torsion-free } T\text{-module such that } ML \text{ is a free } S\text{-module}\}.$$

Notice that $\mathcal{C}_T$ contains all finitely generated torsion-free modules, and it is closed by arbitrary direct sums and direct summands. In addition, $M = S \in \mathcal{C}_T$.

Assume that ML is a free S-module, then we can fix an S-basis $\mathcal{B} = \{v_i\}_{i\in A}$. Then if W is the L-vector space generated by $\mathcal{B}$, $M\mathfrak{c} = W\mathfrak{c}$ and $ML = W \oplus W\mathfrak{c}$. Therefore, by the modular law and by Lemma 4.5.3, $M = V \oplus W\mathfrak{c}$ where $V = W \cap M$.

If M_1 and M_2 are two modules in $\mathcal{C}_T$ then, for $i = 1, 2$,

$$M_iL = W_i \oplus W_i\mathfrak{c} \quad \text{and} \quad M_i = V_i \oplus W_i\mathfrak{c}, \quad \text{where } V_i = W_i \cap M_i.$$

Therefore, if (f, g) is a morphism between the artinian pairs $V_1 \hookrightarrow W_1$ and $V_2 \hookrightarrow W_2$ or, equivalently, f is a K-linear map and g is an L-linear map, and there is a commutative diagram

$$\begin{array}{ccc} V_1 & \xhookrightarrow{\text{incl.}} & W_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V_2 & \xhookrightarrow{\text{incl.}} & W_2 \end{array}$$

then f can be extended to an S-linear map

$$\widetilde{g}\colon M_1L = W_1 \oplus W_1\mathfrak{c} \to M_2L = W_2 \oplus W_1\mathfrak{c}$$

by setting $\widetilde{g}(wx^n) = g(w)x^n$ for any $w \in W_1$ and any $n \geq 0$. Notice that g is an isomorphism if and only if $\widetilde{g}$ is an isomorphism because $\widetilde{g^{-1}} = (\widetilde{g})^{-1}$.

As a consequence, we have the following.

Corollary 4.5.6. *The functor F_T described in Corollary 4.5.5 is full when restricted to the category $\mathcal{C}_T$ and it reflects isomorphisms.*

Therefore,

- **(i)** *two objects M_1 and M_2 of $\mathcal{C}_R$ are isomorphic if and only if $(M_1)_{\mathfrak{m}}$ and $(M_2)_{\mathfrak{m}}$ are isomorphic;*
- **(ii)** *M_T is projective if and only if $M \in \mathcal{C}_T$ and the inclusion $M/M\mathfrak{c} \hookrightarrow ML/M\mathfrak{c}$ sends K-basis of $M/M\mathfrak{c}$ to L-basis of $ML/M\mathfrak{c}$ if and only if M_T is a free T-module.*

Proof. The remarks before the statement prove that the functor F_T restricted to the category $\mathcal{C}_T$ is full and reflects isomorphisms.

The statement **(i)** follows because, by Corollary 4.5.5, $F_R(M) = F_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ for any object M of $\mathcal{C}_R$ and, by the first part of the statement, two modules in $\mathcal{C}_R$ are isomorphic if and only if their corresponding artinian pairs are isomorphic if and only if their localizations at $\mathfrak{m}$ are isomorphic.

To prove **(ii)**, notice that since over $R_{\mathfrak{m}}$ all projective modules are free, it follows by **(i)** that all projective modules over R, since they are modules in $\mathcal{C}_R$, are isomorphic to a free module. It is easy to check that this happens if and only if a K-basis of $V \cong M/M\mathfrak{c}$ is also an L-basis of $W = ML/M\mathfrak{c}$. □

Now we summarize some results on finitely generated torsion-free modules over $R = K + xL[x]$. As observed before, these are always in $\mathcal{C}_T$ as well as infinite direct sums of them.

Lemma 4.5.7. *Let $K \subseteq L$ be a field extension, let R denote the ring $K + xL[x]$, and $\mathfrak{m} = xL[x]$. Let M, N and $\{M_i\}_{i\in I}$ be finitely generated, torsion-free R-modules. Then*

- **(i)** *Every finitely generated indecomposable torsion-free $R_{\mathfrak{m}}$-module has local endomorphism ring.*
- **(ii)** *$M \cong N$ if and only if $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$.*
- **(iii)** *M is indecomposable if and only if $M_{\mathfrak{m}}$ is indecomposable.*
- **(iv)** *If M and $\{M_i\}_{i\in I}$ are indecomposable and M is a direct summand of $\bigoplus_{i\in I} M_i$, then $M \cong M_i$ for some $i \in I$.*

Proof. **(i)** This follows from Lemma 2.2.5, taking $T = L[x]_{\mathfrak{m}}$. Note that $\mathfrak{m}$ is the conductor, so it is different from zero as required in the hypothesis of the Lemma.

Statement **(ii)** is included in Corollary 4.5.6.

(iii) Suppose that $M_{\mathfrak{m}} = A \oplus B$, and that M is indecomposable. Let A' and B' be R-submodules of A and B generated by some finite set of $R_{\mathfrak{m}}$-generators of

A and B, respectively. Then $(A' \oplus B')_{\mathfrak{m}} = A \oplus B = M_{\mathfrak{m}}$, and by **(ii)**, we deduce that $A' \oplus B' \cong M$. Therefore, $A' = 0$ or $B' = 0$, that is, $A = 0$ or $B = 0$, and $M_{\mathfrak{m}}$ is indecomposable. The other implication is clear.

(iv) Suppose that M is a direct summand of $\bigoplus_{i\in I} M_i$. Then $M_{\mathfrak{m}}$ is also a direct summand of $(\bigoplus_{i\in I} M_i)_{\mathfrak{m}} \cong \bigoplus_{i\in I}(M_i)_{\mathfrak{m}}$. Since $(M_i)_{\mathfrak{m}}$ has local endomorphism ring, it satisfies the Krull-Schmidt-Remak-Azumaya property, that is, $M_{\mathfrak{m}} \cong (M_i)_{\mathfrak{m}}$ for some $i \in I$. By **(ii)**, we deduce that $M \cong M_i$ for some $i \in I$. □

Corollary 4.5.8. *Let $K \subseteq L$ be a field extension, let R denote the ring $K + xL[x]$. Then the class of modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands.*

Proof. By Lemmas 4.5.1 and 4.5.7, the result follows from Proposition 4.2.2. □

Now we want to explicitly construct finitely generated, indecomposable, torsion-free T-modules where T denotes either $R = K + xL[x]$ or $R_{\mathfrak{m}}$. First, we will specify better how these modules and their endomorphism rings can look like.

Recall that if $T = R = K + xL[x]$ then $S = L[x]$, and if $T = R_{\mathfrak{m}}$ then $S = L[x]_{\mathfrak{m}}$.

Remark 4.5.9. Let M_T be a finitely generated torsion-free T-module of rank n. As ML is a finitely generated free S-module, we may assume that M is a T-submodule of S^n such that $ML = S^n$, in particular $(xS)^n \subseteq M$. Therefore, $M_T = V + (xS)^n$, where V is a K-subspace of L^n satisfying $VL = L^n$. Notice that this gives us a very explicit construction of a T-module $M \in \mathcal{C}$ such that $F(M)$ (cf. Corollary 4.5.5) has as an image the module over the artinian pair $V \hookrightarrow W$.

Now we can also identify the endomorphism ring of M_T with a subring of $\mathrm{M}_n(Q)$, in fact $\mathrm{End}_T(M_T) = \{A \in \mathrm{M}_n(Q) \mid AM \subseteq M\}$. Since $(xS)^n \subseteq M$, if $A \in \mathrm{M}_n(Q)$ represents an endomorphism of M_T, then $A \in \mathrm{M}_n(S)$, i.e., $\mathrm{End}_T(M_T) = \{A \in \mathrm{M}_n(S) \mid AM \subseteq M\}$. Every matrix $A \in \mathrm{M}_n(S)$ can be uniquely decomposed as the sum of a matrix $B \in \mathrm{M}_n(L)$ and a matrix $C \in \mathrm{M}_n(xS)$. Since $\mathrm{M}_n(xS) \subseteq \mathrm{End}_T(M)$, $A \in \mathrm{End}_T(M)$ if and only if $BV \subseteq V$. This is to say that if $A \in \mathrm{End}_T(M_T)$, then $F(A) = B$.

Lemma 4.5.10. *Let $K \subsetneq L$ be a field extension, let R denote the ring $K + xL[x]$, and let $\mathfrak{m} = xL[x]$. Let M be a finitely generated torsion-free right $R_{\mathfrak{m}}$-module of rank n. Then $\mathrm{M}_n(xL[x]_{\mathfrak{m}}) \subseteq \mathrm{J}(\mathrm{End}_{R_{\mathfrak{m}}}(M))$.*

Proof. Note that $\mathrm{J}(R_{\mathfrak{m}}) = \mathfrak{m}R_{\mathfrak{m}}$. Then, for every $A \in \mathrm{M}_n(xL[x]_{\mathfrak{m}})$, we have

$$M = AM + (I - A)M \subseteq \mathrm{J}(R_{\mathfrak{m}})M + (I - A)M.$$

By Nakayama's Lemma, $M = (I-A)M$ and, by Lemma 2.1.1**(i)**, $I-A$ is bijective. Hence $A \in \mathrm{J}(\mathrm{End}_{R_{\mathfrak{m}}}(M))$. □

The following is a modification of Construction 3.13 in [33] to build indecomposable modules over artinian pairs. Using Remark 4.5.9, this immediately yields the existence of indecomposable finitely generated T-modules of arbitrary finite rank $n \geq 2$.

Construction 4.5.11. *Let $n \geq 2$ be a fixed positive integer, and suppose we have chosen $\alpha, \beta \in L$ with $\{1, \alpha, \beta, \alpha^2, \alpha\beta, \beta^2\}$ linearly independent over K. Let I be the identity $n \times n$ matrix and H be the nilpotent $n \times n$ matrix with 1 below the diagonal and 0 elsewhere. For $t \in K$, we consider the $n \times 2n$ matrix,*

$$\Psi_t := [I \mid \alpha I + \beta(tI + H)] = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & \alpha + t\beta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \beta & \alpha + t\beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \alpha + t\beta \end{array}\right]$$

Let V_t be the K-subspace of L^n spanned by the columns of Ψ_t, and let $A \in \mathrm{Hom}_K(V_t, V_u)$, where

$$\mathrm{Hom}_K(V_t, V_u) = \{A \in \mathrm{M}_n(L) \mid AV_t \subseteq V_u\}.$$

The condition $AV_t \subseteq V_u$ implies that there is a $2n \times 2n$ matrix $\theta \in \mathrm{M}_{2n}(K)$ such that $A\Psi_t = \Psi_u\theta$. Write $\theta = \begin{bmatrix} C & D \\ P & Q \end{bmatrix}$, where $C, D, P, Q \in \mathrm{M}_n(K)$. Then, using the condition $A\Psi_t = \Psi_u\theta$, we have the following two equations:

$$\begin{cases} A = C + \alpha P + \beta(uI + H)P \\ \alpha A + \beta A(tI + H) = D + \alpha Q + \beta(uI + H)Q \end{cases}$$

Substituting the first equation into the second and combining terms, we get the following:

$$\begin{aligned} &-D + \alpha(C - Q) + \beta(tC - uQ + CH - HQ) + \alpha^2 P \\ &\qquad + \alpha\beta(tP + uP + HP + PH) + \beta^2(tuP + HPH + tHP + uPH) = 0. \end{aligned}$$

From the linear independence of $\{1, \alpha, \beta, \alpha^2, \alpha\beta, \beta^2\}$, we have

$$D = P = 0, \quad A = C = Q \quad \text{and} \quad (t - u)A + AH = HA.$$

In particular, we deduce that $A \in \mathrm{M}_n(K)$, and if A is an isomorphism and $t \neq u$, then the third equation above gives a contradiction since the left side is invertible and the right side is not. Thus, if $V_t \cong V_u$, then $t = u$. To see that V_t is indecomposable, we take $u = t$ and suppose that A is idempotent. But $AH = HA$, and it follows that A is in $K[H]$, which is a local ring. Therefore $A = 0$ or I, as desired.

By Remark 4.5.9, $M_t = V_t + (xS)^n$, where V_t is the K-subspace of L^n constructed above, is an indecomposable T-module of rank n (as usual, $T = R = K + xL[x]$ or $T = R_{\mathfrak{m}}$). Notice that, in view of Corollary 4.5.5, there are at least $|K|$ different isomorphism classes of such modules.

Part III

Relatively big projective modules and their applications to direct sum decompositions

In this part, we introduce the language of monoids in order to describe projective modules over a given ring Λ. This is the main topic of Chapter 5, where we introduce the notation and the big projectives with respect to an ideal. In particular, we focus on the monoid of relatively big projective modules, which is the main topic of this dissertation: countably generated projective modules which are the liftings of finitely generated projectives modulo a trace ideal of a projective. Our goal is to enlarge the classes of rings for which every countably generated projective module is relatively big.

In Chapter 6, we present new classes of rings for which every countably generated projective module is relatively big. This includes right noetherian rings satisfying a polynomial identity and locally semiperfect torsion-free algebras over an h-local domain with simple artinian ring of quotients. Note that both rings are without a Krull dimension assumption. Right noetherian rings satisfying a polynomial identity arise when considering endomorphism rings of modules over noetherian rings. Locally semiperfect torsion-free algebras over an h-local domain with simple artinian ring of quotients appear naturally as the endomorphism ring of a finitely generated torsion-free module M when $M_{\mathfrak{m}}$ is a direct sum of modules with local endomorphism ring for every maximal ideal $\mathfrak{m}$ of R.

In Chapter 7, we provide arithmetical conditions on the existence of countably generated projective modules over locally semiperfect torsion-free algebras over an h-local with simple artinian ring of quotients, depending on their genus and their rank. As an application, we provide a complete characterization of those h-local domains of Krull dimension 1 for which every direct summand of $\mathrm{Add}(M)$, where M is a finitely generated torsion-free module, is again a direct sum of finitely generated modules.

In Chapter 8, we consider locally semiperfect, module-finite, torsion-free algebras over an h-local domain with semisimple artinian ring of quotients. Those kinds of algebras appear naturally in the representation theory of finite groups, namely, in group algebras over a suitable Dedekind domain. We provide similar arithmetical conditions as in the previous chapter to ensure the existence of countably generated projective modules over those algebras. As an example, we compute the whole monoid of infinitely generated modules over group algebras of some simple non-abelian groups of low order.

5 The language of monoids

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In this chapter, we provide the basic definitions of monoids of infinitely generated projective modules and notation.

In Section 5.1, we introduce the concept of big projective modules with respect to an ideal. An important result is Lemma 5.1.7, which states that the isomorphism class of a J-big projective module is completely determined by the ideal J and the isomorphism class of the quotient P/PJ.

In Section 5.2, we recall the $(*)$-condition, which was introduced in [46] and is a particular kind of descending chain condition. We relate this condition to the condition of being finitely generated modulo a trace ideal, which characterizes J-big projective modules over a noetherian ring (cf. Theorem 5.2.4).

Finally, in Section 5.3, we examine the monoid of isomorphism classes of relatively big projective modules, its algebraic properties, and explore the right/left (non)symmetry of the monoids (see Example 5.3.7 and Corollary 5.3.9). These topics will play a crucial role in the analysis conducted in the subsequent chapters.

5.1 Relatively big projective modules

Notation 5.1.1. Let Λ be a ring. We will denote by $\mathrm{V}_r(\Lambda)$ the set of representatives of the isomorphism classes of finitely generated projective right Λ-modules, and we will denote by $\mathrm{V}_r^*(\Lambda)$ the set of representatives of the isomorphism classes of countably generated projective right Λ-modules.

If P is a finitely generated (resp. countably generated) projective right Λ-module, we denote its representative in $\mathrm{V}_r(\Lambda)$ (resp. $\mathrm{V}_r^*(\Lambda)$) by $[P]$.

One can define $\mathrm{V}_\ell(\Lambda)$ and $\mathrm{V}_\ell^*(\Lambda)$ for projective left Λ-modules analogously.

Unless stated otherwise, our modules will always be right modules, so we will write $\mathrm{V}(\Lambda)$ and $\mathrm{V}^*(\Lambda)$ instead of $\mathrm{V}_r(\Lambda)$ and $\mathrm{V}_r^*(\Lambda)$, respectively, and leave the previous notation in cases where both right and left modules appear. Every result stated for $\mathrm{V}(\Lambda)$ and $\mathrm{V}^*(\Lambda)$ can be presented in the context of left Λ-modules for $\mathrm{V}_\ell(\Lambda)$ and $\mathrm{V}_\ell^*(\Lambda)$, respectively.

Remark 5.1.2. Both $\mathrm{V}(\Lambda)$ and $\mathrm{V}^*(\Lambda)$ are commutative monoids with the operation induced by the direct sum. Note that $\mathrm{V}(\Lambda)$ is a submonoid of $\mathrm{V}^*(\Lambda)$, and that $\mathrm{V}^*(\Lambda) \setminus \mathrm{V}(\Lambda)$ is a subsemigroup of $\mathrm{V}^*(\Lambda)$.

Recall that a commutative monoid H is *reduced* if $x + y = 0$ implies $x = y = 0$, for $x, y \in H$. An element u in H is an *order-unit* if, for every $x \in H$, there exists $y \in H$ and a non-negative integer $n \in \mathbb{Z}$ such that $x + y = nu$.

Remark 5.1.3. The monoid $\mathrm{V}(\Lambda)$ is a reduced commutative monoid with an order-unit (induced by the isomorphism class of the right module Λ). In their seminal papers, Bergman [8] and Bergman and Dicks [9] show that for hereditary rings, a converse is true: for every reduced commutative monoid H with an order-unit and every field K, there exists a hereditary K-algebra Λ with $\mathrm{V}(\Lambda) \cong H$.

The structure of $\mathrm{V}(\Lambda)$ and $\mathrm{V}^*(\Lambda)$ is particularly neat when Λ is either semisimple artinian or semiperfect (cf. Theorem 1.2.8 and Lemma 1.2.10). In the following chapters, we will explore in more detail algebras over commutative domains R whose localizations at maximal ideals of R are semiperfect and whose ring of quotients is semisimple artinian.

Now we are ready to introduce the auxiliary notion of big projective modules with respect to an ideal. It was first introduced by P. Příhoda in [46] as a way of generalizing Bass' uniformly big projective modules. We motivate the definition of this class of projective modules by comparing it with a characterization of the class of modules generated by a projective module (cf. Section 1.5).

Definition 5.1.4. Let Λ be a ring, let P be a countably generated projective right Λ-module, and let J be an ideal of Λ. The module P is said to be *big with respect to* J, or *J-big*, if for every countably generated projective right Λ-module Q satisfying $QJ = Q$, then Q is a homomorphic image of P.

The following remarks show that the significant case of Definition 5.1.4 is when J is a trace ideal of a projective module.

Remark 5.1.5. Let Λ be a ring, and let J be an ideal of Λ.

(1) If Q is a countably generated projective right Λ-module such that $QJ = Q$, then $\mathrm{Tr}_\Lambda(Q) \subseteq J$. Therefore, if J does not contain a non-zero element of $\mathcal{T}(\Lambda)$, then, by Lemma 1.4.3, Definition 5.1.4 is trivially fulfilled. Thus, any countably generated projective module is J-big or, equivalently, 0-big.

(2) If P is J-big, and K is a non-zero element of $\mathcal{T}(\Lambda)$ such that $K \subseteq J$, then P is K-big (cf. Lemma 1.4.3). Note that

$$J' = \sum_{\substack{K \subseteq J \\ K \in \mathcal{T}(\Lambda)}} K$$

is again a trace ideal of a projective module by Lemma 1.4.3. Then a countably generated projective module is J-big if and only if it is J'-big.

We stress that J' does not need to be the trace of a countably generated projective Λ-module, so it is not necessarily an element in $\mathcal{T}(\Lambda)$.

In the next lemma, we summarize what it means to be big with respect to an ideal in $\mathcal{T}(\Lambda)$.

Lemma 5.1.6. *Let Λ be a ring, and let P, Q be countably generated projective right Λ-modules. Let $J = \mathrm{Tr}_\Lambda(Q)$. Then*

- **(i)** *P is J-big if and only if $P \oplus Q^{(\omega)} \cong P$.*
- **(ii)** *If P is J-big and P/PJ is finitely generated, then J is the minimal ideal of Λ such that P/PJ is finitely generated.*

Proof. **(i)**. Since $J = \mathrm{Tr}_\Lambda(Q) = \mathrm{Tr}_\Lambda(Q^{(\omega)})$, by Lemma 1.4.3, $Q^{(\omega)}J = Q^{(\omega)}$. If P is J-big, then there is an epimorphism $P \longrightarrow Q^{(\omega)}$. Since $Q^{(\omega)}$ is projective, this epimorphism splits, so $Q^{(\omega)}$ is isomorphic to a direct summand of P. Write $P \cong P' \oplus Q^{(\omega)}$. Then, $P \cong P' \oplus Q^{(\omega)} \cong P' \oplus Q^{(\omega)} \oplus Q^{(\omega)} \cong P \oplus Q^{(\omega)}$.

Conversely, if $P \oplus Q^{(\omega)} \cong P$, then $Q^{(\omega)}$ is isomorphic to a direct summand of P, so there exists an epimorphism $P \longrightarrow Q^{(\omega)}$. Any countably generated projective module with trace contained in J is an epimorphic image of $Q^{(\omega)}$ by Lemma 1.5.3, so P is J-big.

Statement **(ii)** is Lemma 3.3 in [25]. □

The following lemma states that the isomorphism class of a J-big projective right Λ-module P satisfying that P/PJ is finitely generated, is determined, up to isomorphism, by the ideal J and the isomorphism class of P/PJ.

Lemma 5.1.7. *Let Λ be a ring, let $J \in \mathcal{T}(\Lambda)$, and let P be a J-big module such that P/PJ is finitely generated. Then:*

- **(i)** *If P' is a projective module isomorphic to P then P' is J-big and $P'/P'J$ is finitely generated.*
- **(ii)** (Proposition 3.4 in [25]) *Let P' be a countably generated projective module that is J'-big for some $J' \in \mathcal{T}(\Lambda)$ and such that $P'/P'J'$ is finitely generated. Then $P \cong P'$ if and only if $J = J'$ and $P/PJ \cong P'/P'J$.*

Now we give the definition of a *relatively big* projective module, which is crucial for the rest of the dissertation.

Definition 5.1.8. Let Λ be a ring, and let P be a countably generated projective right Λ-module. The module P is said to be *relatively big* if there exists a countably generated projective right Λ-module Q with trace ideal J satisfying:

- **(1)** P is J-big, that is, $P \oplus Q^{(\omega)} \cong P$.
- **(2)** P/PJ is a finitely generated projective right Λ/J-module.

We will also use the terminology P is *relatively J-big* when we want to specify the ideal $J \in \mathcal{T}(\Lambda)$.

Example 5.1.9. Let X be a finitely generated projective right Λ-module, and let P be a countably generated projective right Λ-module. Then $X \oplus P^{(\omega)}$ is relatively big: take $Q = P$ and $J = \mathrm{Tr}_\Lambda(P)$ in the Definition 5.1.8.

In particular, taking $X = \{0\}$, we see that $P^{(\omega)}$ is relatively big for any countably generated projective module P. Also, taking $P = \{0\}$ we see that any finitely generated projective module is relatively big.

Example 5.1.9 is a particular instance of the following general construction of relatively big projective modules.

Lemma 5.1.10. *Let Λ be a ring. Let P be a countably generated projective right Λ-module, and let J be the trace ideal of a countably generated projective right Λ-module Q. If P/PJ is finitely generated, then $P \oplus Q^{(\omega)}$ is relatively J-big.*

Proof. Note that $(P \oplus Q^{(\omega)})/(P \oplus Q^{(\omega)})J \cong P/PJ$ is finitely generated. Since $P \oplus Q^{(\omega)} \cong P \oplus Q^{(\omega)} \oplus Q^{(\omega)}$, by Lemma 5.1.6, $P \oplus Q^{(\omega)}$ is J-big. □

Remark 5.1.11. Note that an indecomposable relatively big projective module must be finitely generated.

It is easy to prove that the notion of relatively big projective module behaves well with respect to localization.

Corollary 5.1.12. *Let R be a commutative ring and let Λ be an R-algebra. Let $J \in \mathcal{T}(\Lambda)$. Let Σ be a multiplicatively closed subset of R consisting of non-zero divisors. If P is a relatively J-big projective right Λ-module, then P_Σ is a relatively J_Σ-big projective right Λ_Σ-module.*

Proof. By Lemma 2.1.9, $J_\Sigma \in \mathcal{T}(\Lambda_\Sigma)$. Since P/PJ is finitely generated, so is $(P/PJ)_\Sigma \cong P_\Sigma/P_\Sigma J_\Sigma$. Let P' be a countably generated projective right Λ-module with trace J. Since $P \cong P \oplus Q^{(\omega)}$ as Λ-modules, also $P_\Sigma \cong P_\Sigma \oplus (P')_\Sigma^{(\omega)}$ as Λ_Σ-modules. This finishes the proof of the claim. □

The following result characterizes when every relatively big projective module is isomorphic to a direct sum of finitely generated projective modules.

Proposition 5.1.13. *Let Λ be a ring. Then every relatively big projective right Λ-module is a direct sum of finitely generated projective modules if and only if the following two properties are satisfied:*

(1) *If J is the trace ideal of a projective right Λ-module, then there is a right Λ-module P, which is a direct sum of finitely generated projective modules such that $\mathrm{Tr}_\Lambda(P) = J$.*

(2) *If J is the trace ideal of a countably generated projective right Λ-module and P' is a finitely generated projective right Λ/J-module, then there is a finitely generated projective right Λ-module X such that $P' \cong X/XJ$.*

Proof. Assume that **(1)** and **(2)** are satisfied, and let P be a relatively big projective right Λ-module. Then there exists a countably generated projective right Λ-module Q with trace ideal J such that P is relatively J-big.

By **(1)**, Q can be taken to be a direct sum of finitely generated projective modules. By **(2)** applied to P/PJ, there exists a finitely generated projective right Λ-module X such that $X/XJ \cong P/PJ$. Let $P' = X \oplus Q^{(\omega)}$. By Lemma 5.1.10, P' is relatively big, and by Lemma 5.1.7, $P \cong P'$, so it is a direct sum of finitely generated projective modules.

Assume now that every relatively big projective right Λ-module is a direct sum of finitely generated projective modules. Let P be a projective right Λ-module with trace J. By [29, Theorem 1], $P = \bigoplus_{i\in I} P_i$ is a direct sum of countably generated projective modules. By Example 5.1.9, $P_i^{(\omega)}$ is relatively big. Therefore, $P_i^{(\omega)}$ is isomorphic to a direct sum of finitely generated projective modules for each $i \in I$. Hence, $P^{(\omega)}$ is isomorphic to a direct sum of finitely generated projective modules and $J = \mathrm{Tr}_\Lambda(P^{(\omega)})$. This gives **(1)**.

To prove **(2)**, let Q be a countably generated projective right Λ-module with trace J, and let P' be a finitely generated projective right Λ/J-module. By Lemma 1.5.4, since J is the trace of a countably generated projective module, there exists a countably generated projective right Λ-module P such that $J \subseteq \mathrm{Tr}_\Lambda(P)$, $P/PJ \cong P'$, and $\mathrm{Tr}_\Lambda(P)/J = \mathrm{Tr}_{\Lambda/I}(P')$.

By Lemma 5.1.10, $P \oplus Q^{(\omega)}$ is relatively big, so it is a direct sum of finitely generated projective modules. Write $P \oplus Q^{(\omega)} = \bigoplus_{i\in I} X_i$ with X_i a finitely generated projective Λ-module for each $i \in I$. Since $(\bigoplus_{i\in I} X_i)/(\bigoplus_{i\in I} X_i)J \cong P'$ is finitely generated, there is a finite subset $I_0 \subseteq I$ such that $(\bigoplus_{i\in I_0} X_i)/(\bigoplus_{i\in I_0} X_i)J \cong P'$, so $X = \bigoplus_{i\in I_0} X_i$ is the finitely generated projective Λ-module we were looking for. □

5.2 Finite generation modulo an ideal and the $(*)$-condition

In this section, we take a closer look at condition **(2)** in Definition 5.1.8. Let Λ be a ring, and let M be a right Λ-module. Consider the set

$$\mathcal{I}(M) = \{I \trianglelefteq \Lambda \mid M/MI \text{ is finitely generated}\}$$

We recall the following crucial properties of a descending sequence of right ideals associated with a column finite idempotent matrix. The result is taken from [24], but all the necessary ideas are already in [46].

Lemma 5.2.1 (Lemmas 5.1 and 5.2 in [24])**.** *Let Λ be a ring. Let $A = (a_{i,j})$ be an $\mathbb{N}\times\mathbb{N}$-matrix with entries in Λ, that is a column-finite matrix and such that $A^2 = A$, and set $P = A\Lambda^{(\mathbb{N})}$. For any $k \geq 0$, let $I_k = \sum_{i>k, j\in\mathbb{N}} a_{ij}\Lambda$. Then,*

(i) $\mathrm{Tr}_\Lambda(P) = \Lambda I_0$.

(ii) *For any $k \geq 0$ and any $0 < i \leq k$,*

$$\Big(\sum_{j\in\mathbb{N}} a_{ij}\Lambda + \Lambda I_k\Big)/\Lambda I_k = \Big(\sum_{1\leq\ell\leq k} a_{i\ell}\Lambda + \Lambda I_k\Big)/\Lambda I_k.$$

In particular, if $0 \leq n \leq k$, $(I_n + \Lambda I_k)/\Lambda I_k$ is finitely generated.

(iii) *For any $k \geq 0$, there exists $n_k > k$ such that $I_{n_k} I_k = I_{n_k}$. In particular, the descending sequence of two-sided ideals $(\Lambda I_k)_{k\geq 0}$ has a subsequence $(J_n)_{n\geq 0}$ of two-sided ideals satisfying that*

- **(a)** *for any $n \geq 0$, $J_{n+1}J_n = J_{n+1}$;*
- **(b)** *for any $k \geq 0$, there exists n and ℓ such that $J_n = \Lambda I_{k+\ell} \subseteq \Lambda I_k$.*

(iv) *For any $k \geq 0$, $\Lambda I_k \in \mathcal{I}(P)$. Moreover, for any $I \in \mathcal{I}(P)$, there exists k_0 such that $\Lambda I_{k_0} \leq I$.*

(v) *$\mathcal{I}(P)$ has minimal elements if and only if there exists $k_0 \in \mathbb{N}_0$ such that $\Lambda I_{k_0} = \Lambda I_{k_0+\ell}$ for every $\ell \in \mathbb{N}$. In this case, $I_{k_0} = I_{k_0}^2$, and ΛI_{k_0} is the minimal element of $\mathcal{I}(P)$.*

Statement **(v)** in Lemma 5.2.1 motivates the following definition.

Definition 5.2.2. [46] A ring Λ satisfies the $(*)$-condition provided that every descending chain of two-sided ideals

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k \supseteq \cdots$$

such that $J_{k+1}J_k = J_{k+1}$, for each $k \geq 1$, is eventually stationary.

The following statement explains the role of the $(*)$-condition in our discussion.

Corollary 5.2.3. *Let Λ be a ring. Let P be a countably generated projective right Λ-module. If Λ satisfies the $(*)$-condition, then $\mathcal{I}(P)$ has a minimal element, and this minimal element is an idempotent ideal.*

Proof. By Lemma 5.2.1 **(v)**, $\mathcal{I}(P)$ has a minimal element if and only if the descending sequence of two-sided ideals $(\Lambda I_k)_{k\geq 0}$ is stationary. By Lemma 5.2.1 **(iii)**, this will happen if and only if the partial subsequence $(J_n)_{n\geq 0}$ given by that statement is stationary. Since such a partial subsequence satisfies that $J_{n+1}J_n = J_{n+1}$, for any $n \geq 0$, it is stationary because Λ satisfies the $(*)$-condition.

The fact that the minimal element in $\mathcal{I}(P)$ is an idempotent ideal follows from Lemma 5.2.1 **(v)**. □

Theorem 5.2.4 (Lemma 2.8 in [46])**.** *Let Λ be a noetherian ring. Let P be a countably generated projective right Λ-module such that $\mathcal{I}(P)$ has a minimal element J. Then J is the trace ideal of a countably generated projective right Λ-module, and P is J-big. In particular, if Λ satisfies the $(*)$-condition, then any countably generated projective right module is relatively big.*

Remarks 5.2.5. **(1)** Let Λ be a semilocal ring, and let P be a countably generated projective right Λ-module. Then, by Proposition 5.6 in [24], $\mathcal{I}(P)$ has a minimal element J. Then J is an idempotent ideal, but it does not need to be the trace of a projective module as it can even be contained in the Jacobson radical.

(2) We do not know whether the $(*)$-condition is left-right symmetric, even for the case of noetherian rings.

(3) There are noetherian rings with infinite descending sequences of idempotent ideals, so they do not satisfy the $(*)$-condition. Over such rings, there may be countably generated projective modules that are not relatively big. See, for example, the last section of [24].

5.3 The monoid of relatively big projective modules

Definition 5.3.1. Let Λ be a ring. We denote by $\mathrm{B}(\Lambda)$ the subset of $\mathrm{V}^*(\Lambda)$ of isomorphism classes of relatively big projective right Λ-modules which are not finitely generated. We denote by $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ the disjoint union of $\mathrm{V}(\Lambda)$ and $\mathrm{B}(\Lambda)$, which is also a subset of $\mathrm{V}^*(\Lambda)$ and consists of the isomorphism classes of all the relatively big projective right Λ-modules.

It is important to observe that, by Lemma 5.1.7, if P is a countably generated projective right Λ-module such that $[P] \in \mathrm{B}(\Lambda)$, then P is relatively big, so there exists a non-zero ideal $J \in \mathcal{T}(\Lambda)$ such that P/PJ is finitely generated. Moreover, J and P/PJ uniquely determine $[P]$. We will use this fact freely throughout this dissertation.

Lemma 5.3.2. *Let Λ be a ring. Then*

(i) $\mathrm{B}(\Lambda)$ *is a subsemigroup of* $\mathrm{V}^*(\Lambda) \setminus \mathrm{V}(\Lambda)$.

(ii) $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ *is a submonoid of* $\mathrm{V}^*(\Lambda)$.

Proof. **(i)**. Let $[P_1], [P_2] \in \mathrm{B}(\Lambda)$. Then, for each $i = 1, 2$, there exists a countably generated non-zero projective right Λ-module Q_i with trace ideal J_i such that $P_i \oplus Q_i^{(\omega)} \cong P_i$, and P_i/P_iJ_i is a finitely generated projective right Λ/J_i-module. Take $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$. Then

$$P \oplus Q^{(\omega)} \cong P_1 \oplus P_2 \oplus Q_1^{(\omega)} \oplus Q_2^{(\omega)} \cong P_1 \oplus Q_1^{(\omega)} \oplus P_2 \oplus Q_2^{(\omega)} \cong P_1 \oplus P_2 = P.$$

By Lemma 1.4.3, $J = \mathrm{Tr}_\Lambda(Q) = \mathrm{Tr}_\Lambda(Q_1) + \mathrm{Tr}_\Lambda(Q_2) = J_1 + J_2$, so

$$P/PJ \cong \frac{P_1}{P_1(J_1+J_2)} \oplus \frac{P_2}{P_2(J_1+J_2)} \cong \frac{P_1/P_1J_1}{P_1(J_1+J_2)/P_1J_1} \oplus \frac{P_2/P_2J_2}{P_2(J_1+J_2)/P_2J_2}$$

and P/PJ is a finitely generated projective right Λ/J-module because it is a direct sum of quotients of finitely generated projective modules. Therefore, as P is not finitely generated, $[P] \in \mathrm{B}(\Lambda)$.

(ii) is proved similarly, taking into account that the modules in $V(\Lambda)$ are the relatively 0-big projective modules. □

Lemma 5.3.3. *Let Λ be a ring, and let $I \in \mathcal{T}(\Lambda)$. Then, there is a semigroup embedding $\iota_I \colon \mathrm{V}(\Lambda/I) \hookrightarrow \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$. The image of ι_I is all the isomorphism classes of relatively I-big projective right Λ-modules.*

Proof. Let Q be a countably generated projective right Λ-module with trace ideal I. Let P' be a finitely generated projective right Λ/I-module. By Lemma 1.5.4, there exists a projective right Λ-module P such that $I \subseteq \mathrm{Tr}_\Lambda(P)$, $P/PI \cong P'$, and $\mathrm{Tr}_\Lambda(P)/I = \mathrm{Tr}_{\Lambda/I}(P')$.

By Lemma 5.1.10, $P \oplus Q^{(\omega)}$ is relatively I-big, and then $[P \oplus Q^{(\omega)}] \in \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$. The isomorphism class of $P \oplus Q^{(\omega)}$ is uniquely determined by I and P'. Hence, there is a well-defined embedding $\iota_I \colon \mathrm{V}(\Lambda/I) \longrightarrow \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ defined by $\iota_I([P']) = [P \oplus Q^{(\omega)}]$.

The map ι_I is a semigroup homomorphism because, for P_1' and P_2' in $\mathrm{V}(\Lambda/I)$:

$$\begin{aligned}\iota_I([P_1' \oplus P_2']) &= [P_1 \oplus P_2 \oplus Q^{(\omega)}] = [P_1 \oplus Q^{(\omega)} \oplus P_2 \oplus Q^{(\omega)}] \\ &= [P_1 \oplus Q^{(\omega)}] + [P_2 \oplus Q^{(\omega)}] = \iota_I([P_1']) + \iota_I([P_2']).\end{aligned}$$

where for $i = 1, 2$, P_i denotes a countably generated projective module such that $P_i/P_i \cong P_i'$, which exists because of Lemma 1.5.4.

Let P be a relatively I-big projective right Λ-module. Then P/PI is a finitely generated projective right Λ/I-module, and $\iota_I([P/PI]) = [P]$. □

If $I \neq 0$, the embedding ι_I of Lemma 5.3.3 is not a monoid embedding because $[0]$ maps to the isomorphism class of a non-finitely generated projective module.

Remark 5.3.4. Consider the set $\mathcal{T}(\Lambda)$ and the disjoint union $\bigsqcup_{I \in \mathcal{T}(\Lambda)} \mathrm{V}(\Lambda/I)$. We will denote the elements of $\bigsqcup_{I \in \mathcal{T}(\Lambda)} \mathrm{V}(\Lambda/I)$ by $([P], I)$ with $[P] \in \mathrm{V}(\Lambda/I)$. Note that if $I \subseteq J$, there is a morphism of monoids $f_{JI} \colon \mathrm{V}(\Lambda/I) \longrightarrow \mathrm{V}(\Lambda/J)$ defined by

$f_{JI}([X], I) = ([X/XJ], J)$ with $[X] \in \mathrm{V}(\Lambda/I)$. Therefore, $f_{JI}(\mathrm{V}(\Lambda/I)) \subseteq \mathrm{V}(\Lambda/J)$. Define addition in $\bigsqcup_{I\in\mathcal{T}(\Lambda)} \mathrm{V}(\Lambda/I)$ by

$$\begin{aligned}([P_1], I) + ([P_2], J) &= f_{I+J,I}([P_1], I) + f_{I+J,J}([P_2], J)\\ &= ([P_1/P_1J], I+J) + ([P_2/P_2I], I+J)\\ &= ([P_1/P_1J \oplus P_2/P_2I], I+J)\end{aligned}$$

for each $[P_1] \in \mathrm{V}(\Lambda/I)$, $[P_2] \in \mathrm{V}(\Lambda/J)$. Therefore, the universal property of the disjoint union and Lemma 5.3.3 imply that there exists a monoid embedding

$$\iota\colon \bigsqcup_{I\in\mathcal{T}(\Lambda)} \mathrm{V}(\Lambda/I) \longrightarrow \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda).$$

The following result states that ι is an isomorphism of monoids.

Corollary 5.3.5 (Theorem 3.8 in [25])**.** *Let Λ be a ring. Then*

(i) $\bigsqcup_{I\in\mathcal{T}(\Lambda)} \mathrm{V}(\Lambda/I) \cong \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ *as monoids.*

(ii) $\bigsqcup_{I\in\mathcal{T}(\Lambda)\setminus\{0\}} \mathrm{V}(\Lambda/I) \cong \mathrm{B}(\Lambda)$ *as semigroups.*

Proof. The isomorphism in **(i)** is given by $\iota([X], I) = \iota_I([X])$, where ι_I is the embedding from Lemma 5.3.3. By Lemma 5.1.7, ι is bijective.

To prove **(ii)**, note that elements in $\mathrm{V}(\Lambda)$ can be thought of as elements in $\mathrm{V}(\Lambda/I)$ for $I = \{0\}$, so **(ii)** follows from **(i)**. □

Remark 5.3.6. Let Λ be a ring. It is well known that the functor $(-)^* = \mathrm{Hom}_\Lambda(-, \Lambda)$ induces a duality between finitely generated projective right Λ-modules and finitely generated projective left Λ-modules. So that, $\varphi\colon \mathrm{V}_r(\Lambda) \longrightarrow \mathrm{V}_\ell(\Lambda)$ defined by $\varphi([P]) = [P^*]$, is an isomorphism of monoids. Moreover, P and P^* have the same trace (cf. Exercise 22.1 in [3]).

In the next example, we show that the canonical monoid isomorphism $\mathrm{V}_r(\Lambda) \cong \mathrm{V}_\ell(\Lambda)$ of Remark 5.3.6 does not, in general, extend to an isomorphism between the monoids $\mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda)$ and $\mathrm{V}_\ell(\Lambda) \sqcup \mathrm{B}_\ell(\Lambda)$. We will show that it does extend, provided that $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$.

Example 5.3.7. In Example 3.6 (iv) of [23] it is given an example of a semilocal ring Λ such that $\Lambda/\mathrm{J}(\Lambda) \cong D_1 \times D_2$, with D_1 and D_2 division rings, such that any projective left Λ-module is free but this is not true on the right. Though, on the right, it is still true that $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$.

In such Λ, what happens is that one of the maximal ideals contains a non-zero trace of a countably generated projective right R-module. Therefore $\mathcal{T}_\ell(\Lambda) = \{0, \Lambda\} \subsetneq \mathcal{T}_r(\Lambda)$. Then $\mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda)$ and $\mathrm{V}_\ell(\Lambda) \sqcup \mathrm{B}_\ell(\Lambda)$ are not isomorphic. This cannot happen if $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$.

Assume Λ is a ring such that $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$. We define a map $\mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda) \longrightarrow \mathrm{V}_\ell(\Lambda) \sqcup \mathrm{B}_\ell(\Lambda)$, that extends the isomorphism between the monoids of finitely generated projective modules of Remark 5.3.6, by $([P], I) \mapsto ([P^*], I)$ for any $([P], I) \in V_r(\Lambda) \sqcup B_r(\Lambda)$. We must check that $([P_1], I) + ([P_2], J)$ is mapped to $([P_1^*], I) + ([P_2^*], J)$.

The following technical lemma is crucial for our discussion.

Lemma 5.3.8. *Let Λ be a ring, and let I, J be ideals of Λ. Let P be a finitely generated projective right Λ/I-module. Then*

$$\mathrm{Hom}_{\Lambda/(I+J)}(P/PJ, \Lambda/(I+J)) \cong \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)/((I+J)/I \cdot \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)).$$

Proof. Note that we can identify P/PJ with $P \otimes_{\Lambda/I} \Lambda/(I+J)$. Then

$$\begin{aligned}\mathrm{Hom}_{\Lambda/(I+J)}(P/PJ, \Lambda/(I+J)) &\cong \mathrm{Hom}_{\Lambda/(I+J)}(P \otimes_{\Lambda/I} \Lambda/(I+J), \Lambda/(I+J)) \\ &\cong \mathrm{Hom}_{\Lambda/I}(P, \Lambda/(I+J)).\end{aligned}$$

where in the second isomorphism, we have used the Hom-tensor adjunction. Now, we want to prove that $\mathrm{Hom}_{\Lambda/I}(P, \Lambda/(I+J)) \cong \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)/J\,\mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)$ as left Λ/I-modules.

Consider the map $\varphi\colon \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I) \longrightarrow \mathrm{Hom}_{\Lambda/I}(P, \Lambda/(I+J))$ defined by $\varphi(f) = \pi \circ f$, where $\pi\colon \Lambda/I \longrightarrow \Lambda/(I+J)$ is the projection modulo $(I+J)/I$. Clearly, it is a homomorphism of left Λ/I-modules. Let $g \in \mathrm{Hom}_{\Lambda/I}(P, \Lambda/(I+J))$. Since P is projective, there is a lift $G \in \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)$ making the following diagram commute

$$\begin{array}{ccccc} & & & & P \\ & & \overset{G}{\swarrow} & & \downarrow{\scriptstyle g} \\ \Lambda/I & \xrightarrow{\pi} & \Lambda/(I+J) & \longrightarrow & 0 \end{array}$$

Therefore, φ is surjective. The kernel of this homomorphism is

$$\begin{aligned}\ker\varphi &= \{f \in \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I) \mid \pi \circ f = 0\} \\ &= \{f \in \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I) \mid f(x) \in (I+J)/I \text{ for every } x \in P\} \\ &= \mathrm{Hom}_{\Lambda/I}(P, (I+J)/I).\end{aligned}$$

It remains to prove that $\mathrm{Hom}_{\Lambda/I}(P, (I+J)/I) = (I+J)/I \cdot \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)$.

Let $f \in \mathrm{Hom}_{\Lambda/I}(P, (I+J)/I)$. Consider $(I+J)/I$ as a right Λ/I-module. Then there exists an epimorphism $\pi\colon (\Lambda/I)^{(A)} \longrightarrow (I+J)/I$ for some set A. Since P is projective, there exists a lift $F \in \mathrm{Hom}_{\Lambda/I}(P, (\Lambda/I)^{(A)})$ such that $\pi \circ F = f$, that is, the following diagram commutes

$$\begin{array}{ccccc} & & & & P \\ & & \overset{F}{\swarrow} & & \downarrow{\scriptstyle f} \\ (\Lambda/I)^{(A)} & \xrightarrow{\pi} & (I+J)/I & \longrightarrow & 0. \end{array}$$

Since P is finitely generated, the image of F is contained in a finite number of copies of Λ/I, namely $(\Lambda/I)^{(A_0)}$, with $A_0 \subseteq A$ finite. Let $\{x_i\}_{i\in A}$ be the canonical basis of $(\Lambda/I)^{(A)}$. Then

$$f(x) = \pi \circ F(x) = \pi\Big(\sum_{i\in A_0} x_i a_i\Big) = \sum_{i\in A_0} \pi(x_i)a_i = \sum_{i\in A_0} \pi(x_i) \cdot \varphi_i(F(x)).$$

where $\varphi_i\colon (\Lambda/I)^{(A_0)} \longrightarrow \Lambda/I$ denotes the projection into the i^{th} component, $i \in A_0$. Hence, $f = \sum_{i\in A_0} \pi(x_i) \cdot (\varphi_i \circ F) \in (I+J)/I \cdot \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)$.

Conversely, let $F = \sum_{i=1}^{n} (y_i + I) \cdot F_i \in (I+J)/I \cdot \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)$ for some $y_1, \ldots, y_n \in J$ and $F_1, \ldots, F_n \in \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)$. Then, regarding the product

of elements as a composition of homomorphisms, $F \in \mathrm{Hom}_{\Lambda/I}(P, (I+J)/I)$. Therefore, by the First Isomorphism Theorem,

$$\mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)/((I+J)/I \cdot \mathrm{Hom}_{\Lambda/I}(P, \Lambda/I)) \cong \mathrm{Hom}_{\Lambda/I}(P, \Lambda/(I+J)),$$

so the statement is proved. □

Corollary 5.3.9. *Let Λ be a ring such that $\mathcal{T}(\Lambda) := \mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$. Then there is an isomorphism of monoids between $\mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda)$ and $\mathrm{V}_\ell(\Lambda) \sqcup \mathrm{B}_\ell(\Lambda)$.*

Proof. Let $\varphi\colon \bigsqcup_{I\in\mathcal{T}(\Lambda)} \mathrm{V}_r(\Lambda/I) \longrightarrow \bigsqcup_{I\in\mathcal{T}(\Lambda)} \mathrm{V}_\ell(\Lambda/I)$ be the map defined by $\varphi([P], I) = ([P^*], I)$. By Remark 5.3.6, it is well-defined and bijective. By Lemma 5.3.8, it is a monoid homomorphism since

$$\begin{aligned}
\varphi(([P_1], I) + ([P_2], J)) &= \varphi([P_1/P_1J \oplus P_2/P_2I], I+J) \\
&= ([\mathrm{Hom}_{\Lambda/(I+J)}(P_1/P_1J \oplus P_2/P_2I, \Lambda/(I+J))], I+J) \\
&= ([\mathrm{Hom}_{\Lambda/(I+J)}(P_1/P_1J, \Lambda/(I+J)) \\
&\qquad \oplus \mathrm{Hom}_{\Lambda/(I+J)}(P_2/P_2I, \Lambda/(I+J))], I+J) \\
&= ([\mathrm{Hom}_{\Lambda/I}(P_1, \Lambda/I)/J\,\mathrm{Hom}_{\Lambda/I}(P_1, \Lambda/I)) \\
&\qquad \oplus \mathrm{Hom}_{\Lambda/J}(P_2, \Lambda/J)/I\,\mathrm{Hom}_{\Lambda/J}(P_2, \Lambda/J))], I+J) \\
&= ([\mathrm{Hom}_{\Lambda/I}(P_1, \Lambda/I)], I) + ([\mathrm{Hom}_{\Lambda/J}(P_2, \Lambda/J)], J) \\
&= \varphi([P_1, I]) + \varphi([P_2], J)
\end{aligned}$$

Finally, as φ is a monoid isomorphism, by Corollary 5.3.5, there is also a monoid isomorphism $V_r(\Lambda) \sqcup B_r(\Lambda) \longrightarrow V_\ell(\Lambda) \sqcup B_\ell(\Lambda)$. □

Then, Theorem 5.2.4, together with the fact that over a noetherian ring Λ, $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$ is just the set of idempotent ideals of Λ (cf. [72]), immediately yields.

Theorem 5.3.10 (Theorem 2.12 in [46])**.** *Let Λ be a noetherian ring satisfying condition $(*)$. Then $\mathrm{V}_r^*(\Lambda) = \mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda) \cong \mathrm{V}_\ell(\Lambda) \sqcup \mathrm{B}_\ell(\Lambda) = \mathrm{V}_\ell^*(\Lambda)$.*

In general, it is an open question whether $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$ for Λ a right noetherian ring (cf. Corollary 6.2.9 and the comments after it for further information regarding this question).

6 Relatively big projective modules

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In this chapter, we want to enlarge the classes of rings Λ for which every countably generated projective module is relatively big. In this chapter, we prove Theorem D stated in the introduction:

Theorem D (Corollaries 6.2.9 and 6.2.10, Theorem 6.3.17)**.** *Let Λ be a ring. Then $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$ in the following situations:*

(a) *Λ is a right noetherian PI-ring, in particular, if it is a module-finite algebra over a commutative noetherian ring,*

(b) *Λ is a locally semiperfect torsion-free algebra over an h-local domain with field of fractions Q and such that Λ_Q is simple artinian.*

In Section 6.1 we show that countably generated projective right modules lift modulo a right T-nilpotent ideal, cf. Lemma 6.1.2. We prove that there is a monoid isomorphism between $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ and $\mathrm{V}(\Lambda/N) \sqcup \mathrm{B}(\Lambda/N)$ (cf. Corollary 6.1.4). We apply those results in Section 6.2 to prove **(a)** in Theorem D.

In Section 6.3, we turn our attention to the structure of projective modules over semiperfect rings and prove that there is only a finite number of trace ideals over these rings. Then, we study locally semiperfect algebras over h-local domains with simple artinian ring of quotients and prove **(b)** in Theorem D.

6.1 Lifting projectives modules modulo a right T-nilpotent ideal

Definition 6.1.1. Let Λ be a ring. A subset A of Λ is called *right T-nilpotent* if, for any sequence of elements $\{a_1, a_2, \dots\} \subseteq A$, there exists a positive integer $n \geq 1$ such that $a_n \cdots a_2 a_1 = 0$.

Lemma 6.1.2. *Let Λ be a ring, and let N be a right T-nilpotent ideal of Λ.*

(i) *Let X be a right projective Λ/N-module. Then there exists a projective right Λ-module P such that $P/PN \cong X$. All projective modules with this property are isomorphic. In addition, X is finitely generated (countably generated) if and only if P is finitely generated (countably generated).*

(ii) *Let P and Q be projective right Λ-modules. Let $f\colon P \longrightarrow Q$ be a morphism of Λ-modules. Then f is an isomorphism if and only if the induced map $P/PN \longrightarrow Q/QN$ is an isomorphism of Λ/N-modules.*

(iii) *Let I be an idempotent ideal of Λ such that $(I+N)/N$ is the trace of a countably generated projective right Λ/N-module X. Then I is the trace of any countably generated projective right Λ-module P such that $P/PN \cong X$.*

Proof. **(i)** Let X be a projective right Λ/N-module. As X is a direct sum of countably generated projective right Λ/N-modules, we may assume that X is countably generated. In this case, X is the direct limit of a direct system

$$\overline{F}_1 \longrightarrow \overline{F}_2 \longrightarrow \cdots \longrightarrow \overline{F}_n \longrightarrow \cdots \tag{6.1}$$

of finitely generated free Λ/N-modules, and the canonical presentation of the direct limit

$$0 \longrightarrow \bigoplus_{n\geq 1} \overline{F}_n \longrightarrow \bigoplus_{n\geq 1} \overline{F}_n \longrightarrow X \longrightarrow 0 \tag{6.2}$$

splits. We can lift (6.1) to a countable direct system of finitely generated free Λ-modules

$$F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_n \longrightarrow \cdots \tag{6.3}$$

such that when we apply the functor $-\otimes_\Lambda \Lambda/N$ to (6.3) we obtain a direct system isomorphic to (6.1).

Let P be the direct limit of (6.3), and consider its canonical presentation

$$0 \longrightarrow \bigoplus_{n\geq 1} F_n \xrightarrow{\Phi} \bigoplus_{n\geq 1} F_n \longrightarrow P \longrightarrow 0$$

As (6.2) splits, there exists $\Psi\colon \bigoplus_{n\geq 1} F_n \longrightarrow \bigoplus_{n\geq 1} F_n$ such that $\Psi\circ\Phi = \mathrm{Id} + B$ where B can be thought of as a column finite matrix with entries in N (that is, $B\colon \bigoplus_{n\geq 1} F_n \longrightarrow \bigoplus_{n\geq 1} F_n$ has its image contained in $\bigoplus_{n\geq 1} F_n N$). Since N is right T-nilpotent, B is in the Jacobson radical of the ring of $\mathbb{N}\times\mathbb{N}$ column-finite matrices with coefficients in Λ [62], and therefore, $\mathrm{Id}+B$ is invertible. So that $(\mathrm{Id}+B)^{-1}\circ\Psi$ is a left inverse of Φ, and we can deduce that P is projective.

This shows that any lifting of the direct system (6.1) gives a projective Λ-module P that, in addition, satisfies $P/PN \cong X$. As PN is a small submodule of P by Lemma 28.3 in [3], $P/PN \cong X$ is finitely generated (countably generated) if and only if P is also finitely generated (countably generated).

The uniqueness, up to isomorphism, of P follows from **(ii)**.

(ii) If $f\colon P \longrightarrow Q$ is an isomorphism of Λ-modules, then it has an inverse homomorphism $g\colon Q \longrightarrow P$. Let $\overline{f}\colon P/PN \longrightarrow Q/QN$ and $\overline{g}\colon Q/QN \longrightarrow P/PN$ be the induced maps. Then $\overline{g}$ is the inverse of $\overline{f}$, so both are also invertible morphisms of Λ/N-modules.

Conversely, if $\alpha\colon P/PN \longrightarrow Q/QN$ is an isomorphism of Λ/N-modules. As PN is a small submodule of P by [3, Lemma 28.3], P and Q are projective covers of P/PN. Therefore, any lifting of α to a morphism $P \longrightarrow Q$ is an isomorphism.

(iii) By **(i)**, the module X can be lifted to a countably generated projective right Λ-module P such that $P/PN \cong X$. Let J be the trace of P. Then, by Lemma 1.4.5, $J + N = I + N$. By Corollary 2.9 in [24], $J \subseteq I$ because J is a trace ideal of a projective module, and N, being right T-nilpotent, is contained in the Jacobson radical of Λ. Since $J + N = I + N$ and $I^2 = I$, $IJ + IN = I$. By Lemma 28.3 in [3], $IN \ll I$, which implies $IJ = I$ and, in particular, $I \subseteq J$. Therefore, $I = J$ is the trace of P. □

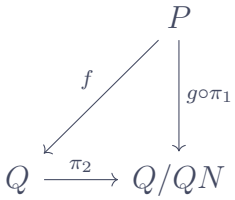
Lemma 6.1.3. *Let Λ be a ring, and let N be a right T-nilpotent ideal of Λ. Let I be an ideal of Λ, and let P be a countably generated right Λ-module. Then, P is I-big if and only if P/PN is $(I + N)/N$-big.*

Proof. Assume that P is I-big. Let X be a countably generated projective right Λ/N-module such that its trace is contained in $(I+N)/N$. By Lemma 6.1.2, there exists a countably generated projective right Λ-module Q such that $Q/QN \cong X$. Therefore, its trace J is contained in $I + N$. Since J is the trace of a countably generated projective, and N is right T-nilpotent, we deduce from Corollary 2.9 in [24] that $J \subseteq I$. By hypothesis, there exists an onto module homomorphism $f\colon P \longrightarrow Q$. Therefore, there is an onto module homomorphism

$$g\colon P \xrightarrow{f} Q \xrightarrow{\pi} Q/QN \cong X$$

where π denotes the canonical projection. As PN is contained in the kernel of g, the homomorphism g factors through an onto homomorphism $P/PN \longrightarrow X$. This shows that P/PN is $(I + N)/N$-big.

To prove the converse, let Q be a countably generated projective right Λ-module such that its trace is contained in I. By Lemma 1.4.5, the trace of Q/QN is contained in $(I + N)/N$, and since P/PN is $(I + N)/N$-big, there exists an onto module homomorphism $g\colon P/PN \longrightarrow Q/QN$. Since P is projective, there exists $f\colon P \longrightarrow Q$ such that the diagram



is commutative, and where $\pi_1\colon P \longrightarrow P/PN$ and $\pi_2\colon Q \longrightarrow Q/QN$ denote canonical projections. Therefore, $Q = f(P) + QN$. By Lemma 28.3 in [3], QN is a small submodule of Q, and we deduce that $f(P) = Q$. This shows that P is I-big. □

Corollary 6.1.4. *Let Λ be a ring, and let N be a right T-nilpotent ideal of Λ. Then the functor $- \otimes_\Lambda \Lambda/N$ induces a monoid isomorphism $\varphi\colon \mathrm{V}^*(\Lambda) \longrightarrow \mathrm{V}^*(\Lambda/N)$ which restricts to monoid isomorphisms $\mathrm{V}(\Lambda) \longrightarrow \mathrm{V}(\Lambda/N)$ and $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) \longrightarrow \mathrm{V}(\Lambda/N) \sqcup \mathrm{B}(\Lambda/N)$.*

Proof. Consider the monoid homomorphism $\varphi\colon \mathrm{V}^*(\Lambda) \longrightarrow \mathrm{V}^*(\Lambda/N)$ defined by $\varphi([P]) = [P/PN]$, where P denotes a countably generated projective right Λ-module. It follows from Lemma 6.1.2 that φ is an isomorphism. More precisely, φ is injective by the claim **(ii)** of Lemma 6.1.2, and it is onto by claim **(i)**. Claim **(i)** of Lemma 6.1.2 also implies that φ restricts to an isomorphism $V(\Lambda) \longrightarrow V(\Lambda/N)$.

It remains to prove that φ restricts to an isomorphism $\mathrm{B}(\Lambda) \longrightarrow \mathrm{B}(\Lambda/N)$.

Let P be a countably generated projective Λ-module such that $[P] \in B(\Lambda)$. That is, there exists I, the trace of a non-zero countably generated projective module Q, such that P is I-big and P/PI is finitely generated.

By Lemma 1.4.5, $(I+N)/N$ is the trace of Q/QN. Notice also that $(I+N)/N \neq \{0\}$. By Lemma 6.1.3, P/PN is $(I+N)/N$-big. Clearly,

$$(P/PN)/(P(I+N)/PN) \cong P/P(I+N)$$

is finitely generated. Hence, $\varphi([P]) \in B(\Lambda/N)$.

Let X be a countably generated projective right Λ/N-module such that $[X] \in \mathrm{B}(\Lambda/N)$. This means that there exists a non-zero countably generated projective right Λ/N-module with trace ideal K, such that X is K-big and X/XK is finitely generated.

By Lemma 1.5.5, there exist countably generated projective Λ-modules P and Q such that $P/PN \cong X$ and $Q/QN \cong Y$. Let I be the trace of Q. By Lemma 1.4.5, $K = (I+N)/N$.

By Lemma 6.1.3, P is I-big. Moreover, P/PI is finitely generated because $P/P(I+N)$ is also finitely generated. This shows that $[P] = \varphi^{-1}([X]) \in \mathrm{B}(\Lambda)$.

This concludes the proof that φ induces an isomorphism $\mathrm{B}(\Lambda) \longrightarrow \mathrm{B}(\Lambda/N)$. □

6.2 The case of right noetherian PI-rings

In this section, we shall prove that countably generated projective right or left modules over a right noetherian PI-ring are always relatively big. The key result will be to show that these rings satisfy the $(*)$-condition introduced in Section 5.2. Since a right noetherian PI-ring R has nilpotent nilradical $\mathrm{N}(R)$, and $R/\mathrm{N}(R)$ is two-sided noetherian, the result will follow by combining Theorem 5.2.4 with the results in Section 6.1.

First, we consider the case of commutative noetherian rings. P. Příhoda and G. Puninski proved in [49] that commutative noetherian rings satisfy the $(*)$-condition. Here, we provide another proof of this result, which we find interesting as it is quite different from the one given in [49].

6.2.1 Commutative noetherian rings

If I is a countably generated left ideal of a ring Λ, I is the union of an ascending sequence of the type (6.5) in Lemma 6.2.1 if and only if I is a pure left ideal (cf. Lemme 2 in [32]).

Over a commutative ring, an ideal is the trace of a projective module if and only if it is pure (see Remark 1.4.6). Since the trace ideal of a countably generated projective module is countably generated, an ideal is the trace of a countably

generated projective module if and only if it is the union of such an ascending sequence of the type (6.5) in Lemma 6.2.1.

Lemma 6.2.1. *Let Λ be a ring. Let $(a_k)_{k\geq 1}$ be a sequence of elements of Λ satisfying $a_k a_{k+1} = a_{k+1}$ for any $k \geq 1$. Set $b_k = 1 - a_k$ for any $k \geq 1$. Then,*

(i) *There is a descending sequence of principal right ideals*

$$a_1\Lambda \supseteq a_2\Lambda \supseteq \cdots \supseteq a_k\Lambda \supseteq \cdots \tag{6.4}$$

(ii) *$b_k b_{k+1} = b_k$ for any $k \geq 1$, so there is an ascending sequence of principal left ideals*

$$\Lambda b_1 \subseteq \Lambda b_2 \subseteq \cdots \subseteq \Lambda b_k \subseteq \cdots \tag{6.5}$$

(iii) *For any $k \geq 1$,*

$$\Lambda b_k \subseteq \mathrm{l.ann}(a_{k+1}) \subseteq \Lambda b_{k+1},$$

and

$$a_k\Lambda \supseteq \mathrm{r.ann}(b_k) \supseteq a_{k+1}\Lambda.$$

(iv) *$I = \bigcup_{k\geq 1} \Lambda b_k$ is a left pure ideal of R and it is also projective.*

(v) *The sequence* (6.4) *is stationary if and only if so is the sequence* (6.5)*. In this case, there exists k_0 such that b_{k_0} and a_{k_0} are idempotent, $I = \Lambda b_{k_0}$ and $\bigcap_{k\geq 1} a_k\Lambda = a_{k_0}\Lambda$.*

Proof. Statements **(i)**, **(ii)**, and **(iii)** follow easily from direct computation.

The purity claimed in statement **(iv)** was proved by Lazard in Lemme 2 in [32]. Then, as Λ/I is a countably presented flat left Λ-module, I is projective. For a self-contained proof, the reader can check Lemma 2 in [16].

Statement **(iii)** readily implies that (6.4) is stationary if and only if (6.5) is also stationary.

If $a_k\Lambda = a_{k+1}\Lambda$, then there exists $\lambda \in \Lambda$ such that $a_{k+1}\lambda = a_k$. Hence, $a_{k+1}\lambda a_{k+1} = a_{k+1}$, which implies that $a_k = a_{k+1}\lambda$ is idempotent.

If $\Lambda b_k = \Lambda b_{k+1}$, then there exists $\mu \in \Lambda$ such that $\mu b_k = b_{k+1}$. As before, this implies that b_{k+1} is idempotent.

Assume the sequences (6.4) and (6.5) are stationary. Then the argument above shows that, taking k_0 large enough, the second part of statement **(v)** holds. □

Lemma 6.2.2. *Let R be a commutative ring. Let $I \supseteq J$ be ideals of R such that $JI = J$. If J is finitely generated, then there exists $a \in I$ such that $xa = x$ for any $x \in J$. In particular $I \supseteq aR \supseteq J$.*

Proof. Let $J = x_1R + \cdots + x_nR$. The equality $JI = J$ implies that there exists $A \in \mathrm{M}_n(I)$ such that $(x_1, \ldots, x_n)A = (x_1, \ldots, x_n)$, or equivalently

$$(x_1, \ldots, x_n)(\mathrm{Id} - A) = (0, \ldots, 0).$$

Multiplying the above identity on the right by the adjoint matrix of $\mathrm{Id} - A$, we deduce that $\det(A)$ annihilates J. To conclude the proof, note that there exists $a \in I$ such that $\det(A) = 1 - a$. □

Proposition 6.2.3 (Lemma 2.1 in [49]). *Every commutative noetherian ring satisfies the $(*)$-condition. This is to say that if R is a commutative noetherian ring and*

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k \supseteq \cdots \tag{6.6}$$

is a descending chain of ideals of R satisfying $J_{k+1}J_k = J_{k+1}$ for each $k \geq 1$, then (6.6) *is stationary.*

In addition, $\bigcap_{k\geq 1} J_k = eR$ where e is an idempotent of R.

Proof. Since R is noetherian, the ideals J_k are finitely generated for every $k \geq 1$. By Lemma 6.2.2, there exists a sequence of elements $(a_k)_{k\geq 1}$ in R such that

$$J_1 \supseteq a_1R \supseteq J_2 \supseteq a_2R \supseteq \cdots \supseteq J_k \supseteq a_kR \supseteq \cdots$$

and such that $a_{k+1}a_k = a_{k+1}$.

By Lemma 6.2.1 and because R is noetherian, the ascending chain of ideals

$$(1-a_1)R \subseteq (1-a_2)R \subseteq \cdots \subseteq (1-a_k)R \subseteq \cdots$$

is stationary and its union is generated by an idempotent of R. By Lemma 6.2.1, the descending sequence

$$a_1R \supseteq a_2R \supseteq \cdots \supseteq a_kR \supseteq \cdots$$

is also stationary, and its intersection is generated by an idempotent of R. The sequence (6.6) is also stationary, and the intersection of these ideals is generated by an idempotent of R. □

In the following result, we recover a result of Bass that shows that, over commutative noetherian rings, the theory of countably generated projective modules reduces to that of finitely generated ones.

Corollary 6.2.4 ([6]). *Let R be a commutative noetherian ring. Let P be a countably generated projective module that is not finitely generated. Then there exists an idempotent $e \in R$ with $P \cong (1-e)P \oplus (eR)^{(\omega)}$ such that $(1-e)P$ is finitely generated.*

Proof. Let P be a countably generated projective module over R. By Proposition 6.2.3, Lemma 5.2.1, and Theorem 5.2.4, there exists an idempotent $e \in R$ such that P/eP is finitely generated, $I = eR$ is an ideal minimal with respect to that property, and P is I-big. Therefore $P \cong P \oplus (eR)^{(\omega)}$.

Since R is commutative

$$P = eP \oplus (1-e)P \cong eP \oplus (1-e)P \oplus (eR)^{(\omega)} \cong (1-e)P \oplus (eR)^{(\omega)},$$

where the last isomorphism follows from the Eilenberg's trick. To conclude the proof of the statement, observe that $(1-e)P \cong P/eP$. □

Corollary 6.2.4 implies that a countably generated projective module over a commutative noetherian domain that is not finitely generated is free.

Not much is known about the structure of infinitely generated projective modules over general commutative domains. To our knowledge, the basic questions

on the topic posed on page 246 of [18] are still widely open, and it seems that to answer them, one needs to go beyond the theory of relatively big projectives.

In general, over a commutative domain R, the only trace ideals are $\{0\}$ and R, so $\mathrm{B}(R)$ contains only the isomorphism class of $R^{(\omega)}$. Therefore, $\mathrm{V}(R) \sqcup \mathrm{B}(R) = \mathrm{V}(R) \sqcup \{[R^{(\omega)}]\}$. We do not know of any reasonable characterization of domains R such that $\mathrm{V}^*(R) = \mathrm{V}(R) \sqcup \mathrm{B}(R)$, nor are there any examples where this fails.

6.2.2 One-sided noetherian PI-rings

In this subsection, we show that countably generated projective right or left modules over any right noetherian, PI-ring Λ are relatively big.

The big difference between a noetherian PI-ring Λ and the commutative case described in Corollary 6.2.4 is that the idempotent ideals are not just the ideals generated by idempotent elements of Λ.

By a result of Small and Robson [64], the set of idempotent ideals of Λ (recall that it coincides with $\mathcal{T}(\Lambda)$) is finite. But the results from [48] and [25] provide plenty of examples showing that, even in the case where Λ is a finitely generated algebra over a commutative noetherian ring, $\mathcal{T}(\Lambda)$ frequently contains elements that are not traces of finitely generated projective modules.

Lemma 6.2.5. *Let Λ be a ring. Let $I \supseteq J$ be ideals of Λ such that either $JI = J$ or $IJ = J$. Let $K_1, \dots, K_n$ be ideals of Λ (not necessarily different) such that $K_1 \cdots K_n \subseteq J$ and $J + K_i = I + K_i$, for $i = 1, \dots, n$. Then $J = J^2$.*

Proof. Using the hypothesis, we deduce that,

$$I^n \subseteq \prod_{i=1}^{n} (I + K_i) = \prod_{i=1}^{n} (J + K_i) \subseteq J.$$

If $JI = J$, then $JI^n = J$. Since $I^n \subseteq J$, we deduce that $J \subseteq J^2$. This shows that $J = J^2$.

A symmetric argument gives the conclusion if $IJ = J$. □

Proposition 6.2.6. *Let Λ be a right noetherian ring. Let*

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots \tag{6.7}$$

be a sequence of ideals of Λ satisfying that $I_{k+1} I_k = I_{k+1}$ for any $k \geq 1$. Let

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k \supseteq \cdots \tag{6.8}$$

be a sequence of ideals of Λ satisfying that $J_k J_{k+1} = J_{k+1}$ for any $k \geq 1$. Then,

- **(i)** *Let $\mathrm{N}(\Lambda)$ denote the prime radical of Λ (that is, the intersection of all prime ideals of Λ). If (6.7) or (6.8) are stationary in $\Lambda/\mathrm{N}(\Lambda)$, then they are also stationary over Λ.*
- **(ii)** *Assume that Λ is semiprime, and that (6.7) is stationary over Λ/P for any minimal prime ideal P of Λ. Then there exists $k_0 \geq 1$ such that $I_{k_0+\ell}^2 = I_{k_0+\ell}$ for any $\ell \geq 0$.*

(iii) *Assume that* Λ *is semiprime, and that* (6.8) *is stationary over* Λ/P *for any minimal prime ideal* P *of* Λ. *Then there exists* $k_0 \geq 1$ *such that* $J_{k_0+\ell}^2 = J_{k_0+\ell}$ *for any* $\ell \geq 0$.

Proof. **(i)** By Theorem 3.11 in [20], $\mathrm{N}(\Lambda)$ is nilpotent. Assume that (6.7) is stationary over $\Lambda/\mathrm{N}(\Lambda)$. Then there exists $k_1 \geq 0$ such that $I_{k_1} + \mathrm{N}(\Lambda) = I_{k_1+\ell} + \mathrm{N}(\Lambda)$ for any $\ell \geq 0$. By Lemma 6.2.5, this implies that $I_{k_1+1+\ell}^2 = I_{k_1+1+\ell}$ for any $\ell \geq 0$. Since idempotent ideals that are equal modulo a nilpotent ideal must be equal, the claim follows.

A similar argument works if (6.8) is stationary.

(ii) Since Λ is a semiprime right Goldie ring, it has a finite number of minimal prime ideals $P_1, \ldots, P_n$, and its product is zero, see Proposition 7.1 in [20].

By hypothesis, there exists k_1 such that $I_{k_1+\ell} + P_i = I_{k_1} + P_i$ for any $\ell \geq 0$ and any $i = 1, \ldots, n$.

As in **(i)**, we use Lemma 6.2.5 to deduce that $k_0 = k_1 + 1$ satisfies the conclusion of the statement.

Statement **(iii)** is proved in a similar way. □

Robson and Small proved in [64] that PI-rings satisfying the ascending chain condition on two-sided ideals have a finite number of idempotent ideals.

We shall prove in Theorem 6.2.8 that, in addition, right noetherian PI-rings satisfy the $(*)$-condition and its left analog, which we call $(**)$. The proof is patterned after that of Robson and Small.

First, we provide some basic definitions needed to follow the proof.

Definition 6.2.7. A ring Λ is said to be a *polynomial identity ring*, a *PI-ring* for short, if there exists n such that Λ satisfies a monic polynomial of $\mathbf{Z}\langle x_1, \ldots, x_n\rangle$. That is, there exists a polynomial in non-commuting variables $x_1, \ldots, x_n$ with coefficients in $\mathbf{Z}$, say $f(x_1, \ldots, x_n)$, such that:

(1) $f(r_1, \ldots, r_n) = 0$ for any $r_1, \ldots, r_n \in \Lambda$,

(2) the coefficient of one of the monomials of the highest degree of f is one.

The polynomials in $\mathbf{Z}\langle x_1, \ldots, x_n\rangle$ of the form $\sum_{\sigma\in\Sigma_n} a(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$, where Σ_n denotes the permutation group of n elements and $a(\sigma) \in \mathbf{Z}$, are said to be multilinear polynomials or multilinear identities of degree n.

Of special importance in the theory is the *standard identity* of degree n

$$s_n = s_n(x_1, \ldots, x_n) = \sum_{\sigma\in\Sigma_n} \varepsilon(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

where $\varepsilon(\sigma)$ denotes the sign of the permutation σ. For example, if R is a non-zero commutative ring, a theorem of Amitsur and Levitzki shows that $M_n(R)$ satisfies the standard identity s_m for any $m \geq 2n$ and does not satisfy s_{2n-1}, cf. Section 13.3.2 in [40]. Moreover, if a PI-ring satisfies s_m, then it satisfies s_t for any $t \geq m$, cf. Section 13.3.3 in [40].

A theorem of Kaplansky shows that if a ring satisfies a monic identity of degree at most d, then it satisfies a monic multilinear identity of degree at most d (cf. Section 13.1.9 in [40]).

It is possible to define the *PI-degree* of a central simple algebra Λ (that is, a simple algebra that is finite dimensional over its center) as the minimum n such that Λ satisfies s_{2n} and not s_{2n-1}. In this case, we write PI-deg$(\Lambda) = n$.

Prime PI-rings Λ localized at the non-zero elements of the center are central simple algebras. Then, by definition, the PI-degree of Λ is the PI-degree of this localization. For details, check Section 13.6.7 in [40].

Theorem 6.2.8. *Let Λ be a right noetherian PI-ring. Then Λ satisfies the $(*)$-condition and its symmetric $(**)$.*

Proof. Assume the statement is not true. By noetherian induction, we may assume that Λ is a right noetherian PI-ring that does not satisfy the $(*)$-condition, but so does any proper quotient ring of Λ.

By Theorem 4 in [64], Λ has only a finite number of idempotent ideals. By Proposition 6.2.6, this implies that Λ is a prime, right noetherian, PI-ring. Therefore, Λ is (right and left) noetherian by a theorem of G. Cauchon (Theorem 13.6.15 in [40]). Now we adapt the steps of the proof of Theorem 4 in [64] to obtain a contradiction.

Let

$$\Lambda \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_k \supsetneq \cdots \tag{6.9}$$

be a sequence of ideals of Λ satisfying that $I_{k+1}I_k = I_{k+1}$ for any $k \geq 1$. Let P be a prime ideal containing I_k for some k and assume that PI-deg $(\Lambda/P) =$ PI-deg (Λ). By [63, Theorem], the set of elements that are regular modulo P consists of regular elements of Λ, forms an Ore set S, and the localization ΛS^{-1} is a local ring with maximal ideal $P\Lambda S^{-1}$. Since I_k is contained in P, so is $I_{k+\ell}$ for any $\ell \geq 1$. But in the local ring ΛS^{-1}, $P\Lambda S^{-1} I_{k+\ell}\Lambda S^{-1} = I_{k+\ell}\Lambda S^{-1}$ for any $\ell \geq 1$. As the ring is noetherian, Nakayama's lemma implies that $I_{k+\ell}\Lambda S^{-1} = \{0\}$ for any $\ell \geq 1$. Therefore, $I_{k+\ell} = \{0\}$ for any $\ell \geq 1$. This contradicts our hypothesis. Therefore, PI-deg $(\Lambda/P) <$ PI-deg (Λ) for any prime ideal P containing I_k for some k.

Let $n =$ PI-deg (Λ). Let $E(\Lambda)$ be the non-zero ideal generated by the evaluations in Λ of s_{2n-2} (cf. [40, 13.6.7 Corollary]). For any fixed k, and for any prime ideal P containing I_k, the ideal $E(\Lambda)$ is contained in P because PI-deg $(\Lambda/P) <$ PI-deg (Λ). Thus, Λ/P must satisfy s_{2n-2}. Then, $E(\Lambda)$ is contained in the prime radical of I_k. Therefore, for any k, there exists m_k such that $E(\Lambda)^{m_k} \subseteq I_k$ (recall that the prime radical of Λ/I_k is nilpotent by [20, Theorem 3.11]).

By hypothesis, $\Lambda/E(\Lambda)$ satisfies the $(*)$-condition. This implies that there exists k_0 such that $I_{k_0+\ell} + E(\Lambda) = I_{k_0} + E(\Lambda)$ for any $\ell \geq 0$. By Lemma 6.2.5, we deduce that $I_{k_0+\ell+1}$ is an idempotent ideal for any $\ell \geq 0$. As a consequence, the sequence (6.9) contains infinitely many different idempotent ideals, which gives the desired contradiction with the statement of [64, Theorem 4]. This concludes the proof of the statement. □

Corollary 6.2.9. *Let Λ be a right noetherian PI-ring. Then:*

- **(i)** *Every idempotent ideal of Λ is the trace of a countably generated projective right Λ-module and the trace of a countably generated projective left Λ-module.*
- **(ii)** $\mathrm{V}_r^*(\Lambda) = \mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda) \cong \mathrm{V}_\ell(\Lambda) \sqcup \mathrm{B}_\ell(\Lambda) = \mathrm{V}_\ell^*(\Lambda)$.

Proof. Statement **(i)** follows from the fact that $\Lambda/\mathrm{N}(\Lambda)$, where $\mathrm{N}(\Lambda)$ denotes the prime radical of Λ, is two-sided noetherian, see 13.6.15 in [40], and that $\mathrm{N}(\Lambda)$ is nilpotent. Therefore, if I is an idempotent ideal of Λ, then $(I + \mathrm{N}(\Lambda))/\mathrm{N}(\Lambda)$ is an idempotent ideal of $\Lambda/\mathrm{N}(\Lambda)$, so it is the trace of a countably generated projective right $\Lambda/\mathrm{N}(\Lambda)$-module and the trace of a countably generated projective left $\Lambda/\mathrm{N}(\Lambda)$-module (cf. [72]). By Lemma 6.1.2, $I \in \mathcal{T}_r(\Lambda)$ and $I \in \mathcal{T}_\ell(\Lambda)$.

By Theorem 6.2.8, Λ and Λ/N satisfy the $(*)$-condition and the symmetric $(**)$-condition. By Theorem 5.3.10, $\mathrm{V}^*(\Lambda/\mathrm{N}(\Lambda)) = \mathrm{V}(\Lambda/\mathrm{N}(\Lambda)) \sqcup \mathrm{B}(\Lambda/\mathrm{N}(\Lambda))$ and from Corollary 6.1.4 we deduce that $\mathrm{V}^*_r(\Lambda) = \mathrm{V}_r(\Lambda) \sqcup \mathrm{B}_r(\Lambda)$ and $\mathrm{V}^*_\ell(\Lambda) = \mathrm{V}_\ell(\Lambda) \sqcup B_\ell(\Lambda)$.

Since, by **(i)**, $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$, the rest of the statement follows from Corollary 5.3.9. □

It is an interesting open question to determine whether there exists a right noetherian ring Λ with an idempotent ideal that is the trace of a (countably generated) projective left Λ-module but not the trace of a projective right Λ-module. Corollary 6.2.9 shows that if such an example exists, it cannot be a PI-ring.

The following corollary was our initial motivation for developing these results. As it shows, the basics of the machinery developed in [25] for the endomorphism ring of finitely generated modules over commutative noetherian rings of Krull dimension 1 are also available in arbitrary Krull dimension. See also Section 6.2.3 for the transfer of information between direct sum decompositions of a finitely generated module and a projective module over its endomorphism ring.

Corollary 6.2.10. *Let R be a commutative noetherian ring, and let Λ be a module-finite R-algebra. Then Λ satisfies the $(*)$-condition and its symmetric $(**)$-condition. Moreover, it has only a finite number of idempotent ideals, and projective modules satisfy the conclusions of Corollary 6.2.9.*

Proof. If Λ is generated by n-elements as an R-module, then Λ is a sub-quotient of $\mathrm{M}_n(R)$, so it is PI and noetherian (cf. Corollary 13.4.9 in [40]). Then, Theorem 6.2.8 implies that Λ satisfies the $(*)$-condition.

The statement about idempotent ideals is a consequence of Theorem 4 in [64]. □

Even for a two-sided noetherian ring Λ, there may be a strict inclusion $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) \subsetneq \mathrm{V}^*(\Lambda)$. See, for example, the last two sections of [24].

6.2.3 An application to direct sums of finitely generated modules

Corollary 6.2.11. *Let R be a commutative noetherian ring, and let Λ be a module-finite R-algebra. Let M be a finitely generated right Λ-module. Then any indecomposable element of* $\mathrm{Add}(M)$ *is finitely generated.*

Proof. As $\mathrm{End}_\Lambda(M)$ is a module-finite R-algebra, we can apply Corollary 6.2.10 to deduce that all countably generated projective modules over $\mathrm{End}_\Lambda(M)$ are relatively big. Hence, by Remark 5.1.11, the indecomposable projectives over $\mathrm{End}_\Lambda(M)$ are finitely generated. The conclusion follows from Proposition 1.2.12. □

Remark 6.2.12. In general, if Λ is a PI-ring, then the endomorphism ring of a finitely generated right Λ-module is also a PI-ring (see the argument of Theorem 2 in [51]).

6.3 Semiperfect rings and locally semiperfect rings

We start this section by describing the monoids of projective modules for semiperfect rings. We follow the notation and conventions of Section 27 in [3].

6.3.1 Trace ideals of projective modules over semiperfect rings

Lemma 6.3.1. *Let Λ be a semiperfect ring, and let I be an ideal of Λ. Let P' be a finitely generated, projective right Λ/I-module. Then there exists a finitely generated projective right Λ-module X such that $X/XI \cong P'$.*

Proof. Since Λ is semiperfect, P' has a projective cover. That is, it fits in an exact sequence

$$0 \longrightarrow \ker f \longrightarrow X \xrightarrow{f} P' \longrightarrow 0,$$

where X is a finitely generated projective right Λ-module, and $\ker f$ is a small submodule of X. Therefore, there are inclusions

$$XI \subseteq \ker f \subseteq XJ(\Lambda)$$

which induce an exact sequence of Λ/I-modules

$$0 \longrightarrow \ker f/XI \longrightarrow X/XI \xrightarrow{f'} P' \longrightarrow 0.$$

which splits because P' is projective. Since $\ker f/XI$ is also a small submodule of X/XI, it must be zero. Therefore $X/XI \cong P'$. □

In the next proposition, we recall the behavior of traces of projective modules for semiperfect rings.

Proposition 6.3.2. *Let Λ be a semiperfect ring, and let $J(\Lambda)$ denote its Jacobson radical. Then:*

- **(i)** *$\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda) := \mathcal{T}(\Lambda)$, and the injective map $\varphi\colon \mathcal{T}(\Lambda) \longrightarrow \mathcal{T}(\Lambda/J(\Lambda))$ defined in Lemma 1.5.1 is bijective.*
- **(ii)** *Let $e_1, \ldots, e_t$ be a basic set of (primitive orthogonal) idempotents of Λ. Let $\mathcal{P}(\{1, \ldots, t\})$ denote the set of all subsets of $\{1, \ldots, t\}$. Then the map*

$$\Gamma\colon \mathcal{P}(\{1, \ldots, t\}) \longrightarrow \mathcal{T}(\Lambda)$$

defined by $\Gamma(A) = \sum_{i \in A} \Lambda e_i \Lambda$, for any $A \in \mathcal{P}(\{1, \ldots, t\})$, is a bijection.

(iii) *Let P be a projective right Λ-module with trace I. Assume that $I/IJ(\Lambda) \cong \sum_{i\in A}((e_i + J(\Lambda))\Lambda/J(\Lambda))^{\ell_i}$ for suitable $\ell_i > 0$ and $A \in \mathcal{P}(\{1,\dots,t\})$. Then $P \cong \bigoplus_{i\in A}(e_i\Lambda)^{(L_i)}$, for suitable non-empty sets, L_i.*

(iv) *Keeping the notation from* **(ii)** *and* **(iii)**, *$\Lambda' := \Lambda/I$ is a semiperfect ring and $e_i' := e_i + I, i \notin A$ is a basic set of idempotents of Λ/I. Moreover, let $P' \cong \bigoplus_{i\notin A}(e_i'\Lambda')^{x_i}$ be a finitely generated projective Λ'-module and let P be a finitely generated projective Λ-module such that $P/PI \cong P'$. Then there are non-negative integers $s_i, i \notin A$ such that $P \cong \bigoplus_{i\notin A}(e_i\Lambda)^{x_i} \oplus \bigoplus_{i\in A}(e_i\Lambda)^{s_i}$.*

Proof. The proof of the four statements is more or less routine. One just has to take into account that any projective right (left) Λ-module is isomorphic to a unique direct sum of the indecomposable projective modules $P_i = e_i\Lambda$ ($Q_i = \Lambda e_i$), for $i = 1,\dots,t$ (see, for example, Theorem 3.10 in [14]). Moreover $\mathrm{Tr}_\Lambda(P_i) = \mathrm{Tr}_\Lambda(Q_i) = \Lambda e_i\Lambda$. □

To give explicit descriptions of the monoids of projective modules over certain classes of rings, we introduce the monoid/semiring $\mathbb{N}_0^*$.

Notation 6.3.3. Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and let $\mathbb{N} = \{1, 2, \dots\}$. We denote by $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$ the monoid with the sum extending the one of $\mathbb{N}_0$ and with the additional rule that $x + \infty = \infty + x = \infty$ for any $x \in \mathbb{N}_0^*$. We also see $\mathbb{N}_0^*$ as a semiring with the product extending the one of $\mathbb{N}_0$ together with the rules $x \cdot \infty = \infty \cdot x = \infty$ for any $x \in \mathbb{N}_0^* \setminus \{0\}$, and $0 \cdot \infty = \infty \cdot 0 = 0$.

Proposition 6.3.4. *Let Λ be a semiperfect ring. Then:*

(i) *The canonical isomorphism between $\mathrm{V}(\Lambda) := \mathrm{V}_r(\Lambda) \cong \mathbb{N}_0^t$ and $\mathrm{V}_\ell(\Lambda)$ extends to an isomorphism between $\mathrm{V}^*(\Lambda) := \mathrm{V}_r^*(\Lambda) \cong (\mathbb{N}_0^*)^t$ and $\mathrm{V}_\ell^*(\Lambda)$.*

(ii) *Let P be a projective right Λ-module. Then there exists a two-sided ideal I of Λ that is minimal with respect to the property that P/PI is finitely generated. Moreover, I is the trace of a finitely generated projective right module.*

(iii) *Every countably generated projective module is relatively big, that is, $\mathrm{V}^*(\Lambda) = \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$.*

Proof. Let $e_1,\dots,e_t$ be a basic set of idempotents of Λ. Then $P_i = e_i\Lambda$ and $Q_i = \Lambda e_i \cong \mathrm{Hom}_\Lambda(P_i, \Lambda)$, for $i = 1,\dots,t$, are representatives of the isomorphism classes of indecomposable projective right and left Λ-modules. Recall that any projective right or left module over Λ is a direct sum of indecomposables, uniquely up to isomorphism.

(i) Let P be a countably generated projective right Λ-module. Then there exist $A_1,\dots,A_t$ countable sets such that $P \cong \bigoplus_{i=1}^t P_i^{(A_i)}$. Then the assignment

$$[P] = \left[\bigoplus_{i=1}^t P_i^{(A_i)}\right] \mapsto \left[\bigoplus_{i=1}^t Q_i^{(A_i)}\right]$$

gives an isomorphism of monoids $V_r^*(\Lambda) \longrightarrow V_\ell^*(\Lambda)$ that extends the canonical isomorphism $V_r(\Lambda) \longrightarrow V_\ell(\Lambda)$ from Remark 5.3.6.

Moreover, the assignment $[P] \mapsto (n_1, \dots, n_t)$, where for $i = 1, \dots, t$,

$$n_i = \begin{cases} |A_i| & \text{if } A_i \text{ is finite} \\ \infty & \text{otherwise,} \end{cases}$$

induces an isomorphism $\mathrm{V}_r^*(\Lambda) \longrightarrow (\mathbb{N}_0^*)^t$ that restricts to an isomorphism $\mathrm{V}_r(\Lambda) \longrightarrow \mathbb{N}_0^t$.

(ii) If P is a projective right Λ-module, then $\{1, \dots, t\} = A \cup B$, where A and B are disjoint, and $P \cong \left(\bigoplus_{i \in A} P_i^{(A_i)}\right) \oplus \left(\bigoplus_{i \in B} P_i^{(B_i)}\right)$, where A_i is a finite set (possibly empty) for any $i \in A$, and B_i is an infinite set for any $i \in B$. Then, by Lemma 1.4.3,

$$I = \sum_{i \in B} \mathrm{Tr}_\Lambda(P_i) = \mathrm{Tr}_\Lambda\Big(\bigoplus_{i \in B} P_i\Big),$$

is the ideal we are looking for.

(iii) By Proposition 6.3.2, $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$, so we can apply Corollary 5.3.9 to deduce that $\mathrm{B}_r(\Lambda) := \mathrm{B}(\Lambda)$ is canonically isomorphic to $\mathrm{B}_\ell(\Lambda)$.

Statement **(ii)** implies that any countably generated projective module is relatively big. Hence, $\mathrm{V}^*(\Lambda) = \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$. □

6.3.2 Locally semiperfect algebras over h-local domains

Definition 6.3.5. Let Λ be an algebra over a commutative ring R. We say that Λ is *locally semiperfect* if $\Lambda_\mathfrak{m}$ is semiperfect for every maximal ideal $\mathfrak{m}$ of R.

The following example explains a particular instance of the appearance of locally semiperfect algebras that, in addition, satisfy that they have a simple artinian ring of quotients. With the additional hypothesis that the domain R is h-local, they will be our main object of study in Section 7.3.

Example 6.3.6. Let R be a commutative domain with field of fractions Q, and let M be a finitely generated torsion-free R-module. Then $\Lambda = \mathrm{End}_R(M)$ is an R-algebra and, by Lemma 2.1.11, $\Lambda_Q = \Lambda \otimes_R Q \cong \mathrm{End}_Q(M \otimes_R Q) = \mathrm{End}_Q(M_Q)$ is simple artinian. By Proposition 3.14 in [14] and Lemma 2.1.11, for every maximal ideal $\mathfrak{m}$ of R, $M_\mathfrak{m}$ is a direct sum of modules with local endomorphism ring if and only if $\Lambda_\mathfrak{m} \cong \mathrm{End}_{R_\mathfrak{m}}(M_\mathfrak{m})$ is a semiperfect ring. In other words, Λ is locally semiperfect if and only if $M_\mathfrak{m}$ is a direct sum of modules with local endomorphism ring for every maximal ideal $\mathfrak{m}$ of R.

Lemma 6.3.7. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Then every trace of a projective right (or left) Λ-module is also the trace of a finitely generated projective right Λ-module. In particular, $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$.*

Proof. Let I be a non-zero two-sided ideal of Λ which is the trace of a projective right Λ-module. Since Λ is torsion-free, $I_\mathfrak{m} \neq \{0\}$ holds for every maximal ideal $\mathfrak{m}$ of R. By Lemma 2.1.9 and Lemma 6.3.2, $I_\mathfrak{m}$ is also the trace of a finitely generated projective right $\Lambda_\mathfrak{m}$-module.

Let S denote a simple right Λ_Q-module, and assume that $\Lambda_Q \cong S^k$ as a right Λ_Q-module. Since Λ_Q is simple artinian and I is non-zero, by Lemma 3.3.2, there

exists a finite set $\mathcal{M}$ of maximal ideals of R such that $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ if $\mathfrak{m} \notin \mathcal{M}$. For any maximal ideal $\mathfrak{m} \in \mathcal{M}$, let $P(\mathfrak{m})$ be a finitely generated projective right $\Lambda_{\mathfrak{m}}$-module with trace ideal $I_{\mathfrak{m}}$. Then $P(\mathfrak{m})_Q \cong S^{t_{\mathfrak{m}}}$ for a suitable positive integer $t_{\mathfrak{m}} \geq 1$. Let $t = \operatorname{lcm}\{t_{\mathfrak{m}} \mid \mathfrak{m} \in \mathcal{M}\}$, write $t = \ell_{\mathfrak{m}} \cdot t_{\mathfrak{m}}$, and let $P'(\mathfrak{m}) := P(\mathfrak{m})^{k\ell_{\mathfrak{m}}}$ for each maximal ideal $\mathfrak{m}$ of $\mathcal{M}$. For each $\mathfrak{m} \notin \mathcal{M}$, take $P'(\mathfrak{m}) = \Lambda_{\mathfrak{m}}^t$. Note that $P'(\mathfrak{m})_Q \cong S^{tk} \cong \Lambda_Q^t$ and $\operatorname{Tr}_{\Lambda_{\mathfrak{m}}}(P'(\mathfrak{m})) = I_{\mathfrak{m}}$ hold for every maximal ideal $\mathfrak{m}$ of R. By Package Deal Theorem 3.3.4 applied to the package $\{P'(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec}(R)\}$, there exists a finitely generated projective right Λ-module P such that $P_{\mathfrak{m}} \cong P'(\mathfrak{m})$ for any maximal ideal $\mathfrak{m}$ of R. By Lemma 2.1.9, $\operatorname{Tr}_{\Lambda}(P)_{\mathfrak{m}} = \operatorname{Tr}_{\Lambda_{\mathfrak{m}}}(P'(\mathfrak{m})) = I_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R. This implies that $I = \operatorname{Tr}_{\Lambda}(P) + I = \operatorname{Tr}_{\Lambda}(P)$.

A similar argument gives the conclusion for the traces of projective left Λ-modules due to the symmetry of the hypotheses.

The equality $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$ follows from the canonical bijection between finitely generated projective right and left Λ-modules induced by the Λ-dual, and because the trace of a finitely generated projective module coincides with the trace of its Λ-dual, cf. Remark 5.3.6. □

Corollary 6.3.8. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. If P is a countably generated projective right Λ-module, then $P^{(\omega)}$ is a direct sum of finitely generated modules.*

Proof. By Lemma 6.3.7, the trace of P coincides with the trace of a finitely generated projective module. To conclude, apply that if P and Q are countably generated projective modules then $P^{(\omega)} \cong Q^{(\omega)}$ if and only if P and Q have the same trace. □

Lemma 6.3.9. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian; so that there exist k and a division ring D such that $\Lambda_Q \cong \mathrm{M}_k(D)$. Then, for almost all maximal ideals $\mathfrak{m}$ of R, the ring $\Lambda_{\mathfrak{m}}$ has the following properties:*

- **(i)** *There is exactly one indecomposable projective $\Lambda_{\mathfrak{m}}$-module up to isomorphism. Moreover, $\Lambda_{\mathfrak{m}} \cong \mathrm{M}_k(T_{\mathfrak{m}})$, where $T_{\mathfrak{m}}$ is a local ring such that $T_{\mathfrak{m}} \otimes_R Q \cong D$.*
- **(ii)** *If U and V are projective $\Lambda_{\mathfrak{m}}$-modules, then $U \cong V$ if and only if $U_Q \cong V_Q$.*

Proof. Let S be a simple Λ_Q-module and assume that $\Lambda_Q \cong S^k$. Let X be a finitely generated Λ-module, which is torsion-free as an R-module, and such that $X_Q \cong S$. By Lemma 2.3.2, $X_{\mathfrak{m}}^k$ is free for almost all maximal ideals. If $\mathfrak{m}$ is such a maximal ideal, then $X_{\mathfrak{m}}^k \cong \Lambda_{\mathfrak{m}}$ gives an indecomposable decomposition of the semiperfect ring $\Lambda_{\mathfrak{m}}$. Hence $X_{\mathfrak{m}}$ is the only indecomposable projective $\Lambda_{\mathfrak{m}}$-module up to isomorphism. As $\Lambda_{\mathfrak{m}}$ is semiperfect, the endomorphism ring of $X_{\mathfrak{m}}$ is a local ring that we will denote by $T_{\mathfrak{m}}$. Hence $\Lambda_{\mathfrak{m}} \cong \operatorname{End}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}^k) \cong \mathrm{M}_k(T_{\mathfrak{m}})$. By Lemma 2.1.11, $T_{\mathfrak{m}} \otimes_R Q \cong D$.

Projective modules over $\mathrm{M}_k(T_{\mathfrak{m}})$ are always a direct sum of copies of the indecomposable projective $X_{\mathfrak{m}}$. Moreover, $U \cong X_{\mathfrak{m}}^{(I)}, V \cong X_{\mathfrak{m}}^{(J)}$ then $U \cong V$ if and only if I and J have the same cardinality, if and only if $U_Q \cong V_Q$. □

Remark 6.3.10. Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian, so that there exist k and a division ring D such that $\Lambda_Q \cong M_k(D)$. In view of Lemma 6.3.9, the maximum spectrum of R can be written as a disjoint union of two sets A and B such that for any $\mathfrak{m} \in A$, $\Lambda_\mathfrak{m}$ is a ring of $k \times k$ matrices over a local ring and B is a finite, non-empty set. In particular, for $\mathfrak{m} \in A$, $\mathcal{T}_r(\Lambda_\mathfrak{m}) = \mathcal{T}_\ell(\Lambda_\mathfrak{m}) = \{0, \Lambda_\mathfrak{m}\}$.

In view of Package Deal Theorem 3.4.3 and Lemma 6.3.7, there is a bijective correspondence

$$\mathcal{T}_r(\Lambda) \setminus \{0\} = \mathcal{T}_\ell(\Lambda) \setminus \{0\} \longrightarrow \prod_{\mathfrak{m} \in B} (\mathcal{T}_r(\Lambda_\mathfrak{m}) \setminus \{0\}) .$$

Corollary 6.3.11. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Then $\mathcal{T}(\Lambda)$ is finite. Moreover, if P is a finitely generated projective Λ-module, then $I = \mathrm{Tr}_\Lambda(P)$ is a minimal non-zero element of $\mathcal{T}(\Lambda)$ if and only if $P_\mathfrak{m}$ is a finite power of an indecomposable projective $\Lambda_\mathfrak{m}$-module for any $\mathfrak{m} \in \mathrm{mSpec}(R)$.*

Proof. Finiteness of $\mathcal{T}(\Lambda)$ follows from Remark 6.3.10. Also, note that the correspondence of the remark preserves inclusions, so $\mathrm{Tr}_\Lambda(P)$ is a minimal element of $\mathcal{T}(\Lambda) \setminus \{0\}$ if and only if $\mathrm{Tr}_{\Lambda_\mathfrak{m}}(P_\mathfrak{m})$ is a minimal element of $\mathcal{T}(\Lambda_\mathfrak{m}) \setminus \{0\}$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$ if and only if $P_\mathfrak{m}$ is a finite power of an indecomposable projective $\Lambda_\mathfrak{m}$-module. □

Proposition 6.3.12. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Let P be a countably generated projective right Λ-module. Then,*

- **(i)** *P is finitely generated if and only if P_Q is a finitely generated Λ_Q-module if and only if there exists a maximal ideal $\mathfrak{m}$ of R such that $P_\mathfrak{m}$ is a finitely generated $\Lambda_\mathfrak{m}$-module.*
- **(ii)** *The set $\mathcal{I}(P) = \{J \trianglelefteq \Lambda \mid P/PJ \text{ is finitely generated}\}$ has a minimal element, which is the trace of a finitely generated projective right Λ-module. If I is such ideal, then it is characterized by the property that, for any maximal ideal $\mathfrak{m}$ of R, $I_\mathfrak{m}$ is the minimal element of $\mathcal{I}(P_\mathfrak{m}) = \{J \trianglelefteq \Lambda \mid P_\mathfrak{m}/P_\mathfrak{m}J \text{ is finitely generated}\}$.*

Proof. Throughout the proof, let S denote a simple right Λ_Q-module.

(i) We claim that if $P_Q \cong S^n$ for some $n \in \mathbb{N}$, then P is finitely generated. Indeed, by hypothesis, there exists a finitely generated Λ-submodule P_0 of P such that $(P_0)_Q = P_Q$. Then we may assume that we have $P_0 \leq P \leq \Lambda^{(\omega)} = P \oplus P'$, and for any $p \in P$ there exists $0 \neq d \in R$ such that $pd \in P_0$. Since Λ is torsion-free as an R-module, this implies that there exists a finite subset F of ω such that $P_0 \leq P \subseteq \Lambda^F$. Hence, $\Lambda^F = P \oplus (P' \cap \Lambda^F)$, so P is finitely generated.

This proves statement **(i)**.

(ii) In the proof, we maintain the notation from Remark 6.3.10 regarding the partition of the maximal spectrum of R.

If P is finitely generated, the statement is clear. Assume that P is a countably generated, projective, right Λ-module that is not finitely generated. By (1),

$P_Q \cong S^{(\omega)}$. Therefore, there are two possible situations regarding the localizations of P at the maximal ideals of R:

(a) $P_{\mathfrak{m}}$ is a free $\Lambda_{\mathfrak{m}}$-module of infinite rank for any $\mathfrak{m} \in A$. Therefore, $\Lambda_{\mathfrak{m}}$ is the only element of the set $\mathcal{I}(P_{\mathfrak{m}}) = \{J \trianglelefteq \Lambda_{\mathfrak{m}} \mid P_{\mathfrak{m}}/P_{\mathfrak{m}}J \text{ is finitely generated}\}$.

(b) $P_{\mathfrak{n}}$ is an infinitely generated projective module over the semiperfect ring $\Lambda_{\mathfrak{n}}$ for any $\mathfrak{n} \in B$. By Proposition 6.3.4, there exists a non-zero ideal $I(\mathfrak{n})$ that is the minimal element of the set $\mathcal{I}(P_{\mathfrak{n}}) = \{J \trianglelefteq \Lambda_{\mathfrak{n}} \mid P_{\mathfrak{n}}/P_{\mathfrak{n}}J \text{ is finitely generated}\}$. In addition, $I(\mathfrak{n})$ is the trace of a finitely generated projective right $\Lambda_{\mathfrak{n}}$-module.

By Package Deal Theorem 3.4.3, there exists an ideal I of Λ such that $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for any $\mathfrak{m} \in A$ and $I_{\mathfrak{n}} = I(\mathfrak{n})$ for any $\mathfrak{n} \in B$. In addition, I is the trace of a (finitely generated, by Lemma 6.3.7) projective right Λ-module. We claim that I is the minimal element of $\mathcal{I}(P)$.

Since $I_Q = \Lambda_Q$, $I \cap R \neq \{0\}$. As R has finite character, there exists a finite set $\mathcal{M}$ of maximal ideals of R, such that $R/(I \cap R) \cong \prod_{\mathfrak{m} \in \mathcal{M}} (R/(I \cap R))_{\mathfrak{m}}$. We may assume that $B \subseteq \mathcal{M}$. Hence,

$$P/PI \cong \prod_{\mathfrak{m} \in \mathcal{M}} P_{\mathfrak{m}}/P_{\mathfrak{m}}I_{\mathfrak{m}} = \prod_{\mathfrak{n} \in B} P_{\mathfrak{n}}/P_{\mathfrak{n}}I(\mathfrak{n})$$

which is finitely generated. Therefore, $I \in \mathcal{I}(P)$.

Let $I' \in \mathcal{I}(P)$ be such that $I' \subseteq I$. Then, because Λ is torsion-free as an R-module, for any maximal ideal $\mathfrak{m}$ of R, $I'_{\mathfrak{m}} \in \mathcal{I}(P_{\mathfrak{m}})$, and $(I')_{\mathfrak{m}} \subseteq I_{\mathfrak{m}} = I(\mathfrak{m})$. The minimality in the choice of $I(\mathfrak{m})$ in **(a)** and **(b)** implies that $(I')_{\mathfrak{m}} = I_{\mathfrak{m}} = I(\mathfrak{m})$ for any maximal ideal $\mathfrak{m}$ of R. Hence, $I' = I$. This shows that I is the minimal element of $\mathcal{I}(P)$.

The proof also shows that for any maximal ideal $\mathfrak{m}$ of R, $I_{\mathfrak{m}}$ is the minimal element of $\mathcal{I}(P_{\mathfrak{m}})$. □

Lemma 6.3.13. *Let R be a commutative domain with field of fractions Q, and let Λ be a torsion-free R-algebra. Let $I \in \mathcal{T}(\Lambda)$. Let P be a countably generated projective Λ-module, and let V be a finitely generated Λ-module which is torsion-free as an R-module. Let $\mathfrak{p}$ be a prime ideal of R such that $V_{\mathfrak{p}} = V_{\mathfrak{p}}I_{\mathfrak{p}}$ and $P_{\mathfrak{p}} = \bigoplus_{i \in \mathbb{N}} U_i$, where each U_i is a finitely generated Λ_p-module, such that infinitely many of these U_i are isomorphic to a $\Lambda_{\mathfrak{p}}$-module U whose trace ideal contains $I_{\mathfrak{p}}$.*

Then, for every finitely generated submodule P_1 of P, there exists a homomorphism $f \in \mathrm{Hom}_{\Lambda}(P, V)$ such that $f(P_1) = 0$, and $f_{\mathfrak{p}}$ is onto.

Proof. Let $\nu \in \mathrm{Hom}_{\Lambda}(P, \Lambda^{(\omega)})$ and $\pi \in \mathrm{Hom}_{\Lambda}(\Lambda^{(\omega)}, P)$ be such that $\pi\nu = 1_P$.

Note that if U is a projective $\Lambda_{\mathfrak{p}}$-module and $I_{\mathfrak{p}} \subseteq \mathrm{Tr}_{\Lambda}(U)$, then U generates any $\Lambda_{\mathfrak{p}}$-module W satisfying $WI_{\mathfrak{p}} = W$. If W is also finitely generated, then there exists $k \in \mathbb{N}$ such that W is a homomorphic image of U^k. Therefore $V_{\mathfrak{p}}$ is a homomorphic image of $\bigoplus_{i \in C} U_i$ whenever C is a cofinal subset of $\mathbb{N}$. In particular, there exists an onto homomorphism $g \in \mathrm{Hom}_{\Lambda_{\mathfrak{p}}}(P_{\mathfrak{p}}, V_{\mathfrak{p}})$ such that $g(P_1) = 0$ (recall that P is embedded in $P_{\mathfrak{p}}$). Since V is finitely generated, there exists P'_0, a finitely generated submodule of $P_{\mathfrak{p}}$, such that $g(P'_0) = V_{\mathfrak{p}}$. Scaling by a suitable element of $R \setminus \mathfrak{p}$ (which is invertible over Λ_p), we observe that there exists a finitely generated $P_0 \subseteq P$ such that $P_0\Lambda_{\mathfrak{p}} = P'_0$.

Let F be a finite subset of ω such that $\nu(P_0+P_1)$ is contained in $\Lambda^{(F)}$. Further, let $\iota \in \mathrm{Hom}_\Lambda(\Lambda^{(F)}, \Lambda^{(\omega)})$ and $\varrho \in \mathrm{Hom}_\Lambda(\Lambda^{(\omega)}, \Lambda^{(F)})$ be the canonical embedding and projection. Since $\Lambda^{(F)}$ is finitely generated over Λ and torsion-free over R, by Lemma 2.1.11, there exist $h \in \mathrm{Hom}_\Lambda(\Lambda^{(F)}, V)$ and $q \in R_\mathfrak{p}$ such that $h \otimes q = g(\pi\iota)_\mathfrak{p}$. Consider $f := h\varrho\nu$. Then $f \otimes q = h \otimes q \circ (\varrho\nu) \otimes 1 = g(\pi\iota\varrho\nu)_\mathfrak{p}$. Since $\Lambda^{(F)}$ contains $\nu(P_0+P_1)$, $\pi\iota\varrho\nu$ is the identity on P_0+P_1, and consequently, $f \otimes q$ is onto and $f \otimes q(P_1 \otimes 1) = 0$. This implies that $f_\mathfrak{p}$ is onto and $f(P_1) = 0$. □

Remark 6.3.14. We are going to apply Lemma 6.3.13 in the following situation: Λ is a locally semiperfect torsion-free algebra over an h-local domain R such that Λ_Q is a simple artinian ring. Assume that P is a countably generated projective Λ-module and $0 \neq I$ is the least element of $\mathcal{I}(P) = \{J \trianglelefteq \Lambda \mid P/PJ \text{ is finitely generated}\}$. Take any $\mathfrak{m} \in \mathrm{mSpec}(R)$ and let us check that $P_\mathfrak{m}$ is a countable direct sum of finitely generated projective $\Lambda_\mathfrak{m}$-modules such that almost all these direct summands have a trace equal to $I_\mathfrak{m}$.

Indeed, since P is not finitely generated, $P_\mathfrak{m}$ is not a finitely generated $\Lambda_\mathfrak{m}$-module by Proposition 6.3.12. Since $\Lambda_\mathfrak{m}$ is semiperfect, $P_\mathfrak{m}$ is a direct sum of finitely generated modules $P_\mathfrak{m} = U_1^{m_1} \oplus \cdots \oplus U_t^{m_t} \oplus (U_{t+1} \oplus \cdots \oplus U_s)^{(\omega)}$, where $U_1, \ldots, U_s$ is a representative set of indecomposable projective $\Lambda_\mathfrak{m}$-modules. Let $U' = U_{t+1} \oplus \cdots \oplus U_s$. As explained in the proof of Proposition 6.3.12**(ii)**, $I_\mathfrak{m} = \mathrm{Tr}_{\Lambda_\mathfrak{m}}(U')$. Therefore $P_\mathfrak{m} = U_1^{m_1} \oplus \cdots \oplus U_t^{m_t} \oplus U'^{(\omega)}$ is a decomposition of $P_\mathfrak{m}$ that has the required properties.

Lemma 6.3.13 can be stated in simpler form when $\mathfrak{p} = (0)$ and $I = \Lambda$.

Lemma 6.3.15. *Let R be a commutative domain with field of fractions Q, and let Λ be a torsion-free R-algebra. Let P be a projective Λ-module such that P_Q is countably generated but not finitely generated free, and let V be a finitely generated Λ-module which is torsion-free over R. Then, for any finitely generated submodule P_1 of P, there exists $f \in \mathrm{Hom}_\Lambda(P, V)$ such that $f(P_1) = 0$, and f_Q is onto.*

Proposition 6.3.16. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Let P be a countably generated projective Λ-module such that*

$$I = \min\{J \trianglelefteq \Lambda \mid P/PJ \text{ is finitely generated}\}$$

exists and is the trace of a countably generated projective module. Then P is I-big.

Proof. We have to prove that if U is a countably generated projective right Λ-module and $\mathrm{Tr}_\Lambda(U) \subseteq I$, then U is a direct summand of P. By Corollary 6.3.8, $U^{(\omega)}$ is a direct sum of finitely generated modules. Thus, we may assume that U is a direct sum of finitely generated modules.

We prove the following claim: If P_0 is a finitely generated submodule of P and V is a finitely generated projective Λ-module with trace contained in I, then there exists an epimorphism $f\colon P \longrightarrow V$ such that $f(P_0) = 0$.

Assume we have proved the claim and let $U = \bigoplus_{i\in\mathbb{N}} U_i$ with each U_i being finitely generated. Let $p_1, p_2, \cdots \in P$ be a countable set of generators of P. Consider the following construction: Let $f_1\colon P \longrightarrow U_1$ be an epimorphism, and let P_1 be a finitely generated submodule of P such that $f_1(P_1) = U_1$ and $p_1 \in P_1$. Assume that $f_1, f_2, \ldots, f_k$ and $P_1, P_2, \ldots, P_k$ are defined such that $f_i\colon P \longrightarrow U_i$ is

an epimorphism, $P_1 \subseteq \cdots \subseteq P_k$ are finitely generated submodules of P such that $f_i(P_{i-1}) = 0$, $p_i \in P_i$, and $f_i(P_i) = U_i$ (we can set $P_0 = 0$). By the claim, there exists an epimorphism $f_{k+1}\colon P \longrightarrow U_{k+1}$ such that $f_{k+1}(P_k) = 0$. Finally, let P_{k+1} be a finitely generated submodule of P satisfying $P_k \subseteq P_{k+1}$, $p_{k+1} \in P_{k+1}$, and $f_{k+1}(P_{k+1}) = U_{k+1}$.

Consider the canonical homomorphism $f = \prod_{k\in\mathbb{N}} f_k\colon P \longrightarrow \prod_k U_k$. Observe that $f_k(p_l) = 0$ if $k > l$, so $\operatorname{Im} f \subseteq \bigoplus_{i\in\mathbb{N}} U_i$. An easy inductive argument also gives $f(P_i) = \bigoplus_{j=1}^{i} U_j$ for any $i \in \mathbb{N}$. Thus f is an epimorphism.

Thus, we are left to prove the claim. First, note that we may assume $I \neq 0$ since any countably generated projective module is 0-big. Then P is not finitely generated. Therefore, by Proposition 6.3.12, P_Q cannot be finitely generated. Hence $P_Q \simeq \Lambda_Q^{(\omega)}$.

By Lemma 6.3.15, there exists $g \in \operatorname{Hom}_\Lambda(P, V)$ such that g_Q is onto and $g(P_0) = 0$. Let T_0 be a finitely generated submodule of P containing P_0 such that $g_Q(T_0\Lambda_Q) = (g(T_0))_Q = V_Q$. Since V is finitely generated, $V_\mathfrak{m} = g(T_0)_\mathfrak{m}$ holds for almost all maximal ideals $\mathfrak{m}$ of R. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k$ be the maximal ideals of R where the equality does not hold. If there are no such ideals, then we are done, since g is onto. So we may assume that $k \geq 1$.

It remains to show that there are homomorphisms $f_1, f_2, \ldots, f_k \in \operatorname{Hom}_\Lambda(P, V)$ and finitely generated submodules $T_1, T_2, \ldots, T_k$ of P such that

(i) $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$

(ii) $f_i(T_{i-1}) = 0$ for $i = 1, \ldots, k$

(iii) $(g + \sum_{j=1}^{i} f_j)_{\mathfrak{m}_i}((T_i)_{\mathfrak{m}_i}) = V_{\mathfrak{m}_i}$ for $i = 1, \ldots, k$

Then $f = g + \sum_{i=1}^{k} f_i \in \operatorname{Hom}_\Lambda(P, V)$ satisfies $f(T_0) = g(T_0)$. Hence, $f_\mathfrak{m}$ is onto if $\mathfrak{m}$ is not among $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$. Moreover, $f(T_i) = (g + \sum_{j=1}^{i} f_j)(T_i)$ shows that $f_{\mathfrak{m}_i}$ is onto for every $i = 1, \ldots, k$. This implies that f is onto.

Assume $f_1, \ldots, f_{\ell-1}$ and $T_0, T_1, \ldots, T_{\ell-1}$ are found. Consider $g_\ell = g + \sum_{i=1}^{\ell-1} f_i$. Recall that $V_{\mathfrak{m}_\ell}$ is a finitely generated projective module over a semiperfect ring $\Lambda_{\mathfrak{m}_\ell}$. Then, $V_{\mathfrak{m}_\ell}/V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell})$ is a finitely generated (semisimple module). Therefore, if $\pi\colon V_{\mathfrak{m}_\ell} \longrightarrow V_{\mathfrak{m}_\ell}/V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell})$ is the canonical projection, $\pi((g_\ell(P))_{\mathfrak{m}_\ell})$ is a finitely generated submodule of $V_{\mathfrak{m}_\ell}/V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell})$. Let $T'_\ell \subseteq P$ be a finitely generated submodule of P containing $T_{\ell-1}$ such that $(g_\ell(T'_\ell))_{\mathfrak{m}_\ell} + V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell}) = (g_\ell(P))_{\mathfrak{m}_\ell} + V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell})$.

By Lemma 6.3.13, there exists $f_\ell \in \operatorname{Hom}_\Lambda(P, V)$ such that $(f_\ell)_{\mathfrak{m}_\ell}$ is onto and $f_\ell(T'_\ell) = 0$. Then the image of $g_\ell + f_\ell$ contains $g_\ell(T'_\ell)$. Hence, $\operatorname{im}(g_\ell + f_\ell)_{\mathfrak{m}_\ell} + V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell})$ also contains $\operatorname{im}(g_\ell)_{\mathfrak{m}_\ell}$, and since $(f_\ell)_{\mathfrak{m}_\ell}$ is onto, $\operatorname{im}(g_\ell + f_\ell)_{\mathfrak{m}_\ell} + V_{\mathfrak{m}_\ell}J(\Lambda_{\mathfrak{m}_\ell}) = V_{\mathfrak{m}_\ell}$. Nakayama's Lemma gives that $(g_\ell + f_\ell)_{\mathfrak{m}_\ell}$ is onto, and therefore there exists a finitely generated submodule T_ℓ of P containing T'_ℓ such that $((g_\ell + f_\ell))_{\mathfrak{m}_\ell}((T_\ell)_{\mathfrak{m}_\ell}) = V_{\mathfrak{m}_\ell}$, that is, **(iii)** holds if $i = \ell$. □

Theorem 6.3.17. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is a simple artinian ring. Then $\mathrm{V}^*_r(\Lambda) = \mathrm{V}_r(\Lambda) \sqcup B_r(\Lambda) \cong \mathrm{V}_\ell(\Lambda) \sqcup B_\ell(\Lambda) = \mathrm{V}^*_\ell(\Lambda)$.*

Proof. Propositions 6.3.12 and 6.3.16 show that $\mathrm{V}^*_r(\Lambda) = \mathrm{V}_r(\Lambda) \sqcup B_r(\Lambda)$. The symmetry of the hypothesis implies that $\mathrm{V}_\ell(\Lambda) \sqcup B_\ell(\Lambda) = \mathrm{V}^*_\ell(\Lambda)$. By Remark 6.3.10, we can deduce from Corollary 5.3.9 that both monoids are isomorphic. □

Hinohara [27] proved that every projective module over an h-local (or, more generally, weakly noetherian) domain is either finitely generated or free. We can extend this result to our setting.

Corollary 6.3.18 ([27])**.** *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. For each maximal ideal $\mathfrak{m}$ of R, assume that $\Lambda_{\mathfrak{m}}$ is Morita equivalent to a local ring. Then any infinitely generated projective Λ-module is free.*

Proof. By Remark 6.3.10, $\mathcal{T}(\Lambda) = \{0, \Lambda\}$, and hence $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}(\Lambda) \sqcup \{[\Lambda^{(\omega)}]\}$. The result then follows from Theorem 6.3.17. □

7 Locally semiperfect algebras and applications to direct sum decompositions

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Throughout this chapter, R is an h-local domain with field of fractions Q, and Λ is a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. In this chapter, we prove Theorems E and F stated in the introduction. The first part of Theorem E was already proven in the previous chapter. We recall it here for completeness:

Theorem E (Lemma 6.3.9, Propositions 7.1.7 and 7.1.9)**.** *Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Then*

(i) *There exists a finite set of maximal ideals $\mathcal{S}(\Lambda)$ such that $\Lambda_{\mathfrak{m}} \cong \mathrm{M}_k(T_{\mathfrak{m}})$ for some local ring $T_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$. In particular, for every maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $\Lambda_{\mathfrak{m}}$ has exactly one indecomposable projective module up to isomorphism.*

(ii) *For each maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, if U and V are projective $\Lambda_{\mathfrak{m}}$-modules, then $U \cong V$ are isomorphic if and only if $U_Q \cong V_Q$*

(iii) *For each maximal ideal $\mathfrak{m} \in \mathcal{S}(\Lambda)$, let $d_{\mathfrak{m},1}, \dots, d_{\mathfrak{m},\ell_{\mathfrak{m}}}$ be the lengths of the indecomposable projective right $\Lambda_{\mathfrak{m}}$-modules, and fix $\boldsymbol{r}_{\mathfrak{m}} \in (\mathbb{N}_0^*)^{\ell_{\mathfrak{m}}}$.*

Then, there exists a countably generated projective right Λ-module P with genus sequence $\{\boldsymbol{r}_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathrm{mSpec}(R)}$ if and only if there exists $r \in \mathbb{N}_0^*$ such that for every maximal ideal $\mathfrak{m}$ of R

$$\sum_{j=1}^{\ell_{\mathfrak{m}}} d_{\mathfrak{m},j} r_{\mathfrak{m},j} = \begin{bmatrix} d_{\mathfrak{m},1} & \cdots & d_{\mathfrak{m},\ell_{\mathfrak{m}}} \end{bmatrix} \begin{bmatrix} r_{\mathfrak{m},1} \\ \vdots \\ r_{\mathfrak{m},\ell_{\mathfrak{m}}} \end{bmatrix} = r.$$

In Section 7.1, we study the rank and the genus of countably generated projective Λ-modules. We view $\mathrm{V}^*(\Lambda)$ as a submonoid of the product $\prod_{\mathfrak{m}\in\mathrm{mSpec}(R)} \mathrm{V}^*(\Lambda_{\mathfrak{m}})$ to define the genus, but also as a submonoid of $\mathrm{V}^*(\Lambda_Q)$ to define the rank. In Proposition 7.1.1, we show that two infinitely generated projective modules are isomorphic if and only if all their localizations at maximal ideals of R are isomorphic. That is, they are isomorphic if and only if they are in the same genus.

These tools allow us to study in Section 7.2 the arithmetical characterization of countably generated projective Λ-modules given in Theorem E. Finally, in Section 7.3, we apply our results taking Λ to be the endomorphism ring of a finitely generated torsion-free module M over an h-local domain R of Krull dimension 1. In Theorem 7.3.2 (Theorem F in the introduction), we determine when any direct summand of $\mathrm{Add}(M)$ is a sum of finitely generated modules. The results in this section complement those obtained in the previous part of this dissertation.

7.1 Rank and genus of countably generated projective modules

We continue with the same setting as in Section 6.3. Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free algebra over R such that Λ_Q is simple artinian. That is $\Lambda_Q \cong \mathrm{M}_k(D)$, for D a division ring.

There are two natural monoid homomorphisms. The first one is the one associated with tensoring with Q:

$$\Psi\colon\ \mathrm{V}^*(\Lambda) \longrightarrow \mathrm{V}^*(\Lambda_Q)$$

defined by $\Psi([P]) = [P_Q] = [S^r]$, where S is a simple right Λ_Q-module. Then we have a monoid homomorphism, which we also call Ψ:

$$\Psi\colon\ \mathrm{V}^*(\Lambda) \longrightarrow \mathbb{N}_0^*$$

defined by $\Psi([P]) = r$ if P is finitely generated and $P_Q \cong S^r$, and $\Psi([P]) = \infty$ if P is not finitely generated. We call $\Psi([P])$ the rank of P.

Associated with the localization at the different maximal ideals, we have the monoid homomorphism

$$\Phi\colon\ \mathrm{V}^*(\Lambda) \longrightarrow \textstyle\prod_{\mathfrak{m}\in \mathrm{mSpec}(R)} \mathrm{V}^*(\Lambda_{\mathfrak{m}})$$

defined by $\Phi([P]) = ([P_{\mathfrak{m}}])_{\mathfrak{m}\in \mathrm{mSpec}(R)}$. We shall refer to Φ as the *genus morphism*, and to $\Phi([P])$ as the genus of $[P]$.

In this section, we want to describe the image of Ψ and Φ.

At this point, we have plenty of information about $\mathrm{V}^*(\Lambda)$. By Theorem 6.3.17, $V^*(\Lambda) = \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$, and by Lemma 6.3.7, any ideal of $\mathcal{T}(\Lambda)$ is a trace ideal of a finitely generated projective module, and by Corollary 6.3.11, $\mathcal{T}(\Lambda)$ is finite.

In general, there may be plenty of finitely generated projective right modules over Λ in the same genus that are not isomorphic. For example, let $R = \Lambda$ be a Dedekind domain. Then, any localization of a projective module at a maximal ideal is free. Therefore, two projective modules are in the same genus if and only if they have the same rank. Certainly, two finitely generated projective modules over a Dedekind domain with the same rank are not, in general, isomorphic.

The situation changes for projective Λ-modules whose isomorphism class is in $B(\Lambda)$, or for finitely generated projective Λ-modules when Λ is a suitable algebra over a semilocal domain R.

Proposition 7.1.1. *Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Then*

$$\Phi(\mathrm{B}(\Lambda)) \subseteq \prod_{\mathfrak{m}\in \mathrm{mSpec}(R)} \mathrm{B}(\Lambda_{\mathfrak{m}}).$$

Moreover, if $[P]$ and $[P']$ are two elements in $\mathrm{B}(\Lambda)$, then $\Phi([P]) = \Phi([P'])$ if and only if $[P] = [P']$.

Proof. The first part of the proof follows from Corollary 5.1.12 and Proposition 6.3.12.

For the second part of the proof, we are going to use the notation of Corollary 5.3.5 that is, an element $[P] \in \mathrm{B}(\Lambda)$ can be thought of as a pair $([P/PI], I)$ where $I \in \mathcal{T}(\Lambda) \setminus \{0\}$, and P is relatively I-big. Hence, $\Phi([P/PI], I) = ([P_{\mathfrak{m}}/P_{\mathfrak{m}}I_{\mathfrak{m}}], I_{\mathfrak{m}})_{\mathfrak{m}\in \mathrm{mSpec}(R)}$.

By Lemma 3.3.2, there exists a finite set of maximal ideals $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell\}$ of R such that $\Lambda/I \cong \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}$. Hence, $P/PI \cong \prod_{i=1}^{\ell} P_{\mathfrak{m}_i}/P_{\mathfrak{m}_i}I_{\mathfrak{m}_i}$, and $P_{\mathfrak{n}} = P_{\mathfrak{n}}I_{\mathfrak{n}}$ is free and $I_{\mathfrak{n}} = \Lambda_{\mathfrak{n}}$ for any $\mathfrak{n} \notin \mathcal{M}$

By Theorem 3.4.3, I is uniquely determined by $I_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, by Lemma 5.1.7, it follows that Φ is injective when restricted to $\mathrm{B}(\Lambda)$. □

Proposition 7.1.2. *Let R be a semilocal domain, and let Λ be a locally semiperfect torsion-free R-algebra. Then:*

- **(i)** *Λ is semilocal;*
- **(ii)** *Assume Λ is module-finite over R, and let P_1 and P_2 be two finitely generated projective right Λ-module. Then $P_1 \cong P_2$ if and only if $\Phi([P_1]) = \Phi([P_2])$.*

Proof. **(i)** Let $\mathrm{mSpec}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell\}$. Consider the embedding

$$\alpha\colon \Lambda \longrightarrow \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}$$

induced by taking the localization map at each component. If $\lambda \in \Lambda$ is such that $\alpha(\lambda)$ is invertible, then for each $i \in \{1, \dots, \ell\}$, there exists $s_i \in R \setminus \mathfrak{m}_i$ and $\mu_i \in \Lambda$ such that $\lambda\mu_i = s_i$. Then $I = \sum_{i=1}^{\ell} s_i R$ is not contained in any maximal ideal of R, so $I = R$. Since $R = I \subseteq \lambda\Lambda$, we deduce that λ is a right invertible element of Λ. By symmetry, λ is an invertible element of Λ. This shows that α is a local ring homomorphism. Since $\prod_{i\in\{1,\dots,\ell\}} \Lambda_{\mathfrak{m}_i}$ is a finite product of semiperfect rings, we deduce that Λ is semilocal (cf. Theorem 4.2 in [14]).

(ii) This statement is proved in Corollary 2.3.5. □

The following straightened version of the Package Deal Theorem for our setting will be the key ingredient to describe $\Phi(\mathrm{V}^*(\Lambda)) = \Phi(\mathrm{V}(\Lambda)) \sqcup \mathrm{B}(\Lambda))$.

Proposition 7.1.3. *Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Let S be a simple right Λ_Q-module.*

Fix $r \geq 1$. For each maximal ideal $\mathfrak{m}$ of R, let $P(\mathfrak{m})$ be a finitely generated projective right $\Lambda_{\mathfrak{m}}$-module such that $P(\mathfrak{m})_Q \cong S^r$. Then there exists a finitely

generated projective right Λ-module P such that $P_{\mathfrak{m}} \cong P(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R.

Notice that, by definition, $\Psi([P]) = r$.

Proof. Let k be such that $\Lambda_Q \cong S^k$. By Lemma 6.3.9, there is a finite set of maximal ideals of R, $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$, such that $\Lambda_{\mathfrak{m}} \cong \mathrm{M}_k(T_{\mathfrak{m}})$ for some local ring $T_{\mathfrak{m}}$ whenever $\mathfrak{m}$ is a maximal ideal not in $\mathcal{S} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$. Note that if $\mathfrak{m} \notin \mathcal{S}$, then $\Lambda_{\mathfrak{m}}$ is isomorphic to a ring of $k \times k$ matrices over a local ring, so it has a unique indecomposable projective module $X(\mathfrak{m})$ such that $X(\mathfrak{m})_Q \cong S$ and any projective $\Lambda(\mathfrak{m})$-module is isomorphic to a suitable number of copies of $X(\mathfrak{m})$. Therefore, for any $\mathfrak{m} \notin \mathcal{S}$, $P(\mathfrak{m}) \cong X(\mathfrak{m})^r$.

Let q be a multiple of k that is greater than r, and let $F = \Lambda^{q/k}$. Further, let U be a finitely generated submodule of F such that $U_Q \cong S^r$. By Lemma 2.3.2, since the embedding $U \hookrightarrow F$ splits over Λ_Q, it splits over almost all maximal ideals of R. Then U is a finitely generated Λ-module which is torsion-free as an R-module, such that $U_{\mathfrak{m}}$ is projective for almost all maximal ideals $\mathfrak{m}$ of R and $U_Q \cong S^r$. So let B be a finite set of maximal ideals of R such that $\mathcal{S} \subseteq B$, and satisfying that $U_{\mathfrak{m}}$ is projective for any $\mathfrak{m} \notin B$. In particular, for any $\mathfrak{m} \notin B$, $U_{\mathfrak{m}} \cong P(\mathfrak{m}) \cong X(\mathfrak{m})^r$.

By Lemma 2.1.15, for any $\mathfrak{m} \in B$, we can choose $U(\mathfrak{m})$ a submodule of $U_{\mathfrak{m}}$ such that $U(\mathfrak{m}) \cong P(\mathfrak{m})$. For $\mathfrak{m} \notin B$, we set $U(\mathfrak{m}) = U_{\mathfrak{m}}$.

Note that for any maximal ideal $\mathfrak{m}$ of R, $U(\mathfrak{m})_Q = U_Q$, since all these modules have length r over Λ_Q. Therefore, we can apply Theorem 3.2.3 and conclude that there exists a finitely generated Λ-module P of U such that $P(\mathfrak{m}) \cong U(\mathfrak{m}) = P_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{mSpec}(R)$. By Corollary 2.1.14, P is a projective module over Λ. □

We introduce the notation necessary to more thoroughly detail the image of the genus map Φ and the rank map Ψ.

Notation 7.1.4. Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. Let S be a simple Λ_Q-module, and let k be the length of Λ_Q, i.e., $\Lambda_Q \cong S^k$, or, equivalently, $\Psi(\Lambda) = k$. By Lemma 6.3.9, there is a finite set of maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$ such that $\Lambda_{\mathfrak{m}} \cong \mathrm{M}_k(T_{\mathfrak{m}})$ for some local ring $T_{\mathfrak{m}}$ whenever $\mathfrak{m}$ is a maximal ideal not in $\mathcal{S} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$.

If $\ell = 0$, then the assumptions of Corollary 6.3.18 are satisfied. So, in this case, the image of the rank map Ψ is $\mathbb{N}_0^*$, and the genus map composed with the natural isomorphism $\mathrm{V}^*(\Lambda_{\mathfrak{m}}) \cong \mathbb{N}_0^*$

$$\Phi\colon\ \mathrm{V}^*(\Lambda) \longrightarrow \prod_{\mathfrak{m}\in\mathrm{mSpec}(R)} \mathrm{V}^*(\Lambda_{\mathfrak{m}}) \cong \prod_{\mathfrak{m}\in\mathrm{mSpec}(R)} \mathbb{N}_0^*$$

just sends $[P]$ to the constant sequence with the value of the rank of P.

So, from now on, we will assume that $\ell \geq 1$.

For $i = 1, \ldots, \ell$, let $U_{i,1}, \ldots, U_{i,t_i}$ be representatives of indecomposable projective right $\Lambda_{\mathfrak{m}_i}$-modules, and let $r_{i,1}, \ldots, r_{i,t_i}$ be the lengths of their localizations over Λ_Q, i.e., $(U_{i,j})_Q \cong S^{r_{i,j}}$.

Note that, for any $i = 1, \ldots, \ell$, if the right $\Lambda_{\mathfrak{m}_i}$-module $\Lambda_{\mathfrak{m}_i}$ is isomorphic to $U_{i,1}^{m_{i,1}} \oplus \cdots \oplus U_{i,t_i}^{m_{i,t_i}}$, then we have the relation

$$\sum_{j=1}^{t_i} r_{i,j} m_{i,j} = \begin{bmatrix} r_{i,1} & \cdots & r_{i,t_i} \end{bmatrix} \begin{bmatrix} m_{i,1} \\ \vdots \\ m_{i,t_i} \end{bmatrix} = k.$$

We proceed to construct examples of finitely generated locally semiperfect torsion-free $\mathbb{Z}$-algebras. In particular, they are algebras over an h-local domain with an infinite number of maximal ideals. These examples show that there are no other restrictions on the parameters $k, \ell, t_i, r_{ij}, m_{ij}$ than those specified by the ℓ equations above.

Examples 7.1.5. (1) Let k be a positive integer and let p be a prime. Assume we are given positive integers $r_1, \ldots, r_t$, $m_1, \ldots, m_t$ such that $\sum_{i=1}^t m_i r_i = k$. Our aim is to construct a semiperfect $\mathbb{Z}_{(p)}$-order Λ in $\mathrm{M}_k(\mathbb{Q})$ such that $\Lambda \cong \bigoplus_{i=1}^t U_i^{m_i}$ as right modules, where $U_1, \ldots, U_t$ is a representative set of indecomposable projective Λ-modules and r_i is the length of $(U_i)_Q$.

Write $\{1, \ldots, k\}$ as a disjoint union of sets $\bigcup_{i=1}^t (A_{i,1} \cup A_{i,2} \cup \cdots \cup A_{i,m_i})$, where $|A_{i,j}| = r_i$. For any i and any $1 \leq a, b \leq m_i$ we fix a bijection $f_{i,a,b} \colon A_{i,a} \longrightarrow A_{i,b}$ such that $f_{i,a,a}$ is the identity of $A_{i,a}$ and $f_{i,b,c} f_{i,a,b} = f_{i,a,c}$ if $1 \leq a, b, c \leq m_i$.

For elements i, a, b, $1 \leq i \leq t$, $1 \leq a, b \leq m_i$ define $e_{i,a,b} \in \mathrm{M}_k(\mathbb{Q})$ by $e_{i,a,b} = \sum_{j \in A_{i,a}} E_{f_{i,a,b}(j),j}$, where $E_{x,y}$ is a matrix unit in $\mathrm{M}_k(\mathbb{Q})$. Note that $e_{i,a,a}^2 = e_{i,a,a}$ and $e_{i,b,c} e_{i,a,b} = e_{i,a,c}$. Further note that $e_{i,a,b} e_{j,c,d} = 0$ if $1 \leq i \neq j \leq k$, $1 \leq a, b \leq m_i$ and $1 \leq c, d \leq m_j$ or if $i = j$ and $1 \leq d \neq a \leq m_i$.

Consider $J = \mathrm{M}_k(p\mathbb{Z}_{(p)})$ and $\Lambda = J + \sum_{i=1}^t \sum_{1 \leq a,b \leq m_i} e_{i,a,b} \mathbb{Z}_{(p)}$. The following facts are easily verified:

(i) Λ is a subring of $\mathrm{M}_k(\mathbb{Q})$, is a finitely generated $\mathbb{Z}_{(p)}$-algebra and $\Lambda_Q = \mathrm{M}_k(\mathbb{Q})$.

(ii) J is an ideal of Λ, $J \subseteq J(\Lambda)$.

(iii) For any i, j $(e_{i,j,j}\Lambda)/(e_{i,j,j}J)$ is a simple Λ-module.

(iv) $e_{i,a,a}\Lambda \cong e_{j,b,b}\Lambda$ if and only if $i = j$

(v) $e_{i,a,a}\Lambda e_{j,b,b} = e_{i,a,a} J e_{j,b,b}$ whenever $1 \leq i \neq j \leq t$, $1 \leq a \leq m_i, 1 \leq b \leq m_j$

From **(iii)** it follows that Λ/J is a direct sum of simple modules. Hence Λ/J is semisimple artinian and therefore $J = J(\Lambda)$. Also note that the canonical projection $e_{i,j,j}\Lambda \longrightarrow e_{i,j,j}\Lambda/e_{i,j,j}J$ is a projective cover of $e_{i,j,j}\Lambda/e_{i,j,j}J$. So Λ is semiperfect and $e_{i,j,j}\Lambda$ are indecomposable projectives. Also $e_{i,a,a}\Lambda \cong e_{j,b,b}\Lambda$ if and only if $i = j$. So we have that $\Lambda = \bigoplus_{i=1}^t \bigoplus_{a=1}^{m_i} e_{i,a,a}\Lambda \cong \bigoplus_{i=1}^t (e_{i,1,1}\Lambda)^{m_i}$. Also, it is easy to check that $e_{i,a,a}\mathrm{M}_k(\mathbb{Q})$ has length r_i since $|A_{i,a}| = r_i$.

(2) Let k, ℓ be positive integers $p_1, \ldots, p_\ell$ pairwise different primes. Assume that for each i positive integers t_i, $r_{i,j}, m_{i,j}$, $j \in \{1, \ldots, t_i\}$ are given such that $\sum_{j=1}^{t_i} r_{i,j} m_{i,j} = k$ for every $i = 1, \ldots, \ell$. By part **(1)** there exists a semiperfect $\mathbb{Z}_{(p_i)}$-order $\Lambda(p_i)$ in $\mathrm{M}_k(\mathbb{Q})$ such that $\Lambda(p_i) \cong \bigoplus_{j=1}^{t_i} U_{i,j}^{m_{i,j}}$ where $U_{i,1}, \ldots U_{i,t_i}$ are pairwise non-isomorphic projective $\Lambda(p_i)$-modules and $(U_{i,j})_Q$ has rank $r_{i,j}$.

Now the Package Deal Theorem (here it is enough to use the noetherian version from [34]) finds a finitely generated $\mathbb{Z}$-submodule $\Lambda \subseteq \mathrm{M}_k(\mathbb{Z})$ such that

$\Lambda_{(p_i)} = \Lambda(p_i)$ for $i = 1, \dots, \ell$ and $\Lambda_{(q)} = \mathrm{M}_k(\mathbb{Z}_{(q)})$ if q is a prime different from $p_1, \dots, p_\ell$. It is easy to see that Λ has to be a subring of $\mathrm{M}_k(\mathbb{Z})$ and that $\Lambda_{(p_i)} \simeq \Lambda(p_i)$ as rings.

Remark 7.1.6. We keep the Notation 7.1.4. First, we observe that

$$\Phi\colon \mathrm{V}^*(\Lambda) \longrightarrow \prod_{\mathfrak{m}\in \mathrm{mSpec}(R)} \mathrm{V}^*(\Lambda_{\mathfrak{m}}) = \prod_{i=1}^{\ell} \mathrm{V}^*(\Lambda_{\mathfrak{m}_i}) \times \prod_{\mathfrak{m}\notin\mathcal{S}} \mathrm{V}^*(\Lambda_{\mathfrak{m}})$$

So, for any $\mathfrak{m} \notin \mathcal{S}$, $\mathrm{V}^*(\Lambda_{\mathfrak{m}}) \cong \mathbb{N}_0^*$, and for any $[P] \in \mathrm{V}^*(\Lambda)$, this isomorphism sends $[P_{\mathfrak{m}}]$ to its rank, which is $\Psi([P])$. As we are assuming that $\ell \geq 1$, the genus map is completely determined by its image in $\prod_{i=1}^{\ell} \mathrm{V}^*(\Lambda_{\mathfrak{m}_i})$.

Then, there is a monoid homomorphism

$$\Phi'\colon \mathrm{V}(\Lambda) \longrightarrow \prod_{i=1}^{\ell} \mathrm{V}(\Lambda_{\mathfrak{m}_i}) \cong \prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right)$$

$$\Phi'([P]) = ([P_{\mathfrak{m}_1}], \dots, [P_{\mathfrak{m}_\ell}]) = \left(\textstyle\sum_{j=1}^{t_1} [U_{1,j}]^{x_{1,j}}, \dots, \sum_{j=1}^{t_\ell} [U_{\ell,j}]^{x_{\ell,j}} \right).$$

Where the isomorphism $\prod_{i=1}^{\ell} V(\Lambda_{\mathfrak{m}_i}) \cong \prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right)$ assigns to $\Phi'([P])$ the element

$$((x_{1,1}, \dots, x_{1,t_1}), \dots, (x_{\ell,1}, \dots, x_{\ell,t_\ell})) \in \prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right).$$

If $P_Q \cong S^r$, then for any $i = 1, \dots, \ell$, we have the identity

$$\sum_{j=1}^{t_i} r_{i,j} x_{i,j} = \begin{bmatrix} r_{i,1} & \cdots & r_{i,t_i} \end{bmatrix} \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,t_i} \end{bmatrix} = r \tag{7.1}$$

Conversely, by Proposition 7.1.3, an element $\mathbf{X} \in \prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right)$ corresponds to a finitely generated projective right Λ-module if and only if there exists r such that the components of $\mathbf{X}$ satisfy the system of equations (7.1). This describes all the possible genus of finitely generated projective Λ-modules.

In summary, we have proved the following.

Proposition 7.1.7. *Let $\mathcal{A} = \Phi'(\mathrm{V}(\Lambda))$ which is a submonoid of $\prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right)$. Then $\mathbf{X} \in \prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right)$ is an element of $\mathcal{A}$ if and only if there exists $r \in \mathbb{N}_0$ such that the components of $\mathbf{X}$ satisfy*

$$\sum_{j=1}^{t_i} r_{i,j} x_{i,j} = \begin{bmatrix} r_{i,1} & \cdots & r_{i,t_i} \end{bmatrix} \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,t_i} \end{bmatrix} = r$$

for any $i = 1, \dots, \ell$.

If P is a finitely generated projective right Λ-module such that $\Phi'([P]) = \mathbf{X}$, then $P_Q \cong S^r$. So

$$\Psi(\mathrm{V}(\Lambda)) = \{r \in \mathbb{N}_0 \mid \textit{there exists } \mathbf{X} \in \textstyle\prod_{i=1}^{\ell} \left(\mathbb{N}_0^{t_i}\right) \textit{ satisfying } (7.1)\}.$$

Remark 7.1.8. If $\ell = 1$ and $U_1, \dots, U_t$ are the indecomposable projective modules of $\Lambda_{\mathfrak{m}}$, up to isomorphism, and $\Psi(U_i) = r_i$ for $i = 1, \dots, t$. Then $\Psi(\mathrm{V}(\Lambda))$ is the submonoid of $\mathbb{N}_0$ generated by $r_1, \dots, r_\ell$. By Proposition 7.1.3, $\Phi'(\mathrm{V}(\Lambda)) \cong \mathbb{N}_0^t$, because for any $i = 1, \dots, \ell$, there exists an (indecomposable) finitely generated projective right Λ-module P_i such that $(P_i)_{\mathfrak{m}_i} \cong U_i$.

Clearly, $\Psi(\mathrm{V}^*(\Lambda)) = \Psi(\mathrm{V}(\Lambda)) \cup \{\infty\}$ and $\Phi'(\mathrm{V}^*(\Lambda)) \cong (\mathbb{N}_0^*)^t$. Notice that, in this case, by Proposition 7.1.1, any projective Λ-module is isomorphic to a direct sum of finitely generated projective Λ-modules.

Now we proceed to extend this description of the genus to countably generated projective modules. In view of Remark 7.1.8, we may assume $\ell \geq 2$.

Proposition 7.1.9. *Extend the monoid homomorphism Φ' to $\mathrm{V}^*(\Lambda)$ and enlarge $\mathbb{N}_0$ to $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$. Then $\mathbf{Y} \in \prod_{i=1}^{\ell} ((\mathbb{N}_0^*)^{t_i})$ is an element in $\Phi'(\mathrm{B}(\Lambda)) = \Phi'(\mathrm{V}^*(\Lambda) \setminus \mathrm{V}(\Lambda))$ if and only if its components are solutions of*

$$\sum_{j=1}^{t_i} r_{i,j} y_{i,j} = \begin{bmatrix} r_{i,1} & \cdots & r_{i,t_i} \end{bmatrix} \begin{bmatrix} y_{i,1} \\ \vdots \\ y_{i,t_i} \end{bmatrix} = \infty \tag{7.2}$$

for any $i = 1, \dots, \ell$.

Proof. By Theorem 6.3.17, $V^*(\Lambda) = \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$, so $\Phi'(\mathrm{B}(\Lambda)) = \Phi'(\mathrm{V}^*(\Lambda) \setminus \mathrm{V}(\Lambda))$.

Let $\mathbf{Y} \in \prod_{i=1}^{\ell} ((\mathbb{N}_0^*)^{t_i})$. For $i = 1, \dots, \ell$, let $S_i(\mathbf{Y}) = \{j \mid y_{i,j} = \infty\}$. Observe that $\mathbf{Y}$ is a solution of the equations (7.2) if and only if $S_i(\mathbf{Y}) \neq \emptyset$ for any $i = 1, \dots, \ell$.

Let P be a countably generated right Λ-module such that it is not finitely generated. Then $\Psi(P) = \infty$, which implies that, for any $i = 1, \dots, \ell$, there exists j such that the indecomposable module $U_{i,j}$ appears infinitely many times in the decomposition of $P_{\mathfrak{m}}$ as a direct sum of indecomposable modules. Therefore, if $\mathbf{Y} = \Phi'([P])$, then $S_i(\mathbf{Y}) \neq \emptyset$ for any $i = 1, \dots, \ell$. So $\mathbf{Y}$ is a solution to the system (7.2).

To prove the converse, let $\mathbf{Y} \in \prod_{i=1}^{\ell} ((\mathbb{N}_0^*)^{t_i})$ such that $S_i(\mathbf{Y}) \neq \emptyset$ for any $i = 1, \dots, \ell$. Then we show that there is a countably generated projective module P such that $\mathbf{Y} = \Phi'([P])$.

For $i = 1, \dots, \ell$, let $P'(\mathfrak{m}_i) = \bigoplus_{j \in S_i(\mathbf{Y})} U_{i,j}$ and let r_i be the rank of $P'(\mathfrak{m}_i)$. Let r be the least common multiple of $r_1, \dots, r_\ell, k$. For $i = 1, \dots, \ell$, set $P(\mathfrak{m}_i) = P'(\mathfrak{m}_i)^{\frac{m}{r_i}}$ and set $P(\mathfrak{m}) = \Lambda_{\mathfrak{m}}^{\frac{r}{k}}$ if $\mathfrak{m}$ is a maximal ideal of R different from $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$. By Package Deal Theorem 3.3.4, there exists a finitely generated projective Λ-module P' such that $P'_{\mathfrak{m}} \cong P(\mathfrak{m})$ for any maximal ideal $\mathfrak{m}$ of R.

Let I be the trace ideal of P'. Then $I_{\mathfrak{m}_i} = \sum_{j \in S_i(\mathbf{Y})} \mathrm{Tr}(U_{i,j})$ for $i = 1, \dots, \ell$, and $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for the other maximal ideals of R. As R is h-local,

$$\Lambda/I \cong \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}.$$

For any $i = 1, \dots, \ell$, set $A_i = \{1, \dots, t_i\} \setminus S_i(\mathbf{Y})$. Then the indecomposable projective modules over $\Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}$ are $\{U_{i,j}/U_{i,j}I_{\mathfrak{m}_i}\}_{j \in A_i}$.

Now consider the finitely generated projective Λ/I-module

$$\overline{P} = \bigoplus_{i=1}^{\ell} \Big(\bigoplus_{j \in A_i} (U_{i,j}/U_{i,j}I_{\mathfrak{m}_i})^{y_{i,j}} \Big).$$

By Theorem 1.5.4, there exists a countably generated projective Λ-module P such that $P/PI \cong \overline{P}$. Then $P \oplus (P')^{(\omega)}$ is an I-big countably generated projective Λ-module such that $\Psi'(P \oplus (P')^{(\omega)}) = \mathbf{Y}$. □

Remark 7.1.10. In Proposition 7.1.9, the condition on $\mathbf{Y} \in \prod_{i=1}^{\ell} ((\mathbb{N}_0^*)^{t_i})$ is equivalent to saying that, for each $i \in \{1, \dots, \ell\}$, at least one of the components of $(y_{i,1}, \dots, y_{i,t_i})$ is ∞.

We have kept the notation of a system of linear equations because it is consistent with the characterization of the monoids that can be realized as $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ for Λ a semilocal ring given in [22]. Namely, the description of $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ for Λ a semilocal ring can always be given in terms of the solutions in $\mathbb{N}_0^*$ of a system of linear equations and linear congruences with coefficients in $\mathbb{N}_0$. However, it is not at all clear which invariants of the semilocal ring determine the coefficients of the system defining the monoid of relatively big projectives.

In our situation, by Proposition 7.1.2, Λ_Σ where $\Sigma = R \setminus \bigcup_{i=1}^{\ell} \mathfrak{m}_i$ is a locally semiperfect semilocal algebra over the semilocal domain R_Σ. We are proving that, at least for the genus, the system of equations that determines the monoid can be deduced from the structural properties of Λ and its localizations at the maximal ideals of R. Note that if Λ is finitely generated over R, then, by Proposition 7.1.2, $\mathrm{V}^*(\Lambda_\Sigma) = \mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) \cong \Phi'(\mathrm{V}^*(\Lambda))$.

7.2 Infinite sums of finitely generated projective modules

In this section, we continue working with an algebra Λ over an h-local domain R, with field of fractions Q, and such that Λ_Q is simple artinian.

We keep using the notation from the previous section. In particular, the one introduced in Notation 7.1.4. As in Proposition 7.1.7, let $\mathcal{A} = \Phi'(\mathrm{V}(\Lambda))$. If ℓ is either 0 or 1, then every projective module is a direct sum of finitely generated ones (cf. Remark 7.1.8). So we assume $\ell \geq 2$.

Let $\mathcal{B}$ be the image of $\mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda) \setminus \mathrm{V}(\Lambda)$ in $\prod_{i=1}^{\ell} ((\mathbb{N}_0^*)^{t_i})$.

By Proposition 7.1.1, the genus classifies the modules of $\mathrm{B}(\Lambda)$ up to isomorphism. Therefore, the modules in $\mathrm{B}(\Lambda)$ that are isomorphic to a direct sum of finitely generated modules are those such that the image of their isomorphism class via Φ' lies in

$$\mathcal{A} + \infty \cdot (\mathcal{A} \setminus \{0\}) = \{a + \infty \cdot a' \mid a, a' \in \mathcal{A} \text{ and } a' \neq 0\}.$$

In this section we will examine the question of whether any projective Λ-module is a direct sum of finitely generated modules. By Proposition 5.1.13 and Lemma 6.3.7, we only need to characterize when finitely generated projective modules modulo an ideal of $\mathcal{T}(\Lambda)$ lift to finitely generated projective modules. Our next lemma shows that we only need to ensure this property for the minimal elements in $\mathcal{T}(\Lambda) \setminus \{0\}$.

Lemma 7.2.1. *Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is simple artinian. The following are equivalent*

(i) *For any $I \in \mathcal{T}(\Lambda)$ and any finitely generated projective Λ/I-module P', there exists a finitely generated projective Λ-module P such that $P' \cong P/PI$.*

(ii) *For any I which is a minimal element of $\mathcal{T}(\Lambda)\setminus\{0\}$ and any finitely generated projective Λ/I-module P', there exists a finitely generated projective Λ-module P such that $P' \cong P/PI$.*

Proof. Of course, only **(ii)** $\Rightarrow$ **(i)** needs to be proved. To prove **(ii)** $\Rightarrow$ **(i)**, let I be a non-zero trace of Λ containing a minimal trace $X \in T(\Lambda) \setminus \{0\}$. Such an X exists by Corollary 6.3.11.

By Lemma 3.3.2, there exist $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ maximal ideals of R such that $\Lambda/X = \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/X_{\mathfrak{m}_i}$. Thus, it is a finite product of semiperfect rings. Therefore, Λ/X is semiperfect.

Let P' be a finitely generated projective Λ/I-module. By Lemma 6.3.1, there exists a finitely generated projective Λ/X-module P_1 such that $P_1/P_1I \cong P'$. Now we can use **(ii)** to find a finitely generated projective Λ-module P such that $P/PX = P_1$. Then $P/PI \cong (P/PX)/(PI/PX) \cong (P_1/P_1I) \cong P'$. □

The next lemma will be useful for checking when the criteria of Lemma 7.2.1 is satisfied.

Lemma 7.2.2. *Let $2 \leq \ell \in \mathbb{N}$. Let $r_1, \dots, r_\ell, s \in \mathbb{N}$. Let $m = \operatorname{lcm}(r_1, \dots, r_{\ell-1}) > 0$. Then the system*

$$\begin{bmatrix} r_1 & 0 & \dots & 0 & -s \\ 0 & r_2 & \dots & 0 & -s \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & r_{\ell-1} & -s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{bmatrix} = \begin{bmatrix} r_\ell \\ r_\ell \\ \vdots \\ r_\ell \end{bmatrix} \tag{7.3}$$

has a solution in $\mathbb{N}$ if and only if the equation $my - sx_\ell = r_\ell$ has a solution in $\mathbb{N}$. If (y, s) is such a solution, then

$$\left(x_1 = \frac{m}{r_1}y, \dots, x_{\ell-1} = \frac{m}{r_{\ell-1}}y, x_\ell\right)$$

is a solution of (7.3)*. Therefore,* (7.3) *has a solution in $\mathbb{N}$ if and only if $\gcd(m, s)$ divides r_ℓ.*

Proof. The claim follows because

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ -1 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} r_1 & 0 & \dots & 0 & -s \\ 0 & r_2 & \dots & 0 & -s \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & r_{\ell-1} & -s \end{bmatrix} = \begin{bmatrix} r_1 & 0 & \dots & 0 & -s \\ -r_1 & r_2 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -r_1 & 0 & \dots & r_{\ell-1} & 0 \end{bmatrix}$$

so the system (7.3) is equivalent to the system

$$\begin{bmatrix} r_1 & 0 & \dots & 0 & -s \\ -r_1 & r_2 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -r_1 & 0 & \dots & r_{\ell-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{bmatrix} = \begin{bmatrix} r_\ell \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which has a solution in $\mathbb{N}$ if and only if the equation $my - sx_\ell = r_\ell$ has a solution in $\mathbb{N}$. Also, it is clear that the solutions are related, as claimed in the statement. □

Proposition 7.2.3. *Under the Notation 7.1.4, the following are equivalent*

(i) *Every projective Λ-module is a direct sum of finitely generated modules.*

(ii) *For every choosing $a_1, a_2, \dots, a_\ell$, $1 \leq a_i \leq t_i$, and for every $j \in \{1, \dots, \ell\}$ and $1 \leq b \neq a_j \leq t_j$ there are non-negative integers $x_1, x_2, \dots, x_\ell$ such that $r_{j,b} + x_j r_{j,a_j} = x_i r_{i,a_i}$ for every $i \neq j$.*

(iii) *For every choosing $a_1, a_2, \dots, a_\ell$, $1 \leq a_i \leq t_i$, and for every $j \in \{1, \dots, \ell\}$ and $1 \leq b \neq a_j \leq t_j$ then $\gcd(\operatorname{lcm}(\{r_{1,a_1}, \dots, r_{\ell,a_\ell}\} \setminus \{r_{j,a_j}\}), r_{j,a_j})$ divides $r_{j,b}$.*

Proof. Assume **(i)** is true, so finitely generated projective modules of Λ modulo a (minimal) element of $\mathcal{T}(\Lambda) \setminus \{0\}$ can be lifted to finitely generated projective Λ-modules.

By Remark 6.3.10, Λ has an ideal $I \in \mathcal{T}(\Lambda)$ such that $I_{\mathfrak{m}_i} = \operatorname{Tr}_{\Lambda_{\mathfrak{m}_i}}(U_{i,a_i})$ for each $i = 1, \dots, \ell$. As Λ_Q is simple artinian and R is h-local, $\Lambda/I \cong \bigoplus_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}$. Therefore, every (finitely generated) projective Λ/I-module is of the form $\bigoplus_{i=1}^{\ell} M_i$, where M_i is a (finitely generated) projective $(\Lambda/I)_{\mathfrak{m}_i}$-module. Note that $(\Lambda/I)_{\mathfrak{m}_i} \cong \Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}$, which is a semiperfect ring, so projective modules over it are described in Proposition 6.3.2 **(iv)**.

Consider a projective Λ/I-module P' such that $P'_{\mathfrak{m}_j} \cong U_{j,b}/U_{j,b}I_{\mathfrak{m}_j}$ and $P'_{\mathfrak{m}_i} = 0$ if $i \neq j$. By assumption, there exists P a finitely generated projective Λ-module such that $P/PI \cong P'$. Then $P_{\mathfrak{m}_i} \cong U_{i,a_i}^{x_i}$ if $i \neq j$ and $P_{\mathfrak{m}_j} \cong U_{j,b} \oplus U_{j,a_j}^{x_j}$. Since $(P_{\mathfrak{m}_i})_Q \cong P_Q \cong (P_{\mathfrak{m}_j})_Q$, we get $x_i r_{i,a_i} = r_{j,b} + x_j r_{j,a_j}$. So **(ii)** holds.

Conversely, let I be a minimal non-zero trace of Λ. Then $I_{\mathfrak{m}_i} = \operatorname{Tr}_{\Lambda_{\mathfrak{m}_i}}(U_{i,a_i})$ for some a_i, $i \in \{1, \dots, \ell_i\}$. Recall that $\Lambda/I \cong \bigoplus_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/I_{\mathfrak{m}_i}$ is a semiperfect ring. Hence, it is sufficient to lift an indecomposable projective Λ/I-module to a finitely generated projective Λ-module. So let P' be a finitely generated indecomposable projective Λ/I-module. Regarding the decomposition of Λ/I, $P'_{\mathfrak{m}_i} = 0$ for all i's with exactly one exception. Assume $P'_{\mathfrak{m}_j} \cong U_{j,b}/U_{j,b}I_{\mathfrak{m}_j}$, where $b \neq a_j$. Take $x_1, \dots, x_\ell \in \mathbb{N}_0$ such that the equations in **(ii)** are satisfied. Let $r = r_{j,b} + x_j r_{j,a_j}$.

For $i = 1, \dots, \ell$ set $U(\mathfrak{m}_i) = U_{i,a_i}^{x_i}$ if $i \neq j$ and $U(\mathfrak{m}_j) = U_{j,b} \oplus U_{j,a_j}^{x_j}$. For $\mathfrak{n} \in \operatorname{mSpec}(R) \setminus \{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell\}$, let $U(\mathfrak{n})$ be a projective $\Lambda_{\mathfrak{n}}$-module such that $U(\mathfrak{n})_Q$ has length r (it exists by Lemma 6.3.9). Because of the choice of $x_1, \dots, x_\ell$, we can use Proposition 7.1.3 to show the existence of a finitely generated projective Λ-module P such that $P_{\mathfrak{m}} \cong U(\mathfrak{m})$ for any maximal ideal $\mathfrak{m}$ of R. Then $P_{\mathfrak{m}_i} I_{\mathfrak{m}_i} = P_{\mathfrak{m}_i}$ if $i \neq j$ and $P_{\mathfrak{m}_j}/P_{\mathfrak{m}_j} I_{\mathfrak{m}_j} \cong U_{j,b}/U_{j,b}I_{\mathfrak{m}_j}$. Therefore $P/PI \cong \bigoplus_{i=1}^{\ell} P_{\mathfrak{m}_i}/P_{\mathfrak{m}_i} I_{\mathfrak{m}_i} \cong P'$. This proves **(ii)** $\Rightarrow$ **(i)**.

The equivalence of **(ii)** and **(iii)** follows from Lemma 7.2.2. □

Example 7.2.4. Under the assumptions in Notation 7.1.4, let $\ell = 2$, and assume that $U_{1,1}, U_{1,2}$ and $U_{2,1}, U_{2,2}$ are representatives of the finitely generated indecomposable projective right modules over $\Lambda_{\mathfrak{m}_1}$ and $\Lambda_{\mathfrak{m}_2}$, respectively. Suppose that

$$r_{1,1} = 2, \quad r_{1,2} = 4, \quad r_{2,1} = 3, \quad r_{2,2} = 9.$$

Since

$$12 = r_{1,1}2 + r_{1,2}2 = r_{2,1} + r_{2,2},$$

Λ can be taken to be an algebra over an h-local domain R with infinitely many maximal ideals, such that

(i) $\Lambda_Q \cong \mathrm{M}_{12}(T_Q)$;

(ii) $\Lambda_{\mathfrak{m}_1}$ is a semiperfect ring such that modulo its Jacobson radical is isomorphic to $\mathrm{M}_2(T_{1,1}) \times \mathrm{M}_2(T_{1,2})$, and $\Lambda_{\mathfrak{m}_2}$ is a semiperfect ring that is a product of two division rings modulo the Jacobson radical;

(iii) for any other maximal ideal $\mathfrak{m}$ of R the localization $\Lambda_{\mathfrak{m}} \cong \mathrm{M}_{12}(T_{\mathfrak{m}})$ where $T_{\mathfrak{m}}$ is a local ring.

By Examples 7.1.5, such a locally semiperfect algebra Λ exists and can be even taken to be a finitely generated $\mathbb{Z}$-algebra.

By Proposition 7.1.7, to compute $\mathcal{A} = \Phi'(\mathrm{V}(\Lambda))$ in $\mathbb{N}_0^2 \times \mathbb{N}_0^2$ we need to find the $((x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2})) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2$ which are solutions of the system of equations:

$$\begin{aligned} 2x_{1,1} + 4x_{1,2} &= r \\ 3x_{2,1} + 9x_{2,2} &= r, \end{aligned} \tag{7.4}$$

for some fixed $r \in \mathbb{N}_0$.

Since $\operatorname{lcm}(\gcd(2,4), \gcd(3,9)) = 6 \mid r$, the image of the rank function Ψ lies in $6\mathbb{N}_0$ and, since there are finitely generated projective modules of rank 6, in fact $\Psi(\mathrm{V}(\Lambda)) = 6\mathbb{N}_0$.

As each $r_{i,j}$ is coprime to $r_{i',j'}$ for every $i' \neq i$ and $j' \neq j$, statement (iii) in Proposition 7.2.3 is satisfied. So every projective Λ-module is a direct sum of finitely generated modules.

Denote an element in $\mathcal{A}$ by

$$((x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2}))_r,$$

where $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, r \in \mathbb{N}_0$ satisfy (7.4). Hence, a minimal set of generators of $\mathcal{A}$ is

$$\begin{gathered} ((3,0),(2,0))_6, \quad ((1,1),(2,0))_6, \quad ((0,3),(4,0))_{12}, \\ ((6,0),(1,1))_{12}, \quad ((4,1),(1,1))_{12}, \quad ((2,2),(1,1))_{12}, \quad ((0,3),(1,1))_{12}, \\ ((9,0),(0,2))_{18}, \quad ((7,1),(0,2))_{18}, \quad ((5,2),(0,2))_{18}, \\ ((3,3),(0,2))_{18}, \quad ((1,4),(0,2))_{18}, \quad ((0,9),(0,4))_{36}. \end{gathered}$$

So the isomorphism class of finitely generated projective Λ-module has as an image in $\mathcal{A}$ a non-negative integral linear combination of elements in this generating set.

Recall that if Λ is taken to be finitely generated over a noetherian domain R then, by Proposition 7.1.2, $\mathcal{A} \cong V(\Lambda)$.

If we extend the monoid homomorphism Φ' to $\mathrm{V}^*(\Lambda)$ and enlarge $\mathbb{N}_0$ to $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$, then we can also compute $\mathcal{B} = \Phi'(\mathrm{B}(\Lambda))$ in $(\mathbb{N}_0^*)^2 \times (\mathbb{N}_0^*)^2$. A minimal set of generators of $\mathcal{B}$ is

$$\begin{gathered} ((\infty,0),(\infty,0)), \quad ((\infty,0),(0,\infty)), \quad ((0,\infty),(\infty,0)), \quad ((0,\infty),(0,\infty)), \\ ((\infty,1),(\infty,0)), \quad ((\infty,1),(0,\infty)), \quad ((1,\infty),(\infty,0)), \quad ((1,\infty),(0,\infty)), \\ ((\infty,0),(\infty,1)), \quad ((\infty,0),(1,\infty)), \quad ((0,\infty),(\infty,1)), \quad ((0,\infty),(1,\infty)). \end{gathered}$$

and, as explained before, all of them correspond to the isomorphism class of a countably generated projective module that is a direct sum of finitely generated modules. For example,

$$((\infty,1),(\infty,0)) = ((1,1),(2,0))_6 + \infty \cdot ((3,0),(2,0))_6.$$

Example 7.2.5. Under the assumptions in Notation 7.1.4, there are examples of algebras Λ where there is a projective Λ-module which is not a direct sum of finitely generated modules. Set now

$$r_{1,1} = 2, \quad r_{1,2} = 8, \quad r_{2,1} = 2, \quad r_{2,2} = 4.$$

By Examples 7.1.5, we know there exist locally semiperfect algebras Λ with such invariants.

To determine $\mathcal{A} = \Phi'(\mathrm{V}(\Lambda))$ we have to solve the following system of linear equations:

$$\begin{aligned} 2x_{1,1} + 8x_{1,2} &= r \\ 2x_{2,1} + 4x_{2,2} &= r. \end{aligned} \tag{7.5}$$

Since $\gcd(\{r_{1,2}, r_{2,2}\} \setminus \{r_{1,2}\}, r_{1,2}) = \gcd(4, 8) = 4 \nmid 2 = r_{1,1}$, Proposition 7.2.3 (iii) fails. Therefore, in this case, there are projective Λ-modules which are not a direct sum of finitely generated modules. We are going to compute an explicit example of such a module.

As in Example 7.2.4, if we extend the monoid homomorphism Φ' to $\mathrm{V}^*(\Lambda)$ and enlarge $\mathbb{N}_0$ to $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$, then we can compute $\mathcal{B} = \Phi'(\mathrm{B}(\Lambda))$ in $(\mathbb{N}_0^*)^2 \times (\mathbb{N}_0^*)^2$. The generating set is the same as in Example 7.2.4, because it is independent of the $r_{i,j}$'s chosen, it only depends on how many of them there are.

Let $[P] \in \mathrm{B}(\Lambda)$ be such that $\Phi'([P]) = ((0, \infty), (1, \infty))$. Then there is no $\mathbf{X} \in \mathbb{N}_0^2 \times \mathbb{N}_0^2$ of the form $\mathbf{X} = ((0, x_{1,2}), (1, x_{2,2}))$ being a solution of (7.5) since $8x_{1,2} \equiv 0 \mod 2$ and $1 + 4x_{2,2} \equiv 1 \mod 2$. Therefore, there is no finitely generated projective Λ-module P' such that $\Phi'([P']) = ((0, x_{1,2}), (1, x_{2,2}))$, so P is a countably generated projective Λ-module which is not a direct sum of finitely generated modules.

Let P_1 be a finitely generated Λ-module such that $\Phi'([P_1]) = ((0, 1), (0, 2))_8$, and let P_2 be such that $\Phi'([P_2]) = ((0, 1), (2, 1))_8$. By Proposition 7.1.1, $P \cong P \oplus P_1^{(\omega)}$ and $P^2 \cong P_2 \oplus P_1^{(\omega)}$. Thus P^2 is a direct sum of finitely generated projective modules.

7.3 Torsion-free modules over h-local domains

In this section, we will extensively use the notation introduced in Subsection 6.2.3 and Proposition 1.2.12. We apply the previous results to $\mathrm{Add}(M)$ for M a finitely generated torsion-free module M over an h-local domain.

Let M_R be a finitely generated torsion-free module of rank k over an h-local domain R. Following Lemma 2.1.22, let $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$ be the list of those maximal ideals of R over which $M_{\mathfrak{m}}$ is not free. Assume further that Λ is locally semiperfect. That is, each $M_{\mathfrak{m}}$ is a direct sum of modules with local endomorphism rings, cf. Example 6.3.6. For $i = 1, \ldots, \ell$, let $M_{\mathfrak{m}_i} \cong \bigoplus_{j=1}^{t_i} M_{i,j}^{m_{i,j}}$, where $M_{i,1}, \ldots, M_{i,t_i}$ are pairwise non-isomorphic indecomposable modules and $m_{i,j} \in \mathbb{N}$. By Proposition 1.2.12, $\Lambda_{\mathfrak{m}_i} \cong \bigoplus_{j=1}^{t_i} U_{i,j}^{m_{i,j}}$, where $U_{i,j} = \mathrm{Hom}_{R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, M_{i,j})$ is an indecomposable decomposition of $\Lambda_{\mathfrak{m}_i}$. Let $r_{i,j}$ be the length of $(U_{i,j})_Q$. As a Q-module, $(U_{i,j})_Q = Q^{kr_{i,j}}$. On the other hand, by Lemma 2.1.11, $(U_{i,j})_Q \cong \mathrm{Hom}_Q(M_Q, (M_{i,j})_Q) \cong (M_{i,j})_Q^k = Q^{k \operatorname{rank}(M_{i,j})}$. Therefore, $r_{i,j} = \operatorname{rank}(M_{i,j})$.

Lemma 7.3.1. *Let R be an h-local domain of Krull dimension* 1*, and let M be a finitely generated torsion-free R-module which is a generator of* $\mathbf{Mod}_R$*, i.e.,* $\mathrm{Tr}_R(M) = R$*. Assume that every module of* $\mathrm{Add}(M)$ *is a direct sum of finitely generated modules. Then*

(i) $\mathrm{End}_R(M)$ *is a locally semiperfect ring.*

(ii) *for every maximal ideal $\mathfrak{m}$ of R, $M_\mathfrak{m}$ contains a direct summand isomorphic to $R_\mathfrak{m}$.*

Proof. The assumption $\mathrm{Tr}_R(M) = R$ is equivalent to $R \in \mathrm{add}(M)$. In particular, $\mathrm{Add}(M)$ contains $R^{(\omega)} \oplus M$, so **(i)** follows from [1, Proposition 6.1]. By Lemma 2.1.12, $\mathrm{Tr}_{R_\mathfrak{m}}(M_\mathfrak{m}) = R_\mathfrak{m}$ for any maximal ideal $\mathfrak{m} \in \mathrm{mSpec}(R)$, hence $R_\mathfrak{m} \in \mathrm{add}(M_\mathfrak{m})$. Since $R_\mathfrak{m}$ is local, $R_\mathfrak{m}$ has to be isomorphic to a direct summand of $M_\mathfrak{m}$ (otherwise, $\mathrm{Tr}_{R_\mathfrak{m}}(M_\mathfrak{m}) \subseteq \mathfrak{m}R_\mathfrak{m}$). □

Now we are ready to prove our main application to direct sum decompositions of modules.

Theorem 7.3.2. *Let R be an h-local domain, and let M be a non-zero finitely generated torsion-free R-module with endomorphism ring $\Lambda = \mathrm{End}_R(M)$. Consider the following statements,*

(i) *Λ is locally semiperfect and if X and Y are indecomposable direct summands of $M_\mathfrak{m}$ and $M_\mathfrak{n}$, respectively, for some $\mathfrak{m} \neq \mathfrak{n} \in \mathrm{mSpec}(R)$. Then* $\mathrm{rank}(X)$ *and* $\mathrm{rank}(Y)$ *are coprime.*

(ii) *Every module in* $\mathrm{Add}(M)$ *is a direct sum of finitely generated modules.*

Then **(i)** *always implies* **(ii)** *and the converse is true provided M is a generator and R has Krull dimension* 1*.*

Proof. Notice that by Lemma 2.1.22, Λ is torsion-free as an R-module, and Λ_Q is simple artinian.

Let us show that **(i)** ⇒ **(ii)**. Assume that **(i)** is true. We will use the notation introduced at the beginning of the section. Note that $r_{i,a}$ and $r_{j,b}$ are coprime whenever $i \neq j$, $a \in \{1, \dots, t_i\}$, $b \in \{1, \dots, t_j\}$. We want to check that **(ii)** of Proposition 7.2.3 is true.

Consider $a_1, \dots, a_\ell$, $a_i \in \{1, \dots, t_i\}$, and $A = \prod_{i=1}^{\ell} r_{i,a_i}$. Fix $1 \leq j \leq \ell$. Since $r_{1,a_1}, \dots, r_{\ell,a_\ell}$ are pairwise coprime, there exists $t \in \{0, \dots, A-1\}$ such that $t \equiv 0 \pmod{r_{i,a_i}}$ if $i \neq j$ and $t \equiv r_{j,b} \pmod{r_{j,a_j}}$. If $t < r_{j,b}$ we can increase t by a suitable multiple of A, so we may assume $t \geq r_{j,b}$. Set $x_i := t/r_{i,a_i}$ if $i \neq j$ and $x_j := (t - r_{j,b})/r_{j,a_j}$. Then $x_1, \dots, x_\ell$ are non-negative integers, and $t = x_i r_{i,a_i}$ if $i \neq j$ and $t = x_j r_{j,a_j} + r_{j,b}$. In other words, the condition **(ii)** of Proposition 7.2.3 is true. Then any projective Λ-module is a direct sum of finitely generated modules. **(ii)** follows from Proposition 1.2.12.

Now, assume that **(ii)** is true, and suppose that $\mathrm{Tr}_R(M) = R$, and R has Krull dimension 1. Then, by Lemma 7.3.1, Λ is locally semiperfect. Keeping the notation of Proposition 7.2.3, we have to prove that $r_{i,a}$ and $r_{j,b}$ are coprime whenever $i \neq j$, $a \in \{1, \dots, t_i\}$, and $b \in \{1, \dots, t_j\}$. Note that it is trivial if either $r_{i,a} = 1$ or $r_{j,b} = 1$. By Lemma 7.3.1, for any $\mathfrak{m} \in \mathrm{mSpec}(R)$, $R_\mathfrak{m}$ is a direct

summand of $M_{\mathfrak{m}}$ (so $M_{\mathfrak{m}}$ has a direct summand of rank 1, cf. the remarks before Lemma 7.3.1). Therefore, if $r_{i,a} > 1$, then there exists $c \in \{1, \ldots, t_i\}$ such that $r_{i,c} = 1$. Then $r_{i,c} + xr_{i,a} = yr_{j,b}$ is solvable only if $r_{i,a}, r_{j,b}$ are coprime. Now **(i)** follows from Proposition 7.2.3. □

8 Locally semiperfect algebras and representations of finite groups

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In this chapter, we prove Theorem G from the introduction.

Theorem G (Corollary 8.2.5)**.** *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect module-finite torsion-free R-algebra such that Λ_Q is semisimple artinian. Let $\{\mathbf{D}_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$ denote the decomposition matrices associated with Λ. For each maximal ideal $\mathfrak{m}$ of R, let $U_{\mathfrak{m},1}, \dots, U_{\mathfrak{m},\ell_\mathfrak{m}}$ be a set of representatives of the indecomposable projective right $\Lambda_\mathfrak{m}$-modules, and fix $\boldsymbol{r}_\mathfrak{m} \in (\mathbb{N}_0^*)^{\ell_\mathfrak{m}}$.*

Then, there exists a countably generated projective right Λ-module P with genus sequence $\{\boldsymbol{r}_\mathfrak{m}\}_{\mathfrak{m}\in\mathrm{mSpec}(R)}$ if and only if for any pair of maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R

$$\mathbf{D}_\mathfrak{m}\boldsymbol{r}_\mathfrak{m}^t = \mathbf{D}_\mathfrak{n}\boldsymbol{r}_\mathfrak{n}^t.$$

In Section 8.1, we extend results from the world of maximal orders over Dedekind domains to locally semiperfect algebras over h-local domains with semisimple artinian ring of quotients. We study the structure of a locally semiperfect algebra modulo a trace ideal of a countably generated projective right Λ-module and conclude with some Package Deal results.

In Section 8.2, we introduce decomposition matrices associated with a locally semiperfect algebra Λ over a commutative domain R with semisimple artinian ring of quotients Λ_Q. In this section, we prove Theorem G.

In Section 8.3, we review known results about idempotent ideals of group algebras of finite groups over a noetherian domain. Over a noetherian ring, trace ideals are precisely the idempotent ideals. We provide an algorithm to compute every full idempotent ideal in a group algebra with respect to either $\mathrm{Aug}(\Lambda)$ or Λ.

In Section 8.4, we apply these results to develop complete classifications, up to genus, of infinitely generated projective modules over group algebras of finite groups. We focus on the first two simple alternating groups: A_5 and A_6.

8.1 Locally semiperfect algebras with semisimple ring of quotients

In this section, our aim is to extend the results of Section 6.3 for Λ a locally semiperfect module-finite algebra over a commutative noetherian domain Λ_Q is semisimple artinian. An important example where this occurs is in the context of orders. Orders extend the notion of the ring of integers in an algebraic number field to non-necessarily commutative algebras. They are studied within the frameworks of algebraic number theory, representation theory, and non-commutative algebra.

8.1.1 Projective modules over maximal orders

We now recall the definition of order.

Definition 8.1.1. Let R be a commutative noetherian domain with field of fractions Q, and let A be a finite dimensional Q-algebra. An *R-order* in the Q-algebra A is a subring Λ of A with $1_\Lambda = 1_A$ such that it is a finitely generated R-submodule in A and $\Lambda_Q = A$. Note that Λ is both left and right noetherian, since Λ is finitely generated over the noetherian domain R.

Remark 8.1.2. Note that in this context, Λ is a module-finite torsion-free R-algebra with $\Lambda_Q = A$.

Examples 8.1.3.

(1) Let R be a commutative noetherian domain with field of fractions Q, and let $A = \mathrm{M}_n(Q)$. Then $\Lambda = \mathrm{M}_n(R)$ is an R-order in A.

(2) Let R be a Dedekind domain with field of fractions Q, and let L be a finite separable extension of Q. Let S be the integral closure of R in L. Then S is an R-order in L (cf. Theorem 4.7 in [55]). In particular, the ring of integers of any algebraic number field L is a $\mathbb{Z}$-order in L.

(3) Let R be a commutative noetherian domain with field of fractions Q, and let $A = QG$, where G is a finite group. Then $\Lambda = RG$ is an R-order in A.

Thus, the study of orders includes the study of rings of integers in algebraic number theory, as well as the integral representation theory of finite groups. We will focus on group algebras at the end of the chapter.

A particularly good situation arises when the algebra A is separable. When A is separable, every order is contained in a maximal order (Corollary 10.4 in [55]), and maximal orders are left and right hereditary (Theorem 21.4 in [55]). We recall the notions of separability and split semisimplicity.

Definition 8.1.4. Let Q be a field, and let A be a Q-algebra. Let M be a right A-module. We say that

(1) A is *separable* if for every field extension E/Q, $A \otimes_Q E$ is semisimple artinian.

(2) M is *absolutely simple* if for every field extension E/Q, $M \otimes_Q E$ is a simple $A \otimes_Q E$-module.

(3) A is *split semisimple* if A is a finite dimensional semisimple artinian Q-algebra and every simple right A-module is absolutely simple.

Clearly, every split semisimple algebra is separable, and every separable algebra is semisimple artinian.

Proposition 8.1.5 (Theorem 3.34 in [11]). *Let Q be a field, and let A be a finite-dimensional semisimple artinian Q-algebra such that $A \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$. Then the following are equivalent:*

(i) *A is a split semisimple Q-algebra;*

(ii) *$A \cong \mathrm{M}_{n_1}(Q) \times \cdots \times \mathrm{M}_{n_k}(Q)$, that is, $D_i \cong Q$ for each $i = 1, \ldots, k$;*

(iii) *$\mathrm{End}_A(S) \cong Q$ for each simple right A-module S.*

Lemma 8.1.6. *Let R be a Dedekind domain with field of fractions Q. Let Λ be an R-order in a separable Q-algebra A. Let $A = e_1A \oplus \cdots \oplus e_kA$, where $e_1, \ldots, e_k$ are the primitive central idempotents of A. Then*

(i) *There exists a finite set $\mathcal{S}(\Lambda)$ of maximal ideals of R such that for any $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $\Lambda_{\mathfrak{m}}$ is a maximal $R_{\mathfrak{m}}$-order in A. In particular, $\Lambda_{\mathfrak{m}} = e_1\Lambda_{\mathfrak{m}} \oplus \cdots \oplus e_k\Lambda_{\mathfrak{m}}$ for every $\mathfrak{m} \notin \mathcal{S}(\Lambda)$.*

(ii) *For each maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, if M and N are finitely generated right $\Lambda_{\mathfrak{m}}$-modules which are torsion-free as $R_{\mathfrak{m}}$-modules, then $M \cong N$ if and only if $M \otimes_{R_{\mathfrak{m}}} Q \cong N \otimes_{R_{\mathfrak{m}}} Q$ as right A-modules.*

(iii) *For each maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, let M be a finitely generated right $\Lambda_{\mathfrak{m}}$-module which is torsion-free as an R-module. Then M is projective as a $\Lambda_{\mathfrak{m}}$-module. Moreover, M is indecomposable if and only if M_Q is a simple A-module.*

(iv) *Let I be a trace ideal of a projective right Λ-module, and let $B \subset \{1, \ldots, k\}$ be such that $\Lambda_Q/I_Q \cong \bigoplus_{i \in B} e_iA$. Then, for any maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $(\Lambda/I)_{\mathfrak{m}} \cong \Lambda_{\mathfrak{m}}/I_{\mathfrak{m}} \cong \bigoplus_{i \in B} e_i\Lambda_{\mathfrak{m}}$.*

Proof. **(i)** Assume that Λ is not a maximal order. Otherwise, the statement is trivial by Theorem 11.5 in [55] (the property of being a maximal order localizes). By Corollary 10.4 in [55], Λ is contained in a maximal R-order Γ in A. Then Γ/Λ is a finitely generated torsion R-module, so there is a non-zero element $d \in R$ such that $d\Gamma \subseteq \Lambda$. Since R has Krull dimension 1, d is contained in only finitely many maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{S}(\Lambda)$ be the set of all maximal ideals of R containing d, which is a finite set. For any maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $d/1$ is a unit, so $\Gamma_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$.

The final statement follows from Theorem 10.5 in [55].

(ii) follows from **(i)** and Theorem 18.10 in [55].

(iii) The first part follows because maximal orders are left and right hereditary (cf. Theorem 21.4 in [55]). The second part is Corollary 21.5 in [55].

(iv) follows from **(i)** and **(iii)**. □

8.1.2 Locally semiperfect algebras modulo a trace ideal

We are going to generalize the situation of Lemma 8.1.6 to locally semiperfect torsion-free algebras over an h-local domain with semisimple ring of quotients. First, we need the following lemma.

Lemma 8.1.7. *Let R be a domain with field of fractions Q. Let Λ be a torsion-free R-algebra, and let I be a two-sided ideal of Λ. Then $I \cap R \neq \{0\}$ if and only if $I \otimes_R Q = \Lambda_Q$.*

Proof. If there is a non-zero $d \in I \cap R$, then it is an invertible element in $I \otimes_R Q$, so $I \otimes_R Q = \Lambda_Q$. On the other hand, if $1 \in I \otimes_R Q$, then there exist $d \in R \setminus \{0\}$ such that $1 = \frac{\lambda}{d}$ for $\lambda \in I$. Therefore, $d \in I \cap R$. □

Lemma 8.1.8. *Let R be an h-local domain with field of fractions Q. Let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable $D_1, \ldots, D_k$ division rings. Then*

- **(i)** *There exists a finite set $\mathcal{S}(\Lambda)$ of maximal ideal of R such that for any $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $\Lambda_\mathfrak{m} \cong \mathrm{M}_{n_1}(\Lambda_1^\mathfrak{m}) \times \cdots \times \mathrm{M}_{n_k}(\Lambda_k^\mathfrak{m})$ with $\Lambda_i^\mathfrak{m}$ a local torsion-free R-algebra such that $\Lambda_i^\mathfrak{m} \otimes_{R_\mathfrak{m}} Q \cong D_i$ for any $i = 1, \ldots, k$.*
- **(ii)** *Let $\mathfrak{m}$ be a maximal ideal of R that is not in $\mathcal{S}(\Lambda)$, and let P be a $\Lambda_\mathfrak{m}$-module. If P is an indecomposable projective $\Lambda_\mathfrak{m}$-module then $P \otimes_{R_\mathfrak{m}} Q$ is an indecomposable projective Λ_Q-module.*
- **(iii)** *Let I be a trace ideal of a projective Λ-module, and let $B \subset \{1, \ldots, k\}$ be such that $\Lambda_Q/I_Q \cong \prod_{i \in B} \mathrm{M}_{n_i}(D_i)$. Then, for any $\mathfrak{m}$ maximal ideal of R that is not in $\mathcal{S}(\Lambda)$, $(\Lambda/I)_\mathfrak{m} \cong \Lambda_\mathfrak{m}/I_\mathfrak{m} \cong \prod_{i \in B} \mathrm{M}_{n_i}(\Lambda_i^\mathfrak{m})$.*

  In particular, $(\Lambda/I)_\mathfrak{m}$ is a torsion-free R-algebra for any $\mathfrak{m} \notin \mathcal{S}(\Lambda)$.

Proof. **(i)** Fix an isomorphism $\varphi\colon \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k) \longrightarrow \Lambda_Q$. Let $d \in R \setminus \{0\}$ be a common denominator for the central idempotents of Λ_Q, as well as for all matrices $\{\varphi(0, \ldots, e_{\ell t}^{i)}, \ldots, 0)\}_{i,\ell,t}$, where $e_{\ell t}^{i)} \in \mathrm{M}_{n_i}(D_i)$ denotes the matrix that has 1 in the (ℓ, t) entry and zero everywhere else.

Let $\mathcal{S}(\Lambda)$ be the set of all maximal ideals of R containing d. Since R is h-local, $\mathcal{S}(\Lambda)$ is a finite set.

Let $\mathfrak{m}$ be a maximal ideal of R that is not in $\mathcal{S}(\Lambda)$. By the choice of d, the central idempotents of Λ_Q and the matrix units of the decomposition of Λ_Q as semisimple artinian ring are in $\Lambda_\mathfrak{m}$. Therefore, $\Lambda_\mathfrak{m} \cong \mathrm{M}_{n_1}(\Lambda_1^\mathfrak{m}) \times \cdots \times \mathrm{M}_{n_k}(\Lambda_k^\mathfrak{m})$, where, for $i = 1, \ldots, k$, $\Lambda_i^\mathfrak{m} \cong \mathrm{End}_{\Lambda_\mathfrak{m}}\left(\varphi(0, \ldots, e_{11}^{i)}, \ldots, 0)\Lambda_\mathfrak{m}\right)$. As $\Lambda_\mathfrak{m}$ is torsion-free and semiperfect and $\varphi(0, \ldots, e_{11}^{i)}, \ldots, 0)\Lambda_\mathfrak{m} \otimes_{R_\mathfrak{m}} Q$ is simple, $\varphi(0, \ldots, e_{11}^{i)}, \ldots, 0)\Lambda_\mathfrak{m}$ is a cyclic, indecomposable, projective $\Lambda_\mathfrak{m}$-module. Thus, its endomorphism ring is a local ring. Therefore, $\Lambda_i^\mathfrak{m}$ is a local R-algebra for any $i = 1, \ldots, k$.

Statements **(ii)** and **(iii)** are a direct consequence of **(i)**. □

Let Λ be an algebra over a commutative domain R, and let X be a right Λ-module. Then its R-torsion submodule

$$\mathrm{t}_R(X) = \{x \in X \mid xr = 0 \text{ for some non-zero } r \in R\}$$

is a Λ-submodule of X.

Recall that if Λ is a module-finite algebra over a commutative noetherian ring R, by Corollary 6.2.10, every countably generated projective module is relatively big. Therefore, we are interested in understanding the structure of Λ/I, where I is the trace ideal of a countably generated projective module, to compute $\bigsqcup_{I\in\mathcal{T}(\Lambda)} \mathrm{V}(\Lambda/I)$, and then $\mathrm{V}^*(\Lambda)$.

Lemma 8.1.9. *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect module-finite torsion-free R-algebra such that Λ_Q is semisimple artinian. Let X be a finitely generated right Λ-module. Then, the following statements are equivalent*

- **(i)** *$X = \mathrm{t}_R(X)$;*
- **(ii)** *X is an artinian Λ-module, so it is a finite-length module as an R-module and as a Λ-module;*
- **(iii)** *$X \otimes_R Q = \{0\}$.*

In particular, for any Λ-module X, $\mathrm{t}_R(X) = \{0\}$ if and only if $\mathrm{Soc}(X_\Lambda) = \{0\}$.

Proof. **(i)** $\Rightarrow$ **(ii)**. Set $X = \sum_{i=1}^n x_i\Lambda$. By assumption, $\mathrm{ann}_R(x_i) \neq \{0\}$. So $I = \bigcap_{i=1}^n \mathrm{ann}_R(x_i) \neq \{0\}$, and X is a finitely generated module over R/I, which is an artinian ring because R is a domain of Krull dimension 1. Hence, X is also artinian as a Λ-module.

(ii) $\Rightarrow$ **(i)**. Assume X is a finitely generated artinian right Λ-module. If for any $r \in R\setminus\{0\}$, multiplication by r is an injective Λ-module homomorphism then it is also bijective (recall that any injective endomorphism of an artinian module is bijective). So X is a Λ_Q-module which is finitely generated as Λ-module and, hence, as an R-module. As R has Krull dimension 1, this implies that $X = \{0\}$.

If X is a finitely generated non-zero artinian right Λ-module, then in the short exact sequence

$$0 \longrightarrow \mathrm{t}_R(X) \longrightarrow X \longrightarrow X/\mathrm{t}_R(X) \longrightarrow 0$$

the module $X/\mathrm{t}_R(X)$ is a finitely generated, torsion-free, artinian right Λ-module. By the first part of the proof, $X/\mathrm{t}_R(X) = \{0\}$. So $X = \mathrm{t}_R(X)$ is a torsion module.

It is clear that **(i)** and **(iii)** are equivalent statements.

The last part of the statement follows because of the equivalence of **(i)** and **(ii)**. □

Lemma 8.1.10. *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect module-finite torsion-free R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$.*

Let $\mathcal{S}(\Lambda)$ be the finite set of maximal ideals of R defined in Lemma 8.1.8. Let $A = \{e_1, \ldots, e_k\}$ be the set of primitive central idempotents of Λ_Q.

Let I be the trace of a projective right Λ-module and assume that $I \cap R = \{0\}$. Let $\Lambda_I = \Lambda/I$, and denote by N_I the prime radical of Λ_I. Then

- **(i)** *N_I is a nilpotent ideal and Λ_I/N_I is semiprime. Moreover, $N_I \cap R = \{0\}$ and N_I is an artinian Λ-module.*

(ii) $\Lambda_I/N_I = \Gamma_1 \times \Gamma_2$ *where* Γ_2 *is semisimple artinian and* Γ_1 *is a semiprime Goldie, torsion-free* R*-algebra.*

(iii) $(\Gamma_2)_{\mathfrak{m}} = \{0\}$ *and* $(N_I)_{\mathfrak{m}} = \{0\}$ *for every maximal ideal* $\mathfrak{m} \notin \mathcal{S}(\Lambda)$.

(iv) $(\Lambda_I)_Q = (\Gamma_1)_Q \cong \prod_{i\in B} \mathrm{M}_i(D_i)$ *where* $B = \{i \in A \mid e_i I \neq \{0\}\}$.

(v) $\mathrm{V}(\Lambda_I) \cong \mathrm{V}(\Gamma_1) \times \mathrm{V}(\Gamma_2)$.

Proof. **(i)** As Λ_I is noetherian, N_I is nilpotent, and Λ_I/N_I is semiprime, see Theorem 3.11 in [20]. On the other hand, $(\Lambda_I)_Q$ is a quotient of Λ_Q. Hence, it is a semisimple artinian ring, and thus it does not have non-zero nilpotent ideals. Hence, $N_I \otimes_R Q = 0$. Lemma 8.1.7 implies that $N_I \cap R = \{0\}$, and by Lemma 8.1.9, N_I is an artinian Λ_I-module.

(ii) Since Λ_I is noetherian, **(i)** implies that Λ_I/N_I is a semiprime Goldie ring. By Theorem 7.15 in [20], $\Lambda_I/N_I = \Gamma_1 \times \Gamma_2$, where Γ_2 is semisimple artinian and Γ_1 has zero socle. By Lemma 8.1.9, Γ_1 is torsion-free as an R-module.

By **(i)**, the central ring embedding $R \longrightarrow \Lambda$ composed with the projection $\Lambda \longrightarrow \Lambda_I/N_I$ is still a central ring embedding. As Γ_1 is non-zero and torsion-free, it is a torsion-free R-algebra.

Since Λ_I/N_I is a semiprime Goldie ring, so is Γ_1.

(iii) By Lemma 8.1.8, $(\Lambda_I)_{\mathfrak{m}}$ is torsion-free for any maximal ideal $\mathfrak{m}$ of R that is not in $\mathcal{S}(\Lambda)$. Therefore, $(N_I)_{\mathfrak{m}} = \{0\}$ and $(\Gamma_2)_{\mathfrak{m}} = \{0\}$ are torsion as R-modules by Lemma 8.1.9.

(iv) By **(iii)**, $(\Lambda_I)_Q = (\Gamma_1)_Q$. The final isomorphism follows from Lemma 8.1.8,

(v) This statement follows from well-known general facts about finitely generated projective modules. □

Remark 8.1.11. Notice that as R has Krull dimension 1, any proper quotient of R is artinian. So the ring Γ_2 in Lemma 8.1.10 satisfies that

$$\Gamma_2 \cong \prod_{\mathfrak{m}\in\mathcal{S}(\Lambda)} (\Gamma_2)_{\mathfrak{m}}.$$

8.1.3 Package Deal Theorems

As we have done in previous sections, we need a way to glue localizations, and this is achieved with a Package Deal Theorem. First, we consider Λ to be torsion-free and then prove the main theorem.

Lemma 8.1.12. *Let* R *be a noetherian domain of Krull dimension* 1 *with field of fractions* Q*. Let* Λ *be a locally semiperfect module-finite torsion-free* R*-algebra such that* Λ_Q *is semisimple artinian, so that* $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ *for suitable division rings* $D_1, \ldots, D_k$*, and let* $S_1, \ldots, S_k$ *be a set of representatives of the isomorphism classes of simple right* Λ_Q*-modules.*

Fix $r = (r_1, \ldots, r_k) \in \mathbb{N}_0^k$*. For each maximal ideal* $\mathfrak{m}$ *of* R*, let* $P(\mathfrak{m})$ *be a finitely generated projective right* $\Lambda_{\mathfrak{m}}$*-module such that* $P(\mathfrak{m})_Q \cong \bigoplus_{i=1}^k S_i^{r_i}$*. Then there exists a finitely generated projective right* Λ*-module* P *such that* $P_{\mathfrak{m}} \cong P(\mathfrak{m})$ *for every maximal ideal* $\mathfrak{m}$ *of* R*.*

Proof. Let $\mathcal{S}(\Lambda)$ be the finite set of maximal ideals of R given by Lemma 8.1.10. Hence, whenever $\mathfrak{m} \in \mathrm{mSpec}(R) \setminus \mathcal{S}(\Lambda)$, $\Lambda_{\mathfrak{m}} \cong \mathrm{M}_{n_1}(\Lambda_1^{\mathfrak{m}}) \times \cdots \times \mathrm{M}_{n_k}(\Lambda_k^{\mathfrak{m}})$ where, for any $i = 1, \ldots, k$, $\Lambda_i^{\mathfrak{m}}$ a local torsion-free R-algebra such that $\Lambda_i^{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} Q \cong D_i$.

Let $\mathfrak{m} \notin \mathcal{S}(\Lambda)$. Let $U_{\mathfrak{m},1}, \ldots, U_{\mathfrak{m},k}$ be a set of representatives of the indecomposable, projective right $\Lambda_{\mathfrak{m}}$-modules such that $U_{\mathfrak{m},i} \otimes_{R_{\mathfrak{m}}} Q \cong S_i$, for $i = 1, \ldots k$. As $\Lambda_{\mathfrak{m}}$ is semiperfect, any projective $\Lambda_{\mathfrak{m}}$-module is isomorphic to a direct sum of such indecomposable projectives. Therefore, for any $\mathfrak{m} \notin \mathcal{S}(\Lambda)$, $P(\mathfrak{m}) \cong \bigoplus_{i=1}^k U_{\mathfrak{m},i}^{r_i}$.

Let q be such that $r_i \leq n_i q$ for any $i = 1, \ldots, k$. Set $F = \Lambda^q$. Further, let U be a finitely generated Λ-submodule of F such that $U_Q \cong \bigoplus_{i=1}^k S_i^{r_i}$. Since the embedding $U \hookrightarrow F$ splits over Λ_Q, it splits over almost all maximal ideals of R, cf. Lemma 2.3.2 (or see also Lemma 2.4 in [34]). Then U is a finitely generated Λ-module, which is torsion-free as an R-module, such that $U_{\mathfrak{m}}$ is projective for almost all maximal ideals $\mathfrak{m}$ of R and $U_Q \cong \bigoplus_{i=1}^k S_i^{r_i}$. So let $\mathcal{S}$ be a finite set of maximal ideals of R such that $\mathcal{S}(\Lambda) \subseteq \mathcal{S}$, and satisfying that $U_{\mathfrak{m}}$ is projective for any $\mathfrak{m} \notin \mathcal{S}$. In particular, for any $\mathfrak{m} \notin \mathcal{S}$, $U_{\mathfrak{m}} \cong P(\mathfrak{m}) \cong \bigoplus_{i=1}^k X_i(\mathfrak{m})^{r_i}$.

By Lemma 2.1.15, for any $\mathfrak{m} \in \mathcal{S}$, we can choose a submodule $U(\mathfrak{m})$ of $U_{\mathfrak{m}}$ such that $U(\mathfrak{m}) \cong P(\mathfrak{m})$. For $\mathfrak{m} \notin \mathcal{S}$, we set $U(\mathfrak{m}) = U_{\mathfrak{m}}$.

Note that for any maximal ideal $\mathfrak{m}$ of R, $U(\mathfrak{m})_Q = U_Q$. Therefore, we can apply Package Deal Theorem 2.9 in [34] and conclude that there exists a finitely generated Λ-submodule P of U such that $P(\mathfrak{m}) \cong U(\mathfrak{m}) = P_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{mSpec}(R)$. As P is finitely presented and locally projective Λ-module, P is a projective Λ-module. □

Package Deal Theorem 8.1.13. *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect module-finite R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$, and let $S_1, \ldots, S_k$ be representatives of the simple right Λ_Q-modules such that $\mathrm{End}_{\Lambda_Q}(S_i) \cong D_i$, for $i = 1, \ldots, k$.*

Fix $r = (r_1, \ldots, r_k) \in \mathbb{N}_0^k$. For each maximal ideal $\mathfrak{m}$ of R, let $P(\mathfrak{m})$ be a finitely generated projective right $\Lambda_{\mathfrak{m}}$-module such that $P(\mathfrak{m})_Q \cong \bigoplus_{i=1}^k S_i^{r_i}$. Then there exists a finitely generated projective right Λ-module P such that $P_{\mathfrak{m}} \cong P(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R.

Proof. Throughout the proof, let $\mathcal{S}(\Lambda)$ be the finite set of maximal ideals of R given in Lemma 8.1.10.

As a first step, we assume that $\mathrm{N}(\Lambda) = 0$. Then, by Lemma 8.1.10, $\Lambda = \Gamma_1 \times \Gamma_2$ with Γ_1 torsion-free and Γ_2 semisimple artinian, such that $(\Gamma_2)_{\mathfrak{m}} = \{0\}$ for any $\mathfrak{m} \notin \mathcal{S}(\Lambda)$ and $(\Gamma_2)_Q = \{0\}$.

Therefore, for any maximal ideal $\mathfrak{m}$ of R, $P(\mathfrak{m}) = P_1(\mathfrak{m}) \oplus P_2(\mathfrak{m})$, where $P_1(\mathfrak{m})$ is a finitely generated projective $(\Gamma_1)_{\mathfrak{m}}$-module, $P_2(\mathfrak{m})$ is a finitely generated projective $(\Gamma_2)_{\mathfrak{m}}$-module, and $P(\mathfrak{m})_Q \cong P_1(\mathfrak{m})_Q$. In addition, that $P_2(\mathfrak{m}) = \{0\}$ for any $\mathfrak{m}$ not in $\mathcal{S}(\Lambda)$.

By Lemma 8.1.12, there exists a finitely generated projective Γ_1 module P' such that $(P')_{\mathfrak{m}} \cong P_1(\mathfrak{m})$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$. Then $P = P' \oplus (\oplus_{\mathfrak{m} \in \mathcal{S}(\Lambda)} P_2(\mathfrak{m}))$ is the projective module we are looking for (cf. Remark 8.1.11).

Now assume that Λ is as stated. Let $N = \mathrm{N}(\Lambda)$ denote the nilradical of Λ. By Lemma 8.1.10, $\Lambda_Q = (\Lambda/N)_Q$, and for any maximal ideal $\mathfrak{m}$, $P(\mathfrak{m})_Q \cong (P(\mathfrak{m})/P(\mathfrak{m})N)_Q$. By the previous step, we know that there exists a finitely

generated projective right Λ/N-module P' such that $P'_Q \cong (P(\mathfrak{m})/P(\mathfrak{m})N)_Q$. Since N is a nilpotent ideal, by Lemma 6.1.2, there exists a finitely generated projective Λ-module P such that $P/PN \cong P'$. So P satisfies the properties claimed in the statement. □

Apart from a Package Deal Theorem for projectives, we also need a Package Deal Theorem for idempotent ideals.

Proposition 8.1.14. *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect, module-finite, torsion-free R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$.*

Let I be an idempotent ideal of Λ_Q. Assume that for each maximal ideal $\mathfrak{m}$ of R, there is an idempotent ideal $I(\mathfrak{m})$ of $\Lambda_{\mathfrak{m}}$ such that $I(\mathfrak{m})_Q = I$. Then there exists a unique idempotent ideal L of Λ such that, for any $\mathfrak{m} \in \mathrm{mSpec}(R)$, $L_{\mathfrak{m}} = I(\mathfrak{m})$.

Proof. Throughout the proof, let $\mathcal{S}(\Lambda)$ be the finite set of maximal ideals of R given by Lemma 8.1.8.

Let $e_1, \ldots, e_k$ be the set of primitive central idempotents of Λ_Q. Then there exists $A \subseteq \{1, \ldots, k\}$ such that $I = \sum_{i \in A} e_i \Lambda_Q$.

Let $d \in R \setminus \{0\}$ be such that $de_i \in \Lambda$ for any $i \in \{1, \ldots, k\}$. Let $U = \sum_{i \in A} de_i \Lambda$. Notice that, since e_i and d are central in Λ_Q, U is a two-sided ideal of Λ.

Since R is a domain of Krull dimension 1, d is contained in only a finite number of maximal ideals of R. Therefore, we can choose a finite subset $\mathcal{S} \subseteq \mathrm{mSpec}(R)$ containing $\mathcal{S}(\Lambda)$ such that for any maximal ideal $\mathfrak{m} \notin \mathcal{S}$, $U_{\mathfrak{m}} = \sum_{i \in A} e_i \Lambda_{\mathfrak{m}}$. By Lemma 8.1.8, it also follows that $U_{\mathfrak{m}} = I(\mathfrak{m})$ for any maximal ideal $\mathfrak{m} \in \mathrm{mSpec}(R) \setminus \mathcal{S}$.

By Lemma 2.1.15, for any $\mathfrak{m} \in \mathcal{S}$, we can choose a submodule $U(\mathfrak{m})$ of $U_{\mathfrak{m}}$ such that $U(\mathfrak{m}) \cong I(\mathfrak{m})$. For $\mathfrak{m} \notin \mathcal{S}$, we set $U(\mathfrak{m}) = U_{\mathfrak{m}}$.

Note that for any maximal ideal $\mathfrak{m}$ of R, $U(\mathfrak{m})_Q = U_Q$. Therefore, we can apply Package Deal Theorem 2.9 in [34] and conclude that there exists a finitely generated Λ-submodule M of U such that $M_{\mathfrak{m}} \cong U(\mathfrak{m}) \cong I(\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{mSpec}(R)$. Finally, by Proposition 10.5 in [25], $L = \mathrm{Tr}_\Lambda(M)$ is the idempotent ideal for which we are looking.

As isomorphic idempotent ideals are equal and determined by their localization, the idempotent ideal L is uniquely determined by the family of idempotent ideals $\{I(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$. □

8.2 Decomposition matrices of locally semiperfect algebras

Notation 8.2.1. Consider a semisimple artinian ring $\Lambda \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$, and let $S_1, \ldots, S_k$ be a set of representatives of the simple right Λ-modules such that, for $i = 1, \ldots, k$, $\mathrm{End}_\Lambda(S_i) \cong D_i$. Then any Λ-module is isomorphic to a direct sum of simple modules, and such a direct sum is uniquely determined up to isomorphism (cf. Theorem 1.2.8).

If $M \cong \bigoplus_{i=1}^{k} S_i^{(B_i)}$ is a countably generated Λ-module, we set the *dimension* of M as $\dim_\Lambda(M) = (r_1, \ldots, r_k) \in (\mathbb{N}_0^*)^k$, where $r_i = \infty$ if the set B_i is infinite, and $r_i = |B_i|$ if B_i is a finite set.

Then $\dim_\Lambda(M)$ (or just $\dim(M)$ if the ring Λ is clear) determines M up to isomorphism. Notice that $\dim\colon \mathrm{V}^*(\Lambda) \longrightarrow (\mathbb{N}_0^*)^k$ is an isomorphism of monoids.

Assume the ring Λ is semiperfect. Let $U_1, \ldots, U_\ell$ be a set of representatives of the indecomposable projective right Λ-modules. Therefore, if P is a countably generated projective right Λ-module, then $P \cong \bigoplus_{i=1}^{\ell} U_i^{(B_i)}$ for suitable sets B_i (cf. Lemma 1.2.10).

As before, we can set $\dim_\Lambda(P) = (r_1, \ldots, r_\ell) \in (\mathbb{N}_0^*)^\ell$ where $r_i = \infty$ if the set B_i is infinite and $r_i = |B_i|$ if B_i is a finite set. We also get a monoid isomorphism $\dim\colon \mathrm{V}^*(\Lambda) \longrightarrow (\mathbb{N}_0^*)^\ell$.

In order to understand how a countably generated projective module over a locally semiperfect algebra with semisimple artinian ring of quotients decomposes locally, for each maximal ideal $\mathfrak{m}$ of R, we need to know how the finitely generated indecomposable projective right $\Lambda_\mathfrak{m}$-modules decompose when we extend them to Λ_Q as a direct sum of simple right Λ_Q-modules.

Remark 8.2.2. Let R be a commutative domain with field of fractions Q, and let Λ be a locally semiperfect R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$.

Since Λ_Q is semisimple artinian, by Wedderburn-Artin Theorem 1.2.8, there are exactly k non-isomorphic simple right Λ_Q-modules, namely $S_1, \ldots, S_k$. Then, for every countably generated projective Λ_Q-module P, there exists a sequence of sets $A_1, \ldots, A_k$, unique up to cardinality, such that $P \cong \bigoplus_{i=1}^{k} S_i^{(A_i)}$.

Since Λ is locally semiperfect, for each maximal ideal $\mathfrak{m}$ of R, by Lemma 1.2.10, there is a finite set of non-isomorphic indecomposable projective right $\Lambda_\mathfrak{m}$-modules, namely $U_{\mathfrak{m},1}, \ldots, U_{\mathfrak{m},\ell_\mathfrak{m}}$. Then, for every maximal ideal $\mathfrak{m}$ of R and every countably generated projective $\Lambda_\mathfrak{m}$-module $P(\mathfrak{m})$, there exists a sequence of sets $A_{\mathfrak{m},1}, \ldots, A_{\mathfrak{m},\ell_\mathfrak{m}}$, unique up to cardinality, such that $P(\mathfrak{m}) \cong \bigoplus_{i=1}^{\ell_\mathfrak{m}} U_{\mathfrak{m},i}^{(A_{\mathfrak{m},i})}$.

Therefore, for each maximal ideal $\mathfrak{m}$ of R, the image of the monoid homomorphism $\varphi_\mathfrak{m}\colon \mathrm{V}^*(\Lambda_\mathfrak{m}) \longrightarrow \mathrm{V}^*(\Lambda_Q)$ defined by $[P] \mapsto [P \otimes_{R_\mathfrak{m}} Q]$ is completely determined by the images of each indecomposable projective right $\Lambda_\mathfrak{m}$-module $[U_{\mathfrak{m},i}]$ in $\mathrm{V}^*(\Lambda_Q)$ (in fact, in $\mathrm{V}(\Lambda_Q)$) and by how they decompose into a direct sum of the simple right Λ_Q-modules $[S_j]$.

For each maximal ideal $\mathfrak{m}$ of R, and for each $i = 1, \ldots, \ell_m$, let

$$\boldsymbol{d}_{\mathfrak{m},i} = \dim_{\Lambda_Q}(U_{\mathfrak{m},i} \otimes_{R_\mathfrak{m}} Q).$$

Consider the matrix $\mathbf{D}_\mathfrak{m} \in \mathrm{M}_{k \times \ell_m}(\mathbb{N}_0)$ with columns $\boldsymbol{d}_{\mathfrak{m},1}, \ldots, \boldsymbol{d}_{\mathfrak{m},\ell_\mathfrak{m}}$, respectively. The family of matrices $\{\mathbf{D}_\mathfrak{m}\}_{\mathfrak{m} \in \mathrm{mSpec}(R)}$ is called the *decomposition matrices* associated with Λ. By Lemma 8.1.8, $\mathbf{D}_\mathfrak{m} = \mathrm{Id}_k$ for any $\mathfrak{m} \notin \mathcal{S}(\Lambda)$. So it suffices to consider the finite sequence $\{\mathbf{D}_\mathfrak{m}\}_{\mathfrak{m} \in \mathcal{S}(\Lambda)}$.

Let P be a countably generated projective right Λ-module. Let $\boldsymbol{r} = \dim_{\Lambda_Q}(P_Q)$. For any $\mathfrak{m} \in \mathrm{mSpec}(R)$, set $\boldsymbol{r}_\mathfrak{m} = \dim_{\Lambda_\mathfrak{m}}(P_\mathfrak{m})$, then

$$\mathbf{D}_\mathfrak{m} \boldsymbol{r}_\mathfrak{m}^t = \boldsymbol{r}^t.$$

Remark 8.2.3. Let G be a finite group, and let R be a Dedekind domain of characteristic 0 with field of fractions Q, such that RG is locally semiperfect.

Let $\mathfrak{p}$ be a prime ideal of R containing a rational prime p and let $k = R/\mathfrak{p}$. The decomposition matrices associated with Λ in Remark 8.2.2 agree with the decomposition matrices associated with the p-modular system (Q, R, k) and a finite group G defined in Definition 16.20 of [11].

In modular representation theory, there exists a commutative diagram called the *Cartan-Brauer triangle*, which relates $\mathrm{K}_0(R_{\mathfrak{m}}G)$, $\mathrm{G}_0(QG)$, $\mathrm{K}_0(kG)$, and $\mathrm{G}_0(kG)$. For a given ring Λ, $\mathrm{G}_0(\Lambda)$ denotes the *Grothendieck group* of Λ, and $\mathrm{K}_0(\Lambda)$ denotes the *projective class group* (as defined in Definition 16.5 in [11]). By Propositions 16.6 and 16.7 in [11], $\mathrm{K}_0(R_{\mathfrak{m}}G)$, $\mathrm{G}_0(QG)$, $\mathrm{K}_0(kG)$, and $\mathrm{G}_0(kG)$ are free abelian groups.

Let $\{[F_1], \ldots, [F_r]\}$ be a basis of $\mathrm{G}_0(kG)$, $\{[U_1], \ldots, [U_r]\}$ be a basis of $\mathrm{K}_0(kG)$, and $\{[Z_1], \ldots, [Z_s]\}$ be a basis of $\mathrm{G}_0(QG)$. By Proposition 16.17 in [11], there is a homomorphism of free abelian groups

$$d\colon \mathrm{G}_0(QG) \longrightarrow \mathrm{G}_0(kG),$$

defined by $[Z_i] \mapsto \sum_{j=1}^{r} d_{ij}[F_j]$ for every $i = 1, \ldots, s$, where d_{ij} is the multiplicity of F_j as a composition factor of the kG-module obtained by reduction mod $\mathfrak{p}$ of a finitely generated right RG-module M_i which is projective as an R-module and such that $(M_i)_Q = Z_i$. The map d is called the *decomposition map* associated with (Q, R, k) and the finite group G, and the $s \times r$ matrix $\mathbf{D} = (d_{ij})$ is called the *decomposition matrix* associated with (Q, R, k) and the finite group G.

By Propositions 16.4 and 16.6 in [11], there is a homomorphism of free abelian groups

$$c\colon \mathrm{K}_0(kG) \longrightarrow \mathrm{G}_0(kG),$$

defined by $[U_i] \mapsto \sum_{j=1}^{r} c_{ij}[F_j]$ for every $i = 1, \ldots, r$. The map c is called the *Cartan map* associated with $(Q, R, \mathfrak{m})$ and the finite group G, and the $r \times r$ matrix $\mathbf{C} = (c_{ij})$ is called the *Cartan matrix* associated with $(Q, R, \mathfrak{m})$ and the finite group G.

There is a homomorphism of free abelian groups

$$\varphi\colon \mathrm{K}_0(R_{\mathfrak{m}}G) \longrightarrow \mathrm{G}_0(QG),$$

defined by $[P] \mapsto [P \otimes_{R_{\mathfrak{m}}} Q]$. By Theorem 18.2 in [11], there is an isomorphism of free abelian groups $\mathrm{K}_0(R_{\mathfrak{m}}G) \cong \mathrm{K}_0(kG)$. By Proposition 18.5 in [11], there is a homomorphism of free abelian groups

$$e\colon \mathrm{K}_0(kG) \longrightarrow \mathrm{G}_0(KG),$$

defined by composing the φ with the isomorphism $\mathrm{K}_0(kG) \longrightarrow \mathrm{K}_0(R_{\mathfrak{m}}G)$. In particular, there exists a commutative diagram of free abelian groups

$$\begin{array}{ccc} \mathrm{K}_0(R_{\mathfrak{m}}G) & \xrightarrow{\varphi} & \mathrm{G}_0(QG) \\ \cong\downarrow & \overset{e}{\nearrow} & \downarrow d \\ \mathrm{K}_0(kG) & \xrightarrow{c} & \mathrm{G}_0(kG). \end{array}$$

By Theorem 18.9 in [11], the matrix $\mathbf{E}$ of the map $e\colon \mathrm{K}_0(kG) \longrightarrow \mathrm{G}_0(KG)$ with respect to the bases $\{[U_1], \ldots, [U_r]\}$ and $\{[Z_1], \ldots, [Z_s]\}$ is $\mathbf{E} = \mathbf{D}^t$. By

Corollary 18.10 in [11], the Cartan matrix is then $\mathbf{C} = \mathbf{E} \cdot \mathbf{D} = \mathbf{D}^t\mathbf{D}$. Note that the Cartan matrix is a symmetric matrix.

The matrix of the decomposition map is the transpose of the decomposition matrix defined in Remark 8.2.2. From our point of view, we need the map φ, which has not yet been mentioned in the literature. However, if we fix an ordering in the bases of $\mathrm{K}_0(R_{\mathfrak{m}}G)$ and $\mathrm{K}_0(kG)$, such that the isomorphism between both abelian groups is the identity map, then φ is the transpose of the decomposition map. Hence, the matrix of φ with respect to the free bases $\{[U_{\mathfrak{m},1}], \dots, [U_{\mathfrak{m},\ell_{\mathfrak{m}}}]\}$ of $\mathrm{V}(\Lambda_{\mathfrak{m}})$ and $\{[S_1], \dots, [S_k]\}$ of $\mathrm{V}(\Lambda_Q)$ is the decomposition matrix.

Package Deal Theorem 8.2.4. *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect module-finite torsion-free R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \dots, D_k$, and let $S_1, \dots, S_k$ be representatives of the simple right Λ_Q-modules. We use the notation introduced in Notation 8.2.1.*

Fix $\boldsymbol{r} = (r_1, \dots, r_k) \in (\mathbb{N}_0^*)^k$. For each maximal ideal $\mathfrak{m}$ of R, let $P(\mathfrak{m})$ be a countably generated projective right $\Lambda_{\mathfrak{m}}$-module such that $\dim_{\Lambda_Q}(P(\mathfrak{m})_Q) = \boldsymbol{r}$. Then there exists a countably generated projective right Λ-module P such that, for every maximal ideal $\mathfrak{m}$ of R, $P_{\mathfrak{m}} \cong P(\mathfrak{m})$.

Proof. Let $\boldsymbol{s} = (s_1, \dots, s_k) \in (\mathbb{N}_0^*)^k$ be defined by $s_i = \infty$ if $r_i = \infty$, and $s_i = 0$ otherwise. If $\boldsymbol{s} = 0$, then $\boldsymbol{r} \in \mathbb{N}_0^k$, and the result follows from Theorem 8.1.13, so we may assume that $\boldsymbol{s} \neq 0$. Let

$$A = \{i \in \{1, \dots, k\} \mid s_i = 0\}.$$

For each $\mathfrak{m} \in \mathrm{mSpec}(R)$, and because $\Lambda_{\mathfrak{m}}$ is semiperfect, $P(\mathfrak{m})$ is a relatively big projective by Proposition 6.3.4. So there exists an idempotent ideal $I(\mathfrak{m})$ of $\Lambda_{\mathfrak{m}}$ such that it is the trace of a countably (finitely) generated projective $\Lambda_{\mathfrak{m}}$-module $P'(\mathfrak{m})$, and such that

$$P(\mathfrak{m}) \cong P(\mathfrak{m}) \oplus (P'(\mathfrak{m}))^{(\omega)}$$

and $P(\mathfrak{m})/P(\mathfrak{m})I(\mathfrak{m})$ is finitely generated. Moreover, $\dim_{\Lambda_Q}(P'(\mathfrak{m})_Q)^{(\omega)} = \boldsymbol{s}$.

Notice that for any $\mathfrak{m} \in \mathrm{mSpec}(R)$, the trace ideal of $P'(\mathfrak{m})_Q$ is the same idempotent ideal I. For any $\mathfrak{m} \in \mathrm{mSpec}(R)$, let $I(\mathfrak{m})$ be the trace ideal of $P'(\mathfrak{m})$ in $\Lambda_{\mathfrak{m}}$. Then, by Lemma 2.1.9, $I(\mathfrak{m})_Q = I$.

By Proposition 8.1.14, there exists an idempotent ideal J of Λ such that $J_{\mathfrak{m}} = I(\mathfrak{m})$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$. As Λ is noetherian, there exists P' countably generated projective Λ-module with trace ideal J. Hence, by Lemma 2.1.9, $(P'_{\mathfrak{m}})^{(\omega)} \cong (P'(\mathfrak{m}))^{(\omega)}$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$.

If $P(\mathfrak{m})/P(\mathfrak{m})I(\mathfrak{m}) = \{0\}$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$, then $\boldsymbol{r} = \boldsymbol{s}$, and $P = (P')^{(\omega)}$ is a countably generated projective module that satisfies the conclusion of the theorem. Therefore, we can assume that there exists some $\mathfrak{m} \in \mathrm{mSpec}(R)$ such that $P(\mathfrak{m})/P(\mathfrak{m})I(\mathfrak{m}) \neq \{0\}$. In particular, this implies $R \cap J \neq R$.

If $I = J_Q = \Lambda_Q$, then by Lemma 8.1.7 and because R has Krull dimension 1, Λ/J is a finitely generated algebra over the non-zero artinian ring $R/J \cap R$.

Let $\mathcal{S} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell\}$ be the set of maximal ideals of R that contain $J \cap R$. Therefore,

$$\Lambda/J \cong \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/J_{\mathfrak{m}_i} = \prod_{i=1}^{\ell} \Lambda_{\mathfrak{m}_i}/I(\mathfrak{m}_i).$$

Notice that $J_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{mSpec}(R) \setminus \mathcal{S}$, so that $(P'_{\mathfrak{m}})^{(\omega)} \cong P(\mathfrak{m})$ for such maximal ideals. So $\bigoplus_{i=1}^{\ell} P(\mathfrak{m}_i)/P(\mathfrak{m}_i)I(\mathfrak{m}_i)$ is a finitely generated projective Λ/J-module.

By Theorem 1.5.4, there exists a countably generated projective Λ module P'' such that $P''/P''J \cong \bigoplus_{i=1}^{\ell} P(\mathfrak{m}_i)/P(\mathfrak{m}_i)I(\mathfrak{m}_i)$. Then $P = P'' \oplus (P')^{(\omega)}$ is a countably generated projective module that satisfies the conclusion of the theorem.

From now on, we can assume that $J \cap R = \{0\}$, so that $J_Q = I \neq \Lambda_Q$. Therefore, $\Lambda_Q/I = \prod_{i\in A} \mathrm{M}_{n_i}(D_i)$. Moreover, $(\Lambda/J)_Q \cong \Lambda_Q/I$.

Let $\pi\colon (\mathbb{N}_0^*)^k \longrightarrow (\mathbb{N}_0^*)^A$ be the canonical projection on the components in A. Then $\pi(\boldsymbol{r}) = \dim_{\Lambda_Q/I}(P(\mathfrak{m})/P(\mathfrak{m})I(\mathfrak{m}))_Q = \dim_{\Lambda_Q/I}(P(\mathfrak{m})_Q/P(\mathfrak{m})_Q I)$.

By Theorem 8.1.13, there exists a finitely generated Λ/J-module $\overline{P}$ such that $\overline{P}_{\mathfrak{m}} \cong P(\mathfrak{m})/P(\mathfrak{m})I(\mathfrak{m})$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$. By Theorem 1.5.4, there exists a countably generated projective Λ-module P'' such that $P''/P''J \cong \overline{P}$. Then $P = P'' \oplus (P')^{(\omega)}$ is a countably generated projective module that satisfies the conclusion of the theorem. This completes the proof of the statement. □

As we show in the next statement, Theorem 8.2.4 can be thought of as a converse of Remark 8.2.2. To simplify the statement, let us call the sequence $\{\dim_{\Lambda_{\mathfrak{m}}}(P)\}_{\mathfrak{m}\in\mathrm{mSpec}(R)}$ the *genus sequence* of the projective module P.

Corollary 8.2.5. *Let R be a noetherian domain of Krull dimension 1 with field of fractions Q. Let Λ be a locally semiperfect module-finite R-algebra such that Λ_Q is semisimple artinian, so that $\Lambda_Q \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$ for suitable division rings $D_1, \ldots, D_k$, and let $S_1, \ldots, S_k$ be representatives of the simple right Λ_Q-modules.*

For each maximal ideal $\mathfrak{m}$ of R, let $U_{\mathfrak{m},1}, \ldots, U_{\mathfrak{m},\ell_{\mathfrak{m}}}$ be a set of representatives of the indecomposable projective right $\Lambda_{\mathfrak{m}}$-modules.

For each maximal ideal $\mathfrak{m}$ of R, fix $\boldsymbol{r}_{\mathfrak{m}} \in (\mathbb{N}_0^*)^{\ell_{\mathfrak{m}}}$. Then, following the notation of Remark 8.2.2, there exists a countably generated projective right $\Lambda_{\mathfrak{m}}$-module P with genus sequence $\{\boldsymbol{r}_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathrm{mSpec}(R)}$ if and only if

$$\mathbf{D}_{\mathfrak{m}}\boldsymbol{r}_{\mathfrak{m}}^t = \mathbf{D}_{\mathfrak{n}}\boldsymbol{r}_{\mathfrak{n}}^t$$

for any pair of maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R.

Remark 8.2.6. Let Λ be as in Corollary 8.2.5. By Remark 8.2.2, $\mathbf{D}_{\mathfrak{m}} = \mathrm{Id}_k$ for almost all maximal ideals $\mathfrak{m}$ of R. Then, we only need to check the condition

$$\mathbf{D}_{\mathfrak{m}}\boldsymbol{r}_{\mathfrak{m}}^t = \mathbf{D}_{\mathfrak{n}}\boldsymbol{r}_{\mathfrak{n}}^t$$

in Corollary 8.2.5 for a finite number of maximal ideals, namely the ones in $\mathcal{S}(\Lambda)$. If P is a countably generated projective module over Λ, then the decomposition of $P_{\mathfrak{m}}$ is the same as the decomposition of P_Q for every maximal ideal $\mathfrak{m} \notin \mathcal{S}(\Lambda)$.

Let G be a finite group, and let R be a commutative noetherian domain of Krull dimension 1. Then $\Lambda = RG$ is a module-finite R-algebra, so by Corollary 6.2.10, every countably generated projective module is relatively big. In order to apply Corollary 8.2.5 when Λ is the group algebra of a finite group, we need to know when Λ is locally semiperfect. We need the following definition.

Definition 8.2.7. Let G be a finite group, let Q be a field, and let E/Q be a field extension. We say that E is a *splitting field for G* if every simple right EG-module is absolutely simple, that is, if EG is a split semisimple E-algebra.

Corollary 8.2.8. *Let G be a finite group, and let E be a field. Then E is a splitting field for G if and only if $EG \cong \mathrm{M}_{n_1}(E) \times \cdots \times \mathrm{M}_{n_k}(E)$.*

Proof. Follows from Proposition 8.1.5. □

Remark 8.2.9. Note that there is not a unique splitting field for a group G. We are not assuming that if E is a splitting field for a group, it must be the *minimal* splitting field for this group. However, from now on, if we write *the* splitting field for a group, we mean the *minimal* splitting field for this group containing $\mathbb{Q}$. In particular, by Corollary 8.2.8, this minimal splitting field for a given group is obtained by taking the extension of $\mathbb{Q}$ generated by the complex values of the character table of the group.

The following result provides a splitting field for each finite group and all its subgroups.

Proposition 8.2.10 (Corollary 15.18 in [11])**.** *Let F be a field of characteristic 0, G a finite group of exponent n, and ω a primitive n-th root of 1 over $\mathbb{Q}$. Then every field E containing $F(\omega)$ is a splitting field for G and all of its subgroups.*

In particular, to obtain the *minimal* splitting field for a given group, it suffices to take the algebraic number field obtained by adding to $\mathbb{Q}$ all the complex values of the character table of the group.

The importance of the notion of a splitting field for a group is that if we take the ring of integers of a splitting field for a finite group, then the group algebra over this ring is locally semiperfect, as the following result states.

Proposition 8.2.11 (Section 6B in [11])**.** *Let G be a finite group, and let R be either*

- **(i)** *a complete local ring in the $\mathfrak{p}$-adic topology, where $\mathfrak{p}$ is a prime ideal of R, or*
- **(ii)** *a discrete valuation ring with field of fractions Q, provided that Q is a splitting field for G such that the characteristic of Q does not divide the order of G.*

Then RG is a semiperfect ring. In particular, if R is the ring of integers of an algebraic number field Q, which is a splitting field for G, that is, the integral closure of $\mathbb{Z}$ in Q, then RG is locally semiperfect.

We close this section by applying the Package Deal Theorem 8.2.4 to the particular case of group algebras.

Corollary 8.2.12. *Let G be a finite group, and let Q be an algebraic number field such that Q is a splitting field for G. Let R be the ring of integers of Q, and let $\Lambda = RG$. Let $S_1, \ldots, S_k$ be representatives of the simple right Λ_Q-modules.*

For each maximal ideal $\mathfrak{m}$ of R, let $U_{\mathfrak{m},1}, \ldots, U_{\mathfrak{m},\ell_\mathfrak{m}}$ be a set of representatives of the indecomposable projective right $\Lambda_\mathfrak{m}$-modules.

For each maximal ideal $\mathfrak{m}$ of R, fix $\boldsymbol{r}_\mathfrak{m} \in (\mathbb{N}_0^*)^{\ell_\mathfrak{m}}$. Then, following the notation of Remark 8.2.2, there exists a countably generated projective right Λ-module P with genus sequence $\{\boldsymbol{r}_\mathfrak{m}\}_{\mathfrak{m} \in \mathrm{mSpec}(R)}$ if and only if

$$\mathbf{D}_\mathfrak{m} \boldsymbol{r}_\mathfrak{m}^t = \mathbf{D}_\mathfrak{n} \boldsymbol{r}_\mathfrak{n}^t$$

for any pair of maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R.

Proof. By Proposition 8.2.11, Λ is locally semiperfect. Since G is finite, Λ is module-finite, and since R is a domain, Λ is torsion-free as an R-module. By Maschke's Theorem, Λ_Q is semisimple artinian, and $\mathcal{S}(\Lambda)$ corresponds to the prime ideals of R containing each rational prime that divides the order of the group. The statement then follows from Package Deal Theorem 8.2.4. □

8.3 Full idempotent ideals of integral group algebras

We start this section by summarizing what is known about projective modules over group algebras when the ring is a Dedekind domain of characteristic 0.

Theorem 8.3.1. *Let R be a Dedekind domain of characteristic 0, and let G be a finite group such that no prime divisor of the order of G is a unit in R. Then*

- **(i)** (Corollary 6.4 and Theorem 8.1 in [66]) *Every finitely generated projective right module over RG is locally free.*
- **(ii)** (Theorem A in [66]) *Every finitely generated projective right module over RG is the direct sum of a free right RG-module and an ideal I of RG.*

If, in addition, G is solvable, then

- **(iii)** (Theorem 7 in [67]) *Every projective module over RG is either finitely generated or free.*
- **(iv)** (Theorem 3 in [57]) *The group ring RG has no non-trivial idempotent ideals.*

Remark 8.3.2. Statement **(iv)** in Theorem 8.3.1 follows from statements **(i)** and **(iii)** of the same theorem. By **(i)**, if I is the trace of a non-zero finitely generated projective module over RG, then $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. By Lemmas 10.3 and 10.4 in [25], two trace ideals with isomorphic localizations at the maximal ideals of R are equal. Therefore, $I = \Lambda$. On the other hand, if G is solvable, since every projective module over RG is finitely generated or free, the only possible trace ideal of a non-finitely generated projective module is RG. Hence, the only possible trace ideals are 0 and RG.

Remark 8.3.3. Let Λ be as in Corollary 8.2.12. For each maximal ideal $\mathfrak{m}$ of R, write $\Lambda_{\mathfrak{m}} \cong U_{\mathfrak{m},1}^{d_{\mathfrak{m},1}} \oplus \cdots \oplus U_{\mathfrak{m},\ell_{\mathfrak{m}}}^{d_{\mathfrak{m},\ell_{\mathfrak{m}}}}$ for some positive integers $d_{\mathfrak{m},1}, \ldots, d_{\mathfrak{m},\ell_{\mathfrak{m}}}$, and let $\boldsymbol{d}_{\mathfrak{m}} = (d_{\mathfrak{m},1}, \ldots, d_{\mathfrak{m},\ell_{\mathfrak{m}}})$. By Theorem 8.3.1**(i)**, every finitely generated projective right Λ-module is locally free. Therefore, if $\{\boldsymbol{r}_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathrm{mSpec}(R)}$ is the genus sequence of a finitely generated projective right Λ-module P and $P_{\mathfrak{m}} \cong \Lambda_{\mathfrak{m}}^n$ for some positive integer n, then $\boldsymbol{r}_{\mathfrak{m}} = n\boldsymbol{d}_{\mathfrak{m}}$.

The converse of Theorem 8.3.1**(iv)** is also true, i.e., if the group ring RG has no non-trivial idempotent ideals, then G is solvable, and is a result by Roggenkamp (cf. Proposition on page 125 of [57]). We include here an alternative proof. Recall that the augmentation ideal of a group algebra RG is the kernel of the augmentation homomorphism $\varepsilon\colon RG \longrightarrow R$ defined by $\varepsilon(\sum_{g\in G}\alpha_g g) = \sum_{g\in G}\alpha_g$.

Proposition 8.3.4. *Let R be a ring, and let G be a group. Then*

(i) *If G is a perfect group, then the augmentation ideal of G is a non-trivial idempotent ideal of RG.*

(ii) *If G has a non-trivial perfect normal subgroup N, then $\mathrm{Aug}(RN) \otimes_{RN} RG$ is a non-trivial idempotent ideal of RG.*

(iii) *If G is finite non-solvable, then there exists a non-trivial perfect normal subgroup of G such that $\mathrm{Aug}(RN) \otimes_{RN} RG$ is a non-trivial idempotent ideal of RG.*

Proof. **(i)** Since G is a perfect group, $[G, G] = G$. Let $I = \mathrm{Aug}(RG)$. Recall that I is generated by elements of the form $g - 1$ with $g \in G$. For any $x, y \in G$, note that

$$[(x-1)(y-1) - (y-1)(x-1)]x^{-1}y^{-1} = xyx^{-1}y^{-1} - 1. \tag{8.1}$$

is in I^2. Let $g - 1 \in I$. Since $G = [G, G]$, we can write

$$g = x_1y_1x_1^{-1}y_1^{-1} \cdots x_ty_tx_t^{-1}y_t^{-1}.$$

Then, applying (8.1)

$$\begin{aligned} n - 1 &= x_1y_1x_1^{-1}y_1^{-1} \cdots x_ty_tx_t^{-1}y_t^{-1} - 1 \\ &= (x_1y_1x_1^{-1}y_1^{-1} - 1)(x_2y_2x_2^{-1}y_2^{-1} \cdots x_ty_tx_t^{-1}y_t^{-1}) \\ &\quad + (x_2y_2x_2^{-1}y_2^{-1} - 1)(x_3y_3x_3^{-1}y_3^{-1} \cdots x_ty_tx_t^{-1}y_t^{-1}) \\ &\quad + \cdots + (x_ty_tx_t^{-1}y_t^{-1} - 1) \end{aligned}$$

Therefore, $g - 1 \in I^2$, so $\mathrm{Aug}(RG)$ is a non-trivial idempotent ideal of RG.

(ii) Let N be a perfect normal subgroup of G. By **(i)**, $\mathrm{Aug}(RN)$ is a non-trivial idempotent ideal of RN. Tensoring with RG, since N is normal in G, $\mathrm{Aug}(RN) \otimes_{RN} RG$ is a non-trivial idempotent ideal of RG.

(iii) Since G is finite and non-solvable, there exists a non-trivial perfect normal subgroup of G. The statement follows from **(ii)**. □

Therefore, because of Theorem 8.3.1**(iii)** and Proposition 8.3.4, we will focus our attention on infinitely generated projective modules over group algebras of finite non-solvable groups. In particular, we want to find all the possible idempotent ideals of $\Lambda = RG$ when R is a Dedekind domain of characteristic 0, and G is a finite non-solvable group.

Definition 8.3.5. Let R be a commutative domain with field of fractions Q, and let Λ be an R-algebra. Let J be an ideal of Λ. An idempotent ideal I of Λ is *full with respect to* J if $I \subseteq J$ and $I_Q = J_Q$. If $J = \Lambda$, we will simply say that I is a full idempotent ideal.

When R is the ring of integers of the splitting field of a finite group G, $\Lambda = RG$ is locally semiperfect, so one has the decomposition matrices associated with $(Q, R, \mathfrak{p})$ and G as invariants of Λ. The following algorithm computes the localizations of the full idempotent ideals with respect to Λ using precisely these decomposition matrices.

Algorithm 8.3.6. Let G be a finite group, and let Q be a splitting field for G of characteristic 0 with ring of integers R. Let $A = QG$, and $\Lambda = RG$. Let p be a rational prime, and let $\mathfrak{m}$ be a maximal ideal of R lying above p. Let $\Lambda_{\mathfrak{m}}$ be the localization of Λ at $\mathfrak{m}$.

Let $\{[S_1], \ldots, [S_k]\}$ be a set of representatives of the isomorphism classes of simple right A-modules, and let $\{[U_1], \ldots, [U_\ell]\}$ be a set of representatives of the isomorphism classes of finitely generated projective right $\Lambda_{\mathfrak{m}}$-modules.

Let $\mathbf{D} = (d_{ij}) \in \mathrm{M}_{k\times\ell}(\mathbb{N}_0)$ be the decomposition matrix associated with $(Q, R, \mathfrak{m})$. Let $\mathcal{K} \leftarrow \emptyset$ be the set of output subsets.

Input:	Decomposition matrix $\mathbf{D} \in \mathrm{M}_{t\times\ell}(\mathbb{N}_0)$.
Output:	A collection $\mathcal{K} = \{J_1, \ldots, J_m\}$ of subsets of $\{1, \ldots, \ell\}$ such that $\mathrm{Tr}\big(\bigoplus_{j\in\{1,\ldots,\ell\}\setminus J_t} U_j\big)$ is a full idempotent ideal for every $t = 1, \ldots, m$.

Step 1. Initialize:

- $I \leftarrow \{1, \ldots, s\}$ (indices of simple A-modules),
- $J \leftarrow \{1, \ldots, r\}$ (indices of projective indecomposable $\Lambda_{\mathfrak{m}}$-modules).

Step 2. Reduce trivial columns:

- For each $i \in I$, do:
  - If there exists a unique $j \in J$ such that $d_{ij} \neq 0$ and $d_{ij'} = 0$ for all $j' \in J \setminus \{j\}$:
    - Set $J \leftarrow J \setminus \{j\}$,
    - Set $I \leftarrow \{i \in I \mid d_{ij} = 0\}$ (remove rows supported by j).

Step 3. Recursive search:

- If $I = \emptyset$, then:
  - Add every subset of the sorted subset J to $\mathcal{K}$ if not already present.
- If $I \neq \emptyset$ and $J = \{j_1, \ldots, j_m\}$, then:
  - For each $j \in J$, define:
    - $J_j := J \setminus \{j\}$,
    - $I_j := \{i \in I \mid d_{ij} = 0\}$ (remove rows supported by j),
    - Recursively apply Step 3 with (I_j, J_j).

Return: The collection $\mathcal{K}$.

Remark 8.3.7. **(1)** Assume that G is perfect. Then, if we assume that the first column is associated with the trivial representation and delete the first column of the decomposition matrix, the previous algorithm is also valid for computing full idempotent ideals with respect to $\mathrm{Aug}(\Lambda)$. Whenever we apply the algorithm to compute the full idempotent ideals with respect to $\mathrm{Aug}(\Lambda)$, we will denote that matrix by $\widehat{\mathbf{D}}$.

(2) In order to reduce notation, after applying Step 2 in the algorithm, we will delete all the columns and rows that do not appear in I or J, respectively, and denote the new matrix by $\mathbf{D_{red}}$.

Example 8.3.8. Let $G = \mathrm{PSL}(2,11)$. The minimal splitting field for G is $Q = \mathbb{Q}(\sqrt{-11}, \sqrt{5})$. Let $R = \mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}, \frac{1+\sqrt{5}}{2}\right]$ be the ring of integers of Q. The decomposition matrix of G for a prime ideal $\mathfrak{p}$ containing $p = 11$ is:

$$\mathbf{D} = \mathbf{D}_{\mathfrak{p}}(RG) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(cf. Appendix A to know how to compute decomposition matrices with GAP).
Step 1. Let $I = \{1, 2, \ldots, 8\}$, and let $J = \{1, 2, \ldots, 6\}$.
Step 2. For $i = 1$, we observe that $d_{1,1} \neq 0$ and $d_{1,j} = 0$ for all $j \in J \setminus \{1\}$. As a result, we update J to be $J = \{2, 3, 4, 5, 6\}$. Furthermore, since $d_{1,1} \neq 0$ and $d_{5,1} \neq 0$, we update I to $I = \{2, 3, 4, 6, 7, 8\}$. This is illustrated by the following matrix:

$$\mathbf{D} = \begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For $i = 2$, we observe that $d_{2,3} \neq 0$ and $d_{2,j} = 0$ for all $j \in J \setminus \{3\}$. As a result, we update J to $J = \{2, 4, 5, 6\}$. Moreover, since $d_{2,3} \neq 0$, $d_{3,3} \neq 0$, and $d_{7,3} \neq 0$, we update I to $I = \{4, 6, 8\}$. This is illustrated by the following matrix:

$$\mathbf{D} = \begin{bmatrix} \cancel{1} & \cancel{0} & 0 & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For $i = 6$, we observe that $d_{6,6} \neq 0$ and $d_{6,j} = 0$ for all $j \in J \setminus \{6\}$. As a result, we update J to $J = \{2, 4, 5\}$. Moreover, since $d_{6,6} \neq 0$ we update I to $I = \{4, 8\}$. This is illustrated by the following matrix:

$$\mathbf{D} = \begin{bmatrix} \cancel{1} & \cancel{0} & 0 & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3. Now we analyze: $I = \{4, 8\}$ and $J = \{2, 4, 5\}$. We will work with the reduced matrix

$$\mathbf{D_{red}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- For $j = 2$, note that $d_{4,2} \neq 0$ and $d_{8,2} \neq 0$. We set $J_2 = J \setminus \{2\}$ and $I_2 = I \setminus \{4, 8\}$. Repeating **Step 3** with (I_2, J_2), since $I_2 = \emptyset$, we add $\mathcal{P}(J_2)$ to $\mathcal{K}$, that is, $\mathcal{K} = \{\emptyset, \{4\}, \{5\}, \{4, 5\}\}$.

$$\mathbf{D_{red}} = \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{0} \\ \cancel{1} & \cancel{0} & \cancel{1} \end{bmatrix}$$

- For $j = 4$, note that $d_{4,4} \neq 0$. We set $J_4 = J \setminus \{4\}$ and $I_4 = I \setminus \{4\}$. Repeating **Step 3** with (I_4, J_4), since $I_4 \neq \emptyset$, we analyze $I_4 = \{8\}$ and $J_4 = \{2, 5\}$.

$$\mathbf{D_{red}} = \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{0} \\ 1 & \cancel{0} & 1 \end{bmatrix}$$

  - For $j = 2$, note that $d_{8,2} \neq 0$. We set $J_{4,2} = J_4 \setminus \{2\}$ and $I_{4,2} = I_4 \setminus \{8\}$. Repeating **Step 3** with $(I_{4,2}, J_{4,2})$, since $I_{4,2} = \emptyset$ but $J_{4,2} = \{5\} \in \mathcal{K}$, we do not add anything to $\mathcal{K}$.

$$\mathbf{D_{red}} = \begin{bmatrix} \cancel{1} & \cancel{1} & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{bmatrix}$$

  - For $j = 5$, note that $d_{8,5} \neq 0$. We set $J_{4,5} = J_4 \setminus \{5\}$ and $I_{4,5} = I_4 \setminus \{8\}$. Repeating **Step 3** with $(I_{4,5}, J_{4,5})$, since $I_{4,5} = \emptyset$ and $J_{4,5} = \{2\} \notin \mathcal{K}$, we add $\mathcal{P}(\{2\})$ to $\mathcal{K}$, that is, $\mathcal{K} = \{\emptyset, \{2\}, \{4\}, \{5\}, \{4, 5\}\}$.

$$\mathbf{D_{red}} = \begin{bmatrix} 1 & 1 & \cancel{0} \\ \cancel{1} & 0 & \cancel{1} \end{bmatrix}$$

- For $j = 5$, note that $d_{8,5} \neq 0$. We set $J_5 = J \setminus \{5\}$ and $I_5 = I \setminus \{8\}$. Repeating **Step 3** with (I_5, J_5), since $I_5 \neq \emptyset$, we analyze: $I_5 = \{4\}$ $J_5 = \{2, 4\}$.

$$\mathbf{D_{red}} = \begin{bmatrix} 1 & 1 & \cancel{0} \\ \cancel{1} & 0 & \cancel{1} \end{bmatrix}$$

  - For $j = 2$, note that $d_{4,2} \neq 0$. We set $J_{5,2} = J \setminus \{2\}$ and $I_{5,2} = I_4 \setminus \{4\}$. Repeating **Step 3** with $(I_{5,2}, J_{5,2})$, since $I_{5,2} = \emptyset$, but $J_{5,2} = \{4\} \in \mathcal{K}$, we do not add anything to $\mathcal{K}$.

$$\mathbf{D_{red}} = \begin{bmatrix} \cancel{1} & 1 & \cancel{0} \\ \cancel{1} & 0 & \cancel{1} \end{bmatrix}$$

  - For $j = 4$, note that $d_{4,4} \neq 0$. We set $J_{5,4} = J_5 \setminus \{4\}$ and $I_{5,4} = I_5 \setminus \{4\}$. Repeating **Step 3** with $(I_{5,4}, J_{5,4})$, since $I_{5,4} = \emptyset$ but $J_{5,4} = \{2\} \in \mathcal{K}$, we do not add anything to $\mathcal{K}$.

$$\mathbf{D_{red}} = \begin{bmatrix} \cancel{1} & \cancel{1} & 0 \\ 1 & \cancel{0} & \cancel{1} \end{bmatrix}$$

Return. $\mathcal{K} = \{\emptyset, \{2\}, \{4\}, \{5\}, \{4,5\}\}$. Therefore, the list of full idempotent ideals with respect to Λ is

$$\begin{aligned}
\Lambda_{\mathfrak{p}} &= \mathrm{Tr}(U_{\mathfrak{p},1a} \oplus U_{\mathfrak{p},3a} \oplus U_{\mathfrak{p},5a} \oplus U_{\mathfrak{p},7a} \oplus U_{\mathfrak{p},9a} \oplus U_{\mathfrak{p},11a}) \\
K_2 &= \mathrm{Tr}(U_{\mathfrak{p},1a} \oplus U_{\mathfrak{p},5a} \oplus U_{\mathfrak{p},7a} \oplus U_{\mathfrak{p},9a} \oplus U_{\mathfrak{p},11a}) \\
K_4 &= \mathrm{Tr}(U_{\mathfrak{p},1a} \oplus U_{\mathfrak{p},3a} \oplus U_{\mathfrak{p},5a} \oplus U_{\mathfrak{p},9a} \oplus U_{\mathfrak{p},11a}) \\
K_5 &= \mathrm{Tr}(U_{\mathfrak{p},1a} \oplus U_{\mathfrak{p},3a} \oplus U_{\mathfrak{p},5a} \oplus U_{\mathfrak{p},7a} \oplus U_{\mathfrak{p},11a}) \\
K_{4,5} &= \mathrm{Tr}(U_{\mathfrak{p},1a} \oplus U_{\mathfrak{p},3a} \oplus U_{\mathfrak{p},5a} \oplus U_{\mathfrak{p},11a})
\end{aligned}$$

Note that $K_{4,5} \subseteq K_4, K_5$ and $K_2, K_4, K_5 \subseteq \Lambda_{\mathfrak{p}}$.

In Table 8.1, we have a summary of the number of full idempotent ideals found in simple non-abelian groups of low order.

Group	$J = \mathrm{Aug}(\Lambda)$	$J = \Lambda$
A_5	1	4
$L_2(7)$	1	2
A_6	4	28
$L_2(8)$	27	64
$L_2(11)$	3	5
$L_2(13)$	5	32
$L_2(17)$	13	84
A_7	8	96

Table 8.1 Number of non-trivial full idempotent ideals with respect to $\mathrm{Aug}(\Lambda)$ and Λ of group algebras of the first eight simple non-abelian groups over the ring of integers of their corresponding minimal splitting field.

8.4 Infinitely generated representations of finite groups

8.4.1 Alternating Group A_5

Notation 8.4.1. Let $G = A_5$ be the alternating group on 5 letters, and let $Q = \mathbb{Q}(\sqrt{5})$ denote the splitting field for A_5. Let $R = \mathbb{Z}[\zeta_5 + \zeta_5^{-1}] = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, where ζ_5 is a primitive 5th root of unity, denote the ring of integers of Q.

Let $\Lambda = RG$. Note that the rational primes dividing the order of G are 5, 3, and 2. We have three prime ideals of R containing 5, 3, and 2. Namely, $\mathcal{S}(\Lambda) = \{\mathfrak{p}_5 = (\sqrt{5}), \mathfrak{p}_3 = (3), \mathfrak{p}_2 = (2)\}$. We will denote by Λ_p the tensor product $\Lambda \otimes_R R_{\mathfrak{p}}$, where $\mathfrak{p}_p$ is a prime ideal containing a rational prime p.

Let $\{S_{1a}, S_{3a1}, S_{3a2}, S_{4a}, S_{5a}\}$ be representatives of the isomorphism classes of the simple modules over Λ_Q, and let $\{U_{5,1a}, U_{5,3a}, U_{5,5a}\}$, $\{U_{3,1a}, U_{3,3a1}, U_{3,3a2}, U_{3,4a}\}$, and $\{U_{2,1a}, U_{2,2a1}, U_{2,2a2}, U_{2,4a}\}$ be representatives of the isomorphism classes of the indecomposable projective right modules over Λ_5, Λ_3, and Λ_2, respectively.

The decomposition matrices associated with Λ are

$$\mathbf{D}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

see Appendix A.

Remark 8.4.2. Let P be a countably generated right Λ-module, and let

$$\boldsymbol{r} = \dim_{\Lambda_Q}(P \otimes_R Q) = (r_{1a}, r_{3a1}, r_{3a2}, r_{4a}, r_{5a}).$$

Let $\{\boldsymbol{r}_{\mathfrak{m}}\}_{\mathfrak{m}\in \mathrm{mSpec}(R)}$ denote the genus sequence of P, where

$$\begin{aligned} \boldsymbol{r}_5 &= (r_{5,1a}, r_{5,3a}, r_{5,5a}) \\ \boldsymbol{r}_3 &= (r_{3,1a}, r_{3,3a1}, r_{3,3a2}, r_{3,4a}) \\ \boldsymbol{r}_2 &= (r_{2,1a}, r_{2,2a1}, r_{2,2a2}, r_{2,4a}) \\ \boldsymbol{r}_p &= \boldsymbol{r} \qquad \text{if } p \neq 5, 3, 2. \end{aligned}$$

By Corollary 8.2.12, the following systems of equations must be satisfied

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{5,1a} \\ r_{5,3a} \\ r_{5,5a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{3,1a} \\ r_{3,3a1} \\ r_{3,3a2} \\ r_{3,4a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{2,1a} \\ r_{2,2a1} \\ r_{2,2a2} \\ r_{2,4a} \end{bmatrix} = \begin{bmatrix} r_{1a} \\ r_{3a1} \\ r_{3a2} \\ r_{4a} \\ r_{5a} \end{bmatrix}$$

We can rewrite the equations in the following way

$$\begin{cases} r_{5,1a} = r_{1a} \\ r_{5,3a} = r_{3a1} \\ r_{5,5a} = r_{5a} \\ r_{1a} + r_{3a1} = r_{4a} \\ r_{3a1} = r_{3a2} \end{cases} \qquad \begin{cases} r_{3,1a} = r_{1a} \\ r_{3,3a1} = r_{3a1} \\ r_{3,3a2} = r_{3a2} \\ r_{3,4a} = r_{4a} \\ r_{1a} + r_{4a} = r_{5a} \end{cases} \qquad \begin{cases} r_{2,1a} = r_{1a} \\ r_{2,2a1} + r_{1a} = r_{3a1} \\ r_{2,2a2} + r_{1a} = r_{3a2} \\ r_{2,4a} = r_{4a} \\ r_{1a} + r_{5a} = r_{3a1} + r_{3a2} \end{cases}$$

Lemma 8.4.3. *There are* 5 *different non-zero idempotent ideals in* Λ*: the augmentation ideal of* Λ *and* 4 *full idempotent ideals with respect to* Λ*.*

Proof. In order to determine the possible full idempotent ideals with respect to an ideal J, we consider the equations in Remark 8.4.2 relating to the elements of the sequence $\boldsymbol{r} = (r_{1a}, r_{3a1}, r_{3a2}, r_{4a}, r_{5a})$:

$$\begin{cases} r_{1a} + r_{3a1} = r_{4a} \\ r_{3a1} = r_{3a2} \\ r_{1a} + r_{4a} = r_{5a} \\ r_{1a} + r_{5a} = r_{3a1} + r_{3a2} \end{cases}$$

Let K be an idempotent ideal of Λ. Then K_Q can be written as the trace of a direct sum of some of the simple right Λ_Q-modules. Each simple right Λ_Q-module S_i appearing in the decomposition of K_Q as the trace of these simple right Λ_Q-modules corresponds to r_i being ∞ in the above system of equations.

We distinguish the following cases:

(1) If $r_{1a} = \infty$, then the 1st equation implies that $r_{4a} = \infty$, the 3rd equation implies that $r_{5a} = \infty$, and the 2nd equation together with the 4th equation implies that $r_{3a1} = r_{3a2} = \infty$. In this case, $K_Q = \Lambda_Q$.

(2) If $r_{3a1} = r_{3a2} = \infty$, then the 1st equation implies that $r_{4a} = \infty$, and the 3rd equation implies that $r_{5a} = \infty$. In this case, $K_Q = \mathrm{Aug}(\Lambda_Q)$ or $K_Q = \Lambda_Q$.

(3) If $r_{4a} = \infty$, then the 3rd equation implies that $r_{5a} = \infty$, and the 2nd equation together with the 4th equation implies that $r_{3a1} = r_{3a2} = \infty$. In this case, $K_Q = \mathrm{Aug}(\Lambda_Q)$ or $K_Q = \Lambda_Q$.

(4) If $r_{5a} = \infty$, then the 2nd equation together with the 4th equation implies that $r_{3a1} = r_{3a2} = \infty$, and the 1st equation implies that $r_{4a} = \infty$. In this case, $K_Q = \mathrm{Aug}(\Lambda_Q)$ or $K_Q = \Lambda_Q$.

Therefore, every idempotent ideal of Λ is a full idempotent ideal either with respect to $\mathrm{Aug}(\Lambda)$ or Λ.

By Algorithm 8.3.6 applied to the following matrices

$$\widehat{\mathbf{D}}_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \widehat{\mathbf{D}}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \widehat{\mathbf{D}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

we can deduce that the only full idempotent ideal of Λ with respect to the augmentation ideal of Λ is $\mathrm{Aug}(\Lambda)$.

By Algorithm 8.3.6 applied to $\mathbf{D}_5$, $\mathbf{D}_3$, and $\mathbf{D}_2$, respectively, we can deduce that the full idempotent ideals of Λ with respect to Λ are in bijective correspondence with the set

$$\{\emptyset\} \times \{\emptyset\} \times \mathcal{P}(\{2a1, 2a2\}).$$

Therefore, there are 4 full idempotent ideals of Λ with respect to Λ. Namely, I, J_1, J_2, and Λ, where I, J_1, and J_2 have the following localizations

$$\begin{gathered} I_{(2)} = \mathrm{Tr}(U_{2,1a} \oplus U_{2,4a}) \quad \text{and} \quad I_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \text{ for every } \mathfrak{p} \neq (2) \\ (J_1)_{(2)} = \mathrm{Tr}(U_{2,1a} \oplus U_{2,2a1} \oplus U_{2,4a}) \quad \text{and} \quad (J_1)_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \text{ for every } \mathfrak{p} \neq (2) \\ (J_2)_{(2)} = \mathrm{Tr}(U_{2,1a} \oplus U_{2,2a2} \oplus U_{2,4a}) \quad \text{and} \quad (J_2)_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \text{ for every } \mathfrak{p} \neq (2). \quad \square \end{gathered}$$

By Corollary 6.2.10, $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) = \mathrm{V}^*(\Lambda)$, and by Lemma 8.4.3, the non-zero idempotent ideals of Λ are $\mathrm{Aug}(\Lambda)$, I, J_1, J_2, and Λ, so

$$\mathrm{B}(\Lambda) \cong \mathrm{V}(\Lambda/\mathrm{Aug}(\Lambda)) \sqcup \mathrm{V}(\Lambda/I) \sqcup \mathrm{V}(\Lambda/J_1) \sqcup \mathrm{V}(\Lambda/J_2) \sqcup \mathrm{V}(\{0\})$$

Clearly, the class of the relatively Λ-big projective Λ-module corresponding to $[\{0\}] \in \mathrm{V}(\{0\})$ is $[\Lambda^{(\omega)}]$. In the following lemmas, we will compute every monoid in the disjoint union.

Lemma 8.4.4. *Every relatively* $\mathrm{Aug}(\Lambda)$*-big projective right* Λ*-module is of the form* $F \oplus P$*, where* F *is a finitely generated free* Λ*-module and* P *is the unique relatively* $\mathrm{Aug}(\Lambda)$*-big projective right* Λ*-module with trace ideal* $\mathrm{Aug}(\Lambda)$*.*

Proof. Note that $\Lambda/\operatorname{Aug}(\Lambda) \cong R$. Since R is a principal ideal domain, every projective Λ-module over R is free. Therefore, $\mathrm{V}(\Lambda/\operatorname{Aug}(\Lambda)) \cong \mathrm{V}(R) \cong \mathbb{N}_0$. Hence, if X is a finitely generated projective R-module, then $X \cong R^n$ for some $n \in \mathbb{N}_0$. The lifting of $[\{0\}] \in \mathrm{V}(R)$ corresponds to a relatively $\operatorname{Aug}(\Lambda)$-big projective right Λ-module P with trace ideal $\operatorname{Aug}(\Lambda)$, while the lifting of $[R^n] \in \mathrm{V}(R)$ for a positive n corresponds to $[\Lambda^n \oplus P]$. Since every relatively J-big projective right Λ-module X is uniquely determined by the ideal J and the isomorphism class X/XJ, by Lemma 5.1.7, P is unique up to isomorphism. □

Lemma 8.4.5. *Every relatively I-big projective right Λ-module is of the form $F \oplus X$, where F is a finitely generated free Λ-module and X is isomorphic to one of the following modules:*

- **(1)** *B, which is the unique relatively I-big projective right Λ-module up to isomorphism with trace ideal I,*
- **(2)** *B_1, which is the unique relatively I-big projective right Λ-module up to isomorphism with trace ideal J_1 such that $(B_1/B_1I)_{(2)} \cong U_{2,2a1}/U_{2,2a1}I_{(2)}$,*
- **(3)** *B_2, which is the unique relatively I-big projective right Λ-module up to isomorphism with trace ideal J_2 such that $(B_2/B_2I)_{(2)} \cong U_{2,2a2}/U_{2,2a2}I_{(2)}$,*
- **(4)** *$B_1^m \oplus B_2^n$ for some $m, n \in \mathbb{N}_0$, which are relatively I-big projective right Λ-modules with possible trace ideals J_1 (if $n = 0$), J_2 (if $m = 0$), or Λ (if $m, n \neq 0$).*

Moreover, $B \oplus B_1 \cong B_1$, $B \oplus B_2 \cong B_2$, and $B_1^2 \oplus B_2^2 \cong \Lambda \oplus B$.

Proof. We first describe the structure of $\Lambda_I := \Lambda/I$. Since

$$I_{(2)} = \operatorname{Tr}(U_{2,1a} \oplus U_{2,4a})$$

and $I_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$ for every $\mathfrak{p} \neq (2)$,

$$(\Lambda_I)_{(2)} \cong (U_{2,2a1}/U_{2,2a1}I_{(2)})^2 \oplus (U_{2,2a2}/U_{2,2a2}I_{(2)})^2 = \overline{U}_{2,2a1}^2 \oplus \overline{U}_{2,2a2}^2,$$

and $(\Lambda_I)_{\mathfrak{p}} = \{0\}$ for every $\mathfrak{p} \neq (2)$. Since $(\Lambda_I)_Q = \{0\}$, $\overline{U}_{2,2a1} \otimes_{R_{(2)}} Q = \{0\}$ and $\overline{U}_{2,2a2} \otimes_{R_{(2)}} Q = \{0\}$, both $\overline{U}_{2,2a1}$ and $\overline{U}_{2,2a2}$ are artinian.

By Lemma 8.1.10, $\Lambda_I/N_I = \Gamma_2 \cong (\Gamma_2)_{(2)}$ is semisimple artinian. Then, $\Lambda_I \cong (\Lambda_I)_{(2)}$, and by Lemma 5.1.7, every relatively I-big projective Λ-module is uniquely determined up to isomorphism. Hence, every finitely generated projective right Λ_I-module is isomorphic to a direct sum of copies of $\overline{U}_{2,2a1}$ and $\overline{U}_{2,2a2}$. Since the decomposition matrices of Λ_I are empty matrices, there are no restrictions on the multiplicities of the number of copies of each indecomposable projective module over Λ_I.

Let X be a relatively I-big projective right Λ-module. Then, the genus sequence of X is given by

$$\begin{gathered}
\boldsymbol{r}_5 = (\infty, \infty, \infty), \\
\boldsymbol{r}_3 = (\infty, \infty, \infty, \infty), \\
\boldsymbol{r}_2 = (\infty, r_{2,2a1}, r_{2,2a2}, \infty), \quad \text{for some } r_{2,2a1}, r_{2,2a2} \in \mathbb{N}_0 \\
\boldsymbol{r}_p = (\infty, \infty, \infty, \infty, \infty) \quad \text{if } p \neq 5, 3, 2.
\end{gathered}$$

Therefore, $\mathrm{V}(\Lambda_I) \cong \mathbb{N}_0^2$ and each relatively I-big projective right Λ-module is uniquely determined by the values $(r_{2,2a1}, r_{2,2a2}) \in \mathbb{N}_0^2$.

Let B denote the unique relatively I-big projective right Λ-module with $(r_{2,2a1}, r_{2,2a2}) = (0,0)$; let B_1 denote the unique relatively I-big projective right Λ-module with $(r_{2,2a1}, r_{2,2a2}) = (1,0)$; and let B_2 denote the unique relatively I-big projective right Λ-module with $(r_{2,2a1}, r_{2,2a2}) = (0,1)$. Then $B/BI = \{0\}$, $B_1/B_1I \cong \overline{U}_{2,2a1}$, and $B_2/B_2I \cong \overline{U}_{2,2a2}$. The isomorphism class of B is the lifting of the neutral element in $\mathrm{V}(\Lambda_I)$, so $B \oplus B_1 \cong B_1$ and $B \oplus B_2 \cong B_2$. Therefore, every relatively I-big projective right Λ-module is either isomorphic to P or to a finite direct sum of the form $B_1^m \oplus B_2^n$ for some $m, n \in \mathbb{N}_0$.

Since $(B_1^2 \oplus B_2^2)/(B_1^2 \oplus B_2^2)I \cong \overline{U}_{2,2a1}^2 \oplus \overline{U}_{2,2a2}^2 \cong (\Lambda_I)_{(2)}$ and each relatively I-big projective Λ-module is unique up to isomorphism, $B_1^2 \oplus B_2^2 \cong \Lambda \oplus B$. □

Lemma 8.4.6. *Every relatively J_1-big projective right Λ-module is of the form $F \oplus X$, where F is a finitely generated free Λ-module and X is isomorphic to one of the following modules:*

(1) *$B_1^{(\omega)}$, which is the unique relatively J_1-big projective right Λ-module with trace ideal J_1, where B_1 is the module defined in Lemma 8.4.5,*

(2) *$B_1^{(\omega)} \oplus B_2$, which is a relatively J_1-big projective right Λ-module with trace ideal Λ, where B_2 is the module defined in Lemma 8.4.5.*

Proof. We first describe the structure of $\Lambda_{J_1} := \Lambda/J_1$. Since

$$(J_1)_{(2)} = \mathrm{Tr}(U_{2,1a} \oplus U_{2,2a1} \oplus U_{2,4a})$$

and $(J_1)_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$ for every $\mathfrak{p} \neq (2)$,

$$(\Lambda_{J_1})_{(2)} \cong (U_{2,2a2}/U_{2,2a2}I_{(2)})^2 = \overline{U}_{2,2a2}^2,$$

and $(\Lambda_{J_1})_{\mathfrak{p}} = \{0\}$ for every $\mathfrak{p} \neq (2)$. Since $(\Lambda_{J_1})_Q = \{0\}$, $\overline{U}_{2,2a2} \otimes_{R_{(2)}} Q = \{0\}$, so $\overline{U}_{2,2a2}$ is artinian.

By Lemma 8.1.10, $\Lambda_{J_1}/N_{J_1} = \Gamma_2 \cong (\Gamma_2)_{(2)}$ is semisimple artinian. Then, $\Lambda_{J_1} \cong (\Lambda_{J_1})_{(2)}$, and by Lemma 5.1.7, every relatively J_1-big projective Λ-module is uniquely determined up to isomorphism. Hence, every finitely generated projective right Λ_{J_1}-module is isomorphic to a direct sum of copies of $\overline{U}_{2,2a2}$. Since the decomposition matrices of Λ_{J_1} are empty matrices, there are no restrictions on the multiplicities of the number of copies of each indecomposable projective over Λ_{J_1}.

Let X be a relatively J_1-big projective right Λ-module. Then, the genus sequence of X is given by

$$\begin{aligned} \boldsymbol{r}_5 &= (\infty, \infty, \infty), \\ \boldsymbol{r}_3 &= (\infty, \infty, \infty, \infty), \\ \boldsymbol{r}_2 &= (\infty, \infty, r_{2,2a2}, \infty), \quad \text{for some } r_{2,2a2} \in \mathbb{N}_0 \\ \boldsymbol{r}_p &= (\infty, \infty, \infty, \infty, \infty) \quad \text{if } p \neq 5, 3, 2. \end{aligned}$$

Therefore, $\mathrm{V}(\Lambda_I) \cong \mathbb{N}_0$ and each relatively J_1-big projective right Λ-module is uniquely determined by the value $r_{2,2a2} \in \mathbb{N}_0$.

Note that the unique relatively J_1-big projective right Λ-module with $r_{2,2a2} = 0$ is isomorphic to $B_1^{(\omega)}$, where B_1 is the unique relatively I-big projective Λ-module with trace ideal J_1 and such that $B_1/B_1I \cong U_{2,2a1}/U_{2,2a1}I$ defined in Lemma 8.4.5. The isomorphism class of $B_1^{(\omega)}$ is the lifting of the neutral element in $\mathrm{V}(\Lambda_{J_1})$, so the other representative of the relatively J_1-big projective right Λ-module is $B_1^{(\omega)} \oplus B_2$, where B_2 is the unique relatively I-big projective right Λ-module with trace ideal J_2 and such that $B_2/B_2I \cong U_{2,2a2}/U_{2,2a2}I$ defined in Lemma 8.4.5. $B_1^{(\omega)} \oplus B_2$ is the unique relatively J_1-big projective right Λ-module with $r_{2,2a2} = 1$. Since $B_1^2 \oplus B_2^2 \cong \Lambda \oplus B$ and $B \oplus B_1 \cong B_1$, $B_1^{(\omega)} \oplus B_2^2 \cong \Lambda \oplus B_1^{(\omega)}$. Therefore, every relatively J_1-big projective right Λ-module is either isomorphic to $B_1^{(\omega)}$ or to $B_1^{(\omega)} \oplus B_2^n$ for some $n \in \mathbb{N}_0$. □

Lemma 8.4.7. *Every relatively J_2-big projective right Λ-module is of the form $F \oplus X$, where F is a finitely generated free Λ-module and X is isomorphic to one of the following modules:*

(1) *$B_2^{(\omega)}$, which is the unique relatively J_2-big projective right Λ-module with trace ideal J_2, where B_2 is the module defined in Lemma 8.4.5,*

(2) *$B_1 \oplus B_2^{(\omega)}$, which is a relatively J_2-big projective right Λ-module with trace ideal Λ, where B_1 is the module defined in Lemma 8.4.5.*

Proof. The proof has a symmetric argument, similar to the proof of Lemma 8.4.6, but changes the roles of J_1 and J_2. □

Remark 8.4.8. One can think that the situation of the projectives that appear in the decomposition of Λ/I is extreme: either they are all artinian or they are not. This is not true, and we will see an example later in this dissertation.

Theorem 8.4.9. *Every countably generated projective right Λ-module in $\mathrm{B}(\Lambda)$ is of the form $F \oplus X$, where F is a free right Λ-module and X is isomorphic to one of the following countably generated projective right Λ-modules:*

(1) *P, which is the unique relatively $\mathrm{Aug}(\Lambda)$-big projective right Λ-module with trace ideal $\mathrm{Aug}(\Lambda)$,*

(2) *B, which is the unique relatively I-big projective right Λ-module with trace ideal I,*

(3) *$B_1^m \oplus B_2^n$ for some $m, n \in \mathbb{N}_0^*$, where B_1 and B_2 are the unique relatively I-big projective right Λ-modules with trace ideals J_1 and J_2, respectively, and such that $(B_1/B_1I)_{(2)} \cong U_{2,2a1}/U_{2,2a1}I_{(2)}$ and $(B_2/B_2I)_{(2)} \cong U_{2,2a2}/U_{2,2a2}I_{(2)}$.*

Moreover, $B \oplus B_1 \cong B_1$, $B \oplus B_2 \cong B_2$, $B_1^2 \oplus B_2^2 \cong \Lambda \oplus B$, and $P \oplus B \cong \Lambda^{(\omega)}$.

Proof. Only the last relation needs a proof because everything has already been proved in Lemmas 8.4.4 – 8.4.7. Since $P/P\,\mathrm{Aug}(\Lambda) = \{0\}$ and $B/BI = \{0\}$, by the monoid operation defined in $\bigsqcup_{I \in \mathcal{T}(\Lambda)\setminus\{0\}} \mathrm{V}(\Lambda/I) \cong \mathrm{B}(\Lambda)$,

$$([0], \mathrm{Aug}(\Lambda)) + ([0], I) = ([0/0I \oplus 0/0\,\mathrm{Aug}(\Lambda)], \mathrm{Aug}(\Lambda) + I) = ([0], \Lambda)$$

Therefore, $P \oplus B \cong \Lambda^{(\omega)}$. □

Remark 8.4.10. Recall now the structure of $\mathrm{B}(\Lambda)$:

$$\mathrm{B}(\Lambda) \cong \mathrm{V}(\Lambda/\operatorname{Aug}(\Lambda)) \sqcup \mathrm{V}(\Lambda/I) \sqcup \mathrm{V}(\Lambda/J_1) \sqcup \mathrm{V}(\Lambda/J_2) \sqcup \mathrm{V}(\{0\})$$

Identifying $\mathrm{V}(\Lambda/J_1) \cong \mathbb{N}_0$ with $\{\infty\} \times \mathbb{N}_0$, $\mathrm{V}(\Lambda/J_2) \cong \mathbb{N}_0$ with $\mathbb{N}_0 \times \{\infty\}$, and $\mathrm{V}(\{0\}) = \{[\Lambda^{(\omega)}]\}$ with $\{(\infty, \infty)\} \in (\mathbb{N}_0^*)^2$, then

$$\mathrm{V}(\Lambda/I) \sqcup \mathrm{V}(\Lambda/J_1) \sqcup \mathrm{V}(\Lambda/J_2) \sqcup \mathrm{V}(\{0\}) \cong (\mathbb{N}_0^*)^2.$$

Similarly, $\mathrm{V}(\Lambda/\operatorname{Aug}(\Lambda)) \sqcup \mathrm{V}(\{0\}) \cong \mathbb{N}_0^*$. Therefore, $\mathrm{B}(\Lambda) \cong ((\mathbb{N}_0^*)^2 \sqcup \mathbb{N}_0^*)/\sim$, where $\sim$ is the congruence defined by $(\infty, \infty) \sim \infty$, and where addition is induced by the addition in each separate monoid and $(0, 0) + 0 := \infty$.

We illustrate the non-finitely generated, relatively big projective Λ-modules in the following figure.

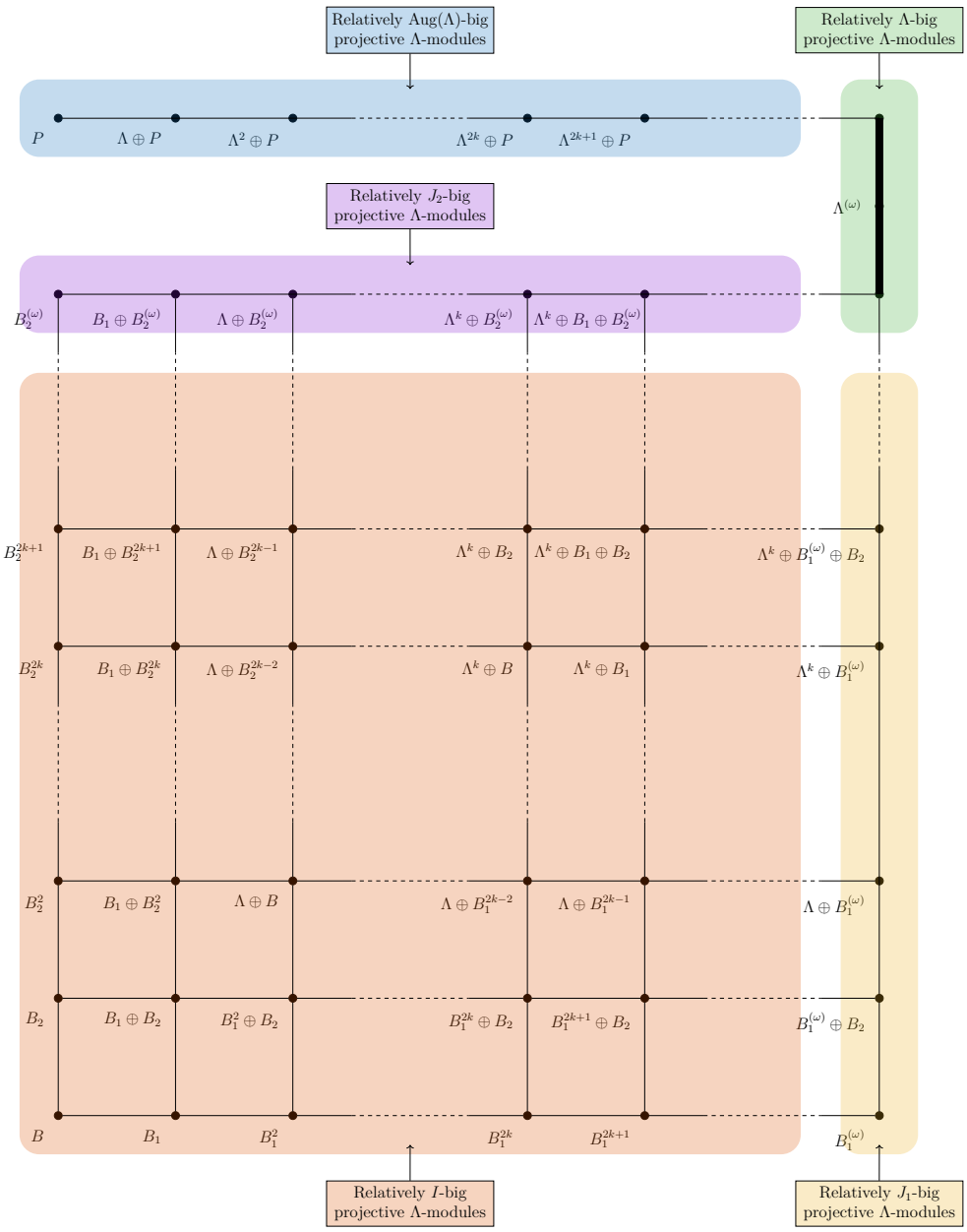


Figure 8.1 The semigroup $\mathrm{B}(\Lambda)$, where $\Lambda = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]A_5$.

In [46], Příhoda classified every relatively big projective $\mathbb{Z}A_5$-module. We will explain how this connects with Theorem 8.4.9 below, but first, we need to reintroduce a little bit of notation.

Notation 8.4.11. Let $G = A_5$ be the alternating group on 5 letters, and let $\mathbb{Z}A_5$. In this case, $\mathcal{S}(\mathbb{Z}A_5) = \{(5), (3), (2)\}$.

Let $\{S'_{1a}, S'_{3a}, S'_{4a}, S'_{5a}\}$ be representatives of the isomorphism classes of the simple modules over $\mathbb{Q}A_5$, and let $\{U'_{5,1a}, U'_{5,3a}, U'_{5,5a}\}$, $\{U'_{3,1a}, U'_{3,3a}, U'_{3,4a}\}$, and $\{U'_{2,1a}, U'_{2,2a}, U'_{2,4a}\}$ be representatives of the isomorphism classes of the indecomposable projective right modules over $\Lambda_{(5)}$, $\Lambda_{(3)}$, and $\Lambda_{(2)}$, respectively. In particular,

$$S'_{xa} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{5}) \cong S_{xa} \text{ for every } x \in \{1, 4, 5\},$$
$$U'_{5,xa} \otimes_{\mathbb{Z}_{(5)}} \mathbb{Z}\left[\tfrac{1+\sqrt{5}}{2}\right]_{(\sqrt{5})} \cong U_{5,xa} \text{ for every } x \in \{1, 3, 5\},$$
$$U'_{3,xa} \otimes_{\mathbb{Z}_{(3)}} \mathbb{Z}\left[\tfrac{1+\sqrt{5}}{2}\right]_{(3)} \cong U_{3,xa} \text{ for every } x \in \{1, 4\},$$
$$U'_{2,xa} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}\left[\tfrac{1+\sqrt{5}}{2}\right]_{(2)} \cong U_{2,xa} \text{ for every } x \in \{1, 4\},$$

but

$$S'_{3a} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{5}) \cong S_{3a1} \oplus S_{3a2},$$
$$U'_{3,3a} \otimes_{\mathbb{Z}_{(3)}} \mathbb{Z}\left[\tfrac{1+\sqrt{5}}{2}\right]_{(3)} \cong U_{3,3a1} \oplus U_{3,3a2},$$
$$U'_{2,2a} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}\left[\tfrac{1+\sqrt{5}}{2}\right]_{(2)} \cong U_{2,2a1} \oplus U_{2,2a2},$$

Theorem 8.4.12 (Proposition 4.12 in [46])**.** *Let $\Lambda = \mathbb{Z}A_5$. The non-zero idempotent ideals of Λ are $\mathrm{Aug}(\Lambda)$, I', and Λ, where I' has the following localizations*

$$I'_{(2)} = \mathrm{Tr}_{\Lambda_{(2)}}(U'_{2,1a} \oplus U'_{2,4a}) \quad \text{and} \quad I'_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \text{ for every } \mathfrak{p} \neq (2).$$

Remark 8.4.13. Note that $I' \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] = I$.

Theorem 8.4.14 (Theorem 4.13 in [46])**.** *Let $\Lambda = \mathbb{Z}A_5$. Then, the elements of $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ are either $[F]$, where F is a free right Λ-module, or $[F \oplus X]$, where X is one of the following countably generated projective right Λ-modules:*

- **(1)** *P', which is the unique relatively $\mathrm{Aug}(\Lambda)$-big projective right Λ-module P with trace ideal $\mathrm{Aug}(\Lambda)$,*
- **(2)** *B', which is the unique relatively I'-big projective right Λ-module with trace ideal I',*
- **(3)** *B_I, which is the unique relatively I'-big projective right Λ-module with trace ideal Λ such that B_I/B_II' is the unique indecomposable projective right Λ/I'-module.*

Moreover, $[B' \oplus B_I] = [B_I]$, and $[B_I \oplus B_I] = [\Lambda \oplus B']$, and $[P' \oplus B_I] = [\Lambda^{(\omega)}]$.

Proof. By [56, Theorem 3.2], the class group of $\mathbb{Z}A_5$ is trivial, so every finitely generated projective right module over the group algebra $\mathbb{Z}A_5$ is free. The rest of the theorem is proved in Theorem 4.13 in [46]. □

Remark 8.4.15. Note that $P' \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \cong P$ and $B' \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \cong B$, where P and B are the relatively big projective right $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]A_5$-modules defined in Lemmas 8.4.4 and 8.4.5, and P' and B' are the relatively big projective right $\mathbb{Z}A_5$-modules defined in Theorem 8.4.14.

Moreover, $B_I \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \cong B_1 \oplus B_2$, where B_I is the relatively I-big projective module over $\mathbb{Z}A_5$ with trace ideal $\mathbb{Z}A_5$ defined in Theorem 8.4.14, and B_1 and B_2 are the relatively big projective modules over $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ with trace ideals J_1 and J_2, respectively, defined in Lemma 8.4.5.

Remark 8.4.16. In this case, the structure of $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$ is as follows:

$$\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) \cong \mathrm{V}(\Lambda) \sqcup \mathrm{V}(\Lambda/\operatorname{Aug}(\Lambda)) \sqcup \mathrm{V}(\Lambda/I') \sqcup \mathrm{V}(\{0\}).$$

Since every finitely generated projective right Λ-module is free, $\mathrm{V}(\Lambda) \cong \mathbb{N}_0 =: H_1$. Similarly, $\mathrm{V}(\Lambda/\operatorname{Aug}(\Lambda)) \cong \mathrm{V}(\mathbb{Z}) \cong \mathbb{N}_0 =: H_2$ and $\mathrm{V}(\Lambda/I') \cong \mathbb{N}_0 := H_3$.

Identify $\mathrm{V}(\{0\}) = \{[\Lambda^{(\omega)}]\}$ with $\{\infty_1\}$, $\{\infty_2\}$, and $\{\infty_3\}$, one copy of infinity for each $\mathbb{N}_0$ in the decomposition of $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$. Then

$$\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda) \cong (\mathbb{N}_0^* \sqcup \mathbb{N}_0^* \sqcup \mathbb{N}_0^*)/\sim,$$

where $\sim$ is the congruence defined by $\infty_1 \sim \infty_2 \sim \infty_3$. Let $H = (\mathbb{N}_0^* \sqcup \mathbb{N}_0^* \sqcup \mathbb{N}_0^*)/\sim$. The identity in H is the element $(0,1) \in H_1$, and the operation is induced by the addition in each separate monoid, together with the following additional rules: $(\ell,1)+(m,2) := (\ell+m,2)$, $(\ell,1)+(n,3) := (\ell+n,3)$, and $(m,2)+(n,3) := \infty$ for each $(\ell,1) \in H_1$, $(m,2) \in H_2$, and $(n,3) \in H_3$.

We illustrate the relatively big projective Λ-modules in the following figure.

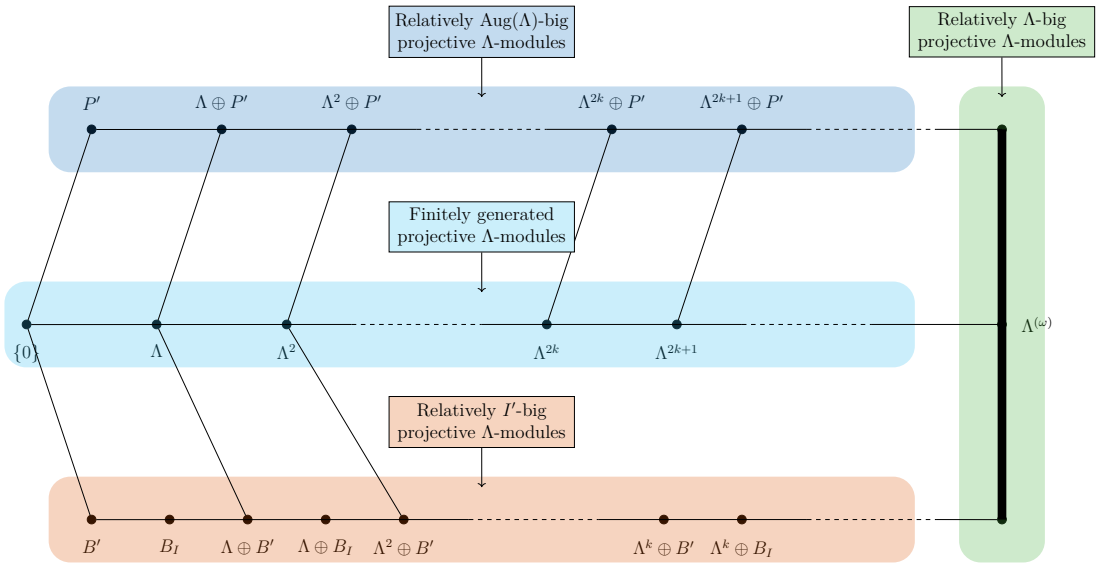


Figure 8.2 The monoid $\mathrm{V}(\Lambda) \sqcup \mathrm{B}(\Lambda)$, where $\Lambda = \mathbb{Z}A_5$.

8.4.2 Alternating Group A_6

Notation 8.4.17. Let $G = A_6$ be the alternating group on 6 letters, and let $Q = \mathbb{Q}(\sqrt{5})$ denote the splitting field for A_6. Let $R = \mathbb{Z}[\zeta_5 + \zeta_5^{-1}] = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, where ζ_5 is a primitive 5th root of unity, denote the ring of integers of Q.

Let $\Lambda = RG$. Note that the rational primes dividing the order of G are 5, 3, and 2. We have three prime ideals of R containing 5, 3, and 2. Namely, $\mathcal{S}(\Lambda) = \{\mathfrak{p}_5 = (\sqrt{5}), \mathfrak{p}_3 = (3), \mathfrak{p}_2 = (2)\}$. We will denote by Λ_p the tensor product $\Lambda \otimes_R R_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal containing a rational prime p.

Let $\{S_{1a}, S_{5a}, S_{5b}, S_{8a1}, S_{8a2}, S_{9a}, S_{10a}\}$ be representatives of the isomorphism classes of the simple modules over Λ_Q. Let $\{U_{5,1a}, U_{5,5a}, U_{5,5b}, U_{5,8a}, U_{5,10a}\}$, $\{U_{3,1a}, U_{3,3a1}, U_{3,3a2}, U_{3,4a}, U_{3,9a}\}$, $\{U_{2,1a}, U_{2,4a}, U_{2,4b}, U_{2,8a1}, U_{2,8a2}\}$, be representatives of the isomorphism classes of the indecomposable projective right modules over Λ_5, Λ_3, and Λ_2, respectively.

The decomposition matrices of Λ are

$$\mathbf{D}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{D}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{D}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}$$

see Appendix A.

The alternating group on 6 letters is the first simple non-abelian group that has proper full idempotent ideals with respect to the augmentation ideal, as the following lemma shows.

Lemma 8.4.18. *There are* 32 *different non-zero idempotent ideals in* Λ*:* 4 *are full with respect to the augmentation ideal of* Λ*, and* 28 *are full with respect to* Λ*.*

Proof. By Algorithm 8.3.6 applied to

$$\widehat{\mathbf{D}}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \widehat{\mathbf{D}}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \widehat{\mathbf{D}}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

we can deduce that the full idempotent ideals of Λ with respect to $\mathrm{Aug}(\Lambda)$ are in bijective correspondence with

$$\{\emptyset\} \times \mathcal{P}(\{3a1, 3a2\}) \times \{\emptyset\}$$

Therefore, there are 4 full idempotent ideals with respect to $\mathrm{Aug}(\Lambda)$.

By Algorithm 8.3.6 applied to $\mathbf{D}_p$ for $p = 5, 3, 2$, the full idempotent ideals of Λ with respect to Λ are in bijective correspondence with the set

$$\{\emptyset\} \times (\mathcal{P}(\{3a1, 3a2, 4a\}) \setminus \{\emptyset\}) \times \mathcal{P}(\{4a, 4b\})$$

Therefore, there are $(8-1) \cdot 4 = 28$ full idempotent ideals with respect to Λ. □

Remark 8.4.19. Let P be a countably generated right Λ-module, and let

$$\boldsymbol{r} = \dim_{\Lambda_Q}(P \otimes_R Q) = (r_{1a}, r_{5a}, r_{5b}, r_{8a1}, r_{8a2}, r_{9a}, r_{10a}).$$

Let $\{\boldsymbol{r}_{\mathfrak{m}}\}_{\mathfrak{m}\in \mathrm{mSpec}(R)}$ denote the genus sequence of P, where

$$\begin{aligned}
\boldsymbol{r}_5 &= (r_{5,1a}, r_{5,5a}, r_{5,5b}, r_{5,8a}, r_{5,10a}) \\
\boldsymbol{r}_3 &= (r_{3,1a}, r_{3,3a1}, r_{3,3a2}, r_{3,4a}, r_{3,9a}) \\
\boldsymbol{r}_2 &= (r_{2,1a}, r_{2,4a}, r_{2,4b}, r_{2,8a1}, r_{2,8a2}) \\
\boldsymbol{r}_p &= \boldsymbol{r} \qquad \text{if } p \neq 5, 3, 2.
\end{aligned}$$

By Corollary 8.2.12, the following systems of equations must be satisfied

$$\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&1&0\\ 1&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} r_{5,1a}\\ r_{5,5a}\\ r_{5,5b}\\ r_{5,8a}\\ r_{5,10a} \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&0&0&1\\ 0&1&1&1&0 \end{bmatrix} \begin{bmatrix} r_{3,1a}\\ r_{3,3a1}\\ r_{3,3a2}\\ r_{3,4a}\\ r_{3,9a} \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&1&1&0&0\\ 2&1&1&0&0 \end{bmatrix} \begin{bmatrix} r_{2,1a}\\ r_{2,4a}\\ r_{2,4b}\\ r_{2,8a1}\\ r_{2,8a2} \end{bmatrix} = \begin{bmatrix} r_{1a}\\ r_{5a}\\ r_{5b}\\ r_{8a1}\\ r_{8a2}\\ r_{9a}\\ r_{10a} \end{bmatrix}$$

We can rewrite the equations in the following way

$$\begin{cases} r_{5,1a} = r_{1a} \\ r_{5,5a} = r_{5a} \\ r_{5,5b} = r_{5b} \\ r_{5,8a} = r_{8a1} \\ r_{5,10a} = r_{10a} \\ r_{8a1} = r_{8a2} \\ r_{1a} + r_{8a1} = r_{9a} \end{cases} \qquad \begin{cases} r_{3,1a} = r_{1a} \\ r_{3,3a1} + r_{5a} = r_{8a1} \\ r_{3,3a2} + r_{5a} = r_{8a2} \\ r_{3,4a} + r_{1a} = r_{5a} \\ r_{3,9a} = r_{9a} \\ r_{5a} = r_{5b} \\ r_{1a} + r_{5a} + r_{10a} = r_{8a1} + r_{8a2} \end{cases} \qquad \begin{cases} r_{2,1a} = r_{1a} \\ r_{2,4a} + r_{1a} = r_{5a} \\ r_{2,4b} + r_{1a} = r_{5b} \\ r_{2,8a1} = r_{8a1} \\ r_{2,8a2} = r_{8a2} \\ r_{1a} + r_{9a} = r_{5a} + r_{5b} \\ r_{5a} + r_{5b} = r_{10a} \end{cases}$$

As an example where the indecomposable projectives in the decomposition of Λ/I are not all artinian or non-artinian, we compute all the big projective modules with respect to one of the full idempotent ideals with respect to the augmentation ideal of Λ.

Lemma 8.4.20. *Let I be the idempotent ideal of Λ such that*

$$I_{(3)} = \mathrm{Tr}_{\Lambda_{(3)}}(U_{3,4a} \oplus U_{3,9a}) \qquad \textit{and} \qquad I_{\mathfrak{p}} = \mathrm{Aug}(\Lambda_{\mathfrak{p}}) \textit{ for every } \mathfrak{p} \neq (3).$$

Then, every relatively I-big projective right Λ-module is of the form $F \oplus X$, where F is a finitely generated free module and X is isomorphic to one of the following modules:

(1) *P, which is the unique relatively I-big projective right Λ-module with trace ideal I,*

(2) *B, which is the unique relatively I-big projective right Λ-module with trace ideal J, where*

$$J_{(3)} = \mathrm{Tr}(U_{3,1a} \oplus U_{3,4a} \oplus U_{3,9a}) \quad \textit{and} \quad J_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \textit{ for every } \mathfrak{p} \neq (3),$$

such that $(B/BI)_{(3)} \cong U_{3,1a}/U_{3,1a}I_{(3)}$ and $P \oplus B \cong B$,

(3) *B_1, which is the unique relatively I-big projective right Λ-module with trace ideal J_1, where*

$$(J_1)_{(3)} = \mathrm{Tr}(U_{3,3a1} \oplus U_{3,4a} \oplus U_{3,9a})$$

and

$$(J_1)_{\mathfrak{p}} = \mathrm{Aug}(\Lambda_{\mathfrak{p}}) \textit{ for every } \mathfrak{p} \neq (3),$$

such that $(B_1/B_1I)_{(3)} \cong U_{3,3a1}/U_{3,3a1}I_{(3)}$ and $P \oplus B_1 \cong B_1$,

(4) *B_2, which is the unique relatively I-big projective right Λ-module with trace ideal J_2, where*

$$(J_2)_{(3)} = \mathrm{Tr}(U_{3,3a2} \oplus U_{3,4a} \oplus U_{3,9a})$$

and

$$(J_2)_{\mathfrak{p}} = \mathrm{Aug}(\Lambda_{\mathfrak{p}}) \textit{ for every } \mathfrak{p} \neq (3),$$

such that $(B_2/B_2I)_{(3)} \cong U_{3,3a2}/U_{3,3a2}I_{(3)}$ and $P \oplus B_2 \cong B_2$,

(5) *$B^k \oplus B_1^m \oplus B_2^n$ for some $k, m, n \in \mathbb{N}_0$, which are relatively I-big projective right Λ-modules with possible trace ideals J (if $m = n = 0$), J_1 (if $k = n = 0$), J_2 (if $k = m = 0$), $J + J_1$ (if $n = 0$), $J + J_2$ (if $m = 0$), $J_1 + J_2$ (if $k = 0$) or Λ, and such that $B \oplus B_1^3 \oplus B_2^3 \cong \Lambda \oplus P$.*

Proof. We first describe the structure of $\Lambda_I := \Lambda/I$. Recall that $\Lambda/\mathrm{Aug}(\Lambda) \cong R$, and since R is a projective R-module, the following short exact sequence of R-modules

$$0 \longrightarrow \mathrm{Aug}(\Lambda)/I \longrightarrow \Lambda/I \longrightarrow R \longrightarrow 0.$$

splits. Therefore, $\Lambda_I \cong R \oplus \mathrm{Aug}(\Lambda)/I$ as R-modules. Since

$$I_{(3)} = \mathrm{Tr}_{\Lambda_{(3)}}(U_{3,4a} \oplus U_{3,9a})$$

and $I_{\mathfrak{p}} = \mathrm{Aug}(\Lambda_{\mathfrak{p}})$ for every $\mathfrak{p} \neq (3)$,

$$\begin{aligned}(\Lambda_I)_{(3)} &\cong U_{3,1a}/U_{3,1a}I_{(3)} \oplus (U_{3,3a1}/U_{3,3a1}I_{(3)})^3 \oplus (U_{3,3a2}/U_{3a2}I_{(3)})^3 \\ &= \overline{U}_{3,1a} \oplus \overline{U}_{3,3a1}^3 \oplus \overline{U}_{3,3a2}^3 \cong R_{(3)} \oplus \overline{U}_{3,3a1}^3 \oplus \overline{U}_{3,3a2}^3,\end{aligned}$$

and

$$(\Lambda_I)_{\mathfrak{p}} \cong U_{\mathfrak{p},1a}/U_{\mathfrak{p},1a}I_{\mathfrak{p}} = \overline{U}_{\mathfrak{p},1a} \cong R_{\mathfrak{p}} \qquad \text{for every } \mathfrak{p} \neq (3).$$

By Lemma 8.1.10, $\Lambda_I/N_I = \Gamma_1 \times \Gamma_2$, where Γ_1 is torsion-free as an R-module and Γ_2 is semisimple artinian, and every finitely generated projective right Λ_I-module is isomorphic to a direct sum of a finitely generated projective right Γ_1-module and a finitely generated projective right Γ_2-module.

Since $\overline{U}_{3,1a} \cong R_{(3)} \hookrightarrow \overline{U}_{3,1a} \otimes_{R_{(3)}} Q \cong Q$, $\overline{U}_{3,1a}$ is torsion-free as an R-module. Since $\overline{U}_{3,3a1} \otimes_{R_{(3)}} Q = \{0\}$ and $\overline{U}_{3,3a2} \otimes_{R_{(3)}} Q = \{0\}$, both $\overline{U}_{3,3a1}$ and $\overline{U}_{3,3a2}$ are artinian. Therefore, $\Gamma_1 \cong R$ and $\Gamma_2 \cong \Gamma_{(3)} \cong (\overline{U}_{3,3a1}^3 \oplus \overline{U}_{3,3a2}^3)/N_I$.

Since R is a PID, every projective R-module is free, so $\mathrm{V}(\Gamma_1) \cong \mathbb{N}_0$. By Lemma 5.1.7, every relatively I-big projective Λ-module X such that $[X/XI] \in \mathrm{V}(\Gamma_1)$ is uniquely determined up to isomorphism. On the other hand, since Γ_2 is semisimple artinian, by Lemma 5.1.7, every relatively I-big projective Λ-module X such that $[X/XI] \in \mathrm{V}(\Gamma_2)$ is uniquely determined up to isomorphism. Therefore, every relatively I-big projective right Λ-module is unique up to isomorphism.

Let X be a relatively I-big projective right Λ-module. Then, the genus sequence of X is given by

$$(r_{5,1a}, \infty, \infty, \infty, \infty)_{(\sqrt{5})}, \quad (r_{3,1a}, r_{3,3a1}, r_{3,3a2}, \infty, \infty)_{(3)}, \quad (r_{2,1a}, \infty, \infty, \infty, \infty)_{(2)},$$

and $(r_{p,1a}, \infty, \infty, \infty, \infty, \infty, \infty)_{\mathfrak{p}}$ for every $\mathfrak{p} \notin \{(\sqrt{5}), (3), (2)\}$. By Corollary 8.2.12, $\mathbf{D}_{\mathfrak{m}} \boldsymbol{r}_{\mathfrak{m}}^t = \mathbf{D}_{\mathfrak{n}} \boldsymbol{r}_{\mathfrak{n}}^t$ for every pair of maximal ideals $\mathfrak{m}, \mathfrak{n}$ of R. By Remark 8.4.19, the only restriction is that $r_{3,1a} = r_{p,1a}$ for every $p \neq 3$. Therefore, $\mathrm{V}(\Lambda_I) \cong \mathbb{N}_0^3$ and each relatively I-big projective right Λ-module is uniquely determined by the values $(r_{3,1a}, r_{3,3a1}, r_{3,3a2}) \in \mathbb{N}_0^3$.

Let P denote the unique relatively I-big projective right Λ-module with $(r_{3,1a}, r_{3,3a1}, r_{3,3a2}) = (0,0,0)$; let B denote the unique relatively I-big projective right Λ-module with $(r_{3,1a}, r_{3,3a1}, r_{3,3a2}) = (1,0,0)$; let B_1 denote the unique relatively I-big projective right Λ-module with $(r_{3,1a}, r_{3,3a1}, r_{3,3a2}) = (0,1,0)$; and let B_2 denote the unique relatively I-big projective right Λ-module with $(r_{3,1a}, r_{3,3a1}, r_{3,3a2}) = (0,0,1)$. Then $P/PI = \{0\}$, $B/BI \cong \overline{U}_{3,1a}$, $B_1/B_1I \cong \overline{U}_{3,3a1}$, and $B_2/B_2I \cong \overline{U}_{3,3a2}$. The isomorphism class of P is the lifting of the neutral element in $\mathrm{V}(\Lambda_I)$, so $P \oplus B \cong B$, $P \oplus B_1 \cong B_1$, and $B \oplus B_2 \cong B_2$. Therefore, every relatively I-big projective right Λ-module is either isomorphic to B or to a finite direct sum of the form $B^k \oplus B_1^m \oplus B_2^n$ for some $k, m, n \in \mathbb{N}_0$.

Since $(B \oplus B_1^3 \oplus B_2^3)/(B \oplus B_1^3 \oplus B_2^3)I \cong \overline{U}_{3,1a} \oplus \overline{U}_{3,3a1}^3 \oplus \overline{U}_{3,3a2}^3 \cong (\Lambda_I)_{(3)}$ and each relatively I-big projective Λ-module is unique up to isomorphism, $B \oplus B_1^3 \oplus B_2^3 \cong \Lambda \oplus P$. $\square$

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List of Publications

1. Álvarez, R.; Herbera, D.; Příhoda, P. Torsion-free Modules over Commutative Domains of Krull Dimension One. *Revista Matemática Iberoamericana*. 2025. 62 pages. On-line first. Available from doi: 10.4171/RMI/1564

2. Álvarez, R.; Herbera, D.; Příhoda, P. Relatively Big Projective Modules and Their Applications to Direct Sum Decompositions. *To appear in Journal of Algebra and Its Applications*. 2026. 50 pages. Available from doi: 10.1142/S0219498825420186

A Character tables and decomposition matrices

All the computations of the character tables and the decomposition matrices have been done with the computer algebra system GAP. The computations of the prime ideals containing rational primes over rings of integers have been done with the computer algebra system SageMath. For example,

```
ordtbl := CharacterTable("PSL(2,11)");
Display(ordtbl);
```

computes the character table of the group $\mathrm{PSL}(2,11)$. With a small modification of this command, we can compute the Brauer simple character table of a group for a given prime p, for example, for $p = 11$, the code

```
p := 11;
modtbl := CharacterTable("PSL(2,11)") mod p;
Display(modtbl);
```

computes the Brauer simple character table of the group $\mathrm{PSL}(2,11)$ for $p = 11$. The decomposition matrix is then computed using this Brauer simple character table. For example,

```
mat := DecompositionMatrix(modtbl);
Display(mat);
```

computes the decomposition matrix of the group $\mathrm{PSL}(2,11)$ over the localization at a prime ideal containing the prime 11 over the ring of integers of a splitting field for $\mathrm{PSL}(2,11)$. The computation of the Cartan matrix is then obtained by computing $\mathbf{D}^t\mathbf{D}$ and the computation of the Brauer projective character table as $\mathbf{D}^t\mathbf{D}\mathbf{M}$, where $\mathbf{M}$ denotes the matrix of the Brauer simple character table.

```
cartan := TransposedMat(mat) * mat;
Display(cartan);
proj := cartan * Irr(modtbl);
Display(proj);
```

Finally, if the splitting field for a group G is given by a polynomial $p(x)$, then the prime ideal containing p can be computed in SageMath. For example, the following code

```
p(x) = x^4 - x^3 + 2*x^2 + 4*x + 3
F.<w> = NumberField(p)
F
R = F.ring_of_integers()
R.gens()
F.prime_above(3)
```

gives a prime ideal containing $p = 3$ of the ring of integers of $\mathbb{Q}[x]/(p(x))$, where $p(x) = x^4 - x^3 + 2x^2 + 4x + 3$.

The information of the splitting fields was also consulted in the database https://beta.lmfdb.org.

A.1 Alternating Group A_5

Let $G = A_5$, the alternating group on 5 letters[1]. The conjugacy classes of A_5 are represented by 1, (1 2)(3 4), (1 2 3), (1 2 3 4 5), and (1 3 5 2 4), with sizes 1, 15, 20, 12, and 12, respectively. The splitting field for G is $K = \mathbb{Q}(\sqrt{5})$, and its ring of integers is $R = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. Let $A = KG$ and $\Lambda = RG$. The complex character table of G is

classes:	1	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	15	20	12	12
χ_{1a}	1	1	1	1	1
χ_{3a1}	3	−1	.	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
χ_{3a2}	3	−1	.	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
χ_{4a}	4	.	1	−1	−1
χ_{5a}	5	1	−1	.	.

where $\zeta_5 = \exp(2\pi i/5)$. The order of the group is $60 = 2^2 \cdot 3 \cdot 5$, and the primes containing 2, 3, and 5 are $(2) = (2)$, $(3) = (3)$, and $(\sqrt{5}) = (\sqrt{5})$, respectively.

Let e_{1a}, e_{3a1}, e_{3a2}, e_{4a}, and e_{5a} be the primitive central idempotents of A. Then

$$A \cong e_{1a}A \oplus e_{3a1}A \oplus e_{3a2}A \oplus e_{4a}A \oplus e_{5a}A$$

as right A-modules. For every prime ideal $\mathfrak{p} \notin \{2, 3, (\sqrt{5})\}$,

$$\Lambda_\mathfrak{p} \cong e_{1a}\Lambda_\mathfrak{p} \oplus e_{3a1}\Lambda_\mathfrak{p} \oplus e_{3a2}\Lambda_\mathfrak{p} \oplus e_{4a}\Lambda_\mathfrak{p} \oplus e_{5a}\Lambda_\mathfrak{p}$$

as right $\Lambda_\mathfrak{p}$-modules.

Localization at $(\sqrt{5})$. There are three 5-conjugacy classes in A_5. The Brauer character table of simple $\Lambda_{(\sqrt{5})}$-modules is obtained from the ordinary character table by deleting the third and fourth rows and the fourth and fifth columns.

classes:	1	(1 2)(3 4)	(1 2 3)
sizes:	1	15	20
ϕ_{1a}	1	1	1
ϕ_{3a}	3	−1	.
ϕ_{5a}	5	1	−1

The Brauer character table of projective $\Lambda_{(\sqrt{5})}$-modules is

classes:	1	(1 2)(3 4)	(1 2 3)
sizes:	1	15	20
η_{1a}	5	1	2
η_{3a}	10	−2	1
η_{5a}	5	1	−1

[1] A_5 is also a simple group of Lie type, in particular $A_5 \cong A_1(4) \cong A_1(5)$, or equivalently, $A_5 \cong L_2(4) \cong L_2(5)$

The decomposition matrix associated with $\Lambda_{(\sqrt{5})}$ is

	ϕ_{1a}	ϕ_{3a}	ϕ_{5a}
χ_{1a}	1	.	.
χ_{3a1}	.	1	.
χ_{3a2}	.	1	.
χ_{4a}	1	1	.
χ_{5a}	.	.	1

The Cartan matrix associated with $\Lambda_{(\sqrt{5})}$ is

	η_{1a}	η_{3a}	η_{5a}
ϕ_{1a}	2	1	.
ϕ_{3a}	1	3	.
ϕ_{5a}	.	.	1

Let $U_{5,1a}$, $U_{5,3a}$, and $U_{5,5a}$ be representatives of the isomorphism classes of the finitely generated indecomposable projective right $\Lambda_{(\sqrt{5})}$-modules. Then

$$\Lambda_{(\sqrt{5})} \cong U_{5,1a} \oplus U_{5,3a}^3 \oplus U_{5,5a}^5$$

as right $\Lambda_{(\sqrt{5})}$-modules. There are 2 blocks: $A_{5,1a} = (1 - e_{5a})\Lambda_{(\sqrt{5})}$ of defect 1, and $A_{5,5a} = e_{5a}\Lambda_{(\sqrt{5})}$ of defect 0, where

$$A_{5,1a} \cong U_{5,1a} \oplus U_{5,3a}^3, \qquad A_{5,5a} \cong U_{5,5a}^5.$$

Localization at (3)**.** There are four 3-conjugacy classes in A_5. The Brauer character table of simple $\Lambda_{(3)}$-modules is obtained from the ordinary character table by deleting the fifth row and third column.

classes:	1	(1 2)(3 4)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	15	12	12
ϕ_{1a}	1	1	1	1
ϕ_{3a1}	3	−1	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
ϕ_{3a2}	3	−1	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
ϕ_{4a}	4	.	−1	−1

The Brauer character table of projective $\Lambda_{(3)}$-modules is

classes:	1	(1 2)(3 4)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	15	12	12
η_{1a}	6	2	1	1
η_{3a1}	3	−1	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
η_{3a2}	3	−1	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
η_{4a}	9	1	−1	−1

The decomposition matrix associated with $\Lambda_{(3)}$ is

	ϕ_{1a}	ϕ_{3a1}	ϕ_{3a2}	ϕ_{4a}
χ_{1a}	1	.	.	.
χ_{3a1}	.	1	.	.
χ_{3a2}	.	.	1	.
χ_{4a}	.	.	.	1
χ_{5a}	1	.	.	1

The Cartan matrix associated with $\Lambda_{(3)}$ is

	η_{1a}	η_{3a1}	η_{3a2}	η_{4a}
ϕ_{1a}	2	.	.	1
ϕ_{3a1}	.	1	.	.
ϕ_{3a2}	.	.	1	.
ϕ_{4a}	1	.	.	2

Let $U_{3,1a}$, $U_{3,3a1}$, $U_{3,3a2}$, and $U_{3,4a}$ be representatives of the isomorphism classes of the finitely generated indecomposable projective right $\Lambda_{(3)}$-modules. Then

$$\Lambda_{(3)} \cong U_{3,1a} \oplus U_{3,3a1}^3 \oplus U_{3,3a2}^3 \oplus U_{3,4a}^4$$

as right $\Lambda_{(3)}$-modules. There are three blocks: $A_{3,1a} = (1 - e_{3a1} - e_{3a2})\Lambda_{(3)}$ of defect 1, $A_{3,3a1} = e_{3a1}\Lambda_{(3)}$ of defect 0, and $A_{3,3a2} = e_{3a2}\Lambda_{(3)}$ of defect 0, where

$$A_{3,1a} \cong U_{3,1a} \oplus U_{3,4a}^4, \qquad A_{3,3a1} \cong U_{3,3a1}^3, \qquad A_{3,3a2} \cong U_{3,3a2}^3.$$

Localization at (2)**.** There are four 3-conjugacy classes in A_5. The Brauer character table of simple $\Lambda_{(2)}$-modules is as follows.

classes:	1	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	20	12	12
ϕ_{1a}	1	1	1	1
ϕ_{2a1}	2	−1	$\zeta_5 + \zeta_5^4$	$\zeta_5^2 + \zeta_5^3$
ϕ_{2a2}	2	−1	$\zeta_5^2 + \zeta_5^3$	$\zeta_5 + \zeta_5^4$
ϕ_{4a}	4	1	−1	−1

The Brauer character table of projective $\Lambda_{(2)}$-modules is

classes:	1	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	20	12	12
η_{1a}	12	.	2	2
η_{3a1}	8	−1	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
η_{3a2}	8	−1	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
η_{4a}	4	1	−1	−1

The decomposition matrix associated with $\Lambda_{(2)}$ is

	ϕ_{1a}	ϕ_{2a1}	ϕ_{2a2}	ϕ_{4a}
χ_{1a}	1	.	.	.
χ_{3a1}	1	1	.	.
χ_{3a2}	1	.	1	.
χ_{4a}	.	.	.	1
χ_{5a}	1	1	1	.

The Cartan matrix associated with $\Lambda_{(2)}$ is

	η_{1a}	η_{2a1}	η_{2a2}	η_{4a}
ϕ_{1a}	4	2	2	.
ϕ_{2a1}	2	2	1	.
ϕ_{2a2}	2	1	2	.
ϕ_{4a}	.	.	.	1

Let $U_{2,1a}$, $U_{2,2a1}$, $U_{2,2a2}$, and $U_{2,4a}$ be representatives of the isomorphism classes of the finitely generated indecomposable projective right $\Lambda_{(2)}$-modules. Then

$$\Lambda_{(2)} \cong U_{2,1a} \oplus U_{2,2a1}^2 \oplus U_{2,2a2}^2 \oplus U_{2,4a}^4$$

as right $\Lambda_{(2)}$-modules. There are two blocks: $A_{2,1a} = (1 - e_{4a})\Lambda_{(2)}$ of defect 2, and $A_{2,4a} = e_{4a}\Lambda_{(2)}$ of defect 0, where

$$A_{2,1a} \cong U_{2,1a} \oplus U_{2,2a1}^2 \oplus U_{2,2a2}^2, \qquad A_{2,4a} \cong U_{2,4a}^4.$$

A.2 Alternating Group A_6

Let $G = A_6$, the alternating group on 6 letters[2]. The conjugacy classes of A_6 are represented by 1, (1 2)(3 4), (1 2 3), (1 2 3)(4 5 6), (1 2 3 4)(5 6), (1 2 3 4 5), and (1 3 5 2 4), with sizes 1, 45, 40, 40, 90, 72, and 72, respectively. The splitting field for G is $K = \mathbb{Q}(\sqrt{5})$, and its ring of integers is $R = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. Let $A = KG$ and $\Lambda = RG$. The complex character table of G is

classes:	1	(1 2)(3 4)	(1 2 3)	(1 2 3)(4 5 6)	(1 2 3 4)(5 6)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	45	40	40	90	72	72
χ_{1a}	1	1	1	1	1	1	1
χ_{5a}	5	1	2	−1	−1	0	0
χ_{5b}	5	1	−1	2	−1	0	0
χ_{8a1}	8	0	−1	−1	0	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
χ_{8a2}	8	0	−1	−1	0	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
χ_{9a}	9	1	0	0	1	−1	−1
χ_{10a}	10	−2	1	1	0	0	0

where $\zeta_5 = \exp(2\pi i/5)$. The order of the group is $360 = 2^3 \cdot 3^2 \cdot 5$, and the primes containing 2, 3, and 5 are $(2) = (2)$, $(3) = (3)$, and $(\sqrt{5}) = (\sqrt{5})$, respectively.

[2] A_6 is also a simple group of Lie type, in particular, $A_6 \cong A_1(9)$, or equivalently, $A_6 \cong L_2(9)$

Let e_{1a}, e_{5a}, e_{5b}, e_{8a1}, e_{8a2}, e_{9a}, and e_{10a} be the primitive central idempotents of A. Then

$$A \cong e_{1a}A \oplus e_{1a}A \oplus e_{5a}A \oplus e_{5b}A \oplus e_{8a1}A \oplus e_{8a2}A \oplus e_{9a}A \oplus e_{10a}A$$

as right A-modules. For every prime ideal $\mathfrak{p} \notin \{2, 3, (\sqrt{5})\}$,

$$\Lambda_{\mathfrak{p}} \cong e_{1a}\Lambda_{\mathfrak{p}} \oplus e_{5a}\Lambda_{\mathfrak{p}} \oplus e_{5b}\Lambda_{\mathfrak{p}} \oplus e_{8a1}\Lambda_{\mathfrak{p}} \oplus e_{8a2}\Lambda_{\mathfrak{p}} \oplus e_{9a}\Lambda_{\mathfrak{p}} \oplus e_{10a}\Lambda_{\mathfrak{p}}$$

as right $\Lambda_{\mathfrak{p}}$-modules.

Localization at $(\sqrt{5})$. There are five 5-conjugacy classes in A_6. The Brauer character table of simple $\Lambda_{(\sqrt{5})}$-modules is obtained from the ordinary character table by deleting the fifth and sixth rows and the sixth and seventh columns.

classes:	1	(1 2)(3 4)	(1 2 3)	(1 2 3)(4 5 6)	(1 2 3 4)(5 6)
sizes:	1	45	40	40	90
ϕ_{1a}	1	1	1	1	1
ϕ_{5a}	5	1	2	−1	−1
ϕ_{5b}	5	1	−1	2	−1
ϕ_{8a}	8	.	−1	−1	.
ϕ_{10a}	10	−2	1	1	.

The Brauer character table of projective $\Lambda_{(\sqrt{5})}$-modules is

classes:	1	(1 2)(3 4)	(1 2 3)	(1 2 3)(4 5 6)	(1 2 3 4)(5 6)
sizes:	1	45	40	40	90
η_{1a}	10	2	1	1	2
η_{5a}	5	1	2	−1	−1
η_{5b}	5	1	−1	2	−1
η_{8a}	25	1	−2	−2	1
η_{10a}	10	−2	1	1	.

The decomposition matrix associated with $\Lambda_{(\sqrt{5})}$ is

	ϕ_{1a}	ϕ_{5a}	ϕ_{5b}	ϕ_{8a}	ϕ_{10a}
χ_{1a}	1	.	.	.	.
χ_{5a}	.	1	.	.	.
χ_{5b}	.	.	1	.	.
χ_{8a1}	.	.	.	1	.
χ_{8a2}	.	.	.	1	.
χ_{9a}	1	.	.	1	.
χ_{10a}	.	.	.	.	1

The Cartan matrix associated with $\Lambda_{(\sqrt{5})}$ is

	η_{1a}	η_{5a}	η_{5b}	η_{8a}	η_{10a}
ϕ_{1a}	2	.	.	1	.
ϕ_{5a}	.	1	.	.	.
ϕ_{5b}	.	.	1	.	.
ϕ_{8a}	1	.	.	3	.
ϕ_{10a}	.	.	.	.	1

Let $U_{5,1a}$, $U_{5,5a}$, $U_{5,5b}$, $U_{5,8a}$, and $U_{5,10a}$ be representatives of the isomorphism classes of the finitely generated indecomposable projective right $\Lambda_{(\sqrt{5})}$-modules. Then

$$\Lambda_{(\sqrt{5})} \cong U_{5,1a} \oplus U_{5,5a}^5 \oplus U_{5,5b}^5 \oplus U_{5,8a}^8 \oplus U_{5,10a}^{10}$$

as right $\Lambda_{(\sqrt{5})}$-modules. There are 4 blocks: $A_{5,1a} = (1 - e_{5a} - e_{5b} - e_{10a})\Lambda_{(\sqrt{5})}$ of defect 1, $A_{5,5a} = e_{5a}\Lambda_{(\sqrt{5})}$ of defect 0, $A_{5,5b} = e_{5b}\Lambda_{(\sqrt{5})}$ of defect 0, and $A_{5,10a} = e_{10a}\Lambda_{(\sqrt{5})}$ of defect 0, where

$$A_{5,1a} \cong U_{5,1a} \oplus U_{5,8a1}^8 \oplus U_{5,8a2}^8 \oplus U_{5,9a}^9,$$
$$A_{5,5a} \cong U_{5,5a}^5, \qquad A_{5,5b} \cong U_{5,5b}^5, \qquad A_{5,10a} \cong U_{5,10a}^{10}.$$

Localization at (3)**.** There are five 3-conjugacy classes in A_6. The Brauer character table of simple $\Lambda_{(3)}$-modules is as follows.

classes:	1	(1 2)(3 4)	(1 2 3 4)(5 6)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	45	90	72	72
ϕ_{1a}	1	1	1	1	1
ϕ_{3a1}	3	−1	1	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
ϕ_{3a2}	3	−1	1	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
ϕ_{4a}	4	.	−2	−1	−1
ϕ_{9a}	9	1	1	−1	−1

The Brauer character table of projective $\Lambda_{(3)}$-modules is

classes:	1	(1 2)(3 4)	(1 2 3 4)(5 6)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	45	90	72	72
η_{1a}	27	3	−1	2	2
η_{3a1}	18	−2	0	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
η_{3a2}	18	−2	0	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$
η_{4a}	36	0	−2	1	1
η_{9a}	9	1	1	−1	−1

The decomposition matrix associated with $\Lambda_{(3)}$ is

	ϕ_{1a}	ϕ_{3a1}	ϕ_{3a2}	ϕ_{4a}	ϕ_{9a}
χ_{1a}	1	.	.	.	.
χ_{5a}	1	.	.	1	.
χ_{5b}	1	.	.	1	.
χ_{8a1}	1	1	.	1	.
χ_{8a2}	1	.	1	1	.
χ_{9a}	.	.	.	.	1
χ_{10a}	.	1	1	1	.

The Cartan matrix associated with $\Lambda_{(3)}$ is

	η_{1a}	η_{3a1}	η_{3a2}	η_{4a}	η_{9a}
ϕ_{1a}	5	1	1	4	.
ϕ_{3a1}	1	2	1	2	.
ϕ_{3a2}	1	1	2	2	.
ϕ_{4a}	4	2	2	5	.
ϕ_{9a}	.	.	.	.	1

Let $U_{3,1a}$, $U_{3,3a1}$, $U_{3,3a2}$, $U_{3,4a}$, and $U_{3,9a}$ be representatives of the isomorphism classes of the finitely generated indecomposable projective right $\Lambda_{(3)}$-modules. Then

$$\Lambda_{(3)} \cong U_{3,1a} \oplus U_{3,3a1}^3 \oplus U_{3,3a2}^3 \oplus U_{3,4a}^4 \oplus U_{3,9a}^9$$

as right $\Lambda_{(3)}$-modules. There are two blocks: $A_{3,1a} = (1 - e_{9a})\Lambda_{(3)}$ of defect 2, and $A_{3,9a} = e_{9a}\Lambda_{(3)}$ of defect 0, where

$$A_{3,1a} \cong U_{3,1a} \oplus U_{3,3a1}^3 \oplus U_{3,3a2}^3 \oplus U_{3,4a}^4, \qquad A_{3,9a} \cong U_{3,9a}^9.$$

Localization at (2)**.** There are five 3-conjugacy classes in A_6. The Brauer character table of simple $\Lambda_{(2)}$-modules is as follows.

classes:	1	(1 2 3)	(1 2 3)(4 5 6)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	40	40	72	72
ϕ_{1a}	1	1	1	1	1
ϕ_{4a}	4	1	−2	−1	−1
ϕ_{4b}	4	−2	1	−1	−1
ϕ_{8a1}	8	−1	−1	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
ϕ_{8a2}	8	−1	−1	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$

The Brauer character table of projective $\Lambda_{(2)}$-modules is

classes:	1	(1 2 3)	(1 2 3)(4 5 6)	(1 2 3 4 5)	(1 3 5 2 4)
sizes:	1	40	40	72	72
η_{1a}	40	4	4	.	.
η_{4a}	24	3	.	−1	−1
η_{4b}	24	.	3	−1	−1
η_{8a1}	8	−1	−1	$-\zeta_5 - \zeta_5^4$	$-\zeta_5^2 - \zeta_5^3$
η_{8a2}	8	−1	−1	$-\zeta_5^2 - \zeta_5^3$	$-\zeta_5 - \zeta_5^4$

The decomposition matrix associated with $\Lambda_{(2)}$ is

	ϕ_{1a}	ϕ_{4a}	ϕ_{4b}	ϕ_{8a1}	ϕ_{8a2}
χ_{1a}	1	.	.	.	.
χ_{5a}	1	1	.	.	.
χ_{5b}	1	.	1	.	.
χ_{8a1}	.	.	.	1	.
χ_{8a2}	.	.	.	.	1
χ_{9a}	1	1	1	.	.
χ_{10a}	2	1	1	.	.

The Cartan matrix associated with $\Lambda_{(2)}$ is

	η_{1a}	η_{4a}	η_{4b}	η_{8a1}	η_{8a2}
ϕ_{1a}	8	4	4	.	.
ϕ_{4a}	4	3	2	.	.
ϕ_{4b}	4	2	3	.	.
ϕ_{8a1}	.	.	.	1	.
ϕ_{8a2}	.	.	.	.	1

Let $U_{2,1a}$, $U_{2,4a}$, $U_{2,4b}$, $U_{2,8a1}$, and $U_{2,8a2}$ be representatives of the isomorphism classes of the finitely generated indecomposable projective right $\Lambda_{(2)}$-modules. Then

$$\Lambda_{(2)} \cong U_{2,1a} \oplus U_{2,4a}^4 \oplus U_{2,4b}^4 \oplus U_{2,8a1}^8 \oplus U_{2,8a2}^8$$

as right $\Lambda_{(2)}$-modules. There are three blocks: $A_{2,1a} = (1 - e_{8a1} - e_{8a2})\Lambda_{(2)}$ of defect 1, $A_{2,8a1} = e_{8a1}\Lambda_{(2)}$ of defect 0, and $A_{2,8a2} = e_{8a2}\Lambda_{(2)}$ of defect 0, where

$$A_{2,1a} \cong U_{2,1a} \oplus U_{2,4a}^4 \oplus U_{2,4b}^4, \qquad A_{2,8a1} \cong U_{2,8a1}^8, \qquad A_{2,8a2} \cong U_{2,8a2}^8.$$