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Bootstrapping Large–Nc QCD and String Amplitudes

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Bootstrapping Large-N_c QCD and String Amplitudes

A thesis presented for the degree of Doctor in Physics by:

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Abstract

This thesis is presented as a compendium of six papers completed during my doctorate, developing the bootstrap of scattering amplitudes along two complementary fronts, applied to two physically distinct ultraviolet completions.

In the first part we use dispersion relations and positivity bounds to constrain large-N_c QCD. We map the allowed region of the chiral Lagrangian Wilson coefficients and show that certain critical kinks correspond to specific UV compleitions. We provide a derivation of Vector Meson Dominance and an explanation of the success of holographic models of QCD. Dressing the pions with external gauge bosons, we extend the dispersive logic into the anomalous sector and derive an analytic upper bound on the chiral anomaly coefficient in terms of the pion polarizabilities, a bound useful as a consistency check on holographic constructions and on the lattice. At higher multiplicity, we develop a notion of *multipositivity* based on a Cauchy–Schwarz statement on factorised planar residues, and find that certain Wilson coefficients of the chiral Lagrangian are bounded from below by the chiral anomaly itself, exposing a tie between the anomalous and nonanomalous sectors that is invisible to four-point scattering.

In the second part we turn the same machinery to the problem of quantum gravity. From a handful of consistency requirements we prove a version of the *uniqueness of strings*: the Veneziano and Virasoro–Shapiro amplitudes emerge as the only solutions, with their spectrum, Regge behaviour, and analytic form delivered as outputs rather than inputs. We then use perturbative gravitational scattering to derive the swampland *completeness* conjecture from the bottom up. Coupling a certain charged sector to gravity, with the graviton being a universal exchange, forces a finite seed of charged states to populate the entire weight lattice of the symmetry group with single-particle states. Imposing the equivalence principle on the high-energy amplitude sharpens this conclusion to a statement of *representation completeness*: single-particle states are forced in every irreducible representation that can be built from a single seed charge, and these deduced states contribute democratically to scattering, with squared couplings independent of their representation.

Complete list of publications during PhD

[1] C. Fernandez, A. Pomarol, F. Riva and F. Sciotti, *Cornering large-N_c QCD with positivity bounds*, *JHEP* **06** (2023) 094 [2211.12488]

[2] T. Ma, A. Pomarol and F. Sciotti, *Bootstrapping the chiral anomaly at large N_c*, *JHEP* **11** (2023) 176 [2307.04729]

[3] Z.-Y. Dong, T. Ma, A. Pomarol and F. Sciotti, *Bootstrapping the chiral-gravitational anomaly*, *JHEP* **05** (2025) 114 [2411.14422]

[4] B. Bellazzini, A. Pomarol, M. Romano and F. Sciotti, *(Super) gravity from positivity*, *JHEP* **03** (2026) 028 [2507.12535]

[5] C. Cheung, G. N. Remmen, F. Sciotti and M. Tarquini, *Strings from almost nothing*, *Phys. Rev. Lett.* (2026)

[6] F. Calisto, C. Cheung, G. N. Remmen, F. Sciotti and M. Tarquini, *Completeness from gravitational scattering*, *Phys. Rev. D* **113** (2026) 106008 [2512.11955]

[7] B. Bellazzini, J. Berman, G. Isabella, F. Riva, M. Romano and F. Sciotti, *Positivity with Long-Range Interactions*, 2512.13780

[8] F. Calisto, C. Cheung, G. N. Remmen, F. Sciotti and M. Tarquini, *On the Inverse Problem in Effective Field Theory*, 2604.15423

[9] F. Calisto, C. Cheung, G. N. Remmen, F. Sciotti and M. Tarquini, *The Equivalence Principle at High Energies Completes the Spectrum*, 2605.20319

[10] C. Cheung, J. Jeong, P. Ko, A. Pomarol, G. N. Remmen and F. Sciotti, *Multipositivity Constrains the Chiral Lagrangian*, 2605.21582

This thesis consists of a collection of six papers completed during my graduate studies: [1], [2], and [10], which concern developments in bootstrapping large-N_c QCD, and [5], [6], and [9], which focus on the program of bootstrapping string amplitudes. They are reproduced throughout the thesis as they appear in their latest formulation.

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1 Introduction

A recurring theme presents itself in the structure of modern theoretical physics: the most fundamental laws of nature reach us through the imprint they leave on physics at far longer distances. Our deepest descriptions of the world are organised as effective expansions where a handful of measurable parameters carry the weight of the physics we have not yet probed. Within such a framework the central theoretical question is not which Lagrangian one can write down, but which Lagrangians are admissible at all: which low-energy theories can be the consistent shadow of a complete and well-defined microscopic description, and what universal features that microscopic physics must possess independently of its details. It is a question that ties the infrared dynamics of pions to the universal structure of any consistent theory of quantum gravity, and it is the question that animates the work of this thesis.

The bootstrap paradigm precisely embodies the picture above. It does not start from a notion of complete theory or of a Lagrangian, but from a short list of structural principles; causality, unitarity, locality, and asks the question: which theories are even consistent with these principles in the first place? Whatever cannot be reconciled with them is, by definition, ruled out, regardless of how plausible the underlying Lagrangian appears. The space of admissible theories turns out to be far narrower than the space of theories one can write down. Before delving into the technicalities of quantum field theory, it is instructive to see how the same logic already manifests itself in a much simpler setting: classical physics.

100 years of Kramers-Kronig. This thesis marks the centenary anniversary of a set of papers by Kramers and Kronig [11, 12] which laid the conceptual groundwork for the modern bootstrap program. At their core, the Kramers–Kronig relations express a powerful idea: causality in the time domain translates into analyticity in the frequency domain[1].

To see this, consider a slab of glass struck at $t = 0$ by a sharp pulse of light. The polarisation responds linearly to the electric field through the susceptibility $\chi(t)$. By definition the slab cannot react before the pulse has reached it, so that causality forces $\chi(t) = 0$ for $t < 0$. Fourier transforming,

$$\chi(\omega) \;=\; \int_0^\infty \chi(t)\, e^{i\omega t}\, dt\,, \tag{1}$$

the integral runs only over $t > 0$, and the factor $e^{-\mathrm{Im}(\omega)\, t}$ ensures convergence for $\mathrm{Im}(\omega) > 0$, so that $\chi(\omega)$ defines an analytic function in the upper half-plane. Equivalently, one may regard Eq. (1) as a Laplace transform in t, analytic in its region of convergence under mild assumptions on the growth of $\chi(t)$. The full equivalence between time-domain causality and upper-half-plane analyticity, including the non-trivial converse, which we have not shown, is the content of Titchmarsh's theorem [14] and follows from a precise statement on the convergence of the square-integral of the function $\chi(t)$.

Given $\chi(\omega)$ analytic in the upper half-plane, consider the integral of $\chi(\omega')/(\omega' - \omega - i\epsilon)$ along a contour C running along the real axis and closing with a large arc in the upper half-plane.

[1] The derivation of the Kramers–Kronig relations based on causal arguments was later formalised in [13].

The simple pole sits at $\omega' = \omega + i\epsilon$, inside the contour, and the residue theorem gives

$$\oint_C \frac{\chi(\omega')}{\omega' - \omega - i\epsilon}\, d\omega' \;=\; 2\pi i\, \chi(\omega + i\epsilon)\,, \tag{2}$$

discarding the contribution on the arc at infinity, assuming that $\chi(\omega) \to 0$ when $\omega \to \infty$, and reading $\chi(\omega)$ as the boundary value $\lim_{\epsilon\to 0^+} \chi(\omega + i\epsilon)$, we find

$$\chi(\omega) \;=\; \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega - i\epsilon}\, d\omega'\,, \tag{3}$$

the $i\epsilon$ prescription keeping the pole displaced infinitesimally above the contour. The susceptibility at any single frequency is determined by an integral over every frequency in the spectrum, including for example the ultraviolet and X-ray bands one may never have measured. The low and high-energy contributions to the response do not influence the answer independently; causality glues them together.

Causality alone fixes the analytic structure but says nothing about signs. Here a second elementary principle enters: a passive medium absorbs energy, never creates it. Applying a monochromatic electric field $E(t) = \mathrm{Re}[E_0\, e^{-i\omega t}]$ to the slab, the time-averaged power dissipated per unit volume is $\dot{W} = \frac{1}{2}\,\varepsilon_0\,\omega\,\mathrm{Im}\,\chi(\omega)\,|E_0|^2$, so passivity ensures that

$$\text{when } \omega > 0 \quad\longrightarrow\quad \mathrm{Im}\,\chi(\omega) \;\geq\; 0\,. \tag{4}$$

Let us now assume the following asymptotic behavior for large frequencies $\omega \to \infty$:

$$\chi(\omega) \sim -\frac{\omega_p^2}{\omega^2}, \tag{5}$$

where $\omega_p^2 = ne^2/(m\varepsilon_0)$. Alternatively, let us take the real part of Eq. (3), which we can express as an integral over its absorptive part

$$\mathrm{Re}\,\chi(\omega) \;=\; \frac{1}{\pi}\,\mathrm{Re}\!\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\chi(\omega')}{\omega' - \omega - i\epsilon}\, d\omega'\,. \tag{6}$$

Expanding Eq. (6) at large ω, where the $i\epsilon$ is harmless (the pole sits far from the integration region), we find

$$\mathrm{Re}\,\chi(\omega) \;\sim\; -\frac{1}{\pi\omega}\int_{-\infty}^{\infty} \mathrm{Im}\,\chi(\omega')\, d\omega' \;-\; \frac{1}{\pi\omega^2}\int_{-\infty}^{\infty} \omega'\,\mathrm{Im}\,\chi(\omega')\, d\omega' \;+\; \mathcal{O}(\omega^{-3})\,. \tag{7}$$

$\chi(t)$ is real-valued in time, so from the Fourier transform we have $\chi(-\omega') = \chi^*(\omega')$. This allows us to fold half of the integral on itself. The leading $1/\omega$ contribution vanishes by parity while the next to leading term survives. Matching to the asymptotic expression Eq. (5) yields

$$1\int_0^{\infty} \omega\,\mathrm{Im}\chi(\omega)\, d\omega = \frac{\pi}{2}\,\omega_p^2\,, \tag{8}$$

the so-called f-sum rule. We found in Eq. (4) that the integrand is nonnegative, therefore this relation states that the total integrated absorption strength is fixed determined by ω_p.

Given a certain ω_p, all frequency bins contribute positively. No material can be transparent at every frequency, and the deficit one observes at any window must be made up elsewhere in the spectrum. A similar argument can be made to show that the principle of causality is at the core of the separation of colours in prisms and the wavelength-dependent spreading of optical signals in diffraction phenomena[2]. The non-negativity of $\mathrm{Im}\chi(\omega)$ here is the classical avatar of unitarity, a condition at the heart of the bootstrap programme and central to everything that follows in this thesis.

The lessons enclosed in this first formulation of dispersive relations in optics, developed a century ago, continue to play a central role in the modern bootstrap program. The thought experiment one should carry forward into the rest of this thesis is therefore the simplest one imaginable. A shutter opens, a light pulse enters a slab, a detector clicks no earlier than it should. From that single inequality follow analyticity, dispersion relations, and an infinite family of positive sum rules linking observables across all energies. It is remarkable that so much structure flows from so little, and it is even more remarkable that the same logic, transplanted into relativistic scattering, continues to do the heavy lifting one hundred years on.

Modern bootstrap: from EFTs to strings. Scattering amplitudes are the central observables of relativistic quantum field theories. They are what scattering experiments measure, and into them fundamental principles such as locality, causality and unitarity are folded in a remarkably compact form. Another key concept is that at sufficiently low energies the framework of effective field theory (EFT) provides a universal description of physics without detailed knowledge of the underlying ultraviolet completion. Rather than attempting to describe all microscopic physics exactly, EFTs parametrize the most general interactions compatible with the symmetries of the system, organized as an expansion in energy. The coefficients of this expansion, called Wilson coefficients, capture the imprint of high-energy physics on low-energy observables. If the underlying high-energy theory is itself consistent with fundamental principles such as unitarity, causality and locality, then these consistency conditions will leave characteristic signatures in the low-energy observables and constrain the space of Wilson coefficients of the effective theory. At the core of the *bootstrap* program is the idea that fundamental principles are far more constraining than what one may think, just as hinted in the context of the Kramers-Kronig relations above, and the goal is therefore to determine, or at least sharply constrain, the space of physically admissible scattering amplitudes and consistent effective field theories without necessarily appealing to a specific microscopic Lagrangian description.

This idea has a long pedigree. Already in the 1940s Heisenberg envisioned an S-matrix-based reformulation of quantum theory [16][3], and through the 1960s it developed into a serious attempt to determine the strong interactions on the basis of analyticity, crossing and unitarity alone[4]. Around the same time, the Veneziano amplitude [22], born in the search for crossing-symmetric, Regge-behaved hadronic amplitudes, opened the road to string theory. The analytic

[2] A brief and insightful review of these ideas was presented in this talk [15].

[3] For an early review and discussion of Heisenberg's S-matrix program, see Ref. [17].

[4] Some of the key historical references are Refs. [18,19], see also the books [20,21] for a comprehensive account of the analytic S-matrix program and its early development.

dispersion relation methods developed in those years, the central technical tool of this thesis, have remained an indispensable element of many subsequent advances in theoretical physics. The bootstrap of that era was eventually overtaken by the discovery of QCD as a field theory of quarks and gluons (and the subsequent discovery of the Standard Model), and by the reformulation of string theory as a worldsheet quantum field theory; the dispersive technology it had produced, however, never went away.

In the 2000s a new wave of bootstrap activity began to reshape the field. The analytic-dispersive viewpoint on effective field theories was revived [23]–a revival made possible by the shift in mindset that Weinberg and Wilson had introduced decades earlier [24,25]. The program of the 1960s had the goal (and hope) of *solving* the strong interactions outright, singling out the one true S-matrix (which perhaps explains why the program lost momentum once QCD was discovered). The modern bootstrap sets itself a humbler but, paradoxically, more powerful goal. Rather than solving for a single theory, it charts the entire *landscape* of admissible ones: every effective field theory that can descend from a healthy ultraviolet completion consistent with quantum mechanics and relativity. Dispersion relations and positivity now enter as sharp *inequalities* that carve out this space. The landscape so delineated is strictly smaller than the set of theories that look acceptable at first sight, yet it remains vast. Originally in [23], the bootstrap technology was applied mainly to cosmology and modified gravity where (quoting) *the pendulum had swung too far in the "anything goes" direction*, and one-sided constraints were derived forcing the leading coefficient of a scalar effective theory to be positive in the forward limit. From this a broad research programme started developing.

On one side it was put to work on different areas of phenomenology, for example constraining the electroweak and Higgs sectors of the Standard Model [26–29] where composite Higgs models were *unbalancing the stability of the "anything goes" pendulum*, or the effective theory of inflation [30], and theories of modified and quantum gravity [31, 32]. On the other hand it moved beyond the forward limit [27, 33, 34], and extended to external states of any spin, from vectors to gravitons [35, 36]. In another influential work [37], via a complementary approach based on the requirement of no Shapiro time advance (rooted in causality) it was shown that a theory's inelastic content is bounded by its elastic one, placing strong constraints on gravity, such as the emergence of an infinite higher-spin tower. A further decisive turn came around 2020, with the realisation that imposing the *full* crossing symmetry of the amplitude promotes one-sided positivity to genuinely two-sided bounds endowed with a rich geometric structure on EFTs. These novel constraints were casted as a moment problem [38], a language which also allowed to determine the masses and couplings of the heavy states directly off the low-energy coefficients[5]. In parallel Wilson coefficients were shown to fill out a convex region [39], dubbed the *EFT-hedron*, within which any consistent ultraviolet completion must lie. Crossing symmetry was distilled into an infinite set of constraints on the UV spectrum, called *null constraints* [40], and dispersive sum rules were turned into sharp two-sided bounds also in the presence of gravity [41].

With the advent of full crossing symmetry, and in particular of the null constraints [40],

[5] in a recent work which I will not be presenting in this thesis [8] we constructed new logarithmic dispersion relations which allow to extract also the spins of the particles from EFT data.

the mindset of the EFT bootstrap program took another turn. While before the main goal had been to exclude theories, the new bounds (now sharp enough to be implemented numerically [42]) shifted the emphasis towards optimisation: searching for the special theories that sit at the boundaries and kinks of the allowed region, in analogy with the success of the 3D Ising model story in the conformal bootstrap [43, 44]. A surprising pattern emerged: known higher-spin ultraviolet models, such as stringy amplitudes, were close to saturating the bounds, a feature already visible in the EFT-hedron [39] and brought into especially sharp focus in the gravitational setting [45–47]. Furthermore throughout the whole space of EFT parameters, it was proven that UV completions are dominated by the lowest spin particle content, meaning that the largest contribution to the Wilson coefficient arises from low spin particles, a feature dubbed *low spin dominance* (this can be seen very simply with null constraints).

Running alongside this dispersive, dual bootstrap is a complementary line, the *primal S-matrix bootstrap*, which constructs the amplitudes themselves rather than the region of their Wilson coefficients [48]. The idea is to scan numerically the space of analytic, crossing-symmetric and unitary amplitudes, finding solutions that saturate the consistency conditions "from the inside". Applied to pion scattering, both with the physical mass spectrum [49] and in the massless limit appropriate to the chiral EFT [50], the method produced extremal amplitudes that come close to the QCD data using little more than chiral symmetry as an input. The primal and dual approaches eventually met: the regions carved out from outside by positivity bounds and those filled in from inside by the primal numerics can be reconciled, and where they meet they share extremal points [51]. Whenever this happens the bounds can be considered converged, and the amplitudes that sit at the boarder can be called extremal.

The emergence of string amplitudes in the bootstrap program revived a much older question: *which* theories are these, and in what precise sense are string amplitudes singled out at all? The question is almost as old as string theory itself. It was born with the very first dual amplitude, written down by Veneziano [22] in the search for a crossing-symmetric, Regge-behaved description of the strong interactions, and almost at immediately people began to ask how unique this amplitude really was. Already in the late 1960s the formula was found to admit deformations, such amplitudes with nonlinear Regge trajectories and an accumulation point in the spectrum [52], or the generalised Veneziano model with the satellite terms of Gross [53]. Even then the amplitude appeared as one point in a larger family of consistent dual models rather than an isolated solution. This circle of ideas was revived, with the tools of the modern amplitudes program, after the mid-2010s, along two complementary fronts. On the first, people started to try and understand the set of assumptions necessary for the Veneziano amplitude to emerge as a solution of the bootstrap problem. Certain conditions were found to give asymptotic uniqueness [54], and in the true spirit of the bootstrap philosophy later on it was shown that when some of the assumptions were relaxed asymptotic uniqueness disappeared [55]. On the second front, the space of admissible deformations was instead mapped out constructively, producing explicit, unitary, consistently ultraviolet-behaved generalisations of the Veneziano amplitude that retain dual resonance for essentially arbitrary spectra [56, 57]. Taken together, these developments turned the uniqueness of strings from a slogan into a sharply posed—and still only partially answered—question.

Whether string amplitudes are truly singled out is, in turn, one facet of a broader question that animates the *swampland* program [58]: which low-energy effective theories can descend from a consistent theory of quantum gravity, and which cannot. That program has crystallised into a web of conjectures, among them the absence of exact global symmetries [59], the weak gravity conjecture [60], and the completeness of the charge spectrum [61]. Once argued largely on formal or stringy grounds, such statements can now be confronted from the bottom up: dispersion relations have become a quantitative tool with which to test and, in favourable cases, to derive the conjectures of the swampland.

Roadmap for the thesis. Everything recounted so far predates the beginning of my doctorate; it is the backdrop against which the present work was carried out. During my PhD both programs advanced along particular fronts, and the six papers gathered in this thesis belong to those advances. Four further works completed in the same period extend the programme along closely related lines and are discussed in the Conclusions, but did not fit into the compendium. On the effective-field-theory side, the positivity bootstrap was used for large-N_c QCD; on the gravitational side, the bootstrap of amplitudes was turned both to the uniqueness of strings and to the proof of swampland conjectures. The thesis is organised accordingly, in two parts.

The first part carries the bootstrap program into large-N_c QCD and the chiral Lagrangian. The opening work [1] charts the parameter space accessible from pion scattering and related properties. The next [2] dresses the pions with external currents to reach the chiral anomaly, converting amplitude consistency into a sharp inequality on the anomaly coefficient. The third [10] ascends to higher-point pion amplitudes, where a new incarnation of positivity binds the anomalous and non-anomalous sectors together, bounding ordinary chiral couplings by the anomaly itself.

The second part crosses over to gravity and strings, taking up the two questions left hanging above. The first work [5] settles the uniqueness of strings: from a handful of requirements, the Veneziano and Virasoro–Shapiro amplitudes emerge as the only solutions, their spectrum and high-energy behaviour delivered as outputs rather than assumed at the outset. The remaining two turn amplitude consistency onto the swampland conjectures themselves; in theories where particles charged under external symmetry groups are coupled to gravity, perturbative gravitational scattering is shown to force the charge spectrum to be complete [6] and to fill out every representation reachable from a single seed charge [9].

These two parts are laid out in Secs. 2 and 3, each opening with a fuller overview of the program and the tools it relies on. Within them every paper is prefaced by a short introduction recalling what was known and explaining what we did, before being reproduced in its published form.

2 Bootstrapping large-N_c QCD

We now take up the first of the two directions announced above: the positivity bootstrap applied to large-N_c QCD. One may wonder: why would we want to study QCD if the full theory is not only known, but also one of the few theories in nature which does not need a UV completion? The reason is that QCD is strongly coupled at the confinement scale, therefore although the Lagrangian is known, calculating anything in the theory is not possible within the framework of perturbative QFT. This is where the bootstrap comes in, offering a handle on precisely the regime that perturbation theory cannot reach. The overview below assembles the ingredients needed to make this programme well defined, and gives context for the three papers introduced later on in the section.

2.1 Overview

The problem of strong coupling. Quantum chromodynamics is, on paper, a complete theory: a gauge field, a handful of quarks, a single coupling. And yet almost none of the physics we actually care about (why quarks are confined, why the proton weighs what it does, why the pions come out so much lighter than everything else) can be pried out of that Lagrangian by the usual means. The trouble is that the coupling grows towards the infrared, so that around the energies where hadrons live the theory is strongly coupled and perturbation theory in the gauge coupling, the workhorse of high-energy physics, simply stops working. We are left with a Lagrangian we cannot solve.

A possible way around this rests on two ideas that act in parallel. The first is to stop expanding in the coupling and expand instead in the number of colours; sending it to infinity reshuffles the theory so that the natural variables are no longer quarks and gluons but the hadrons they bind into, and we end up with what is effectively a theory of mesons. The second is to lean on chiral symmetry: the pions, the lightest hadrons, are governed by the spontaneously broken chiral symmetry of QCD (approximately $SU(3)$, for the three light quarks), and chiral perturbation theory organises their interactions. The work of this first part sits where the two ideas overlap: it is the chiral Lagrangian, but read in the large-N_c limit, where the strong dynamics become clean enough that one can impose consistency conditions on them with real precision.

The large-N_c limit. 't Hooft's resolution [62] is to treat the number of colours, rather than the coupling, as the large quantity: we promote the gauge group to $SU(N_c)$ and send $N_c \to \infty$ while holding the 't Hooft coupling $\lambda = g^2 N_c$ fixed. The diagrams of the theory then reorganise themselves not by the number of loops but by their *topology*, in powers of $1/N_c$. Drawn in the double-line notation that keeps track of colour flow, each diagram tiles a two-dimensional surface, and its weight is set by the genus of that surface: planar diagrams (those that can be drawn on a sphere without any line crossing another) dominate, while every non-planar complication, every handle in the surface and every internal quark loop, costs additional powers of $1/N_c$ and is suppressed. Real QCD has $N_c = 3$, so $1/N_c$ is not especially small, and yet much of the low-energy phenomenology already comes out qualitatively right at leading order,

with the near-linear Regge trajectories of the meson spectrum a textbook illustration. For our purposes the decisive feature is that this limit makes the theory analytically clean enough that we can impose consistency conditions on it sharply.

What the planar expansion describes, as Witten emphasised [63], is a theory of *mesons*. In the planar limit the correlators of colour-singlet quark bilinears are saturated by an infinite tower of stable, non-interacting mesons, with masses of order N_c^0 and couplings that switch off as N_c grows: a trilinear meson vertex scales as $1/\sqrt{N_c}$, a quartic as $1/N_c$, and so on. The resonances become narrow, their widths vanishing in the strict limit, and to leading order they merely propagate and exchange one another at tree level. Baryons, by contrast, behave as heavy solitons with masses of order N_c and decouple from the meson sector. In this precise sense large-N_c QCD is a weakly-coupled theory of an infinite family of mesons, rather than of quarks and gluons. This is the form in which the strongly-coupled theory becomes a target for the bootstrap: a tree-level theory of mesons whose Lagrangian is unknown and contains, in principle, infinitely many terms, but whose amplitudes are nonetheless tightly constrained. None of this requires solving QCD; the problem is no longer to compute the spectrum from first principles, but to ask which spectra are consistent at all.

Chiral Lagrangian. Massless QCD with N_f quark flavours possesses a chiral symmetry $U(N_f)_L \times U(N_f)_R$[6] that the strong dynamics break spontaneously to the diagonal $U(N_f)_V$. The associated Goldstone bosons are the pions, sitting in the adjoint of $SU(N_f)$, together with the flavour-singlet η'. At finite N_c the η' is conspicuously heavier than the pions, but in the planar limit it rejoins the Goldstone multiplet as a genuine massless state. Being Goldstones, the pions are the lightest excitations and decouple from the heavier resonances at low energies: they are the *infrared*, while the tower of massive mesons (the ρ, the f_2, and their ever-higher-spin excitations) is the *ultraviolet* whose structure we wish to reconstruct. The pion dynamics are governed by the chiral Lagrangian, the most general derivative expansion compatible with the broken symmetry, written in terms of $U = \exp(\sqrt{2}\, i\, \pi^a T^a / F_\pi)$ in the bifundamental of $U(N_f)_L \times U(N_f)_R$,

$$\mathcal{L} = -\frac{F_\pi^2}{4}\,\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\Big) + L_3\,\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\,\partial_\nu U^\dagger \partial^\nu U\Big) + L_4\,\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial_\nu U\,\partial^\mu U^\dagger \partial^\nu U\Big) + \dots\,. \quad (9)$$

The leading term, fixed by the single scale F_π (the pion decay constant), governs the two-derivative interactions and is universal; all the model-dependent information sits in the higher-derivative operators, with their dimensionless *Wilson coefficients* $L_3, L_4, \dots$. At large N_c only the single-trace operators survive at leading order, the multi-trace structures being suppressed by further powers of $1/N_c$, so the derivative expansion is itself organised by the same topological counting. These coefficients are the numbers that distinguish QCD's pions from those of any other theory with the same symmetry-breaking pattern, and they are physical: they control the pion scattering lengths, the curvature of the scattering amplitude away from threshold,

[6] At finite N_c the $U(1)_A$ factor is anomalous and the symmetry reduces to $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$; we work throughout in the planar limit, where the anomaly is $1/N_c$-suppressed and the full $U(N_f) \times U(N_f)$ enhancement is restored.

the electromagnetic polarizabilities of the pion, and the rates of the anomalous processes. Determining them, or even bounding them, is the concrete content of the programme, and decades after the chiral Lagrangian was written down it is still not fully known, even in principle, which values of these constants are consistent.

Planarity and flavour stripping. Two features of the planar limit make the dispersive analysis possible. The first is the analytic structure already noted: at leading order in $1/N_c$ the amplitude is meromorphic, its only singularities the simple poles of single-meson exchange, with the multi-particle cuts pushed to order $1/N_c^2$. This lets the contour be collapsed onto a discrete sum over states, and it is special to the planar limit; at finite N_c the cuts return. The second feature is flavour. A connected planar diagram is bounded by a *single* quark line, and the external pions are read off in cyclic order along it, so the only flavour structures that appear are single traces of the symmetry generators. An n-pion amplitude decomposes into *flavour-ordered* partial amplitudes, one for each cyclic ordering,

$$A_{a_1\cdots a_n} = \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\; A^{(n)}\Big(p_{\sigma(1)},\ldots,p_{\sigma(n)}\Big)\,, \tag{10}$$

and the same single-trace structure governs the chiral Lagrangian, whose multi-trace operators carry extra powers of $1/N_c$.

The ordering does real work: it fixes which states sit in which channel. In particular, the four-point amplitude $M(s,u)$ has no poles in the t-channel, where the two pions would form the exotic isospin-two combination that no $q\bar{q}$ meson can carry; such an exchange is non-planar, suppressed by $1/N_c^2$. The amplitude is singular only in the s and u channels, and this is the structure the dispersion relations exploit.

The bootstrap problem. The Wilson coefficients are the imprint of the ultraviolet on the infrared: each is generated by integrating out the heavy mesons, so the low-energy constants store the masses, spins and couplings of the whole tower above. To compute them we would need that spectrum, the very thing the strong coupling hides. The bootstrap turns the question around, asking instead which spectra are consistent with the low-energy data, and the bridge between the two is Cauchy's theorem.

View the flavour-ordered amplitude as an analytic function of s at fixed momentum transfer. Near the origin it is given by its low-energy expansion,

$$M(s,u) = g_{1,0}(s+u) + g_{2,0}(s^2+u^2) + g_{2,1}\,su + \ldots\,, \tag{11}$$

with no constant term, the *Adler zero* [64] that the Goldstone nature of the pion enforces, and coefficients $g_{n,l}$ built from the L_i. A small contour around $s=0$ isolates one such coefficient; Cauchy's theorem then lets us blow the contour up until it wraps the poles in the s and u channels, equating that coefficient to a sum over the mesons exchanged there. The step is true only if the arc at infinity drops out, which means dividing by enough powers of s. Each power is a *subtraction*, and how many are allowed is set by how fast the amplitude grows, which Regge theory ties to the spectrum: the mesons of rising spin lie on roughly linear *Regge*

trajectories $\alpha(t) \simeq \alpha_0 + \alpha' t$, functions on the complex t-plane along which the spectrum sits at the points $J_n = \alpha(m_n^2)$ where the trajectory passes through non-negative integers; the leading *intercept* $\alpha_0 = \alpha(0)$ then fixes the forward growth $M \sim s^{\alpha_0}$. The Froissart bound [65, 66] gives $M/s^2 \to 0$ and two subtractions; since large-N_c QCD has no *Pomeron*, no trajectory with intercept reaching one, its leading intercept lies below one, $M/s \to 0$, and a single subtraction is enough.

When the contour is opened up, what it picks up along the real s axis is the discontinuity of the amplitude, which is proportional to its imaginary part, and this is where unitarity enters. Decomposed into partial waves of definite spin, that imaginary part is a sum of spectral densities $\rho_J(s)$, non-negative because they measure probabilities. Positivity of the ρ_J turns each sum rule into an inequality: the leading coefficients are sums of manifestly positive terms,

$$g_{n,0} = \sum_i \frac{g_{i\pi\pi}^2}{m_i^{2n}} \;\geq\; 0. \tag{12}$$

These are the *positivity bounds*. Crossing symmetry over-determines the system and forces particular combinations of the spectral densities to vanish, the *null constraints*. Together, positivity and the null constraints carve out a bounded region for the L_i, and mapping that region is the bootstrap problem for large-N_c pions.

Higher-point amplitudes. The same flavour ordering carries over to higher multiplicity, and it is again what makes the problem tractable. At first sight going beyond four points looks forbidding: the kinematics proliferate, a six-point amplitude already depending on many invariants that are not independent but tied together by Gram constraints special to four dimensions. What saves the analysis is that planarity simplifies the analytic structure in the same way it did at four points. The flavour-ordered amplitude is meromorphic, and its simple poles appear only in the *planar channels*, the invariants built from cyclically adjacent momenta; exchanges in the remaining channels are non-planar and suppressed by $1/N_c^2$. With the singularities confined to this restricted set of channels, and their residues fixed by factorisation onto lower-point amplitudes, the dispersive logic of the four-point case goes through, now tying amplitudes of different multiplicity to one another. Large N_c buys one more thing, though it is less essential: *Regge boundedness*. With no Pomeron the leading intercepts lie below one, so the planar amplitudes grow slowly at high energy and a once-subtracted dispersion relation in a single channel already suffices.

The anomalous sector. There is one further piece of low-energy data, of an entirely different character from the L_i. Beside the parity-even derivative expansion, the chiral Lagrangian carries a topological Wess–Zumino–Witten term [67, 68] encoding the chiral anomaly,

$$S_{\mathrm{WZW}} = -\frac{\kappa}{240\pi^2} \int d^5x\, \epsilon^{\mu\nu\rho\sigma\tau}\, \mathrm{Tr}(j_\mu j_\nu j_\rho j_\sigma j_\tau)\,, \qquad j_\mu = i\, U^\dagger \partial_\mu U, \tag{13}$$

written as an integral over a five-dimensional manifold whose boundary is spacetime, and governing the parity-odd processes ($\pi^0 \to \gamma\gamma$, the five-pion vertex) that the derivative expansion

alone cannot produce. Its coefficient is special in a way no other Wilson coefficient is. The L_i depend on the details of the heavy spectrum and run; the anomaly does not. It is fixed once and for all by matching to the ultraviolet, and Witten's quantisation argument pins it to the integer $\kappa = N_c$. The anomaly is thus an exact piece of ultraviolet information mapped directly onto the infrared theory. Reaching it, however, requires going beyond pure pion scattering: either by dressing the pions with external currents, so that anomalous amplitudes appear, or by climbing to higher multiplicity, where the five-point amplitude carries the anomaly directly.

2.2 Paper 1: Cornering Large-N_c QCD with Positivity Bounds

This paper is based on the idea of using positivity bounds to target the chiral Lagrangian. At large N_c the strong interactions become a weakly-coupled theory of mesons, and the lightest of them, the pions, are the Goldstone bosons of the spontaneously broken chiral symmetry; their low-energy interactions are governed by the chiral Lagrangian, and the Wilson coefficients carry the imprint of the heavier mesons above. That low-energy theory is not unique to QCD. A whole space of consistent ultraviolet completions flows to the same pion effective description, and large-N_c QCD is only one point in it; positivity bounds turn the consistency of any such completion into inequalities on the Wilson coefficients, carving out the region in which every admissible theory of pions must lie, whatever sits above it.

This strategy was first carried through in [69], where positivity and the simple planar analytic structure inherited from the large-N_c limit were imposed on the $\pi\pi \to \pi\pi$ amplitude to map the allowed region of the leading Wilson coefficients. That region is bounded by smooth borders and sharp *kinks*. The most surprising and exciting result of the paper was that it found a novel kink which appeared stable under numerics and that did not map to any known UV completion (unlike other kinks which are populated by known UV completions such as theories with scalars in the UV). This novel kink was hypothesised to be given by large-N_c QCD, possibly in analogy with the numerical kink that the conformal bootstrap had identified with the 3D Ising model.

One of the central goals of our work [1] was to settle the status of that novel kink, and using analytic techniques we found it to be a numerical artefact. Recasting its location as a one-dimensional moment problem, we proved that it collapses onto a known ultraviolet completion, the theory with a single vector resonance; the kink is therefore neither large-N_c QCD nor a new theory. We are therefore left with a space where every kink turns out to correspond to a known amplitude: either a single vector or scalar pole, or an infinitely degenerate higher-spin tower (the amplitude known as the *su*-model). One region, however, remains to be understood: the curve connecting the single-vector theory to the *su*-model. Along it the ultraviolet completions are towers of higher-spin particles where the location and distribution of the masses is unknown. An well behaved (unitary, crossing etc..) amplitude which lives around this region is the Lovelace–Shapiro amplitude [70, 71], a version of the Veneziano with an Adler zero. It sits nearest the numerical boundary which however has not converged, therefore it is still unclear whether this amplitude actually saturates it or not. Lovelace–Shapiro is therefore the most extremal solution we currently know, and whether the true boundary is actually saturated by it or by some other theory remains an open question.

The same positivity that bounds the coefficients also explains *Vector Meson Dominance* [72], the hypothesis that the spin-1 ρ dominates the chiral couplings. Each Wilson coefficient is a sum of positive contributions weighted by the masses and couplings of the exchanged mesons, and by using null constraints [40] it is possible to simply prove that the lighter the spin, the more a state contributes to the Wilsons. This property was already observed in gravitational scattering [45] and for pions [69], and systematically studied by us and used to explain the success of holographic models in QCD [73, 74]. Using null constraints it is also simple to show that as soon as one higher spin particle enters, every other spin must enter as well.

All of these results presuppose a well-behaved amplitude, $M/s \to 0$ at high energy; we also work out what survives under only the weaker Froissart–Martin bound $M/s^2 \to 0$, where the spin-0 and spin-1 contributions decouple and the higher-spin sector is governed instead by the spin-2 state, a generalisation of the same dominance. We also compare the bounds to the results found in lattice studies of large-N_c [75]; in a parallel direction, experimental pion-scattering data have been fed directly into the amplitude ansatz of the primal bootstrap [76], providing another external benchmark against which our region can in principle be tested.

The most critical open question which remains to be understood following this paper, and that still today is left unanswered, is the precise nature of the boundary. As I mentioned earlier for a specific part of the space: the region where higher spin UV completions work, we only know the two boundary solutions, which are a theory with a spin 1 particle alone[7], and a theory with all higher spin particles at the same mass. What we do not understand is everything in between. Whether any point on the curve corresponds to a known theory like the Lovelace–Shapiro amplitude (which comes close), or if there is a way to make the numerics can converge and reveal something new. Charting the exact edge of the allowed region, and the amplitudes that populate it, is one of the most pressing goals of the program.

2.3 Paper 2: Bootstrapping the Chiral Anomaly at Large N_c

In this paper we pushed the positivity program into the anomalous sector of the chiral Lagrangian. The chiral anomaly is a piece of low-energy data unlike any other: a parity-odd, quantised coefficient κ, carried by the Wess–Zumino–Witten term and fixed once and for all by the ultraviolet, that governs processes such as $\pi^0 \to \gamma\gamma$. It is exactly the kind of sharp, rigid number a bootstrap ought to be able to pin down.

The obstacle was that the bootstrap had so far only seen the parity-even sector. Positivity had bounded the Wilson coefficients of the chiral Lagrangian and the couplings of the heavier mesons, and had shown that high-spin states contribute little, the origin of Vector Meson Dominance and of the success of holographic QCD; but all of it was read off from the $\pi\pi \to \pi\pi$ amplitude. That amplitude is parity-even, and so blind to the anomaly: no dispersive bound had reached it, for the simple reason that pure pion scattering does not produce it.

In this work [2] we bring it within reach by dressing the Goldstones with external gauge

[7] A further detail, already observed in [69], is that the spin one is not truly alone in the spectrum, otherwise it would not satisfy the condition $M/s \to 0$, we must add a tower of higher spin particles. In the *vector model* these particles are decoupled at infinity, as explicitly shown by the amplitude written down.

bosons and studying the amplitudes $W\pi \to W\pi$ and $W\pi \to \eta\pi$, the second of which is proportional at low energy to κ. These amplitudes obey their own meson-exchange selection rules, dictating which states appear in which channel, and working them out is what fixes the signs of the on-shell residues, and therefore the notion of positivity.With positivity under control, the dispersion relations and their null constraints deliver the main result: an analytic upper bound on the anomaly coefficient κ in terms of the pion polarizabilities (the lowest Wilson coefficient of the amplitude $W\pi \to W\pi$, physically the constants that govern how the pion responds to an external electromagnetic field [77]). A quantised piece of ultraviolet data is thereby tied directly to ordinary, measurable low-energy quantities, and for models whose ultraviolet completion is unknown, such as in holographic constructions, the bound is a nontrivial consistency check, as it is for the lattice.

The bound is one-sided, and it rests on the assumption that the amplitude is well behaved at high energy; sharpening it into a two-sided constraint, and understanding whether the bootstrap can see the integer quantisation $\kappa = N_c$ directly rather than only bound it, are left for the future. The same amplitudes should also give access to a broader set of electromagnetic low-energy constants of the pion and the other mesons, and the connection they open between the anomalous and parity-even sectors is taken much further in the work that follows.

2.4 Paper 3: Multipositivity Constrains the Chiral Lagrangian

In this paper we attempt an ambitious task: bootstrapping the chiral Lagrangian beyond four-point scattering. The symmetry of the theory ties pion interactions of different multiplicities together, so the higher-point amplitudes are not independent of the four-point one; there is information in them that the standard $2 \to 2$ bounds cannot reach.

Almost the entire positivity programme had been built on two-to-two scattering, and even for the chiral Lagrangian, among the most studied effective theories in physics, the space of consistent couplings was not fully charted. A first step beyond four points had been taken with *multipositivity* [78], bounds relating amplitudes with different numbers of external legs, but its machinery had not yet been brought to bear on the higher-point structure of the chiral Lagrangian itself.

In this work [10] we close that gap. We work with a dispersion relation in a single kinematic channel—which further simplifies the multipositivity logic presented in [78]—and use that the residue at each pole factorises into lower-point amplitudes; choosing complex forward kinematics turns those residues into modulus squares, to which a Cauchy–Schwarz inequality applies directly. The result is a new family of bounds relating planar amplitudes of different multiplicity, which say in effect that the strength with which a state couples into one multiplicity limits how strongly it can couple into another. The argument rests only on factorisation and unitarity, so it holds for any planar tree-level theory.

Imposed on the chiral Lagrangian, these bounds carve out a sizeable region of the parameter space that four-point scattering leaves untouched. The sharpest consequence concerns the anomaly: in the planar limit certain Wilson coefficients are forced to be bounded from below by the chiral anomaly, so that the parity-odd, quantised datum of the Wess–Zumino–Witten term sets the scale of ordinary parity-even couplings. This exposes a connection between

the anomalous and non-anomalous sectors that neither symmetry nor four-point positivity had revealed, and it turns the one-sided bound on the anomaly of the previous work into a statement about the chiral couplings themselves.

Because the construction asks only for planarity and tree-level factorisation, the most natural thing left to do is to turn it on other theories, from gauge theory to gravity, where higher-point multipositivity is largely unexplored. Within the chiral Lagrangian, which amplitudes saturate the new bounds, and how much further the parameter space can be closed by climbing to still higher multiplicity, remain open. An important direction to address is also extrapolating these bounds beyond planarity, and implementing systematically the same program which had success at four-point: null constraints, numerics, etc.

3 Bootstrapping strings: amplitudes and properties

We turn now to the second direction, where the situation is, in a sense, reversed. In the first part the ultraviolet completion was known: large-N_c QCD is strongly coupled, but it is a definite theory, and the lattice can in principle compute anything the dispersion relations constrain. Here it is whatever theory of quantum gravity completes general relativity at the Planck scale, with no Lagrangian to point to and no experiment able to reach the scales at which it would reveal itself. The one datum we do have is universal: gravity couples to everything, with a strength fixed by the *equivalence principle*, so any two particles that exist, whatever else they do, must scatter through graviton exchange. Many properties of the unknown ultraviolet, it turns out, can be reconstructed from this single fact, fed into the dispersive machinery that ran through Part I. The overview that follows assembles the ingredients the bootstrap leans on here, and gives context for the three papers in the section.

3.1 Overview

The bootstrap of amplitudes. The object of study is the scattering amplitude itself, regarded as an analytic function, and the question is which functions are consistent at all. This is in fact the older incarnation of the bootstrap idea: in the 1960s, before QCD, the analytic S-matrix programme sought to determine strong-interaction amplitudes from analyticity, crossing, and unitarity alone, without a Lagrangian. The programme stalled for the strong interactions but its logic never lost its force, and the modern amplitudes programme has revived it with sharper tools. A tree-level four-point amplitude $A(s,t)$ of a healthy theory is heavily constrained before any dynamics are specified: it must be crossing symmetric, exchanging the roles of the Mandelstam invariants s, t, u that satisfy $s+t+u=\sum m_i^2$; it must be meromorphic, with simple poles at the masses of exchanged single-particle states; and it must not grow too fast at large energy, so that the contour deformations behind dispersion relations are legitimate. These requirements (analyticity, crossing, unitarity, and Regge boundedness) carve out a space of admissible amplitudes, and we may ask what lives at its boundary and whether familiar theories are singled out as special points.

Dual amplitudes and the string. The historical seed of the entire subject is the Veneziano amplitude [22], the first *dual* amplitude, built from gamma functions of a linearly rising *Regge trajectory* $\alpha(s)=\alpha_0+\alpha' s$,

$$A(s,t) \;=\; \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}\,. \tag{14}$$

Its defining property, *duality*, is that the single function reproduces the physics of both channels at once: expanded in s it is a sum over an infinite tower of poles (resonances of ever higher spin and mass lying on the trajectory α), and expanded in t it is the same, with no need to add the two contributions separately. The poles sit at $\alpha(s)=0,1,2,\ldots$, and the residue at level n is a polynomial in the scattering angle of degree n, so that each level carries states up to spin equal to its level: a leading Regge trajectory accompanied by its daughters, the hallmark of a

string spectrum. Historically this was the discovery that launched string theory, the amplitude having been written down to embody s–t duality before the worldsheet was understood to be the object producing it. The closed-string analogue, the Virasoro–Shapiro amplitude [79, 80], has the same structure made fully symmetric in the three channels,

$$A(s,t,u) \;=\; \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))\,\Gamma(-\alpha(u))}{\Gamma(1+\alpha(s))\,\Gamma(1+\alpha(t))\,\Gamma(1+\alpha(u))}\,. \tag{15}$$

These amplitudes are rigid objects: an infinite, evenly spaced spectrum, residues positive at every level, and a fixed relationship between mass and maximal spin. In their original string-theoretic guise these amplitudes come with baggage (a tachyon at the bottom of the spectrum and a preferred number of spacetime dimensions in which the worldsheet is consistent), but viewed purely as functions they are characterised by their poles, residues and high-energy growth alone, and it is this function-space characterisation, divorced from the worldsheet, that a bootstrap can hope to reproduce. It is this rigidity that invites the question of whether they can be derived rather than postulated.

Regge behaviour and softness. What distinguishes dual amplitudes dynamically is their behaviour at high energy, and the language for it is Regge theory. At fixed momentum transfer and large s, the sum over the tower resums into a power law set by the leading trajectory,

$$A(s,t) \;\sim\; f(t)\,(-s)^{\alpha(t)}\,, \qquad s\to\infty\,, \tag{16}$$

so that the exponent governing high-energy growth in one channel is controlled by the spectrum of exchanged states in the crossed channel, the deep content of duality. For a linearly rising trajectory the exponent $\alpha(t)$ marches towards $-\infty$ as the momentum transfer $t\to-\infty$, and the amplitude falls off faster than any power: it is *ultrasoft* in the ultraviolet, softer than any local field theory. This softness is the analytic fingerprint of the string. It is also the property most sensitive to deformation, and therefore the natural handle for a uniqueness argument: a small change in the trajectory that preserves the poles and the positivity of residues can nonetheless spoil the soft falloff, and the consistency conditions of this part are largely conditions on this high-energy behaviour.

A landscape of deformations and uniqueness. The string amplitudes are not the only functions with dual resonance structure and Regge behaviour. There is a whole family of deformations that preserve these features while bending the trajectory or shifting the spectrum: the generalised Veneziano amplitudes with "satellite" terms [53], and most famously the Coon amplitude [52], a q-deformation interpolating between the field-theoretic and stringy regimes whose spectrum, instead of rising linearly forever, accumulates towards a finite point. Recently the program of systematically exploring the space of deformations was revived, such as with hypergeometric and bespoke deformation [56, 57] and also with a numerical primal bootstrap program based on a stringy ansatz [55, 81, 82]. These deformations are consistent as far as the crudest requirements go (they have positive residues and can generally be rewritten in dual form), yet they generically differ from the string in their analytic structure at infinity,

the Coon amplitude, for instance, failing to remain soft as the momentum transfer grows and developing pathologies there. The existence of this family sharpens the central question: among all amplitudes with dual resonance and Regge behaviour, which additional consistency requirement, if any, isolates the string? This is the question of *string uniqueness*: whether "string theory" can be made an output of a bootstrap rather than an input. It is also the topic we explored in [5].

Gravity in the infrared. The second strand of this part trades the abstract space of amplitudes for the concrete demands of gravity. That gravity must couple to everything, universally, is not optional in any theory containing a massless spin-two mediator: Lorentz invariance and S-matrix consistency force such a particle to couple to the total energy-momentum of whatever is present, with a single coupling G, which is the field-theoretic content of the equivalence principle [83]. It fixes one term in any four-point amplitude of charged or uncharged matter, the exchange of a graviton in the channel of small momentum transfer, which near the forward direction reads

$$A(s,t) \;\supset\; -\frac{8\pi G\, s^2}{t}\,. \tag{17}$$

The s^2 in the numerator is the imprint of the graviton's spin two, and it is what makes gravitational dispersion relations delicate: an amplitude growing like s^2 sits exactly at the edge of what a twice-subtracted dispersion relation can control, the would-be boundary term threatening to survive. In a complete theory the graviton itself Reggeises (the spin-two pole is the leading state on a gravitational Regge trajectory, the analogue of the *Pomeron*), and the true high-energy growth is tamed below s^2 by the exchange of the whole tower [84, 85]. Four dimensions are especially subtle: the t-channel graviton pole produces genuine infrared divergences in the forward limit, so the analysis must be carried out at small but finite $t < 0$, with the forward limit approached carefully[8]. Unfortunately at finite t the partial waves stop being manifestly positive, although as we will see positivity is not crucial if one asks the right questions.

Dispersion relations with a graviton. The tool that converts this mandatory contribution into information about the spectrum is the same dispersion relation that ran through Part I. Writing a low-energy expansion coefficient $c_n(t)$ as a contour integral of the amplitude around the origin and deforming the contour onto the physical singularities,

$$c_n(t) \;=\; \frac{1}{2\pi i}\oint_{s=0}\frac{ds}{s^{\,n+1}}\,A(s,t) \;=\; \{s\text{-channel}\} + \{u\text{-channel}\} + b_n(t)\,, \tag{18}$$

we equate a quantity read off from the low-energy theory, on the left, to a sum over the discontinuities of the high-energy spectrum plus a boundary term $b_n(t)$ collected from the arc at infinity. The logic that drives this part is then disarmingly simple. If a given coefficient

[8] Note that in this paper [7] we offered a concrete method to go tame the divergence in the forward limit, by constructing *stripped amplitudes* (i.e. amplitudes where the universal soft behaviour is factored out) and showing that they are unitary, analytic and their high energy behaviour is bounded. This allowed us to construct infrared finite bounds in four-dimensions. The paper will not be part of this thesis.

is forced to be non-zero by the low-energy physics, and if the boundary term vanishes, then the right-hand side cannot be empty: *some* state must exist in the s- or u-channel to account for it. Physically, $c_n(t) \neq 0$ with $b_n(t) = 0$ is the statement that an operator blows up faster in the low-energy theory than in the true amplitude, and must therefore be unitarised by states above the cutoff, the familiar situation in which an effective theory is completed in the ultraviolet. Everything hinges on the boundary term, and it vanishes precisely when the ultraviolet completion is weakly coupled and unitarises the amplitude at tree level. This is the one substantive assumption of the gravitational analysis, and it is well motivated: every weakly-coupled string completion is of this kind, and from the bottom up the diffeomorphism invariance of the graviton is a classical notion, so the states that unitarise gravity should enter at tree level. Gravity then supplies the mandatory coefficient for free, from the universal graviton exchange,

$$c_2(t) \;=\; -\frac{8\pi G}{t} \;\neq\; 0\,, \tag{19}$$

so that, with a vanishing boundary term, the mere existence of a pair of scattering states forces further states into being to unitarise their gravitational interaction. A welcome by-product of the tree-level assumption is that the states deduced are genuine single particles, not multi-particle continua, a sharper conclusion than the dispersive logic alone would give.

The swampland and completeness. The states deduced this way acquire their real interest once charge is brought into the picture, and this is where the analysis makes contact with the swampland programme. Its premise is that not every effective field theory that looks consistent at low energies can be completed into a theory of quantum gravity: the ones that can populate the *landscape*, the rest the *swampland* [58, 86]. A web of conjectures attempts to characterise the boundary between them, several concerning the fate of symmetries: that exact global symmetries are forbidden in quantum gravity [59], that gravity is the weakest force (the *weak gravity conjecture* [60]), and, the conjecture this part engages most directly, *completeness*: the claim that every charge permitted by Dirac quantisation is actually realised by some physical state [61]. Polchinski's argument for it [61] runs as follows: if any Dirac-allowed charge were missing from the spectrum, the transformation acting only on that missing charge would commute with every physical state and would therefore generate an exact global symmetry, which the previous conjecture forbids; every allowed charge must be realised. In modern language, this missing-charge transformation is a higher-form global symmetry [87], and a closely related sharpening of the no-global-symmetries argument in quantum gravity is given in [88]. The conclusion is stated as a property of the full theory and is subtle in another way: in its conventional form the required charged states may be multi-particle, which trivialises part of the claim. The standard heuristics for believing it are gravitational: a black hole carrying gauge charge must be able to discharge as it evaporates, which requires charged states of every magnitude to exist, and the weak gravity conjecture sharpens this into a statement about their masses and couplings. The question taken up here is whether completeness can instead be *derived* bottom-up, as a theorem about scattering, and in the stronger single-particle form: given that gravity forces states to exist, can we show that the states forced into being fill out everything a symmetry allows, as genuine particles?

The three directions. Three ingredients form the shared background of the works reproduced in this part: the bootstrap of consistent tree-level amplitudes and the soft, dual structure that marks the string within it; the universal graviton exchange and the dispersion relation that turns it into a demand for states; and the swampland notions of charge and representation completeness, organised by a symmetry group and its charge lattice. The subsections that follow take them up in turn: the question of whether consistency alone singles out the string amplitudes from their deformations; the derivation of charge completeness from the consistency of perturbative gravitational scattering; and the sharper statement of representation completeness, and the democratic way in which the deduced states contribute, that follows from imposing the equivalence principle on the high-energy amplitude. Each is set against what was known before and what the work here adds. Together they pursue a single ambition that runs parallel to Part I: where the first part used dispersion relations to reconstruct a known ultraviolet from its infrared shadow, the second uses the same relations, and the one universal fact about gravity, to constrain an ultraviolet that no experiment can see.

3.2 Paper 4: Strings from Almost Nothing

This paper is based on the idea that string theory might be an output of the bootstrap rather than an input. If consistency alone forces the scattering amplitude into the stringy form, then the spectrum of masses and spins, the soft high-energy behaviour, and even the explicit Veneziano formula are consequences rather than assumptions.

By this point the amplitude bootstrap had grown into a large enterprise, and within it sat a sharp and old question: is the string singled out, or is it merely one consistent amplitude among many? The most serious attempt to answer it [54] established a form of asymptotic uniqueness for tree-level four-point planar scattering, but at a cost: it assumed a specific property of the distribution of the amplitude's zeros, conjectured to be universal. That property was later found to be violated quite generally [55], which left the uniqueness of strings genuinely open.

In this work [5] we settle it from the fewest assumptions we could find. At the heart of the argument is a fact about every tree-level, crossing-symmetric, Lorentz-invariant, unitary amplitude: writing its mass spectrum as $\mu(n)$, its spins as $J(n)$, and its leading Regge trajectory as $\alpha(t)$, the residues at large level are forced to vanish whenever $\alpha(t)$ passes through certain negative integers. These *Regge zeros* are mandated by self-consistency alone, with no assumption about the spectrum, and they hold for planar and nonplanar amplitudes at four points and beyond.

To isolate the string we add only two modest conditions. The first is *ultrasoftness*: the trajectory is bijective and runs off to $-\infty$ at large momentum transfer, so the amplitude falls faster than any power; this is the property that sets the string apart from its neighbours, the Coon q-deformations diverging at finite momentum transfer and ordinary field theory being only boundedly soft. The second is *minimality*: the residues carry no zeros beyond the Regge zeros that consistency already requires. This is the strongest ingredient necessary for strings to emerge as an output, and we find it to be violated in the various unitary deformations of string amplitudes. Bootstrapping the space of amplitudes with these mandated zeros and nothing more, we land precisely on the Veneziano and Virasoro–Shapiro amplitudes, and on nothing

else; the same logic carries over to five points.

Several things are left open. The argument is tree-level, and whether a similar uniqueness survives once loops are included is unknown; the two assumptions, modest as they are, remain assumptions, and it would be more satisfying to derive ultrasoftness and minimality than to impose them. It would be also interesting to understand if there is a systematic way of imposing higher point constraints, in analogy with the techniques we developed in [10].

3.3 Paper 5: Completeness from Gravitational Scattering

This paper is based on the idea of deriving the completeness hypothesis from the bottom up, as a theorem about scattering rather than a conjecture about the full theory. Completeness is the statement that every charge allowed by Dirac quantisation is realised by some physical state; the question is whether the consistency of low-energy scattering alone is enough to force it.

Completeness has long been believed to hold in any consistent theory of quantum gravity, the theories that obey it populating the landscape and those that fail it the swampland. The usual arguments are top-down and heuristic: a black hole carrying gauge charge must be able to shed it as it evaporates, and the absence of exact global symmetries forbids any allowed charge from going unrealised. Persuasive as they are, these arguments treat completeness as a property of the full theory rather than something derivable from low-energy data, and in their conventional form they allow the required charged states to be multi-particle, which drains much of the content from the claim.

In this work [6] we derive it instead from perturbative gravitational scattering, and in the stronger single-particle form. The dispersion relation of Part I already shows that a mandatory, unitarised coupling forces states into being, and gravity supplies exactly such a coupling, since by the equivalence principle any pair of particles scatters through graviton exchange. Adding an exact symmetry group G whose Cartan subgroup generates the charge lattice, the argument becomes an iteration: start from a finite seed of charged states, usually a single particle in the fundamental, scatter them, use the graviton exchange to deduce further charged states, then rescatter those and repeat. If the spectrum so generated fills the entire weight lattice, the theory is *charge complete*.

Whether the iteration closes on the full lattice turns on how much nonabelian structure G carries. A nonabelian group acts on the lattice through its Weyl group, so each newly deduced state drags an entire orbit of charges into the spectrum, and these orbits are what tile the lattice; we turn this into a constructive proof of charge completeness for $G = SO(N)$ with $N \geq 5$ and $SU(N)$ with $N \geq 3$, from which completeness for $Spin(N)$, $Sp(N)$ and E_8 follows. A by-product is worth recording: the $SU(5)$ and $SO(10)$ grand unified theories carry exactly the minimal charged content the argument needs.

Two limitations point the way forward. The argument has no purchase on a purely abelian symmetry, where the deduced charges add along a single direction and never close into anything new, so the abelian case is left untouched. Secondly, we only establish completeness of the charges, not of the representations they sit in; whether gravity forces every representation built

from a seed to appear, and by a cleaner argument than the group-by-group construction used here, is the question taken up next.

3.4 Paper 6: The Equivalence Principle at High Energies Completes the Spectrum

This paper builds on the biggest unanswered question in Paper 5 [6]. We showed that the charge lattice must be complete for some symmetry groups coupled to gravity. This result is already a version of completeness, but it stops short in two ways: it speaks only of charges, not of the representations those charges sit in, and it leans on a constructive, group-by-group analysis.

In this work [9] we prove the stronger statement, *representation completeness*, by a cleaner route: tree-level gravitational scattering mandates single-particle states in every irreducible representation that can be built from a single seed charge, not merely along the Cartan directions. The new ingredient is to track how charge flows through the graviton pole. Because the graviton is neutral, its exchange is proportional to the identity in charge space, and the identity is not inert: it decomposes into a sum of projectors onto the representations that can be exchanged, $\mathbf{1} = \sum_\rho \mathrm{P}_\rho$, one for each representation ρ in the tensor product of the scattering states. The neutral exchange therefore distributes itself across all of these channels at once, and the question sharpens accordingly: not just whether the charge lattice fills, but whether every representation built from the seed is forced to appear, and with what relative strength.

The one assumption that makes the representation-theoretic statement go through is that the *leading* high-energy behaviour is the universal, charge-blind graviton term. This is the equivalence principle promoted from the infrared graviton pole to the ultraviolet Regge regime, the demand that high-energy scattering not care about the charges of what is scattered. Under it, the dispersion relations force single-particle states in every representation reachable from the seed by tensoring, which is the representation completeness we set out to prove. The argument also yields a corollary we call *representation democracy*: the states whose existence we deduce contribute to scattering with squared couplings independent of their representation, once one averages over the quantum numbers other than G. Since these are the asymptotically high-energy excitations, the correspondence principle suggests this even-handedness may be a low-energy shadow of the equipartition of black-hole microstates across charge sectors.

The mechanism behind this conclusion is to consider two dispersion relations in $z = (s-u)/2$ at fixed t: the symmetric one, even in z and carrying the t-channel graviton pole $-8\pi G z^2/t$, and the antisymmetric one, odd in z, found with one fewer subtraction and containing the gauge-boson exchange which grows only linearly. Manipulating the antisymmetric relation lets us trade away the u-channel contribution of the symmetric one, rewriting the latter as a sum over a single channel. Once the residues sit in a single-channel sum, the projector decomposition $\mathbf{1} = \sum_\rho \mathrm{P}_\rho$ forces every representation in the tensor product to enter with a positive contribution, and all the representations the projectors allow must appear in the spectrum.

Interestingly, neither these arguments, nor the ones in Paper 5, rely on positivity of the partial waves. This is the sole reason we are able to deduce such strong conclusions from

gravitational scattering; we can safely perform dispersion relations at finite t, avoiding any problems of infrared divergences and boundary terms. That being said, a few limitations are that these arguments solely apply to tree level amplitudes, as soon as loops are included any notion of completeness is plagued by multi-particle states. Another limit is that we cannot conclude precisely where the higher charge/representation states must enter. A model-independent Tauberian estimate [89] (see [90] for the underlying Tauberian theory) for any twice-subtracted gravitational dispersion relation gives the scale s_* at which the spectrum must saturate the graviton pole. Working at fixed momentum transfer t, the dispersive integral is dominated by $-\frac{1}{2}\alpha' t = O(1)$, therefore the scale until which states contribute is approximately $s_* \sim e^{1/|\alpha(t)|}$ (where in the last step we used $|\alpha(t)| \simeq \alpha'|t|$ at large momentum transfer). The exponential growth in $1/|\alpha(t)|$ means the states we deduce can in principle hide at parametrically large energies, and is the strongest statement available at tree level from any dispersive argument that leans on the universal graviton exchange.

4 Conclusions

The two halves of this thesis began at opposite ends of the energy spectrum and met on common ground. In each, a single instrument was at work: the analytic dispersion relation, which turns the consistency of a theory into sharp constraints on its low-energy data. And a single question was asked: which theories are admissible at all. What changed from one half to the other was only how much of the ultraviolet was known in advance.

In the first half that ultraviolet was QCD, strongly coupled yet in principle understood, and the bootstrap served as a precision instrument. We targeted the chiral Lagrangian in the large-N_c limit, and identified the simple theories sitting at the boundary of the allowed region. We turned amplitude consistency into a bound on the chiral anomaly. Finally, by formulating novel multipositivity techniques, we tied the anomalous and non-anomalous sectors of the theory together. In the second half the ultraviolet was unknown, and the bootstrap became a reconstruction principle. From little more than the universal coupling of the graviton we forced the charge spectrum of a broad class of consistent gravitational theories to fill out completely, and from a handful of consistency demands we singled out the Veneziano and Virasoro–Shapiro amplitudes as the unique solutions. That those same string amplitudes also crowd against the boundary of the gravitational bounds is perhaps the clearest hint that the two halves are studying one structure from two sides.

The broader message is the one promised at the outset. A tool anchored entirely in the infrared, demanding only that a theory be consistent, reaches surprisingly far into the ultraviolet, from the mesons of QCD to the universal features of quantum gravity. Much remains open on both sides, and the subsections below take up the most pressing directions in turn. But the thread that runs from a slab of glass to the swampland shows no sign of fraying.

4.1 Part I: Bootstrapping large-N_c QCD

Applying bootstrap techniques for the chiral Lagrangian and large-N_c QCD teached two important lessons. The first is that, in a strongly-coupled theory whose ultraviolet completion is in principle known but in practice hidden from perturbation theory, dispersive consistency is precise enough to act as a valide alternative: it carves out the parameter space exactly, identifies the simple amplitudes that populate the boundary, and explains, from first principles, why the lightest resonances saturate so much of the low-energy data. Vector Meson Dominance, long an empirical fact in search of a derivation, is recovered as a corollary of crossing and unitarity in the planar limit. The second is that the anomalous and non-anomalous sectors of the chiral Lagrangian, which conventional chiral perturbation theory treats as separate parts of the same derivative expansion, are in fact tied together: multipositivity forces certain parity-even Wilson coefficients to be at least as large as the quantised parity-odd datum carried by the Wess–Zumino–Witten term. The strong interactions are more rigidly organised than the textbook treatment suggests, and the rigidity comes from a place the bootstrap is uniquely able to see, namely higher-point consistency.

Methodologically, the work also makes a case for the value of analytic control. The spurious kink that the numerical bootstrap had presented as the most exciting feature of the chiral

allowed region turned out, once the right one-dimensional moment problem was identified, to be no feature at all but the single-vector theory in disguise. The numerics had not been wrong, only insufficient: it took the analytic recasting to see what the boundary actually contained, and to confirm that no exotic theory lives there. The same lesson applies, in less dramatic form, to the rest of the boundary, where the bounds we trust most are the ones we can write in closed form and check against explicit ultraviolet completions. The chiral bootstrap is at its best when numerical and analytic tools are brought to bear together, the numerics charting the territory and the analysis settling what lives in it.

Many things remain open. The exact shape of the allowed region still has an important boundary to be mapped out: the curve of higher-spin completions interpolating between the single-vector theory and the *su*-model resists analytic control and whether any physical theory traces it out is a pressing question to answer. The entire programme has been conducted in the strict planar limit, and how the bounds deform under $1/N_c$ corrections, and whether the picture survives at $N_c = 3$ with realistic quark masses, are problems in their own right. The external-current bootstrap of Paper 2 should also extend beyond the electroweak gauge bosons to a wider set of low-energy electromagnetic and gravitational constants of the pion, and the spin-sensitive dispersion relations of [8] should sharpen all of these bounds. As lattice determinations of the higher-derivative L_i improve, the bootstrap and the lattice should begin to meet quantitatively where they have so far only complemented each other in principle.

4.2 Part II: Bootstrapping strings: amplitudes and properties

The second part inverted the setting and asked what consistency alone can be made to say about an ultraviolet that no experiment will reach. The answers are stronger than one would naturally expect. From a handful of structural demands, the Veneziano and Virasoro–Shapiro amplitudes emerge as the unique solutions, with their spectra, their Regge behaviour, and even their closed-form expressions delivered as outputs rather than inputs. From the single universal fact that gravity couples to everything, the same dispersive machinery turns the existence of one charged particle into the necessity of infinitely many, filling out the entire charge lattice and, in the sharper representation form, every irreducible representation that the seed can be tensored into. The completeness conjecture, in its single-particle version, ceases to be a top-down statement and becomes a theorem about scattering. The fact that string amplitudes also crowd against the boundary of the gravitational bounds is the clearest sign that the structure being reconstructed in the two halves of this thesis are not tailored to a particular setup, but generic features.

What makes the gravitational arguments hold their force is how little they ask of the ultraviolet. Beyond the existence of a massless spin-two state and a weakly-coupled tree-level completion, they require essentially nothing: no Lagrangian, no symmetry-breaking pattern, no input from any specific construction of quantum gravity. The conclusions that follow, single-particle charge completeness, representation completeness, the uniqueness of the Veneziano and Virasoro–Shapiro amplitudes, all rest on S-matrix consistency alone. That this is enough to recover statements normally derived from string theory or motivated from black-hole heuristics is a sharp argument for the bottom-up programme as a substitute for experimental input in a

regime where no experiment will ever reach. What is genuinely new in our work is that the infrared alone is enough to mandate properties of single-particle states in the ultraviolet.

The most consequential open questions concern the boundaries of this picture. The arguments are tree-level, and at loop order the multi-particle continuum reopens the weaker form of completeness; the cleanly single-particle conclusion will need to be sharpened against that. The dispersion relations mandate that states exist but do not say at what scale they enter, and pinning the parametric mass of the lightest deduced state would tighten any of these results dramatically. A closely related important program is the extension of these arguments to higher multiplicity, where the same factorisation logic derived in [10], should constrain higher-point gravitational amplitudes– which remarks the importance of moving past planarity. More broadly, sharper statements about the boundary of the swampland, free of the weakly-coupled tree-level assumption, would bring the bottom-up programme into full contact with the top-down web of conjectures it has so far only begun to touch.

4.3 Additional research lines

Alongside the six papers reproduced above, four further works completed during this doctorate were not included in the compendium but extend the same programme along closely related lines, and I record them briefly here.

Bootstrapping the chiral-gravitational anomaly. In [3] we carry the chiral-anomaly bootstrap of Paper 2 beyond external gauge bosons to external gravitons. The chiral-gravitational anomaly, a parity-odd quantised coefficient generated when the chiral current is coupled to a gravitational background, turns out to be reachable by exactly the same dispersive logic once the pion is dressed with stress-energy insertions, and the analysis yields an analytic upper bound on its coefficient set by low-energy data, the gravitational counterpart of the bound on κ obtained from electroweak currents. Two quantised pieces of ultraviolet information, one electroweak and one gravitational, are now tied to the same low-energy constants of the chiral Lagrangian, and the route from one to the other is the same in both cases: external currents brought in, selection rules fixing the signs of residues, and the dispersive sum rules turned into an inequality.

Positivity with long-range interactions. Positivity bounds for gravitational effective field theories have been developed in a substantial line of works [41, 45–47, 91–95], turning causality and unitarity into two-sided constraints on the higher-curvature Wilson coefficients of any EFT coupled to gravity. A long-standing obstacle to these bounds in four dimensions has been the infrared divergence that universal long-range exchanges, gravitons and photons, produce in the forward limit. In [7] we bypass it by constructing *stripped amplitudes*, in which the universal soft behaviour is factored out, and showing that what remains is unitary, analytic, and grows slowly enough to dispersion-relate. The result is a set of positivity bounds that are finite in $D = 4$, removing the main technical obstruction that had restricted the gravitational bootstrap.

(Super)gravity from positivity. In [4] we ask whether an isolated, massive, weakly-coupled spin-3/2 particle admits a consistent S-matrix at all, and find that it does not, unless gravitons are added to the spectrum with a coupling that is, almost supersymmetrically, fixed by gravity itself. Once two such particles form a $U(1)$-charged state, a graviphoton gauging the symmetry is required as well, with couplings characteristic of supergravity and saturating the weak gravity conjecture. Gravity, and very nearly supergravity, are forced on us by S-matrix consistency for a single massive higher-spin excitation, a sharp instance of the broader pattern of this thesis where a minimal infrared datum suffices to demand a great deal of ultraviolet structure. We also study positivity bounds for spin-1/2 particles, the *Goldstinos*, again finding that supersymmetric UV completions saturate the bounds. The work sits within a broader programme of positivity bounds on isolated massive higher-spin states in particular with applications to massive gravity showing its incompatibility with positivity [91].

On the inverse problem in effective field theory. Most bootstrap applications bound the Wilson coefficients of an effective theory; the inverse problem, recovering the masses, couplings, and spins of the heavy states from those coefficients, had been only partially solved, with masses and couplings extractable from a moment problem but the spins not. In [8] we construct a new family of *logarithmic* dispersion relations whose sum rules are spin-sensitive in a controlled way, and show that the spins of the exchanged states can be read off from sufficiently many low-energy data points. The dispersive bootstrap thus comes closer to a genuine ultraviolet reconstruction: given enough infrared coefficients, the full quantum numbers of the heavy spectrum, not only its gross moments, can in principle be recovered.

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5 Papers

The six papers are reproduced here in full, in the order in which they are discussed above.

Paper 1: Cornering Large-N_c QCD with Positivity Bounds

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Cornering Large-N_c QCD with Positivity Bounds

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Abstract

The simple analytic structure of meson scattering amplitudes in the large-N_c limit, combined with positivity of the spectral density, provides precise predictions on low-energy observables. Building upon previous studies, we explore the allowed regions of chiral Lagrangian parameters and meson couplings to pions. We reveal a structure of kinks at all orders in the chiral expansion and develop analytical tools to show that kinks always correspond to amplitudes with a single light pole. We build (scalar- and vector-less) deformations of the Lovelace-Shapiro and Coon UV-complete amplitudes, and show that they lie close to the boundaries. Moreover, constraints from crossing-symmetry imply that meson couplings to pions become smaller as their spin increases, providing an explanation for the success of Vector Meson Dominance and holographic QCD. We study how these conclusions depend on assumptions about the high-energy behavior of amplitudes. Finally, we emphasize the complementarity between our results and Lattice computations in the exploration of large-N_c QCD.

1 Introduction

The understanding of theories at strong coupling is one of the most important challenges of modern particle physics. Besides the pragmatic interest for QCD hadronic physics, such understanding would broaden our perspective on theories beyond the Standard Model and plausibly provide intuition on its shortcomings.

Dispersion relations distill the essential ingredients of quantum field theory, unitarity and causality, into consistency conditions for scattering amplitudes. They have been used as *positivity bounds* to shape the parameter space of effective field theories originating from healthy, albeit strongly coupled, microscopic dynamics, see e.g. [1–7], and have had important applications to QCD and the chiral Lagrangian [8–16].

At the same time, in the context of $SU(N_c)$ gauge theories, the limit of many colors $N_c \to \infty$ has provided one of the most insightful approaches for understanding the strongly-coupled regime [17, 18], even though real-world QCD has only $N_c = 3$. The main consequence of this approximation is that the theory has a dual description in terms of weakly-coupled mesons, rather than quarks and gluons. In spite of this important step forward, the predictions of large-N_c QCD have been limited by the fact that the theory contains an infinite number of mesons of any spin, whose Lagrangian is unknown or contains an infinite number of terms.

Recently, Ref. [19] has combined these two approaches and derived important constraints on the low energy $\pi\pi \to \pi\pi$ scattering amplitude. Despite the higher-spin meson spectrum remains unknown, the simple analytic structure of large-N_c amplitudes – combined with certain assumptions on their high-energy behavior – improves the predictiveness of dispersion relations.

In this article we push forward this approach, combining analytic and numerical methods. One of the most important questions raised in Ref. [19] concerns the understanding of which theories define the kinks (and the bulk) of the allowed regions of Wilson coefficients. In this article we identify these theories. We build UV-complete four-pions amplitudes, as variations of the Lovelace-Shapiro amplitude [20, 21] and the Coon amplitude [22] in which the spin-0 (and spin-1) poles have been removed, and show that they reside close to the boundary. At the kinks, on the other hand, we find theories with an infinitely degenerate higher-spin spectrum, or theories with a unique state at finite mass, either of spin $J = 0$, $J = 1$ or $J = 2$. While some of these results are known [4, 6, 19], in this work we are able to exclude other possibilities. In particular, while the numerical approach suggests the existence of a new kink with a more exotic spectrum, we show that the kink position can be reformulated as a 1D moment-problem whose solution converges to the $J = 1$ theory.

An important emerging property of low-energy QCD is Vector Meson Dominance (VMD) [23], the hypothesis that the spin-1 ρ meson gives the main contribution to the chiral Lagrangian [24]. This property has an empirical origin (it works!), but it lacks a theoretical explanation. We will show that VMD finds its origin in the positivity bounds. Indeed, the chiral Lagrangian coefficients can be written as a sum over positive quantities that depend on the couplings and masses of the different mesons: when the theory contains a ρ, this dominates over the other $J > 1$ states. This is especially prominent when the higher-spin states are heavier than the ρ, as it occurs in real-world QCD. Our analysis also provides insight into why holographic models, which contain only (charged) spin-0 and spin-1 states, have been so successful in predicting

low-energy properties of QCD, see e.g. [25, 26]. Finally, we will show that this reasoning also extends to the heavy-meson couplings to pions: the higher the spin, the smaller the coupling.

These arguments, as well as all those from Ref. [19], rely on assuming that the high-energy amplitudes are particularly well behaved, $\mathcal{M}/s \to 0$ at large $|s|$. In this article we will also discuss the implications of relaxing this assumption, such that the $\pi\pi \to \pi\pi$ amplitude is limited only by the Froissart-Martin bound $\mathcal{M}/s^2 \to 0$ at large energies [27, 28]. In this case, the spin-1 and spin-0 contributions decouple from (and can contribute more than) the $J \geq 2$ ones, that are now dominated by the spin-2 state instead, in a generalisation of VMD.

The paper is organised as follows. In Sec. 2 we review the analytical structure of the $\pi\pi \to \pi\pi$ amplitude and the dispersion relations that lead to positivity constraints. We also present possible UV completions to the chiral Lagrangian, as they will play an important role to understand the boundaries of the Wilson-coefficient allowed regions. In Sec. 3 we consider the case in which the four-pion amplitude at large $|s|$ satisfies $\mathcal{M}/s \to 0$. We determine the allowed regions of the leading Wilson coefficients and show which theories reside at the kinks of these boundaries. We also study the emergence of VMD and derive bounds on the couplings of meson resonances to pions. In Sec. 4 we extend the analysis to the case in which the four-pion amplitude only satisfies the Froissart-Martin bound at high energies. We present several appendices with extended discussions on the numerical bootstrap (Appendix A), on the analytical determination of the kinks (Appendix B), the *su*-models (Appendix C), and the Lovelace-Shapiro and Coon amplitudes (Appendix D and Appendix E respectively).

2 The $\pi\pi \to \pi\pi$ Amplitude in large-N_c QCD

This section contains mostly a review of previous literature on the $2 \to 2$ pion amplitude, in particular the results of Ref. [19]. We will work in the massless quark limit.

Pions are the massless Goldstone bosons associated to the spontaneous breaking of the global $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)$, where N_f is the number of quark flavors in QCD. They transform in the *Adj.* representation of $SU(N_f)$, which for the case $N_f = 2$ corresponds to the Isospin $I = 1$, the triplet $\pi^\pm$ and π^0. This allows us to write the $2 \to 2$ pion amplitude as

$$\mathcal{M}(\pi^a\pi^b \to \pi^c\pi^d) = A(s|t,u)\left[\frac{2}{N_f}\delta_s + d_s\right] + A(t|u,s)\left[\frac{2}{N_f}\delta_t + d_t\right] + A(u|s,t)\left[\frac{2}{N_f}\delta_u + d_u\right], \tag{1}$$

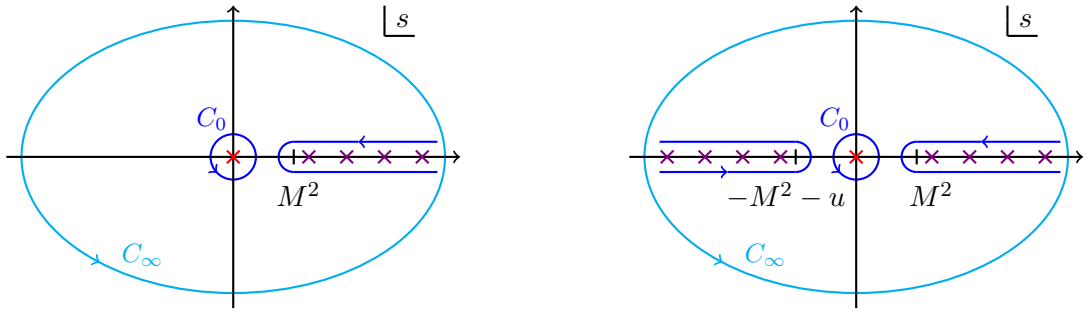
where $s = (p_a + p_b)^2$, $t = (p_a - p_c)^2$, $u = (p_a - p_d)^2$, and

$$\delta_s = \delta_{ab}\delta_{cd}\ ,\ \delta_t = \delta_s(b \leftrightarrow c)\ ,\ \delta_u = \delta_s(b \leftrightarrow d)\,, \tag{2}$$

$$d_s = d_{abe}d_{cde}\ ,\ d_t = d_s(b \leftrightarrow c)\ ,\ d_u = d_s(b \leftrightarrow d)\,, \tag{3}$$

correspond to the various ways of contracting $SU(N_f)$ adjoint indices into singlets. $A(s|t,u)$ is a function of t, u symmetric under their interchange, i.e. $A(s|t,u) = A(s|u,t)$.

In the large-N_c limit, QCD reduces to a theory of weakly coupled mesons, whose trilinear couplings scale as $\sim 1/\sqrt{N_c}$ [17, 18]. In this limit, the $2 \to 2$ pion amplitude is then dominated



(a) $\mathcal{M}(s,u)$ (b) $\mathcal{M}(s,t=-u-s)$

Figure 1: *Analytic structure of* $\mathcal{M}(s,u)$ *and* $\mathcal{M}(s,t=-s-u)$ *for fixed* $u<0$. *We denote by* C_0, C_∞ *(to be taken at* $|s|\to\infty$*) and the discontinuity along the real axis the relevant contours of integration used for dispersion relations.*

by a tree-level meson exchange. Since these mesons are $q\bar{q}$ states with isospin $I=0,1$, the isospin $I=2$ amplitude,

$$\mathcal{M}_s^{I=2}(\pi^a\pi^b\to\pi^c\pi^d)=A(t|u,s)+A(u|s,t)\equiv\mathcal{M}(t,u)\,, \tag{4}$$

(symmetric under $t\leftrightarrow u$) has no poles in the large-N_c limit. All this leads to the following important implications for the analytical structure of $\mathcal{M}$ in the large-N_c limit:

- The only singularities of $\mathcal{M}$ in the complex s-plane are simple poles associated with the tree-level meson exchange (the branch cut along the physical region is at least $O(1/N_c^2)$).
- The absence of $I=2$ meson exchange in the s-channel implies that $\mathcal{M}_s^{I=2}=\mathcal{M}(t,u)$ has no poles for real $s>0$. Since $t=-s-u$, this implies that for fixed $u<0$, there cannot be poles in $\mathcal{M}(t,u)$ on the negative real t axis. Now, by a simple exchange of arguments $(t\to s)$, we come to the conclusion that $\mathcal{M}(s,u)$, for fixed $u<0$, can only have poles on the positive real s axis.
- For fixed $u<0$, $\mathcal{M}(s,t)$ can have poles either on the real positive or negative s axis.

This analytic structure of $\mathcal{M}$ is illustrated in Fig. 1.

At energies below the mass M of the lightest massive meson (corresponding to the position of the first pole in Fig. 1), pions can be well described by an effective theory, corresponding to an expansion in $s/M, u/M\to 0$,

$$\begin{aligned}\mathcal{M}(s,u) &= \sum_{n=1}^{\infty}\sum_{l=0}^{[n/2]} g_{n,l}\, s^{\{n-l}u^{l\}} \\ &= g_{1,0}\,(s+u)+g_{2,0}\,(s^2+u^2)+g_{2,1}su+g_{3,0}\,(s^3+u^3) \\ &+ g_{3,1}\,(s^2u+su^2)+g_{4,0}\,(s^4+u^4)+g_{4,1}\,(s^3u+su^3)+g_{4,2}\,s^2u^2+\dots\,,\end{aligned} \tag{5}$$

where for convenience we have defined the weighed symmetric tensor $s^{\{i}u^{j\}}\equiv s^iu^j+(1-\delta_{ij})s^ju^i$ to avoid double counting of terms with $n=2l$, such as $g_{2,1}$, $g_{4,2}$, etc. A constant term is absent

in Eq. (5) as the amplitude must go to zero for $s, u \to 0$ in order to restore the Adler condition for pions. In the large-N_c limit all Wilson coefficients scale as $g_{n,l} \sim 1/N_c$. For the connection of Eq. (5) with the QCD chiral Lagrangian, see Sec. 3.2.1.

2.1 Dispersion Relations and Sum Rules

Dispersion relations can be derived assuming that the amplitude $\mathcal{M}$ satisfies the following high-energy conditions, for fixed $u < 0$ and for all $k \geq k_{\min}$,

$$\lim_{|s|\to\infty} \frac{\mathcal{M}(s,u)}{s^k} \to 0\,, \qquad \text{(6a)} \qquad\qquad \lim_{|s|\to\infty} \frac{\mathcal{M}(s,-u-s)}{s^k} \to 0\,. \qquad \text{(6b)}$$

Different assumptions about the high-energy behavior of amplitudes are associated with different values of $k_{\min}$. On general grounds, the Froissart-Martin bound [27–29] ensures that in theories with a mass gap Eqs. (6) are satisfied for $k_{\min} = 2$, with similar results for the massless case [7, 30, 31].

On the other hand, Ref. [19] advocated – invoking considerations on the Pomeron Regge trajectory – that the four-pion amplitude might be bounded by $\mathcal{M} \lesssim s$ and therefore $k_{\min} = 1$. In this article we will study both, the case with $k_{\min} = 1$ and $k_{\min} = 2$.

2.1.1 IR-UV Relations

Taking $u < 0$ fixed, we have that the integral of $\mathcal{M}(s,u)/s^{k+1}$ along the contour C_∞ of Fig. 1a vanishes for $k \geq k_{\min}$, due to Eq. (6a). Because of amplitude's analyticity, we can deform C_∞ into the blue contour in Fig. 1a,

$$\oint_{C_0} ds' \, \frac{\mathcal{M}(s',u)}{s'^{k+1}} = 2i \int_{M^2}^{\infty} ds' \, \frac{\text{Im}\mathcal{M}(s',u)}{s'^{k+1}}\,. \tag{7}$$

The amplitude can be expanded in partial waves,

$$\text{Im}\mathcal{M}(s,u) = \sum_J (2J+1)\rho_J(s) P_J\left(1 + \frac{2u}{s}\right), \tag{8}$$

where P_J are the Legendre polynomials and $\rho_J(s)$, the spectral density, must be positive, $\rho_J(s) \geq 0$, due to unitarity of the S-Matrix. Indeed, for large-N_c theories, we have that the spectral density is given by

$$(2J+1)\rho_J(m^2) = \pi \sum_i g_{i\pi\pi}^2 m_i^2 \, \delta(m^2 - m_i^2)\delta_{JJ_i}\,, \tag{9}$$

where i labels mesons of mass m_i, spin J_i and coupling to pions $g_{i\pi\pi}$.

Plugging Eq. (8) into Eq. (7), performing the contour integrals, and expanding around small $u < 0$ we find

$$\begin{aligned}
k=1: \qquad g_{1,0}+g_{2,1}u+g_{3,1}u^2+... &= \left\langle \frac{P_J(1)}{m^2}+2\frac{P_J'(1)}{m^4}u+2\frac{P_J''(1)}{m^6}u^2+... \right\rangle , \\
k=2: \qquad g_{2,0}+g_{3,1}u+g_{4,2}u^2+... &= \left\langle \frac{P_J(1)}{m^4}+2\frac{P_J'(1)}{m^6}u+2\frac{P_J''(1)}{m^8}u^2+... \right\rangle , \\
k=3: \qquad g_{3,0}+g_{4,1}u+g_{5,2}u^2+... &= \left\langle \frac{P_J(1)}{m^6}+2\frac{P_J'(1)}{m^8}u+2\frac{P_J''(1)}{m^{10}}u^2+... \right\rangle , \\
&\vdots
\end{aligned} \tag{10}$$

with the definition of the high-energy average [6],

$$\langle(...)\rangle \equiv \frac{1}{\pi}\sum_J (2J+1)\int_{M^2}^{\infty}\frac{dm^2}{m^2}\rho_J(m^2)(...)\,. \tag{11}$$

Considering equations with $k \geq k_{\min}$ (that we will take later to be $k_{\min}=1,2$), we can relate the IR Wilson coefficients with the UV-averages of derivatives of P_J in the following way:

$$g_{n+l,l} = \frac{2^l}{l!}\left\langle \frac{P_J^{(l)}(1)}{m^{2(n+l)}} \right\rangle , \quad n \geq k_{\min} \text{ and } \; l=0,1,...,\left[\frac{n-1}{2}\right] . \tag{12}$$

Since $P_J^{(l)}(1) \geq 0$, the contributions to Eq. (12) from the different J-states are always additive, and therefore $g_{n+l,l} \geq 0$ – this is a direct consequence of the lack of $s<0$ poles in $\mathcal{M}(s,u)$. Moreover, $P_J^{(l)}(1)=0$ for $l>J$ implying that states with $J \leq l$ do not contribute to $g_{n+l,l}$.

In particular,

$$\begin{aligned}
g_{n,0} &= \left\langle \frac{1}{m^{2n}} \right\rangle = \sum_i \frac{g_{i\pi\pi}^2}{m_i^{2n}} , \\
g_{n+1,1} &= \left\langle \frac{\mathcal{J}^2}{m^{2(n+1)}} \right\rangle = \sum_i \frac{g_{i\pi\pi}^2\, J_i(J_i+1)}{m_i^{2(n+1)}} , \\
g_{n+2,2} &= \frac{1}{4}\left\langle \frac{\mathcal{J}^4-2\mathcal{J}^2}{m^{2(n+1)}} \right\rangle = \sum_i \frac{g_{i\pi\pi}^2\, J_i(J_i-1)(J_i+1)(J_i+2)}{4m_i^{2(n+1)}} ,
\end{aligned} \tag{13}$$

where $\mathcal{J}^2 \equiv J(J+1)$. Notice that only for $k_{\min}=1$ all Wilson coefficients have a dispersive representation in terms of Eq. (12). For $k_{\min}=2$, the couplings $g_{1,0}$ and $g_{2,1}$ are not captured by these dispersion relations.

In a similar way, we can obtain dispersion relations for the $\mathcal{M}(s,t)$ amplitude, whose analytic structure is given in Fig. 1b,

$$\frac{1}{2i}\oint_{C_0} ds' \, \frac{\mathcal{M}(s',-u-s')}{s'^{k+1}} = \int_{M^2}^{\infty} ds' \, \frac{\text{Im}\mathcal{M}(s',-u-s')}{s'^{k+1}} + (-1)^k \int_{M^2}^{\infty} ds' \, \frac{\text{Im}\mathcal{M}(s',-u-s')}{(s'+u)^{k+1}} . \tag{14}$$

These can be expanded as in the previous section, and provide yet more relations [19]. In the case $k_{\rm min} > 1$ these new relations are crucial, as they give access to Wilson coefficients that do not have a dispersive representation in terms of $\mathcal{M}(s,u)$. In particular, for $k_{\rm min} = 2$, the coupling $g_{2,1}$ is not determined by Eq. (12) but appears in Eq. (14),

$$g_{2,1} = 2g_{2,0} - 2\left\langle \frac{(-1)^J}{m^4} \right\rangle , \tag{15}$$

while for $k_{\rm min} = 3$, $g_{3,1}$ can only be determined by

$$g_{3,1} = 3g_{3,0} + \left\langle \frac{(-1)^J(2\mathcal{J}^2-3)}{m^6} \right\rangle . \tag{16}$$

2.2 Null Constraints

The dispersion relations in Eq. (10), and the small-u expansion of Eq. (14), over-determine the Wilson coefficients. This leads to a set of *null constraints*,

$$\left\langle \mathcal{X}_{n,k}(J,m^2) \right\rangle = 0\,, \quad \left\langle \mathcal{Y}_{n,k}(J,m^2) \right\rangle = 0\,, \tag{17}$$

on the high-energy spectral density, with $m^{2n}\mathcal{X}_{n,k}$ and $m^{2n}\mathcal{Y}_{n,k}$ functions of J only. Their compact expression at all orders is provided in Ref. [19]. For the analytic arguments in this article we are only interested in the most relevant null constraints (those involving less powers of $1/m$) and in those with the leading asymptotic $J \to \infty$ behavior at a fixed order n in $1/m^{2n}$.

For $k_{\rm min} = 1$, there is one (and only one) null constraint $\sim O(J^{2(n-1)}/m^{2n})$ at each order n,[1]

$$\begin{aligned}
n=2: \qquad m^4\,\mathcal{Y}_{2,1} &= -2(1-(-1)^J) + \mathcal{J}^2\,,\\
n=3: \qquad m^6\,\mathcal{X}_{3,1} &= -6\mathcal{J}^2 + \mathcal{J}^4\,,\\
n=4: \qquad m^8\,\mathcal{X}_{4,1} &= -24\mathcal{J}^2 - 8\mathcal{J}^4 + \mathcal{J}^6\,,\\
&\vdots\\
(n-1)!^2\; m^{2n}\,\mathcal{X}_{n,1} &= \frac{2^{n-1}}{(n-1)!}P_J^{(n-1)}(1) - \mathcal{J}^2\,.
\end{aligned} \tag{18}$$

The other null constraints have only subleading terms in powers of J w.r.t. these.

When we study larger $k_{\rm min}$, the constraints in Eq. (18) disappear, and subleading null constraints now dominate. For $k_{\rm min} = 2$ this involves null constraints that grow as $O(J^{2(n-2)}/m^{2n})$. There are two of them at each order n, and can be separated into those where the sign of the term $O(J^{2(n-2)}/m^{2n})$ is fixed, and those where this sign oscillates between J-odd and J-even.

[1] We use a slightly different normalization w.r.t. Ref. [19], which has no impact on Eq. (17).

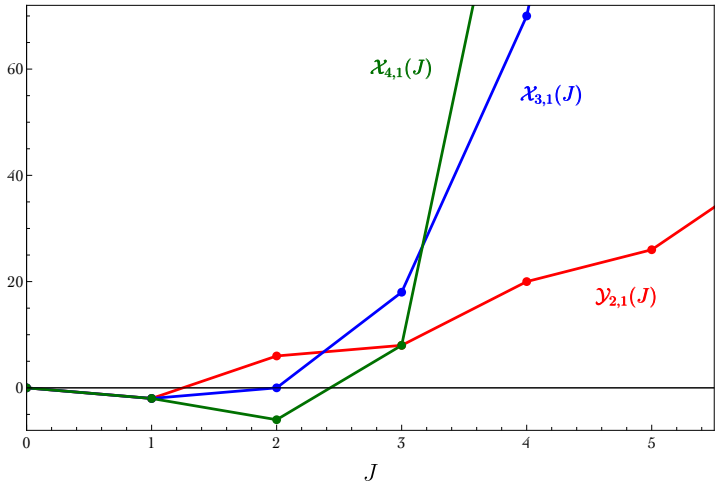


Figure 2: *Null constraints from Eq. (18), as a function of J, for fixed m.*

In the first class we have,

$$\begin{aligned}
n=4: \qquad m^8\big(\mathcal{Y}_{4,2}-\mathcal{Y}_{4,1}\big) &= 8(1-(-1)^J)-10\mathcal{J}^2+\mathcal{J}^4\,,\\
n=5: \qquad m^{10}\,\mathcal{X}_{5,2} &= 30\mathcal{J}^2-17\mathcal{J}^4+\mathcal{J}^6\,,\\
n=6: \qquad m^{12}\,\mathcal{X}_{6,2} &= 144\mathcal{J}^2-46\mathcal{J}^4-20\mathcal{J}^6+\mathcal{J}^8\,,\\
&\vdots\\
(n-2)!^2\;m^{2n}\,\mathcal{X}_{n,2} &= \frac{2^{n-2}}{(n-2)!}P_J^{(n-2)}(1)-2P_J^{(2)}(1)\,.
\end{aligned} \tag{19}$$

In both cases, Eq. (18) and Eq. (19), the $\mathcal{Y}_{n,k}$ null constraints (originating from $\mathcal{M}(s,t)$ dispersion relations) appear only at the lowest order in $1/m^2$, and at higher order the dominant J behavior is controlled by the $\mathcal{X}_{n,k}$ null constraints (originating from $\mathcal{M}(s,u)$ dispersion relations). On the other hand, the most relevant oscillating null constraint is,

$$n=3: \qquad m^6\;\mathcal{Y}_{3,1} \;=\; -6(1-(-1)^J)+2(1-2(-1)^J)\mathcal{J}^2\,, \tag{20}$$

where the sign of the $\mathcal{J}^2$ term oscillates with J.

Notice that for $J=0$ the arguments of all null constraints vanish, as can be easily seen in Eq. (18) and Eq. (19) (and more generally by the expressions in Ref. [19]). This implies that the spin-0 component of the UV spectrum decouples and is not restricted by null constraints. This is related to the fact that models with only $J=0$ states can provide a consistent UV completion of the pion amplitude, satisfying Eq. (6a) and Eq. (6b) for $k_{\rm min}=1$, as we will discuss in section 2.3. For $k_{\rm min}=2$, also $J=1$ states give zero contributions and decouple from the null constraints. This again can be understood from the fact that models with only $J=1$ states provide a pion amplitude that consistently satisfies the Froissart-Martin bound. This pattern persists: for $k_{\rm min}=3$, one finds that the $J=0,1,2$ states decouple from the null constraints, and so on.

It is also instructive to understand how the null constraints can be satisfied, as this tells us information about the mass spectrum of the theories. In Fig. 2 we show the first few null constraints in Eq. (18) as a function of J. We see that for the first two null constraints the contribution from the $J = 1$ is the only opposite in sign to the other ones. As these expressions must average to zero, Eq. (11), this implies that the theory must contain $J = 1$ states. For $\mathcal{X}_{4,1}$, also the $J = 2$ contribution is negative, implying that also $J = 2$ states are needed. As the order n of the null constraint increases, one finds that the number of states with negative coefficients increases: eventually all J are needed to satisfy the null constraints. So, for $k_{\rm min} = 1$, theories in the large-N_c limit either have no $J > 0$ state, or have states with all values of J from 1 to ∞. Similarly, for $k_{\rm min} = 2$, states with spin $J \geq 2$ are either absent, or are all present.

2.3 UV completions of the pion amplitude

Before proceeding to examine the implications from positivity, we would like to discuss simple UV completions for a theory of pions. By this we mean theories that generate consistent crossing-symmetric amplitudes $\mathcal{M}(s, u)$, with simple poles at real $s > 0$, positive spectral density, satisfying the high-energy behavior of Eqs. (6) for some $k_{\rm min}$. Interestingly, these amplitudes will turn out to reside at the kinks of the allowed parameter space, as we will discuss in the next section.

Simple amplitudes describing exchange of a single spin-J particle with mass m_J are characterised by a pole-structure with residue on the associated partial wave,

$$\mathcal{M}^J(s, u) = \frac{m_J^2 P_J(1 + 2u/m_J^2)}{m_J^2 - s} + F(s, u) + (s \leftrightarrow u)\,, \tag{21}$$

where F is an analytic function that defines the theory, but does not contribute to the residue. We will fix F by imposing the Adler's zero $\mathcal{M}^J(0, 0) = 0$ and by requiring that the amplitude satisfies the large-s behavior of Eqs. (6) for the lowest possible value of $k_{\rm min}$.

2.3.1 Theory of scalars

Scalars can provide a consistent UV completion to a theory of pions, via the Higgs mechanism in the linear sigma-model. The $2 \to 2$ pion amplitude mediated by a spin-0 state with $I = 0$ is given by,

$$\mathcal{M}^{(s)}(s, u) = \frac{g_{s\pi\pi}^2 s}{m_s^2 - s} + (s \leftrightarrow u)\,. \tag{22}$$

Expanding Eq. (22) at low energies, $s, u \ll m_s^2$, we obtain,

$$g_{n,0} = \frac{g_{s\pi\pi}^2}{m_s^{2n}} \quad (n \geq 1)\,, \qquad g_{n,l} = 0 \quad (l \neq 0)\,. \tag{23}$$

An important property of Eq. (22) is that it satisfies the high-energy behavior in Eqs. (6a,6b) with $k_{\rm min} = 1$. Therefore the Wilson coefficients Eq. (23) obey the sum rules Eq. (13), as can be easily checked. The fact that a model of only scalars does not need higher-spin states to satisfy Eqs. (6a,6b) with $k_{\rm min} = 1$, explains why the $J = 0$ states decouple from the null constraints, as explained above.

2.3.2 Theory of vectors

Let us now consider a (weakly coupled) spin-1 resonance with isospin $I = 1$ (or, in general, in the *Adj.* representation of $SU(N_f)$), which we will refer to as ρ, in analogy with QCD. From Eq. (21) we have,

$$\mathcal{M}^{(\rho)}(s,u) = \frac{g^2_{\rho\pi\pi} m^2_\rho}{m^2_\rho - s} P_1\left(1 + \frac{2u}{m^2_\rho}\right) + (s \leftrightarrow u)\,, \tag{24}$$

corresponding to the contribution from the transverse components of a massive vector coupling to pions via $g_{\rho\pi\pi} f_{abc} \rho^a_\mu \pi^b \partial^\mu \pi^c$ (minimal coupling) where f_{abc} are the $SU(N_f)$ structure constants.[2] This amplitude can arise in models in which the ρ gets its mass from the Higgs mechanism, or in holographic models where the ρ arises as a Kaluza-Klein state.

Eq. (24) satisfies the Froissart-Martin bound, Eqs. (6a,6b) but only for $k_{\rm min} = 2$. Nevertheless, the high-energy behaviour of Eq. (24) can be improved by the following deformation [19]:

$$\widehat{\mathcal{M}}^{(\rho)}(s,u) = \frac{g^2_{\rho\pi\pi} m^2_\rho}{m^2_\rho - s} P_1\left(1 + \frac{2u}{m^2_\rho}\right) \frac{m^2_\infty}{m^2_\infty - u} + (s \leftrightarrow u)\,, \tag{25}$$

that at high energy $s \gg m^2_\infty$ satisfies Eqs. (6a,6b) with $k_{\rm min} = 1$. By studying the pole structure of Eq. (25), one can see that the amplitude is mediated by states of any J with masses m_∞, but also at $s = m_\rho$ we have now states with $J \neq 1$. Nevertheless, by taking the limit $m_\infty/m_\rho \to \infty$ in Eq. (25), we recover Eq. (24) as well as its low-energy predictions. So, as long as we are only interested in the Wilson coefficients, we can safely use Eq. (24).

At low energies, Eq. (24) leads to

$$g_{1,0} = 3\,\frac{g^2_{\rho\pi\pi}}{m^2_\rho}\,, \quad g_{2,1} = \frac{4}{3}\,\frac{g_{1,0}}{m^2_\rho}\,, \quad g_{n,0} = \frac{1}{3}\,\frac{g_{1,0}}{m^{2n}_\rho}\,, \quad g_{n+1,1} = \frac{2}{3}\,\frac{g_{1,0}}{m^{2(n+1)}_\rho} \quad (n \geq 2)\,, \tag{26}$$

while $g_{n,l} = 0$ for $l \geq 2$. Alternatively, we could compute the Wilson coefficients Eq. (26) from dispersion relations, by using the explicit form of the UV spectral density Eq. (9), with support on $J = 1$ and $m_i = m_\rho$ only. This can help us to appreciate the difference between $\mathcal{M}^{(\rho)}$ and its improved version $\widehat{\mathcal{M}}^{(\rho)}$. Indeed, since the vector amplitude Eq. (24) fulfils Eq. (6a) only for $k_{\rm min} \geq 2$, we can not use the sum rules Eq. (12) for $k_{\rm min} = 1$. In particular, the expressions for $g_{1,0}$ and $g_{2,1}$ from Eq. (13) do not hold – indeed they differ from those obtained directly from the amplitude Eq. (26). On the other hand, Eq. (24) fulfills Eq. (6b) for $k_{\rm min} = 1$ and one can use the prediction for $g_{2,1}$ from Eq. (15), that agrees with Eq. (26).

If instead we use $\widehat{\mathcal{M}}^{(\rho)}$ from Eq. (25) – which has $k_{\rm min} = 1$ high-energy behavior – the extra states beyond the ρ give a nonzero contribution to $g_{1,0}$ and $g_{2,1}$ that makes it to coincide with Eq. (26). These contributions tend to zero in Eq. (15).

[2] A more general function F in Eq. (21) would be associated with the exchange of longitudinal modes (contact terms to the Lagrangian) which would yield in the amplitude multiplicative factors $(s/m^2_\rho)^n$ that worsen the high-energy behavior.

2.3.3 Theory of spin-2 states

The pole structure of four-pion amplitude mediated by a spin-2 state only (in analogy with QCD we refer to it as f_2), can be written as,

$$\mathcal{M}^{(f_2)} = \frac{g^2_{f_2\pi\pi} m^2_{f_2}}{m^2_{f_2} - s} P_2\left(1 + \frac{2u}{m^2_{f_2}}\right) + (s \leftrightarrow u)\,. \tag{27}$$

Since $P_2(x) = (3x^2-1)/2$, this amplitude grows like $\sim s^2/u$ for large s, violating even the Froissart-Martin $k_{\rm min} = 2$ bound. Nevertheless, Eq. (27) can be deformed as in Eq. (25) to improve its high-energy behavior:

$$\widehat{\mathcal{M}}^{(f_2)} = \frac{g^2_{f_2\pi\pi} m^2_{f_2}}{m^2_{f_2} - s} P_2\left(1 + \frac{2u}{m^2_{f_2}}\right) \frac{m^2_\infty}{m^2_\infty - u} + (s \leftrightarrow u)\,, \tag{28}$$

that indeed satisfies the Froissart-Martin bound for $s \gg m^2_\infty$, and leads to Eq. (27) in the limit $m_\infty/m_{f_2} \to \infty$. It is not possible to improve this further, as the resulting amplitude would have negative spectral density, and violates unitarity. For this reason amplitudes with light $J \geq 3$ (which grow as s^J from $P_J(1+2s/m_J^2)$ can not be completed into amplitudes that satisfy the Froissart-Martin bound, see also [32].

Contact terms (the function F in Eq. (21)) would modify Eq. (27). Nevertheless, demanding that they do not grow faster than Eq. (27), leaves only terms up to $O(s)$. These terms can only affect $g_{1,0}$, telling us that this Wilson coefficient cannot be predicted in a theory of spin-2 states. The rest of the Wilson coefficients, however, can be unambiguously derived from Eq. (27) in the low-energy limit:

$$\begin{aligned} &g_{2,0} = 7\,\frac{g^2_{f_2\pi\pi}}{m^4_{f2}}\,, \quad g_{2,1} = \frac{12}{7} g_{2,0} = g_{3,1} m^2_{f_2} = g_{4,2} m^4_{f_2}\,, \quad g_{n,0} = \frac{1}{7}\,\frac{g_{2,0}}{m^{2n-4}_{f_2}} \quad (n \geq 3)\,, \\ &g_{n,1} = \frac{6}{7}\,\frac{g_{2,0}}{m^{2n-4}_{f_2}} \quad (n \geq 4)\,, \quad g_{n,2} = \frac{6}{7}\,\frac{g_{2,0}}{m^{2n-4}_{f_2}} \quad (n \geq 5)\,, \end{aligned} \tag{29}$$

with $g_{n,l} = 0$ for $l > 2$.

Similarly to the spin-1 case, we could calculate some of the Wilson coefficients from dispersion relations, with spectral density for $J = 2$ only. As Eq. (27) satisfies the high-energy limit Eqs. (6) for $k \geq k_{\rm min} = 3$, we can use the sum rules Eq. (12) with $k_{\rm min} = 3$ to obtain all the Wilson coefficients except $g_{1,0}$, $g_{2,0}$, $g_{2,1}$, $g_{3,1}$, $g_{4,2}$ and $g_{5,3}$ – indeed their value via Eq. (12) disagrees with Eq. (29). This can also be understood by realizing that the states of mass m_∞ present in the consistent amplitude Eq. (28) give finite contributions to $g_{1,0}$, $g_{2,0}$, $g_{2,1}$, $g_{3,1}$, $g_{4,2}$ and $g_{5,3}$ when Eq. (12) is used, even in the limit $m_\infty \to \infty$. For the coefficient $g_{3,1}$, however, we can alternatively determine it by using Eq. (16) that indeed agrees with Eq. (29).

2.3.4 The *su*-models

The *su*-models [6, 7] give the simplest four-pion amplitude mediated entirely by higher-spin states. The particularity of these models is that their spectrum is fully degenerate, $\mathcal{M}(s,u) \propto$

$1/(s-m^2)(u-m^2)$, a condition that naturally places these models at the boundary of the allowed parameter space, as we shall see. These *su*-amplitudes and the associated Wilson coefficients are discussed in detail in Appendix C.

2.3.5 The Lovelace-Shapiro and Coon amplitude

There are other four-pion amplitudes mediated by higher-spin states: the Lovelace-Shapiro [20, 21] amplitude and its generalization, the Coon amplitude [22]. These amplitudes originate in the context of string theory, and can provide therefore fully consistent UV completions to a theory of pions. We will see that the Wilson coefficients predicted by these amplitudes lie at the closest point to one of the boundaries of the allowed regions. Therefore they provide information about the mass spectrum of the theories residing on these boundaries. The details of these amplitudes are given in Appendices D and E.

3 Implications of $\mathcal{M}(s,u)/s \to 0$ at large s

In this section, we assume that the pion amplitude $\mathcal{M}(s,u)$ satisfies the conditions Eqs. (6a,6b) for $k_{\min} = 1$, as argued in Ref. [19]. Since all Wilson coefficients scale like $1/N_c$, and because positivity bounds are inherently projective (i.e. only ratios of Wilson coefficients are constrained), we will work with

$$\tilde{g}_{n,l} \equiv \frac{g_{n,l}}{g_{1,0}} M^{2(n-1)} \,, \tag{30}$$

where $g_{1,0}$ is the leading Wilson coefficient and M the EFT cutoff defined in Fig 1. Unless stated, we will take M as the lowest resonance mass. The $\tilde{g}_{n,l}$ are independent of N_c in the large-N_c limit.

We can use the sum rules Eq. (12) to determine the Wilson coefficients as a function of the mass spectrum, which is itself constrained by the null constraints. Our goal will be to shape the boundaries of the EFT parameter space, and identify the UV theories that generate it. When possible we shall use analytic arguments, complemented when necessary by numerics.

3.1 Bounds on the leading Wilson coefficients

Let us start by studying the implications of positivity bounds for $\pi\pi$ scattering at order $O(s^2)$, i.e. for the coefficients $g_{2,0}$ and $g_{2,1}$. As explained in Ref. [19], and recalled in Appendix A, the allowed regions can be obtained by numerical methods, see Fig. 3. These methods work extremely well as explorative tools and give conclusive answers when they rapidly approach known theories, e.g. [33]. Nevertheless, they are limited by computer power, and leave the question open of whether the extrapolation from finite resolution truly reveals a physically meaningful result. Here we will show that for some questions the numerical convergence is too slow, and we proceed by using analytic methods to map as much of the parameter space as possible.

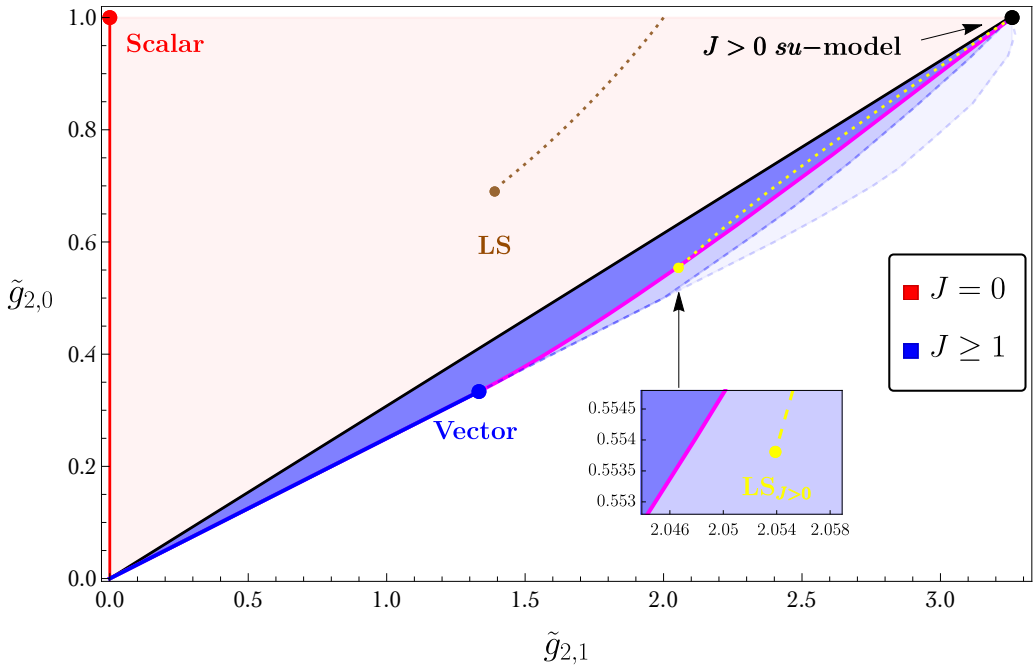


Figure 3: *Allowed regions in the $\tilde{g}_{2,1}$–$\tilde{g}_{2,0}$ plane, divided into regions with $J = 0$ states only (red line) and $J > 0$ states (blue) – using null constraints with $n_{\max} = 3$ (dashed light-blue line) and $n_{\max} = 11$ (dashed blue line). The dark blue area spans Eq. (47) for $0 \leq m_\rho \leq M$ and $m_\rho \leq m \leq \infty$. Moreover, $J = 1$ models lie on the blue line, $J > 0$ su-models on the black line (Eq. (92) with Eq. (94) and $0 \leq m \leq M$), and the magenta line corresponds to Eq. (48). The brown and yellow dot correspond to the Lovelace-Shapiro amplitude with and without scalars (Eq. (113))) respectively. The dashed brown and yellow line are the Coon amplitude (Eq. (118) with $C = 1$) with and without $J = 0$ states respectively.*

Scalar theories and the $\tilde{g}_{2,1} > 0$ boundary. The smallest value of $\tilde{g}_{2,1} = 0$ is saturated by an amplitude mediated by $J = 0$ states, as discussed in section 2.3.1 – see also Ref. [19]. In particular, from Eq. (23), identifying $M = m_s$, we have $(\tilde{g}_{2,1}, \tilde{g}_{2,0}) = (0, 1)$, depicted as a red dot in Fig. 3. This point must clearly be a corner (kink) of the full allowed region, since from Eq. (13) one can see that $\tilde{g}_{2,1} \geq 0$ and $\tilde{g}_{2,0} \leq 1$. When more (non-degenerate) scalars are present, the value of $g_{2,0}/g_{1,0}$ always decreases, since $g_{2,0}$ scales as $\sim 1/m_s^4$ and therefore extra heavy scalars contribute more to $g_{1,0}$ than $g_{2,0}$ (the same is true for $g_{n,0}/g_{n-1,0}$). This is shown in Fig. 3 by a red line.

As discussed in section 2.1, states with $J < k_{\min}$ decouple from the null constraints. For $k_{\min} = 1$, only the $J = 0$ states decouple, meaning that it is the only simple theory that can provide a standalone UV completion of the chiral Lagrangian. All other UV completions must involve infinitely many states with all spins. It is therefore interesting to study the boundary of the region with $J \geq 1$ independently, as we discuss in what follows.

Spin-1 theories, the $\tilde{g}_{2,1}/\tilde{g}_{2,0} < 4$ boundary and its kink. The largest value of $\tilde{g}_{2,1}/\tilde{g}_{2,0}$ can be determined from Eq. (13) and Eq. (15), which gives

$$\frac{\tilde{g}_{2,1}}{\tilde{g}_{2,0}} = \frac{4\left\langle \frac{1}{m^4}\right\rangle_{J-\text{odd}}}{\left\langle \frac{1}{m^4}\right\rangle_{J-\text{odd}} + \left\langle \frac{1}{m^4}\right\rangle_{J-\text{even}}}\,, \tag{31}$$

where the subscript denotes average over even or odd J only. This is saturated by $\tilde{g}_{2,1}/\tilde{g}_{2,0} = 4$ and corresponds to theories with only J-odd states. The simplest theory of this type is that described in Sec. 2.3.2 consisting of a light spin-1 state only, whose amplitude with the improved high-energy behavior is given in Eq. (25) with $m_\infty/m_\rho \to \infty$. From Eq. (26) we have

$$(\tilde{g}_{2,1}, \tilde{g}_{2,0})_{\text{vector}} = (4/3, 1/3)\,, \tag{32}$$

shown in Fig. 3 as a blue dot. Adding extra spin-1 states allows us to move down from Eq. (32) to the origin, along the blue line of Fig. 3. This line must be part of the boundary of the allowed region for $(\tilde{g}_{2,1}, \tilde{g}_{2,0})$, since the ratio $\tilde{g}_{2,1}/\tilde{g}_{2,0} = 4$ takes the largest possible value.

The important question to address now is whether the allowed region can extend along the blue line beyond Eq. (32) or not. The numerical analysis of [19] was able to show that as one increases the number of null constraints, the kink moves towards Eq. (32), but appeared to tend asymptotically to a larger value. Here we will show that the kink resides at Eq. (32), i.e. the extremal theory along the boundary contains only a spin-1 light state.

To show this, we will first argue that at the boundary,

$$\tilde{g}_{2,0} = M^2 \frac{\left\langle \frac{1}{m^4}\right\rangle}{\left\langle \frac{1}{m^2}\right\rangle} \quad \to \quad M^2 \frac{\left\langle \frac{1}{m^4}\right\rangle_{J=1}}{\left\langle \frac{1}{m^2}\right\rangle_{J=1}} \frac{1}{1 + \frac{\left\langle \frac{1}{m^2}\right\rangle_{J>1}}{\left\langle \frac{1}{m^2}\right\rangle_{J=1}}}\,. \tag{33}$$

This can be proven as follows. Eq. (31) implies that at the boundary even-J states must decouple from the average in $g_{2,0}$,

$$\left\langle \frac{1}{m^4}\right\rangle_{J-\text{even}} \to 0\,. \tag{34}$$

Using the Cauchy-Schwarz inequality (where $\left\langle \mathcal{J}^4/m^8\right\rangle$ must be finite because it enters in the null constraint in the first expression of Eq. (19)), we have

$$\sqrt{\left\langle \frac{\mathcal{J}^4}{m^8}\right\rangle_{J-\text{even}} \left\langle \frac{1}{m^4}\right\rangle_{J-\text{even}}} \geq \left\langle \frac{\mathcal{J}^2}{m^6}\right\rangle_{J-\text{even}} \to 0\,. \tag{35}$$

This, together with the null constraint $\left\langle \mathcal{Y}_{3,1}\right\rangle = 0$ Eq. (20), implies,

$$\left\langle \frac{\mathcal{J}^2}{m^6}\right\rangle_{J-\text{even}} = 3\left\langle \frac{\mathcal{J}^2 - 2}{m^6}\right\rangle_{J-\text{odd}} \geq 12\left\langle \frac{1}{m^6}\right\rangle_{J>1-\text{odd}} \to 0\,, \tag{36}$$

because $\mathcal{J}^2 - 2 = 0$ on $J = 1$ and $\mathcal{J}^2 - 2 \geq 4$ for $J \geq 2$. Using Cauchy-Schwarz again (with measure on odd $J > 1$ only), and the fact that $\left\langle \frac{1}{m^2}\right\rangle_{J>1-\text{odd}}$ is finite, we find

$$\sqrt{\left\langle \frac{1}{m^6}\right\rangle_{J>1-\text{odd}} \left\langle \frac{1}{m^2}\right\rangle_{J>1-\text{odd}}} \geq \left\langle \frac{1}{m^4}\right\rangle_{J>1-\text{odd}} \to 0\,, \tag{37}$$

thus proving Eq. (33). The theory that lives on the boundary must therefore consist of spin-1 states at finite mass accompanied by higher-spin states at infinite mass.

Unfortunately, this is not yet enough to claim that it is *exactly* the theory in Eq. (25), since there might be different ways in which the $J \to \infty$ states enter in the spectral density.

To prove that it is, we reformulate the question of finding the maximum value of the kink, as a 1D moment problem. Indeed, the kink is positioned at the maximum value of $\tilde{g}_{2,0}$ along the boundary given by Eq. (33). The first factor $\left\langle \frac{1}{m^4} \right\rangle_{J=1} / \left\langle \frac{1}{m^2} \right\rangle_{J=1} \leq 1$ is maximally saturated when the spectrum contains only one spin-1 particle at the mass M, $M^2 \langle \frac{1}{m^4} \rangle_{J=1} = \langle \frac{1}{m^2} \rangle_{J=1} \equiv g^2_{\rho\pi\pi}/M^2$. Then, from the second factor in Eq. (33) we read that the kink is located at the *minimum* of,

$$\frac{M^2}{g^2_{\rho\pi\pi}} \left\langle \frac{1}{m^2} \right\rangle_{J>1} . \tag{38}$$

This minimum cannot be zero, because the null constraints relate the high-energy averages to $g^2_{\rho\pi\pi}$, in such a way that the ratio is finite. Indeed, along the boundary, the dominant null constraints Eq. (18) can be written as,

$$2(n-1)!^2 = \frac{M^{2n}}{g^2_{\rho\pi\pi}} \left\langle \frac{J^{2(n-1)}}{m^{2n}} \right\rangle_{J>1} , \quad n = \underline{2}, 3, 4, \dots , \tag{39}$$

i.e. all subleading terms J^k with $k < 2(n-1)$ can be neglected. This can be understood from the comment below Eq. (37) – for states with infinite mass, if in the average $\lim_{m\to\infty} J^{2(n-1)}/m^{2n}$ is finite, then all subleading powers of J must vanish.[3] With a change of variables to (the square of) impact parameter $x \equiv J^2 M^2/m^2$, and redefining n, we can write Eq. (39) as

$$2n!^2 = \int_0^\infty d\mu(x) x^n dx \equiv \mu_n \quad , \quad n = 1, 2, 3, \dots , \tag{41}$$

with $d\mu(x)$ a positive distribution. In this language Eq. (38) is μ_0, and the problem of finding the kink position translates into a 1-dimensional moment problem:

finding the minimum of μ_0 *such that* $\{\mu_0, \mu_1, \mu_2, \cdots\}$ *is a moment series from a positive distribution, with* $\mu_n = 2n!^2$ *for* $n \geq 1$. (42)

A sufficient condition for this, is that the Hankel matrix H^0_N, with $(H^k_N)_{ij} = \mu_{i+j+k}$, for

[3] It can also be seen by using the Hölder inequality and $\langle 1/m^{2k} \rangle_{J>1} \to 0$ (for $k \geq 1$) from Eq. (37),

$$\left\langle \frac{1}{m^{2k}} \right\rangle^{\frac{1}{k}}_{J>1} \left\langle \frac{J^{2(n-k-1)}}{m^{2(n-k)}} \right\rangle^{1-\frac{1}{k}}_{J>1} \geq \left\langle \frac{J^{2(n-k-1)}}{m^{2n}} \right\rangle_{J>1} \to 0 . \tag{40}$$

$i, j = 0, \ldots, \lfloor N/2 \rfloor$, be asymptotically positive definite [4, 34],

$$\lim_{N\to\infty} H_N^0 = 2 \lim_{N\to\infty} \begin{pmatrix} \mu_0/2 & 1!^2 & 2!^2 & \cdots & n!^2 \\ 1!^2 & 2!^2 & 3!^2 & \cdots & (n+1)!^2 \\ 2!^2 & 3!^2 & 4!^2 & \cdots & (n+2)!^2 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ n!^2 & (n+1)!^2 & (n+2)!^2 & \cdots & (2n)!^2 \end{pmatrix} \succ 0\,. \tag{43}$$

Equivalently (using Silvester's criterion) this can be rewritten as,

$$1 - \mu_0/2 \leq \lim_{N\to\infty} \frac{\det H_N^0|_{\mu_0=1}}{\det H_N^2}\,. \tag{44}$$

Explicit evaluation of Eq. (44) for fixed N enables us to reach smaller and smaller values, going from $\mu_0 \approx 0.95$ for $N = 10$ (equivalent to $\tilde{g}_{2,0} \approx 0.51$) to $\mu_0 \approx 1.54$ for $N = 700$ (equivalent to $\tilde{g}_{2,0} \approx 0.39$) – to be compared with $\tilde{g}_{2,0} \approx 0.42$ of Ref. [19]. Computing the asymptotic behavior of determinants of this type is an interesting open problem in mathematics, see e.g. [35], motivated by their appearance in random matrix theory (interestingly, also in relation with QCD and chiral perturbation theory [36, 37]). Leaving this for future work, in Appendix B we take a shortcut and, rather that computing the individual determinants, we focus on the most efficient way of computing the ratio Eq. (44), and show that as $n \to \infty$,

$$\mu_0 \to 2 \qquad \text{and} \qquad (\tilde{g}_{2,1}, \tilde{g}_{2,0}) \to (4/3, 1/3)\,. \tag{45}$$

At the kink resides the theory of a single spin-1 state, with the improved high-energy behavior amplitude Eq. (25) with $m_\infty \gg m_\rho$.

The su-model and the boundary for $J \geq 1$ with minimal $\tilde{g}_{2,1}/\tilde{g}_{2,0}$. At the largest value of $\tilde{g}_{2,0} = 1$ must lie theories with a degenerate spectrum, see Eq. (13). Apart from a theory of a scalar (discussed before), the only amplitude with this property is the *su*-model discussed in Appendix C, with amplitude Eq. (92). This amplitude can also be obtained analytically by solving the null constraints. Indeed, for a degenerate spectrum, the null constraints reduce to a system of equations for the couplings $g^2_{i\pi\pi}$. The dominant null constraints Eq. (18), for instance, are linearly independent, and can be solved explicitly for a fixed number of couplings $g^2_{i\pi\pi}$ with $i = 1, \cdots, n$. The solution is a function that can be resummed and converges into the *su*-model prediction.

This *su*-model contains a fraction of scalar residues, controlled by the value of λ in Eq. (93); for the value in Eq. (94) the theory has no scalars. Its amplitude lies at,

$$(\tilde{g}_{2,1}, \tilde{g}_{2,0})_{J>0\ su-\text{model}} = (\approx 3.26, 1)\,, \tag{46}$$

shown by the black dot in Fig. 3. The uniqueness of this amplitude naturally puts it at kink of the $J \geq 1$ region (and its linear combination with the scalar amplitude at the boundary of the $J \geq 0$ region).

As the spectrum becomes heavier, $M/m \to 0$, the *su*-model morphs into the free theory (at the origin of the plot in Fig. 3). Interestingly, this line defines the boundary of the allowed region for theories characterised by resonances of spin $J \geq 1$. In Appendix C we show that this line is indeed a boundary by considering the most generic deformation of the *su*-model and showing that in order to not spoil positivity these deformations must push you in the bulk of the exclusion region (excluding the contribution from scalars).

Boundary between kinks. Therefore we are left with the boundary line joining the two kinks, Eq. (32) and Eq. (46). We have not found an analytical expression for this curve. Nevertheless, as proposed in Ref. [19], we can have a reasonable analytical formula by considering a model that interpolates between a spin-1 model and the $J > 0$ *su*-model. The interpolating amplitude is given by[4]

$$\mathcal{M} = \mathcal{M}_1^{(su)} - \frac{3(\ln 8 - 2)}{g_{\rho\pi\pi}^2}\left(\widehat{\mathcal{M}}^{(\rho)}(m_\rho \to m) - \widehat{\mathcal{M}}^{(\rho)}\right), \tag{47}$$

which corresponds to a $J > 0$ *su*-model in which the spin-1 state of mass m has been subtracted and replaced by a spin-1 state of mass m_ρ. We have used the corrected amplitude $\widehat{\mathcal{M}}^{(\rho)}$ from Eq. (25), to assure that Eq. (47) satisfies the high-energy conditions with $k_{\text{min}} = 1$ for all values of its parameters; nevertheless we will be taking the limit $m_\infty/m_\rho \to \infty$ that corresponds to $\widehat{\mathcal{M}}^{(\rho)} \to \mathcal{M}^{(\rho)}$. From Eq. (47) it is clear that by varying the mass m from m_ρ to infinity, we are effectively pushing up the masses of the $J > 1$ states in the *su*-model, leaving only a $J = 1$ state at low energy. The corresponding Wilson coefficients of Eq. (47) can be easily calculated and one obtains, for $M = m_\rho$,

$$\tilde{g}_{2,0} = \frac{a - (3 - 10a)(r^2 - 1)}{a - (9 - 28a)(r - 1)}, \qquad \tilde{g}_{2,1} = \frac{1 + (36a - 11)(r^2 - 1)}{a - (9 - 28a)(r - 1)}, \tag{48}$$

where $r = m_\rho^2/m^2$ and $a = 1 - \ln 2$. The magenta line in Fig. 3 is obtained by varying $r \in [0, 1]$.

We have determined this boundary numerically (following Ref. [19], see our Appendix A for details), including null constraints with $n_{\text{max}} = 3$ and $n_{\text{max}} = 11$, shown by dashed lines in Fig. 3. Due to the lack of computational power, however, we have not been able to understand how much the true boundary approaches the analytic boundary Eq. (48) as $n \to \infty$. Nevertheless, we can claim that Eq. (48) cannot coincide with the true bound, as we know of another consistent amplitude that lies on the RHS of this line. This is the Lovelace-Shapiro amplitude in which the scalar contribution has been removed (see Appendix D for details). The prediction for this amplitude (see Eq. (115)) is shown by a yellow dot in Fig. 3. In spite of this, the prediction of this amplitude is impressively close to Eq. (47) as can be appreciated by the zoom area in Fig. 3. We also show a version of the scalar-subtracted Coon amplitude, Eq. (118) in Appendix E, with $C = 1$ and q varied from 0 to 1. This line starts at the $J > 0$ *su*-model and goes down to the Lovelace-Shapiro model. It slightly improves Eq. (48) but only close to the Lovelace-Shapiro

[4]There are other possible interpolating amplitudes, but Eq. (47) is the one with the smallest number of states that we have found. Adding more states will give predictions for $(\tilde{g}_{2,1}, \tilde{g}_{2,0})$ that will lie on the left of the magenta line of Fig. 3.

values ($q \simeq 1$). It could be interesting to know if there is another deformation of the Lovelace-Shapiro amplitude that saturates the true boundary. We also show in Fig. 3 the prediction from the full Lovelace-Shapiro and Coon amplitudes (with $C = 1$) without subtracting the scalar contributions.

For models with both $J = 0$ and $J > 0$ states, the allowed region corresponds to the convex hull spanned by the individual boundaries, shown in light red (plus blue) in Fig. 3.

3.2 Emergence of Vector Meson Dominance

We have seen that a theory of a spin-1 state does not satisfy $\mathcal{M}(s,u)/s \to 0$ at large s, fixed $u < 0$, and requires higher-spin states to soften the high-energy behavior. The converse is also true: any model of higher-spin states must contain spin-1 mesons. This can be made explicit by looking at null constraints. For example, the first two null constraints, $\langle \mathcal{Y}_{2,1} \rangle = 0$ and $\langle \mathcal{X}_{3,1} \rangle = 0$ of Eq. (18), lead to

$$\begin{aligned} \left\langle \frac{1}{m^4} \right\rangle \Big|_{J=1} &= 3 \left\langle \frac{1}{m^4} \right\rangle \Big|_{J=2} + 4 \left\langle \frac{1}{m^4} \right\rangle \Big|_{J=3} + \cdots , \\ \left\langle \frac{1}{m^6} \right\rangle \Big|_{J=1} &= 9 \left\langle \frac{1}{m^6} \right\rangle \Big|_{J=3} + 35 \left\langle \frac{1}{m^6} \right\rangle \Big|_{J=4} + \cdots . \end{aligned} \tag{49}$$

Since the RHS is always positive, this identity can only be fulfilled if there are spin-1 states in the theory. These equations also tell us that, at any order in $1/m^n$ in the average, the contributions from any individual $J > 1$ state must always be smaller than the ρ contribution, since the coefficients appearing on the RHS are always bigger than one. Moreover, these coefficients scale with large J as $\sim J^{2n-2}/m^{2n}$, which is faster than how they appear in the low-energy couplings such as $g_{n,0} \sim \langle 1/m^{2n} \rangle$ or $g_{n,1} \sim \langle J^2/m^{2n} \rangle$ (but $g_{2,1} \sim \langle 1/m^4 \rangle$) from Eq. (12).[5]

The property that the ρ meson dominates the low-energy amplitude of pions (or at least that amplitudes with ρ mesons populate the space of consistent pion amplitudes) is referred to as Vector Meson Dominance (VMD) [23, 24]. Despite its poor theoretical motivation, VMD is known to lead to good agreement with QCD experimental data. Here we see that VMD emerges from unitarity and crossing symmetry. For the most relevant couplings, it is illustrated by the alignment between the $J > 0$ allowed region and the spin-1 line observed in Fig. 4. The QCD experimental value, as determined in Ref. [38], is denoted by the red cross in the figure.

As the masses of the $J > 1$ states increase, VMD becomes more manifest. Indeed, from Eqs. (13,15), the contributions from the high-spin states at large mass $m_{J>1} \to \infty$ go as,

$$\frac{g_{2,0}}{g_{1,0}} \sim \frac{g_{2,1}}{g_{1,0}} \sim \frac{1}{m_{J>1}^2} \to 0 ,$$

while the $J = 1$ contribution remains finite. This is also true for the rest of the Wilson coefficients, since for large scale separations between the ρ and higher spins $m_\rho/m_{J>1} \to 0$, The

[5] For couplings $g_{n,l}$ with larger and larger l this argument is more involved, since the $J = 1$ contribution cancels from Eq. (12) and is restored only via null constraints.

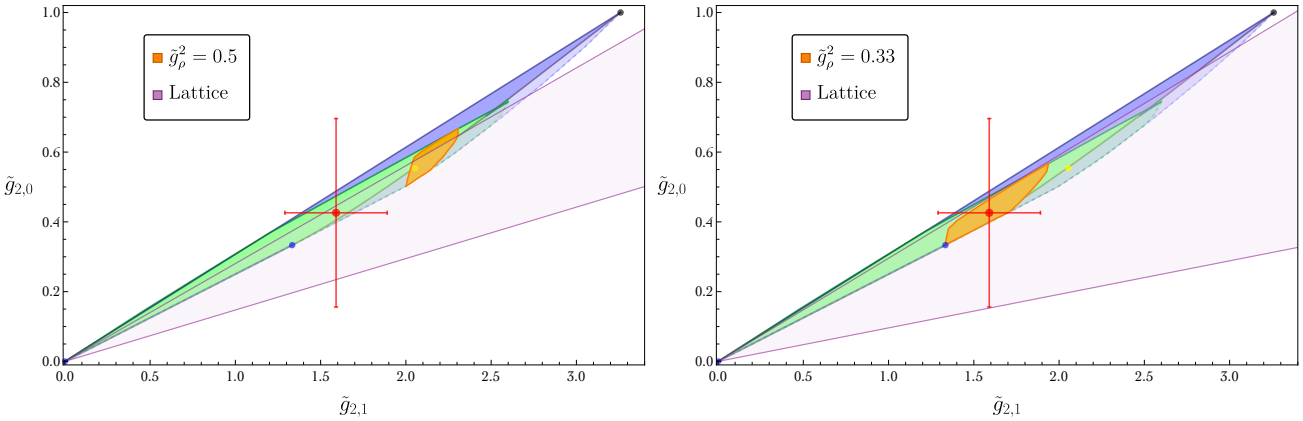


Figure 4: *Allowed regions from positivity with $J \geq 1$ (as in Fig. 3). The green regions have heavier resonances of all spin $J \geq 1$ with masses $M' \geq 1.65\, m_\rho$: the dashed green line uses null constraints up to $n_{\max} = 11$. In orange the allowed region for a fixed $\tilde{g}_\rho = 1/2$ (left) and $\tilde{g}_\rho = 1/3$ (right), with still $M'/m_\rho = 1.65$. In purple, the allowed region from Lattice simulations [39–41]. In red the QCD experimental value from Ref. [38].*

null constraints Eq. (18) require a finite contribution from the $J \to \infty$ states, in such a way that $J^2/m_{J>1}^2$ remains fixed in units of $g_{1,0} \sim 1/m_{J>1}^2$. Using Eq. (13), this implies that the $J > 1$ contribution to the Wilson coefficients scales as,

$$\Delta g_{n+l,l} \sim \frac{J^{2l}}{m_{J>1}^{2(n+l)}} \sim \frac{1}{m_{J>1}^{2n}}\,, \tag{50}$$

and therefore tends to zero for large $m_{J>1}$.

At finite masses, we can study this effect numerically, requiring a finite mass gap between the ρ meson and other resonances $M' > m_\rho$, i.e. we work with the spectral density [19],

$$(2J+1)\rho_J(s) \to \frac{\pi}{2} g_{\rho\pi\pi}^2 \delta_{J,1} \delta(s - m_\rho^2) + (2J+1)\rho'_J(s)\,, \tag{51}$$

where the last term corresponds to extra states with $s \in [M', \infty)$ (see Appendix A). In QCD, the lightest higher-spin resonce is a spin-2 meson f_2 with a mass $m_{f_2} \approx 1.3$ GeV, which implies $M'/m_\rho \sim 1.65$. With this mass gap, the allowed region reduces to the small green strip in Fig. 4, which is more strongly aligned with the the spin-1 prediction, evidencing VMD.

It is instructive to further divide the allowed region of parameters in terms of the contribution of the ρ to the leading Wilson coefficient $g_{1,0}$, i.e.

$$\tilde{g}_\rho^2 \equiv \frac{g_{\rho\pi\pi}^2}{g_{1,0} m_\rho^2}\,, \tag{52}$$

which can be thought to quantify VMD. In Fig. 4 we show in orange the allowed regions in which $\tilde{g}_\rho^2$ matches the experimental QCD value and the value taken in the vector model Eq. (24),

$$(a)\ \tilde{g}_\rho^2 \simeq \frac{1}{2} \quad \text{(QCD)}\,, \qquad (b)\ \tilde{g}_\rho^2 = \frac{1}{3} \quad \text{(spin-1 model)}\,. \tag{53}$$

As we will discuss in Sec. 3.4, $\tilde{g}_\rho^2 = 1/2$ is close to its maximal value $\tilde{g}_\rho^2 \sim 0.78$, which is also where the associated spectral density has the $J > 1$ contributions maximised (saturated by the *su*-model). On the other hand, even for $\tilde{g}_\rho^2 = 1/3$ (for which 2/3 of the leading effects are taken care by higher spin-mesons), the allowed region for $\tilde{g}_{2,0}$ and $\tilde{g}_{2,1}$ still sits close to the spin-1 contribution (blue dot), showing a small effect from the $J > 1$ states.

Our discussion of VMD so far focused on quantifying the contributions from $J > 1$ mesons: these are the difficult ones to model, and for which our arguments are particularly important. On the other hand, since $J = 0$ states decouple from the null constraint, they could indeed dominate $\tilde{g}_{2,0}$ and $\tilde{g}_{2,1}$, as it happens in the Higgs model. Nevertheless, scalars can be easily accommodated in any phenomenological analyses as they have simple UV completions. It is worth noticing, however, that when a spin-1 ρ is assumed to be the lightest meson in the spectrum, as in QCD, the scalar contribution becomes smaller. This property is tied to the fact that contributions to the Wilson coefficients are always positive. For example, taking scalars with masses $\gtrsim 1.65\ m_\rho$, while still fixing $\tilde{g}_\rho^2$ to the values considered above, we find that the resulting allowed regions depicted in Fig. 4 increase in size by only $10 - 25\%$ along the $\tilde{g}_{2,0}$ direction.

3.2.1 Comparison with Lattice QCD

The Wilson coefficients $L_{1,2,3}$, traditionally defined in the $SU(3)$ chiral Lagrangian [42],[6] are related to ours by

$$\tilde{g}_{2,0} = 4(2L_1 + 3L_2 + L_3)\frac{M^2}{F_\pi^2}\,, \qquad \tilde{g}_{2,1} = 16L_2\frac{M^2}{F_\pi^2}\,. \tag{54}$$

In the large-N_c limit, $2L_1 = L_2$ [42], which leads to

$$\frac{\tilde{g}_{2,0}}{\tilde{g}_{2,1}} = \frac{1}{4}\left(1 + \frac{\Delta_L}{L_2}\right), \tag{55}$$

where $\Delta_L = 3L_2 + L_3$. This quantity vanishes for theories with only spin-1 resonances, so VMD predicts $\Delta_L \sim 0$. Moreover, the positivity of the Wilson coefficients implies that $L_2, \Delta_L \geq 0$.

A combination of recent lattice simulation results for large-N_c QCD [39–41], gives $\Delta_L = (-0.12 \pm 0.22) \cdot 10^{-3}$. Unfortunately, we have not found any lattice determination of L_2. Nevertheless, for any given value of $\tilde{g}_\rho$, we have a minimal value for $L_2 \propto g_{2,1}$, which plugged into Eq. (55) with the lattice value of Δ_L, can provide a bound on $\tilde{g}_{2,0}/\tilde{g}_{2,1}$. For the two values of Eq. (53), we find

$$(a)\ 0.15 \lesssim \frac{\tilde{g}_{2,0}}{\tilde{g}_{2,1}} \lesssim 0.28\,, \quad (b)\ 0.10 \lesssim \frac{\tilde{g}_{2,0}}{\tilde{g}_{2,1}} \lesssim 0.30\,, \tag{56}$$

[6] Following the chiral Lagrangian definition in Ref. [42],

$$\mathcal{L} = \frac{F_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + L_1 \mathrm{Tr}^2\left(\partial_\mu U^\dagger \partial^\mu U\right) + L_2 \mathrm{Tr}\left(\partial_\mu U^\dagger \partial_\nu U\right)\mathrm{Tr}\left(\partial^\mu U^\dagger \partial^\nu U\right) + L_3 \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U\right).$$

that correspond to the purple areas in Fig. 4. We notice that in both cases positivity bounds are complementary to bounds from lattice, ruling out different regions. In the particular case (a), the upper bound coming from lattice seems to be more restrictive than the one from positivity, but this is not the case as we decrease $\tilde{g}_\rho^2$. In the future a combination of both approaches can lead to a better determination of the geometry of the allowed regions in parameter space.

3.2.2 Holography

The results in Figs. 4 provide also an explanation for the success of holography for predicting QCD properties [25, 26]. Holographic models consist in weakly-coupled 5D constructions describing 4D strongly-coupled dynamics in both the $N_c \to \infty$ limit, and the limit of a large mass gap between spin-0,1 states and other higher-spin states.[7]

In real QCD, however, the mass ratio between the $q\bar{q}$ mesons of spin-2 and spin-1 is not large, $m_{f_2}/m_\rho \sim 1.65$. Therefore, one would expect holographic models not to provide a good description of low-energy QCD, contrary to what is observed [25, 26]. Nevertheless, the above analysis shows that unitarity, causality and crossing-symmetry suppress the effects of higher-spin states in the QCD Wilson coefficients. Therefore, even if the mass gap m_{f_2}/m_ρ is not large, the low-energy QCD quantities are mostly affected by only spin-0 and spin-1 states, which are the ones captured by holographic models. For this reason they can provide a good fit to real-world QCD.

In particular, in the holographic model of Ref. [43], one can show that the predictions for $\tilde{g}_{2,0}$ and $\tilde{g}_{2,1}$ are very close to those of the vector model $(\tilde{g}_{2,1}, \tilde{g}_{2,0}) = (1.32, 0.33)$.

3.3 Higher order Wilson coefficients

The features that sculpt the allowed region of $\tilde{g}_{2,0}$ and $\tilde{g}_{2,1}$ play a dominant role also in understanding higher-order Wilson coefficients, $\tilde{g}_{n,0}$ versus $\tilde{g}_{n,1}$. From Eq. (13) we have,

$$\frac{\tilde{g}_{n,1}}{\tilde{g}_{n,0}} = \frac{\left\langle \frac{\mathcal{J}^2}{m^{2n}} \right\rangle}{\left\langle \frac{1}{m^{2n}} \right\rangle} , \tag{57}$$

whose minimal value (zero) corresponds to a model with $J = 0$. Focusing instead on $J > 0$ theories, the minimal value arises for models of spin-1 that give $\tilde{g}_{n,1}/\tilde{g}_{n,0} = 2$.[8]

We illustrate this in Fig. 5, where a blue dot corresponds to a model with a single $J = 1$ state:

$$(\tilde{g}_{3,1}, \tilde{g}_{3,0})_{\text{vector}} = (2/3, 1/3) , \tag{58}$$

while theories with many spin-1 states populate the blue line.

[7] The model also has 5D gravitons, but these correspond to glueballs of spin $J \leq 2$, which decouple from the pion amplitudes.

[8] Notice that we could not use this argument for the case $n = 2$, since we cannot use the sum rules in Eq. (57) with $n = 2$ for a theory of $J = 1$ states only, as explained at the end of Sec. 2.3.2. In other words, the infinitely heavy $J > 1$ states give zero contribution to Eq. (57) only when $n > 2$.

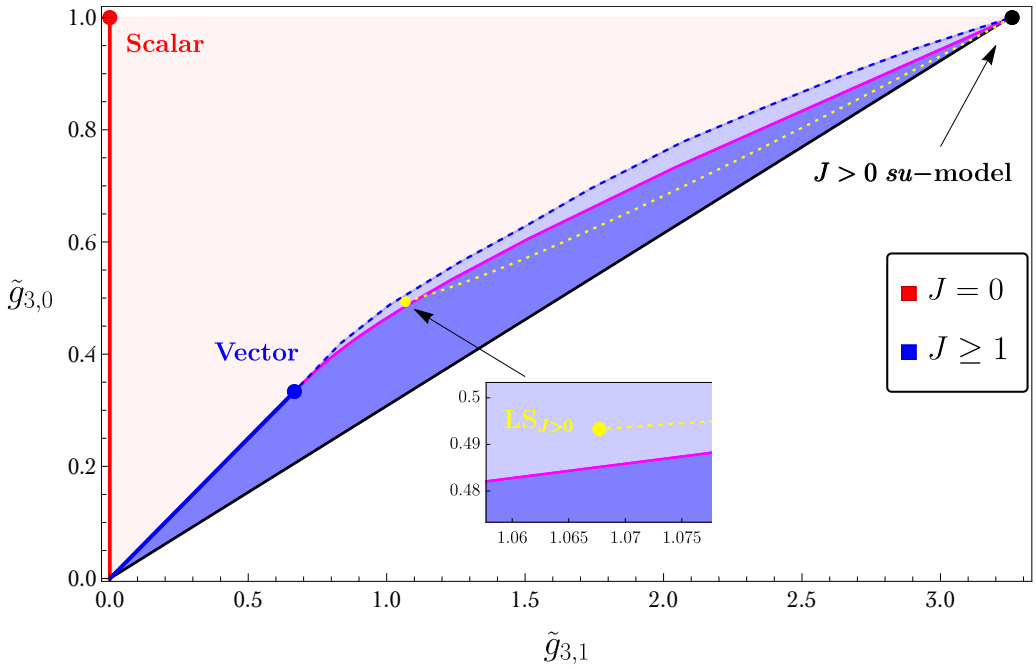


Figure 5: *Allowed region in the* $\tilde{g}_{3,1}$–$\tilde{g}_{3,0}$ *plane from positivity. Same labelling as in Fig. 3, with* $n_{\max} = 11$ *null constraints.*

As for $\tilde{g}_{2,0}, \tilde{g}_{2,1}$, we can show that Eq. (58) corresponds to a kink of the boundary, see Appendix B. The other kink corresponds again to the $J > 0$ *su*-model (the only one with a degenerate spectrum) that gives

$$(\tilde{g}_{3,1}, \tilde{g}_{3,0})_{su-\text{model}} = (\approx 3.26, 1)\,. \tag{59}$$

We have not been able to find an analytic formula for the boundary connecting the two kinks, Eq. (58) and Eq. (59); we illustrate the numerical analysis in Fig. 5. We believe that by adding more null constraints the boundary must approach, but not reach, Eq. (47), consisting of a theory connecting the two kinks (the magenta line in Fig. 5). Nevertheless, as in Sec. 3.1, this line cannot be the true boundary since the Lovelace-Sphapiro model with $J > 0$ states lies at the left of this line, and so does part of the Coon amplitude, Eq. (118) (with $C = 1$ and $q \in [0, 1]$, after subtracting all scalars).

3.4 Bounding the couplings of mesons to pions

So far, we have phrased dispersion relations as UV→ IR vehicles to reformulate microscopic unitarity, causality and crossing-symmetry as predictions for low-energy coefficients. Null constraints, however, provide genuine UV-UV relations, inspired by the same principles. As such, they contain information on the UV meson spectrum and couplings to pions. We define the latter, normalized as,

$$\tilde{g}_i^2 = \frac{g_{i\pi\pi}^2}{g_{1,0}\, m_i^2}\,, \tag{60}$$

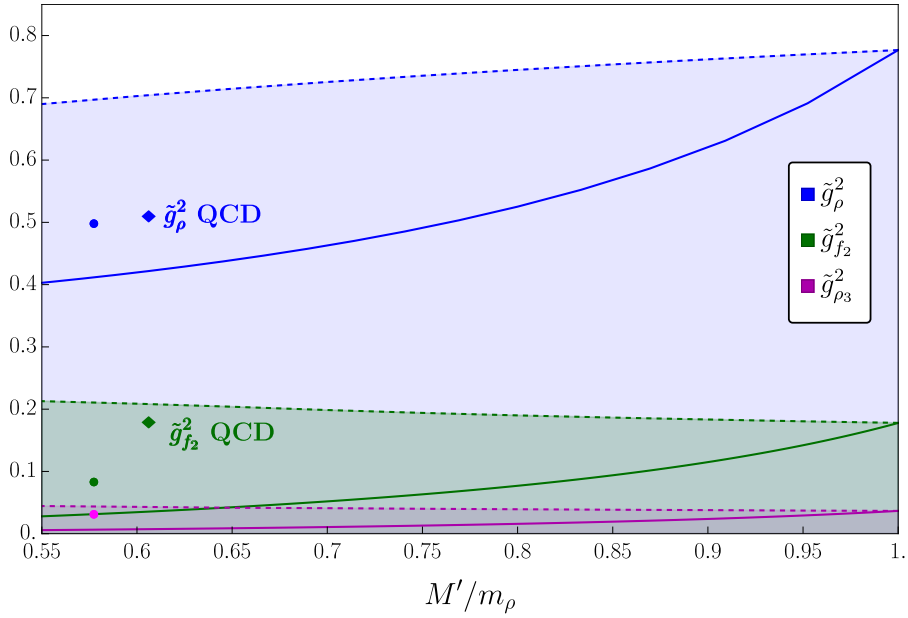


Figure 6: *Upper bound on $\tilde{g}^2_\rho$ (dashed blue line), $\tilde{g}^2_{f_2}$ (dashed green line) and $\tilde{g}^2_{\rho_3}$ (dashed magenta line) as a function of M'/m_ρ using null constraints with $n_{\max} = 7$. The solid lines correspond to the prediction from the interpolating model Eq. (47). The dots correspond to the values of the Lovelace-Shapiro amplitude without scalars, and the diamonds to the QCD experimental values.*

where $i = s, \rho, f_2, \rho_3, ...$ labels $J = 0, 1, 2, 3, ...$ mesons, following the QCD notation of Ref. [44].

Since spin-0 mesons decouple from the null constraints, it is easy to understand that $\tilde{g}^2_s$ is maximised by the smallest possible value of $g_{1,0}$ that, due to its additive property, occurs when the spectrum contains one scalar only:

$$\tilde{g}^2_s \leq 1\,. \tag{61}$$

On the other hand, bounds on the couplings of $J \geq 1$ mesons, involve all null constraints, which we explore numerically, as explained in Appendix A. The results are illustrated in Fig. 6. For instance, the bound on the ρ coupling $\tilde{g}^2_\rho$ (dashed blue line), is obtained as a function of $M' \geq m_\rho$ by singling out this state from the spectral density as in Eq. (51). The bound goes from a maximal value $\tilde{g}^2_\rho \simeq 0.78$, corresponding to the $J > 0$ *su*-model where $M' = m_\rho$, to the minimal value corresponding to the vector model, $\tilde{g}^2_\rho = 1/3$ where $M' \to \infty$. This can be compared with the interpolating model Eq. (47) shown by the solid blue line and with the Lovelace-Shapiro model without scalars Eq. (113) (shown by dots) that lies between the two lines. Interestingly, while all these models give similar predictions in terms of the Wilson coefficients – see Fig. 3 – they differ substantially at the quantitative level in Fig. 6. This provides an interesting experimental handle to differentiate these theories by testing the couplings of pions to the accessible resonances (amplitude's residues).

Similarly for the spin-2 meson f_2, we rewrite the spectral density as,

$$(2J+1)\rho_J(s) \to \frac{\pi}{2} g^2_{\rho\pi\pi}\delta_{J,1}\delta(s-m^2_\rho) + \frac{\pi}{2} g^2_{f_2\pi\pi}\delta_{J,2}\delta(s-M'^2) + (2J+1)\rho'_J(s)\,, \tag{62}$$

and look for the upper bound on $\tilde{g}^2_{f_2}$ as a function of M'/m_ρ for any value of $g^2_{\rho\pi\pi} > 0$. This is shown by the green dashed line in Fig. 6. The resulting bound is much stricter than that for the ρ – another manifestation of VMD.

As we study mesons of higher and higher spin J, the bound on $\tilde{g}^2_i$ becomes stronger and stronger – see the magenta dashed line for $\tilde{g}^2_{\rho_3}$. Interestingly, this pattern is also tracked by the QCD experimental values, $\tilde{g}^2_\rho = 0.51 \pm 0.01$ and $\tilde{g}^2_{f_2} = 0.18 \pm 0.01$, extracted from the widths $\rho, f_2 \to \pi\pi$ [44], as shown by the diamonds in Fig. 6.

4 Implications of $\mathcal{M}(s,u)/s^2 \to 0$ at large s

A more conservative assumption is that the four-pion amplitude satisfies the Froissart-Martin bound, Eqs. (6a,6b) for $k_{\min} = 2$, rather than $k_{\min} = 1$ as in the previous section. Unfortunately in this case we lose the sum rule for the Wilson coefficient $g_{1,0}$ (and $g_{2,1}$ is no longer expressible by Eq. (12) – which comes from $k_{\min} = 1$ – but requires Eq. (15)). So, we will need to choose another Wilson coefficient to normalize the rest and make the predictions N_c-independent. We will use $g_{2,0}$ and define,

$$\bar{g}_{n,l} \equiv \frac{g_{n,l}}{g_{2,0}} M^{2(n-2)} \,. \tag{63}$$

There is another important difference w.r.t. Sec. 3. The amplitude mediated by a spin-1 state, Eq. (24), fulfils $\mathcal{M}^{(\rho)}/s^2 \to 0$ for large $|s|$, and therefore now provides a good UV description of the four-pion amplitude. As a consequence, models of spin-1 states do not require anymore higher-spin states. From the perspective of null constraints this is realised by the decoupling of $J = 1$, when setting $k_{\min} = 2$, as we explained in Sec. 2.1

It is the spin-2 state that now plays an analog role to that of the ρ-meson in Sec. 3. Indeed, the $\langle \mathcal{Y}_{4,2} - \mathcal{Y}_{4,1} \rangle = 0$ null constraint reads,

$$6 \left\langle \frac{1}{m^8} \right\rangle \Big|_{J=2} = 10 \left\langle \frac{1}{m^8} \right\rangle \Big|_{J=3} + 50 \left\langle \frac{1}{m^8} \right\rangle \Big|_{J=4} + \cdots \,, \tag{64}$$

which tells us that spin-2 states need $J > 2$ states and viceversa (for instance, the amplitude Eq. (27) requires $J \geq 3$ states as in Eq. (28), to comply with the Froissart-Martin bound).

4.1 Bounds on Wilson coefficients

At the leading order $O(s^2)$, we only have $\bar{g}_{2,1}$. As discussed before, we have $\bar{g}_{2,1} = 0$ ($\bar{g}_{2,1} = 4$) for $J = 0$ ($J = 1$) models, that are decoupled from higher-spin states. Focusing instead on theories with $J \geq 2$ states, the largest value of $\bar{g}_{2,1}$ comes from the su-model, once we have subtracted not only the scalar but also the $J = 1$ state, whose amplitude is given by Eq. (97) in appendix C, using Eq. (98). Then, from Eq. (100) we find,

$$\bar{g}_{2,1} \leq \frac{18 \ln 2 - 13}{10 \ln 2 - 7} \simeq 7.6 \,. \tag{65}$$

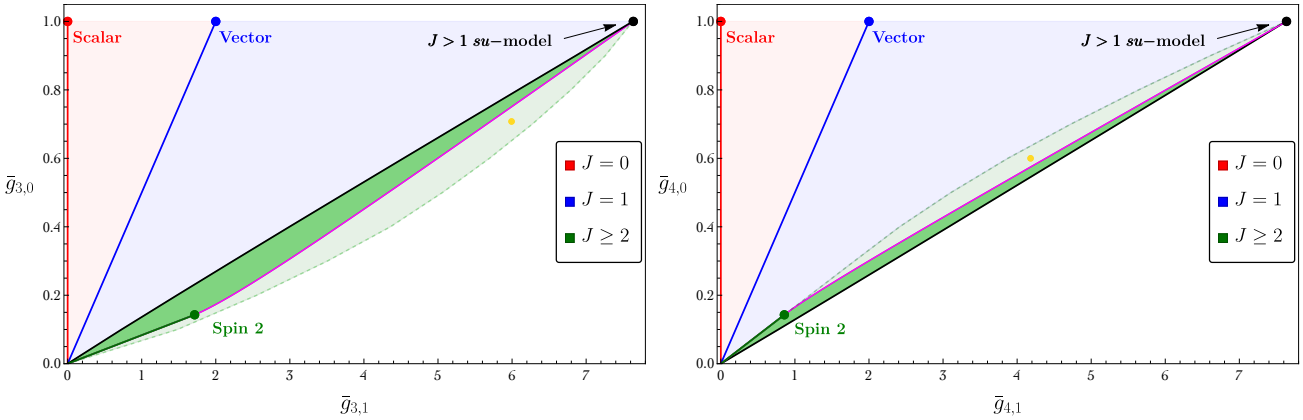


Figure 7: *Allowed region in the $\bar{g}_{3,1}$–$\bar{g}_{3,0}$ plane (left) and $\bar{g}_{4,1}$–$\bar{g}_{4,0}$ plane (right) from positivity. The black lines correspond to the $J>1$ su-model (Eq. (97) with Eq. (98) and $0\leq m\leq M$), the green lines to $J=2$ models, and the magenta lines to Eq. (69) and Eq. (70) for the left and right plot respectively. The green areas limited by the dashed line corresponds to the allowed region with $n_{\text{max}}=7$ null constraints.*

At order $O(s^3)$ and $O(s^4)$ we can also consider $(\bar{g}_{3,1},\bar{g}_{3,0})$ and $(\bar{g}_{4,1},\bar{g}_{4,0})$. The contributions from models of scalars and vectors are given by the lines going respectively from the points $(0,1)$ and $(2,1)$ to the origin, illustrated in Fig. 7 in red and blue.

The allowed regions for $J\geq 2$ are less trivial, and correspond to the green areas in Fig. 7. Again the upper kink is associated with the *su*-model, with its degenerate spectrum that makes $\bar{g}_{n,0}$ maximal. The coefficients are

$$(\bar{g}_{n,1},\bar{g}_{n,0})_{J>1\ \ su-\text{model}}=(\approx 7.6,1)\,, \tag{66}$$

as illustrated by the black dots in Fig. 7. For $m\in[0,M]$, the *su*-model spans the minimum and maximum of the $J\geq 2$ region of $(\bar{g}_{3,1},\bar{g}_{3,0})$ and $(\bar{g}_{4,1},\bar{g}_{4,0})$ respectively (black lines in the figures).

From Eq. (57) the minimal values for $\bar{g}_{n,1}/\bar{g}_{n,0}$ correspond to models with the lowest possible spin. This would mean a theory with $J=2$ mesons only, but we have seen that this is incompatible with the Froissart-Martin bound: $J\geq 2$ states are also needed to satisfy the null constraints. Although these states can be made infinitely heavy (see section 2.3.3), they only decouple in $\bar{g}_{n,1}$ for $n\geq 4$. Therefore the $J=2$ theory would give the minimal value for $\bar{g}_{4,1}/\bar{g}_{4,0}=J(J+1)|_{J=2}=6$, but not for $\bar{g}_{3,1}/\bar{g}_{3,0}$. The amplitude of a single spin-2 meson gives from Eq. (27),

$$(\bar{g}_{4,1},\bar{g}_{4,0})_{\text{spin}-2}=(6/7,1/7)\,. \tag{67}$$

Following a similar analysis as in Appendix B, we expect this to be also a kink in the limit where all null constraints are taken into account. The boundary between this kink Eq. (67) and the $J>1$ *su*-model has be obtained numerically and is illustrated by the green dashed line in Fig. 7.

For $(\bar{g}_{3,1},\bar{g}_{3,0})$, we cannot assure the existence of a kink at the $J=2$ value $(12/7,1/7)$, but the numerical bound (green dashed line) seems to approach this point.

A consistent amplitude that interpolates between the $J > 1$ su-model and the spin-2 model is given by

$$\widehat{\mathcal{M}} = \mathcal{M}_2^{(su)} - \frac{5(13\ln 2 - 9)}{g_{f_2\pi\pi}^2}\left(\widehat{\mathcal{M}}^{(f_2)}(m_{f_2} \to m) - \widehat{\mathcal{M}}^{(f_2)}\right), \tag{68}$$

and corresponds to the $J > 1$ su-model with the spin-2 state of mass m removed and added back at mass m_{f_2}. In the limit $m_\infty \to \infty$, Eq. (68) leads to

$$\bar{g}_{3,0} = \frac{20 - 23r^3 + a(-65 + 75r^3)}{140 - 143r^2 + 5a(-91 + 93r^2)}, \quad \bar{g}_{3,1} = \frac{240 - 245r^3 + a(-780 + 798r^3)}{140 - 143r^2 + 5a(-91 + 93r^2)}, \tag{69}$$

$$\bar{g}_{4,0} = \frac{20 - 23r^4 + a(-65 + 75r^4)}{140 - 143r^2 + 5a(-91 + 93r^2)}, \quad \bar{g}_{4,1} = \frac{120 - 125r^4 + 6a(-65 + 68r^4)}{140 - 143r^2 + 5a(-91 + 93r^2)}. \tag{70}$$

where $r = m_{f_2}^2/m^2$ and $a = 1 - \ln 2$, shown in Fig. 7 as magenta lines, and compared to the predictions from the Lovelace-Shapiro amplitude (yellow dot), that lies – again – outside of the area limited by the magenta lines.

4.2 VMD and Spin-2 dominance

As we have seen, for amplitudes satisfying $\mathcal{M}/s^2 \to 0$ at large energies, spin-1 states decouple and our previous arguments for VMD no longer hold. In this case, spin-1, spin-0 and higher spin contributions are independent phenomenological quantities and VMD would be a mere accident of Nature. In spite of this, we still expect the f_2 coupling to pions to be larger than that for $J \geq 3$ states, in analogy to VMD. To quantify this statement, we define

$$\bar{g}_i^2 = \frac{g_{i\pi\pi}^2}{g_{2,0}\, m_i^4}, \tag{71}$$

and look for their largest allowed values compatible with positivity. For scalars and vectors, that decouple from the null constraints, we have

$$\bar{g}_s^2, \bar{g}_\rho^2 \leq 1. \tag{72}$$

For the spin-2 state, instead, we obtain

$$\bar{g}_{f_2}^2 \lesssim 0.80. \tag{73}$$

Similarly to the spin-1 meson in the case $\mathcal{M}/s^2 \to 0$, this latter bound is saturated by the $J > 1$ su-model. The QCD experimental value (extracted from data in [44]) is $\bar{g}_{f_2}^2 = 0.41 \pm 0.25$ and is much smaller than Eq. (73). This is due to the presence of the ρ-meson that also contributes to the normalization factor $g_{2,0}$ and therefore makes $\bar{g}_{f_2}^2$ smaller. For higher-spin states, we find, taking $m_i/m_{f_2} \sim 1.3$,

$$\bar{g}_{\rho_3}^2 \lesssim 0.14, \quad \bar{g}_{f_4}^2 \lesssim 0.04. \tag{74}$$

Interestingly, if we assume $\bar{g}_\rho \neq 0$ (which is not implied by the null constraints), we can still find a bound on the maximal contribution to the Wilson coefficients arising from $J > 1$ states:

$$\frac{\bar{g}_{2,1}}{\bar{g}_{2,1}|_\rho} \lesssim 1 + 0.17\,\frac{1 - \bar{g}_\rho^2}{\bar{g}_\rho^2}. \tag{75}$$

Here we have used that theories containing $J \geq 2$ mesons have $\langle 1/m^4 \rangle_{J-\text{odd}} / \langle 1/m^4 \rangle \lesssim 0.17$, with the bound saturated by the $J > 1$ su-model. Taking for instance $\bar{g}_\rho^2 \sim 0.5$, we obtain from Eq. (75) that the ρ contributes $\gtrsim 80\%$: it still dominates, in line with VMD.

5 Conclusions

We have studied pion scattering amplitudes in the large-N_c limit, using dispersion relations based on crossing-symmetry, unitarity and causality of the QCD dynamics. Under different assumptions about the high-energy behavior of amplitudes, but agnostic of the specific meson spectral density, we have identified the allowed regions of parameter space for the low-energy Wilson coefficients.

For amplitudes with $\mathcal{M}(s) \lesssim s$ at high-energy, building upon Ref. [19], we have made progress in several directions. By separating theories with $J = 0$ mesons only, from theories with $J \geq 1$, we have revealed a pattern of kinks that characterizes both the leading and the more irrelevant Wilson coefficients. Some kinks are populated by known theories, such as the linear sigma model (involving only a spin-0 meson in the UV completion) and the (scalarless) su-model, that involves a degenerate spectrum of infinitely many higher-spin resonances. The other kinks appear to be associated with a more complex high-energy spectrum, but a numerical exploration of the constraints converges slowly. In these points, we have been able to solve the null constraints analytically, by translating them into a 1D moment problem, and shown that at the kink sits a theory with a single light spin-1 resonance, the ρ meson.

There is one part of the boundary of the allowed region of Wilson coefficients that still eludes analytic methods: here we have found that the scalar-subtracted Lovelace-Shapiro (and part of the Coon) amplitude, provides the best analytical approximation of the constraints – within the numerical bounds but outside the region spanned by simpler theories. This suggests that possibly another deformation of the Lovelace-Shapiro amplitude exists which better approximates the entire boundary.

We have also shown how the soft high-energy behavior of amplitudes implies, via positivity bounds, Vector Meson Dominance and explains the success of holographic QCD. The reason is that although these models do not incorporate higher-spin states, positivity constraints tell us that their contributions in low-energy quantities have to be small.

For amplitudes that are not as soft at high-energy, but that still respect the Froissart-Martin bound $\mathcal{M}(s) \lesssim s \log^2 s$, we have found that analogous results hold. Now, theories with states with $J = 0$ or $J = 1$ decouple from the rest and can be studied in isolation. The remaining theories with $J \geq 2$ also have kinks, populated by the (scalarless and vectorless) su-model or by a theory with a single light spin-2 meson. In this context, we have highlighted a version of "spin-2 meson dominance".

The comparison between our results and large-N_c lattice simulations (see Fig. 4) emphasizes the complementarity between the two methods and lies down an exciting symbiotic program that combines analytic and lattice methods to corner QCD.

The physics captured by the chiral Lagrangian is rich and describes many processes beyond $\pi\pi$ scattering: the program of cornering large-N_c QCD with positivity bounds can move forward

in many directions. An important direction for further study would be processes involving pions interacting with external sources, scalars or photons. Here we expect that including full unitarity of the QCD amplitude, as done by Ref. [45] in the context of the a-anomaly, would reveal an interesting interplay between the chiral anomaly, the QED minimal coupling, and the parameters studied in this article. This direction of research is interesting also in the framework of physics beyond the Standard Model, in particular for the question of whether the Higgs is composite or not. There, Ref. [46] proposed a set of power-counting rules (on of which was dubbed *minimal coupling*) which are realized at weak coupling and in string theory – it would be interesting to prove this in the wider context of dispersion relations.

On a different front, it would be interesting to incorporate more systematically real-world QCD data to shape the meson spectrum, or vice-versa, see e.g. [47]. To this goal, it is essential to first understand finite N_c effects, as discussed in the single-flavor case in Refs. [4, 48] – a task that we will leave for future work [49, 50].

From a more technical point of view, it is important to bring the analytic methods we have used in this work into a more systematic tool to identify all theories at the boundary, e.g. via a clever use of Lagrange multipliers and resummations at all orders. This would allow us to understand if the Lovelace-Shapiro amplitude (with subtracted scalars, or scalars and vectors) is indeed extremal or not. Here it would be useful to further understand which deformations of known amplitudes are consistent with unitarity, along the lines of [51–53].

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A Numerical Bootstrap

In this appendix, we briefly explain the numerical optimization procedure used in this article, following [6, 7, 19]. From Eq. (17) we know that,

$$\langle \mathcal{X}_{n,l}(m^2, J) \rangle = 0 \ , \quad \langle \mathcal{Y}_{n,l}(m^2, J) \rangle = 0 \ , \tag{76}$$

while Eq. (12) implies,

$$g_{n+l,l} = \left\langle g_{n+l,l}(m^2, J) \right\rangle , \qquad \text{where} \qquad g_{n+l,l}(m^2, J) = \frac{2^l}{l!} \frac{P_J^{(l)}(1)}{m^{2(n+l)}} . \tag{77}$$

If we define the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad \vec{v}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} , \quad \vec{v}_{\text{HE}}(m^2, J) = \begin{pmatrix} -g_{1,0}(m^2, J)M^2 \\ -g_{n,l}(m^2, J)M^{2n} \\ \mathcal{Y}_{2,1}(m^2, J) \\ \vdots \end{pmatrix} , \tag{78}$$

then the following equation holds

$$g_{1,0}M^2\vec{v}_1 + g_{n,l}M^{2n}\vec{v}_n + \left\langle \vec{v}_{\text{HE}}(m^2, J) \right\rangle = 0 . \tag{79}$$

This equation can be adjusted to include more Wilson coefficients.Now, multiplying everything by a vector $\vec{\alpha}$ we can solve the following optimization problem:

$$\begin{array}{ll} \text{maximize} & \vec{\alpha} \cdot \vec{v}_1 \\ \text{such that} & \vec{\alpha} \cdot \vec{v}_{\text{HE}}(m^2, J) \geq 0 \\ & \vec{\alpha} \cdot \vec{v}_n = \pm 1 , \end{array} \tag{80}$$

where positivity of the high-energy average and the normalization $\vec{\alpha} \cdot \vec{v}_n = +1$ will yield the upper bound $\tilde{g}_{n,l} \leq -\vec{\alpha}_{(+)} \cdot \vec{v}_1$, while $\vec{\alpha} \cdot \vec{v}_n = -1$ the lower bound $\tilde{g}_{n,l} \geq \vec{\alpha}_{(-)} \cdot \vec{v}_1$.

A problem of this kind is solved using a semidefinite problem solver such as `SDPB` [54]. Notice that in this procedure, we divided everything by the positive term $g_{1,0}M^2$ and for the last term in Eq. (79) we reabsorbed it in the definition of the high-energy average. Furthermore `SDPB` can only impose the positivity condition in Eq. (81) at the level of polynomials, therefore we must perform the change of variables $m^2 \to M^2(1+x)$ with $x \geq 0$ and absorb the common denominator once again in the definition of the high-energy average. We can also implement a mass gap simply by modifying the above change of variables to the one $m^2 \to M'^2(1+x)$, where M'/M is the mass gap. Numerically, Eq. (79) is evaluated on a grid in x and J. The choice of the maximum value J_{max} depends on the number of null constraints we want to consider; as more null constraints are included, a bigger J_{max} must be chosen in order to ensure the convergence of the bounds.

In the specific case of constraining $\tilde{g}_{2,0}, \tilde{g}_{2,1}$, we input fixed values of $\tilde{g}_{2,0}$, and use

$$\vec{v}_1 + \tilde{g}_{2,0}\vec{v}_2 + \left\langle \vec{v}_{\text{HE}}(m^2, J) \right\rangle = 0 . \tag{81}$$

Then we probe the allowed values of $\tilde{g}_{2,0}$ to find bounds on $\tilde{g}_{2,1}$,

$$(1 + \tilde{g}_{2,0})\vec{v}_1 + \tilde{g}_{2,1}\vec{v}_2 + \left\langle \vec{v}_{\text{HE}}(m^2, J) \right\rangle = 0 . \tag{82}$$

To extract a ρ state from the high-energy average and impose a mass gap between the ρ and the spectrum in the UV (as explained in the paragraph above), we define $\vec{v}_\rho = \vec{v}_{\rm HE}(m_\rho^2, 1)$. Then we start by bounding $\tilde{g}_{2,0}$ via,

$$\vec{v}_1 + \tilde{g}_\rho^2 \vec{v}_\rho + \tilde{g}_{2,0}\vec{v}_2 + \left\langle \vec{v}_{\rm HE}(m^2, J)' \right\rangle = 0 \ . \tag{83}$$

We can both reabsorb the term $\tilde{g}_\rho^2\vec{v}_\rho$ in $\vec{v}_1$ if we want to assume a value for the coupling, otherwise we can add it to the high-energy average in the positivity condition Eq. (81). Then we use the allowed values of $\tilde{g}_{2,0}$ to bound $\tilde{g}_{2,1}$ with the equation

$$(1 + \tilde{g}_{2,0})\vec{v}_1 + \tilde{g}_\rho^2 \vec{v}_\rho + \tilde{g}_{2,1}\vec{v}_2 + \left\langle \vec{v}_{\rm HE}(m^2, J)' \right\rangle = 0 \ . \tag{84}$$

The procedure described above gives the plots shown in Figs. 3,4. As we vary $n_{\rm max}$ (the number of null constraints included), features of the exclusion plots vary.

B Minimum of μ_0

A measure that reproduces Eq. (41) is $d\tilde{\mu}(x) \equiv 4K_0(2\sqrt{x})dx$, where K_0 is the Bessel function of the second kind. The (Stieltjes) moment problem associated with Eq. (42) is indeterminate, in the sense that there exist multiple measures $d\mu(x)$ that give rise to this sequence of moments. For instance, asymptotically $K_0(\sqrt{x}) \to \ e^{-\sqrt{x}}$ has the same moments as $e^{-\sqrt{x}}(1 + w\cos\sqrt{x})$ for any w. For this reason, as far as μ_0 is concerned, we will treat the measure as unknown, and employ $d\tilde{\mu}(x)$ only for μ_n, $n \geq 1$.

We separate the integration domain $I \equiv [0, \infty[$ into the origin 0 and $I_L \equiv]0, L]$, where eventually we will take $L \to \infty$. Physically this corresponds to separate out the contribution of states with infinite mass but fixed spin $J \geq 1$. As noted already on page 12, heavy states contribute more to lower, rather than higher, moments; in what follows we will see that in this case infinitely heavy states ($x = 0$) contribute only to the first moment. Indeed, moments μ_n with $n \geq 1$ have no support on 0 and therefore $\mu_n = \mu_n^{I_L}$, where $\mu_n^{I_L} = \int_{I_L} x^n d\mu(x) > 0$ are also moments. Instead μ_0 has support in 0 (we call it $\mu_0^0 \geq 0$), and we can write $\mu_0 = \mu_0^0 + \mu_0^{I_L} \geq \mu_0^{I_L}$.

To prove that $\mu_0^{I_L}$ can be as small as 1, we define a sequence of functions built using positive powers of x, $f_n(x) = \sum_{k=1}^n a_k x^k$, so that

$$0 \leq 1 - f_n(x) \leq 1 \quad x \in I_L \, , \tag{85}$$

and such that f_n converges pointwise to unity $\lim_{n\to\infty} f_n(x) \to 1$ on I_L (for instance, a candidate for $f_n(x)$ is $1/x$ times the Taylor expansion of $1/x$ in L). Then, integrating over the measure,

$$0 \leq \int_{I_L} (1 - f_n(x)) d\mu(x) = \mu_0^{I_L} - \int_{I_L} f_n(x) d\tilde{\mu}(x) \, , \tag{86}$$

where the last integral can be written in terms of higher moments $\sum_{k=1}^n a_k \mu_k^{I_L}$ and provides an expression for the minimum of $\mu_0^{I_L}$ in terms of higher-moments only. Now we can use the dominated convergence theorem to write

$$\mu_0 \geq \mu_0^L \geq \lim_{n\to\infty} \int_{I_L} f_n(x) d\tilde{\mu}(x) = \int_{I_L} d\tilde{\mu}(x) \, . \tag{87}$$

Since x corresponds to (the square of) impact parameter, in theories with a mass gap, the measure $\mu(x)$ falls off exponentially at large x.

Since the measure decreases exponentially at large x, the limit $L \to \infty$ is regular, and the arguments still hold. For $L \to \infty$, the integral in Eq. (87) is known, and we find, $\mu_0 \geq \int d\tilde{\mu}(x) = 2$ which is equivalent to

$$\tilde{g}_{2,0} \leq \frac{1}{3} \,. \tag{88}$$

This implies that the kink must lie at $(\tilde{g}_{2,1}, \tilde{g}_{2,0}) = (4/3, 1/3)$, corresponding to a theory of a spin-1 particle at mass M, and higher-spin states at ∞, as in Eq. (25).

Higher kinks. These arguments can be exploited also to obtain the kink positions in the planes of other Wilson coefficients. In particular, *considering only spins* $J \geq 1$ (i.e. singling out the spin-0 contribution), one finds that the allowed regions of $(\tilde{g}_{n,1}, \tilde{g}_{n,0})$ have also kinks, see Fig. 5. Again, numerically the position of such kinks converges very slowly, but analytically we can prove that their position is at,

$$(\tilde{g}_{n,1}, \tilde{g}_{n,0}) = (2/3, 1/3) \,, \qquad \text{for } n > 2 \,. \tag{89}$$

Indeed, from Eq. (57), we have for $J \geq 1$ that $\frac{g_{n,1}}{g_{n,0}} \geq 2$. Moving along the extremal line $g_{n,1} = 2g_{n,0}$, corresponding to a spin-1 state, the kink resides at the largest value of $g_{n,0}/g_{1,0}$. To find this value one has to ask whether it is possible add to the spin-1 state a spectrum of states that enhance $g_{n,0}/g_{1,0}$ without affecting the ratio $g_{n,1}/g_{n,0}$ ($n = 3, 4, ...$). This latter condition implies, from Eq. (57), $\left\langle \frac{\mathcal{J}^2}{m^{2n}} \right\rangle \to 0$ and therefore also $\left\langle 1/m^{2n} \right\rangle \to 0$. This leads to

$$\tilde{g}_{n,0} = \frac{1}{1 + \frac{M^2}{g^2_{\rho\pi\pi}} \left\langle \frac{1}{m^2} \right\rangle} \,. \tag{90}$$

Following the same reasoning as before, one can obtain that $\frac{M^2}{g^2_{\rho\pi\pi}} \left\langle \frac{1}{\hat{m}^2} \right\rangle \geq 2$, and therefore $\tilde{g}_{n,0} \leq 1/3$, leading to Eq. (89). Also this kink lies at the extremum of the spin-1 line.

C The su-models

Let us consider the most general theory of a degenerate spectrum that contributes to the four-pion amplitude $\mathcal{M}(s,u)$ [6, 7]. This means that all states have equal mass m, and therefore the denominator of this amplitude is fixed to be $\mathcal{M}(s,u) \propto 1/((s-m^2)(u-m^2))$. If we further demand that Eq. (6a) and Eq. (6b) are satisfied for $k_{\text{min}} = 1$, we are led to

$$\mathcal{M}(s,u) = \frac{a_1 m^4 + a_2 m^2 (s+u) + a_3 su}{(s-m^2)(u-m^2)} \,, \tag{91}$$

where a_i are constants. The Adler's zero condition fixes $a_1 = 0$. Then, aside from a global multiplicative factor, the amplitude has only one free parameter. We can write it as

$$\mathcal{M}_1^{(su)}(s,u) = \frac{m^2(s+u) + \lambda su}{(s-m^2)(u-m^2)} \,, \tag{92}$$

where the possible values of λ are determined by unitarity. Indeed, imposing the positivity of the residues of Eq. (92), we obtain

$$-2 \leq \lambda \leq \frac{2\ln 2 - 1}{1 - \ln 2}\,. \tag{93}$$

In the limiting case $\lambda = -2$, the residues of all $J > 0$ states are zero, and we are left with the scalar amplitude Eq. (22). In the other limit,

$$\lambda = \frac{2\ln 2 - 1}{1 - \ln 2} \simeq 1.26\,, \tag{94}$$

the residue of the spin-0 state is zero, leading to an amplitude mediated by an infinite tower of states of spin $J > 0$ and mass m. We will refer to this latter case as the $J > 0$ *su*-model.

Expanding Eq. (92) for $s, u \ll m^2$, we can obtain the Wilson coefficients:

$$g_{n,0} = \frac{1}{m^{2n}}\,, \qquad g_{n,l} = \frac{2+\lambda}{m^{2n}} \quad (n, l > 0)\,. \tag{95}$$

For Eq. (94), the Wilson coefficients, normalized as in Eq. (30) for $M = m$, are given by

$$\tilde{g}_{n,0} = 1\,, \qquad \tilde{g}_{n,l} = \frac{1}{1 - \ln 2} \simeq 3.26 \quad (n > 1, l > 0)\,. \tag{96}$$

These are shown in Figs. 3–5 as a black dot.

We can proceed in a similar way to construct the most general amplitude of degenerate high-spin states, but now satisfying Eq. (6a) and Eq. (6b) for $k_{\min} = 2$. Imposing the Adler's zero condition, such amplitude is given by (up to a multiplicative constant factor)

$$\mathcal{M}_2^{(su)}(s,u) = \frac{m^2(s+u) + \lambda_1 su + \lambda_2(s^2+u^2)}{(s-m^2)(u-m^2)}\,. \tag{97}$$

The values of the constants λ_1 and λ_2 determine different theories according to:

- For $2 + \lambda_1 + 2\lambda_2 = 0$, the residues for $J > 1$ vanish and we have a theory with only a spin-0 and spin-1 state.
- For $\lambda_2 = (2+\lambda_1)\frac{6\ln 8 - 12}{25 - 36\ln 2}$, the residue for the spin-1 state is zero and we have a theory with $J = 0$ and $J > 1$.
- For $\lambda_2 = -1 + \frac{1 - \lambda_1(\ln 4 - 2)}{\ln 16 - 1}$, the residue for the spin-0 vanishes and we have a theory with spins $J > 0$.

Furthermore, for $\lambda_1 = -2$, $\lambda_2 = 0$, we recover the scalar amplitude Eq. (22), while for $\lambda_1 = \lambda_2 = -2/3$ we obtain the spin-1 amplitude Eq. (24). We are interested in the case where the amplitude Eq. (97) contains only $J > 1$ states which corresponds to taking

$$\lambda_1 = \frac{20\ln 2 - 13}{19 - 28\ln 2}\,, \quad \lambda_2 = \frac{6\ln 8 - 12}{19 - 28\ln 2}\,. \tag{98}$$

We will refer to this case as the $J > 1$ *su*-model.

The Wilson coefficients of Eq. (97) are given by

$$\begin{aligned} g_{1,0} &= \frac{1}{m^2}\,, \qquad g_{2,1} = \frac{2+\lambda_1}{m^4}\,, \qquad g_{n,0} = \frac{1+\lambda_2}{m^{2n}} \quad (n>1)\,, \\ g_{n,1} &= \frac{2+\lambda_1+\lambda_2}{m^{2n}}\,, \qquad g_{n,l} = \frac{2+\lambda_1+2\lambda_2}{m^{2n}} \quad (n,l>1)\,. \end{aligned} \tag{99}$$

For the case Eq. (98), normalizing the coefficients as given in Eq. (63), we have

$$\bar{g}_{n,0} = 1\,, \quad \bar{g}_{n,1} = \frac{18\ln 2 - 13}{10\ln 2 - 7} \simeq 7.64\,, \quad \bar{g}_{n,l} = \frac{1}{7-10\ln 2} \simeq 14.59 \quad (n>2, l>1)\,. \tag{100}$$

These are shown in Fig. 7 as a black dot.

C.1 Two-mass *su*-model

The *su*-models define part of the boundaries of the allowed regions of the Wilson coefficients. To see this, we can deform the above *su*-models by introducing an additional pole in the amplitude, i.e., $\mathcal{M}(s,u) \propto 1/((s-m^2)(u-m^2)(s-M^2)(u-M^2))$. In this case the most general amplitude can be written as

$$\mathcal{M}(s,u) = \mathcal{M}_1^{(su)}(m) + \alpha \mathcal{M}_1^{(su)}(M) + \frac{\beta\, s^2 u^2}{(s-m^2)(u-m^2)(s-M^2)(u-M^2)}\,, \tag{101}$$

that corresponds to two *su*-models (Eq. (92)) with mass m and M respectively, and an extra term. Apart from the masses, the amplitude has 4 parameters: the two λ of the *su*-models, α and β. We are interested in this model without the scalars. Removing the scalars in the two *su*-models fixes the λ's to the value Eq. (94). Removing the scalar from the last term of Eq. (101) corresponds to adding to the amplitude the term

$$\beta\left[f(m,M)\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right) + (M \leftrightarrow m)\right]\,, \tag{102}$$

where

$$f(m,M) = \frac{m^4M^2 + m^6\,(\ln 2 - 1) + m^2M^4 \ln\frac{M^2}{m^2+M^2}}{\left(m^2-M^2\right)^2}\,. \tag{103}$$

Requiring the positivity of the spectral function for the $J>0$ states in Eq. (101) leads to $\beta \geq 0$.

Eq. (101) with Eq. (102) leads to

$$\frac{\tilde{g}_{2,1}}{\tilde{g}_{2,0}} = \frac{3.26\left(\frac{1}{m^4}+\frac{a}{M^4}\right)}{\left(\frac{1}{m^4}+\frac{a}{M^4}\right) - \beta\left(\frac{f(m,M)}{m^4}+\frac{f(M,m)}{M^4}\right)}\,. \tag{104}$$

Since $\beta\left(\frac{f(m,M)}{m^4}+\frac{f(M,m)}{M^4}\right)$ is a positive-definite function, we see that the ratio $\tilde{g}_{2,1}/\tilde{g}_{2,0}$ is bounded from below by the *su*-model.

D The Lovelace-Shapiro amplitude

The Lovelace-Shapiro (LS) amplitude for the scattering of four pions is defined as [20, 21]

$$\mathcal{M}^{(\mathrm{LS})}(s,u) = \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(u))}{\Gamma(1-\alpha(s)-\alpha(u))}\,, \tag{105}$$

where $\alpha(s) = \alpha_0 + \alpha' s$ is referred as the Regge trajectory. We will fix the values of α_0 and α' by requiring that Eq. (106) satisfies the Adler zero condition, $\mathcal{M}^{(\mathrm{LS})}(s,u) \to 0$ for $s, u \to 0$, and that the first pole of Eq. (106) occurs for $s = m_\rho^2$. These two conditions lead to $\alpha_0 = 1/2$ and $\alpha' = 1/(2m_\rho^2)$ [55] and then we can write

$$\mathcal{M}^{(\mathrm{LS})}(s,u) = \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2m_\rho^2}\right)\Gamma\left(\frac{1}{2} - \frac{u}{2m_\rho^2}\right)}{\Gamma\left(\frac{t}{2m_\rho^2}\right)}\,. \tag{106}$$

By looking at the poles of Eq. (106), one can see that the LS amplitude corresponds to a theory of higher-spin states with masses

$$m_n^2 = m_\rho^2(2n+1)\,, \quad n = 0, 1, 2, \ldots\,. \tag{107}$$

For a given n, there are at most $n+1$ states with spin $J = 0, 1, ..., n+1$. Furthermore, Eq. (106) satisfies the condition Eq. (6a) and Eq. (6b) with $k_{\min} = 1$.

The first Wilson coefficients arising from Eq. (106) in a low-energy expansion are given by

$$g_{1,0} = \frac{\pi}{2m_\rho^2}\,, \qquad g_{2,0} = \frac{1}{2}g_{2,1} = \frac{\pi \ln 2}{2m_\rho^4}\,, \qquad g_{3,0} = \frac{\pi^3 + 12\pi \ln^2 2}{48 m_\rho^6}\,, \tag{108}$$

$$g_{3,1} = \frac{3\pi \ln^2 2}{4m_\rho^6}\,, \qquad g_{4,0} = \frac{\pi\left(\pi^2 \ln 2 + 4\ln^3 2 + 6\zeta(3)\right)}{48 m_\rho^8}\,, \tag{109}$$

$$g_{4,1} = \frac{\pi\left(\pi^2 \ln 2 + 16\ln^3 2 + 3\zeta(3)\right)}{48 m_\rho^8}\,, \qquad g_{4,2} = \frac{\pi\left(4\ln^3 2 - \zeta(3)\right)}{8 m_\rho^8}\,, \tag{110}$$

with ζ the Riemann zeta function. For the normalized Wilson coefficients defined in Eq. (30) we have, taking $M = m_\rho$,

$$\tilde{g}_{2,0} \simeq 0.69\,,\ \tilde{g}_{2,1} \simeq 1.39\,,\ \tilde{g}_{3,0} \simeq 0.65\,,\ \tilde{g}_{3,1} \simeq 0.72\,,\ \tilde{g}_{4,0} \simeq 0.64\,,\ \tilde{g}_{4,1} \simeq 0.66\,,\ \tilde{g}_{4,2} \simeq 0.03\,, \tag{111}$$

while for the normalized coefficients in Eq. (63), we have

$$\bar{g}_{2,1} \simeq 2\,,\ \bar{g}_{3,0} \simeq 0.94\,,\ \bar{g}_{3,1} \simeq 1.04\,,\ \bar{g}_{4,0} \simeq 0.92\,,\ \bar{g}_{4,1} \simeq 0.95\,,\ \bar{g}_{4,2} \simeq 0.05\,. \tag{112}$$

Since a theory of scalars provides a consistent UV completion of the pion amplitude $\mathcal{M}$, satisfying Eq. (6a) and Eq. (6b) with $k_{\min} = 1$, we can find a new consistent amplitude by subtracting the scalars from Eq. (106). This leads to

$$\mathcal{M}^{(\mathrm{LS})}_{J>0}(s,u) = \mathcal{M}^{(\mathrm{LS})}(s,u) - \sum_{n=0}^{\infty}\left[\frac{m_n^2}{s-m_n^2}\kappa^{\mathrm{LS}}_{s,0} + (s \leftrightarrow u)\right]\,, \tag{113}$$

with

$$\kappa_{s,J}^{\rm LS} = \frac{2J+1}{2}\int_{-1}^{1} dx\ P_J(x) \operatorname*{Res}_{s=m_n^2}\left[\mathcal{M}^{(\rm LS)}(s,u(x))\right]\,, \tag{114}$$

where $u(x) = -s(1-x)/2$. From Eq. (113), we obtain by expanding at small s, u:

$$\tilde{g}_{2,0} \simeq 0.55\,,\ \tilde{g}_{2,1} \simeq 2.05\,,\ \tilde{g}_{3,0} \simeq 0.49\,,\ \tilde{g}_{3,1} \simeq 1.07\,,\ \tilde{g}_{4,0} \simeq 0.48\,,\ \tilde{g}_{4,1} \simeq 0.97\,,\ \tilde{g}_{4,2} \simeq 0.05\,. \tag{115}$$

In a similar way, we can also remove the (infinite) spin-1 states of Eq. (113) to obtain an amplitude that still satisfies the Froissart-Martin condition, Eq. (6a) and Eq. (6b) with $k_{\rm min} = 2$:

$$\mathcal{M}_{J>1}^{(\rm LS)}(s,u) = \mathcal{M}_{J>0}^{(\rm LS)}(s,u) - \sum_{n=0}^{\infty}\left[\frac{m_n^2+2u}{s-m_n^2}\kappa_{s,1}^{\rm LS} + (s\leftrightarrow u)\right]\,. \tag{116}$$

From Eq. (116) we obtain

$$\bar{g}_{2,1} \simeq 0.97\,,\ \bar{g}_{3,0} \simeq 0.71\,,\ \bar{g}_{3,1} \simeq 5.99\,,\ \bar{g}_{4,0} \simeq 0.60\,,\ \bar{g}_{4,1} \simeq 4.18\,,\ \bar{g}_{4,2} \simeq 8.15\,. \tag{117}$$

E The Coon amplitude

The Lovelace-Shapiro amplitude presented in Appendix D can be generalized to a larger class of amplitudes depending on an additional parameter q. This is the so-called Coon amplitude, which was first proposed in [22][9]:

$$\mathcal{M}_q(s,u) = C(\sigma,\tau,q)\prod_{n=0}^{\infty}\frac{(1-q^{n+1})\,(\sigma\tau-q^{n+1})}{(\sigma-q^{n+1})\,(\tau-q^{n+1})}\,, \tag{118}$$

where $\sigma = 1+(q-1)(\alpha_0+\alpha' s)$ and $\tau = 1+(q-1)(\alpha_0+\alpha' u)$. As explained in Appendix D, we take $\alpha_0 = 1/2$ and $\alpha' = 1/(2m_\rho^2)$. The parameter q takes values between 0 and 1, and in the limit $q\to 1$ we recover the LS amplitude Eq. (106). There is some freedom in the choice of the prefactor C, as long as it satisfies $\lim_{q\to 1} C(\sigma,\tau,q) = 1$.

The Coon amplitude has an infinite number of simple poles at

$$s_n = m_\rho^2\frac{1+q-2q^{n+1}}{1-q}\,, \qquad n = 0,1,2,\ldots\,. \tag{119}$$

The corresponding residues are

$$\operatorname*{Res}_{s=s_n}\mathcal{M}_q(s,u) = C(\sigma_n,\tau,q)\,\frac{2q^{n+1}}{1-q}\,\frac{\tau-1}{\tau^{n+1}}\,m_\rho^2\prod_{l=0}^{n-1}\frac{(\tau-q^{l-n})}{(1-q^{l-n})}\,, \tag{120}$$

[9]The idea of the Coon amplitude goes back to an earlier work by Coon [56], where he defined a generalization of the Veneziano amplitude which was slightly different from Eq. (118). Shortly after that he proposed the Coon version of Lovelace-Shapiro amplitude together with Sukhatme and Tran Thanh Van in [22].

where $\sigma_n = \sigma(s = s_n)$. It is important to remark that the spectrum has an *accumulation point* at $s_* = \lim_{n\to\infty} s_n = m_\rho^2 \frac{1+q}{1-q}$. In the limit $q \to 1$, the accumulation point is located at infinity and we recover the evenly-spaced spectrum of the LS amplitude.

It is customary to fix the prefactor $C(\sigma, \tau, q)$ with the further assumption that the residues of the Coon amplitude are polynomials in u, since it is believed that non-polynomial residues lead to problems with the locality of the theory. The prefactor is in this case set to

$$C(\sigma, \tau, q) = q^{\frac{\ln\sigma \ln\tau}{\ln q \ln q}} \ , \tag{121}$$

which reduces to $C(\sigma_n, \tau, q) = \tau^{n+1}$ at the s_n pole. This term cancels the factor τ^{n+1} in the denominator of Eq. (120) and ensures that the residues are polynomials. In this case, we have that for any n, there are $n+1$ states with spin $J = 0, 1, ..., n+1$, as in the LS amplitude.

Using the prefactor Eq. (121) makes however the Coon amplitude Eq. (118) non-meromorphic. In addition to the simple poles, there is a branch cut starting at the accumulation point s_*. Although the physical meaning of this kind of singularities is unclear, amplitudes with branch cuts can still obey the requirements of unitarity, crossing symmetry and Regge boundedness, so it is interesting to include them in our study.

Regarding the high-energy behavior, the amplitude with prefactor Eq. (121) grows at fixed u like $\mathcal{M}_q(s, u) \sim f(u)\ s^{\ln\tau/\ln q}$. For negative u, $\ln\tau/\ln q < 0.5$, so the amplitude obeys Eq. (6a) for $k_{\min} = 1$. At fixed t, the amplitude grows like $\mathcal{M}_q(s, u) \sim s^{\ln((1-q)\alpha' s)/\ln q}$. Since $\ln((1-q)\alpha' s)/\ln q < 0.5$, it also obeys Eq. (6b) for $k_{\min} = 1$.

The last point to address is the unitarity of the Coon amplitude. We have found that the $J = 0$ states have negative residues for all $n > 0$, making the amplitude non-unitary. This result is in agreement with the early study [57][10], which pointed out the presence of ghosts. On the other hand, we have found that the rest of the J states have positive residues inside the scope of our numerical searches. As for the branch cut discontinuity, it can also be expanded in partial waves according to Eq. (8). We have obtained that the spectral density $\rho_J(s)$ is positive for all J except from $J = 0$. Due to this problems, we will not discuss this case further.

On the other hand, relaxing the assumption of polynomial residues, we can simply take $C(\sigma, \tau, q) = 1$, and the amplitude becomes meromorphic (there is no branch cut). We now have an infinite number of spins present for each n. At large s, taking both fixed u and t, we have $\mathcal{M}_q(s, u) \sim constant$. Therefore it satisfies both Eq. (6a) and Eq. (6b) for $k_{\min} = 1$. For $C(\sigma, \tau, q) = 1$ we have also found that the only negative residue corresponds to the $J = 0$, $n = 1$ state only in the small region $0.94 \lesssim q < 1$.

The contribution of the Coon amplitude with $C = 1$ is shown in Fig. 3 for $q \in [0, 1]$. We also show the case in which all scalars are removed from the spectrum. In the limit $q \to 1$ the Coon amplitude approaches the $J > 0$ Lovelace-Shapiro model, while for $q \to 1$, the Coon amplitude is dominated by the $n = 0$ level (the $n > 0$ levels decouple), giving the $J > 0$ *su*-model.

[10] Some recent works [58–60] on the Veneziano version of the Coon amplitude have shown that it is unitary in $D = 4$ dimensions. Our result agrees with the results of Coon and Yu [57], who considered the Lovelace-Shapiro version of the Coon amplitude.

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Paper 2: Bootstrapping the Chiral Anomaly at Large N_c

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Bootstrapping the Chiral Anomaly at Large N_c

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Abstract

The bootstrap approach (demanding consistency conditions to scattering amplitudes) has shown to be quite powerful to tightly constrain gauge theories at large N_c. We extend previous analysis to scattering amplitudes involving pions and external gauge bosons. These amplitudes allow us to access the chiral anomaly and connect low-energy physical quantities to UV properties of the theory. In particular, we are able to obtain an analytic bound on the chiral anomaly coefficient as a function of the pion dipole polarizabilities. This bound can be useful for holographic models whose dual UV completions are not known, and provide a consistency condition to lattice simulations.

1 Introduction

In the large-N_c limit strongly-coupled gauge theories find a dual description as weakly-coupled theories of mesons and glueballs [1, 2]. This has allowed to obtain valuable information on QCD in the strongly-coupled regime. In spite of this, the fact that the theory contains infinite number of states with all spins has made difficult to get quantitative predictions on physical quantities.

Recently [3, 4], it has been shown that a better quantitative understanding of large-N_c theories can be achieved by requiring consistency conditions, such as unitarity and causality, to the scattering amplitudes [5–13]. Thanks to the simple analytical structure of the scattering amplitudes at large-N_c, positivity constraints have provided precise and strong bounds on low-energy quantities, such as Wilson coefficients of the chiral Lagrangian as well as meson couplings. High-spin states contributions to these low-energy physical quantities have shown to be small, giving an understanding of the phenomenological successes of Vector Meson Dominance and holographic QCD [4].

We extend these previous analysis to scattering amplitudes of Goldstones (π and η) interacting with external gauge bosons (W). The main purpose is to access information related to the chiral anomaly and other physical quantities such as pion polarizabilities. In particular, we will analyze the amplitudes $W\pi \to W\pi$ and $W\pi \to \eta\pi$, this later being proportional, at low-energies, to the chiral anomaly. We will understand the meson exchange selection rules that apply to these amplitudes and that will help us to derive the sign of their on-shell residues. By demanding a good high-energy behaviour of the scattering amplitudes, we will derive dispersion relations that will allow us to obtain interesting bounds on low-energy quantities.[1]

Our main result will be the derivation of an upper bound on the coefficient of the chiral anomaly κ:

$$\frac{\kappa}{\sqrt{\mathcal{P}/F_\pi^2}} \leq \sqrt{\frac{1}{2}}\,, \tag{1}$$

where F_π is the pion decay constant, and $\mathcal{P}$, to be specified later, is related to pion polarizabilities. This bound provides a non-trivial connection between low-energy physical quantities and UV properties of the theories.[2] On the other hand, for phenomenological models for QCD, such as a holographic or NJL models, the above upper bound can provide a valuable consistency check, as well as for lattice simulations.

The analysis described here can also be useful to derive constraints on a large number of low-energy physical constants, specially to those related to the electromagnetic properties of pions and other mesons.

The paper is organized as follows. In Sec. 2 we discuss the properties of amplitudes for pions and gauge bosons and derive the dispersion relations and null constraints. In Sec. 3, we analytically derive an upper bound on the chiral anomaly coefficient. We conclude in Sec. 4. The appendices contain details about the classification of mesons and the signs of residues (A), pion polarizabilities (B), numerical bounds on Wilson coefficients (C), and su-models (D).

[1] Previous studies in the context of QCD can be found in [14–22].

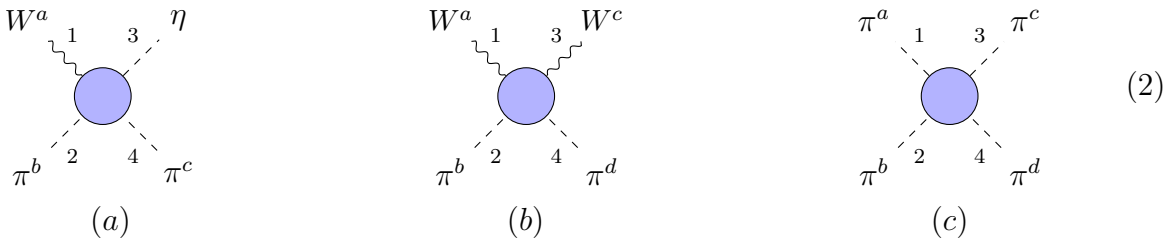
[2] Bounds on the a-anomaly from scattering amplitudes requirements have been derived in [23, 24].

2 Amplitudes for pions and external gauge bosons in the large-N_c limit

We are interested in studying $SU(N_c)$ gauge theories in the large-N_c limit. We will consider that the theory also contains N_f massless quarks (q_L, q_R) in the fundamental representation of $SU(N_c)$, with the following pattern of chiral symmetry breaking: $U(N_f)_L \times U(N_f)_R \to SU(N_f) \times U(1)$. The massless Goldstone bosons associated with the breaking of this global symmetry are the pions (π^a with $a = 1, ..., N_f^2 - 1$ in the adjoint representation of $SU(N_f)$) and the η (singlet). The chiral axial symmetry $U(1)_A$ is anomalous but in the large-N_c limit the corresponding Goldstone, the η, remains massless.

As usual, we will introduce the coupling to external gauge bosons by gauging the global $SU(N_f) \times U(1)$ symmetry. These gauge bosons will be considered (non-propagating) external fields, sitting in the adjoint and singlet representations.

For simplicity, we will concentrate in this paper in the $N_f = 2$ case, but the arguments we make can be extended straightforwardly to a general N_f. For $N_f = 2$ the pions and gauge bosons are isospin triplets, π^a and W^a, and isospin singlets, the η and B.[3] We are interested in the following amplitudes:



(2)

The amplitude (a) is important as, at low-energies, becomes proportional to the chiral anomaly coefficient that we want to bootstrap. For this purpose, as we will see, we will also need the amplitudes (b) and (c). The amplitude (c) was already discussed in detail in [3, 4]. In this section we will then only discuss the amplitudes (b) and (a) in the large-N_c limit for the different polarizations of the gauge bosons $\lambda_W = \pm 1$. In the following, we will derive the relevant dispersion relations necessary to obtain sum rules for the chiral anomaly coefficient and other Wilson coefficients, as well as null constraints necessary for obtaining bounds.

2.1 Exchanged meson states

In the large-N_c limit, a $SU(N_c)$ gauge theory reduces to a theory of weakly-coupled mesons, whose trilinear couplings scale as $\sim 1/\sqrt{N_c}$ [1, 2]. Amplitudes are then mediated by tree-level meson exchange. In the amplitudes (2), the exchanged mesons are colorless $q\bar{q}$ states that, as in the quark model, can be classified according to their quantum numbers: Isospin (I), $G-$parity (G), parity (P) and spin (J). The relation between these quantum numbers is discussed in Appendix A.[4] We have six types of mesons as shown in Table 1. It is important to say that in

[3] We normalize the gauge bosons as $W^a T^a + B\,\mathbb{I}/2$, where $\mathrm{Tr}\,[T^a T^b] = 1/2$, and the gauge coupling $g = 1$.

[4] Fixing I, G and P, the spin J is fixed to either be even or odd, as explicitly shown in Table 1.

	Isospin $I=0$	Isospin $I=1$		n
$G=+1$	$J^+_{\rm odd}\ (f_J)$	$J^+_{\rm odd}\ (a_J)$	$G=-1$	1
	$J^+_{\rm even}\ (f_J)$	$J^+_{\rm even}\ (a_J)$		2
	$J^-_{\rm even}\ (\eta_J)$	$J^-_{\rm even}\ (\pi_J)$		3
$G=-1$	$J^+_{\rm odd}\ (h_J)$	$J^+_{\rm odd}\ (b_J)$	$G=+1$	4
	$J^-_{\rm odd}\ (\omega_J)$	$J^-_{\rm odd}\ (\rho_J)$		5
	$J^-_{\rm even}\ (\omega_J)$	$J^-_{\rm even}\ (\rho_J)$		6

Table 1: *List of $q\bar{q}$ states classified in terms of their Isospin, $G-$parity and J^P. We also provide the names as they are generally referred to in QCD [25]. In green we highlight the states that enter in the amplitude $W\pi \to \eta\pi$ which is related to the chiral anomaly.*

the large-N_c limit, each $I=1$ state (left column) has an $I=0$ associated state (the one next in the right column) of equal mass and related couplings due to the underlying $U(2)$ symmetry [1, 2].

2.2 The $\pi W \to \pi W$ Amplitudes

Let us start with the amplitude (b). For $N_f=2$, the most general structure we can write for this amplitude takes the form

$$\mathcal{M}(W^a,\pi^b,W^c,\pi^d) = A_t(s,u)\delta_t + \tilde{A}(s,u)\delta_s + \tilde{A}(u,s)\delta_u\,, \tag{3}$$

where $s=(p_a+p_b)^2$, $t=(p_a-p_c)^2$, $u=(p_a-p_d)^2$, and $\delta_s=\delta_{ab}\delta_{cd}$, $\delta_t=\delta_{ac}\delta_{bd}$, $\delta_u=\delta_{ad}\delta_{cb}$. We have demanded the amplitude to be invariant under the exchange of the two pions that also requires $A_t(s,u)$ to be $s\leftrightarrow u$ symmetric.

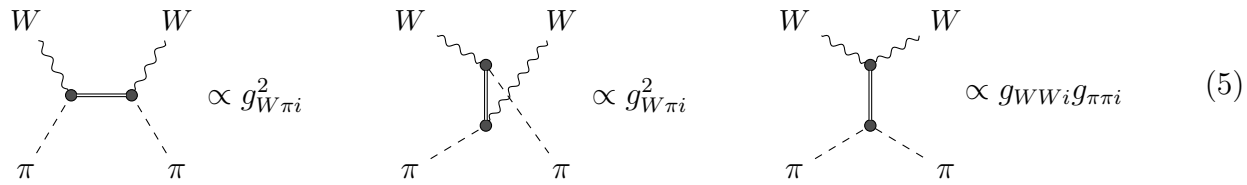
In the large-N_c limit this amplitude is mediated by tree-level exchanges of mesons, either in the $s-$channel, $t-$channel and/or $u-$channel. Since $q\bar{q}$ states cannot have $I=2$, it will be important to focus in $I=2$ eigen-amplitudes. These are given by

$$M_s^{I=2} = A_t(s,u)+\tilde{A}(u,s)\,, \qquad M_t^{I=2} = \tilde{A}(s,u)+\tilde{A}(u,s)\,, \qquad M_u^{I=2} = A_t(s,u)+\tilde{A}(s,u)\,, \tag{4}$$

respectively for the s, t and $u-$channel. These amplitudes cannot have mesons exchanging in the s, t and $u-$channel respectively.

In the amplitude Eq. (3) the $s-$channel and $u-$channel exchanged meson i couples to $W\pi$,

while the $t-$channel exchanged meson couples to WW and $W\pi$:



(5)

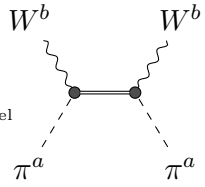
Since the couplings g_{WWi} do not appear in the amplitude $W\pi\eta\pi$ (the one related to the chiral anomaly), we will restrict to situations in which the $t-$channel meson exchange is not present such that we will not have to deal with these couplings g_{WWi} any longer. This can be achieved by either taking $t-$fixed or working with $M_t^{I=2}$. This leaves only 3 possibilities to consider:

1. $M_t^{I=2}$ at $u-$fixed,
2. $M_t^{I=2}$ at $t-$fixed,
3. $M_u^{I=2}$ at $t-$fixed.

Notice that we have not included $M_s^{I=2}$ since it contains the same information as $M_u^{I=2}$ as far as our positivity arguments are concerned.

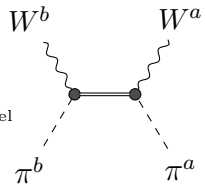
It is also of central importance to understand the sign of the on-shell couplings of the meson being exchanged, or, equivalently, the sign of the residues at the mass poles. Defining these as $R_{I=1,0}$ for the $I = 1, 0$ mesons respectively, we have

$$\mathcal{M}(\delta_t \neq 0, \delta_s = \delta_u = 0) = A_t(s,u) \underset{\substack{s-\text{channel}\\ \text{onshell}}}{\longrightarrow}$$



$$= -\sum_i \frac{(R_{I=1})_i}{s-m_i^2},$$

$$\mathcal{M}(\delta_s \neq 0, \delta_t = \delta_u = 0) = \tilde{A}(s,u) \underset{\substack{s-\text{channel}\\ \text{onshell}}}{\longrightarrow}$$



$$= -\sum_i \frac{(R_{I=0})_i}{s-m_i^2},$$

where the sum over i runs over all the possible intermediate states. Knowing that the amplitudes Eq. (4) cannot have poles at the s, t and $u-$channel respectively, we can derive the relative sign of the residues in the s and $u-$channel, which we show in Table 2.

For $I = 1\,(0)$ mesons, we have 3 types of states, $n = 1, 2, 3\,(4, 5, 6)$, given in Table 1. The signs of their residues depend on the helicity of the external vector states. We have two independent possibilities, either an elastic process ($W^+\pi \to W^+\pi$) which we will refer to as the $+-$ amplitude, or inelastic ($W^+\pi \to W^-\pi$) which we will refer to as the $++$ amplitude. Following the discussion of Appendix A, one finds that the signs of the residues are given by

$$\begin{aligned} R^{+-}_{I=0} &= +g_4^2 + g_5^2 + g_6^2 \qquad & R^{+-}_{I=1} &= +g_1^2 + g_2^2 + g_3^2\,, \\ R^{++}_{I=0} &= -g_4^2 + g_5^2 - g_6^2 \qquad & R^{++}_{I=1} &= -g_1^2 + g_2^2 - g_3^2\,, \end{aligned} \tag{6}$$

Meson channel	$-A_t(s,u)$	$-\tilde{A}(s,u)$	$-\tilde{A}(u,s)$
$s-$channel	$R_{I=1}$	$R_{I=0}$	$-R_{I=1}$
$u-$channel	$R_{I=1}$	$-R_{I=1}$	$R_{I=0}$

Table 2: *Residues in the s and $u-$channel of the amplitudes A_t and $\tilde{A}$.*

where we have defined

$$g_n^2 \equiv g_{\pi W,n}^2 > 0 \ , \quad n = 1, ..., 6 \, , \tag{7}$$

and n labels the different type of mesons as defined in Table 1.

The helicity configuration forces a particular kinematic pre-factor in the amplitude which, in the spinor helicity formalism, can be written as

$$\mathcal{M}^{+-} \propto \langle 12\rangle^2 [23]^2 \propto su \quad , \quad \mathcal{M}^{++} \propto [13]^2 \propto s+u \, , \tag{8}$$

where the numbers label the states as in Fig. 2 (b).

At energies below the mass M of the lightest massive meson, we can expand the amplitudes in powers of $s/M, u/M \to 0$. For the elastic scattering amplitude we have

$$\begin{aligned} M_t^{I=2}(s,u)|_{+-} &= 2 + su\Big(\sum_{m=0}^{\infty}\sum_{l=0}^{[m/2]} g_{m,l}\, s^{\{m-l}u^{l\}}\Big) \qquad (9)\\ &= 2 + su\big(g_{0,0} + g_{1,0}\,(s+u) + g_{2,0}\,(s^2+u^2) + 2g_{2,1}su + ...\big) \ , \\ M_u^{I=2}(s,u)|_{+-} &= -1 - su\Big(\sum_{m=0}^{\infty}\sum_{l=0}^{[m/2]} h^s_{m,l}\, s^{\{m-l}u^{l\}} + \sum_{m=0}^{\infty}\sum_{l=0}^{[m/2]} h^a_{m,l}\, s^{[m-l}u^{l]}\Big) \qquad (10)\\ &= -1 - su\big(h^s_{0,0} + h^s_{1,0}\,(s+u) + h^a_{1,0}\,(s-u) + h^s_{2,0}\,(s^2+u^2) + ...\big) \ , \end{aligned}$$

where Eq. (9) is fully $s \leftrightarrow u$ symmetric, and in Eq. (10) we have separated the symmetric and the antisymmetric part. The first terms of the expansion correspond to the pion exchange whose gauge coupling we normalize to 1. On the other hand, for the inelastic scattering amplitude we have

$$\begin{aligned} M_t^{I=2}(s,u)|_{++} &= (s+u)\Big(\sum_{m=0}^{\infty}\sum_{l=0}^{[m/2]} f_{m,l}\, s^{\{m-l}u^{l\}}\Big) \qquad (11)\\ &= (s+u)\big(f_{0,0} + f_{1,0}\,(s+u) + f_{2,0}\,(s^2+u^2) + f_{2,1}su + ...\big) \ . \end{aligned}$$

We do not need to give the expression for the low-energy expansion of $M_u^{I=2}(s,u)|_{++}$ since, as we will see, this amplitude does not provide any extra information for bounding the chiral anomaly. The low-energy constants $g_{m,l}$, $h^s_{m,l}$, $h^a_{m,l}$ and $f_{m,l}$, referred here as Wilson coefficients, are the low-energy parameters that we want to constrain.

2.3 Dispersion Relations and Null Constraints

In order to derive sum rules and null constraints from our dispersion relations, we will assume that for all $k \geq k_{\min}$, the amplitudes behave as follows at high energies ($k_{\min}$ will be indicated for the various processes):

$$\lim_{|s|\to\infty} \frac{M_t^{I=2}(s,u)}{s^k} \to 0\,, \qquad (12a) \qquad\qquad \lim_{|s|\to\infty} \frac{M_t^{I=2}(s,-t-s)}{s^k} \to 0\,, \quad (12b)$$

$$\lim_{|s|\to\infty} \frac{M_u^{I=2}(s,-s-t)}{s^k} \to 0\,. \quad (12c)$$

The amplitudes can be expanded in partial waves in the physical regions. For the elastic process we have

$$\begin{aligned} \mathrm{Im}M_{u,t}^{I=2}(s,u)\big|_{+-} &= \sum_i (2J+1)\rho_J^{+-,\pm}(s) d^J_{1,1}\big(\cos\theta_s\big)\,, \\ (2J+1)\rho_J^{+-,\pm}(s) &= \pi \sum_i \left(R^{+-}_{I=0} \pm R^{+-}_{I=1}\right)_i m_i^2 \delta(s-m_i^2)\delta_{JJ_i}\,, \end{aligned} \qquad (13)$$

where the u,t subscript refers to the $\pm$ spectral density and $\cos\theta_s = (t-u)/s$. For the inelastic amplitude similarly we have

$$\begin{aligned} \mathrm{Im}M_{u,t}^{I=2}(s,u)\big|_{++} &= \sum_i (2J+1)\rho_J^{++,\pm}(s) d^J_{1,-1}\big(\cos\theta_s\big)\,, \\ (2J+1)\rho_J^{++,\pm}(s) &= \pi \sum_i \left(R^{++}_{I=0} \pm R^{++}_{I=1}\right)_i m_i^2 \delta(s-m_i^2)\delta_{JJ_i}\,. \end{aligned} \qquad (14)$$

2.3.1 Elastic Process

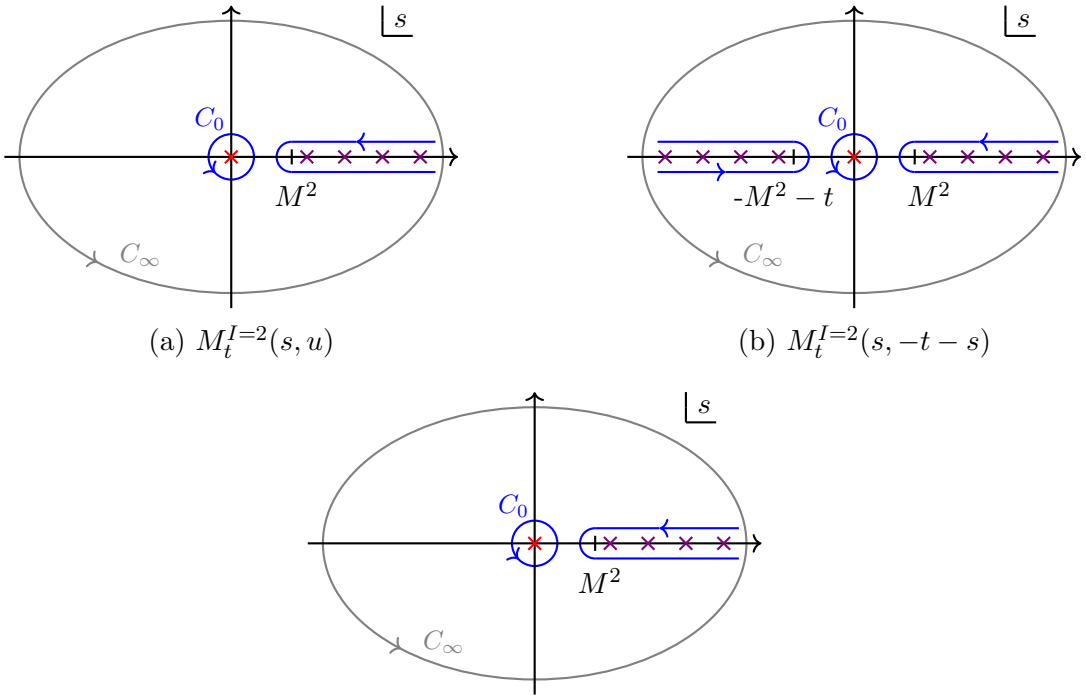
For the elastic process the dispersion relations that we will consider involve $M_t^{I=2}$ at fixed $u<0$ and fixed $t<0$, and $M_u^{I=2}$ at fixed $t<0$ (all in the $s-$plane). The analytic structure of these amplitudes and the contours we choose for the dispersion relations can be found in Fig. 1. For the elastic process we define the high-energy average along the lines of [11] as

$$\big\langle(...)\big\rangle^{\pm}_{+-} \equiv \frac{1}{\pi}\sum_i (2J+1)\int_{M^2}^{\infty}\frac{dm^2}{m^2}\rho_J^{+-,\pm}(m^2)(...)\,. \qquad (15)$$

- $M_t^{I=2}$ $u-$**fixed:**

The integral of $M_t^{I=2}(s,u)/s^{k+1}$ along the contour C_∞ of Fig. 1a vanishes for $k \geq k_{\min}$, due to Eq. (12a). Because of the amplitude's analyticity, we can deform C_∞ into the blue contour in Fig. 1a giving

$$\oint_{C_0} ds' \, \frac{M_t^{I=2}(s',u)}{s'^{k+1}} = 2i\int_{M^2}^{\infty} ds' \, \frac{\mathrm{Im}M_t^{I=2}(s',u)}{s'^{k+1}}\,. \qquad (16)$$



(a) $M_t^{I=2}(s,u)$ (b) $M_t^{I=2}(s,-t-s)$

(c) $M_u^{I=2}(s,-s-t)$

Figure 1: *Analytic structure of $M_t^{I=2}$ at fixed $u<0$ (a), fixed $t<0$ (b) and of $M_u^{I=2}$ at fixed $t<0$ (c). We denote by C_0, C_∞ (to be taken at $|s|\to\infty$) and the discontinuity along the real axis the relevant contours of integration used for the dispersion relations.*

Assuming $k_{\min}=1$, we are able to derive, inserting Eq. (13) into Eq. (16) and expanding for $u\to 0$, the following system of relations:

$$\begin{aligned}
(k=1)\quad g_{0,0}u+g_{1,0}u^2+\ldots &= \left\langle \frac{(-1)^J\mathcal{J}^2}{2m^4}u+\frac{(-1)^J\mathcal{J}^4}{6m^6}u^2+\ldots\right\rangle^-_{+-},\\
(k=2)\quad g_{1,0}u+g_{2,1}u^2+\ldots &= \left\langle \frac{(-1)^J\mathcal{J}^2}{2m^6}u+\frac{(-1)^J\mathcal{J}^4}{6m^8}u^2+\ldots\right\rangle^-_{+-},\\
&\vdots
\end{aligned} \tag{17}$$

where $\mathcal{J}^2=J(J+1)$, $\mathcal{J}^4=(J-1)J(J+1)(J+2)$, etc... . The system above allows us to relate the low-energy Wilson coefficients to the high-energy UV-averages:

$$g_{0,0}=\left\langle\frac{(-1)^J\mathcal{J}^2}{2m^4}\right\rangle^-_{+-},\quad g_{1,0}=\left\langle\frac{(-1)^J\mathcal{J}^2}{2m^6}\right\rangle^-_{+-},\ \ldots\ . \tag{18}$$

Furthermore one can notice that the system is over-constrained and therefore it is possible to find a set of non trivial null constraints, the first of which starts at $\mathcal{O}(1/m^6)$:

$$\left\langle\frac{(-1)^J\mathcal{J}^2}{2m^6}\right\rangle^-_{+-}=\left\langle\frac{(-1)^J\mathcal{J}^4}{6m^6}\right\rangle^-_{+-}. \tag{19}$$

Nevertheless, as we will show later, the null constraints needed to bound the chiral anomaly are only those at $\mathcal{O}(1/m^4)$.

- $M_t^{I=2}$ $t-$**fixed:**

We can repeat the arguments explained above for $M_t^{I=2}(s,-s-t)$ at fixed $t<0$. The integral identity one gets using Cauchy's theorem, following Fig. 1b, is

$$\frac{1}{2i}\oint_{C_0} ds' \, \frac{M_t^{I=2}(s',-t-s')}{s'^{k+1}} = \int_{M^2}^{\infty} ds' \, \frac{\text{Im}M_t^{I=2}(s',-t-s')}{s'^{k+1}} + (-1)^k \int_{M^2}^{\infty} ds' \, \frac{\text{Im}M_t^{I=2}(s',-t-s')}{(s'+t)^{k+1}} \,. \tag{20}$$

This dispersion relation leads to a system of equations [5]

$$\begin{aligned} (k=2) \quad -g_{0,0}+g_{1,0}t+\ldots \;&=\; \left\langle \frac{2}{m^4} \right\rangle_{+-}^{-} + \left\langle \frac{2\mathcal{J}^2-5}{m^6} \right\rangle_{+-}^{-} t + \ldots \,, \\ &\vdots \end{aligned} \tag{21}$$

that is not over-constrained. Therefore, by themselves, these equations do not lead to any null constraint. Neverthless, we can combine it with the $u-$fixed system in Eq. (17) to get a new set of null constraints. We will be interested in the one at $\mathcal{O}(1/m^4)$ which we will need to constrain the anomaly. This is given by

$$\left\langle \frac{(-1)^J\mathcal{J}^2+4}{2m^4} \right\rangle_{+-}^{-} = 0 \,. \tag{22}$$

Notice that the averages are taken with $R_{I=0}^{+-}-R_{I=1}^{+-}$, and therefore positivity is not guaranteed.

- $M_u^{I=2}$ $t-$**fixed:**

The amplitude $M_u^{I=2} = A_t(s,u)+\tilde{A}(s,u)$ at $t-$fixed is the only one whose residues are all positive, as can be seen from Table 2. Its has poles only in the $s-$channel, as can be seen in Fig. 1c. According to Eq. (13) the high-energy average will now have a positive spectral density $\rho_J^{+-,+}(s)>0$. The dispersion relations look identical to Eq. (16), but with the low-energy amplitude given by Eq. (10). Following the steps explained in the previous sections, we can find sum rules for the Wilson coefficients of Eq. (10), as for example (for $k=2$),

$$h_{0,0}^{s} = \left\langle \frac{1}{m^4} \right\rangle_{+-}^{+} \,. \tag{23}$$

The conditions one gets from these dispersion relations (analogous to Eq. (17)) is however not over-constrained, and therefore no additional null constraints are obtained.

[5] We can assume here $k_{\text{min}}=2$ as $k=1$ does not provide additional information.

2.3.2 Inelastic Process

Let us now focus on the inelastic process. In particular, out of the three possible dispersion relations listed in Fig. 1, we will only consider $M_t^{I=2}$ at fixed $u < 0$ and fixed $t < 0$ in the $s-$plane.[6] We can proceed similarly as in the previous sections, but with the low-energy amplitude given in Eq. (11), the partial-wave decomposition given in Eq. (14), and the high-energy average defined by

$$\big\langle (...) \big\rangle_{++}^{\pm} \equiv \frac{1}{\pi} \sum_i (2J+1) \int_{M^2}^{\infty} \frac{dm^2}{m^2} \rho_J^{++,\pm}(m^2)(...) \,. \tag{24}$$

- $M_t^{I=2}$ $u-$**fixed:**

Once again the integral of $M_t^{I=2}(s,u)/s^{k+1}$ along the contour C_∞ of Fig. 1a vanishes for $k \geq k_{\text{min}}$, due to Eq. (12a). We have again Eq. (16) that expanded for small u gives

$$\begin{aligned}
(k=1) \qquad f_{0,0} + 2f_{1,0}u + ... &= \left\langle \frac{(-1)^{J+1}}{m^2} - \frac{(-1)^J(J^2+J-1)}{m^4} u + ... \right\rangle_{++}^{-} , \\
(k=2) \quad f_{1,0} + (f_{2,1}+f_{2,0})u + ... &= \left\langle \frac{(-1)^{J+1}}{m^4} - \frac{(-1)^J(J^2+J-1)}{m^6} u + ... \right\rangle_{++}^{-} , \\
&\vdots
\end{aligned} \tag{25}$$

leading to a new set of null constraints. At $\mathcal{O}(1/m^4)$, we have

$$\left\langle \frac{(-1)^J(\mathcal{J}^2-3)}{2m^4} \right\rangle_{++}^{-} = 0 \,. \tag{26}$$

- $M_t^{I=2}$ $t-$**fixed:**

For fixed $t < 0$, we can obtain new dispersion relations that combined with the system Eq. (25) leads to a second set of null constraints. By doing this, one can notice that the first new null constraint enters at order $\mathcal{O}(1/m^6)$.

2.3.3 Bounds on Wilson coefficients

From the above dispersion relations it is already possible to obtain interesting bounds on the low-energy Wilson coefficients of the $W\pi W\pi$ amplitude, that are related to physical quantities such as the dipole and quadrupole polarizabilities of the pions (see Appendix B). A more detailed discussion is given in Appendix C. For example, at $\mathcal{O}(s^2)$, we find the bounds

$$-2 \leq \frac{g_{0,0}}{h^s_{0,0}} \leq 2 \,, \qquad -1 \leq \frac{f_{1,0}}{h^s_{0,0}} \leq 1 \,. \tag{27}$$

For higher-oder Wilson coefficients a numerical analysis is sometimes needed in order to find the allowed values. In Figure 2 we provide an example (see Appendix C for details).

[6] We will not consider $M_u^{I=2}$ in the inelastic case. The corresponding dispersion relations were only useful for the elastic case to relate Wilson coefficients, such as $h^s_{0,0}$, to sums over positive residues.

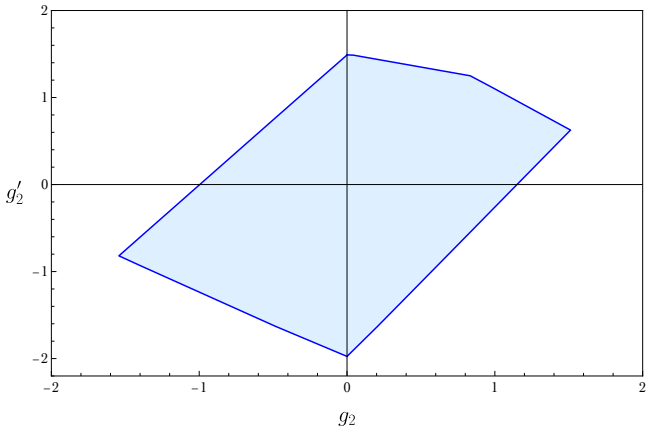


Figure 2: *Exclusion plot for the Wilsons g_2 and g_2' given in Eq. (62).*

2.4 The $W\pi \to \eta\pi$ Amplitude and the Chiral Anomaly

Let us now consider the $W\pi \to \eta\pi$ amplitude. For $\lambda_W = +1$ helicity, we have

$$\mathcal{M}(W^{a+}, \pi^b, \eta, \pi^c) \propto [12]\langle 24\rangle[41] \propto \sqrt{stu}\,. \tag{28}$$

At low-energy, this amplitude can be written as

$$|\mathcal{M}(W^{a+}, \pi^b, \eta, \pi^c)| = |f_{abc}|\mathcal{M}_{W\pi\eta\pi} = \frac{\kappa}{2\sqrt{2}}|f_{abc}|\sqrt{stu} + \dots\,. \tag{29}$$

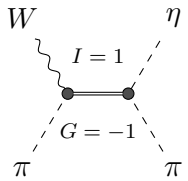
The Wilson coefficient κ is related to the chiral anomaly. Indeed, from the WZW term [26, 27] we have

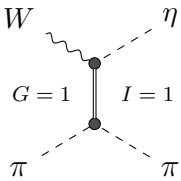
$$-\frac{N_c}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[A_{\mu L}U_{\nu L}U_{\alpha L}U_{\beta L} + \mathrm{L}\to\mathrm{R}\right] \subset \mathcal{L}_{WZW}\,, \tag{30}$$

where $U_{\nu L} = (\partial_\nu U)U^\dagger$, $U_{\nu R} = U^\dagger(\partial_\nu U)$, and $U = \mathrm{Exp}(i\eta/F_\pi)\mathrm{Exp}(2i\pi^a\tau^a/F_\pi)$ with $\tau^a = \sigma^a/2$ and $A_L^a = A_R^a = W^a$. By matching Eq. (29) with Eq. (30), we obtain

$$\kappa = \frac{N_c}{12\pi^2 F_\pi^3}\,. \tag{31}$$

The $W\pi\eta\pi$ amplitude is mediated by the following meson states:

$s/u-$ channel: 

$t-$ channel: 

(32)

For the $s/u-$channel the exchanged mesons must have $G = -1$, $I = 1$, $J =$ even and parity $P = (-1)^J = +1$. These states are classified as $n = 2$ in Table. 1 (highlighted in green). On the other hand, in the $t-$channel the states must have $G = +1$, $I = 1$, $J =$ odd and $P = (-1)^J = -1$. These states are classified as $n = 5$ in Table. 1 (also highlighted in green). We notice therefore that in this amplitude, only 2 out of the 6 possible types of mesons contribute.

2.5 Dispersion Relations

Let us consider the $W\pi\eta\pi$ amplitude without the non-analytical pre-factor coming from the polarization structure, $\mathcal{M}_{W\pi\eta\pi}/\sqrt{stu}$, and assume at fixed $u<0$

$$\lim_{|s|\to\infty}\frac{\mathcal{M}_{W\pi\eta\pi}}{\sqrt{stu}}=0\,, \tag{33}$$

and similarly at fixed $t<0$. In the s-plane with u fixed, we get the following dispersion relation:

$$\oint\frac{ds}{2\pi i}\frac{\mathcal{M}_{W\pi\eta\pi}(s,u)}{\sqrt{-s(s+u)u}}=\sum_i^{J^+_{\rm even}}\frac{g_{W\pi i}g_{\pi\eta i}\,d^J_{1,0}(-1-\frac{2u}{m_i^2})}{\sqrt{-m_i^2(m_i^2+u)u}}+\sum_i^{J^-_{\rm odd}}\frac{g_{W\eta i}g_{\pi\pi i}\,d^J_{1,0}(-1-\frac{2u}{m_i^2})}{\sqrt{-m_i^2(m_i^2+u)u}}\,, \tag{34}$$

where we specified that only J^P mesons with J odd or even are summed over. Expanding the above dispersion relation for $u\to 0$, we obtain

$$\frac{\kappa}{2\sqrt{2}}=\sum_i^{J^-_{\rm odd}}\frac{g_{W\eta i}g_{\pi\pi i}\,\mathcal{J}}{m_i^3}-\sum_i^{J^+_{\rm even}}\frac{g_{W\pi i}g_{\pi\eta i}\,\mathcal{J}}{m_i^3}\,, \tag{35}$$

where $\mathcal{J}=\sqrt{\mathcal{J}^2}$.

Similarly, we can also get another dispersion relation by fixing $t<0$ in the s-plane. We obtain in this case

$$\frac{\kappa}{2\sqrt{2}}=2\sum_i^{J^+_{\rm even}}\frac{g_{W\pi i}g_{\pi\eta i}\,\mathcal{J}}{m_i^3}\,. \tag{36}$$

From Eq. (35) and Eq. (36) we get the null constraint

$$\sum_i^{J^-_{\rm odd}}\frac{g_{W\eta i}g_{\pi\pi i}\,\mathcal{J}}{m_i^3}=3\sum_i^{J^+_{\rm even}}\frac{g_{W\pi i}g_{\pi\eta i}\,\mathcal{J}}{m_i^3}\,. \tag{37}$$

At large N_c, the coupings of singlet η can be related with the π^a couplings:

$$g_{W\eta i}=(g_5)_i\quad(J^-_{\rm odd})\quad,\quad g_{\pi\eta i}=g_{\pi\pi i}\quad(J^+_{\rm even})\,. \tag{38}$$

Using these relations, we can express κ and the null constraint Eq. (37) as

$$\frac{\kappa}{4\sqrt{2}}=\sum_i^{J^+_{\rm even}}\frac{(g_2g_{\pi\pi})_i\mathcal{J}}{m_i^3} \tag{39}$$

$$=\sum_i^{J^-_{\rm odd}}\frac{(g_5g_{\pi\pi})_i\mathcal{J}}{3m_i^3}\,. \tag{40}$$

3 Bounding the Chiral Anomaly

In this section we analytically derive an upper bound on the anomaly coefficient κ using the $W\pi \to W\pi$ null constraints Eq. (22) and Eq. (26). These two null constraints are enough to get a bound on κ. We will however improve the bound by incorporating the null constraint Eq. (37) from $W\pi \to \eta\pi$.

Using the identity (for $a_i, b_i \in \mathbb{R}$)

$$\sum_i a_i b_i \leq \sqrt{\sum_i a_i^2 b_i^2} \leq \sqrt{\big(\sum_i a_i^2\big)\big(\sum_i b_i^2\big)}\,, \tag{41}$$

we can obtain a bound on κ using Eq. (39) and Eq. (40) respectively

$$\begin{aligned}
\frac{\kappa}{4\sqrt{2}} &\leq \sqrt{\Big(\sum_i^{J^+_{\text{even}}} \frac{g^2_{\pi\pi i}}{m_i^2}\Big)\Big\langle \frac{\mathcal{J}^2}{m^4}\Big\rangle_2} \equiv \kappa_2^{\text{UB}}\,, \\
\frac{\kappa}{4\sqrt{2}} &\leq \frac{1}{3}\sqrt{\Big(\sum_i^{J^-_{\text{odd}}} \frac{g^2_{\pi\pi i}}{m_i^2}\Big)\Big\langle \frac{\mathcal{J}^2}{m^4}\Big\rangle_5} \equiv \kappa_5^{\text{UB}}\,.
\end{aligned} \tag{42}$$

where we have defined

$$\langle(...)\rangle_n \equiv \frac{1}{\pi}\sum_i (2J+1)\int_{M^2}^{\infty}\frac{dm^2}{m^2}\rho_J^n(m^2)(...)\,, \tag{43}$$

with

$$(2J+1)\rho_J^n(m^2) = \pi\sum_i (g_n^2)_i m_i^2 \delta(m^2-m_i^2)\delta_{JJ_i}\,, \tag{44}$$

and $g_n^2 \geq 0$ defined in Eq. (7). The bounds Eq. (42) depend on $\mathcal{J}^2/m^4$ high-energy averages which can be written as a function of $1/m^4$ averages by using the null constraints Eq. (22) and Eq. (26), rewritten as

$$\begin{aligned}
&+\Big\langle\frac{\mathcal{J}^2-4}{m^4}\Big\rangle_1 - \Big\langle\frac{\mathcal{J}^2+4}{m^4}\Big\rangle_2 - \Big\langle\frac{\mathcal{J}^2+4}{m^4}\Big\rangle_3 - \Big\langle\frac{\mathcal{J}^2-4}{m^4}\Big\rangle_4 - \Big\langle\frac{\mathcal{J}^2-4}{m^4}\Big\rangle_5 + \Big\langle\frac{\mathcal{J}^2+4}{m^4}\Big\rangle_6 = 0\,, \\
&-\Big\langle\frac{\mathcal{J}^2-3}{m^4}\Big\rangle_1 - \Big\langle\frac{\mathcal{J}^2-3}{m^4}\Big\rangle_2 + \Big\langle\frac{\mathcal{J}^2-3}{m^4}\Big\rangle_3 + \Big\langle\frac{\mathcal{J}^2-3}{m^4}\Big\rangle_4 - \Big\langle\frac{\mathcal{J}^2-3}{m^4}\Big\rangle_5 - \Big\langle\frac{\mathcal{J}^2-3}{m^4}\Big\rangle_6 = 0\,.
\end{aligned} \tag{45}$$

By summing these two relations we get

$$\Big\langle\frac{2\mathcal{J}^2}{m^4}\Big\rangle_2 + \Big\langle\frac{2\mathcal{J}^2}{m^4}\Big\rangle_5 = -\Big\langle\frac{1}{m^4}\Big\rangle_1 - \Big\langle\frac{1}{m^4}\Big\rangle_2 - \Big\langle\frac{7}{m^4}\Big\rangle_3 + \Big\langle\frac{1}{m^4}\Big\rangle_4 + \Big\langle\frac{7}{m^4}\Big\rangle_5 + \Big\langle\frac{7}{m^4}\Big\rangle_6\,. \tag{46}$$

Eq. (46) gives an interesting relation between the two $\mathcal{J}^2/m^4$ averages involved in Eq. (42) that allows to find the following relation

$$\Big\langle\frac{\mathcal{J}^2}{m^4}\Big\rangle_2 + \Big\langle\frac{\mathcal{J}^2}{m^4}\Big\rangle_5 = 3f_{1,0} - 2g_{0,0} \equiv \mathcal{P} \geq 0\,, \tag{47}$$

where we have used

$$g_{0,0} = \left\langle \frac{2}{m^4} \right\rangle_1 + \left\langle \frac{2}{m^4} \right\rangle_2 + \left\langle \frac{2}{m^4} \right\rangle_3 - \left\langle \frac{2}{m^4} \right\rangle_4 - \left\langle \frac{2}{m^4} \right\rangle_5 - \left\langle \frac{2}{m^4} \right\rangle_6 , \tag{48}$$

$$f_{1,0} = \left\langle \frac{1}{m^4} \right\rangle_1 + \left\langle \frac{1}{m^4} \right\rangle_2 - \left\langle \frac{1}{m^4} \right\rangle_3 - \left\langle \frac{1}{m^4} \right\rangle_4 + \left\langle \frac{1}{m^4} \right\rangle_5 + \left\langle \frac{1}{m^4} \right\rangle_6 , \tag{49}$$

obtained from the sum rules Eq. (21) and Eq. (25) respectively.

We can also obtain a relation between the sum over $g^2_{\pi\pi i}/m_i^2$ in Eq. (42) by using sum rules derived from dispersion relations for the $\pi\pi \to \pi\pi$ process in [3, 4]. In particular, we need

$$F_\pi^{-2} = 2\left(\sum_i^{J^+_{\text{even}}} \frac{g^2_{\pi\pi i}}{m_i^2} + \sum_i^{J^-_{\text{odd}}} \frac{g^2_{\pi\pi i}}{m_i^2} \right) . \tag{50}$$

Armed with Eq. (47) and Eq. (50), we can now obtain a bound on κ as a function of F_π^2 and $\mathcal{P}$. Defining

$$X \equiv F_\pi^2 \sum_i^{J^-_{\text{odd}}} \frac{g^2_{\pi\pi i}}{m_i^2} \quad \text{and} \quad Y \equiv \frac{1}{\mathcal{P}} \left\langle \frac{\mathcal{J}^2}{m^4} \right\rangle_5 , \tag{51}$$

we can rewrite κ_5^{UB} and κ_2^{UB} as

$$\frac{\kappa_5^{\text{UB}}}{\sqrt{\mathcal{P}/F_\pi^2}} = \frac{1}{3}\sqrt{XY} , \qquad \frac{\kappa_2^{\text{UB}}}{\sqrt{\mathcal{P}/F_\pi^2}} \leq \sqrt{\left(\frac{1}{2} - X\right)\left(\frac{1}{2} - Y\right)} . \tag{52}$$

Written in this way we notice that as we decrease X and Y to make κ_2^{UB} larger, κ_5^{UB} becomes smaller, and viceversa. Since κ must be smaller than both bounds, the largest value of κ is reached when $\kappa_2^{\text{UB}} = \kappa_5^{\text{UB}}$. This is achieved when

$$\frac{1}{9}XY = \left(\frac{1}{2} - X\right)\left(\frac{1}{2} - Y\right) \quad \to \quad Y = \frac{X - 1/2}{16X/9 - 1} , \tag{53}$$

that inserted in Eq. (52) gives us κ_5^{UB} as a function of only X that is maximized for $X = 3/8$. This corresponds to $Y = 3/8$, and gives $\kappa_5^{\text{UB}}|_{\text{max}} = 1/8$. Taking this value in Eq. (42), we finally get the following upper bound on the chiral anomaly:

$$\frac{\kappa}{\sqrt{\mathcal{P}/F_\pi^2}} \leq \frac{1}{\sqrt{2}} . \tag{54}$$

This is the main result of the paper. This tells us that the anomaly coefficient is bounded by $\mathcal{P}$, a quantity related with the polarizabilities of the pions (see Appendix B). Using the constraints Eq. (27), we can write Eq. (54) as a function of other Wilson coefficients. For example, we get that $\mathcal{P} \leq 7h^s_{0,0}$ that leads to

$$\frac{\kappa}{\sqrt{h^s_{0,0}/F_\pi^2}} \leq \sqrt{\frac{7}{2}} . \tag{55}$$

In the case of $SU(N_c)$ gauge theories with N_f quarks, the anomaly coefficient κ is known, and Eq. (54) provides a bound on the Wilson coefficients. For example, using Eq. (31) and $F_\pi \simeq \sqrt{N_c/3}\, m_\rho/7$ [28], we obtain from Eq. (55)

$$h^s_{0,0} \gtrsim \frac{0.44}{m_\rho^4}\,. \tag{56}$$

As in [3, 4], it is also interesting to obtain the predictions of different models to the ratio Eq. (54), in order to understand how close these are to the upper bound. For example, in models of only scalars, we have that $h^s_{0,0}$ is zero and then $\kappa = 0$. For su-models (introduced in [11] for the first time, and studied in [3, 4, 12] for four-pion amplitudes), we show in Appendix D that

$$\left.\frac{\kappa}{\sqrt{\mathcal{P}/F_\pi^2}}\right|_{su-\text{model}} \lesssim \frac{0.97}{\sqrt{2}}\,, \tag{57}$$

which falls within the bound established in Eq. (54), only $\sim 3\%$ away from saturation.

Eq. (54) gives a non-trivial constraint on phenomenological models for QCD, such as NJL or holographic models. For example, in holographic models the chiral anomaly arises from a Chern-Simons (CS) term. When the corresponding dual 4D theory is not known, the CS coefficient cannot be matched to the UV theory and therefore cannot be determined. Eq. (54) provides in this case a bound on the the size of the CS term. This could also be useful for models of axions.

4 Conclusions

The analyticity, unitarity and the good high-energy behaviour of scattering amplitudes provide severe constraints on the low-energy physical quantities of gauge theories like QCD. In this work we have studied scattering amplitudes involving external gauge fields and goldstones (π and η) in the large-N_c limit to obtain several new constraints.

Our main result has been to provide an analytical bound on the chiral anomaly, Eq. (54). To derive this bound we have first analyzed the $W\pi \to W\pi$ amplitude, providing the selection rules for the mesons exchanges, as well as the sign of their on-shell residues. We have considered the elastic ($W^+\pi \to W^+\pi$) and inelastic ($W^+\pi \to W^-\pi$) processes, as both are needed to derive $\mathcal{O}(1/m^4)$ null constraints on the meson couplings.[7] We have also derived sum rules for the Wilson coefficients of these amplitudes. Of especial use has been the sum rule for $h^s_{0,0}$ that has shown to have all its terms positive.

We have later considered the $W\pi \to \eta\pi$ amplitude, which at low energy yields the chiral anomaly coefficient κ. We have shown, once again via dispersion relations, that there are two different ways to determine κ, implying a non-trivial constraint among the corresponding meson couplings.

Putting all this together, we have been able to derive analytically Eq. (54). This bound has interesting implications for UV completions of these amplitudes, as we have shown for

[7] We have found that adding more null constraints does not improve the bound.

the case of large-N_c QCD. We have also considered su-models (see Appendix D) that seem to almost saturate the bound. We hope that in the future this bound could be tested in lattice simulations. Finally, we have also briefly study the bounds on other Wilson coefficients, showing the EFT-hedron geometry [6] of their allowed parameter space -see for example Fig. 2.

In the future, it could also be interesting to study in more detail which theories saturate these bounds. Similarly, our analysis can be extended to understand implications on composite Higgs models and bounds on dimension six operators such as $H^\dagger H F_{\mu\nu} F^{\mu\nu}$.

Note added: While preparing this article, it was submitted to the archives Ref. [29] that also uses dispersion relations to numerically obtain bounds on the chiral anomaly coefficient.

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A $q\bar{q}$ mesons and the sign of on-shell residues

The $q\bar{q}$ mesons can be classified according to their quantum numbers: mass m, spin J, parity P, Isospin I, and G-parity. Defining by ℓ the orbital angular momentum of the $q\bar{q}$ system, $s = 0, 1$ the total spin, and $I = 0, 1$ the isospin, we have

$$\begin{aligned} J &= \ell + s, ..., \ell - s\,, \\ P &= (-1)^{\ell+1}\,, \\ G &= (-1)^{I+\ell+s}\,. \end{aligned} \tag{58}$$

By taking $\ell = 0, 1, 2, ...$, we can build the meson list of Table 1. Notice that we have only six types of mesons that we label with $n = 1, ..., 6$.

To understand the sign of the residues in Eq. (6) arising from meson exchanges in the process $W\pi \to W\pi$, we proceed in the following way (recalling that the couplings $g_{W\pi i}$ are real):

- For the elastic case $W^+\pi \to W^+\pi$ (amplitude $\mathcal{M}^{+-}$), we obviously have that these are proportional to $g^2_{W\pi i} > 0$ so the sign is always positive.
- For $W^+\pi \to W^-\pi$ (amplitude $\mathcal{M}^{++}$) we can consider the process $W^+\pi \to R_i \to W^-\pi$ in the forward limit $t \to 0$. By performing a P transformation and a spacial rotation of 180^0 degrees, the coupling in $W^+\pi \to R_i$ can be related to that of $R_i \to W^-\pi$. Using that $d^J_{1,0}(-\cos\theta) = (-1)^J d^J_{1,0}(\cos\theta)$, we have then that $W^+\pi \to R_i \to W^-\pi$ is proportional to $g^2_{W\pi i} \times (-1)^{J_i} \times P_i$.

This leads to the signs in Eq. (6) for the different type of mesons.

B Polarizabilities of the pions

Given the $\gamma\gamma \to \pi\pi$ scattering amplitude with the pion exchange subtracted, $\mathcal{M}(\gamma^+, \gamma^\pm, \pi, \pi)$, the pion polarizabilities are defined as the coefficients coming from an expansion in s at fixed $t = m_\pi^2$ [18, 30, 31]:

$$\frac{\alpha}{m_\pi}\mathcal{M}(\gamma^+, \gamma^\pm, \pi, \pi)(s, t = m_\pi^2) = (\alpha_1 \mp \beta_1)_\pi + \frac{s}{12}(\alpha_2 \mp \beta_2)_\pi + \dots . \tag{59}$$

In particular, $(\alpha_1 \mp \beta_1)_\pi$ and $(\alpha_2 \mp \beta_2)_\pi$ are respectively the dipole and quadrupole polarizabilities of the pions $\pi = \pi^\pm, \pi^0$. These quantities have been measured experimentally [18, 30, 31] and also analyzed in lattice simulations [32]. In our convention the photon corresponds to the gauging of $Q = T_3 + B/2$, a subgroup of the global $U(2)$, and we must change $s \leftrightarrow t$ in Eq. (59). The dipole polarizabilities receive then contributions from our $g_{0,0}$ and $h^s_{0,0}$ for the $+-$ helicities of the photon (although they are suppressed by m_π^2), and $f_{0,0}$ for the $++$. On the other hand, $f_{1,0}$ contributes to the quadrupole polarizability.

C Bounds on the Elastic $W\pi W\pi$ Wilson coefficients

Let us show here how to bound the parameter space of the Wilson coefficients appearing in Eq. (9) and Eq. (10). We are considering the elastic process since better bounds can be obtained in this case due to positivity. Nevertheless, the same approach can also be used for the inelastic case.

Let us begin with the simplest analytical bound arising from the sum rules of $g_{0,0}$ and $h^s_{0,0}$, Eq. (21) and Eq. (23) respectively, that gives

$$\frac{g_{0,0}}{h^s_{0,0}} = -\frac{\left\langle \frac{2}{m^4} \right\rangle^-_{+-}}{\left\langle \frac{1}{m^4} \right\rangle^+_{+-}} . \tag{60}$$

The absolute value of this ratio must always be smaller or equal to 2, since the states that enter in $g_{0,0}$ and in $h^s_{0,0}$ are the same, the only difference being that in $h^s_{0,0}$ everything enters additively, while some of these states enter with a minus sign in $g_{0,0}$. The bound will be therefore maximized when the only states exchanged are those who contribute negatively to $g_{0,0}$ (namely states with $n = 1, 2, 3$), while it is minimized when the states that contribute positively are the only ones being exchanged ($n = 4, 5, 6$). This leads to the bound Eq. (27), and similarly for $f_{1,0}$ from its sum rule in Eq. (25).

The same result can be found numerically using `SDPB` [33]. To find this bound we must initially redefine the Wilson coefficients and null constraints (of both the elastic and inelastic processes) in terms of the high-energy average defined Eq. (43), similarly to what was done in Eq. (45). We can then construct the bootstrap equation

$$h_{0,0}\vec{v}_1 + g_{0,0}\vec{v}_2 + \sum_{n=1..6} \langle \vec{v}_n(m^2, J) \rangle_n = 0 , \tag{61}$$

which holds true if $\vec{v}_1 = (1, 0, 0, ..), \vec{v}_2 = (0, 1, 0, ...)$ and for an appropriate choice of the vectors $\vec{v}_n(m^2, J)$ (which contain the null constraints). Following canonical the optimization procedure explained in detail in [3, 11, 12], one can bound numerically the Wilson coefficients. In particular, this method can be replicated for higher-order Wilson coefficients. For example, defining

$$g_2 = \frac{g_{2,0}}{h^a_{2,0} + 2h^s_{2,0} + h^s_{2,1}/2} \qquad \text{and} \qquad g_2' = \frac{g_{2,1}}{h^a_{2,0} + 2h^s_{2,0} + h^s_{2,1}/2}\,, \tag{62}$$

where the normalization has been chosen opportunely for positivity arguments as

$$h^a_{2,0} + 2h^s_{2,0} + h^s_{2,1}/2 = \left\langle \frac{(\mathcal{J}^2 - 8)\mathcal{J}^2 + 14}{4m^8} \right\rangle^{+}_{+-}\,, \tag{63}$$

we can repeat the numerical procedure and find the exclusion plot shown in Figure 2.

D The su-model

The su-models are defined as those leading to healthy amplitudes with a single mass scale m. They often predict Wilson coefficients at the boundaries of the allowed regions [3, 4, 11, 12]. For the $\pi\pi \to \pi\pi$ process the su-amplitude was already presented in [4] where it was shown to take the general form

$$\mathcal{M}(s,u)\Big|_{4\pi} = \frac{m^2(s+u) + \lambda su}{(s-m^2)(u-m^2)}\,, \tag{64}$$

with -2 $\leq \lambda \leq \frac{2\ln 2 - 1}{1 - \ln 2}$. In the limiting case $\lambda = \frac{2\ln 2 - 1}{1 - \ln 2}$, the amplitude Eq. (64) contains no poles associated to $J = 0$ states [4]. In this case, the residues in the s-channel of Eq. (64) are given by

$$g^2_{\pi\pi i}\big|_{4\pi} = \{0.78, 0.18, 0.04, ...\}\,, \tag{65}$$

and correspond to states of $J^P = 1^-, 2^+, ...$ respectively ($n = 5$ and $n = 2$ of states $G = 1$ alternating). The amplitude Eq. (64) predicts the Wilson coefficient

$$F_\pi^{-2} = \frac{2}{m^2}\,. \tag{66}$$

Notice that we have absorbed an overall factor in Eq. (64) into F_π^{-2}.

An su-amplitude for $M_t^{I=2}(s,u)\big|_{+\pm}$ can also be constructed following the conditions:

- It must have a single mass scale m.
- No $t-$channel poles.
- It must be proportional to t and su for $++$ and $+-$ amplitudes respectively.
- It must drop as $M_t^{I=2}(s,u)/s \to 0$ for $|s| \to \infty$ at $t-$fixed, and similarly for $u-$fixed.

The most general amplitude following these criteria takes the form

$$M_t^{I=2}(s,u)\big|_{++} = -\frac{m^2\, t}{(s-m^2)(u-m^2)}\,, \qquad M_t^{I=2}(s,u)\big|_{+-} = -\alpha\frac{su}{(s-m^2)(u-m^2)}\,, \tag{67}$$

where α is a constant. The residues of Eq. (67) in the s-channel are

$$R^{++}_{I=0} - R^{++}_{I=1} = \{0.82, -0.15, 0.03, ...\}\,, \quad R^{+-}_{I=0} - R^{+-}_{I=1} = \alpha\,\{0.58, -0.09, 0.02, ...\}\,. \tag{68}$$

corresponding to $J = 1, 2, 3, ...$ states, where for each $J_{\rm odd}$ ($J_{\rm even}$) they can be of type $n = 1, 4, 5$ ($n = 2, 3, 6$) of Table 1.

From Eq. (67) we obtain

$$f_{1,0} = \frac{1}{m^4}\,, \quad g_{1,0} = -\frac{\alpha}{m^4} \quad \to \quad \mathcal{P} = \frac{3+2\alpha}{m^4}\,. \tag{69}$$

Notice that we have absorbed an overall factor in the $++$ amplitude in Eq. (67) into $f_{1,0}$.

According to Eq. (6), we can obtain the couplings g_2^2 and g_5^2 (which are the ones that enter in the anomaly), by adding the two sets of residues of Eq. (68). We obtain

$$g^2_{W\pi i}\big|_{W^2\pi^2} = \{0.41 + 0.29\alpha, 0.07 + 0.05\alpha, ...\}\,, \tag{70}$$

corresponding to $n = 5$ and $n = 2$ states alternating. To have positive $g^2_{5,2}$, we must demand $-1.41 \leq \alpha \leq 1.41$. The value $\alpha = 1.41$ (-1.41) maximizes (minimizes) $g^2_{5,2}$.

Finally, the $W\pi\eta\pi$ amplitude can be constructed with the following conditions:

- Single mass scale m.
- $s \leftrightarrow u$ crossing symmetric.
- Proportional to $\sqrt{stu}$.
- $s/u-$channel poles associated only to even-spin states (see Eq. (32)).
- It must drop as $\mathcal{M}_{W\pi\eta\pi}/\sqrt{stu} \to 0$ for $|s| \to \infty$ at $t-$fixed, and similarly for $u-$fixed.

The most general amplitude one can construct following these conditions is

$$\mathcal{M}_{W\pi\eta\pi} = -\beta m\sqrt{stu}\,\frac{m^2/2 + t}{(s-m^2)(u-m^2)(t-m^2)}\,, \tag{71}$$

where $\beta > 0$ is a constant. We obtain the following residues in the s-channel:

$$|g_{W\pi i} g_{\pi\pi i}| = \beta\,\{0.48, 0.10, 0.02, ...\}\,, \tag{72}$$

corresponding to states $n = 5$ and $n = 2$ of Table 1 alternating.

From Eq. (29) and Eq. (71) we get the anomaly coefficient

$$\kappa = \frac{\sqrt{2}\beta}{m^3}\,. \tag{73}$$

To maximize this value, we must take the largest possible value of β. Nevertheless, this is constrained by the fact that the couplings in Eq. (72) cannot overcome

$$\sqrt{g^2_{W\pi i}\big|_{W^2\pi^2}\, g^2_{\pi\pi i}\big|_{4\pi}} = \{\sqrt{0.32 + 0.23\alpha}, \sqrt{0.01 + 0.01\alpha}, ...\}\,, \tag{74}$$

coming from Eq. (65) and Eq. (70), as there can always be more states $n = 5, 2$ (for a given J) in Eq. (64) and Eq. (67) than in Eq. (71) (as this latter requires that the interchanged mesons must have both $g_{W\pi i}$ and $g_{\pi\pi i}$ nonzero). This gives

$$\beta_{\max} = \sqrt{1.37 + 0.97\alpha}\,. \tag{75}$$

With Eq. (66), Eq. (69), Eq. (73) and Eq. (75), one finds that $\kappa/\sqrt{\mathcal{P}/F_\pi^2}$ is maximized for $\alpha = 1.41$, leading to Eq. (57).

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Paper 3: Multipositivity Constrains the Chiral Lagrangian

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Multipositivity Constrains the Chiral Lagrangian

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The chiral Lagrangian is a cornerstone of modern particle physics, offering a systematic and quantitative description of low-energy pions. Using tools from the modern scattering amplitudes program, we show that consistent multiparticle dynamics impose novel constraints on the coupling constants of this theory. In the planar limit, these constraints imply that certain Wilson coefficients of the chiral Lagrangian are bounded from below by the chiral anomaly. Our results reveal a subtle connection between the anomalous and nonanomalous sectors of the underlying strong interactions, while introducing a novel formulation of multipositivity bounds that holds for any planar tree-level theory.

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Introduction.—The advent of chiral perturbation theory revolutionized our understanding of the strong interactions. In a single stroke, it reforged the insights of current algebra [1] into the first incarnation of effective field theory (EFT) [2–8], in modern parlance known as the nonlinear sigma model (NLSM). This framework trades the intricate short-distance physics of quantum chromodynamics (QCD) for *parameters* in a long-distance description dictated by symmetry and dimensional analysis. More than half a century later, chiral perturbation theory remains one of the most celebrated, well-studied, and quantitatively successful EFTs in all of physics.

Beyond its phenomenological roots, the NLSM has more recently been a central object of study in the more formal context of the modern scattering amplitudes program. Indeed, NLSM amplitudes are among the handful of models—along with gauge theory and general relativity—that exhibit numerous extraordinary mathematical structures, including soft theorems [9–16], double copy structures [17–24], and remarkable representations for scattering in terms of abstract volumes [25–28].

Given the foundational status of the chiral Lagrangian, one might expect its parameter space to be well-charted territory. Curiously, even decades after its inception, it is not fully understood what is the allowed space of consistent coupling constants, even in principle. In recent years a program has been initiated to systematically answer this question using consistency constraints—or so-called *positivity bounds*—which can be derived solely from general principles like unitarity and causality [29]. Positivity bounds have shown themselves to be a powerful tool to constrain EFTs [30–32], with applications ranging from the Standard Model to quantum gravity [33–122].

Most existing results rely on exploiting positivity properties of two-to-two scattering amplitudes. Recent work has begun to extend these ideas to higher multiplicity [100–106], with Ref. [100] introducing forms of *multipositivity* that relate amplitudes with different numbers of external particles. The methods introduced in this paper build on this direction. We focus on dispersion relations in a single kinematic channel and exploit their factorization structure, allowing us to apply a Cauchy-Schwarz type of inequality directly to residues. This construction leads to a new class of bounds that relate planar amplitudes of different multiplicities in a simple and controlled manner.

This approach is particularly well suited to the chiral Lagrangian, where symmetry relates interactions across different pion multiplicities. As we will show, we find new constraints on chiral EFT data that are invisible to four-point scattering and that carve out a significant portion of the previously allowed parameter space.

Planar Amplitudes in Large-N_c QCD.—In this paper, we derive new constraints on higher-point *planar* amplitudes. The planar limit is particularly convenient because the singularity structure of the amplitude simplifies, as will become clear below. As a concrete and physically relevant arena for these ideas, we will focus on large-N_c QCD, an $SU(N_c)$ gauge theory in the $N_c \to \infty$ limit [123, 124], with N_f massless quarks in the fundamental representation. The theory exhibits the familiar pattern of chiral symmetry breaking, $U(N_f)_L \times U(N_f)_R \to U(N_f)_V$. The massless spectrum consists of Goldstone bosons in the adjoint of $SU(N_f)$, the pions, together with the singlet η'. In the large-N_c limit, the strong dynamics of quarks and gluons enjoy a dual description in terms of the tree-level scattering of weakly interacting mesons.

At leading order in the large-N_c expansion, connected pion amplitudes are generated by planar diagrams with a *single* quark boundary [123, 124]. Since the quarks are in the fundamental representation, this boundary line carries flavor indices. The only structures that can appear in flavor space are therefore single traces [125]. In other words, though N_f is finite, the large-N_c expansion organizes pion amplitudes into single-trace flavor structures that define flavor-ordered partial amplitudes,

$$A^{a_1\cdots a_n} = \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}^{a_{\sigma(1)}\cdots a_{\sigma(n)}} A^{(n)}(p_{\sigma(1)},\ldots,p_{\sigma(n)}), \quad (1)$$

where the momenta p_i are associated to the pions of flavor a_i. Here $\mathrm{Tr}^{a_1\cdots a_n} = \mathrm{Tr}(T^{a_1}\cdots T^{a_n})$ in the nor-

malization convention $\mathrm{Tr}(T^a T^b) = \delta^{ab}$, which is required for planar amplitudes to factorize correctly. Cyclicity of the traces implies that the partial amplitudes are also cyclic, $A^{(n)}(p_1, p_2, ..., p_n) = A^{(n)}(p_2, ..., p_n, p_1)$, which at four-point is nothing more than the $s \leftrightarrow t$ crossing symmetry we are accustomed to. A key property of tree-level pion amplitudes at large N_c is that their singularities correspond to simple poles, associated to the exchanged mesons, that show up *only* in planar channels. The planar partial amplitudes $A^{(n)}$ will be the central objects of study in our higher-point analysis.

Chiral Perturbation Theory.—In chiral perturbation theory, the parity-even, nonanomalous dynamics are encoded by the NLSM Lagrangian in terms of $U = \exp(\sqrt{2} i \pi^a T^a / F_\pi)$, which transforms in the bifundamental of $U(N_f)_L \times U(N_f)_R$:

$$\begin{aligned}\mathcal{L}_{\text{NLSM}} = &-\frac{F_\pi^2}{4}\,\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) \\ &+ L_3 \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U) \\ &+ L_4 \mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U \partial^\mu U^\dagger \partial^\nu U) + \cdots,\end{aligned} \tag{2}$$

where we work in mostly-plus signature. We will ignore higher-trace operators, which are subleading at large N_c, as well as any additional gauge interactions.

Due to the chiral anomaly, there is an additional Wess-Zumino-Witten (WZW) term [126, 127], which breaks $U \to U^\dagger$ invariance while preserving parity, which sends $U \to U^\dagger$ and $\vec{x} \to -\vec{x}$. This topological term is constructed as an integral over a five-dimensional manifold, with physical four-dimensional spacetime as its boundary. Defining the chiral current $j_\mu = i U^\dagger \partial_\mu U$, the leading intrinsic parity-odd, anomalous dynamics come from

$$S_{\text{WZW}} = -\frac{\kappa}{240\pi^2}\int d^5x\, \epsilon^{\mu\nu\rho\sigma\tau}\mathrm{Tr}\left(j_\mu j_\nu j_\rho j_\sigma j_\tau\right), \tag{3}$$

from which Stokes' theorem yields (if $N_f \geq 3$) the anomalous five-point vertex in four dimensions,

$$\mathcal{L}_{\text{WZW}} = \frac{\kappa\,\mathrm{Tr}^{abcde}\epsilon^{\mu\nu\rho\sigma}\pi^a\partial_\mu\pi^b\partial_\nu\pi^c\partial_\rho\pi^d\partial_\sigma\pi^e}{30\sqrt{2}\pi^2 F_\pi^5} + \cdots. \tag{4}$$

Matching to the QCD chiral anomaly [127] precisely fixes $\kappa = N_c$. Since S_{WZW} is odd under $U \to U^\dagger$, it contributes only to odd-point amplitudes.

Positivity from Cauchy-Schwarz.—The logic of our multipositivity constraints is simple. To begin, consider the residue of an $(n+m)$-point amplitude $A^{(n+m)}$ on a factorization channel. Mechanically, $A^{(n+m)}$ exhibits a simple pole at $s = M_k^2$, where s is the kinematic invariant corresponding to the channel and M_k is the mass of the exchanged state labeled by k. The residue on this pole factorizes into a product of lower-point amplitudes,

$$\begin{aligned}&R_k^{(n+m)}(p_1, ..., p_n | q_1, ..., q_m) \\ &\quad = A_k^{(n+1)}(p_1, ..., p_n) A_k^{(m+1)}(q_1, ..., q_m),\end{aligned} \tag{5}$$

where $s \equiv -(p_1 + \cdots + p_n)^2 = -(q_1 + \cdots + q_m)^2$, with all momenta incoming. Since all these amplitudes are planar, their momenta p_i and q_i are cyclically ordered. If there is a degeneracy of states at mass M_k, then the right-hand side is augmented to a sum that will not affect our conclusions. Next, consider the residues of a $2n$-point amplitude $A^{(2n)}$ and a $2m$-point amplitude $A^{(2m)}$,

$$\begin{aligned}R_k^{(2n)}(p_1, ..., p_n | -\bar{p}_n, ..., -\bar{p}_1) &= |A_k^{(n+1)}(p_1, ..., p_n)|^2 \\ R_k^{(2m)}(-\bar{q}_m, ..., -\bar{q}_1 | q_1, ..., q_m) &= |A_k^{(m+1)}(q_1, ..., q_m)|^2,\end{aligned} \tag{6}$$

where bar denotes complex conjugation. We have chosen complex forward kinematics [100], making the residues modulus squares since $A_k^{(n+1)}(-\bar{p}_1, ..., -\bar{p}_n) = \bar{A}_k^{(n+1)}(p_1, ..., p_n)$.

Recasting the lower-point amplitudes as the components of a vector, $\bar{A}_k^{(n+1)} \equiv \langle k|f\rangle M_k^{i+1}$ and $A_k^{(m+1)} \equiv \langle k|g\rangle M_k^{j+1}$, where for future convenience we normalize with different powers of the masses, we find that

$$\begin{aligned}R_k^{(2n)} &= \langle f|k\rangle\langle k|f\rangle M_k^{2i+2} \\ R_k^{(2m)} &= \langle g|k\rangle\langle k|g\rangle M_k^{2j+2} \\ R_k^{(n+m)} &= \langle f|k\rangle\langle k|g\rangle M_k^{(i+j)+2}.\end{aligned} \tag{7}$$

Here $|f\rangle$ and $|g\rangle$ are n- and m-particle states with momenta $p_1, \cdots, p_n$ and $q_1, \cdots, q_m$, respectively, while $|k\rangle$ is the one-particle exchanged state of mass M_k, weighted by any degeneracies.

At low energies, we expand the amplitudes $A^{(r)}$ in terms of Wilson coefficients associated with powers of s:

$$A^{(r)} \to \sum_\ell \lambda_\ell^{(r)} s^\ell, \quad \text{where} \quad \lambda_\ell^{(r)} = \operatorname*{Res}_{s=0}\left(\frac{A^{(r)}}{s^{\ell+1}}\right). \tag{8}$$

Note that the Wilson coefficients $\lambda_\ell^{(r)}$ depend implicitly on all kinematic invariants that are independent of s. Applying Cauchy's theorem to the integral $\oint ds\, A^{(r)}/s^{\ell+1}$ taken over a contour at infinity in the complex s-plane, with all other kinematic invariants fixed, for $(r, \ell) = (2n, i), (2m, j), (n+m, (i+j)/2)$, we obtain

$$\begin{aligned}\lambda_i^{(2n)} &= \operatorname*{Res}_{s=0}\left(\frac{A^{(2n)}}{s^{i+1}}\right) = \sum_k \frac{R_k^{(2n)}}{M_k^{2i+2}} = \langle f|f\rangle, \\ \lambda_j^{(2m)} &= \operatorname*{Res}_{s=0}\left(\frac{A^{(2m)}}{s^{j+1}}\right) = \sum_k \frac{R_k^{(2m)}}{M_k^{2j+2}} = \langle g|g\rangle, \\ \lambda_{(i+j)/2}^{(n+m)} &= \operatorname*{Res}_{s=0}\left(\frac{A^{(n+m)}}{s^{(i+j)/2+1}}\right) = \sum_k \frac{R_k^{(n+m)}}{M_k^{(i+j)+2}} = \langle f|g\rangle.\end{aligned} \tag{9}$$

Here, the planarity of the amplitudes was crucial to relate the residue in $s = 0$ to the sum of residues across a *single* factorization channel. We have assumed that every subtracted amplitude has sufficiently convergent high-energy behavior such that we can drop boundary

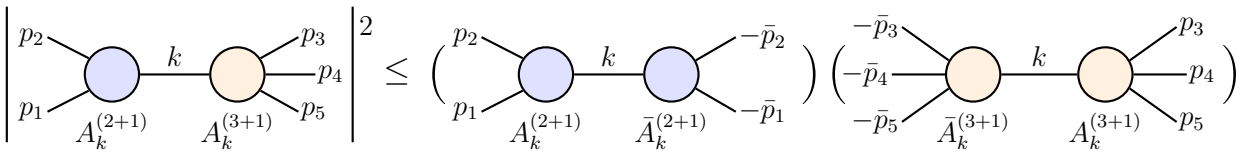


Figure 1. Inequality satisfied by the residues of the four-, five- and six-point scattering amplitudes, which leads to Eq. (10). The state sum on the cut runs over exchanged particles labeled by k.

terms. We will comment later on particular choices of i, j such that this assumption is well motivated.

Applying the Cauchy-Schwarz inequality $|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$ to Eq. (9), we find that

$$|\lambda^{(n+m)}_{(i+j)/2}|^2 \leq \lambda^{(2n)}_i \lambda^{(2m)}_j. \tag{10}$$

The constraint above implicitly depends on a set of s-independent kinematic invariants.

To project Eq. (10) into a particular s-independent kinematic invariant, it is convenient to introduce the following ansatz for the parameterization of momenta

$$p_i \equiv p_i(\{a\}, s) \text{ and } q_i \equiv q_i(\{b\}, s), \tag{11}$$

where $\{a\}$ and $\{b\}$ are sets of parameters that we will define explicitly in the next section. Rerunning the above logic, identifying $\partial_a A^{(n+1)}_k$ and $\partial_b A^{(m+1)}_k$ as the vector components in the Cauchy-Schwarz inequality, we find

$$|\partial_a\partial_b \lambda^{(n+m)}_{(i+j)/2}|^2 \leq \partial_a\partial_{\bar{a}}\lambda^{(2n)}_i \partial_b\partial_{\bar{b}}\lambda^{(2m)}_j. \tag{12}$$

Factorization of the derivatives requires that the left cluster only depend on the momenta p_i and therefore on the parameters $\{a\}$, while the right cluster only on the momenta q_i and therefore on the parameters $\{b\}$. This holds to all derivative orders. The simplest version of Eq. (10) has $n = 2$ and $m = 3$, as shown in Fig. 1, giving

$$\begin{aligned} |\lambda^{(5)}_{(i+j)/2}|^2 &\leq \lambda^{(4)}_i \lambda^{(6)}_j, \\ |\partial_a\partial_b \lambda^{(5)}_{(i+j)/2}|^2 &\leq \partial_a\partial_{\bar{a}}\lambda^{(4)}_i \partial_b\partial_{\bar{b}}\lambda^{(6)}_j, \end{aligned} \tag{13}$$

which constrains the Wilson coefficients of the four-, five-, and six-point amplitudes. We will focus on this system in the rest of the paper and study the implications of Eq. (13) for the chiral Lagrangian.

Parameterization of Momenta.—The bounds we have derived are true only provided that we can construct the states promised in Eq. (7), which allow us to write the dispersion relations in terms of inner products in Eq. (9). We now fulfill this promise. For these vectors to be well defined, we must specify on-shell external momenta p_i and q_i in terms of s and a handful of other parameters as in Eq. (11) such that the four-, five-, and six-point amplitudes are analytic in s and obey momentum conservation. For Eq. (10) to hold, we want $s = -(p_1 + \cdots + p_n)^2$ to activate singularities in a single factorization channel.

The three amplitudes relevant for our multipositivity bounds are $A^{(2n)}(p_1,...,p_{2n})$, $A^{(2m)}(q_1,...,q_{2m})$ and $A^{(n+m)}(p_1,...,p_n,q_1,...,q_m)$. To obtain Eq. (9), we need the residues of $A^{(2n)}$ and $A^{(2m)}$ on the factorization channel to be positive. This requires the complex forward limit of Eq. (6), $(p_{n+1},...,p_{2n}) \to (-\bar{p}_n,...,-\bar{p}_1)$ and $(q_{m+1},...,q_{2m}) \to (-\bar{q}_m,...,-\bar{q}_1)$, for which momentum conservation imposes a constraint,

$$\begin{aligned} (\bar{p}_1 + \cdots + \bar{p}_n) &= (p_1 + \cdots + p_n), \\ (\bar{q}_1 + \cdots + \bar{q}_m) &= (q_1 + \cdots + q_m). \end{aligned} \tag{14}$$

We are now ready to construct explicit momenta for the system comprising four-, five-, and six-point scattering. It will be convenient to introduce a tetrad basis,

$$\ell^\mu_\pm = \frac{1}{2}(1,0,0,\pm 1), \qquad n^\mu_\pm = \frac{1}{2}(0,1,\pm i,0), \tag{15}$$

where the only nonzero contractions are $\ell_+ \cdot \ell_- = -1/2$ and $n_+ \cdot n_- = 1/2$. Following Eq. (11) we define

$$\begin{aligned} p_1 &= \sqrt{s}\,\ell_+ + a n_+ \\ p_2 &= \sqrt{s}\,\ell_- - a n_+ \\ q_1 &= -\sqrt{s}\,\ell_- + b_1 n_- \\ q_2 &= -b_1 b_3/\sqrt{s}\,\ell_+ - (b_1 - b_2) n_- \\ q_3 &= (b_1 b_3/\sqrt{s} - \sqrt{s})\,\ell_+ - b_2 n_-, \end{aligned} \tag{16}$$

where a and $b_{1,2,3}$ are complex numbers. Notice that by construction $(p_1 + p_2) \propto (q_1 + q_2 + q_3) \propto \sqrt{s}$. Momentum conservation in Eq. (14) requires real s on the factorization channel in Eq. (9), where $s = M_k^2$ [128].

Plugging these momenta into the kinematic invariants of each amplitude, we find that for $A^{(4)}(p_1, p_2, -\bar{p}_2, -\bar{p}_1)$,

$$-(p_1 + p_2)^2 = s, \qquad -(p_2 - \bar{p}_2)^2 = a\bar{a}, \tag{17}$$

while for $A^{(5)}(p_1, p_2, q_1, q_2, q_3)$,

$$\begin{aligned} -(p_1 + p_2)^2 &= s, & -(p_2 + q_1)^2 &= a b_1, \\ -(q_1 + q_2)^2 &= b_1 b_3, & -(q_2 + q_3)^2 &= 0, \\ -(q_3 + p_1)^2 &= a b_2, \end{aligned} \tag{18}$$

and for $A^{(6)}(-\bar{q}_3, -\bar{q}_2, -\bar{q}_1, q_1, q_2, q_3)$,

$$\begin{aligned} -(\bar{q}_3 + \bar{q}_2)^2 &= 0, & -(\bar{q}_2 + \bar{q}_1)^2 &= \bar{b}_1 \bar{b}_3, \\ -(\bar{q}_1 - q_1)^2 &= b_1 \bar{b}_1, & -(q_1 + q_2)^2 &= b_1 b_3, \\ -(q_2 + q_3)^2 &= 0, & -(q_3 - \bar{q}_3)^2 &= b_2 \bar{b}_2, \\ -(\bar{q}_2 + \bar{q}_1 - q_1)^2 &= b_1 \bar{b}_2, & -(\bar{q}_1 - q_1 - q_2)^2 &= b_2 \bar{b}_1, \\ -(q_1 + q_2 + q_3)^2 &= s. \end{aligned} \tag{19}$$

One might have expected the five-point amplitude to depend on five independent Mandelstam invariants. However, as visible in Eq. (18), the counting is reduced by one. This happens because the low-energy five-point amplitude coming from the WZW term $\propto \epsilon_{\mu\nu\rho\sigma}\, p_1^\mu p_2^\nu q_1^\rho q_2^\sigma$, when expressed in terms of Mandelstam invariants, is proportional to the square root of $A_5 s^2 + B_5 s + C_5$, where A_5, B_5, and C_5 are functions of the other kinematic invariants. Requiring the amplitude to be analytic in s imposes an additional tuning among the other Mandelstams, $B_5 = \pm 2\sqrt{A_5 C_5}$, which brings the counting of independent Mandelstams down to four, as given in Eq. (18). A similar situation arises at six points, where we would expect nine independent Mandelstams. In four dimensions, however, we have a Gram constraint, which in principle reduces the counting by one. The Gram constraint can be written as $A_6 s^2 + B_6 s + C_6 = 0$, where A_6, B_6, and C_6 are functions of the other kinematic invariants. Satisfying the Gram constraint while keeping the other invariants s-independent—so that a single channel is activated—requires $A_6 = B_6 = C_6 = 0$, reducing the counting of free parameters to six, as visible in Eq. (19).

A Higher-Point Bound in Large-N_c QCD.—We are equipped with a kinematic chart that describes the amplitudes $A^{(4)}$, $A^{(5)}$, $A^{(6)}$ in terms of the same set of variables, in particular allowing us to write their residue sums as inner products as in Eq. (9) and thereby giving us the multipositivity bound in Eq. (13) . By explicit calculation, the planar partial amplitudes—as prescribed in Eq. (1)—associated with the EFT Lagrangian for the NLSM in Eqs. (2) and (3) are

$$
\begin{aligned}
A^{(4)} &= \frac{s+|a|^2}{2F_\pi^2} + \frac{2(L_3+2L_4)(s^2+|a|^4)+8L_4 s|a|^2}{F_\pi^4} + \cdots, \qquad A^{(5)} = \frac{i\kappa}{24\sqrt{2}F_\pi^5\pi^2} a\left[s(b_1-b_2) - b_1^2 b_3\right] + \cdots, \\
A^{(6)} &= \frac{|b_1-b_2|^2 + (b_1 b_3 - b_1\bar{b}_2 b_3/\bar{b}_1 + \text{c.c.})}{4F_\pi^4} - \frac{|b_1|^2|b_3|^2}{4F_\pi^4 s} + \frac{(L_3+2L_4)}{F_\pi^6|b_1|^2}\left[|b_1|^2\left(|b_1|^2+|b_2|^2\right)^2 + |b_1|^4|b_3|^2\right] \\
&\quad - \frac{(L_3+2L_4)}{F_\pi^6|b_1|^2}\left[|b_1|^2(|b_1|^2+|b_2|^2)\bar{b}_1 b_2 - b_1^2 b_3(\bar{b}_1-\bar{b}_2)(|b_2|^2+b_1 b_3) + \frac{|b_1|^4|b_3|^2 b_1 b_3}{s} + \text{c.c.}\right] + \frac{4|b_1-b_2|^2 L_4 s}{F_\pi^6} + \cdots,
\end{aligned}
\tag{20}
$$

where ellipses indicate terms of higher order in the derivative expansion. We now want to determine the number of subtractions we should take, i.e., the choices of i and j that make Eq. (10) and later inequalities true. To do so, we must consider the high-energy limit of the four-, five-, and six-point amplitudes with the other kinematic parameters fixed. The Regge behavior of higher-point amplitudes is highly nontrivial and has not been studied with the same rigor as for $2 \to 2$ processes [129]. The single contour integration in s means that to drop the boundary term we have to study the behavior of $A^{(n)}$ as $s \to \infty$ with all other kinematic invariants kept fixed. In this limit, planar Reggeized amplitudes at tree level obey the structure $A^{(n)} \to s^{\max\{\alpha_1,\alpha_2,\ldots,\alpha_{n-3}\}}$, where $\alpha_1,\ldots,\alpha_{n-3}$ correspond to the intercepts of the Regge trajectories along cuts of a half-ladder diagram [130, 131]. Large-N_c QCD does not have a pomeron in the leading trajectories, so all $\alpha_i < 1$ and $A^{(4)}$, $A^{(5)}$, and $A^{(6)}$ all grow more slowly than s. A once-subtracted dispersion relation is therefore sufficient to drop the boundary term, i.e., we can take $i, j \geq 1$ in Eqs. (10) and (12).

We are interested in the Wilson coefficients linear in s:

$$
\begin{aligned}
\lambda_1^{(4)} &= \frac{1}{2F_\pi^2} + \frac{8L_4}{F_\pi^4}|a|^2 + \cdots \\
\lambda_1^{(5)} &= \frac{i\kappa a(b_1-b_2)}{24\sqrt{2}F_\pi^5\pi^2} + \cdots \\
\lambda_1^{(6)} &= \frac{4L_4|b_1-b_2|^2}{F_\pi^6} + \cdots .
\end{aligned}
\tag{21}
$$

To eliminate contributions from yet higher-derivative terms and ensure that only terms quartic in momenta appear in our bounds, we differentiate with respect to s-independent parameters in Eq. (16) and then pin the parameters to zero,

$$
|\partial_a \partial_{b_1} \lambda_1^{(5)}|^2_{a,b_i=0} \leq \partial_a \partial_{\bar{a}} \lambda_1^{(4)} \partial_{b_1} \partial_{\bar{b}_1} \lambda_1^{(6)}|_{a,b_i=0}, \tag{22}
$$

which leads to

$$
\left|\frac{i\kappa}{24\sqrt{2}F_\pi^5\pi^2}\right|^2 \leq \left(\frac{8L_4}{F_\pi^4}\right)\left(\frac{4L_4}{F_\pi^6}\right). \tag{23}
$$

Recall that in large-N_c QCD, the parameters scale as $F_\pi^2 \propto L_4 \propto N_c$ and $\kappa = N_c$, so N_c cancels on both sides.

To confront this new constraint with bounds previously obtained from four-point positivity, let us define the dimensionless and N_c-independent quantities $\tilde{g}_{2,1} = 16L_4 M^2/F_\pi^2$ and $\tilde{g}_{2,0} = 4(L_3+2L_4)M^2/F_\pi^2$, where M is the mass of the lightest meson exchanged in $\pi\pi$ scattering [33, 34] and $L_4 \geq 0$. The bound in Eq. (23) becomes

$$
\tilde{g}_{2,1} \geq \frac{N_c M^2}{12\pi^2 F_\pi^2} \simeq 1.66, \tag{24}
$$

where in the last relation we have used large-N_c lattice computations [133] to fix the ρ meson mass $M \simeq 8.1\sqrt{3/N_c}F_\pi$ [134]. This leads to a new exclusion region highlighted in blue in Fig. 2. One sees that the bound

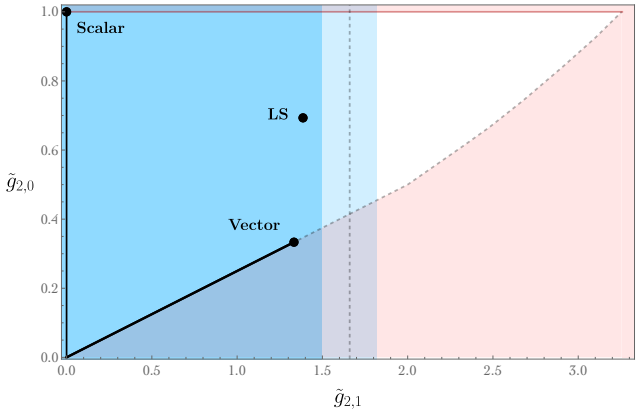


Figure 2. Multipositivity bound in Eq. (24) for large-N_c QCD (in blue) overlaid on the exclusion plot derived from four-point positivity [33, 34] (in red). The band delimiting the exclusion area represents the numerical uncertainty in the lattice computations [132].

severely constrains the scalar contribution to $\tilde{g}_{2,1}$, and is marginally inconsistent with a model where a single vector is exchanged, as well as the stringy Lovelace-Shapiro amplitude (LS), subject to lattice uncertainties.

Discussion.—In this paper, we have derived new constraints on the chiral Lagrangian. At the heart of our analysis is the insight that certain multiparticle scattering processes are mathematically inconsistent. Our results are the first phenomenologically relevant application of this fact. Using the four-, five-, and six-point pion amplitudes of the chiral Lagrangian, we deduced a novel bound on the size of the leading higher-derivative interactions with respect to the WZW term, yielding a remarkable mutual constraint on anomalous and nonanomalous physics. Previous attempts [35, 36] to bound the chiral anomaly relied on, and were therefore limited by, four-point amplitudes, requiring photons as external particles and thus introducing an additional sector. Our methods apply to tree-level planar amplitudes, which are the natural setting for large-N_c QCD.

The present analysis leaves numerous avenues for future exploration. Even in the case of planar tree-level amplitudes, we have only scratched the surface by constraining operators that are leading order in the momentum expansion. Our methods should generalize mechanically to the full tower of higher-derivative interactions in the chiral Lagrangian. While ever more operators will of course appear, with corresponding unknown Wilson coefficients, it may be possible to choose kinematic limits to tame this growth of free parameters. This work opens the door for a systematic numerical study [135] of higher-point constraints.

Since we are analytically continuing in a single Mandelstam variable, it is tempting to generalize beyond tree level. Indeed, if a Lehmann-Källén representation holds on a single cut, then our results immediately hold nonperturbatively, provided that the higher-point amplitude obeys analyticity in s and our Regge boundedness conditions. However, the analytic structure of higher-point amplitudes has not been as axiomatically investigated as the four-point case and is famously subtle, so caution is warranted in extending our results nonperturbatively.

A more technically fraught direction is nonplanar scattering, relevant to finite-N_c QCD. Our construction hinges on detailed control over which factorization channels are activated as the center-of-mass energy is scanned. In nonplanar amplitudes, additional channels and multitrace structures can appear, so it remains to be seen whether these methods extend in a useful form. We leave this question to future investigation.

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Appendix.—For the sake of completeness, we will briefly summarize our calculation of amplitudes in chiral perturbation theory. We employ the alternative formulation derived in Ref. [138], which is highly efficient for our purposes. This formulation was later shown to encode color-kinematics duality directly in terms of fields and currents [139]. In this approach, all dynamics are encoded in the chiral current,

$$j_\mu = i\,U^\dagger \partial_\mu U, \tag{25}$$

which is independent of the choice of embedding of the pion field within U.

The action governing chiral perturbation theory is

$$S = S_{\rm NLSM} + S_{\rm WZW}, \tag{26}$$

where the first and second terms encode the parity-even nonanomalous and parity-odd anomalous dynamics, respectively. In terms of the chiral current, Eq. (2) becomes

$$\mathcal{L}_{\rm NLSM} = -\frac{F_\pi^2}{4}\,{\rm Tr}(j^2) + L_3\,{\rm Tr}(j^2 j^2) + L_4\,{\rm Tr}(j_\mu j_\nu j^\mu j^\nu). \tag{27}$$

Since j_μ takes the form of a pure gauge configuration, its field strength vanishes identically,

$$F_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu - i[j_\mu, j_\nu] = 0. \tag{28}$$

At linearized order, the chiral current is

$$j_\mu = -\frac{\sqrt{2}}{F_\pi}\partial_\mu \pi + O(\pi^2), \tag{29}$$

which will be important later. Note also that

$$-\frac{F_\pi^2}{4}{\rm Tr}(j_\mu j^\mu) = -\frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + O(\pi^3), \tag{30}$$

so the pion kinetic term is canonically normalized.

To incorporate the WZW action, we specify the five-dimensional manifold $\mathcal{M}_5 = \mathcal{M}_4 \times [0,1]$ and take $U(x;y)$ to depend on an auxiliary coordinate y such that the ordinary chiral field corresponds to $y = 1$.

To derive the equations of motion, we apply an infinitesimal variation $U + \delta U = U e^{-i\,\delta\phi}$, so that $\delta U = -i\,U\delta\phi$ and $\delta U^{-1} = i\,\delta\phi\,U^{-1}$. The chiral current varies by

$$\begin{aligned}\delta j_\mu &= i(\delta U^{-1})\partial_\mu U + iU^{-1}\partial_\mu(\delta U)\\ &= \partial_\mu \delta\phi - i[j_\mu, \delta\phi] = D_\mu \delta\phi,\end{aligned} \tag{31}$$

which defines the covariant derivative. Integrating by parts, we obtain the variation of the NLSM action,

$$\delta S_{\rm NLSM} = \int d^4x\,\frac{F_\pi^2}{2}\,{\rm Tr}\left(\delta\phi\, D_\mu \mathcal{J}^\mu\right), \tag{32}$$

where we have defined

$$\mathcal{J}^\mu = j^\mu - \frac{4L_3}{F_\pi^2}\left(j^\mu j^2 + j^2 j^\mu\right) - \frac{8L_4}{F_\pi^2}\,j^\nu j^\mu j_\nu. \tag{33}$$

The identity $F_{\mu\nu} = 0$ implies

$$D_\mu \mathcal{J}^\mu = \partial_\mu \mathcal{J}^\mu, \tag{34}$$

which reduces to the conservation law $\partial_\mu \mathcal{J}^\mu = 0$ in the absence of the WZW action.

Meanwhile, the variation of the WZW action is

$$\delta S_{\rm WZW} = -\frac{\kappa}{48\pi^2}\int d^4x \int_0^1 dy\,\epsilon_{\tau\mu\nu\rho\sigma}\,{\rm Tr}\left((D^\tau \delta\phi) j^\mu j^\nu j^\rho j^\sigma\right). \tag{35}$$

To strip off the covariant derivative acting on $\delta\phi$, we use

$$\epsilon_{\tau\mu\nu\rho\sigma}\,\mathrm{Tr}[\delta\phi\, D^{\tau}(j^{\mu}j^{\nu}j^{\rho}j^{\sigma})] = 0, \tag{36}$$

which follows from the antisymmetry of the Levi-Civita tensor together with $F_{\mu\nu} = 0$. Then we have

$$\delta S_{\mathrm{WZW}} = -\frac{\kappa}{48\pi^2}\int d^4x\,\epsilon_{\alpha\beta\rho\sigma}\mathrm{Tr}\big(\delta\phi\, j^{\alpha}j^{\beta}j^{\rho}j^{\sigma}\big)\Big|_{y=1}, \tag{37}$$

which reduces the WZW variation to a four-dimensional chiral current interaction.

Combining both contributions from the NLSM and WZW actions at $y = 1$ yields

$$\frac{F_\pi^2}{2}\partial_\mu\mathcal{J}^{\mu} - \frac{\kappa}{48\pi^2}\epsilon_{\alpha\beta\rho\sigma}j^{\alpha}j^{\beta}j^{\rho}j^{\sigma} = 0, \tag{38}$$

which we rewrite as

$$\begin{aligned}\partial_\mu j^{\mu} = {}& \frac{1}{F_\pi^2}\partial^{\mu}\Big[4L_3\big(j_\mu j^2 + j^2 j_\mu\big) + 8L_4 j_\nu j_\mu j^{\nu}\Big] \\ & + \frac{\kappa}{24\pi^2 F_\pi^2}\epsilon_{\alpha\beta\rho\sigma}j^{\alpha}j^{\beta}j^{\rho}j^{\sigma},\end{aligned} \tag{39}$$

which is a modified conservation law for the chiral current.

We now reinterpret this equation as a Berends-Giele recursion relation for amplitudes in chiral perturbation theory. To do so, we introduce a source J for the pion,

$$\Box\pi = J + O(\pi^2). \tag{40}$$

In the presence of this source, Eq. (29) gives

$$\partial_\mu j^{\mu} = -\frac{\sqrt{2}}{F_\pi}J + O(\pi^2). \tag{41}$$

Following Ref. [139], we compute the divergence of the flat field strength $\partial^{\mu}F_{\mu\nu} = 0$ to obtain

$$\Box j_\nu - \partial_\nu(\partial_\mu j^{\mu}) - i\partial^{\mu}[j_\mu, j_\nu] = 0. \tag{42}$$

Then, substituting Eq. (39) yields

$$\begin{aligned}-\frac{\sqrt{2}}{F_\pi}\partial_\nu J = {}& \Box j_\nu - i\partial^{\mu}[j_\mu, j_\nu] \\ & - \frac{\partial_\nu\partial^{\mu}}{F_\pi^2}\left[4L_3\left(j_\mu j^2 + j^2 j_\mu\right) + 8L_4 j_\rho j_\mu j^{\rho}\right] \\ & - \frac{\kappa}{24\pi^2 F_\pi^2}\partial_\nu\left(\epsilon_{\alpha\beta\rho\sigma}j^{\alpha}j^{\beta}j^{\rho}j^{\sigma}\right).\end{aligned} \tag{43}$$

Since $\partial^{\mu}F_{\mu\nu} = 0$ is purely geometric and independent of the action, all higher-order interactions enter only through the deformation of the current divergence. Note that the pion source acts as an external polarization vector for the chiral current, $\epsilon_\mu = -\sqrt{2}ip_\mu/F_\pi$.

By iteratively solving this equation, we derive the one-point function of the chiral current at arbitrary perturbative order. The resulting recursive structure is considerably simpler than the standard pion Lagrangian expansion. To compute the n-point scattering amplitude, we compute the one-point function of the chiral current with $n-1$ pion sources, each entering with its own polarization ϵ_μ.

The above construction computes the one-point function of the chiral current rather than the pion field itself. However, the two are related at linearized order by

$$\pi = -\frac{F_\pi}{\sqrt{2}}(q^{\mu}\partial_\mu)^{-1}q_\mu j^{\mu} + O(\pi^2), \tag{44}$$

where q is an arbitrary reference vector and $(q^{\mu}\partial_\mu)^{-1}$ is defined mechanically in momentum space. To extract the pion one-point function, we take the momentum on shell and choose the polarization vector, $\tilde{\epsilon}_\mu = iF_\pi\, q_\mu/\sqrt{2}\,qp$. By dotting the one-point function of the chiral current with $n-1$ pion sources into the final polarization $\tilde{\epsilon}_\mu$, we obtain the n-point pion amplitude.

The explicit amplitudes from our calculation can be found in the supplemental files.

Paper 4: Strings from Almost Nothing

CALT-TH 2025-027

Strings from Almost Nothing

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We argue that string theory emerges inevitably from a few simple assumptions about physical scattering. Consistency alone requires that all tree-level four-point scattering amplitudes exhibit vanishing residues at prescribed values of the momentum transfer. Assuming ultrasoft high-energy behavior, we then prove that the space of minimally consistent amplitudes, whose residues exhibit these mandated zeros and nothing more, are precisely the amplitudes of Veneziano and Virasoro-Shapiro, thus establishing the uniqueness of strings. Similar logic also applies to five-point scattering.

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Introduction.—Are the laws of physics described by string theory? This question has elicited much controversy, despite being a well-posed scientific hypothesis that is either true or false. The good news is that string theory has a sharp prediction: the emergence of very specific higher-spin excitations in gravitational scattering at or below the Planck scale. The bad news is that experiment—the ultimate arbiter of all theories—will not offer any insight into this question in the foreseeable future, or perhaps ever.

In the absence of experimental input, it is tempting to commit faithfully either to string theory or its negation. A less devoted approach is to ask to what extent string theory might follow from more modest or widely accepted assumptions. One instead explores the space of consistent theories by "bootstrapping" all possible scattering amplitudes consistent with fundamental principles like unitarity, locality, and Lorentz invariance. Recent years have witnessed substantial progress in this area, for example on effective field theory constraints [1–35], the primal bootstrap [36–50], semidefinite programming and null constraints [51–70], and bootstrapping string amplitudes and their deformations [71–94].

In this paper, we argue for the uniqueness of string theory from the fewest assumptions possible. All amplitude properties—the spectrum of masses and spins, high-energy behavior, and even its explicit mathematical form—are outputs of this bootstrap construction.

At the heart of our analysis is a remarkable fact about *all tree-level, crossing-symmetric, Lorentz invariant, unitary amplitudes*. Without loss of generality, the spectrum of masses squared and spins is defined by unknown functions $\mu(n)$ and $J(n)$ indexed by integer level n, while the leading Regge behavior is defined by an unknown trajectory $\alpha(t)$ [95]. We will prove that the residue at large n vanishes when $\alpha(t)$ crosses certain negative integers. These "Regge zeros" are mandated by consistency and apply to planar and nonplanar amplitudes at four-point and beyond for any $\mu(n)$, $J(n)$, and $\alpha(t)$. To single out string theory we further assume:

i) *Ultrasoftness.* The Regge trajectory $\alpha(t)$ is bijective for $t<0$ [96] such that the amplitude falls off arbitrarily at infinite momentum transfer, so $\alpha(t)\to-\infty$ for $t\to-\infty$ [97]. As depicted in Fig. 1, ultrasoftness is exhibited by string theory [98] but not its q-deformations [99], which diverge at finite t, nor quantum field theory, whose softness is bounded [100, 101]. Ultrasoftness is equivalent to superpolynomial boundedness [87] together with bijectivity.

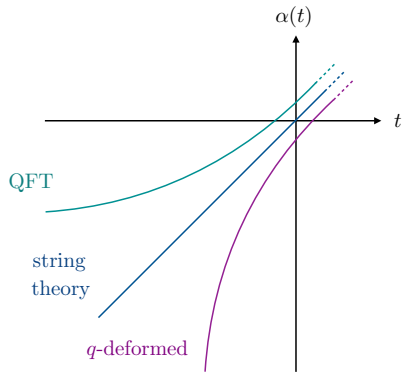


Figure 1. Schematic behavior of Regge trajectories for quantum field theory, string theory, and its q-deformations.

ii) *Minimal Zeros.* All zeros of the residues are Regge zeros mandated by self-consistency. Extra zeros are not physically inconsistent, but they are manifestly nonminimal because they go beyond what is required.

Given these modest conditions, our logic is simple: ultrasoft Regge behavior dictates an infinite set of points at which the residues of the amplitude are zero. Bootstrapping the space of objects that exhibit these required zeros and nothing more, we land precisely on the Veneziano and Virasoro-Shapiro amplitudes of string theory.

Note that our approach is quite distinct from that of Ref. [88], which argued for string uniqueness in tree-level four-point planar scattering while relying on a critical assumption about the distribution of amplitude zeros. Though conjectured to be universal, this property is now understood to be violated quite generally [87], leaving the question of string uniqueness an open one until now.

Regge Zeros.—Consider the four-point amplitude $A(s,t)$ for identical color-ordered scalars interacting via tree-level exchanges. This amplitude exhibits planar crossing symmetry, $A(s,t)=A(t,s)$, with simple poles

at $s,t=\mu(n)$, where $\mu(n)$ parameterizes an arbitrary mass-squared spectrum indexed by the integer level $n\geq 0$. Without loss of generality we take $\mu(n)$ to be monotonically increasing and parameterize the Regge limit at fixed t by some function $\alpha(t)$, which specifies the leading Regge trajectory via

$$A(s,t) \overset{s\to\infty}{\sim} f(t)(-s)^{\alpha(t)}. \tag{1}$$

The prefactor $f(t)$ is analytic except for simple poles at $t=\mu(n)$, whose residues are polynomials in s of degree $J(n)$, the maximal spin at level n [102]. Hence, the spectrum and the leading Regge trajectory satisfy

$$\alpha(\mu(n)) = J(n) \text{ for integer } n\geq 0, \tag{2}$$

which is applicable without any loss of generality.

These meager assumptions imply surprisingly stringent conditions on any tree-level, color-ordered, four-point amplitude. To see why, let us compute the "big arc integral" of $A(s,t)$ along a circle of large radius s_*,

$$\oint^{s_*} ds\, A(s,t) \sim f(t) s_*^{\alpha(t)+1} \frac{\sin\pi\alpha(t)}{\alpha(t)+1}. \tag{3}$$

This follows from Eq. (1) via explicit computation [103], but is intuitive to understand. If $\alpha(t)$ is an integer, then $s^{\alpha(t)}$ is a monomial whose contour integral is one if $\alpha(t)=-1$ and zero if $\alpha(t)\neq -1$. Either way, the contour integral is *independent* of the arc radius s_*. Deforming the contour inward gives a sum over enclosed residues,

$$\oint^{s_*} ds\, A(s,t) \sim \sum_n^{n_*} R(n,t), \tag{4}$$

where $R(n,t) = \lim_{s\to\mu(n)}(\mu(n)-s)A(s,t)$. This result looks very much *dependent* on the radius of the arc $s_*\sim\mu(n_*)$. To reconcile these facts, it must be that the residues vanish for integer $\alpha(t)$. To show this, we differentiate Eqs. (3) and (4) with respect to n_*, approximating the discrete sum over n with a continuous integral over dn, yielding

$$R(n,t) \sim f(t)\mu'(n)\mu(n)^{\alpha(t)} \sin\pi\alpha(t). \tag{5}$$

This holds in the Regge limit, which corresponds mathematically to $|t|\ll\mu(n)$, while also assuming $\mu'(n)\ll\mu(n)$ so the sum is well approximated by an integral.

The $\sin\pi\alpha(t)$ in Eq. (5) implies that $R(n,t)=0$ when $\alpha(t)$ is an integer, subject to two caveats. First, since $R(n,t)$ is nonsingular and $f(t)$ has poles at $t=\mu(n')$ for any integer $n'\geq 0$, those poles must cancel exactly against corresponding zeros in $\sin\pi\alpha(t)$. Thus, a genuine zero of Eq. (5) requires the off-resonance condition $t\neq\mu(n')$ for all $n'\geq 0$. Second, unitarity implies that $R(n,t)=\sum_{\ell=0}^{\infty} a_{n,\ell} G_\ell^{(D)}(1+2t/\mu(n))$ with $a_{n,\ell}\geq 0$. For $\ell>0$, the D-dimensional Gegenbauer polynomials satisfy $G_\ell^{(D)}(x) > G_\ell^{(D)}(1) > 0$ when $x>1$. Since $\mu(n)$, $J(n)>0$ at large n, we see that $R(n,t)>R(n,0)>0$ for $t>0$, so this region has no residue zeros. In summary, in the Regge limit of $|t|\ll\mu(n)$, the residue $R(n,t)$ vanishes on the support of Regge zeros,

$$\alpha(t)+r=0, \tag{6}$$

for any integer r such that the real roots of Eq. (6) satisfy $t<0$ and $t\neq\mu(n')$ for any integer $n'\geq 0$. These Regge zeros are required by consistency in any unitary, tree-level, color-ordered, four-point amplitude. Since Eq. (6) holds in the $|t|\ll\mu(n)$ Regge limit, any Regge zero of $R(n,t)$ will also be a Regge zero of $R(n',t)$ for $n'>n$ since the approximation of the Regge limit only improves as the level grows.

Softer Regge behavior implies more Regge zeros. For example, the Veneziano amplitude of string theory scales as $\Gamma(-s)\Gamma(-t)/\Gamma(-s-t) \overset{s\to\infty}{\sim} s^t$, with unboundedly fast falloff for sufficiently negative t. The expected Regge zeros at $t+r=0$ appear in the string residues,

$$R_{\text{str}}(n,t) = \frac{1}{n!}\prod_{r=1}^{n}(t+r) \sim n^t. \tag{7}$$

Regge zeros are ubiquitous [104], but they are often difficult to ascertain from analytic expressions. Consider three examples: the residues of the simplest deformed worldsheet amplitude in the family of Ref. [105], $\int_0^1 u^{-s-1}(1-u)^{-t-1}e^{gu(1-u)}du$ [106],

$$R_{\text{str}}(n,t)\, {}_2F_2\left[\begin{matrix}-n,-t\\ -\frac{n+t}{2},\frac{1-n-t}{2}\end{matrix};\frac{g}{4}\right], \tag{8}$$

the three-parameter amplitudes explored in Ref. [87],

$$R_{\text{str}}(n,t)\, {}_3F_2\left[\begin{matrix}-n,-t,-c_0-c_1(n+t)\\ -\frac{n+t}{2},\frac{1-n-t}{2}\end{matrix};\lambda\right], \tag{9}$$

and one of the bespoke amplitudes of Ref. [79],

$$2(n+\delta)(R_{\text{str}}(n,-\delta+\sqrt{t}) + R_{\text{str}}(n,-\delta-\sqrt{t})). \tag{10}$$

Here $R_{\text{str}}(n,t)$ appears in all of these residues, but its zeros are either cancelled by poles or scrambled by changes of arguments. Nevertheless, Eqs. (8), (9), and (10) are all polynomials in t that exhibit their expected Regge zeros at large n, as emerges quite vividly via direct numerical evaluation in Fig. 2. Since Eqs. (8) and (9) have the same spectrum and Regge behavior as the Veneziano amplitude, they also have Regge zeros that asymptote to $t+r=0$. In contrast, Eq. (10) describes a modified spectrum, $\mu(n)=(\delta+n)^2$, and Regge behavior, $A(s,t)\overset{s\to\infty}{\sim}(\sqrt{s})^{-\delta+\sqrt{t}}+(\sqrt{s})^{-\delta-\sqrt{t}}$, so its Regge zeros are at $t=(\delta-r)^2$ for t off-resonance, as expected.

Equation (5) implies a quite constraining residue scaling, $R(n,t)\sim\mu(n)^{\alpha(t)}$. In fact, this property rules out any amplitude with a single Regge trajectory. By definition, the residues of any such amplitude describe a single state of spin n state at level n,

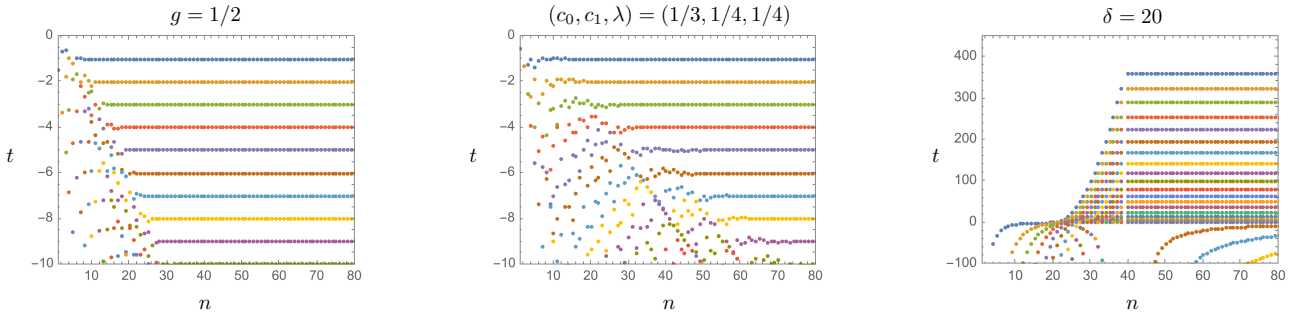


Figure 2. The residues of tree-level four-point amplitudes exhibit Regge zeros at asymptotically large level n. Shown here are the real components of the zeros in t for the residues defined in Eqs. (8), (9), and (10), for levels $n = 1$ to 80.

so $R(n,t) = c(n)\, G_n^{(D)}(1 + 2t/\mu(n))$. This implies that $\partial_t^k \log R(n,t)|_{t=0} \sim (n^2/\mu(n))^k$ at large n, modulo multiplicative factors. However, from the residue scaling we know that $\partial_t^k \log R(n,t)|_{t=0} \sim \log \mu(n)$, indicating the same scaling with n for all k, which is a contradiction. Similar conclusions were obtained via different methods in Refs. [107, 108]. All of these no-go results assume that the single Regge trajectory extends to infinity without any additional high-energy states.

Uniqueness Argument.—We are now equipped to derive a uniqueness principle for string theory, assuming *i*) ultrasoft Regge behavior, and *ii*) residue zeros comprised only of Regge zeros required by consistency. The former imposes an infinite number of Regge zeros at asymptotically large level n. The latter forbids any extra zeros that are not Regge zeros, even at finite n. As noted earlier, Regge zeros necessarily accumulate for increasing n, so *ii*) implies a nested structure whereby every zero of $R(n,t)$ is also a zero of $R(n',t)$ for $n' > n$.

To begin, we take d/dn of the log of Eq. (5) to obtain

$$\alpha(t)\frac{\mu'(n)}{\mu(n)} + \frac{\mu''(n)}{\mu'(n)} \sim \frac{d\log R(n,t)}{dn} \sim \log\frac{R(n,t)}{R(n-1,t)}, \quad (11)$$

where in the final step we have approximated the derivative as a finite difference. Equation (11) implies that, up to a trivial normalization and shift, the Regge trajectory is $\alpha(t) \sim \log P(n,t)$, where $P(n,t) = R(n,t)/R(n-1,t)$ is a rational function of t. Assumption *ii*) says that the residues $R(n,t)$ and $R(n-1,t)$ only exhibit Regge zeros, and we argued earlier that any Regge zeros of $R(n-1,t)$ are also Regge zeros of $R(n,t)$, thus canceling in the ratio and leaving $P(n,t)$ to be a polynomial in t. Without loss of generality, $P(n,t) \sim P(t) + \cdots$, where $P(t)$ is leading in the large-n expansion and the ellipses are subleading.

For nonconstant $P(t)$, at sufficiently large t this polynomial scales as $P(t) \sim t^k$ for some $k > 0$. This implies a logarithmic Regge trajectory $\alpha(t) \sim k \log t$. Plugging into Eq. (2), we learn that $\mu(n) \sim e^{J(n)/k}$, which is precisely the q-deformed spectrum of the Coon amplitude [99], which has $P(t) \sim 1 + (q-1)t$ [109]. Since $k > 0$, we see that $\alpha(t) \to \infty$ as $t \to -\infty$, in violation of assumption *i*). These logarithmic trajectories violate our assumptions, so we do not consider them further.

For constant $P(t)$, we retain the ellipses of $P(n,t) \sim P(t) + \cdots$ to get the leading t-dependence of the Regge trajectory. This happens in the Veneziano amplitude, where $P(n,t) \sim 1 + t/n$. As the subleading terms are polynomial, $\alpha(t)$ is as well. We focus on this case next.

All residues exhibit the Regge zeros in Eq. (6) for $|t| \ll \mu(n)$, but assumption *ii*) enforces the stronger condition that these Regge zeros comprise *all the zeros* of the residue for all t, so $R(n,t)$ is a product of factors of $\alpha(t) + r$. Since unitarity restricts all Regge zeros to the region $t < 0$, assumption *i*) implies that $\alpha(t) < \alpha(0)$, which on the Regge zero requires $r = -\alpha(t) > -\alpha(0)$. Here we assume a minimal value of $r = 1$, since any finite offset will just shift $\alpha(t)$ in a way that leaves our conclusions unchanged. Concretely, we define [110]

$$R(n,t) = c(n) \prod_{r=1}^{N(n)} (\alpha(t) + r)^h, \quad (12)$$

where h is a constant multiplicity of each Regge zero, $c(n)$ is an arbitrary normalization, and we extend the range of r up to $N(n) = J(n)/h\deg(\alpha)$. From Eq. (12) we see that $R(n,t) \sim R_{\rm str}(N(n), \alpha(t))^h \sim N(n)^{h\alpha(t)}$, while Eq. (5) implies that $R(n,t) \sim \mu(n)^{\alpha(t)}$. Comparing these representations, we learn that $\mu(n) \sim N(n)^h \sim J(n)^h$. Plugging into Eq. (2), we determine the Regge trajectory, $\alpha(t) \sim t^{1/h}$. We argued earlier that this function is a polynomial, so $h = 1$. Hence, we arrive at a linear spectrum and Regge trajectory,

$$\mu(n) \sim J(n) \quad \text{and} \quad \alpha(t) \sim t, \quad (13)$$

which is one of the main results of this paper. In conclusion, our assumptions *i*) and *ii*) imply the Chew-Frautschi scaling and linear Regge behavior that are the defining properties and historical raison d'être [98, 111–113] of string theory. Note that it is straightforward to also derive that $J(n) \sim n$, which we detail in App. A.

With $J(n) \sim n$ and $\alpha(t) \sim t$, the residue ansatz in Eq. (12) precisely reduces to the "level truncation" ansatz proposed in Refs. [91, 92], which is solved analytically to obtain a three-parameter space of amplitudes. Within this family, the only object that exhibits arbitrarily soft Regge behavior is $\Gamma(-s)\Gamma(-t)/\Gamma(-s-t)$, which is precisely the Veneziano amplitude [114]. Thus, our

assumptions *i*) and *ii*) actually uniquely bootstrap the Veneziano amplitude in its full mathematical form!

Assumption *ii*) is our least conservative condition, but it can be relaxed by including extraneous zeros in the residue in Eq. (12). For example, if Eq. (12) is augmented by a level-independent factor $g(t)$, as might occur for external states with spin, all of our arguments still apply. For a level-dependent factor, $g(n,t)$, our logic persists if the number of extraneous zeros is parametrically smaller than the number of Regge zeros.

Beyond Planar.—Up until now we have focused on planar amplitudes. However, our logic generalizes straightforwardly to the case of nonplanar scattering. Consider the four-point amplitude $A(s,t)$ for identical massless scalars interacting via tree-level exchanges. The amplitude is crossing symmetric under permutations of s,t,u and has poles at $s,t,u=\mu(n)$. The Regge behavior at fixed t is parameterized by

$$A(s,t) \overset{s\to\infty}{\sim} f(t)(-s)^{\frac{\alpha(t)}{2}}(-u)^{\frac{\alpha(t)}{2}}, \tag{14}$$

which is crossing symmetric under the exchange of s and $u=-s-t$. As before, we compute the big arc integral along a circle of radius s_* in the s plane,

$$\oint^{s_*} d(su)\, A(s,t) \sim f(t) s_*^{\alpha(t)+2} \frac{\sin\frac{1}{2}\pi\alpha(t)}{\alpha(t)+2}, \tag{15}$$

and deform the contour to equate with a weighted sum of residues, $\sum_n^{n_*}(\mu(n)+t/2)R(n,t)$. Differentiating with respect to n_*, we find that the residues take the form

$$R(n,t) \sim f(t)\mu'(n)\mu(n)^{\alpha(t)} \sin\tfrac{1}{2}\pi\alpha(t) \tag{16}$$

in the $|t| \ll \mu(n)$ Regge limit, implying the Regge zeros,

$$\alpha(t) + 2r = 0, \tag{17}$$

for integer r, again with the caveat that any real roots must satisfy $t<0$ and $t\neq\mu(n')$ for any integer $n'\geq 0$.

Equation (16) is nearly identical to Eq. (5), so applying the same logic as in the planar case, we deduce that $\alpha(t)$ is a polynomial. As before, assumption *ii*) says that the residue is composed entirely of Regge zeros, so

$$R(n,t) = c(n) \prod_{r=1}^{N(n)} (\alpha(t)+2r)^h. \tag{18}$$

Since $R(n,t) \sim R_{\rm str}(N(n),\alpha(t)/2)^h \sim N(n)^{h\alpha(t)/2}$ and the asymptotic behavior is $R(n,t)\sim\mu(n)^{\alpha(t)}$, we find that $\mu(n)\sim N(n)^{h/2}\sim J(n)^{h/2}$. Together with Eq. (2), this implies that $\alpha(t)\sim t^{2/h}$, and since $\alpha(t)$ is a polynomial, we deduce that $h=2$. Thus the nonplanar amplitude also has the linear spectrum and Regge trajectory defined in Eq. (13). As in the planar case, App. A implies $J(n)\sim n$, so our nonplanar uniqueness arguments lead to the level truncation ansatz of Ref. [92], whose only solution with ultrasoft Regge behavior is $\Gamma(-s)\Gamma(-t)\Gamma(-u)/\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)$, which is the Virasoro-Shapiro amplitude [115, 116].

Beyond Four-Point.—The Regge behavior of five-point string amplitudes also mandates Regge zeros in their corresponding residues. In fact, these zeros are sufficient to uniquely bootstrap the five-point amplitude.

Here we study a worldsheet integral representation of the five-point string amplitude constructed in Ref. [93],

$$\begin{aligned} A_{\rm str,5} &= \int_0^1 \frac{du_{13}}{u_{13}(1-u_{13})}\frac{du_{14}}{u_{14}(1-u_{14})} F(u_{13},u_{14}), \\ F(u_{13},u_{14}) &= \frac{u_{13}^{X_{13}}u_{14}^{X_{14}}(1-u_{13})^{X_{24}}(1-u_{14})^{X_{35}}}{(1-u_{13}u_{14})^{X_{24}+X_{35}-X_{25}}}, \end{aligned} \tag{19}$$

where $X_{ij}=(p_i+\cdots+p_{j-1})^2$ [117]. In the Regge limit $X_{13},X_{14}\to\infty$, the factors of $u_{13}^{X_{13}}$ and $u_{14}^{X_{14}}$ localize the integral to the saddle at $u_{13}=1-X_{13}^{-1}$ and $u_{14}=1-X_{14}^{-1}$, with scaling $F(1-X_{13}^{-1},1-X_{14}^{-1}) \sim X_{13}^{X_{35}-X_{25}}X_{14}^{X_{24}-X_{25}}(X_{13}+X_{14})^{X_{25}-X_{24}-X_{35}}$. Since we have $F(1-\epsilon,1-\epsilon)\sim\epsilon^{X_{25}}$, the integral converges provided $X_{25}>0$, just like the ${}_3F_2$ closed-form expression for the five-point string amplitude [118], which can be analytically extended to the entire kinematic space using the formulas of Ref. [93]. In terms of the large s-like invariants, $s_1=-X_{13}$ and $s_2=-X_{14}$, and the fixed t-like invariants, $t_{23}=-X_{24}$, $t_{34}=-X_{35}$, and $t_{51}=-X_{25}$, we obtain our final result for the double Regge limit of the five-point string amplitude when $s_1,s_2\to\infty$,

$$A_{\rm str,5} \overset{s_{1,2}\to\infty}{\sim} f(a_1,a_2,a_{12})(-s_1)^{a_1}(-s_2)^{a_2}(-s_1-s_2)^{a_{12}}, \tag{20}$$

where we have defined $a_1=t_{51}-t_{34}$, $a_2=t_{51}-t_{23}$, and $a_{12}=t_{23}+t_{34}-t_{51}$. In analogy with our analysis at four-point, we now compute the double big arc integral, $\oint^{s_{1*}}\oint^{s_{2*}} ds_1 ds_2\, A_{\rm str,5}$, which, by deforming both arc integral contours into a double sum over residues, equals $\sum_{n_1}^{n_{1*}}\sum_{n_2}^{n_{2*}} R_{\rm str,5}(n_1,n_2,t_{23},t_{34},t_{51})$. Inserting Eq. (20), we compute the double big arc integral to be f times

$$\sum_{b=0}^{a_{12}} s_{1*}^{a_1+1+b} s_{2*}^{a_2+1+a_{12}-b} \tbinom{a_{12}}{b} \frac{\sin\pi a_1 \sin\pi(a_2+a_{12})}{(a_1+1+b)(a_2+1+a_{12}-b)}, \tag{21}$$

where have taken $a_{12}\geq 0$ to be a nonnegative integer so that the binomial expansion is finite. Differentiating with respect to n_{1*} and n_{2*}, we obtain

$$\begin{aligned} &\frac{R_{\rm str,5}(n_1,n_2,t_{23},t_{34},t_{51})}{\mu'(n_1)\mu'(n_2)\mu(n_1)^{a_1}\mu(n_2)^{a_2+a_{12}}} \\ &\quad\sim f\sum_{b=0}^{a_{12}} \left(\tfrac{\mu(n_1)}{\mu(n_2)}\right)^b \tbinom{a_{12}}{b} \sin\pi a_1 \sin\pi(a_2+a_{12}), \end{aligned} \tag{22}$$

which stringently constrains the asymptotics and zeros of the five-point residues. By inspection, from Eq. (22) we see that the residue vanishes when a_1, a_2, and a_{12} are integers. Recalling that $X_{25}>0$ for convergence, we deduce the locus of the Regge zeros for five-point residues at large level,

$$-(a_1+a_2+a_{12})\geq 1 \quad\text{and}\quad a_{12}\geq 0, \tag{23}$$

for integers a_1,a_2,a_{12}.

It is straightforward to verify the presence of these Regge zeros in the known expression for the five-point string residue $R_{\text{str},5}(n_1, n_2, t_{23}, t_{34}, t_{51})$ [93],

$$\frac{(1+t_{23})_{n_1}(1+t_{34})_{n_2}}{n_1!n_2!}{}_3F_2\left[\begin{matrix}-n_1, -n_2, -t_{23}-t_{34}+t_{51} \\ -n_1-t_{23}, -n_2-t_{34}\end{matrix}; 1\right]. \quad (24)$$

We see that there are indeed zeros for $a_1+r_1 = a_2+r_2 = a_{12}+r_{12}-r_1-r_2 = 0$, where $|r_1| \leq n_1$, $|r_2| \leq n_2$, and $1 \leq r_{12} \leq r_1+r_2$. This agrees exactly with Eq. (23) at large n_1 and n_2. These Regge zeros can be used to constrain a five-point residue ansatz $R(n_1, n_2, t_{23}, t_{34}, t_{51}) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{\min(n_1-i, n_2-j)} \lambda_{i,j,k}(n_1, n_2) t_{23}^i t_{34}^j t_{51}^k$ that encodes all possible exchanges of particles with maximum spins $n_{1,2}$ on the poles at $s_{1,2} = n_{1,2}$. Enforcing the Regge zeros fixes all $\lambda_{i,j,k}(n_1, n_2)$ to values corresponding *precisely* to the known residues of the string in Eq. (24), modulo a normalization $c(n_1, n_2)$.

Since Regge zeros fix the five-point string residue, it is natural to ask about the structure of its zeros more generally, which we describe in detail in App. B. Setting all three t-like Mandelstams to certain negative integers, one finds that all but a finite number of the residues vanish, reducing an infinite dual resonant sum over all levels (n_1, n_2) down to a finite subset, making the amplitude rational in (s_1, s_2). As shown in Ref. [91], four-point crossing symmetry sculpts out an analytically solvable space of amplitudes that exhibit level truncation. Analogously, in five-point scattering the full cyclic invariance of the external legs produces a similarly solvable algebraic problem. As detailed in App. C, demanding cyclic invariance in fact fixes all $c(n_1, n_2) = 1$. This implies that our bootstrapped amplitude is uniquely fixed by its residue zeros to match the five-point string amplitude. The q-deformed version of our planar analyses is presented in App. D.

Discussion.—We have demonstrated a universal link between the Regge behavior of an amplitude and the zeros of its residues. This result applies to any four-point, tree-level, crossing-symmetric, Lorentz invariant, unitary amplitude. Amplitudes with softer Regge behavior are constrained by more Regge zeros. Assuming a minimality condition that every zero of a residue is a Regge zero mandated by consistency, we proved that any ultrasoft amplitude has a stringy spectrum of masses squared that is linear in spin, $\mu(n) \sim J(n) \sim n$, together with a linear Regge trajectory, $\alpha(t) \sim t$. This reproduces the level truncation ansatz of Ref. [91], from which the exact mathematical formula for the Veneziano amplitude is derived uniquely. While extra zeros are certainly allowed, the notion of minimal zeros gives an operational definition of the "simplest consistent amplitude," which is the Veneziano amplitude itself. Last but not least, we established analogous Regge zero and uniqueness properties beyond the planar four-point limit, successfully bootstrapping the four-point closed-string and five-point open-string amplitudes.

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[95] We work in mostly-plus metric signature throughout, where the Mandelstam invariants are $s = -(p_1 + p_2)^2$, $t = -(p_2 + p_3)^2$, and $u = -(p_1 + p_3)^2$ for cyclically labeled incoming momenta.

[96] Note that the $t > 0$ behavior is unspecified, so a finite or infinite tower of spins is a priori allowed.

[97] Strictly speaking, we impose this property on the real component of $\alpha(t)$, which controls the magnitude of the amplitude in the Regge limit, along the real t axis.

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[102] The case of infinite $J(n)$ at finite n corresponds to an infinite tower of spins residing on an isolated, narrow resonance. This describes a propagating mode of unbounded spatial extent, in violation of locality [10, 52, 76].

[103] Equation (1) exhibits a branch cut that we take to be along the positive real s axis, which is the location of all physical discontinuities. To compute Eq. (3) we transform to polar coordinates, $s = s_* e^{i\theta}$, and integrate around the branch cut in the domain $0 < \theta < 2\pi$.

[104] Note that the hypergeometric [78] and ${}_4F_3$ satellite [90] amplitudes exhibit power-law Regge behavior reminiscent of quantum field theory, where $\alpha(t)$ crosses few if any integers, yielding few if any Regge zeros, as expected.

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$$A(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} {}_2F_2\left[\begin{matrix} -s, -t \\ -\frac{s+t}{2}, \frac{1-s-t}{2} \end{matrix}; \frac{g}{4}\right],$$

by expanding the exponential and resumming.

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[108] C. Eckner, F. Figueroa, and P. Tourkine, "On the number of Regge trajectories for dual amplitudes," arXiv:2405.21057 [hep-th].

[109] The original Coon amplitude [99], whose residues are consistent with unitarity only for $q < 1$, exhibits an accumulation point in the spectrum and is neither dual resonant nor meromorphic, on account of a branch cut dressing factor [78, 119]. While amplitudes of this form are realized in nonperturbative string theoretic constructions [77], they depart from our assumption of tree-level dynamics, so we leave them to future work.

[110] Here we have implicitly assumed that *every* zero of $\alpha(t) + r = 0$ appears in $R(n,t)$ as a block. Said another way, $R(n,t)$ does not include partial branches of solutions where only a subset of the zeros of $\alpha(t) + r = 0$ appear. This structure is justified because of the underlying logic of the Regge zeros: if one solution of $\alpha(t) + r = 0$ appears in $R(n,t)$, then n is sufficiently large to justify the arc integral argument that implies the Regge zeros, and thus the other solutions of $\alpha(t) + r = 0$ should be exhibited as well.

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Appendix A: Linear Spin Spectrum.—Under our stated assumptions, we have shown that $\mu(n) \sim J(n)$. Here we will furthermore deduce that $J(n) \sim n$. To begin, we define t_* to be the locus of the Regge zero at $\alpha(t_*)+r_* = 0$, where $r_* = J(n_*)$ is the maximum spin of the residue at level n_*. We compute the big arc integral,

$$\oint^{s_*} ds\, s^{r-1} A(s,t_*) \sim s_*^{\alpha(t_*)+r} = 0, \tag{A1}$$

which is zero for any r in the range $1 \le r < r_*$. Hence the number of constraints scales as $r_* \sim J(n_*)$ at large level. We can also rewrite the big arc integral as a sum of residues in the usual way,

$$\oint^{s_*} ds\, s^{r-1} A(s,t_*) \sim \sum_{n=0}^{n_*-1} \mu(n)^{r-1} R(n,t_*). \tag{A2}$$

The residue at level n vanishes unless $J(n) < J(n_*)$, since the residues for which $J(n) \ge J(n_*)$ carry vanishing factors of $\alpha(t_*)+r_* = 0$. Since $\mu(n)$ is without loss of generality strictly monotonic, and we have proven that $\mu(n) \sim J(n)$, it follows that $J(n)$ is strictly monotonic as well, and the sum truncates to the region $n < n_*$.

The above sum rules can be interpreted as a set of equations constraining the residues $R(n,t_*)$, which are taken to be variables. The total counting is then

$$\begin{aligned} &\#\text{ of equations} = \#\text{ of sum rules} \sim J(n_*) \\ &\#\text{ of variables} = \#\text{ of nonzero residues} \sim n_*. \end{aligned} \tag{A3}$$

Demanding that the number of constraints not exceed the number of variables, we have

$$J(n_*) \lesssim n_* \tag{A4}$$

at large n_*. Since $J(n)$ is monotonic and integer-valued, $J(n+1)-J(n) \ge 1$, which with Eq. (A4) leaves uniquely

$$J(n) \sim n \tag{A5}$$

at large n. This establishes that the spectrum of spins grows linearly with the level.

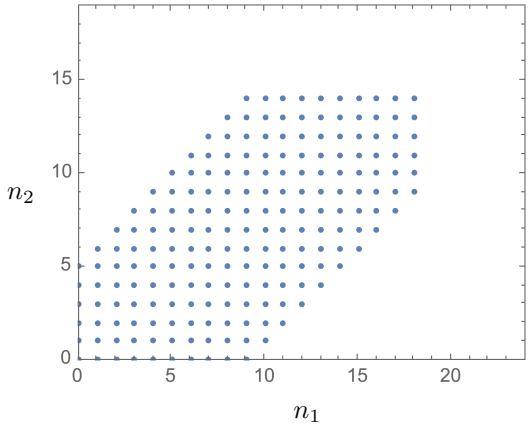


Figure 3. The domain $D(10,6,25)$, where each dot represents a nonzero residue at level (n_1,n_2) in the kinematic configuration $(t_{23},t_{34},t_{51}) = -(10,6,25)$. The number of nonzero residues, and hence the dual resonant sum, is finite.

Appendix B: Five-Point Level Truncation.—Let us write the string amplitude as a dual resonant sum [93] of the five-point string residues in Eq. (24), so

$$\begin{aligned} &A_{\text{str},5}(s_1,s_2,t_{23},t_{34},t_{51}) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{R_{\text{str},5}(n_1,n_2,t_{23},t_{34},t_{51})}{(n_1-s_1)(n_2-s_2)} \\ &= \frac{\Gamma(-s_1)\Gamma(-s_2)\Gamma(-t_{23})\Gamma(-t_{34})}{\Gamma(-s_1-t_{23})\Gamma(-s_2-t_{34})} \times \\ &\quad \times {}_3F_2\left[\begin{matrix} -s_1, -s_2, -t_{23}-t_{34}+t_{51} \\ -s_1-t_{23}, -s_2-t_{34} \end{matrix}; 1\right]. \end{aligned} \tag{A6}$$

The second equality converges when $t_{51} < 0$, but can be extended to arbitrary kinematics via a ${}_6F_5$ function [93]. The five-point residues of the string exhibit a generalization of the level truncation property of four-point amplitudes observed in Ref. [91]. In particular, on the kinematic configuration defined by $(t_{23},t_{34},t_{51}) = -(k_1,k_2,k_3)$ for integers $k_{1,2} \ge 1$ and $k_3 \ge k_1+k_2$, all but a finite number of five-point string residues in Eq. (24) vanish. In particular, the only nonzero residues occur at level (n_1,n_2) in a compact domain $D(k_1,k_2,k_3)$ defined by

$$\left\{ (n_1,n_2) \,\middle|\, \begin{matrix} n_1 < k_3-k_2, n_2 < k_3-k_1, \\ -k_2 < n_1-n_2 < k_1, \\ k_{1,2} \ge 1, k_3 \ge k_1+k_2 \end{matrix} \right\}, \tag{A7}$$

depicted in Fig. 3. On these kinematics, the amplitude in Eq. (A6) reduces to a rational polynomial in (s_1,s_2).

Appendix C: Five-Point Normalization.—We saw earlier how the five-point Regge zeros are sufficient to uniquely fix the five-point residue of the string amplitude up to a level-dependent normalization factor $c(n_1,n_2)$, so

that the five-point bootstrapped amplitude is given by

$$A(s_1,s_2,t_{23},t_{34},t_{51}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{c(n_1,n_2)R_{\text{str},5}(n_1,n_2,t_{23},t_{34},t_{51})}{(n_1-s_1)(n_2-s_2)}. \quad \text{(A8)}$$

Here we will fix these constants as well.

To fix $c(n_1,n_2)$, we exploit the invariance of the amplitude under cyclic permutations of the external legs. Level truncation renders the amplitude a rational function of the kinematics. On this configuration, the constraint of crossing is a solvable algebraic problem. In particular, setting $(s_1,s_2,t_{23},t_{34},t_{51}) = -(k_1,k_4,k_2,k_3,k_5)$ for $D(k_2,k_3,k_5)$ and $D(k_3,k_4,k_1)$ defined in Eq. (A7), we enforce cyclic invariance of the dual resonant form of the five-point amplitude, $A(-k_1,-k_4,-k_2,-k_3,-k_5) = A(-k_2,-k_5,-k_3,-k_4,-k_1)$, that is,

$$\sum_{\substack{(n_1,n_2)\in\\ D(k_2,k_3,k_5)}}\frac{c(n_1,n_2)R_{\text{str},5}(n_1,n_2,-k_2,-k_3,-k_5)}{(n_1+k_1)(n_2+k_4)} = \sum_{\substack{(n_1,n_2)\in\\ D(k_3,k_4,k_1)}}\frac{c(n_1,n_2)R_{\text{str},5}(n_1,n_2,-k_3,-k_4,-k_1)}{(n_1+k_2)(n_2+k_5)}. \quad \text{(A9)}$$

Enforcing this condition for all possible choices of integers yields a unique solution, $c(n_1,n_2)=1$, modulo uniform global normalization. Hence the five-point string amplitude is uniquely specified by its zeros, together with cyclic symmetry.

Appendix D: *q*-Deformations.—The procedure of bootstrapping the four- and five-point amplitudes from their Regge zeros can be generalized from a linear spectrum to a q-spectrum. Relaxing bijectivity of $\alpha(t)$ in assumption *i*) and running through our argument in text for the logarithmic Regge trajectory, we have Regge zeros at $\alpha(t)+r=0$ for integer r, whose root scales as $t\propto 1-q^{-r}$. We choose the normalization of t such that these Regge zeros appear at $t=(1-q^{-r})/(1-q)=[-r]_q$, so the spectrum is given by the q-deformed integers, $\mu(n)=[n]_q$, and we have $\alpha(t)=\log(1+(q-1)t)/\log q$.

Let us first consider four-point scattering. Assuming that the residue exhibits only the Regge zeros, $R(n,t)=c(n)\prod_{r=1}^{n}(t-[-r]_q)$, we construct a dual resonant representation of the four-point amplitude, obtaining

$$A(s,t)=\sum_{n=0}^{\infty}\frac{R(n,t)}{\mu(n)-s}=q^{\alpha(s)\alpha(t)}\frac{\Gamma_q(-\alpha(s))\Gamma_q(-\alpha(t))}{\Gamma_q(-\alpha(s)-\alpha(t))}, \quad \text{(A10)}$$

where the level truncation results of Ref. [91] have been used with crossing symmetry to uniquely fix $c(n)=q^{n(n+3)/2}/\Gamma_q(1+n)$. The above expression is precisely the Coon amplitude, which for $q>1$ is dual resonant on kinematics where the sum in Eq. (A10) converges [91].

The five-point q-deformed amplitude can be written in terms of the q-hypergeometric function ${}_3\phi_2$ as [86, 128]

$$A_{q\text{-str},5}(s_1,s_2,t_{23},t_{34},t_{51}) = q^{\alpha(s_1)\alpha(t_{23})+\alpha(s_2)\alpha(t_{34})}\times \times\frac{\Gamma_q(-\alpha(s_1))\Gamma_q(-\alpha(s_2))\Gamma_q(-\alpha(t_{23}))\Gamma_q(-\alpha(t_{34}))}{\Gamma_q(-\alpha(s_1)-\alpha(t_{23}))\Gamma_q(-\alpha(s_2)-\alpha(t_{34}))}\times \times{}_3\phi_2\left[\begin{matrix}q^{-\alpha(s_1)},q^{-\alpha(s_2)},q^{\alpha(t_{51})-\alpha(t_{23})-\alpha(t_{34})}\\ q^{-\alpha(s_1)-\alpha(t_{23})},q^{-\alpha(s_2)-\alpha(t_{34})}\end{matrix};q;q\right]. \quad \text{(A11)}$$

For $q>1$, this amplitude is dual resonant for kinematics where the sum converges, equaling

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{R_{q\text{-str},5}(n_1,n_2,t_{23},t_{34},t_{51})}{([n_1]_q-s_1)([n_2]_q-s_2)}, \quad \text{(A12)}$$

for the corresponding five-point residue polynomial,

$$R_{q\text{-str},5}(n_1,n_2,t_{23},t_{34},t_{51}) = \frac{(-1)^{n_1+n_2}q^{\frac{n_1(n_1+3)+n_2(n_2+3)}{2}+n_1\alpha(t_{23})+n_2\alpha(t_{34})}}{\Gamma_q(n_1+1)\Gamma_q(n_2+1)}\times \times\frac{\Gamma_q(-\alpha(t_{23}))\Gamma_q(-\alpha(t_{34}))}{\Gamma_q(-n_1-\alpha(t_{23}))\Gamma_q(-n_2-\alpha(t_{34}))}\times \times{}_3\phi_2\left[\begin{matrix}q^{-n_1},q^{-n_2},q^{\alpha(t_{51})-\alpha(t_{23})-\alpha(t_{34})}\\ q^{-n_1-\alpha(t_{23})},q^{-n_2-\alpha(t_{34})}\end{matrix};q;q\right]. \quad \text{(A13)}$$

Simply replacing t_{ij} with $\alpha(t_{ij})$ in the definitions of a_1,a_2,a_{12}, so $a_1=\alpha(t_{51})-\alpha(t_{34})$, $a_2=\alpha(t_{51})-\alpha(t_{23})$, and $a_{12}=\alpha(t_{23})+\alpha(t_{34})-\alpha(t_{51})$, we find that this residue exhibits precisely the same Regge zeros as the five-point string amplitude, namely, $a_1+r_1=a_2+r_2=a_{12}+r_{12}-r_1-r_2=0$, that is, $t_{23}=[r_2-r_{12}]_q$, $t_{34}=[r_1-r_{12}]_q$, and $t_{51}=[-r_{12}]_q$, all for integers $|r_1|\le n_1$, $|r_2|\le n_2$, and $1\le r_{12}\le r_1+r_2$.

Remarkably, we find that the q-deformed five-point amplitude can be constructed uniquely from these zeros, just like the string. Again taking the ansatz residue $R(n_1,n_2,t_{23},t_{34},t_{51}) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{\min(n_1-i,n_2-j)}\lambda_{i,j,k}(n_1,n_2)t_{23}^i t_{34}^j t_{51}^k$ for states of spin up to $n_{1,2}$ on the factorization channel at $s_{1,2}=[n_{1,2}]_q$, we find that the unique solution satisfying the zeros described above is the residue in Eq. (A13), up to some unfixed normalization $c(n_1,n_2)$.

Furthermore, on the special kinematic configuration $(t_{23},t_{34},t_{51})=([-k_1]_q,[-k_2]_q,[-k_3]_q)$, the only nonzero residues are precisely those defined in $D(k_1,k_2,k_3)$ in Eq. (A7). That is, these q-deformed five-point residues exhibit level truncation, so as before we can algebraically fix the $c(n_1,n_2)$ normalizations by demanding cyclic invariance. Here we again employ Eq. (A9), albeit with the residues in Eq. (A13) and with the denominators replaced by factors of $[n]_q-[-k]_q$. This yields a unique solution consistent with cyclic invariance for which $c(n_1,n_2)=1$, modulo global normalization. Thus, the q-deformed five-point amplitude is uniquely constructible from its Regge zeros and cyclic symmetry.

Paper 5: Completeness from Gravitational Scattering

CALT-TH 2025-040

Completeness from Gravitational Scattering

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We prove that symmetry in the presence of gravity implies a version of the completeness hypothesis. For a broad class of theories, we demonstrate that the existence of finitely many charged particles logically necessitates the existence of infinitely many charged particles populating the entire charge lattice. Our conclusions follow from the consistency of perturbative gravitational scattering and require the following ingredients: 1) a weakly coupled ultraviolet completion of gravity, 2) a nonabelian symmetry G, gauged or global, whose Cartan subgroup generates the abelian charge lattice, and 3) a spectrum containing some finite set of charged representations, in the simplest cases taken to be a single particle in the fundamental. Under these conditions, the abelian charge lattice is completely filled by single-particle states for $G = SO(N)$ with $N \geq 5$ and $G = SU(N)$ with $N \geq 3$, which in turn implies completeness for other symmetry groups such as $Spin(N)$, $Sp(N)$, and E_8. Curiously, a corollary of our results is that the $SU(5)$ and $SO(10)$ grand unified theories have precisely the minimal field content needed to derive completeness using our methodology.

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I. INTRODUCTION

What are the constituents of the universe? Ultimately, this is a question to be decided by experiment. At the same time, it is worth noting that mathematical consistency alone severely limits what can exist in nature, even in principle. For example, Wigner famously showed that the menu of conceivable physical states is not arbitrary, but rigidly constrained by unitarity and special relativity [1]. Furthermore, modern developments in quantum field theory and scattering amplitudes have established that the perturbative dynamics of particles are almost entirely fixed by their kinematical properties. The only self-interacting theories of massless particles of spin one and spin two are gauge theory and gravity, while higher-spin massless particles are inconsistent [2–7].

These insights have demonstrated that certain states are mathematically forbidden. On the other hand, the converse possibility—that certain states might actually be mathematically *required*—is equally if not more intriguing. The maximalist incarnation of this idea is the notion of *completeness*, which is the property that all charges permitted by Dirac quantization are explicitly realized by physical states in the spectrum.

It has been conjectured that completeness is a universal feature of all consistent theories of quantum gravity [8, 9], referred to collectively as the landscape. The complement of this space is the swampland, which describes the set of naively sensible gravitational effective field theories that can never actually be realized by any ultraviolet completion [10]. A well-known motivation for the completeness hypothesis is the absence of global symmetries in quantum gravity [11]. In particular, to explicitly break a global higher-form symmetry, one posits the existence of particles of all allowed charges [12, 13].

In this paper, we adopt an entirely different approach to this question. Using bottom-up reasoning, we rigorously prove a version of the completeness hypothesis. Our conclusions apply to a variety of theories and are derived purely from the mathematical consistency of scattering amplitudes. The workhorse of our methodology is the analytic dispersion relation,

$$\begin{aligned} c_n(t) &= \frac{1}{2\pi i}\oint_{s=0}\frac{ds}{s^{n+1}}A(s,t) \\ &= \{s \text{ channel}\} + \{u \text{ channel}\} + b_n(t). \end{aligned} \tag{1}$$

At fixed t, this expression extracts the Wilson coefficient $c_n(t)$ from the four-point scattering amplitude $A(s,t)$ and relates it to a boundary contribution $b_n(t)$ at infinity plus a sum over discontinuities in the s and u channels. This equation is the key ingredient of our analysis: if we *know* that $c_n(t) \neq 0$ and $b_n(t) = 0$, then there *must* be a state in either the s channel or the u channel, or both.

Physically, the conditions $c_n(t) \neq 0$ and $b_n(t) = 0$ imply that there is some operator that blows up faster in the effective field theory than in the full amplitude. This is the usual situation in which the effective field theory dynamics are "unitarized" by the ultraviolet completion. Crucially, Eq. (1) is only useful for deducing the existence of states if there is a mandatory coupling $c_n(t) \neq 0$ that is always unitarized, so that $b_n(t) = 0$.

Remarkably, *quantum gravity* furnishes the exact conditions that we desire [14]. The equivalence principle says that any pair of particles will interact gravitationally. The corresponding graviton exchange contribution appears near the forward limit of the amplitude [15],

$$A(s,t) = -\frac{8\pi G s^2}{t} + \cdots, \tag{2}$$

so $c_2(t) = -8\pi G/t$ is nonzero for any small but finite t. Notably, it has been argued from the bottom up that $b_2(t) = 0$ for $t < 0$ under mild assumptions about gravitational scattering [16]. Furthermore, $b_2(t) = 0$ for $t < 0$ in all known examples from string theory, where an infinite tower of higher-spin particles intervenes to unita-

rize gravitational scattering. The dispersion relation in Eq. (1) then implies that [17]

$$c_2(t) = -\frac{8\pi G}{t} = \{s \text{ channel}\} + \{u \text{ channel}\}\,, \qquad (3)$$

so there has to be a state in the s or u channel.

In the presence of an exact symmetry, Eq. (3) can be used to prove highly nontrivial constraints on the spectrum of corresponding charges [14]. In our setup we will assume an exact symmetry group G whose maximal abelian subgroup is H. We take G to be a finite semisimple Lie algebra, so H is the Cartan subgroup. Whether G is gauged or not will not matter for our analysis.

We then fold Eq. (3) into a simple iterative procedure. In the very first step, we specify some initial set of particles assumed to be in the spectrum, usually taken to be the fundamental of G unless stated otherwise. We then apply Eq. (3) in order to deduce the existence of additional charged states. Then we *rescatter* the states whose existence we just deduced, taking the exchanged particles to be the external states of a new scattering process. Since these particles also interact gravitationally, we can again apply Eq. (3), and so on ad infinitum. By iterating this algorithm, we very generically discover that an infinite tower of charged particles is required purely by mathematical consistency. If the spectrum of states eventually grows to encompass the full lattice of possible charges, or weights, under H, then we say that the theory exhibits "charge completeness."

Remarkably, we are able to prove charge completeness across a broad range of theories, though our precise conclusions depend sensitively on the degree of symmetry in G. For example, if G is abelian, then our methods are unfortunately insufficient to prove anything. The same is true of $G = SO(3)$ and $SO(4) \sim SO(3)^2$, although completeness does follow if we embed the latter in slightly larger group. For $G = SO(N)$ with $N \geq 5$, the additional nonabelian structure accommodates an explicit constructive proof of charge completeness. Meanwhile, for $G = SU(N)$ with $N \geq 3$, we derive completeness for various infinite subsets of charges that depend on what precisely we assume for the starting representations. Our results suggest that spectral completeness is a robust property of any weakly coupled ultraviolet completion of gravity with a sufficiently large nonabelian symmetry and a finite but appropriately chosen initial set of charged particles.

There is an intriguing corollary of our results that pertains to the phenomenologically relevant case of the $SU(5)$ and $SO(10)$ grand unified theories. Famously, in order to fit the standard model field content, these theories require a minimal set of representations that include $\mathbf{5}, \mathbf{10}, \mathbf{24}$ and $\mathbf{16}, \mathbf{10}, \mathbf{45}$, respectively. Curiously, we find that these sets are sufficient to derive charge completeness, while any *smaller* sets are not. This implies that the phenomenologically required field content of the $SO(10)$ and $SU(5)$ grand unified theories is exactly sufficient to guarantee the completeness of electromagnetic charges.

We should emphasize that a critical assumption in our logic is that gravitational scattering is unitarized at tree level. From a top-down perspective, the assumption of a tree-level ultraviolet is well motivated by the litany of examples from string theory, which always describes a tree-level completion whenever it is weakly coupled. From the bottom up, this is natural because the diffeomorphism invariance of the graviton is a classical notion, so perturbative $\hbar$ counting suggests that any unitarizing states should enter at tree level [18]. Notably, the condition of tree-level dynamics, together with tame Regge behavior, is actually sufficient to bootstrap string amplitudes uniquely [19–21].

In any case, the assumption of tree-level scattering plays an outsized role in our conclusions: because of it, the objects whose existence we deduce will be single-particle states. A byproduct of this fact is that our claim of completeness will be both stronger and weaker than the conventional notion of completeness invoked in the swampland literature [8, 9, 12, 13, 22]. In those works, completeness makes no reference to whether the charged states are single-particle or multi-particle. That is, the presence of ultracharged states is essentially trivialized by the existence of multi-particle states, provided there already exist states of some fundamental charge. For this reason the weight of the swampland conjectures falls on whether these fundamentally charged states are present in the first place.

By contrast, our analysis by fiat assumes that some minimally charged states are present, which in a sense weakens our conclusions. On the other hand, our arguments mandate the existence of ultracharged states that are single particles, which is much stronger than is required by the usual swampland conjectures. Perturbative string theory famously exhibits completeness of the spectrum by way of single-particle states, and remarkably, our methodology arrives at the very same conclusion using bootstrap methods.

The remainder of this paper is structured as follows. We begin in Sec. II by outlining a general iterative procedure for constructively deriving completeness of the charge lattice. As a warmup, we apply this algorithm to $U(1)$ symmetry in Sec. III and explain why completeness cannot be derived in that very simplest case. Afterwards, in Secs. IV and V we derive completeness for $SO(N)$ and $SU(N)$ and generalize to $Spin(N)$, $Sp(N)$, and E_8 in Secs. VI, VII, and VIII. We then discuss the implications of these results for grand unified theories in Sec. IX and summarize our conclusions and future directions in Sec. X.

II. COMPLETENESS ALGORITHM

In this section, we outline our constructive procedure for deriving completeness. To begin, we initialize the algorithm by specifying the symmetries of the theory,

together with some finite set of particles assumed to be in the spectrum. Since the full spectrum must be invariant under the action of the symmetry, we can actually exploit the symmetry to generate new charges from old ones. In particular, starting from any given charge we can apply the generators of the symmetry to construct additional families of charges from this state. We will refer to this action as "orbiting" the charge and the set of resulting charges as its "orbit."

We then scatter pairs of particles and apply dispersion relations to deduce the existence of new charges, iterating the algorithm to consider all possible scattering processes for the additional charges that we find. At each step, we orbit the charges in hand to generate as many new ones as possible. By repeating this procedure, we incrementally populate the charge lattice. If the space of charges ultimately spans the full lattice, completeness is established and we claim victory.

A. Initialization

We assume throughout that the dynamics are invariant under an exact symmetry described by a group G. The eigenvectors of its Cartan subgroup H span a quantized lattice Λ defining the abelian charges.

Let us define Q to be the set of charges that we know to be in the spectrum at any given point in this procedure. At the very start of the algorithm, we initialize Q to be some finite set of charges assumed by fiat to be in the spectrum. In all cases, Q will include the graviton, which is by definition a singlet under G, together with an additional particle that will usually be the fundamental. In specific cases we may sometimes use a different initial choice of representations.

The spectrum Q at any given step must be invariant under the action of G. Furthermore, we will sometimes encounter outer automorphisms C that are not contained in G, but nevertheless leave Q invariant. Formally, we express these invariances of the spectrum as the statement that $G(Q) = C(Q) = Q$.

It is essential to our argument that we can orbit a particular charge by acting on it with G and C to generate the span of all related charges. This will be an indispensable trick for generating whole new families of charges from a single seed charge. At a technical level, the orbit of a charge is implemented by the Weyl symmetries of the root system of G, together with the outer automorphisms C. See App. A for a review of various group theory definitions.

We will discuss the specific mechanics of these orbits in detail when we consider explicit examples. For the moment, let us simply note that the action of the orbit has a nice pictorial interpretation in terms of points moving about the charge lattice Λ. A given charge defines a point in Λ, and the Weyl symmetry orbits this point through the vertices at the boundary of a polytope in Λ corresponding to the highest weight states of a certain representation. The analogue of lowering operators in G can then be used to orbit the boundary inwards, filling in swaths of the hull enclosed by the polytope. For reasons we will explain later, the orbit of the original charge carries the same central charge as the seed.

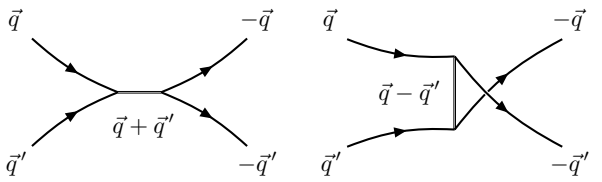
The mechanics of orbiting a particular charge is of course entirely familiar from the $SU(2)$ description of angular momentum in quantum mechanics. Starting from a state of azimuthal angular momentum J_z, we are guaranteed the existence of a whole family of spinning states related by the sequential application of the lowering operator down each rung to $-J_z$. Alternatively, we can instead apply the Weyl group of $SU(2)$ to jump directly from J_z to $-J_z$. The latter corresponds to orbiting through the vertices at the boundary of the spin representation, while the former corresponds to orbiting inwards towards the interior. Note that if we know J_z but we do not know J, we are not guaranteed the existence of new states of larger azimuthal angular momentum, since the raising operator can simply yield zero. This is why we can only orbit inwards and not outwards.

B. Iteration

We are now ready to describe the algorithm for proving completeness. Our end goal is to grow the spectrum Q at each step. If at any point we can show that $Q = \Lambda$, then completeness is established. The steps in this procedure are as follows:

1) Scatter all possible combinations of charged particles. Every possible process is labeled by a pair of charges $\vec{q}, \vec{q}\,' \in Q$. At the level of the four-point scattering amplitude, the external charges are $\vec{q}_1 = -\vec{q}_4 = \vec{q}$ and $\vec{q}_2 = -\vec{q}_3 = \vec{q}\,'$. We require this elastic charge configuration so that the t-channel state is neutral. Only then can the graviton contribute nontrivially to the left-hand side of Eq. (3), which is needed to deduce with certainty the existence of some exchanged state on the right-hand side.

2) Apply the dispersion relation in Eq. (3) to deduce the existence of a particle either in the s or u channel,



whose charges are shown. We will depict the corresponding logical deduction symbolically as

$$\vec{q} \otimes \vec{q}\,' \to (\vec{q}+\vec{q}\,') \vee (\vec{q}-\vec{q}\,'). \tag{4}$$

By CPT invariance, every particle is accompanied by an antiparticle of opposite charge. Consequently, the

above logic also implies the existence of a state of charge either $-\vec{q}-\vec{q}'$ or $-\vec{q}+\vec{q}'$. For brevity we will not explicitly list these CPT conjugates at each step in the algorithm, since they are automatically present.

3) If both $\vec{q}+\vec{q}' \notin Q$ and $\vec{q}-\vec{q}' \notin Q$, then the dispersion relation in Eq. (3) guarantees the existence of a new charge in the spectrum. We refer to such a scattering process as "conclusive." Conversely, if either $\vec{q}+\vec{q}' \in Q$ or $\vec{q}-\vec{q}' \in Q$, then no new charges are strictly required. In such a case we deduce nothing, so the scattering process is deemed "inconclusive."

4) If any scattering process is conclusive, then we update Q to include the required new charges. At this step we compute the orbit of the new charges to generate the full space of charges required by the symmetry G and any outer automorphisms C. The precise mechanics of this manuever will depend on the situation. In some cases $\vec{q}+\vec{q}'$ and $\vec{q}-\vec{q}'$ will be trivially related to each other by the action of G or C, so it does not matter which channel is activated. In other cases, the charges will be distinct, and we must adapt our strategy accordingly in order to determine which channel's charges must be added. Either way, after we have assembled some set of newly deduced charges, we then append them to Q and return to step 1) to iterate.

If at any step in the algorithm we find that all possible scattering processes are inconclusive, then the algorithm halts and completeness is not established.

III. $U(1)$ SYMMETRY

Let us consider the very simplest abelian symmetry, corresponding to $G = H = U(1)$. Furthermore, we assume that the spectrum is composed of a graviton, electron, and positron whose charges are

$$Q = \{0, -1, +1\}. \tag{5}$$

By construction, the spectrum is invariant under charge conjugation C. For graviton-electron scattering we have

$$0 \otimes (-1) \to (-1) \vee (+1), \tag{6}$$

so the s- and u-channel charges are -1 and $+1$. Since both of these are already in Q, no new charged particles are required, and this scattering process is inconclusive. Meanwhile, for electron-electron scattering we have

$$(-1) \otimes (-1) \to (-2) \vee 0, \tag{7}$$

so the s- and u-channel charges are -2 and 0. Since the latter is already in Q, a new charged state is not guaranteed, so the scattering is again inconclusive. Said another way, while a doubly-charged state would have been required in the s channel, that channel need not be activated. If instead only the u channel is present, then the dispersion relation is entirely accounted for by the graviton, which was already in the spectrum. Thus, in this example our algorithm fails to prove completeness.

The situation is similarly bleak for multiple abelian factors, for example considering $G = H = U(1)^2$. Here we assume a spectrum consisting of a graviton, electron, positron, dark electron, and dark positron with charges

$$Q = \{(0,0), (-1,0), (+1,0), (0,-1), (0,+1)\}. \tag{8}$$

Obviously, any scattering involving the graviton will be inconclusive. On the other hand, the scattering of an electron and dark electron yields

$$(-1,0) \otimes (0,-1) \to (-1,-1) \vee (-1,+1). \tag{9}$$

Both the s- and u-channel charges are new, so this pair is actually conclusive. With charge conjugation C, we know that each channel grows the spectrum by the sets $\{(-1,-1),(+1,+1)\}$ or $\{(-1,+1),(+1,-1)\}$.

Concatenating either of these two sets to Q, we obtain two possible and distinct spectra. However, it is easy to verify that in the next iteration of the algorithm, all scattering pairs are inconclusive. That is, even though Eq. (9) implies the existence of new charged particles, all subsequent scattering processes can be accounted for without introducing more charged states on top of this. This difficulty persists for any abelian symmetry.

IV. $SO(N)$ SYMMETRY

The plot thickens when our charges are embedded within an abelian subgroup of a nonabelian symmetry. In this case we can actively exploit the nonabelian generators of the symmetry to transform states of a given charge into those of a different charge.

To begin, we consider the case of a special orthogonal symmetry group $G = SO(N)$, whose Cartan subgroup is $H = U(1)^{\lfloor N/2 \rfloor}$. By definition, every charged particle is a simultaneous eigenstate of the generators of H. The corresponding eigenvalues span the $\lfloor N/2 \rfloor$-dimensional charge lattice of integers,

$$\Lambda = \mathbb{Z}^{\lfloor N/2 \rfloor} = \left\{ \sum_{i=1}^{\lfloor N/2 \rfloor} q_i \vec{e}_i \,\middle|\, q_i \in \mathbb{Z} \right\}, \tag{10}$$

where the $\vec{e}_i$ are orthonormal basis vectors. A charge vector $\vec{q} \in \Lambda$ in the lattice will be denoted by

$$\vec{q} = (q_1, q_2, \ldots, q_{\lfloor N/2 \rfloor}), \tag{11}$$

where each entry is an orthogonal component.

Recall also that the center group is $Z(SO(N)) = \mathbb{1}$ or $\mathbb{Z}_2$ for odd or even N, respectively. In the latter case, the central charge of a given charge vector $\vec{q}$ is

$$z(\vec{q}) = q_1 + q_2 + \cdots + q_{\lfloor N/2 \rfloor} \mod 2, \tag{12}$$

The central charge effectively counts the parity of the number of fundamental indices of a particular representation. We will often find it useful to classify states by their membership in the central charge sectors $z = 0, 1$.

On top of the structures defined in App. A, we will on occasion make use of the outer automorphism group C, which map representations to other representations. The precise nature of C will vary case by case, but it will usually correspond to some version of charge conjugation symmetry.

A. $N = 3$

The smallest nonabelian special orthogonal group is $G = SO(3)$. The Cartan subgroup $H = U(1)$ furnishes a one-dimensional charge lattice defined by the integers $\Lambda = \mathbb{Z}$. The elements of the root system, $\{-1, +1\}$, decrement or increment the charges by unit steps. The symmetries of the root system are encoded by the Weyl group, which acts as multiplication by -1 and swaps the highest and lowest weights states in each representation. The root system modulo the Weyl group yields the single simple root $\{+1\}$. This structure is of course very familiar from the theory of orbital angular momentum.

Starting from the initial spectrum in Eq. (5), we scatter as before to obtain Eq. (7) and are again confronted with the possibility of a u-channel state of vanishing charge. This state is invariant under the Weyl group and functions like a ground state, which can be lowered no more by the roots. Hence the root system and its Weyl symmetries are of little use, and it is still not possible to prove completeness for $G = SO(3)$.

B. $N = 4$

For $G = SO(4)$ with $H = U(1)^2$, the two-dimensional charge lattice spans all pairs of integers, $\Lambda = \mathbb{Z}^2$. The simple roots are $\{(+1, +1), (+1, -1)\}$ and the Weyl group is the set of signed permutations of even signature, so for any charge vector we can use the Weyl group at will to swap its two entries or multiply by the whole vector by -1.

Here we will assume an initial spectrum composed of a fundamental of $SO(4)$, together with the graviton. The charge spectrum coincides precisely with Eq. (8), so we can again scatter particles to obtain Eq. (9). An s-channel state would carry charge $(-1, -1)$, which we map to $(+1, +1)$ using the Weyl group. Using the roots, we then lower this state to $(0, 0)$. Thus, the charge $(-1, -1)$ orbits into $\{(-1, -1), (0, 0), (+1, +1)\}$. Analogous reasoning for the u channel orbits the charge $(-1, +1)$ into $\{(-1, +1), (0, 0), (+1, -1)\}$. These s- and u-channel particles are in the self-dual and anti-self-dual two-form representations of $SO(4)$. While these representations are chiral, there is no inconsistency if the spectrum supports one but not the other. Consequently, it is consistent to augment the spectrum with either the self-dual or anti-self-dual two-form. Iterating the algorithm, one again finds that all allowed scattering processes are inconclusive, so it is not possible to prove completeness.

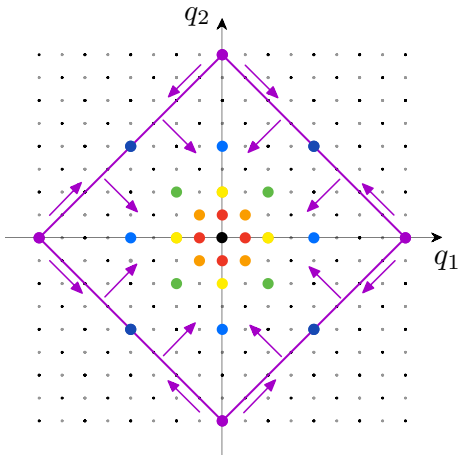


Figure 1. The $SO(4)$ charge lattice, stratified according to the central charge sectors $z = 0$ (black) and $z = 1$ (gray). Overlaid is the sequence of scattering processes in Eq. (13). Starting from an initial spectrum composed of the fundamental (red), we scatter in succession (orange, yellow, green, blue, indigo) to obtain a set of ultracharged states (purple). We then apply lowering operators to generate all charges at the boundary and interior of the diamond (purple).

We can avoid this negative conclusion if we assume more symmetry. Famously, $SO(4) \sim SO(3)^2$ admits an outer automorphism C that swaps each group factor, corresponding to the orientation-reversing elements of $O(4)$. Importantly, the action of the Weyl group and C generate signed permutations of *any* signature, which act on a given charge vector by swapping its two entries or by multiplying any single entry by -1. In what follows we will assume that C is a symmetry of the dynamics.

Armed with both G and C, we realize that our earlier s- and u-channel sets, $\{(-1, -1), (0, 0), (+1, +1)\}$ and $\{(-1, +1), (0, 0), (+1, -1)\}$, are contained within the same orbit. Thus we can add both of these sets to the spectrum Q and repeat the algorithm.

We are now equipped to prove completeness using the following sequence of scattering processes,

$$\begin{aligned}
(1,0) \otimes (0,1) &\to (1,1) \vee (1,-1) \\
(1,1) \otimes (1,-1) &\to (2,0) \vee (0,2) \\
(2,0) \otimes (0,2) &\to (2,2) \vee (2,-2) \\
(2,2) \otimes (2,-2) &\to (4,0) \vee (0,4) \\
&\vdots \\
(2^n,0) \otimes (0,2^n) &\to (2^n,2^n) \vee (2^n,-2^n) \\
(2^n,2^n) \otimes (2^n,-2^n) &\to (2^{n+1},0) \vee (0,2^{n+1}),
\end{aligned} \tag{13}$$

which are depicted in Fig. 1. By design, we have chosen conclusive scattering processes for which the s- and u-

channel states are within the same orbit, which is to say, they are related by a signed permutation. After n iterations, the spectrum contains a set of ultracharged corner states at $\{(2^n,0),(-2^n,0),(0,2^n),(0,-2^n)\}$. These corner states reside in the $z=0$ central charge sector. Lowering these states with the roots, we generate all states with central charge $z=0$ enclosed by these corner charges. Clearly, in large-n limit this process populates all points in the $z=0$ sector of the charge lattice.

To derive completeness in the $z=1$ sector, we take the ultracharged corner states and scatter them against the fundamental. For example, we find

$$\begin{aligned}(2^n,0)\otimes(1,0)&\to(2^n+1,0)\vee(2^n-1,0)\\(0,2^n)\otimes(0,1)&\to(0,2^n+1)\vee(0,2^n-1)\end{aligned}\tag{14}$$

and similarly for the other corner states. Taking the orbits of the s- and u-channel states, we obtain ultracharged corner states residing in the central charge sector $z=1$. Again taking the limit of large n and acting with lowering operators, we generate the full charge lattice for the $z=1$ sector. This concludes our derivation of completeness.

C. $N=5$

For a slightly enlarged symmetry, completeness follows even more simply. Consider the symmetry group $G=SO(5)$, whose Cartan subgroup $H=U(1)^2$ defines the charge lattice $\Lambda=\mathbb{Z}^2$. Here H and Λ are the same as for $G=SO(4)$. However, the Weyl group is enlarged to include all signed permutations of any signature. For any charge vector, we can then swap its two entries or flip the sign of any entry. The manipulations of the previous subsection then follow trivially, since what was an outer automorphism C of $SO(4)$ is now automatically contained in $SO(5)$. Another way to see this fact is that the simple roots of $SO(5)$ are $\{(+1,+1),(+1,0)\}$. The last element can be used to toggle between the self-dual and anti-self-dual two-form representations of $SO(4)$.

D. $N\geq 6$

We are finally ready to study the general case of $G=SO(N)$ with Cartan subgroup $H=U(1)^{\lfloor N/2\rfloor}$ and charge lattice $\Lambda=\mathbb{Z}^{\lfloor N/2\rfloor}$. Here we will reason by induction, assuming that completeness has already been established for all proper special orthogonal subgroups.

To begin, consider the case of odd N. As usual, we assume a starting spectrum that includes the fundamental of $SO(N)$. By extension, the spectrum carries the required fundamentals and antifundamentals of $SO(N-1)$ needed for the induction hypothesis. Since $SO(N-1)$ and $SO(N)$ have the exact same charge lattice and the induction hypothesis assumes completeness of the former, we have completeness of $SO(N)$ as well.

The case of even N is more involved. The induction hypothesis assumes completeness of $SO(N-1)$, so the $SO(N)$ spectrum includes charges of the form $(q_1,\ldots,q_{\frac{N}{2}-1},0)$, where the nonzero entries are integer $SO(N-1)$ charges. The last entry is chosen to be neutral. Using the Weyl group, we can permute this zero entry to any position we like in order to construct any charge vector with one or more vanishing entries.

Next, we construct the following pair of charges and then scatter them,

$$(q_1,\ldots,q_{\frac{N}{2}-2},0,0)\otimes(0,0,\ldots,0,q_{\frac{N}{2}-1},q_{\frac{N}{2}}),\tag{15}$$

which result in the charges $(q_1,\ldots,q_{\frac{N}{2}-2},q_{\frac{N}{2}-1},q_{\frac{N}{2}})$ or $(q_1,\ldots,q_{\frac{N}{2}-2},-q_{\frac{N}{2}-1},-q_{\frac{N}{2}})$ in the s or u channel, respectively. Again using the Weyl group, we apply a signed permutation of even signature to flip the sign of the final two entries, generating $(q_1,\ldots,q_{\frac{N}{2}-2},q_{\frac{N}{2}-1},q_{\frac{N}{2}})$ to establish completeness of the spectrum.

Before moving forward, let us comment briefly on the interplay between completeness and the center of the symmetry group. Obviously, a necessary condition for completeness is that the full spectrum contains at least one particle in all central charge sectors $z=0,1$. For even N, the center is $Z(SO(N))=\mathbb{Z}_2$ and the central charge sectors $z=0,1$ are already represented by the graviton and the fundamental. This is a major reason why completeness arises so straightforwardly in the case of $SO(N)$: the center is tiny, and the starting spectrum already clears the low bar of containing particles with these central charges. We will see shortly that this is not always the case when the center group is larger.

V. $SU(N)$ SYMMETRY

Let us now consider the case of $G=SU(N)$, whose Cartan subgroup is $H=U(1)^{N-1}$. The charge lattice is

$$\Lambda=\left\{\vec{q}=\sum_{i=1}^{N-1}q_i\vec{\mu}_i\,\middle|\,q_i\in\mathbb{Z}\right\},\tag{16}$$

where we emphasize that $\vec{q}$ is *not* expressed in terms of the orthonormal basis $\vec{e}_i$, but rather the nonorthogonal basis of fundamental weights $\vec{\mu}_i$, which satisfy

$$\vec{\mu}_i=\sum_{j=i}^{N-1}\frac{i}{\sqrt{j(j+1)}}\vec{e}_j.\tag{17}$$

In particular, the components of $\vec{q}$ in the basis of fundamental weights are precisely the Dynkin coordinates,

$$\vec{q}=(q_1,q_2,\ldots,q_{N-1}).\tag{18}$$

For this reason, one must take special care when computing dot products. Furthermore, since $\vec{e}_i$ and $\vec{\mu}_i$ are

distinct, the components of $\vec{q}$ are not actually the eigenvalues of the Cartan generators, but are straightforwardly related to them by a linear transformation defined by Eq. (17). In spite of this mismatch, we will abuse nomenclature and glibly refer to $\vec{q}$ throughout this section as the charge vector. Obviously, completeness in the Dynkin coordinates defined by the components of $\vec{q}$ will imply completeness in the bona fide charge lattice.

As described earlier, it will be useful to be able to orbit charges using the action of the symmetry group G and the outer automorphisms C. The former is implemented by the action of the Weyl group, which acts on the simple roots of $SU(N)$. Explicitly, these simple roots are

$$\begin{aligned}\vec{r}_i &= \sqrt{\frac{i+1}{i}}\vec{e}_i - \sqrt{\frac{i-1}{i}}\vec{e}_{i-1} \\ &= -\vec{\mu}_{i-1} + 2\vec{\mu}_i - \vec{\mu}_{i+1},\end{aligned} \tag{19}$$

where $i = 1, 2, \ldots, N-1$ and $\vec{e}_0 = \vec{\mu}_0 = \vec{\mu}_N = 0$, with the normalization chosen so that $\vec{r}_i \cdot \vec{\mu}_j = \delta_{ij}$. In these coordinates, the Weyl group transformation reviewed in Eq. (A.2) maps charge vectors via $\vec{q} \to \vec{q} - q_i\vec{r}_i$, which in Dynkin coordinates sends

$$\begin{gathered}(\ldots, q_{i-1}, q_i, q_{i+1}, \ldots) \\ \downarrow \\ (\ldots, q_{i-1} + q_i, -q_i, q_{i+1} + q_i, \ldots).\end{gathered} \tag{20}$$

This transformation flips the sign of a given entry and adds that entry to its neighbors. On the other hand, charge conjugation C reverses the order of the Dynkin coordinates via

$$(q_1, q_2, \ldots, q_{N-1}) \to (q_{N-1}, q_{N-2}, \ldots, q_1). \tag{21}$$

For example, C swaps the fundamental and antifundamental representations of $SU(N)$.

The center group $Z(SU(N)) = \mathbb{Z}_N$ will play an important role in the subsequent analysis. The corresponding central charges of $SU(N)$ are known as N-ality [23], which are defined as

$$z(\vec{q}) = q_1 + 2q_2 + \cdots + (N-1)q_{N-1} \mod N. \tag{22}$$

Recall that the roots only transform between states within a given central charge sector, which are labeled by $z = 0, 1, \ldots, N-1$. As noted previously, spanning all central charge sectors is a necessary but not sufficient condition for completeness. Obviously, the fundamental and antifundamental, which span the sectors $z = 1$ and $N-1$, are not sufficient on their own to satisfy this criterion. This is the root of the difficulty in establishing completeness for special unitary symmetries.

A. $N = 3$

There is no reason to consider $SU(2)$ because we already considered $SO(3)$ and found that completeness could not be established using our algorithm. We thus move on to $G = SU(3)$, whose Cartan subgroup is $H = U(1)^2$. The corresponding charge lattice is the two-dimensional set of points defined in Eq. (16), As shown in Fig. 2, the charges for the singlet **1**, fundamental **3**, adjoint **8**, and decuplet representation **10** are

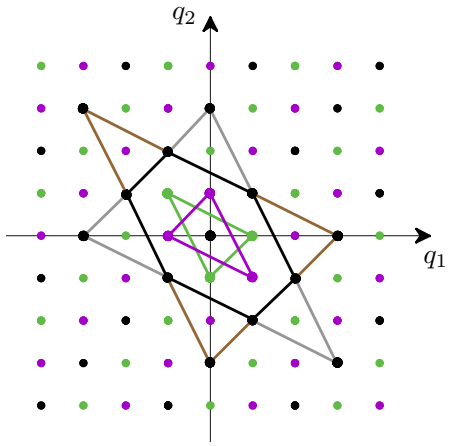


Figure 2. The $SU(3)$ charge lattice, stratified according to the central charge sectors $z = 0$ (black), $z = 1$ (green), and $z = 2$ (purple). The polygons circumscribe the irreducible representations $Q_{\mathbf{3}}$ (green), $Q_{\bar{\mathbf{3}}}$ (purple), $Q_{\mathbf{8}}$ (black), $Q_{\mathbf{10}}$ (brown), and $Q_{\overline{\mathbf{10}}}$ (gray).

$$\begin{aligned}Q_{\mathbf{1}} &= \{(0,0)\} \\ Q_{\mathbf{3}} &= \{(1,0),(-1,1),(0,-1)\} \\ Q_{\mathbf{8}} &= \{(1,1),(-1,2),(-2,1),(-1,-1), \\ &\quad (1,-2),(2,-1),(0,0),(0,0)\} \\ Q_{\mathbf{10}} &= \{(1,1),(-1,2),(-3,3),(-2,1), \\ &\quad (-1,-1),(0,-3),(1,-2),(2,-1), \\ &\quad (3,0),(0,0)\}.\end{aligned} \tag{23}$$

Note that charges of the conjugate representations $Q_{\bar{\mathbf{3}}}$ and $Q_{\overline{\mathbf{10}}}$ are obtained by charge conjugation, which flips various signs in Eq. (23). These charge-conjugate representations are always automatically present, so we will not always explicitly enumerate them.

The **1**, **8**, **10**, and $\overline{\mathbf{10}}$ reside in the $z = 0$ charge sector, the **3** resides in $z = 1$, and the $\bar{\mathbf{3}}$ resides in $z = 2$. We now apply our usual algorithm, which is to scatter all possible states in the assumed spectrum, seeking conclusive processes where a new charged state is guaranteed in both the s and u channel.

For an initial spectrum comprising $Q_{\mathbf{1}}$ and $Q_{\mathbf{3}}$, we see that all possible scattering processes are inconclusive. In particular, for any choice of scattering between $Q_{\mathbf{3}}$ and itself or with $Q_{\bar{\mathbf{3}}}$, there is some choice of s- or u-channel states that are still in the initial spectrum. Hence, completeness cannot be proven.

For the enlarged spectrum comprising $Q_{\mathbf{1}}$, $Q_{\mathbf{3}}$, and $Q_{\mathbf{8}}$, we can actually do better. Scattering appropriately

chosen representatives of $Q_{\bar{\mathbf{3}}}$ with $Q_{\mathbf{8}}$, we obtain

$$(1,-1)\otimes(1,1)\to(2,0)\vee(0,-2), \tag{24}$$

which conclusively implies the existence of a new representation. Again orbiting the charges and applying conjugation, we learn that the spectrum must be augmented to include new representations. In particular, the new representations in Eq. (24) reside in the central charge sector $z=2$, and conjugation yields the corresponding weights in the $z=1$ class.

By repeating this procedure, we accumulate a sequence of progressively larger triangles and hexagons. At every step we orbit each charge to obtain the whole family of charges within the perimeter of the largest polygon we have reached. This algorithm eventually covers all possible states with $z=1,2$.

As discussed in App. B, it is straightforward to verify that the scattering of any states within $Q_{\mathbf{8}}$ is always inconclusive. As noted above, the same is true for the scattering of any state in $Q_{\mathbf{3}}$ with any state in $Q_{\bar{\mathbf{3}}}$. Meanwhile, since all states within $Q_{\mathbf{3}}$ have $z=1$, scattering them will generate $z=2$ or $z=0$, while all states within $Q_{\bar{\mathbf{3}}}$ have $z=2$, so scattering them generates $z=1$ or $z=0$. One ultimately finds that assuming an initial spectrum comprising $Q_{\mathbf{1}}, Q_{\mathbf{3}}, Q_{\mathbf{8}}$ conclusively generates completeness in the central charge sectors $z=1,2$ but not $z=0$.

Last but not least, let us consider an initial spectrum comprising $Q_{\mathbf{1}}$, $Q_{\mathbf{3}}$, and $Q_{\mathbf{10}}$. Pictorially, this spectrum includes the smallest triangular representation in each central charge sector. Scattering states with $Q_{\mathbf{10}}$ turns out to be just enough to cover the entire $z=0$ sector via a sequence of hexagons of increasing size. This then establishes completeness in all central charge sectors $z=0,1,2$, and more importantly, for all points in the full charge lattice. The details of this argument are spelled out in App. B.

B. $N \geq 4$

For the general case of special unitary symmetries, completeness requires a sufficiently large but still finite starting set of charged particles. To see why, let us return to the accounting of central charges.

A spectrum composed of the graviton, fundamental, and antifundamental spans the central charge sectors $z=0,1,N-1$. Scattering any pair of states from this set, we see immediately that either the s or the u channel will contain a state in $z=0,1,N-1$. This implies that we can never conclusively guarantee a new charged particle with central charge $z=2,3,\ldots,N-2$.

Now imagine that we instead augment our starting spectrum to include states residing in $z=0,1,2,N-2,N-1$. By the same logic as before, by scattering these states we can never conclusively guarantee a new charged particle with $z=3,4,\ldots,N-3$, and so on. This reasoning implies that completeness will always require an initial spectrum of states with representatives in all central charge sectors $z=0,1,\ldots,N-1$. Any initial spectrum that fails this criterion will not ensure full charge completeness across all sectors.

Remarkably, the existence of a representative in all central charge sectors $z=0,1,\ldots,N-1$ is a sufficient condition for completeness. This is very much not obvious. To see why this holds, let us start with the example of deriving completeness of the $z=0$ sector assuming that the spectrum contains the adjoint representation. Consider the following scattering process involving states within the adjoint representation,

$$\vec{r}_1\otimes\vec{r}_3\to(\vec{r}_1+\vec{r}_3)\vee(\vec{r}_1-\vec{r}_3). \tag{25}$$

Here we have used that the charges of the adjoint are themselves simple roots. Importantly, both the s- and u-channel exchanges are related by action of the Weyl group. In particular, let us we define w_i to be the Weyl transformation in Eq. (20). Then we see that $(\vec{r}_1+\vec{r}_3)$ and $(\vec{r}_1-\vec{r}_3)$ are related by the action of w_3, so this scattering is conclusive. We then scatter this pair of charges with each other to obtain

$$(\vec{r}_1+\vec{r}_3)\otimes(\vec{r}_1-\vec{r}_3)\to(2\vec{r}_1)\vee(2\vec{r}_3), \tag{26}$$

where the u- and s-channels are related by a sequential composition of Weyl transformations, $w_2\circ w_1\circ w_3\circ w_2$.

After n iterations back and forth between these two classes of scattering processes, we eventually obtain all charges of the form $2^n\,\vec{r}_1$ and $2^n\,\vec{r}_3$, which are charges of the adjoint representation multiplied by 2^n. By orbiting these two charges via the weight strings reviewed in App. A, we obtain all of the adjoint charges, multiplied by 2^n. These sets of charges are concentric polytopes, each twice the size of the previous one, and they reside within an ever larger set of representations. We then use lowering operators of the symmetry group to generate all states enclosed by these charges in the $z=0$ central charge sector. At large n we ultimately recover the entire $z=0$ sector. The procedure here is obviously closely analogous to the one used to derive completeness in $SO(4)$ in Fig. 1. Crucially, we note that the algorithm just outlined requires the existence of $\vec{r}_3$. Since $SU(N)$ has $N-1$ roots, as defined in Eq. (19), this construction highlights why $N=3$ is not enough for completeness.

With completeness in the $z=0$ sector established, it is now straightforward to show that the inclusion of even a single charge with $z=k$ or $z=N-k$ is sufficient to ensure completeness in the entire $z=k$ and $z=N-k$ sectors. This sector-wise charge completeness is achieved as follows. First, we pick any charge in the $z=0$ sector of the form $nN\vec{\mu}_k$. Next, given any charge in the class $z=k$, we are guaranteed the existence of the fundamental weight $\vec{\mu}_k$, since as we have seen we can obtain this charge vector from our starting charge via Weyl transformations and weight string relations. We then scatter them to obtain

$$(nN\vec{\mu}_k)\otimes\vec{\mu}_k\to((nN+1)\vec{\mu}_k)\vee((nN-1)\vec{\mu}_k). \tag{27}$$

If the s-channel exchange is activated, then we obtain the charge $(nN+1)\vec{\mu}_k$, which is manifestly in sector $z=k$. Furthermore, the orbit of this state generates a polytope that is simply a rescaled version of $\vec{\mu}_k$ with vertices expanded outwards to inflate the polytope along all directions. Orbiting these charges into the interior, we then obtain all $z=k$ states inside the polytope, and by charge conjugation C we obtain the same in the $z=N-k$ sector. In the large-n limit, this construction implies completeness in the charge sectors $z=k, N-k$. Alternatively, if we instead have the particle in the u channel with charge $(nN-1)\vec{\mu}_k$, this state is in sector $z=N-k$. In this case, the same conclusion holds, only with the roles of $N-k$ and k swapped. As advertised, this scattering process establishes that the central charge sectors $z=k, N-k$ are complete, provided we already have completeness in the $z=0$ sector and that we have a single charge with either $z=k$ or $z=N-k$. We again emphasize that in the proof we used a fundamental weight of class k. If we started with any other representative, then the weights of the fundamental would automatically be part of the spectrum by the properties of representations discussed in Sec. II A.

VI. *Spin*(N) SYMMETRY

The spin group $Spin(N)$ is the well-known universal cover of the special orthogonal group $SO(N)$. By definition, these groups share the same roots and hence the same Weyl group. They only differ in their charge lattices, with the weight lattice of $Spin(N)$ given by

$$\begin{aligned}
\Lambda &= \Lambda_0 \cup \Lambda_{1/2} \\
\Lambda_0 = \mathbb{Z}^{\lfloor N/2\rfloor} &= \left\{ \vec{q} = \sum_{i=1}^{\lfloor N/2\rfloor} q_i \vec{e}_i \,\middle|\, q_i \in \mathbb{Z} \right\} \\
\Lambda_{1/2} = (\mathbb{Z}+\tfrac{1}{2})^{\lfloor N/2\rfloor} &= \left\{ \vec{q} = \sum_{i=1}^{\lfloor N/2\rfloor} q_i \vec{e}_i \,\middle|\, q_i \in \mathbb{Z}+\tfrac{1}{2} \right\},
\end{aligned} \tag{28}$$

where Λ_0 is just the integer lattice of $SO(N)$ and $\Lambda_{1/2}$ is the same lattice shifted by a half-integer in every direction. The latter correspond to the spinor representations of $Spin(N)$ that are not in $SO(N)$. The group $Spin(N)$ admits the N-dimensional vector representation, together with the $2^{\lfloor (N-1)/2\rfloor}$-dimensional spin representations of multiplicity one and two for N odd and even respectively.

As per our earlier arguments, the existence of a vector will guarantee completeness in the Λ_0 charge lattice, which corresponds to $SO(N)$. Obviously, to ensure completeness in $\Lambda_{1/2}$ as well, we also have to include spinor representations of $Spin(N)$ in the spectrum.

For these reasons, we will assume that our spectrum contains the vector $\vec{e}_i$ and the spinors $\frac{1}{2}\sum_i \sigma_i \vec{e}_i$ for all sign choices $\sigma_i \in \{\pm 1\}$. For even N, the cases where the σ_i sum to even or odd constitute the two different spinors. Equipped with these states, it is then straightforward to generate the full charge lattice.

We take the ultracharged corner states constructed in the previous section and scatter them against the spinors. In the case of $Spin(4)$, this scattering yields

$$\begin{aligned}
(2^n,0)\otimes(\tfrac{1}{2},\tfrac{1}{2}) &\to (2^n+\tfrac{1}{2},\tfrac{1}{2}) \vee (2^n-\tfrac{1}{2},-\tfrac{1}{2}) \\
(0,2^n)\otimes(\tfrac{1}{2},\tfrac{1}{2}) &\to (\tfrac{1}{2},2^n+\tfrac{1}{2}) \vee (-\tfrac{1}{2},2^n-\tfrac{1}{2}).
\end{aligned} \tag{29}$$

For $Spin(N)$, we would use the corresponding corners of the $\lfloor N/2\rfloor$-dimensional orthant. These scattering processes generate ultracharged states in the half-integer lattice, which we can then orbit inwards to generate all enclosed charges within the same central charge sector. Taking the large-n limit, we then derive completeness of the full spectrum across the integers and half-integers. In conclusion, $Spin(N)$ is complete, provided the spectrum includes the vector and spinor representations.

VII. *Sp*(N) SYMMETRY

Consider the compact symplectic group $Sp(N)$, which defines the set of linear transformations that leave invariant the symplectic form in even N dimensions. Here, completeness is an immediate consequence of our earlier results. To see why, note that $Sp(N)$ has the very same charge lattice as $SO(N)$, namely $\Lambda = \mathbb{Z}^{N/2}$. In particular, the vector representation of $Sp(N)$ has the same charges as in $SO(N)$. Furthermore, the root system of $Sp(N)$ contains that of $SO(N)$ as a subset, so the same is true for their respective Weyl groups. This implies that with respect to orbits, $Sp(N)$ is availed of strictly more operations to generate new charges from old charges. Altogether, these relations imply that all of the scattering and orbit operations we performed for $SO(N)$ are equally applicable for $Sp(N)$. This establishes completeness of the symplectic group $Sp(N)$.

VIII. E_8 SYMMETRY

We can make an analogous argument for E_8, the largest exceptional simple Lie group. Unlike $SO(N)$ and $SU(N)$, the smallest nontrivial representation of E_8 is the adjoint **248**. We will therefore assume an initial spectrum Q comprising the graviton and the adjoint. The charge lattice of E_8 is famously even and self-dual,

$$\begin{aligned}
\Lambda &= \Lambda_0 \cup \Lambda_{1/2} \\
\Lambda_0 &= \left\{ \vec{q} = \sum_{i=1}^{8} q_i \vec{e}_i \,\middle|\, q_i \in \mathbb{Z},\ \sum_{i=1}^{8} q_i \in 2\mathbb{Z} \right\} \\
\Lambda_{1/2} &= \left\{ \vec{q} = \sum_{i=1}^{8} q_i \vec{e}_i \,\middle|\, q_i \in \mathbb{Z}+\tfrac{1}{2},\ \sum_{i=1}^{8} q_i \in 2\mathbb{Z} \right\},
\end{aligned} \tag{30}$$

and is equivalent to its root lattice. In particular, the root system comprises R_0, the set of all $\pm\vec{e}_i \pm \vec{e}_j$ with independent signs, along with $R_{1/2}$, the set of all points in $\Lambda_{1/2}$ with all $|q_i| = 1/2$. Together with the point at the origin, R_0 and $R_{1/2}$ are the adjoint weights of E_8.

We immediately recognize Λ_0 and R_0 as precisely the $z = 0$ sector weight lattice and root system of $SO(16)$. For $SO(N)$, the full $z = 0$ sector is generated by our scattering algorithm if we start with the adjoint rather than the vector, as is clear beginning with the second line of Eq. (13), followed by Weyl symmetry and lowering operators as described in that section. In the exact same way, for E_8 we are forced to augment Q to all of Λ_0 by running the scattering algorithm starting from the adjoint weights in R_0.

In particular, we note that the point $2^n(\vec{e}_1 + \vec{e}_2)$ is in Λ_0 and hence in Q. Using the Weyl orbit of this weight, in parallel with our earlier constructions we obtain a polytope composed of the E_8 adjoint weights rescaled by 2^n. Applying the results reviewed in App. A, along with the fact that the center of E_8 is trivial, we conclude that all points in the full lattice Λ contained within this polytope must also be in representations described by the polytope, so these weights must be included in Q as well. As n tends to infinity, we find that $Q = \Lambda$ and hence conclude that charge completeness of E_8 follows from the presence of the adjoint and the graviton.

IX. GRAND UNIFIED THEORIES

The above analysis has direct implications for the charge completeness of grand unified theories. As we will see below, in these theories, the matter and force content of the standard model actually *ensures* charge completeness.

A. $G = SU(5)$

The minimal grand unified theory is the Georgi-Glashow model [24], whose gauge group is $G = SU(5)$. The gauge bosons transform in the **24**, while the Higgs fields that break the electroweak and grand unified symmetries reside in the **5** and **24**. The quarks and leptons of the standard model fit snugly within the $\bar{\mathbf{5}}$ and **10**.

As we saw in Sec. V, the existence of a particle in each central charge sector in the initial spectrum is enough to guarantee full charge completeness. In this case the center group is $Z(SU(5)) = \mathbb{Z}_5$. The sectors $z = 0, 1, 2, 3, 4$ are accounted for by $\mathbf{24}, \mathbf{5}, \mathbf{10}, \overline{\mathbf{10}}, \bar{\mathbf{5}}$, respectively. We thus have completeness of all classes $z = 0, 1, 2, 3, 4$ in the $SU(5)$ grand unified theory.

Curiously, there is an intriguing correlation between the requirements of completeness and those needed for viable phenomenology. In particular, according to our arguments about central charges, any strict subset of the representations $\mathbf{5}, \mathbf{10}, \mathbf{24}$ and their conjugates are *insufficient* to imply completeness. Said another way, a theory with $SU(5)$ symmetry that is missing any of the quarks, leptons, or gauge bosons of the standard model would not furnish enough charged states to ensure completeness using our logic.

B. $G = SO(10)$

An alternative scheme that naturally incorporates all fermions into a single representation is $G = SO(10)$ grand unification [25], or more precisely its double cover $G = Spin(10)$. The gauge bosons reside in the **45**, while the **16** includes the standard model quarks and leptons plus the right-handed neutrino. The Higgs doublet usually resides in the **10**, while the fields responsible for breaking the grand unified symmetry can involve even more representations, depending on the model.

The center of the gauge group is $Z(Spin(10)) = \mathbb{Z}_4$, so to guarantee full charge completeness we need a spectrum that includes states in the equivalence classes $z = 0, 1, 2, 3$. These are provided by $\mathbf{45}, \mathbf{16}, \mathbf{10}, \overline{\mathbf{16}}$, respectively, which are all necessary ingredients to embed the standard model. Remarkably, we see again that any strict subset of $\mathbf{10}, \mathbf{16}, \mathbf{45}$ and their conjugates does not guarantee full charge completeness.

X. DISCUSSION

We have shown that the spectrum of charges is complete across a range of theories under mild assumptions. The cornerstone of our analysis is the dispersion relation for gravitational scattering amplitudes in Eq. (1), first proposed in Ref. [14]. This remarkable formula directly relates the scattering contribution from graviton exchange to a sum over exchanges in the s and u channel. As we have emphasized, the assumption of a tree-level ultraviolet completion of gravity is an important criterion for our analysis. This condition not only zeros out the boundary term in Eq. (1), but also dictates that the unitarizing degrees of freedom are single-particle states.

Our proof exploits the sequential scattering of particles and the action of the symmetry group G itself to generate the full spectrum of charges [26]. For an abelian symmetry G, we are unable to prove completeness in any form. However, for certain choices of nonabelian G, we show that charge completeness is mathematically required as long as the spectrum contains at least some finite handful of charged states, usually taken to be the fundamental. In our context, charge completeness is the property that the full charge lattice of the Cartan subgroup H is populated by single-particle states.

Our results suggest a number of avenues for future work. First and foremost is the question of whether it is possible to derive more general forms of completeness. In particular, given that charge completeness arises relatively straightforwardly, it is natural to ask: are all *irreducible representations* of a nonabelian symmetry G required to be in the spectrum? Of course, charge com-

pleteness necessitates the presence of an infinite collection of representations of arbitrarily high weight. We have initiated a partial investigation into the question of representation completeness, yielding primarily negative results. We elaborate on our various attempts in App. C, based on the strategy of Ref. [14]. We present evidence that in the cases of $G = SO(3), SO(4)$, representation completeness cannot be proven using this methodology. More generally, given that the entirety of our analysis has focused on four-point scattering, it is worth examining whether higher-point processes might afford more leverage. In particular, recent work has shown that positivity constraints on higher-point scattering are exceedingly stringent [27].

A second question relates to the precise nature of the symmetry G. As discussed earlier, the conclusions of the present work are independent of whether G is gauged or global. The latter famously runs afoul of the expectation that quantum gravity forbids exact global symmetries. This does not detract from our logic, however, since we are not claiming that a global symmetry is required. Indeed it would be interesting to study the case where the symmetry is explicitly broken. Conversely, if the symmetry is gauged then any logical implications of the dispersion relation could be relevant to the weak gravity conjecture [28].

Furthermore, while we have assumed throughout that G is internal, another interesting possibility is that G could be a spacetime symmetry, for example the Poincaré group. Since spacetime symmetries also imply conservation laws, it is natural to speculate on the completeness of spacetime charges such as physical spins. Spinning states of this kind are precisely what is needed to explicitly break the higher-form symmetries of gravity [29]. Of course, our assumptions already imply spin completeness: reproducing the $1/t$ pole on the left-hand side of Eq. (1) requires an infinite tower of spins on the right-hand side, since each partial wave is a polynomial in t [30]. A corollary of this fact is that the existence of particle of a given charge implies the existence of an infinite tower of higher-spin cousins with that same charge.

A third line of inquiry concerns whether our results could be strengthened or enriched by relaxing existing assumptions or adding new ones. Clearly, our strongest assumption is that the relevant dynamics are at tree level, so it would be worthwhile to understand the effects of loops. At a technical level, loops would allow for multi-particle states in the dispersion relation in Eq. (1), which naively ensures completeness trivially, starting from some initial seed of charged states. Another central assumption is the presence of the graviton, so it would be interesting to examine how important gravity truly is for our conclusions. What we fundamentally require is a particle that couples universally, so that it shows up in all of our dispersion relations, and whose amplitudes exhibit sufficiently soft high-energy behavior. Studying other setups that share these properties could be illuminating.

Alternatively, since our analysis is relatively conservative it is also reasonable to include stronger assumptions. For example, our arguments make no use of unitarity, let alone detailed kinematic information other than the dispersion relation itself. Unitarity is obviously a very weak assumption, but it would likely provide additional mileage. Furthermore, if we could somehow know for a fact that both the s and u channels of Eq. (1) are always activated, then much stronger claims could be proven. We discuss this possibility briefly in App. C 4.

Last but not least, our results suggest that there is untapped potential in applying the methods of the scattering bootstrap to other conjectures in quantum gravity from the bottom up. This seems especially likely given the close relationships linking the completeness hypothesis to the weak gravity conjecture and the absence of global symmetries. We leave these ideas as compelling avenues for future study.

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[15] Here the ellipses denote all terms in the low-energy amplitude that are not the graviton pole. This includes contributions that are regular in t coming from general relativity, as well as higher-dimension operator corrections in the effective field theory. Notably, all of these terms exhibit different t dependence than the graviton pole, so there are always values of t for which the s^2 coefficient of the low-energy amplitude is nonzero, for instance if t is below the mass gap.

[16] In Ref. [31], Regge bounds were derived for gravitational scattering amplitudes as a consequence of analyticity, unitarity, and the partial wave expansion in various contexts. The most conservative of these bounds, merely assuming dominance of single-graviton exchange at large impact parameter, showed in generality that, for tree-level amplitudes, $A(s,t) < s^{2-\frac{D-7}{2(D-4)}}$ for $t<0$, enabling twice-subtracted dispersion relations in $D>7$. By including eikonalization, even better scaling is possible. More generally, Refs. [32, 33] outlined causality arguments that bound $A(s,t) < s^2$.

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[26] Throughout this work, we will assume that G is a single group factor. For a product group, $G = \otimes_i G_i$, completeness of each G_i factor implies completeness of G under certain conditions. In particular, consider the case in which a starting spectrum Q_i is sufficient to prove completeness for G_i. First, we require an initial spectrum Q for G composed of all states with charges in Q_i under G_i and neutral under all others, for each i. Second, we assume that each G_i has no outer automorphisms, which would induce a relative alignment between charges under different group factors. If these conditions are met, completeness of the product group G is guaranteed.

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[39] Our results on charge completeness can be understood in this language. By scattering the charges $\vec{q}$ and $\vec{q}'$, we have used that there is a state of charge $\vec{q}+\vec{q}'$ or $\vec{q}-\vec{q}'$, corresponding to the s or u channel. This implies that there exists some choice of v for which Eq. (C.4) is satisfied where the set of exchanged representations $\mathcal{R}$ contains neither $\vec{q}+\vec{q}'$ nor $\vec{q}-\vec{q}'$.

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Appendix A: Group Theory Review

In this appendix, we review some of the group theoretic structures needed to establish charge completeness. Consider a finite, semisimple, compact Lie group G with an associated Lie algebra $\mathfrak{g}$. The maximal commuting subalgebra of $\mathfrak{g}$ is the Cartan subalgebra $\mathfrak{h}$, which generates the Cartan subgroup H. The elements of $\mathfrak{h}$ are the generators H_i. The eigenvalues of these generators are defined by $H_i|\vec{q}\rangle = q_i|\vec{q}\rangle$, where q_i are the weights, or charges.

Particularly important to our discussion are the root generators, which are operators $E_{\vec{r}_i}$ satisfying $[H_i, E_{\vec{r}_i}] = \vec{r}_i E_{\vec{r}_i}$. The root generators are each labeled by a vector $\vec{r}_i$, known as the root, which all together form the root system. These roots move between weights of the lattice according to $E_{\vec{r}_i}|\vec{q}\rangle \propto |\vec{q}+\vec{r}_i\rangle$. The roots themselves are also weights, corresponding to the adjoint representation. The group G may have a center $Z(G)$, defining the set of elements that commute with every element of G. A nontrivial center defines charges under the center symmetry for each representation, and we say that representations with the same charges under $Z(G)$ are in the same central charge sector. A pair of weights reside in the same central charge sector if and only if they can be connected by an integer linear combination of roots. Note that the fundamental weights $\vec{\mu}_i$ are defined to be orthogonal to the roots, so $2\,\vec{\mu}_i \cdot \vec{r}_j/|r_j|^2 = \delta_{ij}$. The fundamental representations are the irreducible representations whose maximal weight is a fundamental weight.

For a given charge $\vec{q}$, it is natural to classify each root $\vec{r}$ according to whether it points "towards the origin" or "away from the origin," corresponding to $\vec{q}\cdot\vec{r} < 0$ or $\vec{q}\cdot\vec{r} > 0$, respectively. This classification defines generalized raising and lowering operators that increase or decrease the magnitude of the charge vector. Crucially, if $\vec{q}\cdot\vec{r} < 0$, then the weight string formula [36] says that both $\vec{q}$ and its lowered cousin $\vec{q}+\vec{r}$ reside in the same representation. Concretely, the set of states $\vec{q}+k\vec{r}$ are all in the same representation for all integers $k = -m, -m+1, 0, 1, \ldots, n$, where m and n are nonnegative integers satisfying

$$m - n = \frac{2\vec{q}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}. \tag{A.1}$$

Hence, for $\vec{q}\cdot\vec{r} < 0$, we have $n > m \geq 0$, and so $\vec{q}+\vec{r}$ is in the same representation as $\vec{q}$. Similarly, of course if $\vec{q}\cdot\vec{r} > 0$, then $m > n \geq 0$, and so $\vec{q}-\vec{r}$ is in the same representation as $\vec{q}$.

A second ingredient is that the root system exhibits isometries parameterized by the Weyl group. By definition, the latter reflects any given charge vector $\vec{q}$ through the plane orthogonal to any root $\vec{r}$,

$$\vec{q} \to \vec{q}\,' = \vec{q} - \frac{2(\vec{q}\cdot\vec{r})}{(\vec{r}\cdot\vec{r})}\vec{r}. \tag{A.2}$$

For $SO(N)$, these Weyl transformations act on $\vec{q}$ as a signed permutation of any signature for odd N and of even signature for even N. For $SU(N)$, the action of the Weyl group is more complicated.

Importantly, any two points $\vec{q}$ and $\vec{q}\,'$ related by a Weyl transformation define a line segment,

$$\vec{q}(\lambda) = \lambda\vec{q} + (1-\lambda)\vec{q}\,' = \vec{q} - 2(1-\lambda)\frac{\vec{q}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\vec{r}, \tag{A.3}$$

where $\lambda \in [0,1]$. Going from $\vec{q}$ to $\vec{q}\,'$ moves $\vec{q}$ in the direction of $-\vec{r}$ for $\vec{q}\cdot\vec{r} > 0$ and in the direction of $+\vec{r}$ for $\vec{q}\cdot\vec{r} < 0$. In either case, due to the weight string formula in Eq. (A.1), all charges on this line segment connecting $\vec{q}$ and $\vec{q}\,'$ must lie in the same representation. This mathematical fact allows us to "connect the dots" between charges in order to orbit charges circumscribing the boundaries of the charge sets of representations. In general, the smallest irreducible representation containing a given weight $\vec{q}$ also contains all the weights in the same central charge sector as $\vec{q}$ that reside inside the convex hull defined by this perimeter [37].

Appendix B: $SU(3)$ Symmetry

In the following discussion, we elaborate on the details behind our completeness results for $G = SU(3)$. We will consider the cases in which the initial spectrum is $Q_{\mathbf{1}}, Q_{\mathbf{3}}, Q_{\mathbf{8}}$ and $Q_{\mathbf{1}}, Q_{\mathbf{3}}, Q_{\mathbf{10}}$, where completeness can be derived for $z = 1, 2$ and $z = 0, 1, 2$, respectively. In both cases, after a few scattering processes, we will identify a clear recursive pattern that guarantees completeness in the claimed central charge sectors.

1. $Q_{\mathbf{1}}, Q_{\mathbf{3}}, Q_{\mathbf{8}}$ Spectrum

Starting from $Q_{\mathbf{1}}, Q_{\mathbf{3}}, Q_{\mathbf{8}}$, we consider the following three scattering processes,

$$\begin{aligned} (1,-1)\otimes(1,1) &\to (2,0)\vee(0,-2) \\ (2,-2)\otimes(1,1) &\to (3,-1)\vee(1,-3) \\ (3,-2)\otimes(1,2) &\to (4,0)\vee(2,-4), \end{aligned} \tag{B.1}$$

where we use the Weyl transformation in Eq. (20) and charge conjugation after each process. These scattering processes yield the charges comprising the two triangles in the left panel of Fig. 3, which correspond to the new representations $Q_{\mathbf{15}}$ and $Q_{\overline{\mathbf{15}}}$. In the last line we are guaranteed the existence of a state of charge $(4,0)$ or $(2,-4)$. If the state $(2,-4)$ is activated, then since this state is a representative of the $z = 0$ sector, this implies completeness in $z = 0, 1, 2$ and we claim victory. To be conservative, we instead assume that the other state $(4,0)$ is activated.

Now we are ready to state the iterative procedure, starting from the seed representation $Q_{\overline{\mathbf{15}}}$. The charges at the corners of $Q_{\overline{\mathbf{15}}}$ reside at $\{(4,-4),(0,4),(-4,0)\}$. For later convenience, let us define the sequence of triangles $T_n = \{(n,-n),(0,n),(-n,0)\}$, where T_4 are the

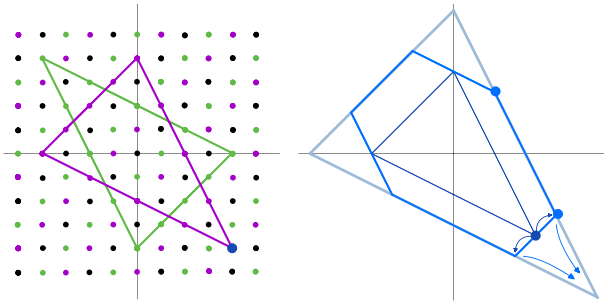


Figure 3. The $SU(3)$ charge lattice. Left: The seed of the iteration. Right: Sequence of polygons allowing iteration, namely, the dark blue triangle T_n, blue hexagon H_n, and light blue triangle T_{n+3}. Relevant points for scattering are denoted with circles.

corners of $Q_{\overline{\mathbf{15}}}$. Note from Eq. (20) that all three corners of T_n are contained within the same Weyl orbit. We then apply the following algorithm:

i) Assuming that we already have the corner charges in T_n, consider the scattering process

$$\begin{array}{c}(n,-n)\otimes(1,1)\\ \downarrow\\ (n+1,-n+1)\vee(n-1,-n-1).\end{array} \tag{B.2}$$

The s- and u-channel states above map are related via the Weyl transformation $w_1\circ w_2\circ w_1$, so the process is conclusive and we append both weights to Q. We use the Weyl group and charge conjugation to construct the full hexagon of charges H_n from T_n. This is depicted geometrically as a blow-up operation in the right panel of Fig. 3. Explicitly, one of the Weyl transformations w_2 sends $(q_1,q_2)\to(q_1+q_2,-q_2)$, which connects the two blue points in the Figure, $(n+1,-n+1)$ and $(2,n-1)$.

ii) Take charges in the hexagon and scatter them,

$$\begin{array}{c}(n+1,-n+1)\otimes(2,n-1)\\ \downarrow\\ (n+3,0)\vee(n-1,-2n+2).\end{array} \tag{B.3}$$

Immediately, we see that the state $(n-1,-2n+2)$ is in the $z=0$ sector, so its presence would ensure completeness in $z=0,1,2$. Again, we choose the more conservative option and assume that the activated channel is $(n+3,0)$. Applying the Weyl group and charge conjugation, we obtain the corners of the new triangle T_{n+3}. This result is shown geometrically by the outermost triangle in the right panel of Fig. 3.

We then repeat these steps ad infinitum to cover the $z=1$ and $z=2$ sectors fully via a sequence of alternating triangles and hexagons.

2. Q_1, Q_3, Q_{10} Spectrum

For $Q_{\mathbf{1}}, Q_{\mathbf{3}}, Q_{\mathbf{10}}$, the initial spectrum includes the brown and gray triangles in Fig. 2. As before, we consider a wisely chosen scattering process to position ourselves for a recursive procedure,

$$(3,-3)\otimes(1,1)\to(4,-2)\vee(2,-4) \tag{B.4}$$

Crucially, via Weyl symmetry, the input of the triangles in Fig. 2 allows us to take the full orbit of the right side of Eq. (B.4) to get all the points in the black hexagon depicted in Fig. 4, in particular the new charge $(2,2)$. This would not have worked assuming only the adjoint.

Next, we define the specific points $P_n=(3+2n,-3-n)$ and $D_n=(2+n,2+n)$, denoted by the large dots in the right panel of Fig. 4. Our starting points will be P_0 and D_0. Now iterate as follows.

i) At step n, consider the scattering process,

$$P_n\otimes D_n\to(5+3n,-1)\vee(1+n,-5-2n). \tag{B.5}$$

Note that the s-channel charge resides in a larger representation than the u-channel charge, since $|5+3n|\geq|5+2n|>|1+n|$. To be conservative, we assume the smaller representation, which corresponds to the latter choice of $(1+n,-5-2n)$. This operation is geometrically understood in Fig. 4 as blowing up each edge of the dark blue hexagon to get the blue one.

ii) Now move to the bottom right charge of the new polygon, P_{n+1}. Scatter this again via

$$P_{n+1}\otimes D_n\to(7+3n,-2)\vee(3+n,-6-2n), \tag{B.6}$$

and choose $(3+n,-6-2n)$ for the same reason as before. This effectively grants us the enclosing light blue hexagon in Fig. 4, so we are back to the shape we started with, but rescaled. Crucially this hexagon contains a higher weight on the diagonal, which means D_{n+1} is ready for use for step $n+1$.

Similarly to the previous algorithm, the resulting infinite sequence of alternating hexagons covers $z=0$ fully. From this construction, we automatically obtain arbitrarily large representatives of $z=1$ and $z=2$ by the process

$$(n,n)\otimes(1,0)\to(n+1,n)\vee(n-1,n) \tag{B.7}$$

Using charge conjugation, we can then obtain both channels. The action of the roots then moves charges towards the origin, allowing us to claim completeness of the full $SU(3)$ lattice.

Appendix C: Representation Completeness

In the main text, we have demonstrated how a non-abelian symmetry group G can be exploited to derive

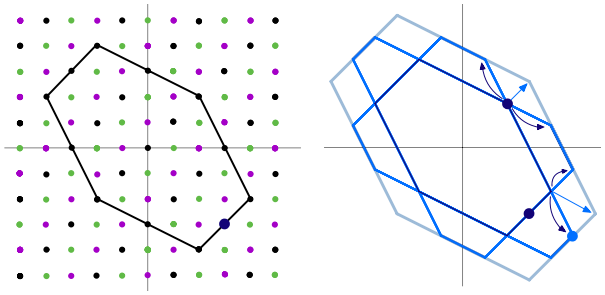


Figure 4. The $SU(3)$ charge lattice. Left: starting hexagon with the starting charge P_0 (dark blue). Right: sequence of hexagons (dark blue, blue, light blue), highlighting the relevant points P_n, (dark blue), D_n (dark blue), and P_{n+1} (blue).

completeness in the abelian charge lattice of the Cartan subgroup H. These results suggest a natural follow-up question: do our assumptions also imply the existence of all possible irreducible nonabelian representations of the symmetry group G? This line of inquiry is particularly well motivated in light of existing statements concerning completeness in the spectrum of irreducible representations [8, 22].

Unfortunately, completeness in the Cartan charge lattice does not, in and of itself, carry any direct implications about the spectrum of nonabelian representations. The reason for this is straightforward. While charge completeness certainly guarantees the existence of arbitrarily large nonabelian representations, this fact tells us absolutely nothing about the precise properties or structure of these representations. So any given abelian charge has the freedom to reside within any of an infinite number of arbitrarily large representations.

In the example of $SU(2)$ angular momentum, this observation corresponds to the fact that a state of a given J_z can appear in any representation of spin $J \geq |J_z|$. Without additional information, there is no way to ascertain the precise value of J.

In this section we briefly summarize some of our attempts at proving representation completeness. In Apps. C 1, C 2, C 3, our analysis will follow that of Ref. [14], which essentially used the full color structure of the dispersion relation in Eq. (1) to deduce powerful constraints on the spectrum of nonabelian representations. There, our results will largely be negative.

We end with Sec. C 4, in which we describe how, with an extra assumption, our algorithm succeeds in yielding representation completeness.

1. Completeness from Proof by Contradiction

The basic strategy of Ref. [14] is to assume certain spectra and then act on Eq. (1) with wisely chosen color projectors to generate a contradiction [38]. Let us now briefly review this approach.

Consider the scattering of a pair of particles in the nonabelian representations R_1 and R_2 of the symmetry group $G = SO(N)$. In terms of the four-point scattering amplitude, the external legs reside in the representations $R_1 = R_4$ and $R_2 = R_3$. The particles in the s or u channels will transform under representations appearing in the direct sum decomposition of $R_1 \otimes R_2$. We denote the set of representations in this direct sum by $\mathcal{S}(R_1, R_2)$.

Given external states with the general nonabelian indices i_1, i_2, i_3, i_4, it is convenient to define the projectors $(\mathcal{P}_R^I)^{i_1 i_2 i_3 i_4}$, for $I = s, t, u$, as a basis of color space. Concretely, $(\mathcal{P}_R^I)^{i_1 i_2 i_3 i_4}$ is the tensor structure induced by a state in the representation R exchanged in each channel. By crossing symmetry, we know that $(\mathcal{P}_R^s)^{i_1 i_2 i_3 i_4} = (\mathcal{P}_R^t)^{i_3 i_2 i_1 i_4} = (\mathcal{P}_R^u)^{i_1 i_3 i_2 i_4}$. The projectors exhibit a natural scalar product, $\langle \mathcal{P}_R^I, \mathcal{P}_{R'}^{I'} \rangle = (\mathcal{P}_R^I)^{i_1 i_2 i_3 i_4} (\mathcal{P}_{R'}^{I'})^{i_1 i_2 i_3 i_4}$, where the projectors are normalized to ensure that $\langle \mathcal{P}_R^I, \mathcal{P}_{R'}^I \rangle = \delta_{RR'}$.

The dispersion relation in Ref. [14] is similar to Eq. (3), but with s-dependent subtraction and expressed in terms of color projectors,

$$-\frac{8\pi G}{t}\mathcal{P}_0^t + \cdots = \sum_{R\in\mathcal{R}} \left[K_R(s,t)\mathcal{P}_R^s + K_R(u,t)\mathcal{P}_R^u\right], \quad \text{(C.1)}$$

where $\mathcal{R} \subseteq \mathcal{S}(R_1, R_2)$ is the set of representations that are exchanged in the s and u channels. To make contact with Ref. [14], we define

$$K_R(X,t) = \sum_J \int_{M^2}^{\infty} ds' \frac{\mathbb{G}_J^{(D)}\left(1+\frac{2t}{s'}\right) \mathrm{Im} f_{J,R}(s')}{\pi s'(s'+t)(s'-X)}, \quad \text{(C.2)}$$

with $\mathbb{G}_J^{(D)}$ the Gegenbauer polynomials. Here, for brevity we write the dispersion relation for massless external states, but introducing masses is straightforward.

To derive a contradiction, we first assume some choice for $\mathcal{R}$. Second, we contract both sides of Eq. (C.1) with a tensor $v^{i_1 i_2 i_3 i_4}$. This object is completely arbitrary, since we have the freedom to choose any external color polarizations. Normalizing this tensor so that $\langle \mathcal{P}_0^t, v \rangle = 1$, we obtain the following equation,

$$-\frac{8\pi G}{t} = \sum_{R\in\mathcal{R}} \left[K_R(s,t)\langle \mathcal{P}_R^s, v\rangle + K_R(u,t)\langle \mathcal{P}_R^u, v\rangle\right], \quad \text{(C.3)}$$

dropping the subleading terms in Eq. (C.1) at small t. Now, if there exists some v for which

$$\langle \mathcal{P}_R^s, v\rangle = \langle \mathcal{P}_R^u, v\rangle = 0, \quad \text{(C.4)}$$

for all $R \in \mathcal{R}$, then Eq. (C.1) cannot be satisfied and $\mathcal{R}$ is not a consistent spectrum. In such a case, there must exist in the spectrum at least one representation in $\mathcal{S}(R_1, R_2)$ that is not in $\mathcal{R}$ in order for the dispersion relation to be consistent [39].

Instead of looking for a general tensor v that solves Eq. (C.4), the authors of Ref. [14] effectively insert the

identity operator in the space of projectors $|\mathcal{P}^t_{R'}\rangle\langle\mathcal{P}^t_{R'}|$ to obtain

$$\begin{aligned}\langle\mathcal{P}^s_R,\mathcal{P}^t_{R'}\rangle\langle\mathcal{P}^t_{R'},v\rangle = \langle\mathcal{P}^u_R,\mathcal{P}^t_{R'}\rangle\langle\mathcal{P}^t_{R'},v\rangle = 0\\ (M_{st})_{RR'}v_{R'} = (M_{ut})_{RR'}v_{R'} = 0,\end{aligned} \tag{C.5}$$

where we introduced the vector $v_{R'} = \langle\mathcal{P}^t_{R'},v\rangle$ and the matrix $(M_{st})_{RR'} = \langle\mathcal{P}^s_R,\mathcal{P}^t_{R'}\rangle$, and similarly for M_{ut}. Note that $(M_{st})_{RR'} = (M_{ut})_{RR'}(-1)^{g_{R'}}$, where $g_{R'} = 0$ for symmetric representations and $g_{R'} = 1$ for antisymmetric ones. The authors of Ref. [14] then solved the linear system of equations in Eq. (C.5) to find inconsistent sets of exchanged representations $\mathcal{R}$. They studied the scattering of fundamentals for both $SO(N)$ and $SU(N)$ and the scattering of adjoints for $SO(N)$.

This approach is challenging to implement for arbitrary initial states and general groups. The obstacle stems from two difficulties. First, computing M_{st} and M_{ut} for general groups and representations is a challenging task. Second, the cardinality of $\mathcal{S}(R_1,R_2)$ increases with the rank of the initial representations R_1, R_2. Scanning over all consistent subsets of $\mathcal{S}(R_1,R_2)$ requires solving of order $2^{|\mathcal{S}(R_1,R_2)|}$ systems of linear equations.

2. *SO*(3) Symmetry

For the symmetry group $G = SO(3)$, the matrices M_{st} and M_{ut} are known in closed form. In particular, these objects are the Racah W-coefficients, which up to a phase are the Wigner 6-j symbols that re-express angular momenta in the t channel in terms of angular momenta in the s and u channel. Explicitly, M_{st} is proportional to $\sqrt{(2J_{12}+1)(2J_{23}+1)}W(j_1j_2j_4j_3;J_{12}J_{23})$ and similarly for M_{tu}. In the terminology of angular momentum, j_1,j_2,j_3,j_4 are the external spins and J_{12},J_{23} are the exchanged spins.

Using this expression, we were able to show that any set of exchanged representations $\mathcal{R}$ comprising a finite set of spins between 0 and N presents no inconsistency, which we have verified explicitly for $N \leq 4$. If any spin between 0 and N is missing, then the spectrum is inconsistent. The case of $N = 1$, which includes spin-zero and spin-one states, was considered in Ref. [14] and shown to be consistent. Importantly, this approach indeed indicates that an infinite number of representations is not needed for scattering consistency, since N need not be infinite.

3. *SO*(4) Symmetry

For $G = SO(4)$, we can also compute M_{st} and M_{ut} analytically. This is possible because $SO(4) = SU(2)^2/\mathbb{Z}_2$, which implies that we can derive the Racah matrices in $SO(4)$ by computing tensor products of Racah matrices of $SU(2)$ (which coincide with those of $SO(3)$). At the same time, given that completeness of $SU(2)$ cannot be proven using this method, it also cannot be proven for $SO(4)$, as discussed in Sec. IV B.

Something more can be said for the group $O(4)$. Each representation of $O(4)$ is of the form $(j_a,j_b)\oplus(j_b,j_a)$, where j_a and j_b are the spins relative to the two $SU(2)$ factors. It is simple to compute Racah matrices for scattering of representations such that either $j_a = j_b$ or $j_b = 0$. However, other than confirming what we already knew from the weight lattice argument, we did not obtain any new general results. In particular, our experiments suggest that completeness at the level of representations cannot be proven using this strategy. We checked that for all the scattering processes we could consider, we could always exclude from a set one arbitrary representation, suggesting that a set that contains all the representations except one is consistent. This conclusion could have been anticipated by simply counting the number of degrees of freedom in Eq. (C.5) and the number of equations. In any case, a systematic study would be required to conclude that this approach cannot be used to prove completeness.

4. Completeness with Both Channels

If, for whatever reason, we are granted knowledge that *both* channels in Eq. (3) are necessarily nonzero, then we can straightforwardly derive much stronger claims of completeness. In this case, the scattering of charges $\vec{q}$ and $\vec{q}'$ will necessarily entail new states in the spectrum of charge $\vec{q}+\vec{q}'$ and also $\vec{q}-\vec{q}'$. Mechanically, we can then deduce the existence of any charge from the sum or difference of other charges. For the case of $G = U(1)$, charge completeness follows trivially from the existence of even a single particle of charge $q = 1$. Scattering $q = 1$ with $q' = 1$ requires a new particle with charge $q+q' = 2$, and so on.

This approach generalizes easily to $G = SO(3)$, assuming the spectrum contains a particle in the fundamental spin-one representation. Scattering a pair of such particles with $q = 1$ and $q' = 1$, we obtain a particle with charge $q+q' = 2$. This state necessarily resides in the spin-two representation, since this is the only representation that carries this charge in the tensor product of spin-one with itself. We then scatter states in the spin-one and spin-two representations with charges $q = 1$ and $q' = 2$, yielding a new particle with $q+q' = 3$. Iterating this process generates all possible spin representations.

The generalization to $SO(N)$ follows easily from Eq. (C.4). The requirement that both channels must be nonzero corresponds to enforcing only one of the two equations. By choosing $v \propto \mathcal{P}^s_{R'}$ the equation $\langle\mathcal{P}^s_R,v\rangle = 0$ is satisfied for all $R \neq R'$, meaning that R' must be included in the spectrum. Given that R' is any representation, representation completeness is proven. Importantly, we also have to check that $\langle\mathcal{P}^t_0,v\rangle$ is different from zero. This follows immediately from the completeness relation $\mathcal{P}^t_0 \propto \sum_R c_R\mathcal{P}^s_R$, with $c_R \neq 0$ [40].

Paper 6: The Equivalence Principle at High Energies Completes the Spectrum

CALT-TH 2026-015

The Equivalence Principle at High Energies Completes the Spectrum

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We prove a version of the completeness hypothesis that follows from the coexistence of symmetry and gravity: tree-level gravitational scattering mandates single-particle states in all possible irreducible representations of the symmetry group constructible from a single seed charge. Our main assumption is that the leading high-energy behavior of scattering is universal irrespective of charge, thus satisfying the equivalence principle. Curiously, we discover that these newly-deduced states contribute democratically—that is, with equal interaction strengths—to scattering.

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Introduction.—What are the universal features of any consistent theory of quantum gravity? By its very nature, this question transcends string theory or any alternatives. Indeed, this challenge pertains to the *full space* of possible ultraviolet completions of general relativity, whatever that may be.

Along these lines, the swampland program [1] has identified a litany of seemingly universal properties of all known string theoretic constructions. These patterns have been articulated as broad conjectures dictating the fate of symmetry in any consistent theory of quantum gravity: exact global symmetries are forbidden [2, 3], gravity is the weakest force [4], and any charged state that is not forbidden is mandatory [5, 6]. The latter is known as the completeness hypothesis.

In parallel with these top-down efforts, the modern scattering amplitudes program has developed bootstrap methods that stringently constrain the space of consistent theories from the bottom up. Axiomatic principles like locality, causality, and unitarity [7] have enjoyed great success bounding a variety of effective field theories (EFTs), even finding purchase in placing bounds on quantum gravity and providing theory-independent justifications for conjectures from the swampland [8–107].

In this paper we present a simple argument that rigorously proves a version of the completeness hypothesis for gravitational EFTs. Our principal assumptions are that there is *i*) an exact symmetry group G, either discrete or continuous, abelian or nonabelian, gauged or nongauged, *ii*) a spectrum including at least one particle charged under G, and *iii*) a weakly coupled ultraviolet completion of gravity in which the leading high-energy behavior satisfies the equivalence principle. Physically, this last condition encodes the reasonable property that the high-energy behavior of gravitationally interacting particles is universal irrespective of charge. Mathematically, it is the property that the leading Regge trajectory is a singlet under G. Note that our assumptions are satisfied by string amplitudes, but they apply more broadly.

Under these assumptions, we use analytic dispersion relations for scattering amplitudes to rigorously derive *representation completeness*: all irreducible representations obtained from the tensor products of a seed charged state must appear as single-particle states in the spectrum. Building on the program first initiated in Ref. [9], the present work establishes a much stronger conclusion than an earlier study [8], which used a distinct and more complicated analysis to derive completeness for abelian subgroups of certain continuous nonabelian symmetries G of sufficiently high rank.

An intriguing corollary of the current analysis is *representation democracy*: the particles whose existence we deduce contribute with squared couplings that are completely independent of their representation after averaging over all quantum numbers distinct from G. Said another way, particles contribute to scattering democratically with respect to their representation. Since these states are the asymptotically high-energy excitations in the spectrum, the correspondence principle [108] suggests that this property may be a manifestation of equipartition of states into charge sectors in a black hole.

Amplitude.—Consider a tree-level four-point amplitude for particles charged under an exact symmetry G in a gravitational EFT. In principle, this symmetry may be global or gauged. We will not consider the case where it is explicitly broken. Without loss of generality, the low-energy amplitude of massless particles is

$$A(z,t) = -\frac{8\pi G_N z^2}{t}\mathbb{1} + \frac{8\pi\alpha z}{t}\mathbb{T} + \sum_{k=0}^{\infty} c_k(t) z^k, \qquad (1)$$

where the center-of-mass energy and momentum transfer are encoded in $z = (s-u)/2$ and $t = -(s+u)$, which are odd and even under crossing the s and u channels. The $1/t$ singularities encode the tree-level exchange of the graviton and, if present, a gauge boson for the symmetry. The $c_k(t)$ are regular functions of t. This expansion generalizes straightforwardly to massive scattering.

Since the external states reside in irreducible representations of the symmetry group, the amplitude and all of its terms carry implicit indices. In particular, the contribution from graviton exchange in Eq. (1) is proportional to the *identity element* $\mathbb{1}$ in charge space, while that of the gauge boson is proportional to a nontrivial tensor $\mathbb{T}$. The former leaves the charges of the scattering states unscathed, in accordance with the equivalence principle.

Case in point, for $G = SO(N)$ and external states in the fundamental, we have the explicit index form,

$$\mathbb{1} = \delta^{i_1 i_4}\delta^{i_2 i_3}. \tag{2}$$

This tensor describes a neutral exchange in the t channel, so the resulting scattering process has the same charge structure as free propagation. Our logic will hinge on the fact that $\mathbb{1}$ can be decomposed into a basis of projectors,

$$\mathbb{1} = \mathbb{P}_0^{(s)} + \mathbb{P}_1^{(s)} + \mathbb{P}_2^{(s)}, \tag{3}$$

for each s-channel irreducible representation,

$$\begin{aligned}
[\mathbb{P}_0^{(s)}]^{i_1 i_2 i_3 i_4} &= \frac{1}{N}\delta^{i_1 i_2}\delta^{i_3 i_4} \\
[\mathbb{P}_1^{(s)}]^{i_1 i_2 i_3 i_4} &= \frac{1}{2}(\delta^{i_1 i_4}\delta^{i_2 i_3} - \delta^{i_1 i_3}\delta^{i_2 i_4}) \\
[\mathbb{P}_2^{(s)}]^{i_1 i_2 i_3 i_4} &= \frac{1}{2}(\delta^{i_1 i_4}\delta^{i_2 i_3} + \delta^{i_1 i_3}\delta^{i_2 i_4}) - \frac{1}{N}\delta^{i_1 i_2}\delta^{i_3 i_4}.
\end{aligned} \tag{4}$$

Physically, Eq. (3) implies that the charge structure associated with the graviton pole is a democratic sum over all irreducible representations.

The form of Eq. (2) is general and independent of the representations of the external states. Furthermore, with appropriate sign conventions for the projectors, it applies to both the s and u channel, so

$$\mathbb{1} = \sum_\rho \mathbb{P}_\rho^{(s)} = \sum_\rho \mathbb{P}_\rho^{(u)}, \tag{5}$$

where ρ labels all irreducible representations in the direct sum decomposition of the tensor product of incoming states in that channel [109]. For instance, if the s-channel fuses states in distinct representations ρ_1 and ρ_2, then $\mathbb{1} = \delta^{i_1 i_4}\delta^{j_2 j_3}$, where $i_1, i_4 = 1, \ldots, \dim \rho_1$ and $j_2, j_3 = 1, \ldots, \dim \rho_2$. The simple mathematical fact defined in Eq. (5) will undergird all of our results.

Dispersion Relation.—We are now equipped to derive a dispersion relation for the EFT coefficients defined by Eq. (1). Doing so will require certain assumptions about the Regge behavior of the amplitude.

The components of the amplitude that are even and odd under s- and u-channel crossing, known as the pomeron and odderon [110, 111], are

$$A^{(\pm)}(z,t) = \frac{1}{2}(A(z,t) \pm A(-z,t)). \tag{6}$$

Any graviton or gauge boson exchanges are encoded in $A^{(+)}$ and $A^{(-)}$, respectively. Recall that at fixed $t < 0$, the Regge behavior of the even amplitude falls off as

$$\lim_{z\to\infty} \frac{A^{(+)}(z,t)}{z^2} = 0. \tag{7}$$

As rigorously derived in Ref. [112], the above asymptotics are a universal feature of all gravitational theories. For example, string amplitudes exhibit $A^{(+)} \sim z^{2+t}$, in which case Eq. (7) is marginal as $t \to 0^-$.

A crucial additional assumption of the present work is that the leading Regge behavior of the amplitude satisfies the equivalence principle. As we will see, this requirement implies that at fixed $t < 0$, the odd amplitude scales as

$$\lim_{z\to\infty} \frac{A^{(-)}(z,t)}{z^{2-\delta}} = 0, \tag{8}$$

for some constant $\delta > 0$ that is independent of t. For string amplitudes, Eq. (8) holds because $A^{(-)} \sim z^{1+t}$.

The appearance of δ in Eq. (8) ensures that the large-z growth of $A^{(-)}$ is strictly subleading to z^2 in such a way that is not even marginal as $t \to 0^-$. If this condition is violated, then $A^{(-)}$ can scale as z^2 or worse as $t \to 0^-$, thus contributing to the leading Regge scaling of the amplitude. The resulting effect is, by definition, odd under crossing, so it cannot be proportional to the identity in charge space. In such a case, the high-energy behavior of the amplitude $A = A^{(+)} + A^{(-)}$ depends explicitly on the charges of the external states, in violation of the equivalence principle. An example of this is $A^{(-)} \sim (s^{2+t} - u^{2+t})/t \sim i(\arg(z) - \arg(-z))z^2$, which is odd under crossing but scales as z^2. Though not strictly pathological—pure consistency only requires the equivalence principle in the t-channel pole mediating the long-range gravitational force—such high-energy behavior is nevertheless very peculiar. In any case, our assumption of a high-energy equivalence principle forbids this possibility, so Eq. (8) applies. See the appendix for a discussion of various putative amplitudes that actually violate Eq. (8) and the equivalence principle.

For $t < 0$, the following contour integral at infinity is zeroed out by the assumption in Eq. (7),

$$\oint_\infty \frac{dz}{z^3} A(z,t) = \oint_\infty \frac{dz}{z^3} A^{(+)}(z,t) = 0. \tag{9}$$

Deforming the contour of integration inwards to encircle all enclosed poles, we then obtain [113]

$$-\frac{8\pi G_N \mathbb{1}}{t} + c_2(t) = \sum_n \frac{R^{(+)}(n,t)}{\mu(n,t)^3}, \tag{10}$$

where we have defined $\mu(n,t) = \mu(n) + t/2$ in terms of the spectrum of nonzero squared masses $\mu(n)$. The amplitude exhibits poles at $s = \mu(n)$ and $u = \mu(n)$ [114], and without loss of generality, the ordering of the label n is chosen so that $\mu(n)$ increases monotonically. We further assume that $\mu(n)$ diverges at large n, thus forbidding the possibility of accumulation point spectra [115, 116]. Here we have also defined the components of the residue that are even and odd under crossing symmetry,

$$\begin{aligned}
R^{(\pm)}(n,t) &= R^{(s)}(n,t) \pm R^{(u)}(n,t) \\
R^{(s)}(n,t) &= \lim_{z\to+\mu(n,t)} (\mu(n,t) - z)A(z,t) \\
R^{(u)}(n,t) &= \lim_{z\to-\mu(n,t)} (\mu(n,t) + z)A(z,t).
\end{aligned} \tag{11}$$

where $R^{(s)}(n,t)$ and $R^{(u)}(n,t)$ are the residues at $s = \mu(n)$ and $u = \mu(n)$, respectively. Note that legs 1 and 4 must be in the same irreducible representation, and similarly for 2 and 3, so that a graviton can be exchanged in the t channel. Locality implies that each of these residues is a polynomial in t, describing a finite set of spins exchanged at that mass level.

Next, we derive a formula for a version of Eq. (10) with $R^{(-)}$ instead of $R^{(+)}$,

$$\tilde{c}_2(t) = \sum_n \frac{R^{(-)}(n,t)}{\mu(n,t)^3}. \tag{12}$$

The power of $\mu(n,t)$ in the denominator is odd rather than even, so $\tilde{c}_2(t)$ does not correspond to a Wilson coefficient that can be extracted from the amplitude. Nonetheless, it can be computed via a line integral along the imaginary z-axis intersecting the origin,

$$\tilde{c}_2(t) = \frac{1}{\pi i}\int_{-i\infty}^{+i\infty} \frac{dz}{z^3}\hat{A}^{(-)}(z,t), \tag{13}$$

where $\hat{A}^{(-)}(z,t) = A^{(-)}(z,t) - z(\partial_{z'}A^{(-)}(z',t)|_{z'=0})$. In the small-$z$ expansion, $\hat{A}^{(-)} = O(z^3)$ is simply $A^{(-)} = O(z^1)$ with its z^1 component subtracted away. Crucially, the integrand of Eq. (13) is regular at $z = 0$, and the integral converges at infinity due to Eq. (8). We thus deform the contour to enclose all poles of $\hat{A}^{(-)}(z,t)$ in the right half-plane, where boundary terms at infinity vanish by Eq. (8). Since $\hat{A}^{(-)}(z,t)$ differs from $A^{(-)}(z,t)$ by an entire function, they share the very same poles at $z = \pm\mu(n,t)$, so

$$A^{(-)}(z,t) = \sum_n \frac{zR^{(-)}(n,t)}{\mu(n,t)^2 - z^2} + \cdots, \tag{14}$$

where the ellipses are nonsingular. Summing the residues in this formula for each $z = +\mu(n,t)$ yields Eq. (12).

If the symmetry is gauged, then $A^{(-)}(z,t)$ exhibits a z/t pole that is precisely cancelled in $\hat{A}^{(-)}(z,t)$. Consequently, the latter is regular as $t \to 0^-$. This regularity of course persists if the symmetry is global. Either way, we learn that $\tilde{c}_2(t)$ is, crucially, finite as $t \to 0^-$ [117].

Completeness.—We are now equipped to derive a remarkable fact about the spectrum of states. Adding and subtracting Eq. (10) and Eq. (12) yields

$$\begin{aligned} -\frac{8\pi G_N \mathbb{1}}{t} + c_2(t) + \tilde{c}_2(t) &= 2\sum_n \frac{R^{(s)}(n,t)}{\mu(n,t)^3} \\ -\frac{8\pi G_N \mathbb{1}}{t} + c_2(t) - \tilde{c}_2(t) &= 2\sum_n \frac{R^{(u)}(n,t)}{\mu(n,t)^3}, \end{aligned} \tag{15}$$

which as $t \to 0^-$ implies that

$$\sum_n \frac{R^{(s)}(n,t)}{\mu(n,t)^3} \sim \sum_n \frac{R^{(u)}(n,t)}{\mu(n,t)^3} \sim -\frac{4\pi G_N \mathbb{1}}{t}, \tag{16}$$

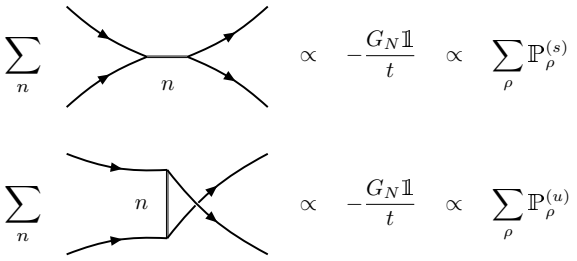


Figure 1. According to Eq. (16), a certain weighted sum of residues in both the s and u channels must resolve to the identity, thus implying completeness: each representation of the symmetry group must be exhibited by a single-particle state in the spectrum. Furthermore, the massive states that dictate the Regge behavior are not only complete, but also appear democratically, with interaction strengths that are independent of their representation.

where we drop all terms that are subleading in t.

Equation (16) implies that the spectrum of irreducible representations of G is complete. To arrive at this conclusion, we appeal to proof by contradiction: if some irreducible representation is completely absent from all s-channel exchanges, then by Eq. (5) it is impossible for the aggregate of s-channel states to sum to $\mathbb{1}$, as required by Eq. (16). Identical logic of course applies to states in the u channel. Thus, as depicted in Fig. 1, all irreducible representations that arise from the tensor product of the external states fusing in either the s or u channel must appear as single-particle states in the spectrum.

Following the logic of Ref. [8], we repeat this process ad infinitum. In particular, we take the s-channel states whose existence we have just deduced and then rescatter them as external states, again assuming Eqs. (7) and (8). Applying the very same dispersion relations as before, we then deduce that all irreducible representations built from these new external states must also be present in the spectrum. Repeating this process forever, we arrive at the conclusion that every irreducible representation of G generated by tensor products of the original seed must be present in the spectrum. Since our amplitudes are tree-level into the ultraviolet, each irreducible representation appears as a single-particle state [118]. In fact, Eq. (16) implies that each irreducible representation appears infinitely many times, in an infinite tower of ever more massive single-particle states.

Note that the logic outlined above is vastly more powerful than that of Ref. [8], which implemented a much more laborious procedure to derive completeness for the abelian charges within certain nonabelian symmetry groups of higher rank. The major technical difference between these works is the high-energy equivalence principle in Eq. (8), which we leverage to derive universality of irreducible representations in both the s *and* u channel. In contrast, the argument of Ref. [8] relied on more conservative assumptions about ultraviolet scaling—requiring only $A \lesssim s^2$ and not Eq. (8)—and

employed a technically involved analysis of the root lattice of various symmetry algebras to deduce new states in either the s *or* u channel. As a result, Ref. [8] only established completeness for the abelian charge lattice, while the present work does so for all irreducible representations of any symmetry.

It is worth noting that G need not be internal, and in fact it can be a symmetry of spacetime. For example, in the forward limit, since all ingoing and outgoing particles are collinear, their transverse polarizations are unambiguously classified as irreducible representations of the D-dimensional massless little group $SO(D-2)$. After recasting little group charge as an internal quantum number [48, 49], the above logic implies completeness of all irreducible representations of $SO(D-2)$. This prediction is borne out by explicit calculation of the spectrum of spinning excitations of the string. For example, see Ref. [119], which demonstrates little group completeness for the bosonic, heterotic, type I, and type II strings, as well as Ref. [120], which tabulates the Young tableaux describing the exotic spin states of the bosonic string. Of course, if the seed particle is a fermion, then we can upgrade to $Spin(D-2)$ to infer completeness among all possible fermionic representations as well.

Democracy.—Let us now consider the Regge limit of the amplitude. At fixed t, we parameterize the behavior of the amplitude at large positive and negative z by

$$\begin{aligned} \lim_{z\to+\infty} A &\sim (-z)^{\alpha^{(s)}(t)}\mathbb{P}_\infty^{(s)} \\ \lim_{z\to-\infty} A &\sim (+z)^{\alpha^{(u)}(t)}\mathbb{P}_\infty^{(u)}. \end{aligned} \tag{17}$$

Here $[\mathbb{P}_\infty^{(s)}]^{i_1j_2j_3i_4}$ and $[\mathbb{P}_\infty^{(u)}]^{i_1j_2j_3i_4}$ are unspecified tensors encoding the symmetry structure of the Regge behavior in the s and u channels. Other tensor structures can appear, but only one will dominate the Regge limit.

Equation (17) exhibits branch cuts along the positive and negative real z-axis. As is well known, these nonanalyticities encode the infinite tower of simple poles in the s and u channel [80]. Said another way, the branch cuts are a smeared approximation of the simple poles, averaged over some resolution in energy squared that we define to be Δz, or equivalently Δn in level spacing. By definition, we take Δz and Δn to be the spacing at which the Regge formula in Eq. (17) is a valid approximation of the actual amplitude.

Next, we compute the weighted integral of the s-channel discontinuity of the amplitude on an interval and equate it to the corresponding sum over residues,

$$\sum_{n'=n}^{n+\Delta n} \frac{R^{(s)}(n',t)}{\mu(n')^3} \sim \frac{1}{2\pi i}\int_z^{z+\Delta z}\frac{dz'}{z'^3}\,\mathrm{disc}(A) \propto \mathbb{P}_\infty^{(s)}, \tag{18}$$

and likewise for the u channel. Assuming that the variation in $\mu(n)$ over Δn is much smaller than $\mu(n)$ itself, it is then natural to define the averaged residues,

$$\overline{R}^{(s)}(n,t) = \frac{1}{\Delta n}\sum_{n'=n}^{n+\Delta n} R^{(s)}(n',t), \tag{19}$$

and similarly for u. Comparing to Eq. (16), we then deduce that $\mathbb{P}_\infty^{(s)} = \mathbb{P}_\infty^{(u)} = \mathbb{1}$, and we arrive at the condition of representation democracy,

$$\overline{R}^{(s)}(n,t) \sim \overline{R}^{(u)}(n,t) \sim \mathbb{1}, \tag{20}$$

where according to Eq. (5) the right-hand side is a democratic superposition of irreducible representations.

By definition, the residue $R^{(s)}(n,t)$ is a product of three-point amplitudes, with the exchanged state summed over all quantum numbers at that level n. Then $\overline{R}^{(s)}(n,t)$ averages this quantity over a range of levels. The result in Eq. (20) thus implies that this average squared coupling is equally distributed among representations, so all representations contribute democratically to the scattering process near level n.

According to the correspondence principle [108], sufficiently high-level resonances of the string should have a dual description as black holes. This transition is expected when the Schwarzschild radius of a given string mode exceeds the string length. Blithely interpreting Eq. (20) from this perspective, it is tempting to conclude that the black hole states produced in scattering exhibit a natural equipartition among all possible charged states. We leave the question of whether this connection can be made rigorous to future work.

Heterotic String Amplitudes.—It is illuminating to verify our claims for a well-behaved ultraviolet completion containing gravity and gauge theory: the heterotic string. The tree-level amplitude for the two-to-two scattering of gauge bosons in heterotic string theory is

$$\begin{aligned} A = &-K\frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(s/2)\Gamma(t/2)\Gamma(u/2)}\times \\ &\times\Bigg[\left(\frac{\mathrm{tr}(T_1T_2)\mathrm{tr}(T_3T_4)}{16s(1+s/2)}+\frac{15\mathrm{tr}(T_1T_2T_3T_4)}{st}\right) \\ &+\begin{pmatrix}1234\to1423\\ s,t,u\to t,u,s\end{pmatrix}+\begin{pmatrix}1234\to1342\\ s,t,u\to u,s,t\end{pmatrix}\Bigg], \end{aligned} \tag{21}$$

where the external states are adjoints of $G = SO(32)$ or $SO(16)\otimes SO(16)\subset E_8\otimes E_8$ [121]. We note the distinctive linear spectrum of squared masses $\mu(n)\propto n$, as evident from the gamma functions. Here the kinematic invariant K encodes the external polarizations and scales as p^4 in the momenta. At low energies, this amplitude describes graviton and gauge boson exchanges,

$$\begin{aligned} A = K\Bigg[&\left(\frac{\mathrm{tr}(T_1T_2)\mathrm{tr}(T_3T_4)}{16s}+\frac{15\,\mathrm{tr}(T_1T_2T_3T_4)}{st}\right) \\ &+\begin{pmatrix}1234\to1423\\ s,t,u\to t,u,s\end{pmatrix}+\begin{pmatrix}1234\to1342\\ s,t,u\to u,s,t\end{pmatrix}\Bigg]+\cdots. \end{aligned} \tag{22}$$

It is straightforward to verify that the Regge behavior of this amplitude at large s and fixed t is

$$\begin{aligned}\lim_{s\to\infty} A^{(+)} &= s^{2+t}\frac{\mathcal{E}\,\mathrm{tr}(T_1T_4)\mathrm{tr}(T_2T_3)}{16(1+t/2)}\\ \lim_{s\to\infty} A^{(-)} &= 15s^{1+t}\mathcal{E}\,[\mathrm{tr}(T_1T_2T_3T_4)-\mathrm{tr}(T_1T_4T_2T_3)]\\ \mathcal{E} &= -\frac{\pi(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)[\cot(\pi s/2)+\cot(\pi t/2)]}{2^{t-1}\Gamma(1+t/2)^2},\end{aligned} \tag{23}$$

taking all polarizations transverse to the momenta. The residues of the heterotic amplitude at large n on the poles at $s=2n$ or $u=2n$ are given by

$$\begin{aligned}R(n,t) &\sim \frac{(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)\mathrm{tr}(T_1T_4)\mathrm{tr}(T_2T_3)}{\Gamma(1+t/2)\Gamma(2+t/2)}n^{2+t}\\ &\pm\frac{120(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)[\mathrm{tr}(T_1T_2T_3T_4)-\mathrm{tr}(T_1T_4T_2T_3)]}{\Gamma(1+t/2)^2}n^{1+t},\end{aligned} \tag{24}$$

where $+$ obtains for s and $-$ for u. From both the trace and polarization structure of the n^{2+t} factor, we see that the leading Regge behavior encodes the identity $\mathbb{1}$ in both the color and polarization indices, so our predictions of representation completeness and democracy are both borne out by the heterotic string. From here we see that the $1/t$ graviton and gauge boson poles at $O(s^2)$ and $O(s)$ arise from $\sum_n n^{-1+t} = -1/t+\cdots$ once $R(n,t)$ is weighted with $1/\mu(n)^3$ or $1/\mu(n)^2$, respectively.

Discussion.—We have argued that a version of the completeness hypothesis follows from certain well-motivated features of gravitational scattering. The cornerstone of our argument is the assumption that the equivalence principle—which mandates universality irrespective of the detailed properties of gravitationally interacting bodies—is robust in the regime of high-energy scattering, as encoded mathematically in Eqs. (7) and (8). From this input we deduce that for any weakly coupled ultraviolet completion of gravity exhibiting a global or gauge symmetry G, the spectrum must contain all possible irreducible representations generated by tensor products of a given seed charge. A remarkable corollary of our analysis is the phenomenon of representation democracy: not only is the spectrum complete, but all representations of G appear in high-energy scattering with squared couplings of precisely the same size.

Since the present work has focused exclusively on tree-level amplitudes, it is natural to wonder if adding loop-level amplitudes will modify our conclusions. In string theory, for example, the perturbative loop corrections to the Regge behavior are quite subtle [122–124]. Fortunately, these concerns are immaterial: in any perturbative ultraviolet completion where the unitarizing states enter at tree level, loop corrections are by definition subleading. If one-loop corrections are as important as tree, then so too are the contributions at two-loop order, and so on. Within such perturbatively calculable theories, we have established completeness of the spectrum.

The results of this paper contribute to a growing body of evidence that the broad features of string theory might actually follow directly from well-motivated physical conditions on scattering [8–10, 80–95]. For example, the seminal work of Ref. [95] argued for the asymptotic uniqueness of strings from unitarity and crossing, although an author of that paper later demonstrated [70] explicit counterexamples to their technical assumptions. In the past year there has been a resurgence of interest in this topic, uniquely bootstrapping strings from alternative assumptions, such as ultrasoftness [80] or supersymmetry [85]. These developments all suggest that there is much promise in characterizing quantum gravity—its states, dynamics, and potentially other swampland conjectures—using the tools of scattering amplitudes.

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[109] Our construction applies extremely broadly, namely to any group G where, given two irreducible representations $\rho_{1,2}$, one can write $\rho_1 \otimes \rho_2$ as a direct sum of irreducible representations of G. This commonsense fact is true for the vast majority of groups of relevance to physics, including abelian and nonabelian simple Lie groups, discrete groups that are either finite or abelian, all finite products thereof, and more. Relevant mathematical results include the theorems of Weyl, Maschke, Peter-Weyl, and Plancherel; see Refs. [125, 126] for a review.
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[114] We can choose our external states so that their masses satisfy $m_1 = m_4$ and $m_2 = m_3$. This choice guarantees that the spectrum of squared masses exchanged in the s and u channel are the same, both described by $\mu(n)$. If the external scattered states are massive, then as in Ref. [37] we should define the subtraction in Eq. (9) as
$$\oint_\infty \frac{dz}{\left(z + \frac{1}{2}\sum_{i=1}^4 m_i^2\right)^3} A^{(+)} = 0,$$
so Eq. (10) is unchanged, and similarly for Eq. (12).
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[117] Mathematically, regularity of $\tilde{c}_2(t)$ in the $t \to 0^-$ limit requires uniform boundedness of Eq. (13). For this reason, we assume that there exist t-independent constants $c > 0$ and $\delta > 0$ such that $|\hat{A}^{(-)}(z,t)| < c|z^{2-\delta}|$ along the full contour of integration in z and on an open set $-\epsilon < t < 0$ for some $\epsilon > 0$. This statement is a more refined version of the assumption in Eq. (8). Physically, this assumption says that the odd amplitude—after subtracting its massless t-channel pole—falls off sufficiently fast that the equivalence principle remains valid at high energies.
[118] Given that an incomplete spectrum is demonstrably inconsistent, it is then natural to ask: At what energy

scale do the spectrum-completing states actually reside? Obviously, if a dispersion relation is only marginally convergent, then one expects states to continue contributing to the dispersion relation up to very high energies. Quantitatively, given Regge behavior of the form $A \sim s^{\alpha(t)}$ for $\alpha(t)$ small and negative, convergence requires summing states at least up to the scale $s \gtrsim e^{1/|\alpha(t)|}$. In all extant works on dispersion relations for scattering, $\alpha(t)$ is taken to be order one or less in units of the gap, in which case one generically expects the dispersion relation to include states that are exponentially heavy relative to that gap [71]. The same is true for us here.

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Appendix: High-Energy Behavior.—It is natural to ask whether democracy holds for the open type I superstring, which has a massless gauge boson but does not exchange a graviton in the planar sector. To investigate this situation, let us replace the ultraviolet scaling assumptions in Eqs. (7) and (8) with lower powers in s, $\lim_{s\to\infty} A^{(-)}/s = 0$ and $\lim_{s\to\infty} A^{(+)} = 0$ for $t < 0$, since the high-energy scattering of massless gauge bosons in open superstring theory goes like $A \sim A^{(-)} \sim s^{1+t}$.

Following a derivation analogous to the gravity case, but with these different ultraviolet conditions, we would find that the averaged residues asymptotically satisfy

$$\lim_{n\to\infty} \overline{R}^{(s)}(n,0) \sim -\lim_{n\to\infty} \overline{R}^{(u)}(n,0) \propto \mathbb{P}^{(t)}_{\text{adj}}, \qquad (25)$$

where the color structure $\mathbb{P}^{(t)}_{\text{adj}}$ is the projector onto the adjoint state in the t channel, $\propto f^{a_1a_4b} f^{a_2a_3b}$.

Writing $\mathbb{P}^{(t)}_{\text{adj}}$ in terms of projectors in the s or u channel when the external particles are gauge bosons does not yield democracy, since it is not $\propto \mathbb{1}$, but it would still imply the existence of more representations than are present in the actual superstring spectrum. At tree level, the open type I superstring contains only adjoints of the $SO(32)$ gauge group, with each end of the string carrying only a single color index [127, 128]. This apparent paradox is reconciled by the fact that, for the superstring amplitude, the assumption $\lim_{s\to\infty} A^{(+)} = 0$ for $t < 0$—which would have implied Eq. (25)—is violated.

It is interesting to understand how this violation arises in this example and compare it to the heterotic string. The explicit four-point open superstring amplitude is

$$A = K\Bigg[\mathrm{tr}(T_1T_2T_3T_4)\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} + \begin{pmatrix} 1234\to1423 \\ s,t,u\to t,u,s \end{pmatrix} + \begin{pmatrix} 1234\to1342 \\ s,t,u\to u,s,t \end{pmatrix}\Bigg], \qquad (26)$$

with $K \sim p^4$ encoding external polarizations. Taking all polarizations transverse to the momenta, the Regge limit of Eq. (26) is a helicity factor $\propto (\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)$ times

$$\Gamma(-t)\left[(-s)^{1+t}(\mathbb{P}^{(t)}_{\text{adj}} + \mathbb{X}) - (-u)^{1+t}(\mathbb{P}^{(t)}_{\text{adj}} - \mathbb{X})\right], \qquad (27)$$

where $\mathbb{P}^{(t)}_{\text{adj}} = [\mathrm{tr}(T_1T_2T_3T_4) - \mathrm{tr}(T_1T_4T_2T_3)]/2$ and $\mathbb{X} = [\mathrm{tr}(T_1T_2T_3T_4) + \mathrm{tr}(T_1T_4T_2T_3)]/2$. The Regge limit of $A^{(+)}$ is therefore $A^{(+)} \sim s^{1+t}\mathbb{X}$, scaling in the same way as $A^{(-)} \sim s^{1+t}\mathbb{P}^{(t)}_{\text{adj}}$. This equivalence is compatible with the t pole being proportional to only $\mathbb{P}^{(t)}_{\text{adj}}$, and not to $\mathbb{X}$, because Eq. (27) reduces to $(s-u)\mathbb{P}^{(t)}_{\text{adj}}/t$ as $t \to 0^-$, thereby reproducing the gauge boson t-channel pole.

On the other hand, for closed strings the amplitude scales as $(su)^{1+t}$ in the Regge limit, with s^2/t scaling in the forward limit signaling massless spin-two exchange. The leading color structure is fixed to $\mathbb{1}$ by crossing symmetry, which forbids any antisymmetric color tensor that could cancel in the small-t limit, and by the fact that the massless spin-two particle is a graviton, i.e., a singlet.

It would be interesting to attempt to construct an amplitude with a graviton that violates completeness or democracy. One possible approach is to take the ansatz from Ref. [70], comprising a sum of stringy satellite terms with arbitrary coefficients, and dress it with color factors. However, it is easy to see that such an amplitude cannot be constructed from this ansatz. In the Regge limit, each satellite becomes $(-s)^{t+a}(-u)^{t+b}$ for some constants a and b. One could dress the satellite terms so that the leading Regge behavior of the full amplitude is $(-s)^{t+a}(-u)^{t+b}(\mathbb{1}+\mathbb{X}) + (-s)^{t+b}(-u)^{t+a}(\mathbb{1}-\mathbb{X})$, which goes like $s^{2t+a+b}[(-1)^a(\mathbb{1}+\mathbb{X}) + (-1)^b(\mathbb{1}-\mathbb{X})]$. This expression is therefore proportional to either $\mathbb{1}$ or $\mathbb{X}$, depending on the relative parity of a and b. However, since this is the leading term, it must reproduce the graviton pole and therefore must be proportional to $\mathbb{1}$, leaving representation completeness and democracy unmodified.