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Representation theory, introduction to finite and Lie cases

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August 2018

Thanks

Many thanks to Wolfgang Pitsch for supervising and guiding me while developing this document and to my friends and family for their moral support.

Contents

1 Introduction.

The goal of this document is to introduce the reader to two aspects of representation theory, namely complex finite dimensional representations of both finite and Lie groups.

The document's contents are organized as follows:

In section 2 we give a definition of complex finite dimensional representations and irreducible representations of a finite group and explore some immediate consequences. Mainly we will prove that any complex finite dimensional representation of a finite group can be uniquely decomposed (in a way that we will define) in irreducible representations.

In section 3 we explain useful techniques for combining representations of a group in order to obtain new representations of that same group.

In section 4 we introduce the reader to character theory and show how to use it to describe the representations of a group in a more manageable way.

In section 5 we provide multiple examples on how to use character theory to describe all irreducible representations of some specific finite groups or sets of finite groups.

In section 6 we use techniques other than the usage of character theory provide a complete description of all irreducible representations of the symmetric group $\mathfrak{S}_d$ for every integer $d \geq 2$.

In section 7 we prove an interesting result valid for finite dimensional complex representations of finite groups. This result, extracted from problem 2.37 of [5], allows us to construct families of representations within which we know all irreducible representations must appear. In other words this result tells us where we should look for the irreducible representations of any given finite group. While the proof of the result is restricted to finite groups the result is often used as a starting point to find irreducible representations of Lie groups.

Section 8 introduces the reader to representation theory in the case of Lie groups and tools for the description of irreducible representations of a specific set of Lie groups are provided.

Finally section 9 shows an example on how to use the introduced tools to describe all representations of the Lie group $SL_2(\mathbb{C})$.

Complementary results can be found in the appendices.

2 Introduction to complex finite dimensional representations of finite groups.

The origins of representation theory of finite groups can be found in a series of letters written on 1896 between the mathematicians F. G. Frobenius and R. Dedekind [8]. In these letters Frobenius explained his ideas for factoring a certain homogeneous polynomial associated with a finite group. More than one hundred years later these ideas have found a big growth, thus giving birth to representation theory not only of finite groups but also of Lie groups and Lie algebras. These last cases have great importance also i fields other than mathematics such as quantum mechanics [6].

In this section and until section 6 we are going to cover the easiest, case of representation theory representation, finite dimensional complex representations of finite groups.

Definition 1. Given G a group and $V \neq \{0\}$ a finite dimensional complex vector space. We call *finite dimensional complex representation* of G (or simply representation of G) any group homomorphism from G to the group of linear automorphisms of V $(\mathrm{GL}(V))$.

This definition can be extended to include representations of the group G on more general K-vector spaces for any field K. However we will focus on the case where V is finite dimensional and $K = \mathbb{C}$.

Notice how representations give us a relation between objects in abstract algebra, such as any finite group, and the much simpler and know linear functions of linear algebra. Here lies the interest in the study of representations.

Since the representation (φ) induces an action of the group G over the vector space V defined by $g \cdot v = \varphi(g)(v)$ for every $g \in G$ and every $v \in V$. We will often use this fact to omit the homomorphism φ when there is no ambiguity. In this case, in order to simplify notation, we will also often refer directly to the vector space V as the representation of G.

Some examples of representations are the trivial representation, that sends every element of a group G to the identity on $\mathrm{GL}(\mathbb{C})$, and the regular representation.

The regular representation (R_G) of a finite group G, with $|G| = n$, is a representation of G on $R_G = \mathbb{C}^n$ defined as follows. First we take $B = \{e_g\}_{g\in G}$ a basis of R_G. Then we define the action of G over the elements of B as $g \cdot e_h = e_{g\cdot h}$. Finally we extend this action by linearity to every element of R_G

$$g \cdot \left(\sum_{h\in G} \lambda_h e_h \right) = \sum_{h\in G} \lambda_h g \cdot e_{g\cdot h} \qquad \text{for every } \lambda_h \in \mathbb{C}, \text{ and } g \in G.$$

Take for example $G = \mathbb{Z}/3\mathbb{Z}$. Following the previously defined steps we have $R_G = \mathbb{C}^3$ with basis

$$B = \{(1, 0, 0) = e_{\overline{0}},\ (0, 1, 0) = e_{\overline{1}},\ (0, 0, 1) = e_{\overline{2}}\},$$

and the action of G over every element of R_G is described by

$$\begin{aligned}
\overline{0} \cdot (a, b, c) &= a\left(\overline{0} \cdot e_{\overline{0}}\right) + b\left(\overline{0} \cdot e_{\overline{1}}\right) + c\left(\overline{0} \cdot e_{\overline{2}}\right) = ae_{\overline{0}} + be_{\overline{1}} + ce_{\overline{2}} = (a, b, c),\\
\overline{1} \cdot (a, b, c) &= ae_{\overline{1}} + be_{\overline{2}} + ce_{\overline{0}} = (c, a, b),\\
\overline{2} \cdot (a, b, c) &= ae_{\overline{2}} + be_{\overline{0}} + ce_{\overline{1}} = (b, c, a).
\end{aligned}$$

In other words the regular representation $\varphi : G \to \mathrm{GL}(R_G)$ is defined by

$$\varphi(\overline{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varphi(\overline{1}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \varphi(\overline{2}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

There are multiple ways of obtaining new representation from known ones, the simplest of them is by taking as new representation the direct sum of previously known representations. A lot of them will be explained in section 3, however, for now, we only need one of them. Namely we need to explain how it is possible to define a representation over a direct sum of representations.

Definition 2. Given a finite group G and two of its representations $\varphi : G \to V$ and $\phi : G \to W$ we can define the representation $\psi : G \to V \oplus W$ as

$$\psi(g)(v, w) = (\varphi(g)v,\ \phi(g)w).$$

This method for constructing new representations immediately raises the question if the inverse process is possible. That is, if given V a representation of a finite group G, do always exist W and W' representations of G such that $W \oplus W' = V$ and $W, W' \neq V$? The answer to this question is affirmative most of the times but in order to be more precise we first need a few more definitions.

Definition 3. Given G a finite group, V a complex vector space and $\varphi : G \to \mathrm{GL}(V)$ a representation of G on V, we say that a sub-vector space $W \subset V$ is a *sub-representation* of V if W remains invariant under the action of G (i.e. $\varphi(G) \cdot W \subseteq W$).

For example, taking $G = \mathbb{Z}/3\mathbb{Z}$ and its regular representation R_G as before, we have that $W = \langle(1, 1, 1)\rangle$ is invariant under the action of G and, therefore, is a sub-representation of R_G. Other examples of sub-representations are given by the trivial ones that is the sub-spaces $\{0\}, V \subseteq V$.

From the definition of sub-representation immediately arises the definition of irreducible representation.

Definition 4. Given G a finite group, $V \neq \{0\}$ a finite dimensional complex vector space and $\varphi : G \to \mathrm{GL}(V)$ a representation of G on V, we say that V is an irreducible representation of G if V has no non-trivial sub-representations.

For example any 1-dimensional representation is irreducible.

We can now give rigorously define a process "inverse" to the one given by definition 2.

Lemma 1. *(Finite reducibility)* Given V a representation of a finite group G and $W \subseteq V$ a sub-representation of V there is a sub-representation W' of V such that $W \oplus W' = V$.

Proof. Since the vector space V is finite dimensional then it is isomorphic to $\mathbb{C}^n$ for some positive integer n and, therefore, it is a Hilbert space. We can thus take H_0 an Hermitian product on V and define the Hermitian product H by averaging H_0 over G

$$H(v, w) = \sum_{g \in G} H_0 \left(gv,\, gw\right).$$

Notice that, thus defined, H is G invariant since for every $v, w \in V$ and every $g \in G$ it satisfies

$$H(g \cdot v, g \cdot w) = \sum_{h \in G} H_0 \left(hg \cdot v,\, hg \cdot w\right) = \sum_{h \in G} H_0 \left(h \cdot v,\, h \cdot w\right) = H(v, w).$$

Finally, take $W' = W^T$ where W^T is the subspace of V perpendicular to W respect to the Hermitian product H.

W^T is G invariant since for every $w \in W^T$, every $v \in W$ and every $g \in G$ we have that

$$H(v,\, g \cdot w) = H(g^{-1}v,\, w) = 0.$$

On the other hand, since W^T is perpendicular to W then $W \oplus W^T = V$ as we wanted to prove.

As an immediate corollary of this lemma we have that. □

Corollary 1. Given a finite group G, any representation of G can be decomposed as a direct sum of irreducible representations of G.

Proof. We can prove this by induction over the dimension of a general representation V.

If $\dim(V) = 1$ then V is irreducible and we are done.

If $\dim(V) = n > 1$ then there are two possibilities. Either V is irreducible, and we are done, or there exists a non-trivial sub-representation $W \subsetneq V$. From the previous lemma this implies the existence of another non-trivial sub-representation $W' \subseteq V$ such that $W \oplus W' = V$. Since $W \neq \{0\}$ then $W' \neq V$ and we can conclude that $\dim(W)$,$\dim(W') < \dim(V) = n$. Therefore, by the induction hypothesis, W and W' can be written as direct sum of irreducible representation and, since $W \oplus W' = V$, then V can also be written as a sum of irreducible representations. □

This decomposition gives us a starting point for describing representations of finite groups. In fact, if we prove this decomposition to be "unique", we could reduce the problem of describing all possible representations of a finite group G to that of describing only its irreducible representation since, from those, we could construct all possible representations of G. This decomposition is indeed "unique", however, in order to prove it, we first need to say in which sense it is "unique". For that we need the following definition.

Definition 5. Given G a finite group and φ and ψ representations of G over finite dimensional complex vector spaces V and W respectively, we call *map between the representations* V and W to any linear map $\phi : V \to W$ such that ϕ commutes with the action of G. That is

$$\phi(\varphi(g)(v)) = \psi(g)(\phi(v))$$

for every $g \in G$ and every $v \in V$. We also say that the map $\phi : V \to W$ is G *invariant.*

We say that two representations V and W of a finite group G are isomorphic if there exists a bijective representation map between V and W. We call this bijective map isomorphism. Now we can clarify that, when we were saying that every representation decomposed "uniquely" as direct sum of irreducible representations, we meant uniquely up to isomorphism. Before proving this uniqueness we first need to prove Schur's lemma which characterizes all maps between irreducible representations.

Lemma 2. *(Schur's Lemma)* Given a finite group G, two of its irreducible representations V and W and φ a representation map from V to W then: either φ is an isomorphism or it is the 0 map. Moreover if $W = V$ then $\varphi = \lambda Id$.

Proof. First notice that, since $g \cdot \varphi(v) = \varphi(g \cdot v)$ and $g \cdot 0 = 0$ for every $g \in G$ and $v \in V$ then both the image and the kernel of φ are G invariant. Therefor the kernel and the image of φ are sub-representations of V and W respectively.

Since W and V are irreducible then either $\ker(\varphi) = V$ and $\text{Im}(\varphi) = 0$ or $\ker(\varphi) = \{0\}$ and $\text{Im}(\varphi) = W$.

In the first case φ is the 0 map.

In the second case, using the first theorem of isomorphisms, we obtain that

$$V \cong V/\{0\} \cong V/\ker(\varphi) \cong \text{Im}(\varphi) = W.$$

We have thus proven the first half of the lemma.

If $W = V$ then, since $\mathbb{C}$ is algebraically closed, there exists λ an eigenvalue of φ. In other words there exists $\lambda \in \mathbb{C}$ such that $\varphi - \lambda Id$ has a non zero kernel. From the irreducibility of V, we deduce that $\ker(\varphi - \lambda Id) = V$ and, therefore, $\varphi - \lambda Id = 0$ which is equivalent to saying that $\varphi = \lambda Id$. □

We are now ready to prove the uniques of representations decompositions.

Proposition 1. (*Representation factorization*) Every finite dimensional complex representation V of a finite group G can be uniquely decomposed (up to isomorphism) as a direct sum of irreducible representations.

Proof. Lemma 1 proves that the decomposition is always possible. Suppose now that a representation V of a finite group G could be decomposed as direct sum of irreducible representations in two different manners, namely

$$V = V_1 \oplus \cdots \oplus V_r = V_1' \oplus \cdots \oplus V_s',$$

then we can take the map isomorphism $\text{Id} : V_1 \oplus \cdots \oplus V_r \to V_1' \oplus \cdots \oplus V_s'$ and restrict it to the sub-representation $V_1 \subset V$. The restriction Id_{V_1} is a representation map between V_1 and V since it is linear and G invariant because Id is. Notice also that the projection map $\pi_i : V \to V_i'$ is also G invariant since it is a diagonal matrix and, therefore, commutes with every other matrix. We can thus conclude that $\pi_i \circ \text{Id}_{V_1}$ is a a representation map between V_1 and V_i' for any $i = 1, \ldots, s$. From Schur's lemma we know that either $\pi_i \circ \text{Id}_{V_1} = 0$ or $V_1 \cong V_i$. If $\pi_i \circ \text{Id}_{V_1} = 0$ for every $i = 1, \ldots, s$ then $\text{Id}_{V_1}(V_1) = 0$. Since $V_1 \neq 0$ by definition and Id_{V_1} is an isomorphism then this result is contradictory. We can thus conclude that exists $i \in \{1, \ldots, s\}$ such that $\pi_i \circ \text{Id}_{V_1}(V_1) = V_i'$ with $\pi_i \circ \text{Id}_{V_1}$ isomorphism. Without loss of generality we can assume that $i = 1$ and, therefore, $V_1 \cong V_1'$. Knowing this we can write

$$V_2 \oplus \cdots \oplus V_r \cong V_1 \oplus \cdots \oplus V_r / V_1 \cong V_1' \oplus \cdots \oplus V_s' / V_1' \cong V_2' \oplus \cdots \oplus V_s'.$$

Repeating the process we obtain that $s \geq r$ and that $V_i \cong V_i'$ for every $i = 1, \ldots, r$. Doing the same process but now from the second decomposition to the first we obtain that $r \geq s$ and, therefore,

$$V_i \cong V_i' \text{ for every } i = 1, \ldots, r = s,$$

thus proving the uniqueness (up to isomorphism) of the decompositions.

Now that we have proved that all representations of a finite group can be uniquely decomposed as a direct sum of irreducible representations we can reduce the problem of describing all representations of a given finite group to describing only its irreducible representations.

This result is quite important since, together with two extra lemmas, allows us to give a constructive method for describing all representations of any finite abelian group. □

Lemma 3. All irreducible representations of any finite abelian group G are one dimensional.

Proof. Take $\varphi : G \to \mathrm{GL}(V)$ any irreducible representation of G. Then, for every $g \in G$, we have that $\varphi(g)$ is a map between the representation V and itself. This can be seen by taking any $h \in G$ and $v \in V$ and noticing that

$$\varphi(g)\left(\varphi(h)(v)\right) = \varphi(gh)(v) = \varphi(hg)(v) = \varphi(h)\left(\varphi(g)(v)\right).$$

Since for every $g \in G$ we have that $\varphi(g)$ is a representation map between an irreducible representation and itself, then, according to Schur's lemma, $\varphi(g) = \lambda Id$. Therefore, every vector of V is an eigenvector for every $g \in G$ which means that every one dimensional sub-space of V is a sub-representation of V. Since V is irreducible then we can conclude that the only 1-dimensional sub-space of V must be V itself. In other words, V is 1-dimensional. □

The second lemma, which we will prove in a later section (see section 8), says that *the regular representation of a finite group G contains all the irreducible representations V_i of G with multiplicity equal to the dimension of V_i.*

We can now find all irreducible representations of A finite abelian group in two simplesteps.

- For any finite abelian group G we can now take a set of $n = |G|$ linearly independent vectors of the regular representation R_G, which are also eigenvectors under the action of every $g \in G$. Using proposition 1 and lemma 3 we can assure that such a set of vectors exists.
- Then for every one of these n eigenvectors we can take the irreducible 1-dimensional by it. According to lemma 8 these are exactly all possible irreducible representations of the abelian group G.

For example we can take, once again, the group $G = \mathbb{Z}/3\mathbb{Z}$. Looking at the, already described, regular representation R_G we see that the vectors

$$v_1 = (1,\, 1,\, 1), \qquad v_2 = (\xi,\, \xi^2,\, 1) \qquad \text{and} \qquad v_3 = (\xi^2,\, \xi^1,\, 1),$$

with $\xi = e^{i\frac{2\pi}{3}}$, are all eigenvectors under the action of every element $g \in G$. Therefore the sub-representations generated by v_1, v_2 and v_3 are all the possible irreducible representations of $\mathbb{Z}/3\mathbb{Z}$.

More in general, knowing that every abelian group G is isomorphic to

$$G \cong \mathbb{Z}/p_1^{n_{1,1}} \times \mathbb{Z}/p_1^{n_{1,2}} \times \cdots \times \mathbb{Z}/p_r^{n_{r,s_r}},$$

where p_i are distinct primes, then we can define the representations $\varphi_{(m_{1,1},\dots,m_{r,s_r})} : G \to \mathbb{C} \cong \mathrm{GL}\,(V)$ as

$$\varphi_{(m_{1,1},\dots,m_{r,s_r})}(l_{1,1},\dots,l_{r,s_r}) = \left(\xi_{p_1^{n_{1,1}}}\right)^{l_{1,1}\cdot m_{1,1}} \cdots \left(\xi_{p_r^{n_{r,s_r}}}\right)^{l_{r,s_r}\cdot m_{r,s_r}},$$

where ξ_n indicates a primitive n-root of unit. It is relatively easy to prove that this represents are well defined. Making the values $m_{i,j}$ range between 0 and $p_i^{n_{i,j}} - 1$ we obtain

$$\left|p_1^{n_{1,1}}\right| \cdots \left|p_r^{n_{r,s_r}}\right| = \left|\mathbb{Z}/p_1^{n_{1,1}}\right| \times \cdots \times \left|\mathbb{Z}/p_r^{n_{r,s_r}}\right| = |G|,$$

distinct representations. Since the number of distinct irreducible representations of any finite abelian group is equal to its cardinality[1] and since all representations found are irreducible for being 1-dimensional we can deduce that the representations $\varphi_{(m_{1,1},\dots,m_{r,s_r})}$ are exactly all the irreducible representations of G. This solves the problem of describing all representations of any finite abelian group G.

For general finite groups the problem of finding all irreducible representations is not as simple as in the abelian case. In order to be able to attack the problem for more general groups we need to develop and make use of a tool called character theory.

[1] This is an immediate consequence of lemma 8, the fact that all irreducible representations of an abelian group are 1-dimensional and the fact that the dimension of the regular representation of any group G is equal to the cardinality of G.

3 Derived representations.

Before proceeding with the study of character theory and in order to avoid unnecessary interruptions later, it is necessary to make a brief parenthesis and explain how, given a group G and two of its representations V and W, they can be used to obtain new representations of G over the vector spaces V^*, $V \otimes W$ and $\mathrm{Hom}(V, W)$. This new obtained representations (specially the tensor product representation) will be used in future sections on many proofs and results. It is therefore very important to make this parenthesis and explain how G acts on these derived representations.

3.1 Dual space V^*.

Given a finite dimensional complex vector space V with basis $B = \{e_i\}_{i \in I}$ we can define its dual vector space V^* as the set of linear functions $f : V \to \mathbb{C}$ together with the point-wise addition and scalar multiplication operations given by

$$(f+g)(v) = f(v) + g(v), \qquad (\lambda f)(v) = \lambda (f(v)),$$

where $f, g \in V^*$, $v \in V$ and $\lambda \in \mathbb{C}$.

For simplicity we will be taking $B^* = \{e_i^*\}_{i \in I}$ as basis for V^* where e_i^* is defined by $e_i^*(e_j) = \delta_{i,j}$. Also for simplicity, given $v = \sum_{i \in I} \lambda_i e_i \in V$, we will denote by v^* the element of the dual space V^* defined as $v^* = \sum_{i \in I} \overline{\lambda_i} e_i^*$. Vice versa, given $v^* = \sum_{i \in I} \overline{\lambda_i} e_i^* \in V^*$ we will denote by v the vector $v = \sum_{i \in I} \lambda_i e_i$.

With this notation we can define the inner product on V as $\langle w;\, v\rangle = w^*(v)$ and, likewise, define $\langle w^*;\, v\rangle = \langle w;\, v\rangle$ for any $v, w \in V$.

Remember now that, according to Reisz representation theorem, given $M \in \mathrm{GL}(V)$ there exists a unique $M^{*\prime} \in \mathrm{GL}(V)$ such that, for every $v, w \in V$ the identity $\langle w;\, M(v)\rangle = \langle M^{*\prime}(w);\, v\rangle$ holds. By the definitions introduced above this result has, as an immediate consequence, the fact that given $M \in \mathrm{GL}(V)$ there exists a unique $M^* \in \mathrm{GL}(V^*)$ (defined as $M^*(v^*) = (M^{*\prime}(v))^*$) such that $\langle w^*;\, M(v)\rangle = \langle M^*(w^*);\, v\rangle$ for every $w^* \in V^*$ and every $v \in V$.

Once all this notation and previous results are introduced we can explain what properties we want the representation on the dual vector space to satisfy.

Given a finite group G, a finite dimensional complex vector space V and a representation $\varphi : G \to \mathrm{GL}(V)$ we would like to define a representation $\varphi^* : G \to \mathrm{GL}(V^*)$ in a way such that the identity

$$\langle \varphi^*(g)(w^*);\, \varphi(g)(v)\rangle = \langle w^*;\, v\rangle$$

holds for every every $v \in V$, $w^* \in V^*$ and $g \in G$.

The representation φ^* that we are looking for is given by the following proposition.

Proposition 2. *Given G a finite group, V a finite complex vector space and $\varphi : G \to GL(V)$ a representation of G on V, then, a representation of G on the dual space V^* can be defined as $\varphi^*(g) = \left(\varphi\left(g^{-1}\right)\right)^*$. This definition is such that the identity*

$$\langle \varphi^*(g)(w^*);\, \varphi(g)(v)\rangle = \langle w^*;\, v\rangle$$

holds for every $v \in V$, $w^* \in V^*$ and $g \in G$.

Proof. By definition we have that, for every $v \in V$, $w^* \in V^*$ and $g \in G$

$$\left\langle \left(\varphi\left(g^{-1}\right)\right)^* w^*, v\right\rangle = \left\langle w^*, \varphi\left(g^{-1}\right) v\right\rangle.$$

Therefore the desired identity holds since

$$\begin{aligned}\langle \varphi^*(g)(w^*);\, \varphi(g)(v)\rangle &= \left\langle \left(\varphi\left(g^{-1}\right)\right)^*(w^*);\, \varphi(g)(v)\right\rangle = \left\langle w^*;\, \varphi\left(g^{-1}\right)(\varphi(g)(v))\right\rangle = \\ &= \left\langle w^*;\, \left(\varphi\left(g^{-1}\right)\varphi(g)\right)(v)\right\rangle = \left\langle w^*;\, \varphi\left(g^{-1}g\right)(v)\right\rangle = \langle w^*;\, \varphi(e)(v)\rangle = \langle w^*;\, v\rangle.\end{aligned}$$

The only thing we are left to prove now is that φ^* is a well defined representation. This is proven by the uniqueness of the dual application and the following identity

$$\begin{aligned}\langle \varphi^*(g)\varphi(h)^* w^*,\, v\rangle &= \left\langle \left(\varphi(g^{-1})\right)^* \left(\varphi(h^{-1})\right)^* w^*,\, v\right\rangle = \left\langle \left(\varphi(g^{-1})\right)^* \left(\varphi(h^{-1})\right)^* w^*,\, v\right\rangle = \\ &= \left\langle \left(\varphi(h^{-1})\right)^* w^*,\, \varphi(g^{-1})(v)\right\rangle = \left\langle w^*,\, \varphi(h^{-1})\varphi(g^{-1})(v)\right\rangle = \\ &= \left\langle w^*,\, \varphi(h^{-1}g^{-1})(v)\right\rangle = \left\langle \left(\varphi(h^{-1}g^{-1})\right)^* (w^*),\, v\right\rangle = \left\langle \left(\varphi((gh)^{-1})\right)^* (w^*),\, v\right\rangle = \\ &= \langle \varphi^*(g\cdot h)(w^*),\, v\rangle\,.\end{aligned}$$

Since the identity holds for every $v \in V$, $w^* \in V^*$ and every $g, h \in G$ then we have that $\varphi^*(g)\varphi^*(h)(w^*) = \varphi^*(g\cdot h)(w^*)$ for every w^* and, therefore, $\varphi^*(g)\varphi^*(h) = \varphi^*(g\cdot h)$ thus proving that φ^* is a well defined group homomorphism and, therefore, a representation. □

3.2 Tensor product $V \otimes W$.

Given two vector spaces V and W over a field K, their tensor product $(V \otimes W)$ is yet another vector space over K that can be defined as follows.

First define the set $S = \{v \otimes w \mid v \in V \text{ and } w \in W\}$.

Second define T as the K vector space freely generated by S.

$$T = \left\{ \sum_{i\in I} \lambda_i s_i \mid |I| < \infty,\ \lambda_i \in K \text{ and } s_i \in S \right\}.$$

Finally define $V \otimes W$ by taking the quotient of T over the equivalence relation $\sim$ given by:

- $v_1 \otimes w + v_2 \otimes w \sim (v_1 + v_2) \otimes w$.
- $v \otimes w_1 + v \otimes w_2 \sim v \otimes (w_1 + w_2)$.
- $\lambda\,(v \otimes w) \sim (\lambda v) \otimes w$.
- $\lambda\,(v \otimes w) \sim v \otimes (\lambda w)$.

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\lambda \in K$.

Given a finite group G, two finite dimensional complex vector spaces V and W and ϕ and φ representations of G over V and W respectively, we can now define the representation $\psi :\, G \to \mathrm{GL}\,(V \otimes W)$ by linear linear extension of

$$\psi(g) \cdot (v \otimes w) = (\phi(g) \cdot v) \otimes (\varphi(g) \cdot w)\,,$$

where $g \in G$, $v \in V$ and $w \in W$.

Notice how the function ψ is well defined since, for every $g \in G$, the map $\psi(g)$ is linear by definition and because $\phi(g)$ and $\varphi(g)$ are linear.

Therefore, to prove that ψ is indeed a representation, we only need to prove that ψ is a group morphism. Given the linearity of ψ, then, in order to prove that $\psi(g\cdot h) = \psi(g)\cdot\psi(h)$ for every $g, h \in G$, we only need to prove that, for every $v \in V$ and every $w \in W$, the identity

$$\psi(g) \cdot (\psi(h) \cdot (v \otimes w)) = (\psi(g) \cdot \psi(h)) \cdot (v \otimes w)\,,$$

holds. This is proven by the following equation

$$\begin{aligned}\psi(g) \cdot (\psi(h) \cdot (v \otimes w)) &= \psi(g) \cdot ((\phi(h) \cdot v) \otimes (\varphi(h) \cdot w))\,, \\ &= (\phi(g) \cdot (\phi(h) \cdot v)) \otimes (\varphi(g) \cdot (\varphi(h) \cdot w))\,, \\ &= (\phi(g\cdot h) \cdot v) \otimes (\varphi(g\cdot h) \cdot w) = \psi\,(g\cdot h) \cdot (v \otimes w)\,.\end{aligned}$$

We can therefore conclude that ψ is a well defined representation of G over the tensor product $V \otimes W$.

3.3 Homomorphism space Hom (V, W).

Given two vector spaces V and W over a field K, then the set $\mathrm{Hom}(V, W)$ of K-linear functions between V and W, can be provided with a vector space structure by point-wise addition and scalar multiplication in an analogous way as we did with V^*.

Given now a finite group G and two of its representations V and W, we can define a representation of G over $\mathrm{Hom}(V, W)$ just by proving the vector space isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$ and making use of the two previous sections.

To prove the isomorphism we claim that the function $\varphi : \mathrm{Hom}(V,\, W) \to V^* \otimes W$ defined as

$$\varphi(u) = \sum_{i=1}^{n} e_i^* \otimes u(e_i),\ \forall u \in \mathrm{Hom}(V,\, W),$$

where $\{e_i\}_{i=1,\dots,n}$ denotes a basis for V, is a vector space isomorphism. In this definition, as well as in the rest of this section, we employ the same notation used in section 3.1.

We have that φ is linear since, given $u_1, u_2 \in \mathrm{Hom}(V,\, W)$ and $\lambda \in K$ the base field of V and W, then

$$\begin{aligned}
\varphi\left(u_1 + \lambda u_2\right) &= \sum_{i=1}^{n} e_i^* \otimes \left(u_1 + \lambda u_2\right)(e_i) = \sum_{i=1}^{n} e_i^* \otimes \left(u_1(e_i) + \lambda u_2(e_i)\right), \\
&= \sum_{i=1}^{n} \left(e_i^* \otimes u_1(e_i) + \lambda e_i^* \otimes u_2(e_i)\right), \\
&= \sum_{i=1}^{n} e_i^* \otimes u_1(e_i) + \lambda \sum_{i=1}^{n} e_i^* \otimes u_2(e_i) = \varphi(u_1) + \lambda\varphi(u_2).
\end{aligned}$$

To prove that φ is bijective, and thus an isomorphism, we first need to prove that.

Lemma 4. *Given K a field and V and W two K-vector spaces, if V (equivalently V^*) is finite dimensional, then every element of $V^* \otimes W$ can be uniquely written as $\sum_{i=1}^{n} e_i^* \otimes w_i$ for some $w_i \in W$.*

Proof. For every $\sum_{i \in I,\, |I| < \infty} v_i^* \otimes w_i' \in V^* \otimes W$ we can write

$$\begin{aligned}
\sum_{i \in I,\, |I|<\infty} v_i^* \otimes w_i' &= \sum_{i \in I,\, |I|<\infty} \left(\sum_{j=i}^{n} \lambda_j e_j^*\right) \otimes w_i' = \sum_{i \in I,\, |I|<\infty} \sum_{j=1}^{n} \lambda_j \left(e_j^* \otimes w_i'\right), \\
&= \sum_{j=1}^{n} \sum_{i \in I,\, |I|<\infty} \lambda_j \left(e_j^* \otimes w_i'\right) = \sum_{j=1}^{n} e_j^* \otimes \left(\lambda_j \sum_{i \in I,\, |I|<\infty} w_i'\right).
\end{aligned}$$

If we now take $w_j = \left(\lambda_j \sum_{i \in I,\, |I|<\infty} w_i'\right)$ we have proven the existence half of the lemma.

Take now $B_W = \left\{e_j^W\right\}_{j \in \Lambda}$ as basis of W and notice that the set $B_{V^* \otimes W} = \left\{e_i^* \otimes e_j^W\right\}_{j \in \Lambda,\, i=1,\dots,n}$ is a basis of $V^* \otimes W$ since, given $s \in V^* \otimes W$, we can write

$$s = \sum_{i=1}^{n} e_i^* \otimes w_i = \sum_{i=1}^{n} e_i^* \otimes \left(\sum_{j \in \Lambda} \lambda_{i,j} e_j^W\right) = \sum_{i=1}^{n} \sum_{j \in \Lambda} \lambda_{i,j} e_i^* \otimes e_j^W.$$

Since $B_{V^* \otimes W}$ is a basis then if $s = 0$ we can deduce that that $\lambda_{i,j} = 0$ for every possible i and j. Therefore, for every $i = 1, \dots, n$, we have that $w_i = \sum_{j \in \Lambda} \lambda_{i,j} e_j^W = 0$. By linearity, this implies that

$$\sum_{i=1}^{n} e_i^* \otimes w_i = \sum_{i=1}^{n} e_i^* \otimes w_i' \Rightarrow \forall i \in \{1, \dots, n\}\ w_i = w_i',$$

which is exactly the uniqueness half of the lemma. □

Using this lemma we can prove that φ is surjective simply by noticing that, for every $s = \sum_{i=1}^{n} e_i^* \otimes w_i \in V^* \otimes W$, there exists $u \in \text{Hom}(V,\, W)$ defined by

$$u\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i w_i,$$

and satisfying $\varphi(u) = s$.

On the other hand, since $\sum_{i=1}^{n} e_i^* \otimes w_i = \sum_{i=1}^{n} e_i^* \otimes w_i'$, if and only if $w_i = w_i'$ for every $i = 1, \ldots, n$, then, given $u, v \in \text{Hom}(V,\, W)$ we have that $\varphi(u) = \varphi(v)$ if and only if $u(e_i) = v(e_i)$ for ever $i = 1, \ldots, n$ and, therefore, since $\{e_i\}_{i=1,\ldots,n}$ forms a basis of V, if and only if $u = v$. This proves that φ is also injective and, therefore, is a vector space isomorphism between $\text{Hom}(V, W)$ and $V^* \otimes W$.

As we have already mentioned the representation isomorphism φ together with the previous two section gives us a representation ψ of G onto $\text{Hom}(V,\, W)$. Now that we have defined the isomorphism we can also explicit the representation ψ as

$$\psi(g) = \varphi^{-1} \circ g \circ \varphi,$$

where the g on right hand side of the equation represents the known action of G on $V^* \otimes W$. The function ψ thus defined is in fact a representation since, for every $g, h \in G$

$$\psi(g)\psi(h) = \varphi^{-1} \circ g \circ \varphi \circ \varphi^{-1} \circ h \circ \varphi = \varphi^{-1} \circ gh \circ \varphi = \psi(gh).$$

Moreover this definition shows that the map φ is in fact a representation isomorphism and, therefore, $\text{Hom}(V, W) \cong V^* \otimes W$ as representations.

Finally we can rewrite the representation ψ in a more manageable way as

$$\psi(g)(f) = \beta(g) \circ f \circ \alpha(g^{-1}),$$

where $g \in G$, $f \in \text{Hom}\,(V,\, W)$ and where α and β are the representations of G over V and W respectively.

4 Character theory.

4.1 Definition and first results.

Up until now we have seen that all representations of finite groups can be uniquely decomposed (up to isomorphism) as a direct sum of irreducible representations. However we have only explored methods to find this decompositions for representations of finite abelian groups. In this section that will result very useful for studying representations over more general finite groups. More precisely, this tool will allow us, once known all irreducible representations of a finite group G, to easily find the decomposition of any representation V of G as a direct sum of such irreducible representations. This tool will also provide us with an easy method for determining if a given representation is irreducible or not and, will allow us to determine the exact number of distinct irreducible representation of a finite group G.

This tool we are referring to is called character theory.

Definition 6. Given a finite group G, a finite dimensional complex vector space V and φ a representation of G over V we define the *character* $\chi_V = \chi_\varphi$ of the representation φ as the complex valued function over G given by

$$\chi_V(g) = \text{Tr}(\varphi(g)).$$

Since the trace is invariant under any change of basis (i.e. conjugation with invertible matrices) we have that

$$\chi_V(g) = \chi_V\left(h\right)^{-1} \cdot \chi_V\left(g\right) \cdot \chi_V\left(h\right) = \chi_V\left(h^{-1}gh\right),$$

for every $g, h \in G$. This means that the complex valued function χ_V is invariant inside every conjugacy class of G (i.e. is a class function over G). We will often use this fact to refer to the character χ_V as a function over the conjugacy classes of G instead of it being over the elements of G.

An example of character is given by taking $G = \mathfrak{S}_3$ and $V = \mathbb{C}^3$ with G acting over V as

$$\varphi \cdot v = \varphi \cdot \left(\sum_{i=1}^{3} \lambda_i e_i \right) = \sum_{i=1}^{3} \lambda_i e_{\varphi(i)},$$

where $B = \{e_i\}_{i=1,2,3}$ is a basis of V, $\varphi \in G$ and φ acts over the set of indexes $\{1, 2, 3\}$ as it arises from the definition of the symmetric group $\mathfrak{S}_3$. This is the so called permutation representation of $\mathfrak{S}_3$. Taking B as basis of V we can explicit the representation element by element as

$$\mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (1,\,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (1,\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$(2,\,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (1,\,2,\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (1,\,3,\,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

Which makes very easy to compute the value of the character χ_V on every conjugacy class as

$$\chi_V(\mathrm{Id}) = 3, \qquad \chi_V((1,\,2)) = 1, \qquad \chi_V((1,\,2,\,3)) = 0.$$

Another example of character can be obtained from the already studied regular representation R of $\mathbb{Z}/3\mathbb{Z}$. Applying what we know about this representation it can be seen that:

$$\chi_R(\overline{0}) = 3 \qquad \chi_R(\overline{1}) = 0 \qquad \chi_R(\overline{2}) = 0.$$

A more general example, that will be used later in proposition 8, is given by the regular representation of any finite group. More precisely

Proposition 3. *Given G a finite group , $e \in G$ its identity and R_G its regular representation we have that $\chi_{R_G}(e) = |G|$ and $\chi_{R_G}(g) = 0$ for all $g \in G \setminus \{e\}$.*

Proof. By definition of regular representation we can take $B = \{e_g\}_{g \in G}$ as basis of R_G and define the action of G on R_G by linear extension of the action of G over B

$$g \cdot e_h = e_{g \cdot h}, \ \forall g, h \in G.$$

Notice now that taking $g, h \in G$ then, by definition, $e_g = e_h$ if and only if $g = h$, and since

$$g \cdot h = h \Rightarrow g = g \cdot (h \cdot h^{-1}) = h \cdot h^{-1} = e,$$

we can conclude that $g \cdot e_h = e_{g \cdot h} = e_h$ for every $h \in G$ if and only if $g = e$.
Therefore every element of $G \setminus \{e\}$ sends every element of the base B to a different element of the same basis and, therefore, $\chi_{R_G}(g) = 0$ by definition of trace. On the other hand e sends every element of B onto itself and therefore, also by definition, $\chi_{R_G}(g) = |B|$. Since $|B| = |G|$ by construction of the regular representation, then $\chi_{R_G}(g) = |G|$ thus completing the proof. □

Notice how in the three previously seen example the character evaluated on the identity element has always been equal to the dimension of the representation. That is in fact always true.

Proposition 4. *Given G a finite group, V a finite dimensional complex vector space and φ a representation of G on V then $\chi_V(e) = dim(V)$ where e is the identity element of G.*

Proof. Since representations are group morphisms, then we have that φ must send the identity element of G to the identity element of $\mathrm{GL}(V)$ or, in other words, $\varphi(e) = \mathrm{Id}$. Since the trace of the identity of $\mathrm{GL}(V)$ is equal to the dimension of V then we can conclude that $\chi_V(e) = \dim(V)$ for every representation V of G. □

A useful property of characters is that, knowing the characters of two representations V and W one can easily compute the characters of their direct sum, tensor product and dual vector space. This allows to easily compute characters of many representations just by knowing the characters of a few of them and without it being necessary to explicit the the representation as we did in two of the previous examples. More precisely we have that.

Lemma 5. *Given V and W representations of a finite group G. Then*

$$\chi_{V\oplus W} = \chi_V + \chi_W, \qquad \chi_{V\otimes W} = \chi_V \cdot \chi_W, \qquad \chi_{V^*} = \overline{\chi_V}.$$

Proof. $\chi_{V\oplus W} = \chi_V + \chi_W$.

Denoting by φ the representation of G on $V \oplus W$ and φ_V and φ_W the representations on V and W then, by definition, we have that, for every $g \in G$ the matrix representation of $\varphi(g), \varphi_V(g)$ and $\varphi_W(g)$ satisfy

$$\varphi(g) = \begin{pmatrix} \varphi_V(g) & 0 \\ 0 & \varphi_W(g) \end{pmatrix},$$

and, therefore,

$$\mathrm{Tr}(\varphi(g)) = \mathrm{Tr}(\varphi_V(g)) + \mathrm{Tr}(\varphi_W(g)),$$

which is equivalent to the statement we wanted to prove.

$\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

To prove this identity it is necessary to notice that, by definition, given G finite group , V a representation of G and $B_V = \{v_i\}_{i\in I}$ a basis of V, then the character $\chi_V(g)$ for any $g \in G$ is equal to the sum $\sum_{i\in I} \lambda_{i,i}$ where the coefficients $\lambda_{i,i}$ are defined by the equation

$$g \cdot v_i = \sum_{j\in I} \lambda_{i,j} v_j.$$

Take now $B_W = \{w_j\}_{j\in J}$ as basis of W and notice that B_V and B_W induce a basis $B = \{v_i \otimes w_j\}_{i\in I, j\in J}$ of $V \otimes W$. Notice now that, by definition of representation on a tensor product we have that

$$\begin{aligned} g \cdot (v_i \otimes w_j) = (g \cdot v_i) \otimes (g \cdot w_j) &= \left(\sum_{i'\in I} \lambda_{i,i'} v_{i'}\right) \otimes \left(\sum_{j'\in J} \lambda_{j,j'} w_{j'}\right), \\ &= \sum_{i'\in I} \lambda_{i,i'} \left(v_{i'} \otimes \left(\sum_{j'\in J} \lambda_{j,j'} w_{j'}\right)\right), \\ &= \sum_{i'\in I} \sum_{j'\in J} \lambda_{i,i'} \lambda_{j,j'} \left(v_{i'} \otimes w_{j'}\right), \end{aligned}$$

for every $g \in G$. For what we have mentioned earlier we can conclude that $\chi_{V\otimes W}(g)$ is equal to $\sum_{i\in I} \sum_{j\in J} \lambda_{i,i}\lambda_{j,j}$ and, likewise, $\chi_V(g) = \sum_{i\in I} \lambda_{i,i}$ and $\chi_W(g) = \sum_{j\in J} \lambda_{j,j}$. Rearranging the elements in the summand and using these last identities we finally obtain

$$\chi_{V\otimes W}(g) = \sum_{i\in I} \sum_{j\in J} \lambda_{i,i}\lambda_{j,j} = \sum_{i\in I} \lambda_{i,i} \sum_{j\in J} \lambda_{j,j} = \chi_V(g) \cdot \chi_W(g),$$

just as we wanted to prove.

$\chi_{V^*} = \overline{\chi_V}$.

By definition of representation on a dual space V^* we know that $\varphi^*(g) = \varphi(g^{-1})^T$, where φ denotes the representation of G over V and φ^* denotes the representation of G over V^*. Since the trace is invariant under transposition then, for every $g \in G$, we have that $\chi_{V^*}(g) = \chi_V(g^{-1})$. Take now a basis for V such that $\varphi(g)$ is in its Jordan form. Since the inverse of a matrix in its Jordan form if formed by the inverse of the Jordan blocks of that matrix, then we can momentarily reduce our study to Jordan blocks. Remember that, given a Jordan block

$$J = \begin{pmatrix} \lambda_i^{-1} & 1 & 0 & \ddots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i^{-1} \end{pmatrix},$$

then its inverse is given by

$$J^{-1} = \begin{pmatrix} \lambda_i^{-1} & -\lambda_i^{-2} & \dots & (-)^{n-1}\lambda_i^{-n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_i^{-2} \\ 0 & \cdots & 0 & \lambda_i^{-1} \end{pmatrix}.$$

Therefore, denoting by λ_i the diagonal entries of the matrix $\varphi(g)$, then the diagonal entries of the matrix $\varphi(g)^{-1}$ must be λ_i^{-1}.

Notice now that, since the group G is finite, then there must be some positive integer n such that $\varphi(g)^n = \mathrm{Id}$. This implies necessarily that all eigenvalues of $\varphi(g)$ (i.e. all its diagonal entries on this basis) are roots of the unit. Therefore we have that

$$\lambda_i^{-1} = \frac{\overline{\lambda_i}}{|\lambda_i|} = \overline{\lambda_i},$$

and, therefore, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ which, since $\chi_{V^*}(g) = \chi_V(g^{-1})$, is equivalent to

$$\chi_{V^*}(g) = \overline{\chi_V(g)},$$

as we wanted to prove. □

Before concluding this section notice that, given G a finite group, $\alpha : G \to V$ and $\beta : G \to E$ two of its representations and $\varphi : V \to W$ an isomorphic representation map then, by definition, for every $g \in G$

$$\varphi^{-1} \circ \alpha(g) \circ \varphi = \beta(g).$$

Since φ is invertible then V and W must have the same dimension and we can think of φ as a basis change. Since traces are invariant under basis change we can conclude that $\chi_\alpha(g) = \chi_\beta(g)$ for every $g \in G$. In other words the character is preserved under representation isomorphisms.

4.2 Inner product of characters.

Given G a finite group and V and W two of its representations, we can define an inner product between these representations as

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_v(g)} \cdot \chi_W(g).$$

In this section we show interesting properties of this product and explain in what manner it can help us in decomposing reducible representations and identifying irreducible representations.

We will start with a definition.

Definition 7. Given G a finite group and V a representation of G we define V^G as the set of all vectors of V invariant under the action of G

$$V^G = \{v \in V : gv = v \ \forall g \in G\}.$$

Notice that $G \cdot V^G = V^G \subset V$ by definition, and that

$$g \cdot (v + w) = g \cdot v + g \cdot w = (v + w),$$
$$g \cdot (\lambda v) = \lambda g \cdot v = \lambda v,$$

for every v, $w \in V^G$, every $g \in G$ and every $\lambda \in \mathbb{C}$. From these facts we can deduce that V^G is a sub-vector-space of V invariant under the action of G or, in other words, V^G is a sub-representation of V. In fact it can be easily proven (see appendix A) that V^G is none other than the direct sum of all copies of the trivial representation appearing in the decomposition of V.
We can now define the representation map φ between V and V^G as

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v \ \forall v \in V.$$

This map is linear since every $g \in G$ acts linearly over V. It is well defined since for every $h \in G$ and every $v \in V$ holds the equation

$$h \cdot \varphi(v) = \frac{h}{|G|} \sum_{g \in G} g \cdot v = \frac{1}{|G|} \sum_{g \in G} h \cdot g \cdot v = \frac{1}{|G|} \sum_{h \cdot g \in G} h \cdot g \cdot v = \frac{1}{|G|} \sum_{g \in G} g \cdot v = \varphi(v),$$

and therefore $\varphi(v) \in V^G$. Finally the map φ is also G invariant since for every $h \in G$ and every $v \in V$

$$h \cdot \varphi(v) = \varphi(v) = \frac{1}{|G|} \sum_{h \cdot g \in G} g \cdot v = \frac{1}{|G|} \sum_{g \in G} g \cdot h \cdot v = \varphi(h \cdot v).$$

We can thus conclude that φ is a well defined representation map.
Notice now that, since for every $v \in G$ the equation

$$\varphi(v) = \frac{1}{|G|} \sum_{h \cdot g \in G} g \cdot v = \frac{1}{|G|} \sum_{h \cdot g \in G} v = \frac{|G|}{|G|} v = v,$$

holds, then φ is a projection of V onto V^G . Since φ is a projection then its trace must satisfy the equation

$$\mathrm{Trace}(\varphi) = \dim(\varphi(V)) = \dim\left(V^G\right).$$

Using all we have just mentioned we can prove that

Lemma 6. *Given G a finite group, V and W two of its representations and $Hom_G(V, W)$ the vector space of* representation *maps between V and W, then*

$$Hom(V, W)^G = Hom_G(V, W),$$

where, reusing the previously defined notation, $Hom(V, W)^G$ denotes the set of functions $f \in Hom(V, W)$ such that $g \cdot f = f$.

Proof. Before proceeding with the proof let us remember that a representation map between the representations V and W is a linear map $f \in \mathrm{Hom}(V, W)$ such that, for every $g \in G$ and every $v \in V$ holds the identity

$$g \cdot (f(v)) = f(g \cdot v).$$

Let us also remember that, as seen in section 4.1, given $f \in \mathrm{Hom}(V, W)$ and $g \in G$, the action of g on f is defined by

$$g \cdot f = \beta(g) \circ f \circ \alpha(g^{-1}),$$

where β is the representation of G on W and α is the representation of G on V.

In order to avoid excessively cumbersome notation, given $v \in V$ we are going to use the convention of writing the previous identity simply as

$$(g \cdot f)(v) = g \cdot \left(f\left(g^{-1} \cdot v\right)\right),$$

where the first dot product on the right hand of the equation denotes the action of G on W while the second dot product denotes the action of G on V.

Now that we have clarified the notation notice that, if $f \in \text{Hom}(V, W)^G$ then, by definition, for every $g \in G$ and every $v \in V$, we have that

$$(g \cdot f)(v) = g \cdot \left(f\left(g^{-1} \cdot v\right)\right) = f(v).$$

Multiplying by g^{-1} on both sides of the identity we obtain that $f \in \text{Hom}(V, W)^G$ if and only if

$$f\left(g^{-1} \cdot v\right) = g^{-1} \cdot (f(v)),$$

for every $v \in V$ and every $g \in G$.

Since the identity is true for every $g \in G$ we can rewrite it as

$$f(g \cdot v) = g \cdot (f(v)),$$

then , by definition, we have that $f \in \text{Hom}(V, W)_G$. We can thus conclude that $f \in \text{Hom}(V, W)^G \Leftrightarrow f \in \text{Hom}(V, W)_G$ and, therefore, $\text{Hom}(V, W)^G = \text{Hom}(V, W)_G$. □

We are now ready to prove the first property of the dot product defined at the beginning of this section.

Proposition 5. *Given G a finite group and V and W representations of G then*

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \cdot \chi_W(g) = dim\left(Hom(V, W)_G\right)$$

.

Proof. From lemma 6 we have that

$$\dim\left(\text{Hom}(V, W)_G\right) = \dim\left(\text{Hom}(V, W)^G\right).$$

On the other hand, since for every representation V the projection φ satisfies that $\varphi(V) = V^G$ and that $\dim\left(V^G\right) = \text{Trace}(\varphi)$, then we can deduce that

$$\dim\left(\text{Hom}(V, W)_G\right) = \dim\left(\varphi\left(\text{Hom}(V, W)\right)\right) = \text{Trace}(\varphi).$$

Using the representation isomorphism between $\text{Hom}(V, W)$ and $V^* \otimes W$ we now can write the trace of φ in terms of the characters of V and W as

$$\text{Trace}(\varphi) = \text{Trace}\left(\frac{1}{|G|} \sum_{g \in G} g\right) = \frac{1}{|G|} \sum_{g \in G} \text{Trace}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{V^* \otimes W}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

Joining the two previous equations we obtain the desired result. □

This proposition, might at first seem obscure and not very useful. However it has a number of interesting corollaries.

Corollary 2. *If V and W are irreducible representations of a finite group G then*

$$\langle \chi_V, \chi_W \rangle = dim\left(Hom(V, W)_G\right) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}$$

Proof. According to Schur's lemma if $V \ncong W$ then the only representation map between V and W is the 0 map. Therefore if $V \ncong W$ then $\dim\left(\operatorname{Hom}(V,W)_G\right) = 0$ since $\operatorname{Hom}(V,W)_G = \{0\}$.

On the other hand, if $V \cong W$ then, also by Schur's lemma, all representation maps between V and W are scalar multiples and, therefore $\dim\left(\operatorname{Hom}(V,W)_G\right) = 1$. The identity between $\dim\left(\operatorname{Hom}(V,W)_G\right)$ and $\langle \chi_V, \chi_W \rangle$ is given by the previous lemma and definition of the inner product $\langle \cdot, \cdot \rangle$. □

Using this corollary we can immediately see, once known its character, if a given representation V isn't an irreducible representation. If the product $\langle \chi_V, \chi_V \rangle$ is not equal to 1 then the representation cannot be irreducible. But we can say even more.

Corollary 3. *If V and W are representation of a finite group G with V irreducible representation then $\langle \chi_V, \chi_W \rangle$ is the multiplicity of V on W. Moreover the value $\langle \chi_W, \chi_W \rangle$ is equal to the sum of squares of the multiplicities with which each possible irreducible representation appears in the decomposition of W.*

Proof. Let's start by writing $W \cong V^{\oplus n} \oplus W_1 \oplus \cdots \oplus W_m$ where W_i are irreducible representations non isomorphic to V. Since, as proven in proposition 5, the character of a direct sum is the sum of characters we can write

$$\begin{aligned}
\langle \chi_V, \chi_W \rangle &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\left(n\chi_V(g) + \sum_{i=1}^{m}\chi_{W_i}(g)\right) = \\
&= n\left(\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_V(g)\right) + \sum_{i=1}^{m}\left(\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_{W_i}(g)\right) = \\
&= n\cdot 1 + \sum_{i=1}^{m} 0 = n,
\end{aligned}$$

which proves the first half of the corollary. For the second half we just have to write $W \cong W_1^{\oplus n_1} \oplus \cdots \oplus W_r^{\oplus n_r}$ with W_i non isomorphic irreducible representations. Then we have that

$$\langle \chi_W, \chi_W \rangle = \sum_{i=1}^{r} n_i \langle \chi_{W_i}, \chi_W \rangle = \sum_{i=1}^{r} n_i^2,$$

thus proving the second half of the corollary and concluding the proof. □

Using this corollary and the previous observations we can deduce that, given a representation V, then $\langle \chi_V, \chi_V \rangle = 1$ if and only if V is irreducible thus obtaining an easy method to determine whether or not a given representation is irreducible.

We can also use this corollary to easily decompose any representation as a direct sum of irreducible representations (given that we know the characters of these). What's more, given a finite group G, the character of irreducible representations of G for an orthonormal set on the vector space of complex valued functions over the conjugacy classes of G.

From this fact we can take other two conclusions.

First, since the characters of irreducible representations are orthonormal, then they are linearly independent. Therefore the character of any representation V is uniquely associated with a linear combination of characters of irreducible representations. Since, as a consequence of corollaries 2 and 3, the coefficient of the character χ_W, with W an irreducible representation, on the mentioned linear combination is exactly the number of copies of the representation W in the decomposition of V, we can conclude that V is unequivocally described (up to isomorphism) by its character.

Second, since the characters of representation are orthonormal then they are linearly independent and, therefore there must necessarily be a number of irreducible representations lower or equal to the number of conjugacy classes of G.

We can in fact be more precise about this second conclusion and say that

Proposition 6. *Given G a finite group, then the number of irreducible representations of G is equal to the number of its conjugacy classes. Equivalently (since characters are linearly independent), the characters of the irreducible representations form an orthonormal basis of $\mathbb{C}_{class}(G)$, the vector space of complex valued functions over the conjugacy classes of G.*

To prove this proposition we first need to prove a related lemma.

Lemma 7. *Set G a finite group and $\alpha : G \to \mathbb{C}$ any function on the group G. For any representation V of G define*

$$\varphi_{\alpha,V} = \sum_{g\in G} \alpha(g) \cdot g : V \to V.$$

Then $\varphi_{\alpha,V}$ is a representation map for all V if and only if α is a class function.

Proof. If α is a class function then , for every $h \in G$ we have that

$$h^{-1}\varphi_{\alpha,V}h = \sum_{g\in G} \alpha(g) \cdot h^{-1}gh = \sum_{g\in G} \alpha\left(h^{-1}gh\right) \cdot h^{-1}gh = \sum \alpha(g) \cdot g = \varphi_{\alpha,V},$$

and, therefore, $\varphi_{\alpha,V}$ is G invariant. Since $\varphi_{\alpha,V}$ is linear, because every g act linearly on V, this last result means it is indeed a representation map from V to V. Since the argument does not depend on the particular V we can conclude that it is valid for every representation V thus proving half of the lemma.

Suppose now that $\varphi_{\alpha,V}$ is a representation map for every representation V of G. Taking R_G the regular representation of G, then, for every $h \in G$ and $v_e \in R_G$, with e the identity element of G, we have that

$$\begin{aligned} h^{-1}\varphi_{\alpha,V}h(v_e) &= \sum_{g\in G} \alpha(g) \cdot h^{-1}gh(v_e) = \sum_{hgh^{-1}\in G} \alpha\left(h \cdot g \cdot h^{-1}\right) g(v_e), \\ &= \sum_{hgh^{-1}\in G} \alpha\left(h \cdot g \cdot h^{-1}\right) v_g, \end{aligned}$$

and

$$h^{-1}\varphi_{\alpha,V}h(v_e) = \varphi_{\alpha,V}(v_e) = \sum_{g\in G} \alpha(g) \cdot g(v_e) = \sum_{g\in G} \alpha(g) v_g,$$

combining both identities we obtain

$$\sum_{hgh^{-1}\in G} \alpha\left(h \cdot g \cdot h^{-1}\right) v_g = \sum_{g\in G} \alpha(g) v_g.$$

Since the set $\{v_g\}_{g\in G}$ forms a basis for R_G then we can conclude from the previous equation that $\alpha\left(h \cdot g \cdot h^{-1}\right) = \alpha(g)$ for every $g, hs \in G$ thus proving that α is a class function. □

We can now continue with the prove of proposition 6.

Proof. (of proposition 6). If we prove that the characters of the irreducible representations of a given finite group G form a basis of $\mathbb{C}_{\text{class}}(G)$ then we will have proven that there are indeed as many characters as conjugacy classes.

Take $\alpha \in \mathbb{C}_{\text{class}}(G)$ such that α is orthogonal to the character χ_V of every irreducible representation V. Using the same notation of the previous lemma we know that $\varphi_{\alpha,V}$ is a representation map between the irreducible representation V and itself. By Schur's lemma this means that $\varphi_{\alpha,V} = \lambda Id$. On the other hand, if we compute the trace of $\varphi_{\alpha,V}$ we obtain

$$\text{trace}(\varphi_{\alpha,V}) = \sum_{g\in G} \alpha(g)\chi_V(g) = \sum_{g\in G} \overline{\chi_{V^*}(g)}\alpha(g) = |G| \langle \chi_{V^*}, \alpha \rangle,$$

were V^* is the representation on the dual space of V. Since α is orthogonal to the character of every irreducible representation and since since V^* is irreducible for being V irreducible[2] we can conclude that

$$\text{trace}(\varphi_{\alpha,V}) = |G| \langle \chi_{V^*}, \alpha \rangle = 0.$$

Therefore, $\varphi_{\alpha,V} = \lambda Id = 0$ for any irreducible representation V. Since the result is true for any irreducible representation then it is true for any representation. In particular, if we take R_g the regular representation and use the same notation used in the proof of lemma 7 we obtain

$$\varphi_{\alpha,R}(v_e) = \sum_{g\in G} \alpha(g) v_g = 0,$$

thus proving that $\alpha = 0$. Since we have just proven the only element of $\mathbb{C}_{\text{class}}(G)$ orthonormal to the character of every irreducible representation is the 0 function, then we can deduce that the vector space generated by the characters of all irreducible representations must span all of $\mathbb{C}_{\text{class}}(G)$. Since these characters are orthonormal and, therefore, linearly independent, we can deduce that they form an orthonormal basis of $\mathbb{C}_{\text{class}}(G)$. As a corollary we can deduce that there are exactly as many irreducible representations (up to isomorphism) of a finite group G as there are conjugacy classes of G. □

To sum up. Given a finite group G, a complex vector space V and a representation $\varphi : G \to V$ we have defined a character of the representation φ as the complex valued function over the conjugacy classes of G satisfying

$$\chi_\varphi(g) = \chi_V(g) = \text{Tr}(\varphi(g)),$$

for every $g \in G$. We have defined the product between the characters of two representations V and W as

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{\chi_V(g)} \cdot \chi_W(g),$$

and we have proven that the characters of irreducible representations form, respect to this product, an orthonormal basis of the vector space $\mathbb{C}_{\text{class}}(G)$ of complex valued functions over the conjugacy classes of G. Two are the main consequences of this last result. First, there are exactly as many irreducible representations as conjugacy classes. Second, given two representations V and W with V irreducible then the value $\langle \chi_V, \chi_W \rangle$ is equal to the number of copies of V appearing in the decomposition of W. Among the other results obtained another result is worth mentioning. That is the identity $\langle \chi_V, \chi_V \rangle = \sum_{i=1}^n n_i^2$, where n_i is the number of copies of the i-th irreducible representations in the decomposition of V.

5 Examples.

This section is dedicated to show multiple examples showing how the previous results can be used to describe, via their character, all irreducible representations of various groups or sets of groups. The main goal of this section is that of consolidating the concepts introduced so far by seeing them in practice. However, some few interesting results such as the already mentioned lemma 8 are shown. The reader that feels confident in her/his understanding of the results shown in the last section can feel free to skim trough this section only paying attention to lemma 8 and corollary 4.

5.1 Symmetric group of three elements $\mathfrak{S}_3$.

Let's start with the, already introduced, symmetric group of three elements $\mathfrak{S}_3$.

First we need to find all conjugacy classes of $\mathfrak{S}_3$ and describe the format we will use to show the results.

As shown in appendix B, the number of conjugacy classes of $\mathfrak{S}_3$ is equal to the number of partitions of 3. On the other hand the integer 3 only has 3 partitions, namely $(1, 1, 1)$, $(2, 1)$ and (3). We can therefore deduce that $\mathfrak{S}_3$ only has the 3 conjugacy classes shown in the table below.

[2]This is proven simply by noticing that, from what we have proven in proposition 5 then $\langle \chi_{V^*}, \chi_{V^*} \rangle = 1 \Leftrightarrow \langle \chi_V, \chi_V \rangle = 1$.

	1	3	2
$\mathfrak{S}_3$	Id	(1, 2)	(1, 2, 3)

The first row of the table shows the number of elements of each conjugacy class while the second row shows a representative of each conjugacy class.

During all this section we will be using tables with format as the one above to describe all irreducible representations of a given group. We will be referring to these tables as "character table". In order to give a complete description of a given group character, the character table should be completed with rows containing the name of the irreducible representation on the first cell and the values of that representation's character on the corresponding conjugacy class on the other cells.

The fact that the first row of the character table shows the number of elements in each conjugacy class is motivated by the more simple expression of

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{C_i} |C_i| \overline{\chi_V(C_i)} \cdot \chi_W(C_i),$$

where C_i iterates over all conjugacy classes of G. This identity is easily deducible from the already mentioned fact that characters are invariant over the same conjugacy class.

Now that we have clarified the notation to be used we can proceed with the description of all irreducible representations of $\mathfrak{S}_3$.

Let's take P the already described permutation representation of $\mathfrak{S}_3$. Computing the product of the character of P with itself we obtain $\langle \chi_P, \chi_P \rangle = 2$. Since the only way to obtain the value 2 as a sum of squared positive integers is $2 = 1^2 + 1^2$, we can deduce that the representation P contains exactly 2 distinct irreducible representations. Since the vector $(1, 1, 1) \in P$ is invariant under the action of G, we can conclude that $U = \langle (1, 1, 1) \rangle$ is one of those irreducible representations. This representation is the trivial one which has character $\chi_U = (1, 1, 1)$. In this notation of the the i-th element in the vector indicates the value of the character over the i-th conjugacy class in order of appearance in the character table.

Since the sum of characters is the character of the direct sum we can obtain the character of the other irreducible representation S in P (known as standard representation) simply by computing $\chi_S = \chi_P - \chi_U$. We can thus update the character table as

	1	3	2
$\mathfrak{S}_3$	Id	(1, 2)	(1, 2, 3)
U	1	1	1
S	2	0	-1

We could now complete the table by noticing that the regular representation of $\mathfrak{S}_3$ contains an irreducible representation non isomorphic to the ones already found and determine exactly how this representation is included in the regular representation. However we are going to prove a more general result that we used in previous proofs and will again be useful later.

Lemma 8. *The regular representation of a finite group G contains all the irreducible representations V_i of G with multiplicity equal to the dimension of V_i. In other words*

$$R_G \cong \bigoplus V_i^{\oplus dim(V_i)}$$

where V_i iterates over all irreducible representations of G.

Proof. Since representations are group morphisms we can deduce that any representation of any finite group G must send the identity element e of G to the identity automorphism. We can therefore conclude that the character of any representation V evaluated on e must be equal to the trace of the identity in $\mathrm{End}(V)$. In other words $\chi_V(e) = \dim(V)$ for any representation V.

On the other hand, as we have already proven in proposition 3, for any finite group G its regular representation is such that $\chi_{R_G}(e) = |G|$ and $\chi_{R_G}(g) = 0$ for every $g \in G \setminus \{e\}$. Thus, for any irreducible representation V of G we have

$$\begin{aligned}\langle V,\, R_G\rangle &= \frac{1}{|G|}\overline{\chi_V(e)}\chi_{R_G}(e) + \frac{1}{|G|}\sum_{g\in G\setminus\{e\}}\overline{\chi_V(g)}\chi_{R_G}(g) = \\ &= \frac{1}{|G|}\dim(V)\,|G| + \frac{1}{|G|}\sum_{g\in G\setminus\{e\}}\overline{\chi_V(g)}0 = \dim(V)\,.\end{aligned}$$

Since for irreducible representation $\langle V,\, R\rangle$ is the number of copies of V appearing in the decomposition of R_G, then, we can conclude the lemma. □

As a corollary we have that.

Corollary 4. *Given a finite group G and $\{V_i\}_{i=1}^n$ the set of all its irreducible representations, then the following formulas hold*

$$\sum_{i=1}^n dim(V_i)\chi_{V_i}(g) = 0 \qquad \textit{for every } g \in G\setminus\{e\}\,,$$

$$\sum_{i=1}^n dim(V_i)^2 = |G|\,.$$

Proof. Since $R_G \cong \bigoplus V_i^{\oplus \dim(V_i)}$, then, from the additivity of characters, we can deduce that

$$\sum_{i=1}^n \dim(V_i)\chi_{V_i}(g) = \chi_{\bigoplus V_i^{\oplus\dim(V_i)}}(g) = \chi_{R_G}(g) \qquad \text{for every } g \in G.$$

Since $\chi_{R_G}(g) = 0$ for every $g \in G\setminus\{e\}$ while $\chi_{R_G}(e) = |G|$ then we can conclude that

$$\sum_{i=1}^n \dim(V_i)^2 = \sum_{i=1}^n \dim(V_i)\chi_{V_i}(e) = \chi_{R_G}(e) = |G|\,,$$

and that

$$\sum_{i=1}^n \dim(V_i)\chi_{V_i}(g) = \chi_{R_G}(g) = 0 \qquad \text{for every } g \in G\setminus\{e\}\,,$$

thus proving the corollary. □

Using this corollary we can now easily complete the character table of $\mathfrak{S}_3$ with the alternate representation U' as.

	1	3	2
$\mathfrak{S}_3$	Id	(1, 2)	(1, 2, 3)
U	1	1	1
U'	1	-1	1
S	2	0	-1

5.2 Standard linear group of size 2 over the field of three elements $\mathbf{SL}_2(F_3)$.

Let's now apply character theory to find all irreducible representations of a more complex group such as $\mathrm{SL}_2(\mathbb{F}_3)$.

The group $\mathrm{SL}_2(\mathbb{F}_3)$ is defined as the group of invertible matrices of $M_{2\times 2}(\mathbb{F}_3)$ with determinant equal to 1. The 24 elements of $\mathrm{SL}_2(\mathbb{F}_3)$ are

$$\begin{pmatrix} 0 & 1 \\ -1 & \pm 1, 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & \pm 1, 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm 1, 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & \pm 1, 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ \pm 1 & -1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 1 \\ 1 & \mp 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & -1 \\ -1 & \mp 1 \end{pmatrix}.$$

Its conjugacy classes are

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \qquad \text{1 element}$$
$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right\}, \text{4 elements}$$
$$\left\{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right\}, \text{4 elements}$$
$$\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\right\} \qquad \text{6 elements}$$

We can now write the beginning of the character table of $\mathrm{SL}_2(\mathbb{F}_3)$ with the trivial representation included.

	1	1	4	4	4	4	6
$SL_2(\mathbb{Z}/3)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
U	1	1	1	1	1	1	1

Using corollary 4 we can easily develop an algorithm (described in appendix C) to obtain all possible dimensions of the irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$.

Applying this algorithm we obtain a single possible configuration for the irreducible representations dimensions. Namely we obtain that the set of irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ is formed by 3 1-dimensional representations, 3 2-dimensional representations and 1 3-dimensional representation.

Take now V the 9-dimensional vector space freely generated by the 9 elements of $\mathbb{F}_3^2$. This vector space has a structure of vector space obtained by linear extensions of the standard action of $\mathrm{SL}_2(\mathbb{F}_3)$ on the basis $B = \left\{v_g : g \in (\mathbb{Z}/3)^2\right\}$.

We can immediately see that $\langle v_{(0,0)} \rangle = V_0 \subset V$ is a sub-representation of V isomorphic to the, already found, trivial representation. We are therefore going to ignore this representation in our following analysis, focusing on the 8-dimensional representation $W = V_0^T \subset V$ orthogonal to V_0 with respect to some $\mathrm{Sl}_2(\mathbb{F}_3)$-invariant hermitian product. It can be easily seen that the $\mathrm{Sl}_2(\mathbb{F}_3)$-invariant hermitian product is, in fact, the standard inner product of $\mathbb{C}^9 \cong V$ and, therefore, the representation W is the sub-vector space of V generated by the set of vectors $B \backslash v_{(0,0)}$.

The character of the representation W is $\chi_W = (8, 0, 2, 0, 2, 0, 0)$. This character satisfies $\langle \chi_W, \chi_W \rangle = 4$. Since 4 can be written as a sum of squared integer only as $4 = 2^2$ or $4 = 1^2 + 1^2 + 1^2 + 1^2$ we can deduce that, either W contains two copies of the same irreducible representation or 1 copy of 4 different irreducible representations. In the first case the only irreducible representation V should be 4-dimensional since we would have $V \oplus V = W$. However, since there are no 4-dimensional representations we reach a contradiction and can thus conclude that W contains 4 distinct irreducible representations.

Since $\langle \chi_U, \chi_W \rangle = 1$ then one of these 4 irreducible representations must be the trivial one.

Since there are exactly 3 1-dimensional representations, 3 2-dimensional representations and 1 3-dimensional and the dimensions of the three remaining representations we are looking for must add up to 7, then we can deduce that the three remaining representations have dimensions 3, 2 and 2.

In order to find this representations first we need to define the matrices

$$\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and notice that this matrices generate the group $\mathrm{SL}_2(\mathbb{F}_3)$.

We can now span W with the linearly independent eigenvectors of α listed below

$$\begin{aligned}
v_1 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } 1, \\
v_2 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } 1, \\
v_3 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } -i, \\
v_4 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i\begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } -i, \\
v_5 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } -1, \\
v_6 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } -1, \\
v_7 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - i\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } i, \\
v_8 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - i\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\begin{pmatrix} -1 \\ 1 \end{pmatrix} & &\text{eigenvalue } i.
\end{aligned}$$

From these eigenvectors we find that v_1, v_2, v_5 and v_6 are also eigenvectors of β

$$\beta(v_1) = v_1, \qquad \beta(v_2) = -v_2, \qquad \beta(v_5) = -v_5, \qquad \beta(v_6) = v_6.$$

Finally, making γ act on these four vectors we obtain

$$\gamma(v_1) = v_1 \qquad \text{and} \qquad \gamma(v_2) = -v_6, \qquad \gamma(v_6) = v_5, \qquad \gamma(v_5) = -v_2.$$

We have thus proven that the sub-spaces $V_3 = \langle v_2, v_5, v_6 \rangle$, $V_1 = \langle v_1 \rangle \subset W$ are invariant under the action of the generators of $\mathrm{SL}_2(\mathbb{F}_3)$ and, therefore, under the action of $\mathrm{SL}_2(\mathbb{F}_3)$. In other words, V_3 and V_1 are sub-representations of W. The representation V_1 is a copy of the trivial representation and, therefore, we are not interested in it. However the representation V_3 has character $\chi_{V_3} = (3, 3, 0, 0, 0, 0, -1)$ and since $\langle \chi_{V_3}, \chi_{V_3} \rangle = 1$ we can deduce that V_3 is the only 3-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{F}_3)$.

Denoting by $V = V_3 \otimes V_3$ the tensor product of V_3 with itself we obtain that $\chi_V = (9, 9, 0, 0, 0, 0, 1)$. This character satisfies $\langle \chi_{V_3}, \chi_V \rangle = 2$ and $\langle \chi_U, \chi_V \rangle = 1$. Therefore we can write $V \cong V_3 \oplus V_3 \oplus U \oplus V'$ with V' some 2-dimensional representation with character $\chi_{V'} = (2, 2, -1, -1, -1, -1, 2)$. Since $\langle \chi_{V'}, \chi_{V'} \rangle = 2$ then we can deduce that V' contains the only two 1-dimensional representations of $\mathrm{SL}_2(\mathbb{F}_3)$ (besides the trivial). To better describe these representations we have to look for elements of V which are eigenvectors of α, β and γ. More specifically, since α and β belong both to the same conjugacy class, then we must look for such eigenvectors that also have the same eigenvalue for both α and β. Since we know that a basis of V formed by eigenvectors of both α and β is given by the set $B = \{v_i \otimes v_j : \ i, j = 2, 5, 6\}$ then we are almost done. From the elements of B the only ones having the same eigenvalue for both α and β are

$$\begin{aligned}
&v_2 \otimes v_6, v_2 \otimes v_6, & &\text{with eigenvalue } -1, \\
&v_2 \otimes v_2, v_5 \otimes v_5, \ v_6 \otimes v_6, & &\text{with eigenvalue } 1.
\end{aligned}$$

If we now define $v_\xi = v_6 \otimes v_6 + \xi v_5 \otimes v_5 + \xi^2 v_2 \otimes v_2$, with ξ a primitive cubic root of unit we have that

$$\alpha(v_\xi) = \beta(v_\xi) = v_\xi \text{ and } \gamma(v_\xi) = \xi v_\xi,$$

and a similar result is obtained by replacing ξ with ξ^2. We can thus conclude that the two 1-dimensional representations we were searching for are $V_1^\xi = \langle v_\xi \rangle$ and $V_1^{\xi^2} = \langle v_{\xi^2} \rangle$. We can now expand the character table as

	1	1	4	4	4	4	6
$SL_2(\mathbb{Z}/3)$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & -1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$
U	1	1	1	1	1	1	1
V_3	3	3	0	0	0	0	-1
V_1^ξ	1	1	ξ^2	ξ^2	ξ	ξ	1
$V_1^{\xi^2}$	1	1	ξ	ξ	ξ^2	ξ^2	1

Let's now return to the representation W. We had proven that we could write $W \cong U \oplus V_3 \oplus V_2^\xi \oplus V_2^{\xi^2}$ with V_2^ξ and $V_2^{\xi^2}$ two unknown irreducible 2-dimensional representations. Since we know the characters of W, U and V_3 we can compute the character of $V_2^\xi \oplus V_2^{\xi^2}$ obtaining $\chi_{V_2^\xi \oplus V_2^{\xi^2}} = (4, -4, 1, -1, 1, -1, 0)$. Since both V_2^ξ and $V_2^{\xi^2}$ have the same dimension we can now use corollary 4 to compute the character of the last 2-dimensional representation V_2. This character is $\chi_{V_2} = (2, -2, -1, 1, -1, 1, 0)$.

If we now compute the tensor product between V_2 and both 1-dimensional non trivial representations we obtain two distinct representations with characters

$$\chi = \left(2, -2, -\xi^2, \xi^2, -\xi, \xi, 0\right) \text{ and } \chi' = \left(2, -2, -\xi, \xi, -\xi^2, \xi^2, 0\right).$$

Since $\langle \chi', \chi' \rangle = \langle \chi, \chi \rangle = 1$ we can deduce that these are the characters of the two remaining 2-dimensional irreducible representations. Setting $\chi_{V_2^\xi} = v$ and $\chi_{V_2^{\xi^2}} = \chi'$ we can finally complete the character table as.

	1	1	4	4	4	4	6
$SL_2(\mathbb{Z}/3)$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & -1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$
U	1	1	1	1	1	1	1
V_3	3	3	0	0	0	0	-1
V_1^ξ	1	1	ξ^2	ξ^2	ξ	ξ	1
$V_1^{\xi^2}$	1	1	ξ	ξ	ξ^2	ξ^2	1
V_2^ξ	2	-2	$-\xi^2$	ξ^2	$-\xi$	ξ	0
$V_2^{\xi^2}$	2	-2	$-\xi$	ξ	$-\xi^2$	ξ^2	0
V_2	2	-2	-1	1	-1	1	0

5.3 Generalization of quaternion group H_{2m+1}.

Up until now we have only described the representations of specific groups. In this section's last example we are going to address the more general problem of describing all irreducible representations of the set $\{H_n\}_{n=1}^\infty$ with n odd.

The groups H_n of this set can be described as follows. First, for every positive integer n, define the set $S_n = \{\pm 1, \pm v_1, \ldots, \pm v_n\}$ with v_i some symbols. Then, for every non empty set $I \subseteq \{1, \ldots, n\}$, take $v_I = v_{i_1} \cdots v_{i_r}$ with $i_1 < \cdots < i_r$, $i_j \in I$ and $r = |I|$. Define now H_n as the set of symbols

$$H_n = \{\pm v_I : |I| \text{ is even}\} \cup \{\pm 1\}.$$

Finally associate to this set a binary operation obtained by extension of $\cdot :\ S_n \times S_n \to S_n$ described by the relations

$$v_i \cdot v_j = -v_j \cdot v_i, \qquad v_i \cdot v_i = -1,$$
$$1 \cdot v_i = v_i \cdot 1 = v_i, \qquad -1 \cdot v_i = v_i \cdot -1 = -v_i,$$
$$-1 \cdot 1 = 1 \cdot (-1) = -1, \qquad -1 \cdot (-1) = 1 \cdot 1 = 1,$$

where i and j are integers in the range $[1, n]$ and the elements of H_n are defined as above by a product of an even number of the symbols v_i and either the value 1 or -1. It is easily seen that the obtained binary operation is well defined, is associative, since it is associative over the set S_n, has identity element 1 and every element $v_I \in H_n$ has as inverse v_I if $|I| = 4n$ and $-v_I$ else. Thus $H_n$is indeed a well defined group for every positive integer n.

It can also be easily proven that the center of H_n is $\{\pm 1\}$ if n is odd and that the conjugacy classes (besides those formed by the elements of the center) are of the form $\{\pm v_I\}$ for some non empty $I \subset \{1, \dots, n\}$.

Notice that, by definition of the product, then, given $I, J \subset \{1, \dots, n\}$ with both $|I|$ and $|J|$ even, we have that $v_I \cdot v_J$ is equal to either $v_{(I\cup J)\setminus(I\cap J)}$ or $-v_{(I\cup J)\setminus(I\cap J)}$ were we are using the convention $v_\emptyset = 1$. This property of the product induces the definition of 2^{n-1} 1-dimensional representations given by the pairs $\left\{I, I^C\right\}$ with I any subset of $\{1, \dots, n\}$. These representations are defined by0

$$\varphi_{\{I,I^C\}}\left(v_J\right) = (-1)^{|J\cap I|}.$$

Notice how each one of these representations is irreducible for being 1-dimensional. Moreover they are all distinct. Suppose they were not, then there would exists two different sets $I, J \subset \{1, \dots, n\}$ such that $J \neq I^C$ and that, for every set $K \subset \{1, \dots, n\}$ with an even number of elements, $|K \cap I|$ and $|K \cap J|$ have both the same parity. Let's suppose now without loss of generality that $|I| \leq |J|$. Then, if $I \cap J = \emptyset$ we can take $j \in J$ and $i \notin J \cup I$ and define $K = \{i, j\}$ thus reaching contradiction. On the other hand, if $I \cap J \neq \emptyset$ then we can take $i \in I \cap J$ and $j \in J\setminus I$ and define $K = \{i, j\}$ thus reaching contradiction. In both cases we reach contradiction, therefore we must conclude that such a pair of sets does not exists and, therefore, $\varphi_{\{I,I^C\}}$ are all distinct irreducible representations of H_n. We can then start the character table by writing

	1	1	2	$\cdots$	2
H_{2m+1}	1	-1	$v_{(1,2)}$	$\cdots$	$v_{(2,\dots 2m+1)}$
$V_{\{\emptyset,(1,\dots,2m+1)\}}$	1	1	1		1
$\vdots$	$\vdots$	$\vdots$			
$V_{\{(m+2,\dots,2m+1),(1,\dots,m+1)\}}$	1	1	1		$(-1)^m$

where the blank cells should be filled with either 1 or -1 according to the representations defined above.

Since we know the number of conjugacy classes that the group H_{2m+1} has and we know that the number of conjugacy classes is equal to the number of irreducible representations then we can deduce that we are missing only one irreducible representation. Using corollary 4 we can immediately deduce that the character of the last representation (called spin representation) must be such that $\chi_S(1) = 2^m$ and $\chi_S(-1) = -2^m$. Since we know that $\langle \chi_S, \chi_S \rangle = 1$ for being irreducible then

$$2^{2m+1} = \sum_{g \in H_{2m+1}} |\chi_S(g)|^2 = 2^{2m+1} + \sum_{g \in H_{2m+1}\setminus\{\pm 1\}} |\chi_S(g)|^2,$$

and we can conclude that $\chi_S(g) = 0$ for every $g \in H_{2m+1}\setminus\{\pm 1\}$.

We can thus complete the character table for the odd case as.

	1	1	2	$\cdots$	2
H_{2m+1}	1	-1	$v_{(1,2)}$	$\cdots$	$v_{(2,\ldots 2m+1)}$
S	2^m	-2^m	0	$\cdots$	0
$V_{\{\emptyset,(1,\ldots,2m+1)\}}$	1	1	1		1
$\vdots$	$\vdots$	$\vdots$			
$V_{\{(m+2,\ldots,2m+1),(1,\ldots,m+1)\}}$	1	1	1		$(-1)^m$

6 Example, representations of $\mathfrak{S}_d$.

6.1 Group algebra.

As we where able to observe in the previous section, character theory is a very powerful tool for finding irreducible representations of finite groups. However, like any other tool, there are situations in which is better to proceed without it. That is the case for example when we are faced with the problem of describing all representations of each group of the set of all finite symmetric groups $\{\mathfrak{S}_d\}_{d=2}^{\infty}$. In this section we will describe all irreducible representations of $\mathfrak{S}_d$ for every integer $d \geq 2$, not by using character theory, but by restricting ourself to the group algebra of these groups.

Definition 8. Given a finite group G we define its group algebra $\mathbb{C}G$ as the Algebra that has as underlying vector space the regular representation of G and whose inner product can be extended by linearity from the following relations between its basis elements

$$e_g \cdot e_h = e_{gh}$$

The property of group algebras that will help us describe all irreducible representations of $\mathfrak{S}_d$ is that, as with finite groups, we can define representations of group algebras.

Definition 9. Given G a finite group, $\mathbb{C}G$ its group algebra and, V a finite dimensional complex vector space, we say that $\varphi : \mathbb{C}G \to \mathrm{End}\,(V)$ is a representation of $\mathbb{C}G$ if φ is an algebra homomorphism. As with finite group representations we will often refer directly to the vector space V as the representation of $\mathbb{C}G$.

Notice how this definition gives us a strict correlation between representations of a finite group G and representations of its group algebra $\mathbb{C}G$. In particular notice how, by restricting to the canonical basis of $\mathbb{C}G$ (which is none other than the group G), we can associate to any representation of a group algebra $\mathbb{C}G$ a representation of the group G. Moreover, since any algebra homomorphism is uniquely determined by the image of the elements at its basis, we can, given a representation $\varphi : G \to \mathrm{Aut}\,(V)$, obtain by linear extension a unique representation $\varphi^* : \mathbb{C}G \to \mathrm{End}\,(V)$. In other words we have a 1 to 1 correspondence between representations of a finite group G and representations of its group algebra.

An interesting example of this 1 to 1 correspondence is given by the representation $V = \bigoplus V_i$ of any finite G obtained by taking the direct sum of all its irreducible representations .

Since G acts on each group separately (i.e. $G \cdot V_i \subset V_i$ for every i) then we can deduce that the induced group algebra representations satisfies $\varphi : \mathbb{C}G \to \bigoplus \mathrm{End}\,(V_i) \subset \mathrm{End}\,(\bigoplus V_i)$.

From corollary 4 we know that the vector spaces $\mathbb{C}G$ and $\bigoplus \mathrm{End}\,(V_i)$ have both dimension $|G|$. Therefore if we prove that φ is injective then we will have that $\mathbb{C}G \cong \bigoplus \mathrm{End}\,(V_i)$ as algebras. To prove the injectiveness of φ take $h = \sum_{g\in G} \lambda_g e_g \in \mathbb{C}G$ such that $\varphi(h) = 0$. Since $\varphi\,(\mathbb{C}G) \subset \bigoplus \mathrm{End}\,(V_i)$ then we can obtain any irreducible representation $\varphi_i : \mathbb{C}G \to \mathrm{End}\,(V_i)$, simply by concatenating the previous φ with the projection P_i onto $\mathrm{End}\,(V_i)$. Then, since $\varphi(h) = 0$, and therefore $\varphi_i(h) = P_i\varphi(h) = 0$ for every i, we can conclude that the image of h for every irreducible representation, and therefore for every representation, is the 0 endomorphism. In particular, taking $R_G = \mathbb{C}G$ the regular representation of $\mathbb{C}G$ we have that $h \cdot e_e = 0$, where e is the identity element of G. However, by definition of the product in $\mathbb{C}G$ we have that e_e is the identity element for the product of the algebra $\mathbb{C}G$ and, therefore, $h \cdot e_e = h$, from which we can conclude that $h = 0$ thus proving that φ is injective and, therefore, an algebra isomorphism.

We are now ready to start with the description of the irreducible representations of $\mathfrak{S}_d$. We will proceed according to the following steps.

First, for any partition λ of a given positive integer d, we will define the Young tableaux and young diagram of that partition.

Then we will associate to each Young diagram an element $c_\lambda \in \mathbb{C}\mathfrak{S}_d$ called symmetrizer.

By making c_λ act on $\mathbb{C}\mathfrak{S}_d$ by right multiplication we will obtain a sub-field $V_\lambda = \mathbb{C}\mathfrak{S}_d \cdot c_\lambda$.

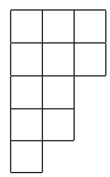
Finally we will prove that the obtained representations V_λ are exactly all irreducible representations of $\mathbb{C}\mathfrak{S}_d$.

6.2 Young Tableaux and Young Diagram.

During what is left of this section we will denote as $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum_{i=1}^{k} \lambda_i = d$ a partition of a given positive integer d.

Definition 10. Given λ a partition of an integer d we define the Young diagram associated to λ as the diagram with λ_i boxes in the i-th row. We will denote such diagram by D_λ.

For example given $(3, 3, 2, 2, 1)$ a partition of 11 its associated Young diagram would be



A Young tableaux is just a Young diagram with numbered cells. By convention we will be referring to the Young tableaux T_λ associated to a given partition λ as the Young tableaux obtained from D_λ by numbering cells from left to right and from top to bottom. For example the Young Tableaux associated to the previous partition would be

$T_{(3,3,2,2,1)} =$

1	2	3
4	5	6
7	8	
9	10	
11		

By changing the order of numbering of the Young diagram we can obtain multiple Young tableaux from a single partition. However, as we will shortly prove, the numbering order of the Young diagram is irrelevant for our purposes.

6.3 Young Symmetrizer and V_λ

Given d a positive integer and λ a partition of d we define the sub-groups P_λ, $Q_\lambda \subset \mathfrak{S}_d$ associated to any tableaux T_λ (not necessarily the standard one) as.

$$P_\lambda = e_g \in \mathbb{C}\mathfrak{S}_d : \ g \text{ preserves every row of } T_\lambda$$
$$Q_\lambda = e_g \in \mathbb{C}\mathfrak{S}_d : \ g \text{ preserves every column of } T_\lambda$$

Observe that, even thought the groups P_λ and Q_λ depend on the numbering of the specific Young tableaux T_λ, this groups will always be isomorphic.

To prove this take two different numberings of a Young diagram. Without loss of generality we can suppose this numberings to be $(1, 2, \dots, d)$ and $(\phi(1), \phi(2), \dots, \phi(d))$ for some $\phi \in \mathfrak{S}_d$. Then, denoting by P_λ and Q_λ the groups associated to the first numbering, and by P'_λ and Q'_λ the groups associated to the second numbering, we have that $\phi \cdot P_\lambda \cdot \phi^{-1} = P'_\lambda$ and $\phi \cdot Q_\lambda \cdot \phi^{-1} = Q'_\lambda$. Therefore all groups P_λ and Q_λ are isomorphic independently of the numbering order and we can, without loss of generality, apply the convention of always referring to the Young tableaux with the standard numbering from left to right and from top to bottom.

We can now use this groups to define the Young symmetrizer c_λ associated to a given partition λ as follows.

Definition 11. Given d a positive integer and λ a partition of d, we define the row-symmetrizer, column-symmetrizer and symmetrizer associated to λ (respectively denoted by $a_\lambda, b_\lambda, c_\lambda \in \mathbb{C}\mathfrak{S}_d$) as

$$a_\lambda = \sum_{\phi \in P_\lambda} \phi$$
$$b_\lambda = \sum_{\phi \in Q_\lambda} \sigma(\phi)\phi$$
$$c_\lambda = a_\lambda b_\lambda$$

Where $\sigma(\phi) \in \{\pm 1\}$ is the sign of the permutation ϕ.

Finally, given a partition λ of a positive integer d, we define the vector space $V_\lambda \subset \mathbb{C}\mathfrak{S}_d$ as $V_\lambda = \mathbb{C}\mathfrak{S}_d c_\lambda$. Notice that V_λ is in fact a sub-representation of the regular representation $R_{\mathfrak{S}_d} = \mathbb{C}\mathfrak{S}_d$ since, for every $g \in \mathfrak{S}_d$, we have that $g \cdot V_\lambda = g \cdot \mathbb{C}\mathfrak{S}_d c_\lambda \subset \mathbb{C}\mathfrak{S}_d c_\lambda = V_\lambda$.

6.4 Irreducible representations of $\mathfrak{S}_d$.

In this subsection we will prove that the defined representations V_λ are all the irreducible representations of $\mathfrak{S}_d$. We will start by proving that V_λ are indeed irreducible. Then we will prove that they are distinct and finally we will count the number of conjugacy classes to deduce (using the results of character theory) that there are no other irreducible representations of $\mathbb{C}\mathfrak{S}_d$ besides the representations V_λ.

First we need three lemmas.

Lemma 9. *Given d a positive integer and λ a partition of d we have that:*

(1) Given $p \in P_\lambda$ then $p \cdot a_\lambda = a_\lambda \cdot p = a_\lambda$

(2) Given $q \in Q_\lambda$ then $q \cdot b_\lambda = b_\lambda \cdot q = \sigma(q) b_\lambda$

(3) Given $p \in P_\lambda$ and $q \in Q_\lambda$ then $p \cdot c_\lambda \cdot q = \sigma(q) c_\lambda$.

(4) Up to multiplication by a scalar, c_λ is the only element of $\mathbb{C}\mathfrak{S}_d$ satisfying (3) for every possible $p \in P_\lambda$ and every $q \in Q_\lambda$.

Proof. (1) By definition of a_λ we have that

$$p \cdot a_\lambda = p \cdot \sum_{g \in P_\lambda} g = \sum_{g \in P_\lambda} p \cdot g = \sum_{p \cdot g \in P_\lambda} p \cdot g = a_\lambda,$$

and that

$$a_\lambda \cdot p = \left(\sum_{g \in P_\lambda} g \right) \cdot p = \sum_{g \in P_\lambda} g \cdot p = \sum_{g \cdot p \in P_\lambda} g \cdot p = a_\lambda.$$

Thus proving (1).

(2) As in the previous proof we have that, by definition of b_λ the following identities hold

$$q \cdot b_\lambda = q \sum_{g \in Q_\lambda} \sigma(g) \cdot g = \sum_{g \in Q_\lambda} \sigma(g) q \cdot g = \sigma(q) \sum_{q \cdot q \in Q_\lambda} \sigma(q \cdot g) q \cdot g = \sigma(q) b_\lambda,$$
$$b_\lambda \cdot q = \left(\sum_{g \in Q_\lambda} \sigma(g) \cdot g \right) \cdot q = \sum_{g \in Q_\lambda} \sigma(g) g \cdot q = \sigma(q) \sum_{g \cdot q \in Q_\lambda} \sigma(g \cdot q) g \cdot q = \sigma(q) b_\lambda.$$

Thus proving (2).

(3) Immediately from (1) and (2) we have that

$$pc_\lambda q = pa_\lambda b_\lambda q = (pa_\lambda)(b_\lambda q) = a_\lambda \sigma(q) b_\lambda = \sigma(q) a_\lambda b_\lambda = \sigma(q) c_\lambda,$$

thus proving (3)

(4) This point is quite trickier to prove than the previous. Let's suppose that $h = \sum_{g\in\mathfrak{S}_d} n_g e_g$ satisfies (3). Then, for every $p \in P_\lambda \subset \mathfrak{S}_d$ and every $q \in Q_\lambda \subset \mathfrak{S}_d$

$$\sum_{g\in\mathfrak{S}_d} \sigma(q) n_g e_g = p \cdot \left(\sum_{g\in\mathfrak{S}_d} n_g e_g\right) \cdot q = \sum_{g\in\mathfrak{S}_d} n_g e_{p\cdot g\cdot q} = \sum_{g\in\mathfrak{S}_d} n_{p^{-1}gq^{-1}} e_g.$$

From the linear independence of the basis elements we can deduce that the previous identity is equivalent to the identities $\sigma(q)n_g = n_{p^{-1}gq^{-1}}$ for every $g \in \mathfrak{S}_d$. Since $\sigma(q) = \sigma(q)^{-1}$ we can rewrite the previous identities as

$$n_g = \sigma(q) n_{p^{-1}gq^{-1}} \qquad \text{for every } g \in \mathfrak{S}_d. \tag{1}$$

As a particular case, taking $g = p \cdot q$ then $n_g = \sigma(q) n_{\mathrm{Id}}$. Therefore, repeating the process for every p and q, we have that

$$\begin{aligned} h &= n_{\mathrm{Id}} \sum_{p\in P_\lambda, q\in Q_\lambda} \sigma(q) e_{p\cdot q} + \sum_{g\in\mathfrak{S}_d\setminus P_\lambda\cdot Q_\lambda} n_g e_g, \\ &= n_{\mathrm{Id}} \left(\sum_{p\in P_\lambda} e_p\right)\left(\sum_{q\in Q_\lambda} \sigma(q) e_q\right) + \sum_{g\in\mathfrak{S}_d\setminus P_\lambda\cdot Q_\lambda} n_g e_g, \\ &= n_{\mathrm{Id}} a_\lambda b_\lambda + \sum_{g\in\mathfrak{S}_d\setminus P_\lambda\cdot Q_\lambda} n_g e_g = n_{\mathrm{Id}} c_\lambda + \sum_{g\in\mathfrak{S}_d\setminus P_\lambda\cdot Q_\lambda} n_g e_g. \end{aligned}$$

If we now prove that $n_g = 0$ for every $g \in \mathfrak{S}_d$ such that $g \notin P_\lambda \cdot Q_\lambda$ then we will have that $h = n_{Id} c_\lambda$. To do this we will prove that for any such g there is a transposition $t \in \mathfrak{S}_d$ such that $p = t \in P_\lambda$ and $q = g^{-1}tg \in Q_\lambda$. By proving this we will obtain that

$$e_t \cdot n_g e_g \cdot e_{g^{-1}t\cdot g} = n_g e_{tgg^{-1}tg} = n_g e_g,$$

where we have used that $t = t^{-1}$ for any transposition t. Combining this result with identity 1 we can deduce that

$$n_g = \sigma(g^{-1}tg) n_{t\cdot g\cdot g^{-1}tg} = \sigma(g)^2 \sigma(t) n_g = 1(-1) n_g = -n_g,$$

where we have used that $\sigma(g \cdot h) = \sigma(g) \cdot \sigma(h)$ and that $\sigma(g) = \sigma(g^{-1})$. Since $n_g = -n_g$ then we must have $n_g = 0$.

We have thus reduced the problem to finding such a transposition.

This transposition can, by definition, be the transposition of any two elements that are in the same row of the Young tableaux T_λ and in the same column of the Young tableaux gT_λ. Here, and from now on, gT_λ will denote the young tableaux numbered from left to right and from top to bottom with the values $(g(1), \ldots, g(d))$.

Let's suppose these two elements do not exist. Then we could take $p_1 \in P_\lambda$ and $q_1 \in gQ_\lambda g^{-1}$ such that the first row of $p_1 T_\lambda$ is equal to the first row of $q_1 g T_\lambda$. Since $P_\lambda = pP_\lambda$ for any $p \in P_\lambda$ and $Q_\lambda = qQ_\lambda$ for every $q \in Q_\lambda$ then we can repeat this process for every one of the r rows of T_λ obtaining the identity of Young tableaux

$$p_r \cdots p_1 T_\lambda = q_r \cdots q_1 \cdot gT_\lambda,$$

which is equivalent to the identity

$$p_r \cdots p_1 = q_r \cdots q_1 \cdot g.$$

Since $P_\lambda$and $Q_\lambda$are subgroups of $\mathfrak{S}_d$ they are closed under multiplication, therefore we have that $p = p_r \cdots p_1 \in P_\lambda$, that $q^{-1} = g^{-1} q_r \cdots q_1 g \in Q_\lambda$ and that

$$p = gq^{-1}g^{-1}g = gq^{-1},$$

which is equivalent to the identity $p \cdot q = g$ with $p \in P_\lambda$ and $q \in Q_\lambda$ contrary to our hypothesis.

Going back this proves that if $g \neq p \cdot q$ for any $p \in P_\lambda$and $q \in Q_\lambda$ then exists a transposition $p \in P_\lambda$ such that $q = g^{-1}pg \in Q_\lambda$ which implies that

$$n_g e_g = \sigma(g^{-1}pg) n_g e_{pgg^{-1}qg} = -n_g,$$

thus proving that $n_g = 0$ for any $g \in \mathfrak{S}_d \backslash P_\lambda Q_\lambda$. Since we have seen that any which proves that any $h = \sum_{g \in \mathfrak{S}_d} n_g e_g \in \mathbb{C}\mathfrak{S}_d$ satisfying $p \cdot c_\lambda \cdot q = \sigma(q) c_\lambda$ also satisfies that

$$h = n_{\mathrm{Id}} c_\lambda + \sum_{g \in \mathfrak{S}_d \backslash P_\lambda \cdot Q_\lambda} n_g e_g,$$

then we can conclude $h = n_{\mathrm{Id}} c_\lambda$ for some $n_{\mathrm{Id}} \in \mathbb{C}$ which is exactly what we wanted to prove. □

We are now one step closer to proving that the representations V_λ are irreducible. We only need two more lemmas much easier to prove than the previous one.

However, before proceeding to exposing the statement and proof of those lemmas we need to introduce an order relation in the set of partitions.

Definition 12. Given a positive integer d and $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ two partitions of d then we say that $\lambda > \mu$ if exists $i \in \{1, \dots, \min(r, s)\}$ such that $\lambda_i > \mu_i$ and $\lambda_j = \mu_j$ for all $j < i$.

We can now proceed with the next lemma.

Lemma 10. *Given d a positive integer and λ, μ partitions of d*

(1) if $\lambda > \mu$ then $a_\lambda \cdot g \cdot b_\mu = 0$ for every $g \in \mathbb{C}\mathfrak{S}_d$. In particular, taking $g = b_\lambda a_\mu$, then $c_\lambda c_\mu = 0$.

(2) For every $g \in \mathbb{C}\mathfrak{S}_d$ the identity $c_\lambda \cdot g \cdot c_\lambda = x c_\lambda$ holds for some $x \in \mathbb{C}$. In particular $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$

Proof. (1) For every $g \in \mathbb{C}\mathfrak{S}_d$ we can write

$$g = \sum_{\phi \in \mathbb{C}\mathfrak{S}_d} \lambda_\phi e_\phi,$$

and therefore

$$a_\lambda g b_\mu = \sum_{\phi \in \mathbb{C}\mathfrak{S}_d} \lambda_\phi a_\lambda e_\phi b_\mu$$

then, if we prove that $a_\lambda \cdot e_\phi \cdot b_\mu = 0$ for every element e_ϕ of the canonical basis of $\mathbb{C}\mathfrak{S}_d$ we will have finished.

Notice now that, by applying the permutation ϕ to the numbering of the Young tableaux T_μ we obtain a young tableaux $e_\phi T_\mu$ with associated column-symmetrizer $b_\mu^\phi = e_\phi b_\mu e_{\phi^{-1}}$. Since $e_{\phi^{-1}}$ is invertible then we have that

$$a_\lambda \cdot e_\phi \cdot b_\mu = 0 \Leftrightarrow a_\lambda \cdot b_\mu^\phi = 0.$$

Notice now that we can take two integer values such that they are in the same row of the Young tableaux T_λ and the same column of the Young tableaux $e_\phi T_\mu = T_\mu^\phi$. These values exist since, otherwise, following the same steps taken in the proof of the previous lemma, we could proof the existence of permutations $\varphi \in P_\lambda$ and $\psi \in Q_\mu^\phi$ [3] such that $e_\varphi T_\lambda = e_\psi T_\mu^\phi$, which is impossible since the partitions λ and μ are different.

Taking t the transposition of these two integers we obtain that $t \in P_\lambda$ and $t \in Q_\mu^\phi$. Then by the previous lemma we have that

[3]Here Q_μ^ϕ denotes the column permutation group of T_μ^ϕ.

$$a_\lambda b_\mu^\phi = a_\lambda \cdot (t \cdot t) \cdot b_\mu^\phi = (a_\lambda \cdot t) \cdot \left(t \cdot b_\mu^\phi\right) = a_\lambda \sigma(t) b_\mu^\phi = -a_\lambda b_\mu^\phi.$$

Since $a_\lambda b_\mu^\phi = -a_\lambda b_\mu^\phi$ then $a_\lambda b_\mu^\phi = a_\lambda \cdot e_\phi \cdot b_\mu = 0$ for every $\phi \in \mathfrak{S}_d$ and therefore $a_\lambda \cdot g \cdot b_\mu = 0$ for every $g \in \mathbb{C}\mathfrak{S}_d$.

(2) For any $p \in P_\lambda$, $q \in Q_\lambda$ and $g \in \mathbb{C}\mathfrak{S}_d$ we have that

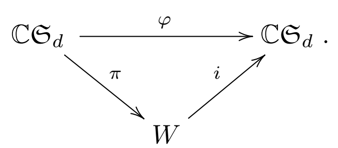
$$p\left(c_\lambda g c_\lambda\right) q = \left(p a_\lambda\right) b_\lambda g a_\lambda \left(b_\lambda q\right) = \sigma(q) c_\lambda g c_\lambda.$$

By point (4) of previous lemma this implies that exists a scalar x such that $c_\lambda g c_\lambda = x c_\lambda$ which is exactly what we wanted to prove. ☐

The last lemmas we need says that

Lemma 11. *Given d a positive integer and V a sub-representation of the representation $\mathbb{C}\mathfrak{S}_d \cong \bigoplus End(W_i)$ with W_i irreducible representations of $\mathfrak{S}_d$. Then, for any sub-representation W of V, exists $a \in W$ such that $a = a \cdot a$ and $W = V \cdot a$*

Proof. Define the $\mathbb{C}\mathfrak{S}_d$ left invariant algebra morphism between $\mathbb{C}\mathfrak{S}_d$ and itself given by the composition of the projection π on W and the natural inclusion i on A



Then we have that $\varphi \circ \varphi = \varphi$. If we now look at the set $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}(\mathbb{C}\mathfrak{S}_d, \mathbb{C}\mathfrak{S}_d)$ of left invariant algebra morphisms between $\mathbb{C}\mathfrak{S}_d$ and itself we obtain that every morphism of this set is uniquely described by the image of the product identity element in $\mathbb{C}\mathfrak{S}_d$. Since the identity can have as image any value in $\mathbb{C}\mathfrak{S}_d$. Then we have a bijection $\psi : \mathbb{C}\mathfrak{S}_d \to \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}(\mathbb{C}\mathfrak{S}_d, \mathbb{C}\mathfrak{S}_d)$ defined by

$$\psi(a) = x \to x \cdot a.$$

Since this bijection satisfies the relation $\psi(a \cdot b) = \psi(a) \circ \psi(b)$ then we can conclude that it describes a semi-group isomorphism between $\mathbb{C}\mathfrak{S}_d$ and $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}(\mathbb{C}\mathfrak{S}_d, \mathbb{C}\mathfrak{S}_d)$.

For the mentioned properties of φ then taking $a = \psi^{-1}(\varphi)$ we will have proven the lemma. ☐

We are now ready to prove that the representations V_λ are distinct irreducible representations of $\mathbb{C}\mathfrak{S}_d$.

Proposition 7. *Given d a positive integer and λ a partition of d*

(1) Each V_λ is an irreducible representation of $\mathfrak{S}_d$.

(2) If $\lambda \neq \mu$ then V_λ and V_μ are not isomorphic.

Proof. (1) By the assertion (2) of lemma 10 we know that

$$c_\lambda V_\lambda = c_\lambda \mathbb{C}\mathfrak{S}_d c_\lambda \subseteq \mathbb{C} c_\lambda.$$

Then, if $W \subseteq V_\lambda$ is a sub-representation of V_λ, we have that

$$c_\lambda W \subseteq c_\lambda V_\lambda \subseteq \mathbb{C} c_\lambda,$$

Which, since $c_\lambda W$ is a vector space, leaves us only two possibilities, either $c_\lambda W = \mathbb{C}\mathfrak{S}_d$ or $c_\lambda W = 0$.

In the case $c_\lambda W = \mathbb{C}\mathfrak{S}_d$, using the previous lemma, we have that $W = \mathbb{C}\mathfrak{S} \cdot a$ for some $a = c_\lambda b \in W$.

If $a \notin \mathbb{C} c_\lambda$ then, using assertion (2) of lemma 10, we have that

$$c_\lambda \cdot \mathrm{Id} \cdot a = c_\lambda \cdot \mathrm{Id} \cdot c_\lambda b = x \cdot c_\lambda b = x \cdot c_\lambda b = x \cdot a \notin \mathbb{C} c_\lambda,$$

for some $x \in \mathbb{C}$. However

$$c_\lambda \cdot \mathrm{Id} \cdot a \in c_\lambda \cdot \mathbb{C}\mathfrak{S} \cdot a = c_\lambda W \subseteq \mathbb{C}c_\lambda,$$

leading us to a contradiction. This proves that $a = x \cdot c_\lambda$for some $x \in \mathbb{C}$ and, therefore,

$$W = \mathbb{C}\mathfrak{S} \cdot x \cdot c_\lambda = \mathbb{C}\mathfrak{S} \cdot c_\lambda = V_\lambda.$$

On the other hand if $c_\lambda W = 0$ then

$$W \cdot W \subseteq V_\lambda \cdot W = \mathbb{C}\mathfrak{S}_d \cdot c_\lambda W = 0.$$

Since, according to the previous lemma, there exists a such that

$$W = \mathbb{C}\mathfrak{S}_d \cdot a = \mathbb{C}\mathfrak{S}_d \cdot a \cdot a,$$

then $a = a \cdot a \in W \cdot W = 0$ which implies that $a = 0$ and therefore $W = 0$.

The same argument also proves that $c_\lambda V_\lambda = \mathbb{C}c_\lambda$ since otherwise $V_\lambda = 0$ which is false since $c_\lambda \in V_\lambda$ and $c_\lambda \neq 0$.

To sum up, we have proven that if there is some sub-representations $W \subset V_\lambda$ then either $W = 0$ or $W = V_\lambda$. Since we have also proven that $V_\lambda \neq 0$ we can conclude that V_λ is an irreducible representation.

(2) Suppose without loss of generality that $\lambda > \mu$. If V_λ and V_μ where isomorphic then $c_\lambda V_\lambda \cong c_\lambda V_\mu$. However from the previous assertion we have that

$$c_\lambda V_\lambda = \mathbb{C}c_\lambda \cong \mathbb{C},$$

while from lemma 10

$$c_\lambda V_\mu = c_\lambda \mathbb{C}\mathfrak{S}_d c_\mu = \{0\}$$

thus proving that $c_\lambda V_\lambda \ncong c_\lambda V_\mu$ and, therefore $V_\lambda \ncong V_\mu$. □

Finally we are just left to prove that the found representations are indeed all possible representations. We can prove this simply by counting representations and conjugacy classes and confirming there are the same number of both.

As it is proven in appendix B, the number of partitions of any positive integer d is equal to the number of conjugacy classes of $\mathfrak{S}_d$. On the other hand, since, by definition, there are as many conjugacy classes as there are irreducible representations V_λ, then applying proposition 6 we can deduce that the representations V_λ are all the possible irreducible representations of $\mathfrak{S}_d$ (up to isomorphism).

7 About irreducible representations of general finite groups.

In the two previous sections we have observed numerous methods to describe all representations of finite groups. There is, however, no known specific method that could be used to describe all representations of any given finite group G. The result, shown in these lectures, that most resembles this ideal method is given by lemma 8. This lemma assures us that all irreducible representations of a given finite group can be found as a sub-representation of the regular representation.

In this short section we will show a result that somehow complements this lemma. More specifically we will prove that

Proposition 8. *Given V a faithful representation of a finite group G, then, any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V.*

Proof. Denote by V_i some irreducible representation of G and remember that, from lemma 5, we have that $\chi_{V^{\otimes n}} = (\chi_V)^n$. Define now $a_n = \langle \chi_{V_i}, \chi_V^n \rangle$ and consider the power series

$$f(t) = \sum_{n=0}^{\infty} a_n t^n = \frac{1}{|G|} \sum_{n=0}^{\infty} \sum_C |C| \, \overline{\chi_{V_i}(C)} \chi_V(C)^n t^n = \frac{1}{|G|} \sum_C \frac{|C| \, \overline{\chi_{V_i}(C)}}{(1 - \chi_V(C)t)}, \tag{2}$$

where C iterates over all conjugacy classes of G and the last identity comes from taking $|t| < \frac{1}{\max(|\chi(C)|)}$.

Let us show that $\max(|\chi(C)|) = \chi(e) = \dim V$ where e is the identity element of G. The second equality is consequence of the fact that every group morphism sends identity element to identity element and, therefore, the image of e by the representation is the identity on $\mathrm{GL}(V)$.

Think now of $\chi_V(C)$ as it would be in its Jordan form. Denoting by λ_i^C and v_i^C respectively the eigenvalue and an eigenvector associated with the i-th Jordan block of $\chi_V(C)$ and by n_i the dimension of that Jordan block then, for every conjugacy class C of G, we have that,

$$|\chi_V(C)| = \left|\sum_i n_i \cdot \lambda_i^C\right| \leq \sum_i n_i \cdot \left|\lambda_i^C\right|. \tag{3}$$

Since the group G is finite then, for any $g \in G$, we have that $g^{|G|} = e$ with e the identity element of G. Therefore, for every v_i^C and every $g \in C$, we have that

$$\left(\lambda_i^C\right)^{|G|} v_i^C = g^{|G|} \cdot v_i^C = e \cdot v_i^C = v_i^C,$$

and therefore λ_i^C is a $|G|$-th root of unit which implies that $\left|\lambda_i^C\right| = 1$. Replacing this in equation 3 we obtain

$$|\chi_V(C)| \leq \sum_i n_i \cdot \left|\lambda_i^C\right| = \sum_i n_i = \dim(V).$$

On the other hand, since e acts on V as the identity then

$$\chi_V(e) = \dim(V) = \max(\chi_V(C)),$$

where C iterates over all conjugacy classes. Moreover, using the same notation employed in 3 , from Schwartz inequality we have that

$$\left|\sum_i \sum_{j=1}^{n_i} 1 \cdot \lambda_i^C\right| \leq \sqrt{\sum_i \sum_{j=1}^{n_i} 1^2} \cdot \sqrt{\sum_i \sum_{j=1}^{n_i} \left|\lambda_i^C\right|^2} = \sum_i \sum_{j=1}^{n_i} 1^2 = \dim(V),$$

with the identity occurring if and only if $(1, \ldots, 1)$ and $(\lambda_i^C, \ldots, \lambda_r^C)$ are linearly dependent[4]. This implies that, for any, conjugacy class C, then $|\chi_V(C)| = \chi_V(e) = \dim(V)$ if and only if the Jordan form of every $g \in C$ contains in the diagonal only a given value λ with modulus 1. Immediately from this we can deduce that $\chi_V(C) = \chi_V(e)$ if and only if the Jordan form of every $g \in C$ contains only ones in the diagonal.

As proven in appendix D, given J a Jordan block with only ones in the diagonal, we have that

$$J^{|G|} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}^{|G|} = \begin{pmatrix} 1 & \binom{|G|}{1} & \binom{|G|}{2} & \cdots & \binom{|G|}{k-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \binom{|G|}{2} \\ \vdots & \ddots & \ddots & \ddots & \binom{|G|}{1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

where k is the size of the matrix and we use the convention $\binom{n}{m} = 0$ whenever $m > n$.

From this we can deduce that if $g \in C$ has a Jordan form with only ones in the diagonal then $g = \mathrm{Id}$ since, otherwise, $g^{|G|} = e \neq \mathrm{Id}$. Since the representation V is faithful then the only element $g \in G$ acting on V as the identity is the identity element e. Therefore we can deduce that $\chi_V(C) = \chi_V(e)$ if and only if $C = \{e\}$.

[4] Here r is the number of Jordan blocks

Going back to equation 2, this proves that,

$$\begin{aligned}\left|\lim_{t\to\frac{1}{\dim(V)}} f(t)\right| &= \left|\lim_{t\to\frac{1}{\dim(V)}} \frac{1}{|G|}\sum_{C}\frac{|C|\,\overline{\chi_{V_i}(C)}}{(1-\chi_V(C)t)}\right|,\\ &= \left|\frac{1}{|G|}\sum_{C\neq\{e\}}\frac{\dim(V)\,|C|\,\overline{\chi_{V_i}(C)}}{(\dim(V)-\chi_V(C))} + \lim_{t\to\frac{1}{\dim(V)}}\frac{1}{|G|}\frac{\overline{\chi_{V_i}(e)}}{(1-\chi_V(e)t)}\right|,\\ &= \left|\frac{1}{|G|}\sum_{C\neq\{e\}}\frac{\dim(V)\,|C|\,\overline{\chi_{V_i}(C)}}{(\dim(V)-\chi_V(C))} + \lim_{t\to\frac{1}{\dim(V)}}\frac{1}{|G|}\frac{\dim(V_i)}{(1-\dim(V)t)}\right| = \infty.\end{aligned}$$

from this we can deduce that $f(t)\neq 0$. Therefore there exists a_n in the series $f(t)=\sum_{n=0}^{\infty} a_n t^n$ such that $a_n\neq 0$. Since, by definition, $a_n=\langle\chi_{V_i},\chi_V^n\rangle=\langle\chi_{V_i},\chi_{V^{\otimes n}}\rangle$, then, by corollary 3 we can conclude that the tensor power $V^{\otimes n}$ contains $a_n>0$ copies of the irreducible representation V_i. Since these arguments are valid for any irreducible representation V_i then the proposition is proven. □

This proposition gives us ideas for where we should look for irreducible representations of a given finite group G. Notice that the given proof strongly uses the fact that G is finite and, therefore, it cannot be extended to the Lie case. However, as we will later see, in many situations, this proposition is used to look for irreducible representations of Lie groups[5] in tensor powers of some specific faithful representation.

8 Representations of Lie groups.

As we mentioned in section 2, representation theory is not limited to the study of representations over finite groups and group algebras associated to those groups. In this section we will in fact explore representation theory in the case of Lie groups.

Definition 13. Given G a differential manifold with a group structure, we say that G is a *Lie group* if the product function $\cdot: G\times G\to G$ and the inverse function $^{-1}: G\to G$ are both differentiable. Analogously we can define complex Lie group just by replacing the words "differentiable manifold" with the words "complex manifold" in the previous definition. In this document we are going to focus on complex Lie groups and, therefore, in order to simplify notation, we are going to refer to complex Lie groups simply as Lie groups.

Many examples of Lie groups can be found as sub-groups of the group of invertible n-dimensional matrices with complex coefficients $\mathrm{GL}_n(\mathbb{C})$. A particular example we will be focusing on in the next section, is given by the special linear group $\mathrm{SL}_2(\mathbb{C})$ of 2×2 invertible matrices with determinant one .

As mentioned earlier, representations of Lie groups can also be defined. Its definition is very similar to that of representations of finite groups with the only additional property that some smoothness in the representation is required. More precisely

Definition 14. Given G a Lie group and V a finite dimensional complex vector space, we say that $\varphi: G\to GL(V)$ is a finite dimensional complex representation (or simply representation) of G over V, if φ is a differentiable group homomorphism.

As in the case of representations of finite groups we will often refer to V as the representation of G and will usually avoid mentioning the homomorphism φ by using the convention $g\cdot v=\varphi(g)(v)$ for every $g\in G$ and every $v\in V$.

As in the finite case, given a Lie group G and a finite dimensional complex vector space V, we can define sub-representation W of V as a sub-vector space $W\subseteq V$ such that $G\cdot W\subseteq W$. This arises in time the definition of an irreducible representation as a representation that does not have any sub-representations besides the trivial ones.

[5]In particular we are going to see this for the Lie group $\mathrm{SL}_2(\mathbb{C})$.

The same results obtained in the finite case are however not always valid in the Lie case. Most important the complete reducibility of representations does no longer hold for representations of every Lie group. That is given a Lie group G and a representation V of G it may not exist any set of $\{W_i\}$ of irreducible representations of G such that V is isomorphic to $\bigoplus W_i$ as a representation[6]. An example where the complete reducibility is violated is given in appendix E. However it can be proven (see appendix C of [5]) that for semi-simple Lie groups [7] such as $\mathrm{SL}_2(\mathbb{C})$ the complete reducibility still holds.

8.1 Lie algebras.

During section 6.1 we introduced the concept of group algebra to help us study representations of all symmetric groups. In the case of Lie groups a similar trick is usually applied. In this case case Lie algebra take the place of group algebras.

Definition 15. A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear skew-symmetric binary function $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that, for every $X, Y, Z \in \mathfrak{g}$, the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$$

is satisfied. The skew-symmetric binary function $[\cdot,\cdot]$ is usually referred to as Lie claudator.

Whenever a dot product is defined in the Lie algebra (for example for Lie algebras injected in matrix vector spaces) the Lie claudator is simply given by the commutator

$$[X,Y] = X\cdot Y - Y\cdot X,$$

for which, it is easily proven, that the Jacobi identity holds.

Similarly to how we did for group algebras it is also possible to define finite complex representation of a Lie algebra. In the case of group algebra we extended the definition of representation of finite group by imposing that the representation should preserve multiplication besides addition. For Lie algebra we extend the definition of representation of Lie group in a similar way by imposing the preservation of of the Lie claudator. More specifically.

Definition 16. Given $\mathfrak{g}$ a Lie algebra and V a finite dimensional complex vector space, we say that $\varphi:\ \mathfrak{g}\to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$ over V if φ is a linear map such that, for every $X, Y \in \mathfrak{g}$, holds the identity

$$\varphi([X,Y]) = \varphi(X)\circ\varphi(Y) - \varphi(Y)\circ\varphi(X).$$

For Lie algebra representations we will use the same notation conventions specified for all other defined representations. The definition of irreducible Lie algebra representation is also analogous to those already seen.

There are two main reasons for which introducing Lie algebras is useful when studying representations of Lie groups.

First, for every connected Lie group G we can define an associated vector space T_eG (the vector space tangent to G at the identity element e) and provide a Lie algebra structure for this vector space. Details on how to provide this Lie algebra structure can be found in appendix F.

Second, there are two theorems (principles) that allow us to relate representations of a Lie group to representations on its associated Lie algebra. These principles are

Theorem 1. *(First principle) Let G and H be Lie groups with G connected, then every Lie group homomorphism (i.e. every differentiable group morphism) $\varphi: G\to H$ is uniquely determined by its differential at the identity element $(d\varphi)_e: T_eG\to T_eH$.*

[6] As in the finite case we say that two representations V and W of a Lie group G are isomorphic if there exists a linear isomorphism $\varphi:\ V\to W$ such that $\varphi(g\cdot v) = g\cdot\varphi(v)$ for every $g\in G$ and every $v\in V$.

[7] A semi-simple Lie group is a Lie group that does not contain non-trivial connected abelian normal groups. Or, equivalently a Lie group whose Lie algebra is semi-simple (i.e. it is a direct sum of simple Lie algebras, algebras whose only ideals are trivial).

and

Theorem 2. *(Second principle) Let G and H be Lie groups with G connected and simply connected, then a linear function $\varphi : T_eG \to T_eH$ is the differential of a differentiable group morphism if and only if φ is a Lie algebra morphism (i.e. preserves the Lie claudator).*

Due to space restrictions, we are just going to enunciate these theorems without proving them. For a proof please refer to lecture 8 of [5].

To understand how these two principles can help us relate Lie algebras representations with Lie groups representations an example is necessary.

Take H the Lie group of linear automorphisms of a finite complex vector space V and G any connected and simply connected Lie group such that a representation $\varphi : G \to \mathrm{Aut}(V) = H$ of G over V exists. Since φ is also a Lie group morphism, then, according to the first principle, since G is connected, the representation φ is uniquely determined by its differential $(d\varphi)_e$ on the identity element e.

Since G is also simply connected, then, according to the second principle, this differential $(d\varphi)_e$ is a representation of the associated Lie algebra T_eG over the vector space V. Moreover, the second principle also tells us that every representation of the Lie algebra T_eG arises like this from a representation of the Lie group G.

In other words, for every connected and simply connected Lie group G the first and second principles give us a 1 to 1 correspondence between the representations of G and the representations of its associated Lie algebra T_eG. This allows us to reduce the problem of finding representations of a connected and simply connected Lie group G to that of finding representations of its associated Lie algebra T_eG.

However we can say more than that. Take G a connected and simply connected Lie group, T_eG its associated Lie algebra, V a finite dimensional complex vector space and $(d\varphi)_e : T_eG \to \mathrm{End}(V)$ a representation of T_eG over V. Then as its explained in appendix G, we can define an exponential function $\exp : T_eG \to G$ such that the following diagram is commutative

$$\begin{array}{ccc} T_eG & \xrightarrow{(d\rho)_e} & \mathrm{End}(V)\ , \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\rho} & \mathrm{Aut}(V) \end{array}$$

where φ is a representation of G over V that arises from $(d\varphi)_e$ as $\varphi(\exp(X)) = \exp((d\varphi)_e(X))$.

For simplicity it is useful to mention that, as explained in appendix G, whenever both the Lie group and its associated Lie algebra have a natural injection in the vector space of complex matrices the exponential function can be defined as

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!},$$

definition that makes clear the origin of its name.

9 Example, representations of $\mathrm{SL}_2(\mathbb{C})$.

As we have previously mentioned the special linear group group $\mathrm{SL}_2(\mathbb{C})$ is defined as the group of 2×2 complex invertible matrices with determinant 1. This group has an obvious structure of complex manifold and, since the functions of product and inverse are differentiable with respect to this structure we can confirm that it is indeed a Lie group. During this section we are going to describe all irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ noticing how, like in the finite case (proposition 8), they can all be found as subgroups of tensor products of a faithful representations.

To describe the representations of $\mathrm{SL}_2(\mathbb{C})$ we are going to proceed as follows:

First we will prove that the conditions to apply the first and second principle and thus reduce he problem to the lie algebra of $\mathrm{SL}_2(\mathbb{C})$ are met.

Then we will describe the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ associated to $\mathrm{SL}_2(\mathbb{C})$ and state all necessary information relative to this algebra.

We will continue by deducing the conditions that all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ must satisfy depending on the dimension of the representation.

Finally we will conclude by finding, for all possible dimensions, concrete examples of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ satisfying the specified conditions.

9.1 Simplification to Lie algebras.

In order to simplify the problem of finding representations of $\mathrm{SL}_2(\mathbb{C})$ to finding representations of its associated Lie algebra we must prove that $\mathrm{SL}_2(\mathbb{C})$ is connected and simply connected. To do this we will prove that the Lie group $\mathrm{SU}_2(\mathbb{C})$ is connected and simply connected and that $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SU}_2(\mathbb{C})$ are homotopy equivalent (we will define what this means). Since homotopy equivalence preserves connectedness and simply connectedness then this two facts will suffice to prove the desired result.

$\mathrm{SU}_2(\mathbb{C})$ is defined as the set of 2×2 complex unitary matrices. That is the set of complex matrices whose columns form an orthonormal basis of $\mathbb{C}^2$. In other words

$$\mathrm{SU}_2(\mathbb{C}) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \ (\alpha,\, \beta \in \mathbb{C}) \ \text{and} \ \left(|\alpha|^2 + |\beta|^2 = 1 \right) \right\}.$$

From this definition it is easy to notice an homeomorphism between the group $\mathrm{SU}_2(\mathbb{C})$ and the complex unit circumference or, equivalently, between $\mathrm{SU}_2(\mathbb{C})$ and the hyper-sphere S^3, given by

$$f\left(\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \right) = (\mathrm{Re}(\alpha),\, \mathrm{Im}(\alpha),\, \mathrm{Re}(\beta),\, \mathrm{Im}(\beta))\,.$$

Since S^3 is connected ad simply connected and these properties are preserved by homeomorphism then we can deduce that $\mathrm{SU}_2(\mathbb{C})$ is connected and simply connected.

We now need only to prove the homotopy equivalence between $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$. That is we must find a pair of continuous functions, $f : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SU}_2(\mathbb{C})$, $g : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ such that $f \circ g$ is homotopic[8] to $\mathrm{Id}_{\mathrm{SU}_2(\mathbb{C})}$ and $g \circ f$ is homotopic to $\mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})}$. Since $\mathrm{SU}_2(\mathbb{C}) \subset \mathrm{SL}_2(\mathbb{C})$ we can chose g to be the natural inclusion. For the function f we can apply the Gram-Schmidth orthonormalization algorithm and define

$$f\left(\begin{pmatrix} a_1 & a_2 \end{pmatrix}\right) = \begin{pmatrix} \frac{a_1}{\|a_1\|} & \frac{a_2'}{\|a_2'\|} \end{pmatrix},$$

where a_1 and a_2 represent column vectors, a_2' is equal to $a_2 - a_1 \frac{\langle a_1, a_2 \rangle}{\langle a_1, a_1 \rangle}$ with $\langle \cdot, \cdot \rangle$ the standard inner product of $\mathbb{C}^2$, and $\|a\| = \sqrt{\langle a, a \rangle}$ is the standard norm of $\mathbb{C}^2$.

By definition of f and g we have that $g \circ f$ is the identity on $\mathrm{SU}_2(\mathbb{C})$. Since a function is trivially homotopic to itself, then we need only to prove that $f \circ g$ is homotopic to $\mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})}$.

Define $h : [0,1] \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ as

$$\begin{aligned} h\left(t, (a_1,\, a_2)\right) &= t \cdot (a_1,\, a_2) + (1-t) f \circ g\,(a_1,\, a_2)\,, \\ &= \left(a_1 \left(t + (1-t)\frac{1}{\|a_1\|} \right), a_2 \left(t + (1-t)\frac{1}{\|a_2'\|} \right) - a_1 \left((1-t)\frac{\langle a_1, a_2 \rangle}{\langle a_1, a_1 \rangle} \right) \right), \\ &= (u_{1,t}, u_{2,t}) \end{aligned}$$

with the same notation used previously. Notice that $u_{1,t}$ and $u_{2,t}$ are linearly independent for every $t \in [0,1]$ since a_1 and a_2 are, and they are well defined since

$$\left(t + (1-t)\frac{1}{\|a_1\|} \right), \left(t + (1-t)\frac{1}{\|a_2'\|} \right) \geq \min\left(1, \frac{1}{\|a_1\|}, \frac{1}{\|a_2'\|}\right) > 0$$

[8] Given two topological sets X and Y, two continuous functions f, g from X to Y are said to be homotopic if there exists a continuous function $h : [0,1] \times X \to Y$ such that $h(0,\cdot) = f$ and $h(1,\cdot) = g$.

for being $\frac{1}{\|a_2'\|}$ and $\frac{1}{\|a_1\|}$ positive values. Therefore the function h is well defined. h is also continuous since it is a linear combination of continuous functions. Finally we have that $h(0,\cdot) = f \circ g(\cdot)$ and that $h(1,\cdot) = \mathrm{Id}_{\mathrm{SL}_2(\mathbb{C})}$ thus completing the poof that $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ are homotopy equivalent.

9.2 Description of associated Lie algebra.

We know that the exponential function is a bijection between a Lie algebra and the Lie group to which it is associated. Therefore to find the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ we need to find all the matrices A such that det(exp (A))=1. To do this first we need a lemma.

Lemma 12. *$det(exp(A)) = e^{tr(A)}$ for every $A \in M_{n\times n}(\mathbb{C})$.*

Proof. Take A to be any matrix in $M_{n\times n}(\mathbb{C})$ and $J = P^{-1}AP$ its Jordan form. By definition we have that

$$\begin{aligned}
\exp(A) &= \sum_{n=0}^{\infty} \frac{A^n}{n!} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{A^n}{n!} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\left(PJP^{-1}\right)^n}{n!}, \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{PJ^nP^{-1}}{n!} = \lim_{N\to\infty} P \left(\sum_{n=0}^{N} \frac{J^n}{n!} \right) P^{-1}, \\
&= P \left(\lim_{N\to\infty} \left(\sum_{n=0}^{N} \frac{J^n}{n!} \right) \right) P^{-1} = P\exp(J)\, P^{-1}.
\end{aligned}$$

Since both the determinant and the trace of a matrix are invariant under conjugation, we can, from the previous formula, reduce our problem to proving the identity for Jordan matrices. Moreover, denoting by J_i the Jordan blocks of a Jordan matrix J we have that

$$\begin{aligned}
\exp(J) &= \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{\begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_d \end{pmatrix}^n}{n!} = \sum_{n=0}^{\infty} \frac{\begin{pmatrix} J_1^n & & 0 \\ & \ddots & \\ 0 & & J_d^n \end{pmatrix}}{n!}, \\
&= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{J_1^n}{n!} & & 0 \\ & \ddots & \\ 0 & & \sum_{n=0}^{\infty} \frac{J_d^n}{n!} \end{pmatrix} = \begin{pmatrix} \exp(J_1) & & 0 \\ & \ddots & \\ 0 & & \exp(J_d) \end{pmatrix}.
\end{aligned}$$

Thus, if the identity is true for Jordan blocks, then it will be true for Jordan matrices since

$$\begin{aligned}
\det(\exp(J)) &= \det\left(\begin{pmatrix} \exp(J_1) & & 0 \\ & \ddots & \\ 0 & & \exp(J_d) \end{pmatrix} \right) = \det(\exp(J_1)) \cdots \det(\exp(J_2)), \\
&= e^{\mathrm{tr}(J_1)} \cdots e^{\mathrm{tr}(J_d)} = e^{\mathrm{tr}(J_1)+\cdots+\mathrm{tr}(J_d)} = e^{\mathrm{tr}\left(\begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_d \end{pmatrix}\right)}, \\
&= e^{\mathrm{tr}(J)}.
\end{aligned}$$

Therefore we need only to prove the identity for Jordan blocks.

Let us write a Jordan block as $J = D + U$ where $D = \lambda\mathrm{Id}$ is a diagonal matrix and U is a strictly upper triangular matrix. Notice how, given A any upper triangular matrix then the product between $A \cdot U$ is also a strictly upper triangular matrix. We can therefore write

$$\exp(J) = \sum_{n=0}^{\infty} \frac{J^n}{n!} = \sum_{n=0}^{\infty} \frac{(D+U)^n}{n!} = \sum_{n=0}^{\infty} \frac{D^n + (\cdots)U}{n!},$$

$$= \sum_{n=0}^{\infty} \left(\begin{pmatrix} \frac{\lambda^n}{n!} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda^n}{n!} \end{pmatrix} + \tilde{A}_n \right) = \begin{pmatrix} e^{\lambda} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda} \end{pmatrix} + \tilde{A},$$

where both $\tilde{A}$ and $\tilde{A_n}$are strictly upper triangular matrices and, therefore, have trace 0. For the identity $(D+U)^n = D^n + (\cdots)U$ we have used that, being D diagonal, then D and U commute.
Matrix $\tilde{A}$ also satisfy the identity

$$\det\left(\begin{pmatrix} e^{\lambda} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda} \end{pmatrix} + \tilde{A} \right) = \det\left(\begin{pmatrix} e^{\lambda} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda} \end{pmatrix} \right).$$

Denoting by n the dimension of a Jordan block J we can thus conclude that

$$\det(\exp(J)) = \det\left(\begin{pmatrix} e^{\lambda} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda} \end{pmatrix} + \tilde{A} \right) = \det\left(\begin{pmatrix} e^{\lambda} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda} \end{pmatrix} \right),$$

$$= \left(e^{\lambda}\right)^n = e^{n\lambda} = e^{\operatorname{tr}\begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}} = e^{\operatorname{tr}(J)},$$

thus proving the lemma. □

From this lemma we can conclude that the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ is formed by the set of traceless complex matrices of dimension 2. In other words we have

$$\mathfrak{sl}_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) : \operatorname{tr}(A) = 0\}.$$

As basis elements of $\mathfrak{sl}_2(\mathbb{C})$ we will take the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which satisfy the convenient relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Unless otherwise specified, in what is left of this section, the symbols H, X and Y will be referring to this matrices.

9.3 Possible representations.

As we have already mentioned the representations of $\mathrm{SL}_2(\mathbb{C})$ satisfy the complete reducibility property. Therefore we can restrict the problem of finding all representations of $\mathrm{SL}_2(\mathbb{C})$ to the problem of finding its irreducible representations and, from the previous section, we can reduce this problem to that of finding the irreducible representation of its lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Before starting however we need to introduce, without proving, one more result.

Proposition 9. *(Preservation of Jordan Decomposition). Let $\mathfrak{g}$ be a semi-simple Lie algebra. For any element $Z \in \mathfrak{g}$, there exist unique Z_s and Z_n in $\mathfrak{g}$ such that Z_s is diagonalizable, Z_n is nihilpotent, both of them commute and $Z = Z_s + Z_n$. Moreover, with the same notation we have that for any representation φ of the Lie algebra $\mathfrak{g}$ hold the identities*

$$\varphi(Z)_s = \varphi(Z_s), \qquad \qquad \varphi(Z)_n = \varphi(Z_n)$$

Prove of this proposition can be found in proposition 9.20 and appendix C of [5].

As a corollary of the previous proposition we have that.

Corollary 5. *Given $\mathfrak{g}$ a Lie algebra, φ any representation of $\mathfrak{g}$ and Z a diagonalizable element of $\mathfrak{g}$ then $\varphi(Z)$ is also diagonalizable.*

Take now V a finite dimensional complex vector space and φ a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on V. Then, for the previous corollary, we have that $\varphi(H)$ is diagonalizable[9]. We can therefore write V as direct sum of one dimensional eigenspaces generated by linearly independent eigenvectors of $\varphi(H)$. In other words we have that $V \cong \oplus V_\lambda^{\oplus n_\lambda}$ where $V_\lambda = \langle v_\lambda \rangle$ are such that $\varphi(H)(v_\lambda) = \lambda v_\lambda$. For simplicity we will be referring to the eigenvectors of $\varphi(H)$ as if they were eigenvectors of H.

According to the commutation relations between the basis elements of $\mathfrak{sl}_2(\mathbb{C})$, for every possible λ we have that

$$HX \cdot v_\lambda = ([H,X] + XH)\, v_\lambda = (2X + XH)\, v_\lambda = 2Xv_\lambda + X\lambda v_\lambda = (\lambda+2)Xv_\lambda,$$
$$HY \cdot v_\lambda = ([H,Y] + YH)\, v_\lambda = (-2Y + YH)\, v_\lambda = -2Yv_\lambda + Y\lambda v_\lambda = (\lambda-2)Yv_\lambda.$$

We can thus deduce that the matrix X sends any eigenvector v_λ to an eigenvector $v_{\lambda+2}$while matrix Y sends v_λ to an eigenvector $v_{\lambda-2}$.

Since V is finite dimensional and eigenvectors of diagonal matrices with different values are linearly independent, there must exists an eigenvector $v_\mu \neq 0$ such that $Xv_\mu = 0$.

We can now define the set B as the set of vectors obtained by making Y act on v_μ any finite number of times and removing the 0 vector from the result

$$B = \{Y^n \cdot v_\mu : \ n \in \mathbb{N} \cup \{0\}\} \setminus \{0\}.$$

Given that V is finite dimensional we can, as before, deduce that exists a non negative integer r such that, $Y^n v_\mu = 0$ for every $n > r$ while $Y^r v_\mu \neq 0$.

Denoting $v_{\mu-2n} = Y^n v_\mu$ we can thus redefine

$$B = \{v_\mu, v_{\mu-2}, \ldots, v_{\mu-2r}\}.$$

From now on we will denote $V_\mu = \langle B \rangle \subset V$ the vector space generated by the vectors in B. By construction, V_μ is invariant under the action of both Y and H. If we now prove that $V_\mu$is also invariant under the action of X then will have that V_μ remains invariant under the action of all elements of the basis of $\mathfrak{sl}_2(\mathbb{C})$ and, therefore, remains invariant under the action of any element of $\mathfrak{sl}_2(\mathbb{C})$. In other words we will have proven that V_μ is a sub-representation of V.

To prove that V_μ is invariant under X we first need the following lemma

Lemma 13. *Using the notation introduced until now, for every $0 \leq n \leq r$ exist values $a_n, b_n \in \mathbb{C}$ such that*

$$XYv_{\mu-2n} = a_n v_{\mu-2n} \qquad \text{and} \qquad YXv_{\mu-2n} = b_n v_{\mu-2n},$$

with $a_n = (n+1)(\mu-n)$ and $b_n = n(\mu-n-1)$.

[9] Remember that we are using the notation introduced at the end of section 9.2.

Proof. We will prove the result by induction over n.

$n = 0$.

in this case we have, by definition, that

$$YXv_\mu = Y0 = 0,$$

and therefore $b_0 = 0 = 0 \cdot (\mu - 0 - 1)$. On the other hand

$$\mu v_\mu = Hv_\mu = [X, Y]\, v_\mu = (XY - YX)\, v_\mu = XYv_\mu - YXv_\mu = XYv_\mu,$$

and, since $a_0 = (0+1)(\mu - 0) = \mu$ we can conclude that the lemma is true for $n = 0$.

$n > 0$.

In this case we have that

$$YXv_{\mu-2n} = YXYv_{\mu-2(n-1)} = Y\left(XYv_{\mu-2(n-1)}\right).$$

Applying the induction hypothesis we obtain

$$YXv_{\mu-2n} = Ya_{n-1}v_{\mu-2(n-1)} = a_{n-1}Yv_{\mu-2(n-1)} = a_{n-1}v_{\mu-2n},$$

and therefore $b_n = a_{n-1}$. Using this result we can write

$$\begin{aligned}(\mu - 2n)v_{\mu-2n} &= Hv_{\mu-2n} = [X, Y]\, v_{\mu-2n} = (XY - YX)\, v_{\mu-2n}, \\ &= XYv_{\mu-2n} - YXv_{\mu-2n} = XYv_{\mu-2n} - a_{n-1}v_{\mu-2n},\end{aligned}$$

and therefore

$$XYv_{\mu-2n} = (a_{n-1} + \mu - 2n)v_{\mu-2n}.$$

This last identity proves the lemma with $a_n = a_{n-1} + \mu - 2n$. If we combine the identity with the already found value $a_0 = \mu$ we can obtain the following closed expression for a_n

$$a_n = a_{n-1} + \mu - 2n = \sum_{j=0}^{n} (\mu - 2j) = (n+1)\mu - 2\frac{n(n+1)}{2} = (n+1)(\mu - n).$$

From this, knowing that $b_n = a_{n-1}$, we deduce that

$$b_n = n(\mu - n - 1),$$

thus completing the proof of the lemma. □

Using the previous lemma we have that, for every $n = 0, \ldots, r$

$$Xv_{\mu-2n} = XYv_{\mu-2(n-1)} = a_{n-1}v_{\mu-2(n-1)} = n(\mu - n - 1)v_{\mu-2(n-1)},$$

where $v_{\mu+2} = 0$ and $a_{-1} = 0$. In other words X sends every element of the basis B to a scalar multiple of another element of the same basis. This implies that $V_\mu = \langle B \rangle$ is invariant under X and, since it is also invariant under Y and H, we can conclude it is invariant under $\mathfrak{sl}_2(\mathbb{C})$. In other words V_μ is a sub-representation of V.

Since we have taken V any representation we can, in particular, take V an irreducible representation. Then, since, $V_\mu \neq 0$ is an irreducible sub-representation of V we must have $V_\mu = V$. In other words all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ can be obtained following are the same procedure used to obtain V_μ.

We can be more specific about the representation V_μ. In fact, we can explicit the value μ depending on the dimension of V_μ.

We know that, for how r is defined, $v_{\mu-2r} \neq 0$ and

$$XYv_{\mu-2r} = X0 = 0.$$

On the other hand, according to the previous lemma we have that

$$0 = XYv_{\mu-2r} = a_r v_{\mu-2r},$$

and we can therefore conclude that $a_r = 0$. From the lemma we know that $a_r = (r+1)(\mu - r)$. Since r is a non negative integer then $r + 1 \neq 0$ and, therefore, we must have $\mu - r = 0$, or, written differently $\mu = r$.

From this we can deduce that V_r is irreducible for every non negative integer r. Suppose in fact that we could write $V_r = W_r \oplus W_r'$. Since H would act diagonally in both W_r and W_r' we could span these representations with eigenvectors of H as we did before with the general representation V. Since $V_r = W_r \oplus W_r'$ the eigenvalues of this eigenvectors must be the same eigenvalues we found when studying the action of H over V_r. We can thus take v_r an eigenvector of either W_r or W_r' with eigenvalue r. Without loss of generality we can suppose $v_r \in W_r$. Repeating now the same procedure used to construct the representations V_μ we could, from v_r, construct a sub-representation $V_r' \subseteq W_r \subseteq V$. We have also proven that both V_r' and V_r are $r+1$ dimensional and, therefore, we can conclude from the previous inclusion that $V_r' = W_r = V$. Since $W_r = V$ then $W_r' = \{0\}$ and since W_r and W_r' are general sub-representations of V_r we can conclude that V_r is irreducible.

In synthesis, during this section we have proven that all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ must be of the form V_r with r and integer such that $r + 1 = \dim(V_r)$. A possible basis of the irreducible representation V_r is given by the set

$$B = \{v_{-r}, v_{-r+2}, \ldots, v_r\},$$

where the the values v_n satisfy

$$Hv_n = nv_n \qquad Yv_n = v_{n-2} \qquad Xv_n = \frac{r-n}{2}(\frac{r+n}{2} - 1)v_{n+2},$$

where $v_{r+1} = v_{-r-1} = 0$. We have also proven that all representations of this form are irreducible.

The only thing we now need to do in order to complete the description of all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ is finding, for every non negative integer r a concrete example of the representation V_r.

9.4 Examples of V_r for every non negative V_r.

For every non negative integer r, we will find a representation V having an eigenvector v_r of H with eigenvalue r satisfying $Xv_r = 0$. Using this eigenvector will be able, following the same steps descried in the previous section, to construct the irreducible representation V_r.

Take $V = \mathbb{C}^2$ the standard representation of $\mathfrak{sl}_2(\mathbb{C})$. That is the representation in which $\mathfrak{sl}_2(\mathbb{C}) \subset M_{2\times 2}(\mathbb{C})$ acts on $\mathbb{C}^2$ by left matrix multiplication. This representation is clearly faithful and, therefore, as proposition 8 suggests for the finite case, we will look for irreducible representations in its tensor powers.

Notice now that, as it is explained in page 110 of [5], the representation of $\mathfrak{sl}_2(\mathbb{C})$ over V induces, for every positive integer n, a representation of $\mathfrak{sl}_2(\mathbb{C})$ over $V^{\otimes n}$ given by

$$Z \cdot (v^1, \ldots, v^n) = \sum_{i=1}^{n} (v^1, \ldots, v^{i-1}, Z \cdot v^i, v^{i+1}, \ldots, v^n) \text{ for every } Z \in \mathfrak{sl}_2(\mathbb{C}).$$

From this definition of the representation over $V^{\otimes n}$ we can deduce that the eigenvectors for H on $V^{\otimes n}$ are all those elements $(v^1, \ldots, v^n) \in V^{\otimes n}$ such that, for every $i = 1, \ldots, n$, the vector v^i is an eigenvector for H on V. If we now take $v_1 = (1, 0)$ and $v_{-1} = (0, 1)$ two linearly independent eigenvectors of V with eigenvalues 1 and -1 respectively we can define the set $B = \{(v_{\pm 1}, \ldots, v_{\pm 1}) \in V^{\otimes n}\}$ of linearly independent vectors of $V^{\otimes n}$. Since $|B| = 2^n = \dim(V)^n = \dim(V^{\otimes n})$ then the eigenvalues associated to each eigenvector of B must be all eigenvalues of H (seen as an automorphism of $V^{\otimes n}$). Since the eigenvalue of the vector $(v_{s_1}, \ldots, v_{s_n}) \in B$ is equal to $\sum_{i=1}^n s_i$ we can deduce then that the eigenvalues of H (as an automorphism of $V^{\otimes n}$) range from $-n$ to n reaching both extremes with vectors $v_{-n} = (v_{-1}, \ldots, v_{-1})$ and $v_n = (v_1, \ldots, v_1)$ respectively. Since there cannot be any eigenvector with eigenvalue greater than n we can conclude that $Xv_n = 0$. We can now follow the steps described in the previous section to obtain from the vector v_n the irreducible representation $V_n \subset V^{\otimes n}$.

The previous argument gives an example of the irreducible representation V_n for every positive integer n. The only representation left is $V_0 = \mathbb{C}$ which is none other that the trivial representation defined by the action $\mathfrak{sl}_2(\mathbb{C}) \cdot V_0 = 0$.

We can however tell a little more about the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. Notice that, if we take $\mathrm{Sym}^n(V) \subset V^{\otimes n}$, then we will have that $\mathrm{Sym}^n(V)$ is a sub-representation of $V^{\otimes n}$. Moreover we can use as basis of $\mathrm{Sym}^n(V)$ the set

$$B^* = \left\{v_{-n} = (v_{-1}, \ldots, v_{-1}), v_1^*, \ldots, v_{n-1}^*, (v_1, \ldots, v_1) = v_n\right\},$$

where v_i^* denotes the element of $V^{\otimes n}$ obtained by adding all vectors of the basis B containing the element v_1 exactly i times.

Since $v_n \in B^*$ then we can deduce that the irreducible representation V_n is contained in $\mathrm{Sym}^n(V)$. On the other hand $\mathrm{Sym}^n(V)$ has dimension $n+1$ which is equal to the dimension of V_n and we can therefore deduce that $\mathrm{Sym}^n(V) = V_n$ for every $n \geq 2$. If we use the convention $\mathrm{Sym}^1(V) = V$ and $\mathrm{Sym}^0(V) = \mathbb{C}$ then the result is true for every non negative integer n. We have thus completed the description of all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ and, therefore, of $\mathrm{SL}_2(\mathbb{C})$.

A Sub-representation V^G.

Take G a finite group, V a complex finite dimensional representation of G and the sub-representation $V^G \subset V$ defined by

$$V^G = \{v \in V : \; gv = v \; \forall g \in G\}.$$

In this section we will prove that V^G contains all copies of the trivial representation appearing in V. During this section we will denote $n = \dim\left(V^G\right)$ and will take $B = \{v_i\}_{i=1,\ldots,n}$ as basis of V^G.

Notice that, since every vector of V^G is invariant under the action of G, then the sub-spaces $V_i = \langle v_i \rangle \subset V^G$ are, in fact, sub-representations of V. Moreover, since B is a basis of V^G, we have that $V^G = \bigoplus V_i$ and since V_i are irreducible for being one dimensional we can deduce that this is the decomposition of V^G as direct sum of irreducible representations. Since G acts over every V_i as the identity and all V_i are 1-dimensional then every V_i is isomorphic (as a representation) to the trivial representation T of G. In other words we have that $V^G \cong T^{\oplus n}$.

Let us suppose now that the representation V is isomorphic as a representation to $V^G \oplus T \oplus W \cong V$ for some (possibly 0-dimensional) representation W. Then it would exist a vector $v \in T$, $v \notin V^G$, such that v is invariant under the action of G. However this would contradict the definition of V^G and it is, therefore, impossible. We can thus conclude that the sub-representation $V^G \subset V$ is the direct sum of exactly all copies of the trivial representation appearing in the decomposition of V.

B Conjugacy classes of $\mathfrak{S}_d$.

In this section we will study the conjugacy classes of the symmetric group $\mathfrak{S}_d$ for any positive integer d. More precisely

Proposition 10. *For any positive integer d the number of conjugacy classes of the group $\mathfrak{S}_d$ is equal to the number of partitions of d. Moreover, given $\varphi, \psi \in \mathfrak{S}_d$ two permutations, then they belong to the same conjugacy class of $\mathfrak{S}_d$ if and only if, in their expression as composition of disjoint cycles, their cycles have the same lengths.*

Proof. For any cycle $(n_1, \ldots, n_m)$, any permutation $\varphi \in \mathfrak{S}_d$ and any $i = 1, \ldots, m$ we have that

$$\varphi \circ (n_1, \ldots, n_m) \circ \varphi^{-1} \left(\varphi(n_i)\right) = \varphi \circ (n_1, \ldots, n_m)\,(n_i) = \varphi\left(n_{i+1}\right),$$

where $n_{m+1} = n_1$. Therefore we can write

$$\varphi \circ (n_1, \ldots, n_m) \circ \varphi^{-1} = \left(\varphi\left(n_1\right), \ldots, \varphi\left(n_m\right)\right).$$

Take now any permutation $\phi \in \mathfrak{S}_d$ and express it like

$$\phi = \left(n_1, \ldots, n_{m_1}\right)\left(n_{m_1+1}, \ldots, n_{m_2}\right) \cdots \left(n_{m_{r-1}+1}, \ldots, n_{m_r}\right),$$

where the positive integers m_i satisfy that $m_i > m_j \Leftrightarrow i > j$ and $m_r = d$. Then we have that all members of the same conjugacy class as ϕ are of the form

$$\begin{aligned}\varphi \circ \phi \circ \varphi^{-1} &= \varphi\left(n_1, \ldots, n_{m_1}\right) \cdots \left(n_{m_{r-1}+1}, \ldots, n_{m_r}\right) \varphi^{-1} = \\ &= \varphi\left(n_1, \ldots, n_{m_1}\right) \varphi^{-1} \varphi \cdots \varphi^{-1} \varphi\left(n_{m_{r-1}+1}, \ldots, n_{m_r}\right) \varphi^{-1} = \\ &= \left(\varphi\left(n_1\right), \ldots, \left(n_{m_1}\right)\right) \cdots \left(\varphi\left(n_{m_{r-1}+1}\right), \ldots, \varphi\left(n_{m_r}\right)\right)\end{aligned}$$

for some $\varphi \in \mathfrak{S}_d$. Therefore all permutations of the same conjugacy class have cycles of the same lengths.

Given now two permutations $\phi_1, \phi_2 \in \mathfrak{S}_d$ with cycles of the same lengths

$$\begin{aligned}\phi_1 &= \left(n_1^1, \ldots, n_{m_1}^1\right) \cdots \left(n_{m_{r-1}+1}^1, \ldots, n_{m_r}^1\right), \\ \phi_2 &= \left(n_1^2, \ldots, n_{m_1}^2\right) \cdots \left(n_{m_{r-1}+1}^2, \ldots, n_{m_r}^2\right),\end{aligned}$$

the permutation $\varphi \in \mathfrak{S}_d$ defined by $\varphi\left(n_i^1\right) = n_i^2$ satisfies that $\varphi \circ \phi_1 \circ \varphi^{-1} = \phi_2$. In other words, two permutations $\phi_1, \phi_2 \in \mathfrak{S}_d$ having cycles of the same lengths belong to the same conjugacy class thus completing the proof of the lemma. □

As a consequence of this lemma and since the lengths $(m_{i+1} - m_i)$ of the cycles (considering cycles of length 1) are positive integers adding up to $m_r = d$, then the number of possible different cycles lengths (and therefore the conjugacy classes of $\mathfrak{S}_d$) coincides with the number of partitions of d. In other words, for any positive integer d the symmetric group $\mathfrak{S}_d$ has as many cycles as there are partitions of d.

C Algorithm, representations dimensions.

Given G a finite group and $\{V_i\}_{i=1}^n$ all its irreducible representations we know, from corollary 4, that

$$\sum_{i=1}^{n} \dim(V_i)^2 = |G|\,.$$

Since we also know that one of the irreducible representations of G is the trivial one, with dimension 1, we can develop the following algorithm for finding the set of all possible dimensions for the irreducible representations

1. Initialize the set of all possible dimensions as $\text{dims} = \emptyset$ and the dimensions of irreducible representations as $\dim(V_i) = 1$ for every i.
2. If $\sum_{i=1}^n \dim(V_i)^2 = |G|$ then we add the dimensions vector $(\dim(V_1), \ldots, \dim(V_n))$ to the set dims of possible dimensions.
3. Add 1 to the value of $\dim(V_1)$ and start $i = 1$.
4. Check if $\dim(V_i)^2 > |G|$.
5. If the previous condition is met then set $\dim(V_i) = 1$ add one to $\dim(V_{i+1})$, add 1 to i and continue from step 4.
6. If $\dim(V_n)$ is equal to 1 we continue from step 2.
7. Define in dims the equivalence relation $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if exists $\varphi \in \mathfrak{S}_n$ such that $a_i = \varphi(b_i)$ for every $i = 1, \ldots, n$.
8. Return a representative for every class of $\text{dims}/\sim$.

More intuitively, this algorithm iterates over the set

$$B = \left\{(\dim(V_1), \ldots, \dim(V_n)) \in \mathbb{N}\setminus\{0\} \ : \dim(V_i) \geq \dim(V_{i+1}) \ \text{and} \ \dim(V_i)^2 \leq |G| \ \text{for every} \ i = 1, \ldots, n-1\right\},$$

where V_n is the trivial representation[10], and returns the set

$$\text{dims} = \left\{(\dim(V_1), \ldots, \dim(V_n)) \in \mathbb{N}\setminus\{0\} \ : \sum_{i=1}^{n} \dim(V_i)^2 = |G|\right\}.$$

Next we show a Python of the described algorithm.

```python
import argparse
from typing import List

from math import sqrt
```

[10]Remember that the trivial representation is one dimensional.

```python
def add_one(dim_list: List[int], max_dim: int):
    dim_list[0] += 1
    for i in range(len(dim_list) - 1):
        if dim_list[i] > max_dim:
            dim_list[i] = 1
            dim_list[i + 1] += 1
    return dim_list


def find_dimensions_list(n_repr: int, group_dim: int):
    max_dim = int(sqrt(group_dim)) + 1
    dimension_list = [1] * n_repr

    possible = []
    while dimension_list[-1] == 1: # the trivial representation has dimension 1.
        dimension_list = add_one(dimension_list, max_dim)
        if sum([x ** 2 for x in dimension_list]) == group_dim:
            found_list = sorted(dimension_list)
            if found_list not in possible:
                possible.append(found_list)

    if len(possible) == 0:
        print('There is no such finite group')
    else:
        print('here is a list of all possible dimension of the group representations')
        for dim in possible:
            print(dim)


def main():
    argument_parser = argparse.ArgumentParser()
    argument_parser.add_argument('--n-repr', help='Number of irreducible representations.',
                                 type=int)
    argument_parser.add_argument('--group-dim', help='Dimension of the group.',
                                 type=int)

    args = argument_parser.parse_args()

    find_dimensions_list(args.n_repr, args.group_dim)


if __name__ == '__main__':
    main()
```

D Power of 1-Jordan blocs.

Take J a Jordan block of dimension k and eigenvector λ. We want to prove that, for every positive integer n, then

$$J^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \binom{n}{k-1}\lambda^{n-(k-1)} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \binom{n}{2}\lambda^{n-2} \\ \vdots & \ddots & \ddots & \ddots & \binom{n}{1}\lambda^{n-1} \\ 0 & \cdots & 0 & 0 & \lambda^n \end{pmatrix},$$

where we use the convention $\binom{n}{m} = 0$ whenever $m > n$ and $0^0 = 1$ while $0^n = 0$ for any non zero integer n.
For $n = 1$ then, since $\binom{1}{1} = 1$ and $\lambda^0 = 1$ then the identity arises from the definition of Jordan block and the previously explained notation.
If it is true for $n - 1 \geq 1$ then

$$J^n = J^{n-1}J = \begin{pmatrix} \lambda^{n-1} & \binom{n-1}{1}\lambda^{n-2} & \binom{n-1}{2}\lambda^{n-3} & \cdots & \binom{n-1}{k-1}\lambda^{n-1-(k-1)} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \binom{n-1}{2}\lambda^{n-3} \\ \vdots & \ddots & \ddots & \ddots & \binom{n-1}{1}\lambda^{n-2} \\ 0 & \cdots & 0 & 0 & \lambda^{n-1} \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix},$$

$$= \begin{pmatrix} \lambda^n & \left(\binom{n-1}{1} + 1\right)\lambda^{n-1} & \left(\binom{n-1}{2} + \binom{n-1}{1}\right)\lambda^{n-2} & \cdots & \left(\binom{n-1}{k-1} + \binom{n-1}{k-2}\right)\lambda^{n-(k-1)} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \left(\binom{n-1}{2} + \binom{n-1}{1}\right)\lambda^{n-2} \\ \vdots & \ddots & \ddots & \ddots & \left(\binom{n-1}{1} + 1\right)\lambda^{n-1} \\ 0 & \cdots & 0 & 0 & \lambda^n \end{pmatrix},$$

Since by definition of the product we have that

$$J^n(i,j) = \lambda J^{n-1}(i,j) + J^{n-1}(i,j-1),$$

where $J^n(i,j)$ denotes the value in row i and column j of matrix J^n and $J^n(i,0) = 0$ for every n. Then since $1 = \binom{n}{0}$ and, for every $1 \leq m \leq n-1$ we have

$$\begin{aligned} \binom{n-1}{m} + \binom{n-1}{m-1} &= \frac{(n-1)!}{m!(n-1-m)!} + \frac{(n-1)!}{(m-1)!(n-1-(m-1))!}, \\ &= \frac{(n-1)!}{m!(n-1-m)!} + \frac{(n-1)!}{(m-1)!(n-m)!}, \\ &= \frac{(n-m)\,(n-1)! + m\,(n-1)!}{m!(n-m)!}, \\ &= \frac{n\,(n-1)!}{m!(n-m)!} = \binom{n}{m}, \end{aligned}$$

while for $m = n$ and $m > n$ we also have

$$\begin{aligned} &\binom{n-1}{m} + \binom{n-1}{m-1} = \binom{n-1}{n} + \binom{n-1}{n-1} = 0 + 1 = 1 = \binom{n}{n} = \binom{n}{m}, \\ &\binom{n-1}{m} + \binom{n-1}{m-1} = 0 + 0 = 0 = \binom{n}{m}, \end{aligned}$$

thus proving that, for any non negative integers m and n then

$$\binom{n-1}{m} + \binom{n-1}{m-1} = \binom{n}{m}.$$

Replacing this identity in the previous product then we have

$$J^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \binom{n}{k-1}\lambda^{n-(k-1)} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \binom{n}{2}\lambda^{n-2} \\ \vdots & \ddots & \ddots & \ddots & \binom{n}{1}\lambda^{n-1} \\ 0 & \cdots & 0 & 0 & \lambda^n \end{pmatrix},$$

thus proving the identity.

E Non complete reducibility of Lie algebras.

A simple example for the non complete reducibility of a Lie algebra is given by the matrix Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} : \ a \in \mathbb{C} \right\}.$$

A representation of this Lie algebra can be found by making $\mathfrak{g}$ act on $V = \mathbb{C}^2$ by matrix multiplication.

If we now look at $W \subsetneq V$ defined by $W = \{(c, 0) : \ c \in \mathbb{C}\}$ we can easily verify that W is $\mathfrak{g}$-invariant and, therefore, is a non trivial sub-representation of V.

Let's now suppose that exists $W' \subsetneq V$ such that W' is also a sub-representation of V and $W \oplus W' = V$. Since V is two dimensional and W is one dimensional then W' must be 1 dimensional. Therefore we can take $w' = (w_1, w_2) \in W'$ such that $W' = \{w' \cdot c : \ c \in \mathbb{C}\}$. Since $W \oplus W' = V$ then we must have that $w = (1, 0)$ and w' are linearly independent. On the other hand since W' is a representation of $\mathfrak{g}$ then there must be some $c \in \mathbb{C}$ such that

$$\begin{pmatrix} w_1 + w_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} c \cdot w_1 \\ c \cdot w_2 \end{pmatrix}.$$

However the above identity can only be true if $c = 1$ and $w_2 = 0$. If this happens then $w' = w_1 \cdot w$ thus contradicting the linear independence between w and w'.

We can therefore deduce that such a representation W' does not exist. Moreover, the previous proof has as a corollary $(c, 0)$ is the only eigenvector of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Therefore there is no 1-dimensional sub-representation of V other than W.

This implies that there exists no couple of (irreducible) sub-representations $W, W' \subsetneq V$ such that $W \oplus W' = V$. Since V is not irreducible this example disproves the complete reducibility of general Lie algebras representations.

F Lie claudator of an associated Lie algebra.

During this section we will denote by G a generic Lie group (complex or not).

For every element $g \in G$ we can define the conjugation function $\Psi_g : G \to G$ as

$$\Psi_g (h) = ghg^{-1}.$$

Since, by definition of Lie group, both the product function and the inverse function are differentiable, then Ψ_g is also differentiable.

Taking the differential of Ψ_g at the origin (the identity element $e \in G$) we can define the function $\mathrm{Ad} : G \to \mathrm{Aut}\,(T_eG)$ as

$$\mathrm{Ad}(g) := (d\Psi_g)_e \,.$$

Taking now the differential at the origin of the function Ad we obtain the function $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ defined as

$$\mathrm{ad}(X) := (d\mathrm{Ad})_e \,(X).$$

Finally, using the function ad we can define a binary operation on T_eG as

$$[X, Y] = \mathrm{ad}(X)(Y).$$

As it is proven in section 8.1 of [5] the operator $[\cdot, \cdot]$ is bilinear, skew-symmetric and satisfies the Jacobi identity. This operator therefore provides the vector space T_eG of a structure of a Lie algebra. We refer to the algebra thus obtained as the Lie algebra associated to the Lie group G.

If G is a sub-group of the general linear group $\mathrm{GL}_n\,(\mathbb{C})$ then T_eG can be seen as a sub-space of the space of linear endomorphisms of an n-dimensional vector space.

Under this conditions, as shown in [5], the operator $[\cdot, \cdot]$ can be extremely simplified as

$$[X, Y] = X \cdot Y - Y \cdot X.$$

G Exponential map.

During this section we will be using the convention of denoting by G a general Lie group (complex or not), by $e \in G$ the identity element of G, by $\mathfrak{g} = T_e G$ its associated Lie algebra, by g and h elements of G and by X and Y elements of $\mathfrak{g}$.

The goal of this section is that of defining the exponential map $\exp : \mathfrak{g} \to G$.

To do this we first need to introduce a few new concepts.

Given $g \in G$ we can define the map $m_g : G \to G$ as

$$m_g(h) := g \cdot h.$$

Using this, we can, for every vector X tangent to G on some $h \in G$, define the map

$$v_X(g) := \left(dm_g\right)_h (X).$$

Which allows us to define $\varphi_X : \mathbb{R} \to G$ as the unique[11] function such that

$$\frac{d\varphi_X(u)}{du}(t) = v_X(\varphi_X(t)) \qquad \text{and} \qquad \varphi_X(0) = e.$$

We can finally define the exponential map as

$$\exp(X) = \varphi_X(1).$$

In the situation where G is a sub-group of $\mathrm{GL}_n(\mathbb{C})$ we can give, without proving, a more explicit definition of exponential map. In fact, under this conditions φ_X is a solution of a well know differential equation and we can write

$$\varphi_X(1) = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}.$$

Therefore the exponential map is none other that the well known exponential function as defined on the matrix vector space

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

As it is proven in section 8.3 of [5] the exponential map has the property that, given G' and $\mathfrak{g}'$ another Lie group and its associated Lie algebra respectively, then, for any differentiable group morphism $\varphi : G \to G'$, the following diagram is commutative

$$\begin{array}{ccc} g & \xrightarrow{(d\rho)_e} & g' \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\rho} & G' \end{array} \quad .$$

[11] Existence and uniqueness is given by the existence and uniqueness of solutions to differential equations.

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