TITLE: TRIGONOMETRIC ADJUSTMENT ON RELATIVE VOLATILITY (TARV)

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ABSTRACT

This paper aims to analyse the common linear market risk measures and to propose a complementary non-linear and non-parametric risk measure named TARV. Data and methodology have their own chapter in this paper, but they basically comprise the study of *ex post* returns of six Exchange Traded Products (ETPs) and two market indexes from 2014 until 2018. The approach is to complement and enhance the quantitative methods used in risk and portfolio management which have very particular profile of securities when applied to high volatile, leveraged, contrarian to market sentiment and non-linear related with the benchmark. Besides these characteristics, TARV can be applied to general traded securities as equity, portfolios, futures, non-linear payoff derivatives and indexes. Non-linear dependence between the security and the benchmark displays financial incoherence in linear risk measures, which can mislead the required capital needed in case of market turmoil and the final decision of investment decided by the manager. Beyond financial classical and coherent risk assumptions that defend risk neutrality of risk measures, TARV can provide risk aversion approach thanks to its magnificent effect on important market movements and a collapse effect on returns considered as *market noise*. TARV's graphical representation is like the Maximum Downward risk indicator but it relates the expected maximum market risk exposure of the security until the last pricing day available. TARV can be used as a unique or complementary gauge in risk and portfolio management but specifically in determining capital requirements, stress tests and setting hedges. An automatic adjustment on high and low volatile market periods and a supply of most outstanding movements the manager should be concerned about, are the value and original core of TARV.

TABLE OF CONTENTS

1. INTRODUCTION

TARV mistrusts the sentence *this could not get any worse* so the history of private financial institutions and rating agencies neither backs. The fact that one event has not appeared in our universe does not mean that it does not exist. TARV overvalues the maximum expected exposure to market risk when the security has high volatility regarding the benchmark. The other way around, TARV undervalues the maximum expected exposure to market risk when the security has low volatility regarding the benchmark.

TARV follows Mr. Buffett's famous quote *be fearful when others are greedy and greedy when others are fearful*, in an alternative way. Linear risk indicators undervalue the capital to be held in case of market turmoil and overvalue it when objectivity governs in capital and monetary markets. These type of indicators as Relative Volatility (RV), ß and Coefficient of Variation (CV) do follow Mr. Buffett's sentence: they are *greedy* when they must be *fearful,* and they are *fearful* in *greedy* times. This sentence defends the classical theory of *contrarian opinion*, which can be useful depending on the context. Since originally this quote was generally thought for equity market, let's observe what happens when one allocates it in a capital requirements field.

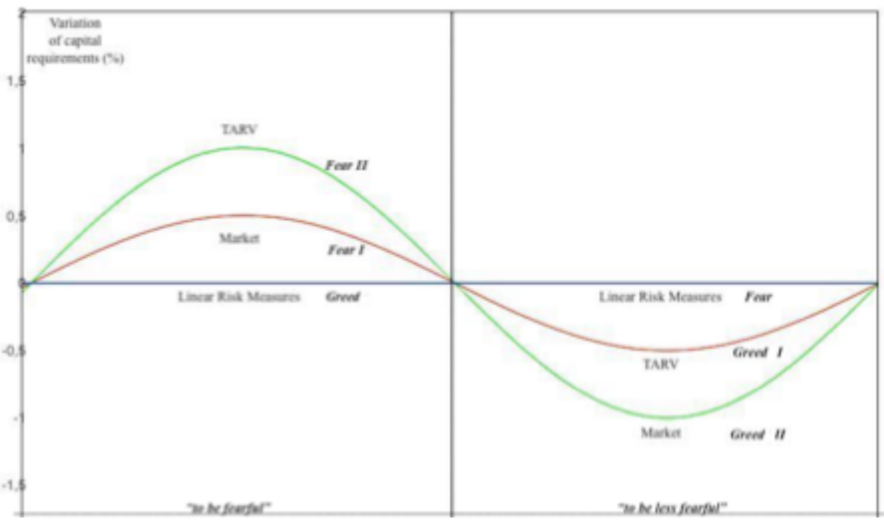
Let's *binarize* our financial world into two situations: *to be fearful* and *to be less fearful* or *greedy*. This division is in honour of the mentioned quote of Mr. Buffett but with the difference that it is more convenient to name the calmest scenario *to be less fearful* rather than *greedy*. The change in the name is due to not allowing the market to catch the manager off guard and, in contrast, hedge some of the highest risky positions. Therefore, insurance through options also pays off or retaining cash in a portfolio is a profitable strategy in market turmoil period.

As individuals we cannot go further than making estimations of what it is expected to occur in the mid-term, or at least, the near-term. The long-term estimations are left for more complex mathematical methods. Those estimations have outliers that can affect our portfolios negatively.

The graph of *Figure 1* perfectly explains what differentiates TARV from other linear risk measures as RV, CV or ß. The vertical axis is the percentage of change of capital requirements. It is logic to increase capital requirements when the market increases its volatility and to decrease capital requirements when the market decreases its volatility. We assume that either increasing or not decreasing capital requirements is a cost for the fund in terms of required rate of return and opportunity cost of investing that capital at the risk-free rate. The horizontal axis represents the volatility of the security. The central value is the expected value of the standard deviation of security that maximizes TARV and markets' change of capital requirement. The relative maximum of TARV as market function is one. TARV overvalues security's maximum exposure to market risk and linear risk measures undervalues security's maximum exposure to market risk depending on market situation. The real market exposure of the security must be positioned between the two approximations: the non-linear and non-parametric one and the linear one.

On the left, the red line states the required capital for assuming market risk from the point of view of the market. On the right, the green line states the required capital for assuming market risk from the point of view of the market. Either in both scenarios, the lines are invisible for managers; they solely assess the exposure of the portfolio to market risk throughout RV or ß, which are linear measures of market risk exposure.

Figure 1. Representation of TARV automatic adjustment.



Source: self-created (2019).

The left part of the graph, as TARV function leaves the expected value for both sides, TARV changes of capital requirement diminishes because the expected value is the best amount of capital requirement given the market requirement function. In other words, projected value of TARV capital function requirement is the best capital requirement change regarding the linear risk measures. This positive capital requirement change ensures the security or the portfolio to be completely hedged in front of any unlikely market movement regarding historical data. Furthermore, in the *to be fearful* scenario, as managers, we cannot view the market capital required line neither the green TARV line (TARV is not used by managers yet). Therefore, we are being *greedy* in a scenario we ought to be *fearful*. Mr. Buffett's quote occurs satisfactorily, but the context is far from being good for the manager; while ignoring the red line, the manager is acting greedily with the capital requirement remaining constant in the scenario *to be fearful*. The market is requiring a positive change of capital requirement i.e. increasing the capital to be left in case of hedge or acquiring new positions. We, as managers, are maintaining the initial capital requirement without any change. A sudden market movement will catch the manager off guard and will provoke losses in the portfolio. The resignation of the manager will be imminent.

On the right part of the graph, as TARV function leaves the expected value on both sides, TARV's change of capital requirement diminishes because the expected value is the best amount of capital requirement given the market requirement function. In other words, expected value in absolute terms of TARV requirement capital function is the best capital requirement change regarding the linear risk measures. The negative capital requirement change still leaves capital that ensures the security or the portfolio to be partly hedged in front of any unlikely market movement regarding historical data. Additionally, in *to be less fearful* scenario, as managers, we cannot view the market capital required line neither the green TARV line (TARV is not used by managers yet). Therefore, we are being *fearful* when we need to be *greedy*. Again, Mr. Buffett's quote occurs satisfactorily, but the context is not good enough for the manager: while ignoring the green line, the manager is acting fearfully with the capital requirement remaining constant in a scenario to be less hesitant. The market is requiring a negative change of capital requirement i.e. decreasing the capital to be left in case of hedge or in case new positions are acquired. As managers, we are maintaining the initial capital requirement without any change. This *unemployed* capital that is left is an opportunity cost for the fund and it can act as a synonym of higher required rates of return. Less contrarian market movements than expected will cause less security or less portfolio returns than the funds that use more accurate models. The resignation of the manager is considered.

The crossing point is different from zero yet TARV and market follow a periodic non-linear function and the horizontal axis must be positive because it represents the standard deviation.

Let's assume that the manager is aware of the existence of TARV as a non-linear and non-parametric risk measure and it is visible indeed. The manager, a strong believer of Mr. Buffett's quote, understands that TARV is perfectly compatible with the famous sentence: when other funds are undervalued contrarian market movements, the manager decides to increase the amount of capital in case of hedging the riskiest positions. The manager also knows that between TARV overvaluation and undervaluation of linear risk measures is where the market optimal capital requirement line displays. Likewise, the manager knows for sure that the maximum exposure to market risk on historical data will be overvalued if using TARV. Although there is a positive opportunity cost of capital retained, it is better than suffering losses for not doing so.

While other funds are overvaluing contrarian market movements, the manager decides to decrease the amount of capital but still maintain some quantity just in case of hedging the riskiest positions. The manager also knows that TARV undervaluation stays between overvaluation of linear risk measures and the market optimal capital requirement line. In the same way, the manager knows for sure that she will be undervaluing the maximum exposure to market risk on historical data. In order to avoid the opportunity cost of maintaining the capital in the portfolio expecting few huge market movements, the manager can plan an investment strategy on the risk-free rate as the minimum required rate of return. At the end of the time period, the amount of capital that was not needed because there were less hedges than expected, has preserved the time value of money.

2. MOTIVATION

This analysis surges from the necessity to create an indicator for market risk without having the disadvantages that the ß coefficient presents, especially for high volatile securities, which traditional theory seems to avoid. Besides the controversial meanings that the ß coefficient has, the investor community ought to use the definition that Sharpe attributed by its time. Every institution that has priced the cost of capital or calculation of required rate of return of an investment has the CAPM in their schedules and Sharpe definition of ß coefficient. Most examples used are made by securities positively linearity related to the benchmark which do not cause any contradiction with the predetermined definition and are easy to understand.

Nevertheless, this method is acceptable in early stages of financial education but harmful in advanced ones yet in capital markets there are many securities that differ from *idyllic* characteristics. Seldomly one can hear ß coefficient in a derivatives class because, as the word states, derivatives rely on the underlying asset and are more complicated when pricing them. Beyond this tough division between financial derivatives and financial assets, surge these types of financial asset which include both worlds: The Exchange Traded Products (ETP). If our commitment occupies, these ETPs rely on Short-Term Futures which simultaneously depend on the VIX Options.

The fact that the payoff of these three-level ETPs is far away from being linear describes the first characteristic that reduces the certainty of ß coefficient calculation. The ß coefficient is the slope of a linear regression and, by definition, it specifies a linear dependence. The second round is about the contrarian direction that ETPs' returns follow regarding the performance of S&P500 Index as benchmark. This direction can be explained by the Pearson correlation coefficient and, surprisingly, it figures in the ß coefficient formula. Since the VIX Options represents the expected volatility at one month, the Short-Term Futures, which also figure the price of that VIX options at one month, stay in a different moment in time regarding the ETPs. ETPs' price is based on the current prices of these futures and indirectly on those options.

Calculating the exact payoff of these ETPs can be such a complex procedure yet it involves many non-linearity factors related between them and in different timing. Following this line, assessing their market risk would be the same penalty because ß coefficient presents controversial issues. Although ß coefficient is not a proper gauge when calculating ETPs’ exposure to market risk, the financial private firm includes this measure in their private and public factsheets. This action is totally legitimate because ß coefficient is the most extended market risk measure and it does not need advanced courses in mathematics to understand and synthesize the information it outputs.

Withstanding this fragile argument about including ß coefficient in their factsheets, it crosses the threshold of academic thoroughness. The contradiction between Sharpe’s definition about ß coefficient and the exposed results are the main objective of this thesis. Once identified the problem it must be complemented with a solution which is exposed in detail throughout these paragraphs. Going further away of just creating and crafting a new indicator based on traditional ß coefficient’s formula, it focuses on assessing the expected maximum market risk exposure of the security until the last pricing day available.

3. DATA AND METHODOLOGY

The database utilized in this empirical assignment is Thomson Reuters with all rights reserved to the Autonomous University of Barcelona. The validity of the data set corresponds to Thomson Reuters.

The information that refers to the profiles of Volatility ETPs are retrieved from their corresponding public prospectus and those issued by Thomson Reuters. The analysis has been done over financial series from 2014 to 2018 (both included) capturing the daily prices of six ETP Short-Term Futures of VIX: UVXY, VIXY, PHDG, VXX, TVIX and SVXY. Besides the products listed before, it was appropriate to also include the VIX Index itself and the S&P500 Index to give TARV a wider application to a non-linear payoff derivative as indexes. The neutrality in TARV's units of measure, which extends its universality, can be applied on securities and indexes thanks to it being a relative value.

Since volatility markets have experienced two collapses in 2018, the first on February 5th when Credit Suisse retired the issued Velocity Shares Daily Inverse VIX Short-Term (XIV) and the second when it deleveraged SVXY and UVXY on February 27th. As a result of the collapse, on February 6th the log return of SVXY was -176.95% because the price changed from $287.28 to $48.96. Since the issuer accepted this sharp change and the price stabilized on that level, it is not included in this analysis. Originally it was included but increased the TARV to irrational levels, yet it is very sensitive to extreme movements. Instead of -176.95% it is replicated the return from February 7th to February 8th, that is, 0.73%. The case of UVXY is smoother because on February 6th the log return was about -40.72%. Indeed, it is a great downward if it is equated with equity but in volatility ETPs it is considered moderate rather than a big movement.

In this analysis the S&P500 Index is used to refer to the market portfolio although the securities analysed, the volatility ETPs, are not components of it. One of the CAPM assumptions is the inclusion of all type of investments. So the ß coefficient does not feel comfortable with contrarian and non-linear payoff derivatives as underlying asset of ETPs.

The entire analysis has been done on daily basis data in order to prove the capability of TARV in the shortest time period as daily basis is. The indicator can be calculated daily, but its statistical relevance is weak regarding the same calculation over five years backward as time horizon. Since TARV and RV are a quotient of two standard deviations, they are annualized as volatility is normally treated. This daily-to-yearly transformation vanishes due to the division, thus both TARV and RV results are treated as annualized results.

In the aim of coinciding with the assumption of normal distribution of returns, therefore, log-normal distribution of prices, the daily change of prices has been calculated by natural logarithms reaching additive and symmetric properties, reduction of heteroscedasticity and serial autocorrelation. All the prices are taken at the closing of the session.

There is a trade-off between having enough data for statistical significance and too much data so the TARV does not properly reflect the current market conditions. According to the general rule of five years, this analysis focuses on the period 2014-2018 rather than the period 1990-2018 due to the lack of available data in many studied securities and for the current market data representation mentioned above. The purpose of the long backward moving in time is to demonstrate the ability of TARV as a systematic crisis indicator apart from already being a properly indicator of market risk exposure.

This study gives priority to short-term ETPs rather than mid-term ETPs because of the higher frequency of rolling over making them more volatile rather the mid-term ETPs. The fact of implementing TARV to volatile securities which might be applied for and against the market, that some of them are leveraged (1.5x and 2x) and in one case for hedged non-linear payoff derivatives, plus forcing the applicability on indexes (S&P500 Index and VIX Index) it is a complex aggregated scenario for TARV. The main intention is to figure a market risk exposure for alternative derivatives and scenarios that escape from traditional boundaries. In other words, to prove TARV with high volatile and atypical securities. Besides this complex situation, TARV can perform satisfactory results across assets and time.

Some financial public web pages exhibit positive and near-to-one ß coefficient of the listed securities above. Despite these public places aware lectors not to make investment decisions regarding their published data, it is still a huge mistake to compute positive or almost unity ß coefficients when in the description of some listed securities clearly figure the word VIX Index. Every security related to VIX cannot be positive correlated with the S&P500 Index in any case.

Each table, figure and calculation of this document is self-created, from Thomson Reuters close price data. Once that is noticed, the lecturer can expect to find the source of information below the tables and figures since every media is self-created and has retrieved basic data from Thomson Reuters.

4. DESCRIPTION

The Capital Asset Pricing Model (CAPM) is a linear risk model used to output the relationship between market risk and expected return for financial assets through expected variables such as the expected return of an investment, the risk-free rate and the well-known β parameter. Likewise, the β of a financial asset is the (linear) proportion of market risk that a specific asset is exposed to, and it is the slope of the linear regression. There is a positive relationship between β and the expected return of an investment; the greater β is, the greater the expected return will be because of the market risk premium. The parameter β would vary around one depending on how much the asset is exposed to market risk; above the threshold of one indicates that the specific asset is riskier than the analysed market. The same meaning in the other way around if the parameter β is under the market threshold. In the case that the asset's market risk exposure is as low as zero it would induce us to diminish the market risk premium, because it is nonsense to include such reward to market risk. That would let the CAPM solely with the constant risk-free rate. One step forward could be questioning the relationship between β and expected return if the derivative's behaviour is highly contrarian to market's variability in the short term; in the same way, as covariance as correlation coefficient with negative sign. It would be the case of a hedging operation in temporary market turmoil period such as ETPs over Short Term Futures of VIX. Holding such characteristics, the parameter β will not even include value since it is negative, but instead it will deduct a determined quantity from the risk-free rate return, yet the market risk premium will also turn negative. The output is a negative expected return for the investor, who will not hesitate to directly invest funds in the risk-free rate before taking a long position in *something* that has negative expected return. The fact is that a derivative with negative covariance and negative correlation coefficient regarding the market does not imply that there will be less risk than in the market because of the negative β and, therefore, negatively affecting the market risk premium and the final expected return of the investment. Following the CAPM, every contrarian derivative to market's behaviour will have negative expected return if the risk-free rate is lower than the price of market risk.

Regarding the theory, a β below one, not specifying if remaining above or below zero, is a value of conservation which is exposed to less market risk than an asset that has a β higher than or equal to one. This leads to fence β in all positive real numbers including the zero. Although β being equal to zero would mean that the covariance of the asset regarding its market is also zero and the asset returns are completely an independent vector if the correlation coefficient is null too. Then, plotting both vector returns, the asset would vary much more than the market even though having virtually zero β. The lack of covariance does not lead to conclude there is no variance, in other words, market risk. Since avoiding negative β from the linear regression point of view will not be totally correct because not the parameter β describes the slope degree in the linear equation. A negative slope is completely financially plausible because the relationship, as mentioned before, can be either positive or negative. In order to downgrade the portfolio's market risk, it is preferable to include securities that are partially or totally negatively correlated among them. Going forward, allocating funds to hedge long positions in case of reverse market movements. Since the optimal relationship among securities in the portfolio owns negative correlation and negative covariance it will result in a negative slope (β) in the CAPM, which is totally deserved.

In fact, the slope of the linear regression and the traditional β formula yield the same result. It seems there is a controversial reality. On the one hand, there exists a security that varies more than the market but, following the traditional formula, it ought to have minimum market risk exposure due to the negative β. A volatile contrarian asset could be interpreted wrongly as a refugee value if solely assessing by the negative sign of its β. So, of this restricting β's domain in the universe of positive numbers including the zero is still financially plausible but will not be totally correct when applying a linear regression. On the other hand, it is correct that a contrarian volatile asset has negative slope regarding the market because their returns are negatively correlated and vary in different ways. Hence, plotting the asset and market returns in a linear regression will output negative slope which will have the exact same result as using the traditional formula.

Extrapolating the widen spread meaning of ß's coefficient to a real situation, one could be: The exposition being shot equals to a positive ß. Hidden in a bunker equal to zero ß, because of the risk to be shot is zero. Could one be negatively exposed to be shot? It is possible if you are overly protected. Then, negative exposure to be shot equal to conservationist securities again and not to contrarian-market linear movements.

The first solution that springs is to interpret negative ßs in absolute value and conclude that if a security has negative slope in SML line is because its behavior is linearly contrarian with the market. Mathematically, it is demonstrated that ß's domain is all the real numbers and, of course, it can be negative. Moreover, if the relationship between the security and the benchmark is not linear, ß's sign meaning loses sense and it requires to focus our attention on the range of change rather than in its direction.

If the main concern is to compute the RV of an investment regarding the volatility of the market, it would be more appropriate to keep aside the formula of ß's and focus the attention on the RV (RV). Since it is the quotient of two standard deviations, the result will always be positive and will solely settle in real positive numbers. The RV measures the range of variability. Therefore, if the RV defends the real definition of market risk exposure and solves the interpretation problem of negative sign, it seems all is tied up.

Either ß or RV are linear measures of market risk, so they take the arithmetic average of price oscillations. In the security universe of non-linear payoffs, taking the arithmetic average could be a mistake of risk undervaluation. Focusing on the computing, the maximum exposure to market risk until the last pricing day available is the main objective of the Trigonometric Adjustment on RV (TARV).

4.1 Volatility ETPs

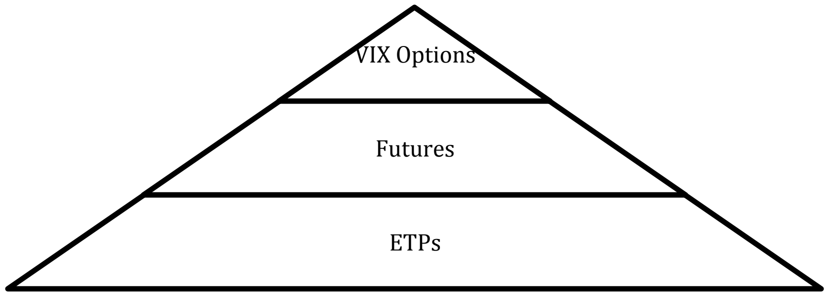
The hedge volatility typologies over investments are structured to cover temporary volatility peaks or to construct equity market neutral VIX portfolios among other well-known strategies. Thus, the VIX Short-Term Futures Index rolls from the one-month VIX Futures contract into the two-month contract maintaining a constant one-month maturity. The VIX Mid-Term Futures Index are four-through-seven-month VIX Futures maintaining a constant five-month maturity. All VIX Futures Indexes are affected by roll costs or roll gains in rolling from shorter contracts to longer contracts, depending on whether the term structure in the VIX Futures market is *contango* or *backwardation* respectively. The basis of volatility hedges is that the VIX Futures are highly negatively correlated with the S&P500 and highly positive correlated with the VIX Index. Then, in the context of market turmoil it is convenient to take advantage and allocate a small part of the portfolio to long VIX Short-Term Futures or volatility ETPs.

The structure of these VIX ETPs can be described as three surfaces. The top level and the inception of payoffs' non-linearity is the option of the market of the S&P500 Index, which trade its averaged 30-day implied volatility and are included in the S&P500 Options portfolio. The non-linearity implies that the payoff depends on the expiration time and the space whether the option is OTM, ITM or ATM[1]. This Index estimates the expected volatility directly from the weighted prices of the S&P500 puts and calls covering a range of strike prices. The entire value of the S&P500 Options portfolio is widely known as the Volatility Index (VIX). A similar definition is found for VIX Index, "Bargadett, Gourier and Leippold stay 'the VIX Index non-parametrically approximates the expected future realized volatility of the S&P500 returns over the next 30 days' (2016:593)." Also "Bargadett, et al. stays 'we need a model that is flexible of both markets over time, but the empirical analysis of such highly nonlinear data poses a significant computational hurdle' (2016:594)." The VIX Index is quoted in percentage points and performs the expected annualized range of oscillation in the S&P500 Index with approximately 68% of probability.

The prices of these options and the VIX maintain a linear relationship between them, but not with the S&P500 Index. The second ring, in this case, represents the Short-Term Futures on the S&P500 Options market. As well known, standard type of Futures has lineal payoffs but, on account of that they trade non-linear payoffs, its function turns non-linear too. The deepest surface belongs to ETPs that usually stand for linear payoffs. Once again, the fact that the core of this figure has non-linear payoffs affects the other levels and turns them into non-linear payoff functions.

[1] Out of The Money (OTM), In The Money (ITM) and At The Money (ATM).

Figure 2. ETPs structure over Futures on VIX Options.



Source: Information retrieved from Thomson Reuters (2019).

4.1.1 Profiles

In this section all the volatility ETPs used for his empirical analysis are explained. All of them are Exchange Traded Funds (ETF) except for the iPath Exchange Traded Notes S&P500 VIX Ftrs A (VXX) which is an Exchange Traded Note (ETN). Both securities are very similar and share low expense ratios. In the case of the ETF, the investment is into a fund that holds positions in financial securities. While for an ETN is an unsecured debt note usually issued by a private institution. This security can be held to maturity. As bonds, in case of bankruptcy of the issuer, the investor is exposed to default risk and credit risk plus market risk as the ETFs. The main difference between ETNs and ETFs is that ETNs does not have tracking error regarding the market.

Volatility ETPs which are long in S&P500 SPVXSP have negative ß coefficient either at one year, three years or five years. Regarding CFA Institute and classical theory, the most reliable ß coefficient is the one that uses data for five years and embraces more statistical significance. The other volatility ETPs which are short in S&P500 SPVXSP, stand for a positive ß coefficient in all the periods.

It is interesting to stress the diversity of volatility ETPs listed above, specially Invesco S&P500 Downside Hedged ETF (PHDG), which is the hedged one. Obviously, there are much more volatility ETPs offered in the market than the listed ones. This analysis focuses on Short-Term Futures and, therefore, on the ETPs that track these Futures. Volatility ETPs that track Mid-Term Futures are ZIV, XVIX, TVIZ, VIXM and VIIZ, to cite a few.

Table 1. Main characteristics of volatility ETPs used in this analysis.

Fund	Goal	Distinctive item	S&P500 SPVXSP position	Exposure to market risk		
				1Y	3Y	5Y
UVXY	Investment results that correspond to 1.5x the performance of the S&P500 SPVXSP during a day	1.5x leveraged on S&P500 SPVXSP	Long	-5.09	-6.56	-7.66
VIXY	Investment results (before fees and expenses) that match the performance of the S&P500 SPVXSP	Standard exposure to S&P500 SPVXSP	Long	-3.64	-4.02	-4.33
VXX	Exposure to daily rolling long position in the Short-Term VIX contracts and reflects the implied volatility of the S&P500 Index at various points along the volatility forward curve	Implied volatility of the S&P500 Index	Long	-3.8	-4.13	-4.40
TVIX	2x the daily return of the S&P500 SPVXSP	2x leveraged on S&P500 SPVXSP	Long	-6.48	-7.31	-8.13
SVXY	Investment results that correspond to -0.5x of the S&P500 SPVXSP during a day	0.5x contrarian moviment of S&P500 SPVXSP	Short	6.97	6.27	5.83
PHDG	positive total returns in rising or falling markets that are not directly correlated to board equity or fixed income market returns. The returns correspond to the performance of the S&P500 Dynamic VEQTOR Index	Hedged	Short	N/A	0.62	N/A

N/A: lack of data. *Source*: Information retrieved from Thomson Reuters and self-created (2019).

4.2 Statistical Outline

The highest movement of VIX from 1990 to 2017 was achieved on 27th of February in 2007 pretty before the sub-prime crisis with hidden evidences of what was coming and just a few expected. The sharp spike of VIX was isolated in that day and achieved a 50% return regarding the previous day. Among all the factors which moved the market that day, the paramount ones where concerns about Chinese interest rate raising and publication of US economy data. The externality over the S&P500 Index was a backward of 3.53%. The majority would have expected a large negative movement in the S&P500 Index since the VIX experienced the biggest one. This statement is partially certain because of the non-linearity dependence, condition the VIX owns with the S&P500 Index. Narrowing the temporal horizon of analysis, from 2017 to 2018 has produced the greatest movement in history of VIX price performance on 5th of February in 2018. It could be interesting to conduct a study regarding the delicate relation of VIX with February.

Table 2. Linear estimation across securities from 2014 to 2018 over logarithmic returns.

2014-2018	2014	2015	2016	2017	2018	5 Years
S&P500 Index	1	1	1	1	1	1
VIX Index	-0.0818	-0.0991	-0.0871	-0.0458	-0.0867	-0.0831
UVXY	-0.0885	-0.0935	-0.0891	-0.0547	-0.1058	-0.0899
VIXY	-0.1749	-0.1841	-0.1751	-0.1097	-0.1927	-0.1750
PHDG	0.8164	1.0953	0.8539	N/A	N/A	N/A
VXX	-0.1756	-0.1854	-0.1758	-0.1097	-0.1922	-0.1755
TVIX	-0.0916	-0.0952	-0.0910	-0.0553	-0.0934	-0.0888
SVXY	0.1732	0.1765	0.1618	0.1034	0.1965	0.1674

N/A: Lack of data. *Source*: market quotation retrieved from Thomson Reuters and self-created (2019).

Despite the high volatility that volatility ETPs own, traditional ß coefficient is not able to capture its high-frequent changes due to almost zero covariance and negative correlated returns between the securities and the market. Regarding Sharpe's instructions, the audience could say that most of the securities listed below are *defensive stocks* because their ß coefficient is below the unity and zero.

Table 3. Principal statistical ratios.

2014-2018	Annualized Arithmetic Average	Annualized Stantard Deviation	Daily Arithmetic Average	Daily Median	Skewness Coefficient	Kurtosis Coefficient
S&P500 Index	0.0629	0.1325	0.0002	0.0004	-0.4957	3.7856
VIX Index	0.1163	1.3032	0.0005	-0.0052	1.3117	8.5496
UVXY	-1.5368	1.2198	-0.0061	-0.0103	0.6719	4.8684
VIXY	-0.5440	0.6386	-0.0022	-0.0055	1.1223	5.3227
PHDG	-0.0089	0.0924	0.0000	0.0002	0.2704	12.7331
VXX	-0.5406	0.6357	-0.0021	-0.0053	1.1271	5.3190
TVIX	-1.5845	1.2420	-0.0063	-0.0105	0.5936	5.5016
SVXY	0.1263	0.6330	0.0005	0.9992	-2.1299	13.7166

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

The high skewness and kurtosis coefficient track the extreme returns, which in this case appear with more probability than expected. The probability distributions of volatility ETPs under analysis are far from being symmetric, thus, they do not follow an elliptical multivariate probability distribution. Daily arithmetic average and daily median are not equal, and, in some cases, there is significant difference. This increases the skewness coefficient and induces again that these volatility ETPs do not follow an elliptical multivariate probability distribution. The performance of volatility ETPs have the characteristics of kurtosis and skewness coefficient from an exponential distribution, a specific case of gamma distributions. The high values of skewness and kurtosis' coefficients are due to SVXY having downgraded almost its entire value at the inception of 2018. As volatility markets have experienced two collapses in 2018, on February 5th Credit Suisse retired the issued Velocity Shares Daily Inverse VIX Short-Term (XIV) and deleveraged SVXY.

Outlined in the assumptions, classical financial statements hold elliptical probability distributions[2] for returns that are independent and identically distributed. Therefore, calculating Pearson's correlation coefficient is financially plausible because of the elliptically distributed underlying data. The point is to establish stochastic stationary process that promotes random variables. Beyond theory, financial tradable assets have an autocorrelation and a heteroscedasticity that invalidates the stochastic stationary process. Not having independent variables appears to be something bad, but it is actually truly good for linear dependence measures as Pearson's correlation coefficient as covariance. There is no purpose when calculating the correlation coefficient between two variables that are not independent. It can bring to causality confusions.

This scenario is done in order to emphasize the application of the limited universe of ß coefficient, therefore, CAPM meaning when it is compared to the trigonometric adjustment on the RV. As explained before, it is possible a negative ß coefficient, but it is not financially plausible when the market and the security do not hold a linear relationship. When the market goes down in one percent, the security can be in steady state or grow a three percent due to its non-linear features of the options as the underlying asset of the Futures. The ß negative coefficient stays as the arithmetic average and slope of observations which can either have a linear or non-linear relationship with the market. The fact that this ß coefficient solely plots the average slope, the distance of the extreme movements of the market and the regression line are left to increase the standard error of the regression.

4.3 Applications

Once stated qualitative and quantitative characteristics of these ETPs, one can determine that a portfolio with no other security would attain huge negative historical returns due to the nature of these securities. The majority track the VIX Index and daily achieve huge market movements that can also require huge amounts of capital requirements to keep open investors' position in the market. Far from risk adverse investors, this portfolio can be perfectly arising whole of volatility ETPs but would not be the generalized case.

[2] Univariate or multivariate probability distributions.

Yet a portfolio full of ETPs is seldomly created, the operations involving volatility ETPs as a mechanism of hedging positions are increasing. In period of market turmoil, it is very useful to have a large position in cash and some percentage of the portfolio allocated in volatility ETPs such as the ones that figure in this analysis. Including these high-volatility securities in an equity portfolio can reduce the exposure to market risk even reducing the specific risk to the maximum achievable in a well-diversified equity portfolio. Therefore, the expected view of the investor is regarding the volatility ETPs as a mechanism of hedging rather than the main investment in the portfolio. As much *hackneyed subject* it could sound, there are infinite combinations of portfolios with diversification in volatility ETPs (i.e. combination of PHDG and UVXY in the same portfolio in order to dynamically hedge different securities). Beyond the classical hedging methodology through derivatives, specially synthetic options, these volatility ETPs provide more diversification in terms of securities involved (i.e. they count on options on the S&P500 Index, short-term futures over that options and finally Exchange Traded Products that rely on those futures, which at the same time perform according to the S&P500 options joined in a portfolio named VIX Index).

4.4 Other Measures of Linear Relationship

4.4.1 Coefficient of variation (CV)

The Coefficient of Variation (CV) could be chosen instead of the RV ratio as a measure of dispersion and homogeneity of returns. The CV normalizes the dispersion by using the average as a scale measure. Estimation of CV using the ratio of the sample standard deviation to the sample mean:

$$CV_t = \frac{\sigma}{|\bar{x}|} \cdot 100$$

Regarding *Table 4*, daily arithmetic average data is around zero for log returns and around one for cosine log returns. If one substitutes these daily data into the CV, the result could not be as expected. In the specific case of daily cosine log returns, the CV would be the standard deviation itself multiplied by one hundred. Such homogeneous transformation does not provide any relevant information. The evidence of log returns either annualized or daily is that the observations highly differ from their average given in absolute terms. Given these extreme values of CV from log returns although they are annualized and the linear relationship that CV assumes, it is appropriate to modify the CV with the application of the cosine. The results are interesting enough to centre this analysis in commenting them. Here arises the question of why not to apply the trigonometric adjustment on CV instead of RV. Since the market risk is the one that cannot be eliminated with diversification, the maximum expected exposure to it is a paramount concern among all risk and portfolio managers. The RV, as the name exposes, *relative*, is a measure of comparison with a benchmark of reference. The CV is simply the standard deviation per unit of average. The idea is to know, or at least, to have an estimation of which is the maximum expected exposure to market risk of a security and not to know how this security differs from its average. Yet this last question is the specific risk or security's volatility and it can be reduced throughout holding a well-diversified portfolio.

Table 4. Comparision of CV between log returns and cosine log returns.

2014-2018	Annualized CV		Daily CV	
	log returns	cosine log returns	log returns	cosine log returns
S&P500 Index	210.7213	0.0005	3,345.0970	0.0083
VIX Index	1,120.4551	0.0670	17,786.6732	1.0634
UVXY	79.3714	0.0476	1,259.9824	0.7555
VIXY	117.3950	0.0135	1,863.5880	0.2144
PHDG	1,034.1731	0.0004	16,416.9895	0.0065
VXX	117.5999	0.0134	1,866.8410	0.2124
TVIX	78.3799	0.0516	1,244.2422	0.8188
SVXY	501.0470	0.0196	7,953.8749	0.3112

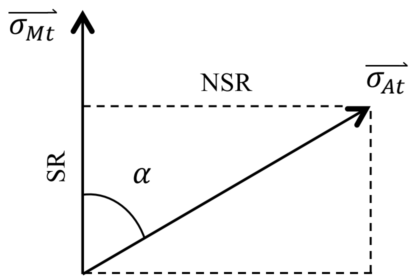
Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Albeit TARV and CV are mostly used monthly rather than daily, it is interesting to stress the extreme value of CV (and the lack of meaning when reaching these extreme values) in its application on daily data. The trigonometric adjustment on CV, CVTA, respects the logarithmic scale of observations on daily data. Beyond the difference of monotonous modification between CV and CVTA, if one calculates the first difference between daily CV and annualized CV on log returns and cosine log returns the same result would arise.

4.4.2 Geometric vision of Pearson's correlation coefficient

Let two random variables be $A_t(A_i, \dots, A_n)$ and $M_t(M_i, \dots, M_n)$ standing $\{A_t, M_t \in \mathbb{R} : A_t, M_t > 0\} \; \forall \infty > n \geq i > 0, \infty > t > 0$ while performing the prices of a determined security, A_t, and the S&P500 Index, M_t, simulatenously at time t. Lets consider their standard deviations $(\sigma_{At}, \sigma_{Mt})$ as vectors in a space at n dimensions, $\overrightarrow{\sigma_{At}} = \{\sigma_{Ai}, \dots, \sigma_{An}\}$ and $\overrightarrow{\sigma_{Mt}} = \{\sigma_{Mi}, \dots, \sigma_{Mn}\}$.

Figure 3. Angle of systematic and non-systematic risks.



Source: self-created (2019).

The coefficient α is the Pearson's correlation coefficient (ρ) that joins the market returns ($\overrightarrow{\sigma_{Mt}}$) and the security returns ($\overrightarrow{\sigma_{At}}$). The systematic risk (SR) is the adjacent leg, which is calculated by multiplying the cosine of α by the vector of standard deviations of the security ($\overrightarrow{\sigma_{At}}$). The specific risk or non-systematic risk (NSR) is the opposite side of the angle that is calculated by multiplying the sine of α by the vector of standard deviations of the security ($\overrightarrow{\sigma_{At}}$).

Both systematic and non-systematic risk can be expressed in linear or in non-linear way as follows:

SR: $\overrightarrow{\sigma_{At}} \cdot \cos(\alpha) = \overrightarrow{\sigma_{At}} \cdot \rho(\overrightarrow{\sigma_{At}}, \overrightarrow{\sigma_{Mt}})$

NSR: $\overrightarrow{\sigma_{At}} \cdot sine(\alpha) = \sqrt{\overrightarrow{\sigma_{At}}^2 \cdot \left(1 - \rho(\overrightarrow{\sigma_{At}}, \overrightarrow{\sigma_{Mt}})\right)^2}$

Despite the geometric vision, from mathematical analysis' point of view, Pearson's correlation coefficient continues to be a lineal measure of relationship of dependence between two variables. Likewise, the assumption of elliptical probability distribution still needs to be satisfied when measuring the relationship of dependence throughout Pearson's correlation coefficient. "Francis conducted an empirical analysis about the stability of ß coefficients and concluded, 'the correlation with the market is the primary cause of changing betas, (…) the standard deviations of individual assets are fairly stable' (1979:994).". Therefore, RV and TARV do not include the correlation coefficient in their calculations.

5. MATHEMATICAL MODEL

5.1 Assumptions

1. Capital markets classical assumptions:
   a. Incomplete information.
   b. Market inefficiency.
   c. Discrete time and prices.
   d. Returns are independent and identically distributed (stationary process).

2. TARV can be applied on all tradeable market securities and indexes.

   Like the CAPM, this indicator can also include all market securities. TARV is specially created for non-conventional securities but it does not mean that the investor community cannot use it for conventional ones. The expected maximum exposure to market risk is the same concept for all tradable assets and if TARV outputs plausible results when applied on complex securities, it can be assumed that it will behave in the same way with simple ones.

   Beyond including all tradable market securities, it can also include market indexes that hold either positive or negative linear or non-linear relationship between them. For instance, in this analysis it is compared the S&P500 Index with volatility ETPs, which would be the case of non-linear relationship between both divisors. Every component of S&P500 Index that has a linear relationship with it will hold a non-linear one when regressed with the VIX Index. Expanding the horizon of study, TARV could be applied on European indexes like the VSTOXX and EURO STOXX 50.

   TARV could also be applied on a tradable fixed income with active secondary market regarding a benchmark of reference. Originally TARV is planned to be applied to securities or indexes with high volatility and abnormal or specific characteristics. TARV overvalues the maximum expected exposure to market risk when the security has high volatility regarding the benchmark. As opposed to this, TARV undervalues the maximum expected exposure to market risk when the security has low volatility regarding the benchmark.

3. Daily expected value is zero for RV and one for TARV.

The daily expected value of securities and benchmark performance is zero when calculating returns throughout first differences to implement them into RV formula. Mathematically, $\mathbb{E}(a_t), \mathbb{E}(m_t) = 0$. The daily expected value of securities and benchmark performance is one when apart from calculating returns throughout first differences, it is added the cosine. This modification makes TARV's output possible. Matematically, $\mathbb{E}(\mathbb{a}_t), \mathbb{E}(\mathbb{m}_t) = 1$. The following empirical evidences corroborate what is assumed beforehand.

Table 5. Daily arithmetic averages of volatility ETPs, VIX Index and S&P500 Index.

2014-2018	Daily Arithmetic Average	Daily Arithmetic Cosine Average
S&P500 Index	0.0002	1.0000
VIX Index	0.0005	0.9967
UVXY	-0.0061	0.9970
VIXY	-0.0022	0.9992
PHDG	0.0000	1.0000
VXX	-0.0021	0.9992
TVIX	-0.0063	0.9969
SVXY	-0.0009	0.9992

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

4. When the security a_t at time *t* is more volatile than the benchmark m_t at time *t*, TARV is considered as a multiple because its value exceeds one standard deviaiton. When the security a_t at time *t* is less volatile than the benchmark m_t at time *t*, TARV is considered as a ratio because its value is below one standard deviation.

In this statement it is assumed that the benchmark would be less volatile than the security, its returns would oscillate closer to its expected value and individual change would produce an infinitesimal effect. From security's point of view, its returns would be more volatile than the benchmark ones and, thus the individual change would describe in a better way the security's behaviour rather than its expected value.

This is assumed because it is preferred a different RV as well as a different TARV than the unity to provide informational advantages on its application. The unity case would be produced in the scenario of comparing the benchmark with an ETF that perfectly tracks benchmark's returns. There is the trivial case of comparing benchmark behaviour with itself, which evidently, will output the unity. Beyond these neutral examples, there is the real application of both RV and TARV. The important element here is the inequality between the numerator and the denominator. However, it is assumed that security would be the most volatile it can be considering the benchmark. This case could be reached by making the inverse of security-is-the-most-volatile assumption. The situation is realized when one is comparing the VIX Index with one of the volatility ETPs listed above.

5. The exposure to market risk is positive or zero. Matematically, $RV_t \geq 0$ and $TARV_t \geq 0$.

Since TARV incorporates a quadratic function, it will always be strictly positive. The sign of the correlation of the securities regarding the reference market does not plan any problem when assessing the RV in extreme scenarios. Provided that RV is, as well as TARV, a quotient of standard deviations it cannot adopt values below zero in any case. Besides it is mathematically demonstrable that ß coefficient can reach negative values because of symmetry, it has a lack of financial meaning when one attempts to figure negative market risk exposure. Once the market risk is hedged, market risk exposure goes to zero and remains there.

5.2 Description

The Relative Volatility (RV) or the volatility ratio is the quotient of the standard deviation of a financial asset at time *t* and the standard deviation of the market portfolio or benchmark also at time *t*. As described before, it compares the degree of volatility of rates of return. Setting the greatest volatile data as the numerator and the lowest volatile data as the denominator, a RV bigger than one means that the upper security is more volatile than the lower security. For RV less than one is the other way around. In the case of neutrality, that is, when the security and the market have the same volatility, the quotient will equal to the unity. The RV quotient maintains the symmetry because it treats the changes of asset's returns in the same way as market's returns. Likewise, another advantage is that RV does not include Pearson's coefficient of correlation between the asset and the market leading to more stable ß coefficients over time.

Citing Sharpe's definition of market risk exposure, his interpretation of ß is consistent with RV formula. Therefore, using the RV instead of ß coefficient to measure the exposure of an investment to its market risk would be the appropriate way.

Normal market conditions refer to the expected value, which is zero in the case of daily returns and one in the case of daily cosine returns, as for the security and the benchmark. This expected value can be used as a significant referent point when assessing the RV under normal market conditions. Shifting the normal scenario to an abnormal one, most total daily returns are far away from the expected value even if this expected value was calculated by the volatile returns. Either positive or negative remarkable market movements are engendered in the market, and they increase the standard deviation of daily returns in a greater manner in the case of the security and with lower implication in the case of the benchmark. The degree of diversification of idiosyncratic risk is what prevents the benchmark from suffering mentioned spike movements. Since it is preferable to analyse the RV regarding the market, that is, to avoid dividing market situations into extreme and normal types, both RV and TARV are implemented from 2014 until 2018 in the case of S&P500 Index and VIX Index. In the case of securities, their studying period also goes from 2014 until 2018. During these five years there are multiple and more complex scenarios that go beyond the classical normal-abnormal binary distinction.

Creating an algorithm that sorts all these situations is beyond the scope of this article. Nevertheless, giving a new direction to the possibility of sorting all market situations is plausible and to simultaneously allocate more weight to extreme returns and subtract weight from returns that are very similar to their expected value. This expected value is calculated using all the elements of the sample, that is, including outliers and normal values. Here surges the necessity to find a function that allocates neutral values to low volatility performances and high values to the ones that are the most volatile. Mentioned procedure is similar to the calculation of a classical weighted average through a system of equations that finds the desired values. From this point of view, TARV could be interpreted as a non-parametric indicator since it allocates different weight depending on the volatility of the returns.

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[3] In this context, an *outlier* is a correct data that represents an uncommon event.

[4] Recovering previous information, β coefficient can be calculated through linear regression (OLS method), traditional formula and trigonometric identities. It can either be plotted by a straight line or geometric representation. Besides there is the option of calculating β coefficient through trigonometric identities, the result will equal to the others and, therefore it will stand the same problems of instability and linear dependency.

TARV follows the same structure of RV because both are the quotient of two standard deviations. Throughout time, the standard deviation is more stable than the correlation coefficient used in ß coefficient's formula. TARV avoids using both correlation coefficient and covariance in its calculation, the securities and the benchmark can be independent variables, or close to be, accordingly it focuses on solely using the standard deviation. In the same line, the randomness of the variables is welcomed, and the indicator could be developed in further studies in case of having cointegrated prices in the securities. Nevertheless, TARV shows more sensitivity regarding market movements rather than RV while provoking instable coefficients throughout time. But this sensitivity is what is actually searched when using TARV instead of RV: taking into account the most relevant market movements than RV does. TARV overvalues its market risk exposure at expenses of less stability in its coefficients.

The result of TARV is the expected maximum market risk exposure provided that the options from VIX's portfolio have an average maturity of thirty days in advance. Therefore, the VIX Index tracks the expected volatility over the S&P500 Index for the following thirty days. TARV counts on the arithmetic average of most significant movements in calculations. This induces that in presence of spike market movements, it is expected to have TARV exposure to market risk according to past data. Furthermore, the formula of TARV acts like normalization as the CV because TARV's output can be expressed as the number of standard deviations the security has regarding the benchmark.

5.3 Calculations

Beyond the geometric interpretation, this analysis focuses on the analytical arguments to defend the trigonometric modification on the RV.

Regarding the Taylor-McLaurin Series, let $P_{a,f,n}(x)$ be a polinomyal so that it is a Taylor expansion settled in zero.

$$P_{a,f,n}(x) = f(a) + df(a)(x-a) + \frac{d^2 f(a)(x-a)}{2} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Following the structure and developing the expansion, the cosine can be expressed as follows

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

Since the cosine is cuadratic, the Taylor-McLaurin series will end in the second order expansion and they are settled in zero, a=0. The error o(x) is linear and dwells in $df(a)$.

$$\cos(0) + \big(-sine(0)\big)(x) - \left(\frac{\cos(x)}{2}\right)(2x^2 - 0) = 1 + 0 - \frac{x^2}{2}$$

$$1 - \frac{x^2}{2} \approx x + o(x)$$

An o(x) of a given function x, is a small part of that function x. Let K(h) and J(h) be any two functions. The limit between K(h) and J(h) when h tends to a particular point it is zero. While h tends to 0, K(h) is becoming infinitesimally smaller than J(h). In the same way, K(h) will be a small part of J(h) which means K(h) = o(J(h)). The main objective is to bring the property of that limit to a more copmplex analysis and that small part accumulates the error of the cosine in the proximities of zero and one.

$$\lim_{h \to 0} \frac{K(h)}{J(h)} = 0 \iff K(h) = o\big(J(h)\big)$$

Aknowledging how individual variations will be represented, setting the domain for RV and TARV is the following step.

For RV, let $a_t(a_i, \dots, a_n)$ and $m_t(m_i, \dots, m_n)$ standing $\{a_t, m_t \in \mathbb{R}\} \forall \infty > n \geq i > 0, \infty > t > 0$ while performing normally distributed returns of an asset, A_i, and the S&P500 Index, M_i, simulatenously at time t. Tacking into account that financial series are likely to have heteroscedasticity and serial autocorrelation problems because of well defined trends, the daily returns are calculated by natural logarithms[5].

$$a_i = ln\left(\frac{A_i}{A_{i-1}}\right)$$

$$m_i = ln\left(\frac{M_i}{M_{i-1}}\right)$$

Assuming $\mathbb{E}(a_t), \mathbb{E}(m_t) = 0$, it is expected that a_t and m_t oscillate around its average value plus a small individual variation defined as $o(a_t)$ and $o(m_t)$ that are given by the random variables a_t and m_t. The functions o(a_t) and o(m_t) include all variations.

$$a_t = \mathbb{E}(a_t) + o(a_t) = 0 + o(a_t)$$

$$m_t = \mathbb{E}(m_t) + o(m_t) = 0 + o(m_t)$$

Then, the RV is expressed as

$$RV_t = \frac{\sigma_{a_t}}{\sigma_{m_t}} = \sqrt{\frac{(a_t - \mathbb{E}(a_t))^2}{\left(m_t - \mathbb{E}(m_t)\right)^2}} = \frac{a_t - \mathbb{E}(a_t)}{m_t - \mathbb{E}(m_t)} = \frac{0 + o(a_t) - 0}{0 + o(m_t) - 0} = \frac{o(a_t)}{o(m_t)}$$

The result is the quotient of the small individual changes subject to their random variables. As a linear measure of risk, it takes the arithmetic average of the security and market returns to calculate both standard deviations. This implies that it will consider all the data in the same way; the high changes will weigh the same as the low changes. When the security has high volatility and it is linearly compared with the volatility of the market, volatility spikes can be hidden by the lower oscillations.

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5 When calculating first differences on data that apparently can have heteroscedastic and serial atutocorrelation as financial series, it is convenient to use the natural logarithm to calculate daily returns that approximates to simple return calculations. Defined P_t as the price for day *t*, $l\,n\left(\frac{P_t}{P_{t-1}}\right) \cong \frac{P_t - P_{t-1}}{P_{t-1}}$.

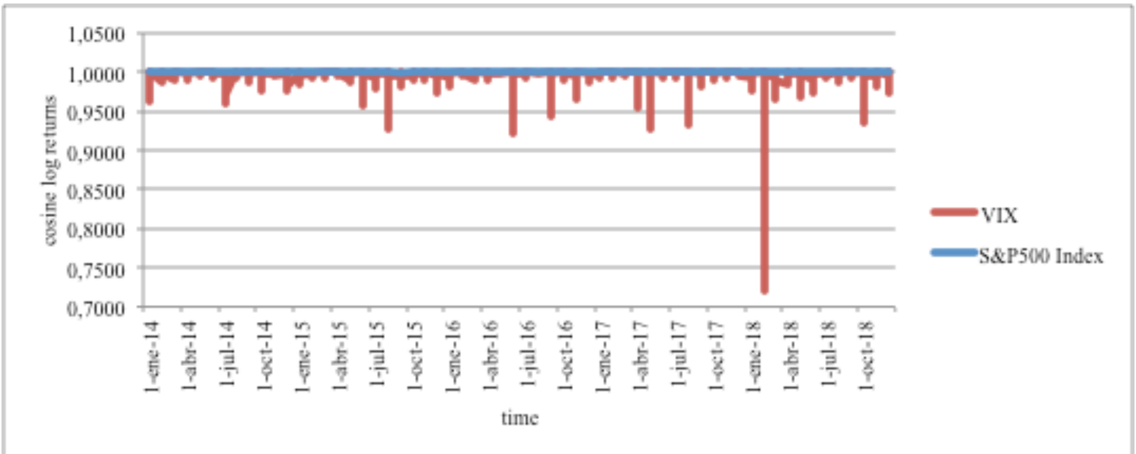
For TARV, let $\mathbb{a}_t(\mathbb{a}_i, \dots, \mathbb{a}_n)$ and $\mathbb{m}_t(\mathbb{m}_i, \dots, \mathbb{m}_n)$ standing $\{\mathbb{a}_t, \mathbb{m}_t \in \mathbb{R} : 0 < \mathbb{a}_t, \mathbb{m}_t \leq 1\} \; \forall \infty > n \geq i > 0, \infty > t > 0$ while performing uniformly distributed returns of an asset, A_i, and the S&P500 Index, M_i, simulatenously at time t.

$$\mathbb{a}_i = cos\left(\ln\left(\frac{A_i}{A_{i-1}}\right)\right)$$

$$\mathbb{m}_i = cos\left(ln\left(\frac{M_i}{M_{i-1}}\right)\right)$$

Assuming $\mathbb{E}(\mathbb{a}_t), \mathbb{E}(\mathbb{m}_t) = 1$, it is expected that $\mathbb{a}_t$ and $\mathbb{m}_t$ oscillate around its average value plus a small individual variation defined as $o(\mathbb{a}_t)$ and $o(\mathbb{m}_t)$ respectively. These small variations are the error given by the random variables, $\mathbb{a}_t$ and $\mathbb{m}_t$, in which the cosine will only output the most significant variations. The cosine acts like a filter against the changes which equal to benchmark's changes. This *magnificent effect* powered by division of cosines plays an important role in revealing the major changes of assets regarding the market. As empirically showed below, almost all $\mathbb{a}_t$ and $\mathbb{m}_t$are settled in the interval $[1,0.99)$ and the arithmetic average of all volatility ETPs, the VIX and the S&P500 throughout five years is 0.9985.

Figure 4. Performance of TA applied on S&P500 Index and VIX Index from 2014 until 2018.



While the VIX Index is oscillating widely far from one, the S&P500 Index remains closer to the unity once applied TA on them. Therefore, the frecuency of the data is sufficiently good approximed to a Uniform probability distribution. *Source*: market quotation retrieved from Thomson Reuters and self-created (2019).

The functions o($\mathbb{a}_t$) and o($\mathbb{m}_t$) are inherit from the cosine which always goes down from 0, because in 0 the cosine finds its maximum of 1. Therefore, o($\mathbb{a}_t$) and o($\mathbb{m}_t$) are always negative and they are added up to $\mathbb{E}(\mathbb{a}_t)$ and $\mathbb{E}(\mathbb{m}_t)$ respectively. As $\mathbb{a}_t$ is more volatile than $\mathbb{m}_t$, it is expected $o(\mathbb{a}_t) > o(\mathbb{m}_t)$. Since the individual variations o($\mathbb{m}_t$) are very small for $\mathbb{m}_t$, it is correct to point that $\mathbb{m}_t$ distributes approximately as a uniform probability distribution with expected value of one and variance zero[6].

$$\mathbb{a}_t = \mathbb{E}(\mathbb{a}_t) - o(\mathbb{a}_t) = 1 + o(\mathbb{a}_t)$$

$$\mathbb{m}_t = \mathbb{E}(\mathbb{m}_t) - o(\mathbb{m}_t) = 1 + o(\mathbb{m}_t)$$

Then,

$$TARV_t = \frac{\sigma_{\mathbb{a}_t}}{\sigma_{\mathbb{m}_t}} = \sqrt{\frac{(\mathbb{a}_t - \mathbb{E}(\mathbb{a}_t))^2}{(\mathbb{m}_t - \mathbb{E}(\mathbb{m}_t))^2}} = \frac{\mathbb{a}_t - \mathbb{E}(\mathbb{a}_t)}{\mathbb{m}_t - \mathbb{E}(\mathbb{m}_t)} = \frac{1 + o(\mathbb{a}_t) - 1}{1 + o(\mathbb{m}_t) - 1} = \frac{o(\mathbb{a}_t)}{o(\mathbb{m}_t)}$$

TARV can provide the maximum RV or the maximum exposure to market risk that a security can achieve regarding the market until the last pricing day available. The application of the cosine goes forward and beyond the geometric relationship between two variables due to it being a sequence of infinite degree in zero so that the function has to go away from zero to maintain its difference. Geometrically it is possible to demonstrate a negative exposure to market risk. Traditional ß formula can achieve negative values when the correlation coefficient or the covariance are below zero. That is, when contrarian-market-trend securities are included in the analysis. Since negative results are completely correct, when these odd securities have non-linear payoffs it produces an undervaluation of the exposure to market risk due to either ß or RV being linear measures of risk. As it is well known, market risk is non-linear, and it is required a non-linear mechanism to properly assess it.

[6] The S&P500 Index's cosine returns has distributed as $E(\mathbb{m}_t) = 1$ and $\sigma(\mathbb{m}_t) = 0.0000000069 \approx 0$ from 2014 to 2018.

As shown in *Figure 4*, the maximum value the cosine can reach is the unity from imputed zero in the function. The most diversified portfolio with zero idiosyncratic risk among the analysed variables is, with no doubt, the S&P500 Index. Empirically, it paralyzes at one because their daily returns are approximately zero and when applied the cosine, it outputs closer values to one. The same occurs in the case of the VIX Index but in the other way around. Its daily returns stand away from zero because it is the most volatile variable of all the sample. Once applied the cosine, these values go straight away from one. *Figure 4* defines very well what is described.

The main objective of TARV is to diminish the importance of small market movements in a determined period and to stress the importance of big market movement in a determined period. In order to reduce the relevance of less volatile returns, it collapses these movements around the unity through the cosine. These changes are very similar to benchmark's ones and, due to search for extreme relative standard deviations, market movements benchmark-like will be non-significative and collapsed around one. As traditional risk measures do, financial industry wants to focus on situations that are out of the normal probability distribution so the VaR[7] is. Following the line of the financial institutions which want to monitor the expected exposure to market risk during market turmoil situations, that is, in non-normal market conditions when the VIX Index spikes and the S&P500 Index deflates. Under normal market conditions most outliers are out of the sample. The expected value is the arithmetic average and the cases can be settled in a confidence interval with statistical significance based in previous data. This arithmetic average is the same value that outputs the regression line that contemplates ß coefficient as market risk exposure. Beyond the confidence interval rest the extreme values or outliers that distort the probability normal distribution through increasing the kurtosis and the skewness coefficient. Thus, this distortion also affects the regression line through increasing the residual sum of squares due to the increment of including outliers in the sample. In the same way, RV also relies on the arithmetic average and allocates the same importance to either big or small market movements. This uniform allocation provokes an undervaluation of the market risk exposure like big market

[7] *Value At Risk* is a statistical measure that quantifies the level of financial risk of a security or corporation over a determined period of time. In investment and commercial banking frame it is used to settle the ratio of potential losses in a market operation.

movements, that is, high standard deviation of return ought to weight more than market benchmark-like movements with lower standard deviation of returns. As mentioned above, there is no surprise in benchmark-like performance because, following the theory, it is the optimal risky portfolio that holds null idiosyncratic risk according to diversification properties and it normally performs inside a *constant* volatility range. Outside this normal performance and constant volatility range rest the extreme situations that surge when the whole market collapses. During these unexpected cases arises the necessity to calculate the expected market risk exposure that a security or portfolio might have. Instead of dividing the sample into two scenarios, normal and abnormal market situation, it is preferable to treat the sample as a whole and let the indicator identify which market momentum, we are immersed in. According to the degree of volatility, TARV will compress that performance into the cosine function and will take more into account the extreme changes rather than the normal changes. Since the normal changes are considered less important than the abnormal ones, it seems that TARV itself filters market movements regarding previous data at a determined period. This process is achieved thanks to the incorporation of the cosine in the RV formula. The cosine settles small returns close to one and structures an indicator. Let's refer to it as a *Constance indicator*. This indicator is plotted in *Figure 4*. The label *Constant* comes from the fact that as further away from one the value is, the more volatility the security has. The expected value either for the security or the market is one since it is assumed investors are operating under normal market conditions. The investing comfort zone can be defined inside the interval of [1,0.95) in terms of TARV. This range begins with the expected value which includes an individual change of almost 0.05. The interval that covers [0.95,0] is where the temporary market turmoil and financial shocks took place since 1990.

By calculating the quartiles, it is obtained the number of observations considered outliers and insiders that stress the *magnificent effect* of applying the cosine on prices.

Table 6. Outsiders of volatility ETPs' log returns.

2014-2018	Outliers	Insiders
SPX	67	1190
VIX	21	1236
UVXY	32	1225
VIXY	24	1233
PHDG	36	1221
VXX	26	1231
TVIX	30	1227
SVXY	65	1192

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Table 7. Outisders and insiders of volatility ETPs' cosine of log returns.

2014-2018	Outliers	Insiders
SPX	149	1108
VIX	133	1124
UVXY	145	1112
VIXY	143	1114
PHDG	133	1124
VXX	145	1112
TVIX	149	1108
SVXY	148	1109

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

In *Table 6* the proportion of outliers over insiders is strictly below 6%, which might not accomplish the higher expected number of outliers regarding their skewness coefficient. In *Table 7*, once applied the cosine on log returns, the number of outliers has increased significantly regarding *Table 6*. Now, the proportion of outliers over insiders is strictly below 14%. This increment in the number of outliers is due to the narrowing of interval in which there are all the observations. The effect of the cosine as positive for high volatile returns and as negative for low volatile returns leads to increase the number of outliers and decrease the number of insiders. The *magnificent effect* that allocates more importance to extreme values is clearly viewed using the quartiles.

5.4 Comparison between RV and TARV

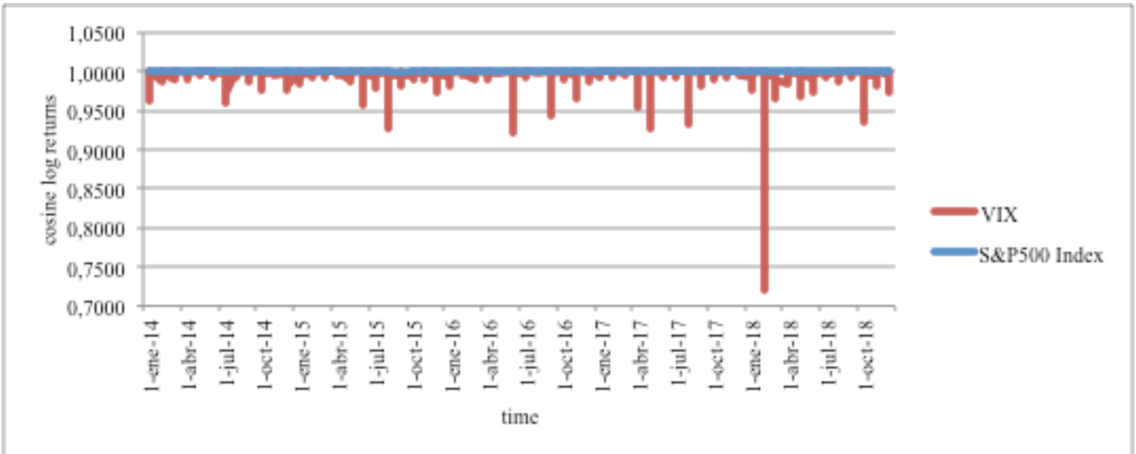
The analytical argument leads to conclude that the core for VR and for TARV are the little individual changes around their expected value.

$$VR_t = \frac{o(a_t)}{o(m_t)}$$

$$TARV_t = \frac{o(\mathbb{a}_t)}{o(\mathbb{m}_t)}$$

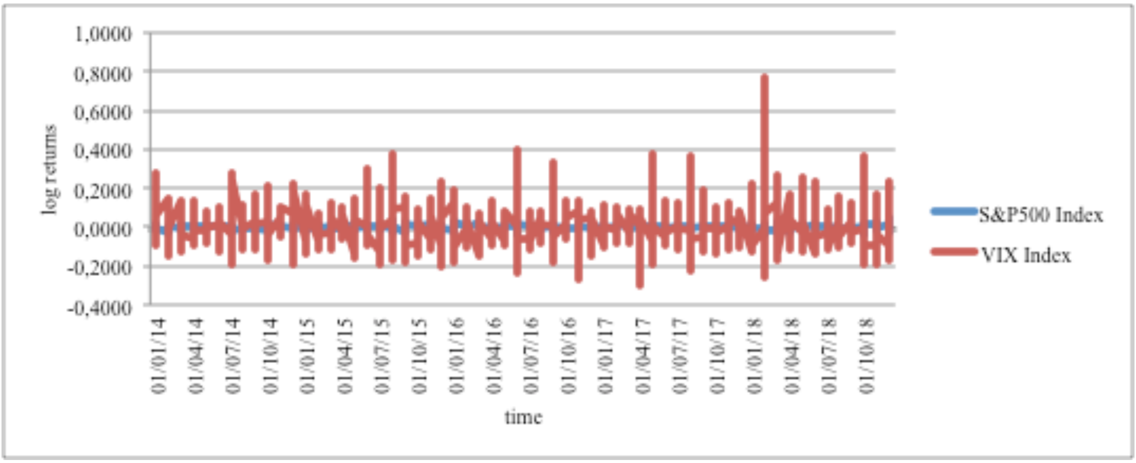
As RV follows, TARV accomplishes the unity when the volatilities compared are the same. Besides the fact that this fundamental point seems basic to achieve, it must be reminded that the coefficient ß does not output the unity when the exposure to market risk is the same as the one the market has.

Figure 5. Performance of TA applied on S&P500 Index and VIX Index from 2014 until 2018.



Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Figure 6. Performance of S&P500 Index and VIX Index returns from 2014 until 2018.

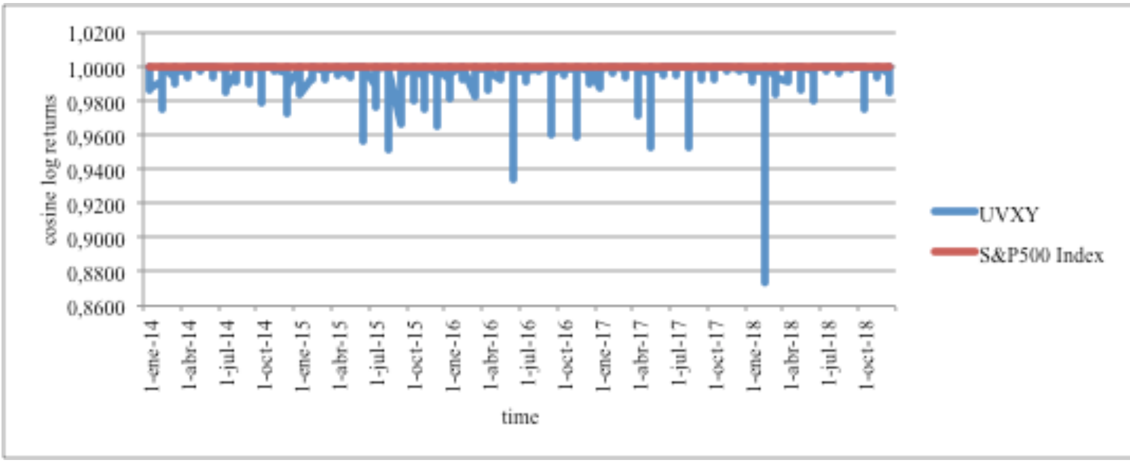


Source: market quotation retrieved from Thomson Reuters and self-created (2019).

In terms of facility, *Figure 5* shows rapidly which moments in time the community of investors should be aware of and preferably have hedged positions. Recovering what is stated above and looking at *Figure 5*, the interval [0.95,0.90] gathers atypical changes that are not enough volatile to be considered extreme changes. It is easy to count the times that the volatility of security's returns has crossed down 0.95 threshold, and which has remained above 0.90. In the case of *Figure 6* the spike-identification becomes more complicated in comparison with *Figure 5*. Evidently in this case it is also possible to count all the spikes, but we must be aware of positive and negative spikes. In contrast of *Figure 5*, the procedure is more complicated in *Figure 6* due to the necessity of mentally analysing every positive and negative spike and comparing them with the others and finally concluding the moments in time with extreme market conditions. Moreover, it is difficult to label a market situation between normal and extreme market conditions. The ability to designate intermediate level market situation with less than half a minute is reserved to *Figure 5*. In the same line, S&P500 Index behaviour is almost non-existent in *Figure 5*, yet the returns are collapsed in one in order to be neutral in calculations and to centralize all our attention to volatile security's returns.

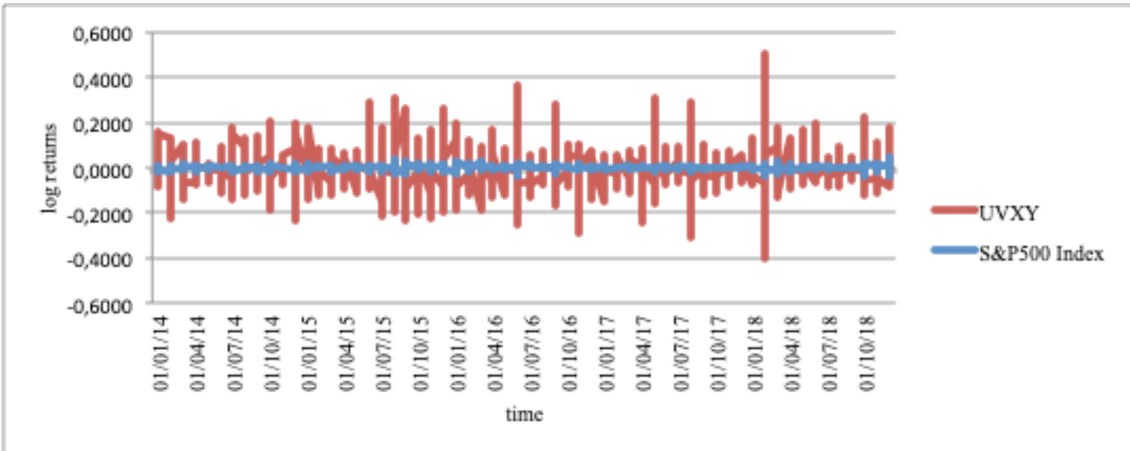
The most significant volatility ETPs are UVXY and PHDG because the first one represents the volatility ETPs that are long on S&P500 SPVXSP and the second one represents the volatility ETPs that are short on S&P500 SPVXSP. All two dimensional graphs are properly attached in the Annex I. The graphical representation of TARV is very similar to the indicator of Maximum Downward.

Figure 7. Cosine log returns of UVXY and the S&P500 Index from 2014 to 2018.



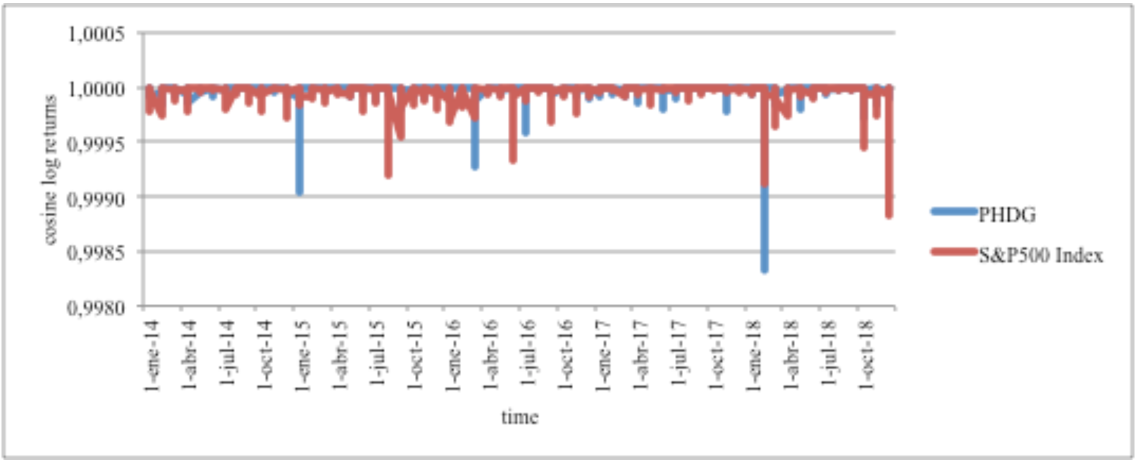
Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Figure 8. Log returns of UVXY and the S&P500 Index from 2014-2018.



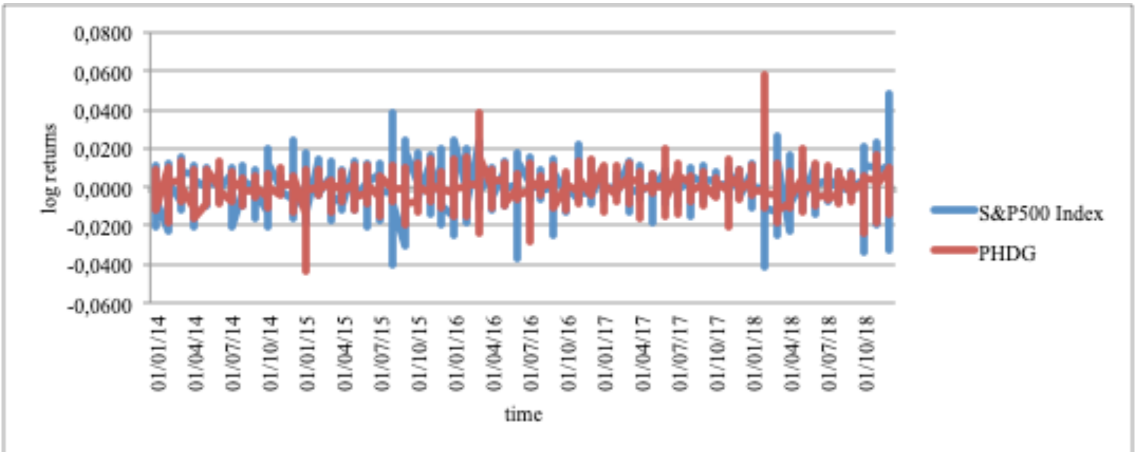
Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Figure 9. Cosine log returns of PHDG and the S&P500 Index from 2014 to 2018.



Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Figure 10. Log returns of PHDG and the S&P500 Index from 2014-2018.



Source: market quotation retrieved from Thomson Reuters and self-created (2019).

5.5 Mathematical Representation

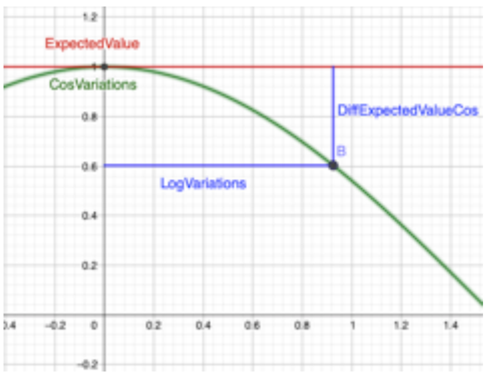
The core of TARV's formula, that is, the cosine function, is plotted in a two-dimensional graph. The vertical axis represents the values the cosine function can take. The horizontal values represent both the values the security and the market can take as performance at time *t*.

The expected value is the only constant variable in the formula that follows a uniform distribution of probability and it the red line that appears as *ExpectedValue*. The difference between the expected value and the cosine, matematically, $\mathbb{E}(\mathbb{a}_t) - \mathbb{a}_t = 1 - cos\left(\ln\left(\frac{A_i}{A_{i-1}}\right)\right)$, is the distance between the superior limit and the inferior limit of cosine's domain. This difference that is perfectly visible in the graph belongs to security performance. In the case of market performance, it is already represented in the graph: its performance is collapsed in one because this difference is very small. If one increases the scale of vertical axis, it is possible to observe market changes and how closer to one they really are. This difference is defined in the graph as *DiffExpectedValuesCos*. Once representated on the graph, one can rapidly find out the parallelism in domain terms between the cosine and the correlation coefficient: both share the range of values that they can take. The label of *LogVariations* is referred to the performance as the security as the benchmark can take at time *t*.

Figure 11. Graphical representation of the cosine.

If the security has low volatility, their returns will be represented by the linear red line. In the case that the security has high volatile returns, the non-linear green line will represent them. *Source*: self-created (2019).

Figure 12. Expanded vision of Figure 11.



Source: self-created (2019).

5.6 Axioms of Coherent Risk Measures

The universal acceptance of a risk measure depends on the coherent degree that it has. This coherent degree can be clearly structured in five statements, which each mathematical demonstration is properly detailed in Annex I.

The main objective of this paragraph is to demonstrate that TARV accomplishes the Coherent Risk Metrics except for the translation invariance assumption. Besides that, the paramount property, which is sub-additivity, is fully accomplished as well as normalised.

Let's define the variables: p[8] as the log return of portfolio or security, R[p] as a function of risk measure on portfolio or security p, TARV[p] as a function of risk measure on portfolio or security p, r[9] as the risk-free rate and k[10] as an amount of capital. The time is not specified because the analysis is done in a single time period that would result of multiplying by the unity. Therefore, the time is not specified throughout the following demonstrations.

1. Translation invariance.

   For all $p \in \mathbb{R}_{\geq 0}$ and every constant $k \in \mathbb{R}$, then

$$R[p + rk] = R[p] - k$$

$$TARV[p + rk] \neq TARV[p] - k$$

   Translation invariance implies the investment of an amount of capital k at the risk-free rate r, which stands for kr, it would reduce the risk by the same amount

[8] Following Markowitz portfolio optimization, the manager wants to maximize the returns of her security or portfolio. Therefore, she would require more than the risk free rate to assume the proportion of market risk given her exposure to it. Specific risk is not considered since it is stated that the manager seeks the optimal portfolio, therefore, it is already well-diversified.
[9] Let's consider non-negative risk free rate, then $r \in \mathbb{R}_{\geq 0}$.
[10] This amount of capital k can be either positive or negative yet the manager can be lender (positive) or borrower (negative) of that amount of capital k.

rk. The gain of investing *k* at the risk-free rate, *kr*, is one on the right part of the expression, therefore it does not appear. Investing in *r*, should lead to less capital requirement in the investment and a reduction in the risk of *p*. That is, a decrease of *k* in the output of the risk measure used, R[*p*].

This property is not accomplished neither by TARV nor by any risk measure that is expressed in relative terms, which are: tracking error, standard deviation (volatility) and ß. The point is that TARV[*p*] cannot be expressed in the same units as *k* is, i.e. *kr* is the amount of capital invested in *r*, therefore it is expressed in absolute terms. The fact that risk measures should be expressed in absolute terms (in capital terms) it is because it better reflects the losses rather than a percentage. Nevertheless, this is not an obstacle, yet one can multiply TARV's output by the market value of the portfolio or the security.

This operation would lead to know the maximum capital requirement that the manager would need to over-hedge *p* in market turmoil period based on historical data. The over-hedge refers to the text from the introduction where it is remarked that when the market of the underlying asset or p is increasing its volatility, TARV will remove the *market noise* and stress the significant movements. Therefore, giving more importance to significant movements and less importance to secondary ones, provokes that the exposure to market risk dictated by the market, whatever it was, would be lower than the exposure to the market risk assessed by TARV when *p* is more volatile than the market. That is, when *p* is riskier than the market. Therefore, the TARV overvalues the exposure to market risk to ensure the manager has hedged *p*.

2. Sub-additivity.

This is probably the most important property a risk measure should really accomplish. Sub-additivity is the base of Markowitz portfolio optimization since it makes sense to diversify a portfolio in order to reduce the specific risk. The main idea is that joined risks help to reduce the global risk by reducing its specific risk. Without sub-additivity, there is no incentive to create well-diversified portfolios and the risk metric would not be used for risk budgeting.

For all $p_1, p_2 \in \mathbb{R}_{\geq 0}$, then

$$TARV[p_1 + p_2] \leq TARV[p_1] + TARV[p_2]$$

The evidence of this property is also given by volatility short-term S&P500 Futures ETPs of VIX Index, which are the ones with the most volatile securities in the market. The introduction of these volatility ETPs in a portfolio can boost the returns in both directions, negative and positive. As mentioned before, TARV gives more importance to the biggest historical p movements and less importance to the lowest historical ones. Since risk evaluation of a well-diversified portfolio without including high volatile securities already accomplishes this purpose, the case of TARV is not far from being different. TARV applied on an isolate security would overvalue the highest movements and, therefore, the market risk exposure will be greater than the market risk exposure of a well-diversified portfolio, which includes that security.

Volatility ETPs are used as partial hedges, it is seldom to encounter a portfolio full of these securities because each one has its own characteristics and specific functions. Professional investors combine well-diversified portfolios with volatility ETPs in aim of not going further than hedging their long positions in the equity market. In the same way, the famous Gamma hedges, which usually are long in volatility ETPs and options and short in the underlying asset when the equity market is bullish, do trade with specific volatility ETPs and not with all simultaneously.

Recalling introduction again, the capital requirement will be greater in the case of having isolated portfolios with one single security than when having well-diversified portfolios which include that specific security and others, with minimum linear dependency between them.

TARV is simply the division of two standard deviations trigonometrically modified. The cosine monotonous modification is neutral when proving these properties. The mathematical representation of standard deviation which accomplishes sub-additivity property is stated in Annex I.

In the general case of quartile-based risk measures this property is achieved but it fails out of the general case when returns are not well distributed following an elliptical normal distribution of probability. The property of symmetry is not always reachable even for some equities.

Let A be an equity well-diversified portfolio with participation of Volatility ETPs in order to hedge specific or general positions. The performance of portfolio A is independent and identically distributed and follows an elliptical normal distribution of probability. In order to follow Markowitz's technique of portfolio optimization and achieve efficient risky portfolio given the market situation, it is necessary that the performance of involved portfolios accomplishes the elliptical distribution of probability. Furthermore, it is preferable that the coefficient of Pearson's correlation becomes a proper measure of dependence when assessing the degree of diversification and, thus, the degree of specific risk of the portfolio. To accomplish that, the values also must be distributed following an elliptical distribution of probability. Besides the correlation coefficient not being used in this analysis since volatility ETPs perform non-linearly regarding the benchmark, when computing the portfolio in general (i.e. including as equity as Volatility ETPs in the computation) it will be used.

3. Positive Homogenity.

This property states that the risk of a security is proportional to the weight it represents in the portfolio. If the risky assets represent an important stake of the portfolio, the TARV will increase.

Let $p = a \cdot z$ for any positive constant a, $a > 0$, then

$$TARV[p] = a \cdot TARV[z]$$

The fact that a risk metric does not provide information about the risk attitude of an investor and, therefore, it defends the risk neutral scenario, is not the case of TARV. Stated the magnificent effect on extreme values, TARV is planned for risk adverse investors who seek to hedge the riskiest positions indicated by TARV.

4. Monotonocy.

Monotony uses weak stochastic dominance[11] as a basis to demonstrate a positive increasing path of the function. The main idea is that if the investment *X* weakly stochastically dominates investment *Y*, then the investment *X* should be assessed as not being riskier than investment *Y* according to TARV.

$$TARV[p_1] \leq TARV[p_2]$$

5.7 Comparison between VaR and TARV

VaR is expressed in absolute terms and under normally distributed returns, therefore it is generally a coherent risk metric because it equally performs the standard deviation of returns. Beyond special assumptions about the distribution of returns, VaR is far from being a coherent risk measure due to fails to be sub-additive because of the quantiles unless the returns have an elliptical probability distribution. "Artzener, Delbaen, Eber and Heath 'defend that VaR does not accomplish the assumption of sub-additive" (1999:203).". "Daníelsson, Jorgensen, Samorodnitsky, Sarma and Vries state, 'in the specific case of normality of returns, a property at odds with stylized facts of financial returns, VaR is known to be coherent below the mean' (2005:2).". Also, "Daníelsson et al. add, 'we demonstrate that VaR is sub-additive for the tails of all fat tailed distributions, provided the tails are not super fat. For most asset classes we do not have to worry about violations of sub-additive' (2005:3).". To determine, if VaR accomplishes the sub-additivity it will be appropriate to focus our attention on the mass

[11] If we have exactly same returns, R, obtained with different investments, *X* and *Y*, the probability of return exceeding any threshold, t, is always strictly bigger with investment *X*. Therefore, investment *X* should be preferred to *Y*. It is said that *X* strictly dominates another investment *Y*. Matematically, $P_x(R > t) > P_y(R > t)$ for any *t*. Investment *X* weakly dominates another investment *Y*.

of the tails in probability distributions of volatility ETPs. The kurtosis coefficient and the excess kurtosis are used to label which probability distributions have enough *fat tails* to consider the application of VaR incoherent as a risk measure on those securities. The lack of symmetry in volatility ETPs probability distributions is provided by the cosine in the formula of TARV.

Table 8. Kurtosis and excess kurtosis coefficients over volatility ETPs during the period 2014-2018.

2014-2018	Kurtosis coefficient	Excess Kurtosis
S&P500 Index	3.7856	0.7856
VIX Index	8.5496	5.5496
UVXY	4.8684	1.8684
VIXY	5.3227	2.3227
PHDG	12.7331	9.7331
VXX	5.3190	2.3190
TVIX	5.5015	2.5015
SVXY	13.7166	10.7166

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Generally, it is used excess kurtosis' threshold of number two as a reference of having *fat tails* in a probability distribution. Following this law and applying it in our case, VIX Index, PHDG and SVXY securities will not be candidates to implement VaR as a coherent risk measure. During this period, there are only three participants that own too much mass in their tails but, there might be other cases if one changes the time period of observation or changes short-term futures for mid-term futures. "Daníelsson et al. continue, 'Options can also be constructed in a way to give super fat tails. In such cases, sub-additivity violations are likely to be a matter of serious concern' (2005:2).". The last one, "Daníelsson et al. also end, 'VaR risk measure is sub-additive, we focus on the case where returns are fat tailed, i.e. are regularly varying. If returns are normal, we know sub-additivity holds, so it is sufficient to focus on the fat tailed case' (2005:7).". Considering that volatility ETPs are such complicated securities because they depend on futures that, at the same time, depend on options, it might be too risky to try to properly assess the risk solely using the VaR itself. Once stated that the securities in this analysis own high coefficients of kurtosis and skewness and excess kurtosis cancels the symmetry property of the probability distribution, it is clear not to use VaR for such securities.

6. RESULTS

After describing, stating and commenting on the mathematical background of TARV, it is time to plot the results and analyse their evolution throughout 2014-2018, including the VIX Index.

Table 9. RV securities and time matrix of results.

RV	VIX	UVXY	VIXY	PHDG	VXX	TVIX	SVXY
2018	9.3235	7.6498	4.4089	0.6628	4.3973	8.8665	3.6768
2017	16.2759	14.0079	7.0106	1.1455	6.9709	13.8059	7.3682
2016	9.3322	9.6429	4.9142	0.7285	4.8888	9.4114	5.2663
2015	8.8527	9.2583	4.7019	0.6182	4.6730	9.0231	4.9019
2014	10.4067	9.5365	4.8171	0.6479	4.8059	9.0999	4.8585

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Table 10. TARV securities and time matrix of results.

TARV	VIX	UVXY	VIXY	PHDG	VXX	TVIX	SVXY
2018	145.6556	87.3310	26.0727	0.8460	25.5128	104.8898	16.4908
2017	393.8237	294.1364	76.1208	1.3301	75.7402	286.6450	173.8362
2016	104.6552	100.2856	29.0582	0.8388	29.0917	94.5168	35.1412
2015	82.3250	76.8475	20.3983	0.6760	20.6618	77.0902	11.7886
2014	115.0995	91.6285	23.1501	0.4278	23.0633	84.2640	27.2470

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

The value arises when it is compared with *Table 10*, in which it figures the maximum exposure to market risk from the point of view of TARV.

Table 11. Absolute differences between TARV and RV matrix.

AD	VIX	UVXY	VIXY	PHDG	VXX	TVIX	SVXY
2018	136.3322	79.6812	21.6638	0.1832	21.1155	96.0233	12.8140
2017	377.5479	280.1285	69.1102	0.1846	68.7693	272.8390	166.4680
2016	95.3229	90.6427	24.1440	0.1103	24.2030	85.1054	29.8749
2015	73.4723	67.5892	15.6963	0.0578	15.9888	68.0671	6.8867
2014	104.6928	82.0920	18.3331	-0.2200	18.2574	75.1641	22.3884

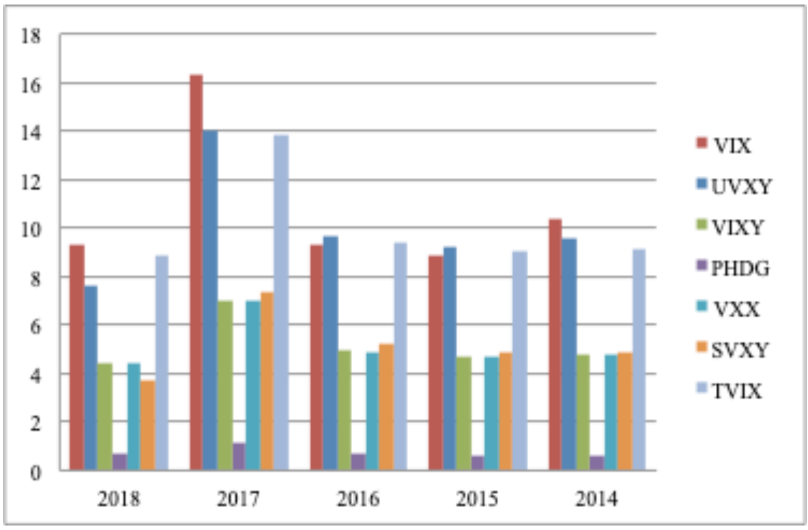
Source: market quotation retrieved from Thomson Reuters and self-created (2019).

The absolute differences (AD) perfectly describe the magnificent effect provided by the quotient of cosines applied on standard deviations of different variables. The colour scale is displayed in the aim of contributing more clearly on perceiving the trigonometric adjustment and how it acts in almost-neutral cases like PHDG. The VIX Index holds the outstanding evidence with the greater absolute difference. This is not a surprise because the VIX Index is the most volatile figure under this study. Below the VIX Index figures the UVXY, TVIX and SVXY. The other cases still own high differences that lead to conclude the huge mistake one could make if assessing the risk of these securities by linear risk measures as RV. Albeit it is true that TARV *magnifies* the extreme values and the real market risk exposure is between TARV's output and linear risk measures, there is still an absolute difference that should worry the manager that uses RV.

Graphically it is more evident that, both TARV and RV give the greatest importance to VIX, UVXY and TVIX. The magnificent effect of TARV is also noticeable when between the highest values and the lowest values it figures a greater gap than the highest and the lowest ones in the RV graph. The best case to see this effect is the couple VXX – SVXY in every year under study. In *Figure 9*, it is more difficult to dictate which of these two has more market risk exposure than in *Figure 10*. Evidently if one stares at the graph it will reach which one has more market risk exposure, but beyond that, it is easier to check using TARV.

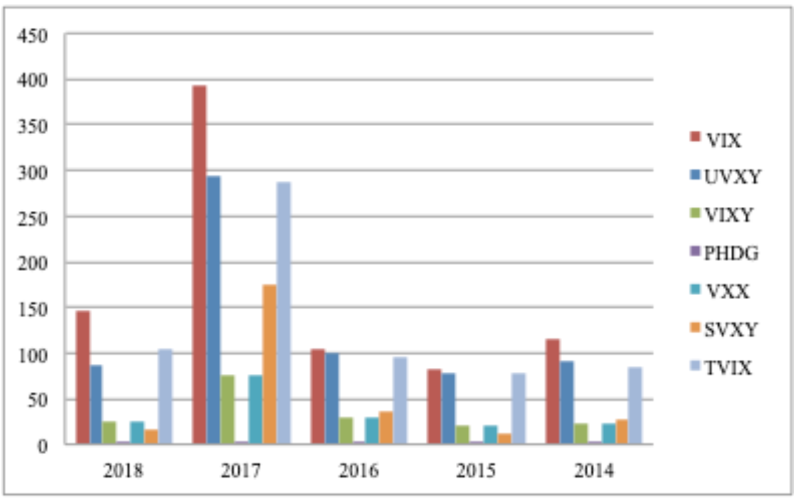
The interesting part is when TARV and RV contradict each other. This is the case of VIX – UVXY during 2016 and 2015. RV states the UVXY has more market risk exposure than the VIX Index either in 2015 or 2016. On the other hand, TARV states that the VIX Index has more market risk exposure in all years under study. Which one should the manager trust? The difference between them are small even in TARV scenario but nonetheless the VIX index stays above UVXY. The risk profile of the manager, the objective of the fund and the market situation will influence in the decision of which to trust too. Provided that TARV can give the upper limit of ex-post maximum market risk exposure, the manager can decide to take lower values than TARV's values and not lower than what is indicated by RV. TARV would be recommended for risk adverse managers.

Figure 13. Evolution of RV applied on different volatility ETPs from 2014 to 2018.



Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Figure 14. Evolution of TARV applied on different volatility ETPs from 2014 to 2018.



Source: market quotation retrieved from Thomson Reuters and self-created (2019).

The time horizon of five years is considered as a good range of time to have statistical significative results. Once again, the non-parametric mechanism of TARV is noticeable if one compares *Table 12* and *Table 13*. TARV allocates less importance to the market risk exposure when the security is less volatile than the benchmark. Considering that the neutral case is the unity, securities that are less volatile than the benchmark are under one. The VIX Index is more volatile than the volatility ETPs.

Table 12. RV accumulated from 2014 to 2018 using as benchmark the S&P500 Index and the VIX Index.

RV							
2014-2018	VIX	UVXY	VIXY	PHDG	VXX	TVIX	SVXY
S&P500	9.8364	9.2064	4.8197	0.6962	4.7980	9.3738	4.7760
VIX	1	0.9360	0.4900	0.0708	0.4878	0.9530	0.4855

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

Table 13. TARV accumulated from 2014 to 2018 using as benchmark the S&P500 Index and the VIX Index.

TARV							
2014-2018	VIX	UVXY	VIXY	PHDG	VXX	TVIX	SVXY
S&P500	127.4047	90.5537	25.7512	0.7751	25.5161	98.1280	26.8898
VIX	1	0.7108	0.2021	0.0061	0.2003	0.7702	0.2111

Source: market quotation retrieved from Thomson Reuters and self-created (2019).

The interpretation of tables form above would be in market standard deviations, that is, normalized. The values that exceed the unity will be considered as whole standard deviations and the values which do not exceed it will be considered as a percentage of a standard deviation. In all the cases that TARV compares security's return with the benchmark, TARV is expressed as an absolute value. For instance, UVXY is 9.2 standard deviations more volatile than the benchmark. In the contrarian extreme, PHDG is 0.77 standard deviations volatile as the benchmark. In other words, PHDG is 77% volatile as benchmark.

Recalling the introductory graph, the non-linear effect plays the central role in stressing relevant market movements and decreasing the importance of the market noise or secondary ones. Incorporating TARV to our risk assessment *disciplines* the final decision of entering or not entering in an investment. Quantitative methods provide mathematical rigour to our financial decisions.

7. CONCLUSIONS

In a world in which finances are biased to mathematical analysis and computation appears the necessity to provide mathematical adjustments to classical financial theory that might be old-fashioned for the times that are passing by. The universe of equity becomes small when one enters in financial derivatives pricing and risk management universes. The possibilities of new analytical methods or, at least indicators as TARV is, are incredible and feasible with just a little inch of creative and open-minded individuals.

The popularity of VIX is increasing thanks to Mr. Trump with the commercial war against China. On April of 2019, the VIX Index has returned to the levels of December 2018, days after Bloomberg had published an article about the huge massive quantity of short positioning in volatility ETPs. That graph did not consider the long Gamma positions neither the hedges, which the majority are long in the volatility ETPs stated in this analysis. Provided that the S&P500 Index has also returned to all-time highs, the volatility is slightly increasing too.

The inception of this analysis was a worrying sentiment when Thomson Reuters spreadsheets of studied volatility ETPs included the ß coefficient. A security that is not linear related, contrarian, which includes options and sometimes it is leveraged cannot describe its market risk exposure with that simply linear gauge. Including ß coefficient is not the worst since it is a worldwide accepted financial indicator. The sign and the interpretation of that coefficient is the inception of this analysis. The conservative negative sign of ß coefficient is completely contrarian to volatility ETPs which achieves high standard deviation values.

The non-linear dependence of volatility ETPs with the benchmark stress to use non-parametric and non-linear risk measure for these volatile contrarian securities. Moreover, the lack of symmetry in their distributions stated in statistical outline paragraph increases the advertisement to avoid Pearson's correlation coefficient. That is, lack of elliptical contours in multivariate normal distributions. The instability of ßs is another argument in favour of TARV so it is the meaning of negativity of coefficients given by Sharpe.

The proposition to use TARV as a non-linear and non-parametric market risk indicator can provide risk managers the certainty that they will completely hedge ex-post all the riskiest positions. Recalling the text from the introduction, the estimations of the real market risk are approximations that can be linear or non-linear. The linear ones will fail in market turmoil period when assessing securities with high number of *outliers* in their observations. In the paragraph where the CV is explained, and it is compared with TARV figure the number of outlier's volatility ETPs securities have. Given that the financial market is always changing, it is difficult to establish a threshold that indicates whether we, as investors or managers, are in high volatile or low volatile times. It is easier to work within an interval that holds an *upper limit* (TARV) and a *lower limit* (linear risk measures) when the security is more volatile than the benchmark.

The S&P500 Index is settled as the benchmark because the portfolio puts and calls on S&P500 Index, which is the VIX Index, is the underlying asset. In the same way, this analysis could be structured using the VIX Index as the benchmark and the TARV will comfortably give financially and mathematically plausible values. The fact that the VIX Index is more volatile than the securities, it will provide TARVs under zero, therefore, TARV expressed as a percentage. In the cases that the security is as volatile as the benchmark is, TARV will provide very similar results to the ones from the linear risk indicators. This property is reflected in the paragraph of the results when comparing TARV and RV over PHDG. The positivity and the neutral case of TARV increase its application to a wider number of securities with particularities.

This analysis is centralized in studying a new risk indicator with more precision than linear risk measures and further investigation about the possibility of positively modifying TARV is left for future analysis.

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