Distortion Theorems for Quasiconformal Mappings

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Vull donar les gràcies a en Xavier Tolsa per l'enorme esforç que ha fet i la paciència que ha tingut amb les meves dificultats. També al tribunal format per en Joan Orobitg, en Jose González i l'Albert Clop per deixar-se embolicar i avaluar aquest treball. Vull fer una especial menció a l'Albert i en Xavier per les seves classes al màster. Cal fer esment de l'Institut-Escola Ramona Calvet, on he treballat durant tot el curs per la seva comprensió així com als alumnes que han sabut entendre les dificultats que he passat aquest any. Vull agrair també a Markus Hohenwarter i la resta del seu equip pel GeoGebra, un programa genial de distribució lliure amb el que s'han fet les il·lustracions. Finalment a la meva família i les meves amistats que han hagut d'aguantar les meves abscències físiques i mentals i, sobretot, a l'Aura que m'ha acompanyat en els moments més durs.
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Contents

Introducció

El treball que teniu a les mans és el resultat del treball de final de màster de l'autor com a alumne del Màster en Matemàtica Avançada 2010-2011 de la Universitat Autònoma de Barcelona sota la direcció de Xavier Tolsa. El treball està escrit en anglès perquè la major part de la literatura sobre la matèria que tracta està en aquesta mateixa llengua.

L'objectiu del treball és familiaritzar l'autor amb el tema de la distorsió de conjunts sota l'acció de transformacions quasiconformes. El resultat fonamental en aquest camp és el conegut teorema de distorsió de l'àrea d'Astala, demostrat l'any 1994. Durant aquesta primera dècada del segle XXI s'han fet avenços molt significatius en aquest camp, quedant tancada ja la màxima distorsió en la mesura de Hausdorff.

L'eix principal del text és donar un cop d'ull al que va ser el pas previ a aquesta fita, concretament la distorsió del contingut de Hausdorff. En el treball no hi ha resultats nous sinó que es fa un recull d'una sèrie de resultats sobre el tema, es revisen les demostracions i s'intenta clarificar-ne els punts que, als ulls de l'autor, resulten més complicats de seguir.

El treball es divideix en dos capítols. En el primer es fa una revisió dels continguts bàsics necessaris per poder dur a terme les demostracions del segon capítol. La primera secció recull les definicions bàsiques així com una sèrie de resultats de diferents contexts que seran d'utilitat.

En la segona secció s'exposen tres teoremes sobre recobriments, el de Besicovitch per quadrats, el de Whitney i el de Vitali. Del primer se'n dóna una demostració completa ja que en la literatura es troba molt fàcilment la versió per cercles però, en canvi, fins allà on l'autor ha pogut llegir, costa trobar una demostració completa del teorema en qüestió. De fet, la demostració que presentem és treta del llibre de Miguel de Guzmán [**Guz75**] on està posat com a exercici amb algunes indicacions. El Teorema del Recobriment de Whitney presenta un problema similar. En la literatura costa trobar-lo enunciat amb prou generalitat i aquí se'n presenta una versió "prêt-à-porter". Els corol·laris són de collita pròpia. Finalment s'enuncia el Teorema de Vitali per boles quíntuples.

En la tercera secció es fa un recull dels resultats més transcendents sobre aplicacions quasiconformes, centrant-nos en aquells que ens seran d'utilitat. Es segueix el llibre d'Astala, Iwaniec i Martin [**AIM**].

Al segon capítol es fa un recull dels principals resultats sobre distorsió de conjunts compactes sota aplicacions quasiconformes. En la primera secció s'estudia la distorsió de l'àrea exposant la demostració del mateix llibre [**AIM**].

En la segona secció es segueix el raonament que porta de la distorsió de l'àrea a la de la dimensió de Hausdorff també seguint els passos del mateix Astala i es comenta el cas dels quasicercles, en els quals es pot obtenir una millor cota tal i com va demostrar S. Smirnov l'any 2000 (tot i que no va ser publicat fins l'any passat a [**Smi10**]).

En la tercera secció, la més extensa del treball, es fa una revisió de la demostració de Lacey, Sawyer i Uriarte-Tuero del Teorema de Distorsió del Contingut de Hausdorff, canviant l'enfocament del seu article [**LSUT**] per remarcar la cota sobre la distorsió enlloc de la conseqüència que té en la conservació de la mesura nul·la. La principal diferència, no obstant, rau en el fet que es demostra que una certa compressió de la transformada de Beurling amb pesos és fitada de manera diferent de l'article original, adaptant una demostració similar de l'article d'Astala, Clop, Tolsa, Uriarte-Tuero i Verdera [**ACTUTV**], encara per publicar. També es fan alguns canvis en el control de les cotes per intentar clarificar alguns passos de la demostració. Finalment apareixen una sèrie de lemes que els articles citats passen per alt degut a la seva senzillesa. En tractar-se del primer contacte de l'autor amb aquests raonaments, ha semblat necessari incloure aquests detalls que el lector experimentat pot passar per alt.

La quarta secció anuncia el resultat rellevant de Tolsa sobre la distorsió de la mesura de Hausdorff que serà publicat aviat a [**ACTUTV**]. En utilitzar continguts generals de Hausdorff, la demostració queda fora de l'abast d'aquest treball.

Per acabar, a la cinquena secció es fa una repassada esquemàtica de l'exemple d'Uriarte-Tuero de distorsió extrema per la mesura de Hausdorff amb il·lustracions, adaptant-la al contingut de Hausdorff però sense entrar en massa detalls.

CHAPTER 1

Background

1.1. Basic Notions

1.1.1. Definitions.

NOTATION 1.1.1. During the whole text, $\mathring{A}$ denotes the interior of A and m denotes Lebesgue planar measure. We will also use the notation $|A| = m(A)$ for any measurable set A. The notation $\mathbb{D}_r = r\,\mathbb{D}$ will stand for the open disk of radius r centered in the complex plane $\mathbb{C}$.

We will write $A \lesssim B$ when there exists a constant C such that $A \leq C \cdot B$. We will write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

We need also some basic definitions about cubes and squares to be used in Section 1.2 and Section 2.3:

DEFINITION 1.1.2. Let x, $y \in \mathbb{R}^n$, $r > 0$. We denote as $\mathrm{dist}_\infty(x, y)$ the usual maximal distance by coordinates,

$$\mathrm{dist}_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

We define the cube $Q(x, r)$ as the closed cubic interval centered at x

$$\{z \in \mathbb{R}^n : \mathrm{dist}_\infty(z, x) \leq r\}.$$

In the case $n = 2$ they can be called also squares. Notice we always refer to cubes with sides parallel to the coordinate axes without stating that fact. We will say that two cubes are *mutually disjoint* if their interiors are. $\ell(Q)$ denotes the side-length of a given cube Q. Notice that $\ell(Q(x, r)) = 2r$. For all $a > 0$ we denote by $a\,Q$ the cube concentric to the cube Q, but such that $\ell(a\,Q) = a\ell(Q)$.

DEFINITION 1.1.3. Consider the lattice of points of $\mathbb{R}^n$ with integer coordinates. This lattice determines a mesh $\mathcal{D}_0$ which is a collection of cubes, namely all cubes of unit length whose vertices are points of the above lattice. The mesh $\mathcal{D}_0$ leads to a two-way infinite chain of such meshes, $\{\mathcal{D}_k\}_{-\infty}^{\infty}$ with $\mathcal{D}_k = 2^{-k}\mathcal{D}_0$. These cubes are the so-called *dyadic cubes*.

$$\overline{\mathcal{D}} = \bigcup_{-\infty}^{\infty} \mathcal{D}_k.$$

Thus, each cube in the mesh $\mathcal{D}_k$ gives rise to 2^n cubes in the mesh $\mathcal{D}_{k+1}$ by dividing its sides by 2. The cubes in the mesh $\mathcal{D}_k$ have each one side-lengths equal to 2^{-k}.

Definition 1.1.4. For a set $E \subset \mathbb{R}^n$, $0 \leq s \leq n$ and $0 < \delta \leq \infty$, one defines

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(E_i)^s : E \subset \bigcup_{i=1}^{\infty} E_i \text{ and } \operatorname{diam}(E_i) \leq \delta\right\},$$

Then one defines the *Hausdorff s-measure* of E to be

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E). \tag{1.1}$$

The quantity $\mathcal{H}^s_\infty(E)$ is usually referred to as the *Hausdorff content of E*. Recall that $\mathcal{H}^s(E) = 0$ if and only if $\mathcal{H}^s_\infty(E) = 0$ [**Mat95**, Theorem 4.6].

It is well known that in the definition of Hausdorff measure, if instead of covering with arbitrary sets, one covers with dyadic cubes, one obtains an equivalent measure (see [**Mat95**]). We rewrite it here in that way as this is the version we will use in Section 2.3.

Definition 1.1.5. For a set $E \subset \mathbb{R}^n$, $0 \leq s \leq n$ and $0 < \delta \leq \infty$, one defines

$$\widetilde{\mathcal{H}}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \ell(Q_i)^s : E \subset \bigcup_{i=1}^{\infty} Q_i \text{ and } \ell(Q_i) \leq \delta, Q_i \in \overline{\mathcal{D}}\right\}.$$

Then one defines $\widetilde{\mathcal{H}}^s(E)$ as in (1.1).

Remark 1.1.6. Being equivalent means that $\widetilde{\mathcal{H}}^s(E) \approx \mathcal{H}^s(E)$.

We need also some definitions from the field of analysis:

Notation 1.1.7. $J(z, f)$ denotes the Jacobian (determinant) of f at z.

Definition 1.1.8. Given any ϕ compactly supported and integrable, the Cauchy transform is defined by

$$(\mathcal{C}\phi)(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(\tau)}{z - \tau} d\tau$$

As long as the derivatives have sense, we have that

$$\mathcal{C}(\partial_{\bar{z}}\phi) = \phi \tag{1.2}$$

Definition 1.1.9. Let

$$(\mathcal{S}f)(z) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}} \frac{\mathrm{f}(\tau)}{(\mathrm{z} - \tau)^2} \mathrm{dm}(\tau),$$

be the *Beurling transform.* Define also the ϵ-truncated Beurling transform as

$$(\mathcal{S}_\epsilon f)(z) = -\frac{1}{\pi}\int_{|z-\tau|>\epsilon} \frac{f(\tau)}{(z - \tau)^2} dm(\tau)$$

and the maximal Beurling transform as

$$(\mathcal{S}_* f)(z) = \sup_{\epsilon>0} |\mathcal{S}_\epsilon f(z)|.$$

The Beurling transform is an example of a standard singular integral bounded on $\mathcal{L}^2(\mathbb{C})$ (see [**AIM**]) defining an isometric operator, that is

$$\|\mathcal{S}\|_{\mathcal{L}^2(\mathbb{C})\to\mathcal{L}^2(\mathbb{C})} = 1 \tag{1.3}$$

In fact, $\mathcal{S}$ and $\mathcal{S}_*$ are bounded on $\mathcal{L}^p$ for $1 < p < \infty$ and of weak type $(1,1)$, and thus the operators $\mathcal{S}_\epsilon$ are uniformly bounded (see [**Ste93**]). We will not prove these facts here, neither the statement that, for any locally integrable function with square integrable distributional derivatives,

$$\mathcal{S}(\partial_{\bar{z}} f) = \partial_z f.$$

Definition 1.1.10. Let $f : \Omega \to \Omega'$ be a measurable mapping. We say that f satisfies Lusin's condition $\mathcal{N}$ if for all $E \subset \Omega$,

$$|E| = 0 \implies |f(E)| = 0$$

We say f satisfies Lusin's condition $\mathcal{N}^{-1}$ if for all $E \subset \Omega'$

$$|E| = 0 \implies |f^{-1}(E)| = 0$$

Definition 1.1.11. Let Ω be a subset of $\mathbb{C}$, $\mathbb{V}$ a vector space equipped with inner product and $\mathcal{D}'(\Omega, \mathbb{V})$ be the space of all distributions $f : \Omega \to \mathbb{V}$. We define

$$W^{k,p}(\Omega, \mathbb{V}) = \left\{ f \in \mathcal{D}' : \|f\|_{W^{k,p}(\Omega,\mathbb{V})} < \infty \right\},$$

where $\|f\|_{W^{k,p}(\Omega,\mathbb{V})} = \left(\sum_{|\alpha|\leq k} \int_\Omega |\partial^\alpha f|^p \right)^{1/p}$. We will consider $W^{k,p}(\Omega) = W^{k,p}(\Omega, \mathbb{C})$ in particular.

We define also the local version $f \in W^{k,p}_{loc}(\Omega, \mathbb{V}) \subset \mathcal{D}'$ if for all $x \in \Omega$, there exist a neighbourhood $x \in U$ such that $f|_U \in W^{k,p}(U, \mathbb{V})$.

1.1.2. Known Facts. We present here some theorems that we will use later on that are of common use in the literature.

The first of them, Jensen's inequality, relates the mean value of a convex function with the image of the mean value. Taking the function concave would invert the inequalities.

Theorem 1.1.12 (Jensen's Inequality). *Let (Ω, A, μ) be a measure space, such that $\mu(\Omega) = 1$. If $g : \Omega \to \mathbb{R}$ belongs to $\mathcal{L}^1(\Omega)$, and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function on the real line, then*

$$\phi\left(\int_\Omega g \, d\mu\right) \leq \int_\Omega \phi \circ g \, d\mu.$$

Corollary 1.1.13. *Let E be a set of positive Lebesgue measure. Given a function $a(z) > 0$, $z \in E$, then*

$$\log \int_E a(z) = \sup_p \left(\int_E p(z) \log \left(\frac{a(z)}{p(z)} \right) \right), \tag{1.4}$$

where the supremum is taken over all the functions p such that $p(z) > 0$ for almost every $z \in E$ and $\int_E p = 1$.

Proof. Just notice that for $B \subset E$, $\int_B p$ defines a measure and, taking into account that the logarithm is convex, we can apply Theorem 1.1.12 with $g = a/p$. Note that the supremum is attained when $p(z) = a(z)/\int_E a$. $\square$

Harnack's inequality is about harmonic functions. Recall that a harmonic function f defined in an open subset $G \subset \mathbb{R}^n$ is a twice continuously differentiable real-valued function such that

$$\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} \equiv 0 \quad \text{in } G.$$

Recall also that the real and imaginary parts of a holomorphic function are harmonic conjugate due to Cauchy-Riemann equations, and that any harmonic function defined on an open subset of $\mathbb{R}^2$ is locally the real part of a holomorphic function.

Theorem 1.1.14 (Harnack's Inequality). *Let u be a harmonic function in a domain G of an n-dimensional Euclidean space, let $B = B(y, r)$ be the ball with center y and radius r. If the closure $\overline{B} \subset G$, then Harnack's inequality*

$$\left(\frac{r}{r+\rho} \right)^{n-2} \frac{r-\rho}{r+\rho} u(y) \leq u(x) \leq \left(\frac{r}{r-\rho} \right)^{n-2} \frac{r+\rho}{r-\rho} u(y),$$

is valid for all $x \in B(y, \rho) \subset B$.

For a proof of this theorem, we refer to [**Kass06**]. Notice that in the complex plane $\mathbb{C}$ this inequality reads as

$$\frac{r-\rho}{r+\rho} u(y) \leq u(x) \leq \frac{r+\rho}{r-\rho} u(y). \tag{1.5}$$

Finally, we will use also the Marcinkiewicz Interpolation Theorem in [**Mar39**]

Theorem 1.1.15 (Marcinkiewicz Interpolation Theorem). *Let $p < q < \infty$. Let T be a bounded operator of weak type (p, p) and at the same time of weak type (q, q). Then T is of strong type (r, r) for any r between p and q, with $\|T\|_{\mathcal{L}^r \to \mathcal{L}^r} \leq \|T\|^{\delta}_{\mathcal{L}^p \to \mathcal{L}^{p,\infty}} \|T\|^{1-\delta}_{\mathcal{L}^q \to \mathcal{L}^{q,\infty}}$, being δ a function of p, q, r.*

If $q = \infty$ the same holds as long as T is of strong type (∞, ∞).

1.2. Some Covering Lemmas

Covering theorems are extremely useful in analysis. In this section we give the proofs for two classical results on coverings. The arguments below are based on the ideas of [**Guz75**] but they are modified to fit into the proofs we will give in Section 2.3.

The first one is due to Besicovitch (see [**Bes45**] and [**Bes46**]). It deals only with certain properties of some coverings by cubes or balls in $\mathbb{R}^n$. It is found in the literature easily for the case of balls, but it is not very frequent to find the proof of its version for cubes. Here we provide it. The second is a very useful lemma due to Whitney about decompositions of open sets into disjoint cubes with diameter proportional to the distance to the border of the set. We give a generalization of the version in [**Guz75**] to be able to fit it later in our proof of the Main Theorem of Chapter 2.

Finally we state the Vitali's Theorem for non-finite coverings, skipping the proof as it does appear often in the literature.

1.2.1. Besicovitch Covering Theorem for Cubes.

Theorem 1.2.1. *Let A be a bounded set in $\mathbb{R}^n$. For each $x \in A$ a closed cubic interval $Q(x)$ centered at x is given. Then one can choose, among the given cubes $\{Q(x)\}_{x\in A}$, a sequence $\{Q_k\}_{k\in\mathbb{N}}$ (possibly finite) such that:*

(1) *The set A is covered by the sequence, i.e. $A \subset \bigcup_k Q_k$*
(2) *No point of $\mathbb{R}^n$ is in more than Θ_n (a number that depends only on the dimension n) cubes of the sequence $\{Q_k\}_{k\in\mathbb{N}}$, i.e. for all $z \in \mathbb{R}^n$,*

$$\sum_{k\in\mathbb{N}} \chi_{Q_k}(z) \leqslant \Theta_n$$

(3) *The sequence $\{Q_k\}_{k\in\mathbb{N}}$ can be distributed in ξ_n (a number that depends only on n) families of disjoint cubes.*

Proof. Let $a_0 = \sup\{\ell(Q(x)) : x \in A\}$. If $a_0 = \infty$, as A is bounded, there exists an $x \in A$ with $\operatorname{diam}(A) < \frac{\ell(Q(x))}{2}$, so $A \subset Q(x)$.

Otherwise, for $a_0 < \infty$, we choose $Q_1 \in \{Q(x)\}_{x\in A}$ with center $x_1 \in A$ such that $\ell(Q_1) > \frac{a_0}{2}$.

Let now, by induction,

$$a_i = \sup\left\{\ell(Q(x)) : x \in A \setminus \bigcup_{j=1}^{i} Q_j\right\} \tag{1.6}$$

and choose $Q_{i+1} \in \{Q(x)\}_{x\in A\setminus\bigcup_{j=1}^{i} Q_j}$ with center $x_{i+1} \in A \setminus \bigcup_{j=1}^{i} Q_j$ such that $\ell(Q_{i+1}) > \frac{a_i}{2}$.

Of course, this definition implies that

$$x_i \notin Q_j, \quad \text{for all } 1 \leq j < i, \tag{1.7}$$

but x_j may belong to Q_i. However, taking into account that $\{a_k\}$ is not increasing and that

$$a_{j-1} = \sup \left\{ \ell(Q(x)) : x \in A \setminus \bigcup_{k=1}^{j-1} Q_k \right\} \geq \ell(Q_j) > \frac{a_{j-1}}{2},$$

we get

$$\ell(Q_j) > \frac{a_{j-1}}{2} \geq \frac{a_{i-1}}{2} \geq \frac{\ell(Q_i)}{2} \tag{1.8}$$

but as $x_i \notin Q_j$,

$$d_\infty(x_i, x_j) > \frac{\ell(Q_j)}{2}.$$

Thus, for all $z \in \mathbb{C}$,

$$\frac{\ell(Q_j)}{6} + \frac{\ell(Q_i)}{6} < \frac{\ell(Q_j)}{3} + \frac{\ell(Q_j)}{6} = \frac{\ell(Q_j)}{2} < d_\infty(x_i, x_j) \leq \operatorname{dist}_\infty(x_i, z) + \operatorname{dist}_\infty(z, x_j),$$

so either $\operatorname{dist}_\infty(x_i, z) > \frac{\ell(Q_i)}{6}$ or $\operatorname{dist}_\infty(x_j, z) > \frac{\ell(Q_j)}{6}$, implying that

$$\frac{1}{3} Q_i \bigcap \frac{1}{3} Q_j = \emptyset. \tag{1.9}$$

Let us prove now that this sequence satisfies the three conditions:

(1) If $\{Q_k\}_k$ was finite, by construction, we would have $A \subset \bigcup_k Q_k$. Suppose that, otherwise, it is non finite. Being A bounded we have that $\bigcup_k Q_k$ is also bounded so

$$\ell(Q_i) \overset{i\to\infty}{\longrightarrow} 0, \tag{1.10}$$

since otherwise the measure of the union of the $\frac{1}{3}$-scaled cubes, by (1.9), would be non-finite, contradicting its boundedness. Let $x_0 \in A \setminus \bigcup_{k=1}^\infty Q_k$. By (1.10) there exists $i \in \mathbb{N}$ such that $\ell(Q(x_0)) > 2\ell(Q_i) > a_{i-1}$, contradicting the definition of a_{i-1} (1.6).

(2) Let us take a fixed point $x \in A$ and let us see how many cubes may contain it. To do so, consider H to be one of the 2^n closed hyperquadrants into which the n hyperplanes through x and parallel to the coordinate hyperplanes divide $\mathbb{R}^n$, and let us find a bound to the number of cubes with center on that hyperquadrant that contain x.

To do so, consider the subcollection $\{Q_i\}_{i\in I}$ with $I = \{i \in \mathbb{N} : x \in Q_i,\ \text{and } x_i \in H\}$. Notice that this implies that the part of the cube centered at x and with the same side-length as a given Q_i is contained in Q_i, that is $H \cap Q\left(x, \frac{\ell(Q_i)}{2}\right) \subset H \cap Q_i$. Let j be the smallest index in I. Then, for all other $i \in I \setminus \{j\}$, and thus $i > j$, we can apply (1.7), so $x_i \notin Q\left(x, \frac{\ell(Q_j)}{2}\right)$, which means that $\frac{\ell(Q_j)}{2} < \operatorname{dist}_\infty(x_i, x) \leq \frac{\ell(Q_i)}{2}$. Using (1.8) we get

$$\ell(Q_j) < \ell(Q_i) < 2\ell(Q_j).$$

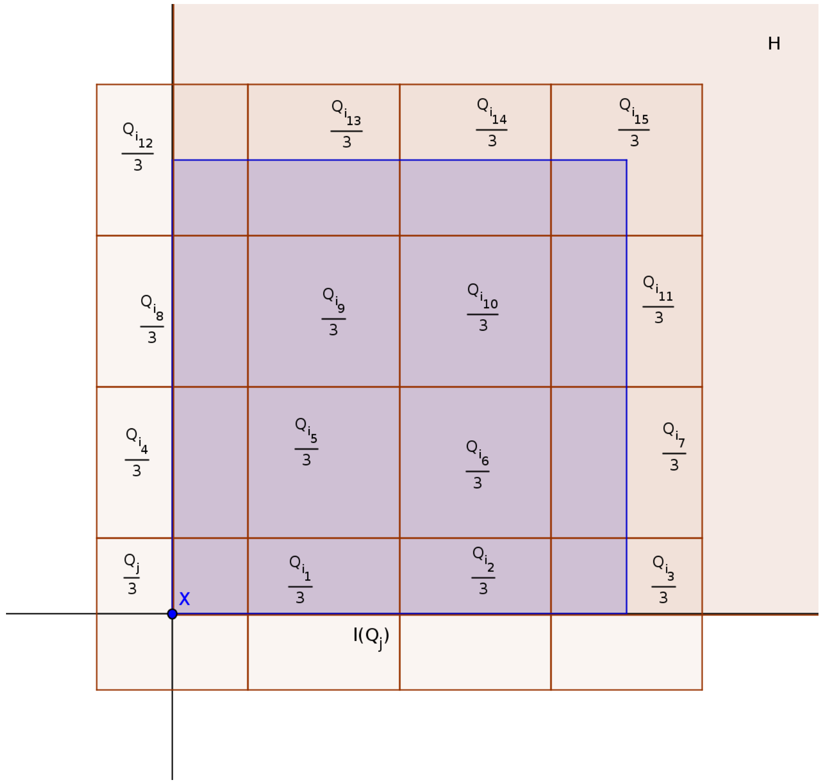


Figure 1.1: The shaded zone shows the area where the centers of the cubes Q_{i_k}, for $i_k \in I \setminus \{j\}$, may be. That is at a distance smaller that $\ell(Q_j)$. This distribution of the $\frac{1}{3}$-scaled cubes for the case $n = 2$ is too tight, as overlapping appears in the borders and the side-lengths are equal. The maximum number of squares intersected by a line parallel to the axis is 3.

As a consequence of (1.9), $\{\frac{Q_i}{3}\}_{i\in I}$ is a disjoint family of closed cubes, each of them of side-length bigger or equal than $\frac{\ell(Q_j)}{3}$ and center close to x at a distance smaller than $\frac{\ell(Q_i)}{2} < \ell(Q_j)$. There are less than 3^n of such cubes (see Figure 1.1).

So we have $\Theta_n \leq 2^n \cdot 3^n = 6^n$. This bound may be improved but it is not our aim here to do so.

(3) Let us consider a fixed $Q_i \in \{Q_k\}$. According to the previous result, at most Θ_n members of the sequence contain a fixed point of Q_i. Each cube of $\{Q_j\}_{j<i}$ is of size bigger than $\frac{Q_i}{2}$, so if $Q_j \cap Q_i \neq \emptyset$, then Q_j contains at least one of the 3^n vertices of the first dyadic sons of Q_i (see Figure 1.2). Hence, for each Q_i there are at most $\xi_n \leq 3^n \cdot \Theta_n$ cubes of the collection $\{Q_j\}_{j<i}$ with non empty intersection with Q_i. Now we split the first $\xi_n + 1$ cubes

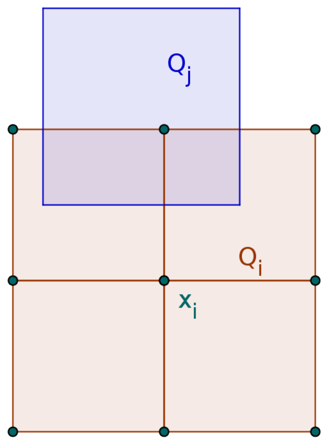


Figure 1.2: First dyadic division of Q_i.

of the collection into different subcollections and then, inductively, we can classify each cube into a subcollection where there is no cube intersecting it.

□

1.2.2. Whitney Covering.

THEOREM 1.2.2. *Let E be a proper closed subset of $\mathbb{R}^n$ and denote Ω its complementary and $c > 1$ be a given real number.*

Then, there exists a collection of cubes $\mathcal{F} = \{Q_k\}_{k\in\mathbb{N}}$ such that

(1) $\bigcup_{k\in\mathbb{N}} Q_k = \Omega$.
(2) *Cubes of $\mathcal{F}$ are mutually disjoint two by two.*
(3) *For all $Q \in \mathcal{F}$, we have*

$$(c-1)\cdot \ell(Q) < \operatorname{dist}_\infty(Q,E) \leq 2c\cdot \ell(Q). \tag{1.11}$$

PROOF. We consider the layers Ω_k defined by

$$\Omega_k = \left\{x \in \Omega : 2^{-k}c < \operatorname{dist}_\infty(x,E) \leq 2^{-k+1}c\right\}.$$

Notice that $\Omega = \biguplus_{k=-\infty}^{\infty} \Omega_k$.

We now make an initial choice of cubes, denoting the resulting collection by $\mathcal{F}_0$. Our choice is made as follows. We consider the dyadic cubes of the mesh $\mathcal{D}_k$ as in Definition 1.1.3, each of size 2^{-k}, and include a cube of this mesh in $\mathcal{F}_0$ if it intersects

Ω_k (the points of the latter are all approximately at a distance 2^{-k} from E). That is

$$\mathcal{F}_0 = \bigcup_{k\in\mathbb{Z}} \{Q \in \mathcal{D}_k : Q \cap \Omega_k \neq \emptyset\}.$$

Then we have

$$\Omega \subset \bigcup_{Q\in\mathcal{F}_0} Q. \tag{1.12}$$

Suppose $Q \in \mathcal{D}_k \cap \mathcal{F}_0$. Then $\ell(Q) = 2^{-k}$ and there exists $x \in Q \cap \Omega_k$. Thus, $\operatorname{dist}_\infty(Q, E) \leq \operatorname{dist}_\infty(x, E) \leq c \cdot 2^{-k+1}$ and $\operatorname{dist}_\infty(Q, E) \geq \operatorname{dist}_\infty(x, E) - \ell(Q) > c \cdot 2^{-k} - 2^{-k} = (c-1) \cdot 2^{-k}$. Summing up,

$$(c-1) \cdot \ell(Q) < \operatorname{dist}_\infty(Q, E) \leq 2c \cdot \ell(Q).$$

For $c > 1$ this grants that cubes in $\mathcal{F}_0$ are disjoint from E. This statement together with (1.12) imply that $\mathcal{F}_0$ satisfy both (1) and (3). Now only remains taking a subcollection of *mutually disjoint* cubes that keeps verifying these assertions.

First of all, as we are treating with dyadic cubes, we know that if two cubes aren't *mutually disjoint*, then one is contained into the other. So let us start with some cube $Q \in \mathcal{F}_0$ and consider the subfamily of cubes of $\mathcal{F}_0$ that contain it. In view of (1.11), for any cube Q' of this family, we have

$$\ell(Q') < \frac{\operatorname{dist}_\infty(Q', E)}{c-1} \leq \frac{\operatorname{dist}_\infty(Q, E)}{c-1} \leq \frac{2c}{c-1}\ell(Q).$$

so the cubes considered have bounded length, and they are totally ordered by inclusion, so the family has a maximal element.

Now take the family $\mathcal{F} \subset \mathcal{F}_0$ of all maximal elements. By the same token they are also disjoint and so they satisfy (1), (2) and (3). □

We shall now make some observations about the family whose existence has just been granted in the last theorem.

Corollary 1.2.3. *Let Q be a cube of the family defined in the previous theorem. Then,*

$$(2c-1)\, Q \cap E = \emptyset$$

and

$$(4c+1)\, Q \cap E \neq \emptyset$$

Proof. Let z be the center of Q. Then, $\operatorname{dist}_\infty(z, E) = \operatorname{dist}_\infty(Q, E) + \frac{\ell(Q)}{2}$, so (1.11) reads as

$$(2c-1) \cdot \frac{\ell(Q)}{2} < \operatorname{dist}_\infty(z, E) \leq (4c+1) \cdot \frac{\ell(Q)}{2}$$

which proves the statement. □

Consider now $x \in \Omega$ and let us see how many cubes of $\mathcal{F}_0$ will reach it when scaling by a factor α.

Corollary 1.2.4. *Let $1 < \alpha < 2c - 3$, with $c > 2$. Then, the family $\{\alpha\, Q_k\}_{k\in\mathbb{N}}$ has finite overlapping, in the sense that $\sum_{k\in\mathbb{N}} \chi_{\alpha\, Q_k} \le C(c, \alpha, n)$, being the last a constant depending only on c, α and the dimension n.*

Proof. Consider $x \in \Omega$ and let $Q_k, Q_j \in \mathcal{F}$ such that $x \in \alpha\, Q_k \cap \alpha\, Q_j$. Notice that $\mathrm{dist}_\infty(Q_j, Q_k) \le \mathrm{dist}_\infty(x, Q_j) + \mathrm{dist}_\infty(x, Q_k) \le \frac{\alpha-1}{2}(\ell(Q_j) + \ell(Q_k))$ (see Figure 1.3). By the triangle inequality one easily gets

$$\mathrm{dist}_\infty(Q_k, E) \le \mathrm{dist}_\infty(Q_j, Q_k) + \mathrm{dist}_\infty(Q_j, E) + \ell(Q_k) + \ell(Q_j)$$

Thus, using (1.11), we get

$$\begin{aligned}\ell(Q_j) &\ge \frac{\mathrm{dist}_\infty(Q_j, E)}{2c}\\ &\ge \frac{\mathrm{dist}_\infty(Q_k, E) - \mathrm{dist}_\infty(Q_j, Q_k) - \ell(Q_j) - \ell(Q_k)}{2c}\\ &> \frac{(c-1)\ell(Q_k) - \frac{\alpha+1}{2}(\ell(Q_j) + \ell(Q_k))}{2c}.\end{aligned}$$

Reagruping, we get

$$\ell(Q_j) > \frac{2c - 3 - \alpha}{4c + \alpha + 1}\ell(Q_k).$$

Thus, for any $1 < \alpha < 2c - 3$ we have got an upper bound for $\ell(Q_k)$ with respect to $\ell(Q_j)$.

By symmetry,

$$\frac{2c - 3 - \alpha}{4c + \alpha + 1}\ell(Q_j) < \ell(Q_k) < \frac{4c + \alpha + 1}{2c - 3 - \alpha}\ell(Q_j). \tag{1.13}$$

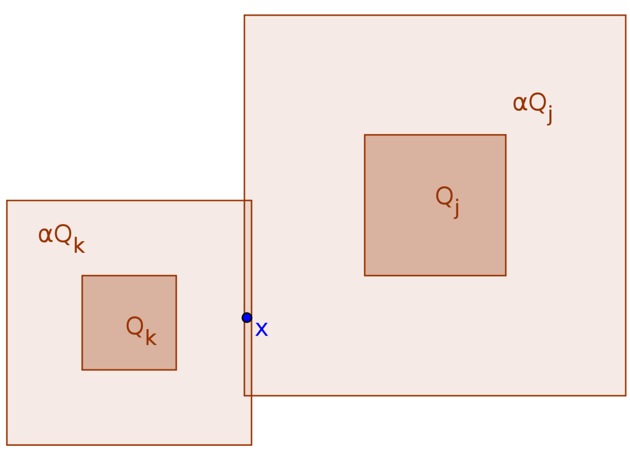


Figure 1.3: A posible configuration for x, Q_j and Q_k.

At the same time, we have

$$\operatorname{dist}_\infty(x, Q_k) \leq \frac{\alpha - 1}{2}\ell(Q_k) < \frac{(\alpha-1)(4c+\alpha+1)}{2(2c-3-\alpha)}\ell(Q_j). \tag{1.14}$$

and, considering in particular Q_j to be the cube containing x, we see that there is only a finite number of such dyadic cubes in the family $\mathcal{F}$, with the bound depending on c, α and the dimension n. □

REMARK 1.2.5. Taking for example $c = 100$ and $\alpha = 10$ we can define the family $\mathcal{F}$ presented in Section 2.3, where less restrictive conditions are required, namely

$$100Q \cap E = \emptyset \tag{1.15}$$

$$1000Q \cap E \neq \emptyset \tag{1.16}$$

$$\sum_{Q_k \in \mathcal{F}} \chi_{10\,Q_k} \leq C \tag{1.17}$$

where we consider, as we will work in dimension 2, that $C = C(100, 10, 2)$, which can be proven using the bounds (1.13) and (1.14) above to be smaller or equal to $42^2 - 3$.

1.2.3. Vitali Covering Lemma.

THEOREM 1.2.6 (Vitali Covering Lemma). *Let X be a boundedly compact metric space and $\mathcal{B}$ a family of closed balls in X such that*

$$\sup\{\operatorname{diam}(B) : B \in \mathcal{B}\} < \infty.$$

Then there is a finite or countable sequence $\{B_i\}_{i\in I} \subset \mathcal{B}$ of disjoint balls such that

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i\in I} 5B_i.$$

We will not provide any proof of this theorem here because it is easy to find in the literature. We refer in particular to [**Mat95**].

1.3. An Overview on Quasiconformal Mappings

Quasiconformal mappings are mappings where we have some control on the derivatives, namely, that the quotient between maximal and the minimal directional derivatives is uniformly bounded almost everywhere. In this section we survey the main known facts that we will use in the next chapter skipping the proofs. We refer the reader to [**AIM**, Chapters 2, 3 and 5] and the articles referred there for the details.

From now on we consider all the sets of points in $\mathbb{C}$.

1.3.1. Basic definitions.

DEFINITION 1.3.1. Let $K \geq 1$. A mapping $f : \Omega \to \Omega'$ is called K-quasiregular if

$$f \in W^{1,2}_{loc}(\Omega),$$

if it is orientation preserving, that is

$$J(z,f) = |\partial_z f|^2 - |\partial_{\bar z} f|^2 \geq 0 \quad \text{almost everywhere,}$$

and if the directional derivatives satisfy

$$\sup_{\alpha\in[0,2\pi)} |\partial_\alpha f(z)| \leq K \inf_{\alpha\in[0,2\pi)} |\partial_\alpha f(z)| \tag{1.18}$$

for almost every $z \in \Omega$. If, in addition, f is a homeomorphism, we say it is K-quasiconformal.

In particular, a mapping is 1-quasiconformal if and only if it is conformal.

REMARK 1.3.2. Being a Sobolev function only implies the existence of the derivatives f_x and f_y almost everywhere. For (1.18) to be meaningful, we just consider the alternate definition

$$\partial_\alpha f(z) = \cos(\alpha) f_x(z) + \sin(\alpha) f_y(z).$$

Anyway, all homeomorphic Sobolev functions are differenciable almost everywhere (see Theorem 1.3.12), so the definition above is equivalent with the more usual

$$\partial_\alpha f(z) = \lim_{r\to 0} \frac{f(z + r\, e^{i\alpha}) - f(z)}{r}$$

when we consider K-quasiconformal mappings.

REMARK 1.3.3. Often it is convenient to reformulate (1.18) as

$$|Df(z)|^2 \leq K J(z,f) \quad \text{for almost every } z \in \Omega, \tag{1.19}$$

or

$$|\partial_z f| + |\partial_{\bar z} f| \leq K(|\partial_z f| - |\partial_{\bar z} f|) \quad \text{for almost every } z \in \Omega,$$

or

$$|\partial_{\bar z} f| \leq k |\partial_z f| \quad \text{for almost every } z \in \Omega$$

for $k = \frac{K-1}{K+1}$.

The smallest constant $K(f)$ for which (1.18) holds is called the *distortion* of the mapping f. Writting $\mu_f(z) = \partial_{\bar{z}} f(z)/\partial_z f(z)$ when $\partial_z f(z) \neq 0$ and $\mu_f(z) = 0$ otherwise, we can rewrite the condition (1.18) as in the next theorem.

THEOREM 1.3.4. *Suppose $f : \Omega \longrightarrow \Omega'$ is a homeomorphic $W^{1,2}_{loc}$ mapping. Then f is K-quasiconformal if and only if*

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu_f(z) \frac{\partial f}{\partial z}(z) \quad \text{for almost every } z \in \Omega, \tag{1.20}$$

where μ_f, called the Beltrami coefficient of f, is a bounded measurable function satisfying

$$\|\mu_f\|_\infty \leq \frac{K-1}{K+1} < 1.$$

REMARK 1.3.5. μ_f is sometimes called *complex dilatation* of f. We can see that, for quasiconformal mappings, $\partial_z f \neq 0$ almost everywhere, implying that the complex dilatation is uniquely defined up to a set of measure zero.

Notice also that

$$\|\mu_f\|_\infty = \frac{K(f)-1}{K(f)+1}$$

and

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

The differential equation in (1.20) is called the *Beltrami equation*. It provides the connections from the geometric theory of quasiconformal mappings to complex analysis and to elliptic PDEs.

1.3.2. Radial Stretchings. There is a class of examples that is important to have at hand as they typically provide extremal examples for some results in quasiconformal mappings. These are the radial stretchings, i.e. mappings $f : \mathbb{D}_R \rightarrow \mathbb{C}$ of the form

$$f(z) = \frac{z}{|z|} \rho(|z|), \quad f(0) = 0.$$

Here the function ρ is assumed to be positive, continuous and strictly increasing. For $\rho(0) = 0$ the mapping is continuous at the origin.

We may calculate the differential and distortions of a radial stretching where the derivative $\dot{\rho}$ exists.

$$\partial_z f(z) = \frac{1}{2} \left(\dot{\rho}(|z|) + \frac{\rho(|z|)}{|z|} \right) \text{ and}$$
$$\partial_{\bar{z}} f(z) = \frac{1}{2} \frac{z}{\bar{z}} \left(\dot{\rho}(|z|) - \frac{\rho(|z|)}{|z|} \right).$$

We obtain

$$|Df(z)| = |\partial_z f(z)| + |\partial_{\bar{z}} f(z)| = \max \left\{ \dot{\rho}(|z|), \frac{\rho(|z|)}{|z|} \right\},$$

$$J(z,f) = |\partial_z f(z)|^2 - |\partial_{\bar z} f(z)|^2 = \frac{\rho(|z|)\,\dot\rho(|z|)}{|z|}, \text{ and}$$
$$\mu_f(z) = \frac{z}{\bar z}\frac{|z|\dot\rho(|z|) - \rho(|z|)}{|z|\dot\rho(|z|) + \rho(|z|)}.$$

In particular, for $\rho(t) = t^K$ and $\rho(t) = t^{1/K}$ we obtain two classical examples, namely

$$f_1(z) = z|z|^{K-1}$$

and

$$f_2(z) = z|z|^{\frac{1}{K}-1}, \tag{1.21}$$

respectively. They arise as extremals for the problems on Hölder continuity, and integrability of the differential (see Theorem 1.3.18, and Remark 2.2.3).

$$|Df_1(z)| = K|z|^{K-1},$$
$$J(z,f_1) = K|z|^{2(K-1)} \text{ and}$$
$$\mu_{f_1}(z) = \frac{z}{\bar z}\frac{K-1}{K+1},$$

so f_1 is K-quasiconformal. Being $f_2 = f_1^{-1}$ the inverse of a K-quasiconformal mapping, it is also K-quasiconformal, as we shall see later in Theorem 1.3.21.

1.3.3. The Area Formula and the Koebe $\frac{1}{4}$-Theorem.

THEOREM 1.3.6 (The Area Formula). *Suppose $f \in W^{1,2}_{loc}(\mathbb{C})$ is analytic outside the disk $\mathbb{D}_r$ and has the expansion*

$$f(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

near ∞. Then

$$\int_{\mathbb{D}_r} J(z,f) = \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right).$$

In particular, if f is orientation-preserving, then

$$\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \le r^2.$$

THEOREM 1.3.7. *Suppose $g : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, which is conformal in the exterior of the unit disk. If g has the developement $g(z) = z + b_0 + b_1 z^{-1} + \cdots$ for $|z| > 1$, then*

$$g(\mathbb{D}) \subset B(b_0, 2).$$

THEOREM 1.3.8 (Koebe $\frac{1}{4}$-Theorem). *Suppose that $\varphi : \mathbb{D} \to \mathbb{C}$ is conformal and normalized by $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then*

$$\mathbb{D}_{\frac{1}{4}} \subset \varphi(\mathbb{D}).$$

We may view Theorem 1.3.7 as the counterpart to Koebe's result at ∞. In bounded domains the following version of the Koebe's $\frac{1}{4}$-Theorem applies in fact to all conformal mappings, independently of their normalization.

THEOREM 1.3.9. *Suppose that f is conformal in a domain $\Omega \subsetneq \mathbb{C}$ with $f(\Omega) = \Omega' \subset \mathbb{C}$. Let $z_0 \in \Omega$. Then*

$$\frac{1}{4}|f'(z_0)|\mathrm{dist}(z_0, \partial\Omega) \le \mathrm{dist}(f(z_0), \partial\Omega') \le |f'(z_0)|\mathrm{dist}(z_0, \partial\Omega).$$

1.3.4. Quasisymmetry. The very definition of quasiconformality supposes the map to be defined in an open set. However, sometimes we have to deal with different configurations that require a more general point of view. As we will see, the definition of quasisymmetry below is locally equivalent to quasiconformality and provides new information which will be basic for the purposes of Chapter 2.

DEFINITION 1.3.10. Let $\eta : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism, $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$ a mapping. We say f is η-quasisymmetric if for each triple $z_0, z_1, z_2 \in A$ we have

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \le \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right). \tag{1.22}$$

Should f be defined on an open set, we will assume that it is orientation preserving and further, we say that f is quasisymmetric if there is some η as above for which f is η-quasisymmetric.

LEMMA 1.3.11. *Let $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$ an η-quasisymmetric mapping. Then we have the following properties:*

- *f is a homeomorphism from A onto its image.*
- *If A is open, $z_0 \in A$ and $B(z_0, r) \subset A$, we have that*

$$\frac{\max_{\{|z-z_0|=r\}} |f(z) - f(z_0)|}{\min_{\{|z-z_0|=r\}} |f(z) - f(z_0)|} \le \eta(1). \tag{1.23}$$

- *f^{-1} is σ-quasisymmetric with*

$$\sigma(t) = \frac{1}{\eta^{-1}(1/t)}. \tag{1.24}$$

- *If f is entire ($A = \mathbb{C}$) then it is a surjection.*
- *If $B \subset A$ is a disk,*

$$\mathrm{diam}(f(B))^2 \le C_0|f(B)|, \tag{1.25}$$

where the constant $C_0 = \frac{4}{\pi}\eta(1)^2$. That is, for all points $z \in B(z_0, s) \subset A$,

$$|f(z) - f(z_0)|^2 \le \frac{4}{\pi}\eta(1)^2|f(B(z_0, s))|.$$

- *If* $\Omega = f(B)$ *is the image of a disk* $B = B(z_0, r)$ *under a quasisymmetric mapping, then*

$$\frac{1}{\eta(1)}\,B\,(w_0, R) \subset \Omega \subset B(w_0, R), \tag{1.26}$$

where $w_0 = f(z_0)$ *and the radius* R *is defined as*

$$R := \max_{|z-z_0|=r} |f(z) - f(z_0)|.$$

- *Let* $s > 0$ *and* $B \subset \Omega$ *a disk for which* $s\,B \subset \Omega$*. Then*

$$|f(B)| \leq \eta^2(1/s)|f(s\,B)|. \tag{1.27}$$

All this properties are quite easy to prove taking (1.22) into account and will be useful later on the text.

1.3.5. The Gehring-Lehto Theorem.

THEOREM 1.3.12 (Gehring Lehto Theorem). *Let* $f : \Omega \to \mathbb{C}$ *be a continuous open mapping. Then* f *is differentiable almost everywhere in* Ω *if and only if* f *has finite first partial derivatives almost everywhere.*

COROLLARY 1.3.13. *Every homeomorphism* $f \in W^{1,1}_{loc}(\Omega)$ *is differenciable almost everywhere.*

1.3.6. Relation Between Quasiconformality and Quasisymmetry. There are three main theorems connecting quasiconformal maps and quasisymmetric ones. Namely that all quasisymmetric maps are quasiconformal, that global quasiconformal maps are quasisymmetric and that both conditions are locally equivalent. We state them below.

THEOREM 1.3.14. *Suppose that* $f : \Omega \to \Omega'$ *is an* η*-quasisymmetric mapping. Then* f *is quasiconformal. In particular,* $f \in W^{1,2}_{loc}(\Omega)$.

THEOREM 1.3.15. *Suppose* $f : \mathbb{C} \to \mathbb{C}$ *is a* K*-quasiconformal homeomorphism. Then* f *is* η_K*-quasisymmetric, where* η_K *depends only on* K.

THEOREM 1.3.16. *Suppose* $f : \Omega \to \Omega'$ *is a homeomorphism and suppose* $z_0 \in \Omega$ *with* $B(z_0, 2r) \subset \Omega$*. Let* $B = B(z_0, r)$*. If* f *is* K*-quasiconformal in* Ω*, then the restriction* $f|_B$ *is* η*-quasisymmetric, where* $\eta = \eta_K$ *depends only on* K*. Conversely, if* f *is* η*-quasisymmetric, then* $f|_B$ *is* K*-quasiconformal, where* $K = K(\eta)$.

1.3.7. Hölder Regularity.

DEFINITION 1.3.17. If we write

$$\mathcal{F} = \{f : \mathbb{C} \to \mathbb{C},\ K\text{-quasiconformal},\ f(0) = 0 \text{ and } f(1) = 1\},$$

then we define the circular distortion of $\mathcal{F}$ as

$$\lambda(K) = \sup\{|f(e^{i\phi})| : f \in \mathcal{F}, 0 \leq \phi \leq 2\pi\}.$$

THEOREM 1.3.18. *Suppose that* $f : \mathbb{C} \to \mathbb{C}$ *is* K*-quasiconformal with* $f(0) = 0$ *and* $f(1) = 1$. *Then*

$$|f(z)| \leq \lambda(K)^2 |z|^{1/K}.$$

The bound is sharp, as f_2 defined as in (1.21) attains the equality.

COROLLARY 1.3.19. *Every* K*-quasiconformal mapping* $f : \Omega \to \Omega'$ *is locally* $\frac{1}{K}$*-Hölder continuous. More precisely, if a disk* $B \subset 2\,B \subset \Omega$, *then*

$$|f(z) - f(w)| \leq C(K)\mathrm{diam}(f(B))\frac{|z-w|^{1/K}}{\mathrm{diam}(B)^{1/K}}, \quad \forall z, w \in B,$$

where the constant $C(K)$ *depends only on* K.

COROLLARY 1.3.20. *Let* f *be a* K*-quasiconformal entire mapping, and let* $\eta(t) = \eta_K(t) = \lambda(K)^{2K} \max\{t^K, t^{1/K}\}$. *Then* f *is an* η*-quasisymmetric mapping.*

1.3.8. Fundamental Properties of Quasiconformal Mappings.

Now, some fundamental properties of the quasiconformal mappings are embodied in the next two theorems:

THEOREM 1.3.21. *Let* $f : \Omega \to \Omega'$ *be a* K*-quasiconformal mapping from the domain* $\Omega \subset \mathbb{C}$ *onto* $\Omega' \subset \mathbb{C}$ *and let* $g : \Omega' \to \mathbb{C}$ *be a* K'*-quasiconformal mapping. Then*

1. $f^{-1} : \Omega' \to \Omega$ *is* K*-quasiconformal.*
2. $g \circ f : \Omega \to \mathbb{C}$ *is* (KK')*-quasiconformal.*
3. *For all measurable sets* $E \subset \Omega$, $|E| = 0$ *if and only if* $|f(E)| = 0$. *In other words,* f *satisfies both conditions* $\mathcal{N}$ *and* $\mathcal{N}^{-1}$.
4. *The Jacobian determinant* $J(z, f) > 0$ *almost everywhere in* Ω. *In particular,*
$$|f(E)| = \int_E J(\cdot, f)$$
for all measurable subsets $E \subset \Omega$.

THEOREM 1.3.22. *Let* $\{f_\nu\}_{\nu \in \mathbb{N}}$ *be a locally bounded sequence of* K*-quasiconformal mappings* $f_\nu : \Omega \to \mathbb{C}$ *defined on* $\Omega \subset \mathbb{C}$. *Then there is a subsequence converging locally uniformly on* Ω *to a mapping* f *and it is either a* K*-quasiconformal mapping or a constant.*

1.3.9. The Measurable Mapping Theorem.

DEFINITION 1.3.23. Suppose that $0 \leq k < 1$ and that $|\mu(z)| \leq k\,\chi_{\mathbb{D}_r}(z)$, $z \in \mathbb{C}$. We say that $f \in W^{1,2}_{loc}(\mathbb{C})$ is a *principal solution* to the Beltrami equation $\partial_{\bar{z}} f(z) = \mu(z)\partial_z f(z)$ if it is normalized by the condition $f(z) = z + \mathcal{O}(1/z)$ near infinity.

The existence of principal solutions for any $|\mu(z)| \leq k\,\chi_{\mathbb{D}_r}(z)$ is proven in [**AIM**, Chapter 5] using (1.2) and giving place to the representations

$$f = z + \mathcal{C}(\partial_{\bar{z}} f),$$

which implies

$$\partial_z f = 1 + \mathcal{S}(\partial_{\bar z} f),$$

and thus

$$\partial_{\bar z} f = \mu\, \partial_z f = \mu + \mu\mathcal{S}(\partial_{\bar z} f).$$

Writing $\partial_{\bar z} f = (\mathrm{Id} - \mu\mathcal{S})^{-1}(\mu)$ and using the standard Neumann series

$$(\mathrm{Id} - \mu\mathcal{S})^{-1} = \mathrm{Id} + \mu\mathcal{S} + \mu\mathcal{S}\,\mu\mathcal{S} + \mu\mathcal{S}\,\mu\mathcal{S}\,\mu\mathcal{S} + \cdots, \tag{1.28}$$

one obtains

$$\partial_{\bar z} f = \mu + \mu\mathcal{S}(\mu) + \mu\mathcal{S}(\mu\mathcal{S}(\mu)) + \cdots. \tag{1.29}$$

The equation (1.3) and the Riesz-Thorin Interpolation Theorem [**Gra08**, p. 72-79, II.4] imply that the map $p \to \|\mathcal{S}\|_{\mathcal{L}^p(\mathbb{C})\to\mathcal{L}^p(\mathbb{C})}$ is continuous in p and so there exist two Hölder conjugate numbers such that $1 < P(k) < 2 < Q(k) < \infty$ with $k\|\mathcal{S}\|_{\mathcal{L}^p(\mathbb{C})\to\mathcal{L}^p(\mathbb{C})} < 1$ for all $p \in (P(k), Q(k))$. The importance of those exponents is related to the expression (1.28) above, which is a series of bounded operators in $\mathcal{L}^p$ that converges in the operators norm when $p \in (P(k), Q(k))$.

Lemma 1.3.24. *Suppose $|\mu|$, $|\nu| \le k\chi_{\mathbb{D}_r}$, where $0 \le k < 1$. Let $f, g \in W^{1,2}_{loc}(\mathbb{C})$ be the principal solutions to the equations*

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z), \quad \partial_{\bar z} g(z) = \nu(z)\,\partial_z g(z)$$

If for a number $s > 1$ we have $2 \le p < ps < P(k)$, then

$$\|\partial_{\bar z} f - \partial_{\bar z} g\|_{\mathcal{L}^p(\mathbb{C})} \le C(p,s,k) r^{\frac{2}{ps}} \|\mu - \nu\|_{\mathcal{L}^{ps/(s-1)}(\mathbb{C})}$$

Theorem 1.3.25 (Measurable Riemann Mapping Theorem). *Let $|\mu| \le k < 1$ be compactly supported and defined on $\mathbb{C}$. Then there is a unique principal solution to the Beltrami equation*

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z) \ \text{ for almost every } z \in \mathbb{C},$$

and the solution $f \in W^{1,2}_{loc}(\mathbb{C})$ is a K-quasiconformal homeomorphism of $\mathbb{C}$.

Remark 1.3.26. We can soften the conditions above, in particular μ doesn't need to be compactly supported. In that case we don't have principal solution to the Beltrami equation but a unique K-quasiconformal solution arises if we force it to be normalized by $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

1.3.10. Factorizations.

Theorem 1.3.27 (Stoilow Factorization). *Let $f : \Omega \to \Omega'$ be a homeomorphic solution to the Beltrami equation*

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z) \ \text{ for almost every } z \in \Omega, \tag{1.30}$$

with $f \in W^{1,1}_{loc}(\Omega)$ and $\|\mu\|_\infty \le k < 1$. Suppose $g \in W^{1,2}_{loc}(\Omega)$ is another solution to (1.30) on Ω. Then there exists a holomorphic function $\Phi : \Omega' \to \mathbb{C}$ such that

$$g(z) = \Phi(f(z)), \quad z \in \Omega.$$

Conversely, if Φ is holomorphic on Ω', then the composition $\Phi \circ f$ is a $W^{1,2}_{loc}$-solution to (1.30) in the domain Ω.

To prove the next theorem one may use the chain rule that we recall here for convenience as we will use it later on. For any f, g functions of complex variable,

$$\partial_z (f \circ g)(z) = \partial_w f(g(z)) \partial_z g(z) + \partial_{\bar{w}} f(g(z)) \overline{\partial_{\bar{z}} g(z)}, \tag{1.31}$$

$$\partial_{\bar{z}} (f \circ g)(z) = \partial_w f(g(z)) \partial_{\bar{z}} g(z) + \partial_{\bar{w}} f(g(z)) \overline{\partial_z g(z)}, \tag{1.32}$$

whenever composition has sense and the appropriate pointwise derivatives exist.

Theorem 1.3.28. *Suppose $f : \Omega \to \Omega'$ is quasiconformal and $g : \Omega \to \mathbb{C}$ is quasiregular, with Beltrami coefficients μ_f and μ_g, respectively. Then the composition $g \circ f^{-1}$ is quasiregular in Ω', with Beltrami coefficient*

$$\mu_{g \circ f^{-1}}(w) = \frac{\mu_g(z) - \mu_f(z)}{1 - \mu_g(z)\overline{\mu_f(z)}} \left(\frac{\partial_z f(z)}{|\partial_z f(z)|} \right)^2, \quad \textit{where } w = f(z).$$

Theorem 1.3.29 (Factoring with Small Distortion). *Let $f : \Omega \to \Omega'$ be K-quasiconformal and let $n \in \mathbb{N}$. Then we can write*

$$f = f_1 \circ f_2 \circ \cdots \circ f_n, \tag{1.33}$$

where each $f_i : f_{i+1} \circ \cdots \circ f_n(\Omega) \to \mathbb{C}$ is $K^{1/n}$-quasiconformal.

1.3.11. Analytic Dependence on Parameters.

Theorem 1.3.30. *Suppose μ is a compactly supported measurable function with $\|\mu\|_\infty = k < 1$. For $\lambda \in \mathbb{D}$, let $f(\lambda, z) = f^\lambda(z)$ be the principal solution to the Beltrami equation*

$$\partial_{\bar{z}} f(z) = \frac{\lambda}{k} \mu(z) \partial_z f(z).$$

If μ vanishes in a neighbourhood U of a point z_0, then the derivative $\partial_z f^\lambda(z)$ of the analytic function $z \mapsto f^\lambda(z)$, $z \in U$, depends holomorphically on $\lambda \in \mathbb{D}$.

CHAPTER 2

On the Distortion of Sets Under Quasiconformal Mappings

2.1. Area Distortion

In this section we focus our attention on what influence has a quasiconformal mapping on the area of a bounded measurable set. Thus we will find a sharp bound related to the Beltrami coefficient of the mapping. To reach a general theorem we will have to study first some easier configurations. In turns out to be useful to study $|f(E)|$ in two complementary situations, known as the conformal inside and the conformal outside. First we assume f is conformal outside.

Theorem 2.1.1. *Suppose f is a K-quasiconformal principal mapping of $\mathbb{C}$ that is conformal outside a compact subset E. Then we have*

$$|f(E)| \leq K|E|.$$

Proof. We shall estimate the area $|f(E)|$ with the help of the identity

$$\partial_z f = 1 + \mathcal{S}(\partial_{\bar z} f),$$

which implies $|\partial_z f|^2 = 1 + |\mathcal{S}(\partial_{\bar z} f)|^2 + 2\mathrm{Re}\langle 1, \mathcal{S}(\partial_{\bar z} f)\rangle$. This gives

$$\begin{aligned} |f(E)| &= \int_E J(z,f) = \int_E (|\partial_z f|^2 - |\partial_{\bar z} f|^2) \\ &= |E| + \int_E 2\mathrm{Re}(\mathcal{S}(\partial_{\bar z} f)) + \int_E (|\mathcal{S}(\partial_{\bar z} f)|^2 - |\partial_{\bar z} f|^2) \\ &\leq |E| + 2\int_E |\mathcal{S}(\partial_{\bar z} f)|. \end{aligned} \tag{2.1}$$

where we used the fact that $\|\mathcal{S}\|_{\mathcal{L}^2(\mathbb{C})} = 1$ when we considered that $\int_E |\mathcal{S}(\partial_{\bar z} f)|^2 \leq \int_{\mathbb{C}} |\mathcal{S}(\partial_{\bar z} f)|^2 \leq \int_{\mathbb{C}} |\partial_{\bar z} f|^2 = \int_E |\partial_{\bar z} f|^2$.

To bound the integral we use the representation of $\partial_{\bar z} f$ as a power series in μ in the equation (1.29)

$$\mathcal{S}(\partial_{\bar z} f) = \mathcal{S}(\mu) + \mathcal{S}(\mu\mathcal{S}(\mu)) + \cdots . \tag{2.2}$$

Observe the inequality we get using Hölder inequality:

$$\int_E |\mathcal{S}(\mu g)| \leq \sqrt{E}\|\mathcal{S}(\mu g)\|_{\mathcal{L}^2(\mathbb{C})} \leq \sqrt{E}\|\mu g\|_{\mathcal{L}^2(\mathbb{C})} \leq \sqrt{E}\|\mu\|_\infty \left(\int_E |g|^2\right)^{1/2}.$$

The inequality is applied inductively to $g = 1$, $g = \mathcal{S}(\mu)$, and so on. Using the triangle inequality in (2.2) together with (2.1),

$$f(E) \leq |E| + 2\|\mu\|_\infty |E| + 2\|\mu\|_\infty^2 |E| + \cdots = |E| \left(1 + \frac{2\|\mu\|_\infty}{1 - \|\mu\|_\infty}\right) = K(f)|E|.$$

□

Now we focus on the case where f is conformal inside E (and also outside the unit disk).

THEOREM 2.1.2. *Suppose f is a principal K- quasiconformal mapping of $\mathbb{C}$ that is conformal outside the unit disk $\mathbb{D}$. Assume also that we are given a measurable set $E \subset \mathbb{D}$ and a non-negative weight w defined on E. If $f|_E$ is conformal, then*

$$\left(\frac{1}{\pi}\int_E w(z)^{1/K}\right)^K \leq \frac{1}{\pi}\int_E J(z,f)\, w(z) \leq \left(\frac{1}{\pi}\int_E w(z)^K\right)^{1/K}. \tag{2.3}$$

PROOF. We first establish the distortion estimate for open sets E and prove the general case subsequently by approximation. Moreover, we need to consider only weigth functions $w \geq 0$ that are bounded away from 0 and ∞ on the set E. The argument for a general w follows by an obvious limiting argument.

Suppose that the weigth $w(z)$ and the mapping f are given, with $\partial_{\bar z} f = 0$ for almost every z in the open set E. Our goal is to show that

$$\frac{1}{\pi}\int_E w(z) J(z,f) \leq \left(\frac{1}{\pi}\int_E w(z)^K\right)^{1/K} \tag{2.4}$$

For this let μ be the complex dilatation of f, so that

$$\partial_{\bar z} f(z) = \mu(z)\partial_z f(z), \quad |\mu| \leq \frac{K-1}{K+1}\chi_{\mathbb{D}\setminus E}. \tag{2.5}$$

Then, for each number $|\lambda| < 1$, we write

$$\mu_\lambda(z) = \lambda \frac{K+1}{K-1}\mu(z),$$

$z \in \mathbb{C}$, and consider the principal solution $f^\lambda \in W^{1,2}_{loc}(\mathbb{C})$ to the Beltrami equation

$$\partial_{\bar z} f^\lambda(z) = \mu_\lambda(z)\partial_z f^\lambda(z).$$

In particular, for $\lambda = k = \frac{K-1}{K+1}$, we have $f^\lambda = f$ and, for $\lambda = 0$, we have $f^\lambda(z) = z$. Furthermore, by Theorem 1.3.30, for each fixed $z \in E$ the function $\lambda \mapsto (f^\lambda)'(z) = \partial_z f^\lambda(z)$ is holomorphic in $\mathbb{D}$. Note also that by conformality on the open set E,

$$(f^\lambda)'(z) \neq 0 \quad \text{for all } z \in E \text{ and } \lambda \in \mathbb{D}. \tag{2.6}$$

In fact, as it is enough to prove the claim for all compact subsets $E' \subset E$, we may assume that on E the derivatives $|(f^\lambda)'(z)|$ are bounded away from 0 and ∞, with constants depending on λ.

We will approach (2.4) by expressing it in a logarithmic form, which allows a decomposition of the integral. In fact, taking

$$a(z) = \frac{1}{\pi} w(z) J(z, f^\lambda) = \frac{1}{\pi} w(z) |(f^\lambda)'(z)|^2$$

for $z \in E$, Jensen's inequality (1.4) gives

$$\begin{aligned} &\log\left(\frac{1}{\pi}\int_E w(z)|(f^\lambda)'(z)|^2\right) \\ &\qquad = \sup_p \left(\int_E p(z)\log w(z) + \int_E p(z)\log\left(\frac{1}{\pi}\frac{|(f^\lambda)'(z)|^2}{p(z)}\right)\right), \end{aligned} \tag{2.7}$$

where the supremum is taken over all the functions p such that $p(z) > 0$ for almost every $z \in E$ and $\int_E p = 1$. If we take a close look at this identity, we see that the latter integral

$$h_p(\lambda) = \int_E p(z) \log\left(\frac{1}{\pi}\frac{|(f^\lambda)'(z)|^2}{p(z)}\right)$$

is harmonic in λ, by (2.6). Moreover, we can use Jensen's inequality (1.4) again and, since f^λ is analytic outside $\mathbb{D} = \mathbb{D}_1$, the area formula Theorem 1.3.6 to deduce

$$h_p(\lambda) \le \log\left(\frac{1}{\pi}\int_E |(f^\lambda)'(z)|^2\right) \le \log\left(\frac{1}{\pi}\int_{\mathbb{D}} J(z, f^\lambda)\right) \le 0.$$

We have thus seen that h_p is harmonic and nonpositive in $\mathbb{D}$.

We are now in position to use Harnack's inequality (1.5),

$$\frac{1-|\lambda|}{1+|\lambda|}(-h_p(0)) \le -h_p(\lambda) \le \frac{1+|\lambda|}{1-|\lambda|}(-h_p(0)). \tag{2.8}$$

Let us take the left hand inequality and use it in (2.7). We get, for $\lambda = k$,

$$\begin{aligned} &\log\left(\frac{1}{\pi}\int_E w(z)|(f^\lambda)'(z)|^2\right) \\ &\le \sup_p \left(\int_E p(z)\log w(z) + \frac{1-|\lambda|}{1+|\lambda|} h_p(0)\right) \\ &= \sup_p \left(\int_E p(z)\log w(z) + \frac{1-|\lambda|}{1+|\lambda|}\int_E p(z)\log\left(\frac{1}{\pi}\frac{1}{p(z)}\right)\right) \\ &= \sup_p \left(\int_E p(z)\frac{1}{K}\log\left(w(z)^K\right) + \int_E \frac{1}{K} p(z)\log\left(\frac{1}{\pi}\frac{1}{p(z)}\right)\right) \\ &= \frac{1}{K}\sup_p \left(\int_E p(z)\log\left(\frac{1}{\pi}\frac{w(z)^K}{p(z)}\right)\right). \end{aligned}$$

It remains to choose the function p, and naturally we take $p(z) = \frac{w(z)^K}{\int_E w^K}$, the function maximizing the last expression. As a result,

$$\log\left(\frac{1}{\pi}\int_E w(z)|(f^\lambda)'(z)|^2\right) \leq \frac{1}{K}\log\left(\frac{1}{\pi}\int_E w(z)^K\right).$$

Exponentiation gives the estimate (2.4) for open sets E.

To deduce the lower bound in the statement, one needs to use the right inequality in (2.8) but otherwise argue in a similar fashion.

We now consider the case of arbitrary measurable sets E for which $f|_E$ is conformal. Choose a decreasing sequence $\{E_n\}_{n=1}^\infty$ of open sets such that $E \subset E_n$ and

$$|E_n \setminus E| \xrightarrow{n\to\infty} 0$$

Set $\mu_n = \mu\chi_{\mathbb{D}\setminus E_n}$, with μ as in (2.5), and let f_n be the principal solution to the Beltrami equation with coefficient μ_n.

It follows from Lemma 1.3.24 that

$$\begin{aligned}
\|\mathcal{S}\partial_{\bar z} f_n - \mathcal{S}\partial_{\bar z} f\|_{\mathcal{L}^2(\mathbb{C})} &\leq \|\partial_{\bar z} f_n - \partial_{\bar z} f\|_{\mathcal{L}^2(\mathbb{C})} \\
&\leq C(2,s,k)\|\mu_n - \mu\|_{\mathcal{L}^{2s/(s-1)}(\mathbb{C})} \\
&\leq C\|\mu\chi_{E_n\setminus E}\|_{\mathcal{L}^{2s/(s-1)}(\mathbb{C})} \\
&\leq C|E_n \setminus E|
\end{aligned}$$

and so

$$\|\partial_z f_n - \partial_z f\|_{\mathcal{L}^2(\mathbb{C})} = \|\mathcal{S}\partial_{\bar z} f_n - \mathcal{S}\partial_{\bar z} f\|_{\mathcal{L}^2(\mathbb{C})} \to 0$$

as $n \to \infty$. Also recall that on the sets E_n and E, we have $J(z, f_n) = |\partial_z f_n(z)|^2$ and $J(z,f) = |\partial_z f(z)|^2$, respectively. Since w can be assumed to be bounded from above on the E_n's, we deduce

$$\int_E w(z)\, J(z,f) = \lim_{n\to\infty}\int_{E_n} w(z)\, J(z,f_n). \tag{2.9}$$

Clearly, for any $q > 0$,

$$\int_E w(z)^q = \lim_{n\to\infty}\int_{E_n} w(z)^q. \tag{2.10}$$

As the inequalities (2.3) hold for each of the open sets E_n, using (2.9) and (2.10) gives the proof of the Theorem 2.1.2. □

In the next theorem we combine both cases to get the general situation where f is principal and conformal outside the unit disk $\mathbb{D}$.

THEOREM 2.1.3. *Suppose f is a K-quasiconformal principal mapping of $\mathbb{C}$ that is conformal outside the unit disk $\mathbb{D}$. Let $E \subset \mathbb{D}$ be measurable. Then,*

$$\frac{|f(E)|}{\pi} \leq K\left(\frac{|E|}{\pi}\right)^{1/K} \tag{2.11}$$

PROOF. First, any measurable set can be approximated from above by open sets G and from below by closed sets F so that $|G\setminus F|$ is arbitrarily small. From Theorem 1.3.21, we know that f satisfies the Lusin condition $\mathcal{N}^{-1}$ and thus the measure $f_{\#}^{-1}m$, defined as $f_{\#}^{-1}m(A) = |f^{-1}(A)|$, is absolutely continuous with respect to the two-dimensional Lebesgue measure. Therefore we may assume that E is in fact a finite union of open disks and complete the proof by an obvious limiting argument. In particular, $|\partial(E)| = 0$.

Next, we shall use the Measurable Riemann Mapping Theorem 1.3.25 to reduce the claim to the last two theorems. Let μ be the complex dilatation of f. We construct $\mu_0 = \mu\,\chi_{\mathbb{C}\setminus E}$. If g is the principal quasiconformal mapping of $\mathbb{C}$ with complex dilatation μ_0, whose existence is granted by the Measurable Riemann Mapping Theorem, then, by Theorem 1.3.28, we have

$$f = h \circ g,$$

where h is K-quasiconformal in $\mathbb{C}$, conformal outside the compact set $g(\bar{E})$ (g is a homeomorphism) and normalized by $h(z) = z + \mathcal{O}(1/z)$.

Now,

$$\begin{aligned} |f(E)| &= |h \circ g(E)| = |h \circ g(\bar{E})| \leq K|g(\bar{E})| \\ &= K|g(E)| \leq K\pi \left(\frac{|E|}{\pi}\right)^{1/K}, \end{aligned}$$

using Lusin condition $\mathcal{N}$ in the second and the fourth steps, Theorem 2.1.1 in the third, and Theorem 2.1.2 in the last one.

The equation (2.11) has been proved for any finite union of open balls. Now, any open set E can be considered as a countable union of open balls $E = \cup_{i=1}^{\infty} B_i$, so

$$\begin{aligned} |f(E)| &= \lim_{n\to\infty} \left| f\left(\bigcup_{i=1}^{n} B_i\right)\right| \\ &\leq \lim_{n\to\infty} K\pi \left(\frac{|\bigcup_{i=1}^{n} B_i|}{\pi}\right)^{1/K} \\ &= K\pi \left(\frac{|E|}{\pi}\right)^{1/K}. \end{aligned}$$

With an analogous argument we get the estimate (2.11) for any measurable set $E \subset \mathbb{D}$. □

Considering the radial stretching $f_0 = z|z|^{1/K-1}\,\chi_{\mathbb{D}} + z\,\chi_{\mathbb{C}\setminus\mathbb{D}}$ together with the set $E = \mathbb{D}_\rho$, with $\rho < 1$, we achieve the equality

$$\frac{|f(E)|}{\pi} = \left(\frac{|E|}{\pi}\right)^{1/K}.$$

Now we can prove the main Theorem of this section:

THEOREM 2.1.4 (Area Distortion Theorem). *For evey $K \geq 1$ there is a constant C_K, depending only on K, such that for any K-quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, for any disk $B \subset \mathbb{C}$ and for any subset $E \subset B$, we have*

$$\frac{1}{C_K}\left(\frac{|E|}{|B|}\right)^K \leq \frac{|f(E)|}{|f(B)|} \leq C_K\left(\frac{|E|}{|B|}\right)^{1/K} \tag{2.12}$$

PROOF. Let us assume that f is a K-quasiconformal mapping and let E be a compact set contained in a ball B. We factorize $f = f_2 \circ f_1$, where f_1, f_2 are both K-quasiconformal maps, with f_1 principal and conformal on $\mathbb{C} \setminus 2\,B$ and f_2 is conformal on $f_1(2\,B)$. Let $g(z) = dz + b$ be the linear function that maps the unit disk to $2\,B$ (so $d = \operatorname{diam}(B)$). The function $h = g^{-1} \circ f_1 \circ g$ verifies the assumptions of Theorem 2.1.3, so that

$$|g^{-1} \circ f_1(E)| \leq C(K)|g^{-1}(E)|^{\frac{1}{K}}.$$

On the other hand,

$$|g^{-1} \circ f_1(E)| = \frac{|f_1(E)|}{\operatorname{diam}(B)^2}, \quad |g^{-1}(E)| = \frac{|E|}{\operatorname{diam}(B)^2}.$$

Using also quasisymmetry equations (1.25) and (1.27) and Theorem 1.3.7, we get that $\operatorname{diam}(f_1(B)) \approx \operatorname{diam}(f_1(2\,B)) \approx \operatorname{diam}(2\,B)$ with constants depending only on K. Hence

$$\frac{|f_1(E)|}{\operatorname{diam}(f_1(B))^2} \leq C(K)\left(\frac{|E|}{\operatorname{diam}(B)^2}\right)^{\frac{1}{K}}.$$

Now, since f_2 is conformal on $f_1(2\,B)$, for each ball B_0 contained in B we can apply Koebe's Distortion Theorem 1.3.9 to both $\Omega = f_1(2\,B)$ and $\Omega = f_1(B_0)$, with z_0 the image of the center of B_0 in both cases. The resultant equations, together with quasisymmetry equation (1.23), easily imply that

$$\frac{\operatorname{diam}(f_2(f_1(B_0)))}{\operatorname{diam}(f_2(f_1(2\,B)))} \approx \frac{\operatorname{diam}(f_1(B_0))}{\operatorname{diam}(f_1(2\,B))}.$$

From this estimate and quasisymmetry again, it is straightforward to check that

$$\frac{\mathcal{H}^2(f_1(E))}{\operatorname{diam}(f_1(B))^2} \approx \frac{\mathcal{H}^2(f(E))}{\operatorname{diam}(f(B))^2}.$$

with constants depending on K, which, taking into account that $\mathcal{H}^2$ is multiple of the Lebesgue measure in $\mathbb{C}$, leads to

$$\frac{|f(E)|}{\operatorname{diam}(f(B))^2} \leq C(K)\left(\frac{|E|}{\operatorname{diam}(B)^2}\right)^{1/K}.$$

By quasisymmetry (1.25), this leads to the left hand side of (2.12)

For the other inequality let us first look at the domain $\Omega = f(B)$. Recall from the last section that as a quasiconformal image of a disk, Ω is roughly of the size of radius $R \approx \operatorname{diam}(\Omega)$. More precisely, if z_0 is the center of B, $w_0 = f(z_0)$, let

$B' = B(w_0, R)$, with $R = \max_{\zeta\in\partial B} |f(\zeta) - w_0|$ and $\delta = \frac{1}{\eta(1)}$, using quasisymmetry (1.26),

$$\delta B' \subset \Omega = f(B) \subset B'.$$

By (1.24), the inverse mapping $g = f^{-1}$ is quasisymmetric with distortion

$$\eta_g(t) = 1/\eta^{-1}(1/t).$$

Thus, by the quasisymmetric property (1.27),

$$|g(B')| \le C_0|g(\delta B')| \le C_0|g(\Omega)|, \quad C_0 = \frac{1}{\eta^{-1}(\delta)^2},$$

where $g(\Omega) = B$. Since we already have the right-hand-side of (2.12), we can apply this to the set $f(E) \subset B'$ and obtain

$$\frac{|E|}{|B|} = \frac{|g(f(E))|}{|B|} \le C_0 \frac{|g(f(E))|}{|g(B')|} \le C_1 \left(\frac{|f(E)|}{|B'|}\right)^{1/K} \le C_1 \left(\frac{|f(E)|}{|f(B)|}\right)^{1/K},$$

using the fact that $f(B) \subset B'$ in the last step. Exponentiating the estimate gives the left-hand-side of (2.12). $\square$

2.2. Distortion of Dimension

In the last section we have obtained a bound on the distortion of the area of a bounded set under a K-quasiconformal mapping. We are now concerned about the distortion of the dimension of sets under the same circumstances. This is in part motivated by the research on the dimension of quasicircles and by the Painlevé's problem, which consists in characterizing geometrically the removable sets under K-quasiregular functions.

First we need some information about integrability of the Jacobian (using the Area Distortion Theorem) and then we will use it to follow the proof that Astala himself gave to the particular case $n = 2$ of a more general conjecture of Iwaniec and Martin for $\mathbb{R}^n$.

2.2.1. Optimal $\mathcal{L}^p$-regularity for Derivatives of Quasiconformal Mappings.

THEOREM 2.2.1. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a K-quasiconformal mapping and let $B \subset \mathbb{C}$ be a disk. Then there is a finite constant C_K, depending only on K, such that for all $t > 0$,*

$$|\{z \in B : J(z, f) > t\}| \leq C_K |B|^{1/(1-K)} \left(\frac{|f(B)|}{t} \right)^{K/(K-1)}$$

and so $J(\cdot, f) \in weak - \mathcal{L}^p_{loc}$.

PROOF. Let $E_t = \{z \in B : J(z, f) > t\} \subset B$. From Theorem 2.1.4 we have

$$t|E_t| \leq \int_{E_t} J(z, f) = |f(E_t)| \leq C_K |f(B)| \left(\frac{|E_t|}{|B|} \right)^{1/K} .$$

Solving for $|E_t|$ proves the required estimate. □

This fact has consequences related to the A_p theory that we will not mention here. We will focus instead on its consequences on $\mathcal{L}^p$- regularity of $J(\cdot, f)$. We will use implicitly that $weak - \mathcal{L}^q_{loc} \subset \mathcal{L}^p_{loc}$ as long as $p < q$.

THEOREM 2.2.2. *Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is K-quasiconformal. Then for all $1 \leq p < \frac{2K}{K-1}$,*

$$f \in W^{1,p}_{loc}(\mathbb{C}).$$

PROOF. Let $B \subset \mathbb{C}$ be a disk. Since $|Df(z)|^2 \leq K\, J(z, f)$ almost everywhere and f is continuous, it suffices to show that $\int_B |J(z, f)|^q < \infty$ for all $0 < q < \frac{K}{K-1}$. The previous Theorem 2.2.1 gave bounds for the distribution function of the

Jacobian derivative. Integrating these, we have for any $0 < T < \infty$,

$$\begin{aligned}\int_B |J(z,f)|^q &= \int_0^\infty q\, t^{q-1} |\{z \in B : |J(z,f)| > t\}| dt \\ &\leq q \int_0^T t^{q-1}|B| + q\, C_K |B|^{1/(1-K)} \int_T^\infty t^{q-1} \left(\frac{|f(B)|}{t}\right)^{K/(K-1)} dt \\ &\leq T^q |B| + C_K' |B|^{1/(1-K)} |f(B)|^{K/(K-1)} \int_T^\infty t^{q-1-K/(K-1)} dt \\ &= T^q|B| - \frac{C_K'}{q - \frac{K}{K-1}} |B|^{1/(1-K)} |f(B)|^{K/(K-1)} T^{q-K/(K-1)} < \infty.\end{aligned}$$

□

REMARK 2.2.3. In fact, for all K we can define f_2 as in (1.21) and then one can check that $f_2 \notin W^{1,2K/K-1}_{loc}(\mathbb{C})$. Nevertheless, we have some configurations where integrability at the borderline arises, namely when f is conformal in E.

THEOREM 2.2.4. *Let $K > 1$ and assume that $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal principal mapping and that f is conformal outside $\mathbb{D}$. Let $E \subset \mathbb{D}$ be a measurable set. If $\partial_{\bar z} f(z) = 0$ for almost every $z \in E$, then*

$$\int_E J(\cdot, f)^{K/(K-1)} \leq \pi.$$

PROOF. We choose a sequence of weigths such that $w_0 = 1$ and

$$w_n = J(\cdot, f)^{1/K + \cdots + 1/K^n},$$

$n \geq 1$. By Theorem 2.1.2,

$$\frac{1}{\pi}\int_E w_n(z) J(z,f) \leq \left(\frac{1}{\pi}\int_E w_n(z)^K\right)^{1/K} = \left(\frac{1}{\pi}\int_E w_{n-1}(z) J(z,f)\right)^{1/K}.$$

for each $n \geq 1$. Using this argument inductively and Theorem 2.1.2 once again with $w(z) = 1$, we arrive at

$$\begin{aligned}\frac{1}{\pi}\int_E J(\cdot,f)^{1+1/K+\cdots+1/K^n} &= \frac{1}{\pi}\int_E w_n(z)J(z,f) \leq \left(\frac{1}{\pi}\int_E w_n(z)^K\right)^{1/K} \\ &\leq \left(\frac{1}{\pi}\int_E J(z,f)\right)^{1/K^n} \\ &\leq \left(\frac{1}{\pi}\int_E 1\right)^{1/K^{n+1}} = \left(\frac{|E|}{\pi}\right)^{1/K^{n+1}}.\end{aligned}$$

With Fatou's lemma we can pass to the limit $n \to \infty$ and thus we get the statement of the theorem. □

COROLLARY 2.2.5. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiregular mapping, $K > 1$. Let $E \subset \mathbb{C}$ be measurable and bounded. If $\partial_{\bar{z}}(z) = 0$ for almost every $z \in E$, then*

$$\int_E |Df(z)|^{2K/(K-1)} < \infty.$$

PROOF. Using a linear change of variables, we first assume that $E \subset \mathbb{D}_{\frac{1}{2}}$. Again we have the decomposition $f = h \circ g$, where g is a K-quasiconformal principal mapping, conformal outside $\mathbb{D}$, and h is K-quasiregular in $\mathbb{C}$ with

$$\partial_{\bar{z}} h(z) = 0 \quad \text{for all } z \in g(\mathbb{D})$$

.

Since h is holomorphic in a neighbourhood of the set $g(E)$, $\sup_{g(E)} |h'(x)| \leq C_h < \infty$ for a constant C_h depending only on the function h. Hence taking into consideration the chain rule (1.31) and (1.32), (1.19) and Theorem 2.2.4, we have

$$\begin{aligned}\int_E |Df(z)|^{\frac{2K}{K-1}} &\leq C_h^{\frac{2K}{K-1}} \int_E |Dg(z)|^{\frac{2K}{K-1}} \\ &\leq K^{\frac{K}{K-1}} C_h^{\frac{2K}{K-1}} \int_E |J(z,f)|^{\frac{K}{K-1}} \\ &\leq C(h,K) < \infty.\end{aligned}$$

□

2.2.2. Distortion of Dimension Theorem.

THEOREM 2.2.6 (Astala's Hausdorff Dimension Distortion Theorem). *For any compact set E and any K-quasiconformal mapping f we have*

$$\frac{1}{K}\left(\frac{1}{\dim_{\mathcal{H}}(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim_{\mathcal{H}}(f(E))} - \frac{1}{2} \leq K\left(\frac{1}{\dim_{\mathcal{H}}(E)} - \frac{1}{2}\right) \tag{2.13}$$

PROOF. Suppose $f \in W^{1,2}_{loc}(\Omega)$ is K-quasiregular and non-constant and let $E \subset \Omega$ be a compact subset with $\dim_{\mathcal{H}}(E) < 2$ (the extremal case, which read as $\dim_{\mathcal{H}}(E) = 2$ if and only if $\dim_{\mathcal{H}}(f(E)) = 2$, fall as a consequence of f and f^{-1} being K-quasiconformal). There is no restriction in assuming $E \subset \mathbb{D}_{1/2}$. Let us first use Stoilow factorization, $f \circ h$, where $g : \mathbb{C} \longrightarrow \mathbb{C}$ is a principal K-quasiconformal mapping, conformal outside $\mathbb{D}$, and h holomorphic in $g(\Omega)$. Since $\dim_{\mathcal{H}}(h(F)) = \dim_{\mathcal{H}}(F)$ for any compact subset of $g(\Omega)$, we can actually assume that $f = g$.

With this setup let us consider a finite number of disks B_i, each of diameter $\operatorname{diam}(B_i) < \delta < 1/4$, that have pairwise disjoint interiors and each of them intersecting E. On the image side quasisymmetry restricts the distortion and tells that

$$\operatorname{diam}(f(B_i))^2 \leq C_0 |f(B_i)|,$$

by (1.25), where the constant only depends on K. Hence for each $t < 1$, using Hölder inequality twice, we obtain that, for $p > 1$,

$$\begin{aligned}\sum_i \operatorname{diam}(f(B_i))^{2t} &\le C_0^t \sum_i |f(B_i)|^t \\ &\le C_0^t \sum_i \left(\int_{B_i} J(z,f)^p\right)^{\frac{t}{p}} |B_i|^{t(1-\frac{1}{p})} \\ &\le C_0^t \left(\sum_i \int_{B_i} J(z,f)^p\right)^{\frac{t}{p}} \left(\sum_i |B_i|^{\frac{p}{p-t}t(1-\frac{1}{p})}\right)^{1-\frac{t}{p}}.\end{aligned}$$

As the disks B_i are all contained in $\mathbb{D}$ with disjoint interiors, for every $1 < p < K/(K-1)$, we get uniform bounds using Theorem 2.2.2 ($J(\cdot, f) \in \mathcal{L}^p_{loc}(\mathbb{C})$):

$$\sum_i \int_{B_i} J(z,f)^p \le \int_{E_\delta} J(z,f)^p \le \|J(\cdot,f)\,\chi_{\mathbb{D}}\|_p^p < \infty,$$

where $E_\delta = \bigcup_{x\in E} B(x, 2\delta)$.

We have therefore shown that for all $p < K/(K-1)$

$$\sum_i \operatorname{diam}(f(B_i))^{2t} \le C_2 \left(\sum_i \operatorname{diam}(B_i)^{\frac{2t(p-1)}{p-t}}\right)^{1-t/p},$$

with C_2 depending on t, p and f and working uniformly for any finite covering of E by disjoint disks with diameter bounded by $1/4$ and thus for any non-finite one without overlapping. In case that overlapping occurs we can apply the Vitali Covering Lemma 1.2.6 and find a subcollection of pairwise disjoint disks with $E \subset \bigcup_k 5B_{i_k}$. By quasisymmetry,

(2.14)
$$\sum_k \operatorname{diam}(f(5B_{i_k}))^{2t} \lesssim \sum_k \operatorname{diam}(f(B_{i_k}))^{2t} \le C_2 \left(\sum_k \operatorname{diam}(B_{i_k})^{\frac{2t(p-1)}{p-t}}\right)^{1-t/p},$$

with constant depending only on the same variables.

Now, if $\dim_{\mathcal{H}}(E) < 2t\frac{p-1}{p-t}$, with a proper choice of the covering B_i the sum of the right-hand-side in (2.14) can be made arbitrarily small and we deduce that $\dim_{\mathcal{H}}(f(E)) < 2t$. Solving the former in terms of t, we find that this happens as long as $t > \frac{p\dim_{\mathcal{H}}(E)}{2(p-1)+\dim_{\mathcal{H}}(E)}$, so

$$\dim_{\mathcal{H}}(f(E)) \le \frac{2p\dim_{\mathcal{H}}(E)}{2(p-1)+\dim_{\mathcal{H}}(E)} \xrightarrow{p\to K/(K-1)} \frac{2K\dim_{\mathcal{H}}(E)}{2+(K-1)\dim_{\mathcal{H}}(E)}$$

This inequality is equivalent to the left hand inequality in (2.13). The right hand inequality follows immediately since the inverse of a K-quasiconformal mapping is also K-quasiconformal. $\square$

Remark 2.2.7. These bounds are optimal, in that equality may occur in either estimate, but we will not prove that fact here. In the last section we give an example of extremal distortion of Hausdorff measure which also implies that fact, although sharpness of Theorem 2.2.6 was proven originally by Astala by means of a simpler example.

2.2.3. The Dimension of Quasicircles. A quasicircle is the image of the unit circle under a quasiconformal homeomorphism of $\mathbb{C}$.

The earlier Theorem 2.2.6 gives the following estimate of dimensional distortion: If $\mathcal{C}$ is a K-quasicircle, it is the image of a set of dimension 1 under a K-quasiconformal mapping and hence

$$\dim_{\mathcal{H}}(\mathcal{C}) \leq 1 + \frac{K-1}{K+1} = 1 + k$$

where $k = \frac{K-1}{K+1}$. This would work for the image of any 1-dimensional bounded set. In [**AIM**] we find for the first time Smirnov's proof of a better bound for this particular case. We state it in the following theorem.

Theorem 2.2.8. *Let $\mathcal{C}_k$ be a $K = \frac{1+k}{1-k}$-quasicircle. Then*

$$\dim_{\mathcal{H}}(\mathcal{C}_k) \leq 1 + k^2$$

Conjecture 2.2.9. *This bound is sharp.*

2.3. Distortion of Hausdorff Content

2.3.1. Introduction.

In this section we follow the approach of the article by M. T. Lacey, E. T. Sawyer and I. Uriarte-Tuero [**LSUT**] to the distortion of content, clarifying some point and changing the approach of the bounds for the Beurling transform, where we take the more classical approach of the article by K. Astala, A. Clop, X. Tolsa, I. Uriarte-Tuero and J. Verdera [**ACTUTV**] instead.

Recall that in Astala's Hausdorff dimension distortion teorem the inequality (2.13)

$$\frac{1}{K}\left(\frac{1}{\dim_{\mathcal{H}}(E)}-\frac{1}{2}\right)\leq\frac{1}{\dim_{\mathcal{H}}(f(E))}-\frac{1}{2}\leq K\left(\frac{1}{\dim_{\mathcal{H}}(E)}-\frac{1}{2}\right)$$

was proved for any K-quasiconformal mapping f defined on the neighbourhood of a compact set E. The question that we study concerns refinement of the inequality above. Can it be improved to the level of Hausdorff measures $\mathcal{H}^t$? Indeed, this is the case. The next theorem, the main result of [**LSUT**], answers in the affirmative Astala's Question 4.4 in [**Ast94**]. In fact, we will find a bound in the Hausdorff content distortion.

Main Theorem 2.3.1. *If ϕ is a planar K-quasiconformal mapping, $0 \leq t \leq 2$ and*

$$t' = \frac{2Kt}{2+(K-1)t},$$

then we have the following implication for all the compact subsets E of a ball B:

$$\frac{\mathcal{H}^{t'}_\infty(\phi(E))}{\operatorname{diam}(\phi(B))^{t'}} \leq C(K,t)\left(\frac{\mathcal{H}^t_\infty(E)}{\operatorname{diam}(B)^t}\right)^{t'/tK}. \tag{2.15}$$

In particular,

$$\mathcal{H}^t(E)=0 \implies \mathcal{H}^{t'}(\phi(E))=0.$$

Since the inverse of a K-quasiconformal mapping is also K-quasiconformal, the following refinement of the right-hand endpoint in (2.13) follows: for a compact set F, we have that $\mathcal{H}^{t'}(F)>0$ implies $\mathcal{H}^t(\phi(F))>0$.

Some instances of this theorem where already known before the publication of [**LSUT**], and have connections to significant further properties of quasiconformal maps (the case $t = 0$ obvious since ϕ is a homeomorphism, the case $t = 2$ as a consequence of the Area Distortion Theorem [**Ast94**], and the case $t' = 1$, related to the Painlevé's problem, whose solution was achieved by Xavier Tolsa in [**Tol03**] and [**Tol05**], and studied in terms of quasiregular functions by Kari Astala, Albert Clop, Joan Mateu, Joan Orobitg and Ignacio Uriarte-Tuero in [**ACMOUT**].

Let us give an overview of this section. We consider the case of small dilatation in Lemma 2.3.13. Thus, we take a compact set E and a K-quasiconformal map ϕ. To provide the conclusion that the t'-Hausdorff content of $\phi(E)$ is bounded by the t-Hausdorff content of E as in (2.15), we will use a covering of $\phi(E)$ by quasidisks that satisfy some related bounds. We show, following [**LSUT**] that this can be done with certain dyadic squares (denoted by $P \in \mathcal{P}$ below) that admit one key additional feature, that they obey a t-packing condition described in (2.16).

Associated with $\mathcal{P}$ is a measure $\omega_{t,\mathcal{P}}$, defined in (2.23), which behaves in a "t-dimensional" way as a consequence of the t-packing condition. We show also that the Beurling operator is bounded on $\mathcal{L}^p(\omega_{t,\mathcal{P}})$ for all $1 < p < \infty$; see Proposition 2.3.3. This proof is taken from [**ACTUTV**] instead of [**LSUT**], as a more classical approach is taken in the former.

The mapping ϕ is then factored into $\phi = \phi_1 \circ h$, where ϕ_1 is the "conformal inside" part and h is the "conformal outside" part. The conformal inside part admits a relevant estimate that can be deduced from [**ACMOUT**], and is proven below. The relevant estimate on the conformal outside part is found in [**LSUT**], and uses in an essential way the two facts just mentioned.

Proposition 2.3.2 is proven in Subsection 2.3.2. In Subsection 2.3.3 the proof of the weighted estimate for the Beurling operator (Proposition 2.3.3) is given. These two propositions are combined in Subsection 2.3.4 to prove Lemma 2.3.14. Finally, we prove Theorem 2.3.1 in Subsection 2.3.5.

2.3.2. Finding a Good Covering for E.

We state our proposition on the approximation of the Hausdorff content with the t-packing condition. Let $\mathcal{P}$ be a finite collection of disjoint dyadic squares in the plane and let $0 < t < 2$. We denote the *t-Carleson packing norm* of $\mathcal{P}$ as follows:

$$\|\mathcal{P}\|_{t-\mathrm{pack}} = \sup_{Q \in \overline{\mathcal{D}}} \left(\sum_{\substack{P \in \mathcal{P} \\ P \subset Q}} \frac{\ell(P)^t}{\ell(Q)^t} \right)^{1/t}, \tag{2.16}$$

where the supremum is taken over all dyadic squares Q. We say that $\mathcal{P}$ satisfies the *t-Carleson packing condition* if $\|\mathcal{P}\|_{t-\mathrm{pack}} < \infty$. That is, for any dyadic square Q,

$$\sum_{\substack{P \in \mathcal{P} \\ P \subset Q}} \ell(P)^t \le \|\mathcal{P}\|_{t-\mathrm{pack}}^t \ell(Q)^t \tag{2.17}$$

Only the case $m = 2$ of the following proposition is used below.

PROPOSITION 2.3.2. *Let $m, M_1 \ge 1$ be integers. Then there is a positive constant C such that, for any compact $E \subset B(0, 2^{M_1}) \subset \mathbb{C}$, $0 < t < 2$ and $\epsilon > 0$, there is a finite collection of closed dyadic squares $\mathcal{P} = \{P_i\}_{i=1}^N$ such that*

(1) $2^m P_i \cap 2^m P_j = \emptyset$ *for* $i \neq j$;
(2) $E \subset \bigcup_{i=1}^N 3 \cdot 2^m P_i$;
(3) $\|\mathcal{P}\|_{t-\mathrm{pack}} \le 1$;
(4) $\sum_{i=1}^N \ell(P_i)^t \le C(\widetilde{\mathcal{H}}^t_\infty(E) + \epsilon)$.
(5) $P_i \subset B(0, 2^{M_1+1})$.

PROOF. Given $\epsilon > 0$, by the definition of dyadic Hausdorff content at dimension t, there exists a (possibly infinite) collection $\{Q_n\}_n$ of closed dyadic squares such that $E \subset \bigcup_n Q_n$, and

$$\sum_n \ell(Q_n)^t \leq \widetilde{\mathcal{H}}^t_\infty(E) + \epsilon. \tag{2.18}$$

By compactness of E, after relabeling indices, there is a finite number M for which

$$E \subset \bigcup_{n=1}^{M} (3\,\overset{\circ}{Q}_n).$$

Since each square of the form $3\,Q_n$ is the union of 9 dyadic squares of the same size as Q_n, we can write, after relabeling, $E \subset \bigcup_{n=1}^{M'} Q_n$, where Q_n are closed dyadic squares (possibly with overlapping and even repeated squares).

By selecting the maximal squares among $\{Q_n\}$, and eliminating those Q_n not intersecting E, we may now assume, after relabeling again, that

$$\sum_{n=1}^{M} \ell(Q_n)^t \leq 9\left(\widetilde{\mathcal{H}}^t_\infty(E) + \epsilon\right), \tag{2.19}$$

and that the squares are dyadic, intersect E, and are pairwise *mutually disjoint* (see Definition 1.1.2).

As $E \subset B(0, 2^{M_1})$, $E \subset \bigcup_{i=1}^{4} \widetilde{Q_i}$, with $0 \in \widetilde{Q_i} \in \overline{\mathcal{D}}$, $\ell(\widetilde{Q_i}) = 2^{M_1}$. Notice that any dyadic square intersecting E will intersect one and only one of those. Let $\min\{\ell(Q_n)\} = 2^{-M_0}$, and call a finite collection of squares $\mathcal{R}$ *admissible*, denoted by $\mathcal{R} \in Adms$, if

(1) $\mathcal{R}$ is a finite collection of dyadic squares that intersect E: $\mathcal{R} = \{R_i\}_{i=1}^{H}$ for some $H \in \mathbb{N}$ and $R_i \cap E \neq \emptyset$ for all i;
(2) $2^{-M_0} \leq \ell(R_i) \leq 2^{M_1}$;
(3) $E \subset \bigcup_{i=1}^{H} R_i$;
(4) they are pairwise *mutually disjoint*.

We have seen that $Adms$ is not-empty (for ϵ small enough to grant an optimal choice, meaning that no square bigger than $\widetilde{Q_1}$ will be chosen in the first covering!). Notice that all these collections satisfy (5). The minimum

$$\min_{\mathcal{R} \in Adms} \sum_{R_i \in \mathcal{R}} \ell(R_i)^t,$$

is achieved, as there are only finitely many admissible collections of squares. Let us choose an admissible collection that achieves the minimum and denote it as $\mathcal{T} = \{T_i\}_{i=1}^{M''}$. By (2.19), we have

$$\sum_{i=1}^{M''} \ell(T_i)^t \leq \sum_{j=1}^{M} \ell(Q_j)^t \leq 9\left(\widetilde{\mathcal{H}}^t_\infty(E) + \epsilon\right). \tag{2.20}$$

Any minimizer also satisfies a local property: for any dyadic square Q such that $2^{-M_0} \leq \ell(Q) \leq 2^{M_1}$,

$$\sum_{T_i \subset Q} \ell(T_i)^t \leq \ell(Q)^t. \tag{2.21}$$

Indeed, if Q intersects E, and this inequality did not hold, the square Q would have been selected instead of the squares T_i with $T_i \subset Q$, contradicting the property of achieving the minimum. If the square Q does not intersect E, then the inequality is trivial as the left side equals zero.

As an immediate consequence, we get that for *any* dyadic square Q,

$$\sum_{T_i \subset Q} \ell(T_i)^t \leq \ell(Q)^t. \tag{2.22}$$

Thus, $\mathcal{T}$ satisfies conditions (3), (4) and (5) of the conclusion. To accomodate (1) and (2) as well, fix an integer $m \in \mathbb{N} \setminus \{0\}$, and fix a square $T_i \in \mathcal{T}$. Subdivide T_i into its 2^{2m+2} dyadic descendants of side-length $2^{-m-1}\ell(T_i)$. Let $\widehat{T_i}$ be the dyadic descendant of T_i of side-length $2^{-m-1}\ell(T_i)$ whose upper right corner is the center of T_i. It is now easy to check that the squares $\{\widehat{T_i}\}$ satisfy (4) in the statement of Proposition 2.3.2 with constant $C = 9 > 2^{(-m-1)t}9$, as well as (1), (2), (3) and (5). □

2.3.3. Weighted Bounds for the Beurling Transform.

Given $0 < t \leq 2$ and a collection $\mathcal{P}$ of pairwise disjoint dyadic squares, we define the measure $\omega_{t,\mathcal{P}}$ associated with $\mathcal{P}$ by

$$\omega(x) = \omega_{t,\mathcal{P}}(x) = \sum_j \ell(P_j)^{t-2}\chi_{P_j}(x), \tag{2.23}$$

We will be concerned with a quasiconformal map f that is conformal outside of $\bar{\mathcal{P}} = \bigcup_{i=1}^N P_i$, and we will need an estimate on the diameters of $f(P_i)$. The map f will have an explicit expression as a von Neumann series involving the Beurling operator. The following proposition gives a weighted norm inequality with respect to the weight $\omega_{t,\mathcal{P}}$ for the compression of $\mathcal{S}$ to the set $\bar{\mathcal{P}}$, i.e. the operator $\chi_{\bar{\mathcal{P}}}\mathcal{S}\chi_{\bar{\mathcal{P}}}$, assuming that $\mathcal{P}$ satisfies a Carleson t-packing condition.

Proposition 2.3.3. *Let $0 < t < 2$ and let $\mathcal{P} = \{P_i\}_{i=1}^N$ be a family of dyadic squares with $\|\mathcal{P}\|_{t-\mathrm{pack}}^t \leq C_{pack}$ such that for all $i \neq j$, $3\,P_i \cap 3\,P_j = \emptyset$ and let $\bar{\mathcal{P}} = \bigcup_{i=1}^N P_i$. If $\omega = \omega_{t,\mathcal{P}}$ is the weight defined by (2.23), then the Beurling transform is bounded in $\mathcal{L}^p(\omega)$, for $1 < p < \infty$, and of weak type (1,1) with respect to ω. That is, for some $C(p, C_{pack}, t) > 0$ depending on p, C_{pack} and t only,*

$$\|\mathcal{S}(f\chi_{\bar{\mathcal{P}}})\|_{\mathcal{L}^p(\omega)} \leq C(p, C_{pack}, t)\|f\|_{\mathcal{L}^p(\omega)} \tag{2.24}$$

for all $f \in \mathcal{L}^p(\omega)$, and

$$\|\mathcal{S}(f\chi_{\bar{\mathcal{P}}})\|_{\mathcal{L}^{1,\infty}(\omega)} \leq C(1, C_{pack}, t)\|f\|_{\mathcal{L}^1(\omega)} \tag{2.25}$$

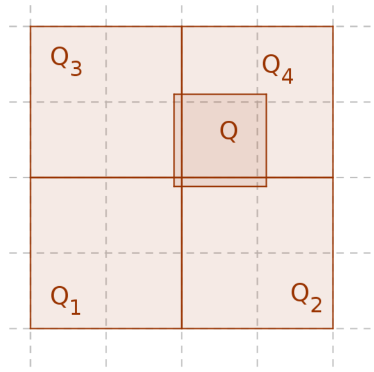


Figure 2.1: The minimal dyadic covering of Q with bigger side-lengths.

for all $f \in \mathcal{L}^1(\omega)$. Moreover, $C(p, C_{pack}, t)$ is increasing with respect to t.

We need to prove some lemmas before we prove the proposition above. The measure ω behaves as a t-dimensional measure, namely,

LEMMA 2.3.4. *If Q is an arbitrary square (dyadic or not), then*

$$\omega(Q) < 16\|\mathcal{P}\|_{t-\text{pack}}^t \ell(Q)^t. \tag{2.26}$$

PROOF. Let Q_i be the dyadic squares such that $\ell(Q) \leq \ell(Q_i) < 2\,\ell(Q)$ with non empty intersection with Q. There are at most four of them (see figure 2.1). We will assume Q and Q_i not included in any $P \in \mathcal{P}$ (otherwise, as P_j are disjoint, there would be only one square to consider and the proof is quite similar taking out the sums in the beginning and changing the bound in the first inequality)

$$\begin{aligned}
\omega(Q) &= \int_Q \sum_j \ell(P_j)^{t-2} \chi_{P_j}(x) dm = \sum_j \int_{P_j \cap Q} \ell(P_j)^{t-2} dm \\
&= \sum_{\mathring{P}_j \cap \mathring{Q} \neq \emptyset} \ell(P_j)^{t-2} \int_{P_j \cap Q} dm \leq \sum_{\mathring{P}_j \cap \mathring{Q} \neq \emptyset} \ell(P_j)^t \\
&= \sum_{\mathring{P}_j \cap \mathring{Q} \neq \emptyset} \left(\frac{\ell(P_j)}{\ell(Q)} \right)^t \ell(Q)^t < \sum_i \sum_{\mathring{P}_j \cap \mathring{Q}_i \neq \emptyset} \left(\frac{\ell(P_j)}{1/2\,\ell(Q_i)} \right)^t \ell(Q)^t \\
&= \sum_i 2^t \sum_{P_j \subset Q_i} \left(\frac{\ell(P_j)}{\ell(Q_i)} \right)^t \ell(Q)^t \leq 16\|\mathcal{P}\|_{t-\text{pack}}^t \ell(Q)^t.
\end{aligned}$$

□

To describe the class of weights we refer to, from now on we suppose that

$$3\,P_i \cap 3\,P_j = \emptyset \tag{2.27}$$

if $i \neq j$ and $\|\mathcal{P}\|^t_{t-\text{pack}} \leq C_{pack}$ in condition (2.17).

LEMMA 2.3.5. *If $\bar{\mathcal{P}} = \bigcup_{i=1}^N P_i$, then for all $Q \subset \mathbb{C}$ square, and for almost every $x \in \bar{\mathcal{P}} \cap Q$,*

$$\frac{\omega(Q)}{\ell(Q)^2} \leq C(C_{pack})\omega(x). \tag{2.28}$$

The constant only depends on C_{pack}.

PROOF. By (2.27), there is only one $P \in \mathcal{P}$ such that $x \in P$. Then,

$$\omega(x) = \ell(P)^{t-2}$$

Now, if $\ell(Q) \geq \ell(P)$, by Lemma 2.3.4, we know $\frac{\omega(Q)}{\ell(Q)^2} \leq C \cdot C_{pack}\,\ell(Q)^{t-2} \leq C\ell(P)^{t-2}$, because $t-2<0$.

Otherwise, let us assume that $\ell(Q) < \ell(P)$. This imples $Q \subset 3\,P$ and, by (2.27), there is no other square in $\mathcal{P}$ intersecting Q. Then,

$$\frac{\omega(Q)}{\ell(Q)^2} = \frac{1}{\ell(Q)^2}\int_{P\cap Q} \ell(P)^{t-2} dm \leq \ell(P)^{t-2} = \omega(x),$$

and we get (2.28). □

COROLLARY 2.3.6. *If $\mathcal{M}$ is the maximal Hardy-Littlewood operator, then, for almost every $x \in \bar{\mathcal{P}}$,*

$$\mathcal{M}\omega(x) \leq C(C_{pack})\omega(x). \tag{2.29}$$

PROOF. Notice that $m(Q(z,r)) = (2r)^2 = \frac{4}{\pi}\pi r^2 = \frac{4}{\pi}m(B(z,r))$, so

$$\mathcal{M}\omega(x) = \sup_{x\in B}\frac{\omega(B)}{m(B)} \leq \frac{4}{\pi}\sup_{x\in Q}\frac{\omega(Q)}{\ell(Q)^2} \leq \frac{4}{\pi}C\omega(x),$$

using (2.28). □

DEFINITION 2.3.7. Given a weight w with support in a domain D, we say that $\omega \in A^{loc}_{1,D}$, the local A_1 Muckenhoupt class, if there exists a constant C such that, for every Q intersecting D,

$$\frac{\omega(Q)}{|Q|} \leq C \operatorname{ess}\inf_{x\in Q\cap D}\omega(x).$$

Then we set

$$|\omega|_{A^{loc}_{1,D}} = \sup_Q \frac{\omega(Q)}{|Q|}\|\omega^{-1}\|_{\mathcal{L}^\infty(Q)}.$$

where the supremum is taken over all the squares Q intersecting D.

REMARK 2.3.8. By Lemma 2.3.5 above, we have that $\omega \in A^{loc}_{1,\bar{\mathcal{P}}}$.

Now, let $\mathcal{M}_\omega$ be the centered Hardy-Littlewood maximal function with respect to ω. That is:

$$\mathcal{M}_\omega f(x) = \sup_{r>0} \frac{1}{\omega(Q(x,r))} \int_{Q(x,r)} |f(y)|\omega(y) dm(y).$$

LEMMA 2.3.9. *$\mathcal{M}_\omega$ is of weak type $(1,1)$ and strong type (p,p) for $1 < p \leq \infty$ with respect to the measure ω, with norm $\|\mathcal{M}_\omega\|_{\mathcal{L}^p \to \mathcal{L}^p} \leq \|\mathcal{M}_\omega\|_{\mathcal{L}^1 \to \mathcal{L}^{1,\infty}} \leq C$.*

PROOF. In virtue of Marcinkiewicz interpolation Theorem 1.1.15, we only need to prove the extremal cases.

Let us assume $f \in \mathcal{L}^\infty(\omega)$. Then,

$$\begin{aligned}\mathcal{M}_\omega f(x) &= \sup_{r>0} \frac{1}{\omega(Q(x,r))} \int_{Q(x,r)} |f(y)|\, d\omega(y) \\ &\leq \|f\|_\infty \sup_{r>0} \frac{1}{\omega(Q(x,r))} \omega(Q(x,r)) = \|f\|_\infty,\end{aligned}$$

and so $\mathcal{M}_\omega f \in \mathcal{L}^\infty(\omega)$, with norm bounded by one.

Assume now $f \in \mathcal{L}^1(\omega)$ and define $E := \{x \in \bar{\mathcal{P}} : \mathcal{M}_\omega f(x) > \lambda\}$, bounded by definition. We want to check that $\omega(E) \leq \frac{C}{\lambda} \|f\|_{\mathcal{L}^1(\omega)}$.

If E is non-empty, for all $x \in E$, there exists a length r_x such that

$$\left| \int_{Q(x,r_x)} f(y) d\omega(y) \right| > \lambda\, \omega(Q(x, r_x)).$$

Besicovitch Covering Theorem 1.2.1 states that there exists C depending only on the dimension of the space (two in the present case) such that there exists a collection of points $\{x_i\}_{i\in\mathbb{N}} \subset E$ with $E \subset \bigcup_{i\in\mathbb{N}} Q_i$ where $Q_i = Q(x_i, r_{x_i})$ and for all $x \in E$, $x \in Q_i$ at most for C different i, i.e. $\sum_{i\in\mathbb{N}} \chi_{Q_i} \leq C$.

Now,

$$\begin{aligned}\omega(E) &\leq \sum_{i\in\mathbb{N}} \omega(Q_i) < \sum_{i\in\mathbb{N}} \left| \frac{1}{\lambda} \int_{Q_i} f(y) d\omega(y) \right| \leq \sum_{i\in\mathbb{N}} \frac{1}{\lambda} \int_{Q_i} |f(y)|\, d\omega(y) \\ &\leq \frac{C}{\lambda} \int_{\mathbb{C}} |f(y)|\, d\omega(y) = \frac{C}{\lambda} \|f\|_{\mathcal{L}^1(\omega)}.\end{aligned}$$

Notice that the constant C remains unchanged during the proof. □

From the following lemma, it follows that the same is also true for $\mathcal{M}$.

LEMMA 2.3.10. *Let ω be as above. There exists a constant C depending only on C_{pack}, such that $\mathcal{M}f(x) \leq C\mathcal{M}_\omega f(x)$ for all $f \in \mathcal{L}^1_{loc}(\bar{\mathcal{P}})$ and for all $x \in \bar{\mathcal{P}}$.*

As a consequence, $\mathcal{M}$ is of weak type $(1,1)$ and strong type (p,p) for $1 < p \leq \infty$ with respect to the measure ω.

PROOF. Let $f \in \mathcal{L}^1_{loc}(\bar{\mathcal{P}})$ and Q a square containing $x \in \bar{\mathcal{P}}$. Consider the maximal square Q' centered at x containing Q. Then, as $\ell(Q') \approx \ell(Q)$, using (2.28) we get

$$\begin{aligned}\frac{1}{m(Q)}\int_Q |f|dm \leq \frac{1}{m(Q)}\int_{Q'} |f|dm \lesssim \frac{1}{\ell(Q')^2}\int_{Q'} |f|dm\\ \lesssim \frac{\inf\{\omega(y) : y \in Q' \cap \bar{\mathcal{P}}\}}{\omega(Q')}\int_{Q'} |f|dm\\ \leq \frac{1}{\omega(Q')}\int_{Q'} |f|\omega dm \leq \mathcal{M}_\omega f(x).\end{aligned}$$

□

To prove the Proposition 2.3.3, we will show the following weak type inequality, which is stronger than (2.25):

$$\|\mathcal{S}_*(f\chi_{\bar{\mathcal{P}}})\|_{\mathcal{L}^{1,\infty}(\omega)} \leq C\|f\|_{\mathcal{L}^1(\omega)}. \tag{2.30}$$

Then, by means of a good lambda inequality, we will deduce that the maximal Beurling transform is bounded in $\mathcal{L}^p(\omega)$, for $1 < p < \infty$, that is

$$\|\mathcal{S}_*(f\chi_{\bar{\mathcal{P}}})\|_{\mathcal{L}^p(\omega)} \leq C\|f\|_{\mathcal{L}^p(\omega)}. \tag{2.31}$$

Clearly, (2.24) follows from (2.31). We prove (2.30) in the next lemma:

Lemma 2.3.11. *For all $f \in \mathcal{L}^1(\omega)$, and $\lambda > 0$, we have*

$$\omega(\{z \in \bar{\mathcal{P}} : |\mathcal{S}_* f(z)| > \lambda\}) \leq \frac{C(C_{pack}, t)}{\lambda}\|f\|_{\mathcal{L}^1(\omega)}$$

with $C(C_{pack}, t)$ depending only on C_{pack} and t and increasing with respect to the latter.

Proof. We have

$$\begin{aligned}\omega(\{z \in \bar{\mathcal{P}} : |\mathcal{S}_* f(z)| > \lambda\}) &= \sum_{i=1}^N \omega(\{z \in P_i : |\mathcal{S}_* f(z)| > \lambda\})\\ &\leq \sum_{i=1}^N \omega(\{z \in P_i : |\mathcal{S}_*(f\chi_{2P_i})(z)| > \frac{\lambda}{2}\})\\ &+ \sum_{i=1}^N \omega(\{z \in P_i : |\mathcal{S}_*(f\chi_{\mathbb{C}\setminus 2P_i})(z)| > \frac{\lambda}{2}\})\\ &=: A + B.\end{aligned}$$

For the first of the two addendi, we have

$$\begin{aligned} A &= \sum_{i=1}^{N} \ell(P_i)^{t-2} m(\{z \in P_i : |\mathcal{S}_*(f\chi_{2\,P_i})(z)| > \frac{\lambda}{2}\}) \\ &\le 2\|\mathcal{S}_*\|_{\mathcal{L}^1 \to \mathcal{L}^{1,\infty}} \sum_{i=1}^{N} \ell(P_i)^{t-2} \frac{1}{\lambda} \int |f\chi_{2\,P_i}| dm \\ &= 2\|\mathcal{S}_*\|_{\mathcal{L}^1 \to \mathcal{L}^{1,\infty}} \frac{\|f\|_{\mathcal{L}^1(\omega)}}{\lambda}, \end{aligned}$$

where the first equality is a consequence of the fact that the squares $3\,P_i$ are disjoint and ω coincides with the Lebesgue measure times $\ell(P_i)^{t-2}$ on every P_i and the inequality follows from the boundedness of $S_* : \mathcal{L}^1 \to \mathcal{L}^{1,\infty}$ with respect to Lebesgue measure.

To bound B notice that, if $\mathcal{P}$ has more that one element, denoting the center of P_k by z_k, if $z \in P_i$ and $\tau \in P_j$ for $j \neq i$, we have that (2.27) implies $3\,\mathrm{dist}(z, z_i) + 3\,\mathrm{dist}(\tau, z_j) \le \sqrt{2}(3\,\mathrm{dist}_\infty(z, z_i) + 3\,\mathrm{dist}_\infty(\tau, z_j)) < \sqrt{2}\mathrm{dist}_\infty(z_i, z_j) \le \sqrt{2}\mathrm{dist}(z_i, z_j)$. This means that

$$\mathrm{dist}(z_i, z_j) \le \mathrm{dist}(z_i, z) + \mathrm{dist}(z, \tau) + \mathrm{dist}(\tau, z_j) < \frac{\sqrt{2}}{3}\mathrm{dist}(z_i, z_j) + \mathrm{dist}(z, \tau).$$

So

$$\frac{3-\sqrt{2}}{3}\mathrm{dist}(z_i, z_j) < \inf\{\mathrm{dist}(z, \tau) : z \in P_i, \tau \in P_j\}.$$

Now, using that fact, for $B = \sum_{i=1}^{N} \omega(\{z \in P_i : |\mathcal{S}_*(f\chi_{\mathbb{C}\setminus 2\,P_i})(z)| > \frac{\lambda}{2}\})$, one has, using Chebyshev inequality and Fubini-Tonelli Theorem, that

$$\begin{aligned} B &\le \frac{2}{\lambda} \sum_{i=1}^{N} \int_{P_i} |\mathcal{S}_*(f\chi_{\mathbb{C}\setminus 2\,P_i})(z)| d\omega(z) \\ &\le \frac{2}{\lambda\pi} \sum_{i=1}^{N} \int_{P_i} \sum_{j\neq i} \int_{P_j} \frac{|f(\tau)|}{|z-\tau|^2} dm(\tau) d\omega(z) \\ &\le \frac{9}{(3-\sqrt{2})^2} \frac{2}{\lambda\pi} \sum_{i=1}^{N} \sum_{j\neq i} \int_{P_i} \int_{P_j} \frac{|f(\tau)|}{|z_j - z_i|^2} dm(\tau) d\omega(z) \\ &\le \frac{C}{\lambda} \sum_{j=1}^{N} \left(\sum_{i\neq j} \frac{\omega(P_i)}{|z_j - z_i|^2} \right) \int_{P_j} |f(\tau)| dm(\tau). \end{aligned}$$

To finish the proof we will use the classical ring decomposition. To do so, notice that, as a consequence of (2.27), for all $z \in P_i$, $\mathrm{dist}_\infty(z, z_j) \le \frac{4}{3}\mathrm{dist}_\infty(z_j, z_i)$, so if $z_i \in 2^k\,P_j$, then $P_i \subset 2^{k+1}\,P_j$. Notice also that $|z_i - z_j|^2 \ge \mathrm{dist}_\infty(z_i, z_j)^2 > \left(2^{k_{min}-1} \cdot \frac{\ell(P_j)}{2}\right)^2$, where k_{min} stands for the minimum k such that $z_i \in 2^k\,P_j$. We can argue, using the *t-Carleson packing condition* (2.17), that

$$\begin{aligned}
\sum_{i\neq j} \frac{\omega(P_i)}{|z_j - z_i|^2} &= \sum_{k=2}^{\infty} \left(\sum_{i\,:\,z_i \in 2^k P_j \setminus 2^{k-1} P_j} \frac{\ell(P_i)^t}{|z_j - z_i|^2} \right) \\
&\leq 16 \sum_{k=2}^{\infty} \left(\sum_{i\,:\,P_i \subset 2^{k+1} P_j} \frac{\ell(P_i)^t}{(2^k \ell(P_j))^2} \right) \\
&= C \sum_{k=2}^{\infty} \frac{1}{(2^k \ell(P_j))^2} \left(\sum_{i\,:\,P_i \subset 2^{k+1} P_j} \ell(P_i)^t \right) \\
&\leq C \sum_{k=2}^{\infty} \frac{C_{pack} \ell(2^{k+1} P_j)^t}{(2^k \ell(P_j))^2} \\
&\leq C \left(\sum_{k=2}^{\infty} 2^{k(t-2)+t} \right) \ell(P_j)^{t-2} \\
&= C \frac{2^{3t}}{1 - 2^{t-2}}\, \ell(P_j)^{t-2}.
\end{aligned}$$

Notice that the constant is increasing on t.

Thus, $B \leq \frac{C}{\lambda} \|f\|_{\mathcal{L}^1(\omega)}$ and the lemma follows since both A and B are bounded by constant multiples of $\frac{1}{\lambda} \|f\|_{\mathcal{L}^1(\omega)}$. $\square$

We will need also the next lemma:

LEMMA 2.3.12. *Given $f \in \mathcal{L}^p(\omega)$, let us denote $\Omega_\lambda = \{z \in \mathbb{C} : \mathcal{S}_* f(z) > \lambda\}$. Then Ω_λ is a bounded open set.*

PROOF. It is bounded because $\mathcal{S}_* f(z)$ tends to 0 as z tends to infinity due to the fact that f has compact support. If it was empty there would be nothing to prove, so we will suppose it is a proper set.

Let us see that it is open. For all $z \in \Omega_\lambda$, it exists an $\epsilon > 0$ such that $|\mathcal{S}_\epsilon f(z)| = \lambda + \mu$ with $\mu > 0$. For all $h \in \mathbb{C}$, and $\epsilon' > 0$,

$$|\mathcal{S}_{\epsilon'} f(z+h)| \geq |\mathcal{S}_\epsilon f(z)| - |\mathcal{S}_\epsilon f(z) - \mathcal{S}_{\epsilon'} f(z+h)|.$$

We will show below that there is some $\delta > 0$ such that for all $h \in \mathbb{C}$ with $|h| < \delta$ there is an $\epsilon' > 0$ with $|\mathcal{S}_\epsilon f(z) - \mathcal{S}_{\epsilon'} f(z+h)| < \mu$ and thus $|\mathcal{S}_{\epsilon'} f(z+h)| > \lambda$. In other words, $B(z, \delta) \subset \Omega_\lambda$, so Ω_λ is open.

Indeed, let us take $\epsilon' = \epsilon - |h|$ with $2|h| < \epsilon$. Now,

$$\begin{aligned}
&\pi|\mathcal{S}_\epsilon f(z) - \mathcal{S}_{\epsilon'} f(z+h)| \\
&\quad = \left| \int_{\mathbb{C}\setminus B(z,\epsilon)} \frac{f(\tau)}{(z-\tau)^2} dm(\tau) - \int_{\mathbb{C}\setminus B(z+h,\epsilon')} \frac{f(\tau)}{(z+h-\tau)^2} dm(\tau) \right| \\
&\quad \leq \int_{B(z,\epsilon)\setminus B(z+h,\epsilon')} \frac{|f(\tau)|}{|z+h-\tau|^2} dm(\tau) + \int_{\mathbb{C}\setminus B(z,\epsilon)} \frac{|f(\tau)||2(z-\tau)h + h^2|}{|z-\tau|^2|z+h-\tau|^2} dm(\tau).
\end{aligned}$$

The first integral can be bounded in the following way:

$$\begin{aligned}
\int_{B(z,\epsilon)\setminus B(z+h,\epsilon')} \frac{|f(\tau)|}{|z+h-\tau|^2} dm(\tau) &\leq \int_{B(z,\epsilon)\setminus B(z+h,\epsilon')} \frac{|f(\tau)|}{(\epsilon - 2|h|)^2} dm(\tau) \\
&\leq \int_{B(z,\epsilon)\setminus B(z,\epsilon-2|h|)} \frac{|f(\tau)|}{(\epsilon - 2|h|)^2} dm(\tau) \\
&= \frac{1}{(\epsilon-2|h|)^2} \int_{B(z,\epsilon)\setminus B(z,\epsilon-2|h|)} |f(\tau)| dm(\tau),
\end{aligned}$$

which is decreasing when $|h|$ tends to zero. Since $f \in \mathcal{L}^p(\omega) \subset \mathcal{L}^1(\omega) \subset \mathcal{L}^1(\mathbb{C})$, it defines a measure m_f in $\mathbb{C}$ and the measured domains $D_{|h|} = B(z,\epsilon) \setminus B(z, \epsilon - 2|h|)$, being totally ordered by inclusion and having finite measure, satisfy that $\lim_{|h|\to 0} m_f(D_{|h|}) = m_f(\bigcap_{|h|\to 0} D_{|h|}) = m_f(\emptyset) = 0$.

The second integral,

$$\begin{aligned}
\int_{\mathbb{C}\setminus B(z,\epsilon)} \frac{|f(\tau)||2(z-\tau)h+h^2|}{|z-\tau|^2|z+h-\tau|^2} dm(\tau) &= |h| \int_{\mathbb{C}\setminus B(z,\epsilon)} \frac{|f(\tau)||2(z-\tau)+h|}{|z-\tau|^2|z+h-\tau|^2} dm(\tau) \\
&\leq |h| \int_{\mathbb{C}\setminus B(z,\epsilon)} \frac{|f(\tau)||z-\tau|}{|z-\tau|^2|z+h-\tau|^2} dm(\tau) \\
&\quad + |h| \int_{\mathbb{C}\setminus B(z,\epsilon)} \frac{|f(\tau)||z+h-\tau|}{|z-\tau|^2|z+h-\tau|^2} dm(\tau) \\
&\leq 2\frac{|h|}{\epsilon(\epsilon-2|h|)^2} \int_{\mathbb{C}\setminus B(z,\epsilon)} |f(\tau)| dm(\tau) \\
&\leq 2\frac{|h|}{\epsilon(\epsilon-2|h|)^2} \|f\|_{\mathcal{L}^1(\mathbb{C})}
\end{aligned}$$

which can be arbitrarilly small for $|h|$ small enough. □

Proof of Proposition 2.3.3. Our main goal is to obtain the following good lambda inequality,

$$\begin{aligned}
&\omega\big(\{z \in \bar{\mathcal{P}} : \mathcal{S}_* f(z) > 10\lambda,\, \mathcal{M}_\omega f(z) \leq \gamma\,\lambda\}\big) \\
&\qquad\qquad \leq C\,\gamma\,\omega\left(\{z \in \bar{\mathcal{P}} : \mathcal{S}_* f(z) > \lambda\}\right)
\end{aligned} \tag{2.32}$$

for every $\lambda > 0$ and some γ small enough. Recall that $\mathcal{M}_\omega$ denotes the centered Hardy-Littlewood maximal function with respect to the ω measure. By standard arguments that will lead to the proposition, as we will see below.

To get (2.32), we will use a Whitney covering of Ω_λ which is a proper open set due to Lemma 2.3.12.

Let

$$\Omega_\lambda = \bigcup_{j=1}^{\infty} Q_j$$

be a Whitney decomposition of Ω_λ such that, as in the Remark 1.2.5, it satisfies $100Q_j \subset \Omega_\lambda$, $1000Q_j \nsubseteq \Omega_\lambda$ and $\sum_j \chi_{10\,Q_j} \leq C$.

Let Q_j be a fixed Whitney square, and assume that there exist $z_j \in Q_j \cap \bar{\mathcal{P}}$ such that $\mathcal{M}_\omega f(z_j) \leq \gamma\,\lambda$ (otherwise there is nothing to prove). Let $t_j \in \mathbb{C} \setminus \Omega_\lambda$ be the closest point to Q_j in $\mathbb{C} \setminus \Omega_\lambda$. Let $B = B(t_j, c_0\ell(Q_j))$, where $c_0 = 506\sqrt{2}$ and $B_j = B(z_j, \ell(Q_j))$ (see figure 2.2).

Notice that

$$3\,B_j \subset 7\,Q_j \subset \Omega_\lambda \tag{2.33}$$

and thus $t_j \notin 3\,B_j$ (recall that $t_j \notin \Omega_\lambda$), but

$$Q_j \subset 2\,B_j. \tag{2.34}$$

Since $\operatorname{dist}(t_j, z_j) \leq 500\sqrt{2}\,\ell(Q_j)$ by (1.16), $10\,Q_j \subset \overline{B\left(z_j, \left(\frac{1}{2}+5\right)\sqrt{2}\,\ell(Q_j)\right)} \subset \overline{B\left(t_j, \left(500+\frac{11}{2}\right)\sqrt{2}\,\ell(Q_j)\right)} \subset B$, so

$$10\,Q_j \subset B. \tag{2.35}$$

As an immediate consequence, by (2.33), $3\,B_j \subset B$. With an analogous reasoning we get $B \subset 1506\sqrt{2}\,Q_j \subset 3000\,Q_j$.

Now, we can decompose $f = f\chi_B + f\chi_{\mathbb{C}\setminus B}$.

For every $z \in Q_j$, the truncated singular integral $\mathcal{S}_\epsilon(f\chi_B)(z)$ can be written as the sum of two terms,

$$\mathcal{S}_\epsilon(f\chi_B)(z) = \mathcal{S}_\epsilon(f\chi_{3\,B_j})(z) + \mathcal{S}_\epsilon(f\chi_{B\setminus 3\,B_j})(z). \tag{2.36}$$

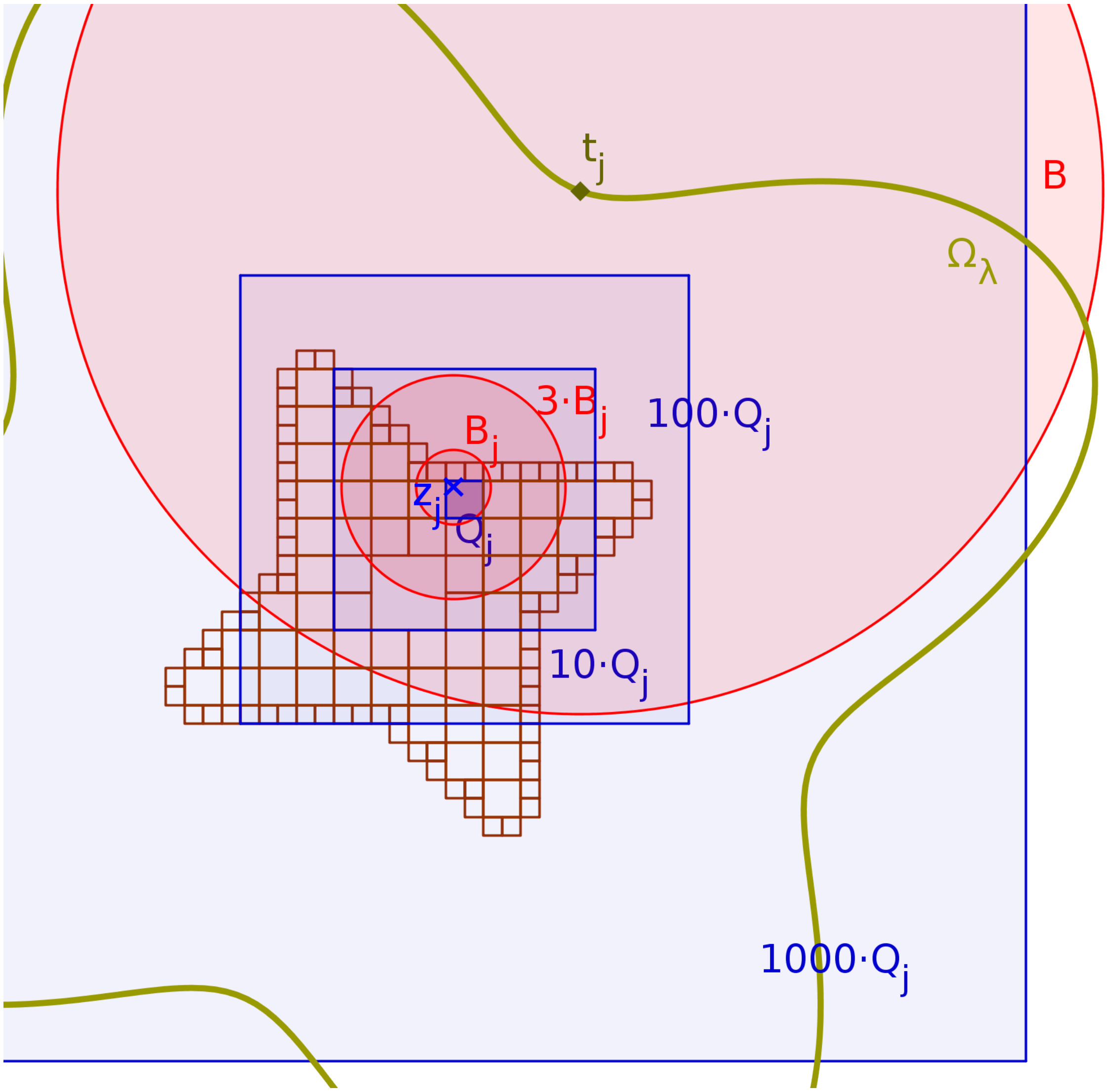


Figure 2.2: Whitney decomposition of Ω_λ and the other elements related.

Taking into account that for all $z \in Q_j$ and $t \in B \setminus 3\,B_j$, by (2.34), we get that $|t-z| \geq \ell(Q_j)$, so we obtain, using also Lemma 2.3.10 and the definition of z_j, that

$$\begin{aligned}
\left|\mathcal{S}_\epsilon(f\chi_{B\setminus 3\,B_j})(z)\right| &= \left|\int_{|t-z|\geq\epsilon} \frac{f(t)\chi_{B\setminus 3\,B_j}(t)}{(t-z)^2}dm(t)\right| \\
&\leq \frac{1}{\ell(Q_j)^2}\int_{B\setminus 3\,B_j}|f(t)|dm(t) \\
&\leq \frac{C}{\ell(3000\,Q_j)^2}\int_{3000\,Q_j}|f(t)|dm(t) \\
&\leq C\,\mathcal{M}f(z_j) \leq C\mathcal{M}_\omega f(z_j) \leq C\,\gamma\,\lambda,
\end{aligned}$$

C depends only on C_{pack}. This inequality is uniform in ϵ, so

$$\mathcal{S}_*(f\chi_{B\setminus 3\,B_j})(z) \le \lambda \tag{2.37}$$

taking γ small enough.

Therefore, since $\mathcal{S}_*f(z) \le \mathcal{S}_*(f\,\chi_B) + \mathcal{S}_*(f\,\chi_{\mathbb{C}\setminus B})$, we have

$$\begin{aligned}
&\omega\left(\{z\in Q_j : \mathcal{S}_*f(z) > 10\lambda,\, \mathcal{M}_\omega f(z) \le \gamma\,\lambda\}\right)\\
&\quad\le \omega\left(\{z\in Q_j : \mathcal{S}_*(f\,\chi_B)(z) > 2\lambda,\, \mathcal{M}_\omega f(z) \le \gamma\,\lambda\}\right)\\
&\qquad+ \omega\left(\{z\in Q_j : \mathcal{S}_*(f\,\chi_{\mathbb{C}\setminus B})(z) > 8\lambda,\, \mathcal{M}_\omega f(z) \le \gamma\,\lambda\}\right)\\
&\quad=: A' + B'.
\end{aligned}$$

But (2.37) implies

$$\mathcal{S}_*(f\chi_B)(z) \le \mathcal{S}_*(f\chi_{3\,B_j})(z) + \mathcal{S}_*(f\chi_{B\setminus 3\,B_j})(z) \le \mathcal{S}_*(f\chi_{3\,B_j})(z) + \lambda,$$

so if $\mathcal{S}_*(f\chi_B)(z)$ is bigger than 2λ, then $\mathcal{S}_*(f\chi_{3\,B_j})(z) > \lambda$. Thus,

$$A' \le \omega\left(\{z\in Q_j : \mathcal{S}_*(f\,\chi_{3\,B_j})(z) > \lambda,\, \mathcal{M}_\omega f(z) \le \gamma\,\lambda\}\right).$$

On the other hand, using Lemma 2.3.11, the equation (2.33) and the fact that $\widetilde{Q_j} := Q(z_j, 3\ell(Q_j)) \subset 7\,Q_j$, one gets

$$\begin{aligned}
\omega\left(\{z\in Q_j : \mathcal{S}_*(f\,\chi_{3\,B_j})(z) > \lambda\}\right) &\le \frac{C}{\lambda}\|f\,\chi_{3\,B_j}\|_{\mathcal{L}^1(\omega)}\\
&= \frac{C}{\lambda}\int_{3\,B_j}|f|d\omega\\
&\le \frac{C}{\lambda}\int_{\widetilde{Q_j}}|f|d\omega\\
&\le \frac{C}{\lambda}\,\omega(7\,Q_j)\mathcal{M}_\omega f(z_j).
\end{aligned}$$

Notice that the constant depends on C_{pack} and t and is increasing with respect to the latter.

Therefore,

$$A' \le C\,\gamma\,\omega(7\,Q_j). \tag{2.38}$$

To estimate B', we will use the triangular inequality

$$\begin{aligned}
\left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z)\right| \le{}& \left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z) - \mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(t_j)\right|\\
&+ \left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(t_j) - \mathcal{S}_\epsilon f(t_j)\right| + \left|\mathcal{S}_\epsilon f(t_j)\right|.
\end{aligned} \tag{2.39}$$

For any $z \in Q_j$,

$$\begin{aligned}\left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z) - \mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(t_j)\right| &\leq \frac{1}{\pi}\int_{\mathbb{C}\setminus B} |f(t)| \left|\frac{1}{(t-z)^2} - \frac{1}{(t-t_j)^2}\right| dm(t) \\ &\leq C\,\ell(Q_j)\int_{\mathbb{C}\setminus B} \frac{|f(t)|\left|t - \frac{z+t_j}{2}\right|}{|t-z|^2|t-t_j|^2} dm(t)\end{aligned}$$

because $\left|\frac{1}{(t-z)^2} - \frac{1}{(t-t_j)^2}\right| = \frac{|(t-t_j)^2-(t-z)^2|}{|t-z|^2|t-t_j|^2} = \frac{|(t_j-z)(t_j+z)-2t\,t_j+2t\,z|}{|t-z|^2|t-t_j|^2} = |t_j - z|\frac{|t_j+z-2t|}{|t-z|^2|t-t_j|^2} \leq 2c_0\ell(Q_j)\frac{\left|\frac{t_j+z}{2}-t\right|}{|t-z|^2|t-t_j|^2}$. Using the fact that $|t-z|$, $|t-t_j|$, $\left|\frac{t_j+z}{2} - t\right|$ and $|t-z_j|$ are between $2c_0\ell(Q_j)$ and $\frac{9\sqrt{2}}{2}\ell(Q_j)$, (2.35) and Lemma 2.3.10, we get

$$\begin{aligned}\left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z) - \mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(t_j)\right| &\leq C\,\ell(Q_j)\int_{\mathbb{C}\setminus B}\frac{|f(t)|}{|t-z_j|^3}dm(t) \\ &\leq C\,\ell(Q_j)\int_{\mathbb{C}\setminus 10\,Q_j}\frac{|f(t)|}{|t-z_j|^3}dm(t) \\ &\leq C\,\ell(Q_j)\sum_{k=4}^\infty\int_{2^k\,Q_j\setminus 2^{k-1}\,Q_j}\frac{|f(t)|}{|t-z_j|^3}dm(t) \\ &\leq C\,\ell(Q_j)\sum_{k=4}^\infty\int_{2^k\,Q_j}\frac{|f(t)|}{(2^{k-3}\ell(Q_j))^3}dm(t) \\ &\leq C\sum_{k=4}^\infty\frac{1}{2^{k-9}}\frac{1}{(2^k\ell(Q_j))^2}\int_{2^k\,Q_j}|f(t)|dm(t) \\ &\leq C\,\mathcal{M}f(z_j) \leq C\,\mathcal{M}_\omega f(z_j).\end{aligned}$$

The constant here depends only on C_{pack}.

Since $B(z, 3\ell(Q_j)) \subset 7\,Q_j \subset B$, it suffices to take $\epsilon > 3\ell(Q_j)$ to compute $\mathcal{S}_*(f\,\chi_{\mathbb{C}\setminus B})(z) = \sup_{\epsilon>0}|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z)|$. Therefore,

$$\begin{aligned}\left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(t_j) - \mathcal{S}_\epsilon(f)(t_j)\right| &= \frac{1}{\pi}\left|\int_{|t-t_j|>\epsilon}\frac{f(t)\,\chi_B(t)}{(t-t_j)^2}dm(t)\right| \\ &\leq \frac{C}{9\ell(Q_j)^2}\int_{|t-t_j|>\epsilon}|f(t)|\chi_B(t)dm(t) \\ &= \frac{C}{\ell(Q_j)^2}\int_{\epsilon<|t-t_j|<c_0\ell(Q_j)}|f(t)|dm(t) \\ &\leq \frac{C}{\ell(Q_j)^2}\int_{|t-z_j|<2c_0\ell(Q_j)}|f(t)|dm(t) \\ &\leq C\,\mathcal{M}f(z_j) \leq C\,\mathcal{M}_\omega f(z_j),\end{aligned}$$

The constant here is also dependent on C_{pack}. Summarizing, we get in (2.39) that

$$\left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z)\right| \le C\,\mathcal{M}_\omega f(z_j) + |\mathcal{S}_\epsilon f(t_j)|$$

for some constant C.

Therefore, if z belongs to $\{z\in Q_j : \mathcal{S}_*(f\,\chi_{\mathbb{C}\setminus B})(z) > 8\lambda,\, \mathcal{M}_\omega f(z) \le \gamma\,\lambda\}$, we get that there exists a real number $\epsilon > 3\ell(Q_j)$ such that

$$\begin{aligned} 8\lambda &< \left|\mathcal{S}_\epsilon(f\,\chi_{\mathbb{C}\setminus B})(z)\right| \le C\,\mathcal{M}_\omega f(z_j) + |\mathcal{S}_\epsilon f(t_j)| \\ &\le C\,\mathcal{M}_\omega f(z_j) + \mathcal{S}_* f(t_j) \le C\,\gamma\,\lambda + \lambda \end{aligned} \tag{2.40}$$

because $t_j \notin \Omega_\lambda = \{z : \mathcal{S}_* f(z) > \lambda\}$. In particular, for small enough γ, this is impossible, so $B' = 0$. Now, with the help of (2.38), we get

$$\omega\left(\{z\in Q_j\cap\bar{\mathcal{P}} : \mathcal{S}_* f(z) > 10\lambda,\, \mathcal{M}_\omega f(z)\le\gamma\,\lambda\}\right) \le C\,\gamma\,\omega\left(7\,Q_j\right). \tag{2.41}$$

Since squares $7\,Q_j$ have bounded overlap, summing in j we obtain

$$\omega\left(\{z\in\bar{\mathcal{P}} : \mathcal{S}_* f(z) > 10\lambda,\, \mathcal{M}_\omega f(z)\le\gamma\,\lambda\}\right) \le C\,\gamma\,\omega\left(\{z\in\bar{\mathcal{P}} : \mathcal{S}_* f(z) > \lambda\}\right)$$

which is (2.32), with C depending on t and C_{pack} and increasing with respect to the former.

Now,

$$\begin{aligned} \frac{1}{10^p}\|\mathcal{S}_* f\|^p_{\mathcal{L}^p(\omega)} &= \frac{1}{10^p}\int_0^\infty p\,\lambda^{p-1}\omega(\{z\in\bar{\mathcal{P}} : \mathcal{S}_* f(z) > \lambda\})d\lambda \\ &= \int_0^\infty p\,\lambda^{p-1}\omega(\{z\in\bar{\mathcal{P}} : \mathcal{S}_* f(z) > 10\,\lambda\})d\lambda \\ &\le \int_0^\infty p\,\lambda^{p-1}\Big(\omega(\{z\in\bar{\mathcal{P}} : \mathcal{S}_* f(z) > 10\lambda),\, \mathcal{M}_\omega f(z)\le\gamma\,\lambda\} \\ &\quad + \omega(\{z\in\bar{\mathcal{P}} : \mathcal{M}_\omega f(z) > \gamma\,\lambda\})\Big)d\lambda \\ &\le \int_0^\infty C\,\gamma\,\omega\left(\{z\in\bar{\mathcal{P}} : \mathcal{S}_* f(z) > \lambda\}\right)p\,\lambda^{p-1}d\lambda \\ &\quad + \int_0^\infty \omega(\{z\in\bar{\mathcal{P}} : \mathcal{M}_\omega f(z) > \gamma\,\lambda\})p\,\lambda^{p-1}d\lambda \\ &= C\,\gamma\|\mathcal{S}_* f\|^p_{\mathcal{L}^p(\omega)} + \gamma^{-p}\|\mathcal{M}_\omega f\|^p_{\mathcal{L}^p(\omega)}, \end{aligned}$$

so

$$\|\mathcal{S}_* f\|^p_{\mathcal{L}^p(\omega)} \le \frac{1}{\gamma^p(10^{-p} - C\,\gamma)}\|\mathcal{M}_\omega f\|^p_{\mathcal{L}^p(\omega)}.$$

For γ small enough, this implies boundedness of $\mathcal{S}_*$, with

$$\|\mathcal{S}_* f\|^p_{\mathcal{L}^p(\omega)} \le C(p, C_{pack}, t)\|f\|^p_{\mathcal{L}^p(\omega)},$$

with $C(p, C_{pack}, t)$ increasing with respect to t. □

2.3.4. Small Dilatation.

We will prove first a restatement of the main Theorem for a specific class of quasiconformal mappings, namely those of small dilatation.

LEMMA 2.3.13. *Let $0 < t \leq t_0 < 2$. Then there is a small constant $0 < \kappa_0 < 1$ ($\kappa_0 = \kappa_0(t_0)$ is a function of t_0) so that the following holds. Let $g : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal principal map and conformal outside $\frac{1}{2}\mathbb{D}$ such that*

$$\frac{K-1}{K+1} \leq \kappa_0.$$

Then we have the following implication for all compact subsets $E \subset \frac{1}{4}\mathbb{D}$:

$$\widetilde{\mathcal{H}}^{t'}_\infty(g(E)) \leq C(t_0)\left(\widetilde{\mathcal{H}}^t_\infty(E)\right)^{t'/tK}, \tag{2.42}$$

where

$$t' = \frac{2Kt}{2+(K-1)t}.$$

We use a familiar scheme, which we recall here. We have already seen how to approximate the t-Hausdorff content of a set E by a finite union of dyadic squares. We can therefore assume that E is in fact one of such unions, and we approximate the Hausdorff content of the image of E. Applying Stoilow factorization methods, a normalized version of the mapping ϕ is written as $\phi = \phi_1 \circ h$, where both h, $\phi_1 : \mathbb{C} \to \mathbb{C}$ are principal K-quasiconformal mappings, such that h is conformal in the complement of the set E and ϕ_1 is conformal on the set $F = h(E)$. One then studies the mapping properties of the two functions ϕ_1 and h separately, referred to the "conformal inside" and the "conformal outside" parts, respectively.

The conformal inside part has already been adressed in [**ACMOUT**] and we proof the relevant result in Theorem 2.3.15 below. The conformal outside part is in [**LSUT**], and the point we turn to now. We make some changes in the original proof to simplify it. In particular we don't use $\mathcal{L}^p$ Hölder inequalities for $0 < p < 1$ to control the weights defined in (2.43) below, but direct calculations and the usual Hölder inequalities.

We will use the following notation. For a finite collection of pairwise disjoint dyadic squares $\mathcal{P} = \{P_j\}_{j=1}^N$, let

$$\beta_j = \frac{\ell(P_j)^{t-2}}{\left(\sum_{i=1}^N \ell(P_i)^t\right)^{1-2/t}}. \tag{2.43}$$

Notice that $\beta_j > 0$ and

$$\sum_{j=1}^{N} \ell(P_j)^2 \cdot \beta_j = \sum_{j=1}^{N} \ell(P_j)^2 \frac{\ell(P_j)^{t-2}}{\left(\sum_{i=1}^{N} \ell(P_i)^t\right)^{1-2/t}} = \frac{\sum_{j=1}^{N} \ell(P_j)^t}{\left(\sum_{i=1}^{N} \ell(P_i)^t\right)^{1-2/t}}$$

$$= \left(\sum_{i=1}^{N} \ell(P_i)^t\right)^{2/t}. \tag{2.44}$$

Now let $E = \bar{\mathcal{P}} = \cup_{j=1}^{N} P_j$ and let

$$\widetilde{\omega} = \sum_{j=1}^{N} \beta_j \, \chi_{P_j}, \tag{2.45}$$

which is a constant multiple of ω, as defined in (2.23).

Lemma 2.3.14. *Let $0 < t \leq t_0 < 2$. There is a positive constant ϵ_0 (which is a function of t_0) so that the following holds.*

Let $\mathcal{P} = \{P_j\}_{j=1}^N$ be a finite collection of dyadic squares which satisfy the t-Carleson packing condition $\|\mathcal{P}\|_{t-\text{pack}}^t \leq C_{pack}$. Assume further that the squares $3\,P_j$ are pairwise disjoint.

Let $E = \bar{\mathcal{P}} = \cup_{j=1}^{N} P_j$ and let $f : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping which is conformal outside the compact set E, with $\frac{K-1}{K+1} < \epsilon_0$.

Then, there is a constant $C(t_0)$ which depends only on t_0, such that

$$\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^t \leq C(t_0) \sum_{j=1}^{N} \ell(P_j)^t. \tag{2.46}$$

Proof. Defining β_j as in (2.43) and $\widetilde{\omega}$ as in (2.45), by quasisymmetry (1.25), and using Hölder inequality for ℓ^p we get

$$\begin{aligned}
\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^t &= \sum_{j=1}^{N} \operatorname{diam}(f(P_j))^t \ell(P_j)^{\frac{t}{2}(t-2)} \ell(P_j)^{t\frac{2-t}{2}} \\
&\leq \left(\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^2 \ell(P_j)^{t-2}\right)^{t/2} \left(\sum_{i=1}^{N} \ell(P_i)^t\right)^{1-\frac{t}{2}} \\
&= \left(\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^2 \, \beta_j\right)^{t/2}.
\end{aligned}$$

Exponentiating, by quasisymmetry we get

$$\left(\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^t\right)^{2/t} \le \sum_{j=1}^{N} \operatorname{diam}(f(P_j))^2\, \beta_j$$
$$\le C(K)\int_E J(z,f)\widetilde{\omega}(z)dm(z). \tag{2.47}$$

We follow Astala's approach for his Area Distortion Theorem ([**AIM**, Chapter 13]), equipped with the results given above. The central role of the Beurling operator is indicated by the identity

$$\partial_z f = 1 + \mathcal{S}(\partial_{\bar z} f). \tag{2.48}$$

Using the trivial inequality $|2\mathrm{Re}(a)| \le 2|a| \le |a|^2+1$, and that $J(z,f) = |\partial_z f|^2 - |\partial_{\bar z} f|^2$, we can estimate

$$\begin{aligned}\int_E J(z,f)\widetilde{\omega}(z)dm(z) &= \int_E (|\partial_z f|^2 - |\partial_{\bar z} f|^2)\widetilde{\omega}(z)dm(z)\\ &= \int_E (1 + 2\mathrm{Re}\mathcal{S}(\partial_{\bar z} f) + |\mathcal{S}(\partial_{\bar z} f)|^2 - |\partial_{\bar z} f|^2)\widetilde{\omega}(z)dm(z)\\ &\le 2\int_E (1+|\mathcal{S}(\partial_{\bar z} f)|^2)\widetilde{\omega}(z)dm(z)\\ &= 2\left(\int_E \widetilde{\omega}(z)dm(z) + \int_E |\mathcal{S}(\partial_{\bar z} f)|^2\widetilde{\omega}(z)dm(z)\right)\\ &= 2(I_1+I_2).\end{aligned} \tag{2.49}$$

Notice that $I_1 = \sum_{j=1}^N \ell(P_j)^2\beta_j$. We shall bound the other term by a multiple of I_1. Indeed, with respect to I_2, since $\widetilde{\omega} = C\omega$, the set of bounded operators in $\mathcal{L}^2(\widetilde{\omega})$ and the equivalent for $\mathcal{L}^2(\omega)$ coincide and the norms of these operators do not change. By Proposition 2.3.3,

$$I_2 = \int_E |\mathcal{S}(\partial_{\bar z} f)|^2\widetilde{\omega}(z)dm(z) \le C(t_0)\int_E |\partial_{\bar z} f|^2\widetilde{\omega}(z)dm(z). \tag{2.50}$$

Let us call the last integral I_3. The Beurling operator is again decisive. Recall the representation of $\partial_{\bar z} f$ as a power series in the Beltrami coefficient μ in (1.29)

$$\partial_{\bar z} f = \mu\partial_z f = \mu + \mu\mathcal{S}(\mu) + \mu\mathcal{S}(\mu\mathcal{S}(\mu)) + \cdots .$$

As we shall see, this series converges in $\mathcal{L}^2(\widetilde{\omega})$ for small (depending on t) $\|\mu\|_\infty$ by Proposition 2.3.3.

Observe the two inequalities

$$\left(\int_E |\mu|^2\widetilde{\omega}(z)dm(z)\right)^{1/2} \le \|\mu\|_\infty \left(\int_E \widetilde{\omega}(z)dm(z)\right)^{1/2} = \|\mu\|_\infty I_1^{1/2}, \tag{2.51}$$

and

$$\left(\int_E |\mu\mathcal{S}(g)|^2\widetilde{\omega}(z)dm(z)\right)^{1/2} \leq \|\mu\|_\infty \|\mathcal{S}\|_{\mathcal{L}^2(\widetilde{\omega})} \left(\int_E |g|^2\widetilde{\omega}(z)dm(z)\right)^{1/2}. \tag{2.52}$$

The second inequality is applied inductively to $g = \mu$, $g = \mu\mathcal{S}(\mu)$, and so on. Using the triangle inequality in (1.29) in the $\mathcal{L}^2(\widetilde{\omega})$ norm gives

$$I_3^{1/2} \leq \|\mu\|_\infty \left(\sum_{n=0}^{\infty} (\|\mu\|_\infty \|\mathcal{S}\|_{\mathcal{L}^2(\widetilde{\omega})})^n\right) I_1^{1/2}. \tag{2.53}$$

The middle term on the right is bounded by two if we demand

$$\|\mu\|_\infty < \epsilon_0 = \min\{(2\,C(2, C_{pack}, t_0)^{-1}, 1/2\} \tag{2.54}$$

using the notation of (2.24). This is the ϵ_0 required in the statement of Lemma 2.3.14. It follows that

$$I_3 \leq I_1. \tag{2.55}$$

From (2.47), (2.49), (2.50) and (2.55), it follows that

$$\left(\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^t\right)^{2/t} \leq C(K)(1 + C(t_0))I_1 \leq C(t_0)I_1 \tag{2.56}$$

It remains to bound I_1 by the right hand side of (2.46).

Taking into account (2.44), we have that

$$I_1 = \sum_{j=1}^{N} \ell(P_j)^2\beta_j = \left(\sum_{j=1}^{N} \ell(P_j)^t\right)^{2/t}, \tag{2.57}$$

which finsihes the proof. □

In [**ACMOUT**] we find how to deal with the "conformal inside" case for the particular case $t' = 1$. The proof is general enough and we adapt it here.

THEOREM 2.3.15. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping which is conformal outside $\mathbb{D}$. Let $\{P_j\}_{j=1}^N$ be a finite family of pairwise disjoint squares and $S_j = f(P_j)$ its images under a unique K-quasiconformal mapping, such that $S_j \subset \mathbb{D}$ and assume that ϕ is conformal in $\Omega = \bigcup_j S_j$. Then, for any $t \in (0, 2]$ and*

$$t' = \frac{2K\,t}{2 + (K-1)t},$$

we have

$$\left(\sum_{j=1}^{N} \operatorname{diam}(\phi(S_j))^{t'}\right)^{1/t'} \leq C(K) \left(\sum_{j=1}^{N} \operatorname{diam}(S_j)^t\right)^{1/t\,K}. \tag{2.58}$$

PROOF. On the image side quasisymmetry restricts the distortion and tells that, as in (1.25),

$$\operatorname{diam}(\phi(S_j)) \approx |\phi(S_j)|^{1/2} = \left(\int_{S_j} J(z,\phi) dm(z) \right)^{1/2},$$

where the constant only depends on K. Hence for each $t \leq 2$, using Hölder inequality twice and Theorem 2.2.4, we obtain

$$\begin{aligned}
\sum_j \operatorname{diam}(\phi(S_j))^{t'} &\leq C(K) \sum_j \left(\int_{S_j} J(z,\phi) dm(z) \right)^{t'/2} \\
&\leq C \sum_j \left(\int_{S_j} J(\cdot,\phi)^{\frac{K}{K-1}} \right)^{\frac{K-1}{K}\frac{t'}{2}} |S_j|^{\frac{t'}{2K}} \\
&\leq C \left(\sum_j \int_{S_j} J(\cdot,\phi)^{\frac{K}{K-1}} \right)^{\frac{t'(K-1)}{2K}} \left(\sum_j |S_j|^{\frac{t'}{2K-(K-1)t'}} \right)^{\frac{2K-(K-1)t'}{2K}} \\
&\leq C \left(\int_{\cup_j S_j} J(\cdot,\phi)^{\frac{K}{K-1}} \right)^{\frac{t'(K-1)}{2K}} \left(\sum_j \operatorname{diam}(S_j)^{\frac{2t'}{2K-(K-1)t'}} \right)^{\frac{t'}{Kt}} \\
&\leq C\pi^{\frac{t'(K-1)}{2K}} \left(\sum_j \operatorname{diam}(S_j)^t \right)^{\frac{t'}{Kt}}.
\end{aligned}$$

□

It should be emphasized that for a general quasiconformal mapping f we have $J(z,f) \in \mathcal{L}^p_{loc}$ granted only for $p < K/(K-1)$. The improved integrability $p = K/(K-1)$ under the extra assumption that $\phi|_\Omega$ is conformal is crucial for the proof of Theorem 2.3.15 above, since we are studying Hausdorff measure rather than dimension.

At this point we prove Astala's conjecture for the case of small dilatation, Lemma 2.3.13.

PROOF OF LEMMA 2.3.13. Consider $\epsilon > 0$ and use Proposition 2.3.2, with $m = 2$ and $M_1 = -2$, that is:

$$4\,P_i \cap 4\,P_j = \emptyset \;\; \forall i \neq j, \tag{2.59}$$

$$E \subset \bigcup_{i=1}^{N} 12\,P_i, \tag{2.60}$$

$$\|\mathcal{P}\|_{t-\text{pack}} \leq 1, \tag{2.61}$$

$$\sum_{i=1}^{N} \ell(P_i)^t \leq C(\widetilde{\mathcal{H}}^t_\infty(E)+\epsilon), \text{ and} \tag{2.62}$$

$$P_i \subset \mathbb{D}_{1/2} \tag{2.63}$$

to obtain a collection of squares $\mathcal{P} = \{P_i\}$ satisfying the conclusions of this proposition with respect to the compact set E. Write

$$\Omega = \left(\bigcup_{i=1}^{N} \mathring{P}_i\right).$$

Let $g = \phi \circ f$, where ϕ and f are both principal K-quasiconformal mappings, f is conformal outside $\bar{\Omega}$ (and thus we can assume, by Theorem 1.3.7, that $f(\frac{1}{2}\mathbb{D}) \subset \mathbb{D}$), and ϕ is conformal in $f(\Omega) \cup (\mathbb{C} \setminus \mathbb{D})$ (this factorization is granted by the Riemann Measurable Mapping Theorem 1.3.25 together with the Composition Formula for Beltrami Coefficients 1.3.28). Recall that Lemma 2.3.14 only applies to quasiconformal mappings with dilatation $\|\mu\|_\infty \leq \epsilon_0$. Taking $\kappa_0 = \epsilon_0$, $\|\mu_f\|_\infty \leq \|\mu_g\|_\infty \leq \epsilon_0$, so Lemma 2.3.14 also applies to it. Now, using (2.60), the equivalence between $\widetilde{\mathcal{H}}^{t'}_\infty$ and $\mathcal{H}^{t'}_\infty$, quasisymmetry, Theorem 2.3.15, Lemma 2.3.14 and (2.62), one gets

$$\begin{aligned}
\widetilde{\mathcal{H}}^{t'}_\infty(g(E)) &\leq \widetilde{\mathcal{H}}^{t'}_\infty\left(g\left(\bigcup_i 12\, P_i\right)\right)\\
&\approx \mathcal{H}^{t'}_\infty\left(g\left(\bigcup_i 12\, P_i\right)\right)\\
&\leq \sum_i \operatorname{diam}(g(12\, P_i))^{t'}\\
&\leq C(K)\sum_i \operatorname{diam}(g(P_i))^{t'}\\
&\leq C(K)\left(\sum_i \operatorname{diam}(f(P_i))^{t}\right)^{t'/t\,K}\\
&\leq C(K,t_0)\left(\sum_i \ell(P_i)^{t}\right)^{t'/t\,K}\\
&\leq C(K,t_0)(\widetilde{\mathcal{H}}^t_\infty(E)+\epsilon)^{t'/t\,K}\\
&\leq C(\kappa_0,t_0)(\widetilde{\mathcal{H}}^t_\infty(E)+\epsilon)^{t'/t\,K}.
\end{aligned}$$

The parameter $\epsilon > 0$ is arbitrary, so the prof is complete. □

2.3.5. Proof of the Main Theorem.

PROOF OF THEOREM 2.3.1. Assume first that f is a K-quasiconformal mapping with

$$\frac{K-1}{K+1} \leq \kappa_0(t_0),$$

and let E be a compact set contained in a ball B. Let $t < t_0$ and t' be as in Theorem 2.3.1.

With an analogous reasoning to the proof of Theorem 2.1.4 we get

$$\frac{\mathcal{H}^{t'}_\infty(f(E))}{\operatorname{diam}(f(B))^{t'}} \leq C(t_0) \left(\frac{\mathcal{H}^t_\infty(E)}{\operatorname{diam}(B)^t} \right)^{t'/t\,K}. \tag{2.64}$$

For arbitrary K, we use the usual factorization of a K-quasiconformal mapping into those with small dilatation. For a fixed K-quasiconformal mapping g, we can write

$$g = g_\lambda \circ \cdots \circ g_2 \circ g_1, \tag{2.65}$$

so that each g_i is K_i-quasiconformal mapping, $K = K_1 \cdots K_\lambda$, and

$$K_i \leq \frac{1+\kappa_0}{1-\kappa_0} \tag{2.66}$$

for all $i = 1, 2, \cdots, \lambda$, with $\kappa_0 = \kappa_0(t')$ (Theorem 1.3.29).

Let us set

$$\tau(t, K) = \frac{2Kt}{2+(K-1)t},$$

and inductively define $\tau_0 = t$, $\tau_{i+1} = \tau(\tau_i, K_{i+1})$. By induction, if $\tau_i = \frac{2K_1 \cdots K_i t}{2+(K_1 \cdots K_i - 1)t}$, then

$$\tau_{i+1} = \frac{2K_{i+1} \frac{2K_1 \cdots K_i t}{2+(K_1 \cdots K_i - 1)t}}{2 + (K_{i+1}-1)\frac{2K_1 \cdots K_i t}{2+(K_1 \cdots K_i - 1)t}} = \frac{2K_1 \cdots K_{i+1} t}{2+(K_1 \cdots K_{i+1} - 1)t}$$

and in particular

$$\tau_\lambda = \frac{2Kt}{2+(K-1)t} = t'.$$

Notice also that $\tau(t, K) > t$ as long as $0 < t < 2$ and $K \geq 1$, implying that the sequence $\{\tau_i\}$ is increasing and, therefore, $\tau_i \leq t'$.

Let $E \subset \mathbb{C}$ be a compact subset of the plane. By (2.66) we apply Lemma 2.3.13 to each g_i individually with $t_0 = t'$ and we get, choosing at each step $B_{i+1} \supset g_i(B_i)$, $B_1 = B$, with diameter comparable,

$$\frac{\mathcal{H}^{\tau_i}_\infty(g_i \circ \cdots \circ g_1(E))}{\operatorname{diam}(g_i(B_i))^{\tau_i}} \leq C(t_0) \left(\frac{\mathcal{H}^{\tau_{i-1}}_\infty(g_{i-1} \circ \circ \cdots g_1(E))}{\operatorname{diam}(B_i)^{\tau_{i-1}}} \right)^{\tau_i/\tau_{i-1}\,K}.$$

It follows from an inductive application of (2.64) that, as the number of steps is controlled by K and t and t_0 depends on t, we have

$$\frac{\mathcal{H}^{\tau_i}_\infty(g_i \circ \cdots \circ g_1(E))}{\operatorname{diam}(g_i \circ \cdots \circ g_1(B))^{\tau_i}} \le C(K,t) \left(\frac{\mathcal{H}^t_\infty(E))}{\operatorname{diam}(B)^t}\right)^{\tau_i/(t\,K_i)}.$$

In particular, for $i = \lambda$, we prove Theorem 2.3.1. $\square$

2.4. Distortion of the Hausdorff Measure

Last year K. Astala, A. Clop, X. Tolsa, I. Uriarte-Tuero and J. Verdera published a preprint [**ACTUTV**] including a similar result to the Main Theorem above but with Hausdorff measure instead of Hausdorff content, obtained by Tolsa. We attach it here for the sake of the completeness of this text.

THEOREM 2.4.1. *Let $0 < t < 2$ and denote $t' = \frac{2Kt}{2+(K-1)t}$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be K-quasiconformal. For any ball B and any compact set $E \subset B$, we have*

$$\frac{\mathcal{H}^{t'}(\phi(E))}{\operatorname{diam}(\phi(B))^{t'}} \leq C(K,t) \left(\frac{\mathcal{H}^t(E)}{\operatorname{diam}(B)^t} \right)^{t'/t\,K}.$$

In particular, if $\mathcal{H}^t(E)$ is finite, then $\mathcal{H}^{t'}(\phi(E))$ is also finite.

The proof of this uses more general Hausdorff contents that the ones we have used in this text, namely, those generated by gauge functions.

2.5. Examples Showing Sharpness of Results

Ignacio Uriarte-Tuero found in 2008 sharp examples for Theorem 2.4.1 (then a conjecture) in [**UT08**]. The same example is valid for Theorem 2.3.1 on Hausdorff content and implies sharpness of Astala's Distortion of Dimension Theorem 2.2.6.

We recall it here and sketch the proof. We refer to the original text [**UT08**] for the details.

2.5.1. Basic Construction. We will construct a K-quasiconformal mapping ϕ as a limit of a sequence ϕ_N of K-quasiconformal mappings and E will be a Cantor-type set. Our objective is that

$$\mathcal{H}^{t'}_\infty(\phi(E)) \approx \mathcal{H}^t_\infty(E) \quad \text{and} \quad \mathcal{H}^{t'}(\phi(E)) \approx \mathcal{H}^t(E)$$

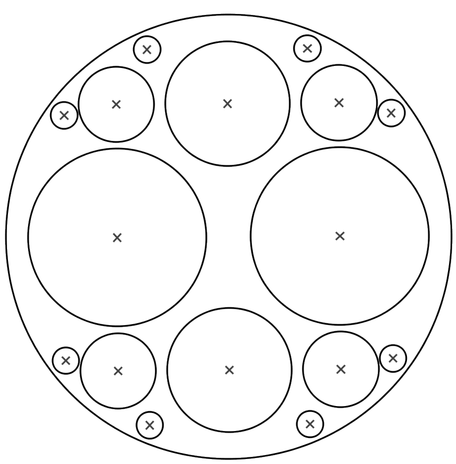
showing thus the sharpness of Theorem 2.3.1 and Theorem 2.4.1.

At the first step we will choose radius R_{1,j_1} and for each of them a different number m_{1,j_1} of disks of that radius.

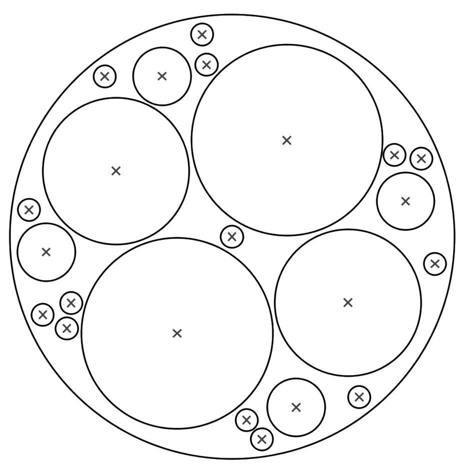
Choose first $m_{1,1}$ disjoint disks $D(z^{i_1}_{1,1}, R_{1,1}) \subset \mathbb{D}$, $i_1 = 1, \cdots, m_{1,1}$ and then, inductively choose m_{1,j_1} disks $D(z^{i_1}_{1,j_1}, R_{1,j_1}) \subset \mathbb{D}$, $i_1 = 1, \cdots, m_{1,j_1}$, disjoint among themselves and with the previous ones, for $j_1 = 2, \cdots, l_1$ so that they cover a big proportion of the area of the disk to be chosen later:

$$c_1 := m_{1,1}(R_{1,1})^2 + \cdots + m_{1,l_1}(R_{1,l_1})^2 = 1 - \epsilon_1.$$

We can assume the radii $R_{1,j_1} < \delta_1$ for a $\delta_1 > 0$ as small as we wish.

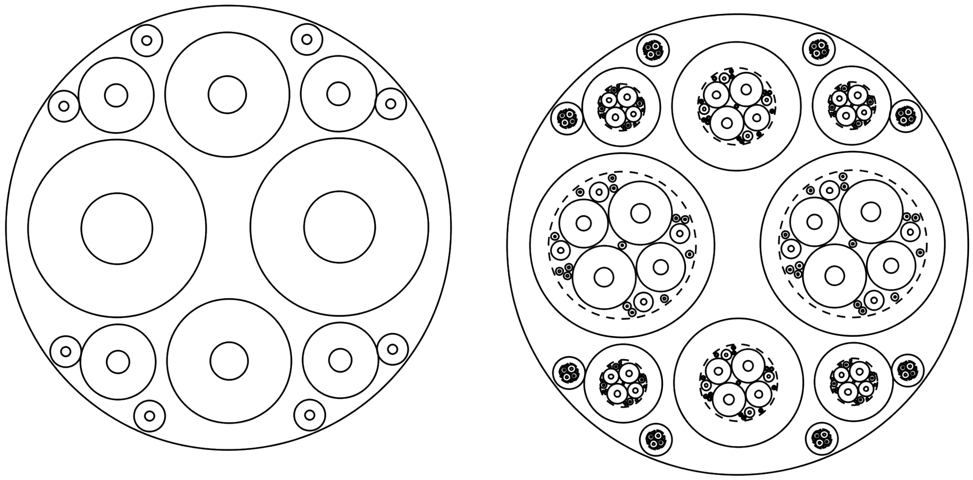


(a) The first generation, $D(z^{i_1}_{1,j_1}, R_{1,j_1})$



(b) The second generation, $D(z^{i_2}_{2,j_2}, R_{2,j_2})$

Figure 2.3: The first two generations of disks $D(z^{i_N}_{N,j_N}, R_{N,j_N})$ inside de unit disk $\mathbb{D}$



(a) The first generation, $D^{(i_1)}_{(j_1)}$ and $\left(D^{(i_1)}_{(j_1)}\right)'$

(b) The second generation, $D^{(i_1,i_2)}_{(j_1,j_2)}$ and $\left(D^{(i_1,i_2)}_{(j_1,j_2)}\right)'$ both contained in $\left(D^{(i_1)}_{(j_1)}\right)''$

Figure 2.4: The first two generations of disks D^I_J.

We associate to each radius R_{1,j_1} a value $\sigma_{1,j_1} < 1$ to be fixed later. We fix for the induction $r_{(j_1)} = R_{1,j_1}$. Let $\varphi^{i_1}_{1,j_1}$ be the homothetic mapping from the unit disk to a neighbourhood of the disk $D(z^{i_1}_{1,j_1}, R_{1,j_1})$ defined by $\varphi^{i_1}_{1,j_1}(z) = z^{i_1}_{1,j_1} + (\sigma_{1,j_1})^K r_{(j_1)} z$ and define

$$D^{(i_1)}_{(j_1)} := \frac{1}{(\sigma_{1,j_1})^K}\varphi^{i_1}_{1,j_1}(\mathbb{D}) = D(z^{i_1}_{1,j_1}, r_{(j_1)}), \text{ and}$$

$$\left(D^{(i_1)}_{(j_1)}\right)' := \varphi^{i_1}_{1,j_1}(\mathbb{D}) = D(z^{i_1}_{1,j_1}, \sigma^K_{1,j_1} r_{(j_1)}) \subset D^{(i_1)}_{(j_1)}.$$

As the first approximation of the mapping, define

$$g_1(z) = \begin{cases} (\sigma_{1,j_1})^{1-K}(z - z^{i_1}_{1,j_1}) + z^{i_1}_{1,j_1}, & z \in \left(D^{(i_1)}_{(j_1)}\right)' \\ \left|\frac{z - z^{i_1}_{1,j_1}}{r_{1,j_1}}\right|^{\frac{1}{K}-1} (z - z^{i_1}_{1,j_1}) + z^{i_1}_{1,j_1} & z \in D^{(i_1)}_{(j_1)} \setminus \left(D^{(i_1)}_{(j_1)}\right)' \\ z, & z \notin \bigcup_{i_1,j_1} D^{(i_1)}_{(j_1)}. \end{cases}$$

This is a K-quasiconformal mapping, conformal outside the union of the annuli $D^{(i_1)}_{(j_1)} \setminus \left(D^{(i_1)}_{(j_1)}\right)'$. It maps $D^{(i_1)}_{(j_1)}$ onto itself and $\left(D^{(i_1)}_{(j_1)}\right)'$ onto $\left(D^{(i_1)}_{(j_1)}\right)'' = D(z^{i_1}_{1,j_1}, \sigma_{1,j_1} R_{1,j_1})$, while the rest of the plane remains fixed. Write $\phi_1 = g_1$.

Now choose, after step N, radii $R_{N,j}$ and for each of them a different number $m_{N,j}$ of disks of that radius.

Choose as before $m_{N,1}$ disjoint disks $D(z_{N,1}^{i_N}, R_{N,1}) \subset \mathbb{D}$, $i_N = 1, \cdots, m_{N,1}$ and then, inductively choose m_{N,j_N} disks $D(z_{N,j_N}^{i_N}, R_{N,j_N}) \subset \mathbb{D}$, $i_N = 1, \cdots, m_{N,j_N}$, disjoint among themselves and with the previous ones, for $j_N = 2, \cdots, l_N$ so that they cover a big proportion of the area of the disk to be chosen later:

$$c_N := m_{N,1}(R_{N,1})^2 + \cdots + m_{N,l_N}(R_{N,l_N})^2 = 1 - \epsilon_N. \tag{2.67}$$

We can assume the radii $R_{N,j_N} < \delta_N$ for a $\delta_N > 0$ as small as we wish.

We associate to each radius R_{N,j_N} a value $\sigma_{N,j_N} < 1$ to be fixed later. Denote, for each multiindex $J = (j_1, \cdots, j_N)$, $r_J = R_{N,j_N}\sigma_{N-1,j_{N-1}}r_{(j_{N-1},\cdots,j_1)}$. Let $\varphi_{N,j_N}^{i_N}$ be the homothetic mapping from the unit disk to a neighbourhood of the disk $D(z_{N,j_N}^{i_N}, R_{N,j_N})$ defined by $\varphi_{N,j_N}^{i_N}(z) = z_{N,j_N}^{i_N} + (\sigma_{N,j_N})^K R_{N,j_N} z$. For any multiindices $J = (j_1, \cdots, j_N)$ and $I = (i_1, \cdots, i_N)$, for $1 \le j_N \le l_N$ and $1 \le i_N \le m_{N,j_N}$, define

$$D_J^I := \phi_{N-1}\left(\frac{1}{(\sigma_{N,j_N})^K}\varphi_{1,j_1}^{i_1} \circ \cdots \circ \varphi_{N,j_N}^{i_N}(\mathbb{D})\right) = D(z_J^I, r_J), \text{ and}$$

$$(D_J^I)' := \phi_{N-1}\left(\varphi_{1,j_1}^{i_1} \circ \cdots \circ \varphi_{N,j_N}^{i_N}(\mathbb{D})\right) = D(z_J^I, \sigma_{N,j_N}^K r_J) \subset D_J^I.$$

Now define

$$g_N(z) = \begin{cases} (\sigma_{N,j_N})^{1-K}(z - z_J^I) + z_J^I, & z \in (D_J^I)' \\ \left|\frac{z-z_J^I}{r_J}\right|^{\frac{1}{K}-1}(z - z_J^I) + z_J^I & z \in D_J^I \setminus (D_J^I)' \\ z, & \text{otherwise.} \end{cases}$$

This is a K-quasiconformal mapping, conformal outside the union of the annuli $D_J^I \setminus (D_J^I)'$. It maps D_J^I onto itself and $(D_J^I)'$ onto $(D_J^I)'' = D(z_J^I, \sigma_{N,j_N} r_J)$, while the rest of the plane remains fixed. Now define $\phi_N = g_N \circ \phi_{N-1}$. Notice that it is also K-quasiconformal.

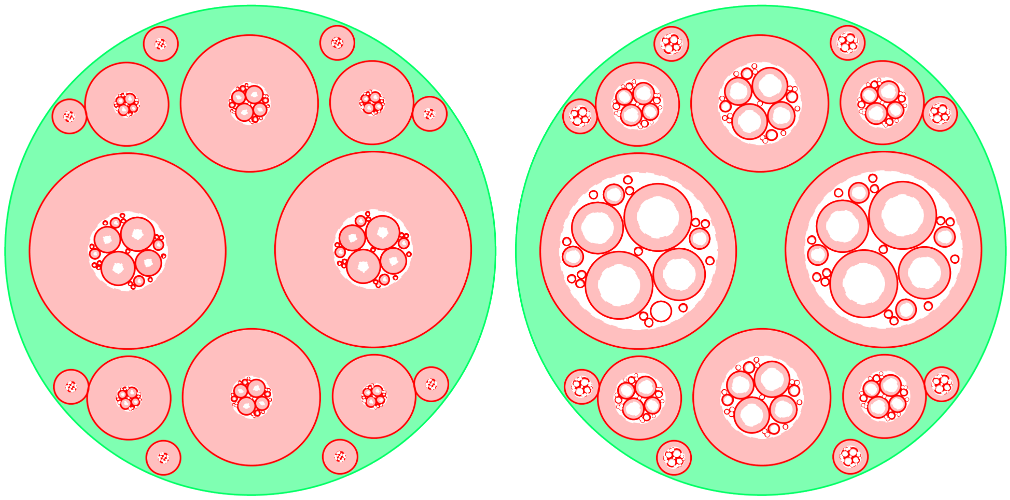
Since each ϕ_N is K-quasiconformal (and the identity outside the unit disk), there is, by Theorem 1.3.22, a partial subsequence converging locally uniformly to a limit ϕ which is also K-quasiconformal. In our case the convergence is uniform.

Now, if we define $\psi_{N,j_N}^{i_N}(z) = z_{N,j_N}^{i_N} + \sigma_{N,j_N} R_{N,j_N} z$, for $1 \le i_N \le m_{N,j_N}$ and $1 \le j_N \le l_N$, we have that ϕ maps the compact set

$$E = \bigcap_{N=1}^{\infty}\left(\bigcup_{\substack{i_1,\cdots,i_N \\ j_1,\cdots,j_N}} \varphi_{1,j_1}^{i_1} \circ \cdots \circ \varphi_{N,j_N}^{i_N}(\overline{\mathbb{D}})\right)$$

to the compact set

$$\phi(E) = \bigcap_{N=1}^{\infty}\left(\bigcup_{\substack{i_1,\cdots,i_N \\ j_1,\cdots,j_N}} \psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{N,j_N}^{i_N}(\overline{\mathbb{D}})\right).$$



(a) In white, the building blocks of E in the 1st and the 2nd steps.

(b) In white, the building blocks of $\phi(E)$ in the 1st and the 2nd steps.

Figure 2.5: The composition $\phi_2 = g_2 \circ \phi_1$. Notice that ϕ_2 is homothetic on the white spots and K-quasiconformal (a radial stretching) in the coloured zones.

Notice that, with our notation, a building block in the Nth step of the construction of E, i.e. a set of the type $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_N}_{N,j_N}(\overline{\mathbb{D}})$, is a disk of radius

$$s_J = \left((\sigma_{1,j_1})^K R_{1,j_1}\right) \cdots \left((\sigma_{N,j_N})^K R_{N,j_N}\right), \tag{2.68}$$

and a building block in the Nth step of the construction of $\phi(E)$ is a disk with radius

$$q_J = (\sigma_{1,j_1} R_{1,j_1}) \cdots (\sigma_{N,j_N} R_{N,j_N}). \tag{2.69}$$

2.5.2. Examples of Extremal Distortion for Hausdorff Content. We want to fix the parameters we haven't fixed above so that E and $\phi(E)$ are both of Hausdorff content comparable to 1, the first t-dimensional and the second t'-dimensional, with $t' = \frac{2Kt}{2+(K-1)t}$.

Consider now equation (2.67). If we want $\mathcal{H}^t_\infty(E) \approx 1$, we should ask, according to (2.68), that

$$c'_N := m_{N,1}((\sigma_{N,1})^K R_{N,1})^t + \cdots + m_{N,l_N}((\sigma_{N,l_N})^K R_{N,l_N})^t \approx 1,$$

and if we want $\mathcal{H}^{t'}_\infty(\phi(E)) \approx 1$, we should ask, according to (2.69), that

$$c''_N = m_{N,1}(\sigma_{N,1} R_{N,1})^{t'} + \cdots + m_{N,l_N}(\sigma_{N,l_N} R_{N,l_N})^{t'} \approx 1,$$

Now, taking

$$(\sigma_{N,j_N})^{tK} = (R_{N,j_N})^{2-t},$$

these three expressions coincide,

$$c_N = c'_N = c''_N = 1 - \epsilon_N,$$

which gives

$$\sum_{j_1,\cdots,j_N} m_{1,j_1}\cdots m_{N,j_N}(s_J)^t = \sum_{j_1,\cdots,j_N} m_{1,j_1}\cdots m_{N,j_N}(q_J)^{t'} = \prod_{n=1}^{N}(1-\epsilon_n).$$

Now take $\epsilon_n \to 0$ so fast that

$$\prod_{n=1}^{\infty}(1-\epsilon_n) \approx 1.$$

Thus, we find coverings such that $\mathcal{H}^t_\infty(E) \lesssim 1$ and $\mathcal{H}^{t'}_\infty(\phi(E)) \lesssim 1$, as well as their respective Hausdorff mesures.

The converse inequality is more technical and we skip the proof here. We just notice that Uriarte-Tuero's proof in [**UT08**] uses a Carleson packing condition together with the Lebesgue number of a finite covering by open disks to find a lower bound for the t-Hausdorff content of E and the t'-Hausdorff content of $\phi(E)$.

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