Counterexamples to some pointwise estimates of the maximal Cauchy transform in terms of the Cauchy transform

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Introduction

The main object of study of this project is the Cauchy transform along a Lipschitz graph Γ, a one-dimensional Calderón-Zygmund operator that we will denote by C. It is known that C satisfies Cotlar's inequality: there exists a constant $c > 0$ such that, for all $f \in L^2(\mathbb{R})$ and all $x \in \mathbb{R}$,

$$C_* f(x) \leq c[M(Cf)(x) + Mf(x)].$$

We find our motivation in the articles [5], [6] and [7] of J. Mateu, J. Orobitg, C. Pérez and J. Verdera, in which the authors prove that, for T a higher order Riesz transform, a new version of Cotlar's inequality holds:

1. $T_* f \lesssim M(Tf)$ if T is even.
2. $T_* f \lesssim M^2(Tf)$ if T is odd.

In both cases, they provide a way of controlling pointwise the maximal singular integral just in terms of the singular integral. In particular, for the Hilbert transform, the result obtained is $H_* f \lesssim M^2(Hf)$.

Since C coincides essentially with H when Γ is a straight line, we considered the problem of establishing a similar way of control of the maximal Cauchy transform in terms of the Cauchy transform. We show here that, unless Γ is a straight line, one cannot have the inequality $C_* f \lesssim M^n(Cf)$ for all $f \in L^2(\mathbb{R})$, for any $n \geq 1$. On the other hand, we show that if T is the Cauchy transform along a sufficiently regular Jordan curve Γ, then $T_* f \lesssim M^2(Tf)$ for all $f \in L^2(\Gamma)$.

One motivation for trying to stablish inequalities like the ones above for these and other operators is the possible relation between them and the David-Semmes problem that we state below, since one could think that having inequalities like those could help to solve it.

Let $0 \leq n < d$, and let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure in $\mathbb{R}^d$. For a fixed Borel set E in $\mathbb{R}^d$, set $\mu = \mathcal{H}^n \lfloor_E$, and consider the n-dimensional Riesz transform with respect to μ, which is defined (formally) for $f \in L^1_{loc}(\mu)$ and $x \in \operatorname{supp}(\mu)$ by

$$R^n_\mu f(x) = \int \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y).$$

David-Semmes conjecture states that the following assertions are equivalent:

1. $\mu(E) < \infty$ and R^n_μ is bounded in $L^2(\mu)$.
2. E is uniformly n-rectifiable.

Chapter 1

Preliminaries.

1.1 Calderón-Zygmund theory.

In this section, we will briefly expose some definitions and results of the classical Calderón-Zygmund theory that will be used throughout the text.

Definition 1.1.1. Let Δ denote the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$,

$$\Delta = \{(x,x) \colon x \in \mathbb{R}^n\}.$$

A **standard kernel** in $\mathbb{R}^n$ is a function $K \colon \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ such that

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \tag{1.1}$$

$$|K(x,y) - K(x,y')| \leq C\frac{|y-y'|^\delta}{|x-y|^{n+\delta}} \quad \text{if } |x-y| > 2|y-y'|, \tag{1.2}$$

$$|K(x,y) - K(x',y)| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \quad \text{if } |x-y| > 2|x-x'|, \tag{1.3}$$

for some constants $C, \delta > 0$.

Examples of standard kernels:

1. In $\mathbb{R}$, the Hilbert kernel is
$$K(x,y) = \frac{1}{\pi}\frac{1}{x-y}.$$
2. In $\mathbb{R}^n$, the Riesz kernels are, for $1 \leq j \leq n$,
$$K_j(x,y) = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}\frac{x_j - y_j}{|x-y|^{n+1}}.$$
3. In $\mathbb{R}^2 \equiv \mathbb{C}$, the Beurling kernel is
$$K(z,w) = -\frac{1}{\pi}\frac{1}{(z-w)^2}.$$

Definition 1.1.2. A **Calderón-Zygmund operator** is an operator T such that

1. T is bounded in $L^2(\mathbb{R}^n)$.

2. There exists a standard kernel K such that for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n \setminus \operatorname{supp}(f)$,
$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy.$$

The kernels of the previous examples lead to three important examples of Calderón-Zygmund operators, namely, the **Hilbert transform**, the **Riesz transforms** and the **Beurling transform**. Notice that all these operators are of convolution type, i.e., their kernels can be expressed in the form

$$K(x,y) = k(x-y),$$

for a function $k\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$.

Calderón-Zygmund operators satisfy certain interesting properties that we summarize in the following theorem. First, we will recall the definition of the space $BMO(\mathbb{R}^n)$.

Definition 1.1.3. Let $f \in L^1_{loc}(\mathbb{R}^n)$. For a cube $Q \subset \mathbb{R}^n$, we will denote by $m_Q f$ the average of f over Q, i.e.,
$$m_Q f = \frac{1}{|Q|}\int_Q f(x)dx.$$
We will say that f has **bounded mean oscillation** if

$$||f||_{BMO} := \sup_Q \frac{1}{|Q|}\int_Q |f(x) - m_Q f|dx < \infty.$$

The space of all functions of bounded mean oscillation in $\mathbb{R}^n$ is denoted $BMO(\mathbb{R}^n)$.

We remark that, with the identification

$$f = g \Leftrightarrow f - g \text{ coincides with a constant almost everywhere}$$

and endowed with the norm $||\cdot||_{BMO}$, $BMO(\mathbb{R}^n)$ turns into a Banach space.

Theorem 1.1.1. Let T be a Calderón-Zygmund operator. Then,

1. T is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

2. T is of weak type $(1,1)$, i.e., it is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.

3. T is bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

1.1.1 Principal values and pointwise estimates. The truncated and maximal operators.

For particular examples of Calderón-Zygmund operators, such as, for instance, the Hilbert transform, it is known that all $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$

$$\text{p.v.}Tf(x) := \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x,y)f(y)dy$$

exists and coincides with $Tf(x)$ for a.e. $x \in \mathbb{R}^n$. However, this kind of results are not immediate and, in fact, they are not true in general. To study the existence of the limit of the right hand side, one is naturally led to study the boundedness of the maximal operator.

Definition 1.1.4. Let K be a standard kernel in $\mathbb{R}^n$. We define the **ϵ-truncated operators** associated with K by

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x,y)f(y)dy$$

and the **maximal operator** by

$$T_* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|.$$

It is known that if T_* is of weak type (p,p) for some p, then the set

$$\mathcal{A} = \{f \in L^p(\mathbb{R}^n) \colon \text{p.v.}Tf(x) \text{ exists for a.e. } x \in \mathbb{R}^n\}$$

is closed in $L^p(\mathbb{R}^n)$. In such a case, one would only need to check the almost everywhere existence of p.v.$Tf(x)$ for functions f belonging to a dense subclass of $L^p(\mathbb{R}^n)$. This is, indeed, the case, since the weak type (p,p) of T_*, and so the closedness of $\mathcal{A}$, follows easily from the well known Cotlar's inequality.

Definition 1.1.5. Let $f \in L^1_{loc}(\mathbb{R}^n)$. The **Hardy-Littlewood maximal function** of f is defined, for $x \in \mathbb{R}^n$, by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

The operator $f \mapsto Mf$ is called the **Hardy-Littlewood maximal operator**.

Recall that M is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$ and it is of weak type $(1,1)$.

Theorem 1.1.2 (Cotlar's inequality)**.** Let T be a Calderón-Zygmund operator with standard kernel K. Then, for every $0 < s \leq 1$, there exists $C_s > 0$ such that for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and all $x \in \mathbb{R}^n$,

$$T_* f(x) \leq C[M(|Tf|^s)(x)^{\frac{1}{s}} + Mf(x)].$$

Corollary 1.1.3. Let T be a Calderón-Zygmund operator with standard kernel K. Then, the maximal operator T_* is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and it is of weak type $(1,1)$.

1.2 Estimating the maximal operator in terms of the operator.

In the papers [5], [6] and [7], Mateu, Orobitg, Pérez and Verdera study the problem of controlling the maximal singular integral T_*f in terms of the singular integral Tf. We retain here the following results contained in those papers.

Definition 1.2.1. A higher-order Riesz transform is a Calderón-Zygmund operator defined, for $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus \operatorname{supp}(f)$, by

$$Tf(x) = \int_{\mathbb{R}^n} \frac{P(x-y)}{|x-y|^{n+d}} f(y)dy,$$

where P is a harmonic homogeneous polynomial of degree $d \geq 1$. We say that T is odd (respectively, even) if d is odd (respectively, even).

Theorem 1.2.1. Let T be a higher order Riesz transform, and let T_ϵ, $\epsilon > 0$ and T_* be the associated truncated and maximal operators. Then,

1. If T is even, then for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,

$$T_*f(x) \lesssim M(Tf)(x).$$

2. If T is odd, then for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,

$$T_*f(x) \lesssim M^2(Tf)(x).$$

Here, the notation $A \lesssim B$ means that there exists a constant $c > 0$, not depending on A or B, such that $A \leq cB$. Also, the notation $A \approx B$ will be equivalent to $A \lesssim B \lesssim A$.

Definition 1.2.2. A **smooth homogeneous Calderón-Zygmund operator** is a Calderón-Zygmund operator whose kernel is of the form

$$K(x,y) = \frac{\Omega(x-y)}{|x-y|^n},$$

where $\Omega\colon \mathbb{R}^n \to \mathbb{C}$ is a homogeneous function of degree 0 whose restriction to the unit sphere $\mathbb{S}^{n-1}$ is of class $\mathcal{C}^\infty$ and satisfies the cancellation property

$$\int_{\mathbb{S}^{n-1}} \Omega(u)d\sigma(u) = 0.$$

We will say that the operator is odd (resp., even) if Ω is odd (resp., even).

Theorem 1.2.2. Let T be a smooth homogeneous Calderón-Zygmund operator, and let T_* be the associated maximal operator. Then,

- If T is even, the following assertions are equivalent:

1. $T_*f(x) \lesssim M(Tf)(x)$ for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$.
2. $||T_*f||_{L^2} \lesssim ||Tf||_{L^2}$ for all $f \in L^2(\mathbb{R}^n)$.

- If T is odd, the following assertions are equivalent:

  1. $T_*f(x) \lesssim M^2(Tf)(x)$ for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$.
  2. $||T_*f||_{L^2} \lesssim ||Tf||_{L^2}$ for all $f \in L^2(\mathbb{R}^n)$.

The statements in the previous two theorems concerning even operators were proved in [6], while those concerning odd kernels were proved in [5]. In this section, we will present a simplified version of the proof of Theorem 1.2.1 for the case of the Hilbert transform. Nevertheless, no attempt at originality is claimed.

1.2.1 Orlicz spaces.

Some elements of the theory of Orlicz spaces are involved in the proof of $H_*f \lesssim M^2(Hf)$. We will expose here some of them, taken from [4] and [8].

Definition 1.2.3. A **Young function** is a function $\Phi\colon [0,\infty) \to [0,\infty)$ that is convex, increasing and satisfies $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

Definition 1.2.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let Φ be a Young function. For a measurable function $f\colon X \to \mathbb{C}$, we define the **Orlicz norm** of f with respect to Φ by

$$||f||_{\Phi(L)(d\mu)} = \inf \left\{ \lambda > 0\colon \int_X \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1 \right\}.$$

The **Orlicz space** $\Phi(L)(d\mu)$ is defined as the space of all measurable functions f on X with $||f||_{\Phi(L)(d\mu)} < \infty$.

It is easy to check that $||\cdot||_{\Phi(L)(d\mu)}$ defines a norm on $\Phi(L)(d\mu)$ that turns it into a Banach space (with the usual identification $f = 0$ if $f(x) = 0$ for a.e. x). Notice that for $\Phi(t) = t^p, p \geq 1$, one recovers the L^p norm and the L^p spaces. Moreover, one can easily check that, for $||f||_{\Phi(L)(d\mu)} > 0$, the infimum in the definition is, actually, a minimum.

Definition 1.2.5. Let Φ be a Young function. We define the **dual function** of Φ by

$$\Phi^*(y) = \sup\{xy - \Phi(x)\colon x \geq 0\}.$$

The following result is nothing but a straightforward computation.

Proposition 1.2.3. Let Φ be a Young function and let Φ^* be its dual function. Then,

1. For all x, y, $xy \leq \Phi(x) + \Phi^*(y)$.
2. Φ^* is also a Young function.

Examples:

1. Take $1 < p < \infty$, and $\Phi(x) = x^p$. Then, $\Phi^*(y) = \frac{y^{p'}}{p'}$, where p' is the conjugate exponent to p.

2. For $\Phi(x) = e^x - 1$, $\Phi^*(y) = (y \log y - y + 1)\chi_{(1,\infty)}(y)$.

Proposition 1.2.4 (Generalized Hölder's inequality)**.** Let Φ be a Young function and let Φ^* be its dual function. Let $f \in \Phi(L)(d\mu)$ and $g \in \Phi^*(L)(d\mu)$. Then, $fg \in L^1(d\mu)$ and

$$\int_X |fg| d\mu \leq 2||f||_{\Phi(L)(d\mu)} ||g||_{\Phi^*(L)(d\mu))}.$$

Proof. If either $||f||_{\Phi(L)(d\mu)}$ or $||g||_{\Phi^*(L)(d\mu)}$ is 0, the result is trivial. Otherwise, we can limit ourselves to the case

$$||f||_{\Phi(L)(d\mu)} = ||g||_{\Phi^*(L)(d\mu)} = 1.$$

By definition of Φ^*, we have, for all $x \in X$,

$$|f(x)g(x)| \leq \Phi(|f(x)|) + \Phi^*(|g(x)|),$$

and so, integrating both sides, we get

$$\int_X |fg| d\mu \leq \int_X \Phi(|f|) d\mu + \int_X \Phi^*(|g|) d\mu = 2,$$

as claimed. □

Definition 1.2.6. Let Φ be a Young function. If $Q \subset \mathbb{R}^n$ is a cube and f is a Lebesgue measurable function in $\mathbb{R}^n$, we define $||f||_{\Phi(L),Q} = ||f||_{\Phi(L)(d\mu)}$, for

$$d\mu(x) = \frac{1}{|Q|}\chi_Q(x) dx,$$

that is,

$$||f||_{\Phi(L),Q} = \inf \left\{ \lambda > 0 \colon \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

We will focus now in a particular example of Orlicz space due to its relation with the space $BMO(\mathbb{R}^n)$. Consider the Young function $\Phi(x) = e^x - 1$. For a cube $Q \subset \mathbb{R}^n$ and a measurable function f, we will denote $||f||_{\Phi(L),Q} = ||f||_{\exp(L),Q}$. First, we want to recall the classical John-Niremberg inequality.

Theorem 1.2.5 (John-Niremberg inequality)**.** Let $f \in BMO(\mathbb{R}^n)$, Q a cube in $\mathbb{R}^n$ and $\lambda > 0$. Then,

$$|\{x \in Q \colon |f(x) - m_Q f| > \lambda\}| \leq C_1 e^{-\frac{C_2 \lambda}{||f||_{BMO}}} |Q|,$$

where C_1 and C_2 are positive constants only depending on n (in fact, one can take $C_1 = \sqrt{2} < 2$, and we will use this fact later).

It is well known that John-Niremberg inequality implies the equivalence of all norms BMO_p, $1 \le p < \infty$. We include the proof here to show the behaviour of the constants involved.

Corollary 1.2.6. Let $f \in BMO(\mathbb{R}^n)$. Then, for $1 < p < \infty$,

$$||f||_{BMO_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - m_Q f|^p dx \right)^{\frac{1}{p}} \le \frac{(C_1 p \Gamma(p))^{\frac{1}{p}}}{C_2} ||f||_{BMO}.$$

Proof. Fix a cube $Q \subset \mathbb{R}^n$. Then,

$$\begin{aligned} \int_Q |f(x) - m_Q f|^p dx &= \int_0^\infty p t^{p-1} |\{x \in Q \colon |f(x) - m_Q f| > t\}| dt \\ &\le C_1 p |Q| \int_0^\infty t^{p-1} e^{-\frac{C_2 t}{||f||_{BMO}}} dt \\ &= C_1 p |Q| \frac{||f||_{BMO}^p}{C_2^p} \int_0^\infty s^{p-1} e^{-s} ds \\ &= C_1 p |Q| \frac{||f||_{BMO}^p}{C_2^p} \Gamma(p), \end{aligned}$$

and the result follows. □

Corollary 1.2.7. Let $f \in BMO(\mathbb{R}^n)$. Then, there exists a constant $c = c_f > 0$ such that for all cubes $Q \subset \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q \exp(c|f(x) - m_Q f|) dx \le 2.$$

Moreover, c can be chosen to satisfy $c > \frac{a(n)}{||f||_{BMO}}$, where $a(n) > 0$ is a constant only depending on n.

Proof. We will expand the Taylor series of the exponential. Notice that the infinite sum and the integral can be interchanged by the Monotone Convergence Theorem. We obtain

$$\begin{aligned} \frac{1}{|Q|} \int_Q \exp(c|f(x) - m_Q f|) dx &= \frac{1}{Q} \int_Q \left(\sum_{k=0}^\infty \frac{c^k |f(x) - m_Q f|^k}{k!} \right) dx \\ &= \sum_{k=0}^\infty \frac{c^k}{k!} \frac{1}{|Q|} \int_Q |f(x) - m_Q f|^k dx \\ &\le \sum_{k=0}^\infty \frac{c^k}{k!} \frac{C_1 k \Gamma(k)}{C_2^k} ||f||_{BMO}^k \\ &= C_1 \sum_{k=0}^\infty (c ||f||_{BMO} C_2^{-1})^k. \end{aligned}$$

A straightforward computation shows that the latter sum is convergent and has sum less than 2 if

$$0 < c < c' = \min \left\{ \frac{C_2}{||f||_{BMO}}, \frac{\left(1 - \frac{C_1}{2}\right) C_2}{||f||_{BMO}} \right\},$$

and so the result follows taking

$$a(n) = \frac{\min\left\{C_2, \left(1 - \frac{C_1}{2}\right) C_2\right\}}{2}$$

and c such that $\frac{a(n)}{||f||_{BMO}} < c < c'$. ☐

Now we have the tools to relate the quantities $||f||_{BMO}$ and $||f||_{\exp(L),Q}$.

Lemma 1.2.8. Let $f \in BMO(\mathbb{R}^n)$. Then, for all cubes $Q \subset \mathbb{R}^n$,

$$||f - m_Q f||_{\exp(L),Q} \lesssim ||f||_{BMO}.$$

Proof. By Corollary 1.2.7,

$$\frac{1}{|Q|} \int_Q \exp\left(\frac{a(n)}{||f||_{BMO}} |f(x) - m_Q f|\right) dx \leq 2,$$

and so, for $\Phi(x) = e^x - 1$,

$$\frac{1}{|Q|} \int_Q \Phi\left(\frac{|f - m_Q f|}{\frac{||f||_{BMO}}{a(n)}}\right) dx \leq 1,$$

yielding

$$||f - m_Q f||_{\Phi(L),Q} \leq \frac{||f||_{BMO}}{a(n)},$$

as desired. ☐

As it has been said before, the dual function of Φ is

$$\Phi^*(y) = (y \log y - y + 1)\chi_{(1,\infty)}(y).$$

Consider now the Young function $\varphi(y) = y \log(e + y)$. For a measurable function f and a cube $Q \subset \mathbb{R}^n$, we will denote

$$||f||_{L \log L,Q} = ||f||_{\varphi,Q}.$$

Lemma 1.2.9. Let f be a measurable function in $\mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ a cube. Then,

$$||f||_{\Phi^*(L),Q} \leq ||f||_{L \log L,Q}.$$

Proof. Observe that, for all $y > 0$,

$$\varphi'(y) = \log(e + y) + \frac{y}{e + y} \geq (\log y)\chi_{(1,\infty)}(y) = (\Phi^*)'(y).$$

This, together with the fact that $\varphi(0) = \Phi^*(0) = 0$, yields that $\varphi \geq \Phi^*$. Then, we have, for all $\lambda > 0$,

$$\frac{1}{|Q|} \int_Q \Phi^*\left(\frac{|f(x)|}{\lambda}\right) dx \leq \frac{1}{|Q|} \int_Q \varphi\left(\frac{|f(x)|}{\lambda}\right) dx.$$

As a consequence,

$$\left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q \Phi^*\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\} \supset \left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q \varphi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\},$$

and so, taking infimum

$$||f||_{\Phi^*(L),Q} \le ||f||_{L\log L,Q},$$

as claimed. □

As a direct application of the generalized Hölder's inequality in Proposition 1.2.4 and Lemmas 1.2.8 and 1.2.9, we obtain the following result.

Corollary 1.2.10. Let $f \in BMO(\mathbb{R}^n)$, g a measurable function in $\mathbb{R}^n$ and Q a cube in $\mathbb{R}^n$. Then,

$$\frac{1}{Q}\int_Q |f(x) - m_Q f||g(x)|dx \le c||f||_{BMO}||g||_{L\log L,Q},$$

where $c > 0$ only depends on n.

Definition 1.2.7. We define the $L\log L$**-Orlicz maximal operator** by

$$M_{L\log L}f(x) = \sup_{x\in Q}||f||_{L\log L,Q},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain x.

The next result exhibits the control of $M_{L\log L}$ by the iterated Hardy-Littlewood maximal operator $M^2 = M \circ M$ (in fact, it can be proved that they are pointwise comparable, but we will only prove one inequality, since it is the one we will need). Recall that the Hardy-Littlewood maximal operator with respect to cubes is defined, for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$M_c f(x) = \sup_Q \frac{1}{|Q|}\int_Q |f(y)|dy,$$

where the supremum is taken over all cubes containing x. Recall also that M_c is pointwise comparable to M.

Theorem 1.2.11. There exists a positive constant $c = c(n)>0$ such that for every cube $Q \subset \mathbb{R}^n$ and every function $f \in L^1_{loc}(\mathbb{R}^n)$ we have

$$||f||_{L\log L,Q} \le \frac{c(n)}{|Q|}\int_Q M_c f(x)dx.$$

As a consequence, there exists another dimensional constant $c' = c'(n)$ such that, for all $f \in L^1_{loc}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,

$$M_{L\log L}f(x) \le c'(n)M^2 f(x),$$

where $M^2 = M \circ M$ and M is the Hardy-Littlewood maximal operator.

Proof. Fix a cube $Q \subset \mathbb{R}^n$, and define, for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in Q$,

$$M_c^Q f(x) = \sup_R \frac{1}{|R|} \int_R |f(y)| dy,$$

where the supremum is taken over all cubes R in $\mathbb{R}^n$ that contain x and are contained in Q. We claim first that, for all $\lambda \geq m_Q|f|$,

$$\frac{1}{\lambda} \int_{Q \cap \{|f|>\lambda\}} |f(x)| dx \leq 2^n |\{x \in Q \colon M_c^Q f(x) > \lambda\}|.$$

Indeed, taking the Calderón-Zygmund decomposition of $|f|$ at height λ relative to Q, we obtain a countable family $\{Q_j\}$ of disjoint dyadic cubes contained in Q such that

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| dx \leq 2^n \lambda \tag{1.4}$$

and $|f(x)| \leq \lambda$ for a.e. $x \in Q \setminus \bigcup_j Q_j$. As a consequence, $Q \cap \{|f| > \lambda\}$ is contained in $\bigcup_j Q_j$ up to a set of measure zero. Thus,

$$\frac{1}{\lambda} \int_{Q \cap \{|f|>\lambda\}} |f(x)| dx \leq \frac{1}{\lambda} \int_{\bigcup_j Q_j} |f(x)| dx.$$

Now, by (1.4), $\bigcup_j Q_j \subset \{x \in Q \colon M_c^Q f(x) > \lambda\}$, and so, if we multiply by $|Q_j|$ and sum over j, we obtain

$$\int_{\bigcup_j Q_j} |f(x)| dx \leq 2^n \lambda \left| \bigcup_j Q_j \right| \leq 2^n \lambda \left| \{x \in Q \colon M_c^Q f(x) > \lambda\} \right|.$$

As a result,

$$\frac{1}{\lambda} \int_{Q \cap \{|f|>\lambda\}} |f(x)| dx \leq \frac{1}{\lambda} \int_{\bigcup_j Q_j} |f(x)| dx \leq 2^n \left| \{x \in Q \colon M_c^Q f(x) > \lambda\} \right|,$$

as claimed.

Now, to prove the first assertion in the theorem, we need to check that for some constant $c > 1$, independent of f, we have

$$\frac{1}{|Q|} \int_Q \frac{|f(x)|}{\lambda_Q} \log \left(e + \frac{|f(x)|}{\lambda_Q} \right) dx \leq 1,$$

where

$$\lambda_Q = \frac{c}{|Q|} \int_Q M_c f(x) dx.$$

Let $g = \frac{|f|}{\lambda_Q}$. Notice that, by Lebesgue's Differentiation Theorem, $0 \leq m_Q g \leq \frac{1}{c}$. We have

$$\begin{aligned} \frac{1}{|Q|} \int_Q \frac{|f(x)|}{\lambda_Q} \log \left(e + \frac{|f(x)|}{\lambda_Q} \right) dx &= \frac{1}{|Q|} \int_Q g(x) \log(e + g(x)) dx \\ &= \int_Q \log(e+g) d\mu = \int_Q \phi(g) d\mu + \int_Q d\mu, \end{aligned}$$

for $\phi(t) = \log(e+t) - 1$ and $d\mu(x) = g(x)\frac{dx}{|Q|}$. Since ϕ is C^1, increasing, and $\phi(0) = 0$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \frac{|f(x)|}{\lambda_Q}\log\left(e + \frac{|f(x)|}{\lambda_Q}\right)dx &= \int_0^\infty \phi'(t)\mu(\{x\in Q\colon g(x)>t\})dt + \int_Q d\mu \\ &= m_Q g + \int_0^\infty \frac{1}{e+t}\mu(\{x\in Q\colon g(x)>t\})dt \\ &= m_Q g + \int_0^\infty \frac{1}{e+t}\left(\frac{1}{|Q|}\int_{Q\cap\{g>t\}} g(x)dx\right)dt \\ &= I + II + III,\end{aligned}$$

where

$$I = m_Q g = \frac{1}{|Q|}\int_Q g(x)dx,$$

$$II = \frac{1}{|Q|}\int_0^{m_Q g} \frac{1}{e+t}\left(\int_{Q\cap\{g>t\}} g(x)dx\right)dt$$

and

$$III = \frac{1}{|Q|}\int_{m_Q g}^\infty \frac{1}{e+t}\left(\int_{Q\cap\{g>t\}} g(x)dx\right)dt.$$

As we said before, $I \leq \frac{1}{c}$. Now,

$$II \leq \frac{1}{|Q|}\int_0^{m_Q g}\frac{1}{e}\left(\int_Q g(x)dx\right)dt \leq \frac{1}{e}(m_Q g)^2 \leq \frac{1}{ec^2}.$$

Finally,

$$\begin{aligned}III &= \frac{1}{|Q|}\int_{m_Q g}^\infty \frac{1}{e+t}\left(\int_{Q\cap\{g>t\}} g(x)dx\right)dt \\ &\leq \frac{1}{|Q|}\int_{m_Q g}^\infty \frac{1}{e+t}2^n t|\{x\in Q\colon M_c^Q g(x)>t\}|dt \\ &= \frac{2^n}{|Q|}\int_{m_Q g}^\infty \frac{t}{e+t}|\{x\in Q\colon M_c^Q g(x)>t\}|dt \\ &\leq \frac{2^n}{|Q|}\int_0^\infty |\{x\in Q\colon M_c^Q g(x)>t\}|dt \\ &= \frac{2^n}{|Q|}\int_Q M_c^Q g(x)dx = \frac{2^n}{|Q|\lambda_Q}\int_Q M_c^Q f(x)dx \leq \frac{2^n}{c}.\end{aligned}$$

Putting all together, we obtain

$$\frac{1}{|Q|}\int_Q \frac{|f(x)|}{\lambda_Q}\log\left(e+\frac{|f(x)|}{\lambda_Q}\right)dx \leq \frac{1}{c} + \frac{1}{ec^2} + \frac{2^n}{c} \leq 1,$$

provided c is large enough. This concludes the proof of

$$||f||_{L\log L,Q} \leq \frac{c(n)}{|Q|}\int_Q M_c f(x)dx.$$

From this, we obtain

$$\begin{aligned} M_{L\log L}f(x) &= \sup_{x\in Q} ||f||_{L\log L,Q} \\ &\le \sup_{x\in Q}\left(\frac{c(n)}{|Q|}\int_Q M_c f(y)dy\right) \\ &= c(n)M_c^2 f(x) \approx M^2 f(x), \end{aligned}$$

since M_c and M are pointwise comparable. □

As a direct consequence of this result and Corollary 1.2.10, we obtain the following result.

Corollary 1.2.12. Let $f \in BMO(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$. Then, for all $x \in \mathbb{R}^n$ and all cubes Q containing x,

$$\frac{1}{|Q|}\int_Q |f(y) - m_Q f||g(y)|dy \le c||f||_{BMO}M^2 g(x),$$

where $c > 0$ only depends on n.

1.2.2 A proof of $H_* f \lesssim M^2(Hf)$.

It is clear, by translating, that we can limit ourselves to the case $x = 0$ (in fact, we can even consider only the case $\epsilon = 1$, but we will not, since it will not be possible to do so in the case we will study later).

Write, for $\epsilon > 0$ and $y \in \mathbb{R}$,

$$K_\epsilon(y) = -\frac{1}{\pi y}\chi_{\mathbb{R}\setminus(-\epsilon,\epsilon)}(y),$$

so that, for $f \in L^2(\mathbb{R})$,

$$H_\epsilon f(0) = -\frac{1}{\pi}\int_{|y|>\epsilon}\frac{f(y)}{y}dy = \int_{\mathbb{R}} f(y)K_\epsilon(y)dy.$$

It is immediate that $K_\epsilon \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $||K_\epsilon||_{L^2} = \sqrt{\frac{2}{\pi\epsilon}}$ and $||K_\epsilon||_{L^\infty} = \frac{1}{\pi\epsilon}$.

Recall that, for all $g, h \in L^2(\mathbb{R})$,

- $\int_{\mathbb{R}} (Hg)h = -\int_{\mathbb{R}} g(Hh)$.
- $H^2(h) = -h$.

As a consequence,

$$K_\epsilon = -H(HK_\epsilon) = H(g_\epsilon),$$

for $g_\epsilon = -H(K_\epsilon)$. With this notation, we have

$$H_\epsilon f(0) = \int_{\mathbb{R}} f(y)K_\epsilon(y)dy = \int_{\mathbb{R}} f(y)H(g_\epsilon)(y)dy = -\int_{\mathbb{R}} Hf(y)g_\epsilon(y)dy.$$

As a consequence,

$$
\begin{aligned}
-H_\epsilon f(0) &= \int_{\mathbb{R}} Hf(y) g_\epsilon(y) dy \\
&= \int_{-2\epsilon}^{2\epsilon} Hf(y) g_\epsilon(y) dy + \int_{|y|>2\epsilon} Hf(y) g_\epsilon(y) dy \\
&= \int_{-2\epsilon}^{2\epsilon} Hf(y)[g_\epsilon(y) - m_{(-2\epsilon,2\epsilon)} g_\epsilon] dy + m_{(-2\epsilon,2\epsilon)} g_\epsilon \int_{-2\epsilon}^{2\epsilon} Hf(y) dy + \int_{|y|>2\epsilon} Hf(y) g_\epsilon(y) dy \\
&= I + II + III.
\end{aligned}
$$

We will show now that $|I| \lesssim M^2(Hf)(0)$, $|II| \lesssim M(Hf)(0)$ and $|III| \lesssim M(Hf)(0)$, and so we will be done.

- $|I| \lesssim M^2(Hf)(0)$.

  Applying Corollary 1.2.12,

$$
|I| \leq \int_{-2\epsilon}^{2\epsilon} |Hf(y)||g_\epsilon(y) - m_{(-2\epsilon,2\epsilon)} g_\epsilon| dy \lesssim 4\epsilon ||g_\epsilon||_{BMO} M^2(Hf)(0).
$$

  Now, by the $L^\infty \to BMO$-boundedness of H,

$$
\epsilon ||g_\epsilon||_{BMO} = \frac{\epsilon}{\pi^2} ||HK\epsilon||_{BMO} \leq \frac{\epsilon}{\pi^2} ||H||_{L^\infty \to BMO} ||K_\epsilon||_{L^\infty} = \frac{||H||_{L^\infty \to BMO}}{\pi^3},
$$

  and so $|I| \lesssim M^2(Hf)(0)$ follows.

- $|II| \lesssim M(Hf)(0)$.

$$
\begin{aligned}
|II| &= \left| \frac{1}{4\epsilon} \int_{-2\epsilon}^{2\epsilon} g_\epsilon(y) dy \int_{-2\epsilon}^{2\epsilon} Hf(y) dy \right| \\
&\leq \left(\int_{-2\epsilon}^{2\epsilon} |g_\epsilon(y)| dy \right) \left(\frac{1}{4\epsilon} \int_{-2\epsilon}^{2\epsilon} Hf(y) dy \right) \\
&\leq \left(\int_{-2\epsilon}^{2\epsilon} |g_\epsilon(y)| dy \right) M(Hf)(0).
\end{aligned}
$$

  Now, applying the Cauchy-Schwarz inequality and taking into account the L^2-boundedness of H,

$$
\begin{aligned}
\int_{-2\epsilon}^{2\epsilon} |g_\epsilon(y)| dy &\leq \left(\int_{-2\epsilon}^{2\epsilon} |g_\epsilon(y)|^2 \right)^{\frac{1}{2}} \sqrt{4\epsilon} \leq 2\sqrt{\epsilon} ||g_\epsilon||_{L^2} = \frac{2\sqrt{\epsilon}}{\pi^2} ||H(K_\epsilon)||_{L^2} \\
&\leq \frac{2\sqrt{\epsilon}}{\pi^2} ||H||_{L^2 \to L^2} ||K_\epsilon||_{L^2} = \frac{2\sqrt{\epsilon}}{\pi^2} \sqrt{\frac{2}{\pi\epsilon}} = \frac{2\sqrt{2}}{\pi^{\frac{5}{2}}},
\end{aligned}
$$

  and so $|II| \lesssim M(Hf)(0)$ follows.

- $|III| \lesssim M(Hf)(0)$. We claim now that

$$
|g_\epsilon(y)| \lesssim \frac{\epsilon}{y^2}, \ |y| > 2\epsilon. \tag{1.5}
$$

Let us assume this to be true. Then, we have

$$\begin{aligned}
|III| &= \left|\int_{|y|>2\epsilon} Hf(y)g_\epsilon(y)dy\right| \lesssim \epsilon\int_{|y|>2\epsilon}|Hf(y)|\frac{dy}{|y|^2}\\
&= \epsilon\sum_{k=1}^{\infty}\int_{2^k\epsilon<|y|<2^{k+1}\epsilon}|Hf(y)|\frac{dy}{|y|^2}\\
&\le \epsilon\sum_{k=1}^{\infty}\int_{2^k\epsilon<|y|<2^{k+1}\epsilon}|Hf(y)|\frac{dy}{(2^k\epsilon)^2}\\
&\le \sum_{k=1}^{\infty}\frac{1}{2^{k-2}}\left(\frac{1}{2\cdot 2^{k+1\epsilon}}\int_{|y|<2^{k+1\epsilon}}|Hf(y)dy|\right)\\
&\le \left(\sum_{k=1}^{\infty}\frac{1}{2^{k-2}}\right)M(Hf)(0) = 4M(Hf)(0),
\end{aligned}$$

as desired. It remains to prove (1.5). To do so, we will compute explicitly $g_\epsilon(y)$ for $|y| > 2\epsilon$. From now on, we will assume $y > 0$, and the case $y < 0$ is treated analogously.

Recall that $g_\epsilon = -H(K_\epsilon)$. Now,

$$H(K_\epsilon)(y) = \text{p.v.}_y\int_{\mathbb{R}}\frac{K_\epsilon(t)}{t-y} = \text{p.v.}_y\int_{|t|>\epsilon}\frac{dt}{t(t-y)}.$$

Now, due to the quadratic decay of the denominator, and taking into account that the integral is a principal value around y, we get

$$\begin{aligned}
H(K_\epsilon)(y) &= \lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\substack{\epsilon<|t|<R\\ \delta<|t-y|}}\frac{dt}{t(t-y)} = \frac{1}{y}\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\substack{\epsilon<|t|<R\\ \delta<|t-y|}}\left(\frac{1}{t-y}-\frac{1}{t}\right)dt\\
&= \frac{1}{y}\lim_{\substack{R\to\infty\\ \delta\to 0}}\left(\int_{-R}^{-\epsilon}\left(\frac{1}{t-y}-\frac{1}{t}\right)dt + \int_{\epsilon}^{y-\delta}\left(\frac{1}{t-y}-\frac{1}{t}\right)dt + \int_{y-\delta}^{R}\left(\frac{1}{t-y}-\frac{1}{t}\right)dt\right)\\
&= \frac{1}{y}\lim_{\substack{R\to\infty\\ \delta\to 0}}\left(\log\left|\frac{y+\epsilon}{y-\epsilon}\right| + \log\left|\frac{y-\delta}{y+\delta}\right| + \log\left|\frac{R-y}{R+y}\right|\right)\\
&= \frac{1}{y}\log\frac{y+\epsilon}{y-\epsilon} = \frac{1}{y}\log\left(1+\frac{2\epsilon}{y-\epsilon}\right).
\end{aligned}$$

Observe that, for $y > 2\epsilon$, $0 < \frac{2\epsilon}{y-\epsilon} < 2$. Then, taking into account that the function $t \mapsto \frac{\log(1+t)}{t}$ is positive and bounded in $(0,\infty)$, we get

$$H(K_\epsilon)(y) = \frac{1}{y}\log\left(1+\frac{2\epsilon}{y-\epsilon}\right) \le \frac{C}{y}\frac{2\epsilon}{y-\epsilon} \le \frac{4C\epsilon}{y^2},$$

finishing the proof.

Chapter 2

Pointwise estimates for the maximal Cauchy transform along a Lipschitz graph.

2.1 Introduction

Let $A\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, and let $\Gamma \subset \mathbb{R}^2 \equiv \mathbb{C}$ be its graph,

$$\Gamma = \{z(x) = x + iA(x) \colon x \in \mathbb{R}\}.$$

Recall that the Lipschitz character of A means that there exists $\Lambda_1 > 0$ such that, for all $x, y \in \mathbb{R}$

$$|A(x) - A(y)| \leq \Lambda_1 |x - y|.$$

It is known that, under this conditions, A is differentiable almost everywhere, $A' \in L^\infty(\mathbb{R})$ and $||A'||_{L^\infty} \leq \Lambda_1$.

We consider now the Cauchy transform along Γ. This operator is defined, at least, for $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and $x \in \mathbb{R} \setminus \operatorname{supp}(f)$ by

$$Cf(x) = \int_{\mathbb{R}} \frac{f(y)}{z(y) - z(x)} dy.$$

This is an example of a one-dimensional Calderón-Zygmund singular integral operator, with kernel

$$K(x, y) = \frac{1}{z(y) - z(x)}.$$

One has to be careful when trying to understand the boundedness of C in some space of functions (say, for example, in $L^2(\mathbb{R})$). Indeed, we have just defined $Cf(x)$ for $f \in \mathcal{C}_c^\infty(\mathbb{R})$

and $x \in \mathbb{R}\backslash\text{supp}(f)$, and, in order to study the L^2-boundedness of C we need, in principle, a *richer* definition. Notice that, in general, the integral

$$\int_{\mathbb{R}} \frac{f(y)}{z(y) - z(x)} dy \tag{2.1}$$

will not be absolutely convergent for $x \in \text{supp}(f)$. Then, one considers the truncated operators, which are defined, for $\epsilon > 0$, by

$$C_\epsilon f(x) = \int_{|y-x|>\epsilon} \frac{f(y)}{z(y) - z(x)} dy.$$

Now, if $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\epsilon > 0$, $C_\epsilon f(x)$ is well defined for all $x \in \mathbb{R}$. One can then understand the integral in (2.1) as a principal value around x. Indeed, it can be proved that, if $f \in \mathcal{C}_c^\infty(\mathbb{R})$, then

$$\text{p.v.}Cf(x) := \lim_{\epsilon \to 0} C_\epsilon f(x)$$

exists for a.e. $x \in \mathbb{R}$. Then, the boundedness of C in $L^2(\mathbb{R})$ can be understood as the existence of a constant $c > 0$ such that, for all $f \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$||\text{p.v.}Cf||_{L^2} \leq c||f||_{L^2},$$

since, from this, one would be able to extend p.v.C to the whole of $L^2(\mathbb{R})$ by a density argument, and thus to define C as a bounded operator in $L^2(\mathbb{R})$.

In [1], Calderón proved that C is bounded in $L^2(\mathbb{R})$ when $||A'||_\infty$ is sufficiently small. Later, in [2], Coifman, McIntosh and Meyer proved that C is bounded in $L^2(\mathbb{R})$ for every Lipschitz function A. As a consequence, from the classical Calderón-Zygmund theory, we obtain that

- C is bounded in $L^p(\mathbb{R})$ for all $1 < p < \infty$.
- C is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$.
- C is bounded from $L^\infty(\mathbb{R})$ to $BMO(\mathbb{R})$.

Consider also the maximal Cauchy transform, which is defined by

$$C_* f(x) = \sup_{\epsilon > 0} |C_\epsilon f(x)|.$$

Again, the L^2 boundedness of C implies that C_* satisfies the classical Cotlar's inequality, i.e., for all $f \in L^2(\mathbb{R})$ and all $x \in \mathbb{R}$,

$$C_* f(x) \lesssim M(Cf)(x) + Mf(x).$$

Motivated by the work of Mateu, Orobitg, Pérez and Verdera in [5], [6] and [7], we considered the problem of controlling the maximal Cauchy transform just in terms of the

Cauchy transform. We only consider the problem of giving a pointwise estimate of the form

$$C_* f(x) \lesssim M^n(Cf)(x),$$

since the inequality

$$||C_* f||_{L^2(\mathbb{R})} \lesssim ||Cf||_{L^2(\mathbb{R})}$$

is almost trivial, as we will show later.

Notice that the Cauchy transform along a Lipschitz graph Γ coincides with a constant multiple of the Hilbert transform when Γ is a straight line, and this is a reason why one could think that the pointwise estimate $C_* f \lesssim M^n(Cf)$ could hold for the Cauchy transform along, at least, some class of graphs Γ. We will show that one cannot have a similar inequality for the Cauchy transform, unless Γ is a straight line. More precisely, we will prove the following results:

Theorem 2.1.1. Consider the Lipschitz function $A(x) = |x|$, and let C denote the Cauchy transform along Γ, the graph of A. Then, there exists $f \in L^2(\mathbb{R})$ such that for all $c > 0$ and all $n \geq 1$, there exists $\epsilon > 0$ such that

$$|C_\epsilon f(0)| > cM^n(Cf)(0).$$

This theorem can be easily generalized to Lipschitz graphs Γ with angles, meaning with this points x where A' has a jump discontinuity, as we will show later.

After obtaining this result, we thought that maybe we would be able to stablish the inequality $C_* f \lesssim M^n(Cf)$ imposing some restrictions on the smoothness of A. This is not the case, as the next theorem shows.

Theorem 2.1.2. Let A be a Lipschitz function with compact support, and let C denote the Cauchy transform along Γ, the graph of A. Suppose A is not identically null, or, equivalently, that Γ is not a straight line. Then, there exists $x \in \mathbb{R}$ such that for all $c > 0$ there exists $f \in L^2(\mathbb{R})$ with

$$C_* f(x) > cM^n(Cf)(x)$$

for all $n \geq 1$.

We want to remark that the points x mentioned in this last theorem are ‘easy’ to find. For example, when A is of class C^2, any point x with $A''(x) \neq 0$ will do the job. Notice also that in the case when A has compact support and Γ has an angle at a point x, the failure of the inequality $C_* f(x) \leq cM^n(Cf)(x)$ for all $f \in L^2(\mathbb{R})$ can also be deduced from this result, but the previous one is stronger in this setting, since the argument used there provides a single function f for which the previous inequality fails for all possible constants $c > 0$.

2.2 Another version of the Cauchy transform.

We define a new operator, which, abusing language, we will also call the Cauchy transform along Γ, by

$$Tf(x) = \frac{1}{\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{z(y)-z(x)}dz(y),$$

where $dz(y) = (1 + iA'(y))dy$. As before, associated with it, we will have the truncated operators T_ϵ and the maximal operator T_*. This operator is very closely related to C. Indeed,

$$\begin{aligned} Tf(x) &= \frac{1}{\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{z(y)-z(x)}dz(y) \\ &= \frac{1}{\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)z'(y)}{z(y)-z(x)}dy = \frac{1}{\pi i}C(f\cdot z')(x). \end{aligned} \tag{2.2}$$

Analogously,

$$Cf(x) = \pi i T\left(\frac{f}{z'}\right)(x). \tag{2.3}$$

It is clear that T satisfies the same boundedness properties that C satisfies (with different multiplicative constants). Moreover, by equations (2.2) and (2.3), and taking into account that $z' \in L^\infty$ and $|z'| \approx 1$, we can limit ourselves to prove the Theorems 2.1.1 and 2.1.2 substituting C by T, C_ϵ by T_ϵ and C_* by T_*.

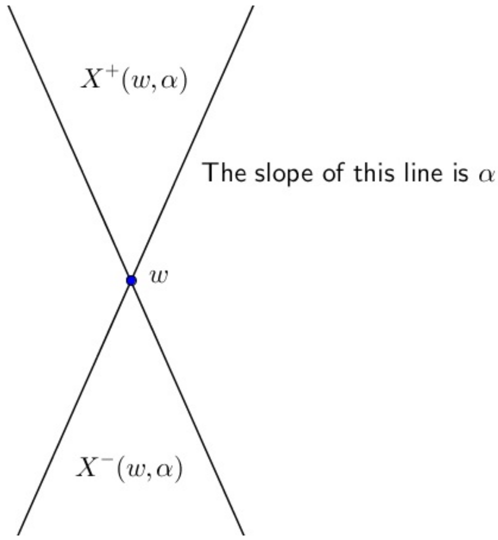
The main reason for using this version of the Cauchy transform is contained in the following result, which we learnt from Luis Escauriaza.

Lemma 2.2.1. If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then $T^2 f = f$.

Proof. For $w \in \mathbb{C}$ and $\alpha > 0$, we define the upper and lower half cones with vertex at w and generatrix slope α, respectively, by

$$X^+(w,\alpha) = \{z \in \mathbb{C}\colon |\text{Re } z - \text{Re } w| < \alpha(\text{Im } z - \text{Im } w)\}$$

$$X^-(w,\alpha) = \{z \in \mathbb{C}\colon |\text{Re } z - \text{Re } w| < \alpha(\text{Im } w - \text{Im } z)\}.$$



It is immediate that for all $w \in \Gamma$ and all $0 < \alpha < \frac{1}{||A'||_\infty}$,

$$X^+(w,\alpha) \subset \{x + iy \in \mathbb{C} \colon y > A(x)\}$$

and

$$X^-(w,\alpha) \subset \{x + iy \in \mathbb{C} \colon y < A(x)\}.$$

Fix $0 < \alpha < \frac{1}{||A'||_\infty}$. Let $f \in L^p(\mathbb{R})$, and let us define, for $x \in \mathbb{R}$,

$$T_+f(x) = \lim_{\substack{w \to z(x) \\ w \in X^+(z(x),\alpha)}} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(y)}{w - z(x)} dz(y),$$

$$T_-f(x) = \lim_{\substack{w \to z(x) \\ w \in X^-(z(x),\alpha)}} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(y)}{w - z(x)} dz(y).$$

From the Plemelj's formulas, we obtain

$$T_+f(x) = Tf(x) + f(x); \;\; T_-f(x) = Tf(x) - f(x)$$

for a.e. $x \in \mathbb{R}$. In particular, $T = T_+ - Id$. Hence,

$$T^2 = (T_+ - Id)^2 = (T_+)^2 - 2T_+ + Id.$$

A direct application of Cauchy's integral formula gives $(T_+)^2 = 2T_+$. As a consequence, $T^2 = Id$, as desired. □

Corollary 2.2.2. Let $f \in L^2(\mathbb{R})$. Then, $||T_* f||_{L^2} \lesssim ||Tf||_{L^2}$.

Proof. Recall that, by Corollary 1.1.3, T_* is bounded in $L^2(\mathbb{R})$. As a result, taking also into account that T is bounded in $L^2(\mathbb{R})$, we get, for $f \in L^2(\mathbb{R})$,

$$||T_* f||_{L^2} \lesssim ||f||_{L^2} = ||T(Tf)||_{L^2} \lesssim ||Tf||_{L^2}.$$

□

Lemma 2.2.3. Let $1 < p < \infty$, p' the conjugate exponent to p and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$. Then,

$$\int_{\mathbb{R}} Tf(x)g(x)dz(x) = -\int_{\mathbb{R}} f(y)Tg(y)dz(y).$$

Proof. Clearly, it is enough to prove that, for all $\epsilon > 0$,

$$\int_{\mathbb{R}} T_\epsilon f(x)g(x)dz(x) = -\int_{\mathbb{R}} f(y)T_\epsilon g(y)dz(y).$$

To prove this, assume first that $f, g \in \mathcal{C}_c^\infty(\mathbb{R})$. We have

$$\int_{\mathbb{R}} T_\epsilon f(x)g(x)dz(x) = \int_{\mathbb{R}} \left(\frac{1}{\pi i} \int_{|y-x|>\epsilon} \frac{f(y)}{z(y)-z(x)} dz(y) \right) g(x)dz(x)$$

We will apply Fubini's theorem to invert the order of integration. Taking into account that $|z'| \leq (1+\Lambda_1)$ and $\frac{1}{|z(y)-z(x)|} \leq \frac{1}{\epsilon}$ for $|x-y| > \epsilon$, we get

$$\int_{\mathbb{R}} \left(\int_{|y-x|>\epsilon} \frac{|f(y)|}{|z(y)-z(x)|} |z'(y)|dy \right) |g(x)||z'(x)|dx \leq \frac{(1+\Lambda_1)^2}{\epsilon} ||f||_{L^1} ||g||_{L^1} < \infty.$$

As a consequence, by Fubini's Theorem,

$$\begin{aligned}
\int_{\mathbb{R}} T_\epsilon f(x)g(x)dz(x) &= \int_{\mathbb{R}} \left(\frac{1}{\pi i} \int_{|y-x|>\epsilon} \frac{f(y)}{z(y)-z(x)} dz(y) \right) g(x)dz(x) \\
&= \int_{\mathbb{R}} f(y) \left(\frac{1}{\pi i} \int_{|y-x|>\epsilon} \frac{g(x)}{z(y)-z(x)} dz(x) \right) dz(y) \\
&= -\int_{\mathbb{R}} f(y)T_\epsilon g(y)dz(y).
\end{aligned}$$

The general case follows by approximation, taking into account that $\mathcal{C}_c^\infty(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$. □

2.3 The proofs.

In the beginning, we tried to adapt the proof of $H_* f \lesssim M^2(Hf)$ in [5] to the case of the Cauchy transform, and this led us to build a couple of counterexamples to the analogous inequality $T_* f \lesssim M^n(Tf)$. We discuss here this process.

Let $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$ and $\epsilon > 0$. We have

$$T_\epsilon f(x) = \frac{1}{\pi i} \int_{|y-x|>\epsilon} \frac{f(y)}{z(y) - z(x)} dz(y).$$

For $x \in \mathbb{R}$ and $\epsilon > 0$, define

$$K_{x,\epsilon}(y) = \frac{1}{\pi i} \frac{1}{z(y) - z(x)} \chi_{|y-x|>\epsilon}(y),$$

so that

$$T_\epsilon f(x) = \int_{\mathbb{R}} f(y) K_{x,\epsilon}(y) dz(y).$$

Lemma 2.3.1. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Then,

1. $K_{x,\epsilon} \in L^\infty(\mathbb{R})$ and $||K_{x,\epsilon}||_{L^\infty} \leq \frac{1}{\epsilon}$.
2. $K_{x,\epsilon} \in L^2(\mathbb{R})$ and $||K_{x,\epsilon}||_{L^2} \leq \frac{1}{\sqrt{\epsilon}}$.

Proof. Clearly, for $y \in \mathbb{R}$, $|y - x| > \epsilon$

$$|K_{x,\epsilon}(y)| = \frac{1}{\pi |z(y) - z(x)|} \leq \frac{1}{\pi |y - x|} \leq \frac{1}{\pi \epsilon},$$

so (1) follows. On the other hand,

$$\begin{aligned} ||K_{x,\epsilon}||^2_{L^2} = \int_{\mathbb{R}} |K_{x,\epsilon}(y)|^2 dy = \frac{1}{\pi^2} \int_{|y-x|>\epsilon} \frac{1}{|z(y) - z(x)|^2} \\ \leq \frac{1}{\pi^2} \int_{|y-x|>\epsilon} \frac{dy}{|y - x|^2} = \frac{1}{\pi^2} \int_{|t|>\epsilon} \frac{dt}{t^2} = \frac{2}{\pi^2 \epsilon}, \end{aligned}$$

so (2) follows. ☐

Let $g_{x,\epsilon} = T(K_{x,\epsilon})$. Since $K_{x,\epsilon} \in L^2(\mathbb{R})$, $K_{x,\epsilon} = T^2(K_{x,\epsilon}) = T(T(K_{x,\epsilon})) = T(g_{x,\epsilon})$. Then, we have,

$$\begin{aligned} T_\epsilon f(x) &= \int_{\mathbb{R}} f(y) K_{x,\epsilon}(y) dz(y) = \int_{\mathbb{R}} f(y) T(g_{x,\epsilon})(y) dz(y) \\ &= -\int_{\mathbb{R}} Tf(y) g_{x,\epsilon}(y) dz(y). \end{aligned}$$

Let us fix now $N > 0$ to be chosen later, and denote, for $a \in \mathbb{R}$ and $r > 0$,

$$I(a, r) = (a - r, a + r).$$

Then, we have,

$$\begin{aligned} -T_\epsilon f(x) &= \int_{\mathbb{R}} Tf(y) g_{x,\epsilon}(y) dz(y) \\ &= \int_{|y-x|<N\epsilon} Tf(y) g_{x,\epsilon}(y) dz(y) + \int_{|y-x|>N\epsilon} Tf(y) g_{x,\epsilon}(y) dz(y) \\ &= \int_{I_{x,N\epsilon}} Tf(y)[g_{x,\epsilon}(y) - m_{I_{x,N\epsilon}}(g_{x,\epsilon})] dz(y) + m_{I_{x,N\epsilon}}(g_{x,\epsilon}) \int_{I_{x,N\epsilon}} Tf(y) dz(y) \\ &\quad + \int_{|y-x|>N\epsilon} Tf(y) g_{x,\epsilon}(y) dz(y) = I + II + III. \end{aligned}$$

A slight modification of the argument in [5] will show below that $|I| \lesssim M^2(Tf)(x)$ and $|II| \lesssim M(Tf)(x)$, and we will show later that this type of control is not possible for III.

2.3.1 Estimates for I and II.

Let us start proving that $|I| \leq M^2(Tf)(x)$. Applying Corollary 1.2.12, we obtain

$$\begin{aligned}|I| &\leq (1+\Lambda_1)\int_{I_{x,N\epsilon}} |Tf(y)||g_{x,\epsilon}(y) - m_{I_{x,N\epsilon}}(g_{x,\epsilon})|dy \\ &\lesssim (1+\Lambda_1)2N\epsilon||g_{x,\epsilon}||_{BMO(\mathbb{R})}M^2(Tf)(x),\end{aligned}$$

so we need to show that $\epsilon||g_{x,\epsilon}||_{BMO(\mathbb{R})}$ is bounded independently of x and ϵ. This will follow from the $L^\infty \to BMO$-boundedness of T and (1) of Lemma 2.3.1. Indeed,

$$\begin{aligned}\epsilon||g_{x,\epsilon}||_{BMO(\mathbb{R})} &= \epsilon||T(K_{x,\epsilon})||_{BMO(\mathbb{R})} \leq \epsilon||T||_{L^\infty\to BMO}||K_{x,\epsilon}||_{L^\infty(\mathbb{R})} \\ &\leq \epsilon||T||_{L^\infty\to BMO}\frac{1}{\epsilon} = ||T||_{L^\infty\to BMO},\end{aligned}$$

as desired.

Let us prove now that $|II| \lesssim M(Tf)(x)$. Observe that

$$\begin{aligned}|II| &= \left| m_{I_{x,N\epsilon}}(g_{x,\epsilon})\int_{I_{x,N\epsilon}} Tf(y)dz(y)\right| \\ &\leq \frac{1}{|I_{x,N\epsilon}|}\left|\int_{I_{x,N\epsilon}} g_{x,\epsilon}(y)dy\right|\int_{I_{x,N\epsilon}} |Tf(y)||z'(y)|dy \\ &\leq (1+\Lambda_1)\left|\int_{I_{x,N\epsilon}} g_{x,\epsilon}(y)dy\right|\frac{1}{|I_{x,N\epsilon}|}\int_{I_{x,N\epsilon}} |Tf(y)|dy \\ &\leq (1+\Lambda_1)\left|\int_{I_{x,N\epsilon}} g_{x,\epsilon}(y)dy\right| M(Tf)(x),\end{aligned}$$

so if we prove that

$$\left|\int_{I_{x,N\epsilon}} g_{x,\epsilon}(y)dy\right|$$

is bounded independently of x and ϵ, we will be done. In fact, this is the case, taking into account the L^2-boundedness of T. Indeed, applying the Cauchy-Schwarz's inequality, and (2) of Lemma 2.3.1, we get

$$\begin{aligned}\left|\int_{I_{x,N\epsilon}} g_{x,\epsilon}(y)dy\right| &= \left|\int_{I_{x,N\epsilon}} T(K_{x,\epsilon})(y)dy\right| \\ &\leq |I_{x,N\epsilon}|^{\frac{1}{2}}||T(K_{x,\epsilon})||_{L^2(\mathbb{R})} \\ &\leq \sqrt{2N\epsilon}||T||_{L^2\to L^2}||K_{x,\epsilon}||_{L^2(\mathbb{R})} \\ &\leq \sqrt{2N\epsilon}||T||_{L^2\to L^2}||\frac{1}{\sqrt{\epsilon}} = \sqrt{2N}||T||_{L^2\to L^2},\end{aligned}$$

as claimed.

2.3.2 Estimates for III.

Let us study III now. Recall that

$$III = \int_{|y-x|>N\epsilon} Tf(y) g_{x,\epsilon}(y) dz(y) = \int_{|y-x|>N\epsilon} Tf(y) T(K_{x,\epsilon}(y)) dz(y).$$

Lemma 2.3.2. Fix $x \in \mathbb{R}$ and $\epsilon > 0$. Then, for almost every $y \in \mathbb{R}$ with $|y - x| > \epsilon$, we have

$$T(K_{x,\epsilon})(y) = \frac{1}{\pi i} \frac{1}{z(y) - z(x)} \left[B(x,\epsilon) + G_{x,\epsilon}(y) \right],$$

where

$$B(x,\epsilon) = \log \frac{|z(x+\epsilon) - z(x)|}{|z(x-\epsilon) - z(x)|} + i\Big(\pi + \arg[z(x+\epsilon) - z(x)] - \arg[z(x-\epsilon) - z(x)]\Big)$$

and

$$G_{x,\epsilon}(y) = \log \frac{|z(x-\epsilon) - z(y)|}{|z(x+\epsilon) - z(y)|} + i\Big(\arg[z(x-\epsilon) - z(y)] - \arg[z(x+\epsilon) - z(y)]\Big),$$

where, for a complex number $w \neq 0$, we consider $-\frac{\pi}{2} \leq \arg(w) < \frac{3\pi}{2}$.

Proof. Let $x \in \mathbb{R}$, $\epsilon > 0$ and $y \in \mathbb{R}$ with $|y - x| > \epsilon$. We will assume that $y > x$ (the case $y < x$ is treated analogously) and also that A is differentiable at y. For a set $I \subset \mathbb{R}$, denote

$$\Gamma(I) = \{z(t) \colon t \in I\}.$$

We have

$$\begin{aligned} T(K_{x,\epsilon})(y) &= \frac{1}{\pi i} \text{p.v.} \int_{\mathbb{R}} \frac{K_{x,\epsilon}(t)}{z(t) - z(y)} dz(t) \\ &= \frac{1}{\pi i} \text{p.v.}_y \int_{|t-x|>\epsilon} \frac{1}{z(t) - z(x)} \frac{1}{z(t) - z(y)} dz(t) \\ &= \lim_{\delta \to 0^+} \frac{1}{\pi i} \int_{\Gamma(\{t \colon |t-x|>\epsilon, |t-y|>\delta\})} \frac{dw}{(w - z(x))(w - z(y))}. \end{aligned}$$

For a complex number $w \neq 0$, let $\text{Log}\,(w) = \log|w| + i\arg(w)$. Taking into account the quadratic decay at infinity of the integrand, we can write the last integral as the limit of the integrals in $\Gamma(\{t \colon |t - x| > \epsilon, |t - y| > \delta, |t| < R\})$ as $R \to \infty$ and $\delta \to 0$. Moreover, since

$$\frac{1}{(z - z(x))(z - z(y))} = \frac{1}{z(y) - z(x)} \left(\frac{1}{z - z(y)} - \frac{1}{z - z(x)} \right),$$

we obtain

$$T(K_{x,\epsilon})(y) = \frac{1}{\pi i (z(y) - z(x))} \lim_{\substack{R \to \infty \\ \delta \to 0}} (I_{R,\delta} + II_{R,\delta} + III_{R,\delta}),$$

where, for sufficiently small $\delta > 0$ and sufficiently big $R > 0$,

$$\begin{aligned} I_{R,\delta} &= \int_{\Gamma((-R,x-\epsilon))} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\ &= \text{Log } [z(x-\epsilon) - z(y)] - \text{Log } [z(-R) - z(y)] \\ &\quad - \text{Log } [z(x-\epsilon) - z(x)] + \text{Log } [z(-R) - z(x)], \\ II_{R,\delta} &= \int_{\Gamma((x+\epsilon,y-\delta))} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\ &= \text{Log } [z(y-\delta) - z(y)] - \text{Log } [z(x+\epsilon) - z(y)] \\ &\quad - \text{Log } [z(y-\delta) - z(x)] + \text{Log } [z(x+\epsilon) - z(x)] \end{aligned}$$

and

$$\begin{aligned} III_{R,\delta} &= \int_{\Gamma((y+\delta,R))} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\ &= \text{Log } [z(R) - z(y)] - \text{Log } [z(y+\delta) - z(y)] \\ &\quad - \text{Log } [z(R) - z(x)] + \text{Log } [z(y+\delta) - z(x)]. \end{aligned}$$

Gathering the previous identities, we obtain

$$\begin{aligned} \text{Re } (I_{R,\delta} + II_{R,\delta} + III_{R,\delta}) &= \log \frac{|z(x-\epsilon) - z(y)||z(x+\epsilon) - z(x)|}{|z(x+\epsilon) - z(y)||z(x-\epsilon) - z(x)|} \\ &+ \log \frac{|z(-R) - z(x)||z(R) - z(y)|}{|z(-R) - z(y)||z(R) - z(x)|} \\ &+ \log \frac{|z(y-\delta) - z(y)||z(y+\delta) - z(x)|}{|z(y+\delta) - z(y)||z(y-\delta) - z(x)|}. \end{aligned}$$

Now, we have

$$\lim_{R\to\infty} \log \frac{|z(-R) - z(x)||z(R) - z(y)|}{|z(-R) - z(y)||z(R) - z(x)|} = 0$$

and

$$\lim_{\delta\to 0} \log \frac{|z(y-\delta) - z(y)||z(y+\delta) - z(x)|}{|z(y+\delta) - z(y)||z(y-\delta) - z(x)|} = 0,$$

since A is differentiable at y. As a consequence,

$$\lim_{\substack{R\to\infty \\ \delta\to 0}} \text{Re } (I_{R,\delta} + II_{R,\delta} + III_{R,\delta}) = \log \frac{|z(x-\epsilon) - z(y)||z(x+\epsilon) - z(x)|}{|z(x+\epsilon) - z(y)||z(x-\epsilon) - z(x)|}.$$

On the other hand,

$$\begin{aligned} \text{Im } (I_{R,\delta} + II_{R,\delta} + III_{R,\delta}) &= (\arg[z(x-\epsilon) - z(y)] - \arg[z(x-\epsilon) - z(x)] \\ &- \arg[z(x+\epsilon) - z(y)] + \arg[z(x+\epsilon) - z(x)]) \\ &+ (-\arg[z(-R) - z(y)] + \arg[z(-R) - z(x)] \\ &+ \arg[z(R) - z(y)] - \arg[z(R) - z(x)]) \\ &+ (\arg[z(y-\delta) - z(y)] - \arg[z(y-\delta) - z(x)] \\ &- \arg[z(y+\delta) - z(y)] + \arg[z(y+\delta) - z(x)]). \end{aligned}$$

Now, we have

$$\lim_{R\to\infty}(-\arg[z(-R)-z(y)]+\arg[z(-R)-z(x)] \\ +\arg[z(R)-z(y)]-\arg[z(R)-z(x)])=0$$

and

$$\lim_{\delta\to 0}(\arg[z(y-\delta)-z(y)]-\arg[z(y-\delta)-z(x)] \\ -\arg[z(y+\delta)-z(y)]+\arg[z(y+\delta)-z(x)]) \\ =\lim_{\delta\to 0}(\arg[z(y-\delta)-z(y)]-\arg[z(y+\delta)-z(y)])=\pi$$

again because A is differentiable at y. As a consequence,

$$\lim_{\substack{R\to\infty\\ \delta\to 0}} \text{Im } (I_{R,\delta}+II_{R,\delta}+III_{R,\delta})=\pi+(\arg[z(x-\epsilon)-z(y)]-\arg[z(x-\epsilon)-z(x)] \\ -\arg[z(x+\epsilon)-z(y)]+\arg[z(x+\epsilon)-z(x)]).$$

Gathering again, we have proved that, for all points y with $|y-x|>\epsilon$ such that A is differentiable at y,

$$\lim_{\substack{R\to\infty\\ \delta\to 0}} (I_{R,\delta}+II_{R,\delta}+III_{R,\delta})=G_{x,\epsilon}(y)+B(x,\epsilon),$$

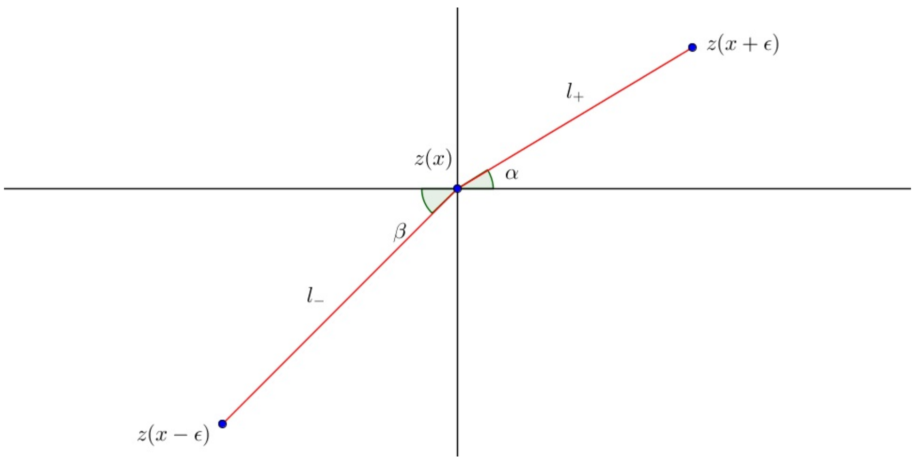
and so the desired conclusion follows. □

Remark: The function $B=B(x,\epsilon)$ must be understood as a way of quantifying the curvature or convexity of Γ around x. Indeed, it is easy to check that $B(x,\epsilon)=0$ if, and only if, the points $z(x-\epsilon)$, $z(x)$ and $z(x+\epsilon)$ are collinear, as the following lemma states.

Lemma 2.3.3. Let $x\in\mathbb{R}$ and $\epsilon>0$. Then, the following assertions are equivalent:

1. $B(x,\epsilon)=0$.
2. $\text{Im } B(x,\epsilon)=0$.
3. The points $z(x-\epsilon)$, $z(x)$ and $z(x+\epsilon)$ are collinear.

Proof. Take a look at the following scheme:



Here, $\alpha = \arg[z(x+\epsilon) - z(x)]$, $\pi + \beta = \arg[z(x-\epsilon) - z(x)]$, $l_- = |z(x-\epsilon) - z(x)|$ and $l_+ = |z(x+\epsilon) - z(x)|$, and with this notation,

$$B(x,\epsilon) = \log \frac{l_+}{l_-} + i(\pi + \alpha - (\pi + \beta)).$$

As a consequence,

$$\text{Im } B(x,\epsilon) = \pi + \alpha - (\pi + \beta) = \alpha - \beta.$$

Now, (1) $\Rightarrow$ (2) is trivial. To prove (2) $\Rightarrow$ (3), observe that

$$\text{Im } B(x,\epsilon) = 0 \Rightarrow \alpha = \beta,$$

and so $z(x-\epsilon)$, $z(x)$ and $z(x+\epsilon)$ are collinear. Finally, (3) $\Rightarrow$ (1) is again trivial. □

Applying Lemma 2.3.2, we get

$$\begin{aligned} III &= \frac{1}{\pi i} \int_{|y-x|>N\epsilon} Tf(y) \frac{1}{z(y) - z(x)} \left[B(x,\epsilon) + G_{x,\epsilon}(y) \right] dz(y) \\ &= \frac{1}{\pi i} \left[B(x,\epsilon) \int_{|y-x|>N\epsilon} Tf(y) \frac{dz(y)}{z(y) - z(x)} + \int_{|y-x|>N\epsilon} Tf(y) \frac{G_{x,\epsilon}(y) dz(y)}{z(y) - z(x)} \right] \qquad (2.4) \\ &= B(x,\epsilon) T_{N\epsilon}(Tf)(x) + IV. \end{aligned}$$

We will see now that, for an appropriate choice of N, one has $|IV| \lesssim M(Tf)(x)$.

Lemma 2.3.4. Choose $N > 1 + 4(1 + \Lambda_1)$. Then for $|y - x| > N\epsilon$,

$$|G_{x,\epsilon}(y)| \lesssim \frac{\epsilon}{|y - x|}.$$

Proof. Let

$$u_{x,\epsilon}(y) = \text{Re } G_{x,\epsilon}(y) = \log \frac{|z(x-\epsilon) - z(y)|}{|z(x+\epsilon) - z(y)|}$$

and

$$v_{x,\epsilon}(y) = \text{Im } G_{x,\epsilon}(y) = \arg[z(x-\epsilon) - z(y)] - \arg[z(x+\epsilon) - z(y)].$$

Recall that, for $w \in \mathbb{C}$, $|w| < \frac{1}{2}$,

$$|\text{Log } (1 + w)| \leq 2|w|,$$

where Log is the complex logarithm defined by

$$\text{Log } (z) = \log |z| + i \arg(z), \; z \neq 0, \, -\frac{\pi}{2} \leq \arg(z) < \frac{3\pi}{2}.$$

Now, for $|y - x| > N\epsilon$, we have

$$\frac{z(x-\epsilon) - z(y)}{z(x+\epsilon) - z(y)} = 1 + \frac{z(x-\epsilon) - z(x+\epsilon)}{z(x+\epsilon) - z(y)},$$

and

$$\left|\frac{z(x-\epsilon)-z(x+\epsilon)}{z(x+\epsilon)-z(y)}\right| \leq \frac{(1+\Lambda_1)2\epsilon}{|y-(x+\epsilon)|} \leq \frac{(1+\Lambda_1)2\epsilon}{\frac{N-1}{N}|y-x|}$$
$$\leq \frac{(1+\Lambda_1)2\epsilon}{\frac{N-1}{N}N\epsilon} = \frac{2(1+\Lambda_1)}{N-1} \leq \frac{1}{2},$$

where the last inequality holds precisely because of the choice of N. Then,

$$\begin{aligned}
|u_{x,\epsilon}(y)| &= \left|\log\frac{|z(x-\epsilon)-z(y)|}{|z(x+\epsilon)-z(y)|}\right| = \left|\log\left|1+\frac{z(x-\epsilon)-z(x+\epsilon)}{z(x+\epsilon)-z(y)}\right|\right| \\
&= \left|\text{Re Log }\left(1+\frac{z(x-\epsilon)-z(x+\epsilon)}{z(x+\epsilon)-z(y)}\right)\right| \\
&\leq \left|\text{Log }\left(1+\frac{z(x-\epsilon)-z(x+\epsilon)}{z(x+\epsilon)-z(y)}\right)\right| \leq 2\left|\frac{z(x-\epsilon)-z(x+\epsilon)}{z(x+\epsilon)-z(y)}\right| \\
&\leq 2\frac{(1+\Lambda_1)2\epsilon}{\frac{N-1}{N}|y-x|} = \frac{4N(1+\Lambda_1)}{N-1}\frac{\epsilon}{|y-x|} \lesssim \frac{\epsilon}{|y-x|}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|v_{x,\epsilon}(y)| &= |\arg[z(x-\epsilon)-z(y)] - \arg[z(x+\epsilon)-z(y)]| \\
&= |\text{Im }(\text{Log }[z(x-\epsilon)-z(y)] - \text{Log }[z(x+\epsilon)-z(y)])| \\
&\leq |\text{Log }[z(x-\epsilon)-z(y)] - \text{Log }[z(x+\epsilon)-z(y)]| \\
&= \left|\int_{\Gamma((x-\epsilon,x+\epsilon))}\frac{dz}{z-z(y)}\right| \leq \text{length}(\Gamma((x-\epsilon,x+\epsilon)))\max_{|t-x|\leq\epsilon}\frac{1}{|z(t)-z(y)|} \\
&\leq \frac{2(1+\Lambda_1)\epsilon}{|y-x|} \lesssim \frac{\epsilon}{|y-x|}.
\end{aligned}$$

Putting all together, we obtain

$$|G_{x,\epsilon}(y)| \lesssim \frac{\epsilon}{|y-x|},$$

as desired. □

From now on, we fix $N > 1+4(1+\Lambda_1)$, so that the conditions of the previous lemma hold. Then, we have

$$\begin{aligned}
|IV| &\leq \frac{1}{\pi}\int_{|y-x|>N\epsilon}|Tf(y)|\frac{|G_{x,\epsilon}(y)||z'(y)|}{|z(y)-z(x)|}dy \\
&\lesssim (1+\Lambda_1)\epsilon\int_{|y-x|>N\epsilon}|Tf(y)|\frac{dy}{|y-x|^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{|y-x|>N\epsilon} |Tf(y)| \frac{dy}{|y-x|^2} &= \sum_{k=0}^{\infty} \int_{2^k N\epsilon<|y-x|<2^{k+1}N\epsilon} |Tf(y)| \frac{dy}{|y-x|^2} \\
&\leq \sum_{k=0}^{\infty} \int_{|y-x|<2^{k+1}N\epsilon} |Tf(y)| \frac{dy}{(2^k N\epsilon)^2} \\
&\leq \sum_{k=0}^{\infty} \frac{2\cdot 2^{k+1}N\epsilon}{(2^k N\epsilon)^2} \left(\frac{1}{2\cdot 2^{k+1}N\epsilon} \int_{|y-x|<2^{k+1}N\epsilon} |Tf(y)|dy \right) \qquad (2.5) \\
&\leq \left(\sum_{k=0}^{\infty} \frac{2\cdot 2^{k+1}N\epsilon}{(2^k N\epsilon)^2} \right) M(Tf)(x) \\
&= \left(\frac{4}{N\epsilon} \sum_{k=0}^{\infty} 2^{-k} \right) M(Tf)(x) = \frac{8}{N\epsilon} M(Tf)(x),
\end{aligned}$$

we get

$$|IV| \lesssim (1+\Lambda_1)\epsilon \frac{8}{N\epsilon} M(Tf)(x) \lesssim M^2(Tf)(x), \qquad (2.6)$$

as wished.

Summing up, we have the following result, which follows directly from (2.4) and (2.6).

Lemma 2.3.5. Fix $N > 1 + 4(1+\Lambda_1)$. Then, for all $f \in L^2(\mathbb{R})$, all $x \in \mathbb{R}$ and all $\epsilon > 0$,

$$|T_\epsilon f(x) + B(x,\epsilon) T_{N\epsilon}(Tf)(x)| \lesssim M^2(Tf)(x).$$

2.3.3 The proof of Theorem 2.1.1.

The following example will show that, when Γ has angles, the inequality

$$T_* f(x) \lesssim M^n(Tf)(x)$$

does not hold in general.

Fix the Lipschitz function $A(x) = |x|$. In this case,

$$B(0,\epsilon) = \log \frac{|z(\epsilon)-z(0)|}{|z(-\epsilon)-z(0)|} + i\Big(\pi + \arg[z(\epsilon)-z(0)] - \arg[z(-\epsilon)-z(0)]\Big) = \frac{\pi i}{2}.$$

Assume that the inequality

$$T_* f(x) \lesssim M^n(Tf)(x) \quad \text{for all } f \in L^2(\mathbb{R})$$

were true for some $n \geq 2$. Then, applying Lemma 2.3.5, this would yield

$$|B(x,\epsilon) T_{N\epsilon}(Tf)(x)| \lesssim M^n(Tf)(x)$$

for all $f \in L^2(\mathbb{R})$. Now, taking into account that $T^2 = Id$, and taking $x = 0$, the latter implies

$$|T_{N\epsilon} f(0)| \lesssim M^n f(0), \qquad (2.7)$$

for all $f \in L^2(\mathbb{R})$, and this is false for $f = \chi_{[0,1]}$. Indeed, $M^n f(0) \leq 1$, while for $0 < N\epsilon < 1$,

$$\begin{aligned} T_{N\epsilon} f(0) &= \frac{1}{\pi i} \int_{|y|>N\epsilon} \chi_{[0,1]}(y) \frac{dz(y)}{z(y) - z(0)} \\ &= \frac{1}{\pi i} \int_{N\epsilon}^{1} \frac{1+i}{y+iy} dy \\ &= \frac{1}{\pi i} \int_{N\epsilon}^{1} \frac{dy}{y} = -\frac{1}{\pi i} \log(N\epsilon), \end{aligned}$$

so

$$\lim_{\epsilon \to 0} |T_{N\epsilon} f(0)| = \infty,$$

yielding a contradiction with (2.7).

This counterexample can be generalized in the following way. Suppose Γ has an angle at a point $z(x)$, $x \in \mathbb{R}$, meaning with this that A' has a jump discontinuity at x, i.e.,

$$\lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} = A'_+(x) \neq A'_-(x) = \lim_{h \to 0^-} \frac{A(x+h) - A(x)}{h}.$$

A straightforward computation shows now that

$$\lim_{\epsilon \to 0} \operatorname{Im} B(x, \epsilon) = \arctan(A'_+(x)) - \arctan(A'_-(x)) \neq 0,$$

and so $B(x, \epsilon)$ stays away from 0 as $\epsilon \to 0$. The same argument that was used above, substituting $\chi_{[0,1]}$ by $\chi_{[x,x+1]}$, will show that the inequality

$$T_* f(x) \lesssim M^n(Tf)(x)$$

cannot hold.

2.3.4 The proof of Theorem 2.1.2.

We will study now the term $T_{N\epsilon}(Tf)(x)$ to give more light to this subject. This will lead us to prove that, when A has compact support, the inequality

$$T_* f(x) \lesssim M^n(Tf)(x)$$

can only hold when $A = 0$, i.e., when Γ is a straight line, which is a case already known since T is, essentialy, the Hilbert transform.

Assume that A has compact support, say $\operatorname{supp}(A) \subset [-L, L]$, $L > 0$. Let $f \in L^2(\mathbb{R})$, and write $g = (Tf)\chi_{[-2L,2L]}$, $h = (Tf)\chi_{\mathbb{R}\setminus[-2L,2L]}$, so that $Tf = g + h$ and

$$T_{N\epsilon}(Tf)(x) = T_{N\epsilon} g(x) + T_{N\epsilon} h(x).$$

Fix $x \in [-L, L]$. Observe that

$$\begin{aligned} i\pi T_{N\epsilon} g(x) &= \int_{|y-x|>N\epsilon} \frac{g(y)}{z(y)-z(x)} dz(y) \\ &= \sum_{k=0}^{\infty} \int_{2^k N\epsilon<|y-x|<2^{k+1}N\epsilon} \frac{g(y)}{z(y)-z(x)} dz(y). \end{aligned}$$

Now, taking into account that $\mathrm{supp}(g) \subset [-2L, 2L]$, one gets that, when $2^k N\epsilon > 4L$,

$$\int_{2^k N\epsilon<|y-x|<2^{k+1}N\epsilon} \frac{g(y)}{z(y)-z(x)} dz(y) = 0.$$

This yields that only the first $M_{L,\epsilon}$ terms of the sum above do not vanish, where

$$M_{L,\epsilon} = \left\lceil \frac{\log\left(\frac{4L}{N\epsilon}\right)}{\log 2} \right\rceil$$

(by $\lceil M_{L,\epsilon} \rceil$ we denote the smallest integer n such that $M_{L,\epsilon} \leq n$).

Furthermore, for each $k \geq 0$,

$$\begin{aligned} \left| \int_{2^k N\epsilon<|y-x|<2^{k+1}N\epsilon} \frac{g(y)}{z(y)-z(x)} dz(y) \right| &\leq (1+\Lambda_1) \int_{2^k N\epsilon<|y-x|<2^{k+1}N\epsilon} \frac{|g(y)|}{|y-x|} dy \\ &\leq (1+\Lambda_1) \frac{1}{2^k N\epsilon} \int_{|y-x|<2^{k+1}N\epsilon} |g(y)| dy \\ &\leq 4(1+\Lambda_1) Mg(x). \end{aligned}$$

Putting all together, and taking into account that $Mg \leq M(Tf)$, we obtain

$$|T_{N\epsilon} g(x)| \leq 4\pi(1+\Lambda_1) \left(1 + \left| \frac{\log\left(\frac{4L}{N\epsilon}\right)}{\log 2} \right| \right) M(Tf)(x).$$

On the other hand, taking into account that $A = 0$ on $\mathrm{supp}(h)$, we get

$$i\pi T_{N\epsilon} h(x) = \int_{|y-x|>N\epsilon} \frac{h(y)}{z(y)-z(x)} dz(y) = \int_{|y-x|>N\epsilon} \frac{h(y)}{y-z(x)} dy.$$

Now, for $|y-x| > N\epsilon$,

$$\frac{1}{y-z(x)} = \frac{1}{y-x} + \left(\frac{1}{y-z(x)} - \frac{1}{y-x} \right) = \frac{1}{y-x} + D(x,y),$$

and so

$$i\pi T_{N\epsilon} h(x) = H_{N\epsilon} h(x) + \int_{|y-x|>N\epsilon} h(y) D(x,y) dy.$$

Observe now that, for $x \neq y$,

$$|D(x,y)| = \left| \frac{1}{y-z(x)} - \frac{1}{y-x} \right| = \left| \frac{iA(x))}{(y-x)(y-z(x))} \right| \leq \frac{|A(x)|}{|y-x|^2}.$$

Then, taking into account that $h = 0$ on $[-2L, 2L]$, and recalling that $|x| \leq L$, one gets

$$\left| \int_{|y-x|>N\epsilon} h(y) D(x,y) dy \right| \leq |A(x)| \int_{|y-x|>L} \frac{|h(y)|}{|y-x|^2} dy.$$

Splitting the last integral into the regions $\{2^k L < |y - x| \leq 2^{k+1} L\}$, and taking into account that $M(h) \leq M(Tf)$, we get, arguing as in (2.5),

$$\left| \int_{|y-x|>N\epsilon} h(y) D(x,y) dy \right| \leq \frac{8}{L} |A(x)| M(Tf)(x).$$

The previous discussion shows that

$$T_{N\epsilon}(Tf)(x) = \frac{1}{\pi i} H_{N\epsilon} h(x) + V,$$

where

$$|V| \leq c(x, \epsilon, N, L) M(Tf)(x)$$

and $0 < c(x, \epsilon, N, L) < \infty$. Recall now that, by Lemma 2.3.5, we have

$$|T_\epsilon f(x) + B(x,\epsilon) T_{N\epsilon}(Tf)(x)| \lesssim M^2(Tf)(x).$$

Then, it follows that

$$\left| T_\epsilon f(x) + \frac{1}{\pi i} B(x,\epsilon) H_{N\epsilon} h(x) \right| \leq c'(x, \epsilon, N, L) M^2(Tf)(x),$$

where $0 < c'(x, \epsilon, N, L) < \infty$.

Assume A is not identically null, and suppose that the inequality $T_* f(x) \lesssim M^n(Tf)(x)$ holds. Applying Lemma 2.3.3, we may pick $x \in [-L, L]$ and $\epsilon > 0$ as small as we want such that $B(x, \epsilon) \neq 0$. Then, it follows that

$$|B(x,\epsilon)| |H_{N\epsilon}((Tf)\chi_{\mathbb{R}\setminus[-2L,2L]})(x)| \leq c''(x, \epsilon, N, L) M^n(Tf)(x),$$

with $0 < c''(x, \epsilon, N, L) < \infty$.

Now, for each $k = 3, 4, \ldots$, pick $f_k \in L^2(\mathbb{R})$ such that $Tf_k = \chi_{[0,kL]}$, and so $(Tf_k)\chi_{\mathbb{R}\setminus[-2L,2L]} = \chi_{(2L,kL]}$. Applying the previous inequality for each f_k, and taking into account that $M^n(Tf_k) \leq 1$, we obtain

$$|B(x,\epsilon)| |H_{N\epsilon}(\chi_{(2L,kL]})(x)| \leq c''(x, \epsilon, N, L).$$

Finally, observe that

$$H_{N\epsilon}(\chi_{(2L,kL]})(x) = \int_{2L}^{kL} \frac{dy}{y - x} = \log \frac{kL - x}{2L - x},$$

and so

$$|B(x,\epsilon)| \log \frac{kL - x}{2L - x} \leq c''(x, \epsilon, N, L),$$

yielding a contradiction, since the left hand side tends to ∞ as $k \to \infty$.

2.4 Another version of the truncated and maximal operators.

Let us consider now another version of the truncated operators. Define, for $\epsilon > 0$ and $x \in \mathbb{R}$,

$$\tilde{T}_\epsilon f(x) = \frac{1}{\pi i} \int_{|z(y)-z(x)|>\epsilon} \frac{f(y)}{z(y) - z(x)} dz(y)$$

and the associated maximal operator $\tilde{T}_* f(x) = \sup_{\epsilon>0} |\tilde{T}_\epsilon f(x)|$. This is a truncation over balls of radius ϵ, while the one for T_ϵ was a truncation over strips of width 2ϵ.

We consider now the same problem as before: that of giving an estimate of the form

$$\tilde{T}_* f(x) \lesssim M^n(Tf)(x),$$

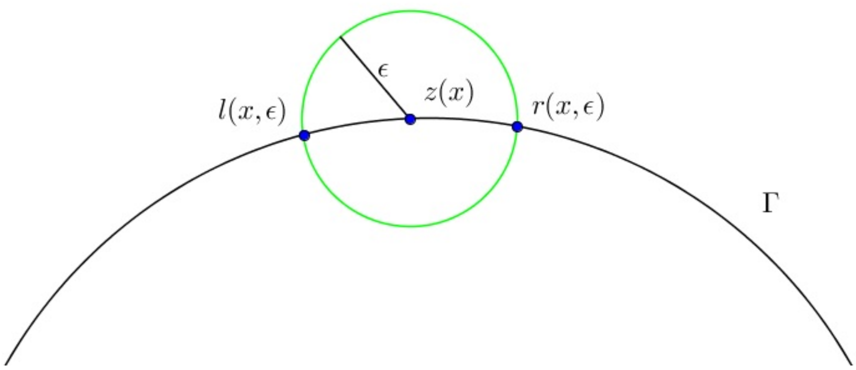
and the same arguments employed before will work here. Indeed, if we define $l(x, \epsilon) = z(x_-)$, $r(x, \epsilon) = z(x_+)$, where

$$x_- = \sup\{t < x \colon |z(t) - z(x)| = \epsilon\}$$

and

$$x_+ = \inf\{t > x \colon |z(t) - z(x)| = \epsilon\},$$

then $l(x, \epsilon)$ and $r(x, \epsilon)$ will play the same role that $z(x - \epsilon)$ and $z(x + \epsilon)$ played before. Precisely, $l(x, \epsilon)$ is the *last* point of Γ to the left of $z(x)$ that belongs to the circle centered at $z(x)$ with radius ϵ, and $r(x, \epsilon)$ is the analog of this at the right.



Taking into account that the quantities $|y - x|$ and $|z(y) - z(x)|$ are comparable, one can repeat the arguments used before to get an analogous of Lemma 2.3.5, which will be stated now as

$$|\tilde{T}_\epsilon f(x) - \tilde{B}(x, \epsilon)\tilde{T}_{N\epsilon} f(x)| \lesssim M^2(Tf)(x),$$

where

$$\tilde{B}(x,\epsilon) = \log\frac{|r(x,\epsilon)-z(x)|}{|l(x,\epsilon)-z(x)|} + i\Big(\pi + \arg[r(x,\epsilon)-z(x)] - \arg[l(x,\epsilon)-z(x)]\Big).$$

As in Lemma 2.3.3, $\tilde{B}(x,\epsilon)=0$ if, and only if, $l(x,\epsilon)$, $z(x)$ and $r(x,\epsilon)$ are collinear.

With this tools at hand , one can prove the following results, which are the analogs to Theorems 2.1.1 and 2.1.2 in this setting.

Theorem 2.4.1. Consider the Lipschitz function $A(x)=|x|$. Then, there exists $f\in L^2(\mathbb{R})$ such that for all $c>0$ and all $n\geq 1$, there exists $\epsilon>0$ such that

$$|\tilde{T}_\epsilon f(0)| > cM^n(Tf)(0).$$

To prove this, one can mimic the argument in section 2.3.3, since here we have again $\tilde{B}(0,\epsilon)=i\frac{\pi}{2}$.

Theorem 2.4.2. Let A be a Lipschitz function with compact support. Suppose A is not identically null, or, equivalently, that Γ is not a straight line. Then, there exists $x\in\mathbb{R}$ such that for all $c>0$ there exists $f\in L^2(\mathbb{R})$ with

$$\tilde{T}_* f(x) > c\tilde{T}^n(Cf)(x)$$

for all $n\geq 1$.

Again, the argument in Section 2.3.4 adapts trivially to this case, by just taking into account that, if A is not identically null, one can find $x\in\mathbb{R}$ and $\epsilon>0$ as small as needed such that $l(x,\epsilon)$, $z(x)$ and $r(x,\epsilon)$ are not collinear.

2.5 A positive result for the case of Jordan curves.

Let Γ be a Jordan curve in the plane, parametrized by a periodic function $\gamma\colon\mathbb{R}\to\mathbb{C}$. We will pose, for the moment, the following assumptions on γ:

- γ is of class $\mathcal{C}^1$.
- γ is L-periodic, $\gamma([0,L))=\Gamma$.
- γ is injective on $[0,L)$.
- $|\gamma'(t)|=1$ for all t.
- ω is the modulus of continuity of γ' (this means that ω is a non-negative and increasing continuous function in $[0,\infty)$ with $\omega(0)=0$ and such that $|\gamma'(s)-\gamma'(t)|\leq\omega(|s-t|)$ for all $s,t\in\mathbb{R}$).

We denote by μ the arc-length measure on Γ. We have, for a Borel set $I \subset [0, L)$,

$$\mu(\gamma(I)) = \int_I |\gamma'(t)| dt = |I|.$$

For a point $z \in \Gamma$ and $r > 0$, denote

$$\Gamma_{z,r} = \gamma(\{t : |t - x| < r\}),$$

where $z = \gamma(x)$, $x \in \mathbb{R}$.

The Hardy-Littlewood maximal function of a function $f \in L^1(\Gamma, \mu)$ is defined, for $z \in \Gamma$, by

$$Mf(z) = \sup_{r>0} \frac{1}{\mu(\Gamma_{z,r})} \int_{\Gamma_{z,r}} |f| d\mu = \sup_{r>0} \frac{1}{2r} \int_{\Gamma_{z,r}} |f| d\mu$$

The Cauchy transform of a function $f \in L^2(\Gamma, d\mu)$ is defined, for $z \in \Gamma$, as the principal value integral

$$Tf(z) = \lim_{\epsilon \to 0} T_\epsilon f(z),$$

where

$$T_\epsilon f(z) = \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma_{z,\epsilon}} \frac{f(\xi)}{\xi - z} d\xi.$$

We consider as well the maximal operator associated with T,

$$T_* f(z) = \sup_{\epsilon > 0} |T_\epsilon f(z)|.$$

In this section we will prove that, if γ is regular enough (we will specify later how much regularity is needed), then

$$T_* f(z) \lesssim M^2(Tf)(z) \text{ for all } f \in L^2(\Gamma, \mu).$$

We will follow, essentially, the same steps we have taken in Section 2.3 for the case of Lipschitz graphs. Most of the arguments there will be valid in this setting, and so we will not enter into many details. First of all, we remark that the analogues of Lemmas 2.2.1 and 2.2.3 hold now:

Lemma 2.5.1. If $f \in L^2(\Gamma, \mu)$, $T^2 f = f$.

Lemma 2.5.2. If $f, g \in L^2(\Gamma, \mu)$, then

$$\int_\Gamma Tf(z) g(z) dz = -\int_\Gamma f(z) Tg(z) dz.$$

We argue now as in Section 2.3. Fix $f \in L^2(\Gamma, \mu)$, $z \in \Gamma$ and $\epsilon > 0$. Then, we have

$$T_\epsilon f(z) = \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma_{z,\epsilon}} \frac{f(\xi)}{\xi - z} d\xi = \int_\Gamma f(\xi) K_{z,\epsilon}(\xi) d\xi,$$

where

$$K_{z,\epsilon}(\xi) = \frac{1}{\pi i(\xi - z)} \chi_{\Gamma \setminus \Gamma_{z,\epsilon}}(\xi).$$

It is easy to check that $K_{z,\epsilon} \in L^2(\Gamma, \mu) \cap L^\infty(\Gamma, \mu)$, and moreover

$$||K_{z,\epsilon}||_{L^2} \lesssim \frac{1}{\sqrt{\epsilon}}, \quad ||K_{z,\epsilon}||_{L^\infty} \lesssim \frac{1}{\epsilon}.$$

Since $K_{z,\epsilon} \in L^2(\Gamma, \mu)$, $K_{z,\epsilon} = T^2(K_{z,\epsilon}) = T(g_{z,\epsilon})$, for $g_{z,\epsilon} = T(K_{z,\epsilon})$. Then, we have

$$T_\epsilon f(z) = \int_\Gamma f(\xi) K_{z,\epsilon}(\xi) d\xi = \int_\Gamma f(\xi) T(g_{z,\epsilon})(\xi) d\xi = -\int_\Gamma Tf(\xi) g_{z,\epsilon}(\xi) d\xi,$$

and, as a consequence,

$$\begin{aligned}
-T_\epsilon f(z) &= \int_\Gamma Tf(\xi) g_{z,\epsilon}(\xi) d\xi \\
&= \int_{\Gamma_{z,2\epsilon}} Tf(\xi) g_{z,\epsilon}(\xi) d\xi + \int_{\Gamma \setminus \Gamma_{z,2\epsilon}} Tf(\xi) g_{z,\epsilon}(\xi) d\xi \\
&= \int_{\Gamma_{z,2\epsilon}} Tf(\xi)[g_{z,\epsilon}(\xi) - m_{\Gamma_{z,2\epsilon}}(g_{z,\epsilon})] d\xi + m_{\Gamma_{z,2\epsilon}}(g_{z,\epsilon}) \int_{\Gamma_{z,2\epsilon}} Tf(\xi) d\xi + \int_{\Gamma \setminus \Gamma_{z,2\epsilon}} Tf(\xi) g_{z,\epsilon}(\xi) d\xi \\
&= I + II + III,
\end{aligned}$$

where, for a function $h \in L^1(\Gamma, \mu)$ and a Borel set $E \subset \Gamma$ with $\mu(E) > 0$,

$$m_E h = \frac{1}{\mu(E)} \int_E h d\mu.$$

Arguing essentially as in Section 2.3.1, one can prove that $|I| \lesssim M^2(Tf)(z)$ and $|II| \lesssim M(Tf)(z)$. Let us study III now.

$$III = \int_{\Gamma \setminus \Gamma_{z,2\epsilon}} Tf(\xi) g_{z,\epsilon}(\xi) d\xi = \int_{\Gamma \setminus \Gamma_{z,2\epsilon}} Tf(\xi) T(K_{z,\epsilon})(\xi) d\xi.$$

A similar argument to the one used in Lemma 2.3.2 yields the following result.

Lemma 2.5.3. For $\xi \in \Gamma \setminus \Gamma_{z,2\epsilon}$,

$$T(K_{z,\epsilon})(\xi) = \frac{1}{\pi i} \frac{1}{z - \xi} [B(z, \epsilon) + G_{z,\epsilon}(\xi)],$$

where

$$G_{z,\epsilon}(\xi) \lesssim \frac{\epsilon}{|z - \xi|}$$

and

$$|B(z, \epsilon)| \lesssim \omega(2\epsilon).$$

Remark: The expressions of $G_{z,\epsilon}(\xi)$ and $B(z,\epsilon)$ are totally analogous to the ones for $G_{x,\epsilon}(y)$ and $B(x,\epsilon)$ in Lemma 2.3.2, for suitably chosen branches of $\arg(w-z)$ and $\arg(w-\xi)$.

From this, it follows that

$$\begin{aligned}
III &= \int_{\Gamma\setminus\Gamma_{z,2\epsilon}} Tf(\xi)T(K_{z,\epsilon})(\xi)d\xi \\
&= B(z,\epsilon)\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} Tf(\xi)\frac{1}{z-\xi}d\xi + \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} Tf(\xi)\frac{G_{z,\epsilon}(\xi)}{z-\xi}d\xi \\
&= B(z,\epsilon)T_{2\epsilon}(Tf)(z) + \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} Tf(\xi)\frac{G_{z,\epsilon}(\xi)}{z-\xi}d\xi \\
&= III_1 + III_2.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
|III_2| &\leq \frac{1}{\pi}\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} |Tf(\xi)|\frac{|G_{z,\epsilon}(\xi)|}{|z-\xi|}d\mu(\xi) \\
&\lesssim \epsilon\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} \frac{|Tf(\xi)|}{|z-\xi|^2}d\mu(\xi) \lesssim M(Tf)(z)
\end{aligned}$$

where the last inequality is shown by splitting the integral over the sets

$$\Gamma_{z,2^{k+1}\epsilon}\setminus\Gamma_{z,2^k\epsilon}, \quad k=1,2,3,\ldots$$

On the other hand

$$|III_1| = |B(z,\epsilon)|\frac{1}{\pi}\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} \frac{|Tf(\xi)|}{|\xi-z|d\mu(\xi)} \lesssim \omega(2\epsilon)\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} \frac{|Tf(\xi)|}{|z-\xi|}d\mu(\xi).$$

To estimate the last integral, we also split it over the sets

$$\Gamma_{z,2^{k+1}\epsilon}\setminus\Gamma_{z,2^k\epsilon}, \quad k=1,2,3,\ldots$$

Notice that, for k big enough, $\Gamma_{z,2^k\epsilon}=\Gamma$, and so $\Gamma_{z,2^{k+1}\epsilon}\setminus\Gamma_{z,2^k\epsilon}=\emptyset$. Precisely, this holds for all k such that $2^k\epsilon > 2L$, which is equivalent to

$$k > \frac{\log\frac{2L}{\epsilon}}{\log 2}.$$

As a result, if we denote by $k_0(\epsilon)$ the smallest integer k that satisfies the previous inequality, we have

$$\begin{aligned}
\int_{\Gamma\setminus\Gamma_{z,2\epsilon}} \frac{|Tf(\xi)|}{|z-\xi|}d\mu(\xi) &= \sum_{k=1}^{k_0(\epsilon)}\int_{\Gamma_{z,2^{k+1}\epsilon}\setminus\Gamma_{z,2^k\epsilon}} \frac{|Tf(\xi)|}{|z-\xi|}d\mu(\xi) \\
&\lesssim \sum_{k=1}^{k_0(\epsilon)}\frac{1}{2^k\epsilon}\int_{\Gamma_{z,2^{k+1}\epsilon}\setminus\Gamma_{z,2^k\epsilon}} |Tf(\xi)|d\mu(\xi) \\
&\leq 4\sum_{k=1}^{k_0(\epsilon)}\frac{1}{2\cdot 2^k\epsilon}\int_{\Gamma_{z,2^{k+1}\epsilon}} |Tf(\xi)|d\mu(\xi) \\
&\leq 4k_0(\epsilon)M(Tf)(z).
\end{aligned}$$

As a result,

$$|III_1| \lesssim \omega(2\epsilon)k_0(\epsilon)M(Tf)(z) \lesssim \omega(2\epsilon)\left|\log \frac{2L}{\epsilon}\right| M(Tf)(z).$$

Gathering the estimates for $|I|$, $|II|$, $|III_1|$ and $|III_2|$, we have

$$|T_\epsilon f(z)| \lesssim M^2(Tf)(z) + \omega(2\epsilon)\left|\log \frac{2L}{\epsilon}\right| M(Tf)(z).$$

From this, it follows that, if ω is such that $\omega(2\epsilon)|\log \epsilon|$ stays bounded as $\epsilon \to 0$, then we have

$$|T_\epsilon f(z)| \lesssim M^2(Tf)(z).$$

Thus, we have proved the following result:

Theorem 2.5.4. With the notation established in this section, suppose γ' has a modulus of continuity ω such that $\omega(\epsilon)|\log \epsilon|$ stays bounded as $\epsilon \to 0$ (this happens, for example, if $\gamma \in \mathcal{C}^{1+\delta}$ for some $\delta > 0$). Then, there exists a constant $c > 0$ such that, for all $f \in L^2(\Gamma, d\mu)$ and all $z \in \Gamma$,

$$T_* f(z) \leq cM^2(Tf)(z).$$

We want to remark, finally, that a totally analogous result holds if one considers the truncated operators given by

$$\tilde{T}_\epsilon f(z) = \frac{1}{\pi i}\int_{\Gamma\setminus B(z,\epsilon)} \frac{f(\xi)}{\xi - z} d\xi.$$

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