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# An Approach to Asset-Liability Risk Control Through Asset-Liability Securities

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# "AN APPROACH TO ASSET-LIABILITY RISK CONTROL THROUGH ASSET-LIABILITY SECURITIES"[1]

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ABSTRACT

Asset-liability strategies are thought to manage the relationship between the values of assets and liabilities of financial institutions by mixing a portfolio of risky assets with a portfolio as similar as possible to the liabilities of the institution. In this work we explore the consequences of a strategy that takes three main steps: i) create synthetic securities - that we call "asset-liability risky securities" - which consist of a long position in risky security and a short position in a default free security equivalent to the liabilities of the institution; ii) build the efficient frontier of risky portfolios from the synthetic securities that we have just created; and iii) build a risk free asset-liability security and combine it with the efficient frontier of asset-liability securities.

Relying on this background we try to answer three questions: i) what is the optimal composition of the risky portfolio of asset-liability securities; ii) how can the separation theorem be applied to asset-liability securities; and iii) how does the risky portfolio of asset-liability securities behave compared to a traditional portfolio of risky securities, and specifically under which circumstances the weights of risky securities have the same values in both portfolios. We relate our results to the CAPM to find out what CAPM attributes continue to hold for asset-liability strategies. We also study the features of a two limit strategy which relative value remains between an upper and a lower limit previously chosen.

KEYWORDS: asset-liability strategies, financial institutions investment policy.

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## CONTENTS

## 1. INTRODUCTION

In recent years asset-liability strategies have been widely studied and recommended for the management of pension funds and other financial institutions. An asset-liability strategy is thought to manage the relationship between the values of assets and liabilities of a financial institution: It consists of mixing a portfolio of risky assets - stocks or risky debt for instance - with a portfolio as similar as possible to the liabilities of the institution. In these strategies there are two main steps: first building a portfolio of risky securities, and second mixing that portfolio with another as similar as possible to the liabilities of the institution. Following Bookstaber and Gold (1988) we call the latter portfolio "liability asset".

Relying on this background in this work we explore the consequences of a strategy that takes three main steps: first create synthetic securities each one of them consisting of a long position in risky security and a short position in a default free security equivalent to the liabilities of the institution. We call them "asset-liability risky securities". Second build the efficient frontier of risky portfolios from the synthetic securities that we have just created. Third build a risk free asset-liability security and combine it with the efficient frontier of the asset-liability securities. Throughout this work we assume that short sales are allowed since they are often needed to implement hedging strategies.

We try to answer the three following questions: i) what is the optimal composition of the risky portfolio of asset-liability securities; ii) how can the separation theorem be applied to asset-liability securities; and iii) how does the risky portfolio of asset-liability securities behave compared to a traditional portfolio of risky securities, and specifically under which circumstances the portfolio weights of risky securities have the same values in both portfolios.

We relate to the CAPM the results that we have obtained in order to find out what CAPM attributes continue to hold for asset-liability strategies. Besides we study the features of a particular strategy which grants that, at any moment, the relationship between assets and liabilities remains between to values or limits previously chosen: the lower limit is regarded as a protection while the upper limit reduces the cost of that protection.

The paper is organized in the following way: first we study the basic features of asset-liability securities and their portfolios (sections 2 and 3). Next we introduce lending and borrowing in an asset-liability context (section 4). Relying on this background we study some properties of strategies built up using asset-liability securities (section 5). Finally we compare the properties of the portfolios of asset-liability securities with the properties of the asset-liability portfolios (section 6), the latter stemming from the two step strategies that we have mentioned at the beginning of this introduction.

## 2. ASSET-LIABILITY SECURITY

### 2.1. Concept

We define an asset-liability security as a quotient between a security - which may be of variable or fixed income - and a default free security. Since it is a quotient, it can be qualified as a "relative security". Notice that the value of a default free security varies according to changes in default free interest rate.

We call the security on the numerator "asset" and the security on the denominator "liability" and use the following notation:

$S_{jt}$ = value of the "asset security" $j$ at the moment $t$

$L_{jt}$ = value of the "liability security" $j$ at the moment $t$

$s_{jt}$ = value of the asset-liability security $j$ at the moment $t$

Hence the value of the asset-liability security is: $s_{jt} = \dfrac{S_{jt}}{L_{jt}}$ [ 1 ]

We assume the following initial accounting condition:

$$S_{j0} = L_{j0} \qquad [\ 2\ ]$$

So: $s_{j0} = 1$ [ 3 ]

which means that, at the moment when the asset-liability security is created, the value of the asset equals the value of the liability, and therefore the value of the asset-liability security is one.

For the purposes of this work $L_{jt}$ is thought to be a portion of the liabilities of a financial institution, which at $t = 0$ equates the value of the asset j at that moment, $S_{j0}$.

## 2.2. Return of the asset-liability security

Using the following notation:

$R_j$ = return of the asset security $j$

$i$ = default free interest rate that at the same time is the return of the liability security

$r_j$ = return of the asset-liability security $j$

and under the hypothesis that the asset neither pays dividend nor interest (or in case it does, the corresponding amounts are systematically reinvested in the same asset) we can write : $S_{j1} = S_{jo}\, e^{R_j}$ [ 4 ]

$$L_{j1} = L_{j0}\, e^{i} \qquad [\ 5\ ]$$

Taking [ 1 ] into account: $s_{j1} = \dfrac{S_{j0}\, e^{R_j}}{L_{j0}\, e^{i}}$ [ 6 ]

and the initial accounting condition stated in [ 2 ] it is possible to write:

$$s_{j1} = e^{R_j - i} \qquad [\,7\,]$$

Applying the concept of profitability to the asset-liability security we have:

$$s_{j1} = s_{jo}\, e^{r_j} \qquad [\,8\,]$$

and remembering that $s_{j0} = 1$ according to [ 3 ]:

$$s_{j1} = e^{r_j} \qquad [\,9\,]$$

Equating [ 9 ] and [ 7 ] we obtain:

$$e^{r_j} = e^{R_j - i} \qquad [\,10\,]$$

and therefore: $r_j = R_j - i$ [ 11 ]

In consequence it can be said that the expected return of the asset-liability security is the difference between the expected return of the asset security and the liability security:

$$E(r_j) = E(R_j) - E(i) \qquad [\,12\,]$$

**2.3. Variance and covariance of the asset-liability security**

Let us call: $\sigma_j^2$ = variance of the asset security $j$ return

$\sigma_i^2$ = variance of the default free interest rate

$d_j^2$ = variance of the asset-liability security $j$ return

Taking into account [ 11 ] and the expression of the variance of a sum of random variables, we can write:

$$d_j^2 = \sigma_j^2 + \sigma_i^2 - 2\sigma_{ji} \qquad [\,13\,]$$

### 2.4. Covariance of the returns of two asset-liability securities

In order to develop the expression of $Cov\ (r_j\ ,\ r_q)$ we write:

$$Cov(r_j,r_q) = Cov(R_j - i, R_q - i) \qquad [\ 14\ ]$$

and from the additive property of covariances we obtain:

$$Cov(R_j - i, R_q - i) = Cov(R_j, R_q) - Cov(R_j, i) - Cov(i, R_q) + Cov(i,i) \qquad [\ 15\ ]$$

Adopting the simplified notation $Cov(x,y) = \sigma_{xy}$ for standard securities and $Cov(x,y) = d_{xy}$ for asset-liability securities, equation [ 15 ] can be written as:

$$d_{jq} = \sigma_{jq} + \sigma_i^2 - \sigma_{ji} - \sigma_{iq} \qquad [\ 16\ ]$$

## 3. BASIC PROPERTIES OF PORTFOLIOS OF ASSET-LIABILITY SECURITIES

### 3.1. Mean and variance of a portfolio of asset-liability securities

In this section we study the properties of the portfolios that have been built using asset-liability securities. We call these portfolios "portfolios of asset-liability securities".

Be $w_j$ the weight or percentage of the asset-liability security $j$ in the portfolio, and $n$ the number of securities in the portfolio. Then the return of the portfolio of asset-liability securities takes the expression :

$$\tilde{r} = \sum_{j=1}^{n} w_j \tilde{r}_j \qquad [\ 17\ ]$$

where

$$\sum_{j=1}^{n} w_j = 1 \qquad [\ 18\ ]$$

Bearing in mind the expression of the expected value and the variance of a sum of random variables multiplied by constants, the expected return for a portfolio of asset-liability securities can be written as:

$$E(r) = \sum_{j=1}^{n} w_j \, E(r_j) \qquad [\ 19\ ]$$

while the variance is:

$$d^2 = \sum_{j=1}^{n} w_j^2 d_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} w_j w_q \, d_{jq} \qquad [\ 20\ ]$$

Taking [ 11 ] into account we can write:

$$\tilde{r} = \sum_{j=1}^{n} w_j \tilde{R}_j - \tilde{i} \qquad [\ 21\ ]$$

and therefore [ 19 ] and [ 20 ] become the following expressions:

$$E(r) = \sum_{j=1}^{n} w_j \, E(R_j) - E(i) \qquad [\ 22\ ]$$

and

$$d^2 = \sum_{j=1}^{n} w_j^2 \sigma_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} w_j w_q \, \sigma_{jq} + \sigma_i^2 - 2 \sum_{j=1}^{n} w_j \sigma_{ji} \qquad [\ 23\ ]$$

Notice that [ 22 ] and [ 23 ] state the parameters of the portfolio of asset-liability securities as a function of the parameters of the ordinary securities [ $E(R_j)$, $\sigma_j$, $\rho_{jq}$ ] and the default free interest rate [ $E(i)$, $\sigma_i$, $\rho_{ij}$ ].

### 3.2. The efficient frontier of risky asset-liability securities

To calculate the efficient frontier of risky asset-liability securities we apply the Markowitz model. Recalling [ 22 ] and [ 23 ] we can write the following problem:

$$Min\ d^2 = \sum_{j=1}^{n} w_j^2\sigma_j^2 + \sum_{\forall j\neq q}\sum_{j=1}^{n}\sum_{q=1}^{n} w_j w_q \sigma_{jq} + \sigma_i^2 - 2\sum_{j=1}^{n} w_j\sigma_{ji} \qquad [\ 24\ ]$$

subject to:

$$\sum_{j=1}^{n} w_j E(R_j) = R^* \qquad [\ 25\ ]$$

$$\sum_{j=1}^{n} w_j = 1 \qquad [\ 26\ ]$$

The restriction [ 25 ] stems from a manipulation of :

$$\sum_{j=1}^{n} w_j E(r_j) = r^* \qquad [\ 27\ ]$$

where we substitute $E(r_j)$ for $[E(R_j)-E(i)]$ and $r^*$ for $[R^*-E(i)]$.

This form of the problem assumes that short sales are allowed since it does not include the restriction $w_j \geq 0$ . We choose this form because in section 5.2. we develop another strategy that needs short sales.

Then we can write the Lagrangian as follows: [ 28 ]

$$LG = \sum_{j=1}^{n} w_j^2\sigma_j^2 + \sum_{\forall j\neq q}\sum_{j=1}^{n}\sum_{q=1}^{n} w_j w_q \sigma_{jq} + \sigma_i^2 - 2\sum_{j=1}^{n} w_j\sigma_{ji} + \lambda\left(\sum_{j=1}^{n} w_j E(R_j) - R^*\right) + \theta\left(\sum_{j=1}^{n} w_j - 1\right)$$

Taking the derivatives, equating them to zero and operating we obtain:

$$\begin{array}{l}
2w_1(\sigma_1)^2 + 2w_2\sigma_{12} + \ldots + 2w_j\sigma_{1j} + \ldots + 2w_n\sigma_{1n} + \lambda E(R_1) + \theta = 2\sigma_{1i} \\
2w_1\sigma_{21} + 2w_2(\sigma_2)^2 + \ldots + 2w_j\sigma_{2j} + \ldots + 2w_n\sigma_{2n} + \lambda E(R_2) + \theta = 2\sigma_{2i} \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
2w_1\sigma_{j1} + 2w_2\sigma_{j2} + \ldots + 2w_j(\sigma_j)^2 + \ldots + 2w_n\sigma_{jn} + \lambda E(R_j) + \theta = 2\sigma_{ji} \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
2w_1\sigma_{n1} + 2w_2\sigma_{n2} + \ldots + 2w_j\sigma_{nj} + \ldots + 2w_n(\sigma_n)^2 + \lambda E(R_n) + \theta = 2\sigma_{ni} \\
w_1E(R_1) + w_2E(R_2) + \ldots + w_jE(R_j) + \ldots + w_nE(R_n) = R^* \\
w_1 + w_2 + \ldots + w_j + \ldots + w_n = 1
\end{array} \qquad [\ 29\ ][1]$$

[1] See appendix for the efficient frontier of risky asset-liability securities when short sales are not allowed.

Comparing this set of equations to the standard case of the Markowitz model we notice that the difference between both lies on the fact that the coefficients on the right hand side of the first $n$ equations (here $\sigma_{1i}$ ,$\sigma_{2i}$ , ... , $\sigma_{ni}$ ) are zero in the Markowitz case.

## 4. LENDING AND BORROWING

### 4.1. Lending and borrowing in an asset-liability context

In our asset-liability model risk-free lending and borrowing can be introduced. Let us begin with the meaning of "risk-free" from an asset-liability point of view. In this section we focus our attention on the relationship between assets and liabilities, risk being therefore the variability of this relationship. Following Margrabe (1978), we build an asset-liability risk free security by investing in an asset that behaves exactly as liabilities do. Hence, at any moment the numerator of this relative security equals its denominator, and the value of their relationship is always one. Notice that this asset has already been mentioned in the introduction and called "liability-asset" following Bookstaber and Gold (1988). It is very important to stress - as it was pointed out by Margrabe (1978) in his work about exchange options - that the return of the asset-liability risk free security - namely the liability-asset - is always zero, because the value of this relative security is always one. Therefore in this case riskless lending means a long position in the risk free relative asset, while borrowing means a short position in the risk free relative asset, namely the liability-asset.

### 4.2. Portfolios of asset-liability securities and the separation theorem

The separation theorem continues to hold in an asset-liability context. Comparing the present situation to the mean-variance model that leads to the CAPM, both turn out to be similar. In both cases there exists a concave frontier of risky securities and a risk-free security. Hence the process that leads to the separation theorem in the traditional mean-variance model also leads to the separation theorem in our model. This process consists of determining the optimal risky portfolio: a portfolio on the risky efficient frontier identified by the tangency line from the risk-free interest rate - now zero - to that frontier.

Bearing this in mind, and taking again into account that now the risk-free interest rate is zero, the equation of the efficient frontier of an asset-liability strategy when riskless lending and borrowing are allowed can be written as:

$$E(r) = \frac{E(r_p)}{d_p} \cdot d \qquad [30]$$

where [ $E(r_p)$, $d_p$ ] are parameters of the separation portfolio.

However, there is a difference related to the hypothesis: in the tradional mean-variance model a constant risk-free rate has to be assumed; in this model the constant risk-free rate equal to zero for the risk-free relative asset - or liability-asset - is simply a logical consequence of the risk-free concept.

It can be shown (see appendix) that when covariances between the return of any risky-security and default free interest rate are constant - namely have the same value - the separation portfolio for asset-liability securities has the same composition than the separation portfolio for ordinary securities, namely:

$$\sigma_{ji} = \sigma_i^* \;\; \forall j \;\Rightarrow\; \hat{w}_j = \hat{x}_j \qquad [31]$$

where $\hat{w}_j$ and $\hat{x}_j$ denote, respectively, the weights of the asset-liability security $j$ and the asset $j$ in their separation portfolios.

This result can be regarded as a variant of the traditional mean-variance model for the asset-liability case. Provided that the conditions allowing to pass from the separation theorem to the CAPM also hold in this context, this result becomes a variant of the CAPM, namely an asset-liability CAPM. It is worth noticing that this result does not rely on the concrete value of the default free interest rate. However, it depends on the default free interest rate expectations, for the efficient frontier is a function of the covariances between returns of risky securities and default free interest rate.

### 4.3. A comparison with a mean variance model with stochastic interest rate

As we have seen any efficient asset-liability portfolio when lending and borrowing the liability-asset are allowed consists of a combination of asset-liability securities with the liability-asset. It can also be said that they consist of a combination of ordinary risky assets with the default free security that takes into account the correlation between each risky security and the default free security. This statement relies on the fact that each asset-liability securtiy involves a short position in the default free security.

This model can be related to a mean-variance model with stochastic interest rate. It can be shown that both models lead to the same portfolios, that is to say, for the same level of risk the optimal portfolio has the same composition of risky securities and the default free security.

Let us consider the latter model. Be $x_j$ the proportion of the pure risky asset $j$ invested in a portfolio of pure risky assets, while $(1-u)$ is the percentage invested in the default free asset and $u$ the percentage invested in the portfolio of pure risky assets. If we build a portfolio combining ordinary risky assets and the default free asset its rate of return is:

$$\tilde{R} = (1-u)\tilde{i} + u\sum x_j \tilde{R}_j \qquad [32]$$

while the rate of return of the equivalent portfolio in the asset-liability case is:

$$\tilde{r} = u\sum w_j \tilde{r}_j \qquad [\ 33\ ]$$

According to what we know equation [ 33 ] can also be written in the following form:

$$\tilde{R} - \tilde{i} = (1-u)(\tilde{i} - \tilde{i}) + u\sum w_j (\tilde{R}_j - \tilde{i}) \qquad [\ 34\ ]$$

or after some operations as: $\tilde{R} = (1-u)\tilde{i} + u\sum w_j \tilde{R}_j$ [ 35 ]

From [ 32 ] and [ 35 ] we get the same optimization problem[2] and therefore the composition of the optimal portfolio has to be the same. Hence:

$$\hat{x}_j = \hat{w}_j \qquad [\ 36\ ]$$

Notice that the optimization problem stemming from [ 35 ] for the asset-liability case lead us directly to the portfolios on the efficient frontier of an asset-liability strategy when riskless lending and borrowing are allowed (stated in equation [ 30 ] ).

## 5. THE ASSET-LIABILITY SECURITIES STRATEGY

### 5.1. One limit strategy

Let us have a closer look on how an asset-liability security strategy works, when lending and borrowing the liability-asset are allowed. Be a financial institution which liabilities have a present value equal to $L_0$. This financial institution implements an asset-liability strategy.

(1-α) is the percentage of initial budget invested in the liability-asset, and α the percentage invested in the risky portfolio. Therefore $(1-\alpha) > 0$ (and consequently $\alpha < 1$) states a long position in the liability-asset, that is to say

[2] We use the same procedure in the appendix (see 8.2., footnote 3) where we show its connection to an Elton and Gruber proposition.

"lending", and $(1-\alpha) < 0$ (and therefore $\alpha > 1$ ) denotes a short position in the liability-asset, namely "borrowing".

Be: $INV_{Lt}$ = investment in the liability-asset at time $t$.

$INV_{Pt}$ = investment in the risky portfolio at time $t$.

At the beginning of the strategy (t = 0) the investment budget is equal to the present value of the liabilities ($L_0$). It has to be distributed between the liability-asset and the risky portfolio according to the percentages (1-$\alpha$) and $\alpha$ . To invest in the risky portfolio requires to have previously calculated the weights of risky securities in this portfolio. More specifically: first we have to build the asset-liability securities, then calculate their efficient frontier, and finally calculate the separation portfolio parameters from the efficient frontier and the zero interest rate. Thus investments at $t = 0$ are:

$$INV_{L0} = (1-\alpha) L_0 \qquad [37]$$

$$INV_{P0} = \alpha\, L_0 \sum_{j=1}^{n} \hat{w}_j s_{j0} \qquad [38]$$

Notice that at the same time: $INV_{P0} = \alpha\, L_0$ [ 39 ]

because:

$$\sum_{j=1}^{n} \hat{w}_j s_{j0} = 1 \qquad [40]$$

which stems from $s_{j0} = 1$ and $\sum_{j=1}^{n} \hat{w}_j = 1$.

Future changes in the value of investment in risky portfolio will be caused by changes in $s_j$, that is the random variable in the equation of risky investment:

$$INV_{Pt} = \alpha\, L_0 \sum_{j=1}^{n} \hat{w}_j s_{jt} \qquad [41]$$

Be $\gamma_t$ the relative value of the strategy - that is to say the value of the asset-liability ratio - at time $t$. Its expression is:

$$\gamma_t = \frac{(1-\alpha)\, L_t + \alpha\, L_0 \sum_{j=1}^{n} \hat{w}_j s_{jt}}{L_t} \qquad [42]$$

that is to say:

$$\gamma_t = (1-\alpha) + \frac{\alpha\, L_0 \sum_{j=1}^{n} \hat{w}_j s_{jt}}{L_t} \qquad [43]$$

Notice that $\gamma_t \geq (1-\alpha)$ and $\gamma_0 = 1$.

The relative value of the asset-liability security strategy can be expressed as a linear combination of the relative values of the following two strategies:

• pure hedging strategy: it consists of investing all the budget in the asset-liability security. Therefore its relative value is always one.

• pure risky portfolio strategy: it consists of investing all the budget in the risky portfolio of asset-liability securities. Be $\Lambda$ the relative value of this strategy. Then:

$$\Lambda_t = \frac{L_0 \sum_{j=1}^{n} \hat{w}_j s_{jt}}{L_t} \qquad [44]$$

Hence:

$$\gamma_t = (1-\alpha) + \alpha\, \Lambda_t. \qquad [45]$$

It can be said that $(1\text{-}\alpha) > 0$ means a hedging position because:

$$(1\text{-}\alpha) > 0 \quad \Rightarrow \quad \gamma_t > (1-\alpha) \;\; \forall t \qquad [46]$$

and $(1\text{-}\alpha) < 0$ denotes a speculative position because:

$$(1-\alpha) < 0 \quad \Rightarrow \quad [\gamma_t > \Lambda_t \Leftrightarrow \Lambda_t > 1] \qquad [47]$$

$$(1-\alpha) < 0 \quad \Rightarrow \quad [\gamma_t < \Lambda_t \Leftrightarrow \Lambda_t < 1] \qquad [48]$$

Let us sum up the steps that have to be taken to implement an asset-liability security strategy:

i) build the liability-asset;

ii) define the risky asset-liability securities from the standard securities and the liabilities (of the institution);

iii) calculate the efficient frontier of risky asset-liability securities;

iv) calculate the separation portfolio of risky asset-liability securities; and

v) choose a linear combination between the separation portfolio and the liability-asset.

**5.2.. Two limit strategy**

Strategies considered in previous sections of this paper allow financial institutions to settle a lower limit for the value of the strategy, that is to say, a lower limit for its asset-liability ratio. To get this lower limit the institution has to place a percentage (1-$\alpha$) of its investment budget in the liability asset. The managers of a financial institution would probably be able to give up the higher values of the asset-liability ratio if that renunciation allowed them to finance the hedging strategy. In other words, they may accept an upper limit for their asset-liability ratio so as to finance their lower limit protection. In this section we develop a double limit strategy in which the upper limit finances the lower limit, and therefore the financial institution can invest its whole initial budget in the risky portfolio.

Let us call $P_0$ the initial value of the risky portfolio equal to the liabilities of the institution. That is to say:

$$P_0 = L_0 \qquad [49]$$

If the institution invested its whole budget in the risky portfolio, at time $t$ the value of the strategy (asset-liability ratio) would be:

$$\gamma_t = \frac{P_t}{L_t} \qquad [50]$$

Suppose that the institution wants a lower limit for its strategy (asset-liability ratio) equal to (1-α) and it is able to accept an upper limit. Suppose also that at $t = 0$ the institution takes a long position in the liability asset equal to $(1-\alpha)\cdot L_0$ , and that this position is financed through an equivalent short position in the risky portfolio, that is to say, $(1-\alpha)\cdot P_0$ . We have:

$$(1-\alpha)\cdot L_0 = (1-\alpha)\cdot P_0 \qquad [51]$$

On the other hand the institution invests its initial budget in the risky portfolio. At time $t$ the value of assets is $(1-\alpha)\cdot L_t + P_t$ while the value of liabilities is $(1-\alpha)\cdot P_t + L_t$ .

Therefore the relative value of the strategy (or asset-liability ratio) is:

$$\gamma_t = \frac{(1-\alpha)L_t + P_t}{(1-\alpha)P_t + L_t} \qquad [52]$$

It is straightforward to proof that: $lim_{P\to 0}\ \gamma_t = 1-\alpha$ [53]

and $lim_{P\to\infty}\ \gamma_t = \frac{1}{1-\alpha}$ [54]

As a consequence the two limit strategy becomes:

$$\frac{1}{1-\alpha} \geq \gamma \geq 1-\alpha \qquad [55]$$

and at $t = 0$ it is built by taking:

- a long position in $P_0$
- a long position in $(1\text{-}\alpha)\cdot L_0$
- a short position in $(1\text{-}\alpha)\cdot P_0$ ,

and the net investment at the initial moment is simply the initial budget $L_0$ .

## 6. PORTFOLIOS OF ASSET-LIABILITY SECURITIES VERSUS ASSET-LIABILITY PORTFOLIOS

### 6.1. Asset-liability portfolios

In this section we compare the properties of portfolios of asset-liability securities, which have been explained in sections 3 and 4, with the properties of asset-liability portfolios. We define an asset-liability portfolio as the result of combining a portfolio of risky securities with the liability asset.

Suppose we have an efficient portfolio of ordinary risky securities, calculated according to the Markowitz model. Relying on it we build an asset-liability security or, more precisely, an asset-liability portfolio following the process that we have applied to build the asset-liability security. We denote by $P$ the value of the portfolio of risky securities and by $p$ the value of the asset-liability portfolio. Then:

$$p_{jt} = \frac{P_{jt}}{L_{jt}} \qquad [56]$$

and

$$P_{j0} = L_{j0} \qquad [57]$$

$$p_{j0} = 1 \qquad [58]$$

Be $R$ the return of the portfolio of risky securities, $x_j$ the weight of security $j$ in this portfolio, and $r^*$ the return of the asset-liability portfolio. For what we know, we can write:

$$\tilde{R} = \sum_{j=1}^{n} x_j \tilde{R}_j \qquad [\ 59\ ]$$

$$\tilde{r}^* = \sum_{j=1}^{n} x_j \tilde{R}_j - \tilde{i} \qquad [\ 60\ ]$$

Calculating the expected value, $E(r^*)$, and the variance, $d^{*2}$, of the asset-liability portfolio from [ 60 ] we obtain:

$$E(r^*) = \sum_{j=1}^{n} x_j E(R_j) - E(i) \qquad [\ 61\ ]$$

and

$$d^{*2} = \sum_{j=1}^{n} x_j^2 \sigma_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} x_j x_q \sigma_{jq} + \sigma_i^2 - 2 \sum_{j=1}^{n} x_j \sigma_{ji} \qquad [\ 62\ ]$$

Comparing [ 61 ] to [ 22 ] and [ 62 ] to [ 23 ] we notice that the expected value and the variance of the portfolio of asset-liability securities and the asset-liability portfolio have the same values when the weights of their risky assets are the same.

### 6.2. Comparison of properties

Let us compare the properties of the portfolios of asset-liability securities with the properties of asset-liability portfolios. Concerning their mean and variance we have obtained that the expected values and variances of both portfolios have the same values when the weights of their risky assets are the same.

Their main difference lies on the calculus of the efficient frontier of risky securities. The asset-liability securities strategy takes into account the incidence of the default free interest rate on the returns of risky securities. Namely, the default free interest rate is handled as the random variable that it is indeed. Hence, as we have seen, the calculus of the efficient frontier of risky securities pursues to minimize equation [ 24 ], subject to [ 25 ] and [ 26 ]. It is important to point out that the incidence of the random default free interest rate appears on the following term in [24]:

$$\sigma_i^2 - 2\sum_{j=1}^{n} w_j \sigma_{ji}$$

On the other hand an asset-liability portfolio does not take into account the relationship between default free interest rate and returns on risky securities. Consequently, the calculus of the efficient frontier consists of minimizing:

$$Min\ \sigma^2 = \sum_{j=1}^{n} x_j^2 \sigma_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} x_j x_q \sigma_{jq} \qquad [\ 63\ ]$$

subject to:

$$\sum_{j=1}^{n} x_j E(R_j) = R^* \qquad [\ 64\ ]$$

$$\sum_{j=1}^{n} x_j = 1 \qquad [\ 65\ ]$$

where the term [ $\sigma_i^2 - 2\sum_{j=1}^{n} w_j \sigma_{ji}$ ] is omitted. Subsequently one point of the efficient frontier of risky securities is combined with the liability-asset in both strategies.

Then it becomes clear that an asset-liability portfolio does not lead to optimal combinations of risky securities. However, as section 6.3. states, when the covariances between the returns of risky assets and default freee interest rate are constant, both optimization problems become identical.

Furthermore, it is possible to work directly with an asset-liability portfolio if the random relationship with risky securities and the default free interest rate is taken into account. That is to say, in the optimization problem expression [ 62 ] becomes the objetive function and equation [ 61 ] the restriction, although this procedure requires the same data and calculus as the asset-liability security case.

### 6.3. The efficient frontier of risky asset-liability securities when the covariances between the returns on assets and default free interest rate are constant

Suppose that the covariances between the returns on assets and default free interest rate are constant, that is to say, all of them have the same value. Be $\sigma_i^*$ the value of this covariance. In consequence the objective function of our restricted optimization problem becomes:

$$d^2 = \sum_{j=1}^{n} w_j^2 \sigma_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} w_j w_q \sigma_{jq} + \sigma_i^2 - 2\sigma_i^* \sum_{j=1}^{n} w_j \qquad [\ 66\ ]$$

and since $\sum_{j=1}^{n} w_j = 1$ [ 26 ], the optimization problem adopts the following expression:

$$Min\ d^2 = \sum_{j=1}^{n} w_j^2 \sigma_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} w_j w_q \sigma_{jq} + \sigma_i^2 - 2\sigma_i^* \qquad [\ 67\ ]$$

subject to:

$$\sum_{j=1}^{n} w_j E(R_j) = R^* \qquad [\ 68\ ]$$

$$\sum_{j=1}^{n} w_j = 1 \qquad [\ 69\ ]$$

Writing the Lagrangian, taking the derivatives and equating then to zero we obtain:

$$2w_1(\sigma_1)^2 + 2w_2\sigma_{12} + ... + 2w_j\sigma_{1j} + ... + 2w_n\sigma_{1n} + \lambda E(R_1) + \theta = 0$$

$$2w_1\sigma_{21} + 2w_2(\sigma_2)^2 + ... + 2w_j\sigma_{2j} + ... + 2w_n\sigma_{2n} + \lambda E(R_2) + \theta = 0$$

$$... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ...$$

$$2w_1\sigma_{j1} + 2w_2\sigma_{j2} + ... + 2w_j(\sigma_j)^2 + ... + 2w_n\sigma_{jn} + \lambda E(R_j) + \theta = 0$$

$$... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ...$$

$$2w_1\sigma_{n1} + 2w_2\sigma_{n2} + ... + 2w_j\sigma_{nj} + ... + 2w_n(\sigma_n)^2 + \lambda E(R_n) + \theta = 0$$

$$w_1E(R_1) + w_2E(R_2) + ... + w_jE(R_j) + ... + w_nE(R_n) = R^*$$

$$w_1 + w_2 + ... + w_j + ... + w_n = 1 \qquad [70]$$

which is exactly the set of equations that stems from the Markowitz model. Remember that the weights of standard risky securities in the asset-liability portfolio are the ones of the Markowitz case. So we have proved that when the covariances between the returns on risky assets and default free interest rate are constant, the weights of risky securities in an efficient portfolio of asset-liability securities and the weights in an asset-liability portfolio have the same values. Consequently:

$$\sigma_{ji} = \sigma_i^* \;\; \forall j \Rightarrow w_j = x_j \qquad [71]$$

## 6.4. An interpretation for constant covariances

A constant covariance between securities rate of return and default free interest rate implies constant beta between any security and the default free interest rate. This property stems directly from:

$$\beta_{ji} = \frac{\sigma_{ji}}{\sigma_i^2} \qquad [72]$$

from where:

$$\sigma_{ji} = \sigma_i^* \;\; \forall j \Rightarrow \beta_{ji} = \beta_i^* \;\; \forall i \qquad [73]$$

The reciprocal property is straightforward.

On the other hand, it can be said that a sufficient, although not necessary condition for constant covariances is the independence between risk premia embeded in the security asset returns and default free interest rate. The risk premium is $(\tilde{R}-\tilde{i})$ , and the independence between this premium and the default free interest rate requires:

$$\rho(\tilde{R}-\tilde{i},\tilde{i}) = 0 \qquad [\,74\,]$$

that is to say:

$$Cov(\tilde{R}-\tilde{i},\tilde{i}) = 0 \qquad [\,75\,]$$

Taking into account the additive property of covariances, we can write:

$$Cov(\tilde{R}-\tilde{i},\tilde{i}) = Cov(\tilde{R},\tilde{i}) - \sigma_i^2 \qquad [\,76\,]$$

and for [ 70 ] and [ 71 ]:

$$Cov(\tilde{R},\tilde{i}) = \sigma_i^2 \qquad [\,77\,]$$

which also causes that:

$$\beta_i = 1 \qquad [\,78\,]$$

To sum up, constant betas is a necessary and sufficient condition for constant covariances. Besides a sufficient condition is the independence between risk premia and default free interest rate, that leads to a constant beta equal to one.

## 7. CONCLUDING REMARKS

In this paper we have studied some features of asset-liability strategies. Our starting point has been the concept of asset-liability security: an asset-liability security consists of a quotient between a risky security, which may be of variable or fixed income, and a default free security. We call the strategies that use these securities "asset-liability security strategies". With the help of the concept of asset-liability security the following conclusions have been reached:

• Using asset-liability securities it is possible to build an efficient frontier of risky securities that takes into account the effects of the random default free interest rate on efficient portfolios.

• The separation theorem also holds in an asset-liability context. This conclusion relies on two previous results (Margrabe, 1978): i) in such a framework the risk free security is a relative security which consists of a quotient of the default free asset with itself; and ii) the return of this relative asset is zero because its value is always one. Hence, an asset-liability separation portfolio can be built combining the efficient frontier of risky asset-liability securities with the risk free relative asset.

• The asset-liability separation portfolio does not change if the default free interest rate changes because using this methodology we come up with a relative risk free interest rate which is always zero. Nevertheless the asset-liability portfolio changes when the expectations about the default free interest rate change, because on the efficient frontier of risky securities the weights of risky asset-liability securities depend on the covariances between returns on risky securities and default free interest rate.

• We have also developed a two limit asset-liability strategy. The strategies previously considered in this paper were lower limit asset-liability strategies. The two limit strategy allows to finance the lower limit protection with the funds that come from incorporating an upper limit to the asset liability value.

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## 8. APPENDIX

### 8.1. The efficient frontier of risky asset-liability securities when short sales are not allowed

If short sales are not allowed the condition $x_j \geq 0 \ \ \forall j$ has to be added to the problem. Thus the Kuhn and Tucker conditions have to be applied in order to find the optimum. Then the equation set [ 29 ] becomes:

$$2w_1\sigma_{j1} + 2w_2\sigma_{j2} + \dots + 2w_j(\sigma_j)^2 + \dots + 2w_n\sigma_{jn} + \lambda E(R_j) + \theta \geq 2\sigma_{ji} \quad \forall j$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$[\, 2w_1\sigma_{j1} + 2w_2\sigma_{j2} + \dots + 2w_j(\sigma_j)^2 + \dots + 2w_n\sigma_{jn} + \lambda E(R_j) + \theta - 2\sigma_{ji} \,]\, w_j = 0 \quad \forall j$$

$$w_1E(R_1) + w_2E(R_2) + \dots + w_jE(R_j) + \dots + w_nE(R_n) - R^* = 0$$

$$w_1 + w_2 + \dots + w_j + \dots + w_n = 1$$

$$w_j \geq 0 \quad \forall j \qquad [\text{A.1}]$$

where it can be noticed again that the difference between this case and the standard one lies on the coefficients $\sigma_{ji}$ which do not appear in the standard case.

### 8.2. Coincidence of separation portfolios when covariances are constant

Proposition: The separation portfolio of asset-liability securities when the covariances between the returns of risky securities and the default free interest rate are constant has the same composition as the separation portfolio of risky securities in the standard case:

$$\sigma_{qi} = \sigma_i^* \ \ \forall q \ \Rightarrow \ \hat{x}_q = \hat{w}_q \qquad [\text{A.2}]$$

(where the upper-script ^ denotes optimal value).

Proof: Be (1-$u$) the percentage of the risk free asset in the porfolio of the standard case and, at the same time, the percentage of the liability-asset in the asset-liability securities case. Thus the percentage of the risky portfolio in both cases is $u$ . The optimazation problems can be written as follows:

Standard case:

$$Min\ \sigma^2 = u^2\left[\sum_{j=1}^{n} x_j^2\sigma_j^2 + \sum_{\forall j\neq q}\sum_{j=1}^{n}\sum_{q=1}^{n} x_j x_q \sigma_{jq}\right] \qquad [\text{A.3}]$$

subject to: $$(1-u)i + u\sum_{j=1}^{n} x_j E(R_j) = R^* \qquad [\text{A.4}]$$

$$\sum_{j=1}^{n} x_j = 1 \qquad [\text{A.5}]$$

Asset-liability case:

$$Min\ d^2 = u^2\left[\sum_{j=1}^{n} w_j^2\sigma_j^2 + \sum_{\forall j\neq q}\sum_{j=1}^{n}\sum_{q=1}^{n} w_j w_q \sigma_{jq} + \sigma_i^2 - 2\sigma_i^*\right] \qquad [\text{A.6}]$$

subject to: $$(1-u)i + u\sum_{j=1}^{n} w_j E(R_j) = R^* \qquad [\text{A.7}]$$

$$\sum_{j=1}^{n} w_j = 1 \qquad [\text{A.8}]$$

Now let us parametrize the value of $u$ giving a concrete value to that variable:

$$u = \mu \qquad [\text{A.9}]$$

The objective functions take the following forms:

$$\sigma^2 = \mu^2\left[\sum_{j=1}^{n} x_j^2\sigma_j^2 + \sum_{\forall j\neq q}\sum_{j=1}^{n}\sum_{q=1}^{n} x_j x_q \sigma_{jq}\right] \qquad [\text{A.10}]$$

$$d^2 = \mu^2\left[\sum_{j=1}^{n} w_j^2\sigma_j^2 + \sum_{\forall j\neq q}\sum_{j=1}^{n}\sum_{q=1}^{n} w_j w_q \sigma_{jq}\right] + \mu^2\left[\sigma_i^2 - 2\sigma_i^*\right] \qquad [\text{A.11}]$$

We notice that both optimization problems become identical except for the constant $\mu^2\left[\sigma_i^2 - 2\sigma_i^*\right]$ placed in the objective function of the asset-liability case [ A.11 ]. Thus their solutions must be identical:

$$u = \mu \quad \Rightarrow \quad \hat{w}_j = \hat{x}_j \qquad [\text{ A.12 }]$$

It can be shown that the values of the percentages of risky securities (that is to say $x$ and $w$ ) are independent in both problems from the values of $u$. To prove that assertion it is just necessary to take into account that the variable $u$ can be eliminated by substituing its value from the first restriction in the objective function[3]. Consequently, if the optimal values of $x_j$ and $w_j$ are on the one hand equal for a specific value of $u$, and on the other hand independent from $u$, they are always equal to each other:

$$\hat{w}_j = \hat{x}_j \qquad [\text{ A.13 }] \text{ q.e.d.}$$

Notice that if short sales are not allowed (that is to say, if the optimization problem incorporates the restrictions $x_j \geq 0$ and $w_j \geq 0$ ), this proof does not change.

[3] Clearing $u$ in equation [ A.4 ] and substituing its value in the objective function we obtain a new objective function:

$$Min \frac{\sigma_p^2}{\left[E(R_p) - i\right]^2}$$

where $\sigma_p^2 = \sum_{j=1}^{n} x_j^2 \sigma_j^2 + \sum_{\forall j \neq q} \sum_{j=1}^{n} \sum_{q=1}^{n} x_j x_q \sigma_{jq}$

and $E(R_p) = \sum_{j=1}^{n} x_j E(R_j)$.

This result leads immediatey to the Elton and Gruber (1995, 98-104) proposition to find the composition of the efficient portfolio which consists of maximazing $\frac{E(R_p) - i}{\sigma_p}$.

An equivalent result is found for the asset-liability case.