

Coalition Formation in a Contest Game with Three Heterogenous Players

Marc Vorsatz*

April 7, 2003

Abstract

We analyze the incentives for cooperation of three players differing in their efficiency of effort in a contest game. We concentrate on the non-cooperative bargaining foundation of coalition formation, and therefore, we adopt a two-stage model. In the first stage, individuals form coalitions following a bargaining protocol similar to the one proposed by Gul [7]. Afterwards, coalitions play the contest game of Esteban and Ray [5] within the resulting coalition structure of the first stage. We find that the grand coalition forms whenever the distribution of the bargaining power in the coalition formation game is equal to the distribution of the relative efficiency of effort. Finally, we use the case of equal bargaining power for all individuals to show that other types of coalition structures may be observed as well.

Journal of Economic Literature Classification Numbers: D72, D74

Keywords: Contest, Rent-Seeking, Coalition Formation.

*I am very grateful to Joan Esteban for indicating the idea of the paper. I want to thank Francis Bloch, Jordi Massó, and Santiago Sánchez-Pagés for their constant help, patience and encouragement. This research was undertaken with support from the fellowship 2001FI 00451 of the Generalitat de Catalunya and HI2001-0039 of the Ministerio de Ciencia y Tecnología of Spain. The usual disclaimer applies. Address of contact: International Doctorate in Economic Analysis, Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallés, Spain. E-Mail: mvorsatz@idea.uab.es

1 Introduction

A contest is a socio-economic environment in which players spend valuable resources in order to raise their probabilities of winning a fixed prize. In this paper, we analyze the incentives for cooperation of three players in the presence of the strong non-cooperative threat of a contest. In particular, we consider players who differ in their efficiency of effort or represent exogenously given groups of different size. Our main result states that a society-wide agreement may not be reached if the discrepancy between the distribution of the exogenously given group sizes and the distribution of the relative bargaining power is too high.

Economists study contest games since the seminal work on rent-seeking by Tullock [11]. In the rent-seeking literature the individual expenditure is usually interpreted as lobbying effort in terms of time or money and the prize is taken to be a monopoly right or a license. But “lobbying” is far from being the only example, because contest games have been applied to patent races by Pérez-Castrillo and Verdier [8], to market share competition by Schmalensee [9], and to financial institutions and money by Shapley and Shubik [10]. One of the latest developments is due to Esteban and Ray [5] who concentrate on the relationship between the distribution of society into interest groups and the level of conflict defined as the social loss induced by the non-productive efforts.

Our contribution is to consider coalitions among players in a contest game. This extension is of special interest, because there is empirical evidence for the formation of coalitions in contest environments. Consider for instance a country in transition to democracy which

is split into ethnic or religious groups. Often, all groups know that socially it would be best to agree on a new constitution and divide the political power, but finally, negotiation fails and a conflict between the groups emerges. A possible explanation is that if one group has a high bargaining power but is relatively small in size, then the other groups prefer to stop negotiations and fight for their political influence instead of signing an agreement in favor of the small group. The next example is an application to patent races. We observe that firms form joint ventures in research and development in order to share their knowledge and become more efficient. Said differently, firms can raise through cooperation the probability of making the next invention which gives them access to a monopoly for at least some time. Finally, remember the latest Soccer World Cup in Japan and South Korea which is just one example of administrations bidding jointly for the concession of a big cultural or sporting event.

Since we want to be explicit about the non-cooperative bargaining foundations of coalition formation, we use the common approach of dividing the model into two stages. In the first stage, players form coalitions and negotiate about the sharing rules of the cooperative payoffs. We use a random protocol which is a modified version of the partnership game by Gul [7]. In the second stage, coalitions play a contest game similar to the one proposed by Esteban and Ray [5] within the resulting coalition structure of the first stage. Finally, coalitions divide their obtained payoffs according to the sharing rules negotiated in the coalition formation game.

We solve the two stages by backward induction and determine at first how much expected utility each coalition can assure itself in every possible coalition structure. Our

natural prediction for this value is the unique Nash equilibrium payoff of the contest game played within the considered coalition structure. We prove in Proposition 1 and specially in Corollary 1 that the expected utility of a player who faces two single players is different from the expected utility he would get if the other two players have formed a coalition. Because of this externality the contest game is a partition function game. In the next step, we solve the coalition formation game for the equilibrium coalition structures and the equilibrium expected utilities using stationary strategies. We show in Proposition 2 that if the relative efficiency of effort is distributed in the same way as the relative bargaining power, then the grand coalition forms. Moreover, if players are sufficiently patient, then every player receives in equilibrium his relative efficiency of effort. We prove in Proposition 3 that the grand coalition is no longer the unique equilibrium coalition structure if every player has the same bargaining power. Therefore, it is possible to observe a strictly positive level of conflict in equilibrium.

Few papers have analyzed the question of coalition formation in contest games. Baik and Lee [1] and [2] study a rent-seeking model with a linear cost function. They use the open membership game as coalition formation game and obtain that coalitions with about fifty percent of the individuals are formed. Esteban and Sákovics [6] consider a model of repeated conflict and bilateral coalition formation. For the case of three individuals their results predict that one coalition of size two forms. Bloch et al. [4] study the endogenous formation of coalitions in a simplified version of the contest game of Esteban and Ray [5] with a quadratic cost function. Their main result states that the grand coalition is the unique equilibrium coalition structure of the size announcement game by Bloch [3]. Since a

finite number of individuals are assumed to be identical in their model, two individuals who belong to the same coalition receive the same share of the coalitional payoff. Therefore, the prize is divided equally among all individuals.

The remainder of the paper is structured as follows: In the next Section, we introduce the contest game and derive the partition function game. In Section 3 we solve the bargaining model, and in Section 4, we discuss some of our modelling choices.

2 Contest as a Partition Function Game

Consider three individuals who fight over a prize with a value normalized to 1. Let $N = \{1, 2, 3\}$ be the set of individuals. A coalition C is a nonempty subset of N . A coalition structure π is a partition of N . The set of all coalition structures is denoted by Π . Let $V(C, \pi)$ be the worth of coalition C in π .

We start by describing the expected utility maximization problem of the generic individual in the coalition structure $\pi = \{1 | 2 | 3\}$. Every individual i makes at the same time and independently of the others a non-productive effort $r_i \in \mathbb{R}_+$. The efficiency of effort of individual i is common knowledge and denoted by $n_i > 0$. We order the individuals by assuming that $n_1 \geq n_2 \geq n_3$ and normalize the parameter vector $\mathbf{n} = (n_1, n_2, n_3)$ to $\sum_{i=1}^3 n_i = 1$. We denote the three-dimensional unit simplex by Δ^3 and define the total level of conflict as $R = \sum_{j=1}^3 n_j r_j$. The probability that individual i wins the contest is a function $p_i : \mathbb{R}_+ \times \Delta^3 \rightarrow [0, 1]$ satisfying the condition $\sum_{j=1}^3 p_j(\mathbf{r}; \mathbf{n}) = 1$ for all $(\mathbf{r}; \mathbf{n}) \in \mathbb{R}_+ \times \Delta^3$. We consider throughout the probability measure of the proportional

form

$$p_i(\mathbf{r}; \mathbf{n}) = \frac{n_i r_i}{\sum_{j=1}^3 n_j r_j},$$

and use the convention that if $r_j = 0$ for all j , then $p_i(\mathbf{r}; \mathbf{n}) = n_i$. Moreover, let the cost of effort be equal to the level of effort.¹ Therefore, the expected utility maximization problem for individual i given r_{-i} is to choose $r_i \in \mathbb{R}_+$ in order to

$$\max_{r_i \geq 0} \left(\frac{n_i r_i}{\sum_{j=1}^3 n_j r_j} - r_i \right). \quad (1)$$

This part of our model is similar to a special case of the contest model proposed by Esteban and Ray [5].² In their model, the cost of effort is a convex function, whereas we consider the linear cost function suggested by Tullock [11]. The difference between our model and the latter one stems from the fact that Tullock's analysis is restricted to the case of identical individuals.

Consider now the coalition structure $\pi = \{ij | k\}$. We assume that whenever a coalition forms, then all members of the coalition merge.³ Therefore, the coalition $\{i, j\}$ has the following expected utility maximization problem: given r_k , choose $r_{ij} \in \mathbb{R}_+$ in order to

$$\max_{r_{ij} \geq 0} \left(\frac{(n_i + n_j) r_{ij}}{(n_i + n_j) r_{ij} + n_k r_k} - r_{ij} \right). \quad (2)$$

¹We make this assumption for purely technical reasons, because if we had considered a convex cost function, then the calculation of the Nash equilibrium in the coalition structure $\pi = \{1 | 2 | 3\}$ would have become far too complicated. The disadvantage of using a linear cost function is that it implies the existence of corner equilibria.

²Our interpretation of the parameter vector \mathbf{n} is not the same as the one of Esteban and Ray [5], because they regard n_i as the relative size of the exogenously given group i . We return to this point in the last Section of the paper.

³It would also be reasonable to define the objective function of the coalition $\{i, j\}$ by assuming that individuals do not merge, an argument which has been brought forward by Bloch et al. [4]. In the last Section of our paper we provide evidence that our main results are invariant to our modeling choice.

Accordingly, the expected utility maximization problem for individual k is: given r_{ij} , choose $r_k \in \mathbb{R}_+$ in order to

$$\max_{r_k \geq 0} \left(\frac{n_k r_k}{(n_i + n_j) r_{ij} + n_k r_k} - r_k \right). \quad (3)$$

We use the convention that if $r_k = r_{ij} = 0$, then $p_{ij}(r_{ij}, r_k; \mathbf{n}) = n_i + n_j$ and $p_k(r_{ij}, r_k; \mathbf{n}) = n_k$.⁴

So far we have described two of the three possible types of coalition structures. It is optimal for the grand coalition to put zero effort, because it receives the private good anyway. Therefore, it has a worth of one.

Suppose that the coalition structure π is the outcome of the coalition formation game. Since the contest game is a simultaneous move game, we take the Nash equilibrium of the contest game played within π as the natural prediction of the effort vector. Proposition 1 characterizes the unique Nash equilibrium for every non-trivial coalition structure.

Proposition 1 (a) *The unique Nash equilibrium $\mathbf{r} = (r_1^*, r_2^*, r_3^*)$ in the coalition structure $\pi = \{1 | 2 | 3\}$ is as follows: (a.1) if $n_3 > 0.25$, then $r_i^* = \frac{2n_j n_k (n_i n_j + n_i n_k - n_j n_k)}{(n_i n_j + n_i n_k + n_j n_k)^2}$ for all i ; (a.2) if $n_3 \leq 0.25$, then $r_3^* = 0$ and $r_1^* = r_2^* = \frac{n_1 n_2}{(n_1 + n_2)^2}$. (b) *The unique Nash equilibrium (r_{ij}^*, r_k^*) in the coalition structure $\pi = \{ij | k\}$ is $r_{i,j}^* = r_k^* = (n_i + n_j) n_k$.**

Proof: See the Appendix. ■

We derive the partition function game V from Proposition 1 by plugging the equilibrium efforts for every type of coalition structure into the corresponding objective functions.

⁴We are aware that the probabilities $p_{ij}(r_{ij}, r_k; \mathbf{n})$ and $p_k(r_{ij}, r_k; \mathbf{n})$ have not been defined formerly, but nonetheless, this failure should not cause any kind of misunderstanding.

Corollary 1 *The partition function game V is equal to*

$$\begin{aligned}
V(123, \{123\}) &= 1 \\
V(ij, \{ij | k\}) &= (n_i + n_j)^2 \\
V(k, \{ij | k\}) &= n_k^2 \\
V(k, \{i | j | k\}) &= \begin{cases} \left(1 - \frac{2n_i n_j}{n_i n_j + n_i n_k + n_j n_k}\right)^2 & \text{if } n_3 > 0.25 \\ \left(\frac{n_k}{n_1 + n_2}\right)^2 & \text{if } n_3 \leq 0.25 \text{ and } k \neq 3 \\ 0 & \text{if } n_3 \leq 0.25 \text{ and } k = 3. \end{cases}
\end{aligned}$$

3 The Coalition Formation Game

Since V summarizes all necessary information of the contest game, we are ready to address the question of coalition formation. Our coalition formation game is inspired by the partnership game of Gul [7]. The game is parameterized by a common discount factor $0 < \delta < 1$ and an exogenously given probability vector $\mathbf{q} = (q_1, q_2, q_3)$ which represents the relative bargaining power of the players.

The Bilateral Bargaining Game

Period 0:

Players decide sequentially and publicly whether to stay or exit the game according to the ordering 1, 2, 3.⁵ Let S_0 be the set of players that decide to stay and denote by s_0 the cardinality of S_0 . If $s_0 \leq 1$, then the contest game is played within the coalition structure $\{1 | 2 | 3\}$. If $s_0 \geq 2$, then a randomly selected bilateral meeting among the players in S_0 takes place. We assume that every possible meeting occurs with equal probability. Suppose

⁵A change in the ordering of the players does not have any influence on Proposition 2. But, a different ordering requires slight modifications in Proposition 3. Nonetheless, the general insights do not change.

that i and j meet each other. Player i is chosen with probability $q_{i,j} = \frac{q_i}{q_i+q_j}$ to make an offer $x_{i,j}^0 \in \mathbb{R}_+$ which can be accepted or rejected by j . The offer describes j 's share of the payoff $V(ij, \{ij|k\})$. If j rejects $x_{i,j}^0$, then we set $\pi^0 = \{1|2|3\}$ and pass to the next period. If j accepts $x_{i,j}^0$, then the coalition $\{i,j\}$ forms and the actual coalition structure becomes $\pi^0 = \{ij|k\}$. If $k \notin S_0$, then the process of coalition formation stops and the contest game is played within the coalition structure π^0 . If $k \in S_0$, then we pass to the next period.

Period t :

The game arrives at period t if (a) $\pi^{t-1} = \{1|2|3\}$ and $s_{t-1} \geq 2$, or if (b) $\pi^{t-1} = \{ij|k\}$ and $s_{t-1} = 3$. Players in S_{t-1} decide sequentially and publicly according to the ordering 1, 2, 3 restricted to S_{t-1} whether to stay or to leave the game. Let S_t be the set of players that decide to stay and denote by s_t the cardinality of S_t .

Suppose that $\pi^{t-1} = \{1|2|3\}$. If $s_t \leq 1$, then the contest game is played within the coalition structure π^{t-1} . If $s_t \geq 2$, then a randomly selected bilateral meeting among players in S_t takes place. Every possible meeting occurs with equal probability. Suppose that i and j meet each other. Player i is chosen with probability $q_{i,j} = \frac{q_i}{q_i+q_j}$ to make an offer of $x_{i,j}^t \in \mathbb{R}_+$ which can be accepted or rejected by j . If j rejects $x_{i,j}^t$, then we set $\pi^t = \pi^{t-1}$ and pass to the next period. If j accepts $x_{i,j}^t$, then the coalition $\{j,k\}$ forms and the actual coalition structure becomes $\pi^t = \{ij|k\}$. If $k \notin S_t$, then the process of coalition formation stops and the contest game is played within the coalition structure π^t . If $k \in S_t$, then we pass to the next period.

Suppose that $\pi^{t-1} = \{ij|k\}$. If $s_t < 3$, then the contest game is played within the coalition structure π^{t-1} . If $s_t = 3$, then player i , who represents the coalition $\{i, j\}$ and is the first in the ordering between i and j generated by 1,2,3, and player k meet. Player i is chosen with probability $q_{ij,k} = q_i + q_j$ to make an offer $x_{ij,k}^t \in \mathbb{R}_+$ which can be accepted or rejected by k . The offer describes k 's share of the payoff $V(123, \{123\})$. If k rejects $x_{ij,k}^t$, then we set $\pi^t = \pi^{t-1}$ and pass to the next period. If k accepts $x_{ij,k}^t$, then the grand coalition forms and payoffs are assigned accordingly.

Figure 1 below represents the Bilateral Bargaining game for $t = 0$ and $t = 1$.

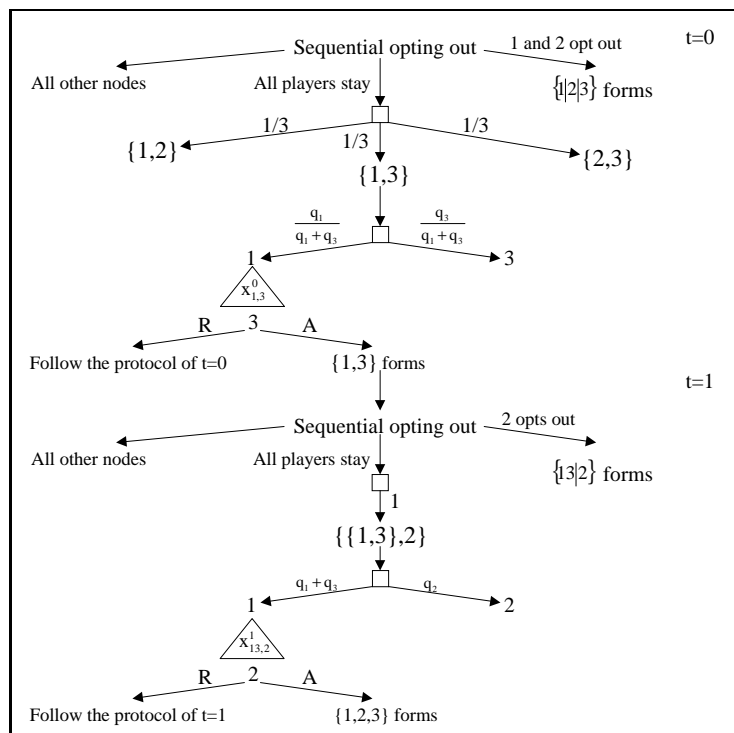


Figure 1: The Bilateral Bargaining Game

The main difference between our coalition formation game and the partnership game is that we allow players to exit the bargaining process. This is the reason why in the partnership game the grand coalition forms for any superadditive game in characteristic function form, whereas we may observe in equilibrium other coalition structures as well. The other differences are of minor importance, e.g. the partnership game is defined in continuous time and analyzed for the vector $\mathbf{q} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

We would like to characterize the equilibrium coalition structures and utilities for any arbitrary \mathbf{q} . Unfortunately, our parameter space would be enlarged too much, because our results have to rely partly on a graphical analysis even for a fixed vector \mathbf{q} . Therefore, we analyze the bilateral bargaining game for the probability vectors (n_1, n_2, n_3) and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The first probability vector is focal, because it reflects symmetry between bargaining power and efficiency of effort, whereas the second probability vector is the one used by Gul [7].

Proposition 2 *Suppose that $\mathbf{q} = \mathbf{n}$. For all $\delta < 1$, the bilateral bargaining game has an unique stationary subgame perfect equilibrium outcome. Let $U_i(\delta)$ be the expected utility of player i in the equilibrium corresponding to δ . Then $\lim_{\delta \rightarrow 1} U_i(\delta) = n_i$. Moreover, the grand coalition is the unique equilibrium coalition structure.*

Proof: See the Appendix. ■

Proposition 2 states that if $\mathbf{q} = \mathbf{n}$, then no individual opts out in equilibrium. But this result does not hold any longer for different probability vectors \mathbf{q} . In particular, this is the case for $\mathbf{q} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Since the calculations for characterizing the stationary subgame perfect equilibrium outcome of the corresponding bilateral bargaining game become much

longer, we refer to the author's homepage for a complete proof of the next Proposition.⁶

Here we provide a graphical overview of its statement.

Proposition 3 *Suppose that $\mathbf{q} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The grand coalition is an equilibrium coalition structure (a) with probability 1 if and only if $n_1 \leq 0.578$ and (a.1) $n_3 > 0.25$, or (a.2) $0.184 \leq n_3 \leq 0.25$ and $\frac{1}{2}(n_2 + n_3)^2 \geq \left(\frac{n_2}{n_1+n_2}\right)^2$, or (a.3) $0.184 \leq n_3 \leq 0.25$ and $\frac{1}{3} \geq \left(\frac{n_1}{n_1+n_2}\right)^2$; (b) with probability $\frac{2}{3}$ if and only if (b.1) $n_1 > 0.578$ and $n_3 \geq 0.184$, or (b.2) $n_1 \leq 0.578$, $n_2 \geq 0.184$, $n_3 < 0.184$ and $\frac{2}{9} + \frac{2}{9}(n_1 + n_2)^2 - \frac{1}{9}n_3^2 \geq \left(\frac{n_1}{n_1+n_2}\right)^2$; (c) with probability $\frac{1}{3}$ if and only if $n_1 > 0.578$, $n_2 \geq 0.184$, $n_3 < 0.184$ and (c.1) $\frac{1}{9} - \frac{2}{9}n_3^2 - \frac{2}{9}n_1n_2 + \frac{1}{9}n_2^2 + \frac{4}{9}n_2n_3 - \frac{2}{9}n_1^2 \geq 0$, or (c.2) $\frac{1}{2}(n_2 + n_3)^2 \geq \left(\frac{n_2}{n_1+n_2}\right)^2$; (d) with probability 0 in all other cases.*

In Figure 2, the set of points (n_1, n_2, n_3) fulfilling the constraints $n_1 \geq n_2 \geq n_3$ and $n_1 + n_2 + n_3 = 1$ are the ones lying within the triangle indicated by the thicker lines. The grand coalition forms for sure for all combinations of points (n_1, n_2, n_3) lying in non-shaded area of the triangle. The lightly grey shaded area within the triangle corresponds to the set of points (n_1, n_2, n_3) for which the grand coalition forms with probability $\frac{2}{3}$. The grand coalition forms with probability $\frac{1}{3}$ for the set of points (n_1, n_2, n_3) lying in the darkly grey shaded area within the triangle. Finally, the black shaded area within the triangle corresponds to set of points (n_1, n_2, n_3) for which the grand coalition does not form.⁷

⁶The current homepage is <http://idea.uab.es/~mvorsatz/>.

⁷Proposition 3 is silent on the question which coalition structure may be observed when the grand coalition does not form. From the proof of Proposition 3 it becomes clear that in some parts of the black shaded area the coalition structure $\pi = \{ij|k\}$ forms, whereas in other parts the coalition structure $\pi = \{1|2|3\}$ is the unique equilibrium coalition structure. Hence, any type of coalition structure may be sustained in equilibrium.

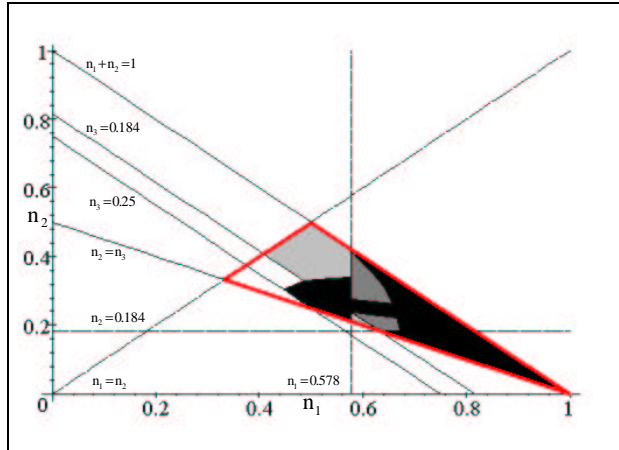


Figure 2: Equilibrium Coalition Structures

4 Discussion

It is difficult to generalize our findings to the case of $n > 3$ players, because we cannot write the worth of a coalition in a coalition structure as a function of the size of coalitions. In the literature this special function is termed “valuation”. In order to a valuation function, one has to concentrate on the case of identical individuals as it has been done by Bloch et al. [4]. This is the main reason why we restrict our analysis to the case of three individuals.

We turn next to the interpretation of the parameter vector \mathbf{n} . So far we have considered a model with three individuals who differ in the efficiency of effort. Esteban and Ray [5] assume that the prize is an excludable public good and define the parameter n_i as the relative size of the exogenously given group i . Following their assumption that all individuals who belong to the same group i are enforced by a binding agreement to make the same level of effort r_i , we can interpret equation (1) as the expected utility maximization problem of the representative individual of group i . Accordingly, if two groups i and j form

a coalition, then the relative group size of coalition $\{i, j\}$ becomes $n_i + n_j$. Hence, equation (2) states the expected utility maximization problem of the representative individual of coalition $\{i, j\}$.

Finally, we want to introduce a different objective function for coalition $\{i, j\}$. Bloch et al. [4] analyze a model with a finite number of homogeneous individuals. In their model individual i and j do not merge after the formation of coalition $\{i, j\}$ and decide cooperatively on the optimal effort levels. The new expected utility maximization problem of coalition $\{i, j\}$ is to take r_k as given and to choose r_i and r_j in order to

$$\max_{r_i, r_j \geq 0} \left(\frac{n_i r_i + n_j r_j}{n_i r_i + n_j r_j + n_k r_k} - r_i - r_j \right). \quad (4)$$

Similarly, the new expected utility maximization problem for individual k is to take r_i and r_j as given and to choose r_k in order to

$$\max_{r_k \geq 0} \left(\frac{n_k r_k}{n_i r_i + n_j r_j + n_k r_k} - r_k \right). \quad (5)$$

One sees that if $n_i > n_j$, then $r_j^* = 0$, a result which can be interpreted as a buy-out of individual j by individual i . Making all the necessary calculations we establish that the corresponding values of the partition function game become

$$V(ij, \{ij | k\}) = \left(\frac{\max\{n_i; n_j\}}{\max\{n_i; n_j\} + n_k} \right)^2 \quad \text{and} \quad V(k, \{ij | k\}) = \left(\frac{n_k}{\max\{n_i; n_j\} + n_k} \right)^2. \quad (6)$$

We omit a formal proof of showing that Proposition 2 does not change due to the new values. The key point is to check whether a player wants to opt out of the coalition formation game after the formation of coalition $\{i, j\}$. All players stay in the game if and only if $n_i + n_j \geq \left(\frac{\max\{n_i; n_j\}}{\max\{n_i; n_j\} + n_k} \right)^2$ and $n_k \geq \left(\frac{n_k}{\max\{n_i; n_j\} + n_k} \right)^2$. We rewrite the first

weak inequality as $(1 - n_k)(1 - n_j)^2 \geq (1 - n_j - n_k)^2$ and reduce it to $n_i + n_j \geq n_j^2$. Since $n_j \geq n_j^2$, the weak inequality holds. We rewrite the second inequality as $n_i^2 \geq n_k(1 - n_k - 2n_i) = n_k(n_j - n_i)$. Since $n_i \geq n_j$ by assumption, the weak inequality holds as well.

References

- [1] K.H. Baik and S. Lee, *Collective Rent-Seeking with Endogeneous Group Sizes*, European Journal of Political Economy **13** (1997), 113–126.
- [2] ———, *Strategic Groups and Rent Dissipation*, Economic Inquiry **39** (2001), 672–684.
- [3] F. Bloch, *Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division*, Games and Economic Behavior **14** (1996), 90–123.
- [4] F. Bloch, S. Sánchez-Pagés, and R. Soubeyran, *When does Universal Peace Prevail? Secession and Group Formation in Rent Seeking Contests and Policy Conflicts*, Unpublished manuscript (2002).
- [5] J. Esteban and D. Ray, *Conflict and Distribution*, Journal of Economic Theory **87** (1999), 379–415.
- [6] J. Esteban and J. Sákovics, *Olson vs. Coase: Coalitional Worth in Conflict*, Unpublished manuscript (2002).
- [7] F. Gul, *Bargaining Foundations of the Shapley Value*, Econometrica **57** (1989), 81–95.

- [8] D. Pérez-Castrillo and T. Verdier, *La structure industrielle dans une course au brevet avec coûts fixes et coûts variables*, *Revue Economique* **42** (1991), 1111–1140.
- [9] R. Schmalensee, *A Model of Promotional Competition in Oligopoly*, *Review of Economic Studies* **43** (1976), 71–76.
- [10] L. Shapley and M. Shubik, *Trade Using one Commodity as a Means of Payment*, *Journal of Political Economy* **85** (1977), 937–967.
- [11] G. Tullock, *The Welfare Costs of Tariffs, Monopolies and Theft*, *Western Economic Journal* **5** (1967), 224–232.

Appendix

Proof of Proposition 1: (a) Consider the coalition structure $\pi = \{1 | 2 | 3\}$. We start by showing that at most one individual makes zero effort in equilibrium. Suppose that $\mathbf{r}^* = 0$. By assumption the expected utility of individual i equals n_i . But if he made an effort of $\varepsilon > 0$, with ε small, then his final utility would be $1 - \varepsilon > n_i$. This is a contradiction to $\mathbf{r}^* = 0$ being an equilibrium. Suppose next that $r_j^* = r_k^* = 0$. Then individual i wants to make the smallest strictly positive effort. Therefore, he does not have a best response and we conclude that there is no equilibrium with $r_j^* = r_k^* = 0$.

Consider now the maximization problem (1). From the first order condition we obtain

$$\frac{n_i R - n_i^2 r_i}{R^2} - 1 = 0 \Leftrightarrow \frac{n_i}{R} \left(\frac{R}{R} - \frac{n_i r_i}{R} \right) = 1 \Leftrightarrow p_i = 1 - \frac{R}{n_i}. \quad (7)$$

Since the level of conflict in equilibrium is implicitly given by the equation $\sum_{i=1}^3 p_i^* = 1$,

the equation $\sum_{i=1}^3 \left(1 - \frac{R^*}{n_i}\right) = 1$ must be satisfied. Straightforward calculus yields

$$R^* = \frac{2n_1n_2n_3}{n_1n_2 + n_1n_3 + n_2n_3} = \frac{2}{m},$$

where $m \equiv \frac{n_1n_2+n_1n_3+n_2n_3}{n_1n_2n_3}$ is the harmonic mean of \mathbf{n} . Plugging R^* into the last equation of (7) yields $p_i^* = \frac{n_in_j+n_in_k-n_jn_k}{n_in_j+n_in_k+n_jn_k}$. From rewriting the first equation of (7) in terms of r_i we deduce that

$$r_i^* = \frac{n_iR^* - (R^*)^2}{n_i^2} = \frac{n_i2n_in_jn_k(n_in_j+n_in_k+n_jn_k) - 4n_i^2n_j^2n_k^2}{n_i^2(n_in_j+n_in_k+n_jn_k)^2} = \frac{2n_jn_k(n_in_j+n_in_k-n_jn_k)}{(n_in_j+n_in_k+n_jn_k)^2}. \quad (8)$$

Hence, r_i^* is positive if and only if $n_in_j+n_in_k-n_jn_k > 0$. The condition for individual 1 and 2 is $n_1n_2 + n_3(n_1 - n_2) > 0$ and $n_2n_3 + n_1(n_2 - n_3) > 0$, respectively. Since $n_1 \geq n_2 \geq n_3$, both conditions are satisfied. Finally, consider the corresponding inequality for individual 3 which can be stated as $n_3 > \frac{n_1n_2}{n_1+n_2}$. Since the weak inequality $\frac{1}{4}(t_1 + t_2) \geq t_1t_2$ holds for any $t_1, t_2 \in [0, 1]$, we have as a particular case $\frac{1}{4}(n_1 + n_2) \geq n_2n_1$. Hence, $n_3 > 0.25$ is a necessary condition for $\mathbf{r}^* \gg 0$. Evaluating the second order condition at the unique critical point yields

$$\frac{(n_i^2 - n_i^2)(R^*)^2 - 2R^*n_i(n_iR^* - n_i^2r_i^*)}{R^4} < 0,$$

where the inequality holds because of $R^* - n_ir_i^* = \sum_{j \neq i} n_jr_j^* > 0$. Hence, if $n_3 > 0.25$, then $r_i^* = \frac{2n_jn_k(n_in_j+n_in_k-n_jn_k)}{(n_in_j+n_in_k+n_jn_k)^2}$ for all i constitutes the unique Nash equilibrium of the contest game in the coalition structure $\pi = \{1 | 2 | 3\}$.

Suppose that $n_3 \leq 0.25$. Then the first order condition of the maximization problem (1) given $r_3^* = 0$ and r_j is $\frac{n_i}{n_ir_i+n_jr_j}(1 - p_i) = 1$. We rewrite it as $p_i^*(1 - p_i^*) = r_i^*$ and use $p_3^* = 0$ in order to obtain $p_1^*p_2^* = r_1^* = r_2^*$. Finally, because of $p_i^* = \frac{n_ir_i^*}{n_ir_i^*+n_jr_j^*}$, we verify that

$r_1^* = r_2^* = \frac{n_1 n_2}{(n_1 + n_2)^2}$. Since the first order condition is as well sufficient we have shown that if $n_3 \leq 0.25$, then the vector $(r_1^*, r_2^*, r_3^*) = \left(\frac{n_1 n_2}{(n_1 + n_2)^2}, \frac{n_1 n_2}{(n_1 + n_2)^2}, 0 \right)$ constitutes the unique Nash equilibrium of the contest game in the coalition structure $\pi = \{1 | 2 | 3\}$.

(b) We turn now to the coalition structure $\pi = \{ij | k\}$. The proof that r_{ij}^* and r_k^* are strictly positive uses similar arguments as the proof of the former part. The first order conditions of the maximization problems (2) and (3) are

$$\frac{n_i + n_j}{(n_i + n_j)r_{ij}^* + n_k r_k^*} (1 - p_{ij}) - 1 = 0 \quad \text{and} \quad \frac{n_k}{(n_i + n_j)r_{ij}^* + n_k r_k^*} (1 - p_k) - 1 = 0. \quad (9)$$

We multiply the first equation of (9) by r_{ij} and the second one by r_k to yield the condition $p_{ij}^* p_k^* = r_{ij}^* = r_k^*$. In the next step, we deduce that $r_{ij}^* = (n_i + n_j) n_k$ by using the definitions of p_{ij} and p_k . Finally, we evaluate the second order conditions in the corresponding critical points. Since the conditions $-\frac{2(n_i + n_j)^2 n_k r_k^*}{((n_i + n_j)r_{ij}^* + n_k r_k^*)^3} < 0$ and $-\frac{(n_i + n_j)n_k^2 r_{ij}^*}{((n_i + n_j)r_{ij}^* + n_k r_k^*)^3} < 0$ hold, we have shown that the vector $(r_{ij}^*, r_k^*) = ((n_i + n_j) n_k, (n_i + n_j) n_k)$ constitutes the unique Nash equilibrium in the coalition structure $\pi = \{ij | k\}$. **q.e.d.**

We prove Proposition 2 in a series of Lemmata. Since we restrict ourselves to stationary strategies, let $x_{S,T}$ be the offer made by S to T at any t . Furthermore, the partition function game V is said to be strictly superadditive if for all $\pi \in \Pi$ and for all $S, T \in \pi$ we have $V(S \cup T, \{(\pi \setminus T \setminus S) \cup (S \cup T)\}) > V(S, \{\pi\}) + V(T, \{\pi\})$.

Lemma 1 *The partition function game V is strictly superadditive.*

Proof of Lemma 1: We prove at first with a geometric argument that $\tilde{V}(\mathbf{n}) \equiv V(ij, \{ij | k\}) - V(i, \{i | j | k\}) - V(j, \{i | j | k\}) > 0$ for all \mathbf{n} . Suppose that $n_3 > 0.25$.

In Figure 3 we draw the level curve $\tilde{V}(\mathbf{n}) = 0$ when $\{i, j\} = \{1, 2\}$. The three straight lines correspond to the set of points satisfying the conditions $n_1 = 0.25$, $n_2 = 0.25$ and $n_1 + n_2 = 0.75$. Therefore, the shaded triangle in the center of the figure is the set of points (n_1, n_2, n_3) where $n_1 > 0.25$, $n_2 > 0.25$ and $n_1 + n_2 < 0.75$. Notice that this area is bounded away from the level curve at zero. Since $\tilde{V}(\mathbf{n})$ is a continuous function in \mathbf{n} , the result follows if we find a point (n_1, n_2, n_3) in the shaded area for which $\tilde{V}(\mathbf{n}) > 0$. Take $n_i = \frac{1}{3}$ for all i , then $\tilde{V}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{4}{9} - \frac{2}{9} = \frac{2}{9}$. Hence, $\tilde{V}(\mathbf{n})$ takes positive values all over the area of interest.

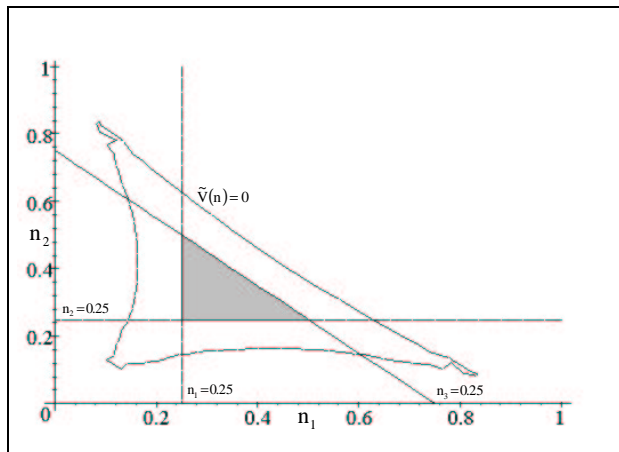


Figure 3: $\tilde{V}(\mathbf{n}) = 0$ when $n_3 > 0.25$

Suppose now that $n_3 \leq 0.25$. By Corollary 1, taking $k = 3$, we have to prove that the inequality $\tilde{V}(\mathbf{n}) = (n_1 + n_2)^2 - \frac{n_1^2 + n_2^2}{(n_1 + n_2)^2} > 0$ holds. We use Figure 4 to establish the result. The four straight lines correspond to the conditions $n_1 + n_2 = 0.75$, $n_1 + n_2 = 1$, $n_1 = n_2$ and $n_2 = n_3$. The shaded area indicates the set of points (n_1, n_2, n_3) satisfying the conditions $0.75 < n_1 + n_2 \leq 1$ and $n_1 \geq n_2 \geq n_3$. Using the point $(n_1, n_2, n_3) = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$

we establish that $\tilde{V}(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}) = 0.409 - 0.32 > 0$. Hence, $\tilde{V}(\mathbf{n})$ takes strictly positive values in this area whenever $n_l > 0$ for all l .

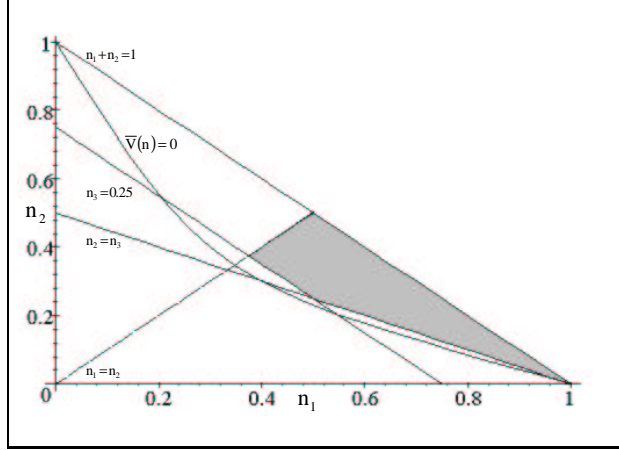


Figure 4: $\tilde{V}(\mathbf{n}) = 0$ when $n_3 \leq 0.25$

Similarly, by Corollary 1 and taking $i = 3$, we have to prove that $(n_3 + n_j)^2 > \left(\frac{n_j}{1-n_3}\right)^2$. We rewrite the inequality as $(n_3 + n_j)(1 - n_3) > n_j$ and perform all the necessary calculus to yield $n_3(1 - n_3 - n_j) = n_3 n_k > 0$. Hence, individual 3 and j profit from forming a coalition. Finally, we have to check that the inequality $V(123, \{123\}) > V(ij, \{ij|k\}) + V(k, \{ij|k\})$ holds for all possible permutations of the set of players. This follows, because the equilibrium level of conflict R^* is strictly positive in any coalition structure of the type $\pi = \{ij|k\}$. **q.e.d.**

Lemma 2 *Let $\mathbf{q} = \mathbf{n}$ and suppose that $x_{i,j}$ has been accepted at t . If $s_{t+1} = 3$, then $x_{k,ij}^*$ and $x_{ij,k}^*$ are acceptable offers.*

Proof of Lemma 2: Suppose that $x_{k,ij}^*$ is not an acceptable offer. If $x_{ij,k}^*$ is not an acceptable offer either, then coalition $\{i, j\}$ and player k will negotiate for ever, because by

stationarity $s_{t+m} = 3$ for all $m \geq 2$. In this case every player gets zero utility. But if player k had left the game before, then he would have received an utility of $n_k^2 > 0$ from playing the contest game within the coalition structure $\pi = \{ij | k\}$. Hence, we have reached a contradiction to $s_{t+1} = 3$ and conclude that $x_{ij,k}^*$ must be an acceptable offer. Since $x_{k,ij}^*$ is not acceptable by assumption, the one-period discounted expected utility of player k is equal to

$$\begin{aligned} U_k(\delta, x_{i,j}) &= \delta \left((n_i + n_j) x_{ij,k}^* + \delta n_k (n_i + n_j) x_{ij,k}^* + \delta^2 n_k^2 (n_i + n_j) x_{ij,k}^* + \dots \right) \\ &= \delta (n_i + n_j) x_{ij,k}^* \sum_{\tau=0}^{\infty} \delta^\tau n_k^\tau = \delta \frac{(n_i + n_j) x_{ij,k}^*}{1 - \delta n_k}. \end{aligned}$$

Player k accepts the offer $x_{ij,k}$ if and only if it is at least as high as the discounted value of the expected continuation utility from rejecting it; that is $x_{ij,k}^* \geq \delta U_k(\delta, x_{i,j})$. On the other hand player i will not offer more than $U_k(\delta, x_{i,j})$. Thus, $x_{ij,k}^* = \frac{\delta(n_i+n_j)}{1-\delta n_k} x_{ij,k}^*$. Since $\frac{\delta(n_i+n_j)}{1-\delta n_k} \neq 1$ for all $\delta < 1$, we must have $x_{ij,k}^* = 0$ which implies $U_k^*(\delta, x_{i,j}) = 0$. This is a contradiction to $s_{t+1} = 3$, because player k can get strictly more by leaving the game and playing the contest game within the coalition structure $\pi = \{ij | k\}$. **q.e.d.**

Lemma 3 *Let $\mathbf{q} = \mathbf{n}$ and suppose that $x_{i,j}$ has been accepted at t . If $s_t = 3$, then the grand coalition forms in the next period and the one-period discounted expected utilities are equal to $(U_i^*(\delta, x_{i,j}), U_j^*(\delta, x_{i,j}), U_k^*(\delta, x_{i,j})) = (\delta(n_i + n_j)(1 - x_{i,j}), \delta(n_i + n_j)x_{i,j}, \delta n_k)$.*

Proof of Lemma 3: Suppose that $s_{t+1} = 3$. Since we know from Lemma 2 that $x_{k,ij}^*$ and $x_{ij,k}^*$ are acceptable offers, the one-period discounted expected utilities are given by

$$\begin{aligned} U_i^*(\delta, x_{i,j}) &= \delta \left((n_i + n_j) (1 - x_{i,j}) (1 - x_{ij,k}^*) + n_k (1 - x_{i,j}) x_{k,ij}^* \right) \\ U_j^*(\delta, x_{i,j}) &= \delta \left((n_i + n_j) x_{i,j} (1 - x_{ij,k}^*) + n_k x_{i,j} x_{k,ij}^* \right) \\ U_k^*(\delta, x_{i,j}) &= \delta \left((n_i + n_j) x_{ij,k}^* + n_k (1 - x_{k,ij}^*) \right). \end{aligned} \tag{10}$$

Player i accepts $x_{k,ij}^*$ if and only if $(1 - x_{i,j}) x_{k,ij}^* \geq \delta U_i^*(\delta, x_{i,j})$. Therefore, in equilibrium the equation must be satisfied with equality. Using a similar argument we establish that $x_{ij,k}^* = \delta U_k^*(\delta, x_{i,j})$. The solution of the system of linear equations (10), given $x_{i,j}$, is

$$\begin{aligned} (x_{ij,k}^*, x_{k,ij}^*) &= (\delta^2 n_k, \delta^2 (n_i + n_j)) \\ (U_i^*(\delta, x_{i,j}), U_j^*(\delta, x_{i,j}), U_k^*(\delta, x_{i,j})) &= (\delta (n_i + n_j) (1 - x_{i,j}), \delta (n_i + n_j) x_{i,j}, \delta n_k). \end{aligned} \quad (11)$$

Equation (11) states the unique stationary subgame equilibrium payoff of the continuation game after $x_{i,j}$ if it is optimal for every player to stay in the game. If player k had left the game before, then he would have received an expected utility of δn_k^2 which is strictly less than δn_k . If player j had opted out, then he would have received an expected utility of $\delta (n_i + n_j)^2 x_{i,j}$ which is strictly less than $U_j^*(\delta, x_{i,j})$. Finally, player i does not to opt out either, because if he did so, then his expected utility would be equal to $\delta (n_1 + n_2)^2 (1 - x_{i,j})$. But this is strictly less than $U_i^*(\delta, x_{i,j})$. **q.e.d.**

Lemma 4 *Let $\mathbf{q} = \mathbf{n}$ and suppose that $s_t = 3$. Then the one-period discounted expected utility of player l is equal to $U_l^*(\delta) = \delta n_l$ for all $l = 1, 2, 3$.*

Proof of Lemma 4: If player j accepts the offer $x_{i,j}$ at t , then $U_i^*(\delta, x_{i,j}) + U_j^*(\delta, x_{i,j}) = \delta (n_i + n_j)$. The corresponding stand alone expected utility of individual k is $U_k^*(\delta, x_{i,j}) = \delta n_k$. Since the final utility of coalition $\{i, j\}$ is independent of the applied sharing rule, player i selects the offer that makes individual j indifferent between accepting and rejecting it. That is $\delta (n_i + n_j) x_{i,j}^* = \delta U_j^*(\delta)$, where $U_j^*(\delta)$ is the expected utility of player j at the beginning of stage t . Hence, the share which remains for player i is equal to $\delta (n_i + n_j) (1 - x_{i,j}^*) = \delta (n_i + n_j) - \delta U_j^*(\delta)$. Player i meets player j and is chosen to make

the offer with probability $\frac{1}{3} \frac{n_i}{n_i+n_j}$. Player j and k meet with probability $\frac{1}{3}$. In this case player i gets his stand alone value δn_i . Finally, player i meets j in the role of the responder with probability $\frac{1}{3} \frac{n_j}{n_i+n_j}$. Therefore, the expected utilities of the players are equal to

$$\begin{aligned}
U_1^*(\delta) &= \frac{1}{3} \frac{n_1}{n_1+n_2} [\delta (n_1 + n_2) - \delta U_2^*(\delta)] + \frac{1}{3} \frac{n_1}{n_1+n_3} [\delta (n_1 + n_3) - \delta U_3^*(\delta)] + \frac{1}{3} \delta n_1 + \\
&\quad + \frac{1}{3} \left(\frac{n_2}{n_1+n_2} + \frac{n_3}{n_1+n_3} \right) \delta U_1^*(\delta) \\
U_2^*(\delta) &= \frac{1}{3} \frac{n_2}{n_1+n_2} [\delta (n_1 + n_2) - \delta U_1^*(\delta)] + \frac{1}{3} \frac{n_2}{n_2+n_3} [\delta (n_2 + n_3) - \delta U_3^*(\delta)] + \frac{1}{3} \delta n_2 + \\
&\quad + \frac{1}{3} \left(\frac{n_1}{n_1+n_2} + \frac{n_3}{n_2+n_3} \right) \delta U_2^*(\delta) \\
U_3^*(\delta) &= \frac{1}{3} \frac{n_3}{n_1+n_3} [\delta (n_1 + n_3) - \delta U_1^*(\delta)] + \frac{1}{3} \frac{n_3}{n_2+n_3} [\delta (n_2 + n_3) - \delta U_2^*(\delta)] + \frac{1}{3} \delta n_3 + \\
&\quad + \frac{1}{3} \left(\frac{n_1}{n_1+n_3} + \frac{n_2}{n_2+n_3} \right) \delta U_3^*(\delta).
\end{aligned}$$

The solution of the system of three linear equation and three unknowns is $U_l^*(\delta) = \delta n_l$ for all $l = 1, 2, 3$. **q.e.d.**

Lemma 5 *Let $\mathbf{q} = \mathbf{n}$. Then, no player leaves the game at t .*

Proof of Lemma 5: Assume that only player k leaves the game at t . Since the partition function game V is strictly superadditive by Lemma 1, we can apply similar arguments to the ones used in Lemmata 2 and 3 to show that player i and j form a coalition and adopt the sharing rules $(x_{i,j}^*, x_{j,i}^*) = \left(\delta^2 \frac{n_j}{n_i+n_j}, \delta^2 \frac{n_i}{n_i+n_j} \right)$ in the unique stationary subgame perfect equilibrium of the continuation game. This implies that the expected utilities of the players in this subgame are equal to $(U_i^*(\delta), U_j^*(\delta), U_k^*(\delta)) = (\delta n_i (n_i + n_j), \delta n_j (n_i + n_j), \delta n_k^2)$. Using all payoffs of the stationary equilibrium of the continuation games that we have already obtained, we represent in Figure 5 the game tree at t .

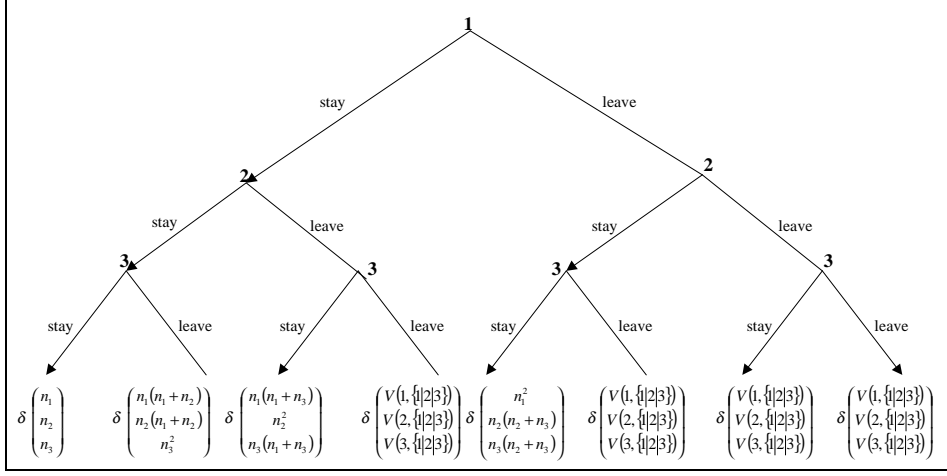


Figure 5: The Opting Out Game at t

We prove in the next step that player 3 may only leave the game whenever player 1 and 2 have left the game before. If $n_3 \leq 0.25$, then player 3 can do stay, because in this case $V(3, \{1|2|3\}) = 0$. Assume now that $n_3 > 0.25$. In Figure 6, we draw the level curve $\hat{V}(\mathbf{n}) \equiv n_i(n_i + n_j) - \left(1 - \frac{2n_j n_k}{n_i n_j + n_i n_k + n_j n_k}\right)^2 = 0$. The shaded area corresponds to the set of points (n_1, n_2, n_3) where $n_i \geq 0.25$. We check that $\hat{V}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9} > 0$. Hence, by continuity of $\hat{V}(\mathbf{n})$ and since the shaded area and the indifference curve do not intersect, we have that $\hat{V}(\mathbf{n})$ takes positives values all over the area of interest. Taking $i = 3$ we conclude that player 3 stays in the game.

We turn now to the stay or exit decision of player 2 given that player 3 stays in the game afterwards. Since $n_2 > n_2^2$, he stays in the game whenever player 1 has decided to stay in the game before. Suppose that player 1 has left the game and that $n_3 > 0.25$. Player 2 does not leave the game, because by taking $i = 2$ and $j = 3$ in Figure 6, we can prove that he gains from staying. Suppose now that $n_3 \leq 0.25$. Player 2 stays in the

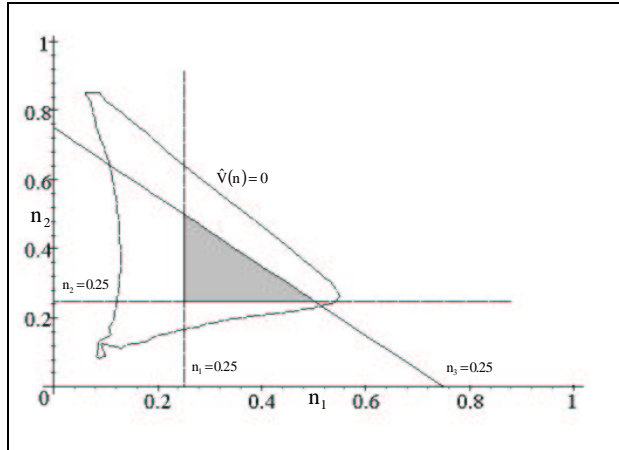


Figure 6: $\hat{V}(\mathbf{n})$ when $n_3 > 0.25$

game if and only if the condition $n_2(n_2 + n_3) \geq \left(\frac{n_2}{n_2 + n_1}\right)^2$ is satisfied. We restate the weak inequality in the form $(1 - n_1)(1 - n_3)^2 \geq n_2 = 1 - n_1 - n_3$. This condition is equivalent to $(1 - n_1 - n_3 + n_1n_3)(1 - n_3) \geq 1 - n_1 - n_3$. We perform all the necessary multiplications to arrive at $n_1(1 - n_3) - n_3(1 - n_1 - n_3) \geq 0$. Since $n_1 \geq n_3$ and $n_3^2 > 0$, we have shown that it is optimal for player 2 to stay in the game. Finally, player 1 decides to stay given that player 2 and 3 do not leave the game afterwards, because his payoff from leaving is $\delta n_1^2 < \delta n_1$. **q.e.d.**

Proof of Proposition 2: By Lemma 5 we have that $s_t = 3$. Therefore, we can apply Lemma 4 to get that $U_l^*(\delta) = \delta n_l$ for all l which reduces $U_l^*(\delta) = n_l$ as $\delta \rightarrow 1$. Moreover, the grand coalition forms independently of the Nature moves, because it has been seen in Lemmata 2 and 4 that every offer is accepted in equilibrium. **q.e.d.**