Increasing quasiconcave production and utility functions with diminishing returns to scale

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Abstract

In microeconomic analysis functions with diminishing returns to scale (DRS) have frequently been employed. Various properties of increasing quasiconcave aggregator functions with DRS are derived. Furthermore duality in the classical sense as well as of a new type is studied for such aggregator functions in production and consumer theory. In particular representation theorems for direct and indirect aggregator functions are obtained. These involve only small sets of generator functions. The study is carried out in the contemporary framework of abstract convexity and abstract concavity.

Key Words: aggregator functions, diminishing returns to scale, abstract convexity, duality

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1 INTRODUCTION

In production theory it is commonly assumed that the production function is increasing and quasiconcave. Likewise in consumer theory one often assumes that the utility function has these properties; e.g., [2], [5]. In this paper we study such production and utility functions under the additional property of diminishing returns to scale (DRS) [10], [16], [19]. Production and utility functions with DRS are of particular interest.

In production theory this property is related to 'convex technologies' [16]. In the context of a homogeneous production technology the DRS property is equivalent to increasing average cost. Then a special decomposition of the cost function is possible, using Shephard's decomposition theorem [8], [18].

In consumer theory functions with DRS are of more limited use. This is due to the fact that the same preference ordering can be expressed by different utility functions obtained through monotone transformation, i.e., through different intensity of the preference order. But for special utility functions it is of interest, for instance in von Neumann-Morgenstern utility theory [3]. Also in the context of the Arrow-Pratt risk aversion the property plays a role [1], [19].

Duality of direct and indirect utility functions is a central concept in consumer theory. Likewise a duality relationship between a producer's production function and indirect production function has been established [2], [5], [13]. Following the unifying approach in [5], we will refer below to the "aggregator" function encompassing both a utility and production function. The "indirect aggregator" function is then the indirect utility function in consumer theory and the indirect production function in producer theory.

An aggregator function $u(x)$ has diminishing returns to scale (DRS) if $u(\beta x) \leq \beta u(x)$ for all $\beta \geq 1$ on the nonnegative orthant. In case of a production function it means that doubling the inputs will not more than double the output. We note that the DRS property is a property 'along rays', in contrast to increasingness and (quasi)concavity.

DRS aggregator functions have been studied mostly in special cases so far. Examples are concave-along-rays production functions or homogeneous with degree $\delta \leq 1$ functions; e.g.,[6], [7]. Furthermore functions with (eventually) diminishing marginal returns or utility are related to DRS aggregator functions.

In light of duality in microeconomic theory we will also consider inverse DRS functions. A function $v(p)$ is called inverse DRS if $v(\alpha p) \leq \dfrac{v(p)}{\alpha}$ for all $\alpha \in (0,1]$ on the nonnegative orthant. In production theory it means that the amount of output which can be produced if all input prices are divided by two is not greater than twice the amount which could be produced before the prices changed. Among others it will be shown that under certain regularity assumptions an aggregator function has DRS if and only if the associated indirect aggregator function is an inverse DRS function.

Some of the major results of this study will be presented in the relatively new framework of abstract convexity [17]. In convex analysis one of the main results asserts that every lower semicontinuous convex function is the upper envelope (point-wise supremum) of a set of affine functions. Many results in convex analysis easily follow from this fact. It is well known that similar results hold in quasiconvex analysis: each lower semicontinuous quasiconvex function can be represented as the upper envelope of a set of quasiaffine functions [12].

More generally, let H be a set of 'elementary functions'. A function f is called abstract convex with respect to H if f can be represented as the upper envelope of a subset of H. The set H is called a supremal generator of a set P of functions if $H \subseteq P$ and each function $f \in P$ is abstract convex with respect to H. In a similar manner we can define abstract concave functions with respect to H and infimal generators of a set of functions. We will study some classes of functions in mathematical economics from the point of view of abstract convexity and abstract concavity. In particular we will identify some small supremal/infimal generators in the context of duality relationships for DRS aggregator functions.

2 FUNCTIONS WITH DRS AND SOME OF THEIR PROPERTIES

We consider the cone $\mathbb{R}^n_+$ of all nonnegative n-vectors with the coordinate-wise order relation $\geq$. In addition, let $\mathbb{R}^n_{++}$ denote the cone of all positive n-vectors. A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is called *increasing* (*decreasing*) if $x \geq y$ implies $f(x) \geq f(y)$ ($f(x) \leq f(y)$). We also use the following notation: $\mathbb{R}_+ = \mathbb{R}^1_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\overline{\mathbb{R}}_+ = [0, +\infty] = \mathbb{R}_+ \cup \{+\infty\}$.

Definition 1 *A function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ is said to have diminishing returns to scale (briefly, DRS) if*

$$u(\beta x) \leq \beta u(x) \qquad \forall \beta \geq 1, \quad \forall x \in \mathbb{R}^n_+. \tag{1}$$

It is easy to see that a function u has DRS if and only if

$$u(\alpha x) \geq \alpha u(x) \qquad \forall \alpha \in (0, 1], \quad \forall x \in \mathbb{R}^n_+. \tag{2}$$

Functions with property (2) are also called co-radiant; see, for example [17].

Denote by U the set of all functions with DRS. We assume that U is equipped with the pointwise order relation. Let us give some properties of the set U.

1) If $u_1, u_2 \in U$, then also $u_1 + u_2 \in U$.

2) If $u \in U$ and $\lambda > 0$, then also $\lambda u \in U$.

3) Let $(u_t)_{t \in T}$ be a net in U and $u(x) = \lim_{t \in T} u_t(x)$ $\left(x \in \mathbb{R}^n_+\right)$. Then $u \in U$.

4) Let $(u_t)_{t \in T}$ be a family of functions in U and $\underline{u}(x) = \inf_{t \in T} u_t(x)$, $\bar{u}(x) = \sup_{t \in T} u_t(x)$ $\left(x \in \mathbb{R}^n_+\right)$. Then $\underline{u}$, $\bar{u} \in U$.

It follows from 1)-3) that U is a closed convex cone in the space of all nonnegative functions $\mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$. Due to 4) we conclude that U is a complete lattice. We now give some examples of functions with DRS.

Example 2 *1) If u is a concave function and $u(0) \geq 0$, then $u \in U$. Indeed, let $\alpha \in [0,1]$ and $x \in \mathbb{R}^n_+$. Then $u(\alpha x) = u(\alpha x + (1-\alpha)0) \geq \alpha u(x) + (1-\alpha)u(0) \geq \alpha u(x)$.*

2) Each nonnegative decreasing function u has DRS. Indeed, for $\alpha \in [0,1]$ we have

$$u(\alpha x) \geq u(x) \geq \alpha u(x) \qquad \forall x \in \mathbb{R}^n_+.$$

3) A function u is called positively homogeneous of degree δ if $u(\alpha x) = \alpha^\delta u(x)$ for all $x \in \mathbb{R}^n_+$. If $0 < \delta \leq 1$, then a positively homogeneous function of degree δ has DRS.

4) Let $p(x)$ be a polynomial of degree m with nonnegative coefficients. Then the function $u(x) = (p(x))^{\frac{1}{m}}$ $\left(x \in \mathbb{R}^n_+\right)$ has DRS. See [17] for details.

We now give a more complicated example.

Let $F : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ be a mapping with increasing returns to scale: $F(\beta x) \geq \beta F(x)$ for all $x \in \mathbb{R}^n_+$ and $\beta \geq 1$. Let $p \in \mathbb{R}^n_+$ and

$$u(y) = \inf\{\langle p, x\rangle : F(x) \geq y\} \qquad \left(y \in \mathbb{R}^m_+\right).$$

Since

$$\left\{x : F\left(\frac{1}{\alpha}x\right) \geq y\right\} \supseteq \left\{x : \frac{1}{\alpha}F(x) \geq y\right\}$$

for $\alpha \in (0,1]$ and $y \in \mathbb{R}^m_+$, it follows that

$$\begin{aligned} u(\alpha y) &= \inf\{\langle p, x\rangle : F(x) \geq \alpha y\} = \inf\left\{\langle p, x\rangle : \frac{1}{\alpha}F(x) \geq y\right\} \\ &\geq \inf\left\{\langle p, x\rangle : F\left(\frac{1}{\alpha}x\right) \geq y\right\} = \inf\{\langle p, \alpha x'\rangle : F(x') \geq y\} \\ &= \inf\{\alpha\langle p, x'\rangle : F(x') \geq y\} = \alpha u(y). \end{aligned}$$

Thus u has DRS. We now give a simple economic interpretation of the example under consideration; see [2] for details.

Example 3 *Let F be a production mapping and p be a price vector. Let y be a required output vector. Then $u(y)$ is the minimal cost of inputs that provide the production of the output y. Thus the increasing returns to scale of a production mapping implies the DRS property of the cost of outputs.*

We now describe some properties of functions with DRS.

Note that DRS is a so-called "property along-rays": in order to verify that a function u possesses this property we need to consider only the restriction of u to each ray $R_x = \{\alpha x : \alpha \geq 0\}$ starting from zero. Let $x \in \mathbb{R}^n_+$, $x \neq 0$, and u_x be the function of one variable defined on $[0, +\infty)$ by $u_x(\alpha) = u(\alpha x)$. The definition of DRS relates only to properties of the functions u_x, and there is no link between these functions for non-proportional x. Clearly this property is very weak. So we have to consider it together with other properties that are not "along rays". Nevertheless we can extract some information about functions with DRS without any additional properties.

Let u be a nonnegative function which has DRS. Consider the domain of this function $dom\ u = \left\{x \in \mathbb{R}^n_+ : u(x) < +\infty\right\}$, the complement to the domain $\mathcal{C}(dom\ u) = \left\{x \in \mathbb{R}^n_+ : u(x) = +\infty\right\}$ and the null set $\left\{x \in \mathbb{R}^n_+ : u(x) = 0\right\}$ of u. The domain is co-radiant, that is, it enjoys the following property: $x \in dom\ u \Longrightarrow \beta x \in dom\ u$ for all $\beta > 1$ which follows from (1). The null set is also co-radiant. The set $\mathcal{C}(dom\ u)$ is radiant: $x \in \mathcal{C}(dom\ u), \alpha \in (0,1] \Longrightarrow \alpha x \in \mathcal{C}(dom\ u)$ which follows from (2).

We now describe differential properties of the functions $u \in U$.

Let $x \in dom\ u$. Since $\beta x \in dom\ u$ for $\beta > 1$, the lower Dini derivative $u^{\downarrow}(x,x)$ of the function u at the point x in the direction x is well defined. Recall that, by definition,

$$u^{\downarrow}(x,x) = \lim\inf_{\alpha \to +0} \frac{1}{\alpha}\left(u(x + \alpha x) - u(x)\right) = (u_x)'_+(1),$$

where $(u_x)'_+(\gamma)$ is the right derivative of the function u_x at the point γ.

Proposition 4 *A function $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ has DRS if and only if*

$$u(x) \geq u^{\downarrow}(x,x) \qquad \forall x \in \mathbb{R}^n_+. \tag{3}$$

Proof. It is easy to show that u has DRS if and only if the function ϕ_x defined by $\phi_x(\gamma) = \dfrac{u_x(\gamma)}{\gamma}$ is decreasing for all x. Indeed, let u have DRS and $\alpha \geq \beta > 0$. Then

$$\phi_x(\beta) = u(\beta x)\frac{1}{\beta} = u\left(\frac{\beta}{\alpha}(\alpha x)\right)\frac{1}{\beta} \geq \frac{\beta}{\alpha}u(\alpha x)\frac{1}{\beta} = u(\alpha x)\frac{1}{\alpha} = \phi_x(\alpha).$$

Assume now that ϕ_x is decreasing for all x. Then $\phi_x(\alpha) \geq \phi_x(1)$ for all $x \in \mathbb{R}^n_+$ and $\alpha \in (0,1)$, so u has DRS.

Since $\phi_x(\lambda) = \dfrac{u(\lambda x)}{\lambda}$, it follows that

$$(\phi_x)'_+(\lambda) = \frac{u^{\downarrow}(\lambda x, x)\lambda - u(\lambda x)}{\lambda^2} \qquad \forall \lambda > 0, \tag{4}$$

so $(\phi_x)'_+(1) = u^\downarrow(x,x) - u(x)$. Since ϕ_x is decreasing, it follows that $(\phi_x)'_+(1) \leq 0$. So (3) holds. Assume now that (3) holds. Then $(\phi_x)'_+(1) \leq 0$ for all $x \in \mathbb{R}^n_+$. Let $y = \lambda x$. We have $u_x(\lambda) = u(\lambda x) = u(y) = u_y(1)$. Then

$$\begin{aligned} u^\downarrow(\lambda x, x) &= \lim_{t\to+0}\inf \frac{1}{t} u((\lambda+t)x - u(\lambda x)) \\ &= \lim_{t\to+0}\inf \frac{1}{\lambda}\frac{\lambda}{t} u\left(y + \frac{t}{\lambda} y\right) - u(y) = \frac{1}{\lambda}(\phi_y)'_+(1). \end{aligned}$$

It follows from (4) that

$$(\phi_x)'_+(\lambda) = \frac{u^\downarrow(\lambda x, x)\lambda - u(\lambda x)}{\lambda^2} = \frac{(\phi_y)'_+(1) - \phi_y(1)}{\lambda^2} \leq 0$$

for all $\lambda > 0$. Hence the function ϕ_x is decreasing. Thus the result follows. ■

We note that in general u is not differentiable or even continuous.

Corollary 5 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ be differentiable on $\mathbb{R}^n_{++}$ and continuous on $\mathbb{R}^n_+$. Then u has DRS if and only if*

$$u(x) \geq \langle \nabla u(x), x\rangle \qquad \forall x \in \mathbb{R}^n_{++}.$$

Denote by V the class of all functions of the form $v(x) = \dfrac{1}{u(x)}$ where $u \in U$. We assume that $\dfrac{1}{0} = +\infty$, $\dfrac{1}{+\infty} = 0$. Clearly $v \in V$ if and only if

$$v(\alpha x) \leq \frac{v(x)}{\alpha} \qquad \forall \alpha \in (0,1], \tag{5}$$

which is equivalent to

$$v(\beta x) \geq \frac{v(x)}{\beta} \qquad \forall \beta \geq 1.$$

If $v \in V$, then the set *dom* v is radiant. The null set $\{x \in \mathbb{R}^n_+ : v(x) = 0\}$ is also radiant. The differential properties of $v = \dfrac{1}{u}$ can easily be derived from the differential properties of u.

3 Increasing functions with DRS

For applications we need to study increasing quasiconcave functions with DRS as well as decreasing functions in the set V. Let

$$U_i = \{u \in U : u \text{ is increasing}\}, \qquad V_d = \{v \in V : v \text{ is decreasing}\}, \tag{6}$$

$$U_{iq} = \{u \in U_i : u \text{ is quasiconcave}\}. \tag{7}$$

In this section we only consider functions in U_i and V_d. Note that $u \in U_i$ if and only if $\frac{1}{u} \in V_d$ where $\frac{1}{u}(x) = \frac{1}{u(x)}$ for all $x \in \mathbb{R}^n_+$.

Let $f : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ be a function such that $f(x) = +\infty$ for some $x \in \mathbb{R}^n_+$. If f is lower semicontinuous (upper semicontinuous, continuous), then the function $\frac{1}{f}$ is upper semicontinuous (lower semicontinuous, continuous).

We now describe some properties of functions $u \in U_i$ and $v \in V_d$.

Proposition 6 *Let u be an increasing function with DRS.*

1) If there exists $y \in \mathbb{R}^n_{++}$ such that $u(y) = 0$, then $u = 0$.

2) If there exists $y \in \mathbb{R}^n_+$ such that $u(y) = +\infty$, then $u(x) = +\infty$ for all $x \in \mathbb{R}^n_{++}$.

3) u is continuous on $\mathbb{R}^n_{++}$.

Proof. 1) We have $u(\beta y) \leq \beta u(y) = 0$ for all $\beta \geq 1$. For each $x \in \mathbb{R}^n_+$ there exists $\beta > 1$ such that $x \leq \beta y$; so $u(x) \leq u(\beta y) = 0$.

2) For each $x \in \mathbb{R}^n_{++}$ there exists $\alpha \in (0, 1]$ such that $\alpha y \leq x$; so $u(x) \geq u(\alpha y) \geq \alpha u(y) = +\infty$.

3) Due to 2), we can assume that *dom* $u = \mathbb{R}^n_{++}$. Let $x \in \mathbb{R}^n_{++}$ and $x_k \to x$. Then for each $\epsilon > 0$ we have $(1 - \epsilon) x \leq x_k \leq (1 + \epsilon) x$ for sufficiently large k; so $u((1 - \epsilon) x) \leq u(x_k) \leq u((1 + \epsilon) x)$. We also have $(1 - \epsilon) u(x) \leq u((1 - \epsilon) x)$ if $\epsilon < 1$ and $u((1 + \epsilon) x) \leq (1 + \epsilon) u(x)$. Thus the result follows. ∎

The following assertion follows immediately from Proposition 6 and the definition of the class V_d.

Proposition 7 *Let $v \in V_d$.*

1) If there exists $y \in \mathbb{R}^n_{++}$ such that $v(y) = +\infty$, then $v(x) = +\infty$ for all $x \in \mathbb{R}^n_+$.

2) If there exists $y \in \mathbb{R}^n_+$ such that $v(y) = 0$, then $v(x) = 0$ for all $x \in \mathbb{R}^n_{++}$.

3) v is continuous on $\mathbb{R}^n_{++}$.

We also need the following simple proposition.

Proposition 8 *1) If u is an upper semicontinuous increasing function on $\mathbb{R}^n_+$, then it is continuous;*

2) If v is a lower semicontinuous decreasing function on $\mathbb{R}^n_+$, then it is continuous.

Proof. We shall prove only 1). Let $x \in \mathbb{R}^n_+$. Since u is increasing, it follows that

$$\lim\sup_{x' \to x} u(x') \geq \lim \sup_{x' \to x,\, x' \geq x} u(x') \geq u(x).$$

On the other hand, $\lim\sup_{x' \to x} u(x') \leq u(x)$ due to upper semicontinuity of u. ∎

The following simple example shows that there exist lower semicontinuous increasing functions with DRS which are not continuous.

Example 9 *The function u defined on $\mathbb{R}^n_+$ by*

$$u(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^n_{++} \\ 0 & \text{if } x \notin \mathbb{R}^n_{++} \end{cases},$$

is increasing and has DRS.

We now present some known results related to abstract convexity and abstract concavity of increasing functions with DRS.

Let $I = \{1, \ldots, n\}$. For a vector $l \in \mathbb{R}^n_+$, define the set $I_+(l) = \{i \in I : l_i > 0\}$. For each $l \in \mathbb{R}^n_+$ and $c \in \overline{\mathbb{R}}_+$, consider the functions $h^-_{l,c}$ and $h^+_{l,c}$ defined on $\mathbb{R}^n_+$ by

$$h^-_{l,c}(x) = \min\left\{\min_{i \in I_+(l)} l_i x_i, c\right\}, \qquad h^+_{l,c}(x) = \max\left\{\max_{i \in I_+(l)} l_i x_i, c\right\}.$$

Note that $h^+_{l,c}(x) = \max\{\max_{i \in I} l_i x_i, c\}$. Let

$$H_- = \left\{h^-_{l,c} : l \in \mathbb{R}^n_+, c \in \overline{\mathbb{R}}_+\right\}, \qquad H_+ = \left\{h^+_{l,c} : l \in \mathbb{R}^n_+, c \in \overline{\mathbb{R}}_+\right\}.$$

The following statement holds [17].

Theorem 10 *1) A function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ belongs to U_i and is lower semicontinuous if and only if it is H_--convex.*

2) A function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ belongs to U_i and is continuous if and only if it is H_+-concave.

In order to present corresponding results for functions $v \in V_d$, we consider the following sets:

$$H^{-1}_- = \left\{\frac{1}{h^-_{l,c}} : h^-_{l,c} \in H_-\right\}, \qquad H^{-1}_+ = \left\{\frac{1}{h^+_{l,c}} : h^+_{l,c} \in H_+\right\}.$$

Let $l \in \mathbb{R}^n_+$. We adopt the following notation:

$$l^{-1} = \frac{1}{l} = \begin{cases} \frac{1}{l_i} & \text{if } i \in I_+(l) \\ 0 & \text{if } i \notin I_+(l) \end{cases}. \tag{8}$$

Proposition 11 *1) $h \in H^{-1}_-$ if and only if there exist $m \in \mathbb{R}^n_+$ and $k \in \overline{\mathbb{R}}_+$ such that*

$$h(x) = \max\left\{\max_{i \in I_+(m)} \frac{m_i}{x_i}, k\right\} \qquad \forall x \in \mathbb{R}^n_+. \tag{9}$$

2) $h \in H^{-1}_+$ if and only if there exist $m \in \mathbb{R}^n_+$ and $k \in \overline{\mathbb{R}}_+$ such that

$$h(x) = \min\left\{\min_{i \in I_+(m)} \frac{m_i}{x_i}, k\right\} \qquad \forall x \in \mathbb{R}^n_+. \tag{10}$$

Proof. 1) Let $h \in H_-^{-1}$. Then there exist a vector $l \in \mathbb{R}_+^n$ and a number $c \in \overline{\mathbb{R}}_+$ such that

$$\begin{aligned} h(x) &= \frac{1}{\min\left\{\min_{i \in I_+(l)} l_i x_i, c\right\}} = \max\left\{\frac{1}{\min_{i \in I_+(l)} l_i x_i}, \frac{1}{c}\right\} \\ &= \max\left\{\max_{i \in I_+(l)} \frac{1}{l_i x_i}, \frac{1}{c}\right\} \qquad \forall x \in \mathbb{R}_+^n. \end{aligned}$$

Let $m = l^{-1}$, $k = c^{-1}$. Then $h(x) = \max\left\{\max_{i \in I_+(l)} \dfrac{m_i}{x_i}, k\right\}$. The same argument shows that a function of the form (9) belongs to H_-^{-1}.

2) Let $h \in H_+^{-1}$. Then there exist a vector $l \in \mathbb{R}_+^n$ and a number $c \in \overline{\mathbb{R}}_+$ such that

$$\begin{aligned} h(x) &= \frac{1}{\max\max_{i \in I_+(l)}\{l_i x_i, c\}} = \min\left\{\frac{1}{\max_{i \in I_+(l)} l_i x_i}, \frac{1}{c}\right\} \\ &= \min \min_{i \in I_+(l)} \left\{\frac{1}{l_i x_i}, \frac{1}{c}\right\} \qquad \forall x \in \mathbb{R}_+^n. \end{aligned}$$

Let $m = l^{-1}$ and $k = c^{-1}$. Then h has the form (10). The same argument shows that the function h defined by (10) belongs to H_+^{-1}. ∎

Theorem 12 *1) A function $v : \mathbb{R}_+^n \to \overline{\mathbb{R}}_+$ belongs to V_d and is upper semicontinuous if and only if it is H_-^{-1}-concave.*

2) A function $v : \mathbb{R}_+^n \to \overline{\mathbb{R}}_+$ belongs to V_d and is continuous if and only if it is H_+^{-1}-convex.

Proof. We shall prove only part 1). Let $v \in V_d$ be upper semicontinuous. Then there exists a lower semicontinuous function $u \in U_i$ such that $v = 1/u$. It follows from Theorem 10 that there exists a set $D \subseteq H_-$ such that $u(x) = \sup\{h(x) : h \in D\}$ for all $x \in \mathbb{R}_+^n$. Let $D^{-1} = \left\{h^{-1} : h \in D\right\}$. Then $D \subseteq H_-^{-1}$. We have

$$\begin{aligned} v(x) &= \frac{1}{u(x)} = \frac{1}{\sup\{h(x) : h \in D\}} = \inf\left\{\frac{1}{h(x)} : h \in D\right\} \\ &= \inf\left\{h'(x) : h' \in D^{-1}\right\}. \end{aligned}$$

The fact that every H_-^{-1}-concave function is upper semicontinuous and belongs to V_d is obvious. ∎

In this section we have seen that only small supremal/infimal generators are needed to represent the functions in U_i/V_d in the sense of abstract convexity/concavity.

4 Increasing quasiconcave functions with DRS

Consider the set U_{iq} of increasing quasiconcave functions with DRS. In contrast to the set U of all functions with DRS and the set U_i of all increasing functions with DRS, the set U_{iq} is not closed under addition. However U_{iq} is a conic set: if $u \in U_{iq}$ and $\lambda > 0$, then $\lambda u \in U_{iq}$. The maximum of two functions in U_{iq} does not necessarily belong to this set. So we shall not discuss abstract convexity of the functions in U_{iq}. On the other hand, U_{iq} is a lower semilattice: if $(u_t)_{t\in T}$, $u_t \in U_{iq}$ for all $t \in T$ and $\underline{u}(x) = \inf_{t\in T} u_t(x)$ $\left(x \in \mathbb{R}^n_+\right)$, then $\underline{u} \in U_{iq}$. This property allows us to discuss abstract concavity of increasing quasiconcave functions with DRS. We shall consider infimal generators of this set that consist of continuous functions, so infima of sets of such functions are upper semicontinuous. However, as it follows from Proposition 6, each upper semicontinuous increasing function with DRS is continuous. Hence it suffices to consider only continuous functions in U_{iq}.

Consider the following sets of functions defined on $\mathbb{R}^n_+$:

$$H_+ = \left\{ h : \exists\, l \in \mathbb{R}^n_+,\ k \in \overline{\mathbb{R}}_+ : h(x) = \max\left\{ \max_{i\in I} l_i x_i, k \right\} \right\};$$

$$H = \left\{ h : \exists\, l \in \mathbb{R}^n, k \in \mathbb{R} : h(x) = \max\left\{ \sum_{i\in I} l_i x_i, k \right\} \right\};$$

$$\mathcal{H} = \left\{ h : \exists\, l \in \mathbb{R}^n_+, k \in \mathbb{R}_+ : h(x) = \max\left\{ \sum_{i\in I} l_i x_i, k \right\} \right\}.$$

It follows from Theorem 10 that the set H_+ is an infimal generator of the set of continuous increasing functions with DRS.

We shall now show that the set $\mathcal{H}$ is an infimal generator of the set of all continuous quasiconcave increasing functions with DRS.

Theorem 13 *A function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ is nonnegative, increasing, quasiconcave, continuous and has DRS if and only if there is a nonempty set $S \subseteq \mathbb{R}^n_+ \times \mathbb{R}_+$ such that*

$$u(x) = \inf_{(x^*,k)\in S} \max\{\langle x, x^*\rangle, k\} \qquad \forall\, x \in \mathbb{R}^n_+. \tag{11}$$

Proof. Suppose first that u has a representation of the form (11). Then u is the pointwise infimum of a family of functions $\{\max\{\langle \cdot, x^*\rangle, k\}\}_{(x^*,k)\in S}$ which are nonnegative, increasing, quasiconcave, continuous and have DRS. Therefore u has these properties.

To prove the converse, we shall show that (11) holds for

$$S = \left\{ (x^*, k) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : \max\{\langle x, x^*\rangle, k\} \geq u(x) \quad \forall x \in \mathbb{R}^n_+ \right\}$$

which implies that $S \neq \emptyset$.

Clearly, the inequality $\leq$ holds in (11). To prove the reverse inequality, we consider two cases for a given $x \in \mathbb{R}^n_+$.

a) The point x is a global maximum of u.

In this case one can easily see that the infimum on the right hand side of (11) is attained at $(x^*, k) = (0, u(x)) \in S$. This clearly implies (11) for the particular point x.

b) The point x is not a global maximum of u.

In this case we consider any $k > u(x)$ such that $u^{-1}([k, +\infty]) \neq \emptyset$. Since this set is convex and closed and does not contain x, by the separation theorem there exist $y^* \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $\langle y, y^* \rangle \geq t > \langle x, y^* \rangle$ for all $y \in u^{-1}([k, +\infty])$. The increasingness of u implies that $y^* \in \mathbb{R}^n_+$, and hence $\langle x, y^* \rangle \geq 0$. Take $t' \in (\langle x, y^* \rangle, t)$, and define $x^* = \frac{k}{t'} y^*$. Obviously, $(x^*, k) \in \mathbb{R}^n_+ \times \mathbb{R}_+$. We shall prove that $(x^*, k) \in S$. To this aim, let $y \in \mathbb{R}^n_+$. Suppose that $k < u(y)$. Then we have $\langle y, y^* \rangle > t'$. Consider the point $\frac{t'}{\langle y, y^* \rangle} y \in \mathbb{R}^n_+$. Since $\left\langle \frac{t'}{\langle y, y^* \rangle} y, y^* \right\rangle = t' < t$, we have $\frac{t'}{\langle y, y^* \rangle} y \notin u^{-1}([k, +\infty])$. Hence $k > u\left(\frac{t'}{\langle y, y^* \rangle} y \right) \geq \frac{t'}{\langle y, y^* \rangle} u(y)$. Therefore $u(y) \leq \frac{k \langle y, y^* \rangle}{t'} = \langle y, x^* \rangle$. We have thus proved that for any $y \in \mathbb{R}^n_+$ either $k \geq u(y)$ or $\langle y, x^* \rangle \geq u(y)$, that is, $\max \{ \langle y, x^* \rangle, k \} \geq u(y)$. In other words, $(x^*, k) \in S$. To finish the proof, it only remains to observe that $\max \{ \langle x, x^* \rangle, k \} = \max \left\{ \left\langle x, \frac{k}{t'} y^* \right\rangle, k \right\} = k \max \left\{ \frac{\langle x, y^* \rangle}{t'}, 1 \right\} = k$. ■

As for U_i in the previous section, we need only a small infimal generator to represent functions in U_{iq} in the sense of abstract concavity. We turn now to duality in microeconomic theory.

5 Duality: the first scheme

In this section we follow the classical approach [2], [5], [13] of duality in microeconomic analysis.

We recall that the indirect aggregator function associated with $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ is $u^* : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$u^*(l) = \sup \{ u(x) \ : \langle x, l \rangle \leq 1 \}.$$

It follows directly from the definition that u^* is a decreasing function for an arbitrary function u.

According to Corollary 2.3 in [13] one can recover u from u^* by means of the classical formula

$$u(x) = \inf \{ u^*(l)) \ : \langle x, l \rangle \leq 1 \} \qquad \forall x \in \mathbb{R}^n_+ \tag{12}$$

if and only if u is increasing, evenly quasiconcave[1] and satisfies

$$u(x) \geq \lim_{\alpha \to 1^-} \underline{u}(\alpha x) \qquad \forall x \in bd\ \mathbb{R}^n_+,$$

with $\underline{u}$ and $bd\ \mathbb{R}^n_+$ denoting the smallest upper semicontinuous majorant of u and the boundary of $\mathbb{R}^n_+$, respectively. An aggregator function u satisfying these properties will be called *regular.* One can easily see that every upper semicontinuous increasing quasiconcave function is regular.

Theorem 14 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a regular aggregator function, and let $u^* : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ be the associated indirect aggregator function. Then u has DRS if and only if*

$$u^*(\alpha x^*) \leq \frac{u^*(x^*)}{\alpha} \qquad \forall\ \alpha \in (0,1] \quad \forall\ x^* \in \mathbb{R}^n_+; \tag{13}$$

i.e., u^* is inverse DRS.

Proof. If u has DRS, then

$$\begin{aligned} u^*(\alpha x^*) &= \sup\{u(x)\ : \langle x, \alpha x^* \rangle \leq 1\} = \sup\{u(x)\ : \langle \alpha x, x^* \rangle \leq 1\} \\ &\leq \sup\left\{\frac{u(\alpha x)}{\alpha}\ : \langle \alpha x, x^* \rangle \leq 1\right\} = \frac{1}{\alpha}\sup\{u(y) : \langle y, x^* \rangle \leq 1\} \\ &= \frac{u^*(x^*)}{\alpha}. \end{aligned}$$

The converse implication can be proved similarly by using (12). ■

Corollary 15 *Let u be a regular aggregator function. Then $u \in U_{iq}$ if and only if $u^* \in V_d$.*

Proof. Let $u \in U_{iq}$. Then $u^* \in V$ due to Theorem 14. Since u^* is decreasing, it follows that $u^* \in V_d$. On the other hand, if $u^* \in V_d$, then u has DRS by Theorem 14. Since $u(x) = \inf\{u^*(l) : \langle x, l \rangle \leq 1\}$, it follows that u is increasing. Hence $u \in U_{iq}$. ■

According to Theorem 12, the set H_-^{-1} of all functions of the form $x \mapsto \max\left\{\max_{i \in I} \frac{m_i}{x_i}, k\right\}$ is an infimal generator of the set of all upper semicontinuous functions in V_d which consists of the functions that are "dual" to the functions in U_{iq} (Corollary 15). On the other hand, the set $\mathcal{H}$ which consists of all functions of the form $x \mapsto \max\{\langle x, l \rangle, k\}$ is an infimal generator of U_{iq} (see Theorem 13). We next show that the set H_-^{-1} is "dual" to $\mathcal{H}$.

[1]A function u is said to be evenly quasiconcave when all of its upper level sets $u^{-1}([\lambda, +\infty])$, $\lambda \in \mathbb{R}$, are evenly convex (that is, intersections of open half-spaces). Evenly convex sets were introduced by Fenchel [9], and functions with evenly convex (lower) level sets were considered for the first time in [11] and [15]. For an analytic characterization of these functions, see [4].

Proposition 16 *Let u be a regular function. Then $u \in \mathcal{H}$ if and only if $u^* \in H_-^{-1}$.*

Proof. Let $u \in \mathcal{H}$, that is, $u(x) = \max\left\{\left\langle x, l^0 \right\rangle, k^0\right\}$ where $l^0 \in \mathbb{R}^n_+$ and $k^0 \in \mathbb{R}_+$. Then

$$\begin{aligned} u^*(l) &= \sup\{u(x) : \langle x, l\rangle \le 1\} \\ &= \sup_{x\ge 0,\, \langle x,l\rangle \le 1} \max\left\{\left\langle x, l^0\right\rangle, k^0\right\} \\ &= \max\left\{\sup_{x\ge 0,\, \langle x,l\rangle\le 1} \left\langle x, l^0\right\rangle, k^0\right\}. \end{aligned}$$

We now calculate

$$\begin{aligned} &\sup_{x\ge 0,\, \langle x,l\rangle\le 1} \left\langle x, l^0\right\rangle \\ &= \sup\left\{\sum_{i\in I_+(l^0)} x_i l_i^0 : x = (x_i)_{i\in I_+(l)}, x_i \ge 0 \ \ (i \in I_+(l)), \sum_{i\in I_+(l)} x_i l_i \le 1\right\}. \end{aligned}$$

First assume that $I_+\left(l^0\right) \subseteq I_+(l)$. Then the function $x \mapsto \sum_{i\in I_+(l^0)} x_i l_i^0$ is bounded on the set $S_l = \left\{(x_i)_{i\in I_+(l)} : x_i \ge 0 \ (i \in I_+(l)), \sum_{i\in I_+(l)} x_i l_i \le 1\right\}$ and attains its maximum at a nonzero extreme point of this set. Note that the nonzero extreme points of S_l are $e_i = 1/l_i,\ i \in I_+(l)$, hence

$$\sup_{x\ge 0,\, \langle x,l\rangle\le 1} \sum_{i\in I} x_i l_i^0 = \max_{i\in I_+(l)} \frac{l_i^0}{l_i} = \max_{i\in I_+(l^0)} \frac{l_i^0}{l_i}.$$

Assume now that $I_+\left(l^0\right) \subseteq I_+(l)$. Then there exists i such that $l_i^0 > 0$ and $l_i = 0$. In such a case

$$\sup_{x\ge 0,\, \langle x,l\rangle\le 1} \left\langle x, l^0\right\rangle = +\infty = \max_{i\in I_+(l^0)} \frac{l_i^0}{l_i}.$$

Thus

$$u^*(l) = \max\left\{\max_{i\in I_+(l^0)} \frac{l_i^0}{l_i}, k^0\right\}. \tag{14}$$

It follows from Proposition 11 that $u^* \in H_-^{-1}$.

Assume now that $u \in U_{iq}$ is a function such that $u^* \in H_-^{-1}$. Then, due to Proposition 11, there exist a vector $l^0 \in \mathbb{R}^n_+$ and a number $k_0 \in \overline{\mathbb{R}}_+$ such that $u^*(l)$ has the form (14). It follows from the first part of the proof that $u^*(l) = u_1^*(l)$ where $u_1(x) = \max\left\{\left\langle x, l^0\right\rangle, k_0\right\}$. Since u is regular, we can conclude that $u = u_1$. ■

Proposition 17 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a regular aggregator function, and let u^* be the associated indirect aggregator function. Assume that u^* is finite. Then u has DRS if and only if*

$$u^*(x^*) + (u^*)^{\downarrow}(x^*, x^*) \geq 0 \qquad \forall x^* \in \mathbb{R}^n_+.$$

Proof. The proof is similar to that of Proposition 4. ■

Corollary 18 *Let u and u^* be as in Prop.17. If u^* is differentiable on $\mathbb{R}^n_{++}$ and continuous on $\mathbb{R}^n_+$, then u has DRS if and only if*

$$u^*(x^*) + \langle \nabla u^*(x^*), x^* \rangle \geq 0 \qquad \forall x^* \in \mathbb{R}^n_{++}.$$

Theorem 19 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a regular aggregator function. If the associated indirect aggregator function u^* is lower semicontinuous, then u has DRS if and only if there is a nonempty set $S \subseteq \mathbb{R}^n_+ \times \mathbb{R}_+$ such that*

$$u^*(x^*) = \sup_{(x^*,k) \in S} \min \left\{ \frac{1}{\langle x, x^* \rangle}, k \right\} \qquad \forall x^* \in \mathbb{R}^n_{++} \tag{15}$$

with the convention $\frac{1}{0} = +\infty$.

Proof. We shall first show that u^* takes the value 0 only if it is identically 0. Let $x_0^* \in \mathbb{R}^n_+$ be such that $u^*(x_0^*) = 0$, and let $x^* \in \mathbb{R}^n_{++}$. Take $\alpha \in (0,1]$ such that $\frac{1}{\alpha} x^* \geq x_0^*$. Since u^* is nonnegative and decreasing, by Corollary 15 one has

$$0 \leq u^*(x^*) \leq u^*(\alpha x_0^*) \leq \frac{u^*(x_0^*)}{\alpha} = 0.$$

Hence $u^*(x^*) = 0$. This shows that u^* is identically 0 on $\mathbb{R}^n_{++}$. Since it is nonnegative and lower semicontinuous, it is identically 0 on the whole of $\mathbb{R}^n_+$.

To prove the existence of $S \subseteq \mathbb{R}^n_+ \times \mathbb{R}_+$ such that (15) holds, we consider two cases:

a) The function u^* is identically 0. In this case one can take $S = \{0\} \times \{0\}$.

b) The function u^* is not identically 0. By the above reasoning u^* does not take on the value 0 at all. Consider the function $w = \frac{1}{u^*}$ with the convention $\frac{1}{+\infty} = 0$. By (13) w has DRS, and hence by Theorem 13 there is a set $T \subseteq \mathbb{R}^n_+ \times \mathbb{R}_+$ such that

$$w(x) = \inf_{(x^*,k) \in T} \max \{ \langle x, x^* \rangle, k \} \qquad \forall x \in \mathbb{R}^n_+.$$

Since $u^* = \frac{1}{w}$ with the convention $\frac{1}{0} = +\infty$, from the preceding equality we see that (15) holds with $S = \left\{ (x^*, k) \in \mathbb{R}^n_+ \times (\mathbb{R}_+ \cup \{+\infty\}) : \left(x^*, \frac{1}{k} \right) \in T \right\}$.

To conclude, we observe that by $\min\left\{\frac{1}{\langle x,x^*\rangle},+\infty\right\}=\sup_{k\in\mathbb{R}_+}\min\left\{\frac{1}{\langle x,x^*\rangle},k\right\}$, we can assume without loss of generality that $S\subseteq\mathbb{R}^n_+\times\mathbb{R}_+$. ■

Theorem 19 demonstrates that the indirect aggregator function u^* is abstract convex with a small supremal generator. This result parallels the one in Theorem 13 which proves that the aggregator function u is abstract concave with a small infimal generator. In both results the DRS property is crucial.

6 Duality: the second scheme

In this section we consider a second duality approach. We refer to [14] where a general class of dualities between complete lattices is introduced. It encompasses the one underlying the approach in this section.

Definition 20 *The secondary aggregator function associated with an aggregator function $u:\mathbb{R}^n_+\to\overline{\mathbb{R}}_+$ is $u^{\#}:\mathbb{R}^n_+\to\overline{\mathbb{R}}_+$, defined by*

$$u^{\#}(x^*)=\sup\{u(x):\langle x,x^*\rangle<u(x)\}\qquad\left(x^*\in\mathbb{R}^n_+\right),\tag{16}$$

with the convention sup $\emptyset=0$.

We give the following interpretation of $u^{\#}$ for the case of a production function. The function u assigns to each input vector x the amount of output $u(x)$, measured in monetary units, which can be produced by these inputs. Let x^* be a price vector. If it represents the market prices for the inputs, the owner of these inputs (the producer) can choose between selling them on the market which would yield $\langle x,x^*\rangle$ monetary units or using them to obtain an amount $u(x)$ of output. The second possibility will be chosen for sure if $\langle x,x^*\rangle<u(x)$. We assume that the producer prefers selling to producing if $\langle x,x^*\rangle=u(x)$, but he prefers producing to selling if he can obtain some strictly positive net profit $u(x)-\langle x,x^*\rangle$, even if it is very small. Thus $u^{\#}(x^*)$ is the largest possible amount of output that can be produced under the input price vector x^* subject to that "rationality" constraint. In this interpretation the producer is an output rather than profit maximizer.

Theorem 21 *Let $w:\mathbb{R}^n_+\to\overline{\mathbb{R}}_+$. There exists $u:\mathbb{R}^n_+\to\overline{\mathbb{R}}_+$ such that w is the secondary aggregator function associated with u if and only if w is quasiconvex, decreasing and lower semicontinuous. In this case u can be taken as a quasiconcave increasing upper semicontinuous function and with DRS. Under these conditions u is unique, namely u is the largest function such that $u^{\#}=w$. Furthermore it satisfies*

$$u(x)=\inf_{x^*\in\mathbb{R}^n_+}\max\{\langle x,x^*\rangle,w(x^*)\}\qquad\forall x\in\mathbb{R}^n_+.\tag{17}$$

Hence the mapping $u\longmapsto u^{\#}$ is a bijection from the set of upper semicontinuous increasing quasiconcave functions $u:\mathbb{R}^n_+\to\overline{\mathbb{R}}_+$ that have DRS onto *the set of lower semicontinuous decreasing quasiconvex functions* $w : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$. *The inverse mapping is the one assigning to each* w *in this set the function* u *of (17).*

Proof. "Only If". From (16) it easily follows that $u^{\#}$ is decreasing and that its lower level sets are intersections of closed halfspaces, hence convex and closed:

$$u^{\#-1}([-\infty, \lambda]) = \bigcap_{x \text{ s.t. } u(x)>\lambda} \left\{x^* \in \mathbb{R}^n_+ : \langle x, x^* \rangle \geq u(x)\right\} \qquad \forall \lambda \in \mathbb{R}.$$

Therefore $u^{\#}$ is quasiconvex and lower semicontinuous.

"If". Suppose that w is quasiconvex, decreasing and lower semicontinuous, and define u by (17). This function is quasiconcave, increasing and upper semicontinuous since it is the pointwise infimum of the set of functions $\{\max\{\langle \cdot, x^* \rangle, w(x^*)\}\}_{x^* \in \mathbb{R}^n_+}$ each of which is quasiconcave, increasing and continuous. We shall prove that w is the secondary aggregator function associated with u, that is

$$w(x_0^*) = \sup\left\{\inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} : \langle x, x_0^* \rangle < \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\}\right\} \tag{18}$$

for all $x_0^* \in \mathbb{R}^n_+$. Given $x_0^* \in \mathbb{R}^n_+$, if $x \in \mathbb{R}^n_+$ is such that

$$\langle x, x_0^* \rangle < \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\},$$

then from

$$\inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} \leq \max\{\langle x, x_0^* \rangle, w(x_0^*)\}$$

we deduce

$$\inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} \leq w(x_0^*)$$

which proves that the inequality $\geq$ holds in (18). Actually, no property of w is required for the validity of this argument. Let us now prove the reverse inequality. Since the right hand side of (18) is not less than $\inf_{x^* \in \mathbb{R}^n_+} w(x^*)$, we only need to consider the case where $w(x_0^*) > \inf_{x^* \in \mathbb{R}^n_+} w(x^*)$. Let $\lambda \in \left(\inf_{x^* \in \mathbb{R}^n_+} w(x^*), w(x_0^*)\right)$. Notice that $\lambda > 0$, given that w is nonnegative. Since $x_0^* \notin w^{-1}([-\infty, \lambda])$ and this level set is convex and closed, by the separation theorem there exist $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{R}$ such that $\langle x_0, x_0^* \rangle < k \leq \langle x_0, x^* \rangle$ for all $x^* \in w^{-1}([-\infty, \lambda])$. From $w^{-1}([-\infty, \lambda]) \neq \emptyset$ and the fact that w is decreasing, one can easily deduce that $x_0 \in \mathbb{R}^n_+$. Hence $k > 0$. Thus for $x_1 = \dfrac{\lambda}{k} x_0$ one has $x_1 \in \mathbb{R}^n_+$ and $\langle x_1, x_0^* \rangle < \lambda \leq \langle x_1, x^* \rangle$ for all $x^* \in w^{-1}([-\infty, \lambda])$. For every

$x^* \in \mathbb{R}^n_+$ we have either $w(x^*) > \lambda$ or $x^* \in w^{-1}([-\infty, \lambda])$, in which case $\langle x_1, x^* \rangle \geq \lambda$. Therefore $\inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x_1, x^* \rangle, w(x^*)\} \geq \lambda > \langle x_1, x^*_0 \rangle$, that is x_1 satisfies the constraint under the supremum in the right hand side of (18). It follows that

$$\begin{aligned}\sup &\left\{ \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} : \langle x, x^*_0 \rangle < \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} \right\} \\ &\geq \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x_1, x^* \rangle, w(x^*)\} > \lambda.\end{aligned}$$

Since λ is an arbitrary number in the interval $\left(\inf_{x^* \in \mathbb{R}^n_+} w(x^*), w(x^*_0)\right)$, this proves the inequality $\leq$ in (18). Consequently, (18) holds.

We shall now prove that u of (17) is the largest function with which w is associated. Let $\widetilde{u} : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ be any function satisfying

$$w(x^*) = \sup\{\widetilde{u}(x) : \langle x, x^* \rangle < \widetilde{u}(x)\} \qquad \forall x^* \in \mathbb{R}^n_+, \tag{19}$$

and let $x \in \mathbb{R}^n_+$. Then for every $x^* \in \mathbb{R}^n_+$ we have either $\langle x, x^* \rangle < \widetilde{u}(x)$ in which case by (19) $\widetilde{u}(x) \leq w(x^*)$ or $\widetilde{u}(x) \leq \langle x, x^* \rangle$, and so $\widetilde{u}(x) \leq \max\{\langle x, x^* \rangle, w(x^*)\}$. Thus we conclude that

$$\widetilde{u}(x) \leq \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} = u(x)$$

which proves that $\widetilde{u} \leq u$.

Finally, we have to verify that among the functions with which w is associated the only one which is quasiconcave, increasing, upper semicontinuous and has DRS is u in (17). Let $\widetilde{u} : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ be any function satisfying these properties and having w as its secondary aggregator function. We have $w(x^*) = \sup\{\widetilde{u}(x) : \langle x, x^* \rangle < \widetilde{u}(x)\}$ and by Theorem 13

$$\widetilde{u}(x) = \inf_{(x^*, k) \in S} \max\{\langle x, x^* \rangle, k\} \qquad \forall x \in \mathbb{R}^n_+$$

for some nonempty set $S \subseteq \mathbb{R}^n_+ \times \mathbb{R}_+$. It follows that

$$\widetilde{u}(x) = \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, \widetilde{w}(x^*)\} \qquad \forall x \in \mathbb{R}^n_+ \tag{20}$$

with $\widetilde{w} : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ being the function defined by

$$\widetilde{w}(x^*) = \inf\{k \in \mathbb{R}_+ : (x^*, k) \in S\} \qquad \left(x^* \in \mathbb{R}^n_+\right).$$

Let u be the function defined by (17). We know that $\widetilde{u} \leq u$. On the other hand, $\widetilde{w} \geq w$ which we show next. Indeed, if $x^*, x \in \mathbb{R}^n_+$ are such that $\langle x, x^* \rangle < \widetilde{u}(x)$, then by (20) one has $\widetilde{u}(x) \leq \widetilde{w}(x^*)$. Hence

$$w(x^*) = \sup\{\widetilde{u}(x) : \langle x, x^* \rangle < \widetilde{u}(x)\} \leq \widetilde{w}(x^*).$$

Therefore, from (20) and (17) we obtain $\widetilde{u}(x) \geq \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} = u(x)$. We have thus proved that $\widetilde{u} = u$. ■

Now we give an interpretation of (17) for production functions. Let $x \in \mathbb{R}^n_+$ be an input vector. Given a price vector $x^* \in \mathbb{R}^n_+$, the maximum amount of output, measured in monetary units, that the producer can obtain subject to the "rationality" constraint of obtaining a strictly positive net profit is $w(x^*)$. By selling the input vector x under those prices he would obtain $\langle x, x^* \rangle$ monetary units. Hence, under the price vector x^*, the maximum amount of money available is $\max\{\langle x, x^* \rangle, w(x^*)\}$. Thus the right hand side of (17) gives the amount of money that is attainable under any price vector when the input vector x is available. Equation (17) says that this amount of money coincides with the amount of output $u(x)$, measured in monetary units, that can be produced by this input vector x.

Corollary 22 *Let $w : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ and let $w_0 : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ be defined by*

$$w_0(x^*) = \sup\{u(x) : \langle x, x^* \rangle < u(x)\} \qquad \left(x^* \in \mathbb{R}^n_+\right),$$

with u of (17). Then w_0 is the largest lower semicontinuous decreasing quasiconvex minorant of w.

Proof. By Theorem 21, w_0 is quasiconvex, decreasing and lower semicontinuous. To prove that $w_0 \le w$, let $x^* \in \mathbb{R}^n_+$ and $x \in \mathbb{R}^n_+$ be such that $\langle x, x^* \rangle < u(x)$. From this inequality and (17) it follows that $\langle x, x^* \rangle < w(x^*)$. By using (17) again we deduce that $u(x) \le \max\{\langle x, x^* \rangle, w(x^*)\} = w(x^*)$. Therefore $w_0(x^*) = \sup\{u(x) : \langle x, x^* \rangle < u(x)\} \le w(x^*)$, so that $w_0 \le w$. It only remains to prove that any lower semicontinuous decreasing quasiconvex minorant w_1 of w satisfies $w_1 \le w_0$. By Theorem 21, for every $x^* \in \mathbb{R}^n_+$ one has

$$w_1(x^*) = \sup\left\{\inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w_1(x^*)\} : \langle x, x^* \rangle < \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w_1(x^*)\}\right\}.$$

Since the right hand side of this inequality is increasing in w_1,

$$\begin{aligned} w_1(x^*) &\le \sup\left\{\inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\} : \langle x, x^* \rangle < \inf_{x^* \in \mathbb{R}^n_+} \max\{\langle x, x^* \rangle, w(x^*)\}\right\} \\ &= \sup\{u(x) : \langle x, x^* \rangle < u(x)\} = w_0(x^*). \end{aligned}$$

Thus $w_1 \le w_0$. ■

Theorem 23 *Let $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$. There exists $w : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ such that (17) holds if and only if u is quasiconcave, increasing, upper semicontinuous and has DRS. In this case w can be taken as a quasiconvex decreasing lower semicontinuous function. Under these conditions w is unique, namely w is the smallest function such that (17) holds. Furthermore it is the secondary aggregator function associated with u.*

Proof. "Only If". If (17) holds, then u is the pointwise infimum of the family $\{\max\{\langle \cdot, x^* \rangle, w(x^*)\}\}_{x^* \in \mathbb{R}^n_+}$ which consists of increasing quasiaffine continuous functions with DRS. Hence it is quasiconcave, increasing, upper semicontinuous and has DRS.

"If". If u is quasiconcave, increasing, upper semicontinuous and has DRS, then by Theorem 13 (17) holds with $w = u^{\#}$. By the same theorem $u^{\#}$ is quasiconvex, decreasing and lower semicontinuous.

Finally, if u is quasiconcave, increasing, upper semicontinuous and has DRS and w is a lower semicontinuous decreasing quasiconvex function for which (17) holds, then by Theorem 13 $w = u^{\#}$ so that w is unique. ■

Corollary 24 *Let $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ and let $u^0 : \mathbb{R}^n_+ \to \overline{\mathbb{R}}_+$ be defined by*

$$u^0(x) = \inf_{x^* \in \mathbb{R}^n_+} \max \left\{ \langle x, x^* \rangle, u^{\#}(x^*) \right\} \qquad \left(x \in \mathbb{R}^n_+ \right). \tag{21}$$

Then u^0 is the smallest upper semicontinuous increasing quasiconcave majorant of u that has DRS.

Proof. By Theorem 23 u^0 is quasiconcave, increasing, upper semicontinuous and has DRS. To prove that $u^0 \geq u$, let $x, x^* \in \mathbb{R}^n_+$. By (16), if $\langle x, x^* \rangle < u(x)$, then $u(x) \leq u^{\#}(x)$. Therefore $\max \left\{ \langle x, x^* \rangle, u^{\#}(x) \right\} \geq u(x)$. Hence from (21) we deduce that $u^0(x) \geq u(x)$. Thus u^0 is a majorant of u. It only remains to prove that any upper semicontinuous increasing quasiconcave majorant u^1 of u that has DRS satisfies $u^1 \geq u^0$. By Theorem 23, for every $x \in \mathbb{R}^n_+$ one has

$$u^1(x) = \inf_{x^* \in \mathbb{R}^n_+} \max \left\{ \langle x, x^* \rangle, \sup \left\{ u^1(x) : \langle x, x^* \rangle < u^1(x) \right\} \right\}.$$

Since the right hand side of this inequality is increasing in u^1, we have

$$\begin{aligned} u^1(x) &\geq \inf_{x^* \in \mathbb{R}^n_+} \max \left\{ \langle x, x^* \rangle, \sup \left\{ u(x) : \langle x, x^* \rangle < u(x) \right\} \right\} \\ &= \inf_{x^* \in \mathbb{R}^n_+} \max \left\{ \langle x, x^* \rangle, u^{\#}(x) \right\} = u^0(x). \end{aligned}$$

Thus $u^1 \geq u^0$. ■

In this last section we introduced a secondary aggregator function. It has an interesting application in case of production functions. As in previous sections we find that the possibility of representing u with a small infimal generator is closely related to the DRS property of u.

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