Free Triples, Large Indifference Classes and the Majority Rule*

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April 2002

Abstract

We present a new domain of preferences under which the majority relation is always quasi-transitive and thus Condorcet winners always exist. We model situations where a set of individuals must choose one individual in the group. Agents are connected through some relationship that can be interpreted as expressing neighborhood, and which is formalized by a graph. Our restriction on preferences is as follows: each agent can freely rank his immediate neighbors, but then he is indifferent between each neighbor and all other agents that this neighbor "leads to". Hence, agents can be highly perceptive regarding their neighbors, while being insensitive to the differences between these and other agents which are further removed from them. We show quasi-transitivity of the majority relation when the graph expressing the neighborhood relation is a tree. We also discuss a further restriction allowing to extend the result for more general graphs. Finally, we compare the proposed restriction with others in the literature, to conclude that it is independent of any previously discussed domain restriction.

*We acknowledge useful comments and suggestions by Gabrielle Demange and Matt Jackson. Barberà acknowledges the support of DGICYT and the Direcció General de Recerca through grants PD-98-0870 and 2001SGR-00162. This paper was written while the second author was a Marie Curie Research Fellow at C.O.D.E. at the Universitat Autònoma de Barcelona. He acknowledges the hospitality of the Departament d'Economia i d'Història Econòmica and financial support from the European Union under Grant No. HPMT-CT-2000-0017.

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JEL classification: C71, D71.
Keywords: Voting, Tree, Majority Rule.

1 Introduction

We present a new domain of preferences for which majority rule is always quasi-transitive, and hence Condorcet winners always exist. This new domain contains preferences which are clearly excluded by standard domain restrictions like single-peakedness, value restriction, and intermediateness. Our preferences arise naturally in certain voting contexts where agents may freely rank some alternatives, but then cannot clearly discriminate among many others. We hope that the new ideas underlying the proposed domains can play a role in proving the existence of voting equilibria within contexts where traditional restrictions on preferences do not hold.

To fix ideas, consider the following example. Assume that agents can be linearly ordered. Each agent has then one neighbor to his left, and one to his right (except for those at the extremes, who only have one neighbor). We allow each agent to freely order himself and its immediate neighbors. Some agent may prefer to be elected rather than seeing his neighbors elected. Others may prefer their neighbor to the right (or left) to be the winner. Likewise, second and third positions in the ranking are free. Hence, our preferences allow for free triples of alternatives. Yet, we also assume that agents cannot clearly distinguish between a victory by their neighbor in the right and the victory of any other candidate in the same direction (this may be due to myopia, or else result from rational calculations). He is indifferent among all candidates to his right, and also among all candidates to his left. In this example, then, each agent has (at most) three indifference classes: all people to the left of the voter form a class, all people to the right of the voter form a second class, and the voter himself is a third class. These classes can be ordered in any possible way. We shall prove that if all agents have preferences of this type, the majority relation associated to any profile of opinions by any number of voters is quasi-transitive (i.e., the strict majorities among alternatives respect transitivity). This is sufficient for the existence of Condorcet winners. It also allows to guarantee the existence of path independent selections (Plott, 1973) from the set of Condorcet winners: elections can be organized sequentially without any fear for agenda manipulation.

Our example can be extended. The linear structure is unnecessarily narrow. Agents may be at the nodes of any tree. Therefore, each agent may have a different number of neighbors, as many as the number of branches starting from or arriving to that node. An agent with k neighbors may now freely rank $k+1$ classes of alternatives, including oneself as a singleton class. Each of the remaining classes includes one neighbor and all other candidates which are further out than this neighbor within the tree structure. This is a very substantial extension of the domain: very perceptive people can have many neighbors and thus freely rank many classes of candidates. Other agents may be restricted by their positions to only rank a few groups. Many structures are allowed, provided they can be represented by a tree, of any form. The tree structure is essential. It will be proved that preferences defined in a similar way on graphs that include loops will always lead to majority cycles for some distributions of opinion. We shall also prove, however, that our basic result still holds if we impose further requirements on the preferences associated with general graphs.

Our general domain restriction can be respected by many different sets of preference profiles, depending on the underlying graph. Let us notice that the preferences which are admissible for one agent are in general not the same as those admissible for others. For the same reason, the set of triples which are free for one agent need not coincide with that of triples which are free for another. Because of these special features, our condition on preference profiles is independent of other domain restrictions in the literature. We shall discuss the connections between several classical domains and our own in Section 5. Before that, the paper is structured as follows. In Section 2 we present our basic model, where individuals have preferences defined on themselves and on other members of society. The connections among members of society, which will be later used to restrict their preferences on others, are also modelled, after the necessary introduction of some concepts from the theory of graphs. In Section 3 we prove our main result: when the graph restricting the agent's preferences is a tree, then the majority relation for any society satisfying our domain condition is always quasi-transitive: hence, a majority (Condorcet) winner always exists. We also argue in this section that not much more can be learnt from the model regarding the position of these winners in the tree. Section 4 deals with the possibility of extending the positive results of Section 3 to a larger class of graphs. We shall discuss the connections between several classical domains and our own in Section 5.

2 Agents, Networks and Preferences

Let N denote the finite set of agents. We always assume that $|N| \geq 3$. Agents have preferences on themselves and on other agents, which are given by complete, reflexive, transitive binary relations on N. The preferences of agent i are denoted by R_i, his strict preferences by P_i and his indifference relation by I_i. The restriction of a binary relation R_i on a subset N' of N will be denoted by $R_i|N'$. We shall concentrate on cases where the preferences of agents on themselves and on other individuals are partially determined by the structure of the set of relationships among agents. Specifically, we shall assume in the next section that the relationships which are relevant for our purposes can be represented by a tree. This will be relaxed later, in Section 4.

Before anything else, let us introduce some necessary pieces of language from graph theory. An *edge* is an element $(ij) \in N \times N$ such that $i \neq j$. Each edge is undirected, i.e. for all $(ij) \in N \times N$, $(ij) = (ji)$. A *network* is a pair (N, E) where E is a set of edges. Given $i, j \in N$ such that $i \neq j$, a *path from i to j* in (N, E) is a sequence $(k_l)_{l \in \{1,\dots,t\}}$ such that (i) $k_1 = i$ and $k_t = j$, (ii) $|\{k_l \,|\, l \in \{1, \dots, t\}\}| = t$, and (iii) for all $l \in \{2, \dots, t\}$, $(k_{l-1}k_l) \in E$. Note that (ii) ensures that a path does not contain any cycles. A network (N, E) is *connected* if for all $i, j \in N$ such that $i \neq j$, there exists a path from i to j in (N, E). A network (N, E) is called a *tree* if (N, E) is connected and for all $i, j \in N$ such that $i \neq j$, if $(k_l)_{l \in \{1,\dots,t\}}$ and $(k'_l)_{l \in \{1,\dots,t'\}}$ are paths from i to j in (N, E), then $t = t'$ and for all $l \in \{1, \dots, t\}$, $k_l = k'_l$. Let (N, E) be a network. In case (N, E) is a tree, we denote by $[i, j]$ the set of agents belonging to the unique path from i to j in (N, E). Given $i \in N$, let $E(i)$ denote the set of agents with whom i forms an edge including himself, i.e. $E(i) \equiv \{j \in N \backslash \{i\} \,|\, (ij) \in E\} \cup \{i\}$.

A *cycle* in (N, E) is a sequence $(k_l)_{l \in \{1,\dots,t\}}$ such that (i) $|\{k_l \,|\, l \in \{1, \dots, t\}\}| = t \geq 3$, (ii) for all $l \in \{1, \dots, t-1\}$, $(k_l k_{l+1}) \in E$, and (iii) $(k_t k_1) \in E$.

A *cyclical component* of the network (N, E) is a set $C \subseteq N$ such that (i) $|C| \geq 3$, (ii) for all $i, j \in C$ there exist two paths $(k^1_l)_{l \in \{1,\dots,t^1\}}$ and $(k^2_l)_{l \in \{1,\dots,t^2\}}$ from i to j in (N, E) such that $\{k^1_l \,|\, l \in \{1, \dots, t^1\}\} \subseteq C$, $\{k^2_l \,|\, l \in \{1, \dots, t^2\}\} \subseteq C$, and $\{k^1_l \,|\, l \in \{1, \dots, t^1\}\} \cap \{k^2_l \,|\, l \in \{1, \dots, t^2\}\} = \{i, j\}$, and (ii) C is maximal with respect to inclusion, i.e. if $C' \subseteq N$ satisfies (ii) and $C' \supseteq C$, then $C' = C$. Given $i \in N$, let $C(i)$ denote the cyclical components to which i belongs to, i.e.

$$C(i) \equiv \{C \subseteq N \,|\, i \in C \text{ and } C \text{ is a cyclical component in } (N, E)\}.$$

Let $\mathcal{C} \equiv \cup_{i \in N} C(i)$ denote the set of all cyclical components in (N, E).

We shall interpret that the network (N, E) represents the set of connections among the agents which are relevant to form their preferences. Based on this interpretation, we shall describe the additional restrictions on preferences which we consider natural in our context. To begin with, take the case where (N, E) is a tree. We shall then find that the permissible preferences for agent i are those that express any ranking of i and its immediate neighbors $E(i)$, and then declare each neighbor j indifferent to all other agents k such that j is between i and k. Formally, an *admissible preference relation of i in a tree* (N, E) is a transitive relation R_i such that for all $j, h \in N \backslash \{i\}$, if there exists $g \in E(i) \backslash \{i\}$ such that $g \in [i, j] \cap [i, h]$, then $j I_i h$. Thus, R_i is determined by the restriction of R_i to the set $E(i)$. Let $\mathcal{R}_i$ denote the set of i's preference relations in (N, E). A *(preference) profile* is a list $R \equiv (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}_i$. Let $\mathcal{R}_N \equiv \times_{i \in N} \mathcal{R}_i$ denote the set of admissible profiles.

3 Existence of Condorcet Winners

Suppose that society must choose one agent from N, when each of its members is endowed with preferences in a domain $\mathcal{R}_i$ restricted by a tree (N, E). One possibility is to choose by majority. For each profile belonging to $\mathcal{R}_N$ the majority relation is defined as follows: Given $R \in \mathcal{R}_N$ and $i, j \in N$, let $N(i \succ_R j) \equiv \{h \in N \mid i P_h j\}$. With each profile $R \in \mathcal{R}_N$ we associate its majority relation, denoted by R^m: for all $i, j \in N$, $i R^m j$ if and only if

$$|N(i \succ_R j)| \geq |N(j \succ_R i)|.$$

It is well known that the majority relation derived from an arbitrary set of preferences may be cyclical. When this occurs, the choices by majority are not well defined. We are thus particularly interested in conditions guaranteeing that this problem is avoided. For each profile $R \in \mathcal{R}_N$, an agent is a Condorcet winner if he is not beaten (in the strict sense) by another agent under the majority relation, i.e. $j \in N$ is a Condorcet winner of R if for all $i \in N$, $j R^m i$. We denote by $C_W(R)$ the set of all Condorcet winners of R. The existence of Condorcet winners is guaranteed at those profiles where the majority relation is quasi-transitive (i.e. the strict relation associated with the majority relation is transitive). This justifies our interest in the following Theorem and its Corollary.

Theorem 1 *Let (N, E) be a tree and $R \in \mathcal{R}_N$. Then the majority relation associated with R is quasi-transitive.*

Corollary 1 *Let (N, E) be a tree. For all $R \in \mathcal{R}_N$, the set of Condorcet winners of R is non-empty.*

Proof of Theorem 1. Let $R \in \mathcal{R}_N$. First note the following fact. For all $i, j \in N$ such that $i \neq j$ we have

$$N(i \succ_R j) \subseteq [i, j] \text{ and } N(j \succ_R i) \subseteq [i, j]. \quad (1)$$

Let $a, b, c \in N$ be such that $aP^m b$ and $bP^m c$. We have to show that $aP^m c$.

Because (N, E) is a tree and $|\{a, b, c\}| = 3$, there is $d \in N$ such that $\{d\} = [a, b] \cap [a, c] \cap [b, c]$. Note that $d = a$, $d = b$, or $d = c$ is possible.

Because $[a, d[\cap \{a, b, c\} \subseteq \{a\}$, for all $i \in [a, d[$ we have $aP_i bI_i c$ (call this preference over $\{a, b, c\}$ the Type-I-preference), $bI_i cP_i a$ (Type-II-preference), or $aI_i bI_i c$. Let n_1 denote the number of agents belonging to $[a, d[$ who have Type-I-preference over $\{a, b, c\}$. Let n_2 denote the number of agents belonging to $[a, d[$ who have Type-II-preference over $\{a, b, c\}$.

Because $[b, d[\cap \{a, b, c\} \subseteq \{b\}$, for all $i \in [b, d[$ we have $bP_i aI_i c$ (Type-III-preference), $aI_i cP_i b$ (Type-IV-preference), or $aI_i bI_i c$. Let n_3 denote the number of agents belonging to $[b, d[$ who have Type-III-preference over $\{a, b, c\}$. Let n_4 denote the number of agents belonging to $[b, d[$ who have Type-IV-preference over $\{a, b, c\}$.

Because $[c, d[\cap \{a, b, c\} \subseteq \{c\}$, for all $i \in [c, d[$ we have $cP_i aI_i b$ (Type-V-preference), $aI_i bP_i c$ (Type-VI-preference), or $aI_i bI_i c$. Let n_5 denote the number of agents belonging to $[c, d[$ who have Type-V-preference over $\{a, b, c\}$. Let n_6 denote the number of agents belonging to $[c, d[$ who have Type-VI-preference over $\{a, b, c\}$.

For all $i, j \in N$, let $1_{iP_d j} \equiv 1$ if $iP_d j$ and $1_{iP_d j} \equiv 0$ otherwise. By $[a, b] = [a, d[\cup \{d\} \cup [b, d[$, (1), and $aP^m b$, we have

$$n_1 + n_4 + 1_{aP_d b} > n_2 + n_3 + 1_{bP_d a}. \tag{2}$$

By $[b, c] = [b, d[\cup \{d\} \cup [c, d[$, (1), and $bP^m c$, we have

$$n_3 + n_6 + 1_{bP_d c} > n_4 + n_5 + 1_{cP_d b}. \tag{3}$$

From (2) and (3) we obtain

$$n_1 + n_6 + 1_{aP_d b} + 1_{bP_d c} > n_2 + n_5 + 1_{bP_d a} + 1_{cP_d b} + 1. \tag{4}$$

By (1) and $[a, c] = [a, d[\cup \{d\} \cup [c, d[$, to show $aP^m c$, we have to prove that

$$n_1 + n_6 + 1_{aP_d c} > n_2 + n_5 + 1_{cP_d a}. \tag{5}$$

We distinguish three cases.

Case 1: $1_{aP_d b} + 1_{bP_d c} = 0$.

Then $bR_d a$ and $cR_d b$. Thus, by transitivity of R_d, $cR_d a$ and $1_{aP_d c} = 0$. Because $1_{bP_d a} + 1_{cP_d c} + 1 \geq 1_{cP_d a}$, (5) follows from (4).

Case 2: $1_{aP_d b} + 1_{bP_d c} = 1$.

Then $1_{bP_d a} + 1_{cP_d b} \leq 1$. Thus, from (4) we obtain $n_1 + n_6 > n_2 + n_5 + 1$. Then (5) follows.

Case 3: $1_{aP_db} + 1_{bP_dc} = 2$.

Then aP_db and bP_dc. Thus, by transitivity of P_d, aP_dc and $1_{aP_dc} = 1$. From (4) and $1_{bP_da} + 1_{cP_db} = 0$ we obtain $n_1 + n_6 + 1 > n_2 + n_5$. Because $1_{aP_dc} = 1$, (5) follows from the previous inequality. □

Having proved our basic result, the rest of the Section is devoted to introduce a number of relevant qualifications. First, notice that our domain condition is based in identifying each of the agents with one node in the graph. This is consistent with the interpretation that individuals are at the same time the voters and the candidates for choice. It also requires that the position of each of the individuals is exclusive. One could think of a closely related but more general setup, where each node would stand for a generic position in society, each position could be shared by several agents, and a domain restriction on preferences based on the same principles would apply. Unfortunately, Theorem 1 is not robust to such an extension, as shown by the following example.

Example 1 Let $N \equiv \{a, b, c, d, e\}$ and $E \equiv \{(ab), (bc), (cd), (de)\}$. Given $i \in N$, let q_i denote the number of agents which are located at node i. Let $q_a \equiv q_e \equiv 4$, $q_b \equiv q_d \equiv 6$, and $q_c \equiv 3$. Let $R \in \mathcal{R}_N$ be such that

R_a	R_b	R_c	R_d	R_e
a	ce	a	e	ac
ce	a	c	ac	e
	b	e	d	

Then $N(a \succ_R c) = \{a, c\}$ and $N(c \succ_R a) = \{b\}$. Because $q_a + q_c = 7 > 6 = q_b$, it follows that aP^mc. Then $N(c \succ_R e) = \{c, e\}$ and $N(e \succ_R c) = \{d\}$. Because $q_c + q_e = 7 > 6 = q_d$, it follows that cP^me. Then $N(e \succ_R a) = \{b, d\}$ and $N(a \succ_R e) = \{a, c, e\}$. Because $q_b + q_d = 12 > 11 = q_a + q_c + q_e$, it follows that eP^ma. Thus, P^m is cyclic over $\{a, c, e\}$. Moreover, it is straightforward to check that aP^mb and cP^md, and thus $C_W(R) \subseteq \{a, c, e\}$. Hence, the set of Condorcet winners is empty at R. ◁

We now turn attention to the possible implications of our domain restriction, beyond quasi-transitivity of the majority relation. As we shall see, it does not imply any particular constraints on the location of the Condorcet winners within the tree that originates the domain restriction. This is in contrast with the results attached to other domains. For example, under single-peaked preferences (see Section 5), the Condorcet winners are to be found in median positions. To be specific on the preceding point, we present two results. The first one shows that any subset of N can be obtained as the set of Condorcet winners associated with some admissible preferences, for any tree (N,E). The second result shows that full transitivity is not to be expected in general. This is illustrated by showing that under two natural subdomains within our class, the typical situation is to obtain quasi-transitive but not transitive majority relations.

Proposition 1 *Let (N,E) be a tree and $\emptyset \neq S \subseteq N$. Then there exists $\bar{R} \in \mathcal{R}_N$ such that $C_W(\bar{R}) = S$.*

Proof. Let $\bar{R} \in \mathcal{R}_N$ be such that (i) for all $i \in S$ and all $j,k \in N\backslash\{i\}$, $i\bar{P}_i j\bar{I}_i k$ and (ii) for all $i \in N\backslash S$ and all $j \in N$, $i\bar{I}_i j$. It is straightforward that $C_W(\bar{R}) = S$. □

Any arbitrary transitive ranking of any three alternatives can be obtained as the result of majority voting on our domains.[1] However, as the following example shows, not any transitive relation on a tree can be obtained as the majority relation on our preference domain.

Example 2 Let $N \equiv \{a,b,c,d,e,f\}$ and $E \equiv \{(ab),(bc),(cd),(de),(ef)\}$. Let R_0 be the relation on N such that $R_0 : f,a,b,c,d,e$ (i.e. $fP_0aP_0bP_0cP_0dP_0e$). We show that R_0 cannot be obtained as a majority preference relation on the domain $\mathcal{R}_N$.

Suppose that there exists $R \in \mathcal{R}_N$ such that $R^m = R_0$. Then by $aP^m b$, we have ($aP_a b$ and $aR_b b$) or ($aR_a b$ and $aP_b b$). By $bP^m c$, $bR_b c$ and $bR_c c$. Thus, by transitivity of R_b and $R \in \mathcal{R}_N$, we have ($aP_a f$ and $aR_b f$) or ($aR_a f$ and $aP_b f$). Hence, $|N(a \succ_R f) \cap \{a,b\}| \geq 1$ and $a,b \notin N(f \succ_R a)$. Similar arguments together with $bR_c c$ and $dR_d e$ (by $dP^m e$) yield $|N(a \succ_R f) \cap \{c,d\}| \geq 1$ and $c,d \notin N(f \succ_R a)$. Therefore, $|N(a \succ_R f) \cap \{a,b,c,d\}| \geq 2$ and $N(f \succ_R a) \subseteq \{e,f\}$. Hence, by definition of R^m, $aR^m f$, which contradicts $fP_0 a$ and $R^m = R_0$. ◁

There are natural additional domain restrictions for which only in rare cases majority rule is transitive. We define two subdomains of $\mathcal{R}_N$ and identify for each domain the unique profile for which the majority rule is transitive.

Given $i \in N$, let $\mathcal{R}_i^t \subseteq \mathcal{R}_i$ denote the preferences in $\mathcal{R}_i$ under which i ranks himself as the unique best element, i.e. $R_i \in \mathcal{R}_i^t$ if and only if $R_i \in \mathcal{R}_i$

[1] Given any three alternatives and a fixed ranking of these three alternatives, there exists an $i \in N$ such that these three alternatives are a free triple for him. Then choose the profile for which the restriction of i's relation to these three alternatives is the fixed ranking and the other agents are indifferent between all alternatives.

and for all $j \in N\backslash\{i\}$, iP_ij. Let $\mathcal{R}_N^t \equiv \times_{i\in N}\mathcal{R}_i^t$. The domain $\mathcal{R}_N^t$ applies to situations in which each agent would like to be elected. Dutta, Jackson, and Le Breton (2001) consider such domains for each agent running as a candidate in an election.

There are also situations in which no agent would like to be elected. Each agent strictly prefers another agent being elected to himself being elected. Given $i \in N$, let $\mathcal{R}_i^b \subseteq \mathcal{R}_i$ denote the preferences in $\mathcal{R}_i$ under which i ranks himself as the unique worst element, i.e. $R_i \in \mathcal{R}_i^b$ if and only if $R_i \in \mathcal{R}_i$ and for all $j \in N\backslash\{i\}$, jP_ii. Let $\mathcal{R}_N^b \equiv \times_{i\in N}\mathcal{R}_i^b$.

Proposition 2 *Let (N, E) be a tree and $R \in \mathcal{R}_N^t \cup \mathcal{R}_N^b$. The majority relation associated with R is transitive if and only if for all $i \in N$ and all $j, h \in E(i)\backslash\{i\}$, jI_ih (and therefore, R^m is the trivial relation, i.e. for all $j, h \in N$, jI^mh).*

Proof. Let $R \in \mathcal{R}_N^t \cup \mathcal{R}_N^b$. Then for all $i, j \in N$, iR^mj if and only if

$$|N(i \succ_R j) \cap]i,j[| \geq |N(j \succ_R i) \cap]i,j[|. \quad (6)$$

The only-if statement is trivial. In proving the if-statement, suppose that R^m is transitive.

Let $i \in N$ and $j, h \in E(i)\backslash\{i\}$. By (6), jI^mi and iI^mh. Thus, by transitivity of R^m, jI^mh. By (6), $R^m|\{j,h\} = R_i|\{j,h\}$. Hence, jI_ih, the desired conclusion. □

4 Voting on Connected Networks

In this section we consider connected networks which are not necessarily trees. If (N, E) is not a tree, then (N, E) contains cycles.

There are several ways of extending preferences from trees to general networks. For instance, consider a cycle containing four nodes. Then each agent has a ranking over his neighbors and himself, and there is exactly one agent whom he does not see. How should he rank this agent? One possibility is to rank this agent indifferent to one of his direct neighbors. But this will not be sufficient to avoid majority cycles. Just to check, we propose the following extension of preferences: let (N, E) be a connected network and $i, j \in N$ be such that $j \notin E(i)$. Then i considers j to be indifferent to one of his direct neighbors who belongs to a shortest path from i to j in the network. This extension is conform with the definition of preferences in Section 2.

Each agent's preference is determined by his preference over $E(i)$ in the following way: $R_i \in \mathcal{R}_i$ if and only if for all $j \in N\backslash\{i\}$, there exists $h \in$

$E(i)\backslash\{i\}$ such that (i) jI_ih and (ii) there exists a path $(k^1_l)_{l\in\{1,\dots,t^1\}}$ from i to j in (N,E) such that $k^1_2=h$ and (iii) for all paths $(k^2_l)_{l\in\{1,\dots,t^2\}}$ from i to j in (N,E), we have $t^2\geq t^1$.

Condition (i) says that under R_i, j is indifferent to a direct neighbor h of i, and (ii) and (iii) say that h must belong to one of the shortest paths from i to j.

Proposition 3 *Let (N,E) be a connected network. If (N,E) is not a tree, then the set of Condorcet winners might be empty on the domain $\mathcal{R}_N$.*

Proof. Because (N,E) is not a tree, (N,E) contains cycles. Let $(k_l)_{l\in\{1,\dots,t\}}$ be a cycle in (N,E) of minimal length, i.e. for all cycles $(k'_l)_{l\in\{1,\dots,t'\}}$ in (N,E) we have $t'\geq t$. Because $(k_l)_{l\in\{1,\dots,t\}}$ is a cycle of minimal length in (N,E), we can choose $a,b,c\in\{k_l\mid l\in\{1,\dots,t\}\}$ such that $\{a,b,c\}$ is a free triple in $\mathcal{R}_a$, $\mathcal{R}_b$, and $\mathcal{R}_c$.

Let $R\in\mathcal{R}_N$ be such that

- for all $i\in N\backslash\{a,b,c\}$ and all $j,h\in N\backslash\{i\}$, jI_ihP_ii,
- aP_abP_ac and for all $i\in N\backslash\{a,b,c\}$, bR_ai,
- bP_bcP_ba and for all $i\in N\backslash\{a,b,c\}$, cR_bi, and
- cP_caP_cb and for all $i\in N\backslash\{a,b,c\}$, aR_ci.

We show that $C_W(R)=\emptyset$.

Let $i\in N\backslash\{a,b,c\}$. By definition of R, $N(a\succ_R i)\supseteq\{a,i\}$ and $N(i\succ_R a)\subseteq\{b\}$. Thus, aP^mi and $i\notin C_W(R)$. Hence, $C_W(R)\subseteq\{a,b,c\}$. By definition of R, for all $i\in N\backslash\{a,b,c\}$, aI_ibI_ic. Thus, aP^mb, bP^mc, and cP^ma. Hence, $a,b,c\notin C_W(R)$ and $C_W(R)=\emptyset$, the desired conclusion. □

The domain $\mathcal{R}_N$ was too large to avoid majority cycles. We now present a more stringent restriction, which still reduces to the condition proposed in Section 2 when applied to trees, and which now will guarantee that the majority relation is quasi-transitive, and thus a majority winner exists. For that, we assume that each agent considers all other agents belonging to a cyclical component of $C(i)$ to be indifferent.

Given a connected network (N,E) and $i\in N$, let $\bar{\mathcal{R}}_i$ denote the set of all preferences $R_i\in\mathcal{R}_i$ satisfying for all $j,h\in N\backslash\{i\}$, if for some $C\in C(i)$, $j,h\in C$, then jI_ih. Let $\bar{\mathcal{R}}_N\equiv\times_{i\in N}\bar{\mathcal{R}}_i$.

The idea of the proof is to successively delete links from the network until we obtain a tree. Then we show that we are in position to apply Theorem 1.

Theorem 2 *Let (N,E) be a connected network and $R \in \bar{\mathcal{R}}_N$. Then the majority relation associated with R is quasi-transitive.*

Corollary 2 *Let (N,E) be a connected network. For all $R \in \bar{\mathcal{R}}_N$, the set of Condorcet winners of R is non-empty.*

Proof of Theorem 2. If (N,E) is a tree, then Theorem 1 implies that R^m is quasi-transitive. Suppose that (N,E) is not a tree. Then (N,E) contains cycles. Let $E^0 \equiv E$. Starting from E^0 we successively delete edges from E^0 until we obtain a network which does not contain any cycles. Let $q \in \mathbb{N} \cup \{0\}$. If (N,E^q) contains cycles, then choose a cycle $(k_l^q)_{l\in\{1,\ldots,t\}}$ in (N,E^q) of minimal length and define $E^{q+1} \equiv E^q \backslash \{(k_1^q k_2^q)\}$. Since E^0 is finite, there exists $Q \in \mathbb{N}$ such that (N,E^Q) does not contain any cycles.

We show that for all $q \in \{0,1,\ldots,Q-1\}$, if (N,E^q) is connected, then (N,E^{q+1}) is connected. Because (N,E^q) is connected and $(k_l^q)_{l\in\{1,\ldots,t\}}$ is a cycle in (N,E^q) of minimal length, for all $i \in N$, there exists a path from k_1^q to i in (N,E^{q+1}). Thus, for all $i,j \in N$, there exists a path from k_1^q to i in (N,E^{q+1}) and a path from k_1^q to j in (N,E^{q+1}). Therefore, there exists a path from i to j in (N,E^{q+1}). Hence, (N,E^{q+1}) is connected.

Since (N,E^Q) is connected and does not contain any cycles, (N,E^Q) is a tree. We show that for all $i \in N$, R_i is an admissible preference relation of i in the tree (N,E^Q). Let $j,g \in N$ be such that $g \in (E^Q(i)\backslash\{i\}) \cap [i,j]^Q$ ($[i,j]^Q$ refers to the path from i to j in (N,E^Q)). It suffices to show that jI_ig. By $R_i \in \mathcal{R}_i$, there is a shortest path $(k_l)_{l\in\{1,\ldots,t\}}$ from i to j in (N,E) such that jI_ik_2. If g belongs to all shortest paths from i to j in (N,E), then $g = k_2$ and jI_ig. If there is a cyclical component $C \in C(i)$ such that $j,g \in C$, then $R_i \in \bar{\mathcal{R}}_i$ implies jI_ig. If the previous two cases do not apply and $k_2 \neq g$, then there is a cyclical component $C \in C(i)$ such that $g \in C$. Suppose $k_2 \notin C$. Because $E^Q \subseteq E$, $[i,j]^Q$ is also a path from i to j in (N,E). Therefore, $k_2 \notin [i,j]^Q$, (g,i,k_2) is a path from g to k_2 in (N,E), and there is a path in $([i,j]^Q \cup \{k_l \,|\, l \in \{1,\ldots,t\}\})\backslash\{i\}$ from g to k_2 in (N,E). This is a contradiction to $g \in C$ and $k_2 \notin C$. Thus, $g,k_2 \in C$. Since $R_i \in \bar{\mathcal{R}}_i$, we have k_2I_ig. Because jI_ik_2, the transitivity of I_i implies jI_ig.

Hence, for all $i \in N$, R_i is an admissible preference relation of i in the tree (N,E^Q). By Theorem 1, R^m is transitive, the desired conclusion. □

5 Conclusion

There exists a vast literarture on domain restrictions and their implications under different rules of preference aggregation. A very complete monograph is due to Gaertner (2001). The most studied aggregation rule is simple majority, and the standard properties which are sought from the majority relation are transitivity, quasi-transitivity, acyclicity or the existence of a maximal element of the relation. Among the many restrictions which have been studied, the most popular is still that of single-peakedness (Black, 1948). Other domains were analyzed by Inada (1964,1969), Sen and Pattanaik (1969), Demange (1982), and Grandmont (1978).[2] Our domains are different than any of those we just mentioned, and they do not seem to have been considered by the previous literature. We now comment briefly on the analogies and the difference between our type of restrictions and those proposed by other authors. First of all, the restrictions we describe are limited in scope, since they only apply to the case where each distinct voter can be identified with a distinct alternative. Its applicability is thus limited to cases where there are as many voters as there are alternatives. Now, even in this particular case, there is a fundamental difference between our setup and all the others we mention (with the exception of Grandmont's). In our case, the set of orders of the alternatives which are admissible is different for each of the agents. Indeed, each voter is allowed to have at most as many indifference classes as the number of its immediate neighbors plus one (we refer to restrictions generated by trees, those considered in Section 2). Hence, agents with different numbers of immediate neighbors will be allowed different sets of preferences. Moreover, even those agents with the same number of neighbors will be in such relationship with different sets of people, and this also accounts for differences among their admissible sets of preferences. In short: under our restrictions, admissible domains are personalized. By contrast, the classical restrictions we now briefly review do limit the set of preferences which are admissible, but then allow all agents to exhibit any of the preferences in this common pool. Inada (1964) considered the case where each agent can classify the set of alternatives into two groups, and then will consider all alternatives within the same group as indifferent. Our conditions also rely on the establishment of "large" indifference classes, but the analogy stops here.[3]

Another interesting set of restrictions were proposed by Sen and Pattanaik (1969) and Inada (1969). A profile $R \in \mathcal{R}_N$ satisfies *value restriction* if for

[2]Inada (1964,1969) and Grandmont (1978) were concerned with the transitivity of the majority relation, Sen and Pattanaik (1969) with the quasi-transitivity of the majority relation, and Demange (1982) with the existence of a maximal element.

[3]Not any classifications of Inada (1964) are included in our model. For example, let $N \equiv \{1, 2, 3, 4\}$ and each agent classifies N into two sets of cardinality 2. If there is a tree such that $\mathcal{R}_N$ includes Inada's (1964) preferences, then the agents at the terminal nodes must classify N into a set of cardinality 1 and a set of cardinality 3, which is impossible.

every triple of alternatives $\{a, b, c\} \subseteq N$ there is one (say a) that is not ranked worst (or best or medium) by all individuals who are not indifferent between a, b, and c (i.e. (for all $i \in N$ such that $\neg a I_i b I_i c$, $a P_i b \vee a P_i c$) or (for all $i \in N$ such that $\neg a I_i b I_i c$, $b P_i a \vee c P_i a$) or (for all $i \in N$ such that $\neg a I_i b I_i c$, $(a P_i b \wedge a P_i c) \vee (b P_i a \wedge c P_i a)$)). A profile $R \in \mathcal{R}_N$ satisfies *extremal restriction* if for every triple of alternatives $\{a, b, c\} \subseteq N$ and if for some $i \in N$, $b P_i a P_i c$, then for all $j \in N$, if $c P_j b$, then $c P_j a P_j b$. A profile $R \in \mathcal{R}_N$ satisfies *limited agreement* if for every triple of alternatives $\{a, b, c\} \subseteq N$ there are two alternatives, say a and b, such that for all $i \in N$, $a R_i b$.

These restrictions define domains under which the majority rule and other forms of binary comparisons will be well behaved. As already noted, there is no reason to expect that conditions of the above type would either imply or be implied by ours, for any tree. The following set of preferences for three agents and three candidates confirm that fact.

Example 3 Let $N \equiv \{a, b, c\}$ and $E \equiv \{(ab), (bc)\}$. Let $R \in \mathcal{R}_N$ be such that

R_a	R_b	R_c
a	b	c
bc	a	ab
	c	

Then R violates value restriction, extremal restriction, and limited agreement. ◁

Sen and Pattanaik (1969, Theorem V) show that a necessary and sufficient condition for the majority rule to be quasi-transitive is that a profile of preference orderings satisfies for each triple of alternatives at least one of the conditions, value restriction, extremal restriction or limited agreement. However, in their result the number of individuals is variable and therefore it does not apply to our model. According to Theorem 1, the majority relation is quasi-transitive in Example 3. Of course, if we allow several agents to be located at one node, then Sen and Pattanaik's result applies, as also shown in Example 1.

Demange (1982) proposed an extension of single-peakedness based on the relative positions of alternatives in the vertices of a tree (the original notion of single-peakedness is based on their position on a line, which is a very special tree). Demange's proposal bears a resemblance with ours in that it builds from a set of a priori given connections among alternatives which can be formalized as a tree. But, once again, the analogy stops here, for the reasons we already mentioned above. Finally, Grandmont (1978) proposed a notion of intermediate preferences leading to attractive and quite different domain restrictions. This notion is based upon the possibility of defining when an agent is in between two others. It requires that, if the two initial agents agree on how to rank certain subset of alternatives, than any agent who is intermediate between them also shares these common preferences. Our domain restriction is also based on the relative positions of agents, but the type of limitations it imposes on preferences is of completely different nature.

As for applications, let us mention one instance where restrictions of the type we just proposed do apply. In a recent paper, Demange (2000) has studied the distribution of profits from cooperation in games with hierarchies. Hierarchies are described by a tree, which describes connections between agents, and by a specific individual, among all the agents, who plays the role of the principal. Demange (2000) shows that if only coalitions which are properly connected can form (she calls them teams), then the (restricted) core of the cooperative game among these agents is nonempty and easy to describe. The admissible teams (and thus, the resulting core distribution) depend on the tree and also on the specific agent who plays the role of principal. One may extend the analysis of Demange (2000) by separating these two ingredients, and by allowing all possible hierarchies which arise from the same tree, as the principal changes. In Demange's interpretation, the tree expresses possible channels of communication among agents. Suppose that these channels are technologically determined, but that the directions of hierarchical communication may be chosen. For example, all agents may vote on who is going to play the role of the principal. Their preferences will depend on the payoffs that they will get in the core, depending on who is the principal (for a given tree). It turns out that agents will get the same payoff for all principals who are on the same branch away from some of their immediate neighbors. That is, preferences induced by the proposed extension of Demange's model satisfy our requirements. Hence, the majority rule would always determine (at least) one winner if agents in that context would vote for a principal. This example is mentioned to illustrate that the restriction arises even in rather unexpected contexts. It suggests that other situations where someone must be chosen to play a special role, may give rise to similar conditions. Of course, voting is only one of the possible methods to choose an agent to play a role. In certain contexts, especially if side payments are possible, these roles may be auctioned (see Pérez-Castrillo and Wettstein (2002)). But voting is, to say the least, one of the most prevailing methods to choose agents, and it is good to know about conditions where its simplest version, simple majority) will work properly.

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