Constrained School Choice*

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Abstract: Recently, several school districts in the US have adopted or consider adopting the Student-Optimal Stable Mechanism or the Top Trading Cycles Mechanism to assign children to public schools. There is clear evidence that for school districts that employ (variants of) the so-called Boston Mechanism the transition would lead to efficiency gains. The first two mechanisms are strategy-proof, but in practice student assignment procedures impede students to submit a preference list that contains all their acceptable schools. Therefore, any desirable property of the mechanisms is likely to get distorted. We study the non trivial preference revelation game where students can only declare up to a fixed number (quota) of schools to be acceptable. We focus on the stability of the Nash equilibrium outcomes. Our main results identify rather stringent necessary and sufficient conditions on the priorities to guarantee stability. This stands in sharp contrast with the Boston Mechanism which yields stable Nash equilibrium outcomes, independently of the quota. Hence, the transition to any of the two mechanisms is likely to come with a higher risk that students seek legal action as lower priority students may occupy more preferred schools.

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1 Introduction

School choice is referred in the literature on education as giving parents a say in the choice of the schools their children will attend. A recent paper by Abdulkadiroğlu and Sönmez (2003) has lead to an upsurge of enthusiasm in the use of matching theory for the design and study

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of school choice mechanisms.[1] Abdulkadiroğlu and Sönmez (2003) discuss critical flaws of the current procedures of some school districts in the US to assign children to public schools. They point out that the widely used Boston Mechanism has the serious shortcoming that it is not in the parents' best interest to reveal their true preferences. Using a mechanism design approach, they propose and analyze two alternative student assignment mechanisms that do not have this shortcoming: the Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism.

Real-life school choice situations typically involve a large number of participants and a relatively small number of school programs. For instance, in the school district of New York city each year more than 90,000 students have to be assigned to about 500 school programs (Abdulkadiroğlu *et al.*, 2005). Parents are asked to elicit a preference list containing only a limited number of schools (currently up to 12). This restriction is reason for concern. Since complete revelation of one's true preferences is typically no longer an option in this case, the argument that Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism are strategy-proof is no longer valid. Imposing a curb on the length of the submitted lists, though certainly having the merit of "simplifying" matters, has the perverse effect of forcing participants not to be truthful, and eventually compel them to adopt a strategic behavior when choosing which ordered list to submit. In other words, we are back in the situation of the Boston Mechanism where participants are forced to play a complicated admission game. Participants may adopt strategic behavior because the "quantitative" effect, *i.e.*, participants cannot reveal their complete preference lists, is likely to have a "qualitative" effect, that is, participants may self-select by not declaring their most preferred options. For instance, if a participant fears rejection by his most preferred programs, it can be advantageous not to apply to these programs and use instead its allowed application slots for less preferred programs.

The goal of this paper is to scrutinize the effects of imposing a *quota* (*i.e.*, a maximal length of submittable preference lists) on the strategic behavior of students. Thereby we revive an issue that was initially discussed by Romero-Medina (1998).[2] To this end, we study school choice problems (Abdulkadiroğlu and Sönmez, 2003) where a number of students has to be assigned to a number of schools, each of which has a limited capacity of seats. Students have preferences over schools and remaining unassigned and schools have exogenously given priority rankings over students.[3] We introduce a non trivial preference revelation game where students can only declare up to a fixed number (the quota) of schools to be acceptable. Each possible quota, from 1 up to the total number of schools, together with a student assignment mechanism induces a strategic "quota-game." We analyze the Nash equilibria and focus on the stability

[1] Recent papers include Abdulkadiroğlu (2005), Abdulkadiroğlu *et al.* (2005), Abdulkadiroğlu *et al.* (2006), Chen and Sönmez (2006), Ergin and Sönmez (2006), Kesten (2006b), Kojima (2006).

[2] To the best of our knowledge, Romero-Medina (1998) is the only paper that explicitly analyzes restrictions on the length of submitted preference lists. He focuses exclusively on the Student-Optimal Stable Mechanism and establishes that the set of stable matchings is implemented in Nash equilibria, independently of the quota (Romero-Medina, 1998, Theorem 7 and Corollary 8). It is true that any stable matching can be sustained at some Nash equilibrium (the first inclusion in Proposition 6.1). However, in general there are also unstable Nash equilibrium outcomes (Examples 6.4 and 8.3).

[3] Very often local or state laws determine the priority rankings. Typically, students who live closer to a school or have siblings attending a school have higher priority to be admitted at the school. In other situations, priority rankings may be determined by one or several entrance exams. Then students who achieve higher test scores in the entrance exam of a school have higher priority for admission at the school than students with lower test scores.

of the induced outcomes. Stability is the central concept in the two-sided matching literature[4] and does not lose its importance in the closely related model of school choice. Loosely speaking, stability of an assignment obtains when, for any student, all the schools he prefers to the one he is assigned to have exhausted their capacity with students that have higher priority. Hence, if an assignment is not stable then a student can seek legal action against the school district authorities for not getting assigned a seat which is either unfilled or filled by a student with a lower priority. Moreover, violations of stability are rather easily detectable; one does not need to consider larger groups of students or schools.

Our main findings can be summarized as follows. For all three mechanisms Nash equilibrium and for any quota, Nash equilibria in pure strategies exist. In fact, a straightforward extension of a result due to Ergin and Sönmez (2006) says that the Boston mechanisms implements the set of stable matching, independently of the quota. For the Student-Optimal Stable Mechanism existence of Nash equilibria in pure strategies was proved by Romero-Medina (1998). For the Top Trading Cycles Mechanism the proof of existence of Nash equilibria in pure strategies is more tortuous. We first show that the Nash equilibrium outcomes do not vary with the quota, and then invoke the strategy-proofness of the mechanism for the unconstrained case.

Next, given the direct implementation result for the Boston Mechanism we only need to analyze the Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism. We first establish that the associated quota-games have a common feature: the equilibria are nested with respect to the quota. More precisely, given a quota any Nash equilibrium is also a Nash equilibrium under any less stringent quota. This leads to the following important observation: If a Nash equilibrium outcome in a quota-game has an undesirable property then this is not simply due to the presence of a constraint on the size of submittable lists. The two mechanisms are different in another aspect: unlike the Top Trading Cycles Mechanism, under the Student-Optimal Stable Mechanism any stable matching can be sustained at some Nash equilibrium, independently of the quota. Yet, in general, under both mechanisms there are also unstable Nash equilibrium outcomes. We exhibit a school choice problem with a (strong) Nash equilibrium in "intuitive" undominated truncations that yields an unstable matching. On the positive side we identify for each of the two mechanisms a necessary and sufficient condition on the priorities to guarantee stable Nash equilibrium outcomes. In the case of the Student-Optimal Stable Mechanism this turns out to be Ergin's (2002) acyclicity condition. For the Top Trading Cycles Mechanism the necessary and sufficient condition is Kesten's (2006a) acyclicity condition. In other words, the two acyclicity condition are necessary and sufficient conditions on the priority structure for the implementation of the set of stable matchings in the two direct preference revelation games.

As a policy implication, our results suggest on the positive side that stability in the restrictive procedure is obtained through strategic interaction if the assignment of students is based on a common priority ranking. On the negative side, in view of the implementation result via the Boston Mechanism and the restrictiveness of both acyclicity conditions, the transition to either the Student-Optimal Stable Mechanism or the Top Trading Cycles Mechanism is likely to come with a higher risk that students seek legal action as lower priority students may occupy more preferred schools.

[4] In many centralized labor markets, clearinghouses are most often successful if they produce stable matchings. Empirical evidence is given in Roth (1984, 1990, 1991) and Roth and Xing (1994).

Besides its policy implications, our paper gives additional strength to Ergin's (2002) and Kesten's (2006a) acyclicity conditions. Ergin (2002) showed that his acyclicity condition on the priority structure is sufficient for Pareto-efficiency, group strategy-proofness, and consistency of the Student-Optimal Stable Mechanism as well as necessary for each of these conditions separately. Maybe somewhat surprisingly, the same acyclicity condition also serves to guarantee the stability of the Nash equilibrium outcomes under the Student-Optimal Stable Mechanism. Kesten (2006a) showed that his acyclicity on the priority structure is sufficient for resource monotonicity, population monotonicity, and stability of the Top Trading Cycles Mechanism as well as necessary for each of these conditions separately. He also proved that the Top Trading Cycles Mechanism coincides with the Student-Optimal Stable Mechanism if and only if the priority structure is acyclic. We show that exactly the same condition also guarantees the stability of the Nash equilibrium outcomes under the Top Trading Cycles Mechanism.

The remainder of the paper is organized as follows. In Section 2, we recall the model of school choice. In Section 3 we describe the three mechanisms and provide an illustrative example. In Section 4, we introduce the strategic game induced by the imposition of a quota on the revealed preferences. In Sections 5, 6, and 7 we present our results on the existence, nestedness, and stability of the Nash equilibrium outcomes for the quota-game under the Boston, Student-Optimal Stable, and Top Trading Cycles Mechanism, respectively. In Section 8 we study Nash equilibria of undominated truncations for the Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism. Finally, in Section 9 we discuss the policy implications of our results and possible future research directions. All proofs are relegated to the Appendices.

2 School Choice

In a school choice problem (Abdulkadiroğlu and Sönmez, 2003), there are a number of schools and a number of students each of which has to be assigned a seat at not more than one of the schools. Each student is assumed to have strict preferences over his acceptable set of schools. Each school is endowed with a strict priority ordering of all students and a fixed capacity of seats that can be filled.

Formally, a *school choice problem* is a 5-tuple (I, S, q, P, f) that consists of

1. a set of *students* $I = \{i_1, \ldots, i_n\}$;
2. a set of *schools* $S = \{s_1, \ldots, s_m\}$;
3. a *capacity* vector $q = (q_1, \ldots, q_m)$;
4. a profile of strict *student preferences* $P = (P_{i_1}, \ldots, P_{i_n})$;
5. a strict *priority structure* of the schools over the students $f = (f_{s_1}, \ldots, f_{s_m})$.

We denote by i and s a generic student and a generic school, respectively. An agent is an element of $V := I \cup S$. A generic agent is denoted by v. With a slight abuse of notation we write v for singletons $\{v\} \subseteq V$.

The preference relation P_i of student i is a linear order over $S \cup i$, where i denotes the option of remaining unassigned. Student i is said to prefer school s to school s' if sP_is'. School s is *acceptable* to i if sP_ii. Henceforth, when describing a particular preference relation of a student we will only represent acceptable schools. For instance, $P_i = s, s'$ means that student i's most preferred school is s, his second best s', and any other school is unacceptable. For the sake of convenience, if all schools are unacceptable for i then we sometimes write $P_i = i$ instead of $P_i = \emptyset$. Let R_i denote the weak preference relation associated with the preference relation P_i.

The *priority ordering* f_s of school s assigns ranks to students according to their priority for school s. The *rank* of student i for school s is $f_s(i)$. Then, $f_s(i) < f_s(j)$ means that *student i has higher priority (or lower rank) for school s than student j*. For $s \in S$ and $i \in I$, we denote $U_s^f(i)$ for the set of students that have higher priority than student i for school s, *i.e.*, $U_s^f(i) = \{j \in I : f_s(j) < f_s(i)\}$.

Throughout the paper we fix the set of students I and the set of schools S. Hence, a *school choice problem* is given by a triple (P, f, q), and simply by P when no confusion is possible.

School choice is closely related to the college admissions model (Gale and Shapley ,1962). The only but key difference between the two models is that in school choice schools are mere "objects" to be consumed by students, whereas in the college admissions model (or more generally, in two-sided matching) both sides of the market are agents with preferences over the other side. In other words, a college admissions problem is given by 1–4 above and 5' below:

5'. a profile of strict school preferences $P_S = (P_{s_1}, \ldots, P_{s_m})$,

where P_s denotes the strict preference relation of school $s \in S$ over all students.

Priority orderings in school choice can be reinterpreted as school preferences in the college admissions model. Therefore, many results or concepts for the college admissions model have their natural counterpart for school choice.[5] In particular, an outcome of a school choice or college admissions problem is a *matching* $\mu : I \cup S \to 2^I \cup S$ such that for any $i \in I$ and any $s \in S$,

- $\mu(i) \in S \cup i$;
- $\mu(s) \in 2^I$;
- $\mu(i) = s$ if and only if $i \in \mu(s)$;
- $|\mu(s)| \leq q_s$.

For $v \in V$, we call $\mu(v)$ agent v's allotment, respectively. For $i \in I$, if $\mu(i) = s \in S$ then student i is said to be *assigned* a seat at school s under μ. If $\mu(i) = i$ then student i is said to be *unassigned* under μ. For convenience we often write a matching as a collection of sets. For instance, $\mu = \{\{i_1, i_2, s_1\}, \{i_3\}, \{i_4, s_2\}\}$ denotes the matching in which students i_1 and i_2 each are assigned a seat at school s_1, student i_3 is unassigned, and student i_4 is assigned a seat at school s_2.

A key property of matchings in the two-sided matching literature is stability. Informally, a matching is stable if there is no blocking pair school-student such that the student prefers to occupy a seat of the school, and the school reciprocally prefers to let the student occupy a seat (by possibly dismissing one of its current students). Stability does not lose its importance in the context of school choice. The reason is that if a matching is not stable then a student can seek

[5] See, for instance, Balinski and Sönmez, 1999; Ehlers and Klaus, 2006a,b; Ergin, 2002; Ergin and Sönmez, 2006; Kesten, 2006a,b.

legal action against the school district authorities for not getting assigned a seat which is either unfilled or filled by a student with a lower priority. Formally, let P be a school choice problem. A matching μ is stable if

- it is *individually rational*, *i.e.*, for all $i \in I$, $\mu(i) R_i i$;
- it is *non-wasteful* (Balinski and Sönmez, 1999), *i.e.*, for all $i \in I$ and all $s \in S$, $s P_i \mu(i)$ implies $|\mu(s)| = q_s$;
- there is no *justified envy*, *i.e.*, for all $i, j \in I$ with $\mu(j) = s \in S$, $s P_i \mu(i)$ implies $f_s(j) < f_s(i)$.

It will be convenient to denote the set of individually rational matchings by $IR(P)$, the set of non-wasteful matchings by $NW(P)$, and the set of stable matchings by $S(P)$.

Another desirable property for a matching is Pareto-efficiency. In the context of school choice, the schools are mere "objects." Therefore, to determine whether a matching is Pareto-efficient we should only take into account the students. A matching ν Pareto dominates a matching μ if all students prefer ν to μ and there is at least one student that strictly prefers ν to μ. Formally, ν *Pareto dominates* μ if $\nu(i) R_i \mu(i)$ for all $i \in I$, and $\nu(i') P_{i'} \mu(i')$ for some $i' \in I$. A matching is *Pareto-efficient* if it is not Pareto dominated by any other matching.

A (student assignment) *mechanism* systematically selects a matching for each school choice problem. A mechanism is individual rational if it always selects an individually rational matching. Similarly, one can speak of non-wasteful, stable, or Pareto-efficient mechanisms.

A mechanism is *strategy-proof* if no student can ever benefit by unilaterally misrepresenting his preferences. A mechanism is non-bossy (Satterthwaite and Sonnenschein, 1981) if no student can maintain his allotment and cause a change in the other students' allotments by reporting different preferences. Formally, a mechanism φ is *non-bossy* if for all $i \in I$, $Q_i, Q'_i \in \mathcal{Q}(m)$, and $Q_{-i} \in \mathcal{Q}(m)^{I \setminus i}$, $\varphi(Q'_i, Q_{-i})(i) = \varphi(Q_i, Q_{-i})(i)$ implies $\varphi(Q'_i, Q_{-i}) = \varphi(Q_i, Q_{-i})$.

3 The Competing Mechanisms

In this section we describe the mechanisms that we study in the context of constrained school choice: the Boston Mechanism, the Gale-Shapley Student-Optimal Stable Mechanism, and the Top Trading Cycles Mechanism. The three mechanisms are direct mechanisms and for any priority structure and reported students' preferences they find a matching via the following algorithms. Let (I, S, q, P, f) be a school choice problem. Set $q_s^1 := q_s$ for all $s \in S$. We sometimes use an additional superindex P and hence write $q_s^{P,1}$, *etc.* to avoid possible confusion.

The Boston Algorithm

STEP 1: Each student i proposes to his most preferred school in P_i (if i finds all schools unacceptable he remains unassigned). Each school s assigns up to q_s^1 seats to its proposers one at a time following the priority order f_s. Remaining students are rejected. Let q_s^2 denote the number of available seats at school s. If $q_s^2 = 0$ then school s is removed.

STEP l, $l \geq 2$: Each student i that is rejected in Step $l-1$ proposes to his next preferred school in P_i (if i finds all remaining schools unacceptable he remains unassigned). School s assigns up to q_s^l seats to its (new) proposers one at a time following the priority order f_s. Remaining students are rejected. Let q_s^l denote the number of available seats at school s. If $q_s^l = 0$ then school s is removed.

The algorithm stops when no student is rejected or all schools have been removed. Any remaining student remains unassigned. Let $\beta(P)$ denote the matching. The mechanism β is the Boston Mechanism.

The Boston Mechanism is individually rational, non-wasteful, and Pareto-efficient. However, it is not stable nor strategy-proof.

The Gale-Shapley Deferred Acceptance (DA) Algorithm

Step 1: Each student i proposes to his most preferred school in P_i (if i finds all schools unacceptable he remains unassigned). Each school s tentatively assigns up to q_s seats to its proposers one at a time following the priority order f_s. Remaining students are rejected.

Step l, $l \geq 2$: Each student i that is rejected in Step $l-1$ proposes to his next preferred school in P_i (if i finds all remaining schools unacceptable he remains unassigned). Each school s considers the new proposers and the students that have a (tentative) seat at s. School s tentatively assigns up to q_s seats to these students one at a time following the priority order f_s. Remaining students are rejected.

The algorithm ends when no student is rejected. Each student is assigned to his final tentative school. Let $\gamma^I(P) = \gamma(P)$ denote the matching. The mechanism γ is the Student-Optimal Stable Mechanism.

The Student-Optimal Stable Mechanism is a stable mechanism that is Pareto superior to any other stable matching mechanism (Gale and Shapley, 1962). An additional important property of the Student-Optimal Stable Mechanism is that it is strategy-proof (Dubins and Freedman, 1981; Roth, 1982). Ergin (2002) showed that "weak acyclicity" of the priority structure is sufficient for Pareto-efficiency, group strategy-proofness, and consistency of the Student-Optimal Stable Mechanism as well as necessary for each of these conditions separately.[6] Finally, by letting the schools propose in the DA-algorithm we obtain, from the students' point of view, the worst stable matching, the School-Optimal Stable Matching, denoted by $\gamma^S(P)$.[7]

The Top Trading Cycles (TTC) Algorithm

Step 1: Each student i points to his most preferred school in P_i (if i finds all schools unacceptable he points to himself, *i.e.*, he forms a *self-cycle*). Each school s points to the student that has the highest priority in f_s. There is at least one cycle. If a student is in a cycle he is assigned a seat at the school he points to (or to himself if he is in a self-cycle). Students that are assigned are removed. If a school s is in a cycle and $q_s^1 = 1$, then the school is removed. If a school s is in a cycle and $q_s^1 > 1$, then the school is not removed and its capacity becomes $q_s^2 := q_s^1 - 1$.

Step l, $l \geq 2$: Each student i that has not been removed yet points to his most preferred school in P_i that has not been removed at some step r, $r < l$, or points to himself if he finds all

[6]Ergin (2002) used the terminology of cycles and acyclicity. However, since we will need to introduce another cyclicity concept due to Kesten (2006a) we slightly change the terminology conveniently.

[7]The Student-Optimal Stable Mechanism is employed in several real-life two-sided matching markets. For instance, the National Resident Matching Program, which assigns medical graduates to hospitals in the US, was redesigned in 1998 and it was decided to switch from the School-Optimal to the Student-Optimal Stable Mechanism (Roth and Peranson, 1999; Roth, 2002).

remaining schools unacceptable. Each school s points to the student with the highest priority in f_s among the students that have not been removed at a step r, $r < l$. There is at least one cycle. If a student is in a cycle he is assigned a seat at the school he points to (or to himself if he is in a self-cycle). Students that are assigned are removed. If a school s is in a cycle and $q_s^l = 1$, then the school is removed. If a school s is in a cycle and $q_s^l > 1$, then the school is not removed and its capacity becomes $q_s^{l+1} := q_s^l - 1$.

The algorithm stops when all students or all schools have been removed. Any remaining student is assigned to himself. Let $\tau(P)$ denote the matching. The mechanism τ is the Top Trading Cycles Mechanism.

The Top Trading Cycles Mechanism was introduced by Abdulkadiroğlu and Sönmez (2003).[8] The Top Trading Cycles Mechanism is a Pareto-efficient and strategy-proof mechanism (see Abdulkadiroğlu and Sönmez, 2003, for proofs in the context of school choice). The mechanism is also individually rational and non-wasteful. Kesten (2006a) showed that "acyclicity" of the priority structure is sufficient for resource monotonicity, population monotonicity, and stability of the Top Trading Cycles Mechanism as well as necessary for each of these conditions separately. He also proved that the Top Trading Cycles Mechanism coincides with the Student-Optimal Stable Mechanism if and only if the priority structure is acyclic.[9]

We illustrate the working of the three mechanisms in the following example.

Example 3.1 Let $I = \{i_1, i_2, i_3, i_4\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 2, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. So, for instance, $P_{i_1} = s_2, s_1$ and $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3) < f_{s_1}(i_4)$.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	f_{s_1}	f_{s_2}	f_{s_3}
s_2	s_1	s_1	s_2	i_1	i_3	i_4
s_1	s_2	s_2	s_3	i_2	i_4	i_1
	s_3		s_1	i_3	i_1	i_2
				i_4	i_2	i_3

If the students truthfully report their preference lists, then the mechanisms yield the following matchings.

The Boston Mechanism.

In Step 1 of the Boston Algorithm each student proposes to his most preferred school. So, school s_1 receives a proposal from i_2 and i_3. Student i_2 has a higher priority, so i_3's proposal is rejected and i_1 is assigned the unique seat at s_1. School s_2 receives a proposal from i_1 and i_4. Since school s_2 has 2 seats each of the students i_1 and i_4 is assigned a seat at s_2.

[8] The Top Trading Cycles Mechanism was inspired by Gale's Top Trading Cycles Algorithm which was used by Roth and Postlewaite (1977) to obtain the unique core allocation for housing markets (Shapley and Scarf ,1974).

[9] A variant of the Top Trading Cycles Mechanism was introduced by Abdulkadiroğlu and Sönmez (1999) for a model of house allocation with existing tenants. Pápai (2000) introduced the class of hierarchical exchange rules of which the Top Trading Cycles Mechanism is a special case. She characterized the class of hierarchical exchange rules to be the only mechanisms that are Pareto-efficient, group strategy-proof (*i.e.*, immune to preference misrepresentations by groups of agents), and reallocation-proof (*i.e.*, immune to manipulations by mispresenting preferences and swapping the assigned objects ex post by pairs of agents).

Schools s_1 and s_2 have filled all their seats and hence are removed. The tentative matching is $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$.

In Step 2 student i_3 cannot propose to his next preferred school, s_2. Since he finds school s_3 unacceptable he is removed and remains unassigned. So, the final matching is given by

$$\beta(P) = \{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}.$$

The Student-Optimal Stable Mechanism.

In Step 1 of the DA algorithm each student proposes to his most preferred school. So, school s_1 receives a proposal from i_2 and i_3. Student i_2 has a higher priority, so i_3's proposal is rejected. School s_2 receives a proposal from i_1 and i_4. Since school s_2 has 2 seats it does not reject any of the two students. The tentative matching is $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$.

In Step 2 student i_3 proposes to school s_2. So, now school s_2 has two (tentatively) accepted students, i_1 and i_4, and one new proposal, from i_3. Since school s_2 has 2 seats it rejects i_1, the student with the lowest priority. The tentative matching becomes $\{\{s_1, i_2\}, \{s_2, i_3, i_4\}, \{s_3\}, \{i_1\}\}$.

In Step 3 student i_1 proposes to school s_1. The unique seat of school s_1 is tentatively occupied by i_2. Since i_1 has a higher priority than student i_2, the latter is rejected. The tentative matching becomes $\{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3\}, \{i_2\}\}$.

In Step 4 student i_2 proposes to school s_3. Since school s_3's unique seat is available, student i_2 is accepted. No student has been rejected in this step, so the tentative matching is the final matching and is given by

$$\gamma(P) = \{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3, i_2\}\}.$$

The Top Trading Cycles Mechanism.

In Step 1 of the TTC algorithm each student points to his most preferred school, and each school points to the student with highest priority. There is a unique cycle that is given by (i_1, s_2, i_3, s_1). So, students i_1 and i_3 are assigned a seat at schools s_2 and s_1, respectively. Students i_1 and i_3 are removed. Since school s_1 had only 1 available seat it is also removed. School s_2 still has an available seat and is therefore not removed. The tentative matching is $\{\{s_1, i_3\}, \{s_2, i_1\}, \{s_3\}, \{i_2\}, \{i_4\}\}$.

In Step 2 there is a unique cycle given by (i_4, s_2). So, student i_4 is assigned the remaining seat at school s_2. Both student i_4 and school s_2 are removed. The tentative matching is $\{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_2\}\}$.

In Step 3 only student i_2 and school s_3 remain. Since i_2 finds school s_3 acceptable, he points to the school. Since i_2 is the only remaining student, school s_3 points to i_2. This creates a cycle and hence i_2 is assigned a seat at school s_3. So, the final matching is

$$\tau(P) = \{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3, i_2\}\}.$$

Note that for the school choice problem above the three mechanisms generate different matchings. Also, the obtained matchings illustrate directly some of the "problems" of the mechanisms. For instance, $\beta(P)$ is Pareto-efficient but not stable because student i_3 has justified envy with respect to school s_2 and any of the students that occupy a seat. In fact, one readily sees that

β is not strategy-proof. (Would student i_3 have announced the list that only contains school s_2 he would have guaranteed a seat at this school.) Similarly, one easily verifies that $\gamma(P)$ is stable but not Pareto-efficient and that $\tau(P)$ is Pareto-efficient but not stable. More importantly, note that if in a direct revelation game under γ or τ students could only submit a list of 2 schools, student i_2 would remain unassigned (and the other students unaffected), provided that each student submits the list with his two most preferred schools. Therefore, if students can only submit short preference lists, then (at least) student i_3 would have to strategize. $\diamond$

4 Constrained Preference Revelation: the Quota-Game

Fix the priority ordering f and the capacities q. We consider the following school choice procedure. Students are asked to submit (simultaneously) preference lists $Q = (Q_{i_1}, \ldots, Q_{i_n})$ of "length" at most k (*i.e.*, preference lists with at most k acceptable schools). Here, k is a positive integer, $1 \leq k \leq m$, and is called the *quota*. Subsequently, a mechanism φ is used to obtain the matching $\varphi(Q)$ and for all $i \in I$, student i is assigned a seat at school $\varphi(Q)(i)$.

It is clear that the above procedure induces a strategic-form game, the *Quota-Game* $\Gamma^{\varphi}(P,k) = \langle I, \mathcal{Q}(k)^I, P \rangle$. The set of players is the set of students I. The strategy set of each student is the set of preference lists with at most k acceptable schools and is denoted by $\mathcal{Q}(k)$. Let $\mathcal{Q} = \mathcal{Q}(m)$. Outcomes of the game are evaluated through the true preferences $P = (P_{i_1}, \ldots, P_{i_n})$, where with some abuse of notation P denotes the straightforward extension of the preference relation over schools and the null school to matchings. That is, for all $i \in I$ and matchings μ and μ', $\mu P_i \mu'$ if and only if $\mu(i) P_i \mu'(i)$.

For any profile of preferences $Q \in \mathcal{Q}^I$ and any $i \in I$, we write Q_{-i} for the profile of preferences that is obtained from Q after leaving out preferences Q_i of student i. A profile of submitted preference lists $Q \in \mathcal{Q}(k)^I$ is a Nash equilibrium in the game $\Gamma^{\varphi}(P,k)$ (or *k-Nash equilibrium* for short) if for all $i \in I$ and all $Q'_i \in \mathcal{Q}(k)$, $\varphi(Q_i, Q_{-i}) R_i \varphi(Q'_i, Q_{-i})$. Let $\mathcal{E}^{\varphi}(P,k)$ denote the set of k-Nash equilibria. Let $\mathcal{O}^{\varphi}(P,k)$ denote the set of k-Nash equilibrium outcomes, *i.e.*, $\mathcal{O}^{\varphi}(P,k) = \{\varphi(Q) : Q \in \mathcal{E}^{\varphi}(P,k)\}$.

5 Boston Mechanism

Our first result that will serve as a benchmark for the other two mechanisms states that the Boston Mechanism implements the set of stable matchings, independently of the quota. This follows directly from a straightforward adaptation of the proof of Theorem 1 in Ergin and Sönmez (2006).[10] Its proof is therefore omitted.

Theorem 5.1 *Let $1 \leq k \leq m$. Then, for any school choice problem P, the game $\Gamma^{\beta}(P,k)$ implements $S(P)$ in Nash equilibria.*

[10] Kojima (2006) shows that the implementation result of Ergin and Sönmez (2006) can also be extended to situations in which schools have more general priority structures.

6 Student-Optimal Stable Mechanism

Our first concern is the existence of Nash equilibria in pure strategies. This question is easily settled in the next proposition. In fact, the next proposition states that any stable matching is the outcome of some Nash equilibrium, and each outcome of some Nash equilibrium is at least individually rational and non-wasteful.

Proposition 6.1 *For any school choice problem P and quota k, $\emptyset \neq S(P) \subseteq \mathcal{O}^{\gamma}(P,k) \subseteq IR(P) \cap NW(P)$.*

We are in position to prove a first important result: the equilibria of the Quota-Games are nested in the sense that for $1 \leq k < m$ any k-Nash equilibrium is also a $(k+1)$-Nash equilibrium. In other words, if a strategy profile is a Nash equilibrium for some quota then it is also a Nash equilibrium for all less stringent quotas. Obviously, this also implies that the sets of Nash equilibrium outcomes are nested. An important consequence of this result is that violation of stability in equilibrium is not simply caused by the presence of a quota imposed on the students' preference lists. Indeed, if for some quota k a Nash equilibrium outcome is unstable then this unstable matching is also supported at (the same) equilibrium when the quota is set to m (the number of school) which is equivalent to not imposing any quota.

Proposition 6.2 *For any school choice problem P and quotas $k < k'$, $\mathcal{E}^{\gamma}(P,k) \subseteq \mathcal{E}^{\gamma}(P,k')$.*

Now we will explore whether the first inclusion of Proposition 6.1 can be reversed. In other words: is the outcome of each Nash equilibrium in the game $\Gamma^{\gamma}(P,k)$ free of justified envy with respect to the true preferences P? We start off with a positive answer for the restrictive case of $k = 1$.

Proposition 6.3 *For any school choice problem P, $S(P) = \mathcal{O}^{\gamma}(P,1)$.*

In Implementation Theory jargon, Proposition 6.3 says that the game $\Gamma^{\gamma}(P,1)$ implements the set of stable matchings for P in Nash equilibria. The following example shows that for any quota $k \neq 1$ this is in general not true.

Example 6.4 ***An Unstable Nash Equilibrium Outcome in $\mathbf{\Gamma^{\gamma}(P,k)}$ for quota $\mathbf{k \neq 1}$***
Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2\}$ be the set of schools, and $q = (1,1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}
s_2	s_1	s_1	i_1	i_3
s_1	s_2	s_2	i_2	i_1
			i_3	i_2

Using the DA-algorithm one finds $\gamma^I(P) = \gamma^S(P) = \{\{i_1, s_1\}, \{i_3, s_2\}, \{i_2\}\}$. Hence, $S(P) = \{\gamma^I(P)\}$ consists of the unique stable matching in which i_1 and i_3 are assigned to s_1 and s_2, respectively, and i_2 remains unassigned.

Let $k = 2$ be the quota. Consider the truth-telling strategy for students i_1 and i_3, *i.e.*, let $Q_{i_1} = P_{i_1}$ and $Q_{i_3} = P_{i_3}$, respectively. Regarding student i_2, consider his 5 possible strategies in $\mathcal{Q}(2) = \{Q^a, Q^b, Q^c, Q^d, Q^e\}$, where $Q^a = s_1$, $Q^b = s_2$, $Q^c = s_1, s_2$, $Q^d = s_2, s_1$, and $Q^e = \emptyset$.

One easily verifies that for any profile $Q = (Q_{i_1}, Q_{i_2}, Q_{i_3})$ with $Q_{i_2} \in \mathcal{Q}(2)$, $\gamma(Q)(i_2) = i_2$. Now note that for $Q^* := (Q_{i_1}, Q^b, Q_{i_3})$, $\gamma(Q^*) = \{\{i_1, s_2\}, \{i_3, s_1\}, \{i_2\}\}$. Hence, at $\gamma(Q^*)$ students i_1 and i_3 are assigned to their most preferred school. So, neither i_1 nor i_3 has a profitable deviation. Hence, $Q^* \in \mathcal{E}^\gamma(P, 2)$. However, $\gamma(Q^*) = \{\{i_1, s_2\}, \{i_3, s_1\}, \{i_2\}\}$ is not stable for P. (Student i_2 has justified envy for school s_1, since $\gamma(Q^*)(i_3) = s_1$, $s_1 P_{i_2} \gamma(Q^*)(i_2)$, and $f_{s_1}(i_2) < f_{s_1}(i_3)$.)

Note that for $k > 2$, $n > 3$, and/or $m > 2$ one can obtain an unstable k-Nash equilibrium outcome by making schools s_1 and s_2 unacceptable in the other students' preferences and the other schools unacceptable for students i_1, i_2, and i_3. ◇

In Example 6.4 student i_2 can block a potential settlement between the other two students without affecting his own position: $\gamma(Q_{i_1}, Q^c, Q_{i_3}) = \{\{i_1, s_1\}, \{i_3, s_2\}, \{i_2\}\}$ but $\gamma(Q_{i_1}, Q^e, Q_{i_3}) = \{\{i_1, s_2\}, \{i_3, s_1\}, \{i_2\}\}$. (This in fact shows that γ is bossy.) Moreover, the restricted capacities of the schools makes that there is competition for the schools. These are the two ingredients that generate an unstable Nash equilibrium outcome. In fact, the example exhibits a strongly cyclic priority structure (Ergin, 2002).

Definition 6.5 ***Strong Cycles and Weak Acyclicity (Ergin, 2002)***
Given a priority structure f, a *strong cycle* is constituted of distinct $s, s' \in S$ and $i, j, l \in I$ such that the following two conditions are satisfied:
cycle condition $f_s(i) < f_s(j) < f_s(l)$ and $f_{s'}(l) < f_{s'}(i)$ and
c-scarcity condition there exist disjoint sets $I_s, I_{s'} \subseteq I \backslash \{i, j, l\}$ (possibly $I_s = \emptyset$ or $I_{s'} = \emptyset$) such that $I_s \subseteq U_s^f(j)$, $I_{s'} \subseteq U_{s'}^f(i)$, $|I_s| = q_s - 1$, and $|I_{s'}| = q_{s'} - 1$.
A priority structure is *weakly acyclic* if no strong cycles exist. △

Ergin (2002) showed that weak acyclicity of the priority structure is sufficient for Pareto-efficiency, group strategy-proofness, and consistency of the Student-Optimal Stable Mechanism as well as necessary for each of these conditions separately. Our first main result says that weak acyclicity is also a necessary and sufficient condition to guarantee that for any profile of student preferences all Nash equilibrium outcomes are stable matchings:

Theorem 6.6 *Let $k \neq 1$. Then, f is a weakly acyclic priority structure if and only if for any school choice problem P, the game $\Gamma^\gamma(P, k)$ implements $S(P)$ in Nash equilibria.*

7 Top Trading Cycles Mechanism

For starters, the Nash equilibrium of the Quota-Game induced by the Top Trading Cycles mechanism are at least individually rational and non-wasteful.

Proposition 7.1 *For any school choice problem P and quota k, $\mathcal{O}^\tau(P, k) \subseteq IR(P) \cap NW(P)$.*

Following the structure of Section 6 and before turning to the implementation of the set of stable matchings, we first turn to the counterpart of Proposition 6.2. The following definition introduces a property that guarantees that a mechanism has nested Nash equilibria.

Definition 7.2 ***Individually Idempotent Mechanism***
Let φ be a mechanism. We say that φ is *individually idempotent* if for any $Q \in \mathcal{Q}^I$, any $i \in I$, $\widetilde{Q}_i = \varphi(Q)(i) \in \mathcal{Q}(1)$ implies $\varphi(\widetilde{Q}_i, Q_{-i}) = \varphi(Q)$. △

Proposition 7.3 *Let φ be an individually idempotent mechanism. For any school choice problem P and quotas $k < k'$, $\mathcal{E}^\varphi(P, k) \subseteq \mathcal{E}^\varphi(P, k')$.*

Lemma 7.4 *Mechanism τ is individually idempotent.*

Corollary 7.5 *For any school choice problem P and quotas $k < k'$, $\mathcal{E}^\tau(P, k) \subseteq \mathcal{E}^\tau(P, k')$.*

Note that a result due to Roth (1982) [Lemma A.1 in the Appendix] may suggest that γ is individually idempotent: for any $Q \in \mathcal{Q}^I$, any $i \in I$, $\widetilde{Q}_i = \gamma(Q)(i) \in \mathcal{Q}(1)$ implies $\gamma(\widetilde{Q}_i, Q_{-i})(i) = \gamma(Q)(i)$. However, in Example 6.4 we have $\gamma(Q_{i_1}, Q^c, Q_{i_3}) = \{\{i_1, s_1\}, \{i_3, s_2\}, \{i_2\}\}$ but $\gamma(Q_{i_1}, Q^e, Q_{i_3}) = \{\{i_1, s_2\}, \{i_3, s_1\}, \{i_2\}\}$, which shows that γ is not individually idempotent.

The next proposition reverses in a certain way Corollary 7.5: any matching that is sustained at some Nash equilibrium for a particular quota can be sustained at some Nash equilibrium for a more stringent quota.

Proposition 7.6 *For any school choice problem P and quota k, $\mathcal{O}^\tau(P, k) = \mathcal{O}^\tau(P, 1) \neq \emptyset$.*

Given the result above, the next two examples show that the set of equilibrium outcomes is not a subset nor a superset of the set of stable matchings. In fact, Example 7.8 shows that the set of stable matchings and the set of Nash equilibrium outcomes can be disjoint sets.

Example 7.7 $\mathcal{O}^\tau(P, 1) \nsubseteq S(P)$
Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}	f_{s_3}
s_1	s_1	s_2	i_3	i_2	i_3
s_2	s_2	s_1	i_1	i_1	i_1
s_3	s_3	s_3	i_2	i_3	i_2

Let $k = 1$ be the quota. Consider the profile $Q = (s_3, s_1, s_2) \in \mathcal{Q}(1)^I$. Clearly, $\tau(Q) = \{\{i_1, s_3\}, \{i_2, s_1\}, \{i_3, s_2\}\}$. Notice that students i_2 and i_3 obtain their most preferred school, so student i is the only possible student that may profitably deviate.

For any strategy student i_1 may choose, observe that the cycle $(s_1, i_3, s_2, i_2, s_1)$ forms in the first step of the execution of the TTC algorithm. It follows that for any $\widehat{Q}_{i_1} \in \mathcal{Q}(1)$, $\tau(\widehat{Q}_{i_1}, Q_{-i_1})(i_1)$ is either i_1 or school s_3. Hence, $Q \in \mathcal{E}^\tau(P, 1)$.

Yet, $\tau(Q)$ is not stable: student i_1 has a higher priority than student i_2 for school $s_1 = \tau(Q)(i_2)$ and student i_1 prefers school s_1 to s_3, his matching under $\tau(Q)$. ◇

Example 7.8 $S(P) \nsubseteq \mathcal{O}^\tau(P, 1)$
Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}
s_1	s_1	s_2	i_1	i_3
			i_2	i_2
			i_3	i_1

It is easy to check that the unique stable matching is $\mu = \{\{i_1\}, \{i_2, s_1\}, \{i_3, s_2\}\}$. We show that μ cannot be sustained at any Nash equilibrium of the game $\Gamma^\tau(P, 1)$. Suppose to the contrary that μ can be sustained at some Nash equilibrium. In other words, there is a profile $Q \in \mathcal{Q}(1)^I$ such that $\tau(Q) = \mu$ and $Q \in \mathcal{E}^\tau(P, 1)$. Since $\tau(Q) = \mu$, $Q_{i_2} = s_1$ and $Q_{i_3} = s_2$. If $Q_{i_1} = s_1$, then $\tau(Q)(i_1) = s_1 \neq \mu(i_1)$. If $Q_{i_1} = s_2$, then $\tau(Q)(i_1) = s_2 \neq \mu(i_1)$. So, $Q_{i_1} = i_1$. However, it is clear that student i_1 is strictly better off by reporting $Q'_{i_1} = s_1$. Hence, $Q \notin \mathcal{E}^\tau(P, 1)$, a contradiction. $\diamond$

Next, we show that Kesten's (2006a) acyclicity condition is a necessary and sufficient condition for the Top Trading Cycles mechanism to implement the set of stable matchings in Nash equilibria.

Definition 7.9 ***Cycles and Acyclicity (Kesten, 2006a)***
Given a priority structure f, a *cycle* is constituted of distinct $s, s' \in S$ and $i, j, l \in I$ such that the following two conditions are satisfied:
cycle condition $f_s(i) < f_s(j) < f_s(l)$ and $f_{s'}(l) < f_{s'}(i), f_{s'}(j)$ and
c-scarcity condition there exists a (possibly empty) set $I_s \subseteq I \backslash \{i, j, l\}$ with $I_s \subseteq U_s^f(i) \cup (U_s^f(j) \backslash U_{s'}^f(l))$ and $|I_s| = q_s - 1$.
A priority structure is *acyclic* if no cycles exist. $\triangle$

Kesten (2006a) showed that acyclicity of the priority structure is sufficient for resource monotonicity, population monotonicity, and stability of the Top Trading Cycles Mechanism as well as necessary for each of these conditions separately. He also proved that the Top Trading Cycles Mechanism coincides with the Student-Optimal Stable Mechanism if and only if the priority structure is acyclic. The latter result and Proposition 7.6 together with our first main result (Theorem 6.6) are the driving force behind our second main result:

Theorem 7.10 *Let $1 \leq k \leq m$. Then, f is an acyclic priority structure if and only if for any school choice problem P, the game $\Gamma^\tau(P, k)$ implements $S(P)$ in Nash equilibria.*

8 Equilibria of Truncations

In this section we first strengthen Theorems 6.6 and 7.10 by exhibiting a school choice problem with a (strong) Nash equilibrium in "intuitive" undominated strategies that yields an unstable matching. Next, we will show that in general there is also no relation between the set of unassigned students at equilibrium and the set of unassigned students in stable matchings. However, for Nash equilibria in "intuitive" undominated strategies we do obtain positive results in this respect.

One piece of advise about which preference list a student should submit follows from the strategy-proofness of the Student-Optimal Stable Mechanism γ in the unrestricted case: it does not pay off to submit a list of schools that does not respect the true order. More precisely, a list that does not respect the order of a student's true preferences is weakly dominated by listing the same schools in the "true order." Let φ be a mechanism. Student i's strategy $Q_i \in \mathcal{Q}(k)$ in the game $\Gamma^{\varphi}(P,k)$ is *weakly k-dominated* by another strategy $Q'_i \in \mathcal{Q}(k)$ if $\varphi(Q'_i, Q_{-i}) R_i \varphi(Q_i, Q_{-i})$ for all $Q_{-i} \in \mathcal{Q}(k)^{I \backslash i}$.

Lemma 8.1 *Let P be a school choice problem. Let $1 \leq k \leq m$. Let $i \in I$ be a student. Consider two strategies $Q_i, Q'_i \in \mathcal{Q}(k)$ such that (a) Q_i and Q'_i contain the same set of schools, and (b) for any two schools s and s' listed in Q_i (or Q'_i), $sQ'_is' \Rightarrow sP_is'$. Then, Q_i is weakly k-dominated by Q'_i in the games $\Gamma^{\gamma}(P,k)$ and $\Gamma^{\tau}(P,k)$.*

The message of Lemma 8.1 is clear: a student cannot lose (and may possibly gain) by submitting the same set of schools in the true order. A special type of strategies that satisfy this condition are the so-called truncations. A truncation of a preference list is a list obtained from the preference list by deleting some specific school and all less preferred acceptable schools. Formally, a *truncation* of a preference list P_i is a list P'_i such that the schools in P'_i are contained in P_i and sP'_is' implies sP_is'. The following lemma says that in the games $\Gamma^{\gamma}(P,k)$ and $\Gamma^{\tau}(P,k)$ submitting a truncation "as long as possible" is k-undominated. Formally, student i's strategy $Q_i \in \mathcal{Q}(k)$ is *k-dominated* by another strategy $Q'_i \in \mathcal{Q}(k)$ if $\varphi(Q'_i, Q_{-i}) R_i \varphi(Q_i, Q_{-i})$ for all $Q_{-i} \in \mathcal{Q}(k)^{I \backslash i}$ and $\varphi(Q'_i, Q'_{-i}) P_i \varphi(Q_i, Q'_{-i})$ for some $Q'_{-i} \in \mathcal{Q}(k)^{I \backslash i}$. A strategy in $\mathcal{Q}(k)$ is *k-undominated* if it is not k-dominated by any other strategy in $\mathcal{Q}(k)$.

Lemma 8.2 *Let P be a school choice problem. Let $1 \leq k \leq m$. Let $i \in I$ be a student. Denote the number of (acceptable) schools in P_i by $|P_i|$. Then, the strategy P_i^k of submitting the first $min\{k, |P_i|\}$ schools of the true preference list P_i in the true order is k-undominated in the games $\Gamma^{\gamma}(P,k)$ and $\Gamma^{\tau}(P,k)$.*

Although the strategy profile P^k is a profile of k-undominated strategies, it is not necessarily a Nash equilibrium in the game $\Gamma^{\varphi}(P,k)$. In case it is a Nash equilibrium it may still induce an unstable matching as the following example shows. This clearly strengthens Examples 6.4 and 7.7 in the sense that it shows that unstability is not simply due to the choice of a pathological equilibrium.

Example 8.3 ***For both γ and τ: Strong Nash Equilibrium in (Undominated) Truncations yields Unstable Matching.***
Let $I = \{i_1, i_2, i_3, i_4\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1,1,1,1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	f_{s_1}	f_{s_2}	f_{s_3}
s_1	s_2	s_3	s_1	i_3	i_1	i_2
s_2	s_3	s_1	s_2	i_1	i_2	i_4
s_3	s_1	s_2	s_3	i_2	i_3	i_3
				i_4	i_4	i_1

Let $\varphi = \gamma, \tau$. Let $k = 2$ be the quota. Consider the strategy profile $Q = P^2 \in \mathcal{Q}(2)^I$ of 2-undominated truncations:

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_1	s_2	s_3	s_1
s_2	s_3	s_1	s_2

One easily verifies that $\varphi(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}, \{i_4\}\}$. Since student i_4 has justified envy for school s_3, $\varphi(Q) \notin S(P)$. It remains to show that Q is a strong Nash equilibrium (*cf.* Aumann, 1959) in $\Gamma^{\varphi}(P, 2)$. Since students i_1, i_2, and i_3 are assigned a seat at their favorite school, it is sufficient to check that student i_4 has no profitable deviation. Notice that the only possibility for student i_4 to change the outcome of the mechanism is by listing school s_3. So, the only strategies that we have to check are given by $\bar{\mathcal{Q}}(2) = \{Q^a, Q^b, Q^c, Q^d, Q^e\}$, where $Q^a = s_3$, $Q^b = s_1, s_3$, $Q^c = s_2, s_3$, $Q^d = s_3, s_1$, and $Q^e = s_3, s_2$. By Lemma 8.1, Q^d and Q^e are weakly 2-dominated by Q^b and Q^c, respectively. So in fact we only have to consider strategies Q^a, Q^b, and Q^c. Given the other students' strategies Q_{-i_4} and the priority orderings of s_1 and s_2, for any of these three strategies for student i_4, in the DA-algorithm i_4 is never tentatively assigned to s_1 or s_2. Hence, $\gamma(Q^a, Q_{-i_4})(i_4) = \gamma(Q^b, Q_{-i_4})(i_4) = \gamma(Q^c, Q_{-i_4})(i_4)$. Routine computations show that $\gamma(Q^a, Q_{-i_4})(i_4) = i_4$. One easily checks that $\tau(Q^a, Q_{-i_4})(i_4) = \tau(Q^b, Q_{-i_4})(i_4) = \tau(Q^c, Q_{-i_4})(i_4) = i_4$ since student i_4 cannot break the cycle $(i_1, s_1, i_3, s_3, i_2, s_2)$ that forms in the first step of the TTC-algorithm. Hence, student i_4 does not have a profitable deviation for either γ or τ. $\diamond$

A straightforward translation of the results of McVitie and Wilson (1970) and Roth (1984) from college admissions to school choice gives that for any school choice problem, the set of unassigned students is the same for all stable matchings.[11] In other words, for $\mu, \mu' \in S(P)$, $\mu(i) = i$ implies $\mu'(i) = i$. Given the restrictiveness of the acyclicity condition to guarantee stable Nash equilibrium outcomes, one may wonder whether at least always the set of unassigned students at equilibrium coincides with the set of unassigned students in stable matchings. In fact, a less ambitious idea would be to establish that at equilibrium the *number* of unassigned students equals the number of unassigned students in stable matchings. The following two examples show that in general this is not true. In other words, the number of unassigned students at equilibrium differs from the number of unassigned students in stable matchings. Given the first inclusion in Proposition 6.1, this in particular implies for the Student-Optimal Stable Mechanism that the number of unassigned students can vary from one equilibrium outcome to another.

Example 8.4 ***For both γ and τ: Less Assigned Students at Equilibrium than in Stable Matchings***
Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}	f_{s_3}
s_1	s_3	s_3	i_3	i_2	i_1
s_3	s_1	s_2	i_1	i_3	i_2
s_2		s_1	i_2	i_1	i_3

[11] A generalization of this result is known in the two-sided matching literature as the "Rural Hospital Theorem" (Roth, 1986) and says that the degree of occupation and quality of interns at typically less demanded rural hospitals in the US is not due to the choice of a specific stable matching.

One easily verifies that strategy profile Q given below is a Nash equilibrium in $\Gamma^{\gamma}(P,2)$ and $\Gamma^{\tau}(P,2)$.

Q_{i_1}	Q_{i_2}	Q_{i_3}
s_1	s_1	s_3
s_3		s_1

Since $\gamma(Q) = \tau(Q) = \{\{i_1, s_1\}, \{i_3, s_3\}, \{i_2\}, \{s_2\}\}$ and $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$, there are less assigned students at $\gamma(Q) = \tau(Q)$ than in any stable matching. ⋄

Example 8.5 ***For both γ and τ: More Assigned Students at Equilibrium than in Stable Matchings.***
Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}		f_{s_1}	f_{s_2}	f_{s_3}
s_2	s_3	s_3		i_3	i_2	i_1
	s_2	s_2		i_1	i_3	i_2
	s_1	s_1		i_2	i_1	i_3

One easily verifies that strategy profile Q given below is a Nash equilibrium in $\Gamma^{\gamma}(P,2)$ and $\Gamma^{\tau}(P,2)$.

Q_{i_1}	Q_{i_2}	Q_{i_3}
s_2	s_3	s_1
s_3	s_2	s_2

Since $\gamma(Q) = \tau(Q) = \{\{i_1, s_2\}, \{i_2, s_3\}, \{i_3, s_1\}\}$ and $\gamma(P) = \{\{i_2, s_3\}, \{i_3, s_2\}, \{i_1\}, \{s_1\}\}$, there are more assigned students at $\gamma(Q) = \tau(Q)$ than in any stable matching. ⋄

We obtain a positive result for γ if we restrict ourselves to equilibria in truncations. More precisely, the following proposition says that if a profile of truncations is a Nash equilibrium in the game $\Gamma^{\gamma}(P,k)$ then the set of assigned students at the equilibrium coincides with the set of assigned students at any stable matching. In fact, each Nash equilibrium in truncations in the game $\Gamma^{\gamma}(P,k)$ yields a matching that is either the Student-Optimal matching $\gamma(P)$ or Pareto dominates $\gamma(P)$. For a matching μ, denote $M(\mu)$ for the set of assigned students, *i.e.*, $M(\mu) = \{i \in I : \mu(i) \neq i\}$.

Proposition 8.6 *Let P be a school choice problem. Let $1 \leq k \leq m$. If the profile of truncations $P^k := (P_i^k)_{i \in I}$ is a Nash equilibrium in $\Gamma^{\gamma}(P,k)$, then $\gamma(P^k)(i) R_i \gamma(P)(i)$ for all $i \in M(\gamma(P))$. Hence, $M(\gamma(P^k)) = M(\gamma(P))$.*

For τ we cannot obtain a similar result as the following proposition shows.

Proposition 8.7 *Let P be a school choice problem. Let $1 \leq k \leq m$. If the profile of truncations $P^k := (P_i^k)_{i \in I}$ is a Nash equilibrium in $\Gamma^{\tau}(P,k)$, then possibly $|M(\tau(P^k))| > |M(\gamma(P))|$ or $|M(\tau(P^k))| < |M(\gamma(P))|$.*

9 Discussion

In this section we first summarize our main findings. Next, we discuss their policy implications and our contribution to the literature on school choice. Finally, we conclude with some extensions of our results and possible directions for future research.

We have analyzed three prominent mechanisms to assign children to public schools on the basis of priority rankings. The main feature of our analysis is that the assignment procedure impedes students to fully reveal their true preferences. The Boston Mechanism, which in several school districts in the US is on the verge of being replaced by either one of (the other) two mechanisms proposed by Abdulkadiroğlu and Sönmez (2003), is robust in the sense that stability is guaranteed in equilibrium, no matter the imposed quota on the length of the submittable preference lists. The other two mechanisms, which have desirable properties in the unconstrained case, do not perform as well. In the first place, we show that both mechanisms allow for equilibria in undominated strategies that induce unstable outcomes. In the second place, we identify two acyclicity conditions on the priority structure that are necessary and sufficient for the implementation of the set of stable matchings.

To fully understand the policy implications of our results, we first note that both acyclicity conditions are quite restrictive.[12] Stability of the equilibrium outcomes, though, is assured for both the Student-Optimal Mechanism and the Top Trading Cycles Mechanism if the assignment of students is based on a common priority ranking. In practice, however, multiple exogenous criteria are employed: geographic distance, social origin, the number of siblings attending the same school, *etc.* Hence, the transition of the Boston Mechanism to either of the two mechanisms is likely to come with a higher risk that students seek legal action as lower priority students may occupy more preferred schools. Therefore, for policy makers opting for this transition possible efficiency gains (Chen and Sönmez, 2006 and Ergin and Sönmez, 2006) should outweigh an increasing risk of violations of stability. Clearly, if the quota of the assignment procedure is not very restrictive (relative to the number of schools and seats), then most students can submit a truncation of their true preferences. In that case, the likelihood of problems due to unstability may remain small.

Apart from the policy implications of our results and providing an additional dimension to the acyclicity conditions due to Ergin (2002) and Kesten (2006a), we also contribute to the theory of implementation in matching markets. To the best of our knowledge, the current paper provides the first complete analysis of the equilibria in the preference revelation game induced by the Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism. All previous studies, except Romero-Medina (1998), assumed preference revelation to be unconstrained. Given the strategy-proofness of both mechanisms in the unconstrained case, an analysis of all (other) equilibria was therefore in some sense not necessary. It is well-known that in the context of two-sided matching, preference revelation induced by stable mechanism may have unstable equilibrium outcomes (Alcalde, 1996 and Sönmez, 1997).[13] In the context of school choice, where only one side of the market is strategic, Ergin and Sönmez (2006) showed that the negative result above can be avoided by using the Boston Mechanism. We show that in this sense also the

[12] See Ergin (2002) and Kesten (2006a) for further illustration and discussion.

[13] Other recent papers on implementation in a various of settings of two-sided matching include Alcalde and Romero-Medina (2000), Kara and Sönmez (1996,1997), Ma (1995), Peleg (1997), Shin and Suh (1996), Shinotsuka and Takamiya (2003), Sotomayor (2003), Suh (2003), Tadenuma and Toda (1998).

Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism can be employed, as long as the priority structure is acyclic.

We have analyzed three prominent student assignment mechanisms that are used or at the point of being adopted in several school districts in the US. As was pointed out by Abdulkadiroğlu and Sönmez (2003) the Student-Optimal Stable Mechanism and the Top Trading Cycles Mechanism have desirable properties (at least in the unconstrained case). Still, one could consider alternative mechanisms. An obvious candidate is the School-Optimal Stable Mechanism γ^S. Theorems 3 and 4 in Sotomayor (1996) imply the implementation of the stable set via γ^S in Nash equilibria in the context of unconstrained school choice. A proof of the same result in the context of constrained school choice can be obtained by a straightforward adaptation of the proofs. Alternatively, one can adapt the proofs of Theorems 4.15 and 4.16 in Roth and Sotomayor (1990), which were originally proven in Gale and Sotomayor (1985a, Theorem 2) and Roth (1984).[14] In spite of this positive implementation result, one should be aware that from the students' point of view the outcomes of the School-Optimal Stable Mechanism are worse than those of the Student-Optimal Stable Mechanism. Again in the context of unconstrained school choice, Ergin and Sönmez (2006, Theorem 4) extended their implementation result of the Boston Mechanism to the class of so-called monotonic priority mechanisms. One can verify that this result can be further generalized to the case of constrained school choice. Finally, our results also hold in the model where policy makers can impose different quotas on different students.

Throughout our analysis we have assumed a complete information environment. Ergin and Sönmez (2006, Example 4) showed that the results for the Boston Mechanism do not carry over to incomplete information environments. Therefore, an important direction for future research would be to determine to what extent the predictions and results under the complete information assumption are robust to changes in the level of information. Analysis of field data and experimental studies may be very helpful.

A Appendix: Proofs Section 6

Proof of Proposition 6.1:
By Gale and Shapley (1962), $S(P) \neq \emptyset$. We now prove that $S(P) \subseteq \mathcal{O}^\gamma(P,k)$. Let $\mu \in S(P)$. Define $Q_i = \mu(i) \in \mathcal{Q}(k)$ for all $i \in I$. Since in the first step of the DA-algorithm for Q no student is rejected, $\gamma(Q) = \mu$. It remains to prove that Q is a Nash equilibrium in the Quota-Game $\Gamma^\gamma(P,k)$. Suppose to the contrary that $Q \notin \mathcal{E}^\gamma(P,k)$. Then there exists a student i and a strategy $Q_i' \in \mathcal{Q}(k)$ such that $\gamma(Q_i', Q_{-i}) P \gamma(Q) = \mu$. Since $\gamma(Q) = \mu \in S(P)$, $\gamma(Q) \in IR(P)$. Hence, $\gamma(Q_i', Q_{-i})(i) \in S$. Denote $s = \gamma(Q_i', Q_{-i})(i)$. Note $i \notin \mu(s)$. Consider the DA-algorithm for (Q_i', Q_{-i}). Of the students in $I \backslash i$, only the students in $\mu(s)$ make their unique proposal to s; all other students make either a unique proposal to another school or make no proposal at all. Since $\gamma(Q_i', Q_{-i})(i) = s$, it follows that student i starts making proposals but gets rejected until he proposes to s and get assigned a seat at s (now the DA-algorithm ends since no new proposals are made). Since under (Q_i', Q_{-i}) school s accepts i it must be that $|\mu(s)| < q_s$ or there is a student $j \in \mu(s)$ with $f_s(j) > f_s(i)$. In the first case, μ is wasteful for P, contradicting $\mu \in S(P)$. In the

[14] A noticeable exception to the use of quotas in school choice is the assignment of students to secondary schools in Singapore where students have to submit a list that contains *all* schools. Teo *et al.* (2001) show that even for the school-optimal stable mechanism this leaves little room for profitable manipulation.

second case, μ is not stable for P (student i has justified envy), also contradicting $\mu \in S(P)$. So, $Q \in \mathcal{E}^\gamma(P,k)$.

Next, we prove that $\mathcal{O}^\gamma(P,k) \subseteq IR(P) \cap NW(P)$. Let $Q \in \mathcal{E}^\gamma(P,k)$. It is immediate that $\gamma(Q) \in IR(P)$. We prove that $\gamma(Q) \in NW(P)$. Suppose to the contrary that $\gamma(Q) \notin NW(P)$. Then, there is a student $i \in I$ and a school $s \in S$ with $sP_i\gamma(i)$ and $|\gamma(Q)(s)| < q_s$. Let $\bar{Q}_i$ be the empty list. Let $\bar{Q} = (\bar{Q}_i, Q_{-i})$. By a result of Gale and Sotomayor (1985, Theorem 2) extended to the college admissions model (Roth and Sotomayor, 1990, Theorem 5.34), for each $j \in I \backslash i$, either $\gamma(\bar{Q})(j) = \gamma(Q)(j)$ or $\gamma(\bar{Q})(j)Q_j\gamma(Q)(j)$. Hence, the set of schools to which each $j \in I \backslash i$ proposes in DA($\bar{Q}$) is a subset of the schools to which he proposes in DA(Q). Since moreover $\bar{Q}_i$ is the empty list, each school receives in DA($\bar{Q}$) only a subset of the proposals of DA(Q). For school s this immediately implies that $|\gamma(\bar{Q})(s)| \leq |\gamma(Q)(s)| < q_s$. So, if we take $Q'_i = s$ then $\gamma(Q'_i, Q_{-i})(i) = sP\gamma(Q)$, *i.e.*, Q'_i is a profitable deviation for i at Q in $\Gamma^\gamma(P,k)$. So, $Q \notin \mathcal{E}^\gamma(P,k)$, a contradiction. Hence, $\gamma(Q) \in NW(P)$. □

We will make use of the following result from two-sided matching to prove Proposition 6.2.

Lemma A.1 ***(Roth, 1982, Lemma 1; cf. Roth and Sotomayor 1990, Lemma 4.8)*** *Let P and P' be two school choice problems. Let $i \in I$. Suppose $P_l = P'_l$ for all $l \in I \backslash i$. Suppose P'_i is a preference list whose first choice is $\gamma(P)(i)$ if $\gamma(P)(i) \neq i$, and the empty list otherwise. Then, $\gamma(P')(i) = \gamma(P)(i)$.*

Proof of Proposition 6.2:
It suffices to prove the proposition for $k' = k+1$. Let $Q \in \mathcal{E}^\gamma(P,k)$ and suppose that $Q \notin \mathcal{E}^\gamma(P,k+1)$. Hence, there is a student i and a strategy $Q'_i \in \mathcal{Q}(k+1)$ such that $\gamma(Q'_i, Q_{-i})P_i\gamma(Q_i, Q_{-i})$. By individual rationality of $\gamma(Q)$ for P (Proposition 6.1), $\gamma(Q'_i, Q_{-i})(i) \in S$. Note also that Q'_i must be a list containing exactly $k+1$ schools, for otherwise it would also be a profitable deviation in $\Gamma^\gamma(P,k)$, contradicting $Q \in \mathcal{E}^\gamma(P,k)$.

Let s be the last school listed in Q'_i. We claim that $\gamma(Q'_i, Q_{-i})(i) = s$. Suppose not. Consider the truncation of Q'_i after $\gamma(Q'_i, Q_{-i})(i)$ and denote this list by Q''_i. In other words, Q''_i is the list obtained from Q'_i by making all schools listed after $\gamma(Q'_i, Q_{-i})(i)$ unacceptable. Note that Q''_i is a list with at most k schools, *i.e.*, $Q''_i \in \mathcal{Q}(k)$. It follows from the DA-algorithm that $\gamma(Q''_i, Q_{-i}) = \gamma(Q'_i, Q_{-i})$. Hence, Q''_i is a profitable deviation for i at Q in $\Gamma^\gamma(P,k)$, a contradiction. So, $\gamma(Q'_i, Q_{-i})(i) = s$.

From Lemma A.1, it follows that with $\widehat{Q}_i = s$ we have $\gamma(\widehat{Q}_i, Q_{-i})(i) = s$. Finally, observe that $\widehat{Q}_i \in \mathcal{Q}(k)$. Hence, $\widehat{Q}_i$ is a profitable deviation for i at Q in $\Gamma^\gamma(P,k)$, a contradiction. Hence, $Q \in \mathcal{E}^\gamma(P,k+1)$. □

Proof of Proposition 6.3:
The inclusion $S(P) \subseteq \mathcal{O}^\gamma(P,1)$ follows from Proposition 6.1. We prove $\mathcal{O}^\gamma(P,1) \subseteq S(P)$. Suppose to the contrary that $Q \in \mathcal{E}^\gamma(P,1)$ but $\gamma(Q) \notin S(P)$. From Proposition 6.1 it follows that at $\gamma(Q)$ some student has justified envy. So, there are two students $i, j \in I$, $i \neq j$, with $\gamma(Q)(j) = s \in S$, $sP_i\gamma(Q)(i)$, and $f_s(i) < f_s(j)$. Now consider the strategy $Q'_i = s$. Since $(Q'_i, Q_{-i}) \in \mathcal{Q}(1)^I$, $\gamma(Q'_i, Q_{-i})(i) = s$, *i.e.*, Q'_i is a profitable deviation for student i at Q in $\Gamma^\gamma(P,1)$. Hence, $Q \notin \mathcal{E}^\gamma(P,1)$, a contradiction. So, $\mathcal{O}^\gamma(P,1) \subseteq S(P)$. □

Lemma A.2 *Let f be a strongly cyclic priority structure. Let $2 \leq k \leq m$. Then, there is a school choice problem P with an unstable equilibrium outcome in the game $\Gamma^\gamma(P,k)$,* i.e., *for some $Q \in \mathcal{E}^\gamma(P,k)$, $\gamma(Q) \notin S(P)$.*

Proof: Since f is strongly cyclic, we may assume, without loss of generality, that students $\{i_1, i_2, i_3\}$ and schools $\{s_1, s_2\}$ constitute a strong cycle. In fact, we may assume, without loss of generality, that
(a) $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3)$ and $f_{s_2}(i_3) < f_{s_2}(i_1)$,
(b) for $j \in \{4, q_{s_1} + 3\}$, $f_{s_1}(i_j) < f_{s_1}(i_2)$, and
(c) for $j \in \{q_{s_1} + 4, q_{s_1} + q_{s_2} + 2\}$, $f_{s_2}(i_j) < f_{s_2}(i_1)$.
Consider the students' preferences P given below.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	$\cdots$	$P_{i_{q_{s_1}+3}}$	$P_{i_{q_{s_1}+4}}$	$\cdots$	$P_{i_{q_{s_1}+q_{s_2}+2}}$	$P_{i_{q_{s_1}+q_{s_2}+3}}$	$\cdots$	P_{i_n}
s_2	s_1	s_1	s_1	s_1	s_1	s_2	s_2	s_2			
s_1		s_2									

Unacceptable schools are not depicted.

There are three possibilities for the priority ordering f_{s_2} of school s_2:
(i) $f_{s_2}(i_2) < f_{s_2}(i_3) < f_{s_2}(i_1)$,
(ii) $f_{s_2}(i_3) < f_{s_2}(i_2) < f_{s_2}(i_1)$, or
(iii) $f_{s_2}(i_3) < f_{s_2}(i_1) < f_{s_2}(i_2)$.
We apply the DA-algorithm (with students proposing) to P. First note that by the construction of P and (b) and (c), all students in $\{i_4, i_{q_{s_1}+q_{s_2}+1}\}$ are assured (and in fact are assigned) a seat at their most preferred school. Since for each $j \in \{q_{s_1} + q_{s_2} + 2, \ldots, n\}$, student i_j finds all schools unacceptable, one seat of each of the schools s_1 and s_2 remains to be assigned to the students in $\{i_1, i_2, i_3\}$. One easily verifies that the DA-algorithm in each of the three cases (i), (ii), and (iii), assigns students i_1 and i_3 to schools s_1 and s_2, respectively. We obtain the same matching if we apply the DA-algorithm to P with schools proposing. Hence, there is a unique stable matching $\mu^* = \mu_I[P] = \mu_S[P]$ for P in which students i_1 and i_3 are assigned to schools s_1 and s_2, respectively (and student i_2 remains unassigned).

Consider the strategy profile $Q \in \mathcal{Q}(k)^I$ given below. We will show that $\gamma(Q) \notin S(P)$ and $Q \in \mathcal{E}^\gamma(P,k)$.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}	$\cdots$	$Q_{i_{q_{s_1}+2}}$	$Q_{i_{q_{s_1}+3}}$	$\cdots$	$Q_{i_{q_{s_1}+q_{s_2}+2}}$	$Q_{i_{q_{s_1}+q_{s_2}+3}}$	$\cdots$	Q_{i_n}
s_2		s_1	s_1	s_1	s_1	s_2	s_2	s_2			
s_1		s_2									

We apply the DA-algorithm (with students proposing) to Q. Similarly as for P, all students in $\{i_4, i_{q_{s_1}+q_{s_2}+2}\}$ are assigned a seat at their most preferred school. One seat of each of the schools s_1 and s_2 remains to be assigned to the students in $\{i_1, i_2, i_3\}$. Since Q_{i_2} is the empty list, and students i_1 and i_3 have different favorite schools at Q, the DA-algorithm assigns in each of the three cases (i), (ii), and (iii), students i_1 and i_3 to schools s_2 and s_1, respectively. So, $\gamma(Q) \neq \mu^*$. Since $S(P) = \{\mu^*\}$, $\gamma(Q) \notin S(P)$.

Finally, we check that $Q \in \mathcal{E}^\gamma(P,k)$. Note that at $\gamma(Q)$ each of the students i_1 and i_3 is assigned a seat at his/her favorite school. So, nor student i_1 nor i_3 has a profitable deviation from his/her strategy Q_{i_1} and Q_{i_3}, respectively. It is easy to check that in any of the cases (i), (ii), and (iii), and for any strategy $Q'_{i_2} \in \mathcal{Q}(k)$, $\gamma(Q) R_{i_2} \gamma(Q_{i_1}, Q'_{i_2}, Q_{i_3})$. In other words, student i_2 does not have a profitable deviation from Q_{i_2}. Hence, $Q \in \mathcal{E}^\gamma(P, k)$. □

Lemma A.3 *Let f be a weakly acyclic priority structure. Let $2 \leq k \leq m$. Then, for any school choice problem P all equilibrium outcomes in the game $\Gamma^\gamma(P, k)$ are stable,* i.e.*, for all $Q \in \mathcal{E}^\gamma(P, k)$, $\gamma(Q) \in S(P)$.*

Before we can prove Lemma A.3 we need to recall a result on the Student-Optimal Stable Mechanism.

Lemma A.4 ***(Ergin, 2002, Theorem 1,** $(iv) \to (iii)$ **and proof of** $(iii) \to (ii)$**)***
Let f be a weakly acyclic priority structure. Then, the Student-Optimal Stable Mechanism γ is non-bossy.

Proof of Lemma A.3:
Suppose to the contrary that $Q \in \mathcal{E}^\gamma(P, k)$ but $\gamma(Q) \notin S(P)$. By Proposition 6.1, $\gamma(Q) \notin F(P, k)$. So, there are two students $i, j \in I$, $i \neq j$ and a school $s \in S$ such that $\gamma(Q)(j) = s$, $sP_i\gamma(Q)(i)$, and $f_s(i) < f_s(j)$.

Since γ is strategy-proof when there are no restrictions on the length of the (revealed) preference lists (*i.e.*, when the quota equals m, the number of schools), $\gamma(P_i, Q_{-i}) R_i \gamma(Q_i, Q_{-i})$. Let $P'_i = \gamma(P)(i)$. Clearly, $P'_i \in \mathcal{Q}(1) \subseteq \mathcal{Q}(k)$. By Lemma A.1, $\gamma(P'_i, Q_{-i})(i) = \gamma(P_i, Q_{-i})(i)$. Hence, $\gamma(P'_i, Q_{-i}) R_i \gamma(Q_i, Q_{-i})$. If $\gamma(P'_i, Q_{-i}) P_i \gamma(Q_i, Q_{-i})$, then $Q \notin \mathcal{E}^\gamma(P, k)$, a contradiction. Hence, $\gamma(P'_i, Q_{-i})(i) = \gamma(Q_i, Q_{-i})(i)$.

By Lemma A.4, γ is non-bossy. Hence, $\gamma(P_i, Q_{-i}) = \gamma(P'_i, Q_{-i}) = \gamma(Q)$. In particular, $\gamma(P_i, Q_{-i})(j) = \gamma(Q)(j) = s$. Since $sP_i\gamma(Q)(i) = \gamma(P_i, Q_{-i})(i)$, student i has justified envy at $\gamma(P_i, Q_{-i})$, contradicting $\gamma(P_i, Q_{-i}) \in S(P_i, Q_{-i})$. Hence, $\gamma(Q) \in S(P)$. □

Proof of Theorem 6.6:
Follows immediately from Proposition 6.1 and Lemmas A.2 and A.3. □

B Appendix: Proofs Section 7

The following observation on the Top Trading Cycles algorithm is key for the results in Section 7.

Observation B.1 In the TTC algorithm, once a student points to a school it will keep on pointing to the school in subsequent steps until he is assigned to a seat at the school or until the school has no longer available seats. Similarly, once a school points to a student it will keep on pointing to the student in subsequent steps until the student is assigned to a seat at this or some other school.

Next, we introduce the following graph-theoretic notation to provide concise proofs of our results. Let $Q \in \mathcal{Q}^I$. Suppose the TTC algorithm is applied to Q and suppose it terminates in no less than l steps. We denote by $G^\tau(Q, l)$ the (directed) graph that corresponds to step l. In this graph, the set of vertices $V^\tau(Q, l)$ is the set of agents present in step l. For any

$v \in V^\tau(Q,l)$ there is a (unique) directed edge in $G^\tau(Q,l)$ from v to some $v' \in V^\tau(Q,l)$ (possibly $v' = v$ if $v \in I$) if agent v points to agent v', which will also be denoted by $e(Q,l,v) = v'$. By the TTC algorithm, for any student $i \in V^\tau(Q,l) \cap I$, if i points to v', then Q_i ranks v' higher than any other agent in $(V^\tau(Q,l) \cap S) \cup i$. Similarly, for any school $s \in V^\tau(Q,l) \cap S$, if s points to student i, then i has a higher priority for s than any other student in $V^\tau(Q,l) \cap I$. A path (from v_1 to v_p) in $G^\tau(Q,l)$ is an ordered list of agents $(v_1, v_2, \ldots, v_p)$ such that $v_r \in V^\tau(Q,l)$ for all $r = 1, \ldots, p$ and each v_r points to v_{r+1} for all $r = 1, \ldots, p-1$. A self-cycle (i) of a student i is a degenerate path: i points to himself in $G^\tau(Q,l)$. An agent $v' \in V^\tau(Q,l)$ is a follower of an agent $v \in V^\tau(Q,l)$ if there is a path from v to v' in $G^\tau(Q,l)$. The set of followers of v is denoted by $F^\tau(Q,l,v)$. An agent $v' \in V^\tau(Q,l)$ is a predecessor of an agent $v \in V^\tau(Q,l)$ if there is a path from v' to v in $G^\tau(Q,l)$. The set of predecessors of v is denoted by $P^\tau(Q,l,v)$. A cycle in $G^\tau(Q,l)$ is a path $(v_1, v_2, \ldots, v_p)$ such that also v_p points to v_1. Note that a self-cycle is a special case of a cycle. With a slight abuse of notation we sometimes refer to a cycle as the corresponding non ordered set of involved agents. Finally, for $v \in I \cup S$, let $\sigma^\tau(Q,v)$ denote the step of the TTC algorithm at which agent v is removed.

For the sake of convenience, we relegate the proof of Proposition 7.1. For the moment we only need and state the following lemma.

Lemma B.2 *Let P be a school choice problem. For any $1 \le k \le m$, $\mathcal{O}^\tau(P,k) \subseteq IR(P)$.*

Proof of Proposition 7.3:
Let $Q \in \mathcal{E}^\varphi(P,k)$. Suppose $Q \notin \mathcal{E}^\varphi(P,k')$. Then there exists a student, say i, and a list $Q_i' \in \mathcal{Q}(k')$ such that $\varphi(Q_i', Q_{-i}) P_i \varphi(Q_i, Q_{-i})$. Let $\widetilde{Q}_i' = \varphi(Q_i', Q_{-i})(i)$. Clearly, $\widetilde{Q}_i' \in \mathcal{Q}(1) \subseteq \mathcal{Q}(k)$. Since φ is individually idempotent, $\varphi(\widetilde{Q}_i', Q_{-i}) = \varphi(Q_i', Q_{-i})$. So, $\varphi(\widetilde{Q}_i', Q_{-i}) P_i \varphi(Q_i, Q_{-i})$, contradicting $Q \in \mathcal{E}^\varphi(P,k)$. Hence, $Q \in \mathcal{E}^\varphi(P,k')$. □

To prove Lemma 7.4 we need the following lemma.

Lemma B.3 *Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and $\widetilde{Q}_i \in \mathcal{Q}$. Suppose that $\tau(Q)(i) \neq \tau(\widetilde{Q})(i)$. Let p and $\widetilde{p}$ be the steps at which student i is assigned in $TTC(Q)$ and $TTC(\widetilde{Q})$, respectively. Let $r = \min\{p, \widetilde{p}\}$. Then, at steps $g = 1, \ldots, r-1$,*

- *(a) at steps $1, \ldots, r-1$, the same cycles form in $TTC(Q)$ and $TTC(\widetilde{Q})$;*
- *(b) $i \in V^\tau(Q,r) = V^\tau(\widetilde{Q},r)$ and for each school $s \in V^\tau(Q,r) \cap S$, $q_s^{Q,r} = q_s^{\widetilde{Q},r}$;*
- *(c) $e(Q,r,v) = e(\widetilde{Q},r,v)$ for each agent $v \in V^\tau(Q,r)$, $v \neq i$;*
- *(d) there is a cycle C with $i \in C$ in either $G^\tau(Q,r)$ or $G^\tau(\widetilde{Q},r)$ (but not both).*

Proof: Item (a) follows from the proof of a result in Abdulkadiroğlu and Sönmez (1999, Lemma 1) or Abdulkadiroğlu and Sönmez (2003, Lemma). As for Item (b), from the definition of r, $i \in V^\tau(Q,r) \cap V^\tau(\widetilde{Q},r)$. The remainder of Item (b) follows directly from Item (a). Item (c) follows from Items (a), (b), and the fact that $\widetilde{Q}_j = Q_j$ for all students $j \in I \backslash i$. By definition of r, there is a cycle C with $i \in C$ in $G^\tau(Q,r)$ or $G^\tau(\widetilde{Q},r)$. By the assumption that

$\tau(Q)(i) \neq \tau(\widetilde{Q})(i)$, $e(Q,r,i) \neq e(\widetilde{Q},r,i)$. In particular, C is not a cycle in both $G^\tau(Q,r)$ and $G^\tau(\widetilde{Q},r)$. This proves Item (d). □

Proof of Lemma 7.4:
Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and define $\widetilde{Q}_i = \tau(Q)(i) \in \mathcal{Q}(1)$. We have to show that $\tau(\widetilde{Q}_i, Q_{-i}) = \tau(Q)$. By non-bossiness of τ, it is sufficient to show that $\tau(\widetilde{Q}_i, Q_{-i})(i) = \tau(Q)(i)$. If $\tau(Q)(i) = i$, then from the definition of the TTC algorithm $\tau(\widetilde{Q})(i) = i = \tau(Q)(i)$.

So, suppose $\tau(Q)(i) = s^* \in S$. Suppose to the contrary that $\tau(\widetilde{Q})(i) \neq \tau(Q)(i)$. Then, since $\widetilde{Q}_i = \tau(Q)(i) = s^*$, student i remains unassigned under $\widetilde{Q}$, *i.e.*, $\tau(\widetilde{Q})(i) = i$. Let p and $\widetilde{p}$ be the steps at which student i is assigned in $TTC(Q)$ and $TTC(\widetilde{Q})$, respectively. Let $r = \min\{p, \widetilde{p}\}$. By Lemma B.3(d), there is a cycle C with $i \in C$ in either $G^\tau(Q,r)$ or $G^\tau(\widetilde{Q},r)$ (but not both).
Case 1: Cycle C is in $G^\tau(Q,r)$ but not in $G^\tau(\widetilde{Q},r)$.
Since student i is assigned through cycle C and $\tau(Q)(i) = s^*$, $e(Q,r,i) = s^*$. Since $e(\widetilde{Q},r,i) \neq e(Q,r,i)$ and $\widetilde{Q}_i = \tau(Q)(i) = s^*$, $e(\widetilde{Q},r,i) = i$. Hence, at the beginning of step r of $TTC(\widetilde{Q})$, school s^* has no available seats, *i.e.*, $q_{s^*}^{\widetilde{Q},r} = 0$. By Lemma B.3(b), $q_{s^*}^{Q,r} = q_{s^*}^{\widetilde{Q},r} = 0$. So, $e(Q,r,i) \neq s^*$, a contradiction.
Case 2: Cycle C is in $G^\tau(\widetilde{Q},r)$ but not in $G^\tau(Q,r)$.
If $e(\widetilde{Q},r,i) = s^*$, then $\tau(\widetilde{Q})(i) = s^*$, a contradiction with $\tau(\widetilde{Q})(i) \neq \tau(Q)(i)$. So by $\widetilde{Q}_i = \tau(Q)(i) = s^*$, $e(\widetilde{Q},r,i) = i$, *i.e.*, $C = (i)$ is a self-cycle. Since $i \in V^\tau(Q,r)$ and $\tau(Q)(i) = s^*$, $q_{s^*}^{Q,r} > 0$. By Lemma B.3(b), $q_{s^*}^{\widetilde{Q},r} = q_{s^*}^{Q,r} > 0$. But then by $\widetilde{Q}_i = \tau(Q)(i) = s^*$, $e(\widetilde{Q},r,i) \neq i$, a contradiction.
Since both Case 1 and Case 2 yield a contradiction, we conclude $\tau(\widetilde{Q})(i) = \tau(Q)(i)$. □

We need Lemmas B.4–B.7 to prove Proposition 7.6.

Lemma B.4 *Let $\bar{Q} \in \mathcal{Q}^I$. Let $v, v' \in I \cup S$, $v \neq v'$. Suppose at some step of the TTC algorithm applied to $\bar{Q}$ there is a path from v' to v. Then, $\sigma^\tau(\bar{Q},v) \leq \sigma^\tau(\bar{Q},v')$ and [$\sigma^\tau(\bar{Q},v) = \sigma^\tau(\bar{Q},v')$ only if v and v' are removed in the same cycle].*

Proof: By Observation B.1, each agent in the path from v' to v will keep on pointing to the same agent at least until the step in which agent v is removed, *i.e.*, step $\sigma^\tau(\bar{Q},v)$. Hence, $\sigma^\tau(\bar{Q},v) \leq \sigma^\tau(\bar{Q},v')$. Suppose $\sigma^\tau(\bar{Q},v) = \sigma^\tau(\bar{Q},v')$. In other words, agent v' is removed at the same step as agent v. Then, all agents in the path from v' to v form part of a cycle at this step. In particular, v and v' are removed in the same cycle. □

Lemma B.5 *Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and $Q'_i \in \mathcal{Q}$. Suppose $\tau(Q)(i) \neq \tau(Q')(i)$ and $\sigma^\tau(Q,i) \leq \sigma^\tau(Q',i)$. For each step l, $\sigma^\tau(Q,i) \leq l \leq \sigma^\tau(Q',i)$, if $v \in V^\tau(Q',l) \backslash (P^\tau(Q',l,i) \cup i)$, then $v \in V^\tau(Q,l)$ and $F^\tau(Q,l,v) = F^\tau(Q',l,v)$.*

Proof: Let $p = \sigma^\tau(Q,i)$ and $r' = \sigma^\tau(Q',i)$. From Lemma B.3(b), $V^\tau(Q,p) = V^\tau(Q',p)$ and

$$q_s^{Q,p} = q_s^{Q',p} \text{ for each school } s \in V^\tau(Q,p) \cap S. \tag{1}$$

With a slight abuse of notation, for each l, $p \leq l \leq r'$, denote $P_l = P^\tau(Q',l,i) \cup i$. From Observation B.1,

$$P_p \subseteq P_{p+1} \subseteq \cdots \subseteq P_{r'-1} \subseteq P_{r'}. \tag{2}$$

Also note

$$V^\tau(Q',r') \subseteq V^\tau(Q',r'-1) \subseteq \cdots \subseteq V^\tau(Q',p+1) \subseteq V^\tau(Q',p). \tag{3}$$

We are done if we prove the following Claim(l) for each l, $p \leq l \leq r'$.
Claim(l): If $v \in V^\tau(Q',l)\backslash P_l$, then $v \in V^\tau(Q,l)$ and $e(Q,l,v) = e(Q',l,v)$.
Indeed Claim (l) immediately implies the following Consequence(l):
Consequence(l): If $v \in V^\tau(Q',l)\backslash P_l$, then $v \in V^\tau(Q,l)$ and $F^\tau(Q,l,v) = F^\tau(Q',l,v)$.
We now prove by induction that Claim(l) is true for each l, $p \leq l \leq r'$. By Lemma B.3 (b) and (c), $V^\tau(Q,p) = V^\tau(Q',p)$ and $e(Q,p,v) = e(Q',p,v)$ for each agent $v \in V^\tau(Q,p)\backslash i$. Hence, Claim($p$) is true.

If $r' = p$ we are done. So, suppose $r' \neq p$. Let l be a step such that $p < l \leq r'$. Assume Claim(g) is true for all g, $p \leq g < l \leq r'$. We prove that Claim(l) is true. Let $v \in V^\tau(Q',l)\backslash P_l$. By (2) and (3), $v \in V^\tau(Q',l-1)\backslash P_{l-1}$. From Consequence($l-1$),

$$F^\tau(Q,l-1,v) = F^\tau(Q',l-1,v). \tag{4}$$

From (2) and (3), $v \in V^\tau(Q',g)\backslash P_g$ for each step g, $p \leq g < l$. From Consequence(g) ($p \leq g < l$),

$$F^\tau(Q,g,v) = F^\tau(Q',g,v) \text{ for each step } g,\ p \leq g < l.$$

Together with (8) this implies

$$q_s^{Q,l-1} = q_s^{Q',l-1} \text{ for } s \in S \cap v. \tag{5}$$

Since $v \in V^\tau(Q',l)$, v is not removed at the end of step $l-1$ in $TTC(Q')$. Then by (4) and (5), v is also not removed at the end of step $l-1$ in $TTC(Q)$. Hence, $v \in V^\tau(Q,l)$.

Assume Claim(l) is not true, *i.e.*, $e(Q,l,v) \neq e(Q',l,v)$. Let $x = e(Q,l,v)$ and $x' = e(Q',l,v)$. Since $v \notin P_l$, $x' \notin P_l$. By (2), $x' \notin P_{l-1}$. By (3) and $x' \in V^\tau(Q',l)$, $x' \in V^\tau(Q',l-1)$. By Claim(l), $x' \in V^\tau(Q,l-1)$. We distinguish between the following two subcases.
Case 2a: Agent x' is removed at the end of step $l-1$ in $TTC(Q)$.
By $x' \in V^\tau(Q',l-1)\backslash P_{l-1}$ and Consequence($l-1$), $F^\tau(Q,l-1,x') = F^\tau(Q',l-1,x')$. Since at step $l-1$ of $TTC(Q)$ agent x' is removed, x' is in some cycle C in $G^\tau(Q,l-1)$. Hence, C is also a cycle in $G^\tau(Q',l-1)$ and

$$C = F^\tau(Q,l-1,x') = F^\tau(Q',l-1,x'). \tag{6}$$

From (2) and (3), $x' \in V^\tau(Q',g)\backslash P_g$ for each step g, $p \leq g < l$. From Consequence(g) ($p \leq g < l$),

$$F^\tau(Q,g,x') = F^\tau(Q',g,x') \text{ for each step } g,\ p \leq g < l.$$

Together with (8) this implies

$$q_s^{Q,l-1} = q_s^{Q',l-1} \text{ for } s \in S \cap x'. \tag{7}$$

Recall that x' is removed at the end of step $l-1$ in $TTC(Q)$. Then, by (6) and (7), x' is also removed at the end of step $l-1$ in $TTC(Q')$. Hence, $x' \notin V^\tau(Q',l)$, a contradiction with $x' = e(Q',l,v)$.
Case 2b: Agent x' is *not* removed at the end of step $l-1$ in $TTC(Q)$.
Then, $x' \in V^\tau(Q,l)$. Since $e(Q,l,v) = x$ and $x \neq x'$, we have xQ_vx'. Since $v \notin P_l$, $v \neq i$. Hence,

since $e(Q', l, v) = x'$, $x \notin V^\tau(Q', l)$. So, agent x was removed in some step g^*, $1 \leq g^* \leq l-1$, in $TTC(Q')$. In fact, by Lemma B.3(a), $p \leq g^* \leq l-1$. Note that no agent in $P_{r'}$ is removed before the end of step r'. By (2), $x \notin P_{g^*}$. Hence, $x \in V^\tau(Q', g^*)\backslash P_{g^*}$. By an argument similar to that of Case 2b, one shows that agent x is also removed at the end of step g^* in $TTC(Q)$. Hence, $x \notin V^\tau(Q, l)$, a contradiction with $x = e(Q, l, v)$. □

Lemma B.6 *Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and $Q'_i \in \mathcal{Q}$. Suppose there exists a student $j \in I\backslash i$ such that $\tau(Q)(j) \neq \tau(Q')(j)$. Then,*

(a) $\sigma^\tau(Q, i) \leq \sigma^\tau(Q, j)$ and [$\sigma^\tau(Q, i) = \sigma^\tau(Q, j)$ only if i and j are assigned in the same cycle in $TTC(Q)$], and

(b) $\sigma^\tau(Q', i) \leq \sigma^\tau(Q', j)$ and [$\sigma^\tau(Q', i) = \sigma^\tau(Q', j)$ only if i and j are assigned in the same cycle in $TTC(Q')$].

Proof: By non-bossiness of τ, $\tau(Q)(i) \neq \tau(Q')(i)$. Let p and p' be the steps at which student i is assigned in $TTC(Q)$ and $TTC(Q')$, respectively. Assume, without loss of generality, $p \leq p'$. By Lemma B.3(b), $V^\tau(Q, p) = V^\tau(Q', p)$ and

$$q_s^{Q,p} = q_s^{Q',p} \text{ for each school } s \in V^\tau(Q, p) \cap S. \tag{8}$$

By definition of p, $p \leq p'$, and Lemma B.3(d), there is a cycle C with $i \in C$ in $G^\tau(Q, p)$ but not in $G^\tau(Q', p)$.

We first prove (a). By Lemma B.3(b), for each student $h \in I\backslash i$ with $\sigma^\tau(Q, h) < p$ or $\sigma^\tau(Q', h) < p$, $\tau(Q)(h) = \tau(Q')(h)$. Since $\tau(Q)(j) \neq \tau(Q')(j)$, there are $r, r' \geq p$ (possibly $r \neq r'$) with $r = \sigma^\tau(Q, j)$ and $r' = \sigma^\tau(Q', j)$. So, $\sigma^\tau(Q, i) = p \leq r = \sigma^\tau(Q, j)$. Suppose $\sigma^\tau(Q, i) = \sigma^\tau(Q, j)$. We have to show that $j \in C$. Suppose to the contrary that $j \notin C$. Then, $j \in C^*$ for some cycle, say C^*, $C^* \neq C$, of $G^\tau(Q, p)$. Note $i \notin C^*$. Since $e(Q, p, v) = e(Q', p, v)$ for each agent $v \in V^\tau(Q, p)\backslash i$, C^* is also a cycle in $G^\tau(Q', p)$. In particular, $\tau(Q)(j) = \tau(Q')(j)$, a contradiction. This completes the proof of (a).

We now prove (b). We distinguish between two cases.

Case 1: $j \in P^\tau(Q', p, i)$.

Then, (b) follows directly from Lemma B.4 with $\bar{Q} = Q'$, $v' = j$, and $v = i$.

Case 2: $j \notin P^\tau(Q', p, i)$.

Assume that (b) is not true. In other words, assume that $\sigma^\tau(Q', i) > \sigma^\tau(Q', j)$ or [$\sigma^\tau(Q', i) = \sigma^\tau(Q', j)$ and i and j are assigned in different cycles in $TTC(Q')$].

Note $\sigma^\tau(Q, i) = p \leq r' = \sigma^\tau(Q', j) \leq \sigma^\tau(Q', i) = p'$. So, by Lemma B.5, if $v \in V^\tau(Q', r')\backslash(P^\tau(Q', r', i) \cup i)$, then $v \in V^\tau(Q, r')$ and $F^\tau(Q, r', v) = F^\tau(Q', r', v)$.

By definition of r', $j \in V^\tau(Q', r')$. By Lemma B.4, $j \notin (P^\tau(Q', r', i) \cup i)$. Hence, $j \in V^\tau(Q, r')$ and $F^\tau(Q, r', j) = F^\tau(Q', r', j)$. Since $\sigma^\tau(Q', j) = r'$, student j forms part of a cycle, say C', in $G^\tau(Q', r')$. Hence, $C' = F^\tau(Q', r', j)$. So, also $C' = F^\tau(Q, r', j)$. Hence, student j is assigned to the same school (or himself) in $TTC(Q)$ and $TTC(Q')$, contradicting $\tau(Q)(j) \neq \tau(Q')(j)$. This completes the proof of (b). □

Lemma B.7 *Let P be a school choice problem. Let $2 \leq k \leq m$. Let $Q \in \mathcal{E}^\tau(P,k)$. Define $\bar{Q}_i := \tau(Q)(i)$ for all $i \in I$. Then, $\bar{Q} \in \mathcal{E}^\tau(P,1)$ and $\tau(\bar{Q}) = \tau(Q)$. In other words, $\mathcal{O}^\tau(P,k) \subseteq \mathcal{O}^\tau(P,1)$.*

Proof It is sufficient to prove the following claim:
CLAIM: Let P be a school choice problem. Let $k \geq 2$, $Q \in \mathcal{E}^\tau(P,k)$, and $j \in I$. Let $\widetilde{Q}_j = \tau(Q)(j)$. Then, $(\widetilde{Q}_j, Q_{-j}) \in \mathcal{E}^\tau(P,k)$.

Indeed, if the above result holds true we can pick students one after another and eventually obtain a profile $\widetilde{Q} \in \mathcal{E}^\tau(P,k)$ where for all $j \in I$, $\widetilde{Q}_j = \tau(Q)(j)$. By construction, $\widetilde{Q} \in \mathcal{Q}(1)^I$. So, $\widetilde{Q} \in \mathcal{E}^\tau(P,1)$. By repeated use of Lemma 7.4, $\tau(\widetilde{Q}) = \tau(Q)$. This proves that $\mathcal{O}^\tau(P,k) \subseteq \mathcal{O}^\tau(P,1)$.

Let $\widetilde{Q} = (\widetilde{Q}_j, Q_{-j})$. Suppose to the contrary that $\widetilde{Q} \notin \mathcal{E}^\tau(P,k)$. Then there exist a student, say i, and a list $Q'_i \in \mathcal{Q}(k)$ such that

$$\tau(Q'_i, \widetilde{Q}_{-i}) P_i \tau(Q_i, \widetilde{Q}_{-i}). \tag{9}$$

By Lemma 7.4, $\tau(\widetilde{Q}) = \tau(Q)$. We claim that $i \neq j$. Suppose $i = j$. Then $\widetilde{Q}_{-i} = \widetilde{Q}_{-j} = Q_{-j}$. Hence, (9) becomes $\tau(Q'_j, Q_{-j}) P_j \tau(Q_j, Q_{-j})$ contradicting $Q \in \mathcal{E}^\tau(P,k)$. So, $i \neq j$.

Let $\widetilde{Q} = (Q_i, \widetilde{Q}_j, Q_{-ij})$, $\widetilde{Q}' = (Q'_i, \widetilde{Q}_j, Q_{-ij})$, and $Q' = (Q'_i, Q_j, Q_{-ij})$. We can rewrite (9) as

$$\tau(\widetilde{Q}') = \tau(Q'_i, \widetilde{Q}_j, Q_{-ij}) P_i \tau(Q_i, \widetilde{Q}_j, Q_{-ij}) = \tau(\widetilde{Q}). \tag{10}$$

By $Q \in \mathcal{E}^\tau(P,k)$ and Lemma B.2, $\tau(Q) \in IR(P)$. So, $\tau(\widetilde{Q}) = \tau(Q) \in IR(P)$. By (10), $\tau(\widetilde{Q}')(i) \in S$. Let $s = \tau(\widetilde{Q}')(i)$. We distinguish between two cases.
CASE 1: $\tau(Q')(j) = \tau(Q)(j)$. Recall that $\widetilde{Q}_j = \tau(Q)(j)$. So, $\widetilde{Q}_j = \tau(Q')(j)$. Hence, Lemma 7.4 implies $\tau(Q'_i, \widetilde{Q}_j, Q_{-ij}) = \tau(Q'_i, Q_j, Q_{-ij})$ and $\tau(Q_i, \widetilde{Q}_j, Q_{-ij}) = \tau(Q_i, Q_j, Q_{-ij})$. The left hand side and right hand side of (10) can then be replaced to obtain $\tau(Q'_i, Q_j, Q_{-ij}) P_i \tau(Q_i, Q_j, Q_{-ij})$. So, $Q \notin \mathcal{E}^\tau(P,k)$, a contradiction.
CASE 2: $\tau(Q')(j) \neq \tau(Q)(j)$. We claim that $\tau(Q')(i) \neq \tau(\widetilde{Q}')(i)$. To prove this, suppose to the contrary that $\tau(Q')(i) = \tau(\widetilde{Q}')(i)$. Since $\tau(\widetilde{Q}) = \tau(Q)$, (10) boils down to $\tau(Q') P_i \tau(Q)$, which implies that $Q \notin \mathcal{E}^\tau(P,k)$, a contradiction. So, $\tau(Q')(i) \neq \tau(\widetilde{Q}')(i)$.

Notice that for any student $h \neq i$, $Q'_h = Q_h$. So, by Lemma B.6, $\sigma^\tau(Q', i) \leq \sigma^\tau(Q', j)$. Notice also that for any student $h \neq j$, $\widetilde{Q}'_h = Q'_h$. So, by Lemma B.6, $\sigma^\tau(Q', j) \leq \sigma^\tau(Q', i)$. So, $\sigma^\tau(Q', j) = \sigma^\tau(Q', i)$. From Lemma B.6 it follows that i and j are in the same cycle when executing the TTC algorithm with the list profile Q'. So, i and j are not in self-cycles. In particular, i is assigned to a school. Since $Q'_i = s$ we have $\tau(Q')(i) = s$. By definition, $s = \tau(\tilde{Q}')(i)$. So, $\tau(Q')(i) = \tau(\tilde{Q}')(i)$, a contradiction.

Since both CASE 1 and CASE 2 yield a contradiction, we conclude that $\widetilde{Q} \in \mathcal{E}^\tau(P,k)$. □

Proof of Proposition 7.6:
Mechanism τ is strategy-proof. Hence, $P \in \mathcal{E}^\tau(P,m)$. By Lemma B.7, $\tau(P) \in \mathcal{O}^\tau(P,k)$ for any $1 \leq k \leq m$. □

Now Proposition 7.1 follows immediately.

Proof of Proposition 7.1:

From Lemma B.2, $\mathcal{O}^\tau(P,k) \subseteq IR(P)$. We now prove that also $\mathcal{O}^\tau(P,k) \subseteq IR(P) \cap NW(P)$. By Proposition 7.6 we may assume $k = 1$. Let $Q \in \mathcal{E}^\tau(P,1)$. Suppose to the contrary that $\tau(Q)$ is wasteful, *i.e.*, there is a student $i \in I$ and a school $s \in S$ such that $|\tau(Q)(s)| < q_s$ and $sP_i\tau(Q)(i)$. Let $Q'_i = s$. We show that $\tau(Q'_i, Q_{-i})(i) = s$. (Since this contradicts $Q \in \mathcal{E}^\tau(P,1)$ we are done.)

First observe that $Q'_i = s$ contains only school s. Hence, if $\tau(Q'_i, Q_{-i})(i) \in S$, then $\tau(Q'_i, Q_{-i})(i) = s$. So, suppose $\tau(Q'_i, Q_{-i})(i) = i$. By definition of the TTC algorithm, in $TTC(Q')$ all seats of school s are assigned to other students. So, $|\tau(Q)(s)| < q_s = |\tau(Q')(s)|$. Hence, there exists a student $j \in S$, $j \neq i$, such that $\tau(Q')(j) = s$ and $\tau(Q)(j) \neq s$. Since $Q'_j = Q_j \in \mathcal{Q}(1)$ and $\tau(Q')(j) = s$, we have $Q_j = Q'_j = s$. Hence, $\tau(Q)(j) = j$. By definition of the TTC algorithm, in $TTC(Q)$ all seats of school s are assigned to other students. In other words, at the end of of $TTC(Q)$ school s has no available seats. So, $|\tau(Q)(s)| = q_s$, a contradiction. □

In order to prove Theorem 6.6 we need the following lemmas.

Lemma B.8 *Let f be a cyclic priority structure. Let $1 \leq k \leq m$. Then, there is a profile of student preferences P with an unstable equilibrium outcome in the game $\Gamma^\tau(P,k)$,* i.e.*, for some $Q \in \mathcal{E}^\tau(P,k)$, $\tau(Q) \notin S(P)$.*

Proof : By Theorem 1 of Kesten (2006a), there is a school choice problem P such that $\tau(P)$ is not stable. Since τ is group strategy-proof, $P \in \mathcal{E}^\tau(P,m)$. Hence, by Corollary 7.6, $\tau(P) \in \mathcal{O}^\tau(P,m) = \mathcal{O}^\tau(P,k)$. Hence, there is a list profile $Q \in \mathcal{Q}(k)^I$ such that $Q \in \mathcal{E}^\tau(P,k)$ and $\tau(Q) = \tau(P) \notin S(P)$. □

Lemma B.9 *Let f be an acyclic priority structure. Let $1 \leq k \leq m$. Then, for any profile of student preferences P all equilibrium outcomes in the game $\Gamma^\tau(P,k)$ are stable,* i.e.*, for all $Q \in \mathcal{E}^\tau(P,k)$, $\tau(Q) \in S(P)$. Moreover, $S(P) \subseteq \mathcal{O}^\tau(P,k)$.*

In order to prove Lemma B.9 we need the following result due to Kesten (2006).

Lemma B.10 ***(Kesten, 2006a, Lemma 1)***
If a priority structure contains a strong cycle, then it also contains a cycle.

Proof of Lemma B.9:
Let $Q \in \mathcal{E}^\tau(P,k)$. By Theorem 1 of Kesten (2006), $\tau = \gamma$. Hence, $Q \in \mathcal{E}^\gamma(P,k)$ and $\tau(Q) = \gamma(Q)$. By Lemma B.10, f does not contain a strong cycle. Hence, by Proposition 6.3 (for $k = 1$) or Theorem A.3 (for $k \geq 2$), $\gamma(Q) \in S(P)$. Finally, from $\tau = \gamma$ and Proposition 6.1 it follows that $S(P) \subseteq \mathcal{O}^\tau(P,k)$. □

Proof of Theorem 6.6:
Follows immediately from Lemmas B.8 and B.9. □

C Appendix: Proofs Section 8

Proof of Lemma 8.1:
Let $\varphi = \gamma, \tau$. The result follows directly from the strategy-proofness of γ (Dubins and Freedman, 1981; Roth, 1982) and τ (Abdulkadiroğlu and Sönmez, 2003) by using Q'_i as student i's "true preferences:" $\varphi(Q'_i, Q_{-i})(i)$ is ranked higher than $\varphi(Q_i, Q_{-i})(i)$ by Q'_i, hence $\varphi(Q'_i, Q_{-i})(i)$ is ranked higher than $\varphi(Q_i, Q_{-i})(i)$ by P_i. □

Proof of Lemma 8.2:
Let $\varphi = \gamma, \tau$. From Lemma 8.1 it follows that P_i^k is weakly k-dominates any strategy that is obtained from P_i^k by interchanging the positions of the (acceptable) schools. So, let $Q_i \in \mathcal{Q}(k)$ be any other strategy. Note that Q_i contains a school that is not in P_i^k. In fact, by Lemma 8.1 we can assume that the schools in Q_i are listed in the true order (*i.e.*, for any two schools s and s' listed in Q_i, $sQ'_is' \Rightarrow sP_is'$). We claim that either $\varphi(P_i^k, Q_{-i})(i) = \varphi(Q)(i)$ for all $Q_{-i} \in \mathcal{Q}(k)^{I\setminus i}$ or $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$ for some $Q'_{-i} \in \mathcal{Q}(k)^{I\setminus i}$. (This completes the proof as this shows that no strategy k-dominates P_i^k.)

Suppose that *not* $\varphi(P_i^k, Q_{-i})(i) = \varphi(Q)(i)$ for all $Q_{-i} \in \mathcal{Q}(k)^{I\setminus i}$. We have to show that $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$ for some $Q'_{-i} \in \mathcal{Q}(k)^{I\setminus i}$. We know that for some $Q_{-i} \in \mathcal{Q}(k)^{I\setminus i}$, $\varphi(P_i^k, Q_{-i})(i) \neq \varphi(Q)(i)$. Clearly, if $\varphi(P_i^k, Q_{-i})P_i\varphi(Q_i, Q_{-i})$, then we are done. So, suppose $\varphi(Q_i, Q_{-i})P_i\varphi(P_i^k, Q_{-i})$. Then, $\varphi(Q_i, Q_{-i})(i) \in S$. Let $s' = \varphi(Q_i, Q_{-i})(i)$ and $S' = \{s \in S : sQ_is'\}$. By definition of the DA-algorithm/TTC-algorithm, there are at least $n' = \sum_{s\in S'} q_s$ students. Let $Q'_{-i} \in \mathcal{Q}(k)^{I\setminus i}$ be such that for each $s \in S'$, exactly q_s students have a list that consists of s only. The other $n - n'$ students have the empty list. Now one easily verifies that $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$. This completes the proof. □

Proof of Proposition 8.6:
By definition of the DA-algorithm, $|M(\gamma(P^k))| \leq |M(\gamma(P))|$. We complete the proof by showing that if $i \in M(\gamma(P))$, then $\gamma(P^k)(i)R_i\gamma(P)(i)$. (Since $\gamma(P) \in IR(P)$, $\gamma(P^k)(i) \in S$. Hence, $i \in M(\gamma(P^k))$. But then $M(\gamma(P^k)) = M(\gamma(P))$.)

Let $i \in M(\gamma(P))$. Denote $s = \gamma(P)(i) \in S$. Suppose to the contrary that $sP_i\gamma(P^k)(i)$. Let $Q'_i = s$. By Lemma A.1, $\gamma(Q'_i, P_{-i})(i) = s$. By a result of Gale and Sotomayor (1985, Theorem 2) extended to the college admissions model (Roth and Sotomayor, 1990, Theorem 5.34), Q'_i ranks $\gamma(Q'_i, P^k_{-i})(i)$ weakly higher than $\gamma(Q'_i, P_{-i})(i)$. Hence, $\gamma(Q'_i, P^k_{-i})(i) = s$, contradicting the assumption that $P^k \in \mathcal{E}(P, k)$. So, $\gamma(P^k)(i)R_is = \gamma(P)(i)$. □

Proof of Proposition 8.7:
In Example 8.5, $\gamma(P) = \{\{i_2, s_3\}, \{i_3, s_2\}, \{i_1\}, \{s_1\}\}$ and $\tau(P) = \{\{i_1, s_2\}, \{i_2, s_3\}, \{i_3, s_1\}\}$. So, $|M(\tau(P))| = 3 > 2 = |M(\gamma(P))|$.

In Example 8.4, $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$ and $\tau(P) = \{\{i_1, s_1\}, \{i_3, s_3\}, \{i_2\}, \{s_2\}\}$. So, $|M(\tau(P))| = 2 < 3 = |M(\gamma(P))|$. □

References

Abdulkadiroğlu, A. (2005) "College Admissions with Affirmative Actions," *International Journal of Game Theory*, **33**, 535-549.

Abdulkadiroğlu, A., P.A. Pathak, and A.E. Roth (2005) "The New York City High School Match," *American Economic Review, Papers and Proceedings*, **95**, 364-367.

Abdulkadiroğlu, A., P.A. Pathak, A.E. Roth, and T. Sönmez (2006) "Changing the Boston School Choice Mechanism: Strategy-proofness as Equal Access," Mimeo, Harvard University.

Abdulkadiroğlu, A. and T. Sönmez (1999) "House Allocation with Existing Tenants," *Journal of Economic Theory*, **88**, 233-260.

Abdulkadiroğlu, A. and T. Sönmez (2003) "School Choice: A Mechanism Design Approach," *American Economic Review*, **93**, 729-747.

Alcalde, J. (1996) "Implementation of Stable Solutions to Marriage Problems," *Journal of Economic Theory*, **69**, 240-254.

Alcalde, J. and A. Romero-Medina (2000) "Simple Mechanisms to Implement the Core of College Admissions Problems," *Games and Economic Behavior*, **31**, 294-302.

Aumann, R.J. (1959) "Acceptable Points in General Cooperative n-Person Games," in Volume IV of Contributions to the Theory of Games, Princeton University Press.

Balinski, M. and T. Sönmez (1999) "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, **84**, 73-94.

Chen, Y. and T. Sönmez (2006) "School Choice: An Experimental Study," *Journal of Economic Theory*, **127**, 202-231.

Dubins, L.E. and D.A. Freedman (1981) "Machiavelli and the Gale-Shapley Algorithm," *American Mathematical Monthly*, **88**, 485-494.

Ehlers, L. and B. Klaus (2006a) "Consistent House Allocation," *Economic Theory*, forthcoming.

Ehlers, L. and B. Klaus (2006b) "Efficient Priority Rules," *Games and Economic Behavior*, 55, 372-384.

Ergin, H. İ. (2002) "Efficient Resource Allocation on the Basis of Priorities," *Econometrica*, **70**, 2489-2497.

Ergin, H. İ. and T. Sönmez (2006) "Games of School Choice under the Boston Mechanism," *Journal of Public Economics*, **90**, 215-237.

Gale, D. and L.S. Shapley (1962) "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, **69**, 9-15.

Gale, D. and M.A.O. Sotomayor (1985a) "Ms. Machiavelli and the Stable Matching Problem," *American Mathematical Monthly*, **92**, 261-268.

Gale, D. and M.A.O. Sotomayor (1985b) "Some Remarks on the Stable Matching Problem," *Discrete Applied Mathematics*, **11**, 223-232.

Kara, T. and T. Sönmez (1996) "Nash Implementation of Matching Rules," *Journal of Economic Theory*, **68**, 425-439.

Kara, T. and T. Sönmez (1997) "Implementation of College Admission Rules," *Economic Theory*, **9**, 197-218.

Kesten, O. (2006a) "On Two Competing Mechanisms for Priority-Based Allocation Problems," *Journal of Economic Theory*, **127**, 155-171.

Kesten, O. (2006b) "Student Placement to Public Schools in the US: Two New Solutions," Mimeo, Harvard University.

Kojima, F. (2006) "Games of School Choice under the Boston Mechanism with General Priority Structures," Mimeo, Harvard University.

Ma, J. (1995) "Stable Matchings and Rematching-Proof Equilibria in a Two-Sided Matching Market," *Journal of Economic Theory*, **66**, 352-369.

McVitie, D.G. and L.B. Wilson (1970) "Stable Marriage Assignment for Unequal Sets," *BIT*, **10**, 295-309.

Pápai, S. (2000) "Strategyproof Assignment by Hierarchical Exchange," *Econometrica*, **68**, 1403-1433.

Peleg, B. (1997) "Implementation of the Core of Marriage Problem," DP132, Hebrew University of Jerusalem.

Romero-Medina, A. (1998) "Implementation of Stable Solutions in a Restricted Matching Market," *Review of Economic Design*, **3**, 137-147.

Roth, A.E. (1982) "The Economics of Matching: Stability and Efficiency," *Mathematics of Operations Research*, **92**, 617-628.

Roth, A.E. (1984) "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, **92**, 991-1016.

Roth, A.E. (1986) "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," *Econometrica*, **54**, 425-428.

Roth, A.E. (1989) "Two-Sided Matching with Incomplete Information about Others' Preferences," *Games and Economic Behavior*, **1**, 191-209.

Roth, A.E. (1990) "New Physicians: A Natural Experiment in Market Organization," *Science*, **250**, 1524-1528.

Roth, A.E. (1991) "A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom," *American Economic Review*, **81**, 415-440.

Roth, A.E. (2002) "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," *Econometrica*, **70**, 1341-1378.

Roth, A.E. and E. Peranson (1999) "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, **89**, 748-780.

Roth, A.E. and A. Postlewaite (1977) "Weak versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics*, **4**, 131-137.

Roth, A.E. and M.A.O. Sotomayor (1990) Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Econometric Society Monograph Series. New York: Cambridge University Press.

Roth, A.E. and X. Xing (1994) "Jumping the Gun: Imperfections and Institutions related to the Timing of Market Transactions," *American Economic Review*, **84**, 992-1044.

Satterthwaite, M.A. and H. Sönnenschein (1981) "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies*, **48**, 587-597.

Shapley, L.S and H. Scarf (1974) "On Cores and Indivisibility," *Journal of Mathematical Economics*, **1**, 23-37.

Shin, S. and S-C. Suh (1996) "A Mechanism Implementing the Stable Rule in Marriage Problems," *Economics Letters*, **51**, 185-189.

Sönmez, T. (1997) "Games of Manipulation in Marriage Problems," *Games and Economic Behavior*, **26**, 429-439.

Sotomayor, M.A.O. (1996) "Admission Mechanisms of Candidates to Institutions. A Game-Theoretic Modeling and Analysis," (in Portuguese) *Brazilian Review of Econometrics*, **16**, 25-63.

Sotomayor, M.A.O. (2003) "Reaching the Core of the Marriage Market through a Non-Revelation Matching Mechanism," *International Journal of Game Theory*, **32**, 241-251.

Suh, S-C. (2003) "Games Implementing the Stable Rule of Marriage Problems in Strong Nash Equilibria," *Social Choice and Welfare*, **20**, 33-39.

Tadenuma, K. and M. Toda (1998) "Implementable Stable Solutions to Pure Matching Problems," *Mathematical Social Sciences*, **35**, 121-132.

Teo, C-P., J. Sethuraman, and W-P. Tan (2001) "Gale-Shapley Stable Marriage Problem Revisited: Strategic Issues and Applications," *Management Science*, **47**, 1252-1267.