Bargaining one-dimensional policies and the efficiency of super majority rules

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Abstract

We consider negotiations selecting one-dimensional policies. Individuals have single-peaked preferences, and they are impatient. Decisions arise from a bargaining game with random proposers and (super) majority approval, ranging from the simple majority up to unanimity. The existence and uniqueness of stationary subgame perfect equilibrium is established, and its explicit characterization provided. We supply an explicit formula to determine the unique alternative that prevails, as impatience vanishes, for each majority. As an application, we examine the efficiency of majority rules. For symmetric distributions of peaks unanimity is the unanimously preferred majority rule. For asymmetric populations rules maximizing social surplus are characterized.

1 Introduction

The genesis, efficiency and stability of democratic institutions are at the center of the public debate. Recent historical developments - notably the emergence of new states after the fall of the Berlin Wall in 1989, and the constitutional construction of the European Union - have fueled an intense debate on the links between political institutions and welfare,[1] and have renewed attention to the classical problems of constitutional design.

This paper contributes to the positive analysis of (super) majority rules with the tools of bargaining theory. As remarked in the literature on bargaining in legislatures initiated by Baron and Ferejohn (1989), collective decisions in democratic polities are often the result of processes where bargaining and majority voting are combined. In this paper we examine this type of negotiations under the assumption that policies must be selected from a continuous one-dimensional set, where individuals have single-peaked and concave utilities, and they are heterogeneous only in the locations of peaks. This simple set up is a classical formulation in the social choice and political economy literatures. Examples are the location of a facility, the election of a public official, the choice of tax rates or minimum wages, or the budget allocated to a specific project. We assume that decisions must be negotiated over time (individuals are impatient), and that the approval of a (super) majority of the group is required for an agreement. Our aim is to examine the outcomes of these negotiations, and to describe how they depend on a) the demographics of the group, namely the number of individuals and the distribution of individual preferences, and b) the institutions for consensus building, notably the size of the (super)majority required to settle a choice.

Our main contribution is to provide a tractable model where the effects of demographical and institutional parameters over collective decisions can be described transparently.

[1] See, for example, Persson and Tabellini (2000), Persson (2002) and Aghion, Alesina and Trebbi (2004).

Our model delivers a unique equilibrium, and we provide its explicit characterization. Hence, we supply a precise prediction for what alternatives prevail in negotiations under each majority rule. This allows tractable comparative statics, which are the tool to address a wide range of applications. The application that we explore in this paper examines the efficiency of majority rules.

More precisely, our first contribution is a complete description of stationary subgame perfect equilibrium[2] outcomes under the standard random proposers protocol: At the beginning of each round, an agent is selected at random to make a proposal which is approved if it obtains the favorable vote of a (super) majority. Upon approval, the selected alternative is implemented and the game ends. If the proposal is not approved, a new round of bargaining begins in the following period. Under the assumption that individuals are impatient, for each profile of peaks and each majority requirement, we explicitly characterize the subgame perfect equilibrium in stationary strategies, and we show its existence and uniqueness. The unique equilibrium is fully described by the approval set, the (unique) subinterval of alternatives that are accepted by the required majority. The size of this majority matters a lot in determining the approval set. We also identify the unique limit equilibrium as players become infinitely patient. We establish that in the limit the approval set shrinks to a unique alternative, and we supply the explicit formula that determines this alternative. Thus, we supply a natural selection criterion to select a single policy among the (potentially very large) set of equilibrium alternatives in environments with infinitely patient individuals.[3]

A unique equilibrium induces a unique distribution of approved alternatives and unique individual (expected) benefits for each majority rule. As an application, we can evaluate the expected benefits that determine the players' preferences over the different majority

[2]Henceforth equilibrium. The restriction to stationary strategies is standard in multilateral bargaining games, as these games are known to have very large sets of subgame perfect equilibria.

[3]Banks and Duggan [2000] establish that every policy in the core of the voting rule is an equilibrium. For supermajorities this is a full interval alternatives.

requirements, and asses their efficiency properties. Very strong results apply for populations with a symmetric distribution of peaks. In these environments, weakening the majority requirement spreads the range of equilibrium alternatives while preserving the mean. When utilities are strictly concave, this implies that all individuals have a strict preference for unanimity over any other majority rule. The conclusion is that in symmetric populations unanimity is the unique Pareto efficient majority requirement. For general asymmetric populations, where many majority rules are Pareto efficient, we examine rules that maximize social surplus in the limit equilibrium. For large populations we show that the first best policy is generically attainable as the limit equilibrium of some majority rule. When utilities are tent-shaped (that is, when the cost of selecting an alternative different from the peak is linear in its distance to the peak) the simple majority (and generically no other rule) delivers the first best policy irrespective of the distribution of peaks. For strictly concave utilities (the cost of selecting an alternative different from the peak is is strictly convex in the distance) and distributions of peaks satisfying a mild regularity condition, the optimal rule is a strict super-majority. This super-majority is weaker than unanimity under natural specifications.

The remainder of the paper is organized as follows. Section 2 discusses the related literature. The environment and the bargaining game are presented in section 3. In section 4 we characterize the stationary subgame perfect equilibrium and we establish its existence and uniqueness. Section 5 examines Pareto Optimal rules under symmetric distributions of peaks. Section 6 characterizes the unique asymptotic equilibrium outcome, and we discuss the asymptotic efficiency of majority rules. Section 7 contains final remarks. Proofs omitted from the main text are in the Appendix.

2 Relation to the literature

The present paper contributes to the literature that addresses multilateral bargaining over policy choices initiated by Baron and Ferejohn (1989). This literature has mostly focussed to situations in which a unit of surplus must be distributed, and decisions require a simple majority. Eraslan (2002) extends the analysis to set ups with heterogeneous discounts and recognition probabilities, and considers the full range of (super) majority rules. She establishes the uniqueness of equilibrium payoffs under linear utilities. Our approach complements this literature by examining the opposite polar case: rather than examining the transfers that are necessary for agreement, we examine negotiations where agreements must lie in an interval, so that transfers of resources among the parties are impossible.[4] A general model that covers both approaches as particular cases is due to Banks and Duggan (2000, 2006). They assume that alternatives are selected from arbitrary compact convex subsets of an Euclidean space, and they examine bargaining protocols where the proposer is selected at random and approval is determined by voting rules in a general family. They prove existence of equilibria under very general conditions, and they establish sufficient conditions for core equivalence. For set-ups where alternatives are in an interval, they show that equilibria (in pure strategies) exist; and that for perfectly patient players they are equivalent to core outcomes. Their results, however, do not provide an explicit characterization of equilibria for impatient players, nor a discussion of conditions for uniqueness. The issue of equilibrium uniqueness for one-dimensional problems is addressed in Cho and Duggan (2003) and Cardona and Ponsati (2007). Under the assumption of quadratic utilities Cho and Duggan (2003) characterize the equilibrium and establish its uniqueness for games with random proposers and a set of decisions rules that includes the simple majority, but not stronger super-majorities. For negotiations that follow a deterministic protocol, the uniqueness of the equilibrium for all majority rules is established in Cardona and Ponsati (2007). They also prove that, as players

[4] See also Eraslan and Merlo (2002), Jackson and Moselle (2002).

become arbitrarily patient, the equilibrium converges to a single alternative, which is independent of the protocol.[5] These results apply to a very rich class of populations, since peaks and utilities can vary across individuals. But this generality comes as the expense of tractability. The characterization of equilibrium is rather involved and, unlike in the present paper, the result are not easily ready for applications.

We remark that most of the literature concerns situations where an alternative is selected only once and for all, none can be implemented in disagreement, and disagreement is the worst outcome for everyone. Some important collective decisions are of this nature; for example the location of a public facility or the appointment of public officials. Other policies, such as tax rates or minimum wages are chosen repeatedly over long time horizons, and thus may be subject to recurrent re-negotiation. Bargaining when the status-quo is not the worst outcome for all individuals is addressed in Banks and Duggan (2006) for general set ups, and also considered in Cho and Duggan (2003). Baron (1996) addresses situations where one dimensional policies are chosen repeatedly under simple majority rule, and decisions become the status quo for future negotiations. Bucovetsky (2003) discusses the effects of super-majority requirements in these environments.

Our results on the efficiency of (super)majority rules contribute to the literature on the endogenous emergence, efficiency and stability of majority rules. The general analysis of social choices over social choice rules is a classical problem. Its modern formalization starts with the discussion of the distinctive role of unanimous consent by Buchanan and Tullock (1963). Collective choices of majority rules under majoritarian regimes are discussed in Greenberg (1979), Caplin and Nalebuff (1988), and Barberà and Jackson (2004), among others. The reader is referred to the later for a discussion of this literature. The efficiency of super-majority rules in polities that choose within an interval is addressed in Aghion and Bolton (2003) and Holden (2005). Their approach is to asses the expected social benefits of different rules at a stage where individuals are "under the veil of ignorance",

[5]Predtetchinski (2007) establishes asymptotic uniqueness for games with random proposals.

i.e. they still do not know what heir preferred policy will be. In contrast, we consider individual preferences and collective choice of consensus rules after individual preferences on alternatives are known. A very different model that discusses the choice of voting rules among heterogeneous individuals is presented in Messner and Polborn (2004). They examine voting rules to implement policy changes in an overlapping generations model and argue that - because reforms benefit the young more than the old - the median voter prefers a super-majority.

3 The model

A group of individuals $I \subset [0,1]$ must collectively select an alternative within the one dimensional policy space $[0,1]$. They negotiate over discrete time, $t = 0, 1, 2....$ with a procedure that combines alternating proposals and voting. This environment is formally described next.

Individual Payoffs. Individuals have single peaked utilities over policies and are impatient. Upon a collective decision that selects alternative $x \in [0,1]$ at date t, individual i obtains utility $\delta^t u(x,i)$, where

$$u(x,i) = v(|i-x|),$$

v is twice differentiable, decreasing and concave, with $v(0) > v(1) \geq 0$, and $\delta \in (0,1)$ is the common discount rate. Note that the rigth and left derivatives $u_x^+(x,i)$ and $u_x^-(x,i)$, are always well defined, and that they coincide for $|i-x| > 0$. Specifying a functional form for the utility will be useful to examine examples that illustrate our results; the main examples are *tent-shaped utilities*, i.e. $u(x,i) = 1 - |x-i|$, and *quadratic utilities*, i.e. $u(x,i) = 1 - (x-i)^2$.

Distributions of Peaks. The different locations of the peaks are the only source of heterogeneity within the population. Each $i \in I$ denotes both a generic individual and the location of her peak, so that all the information regarding heterogeneity within the population is embedded in the cumulative distribution function of peaks, denoted by F. Given F, let $(I, \mathcal{B}, \mu)$ denote the probability space where $\mathcal{B}$ is a $\sigma-$algebra of subsets of I, and μ the probability measure induced by F. We will focus our analysis to the following two distinct classes of populations. i) *Small populations,* where $I = \{i_1, ..., i_n\}, 0 = i_1 < i_2 < ... < i_n = 1$, and n odd. ii) *Large populations,* where $I = [0,1]$, and F has a positive density f on $(0,1)$. The restrictive assumptions defining these two categories are for expositional convenience, and can be relaxed. In particular, the results of section 4 and 5 directly apply to general populations characterized by any cumulative distribution F.

Salient Policies. Within the policy space the following alternatives are specially relevant. i) The *median policy* i^m is the alternative that coincides with the peak of the median individual in I, i.e. $i^m = i_{\frac{n+1}{2}}$, and $F(i^m) = 1/2$, respectively for a small and a large population. ii) The *mean policy* i^e is the alternative that coincides with the average of individual peaks, i.e. $i^e = \int_0^1 idF(i)$. iii) The *first best policy* x^{fb} is the alternative that maximizes social surplus, $S(x) = \int_0^1 u(x,i)dF(i)$, over the set $[0,1]$. It is straightforward that the concavity of utilities implies the strict concavity of S. Since $S'(0) > 0$ and $S'(1) < 0$, for large populations the unique first best policy x^{fb}, is given by the the unique solution to the first order condition $S'(x) = \int_0^1 u_x(x,i)dF(i) = 0$.

Symmetry and Regularity. The symmetry properties of distributions play an important role in our analysis. A distribution is *symmetric* if for every individual $i \in [0,1/2) \cap I$ there is an individual $j = 1 - i \in I$, i.e. $F(i) = 1 - F(1-i) + \mu(i)$ for all $i \in I$.

For non-symmetric populations, the following regularity properties will also be useful. Given a positive density f on $(0,1)$ we define its *induced symmetric density* $\widehat{f}$ as follows: a) If f is such that $i^m \leq 1/2$ then $\widehat{f}(x) = f(x)$ for $x \in [0, i^m]$, $\widehat{f}(x) = f(2i^m - x)$ for $x \in [i^m, 2i^m]$, and $\widehat{f}(x) = 0$ for $x \in [2i^m, 1]$. b) If f is such that $i^m \geq 1/2$ then $\widehat{f}(x) = f(x)$ for $x \in [i^m, 1]$, $\widehat{f}(x) = f(2i^m - x)$ for $x \in [1 - 2i^m, i^m]$, and $\widehat{f}(x) = 0$ for $x \in [0, 1 - 2i^m]$.

We say that F is *symmetry regular* if f is positive over $[0,1]$ and $\widehat{f}$ is constructed by transferring mass towards the median: That is, $\widehat{f}$ crosses f only once, from above if $i^m \leq 1/2$, and from below if $i^m \geq 1/2$. (See Figure 1, where at $\widehat{f}$ agents with higher peaks are substituted with the agents with lower peaks, from A to B.) We say that a distribution F is *mean-median regular* if either a) $i^m < i^e < 1/2$ and $F(i^e) + i^e < 1$, or b) $i^m > i^e > 1/2$ and $F(i^e) + i^e > 1$. Both conditions are mild, and hold under most common specifications. [6]

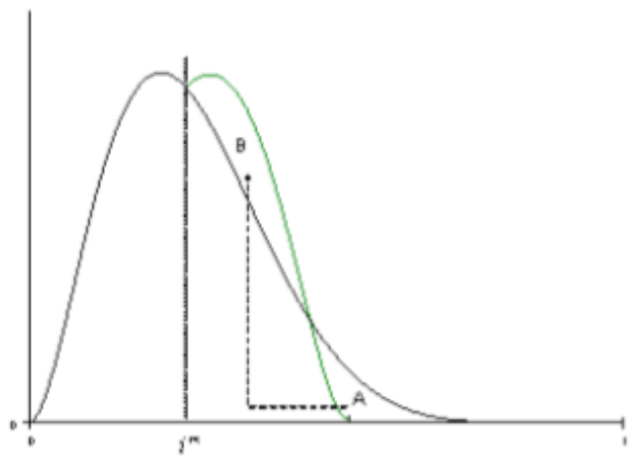

Figure 1: A regular density f and its induced symmetric density $\widehat{f}$.

Majority Rules and Boundary Players. $Q \subset [1/2, 1]$ denotes the set of admissible majority rules, which is $Q = \{k/n : k \in \{(n+1)/2, ..., n\}\}$ for small populations, and $Q = [1/2, 1]$ when the population is large. For each $q \in Q$, $\mathcal{W}(q)$ denotes the set of winning coalitions under q, that is

$$\mathcal{W}(q) = \{S \in \mathcal{B} : \mu(S) \geq q\}.$$

[6] In particular it is easy to check the conditions hold in Triangular, Beta, and Standard Two-Sided Power distributions.

A collective decision under majority $q \in Q$ requires the support of a subset $S \in \mathcal{W}(q)$.

Given F and q the *boundary players* play a crucial role in the bargaining game. They are the two individuals, l and r, that can constitute (tight) q majority with all the individuals on their right and on their left, respectively; i.e. $F(l) = 1 - q$ and $F(r) = q$.

The Bargaining Game. For each environment, i.e. a tuple (F, u, δ, q), which is common knowledge, the negotiation begins at $t = 0$ and proceed as follows. At each $t \geq 0$ an individual is selected at random (all with equal probability) to make a proposal. Then, she chooses an alternative in $[0, 1]$ and all other players, sequentially in the natural order, reply with acceptance or rejection. The proposal is approved if the subset of players that accept it (including the proposer) is a winning coalition $S \in \mathcal{W}(q)$. Upon approval, the agreed alternative is implemented and the game ends. Otherwise, the game moves to $t + 1$, a new proposer is selected, and so on.

An individual *strategy* specifies actions - a proposal, and an acceptance/rejection rule - for each subgame. At a *stationary strategy* each individual makes the same proposal whenever she is selected and always accepts proposals that are no further away from her peak than some given threshold. A *stationary subgame perfect equilibrium* (henceforth an *equilibrium*) is a profile of stationary strategies that are mutually best responses at each subgame, and such that no individual uses weakly dominated actions.[7]

In the sequel the existence of a unique equilibrium for each environment (F, u, δ, q), is established, and its explicit characterization is provided. Hence, the individual and collective expected benefits can be evaluated and compared over the different values of q. This comparative statics exercise allows to examine efficiency of different majority rules.

[7] The later requirement rules out equilibria sustained by weakly dominated acceptances at the voting stage. This is a superfluous restriction when the population is small (since acceptance/rejection moves are assumed sequential), but not when it is large.

4 Equilibrium

In this section we characterize the equilibrium for each environment (F, u, δ, q) and we establish existence and uniqueness. Our first result, Proposition 1, states that an equilibrium is fully described by the the *approval set* - the subset of alternatives that get the acceptance of a q majority - and establishes existence and uniqueness. Our second result, Proposition 2, supplies necessary and sufficient conditions to determine the approval set. This later result is crucial in two respects: it is instrumental to prove the the existence and uniqueness of the equilibrium, and at the same time it provides the tool for an explicit computation of the equilibrium that allows to carry out comparative statics.

Proposition 1 EQUILIBRIUM CHARACTERIZATION, EXISTENCE AND UNIQUENESS. *Fix an environment* (F, u, δ, q).

1. *An equilibrium is fully characterized by the approval set, that is, the set of all alternatives that are accepted at least by a majority q. The approval set is an interval $[\underline{x}, \overline{x}], 0 \leq \underline{x} < \overline{x} \leq 1$, and the payoffs and actions of a generic individual i are as follows:*

   (a) *Expected payoffs are*

   $$U_i[\underline{x}, \overline{x}] = F(\underline{x})\, u(\underline{x}, i) + \int_{\underline{x}}^{\overline{x}} u(z, i)\, dF(z) + (1 - F(\overline{x}))\, u(\overline{x}, i)\,.$$

   (b) *A proposal x is accepted by i if and only if $u(x, i) \geq \delta U_i[\underline{x}, \overline{x}]$*

   (c) *When i is appointed to propose, she proposes her peak $x_i = i$ if $i \in [\underline{x}, \overline{x}]$; otherwise she proposes $x_i = \underline{x}$ if $i < \underline{x}$, or $x_i = \overline{x}$ if $i > \overline{x}$.*

2. *An equilibrium exists and it is unique.*

Overview of the proof. Because stationary subgame perfect equilibria involve no delay, every proposal arising in equilibrium must receive the favorable vote of (at least) a majority q. An individual votes for a proposal only if it lies in her acceptable set, i.e. in the subset of alternatives that are better than delaying play for one period. And a proposal is approved only if it lies in the approval set, the subset of alternatives that are in the acceptable sets of some majority of size at least q. Therefore, the proposer puts forward her most preferred alternative within the approval set. Characterizing an equilibrium is tantamount to providing the necessary and sufficient conditions to determine the approval set. These conditions require a fixed point: From an interval of alternatives that receive approval, we compute the associated expected payoffs, which determine the individual acceptance sets, and in turn induce an interval of approval. In equilibrium, the later approval set must coincide with the former. This condition is described precisely in Proposition 2, where we also observe that it delivers one, and only one solution. Hence, the existence and uniqueness of the equilibrium follows.

Proof. Fix an equilibrium and let us address its properties in detail.

Random Equilibrium Outcome. It is well known that, in the present framework, all stationary subgame perfect equilibria must be in pure strategies and involve no delay. (See Banks and Duggan (2000) for a formal discussion of this result). Hence, each individual makes the same proposal whenever she is selected. Denote by x_i the (time independent) proposal of a typical player i. Given the uniform random selection of proposers, the alternative that prevails as the equilibrium outcome arises as a random draw $x_i \sim [\underline{x}, \overline{x}]$, with some given distribution induced by F. Let U_i denote the time invariant expected utility of player i (prior to appointing the proposer) and let x^e denote the expected equilibrium alternative.

Individual Acceptance Sets. Note that (as $v'' \leq 0$) $U_i \leq u(x^e, i) \leq u(i, i)$. Player i accepts a proposal x if and only if $x \in A_i = \{z \in [0, 1] : u(z, i) \geq \delta U_i\}$. Since preferences are single-peaked, this set is non-empty and connected; i.e., player i's acceptance set takes

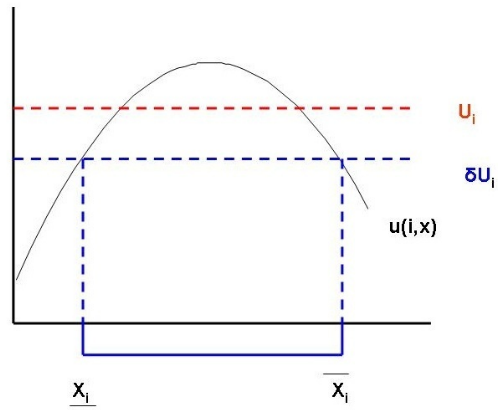

Figure 2: Individual acceptance set

the form $A_i = [\underline{x}_i, \overline{x}_i]$ and $x^e \in A_i$ for all $i \in I$. Figure 2 displays the construction of individual acceptance sets.

Coalition Acceptance Sets. Given the individual acceptance sets, the set of alternatives that is accepted by a coalition $S \in \mathcal{B}$ is

$$A_S = \{z \in [0,1] : u(z,i) \geq \delta U_i \text{ for all } i \in S\} = \cap_{i \in S} A_i.$$

Since is $x^e \in A_i$ for all $i \in I$ this intersection non-empty and connected. That is, it takes the form of $A_S = [\underline{x}_S, \overline{x}_S]$ where

$$\underline{x}_S = \max\{\underline{x}_i : i \in S\} \text{ and } \overline{x}_S = \min\{\overline{x}_i : i \in S\}.$$

Approval Set. Consider the approval set, i.e. the set of all the alternatives that receive approval. An alternative in the approval set must lie in the coalition acceptance set for some winning coalition under majority q. Hence the approval set is

$$A = \{z \in [0,1] : z \in A_S \text{ for some } S \in \mathcal{W}(q)\}.$$

Define

$$\underline{x} = \min\{\underline{x}_S : S \in \mathcal{W}(q)\}, \text{ and } \overline{x} = \max\{\overline{x}_S : S \in \mathcal{W}(q)\}.$$

Now observe that the approval set defined above must be an interval $A = [\underline{x}, \overline{x}]$. Clearly $\underline{x}$ and $\overline{x}$ lie in the approval set by definition. Furthermore, for any $x \in [\underline{x}, \overline{x}]$ there exists $S \in \mathcal{W}(q)$ such that $x \in A_S$. To prove this claim let $S_1, S_2 \in \mathcal{W}(q)$ be such that

$$\underline{x} = \underline{x}_{S_1} \text{ and } \overline{x} = \overline{x}_{S_2}.$$

We know that $x^e \in A_{S_1} \cap A_{S_2}$, implying that $\underline{x}_{S_2} \leq \overline{x}_{S_1}$. Thus, as A_{S_1} and A_{S_2} are connected sets, we have that for any $x \in [\underline{x}, \overline{x}]$ either $x \in A_{S_1}$ or $x \in A_{S_2}$.

Equilibrium Proposals and expected payoffs. Proposals are approved if and only if they lie in the approval set, and this set is an interval $[\underline{x}, \overline{x}]$. Hence, players propose the alternative that they like best within the approval set. That is, $x_i = \underline{x}$ for players $i < \underline{x}$, $x_i = \overline{x}$ for players $i > \overline{x}$, and players $i \in [\underline{x}, \overline{x}]$ must propose their peak. Congruently, the expected payoff (prior to appointment of the proposer) of a typical player i is $U_i[\underline{x}, \overline{x}]$.

We have thus proved 1.

To prove 2 we examine further the conditions that determine the approval set $[\underline{x}, \overline{x}]$.

Determining the Approval Set. Consider an interval of alternatives $[\underline{x}, \overline{x}]$, and assume that it is the approval set. We can compute the associated expected payoffs $U_i[\underline{x}, \overline{x}]$, and determine the corresponding individual acceptance sets A_i. From the collection individual acceptance sets, we can in turn construct the induced approval set. In equilibrium the later must coincide with $[\underline{x}, \overline{x}]$. The necessary and sufficient condition assuring that an interval $[\underline{x}, \overline{x}]$ is the approval set of an equilibrium is stated in Proposition 2. This proposition shows that the approval set is the intersection of the acceptance sets of the two boundary players l and r, it describes how to compute the values of $\underline{x}$ and $\overline{x}$, and it establishes that this computation admits one and only one solution. The existence and uniqueness of the approval set implies in turn the existence and uniqueness of an equilibrium. And the proof of Proposition 1 is complete. ■

The precise characterization of the approval set, that we present in Proposition 2, requires some additional notation. For an arbitrary interval $[a, b] \subset [0, 1]$, let $U_i[a, b] =$

$F(a)u(a,i)+\int_a^b u(x,i)dF(x)+(1-F(b))u(b,i)$. For each pair $(x,i)\in[0,1]^2$, if the equation $u(z,i)=\delta U_i[x,z]$ admits a solution $z\in(x,1]$, then we denote it by $\overline{s}(x,i)$. Similarly if $u(z,i)=\delta U_i[z,x]$ admits a solution $z\in[0,x)$, then we denote it by $\underline{s}(x,i)$. Lemma 10 (in the Appendix) establishes the existence of threshold values $\underline{a}(i),\overline{a}(i),\underline{b}(i),\overline{b}(i)\in[0,1]$ such that $\overline{s}(x,i)$ exists if and only if $x\in[\underline{a}(i),\overline{a}(i)]$ and $\underline{s}(x,i)$ exists if and only if $x\in\left[\underline{b}(i),\overline{b}(i)\right]$. Furthermore, when $\overline{s}(x,i)$ or $\underline{s}(x,i)$ exist, they are unique. We may now define $\overline{\zeta}:[0,1]^2\to[0,1]$, and $\underline{\zeta}:[0,1]^2\to[0,1]$, as follows:

$$\overline{\zeta}(x,i)\equiv\begin{cases}\overline{s}(x,i), & \text{if } x\in[\underline{a}(i),\overline{a}(i)]\\ 1, & \text{otherwise;}\end{cases}$$

and

$$\underline{\zeta}(x,i)\equiv\begin{cases}\underline{s}(x,i), & \text{if } x\in\left[\underline{b}(i),\overline{b}(i)\right],\\ 0, & \text{otherwise.}\end{cases}.$$

We are now ready to state the explicit characterization of the approval set. It is obtained as a fixed point of the function $\overline{\zeta}(.,l),\underline{\zeta}(.,r):[0,1]^2\to[0,1]^2$.

Proposition 2 Characterization of the Approval Set. *Fix an environment (F,u,δ,q) and consider an equilibrium. Then the approval set is the intersection of the acceptance sets of the boundary players l and r, that is $[\underline{x},\overline{x}]=[\underline{x}_r,\overline{x}_l]$. Furthermore, the values of $\underline{x}$ and $\overline{x}$ are given by*

$$\underline{x}=\underline{\zeta}(\overline{x},r)\ \text{and}\ \overline{x}=\overline{\zeta}(\underline{x},l), \tag{1}$$

and these conditions yield one and only one solution.

Proof. Note that the first equation in (1) translates as

$$u(\underline{x},r)=\delta U_r \text{ or } \underline{x}=0 \text{ if } u(0,r)\geq\delta U_r.$$

We show by contradiction that this condition is necessary. Suppose that $u(\underline{x},r)>\delta U_r$ and $u(0,r)<\delta U_r$. By continuity, there exists $y<\underline{x}$ such that $u(y,r)=\delta U_r$, and therefore $u(y,i) > \delta U_i$ for all $i < r$ (see Lemma 11 in the Appendix for a formal proof). Thus, y is accepted by all players $i \leq r$. Hence, a $q-$majority accepts $y < \underline{x}$, contradicting that $\underline{x}$ is the lower bound of the approval set. Similarly, if $u(\underline{x},r) < \delta U_r$ then (again by Lemma 11) more than a fraction q of players reject $\underline{x}$, contradicting that $\underline{x}$ lies in the approval set. A similar argument applies to show that $\overline{x} = \overline{\zeta}(\underline{x},l)$ is necessary.

It is immediate that the condition is also sufficient, since all proposals in $[\underline{x},\overline{x}]$ are accepted by a $q-$majority of players, and all that lie outside are rejected by a by a $q-$majority of players.

Lemma 12 in the Appendix establishes the existence and uniqueness of a solution pair to condition (1). ■

Proposition 2 supplies the tools for an explicit computation. Figure 3 displays the determination of the approval set for an example.

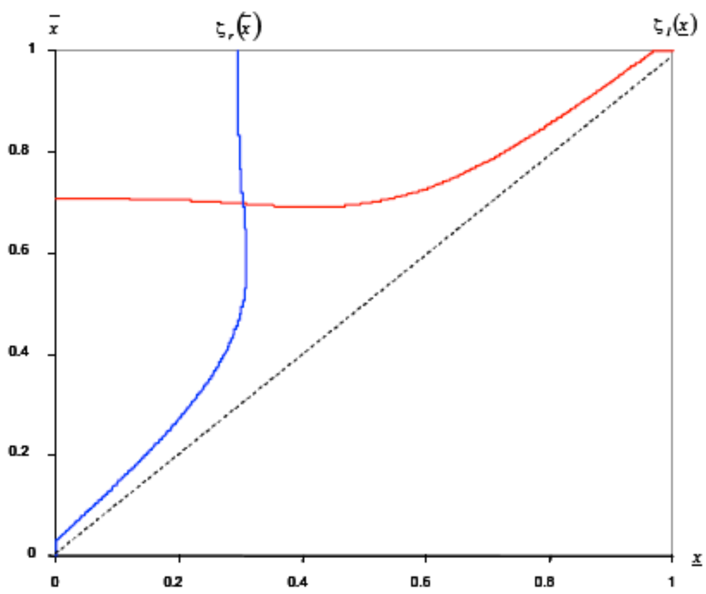

Figure 3: Solving equation (1) for $u(i,x) = 1-(x-i)^2$, $\delta = .95$, $q = 0.65$. and F such that $l = 0.419$ and $r = .581$. The approval set is $[0.303, 0.697]$.

Thanks to the explicit computation of the approval set we can examine the variation of outcomes across different majority requirements. As an illustration Figures 4 and 5 display the approval sets as a function of q for some examples with $u(x,i) = 1 - (i-x)^2$, and $\delta = .99$.

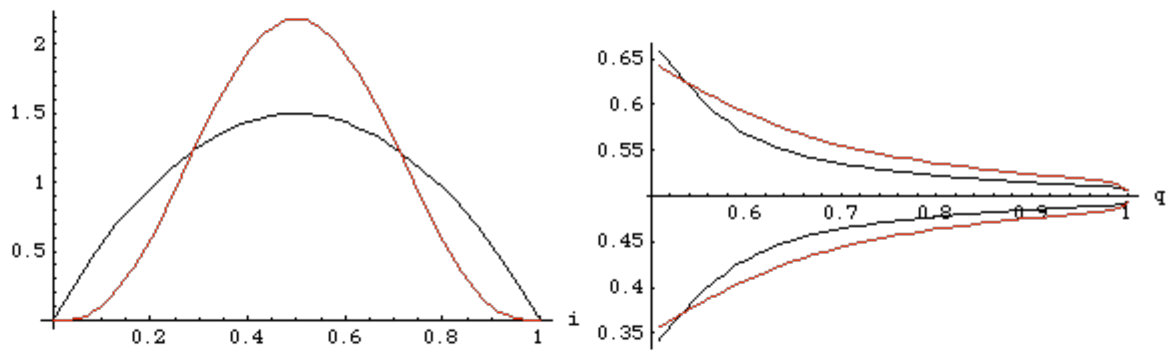

Figure 4: Two symmetric distributions of peaks, and the corresponding approval sets as a function of q.

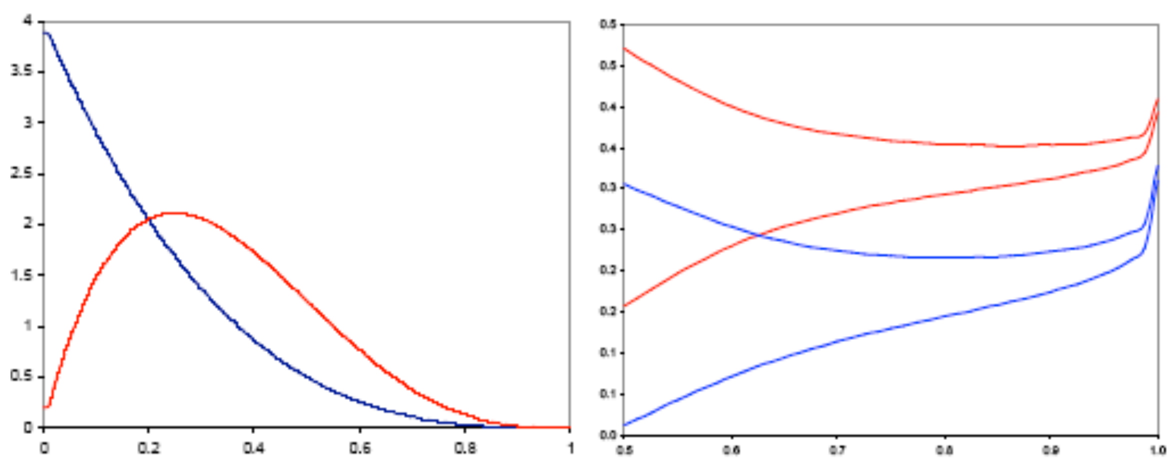

Figure 5: Two asymmetric distributions of peaks, and the corresponding approval sets as a function of q.

5 Pareto Optimal rules, and the efficiency of unanimity in symmetric populations.

The uniqueness of the equilibrium for each majority rule opens the door to comparative statics exercises that measure the effects of changing q over a given environment (F, u, δ). To emphasize that the unique equilibrium clearly depends on what majority rule applies, we will write the approval set under majority rule q as $[\underline{x}(q), \overline{x}(q)]$. We can now consider the preferences of individual i over $q \in Q$, which are naturally given by $U_i(q) \equiv U_i[\underline{x}(q), \overline{x}(q)]$. With these individual preferences well specified for all $i \in I$, we can asses the efficiency of the rules in Q.

The minimal requirement of efficiency is *Pareto Optimality.* We will say that $q \in Q$ is *a Pareto optimal majority rule* for (u, F, δ) if there is no $q' \in Q$, $q' \neq q$ such that $U_i[\underline{x}(q'), \overline{x}(q')] \geq U_i[\underline{x}(q), \overline{x}(q)]$ for all $i \in I$, with strict inequality for a subset $S \subset I$, such that $\mu(S) > 0$.

Recall the equilibrium outcomes displayed in Figure 4, and note that the approval sets shrink symmetrically around 1/2 as the consensus requirement increases. For concave utilities this implies that individual benefits increase in q, and hence stronger majorities dominate weaker majorities. We will argue next that the features displayed in the example are general in symmetric populations. In fact, for every symmetric population the approval set shrinks symmetrically around 1/2 as q increases and therefore individual payoffs $U_i(q)$ are strictly increasing in q for all $i \in I$. Consequently, if F is symmetric, $q = 1$ is uniquely Pareto optimal for all (u, δ).

Proposition 3 establishes in full generality that weakening q induces a mean preserving spread of the approval set.

Proposition 3 *(Weakening q induces a mean preserving spread of the outcome distribution). Fix two environments (F, u, δ, q) and (F, u, δ, q') where F is symmetric and $q < q'$,* *then* $\underline{x}(q) < \underline{x}(q') < \overline{x}(q') < \overline{x}(q)$, *and the distribution of equilibrium outcomes under* q *is a mean preserving spread of the distribution under* q'.

Proof. $\overline{\zeta}(\underline{x}, l)$ is increasing in l (see Lemma 15 in the Appendix). The symmetry of F implies that the boundary players are symmetric, $r = 1 - l$, and consequently the approval set is symmetric, $\underline{x} = 1 - \overline{x}$. On the other hand, as the fraction of votes required for approval increases from q to q', l (strictly) decreases and r (strictly) increases.

For each q, $\underline{x}$ and $\overline{x}$ solve (1), i.e. $\underline{x} = \underline{\zeta}(\overline{x}, r)$ and $\overline{x} = \overline{\zeta}(\underline{x}, l)$, for $l = F^{-1}(1-q)$ and $r = F^{-1}(q)$. Now, since $1 - \underline{x} = \overline{x}$, it is also necessary that $1 - \underline{x} = \overline{\zeta}(\underline{x}, l)$. Hence, an increase in q, decreases $l = F^{-1}(1-q)$ and must therefore also decrease $\overline{\zeta}(\underline{x}, l)$ for any $\underline{x}$. Hence, if $q' > q$, then $\overline{x}(q') \leq \overline{x}(q)$ and $\underline{x}(q') = 1 - \overline{x}(q') \geq 1 - \overline{x}(q) = \underline{x}(q)$. Since, $[\underline{x}(q), \overline{x}(q)] \neq [\underline{x}(q'), \overline{x}(q')]$ we conclude that $\overline{x}(q') < \overline{x}(q)$ and $\underline{x}(q') > \underline{x}(q)$.

By the symmetry of F, we conclude that the distribution of equilibrium outcomes under q is a mean preserving spread of the distribution of equilibrium outcomes under q'.
■

It is well known that, for any strictly concave utility, a mean preserving spread induces a decrease in expected utility. Hence when $v'' < 0$ all $i \in I$ prefer q' over q. When $v'' = 0$, the utilities remain strictly concave for players that have their peak in $[\underline{x}(1), \overline{x}(1)]$, for other players the utility is linear and thus a mean preserving spread leaves them indifferent. Hence the following holds.

Proposition 4 *For every* (F, u, δ) *where* F *is symmetric the unanimity rule,* $q = 1$, *is the unique Pareto optimal rule.*

A remark on bargaining to distribute surplus. It is perhaps useful to point out that a result similar to Proposition 4 holds in negotiations to split surplus *à la* Baron and Ferejohn (1989). When players must share one unit of surplus unanimity is also uniquely

Pareto optimal, provided that individuals are identical and utilities are strictly concave.[8] See Proposition 9 in the Appendix.

Pareto Optimality in Asymmetric populations. When the distribution of peaks is not symmetric Pareto Optimality no longer selects a unique majority rule, and many (if not all) $q \in Q$ are Pareto optimal. This is easy to see in Figure 5, that displays the approval sets under the different majority rules for a large asymmetric population. Note that as q increases the approval set shrinks, and the average outcome drifts. So that a decrease of q leads to a spread which is not mean preserving. Hence some types are better off at weak majorities while others prefer large supermajorities. For these environments a natural optimality criterion to discriminate within Pareto Optimal rules is to evaluate the social surplus that they deliver. This is what we do next.

6 Rules that maximize (asymptotical) social surplus

Our goal in this section is to explore what rules maximize surplus for asymmetric populations. Since it seems sensible to examine the performance of majority rules in a way that does not depend on δ, we first address the characterization of equilibrium outcomes in the limit as $\delta \to 1$.

6.1 Asymptotic equilibrium outcomes

In games with perfectly patient players, i.e. $\delta = 1$, every outcome in the core - that is in the set $[l, r]$ - can be sustained as an equilibrium. And this set is large for supermajorities $q > 1/2$. However, if we consider a sequence of games with $\delta < 1$, as players become perfectly patient $\delta \to 1$, the approval set converges to a single alternative, and the asymptotic equilibrium outcome is unique. This result is established in Proposition 5 that we state

[8]Concavity is not sufficient, with linear utilities all majority rules deliver exactly the same payoffs.

next. Proposition 5 also supplies a straightforward closed form characterization of the limit outcome for each (u, F, q).

Proposition 5 Unique asymptotic outcome. *Consider a sequence of environments (F, u, q, δ_k), where $\delta_k \to 1$. As $\delta_k \to 1$ the approval set converges to a unique asymptotic equilibrium outcome*

$$\lim_{\delta_k \to 1} \underline{x} = \lim_{\delta_k \to 1} \overline{x} = x^*.$$

1. *The asymptotic equilibrium outcome x^* is the unique solution to*

$$K(x) \equiv F(x) \frac{u_x^+(x, l)}{u(x, l)} + [1 - F(x)] \frac{u_x^-(x, r)}{u(x, r)} = 0, \tag{2}$$

   whenever this equation admits a solution $x^* \in [l, r]$.

2. *Otherwise, there exists a unique individual $i^* \in I \cap [l, r]$ such that $K(x) > 0$ for $x \in [l, i^*)$ and $K(x) < 0$ for $x \in [i^*, r]$, and $x^* = i^*$.*

Proof. The result follows by Lemmata 13 and 14 which are proved in the Appendix. By Lemma 13 the approval set converges to a singleton. Lemma 14 assures that the proposed limit alternative lies in the approval set for all $\delta < 1$. ■

For a symmetric population the previous result implies that $\lim_{\delta \to 1} \underline{x} = \lim_{\delta \to 1} \overline{x} = x^* = 1/2$ for all q. When the population is not symmetric the asymptotic outcome coincides with the median i^m at $q = 1/2$ and varies over a wide range as q increases.

To examine this variation, we write $x(q)$ to denote the different values of x^* as a function of q, for (u, F) given. For a large populations, where equation (2) applies, computing $x(q)$ is straight forward. As an illustration we have carried out the exercise for tent shaped and quadratic utilities, under two illustrative specifications of F. The results are displayed in Figures 6 and 7. In the example displayed in Figure 6, $x(q)$ is monotonically increasing. The example of Figure 7 shows that $x(q)$ can be non monotonic.

Next we turn to the analysis of surplus maximizing rules in the limit as $\delta \to 1$.

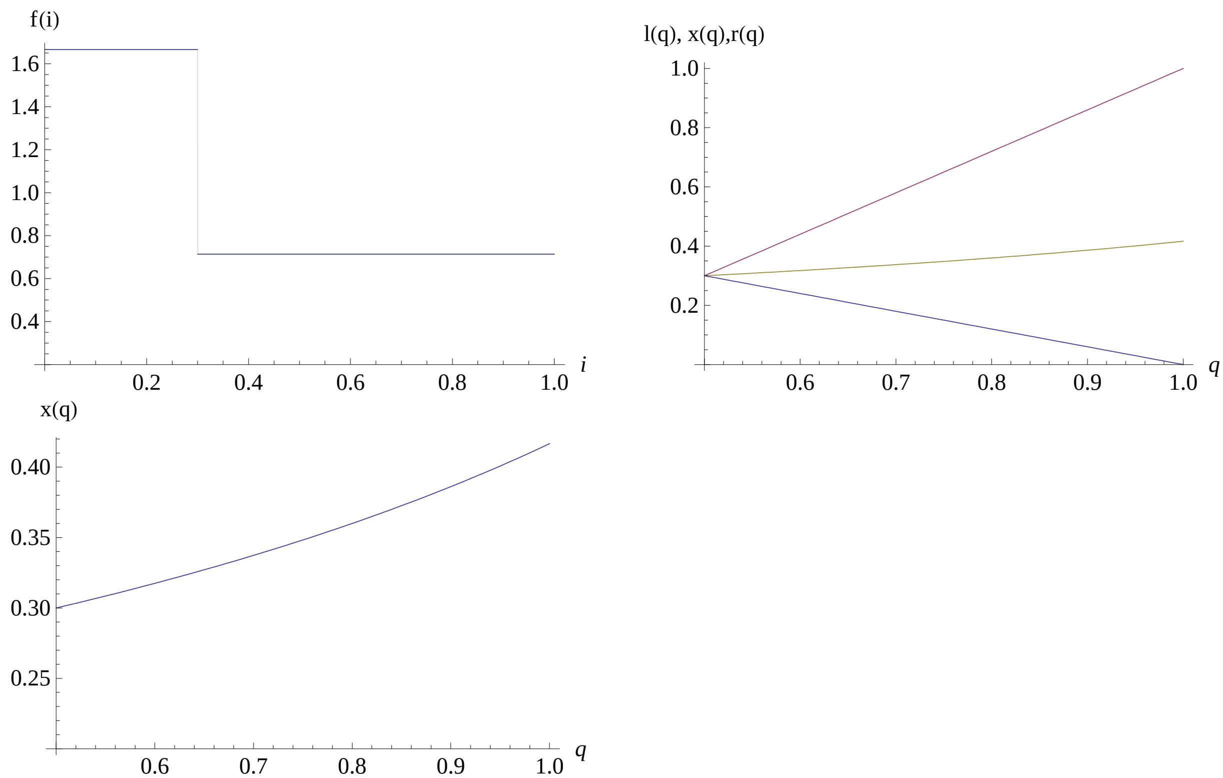

Figure 6: An example where $x(q)$ increases in q. Utilities are tent-shaped.

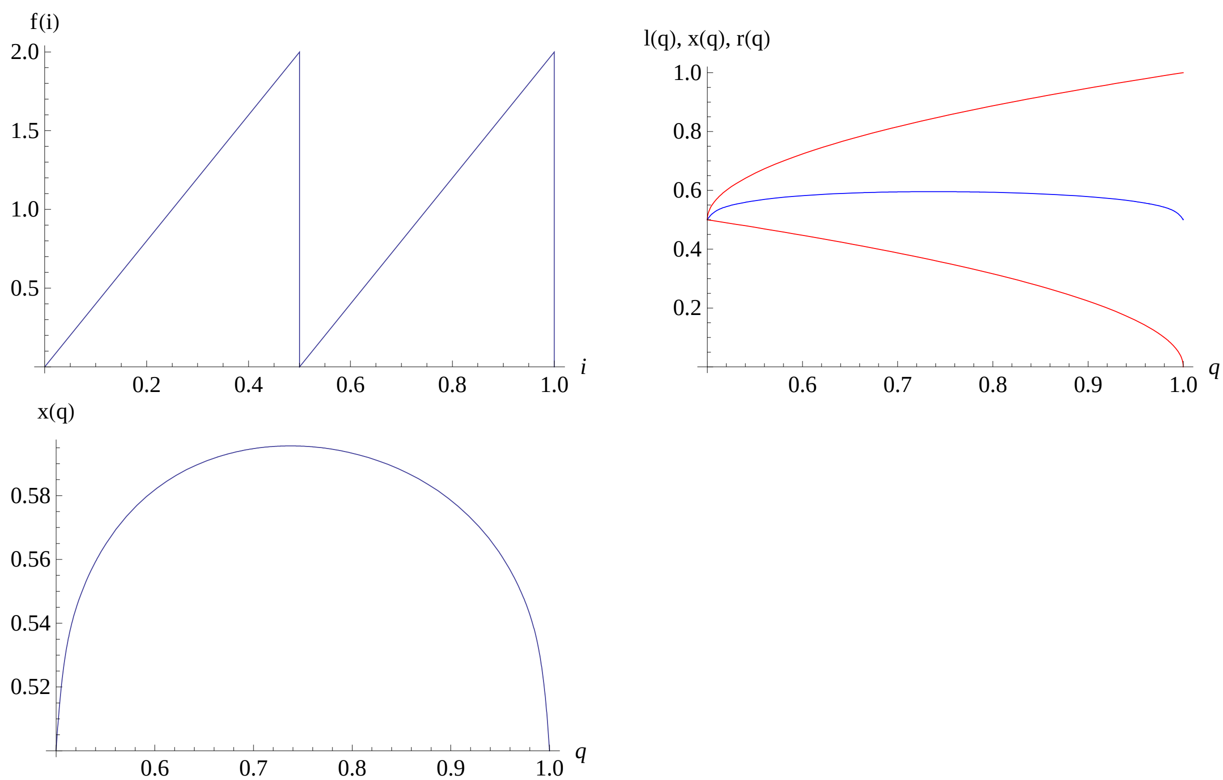

Figure 7: An example where $x(q)$ is non monotonic in q. Utilities are quadratic.

6.2 (Asymptotically) Efficient rules

Given (u, F), the approval set converges to the single alternative $x(q)$, and the asymptotic payoff for individual i is $u(x(q), i)$. Hence, the asymptotic social surplus delivered by q is

$$W^{AS}(q) = \int_0^1 u(x(q), i) dF(i).$$

The set of (asymptotically) feasible alternatives, i.e. those that arise as $x(q)$ for some $q \in Q$, is a strict subset of $[0, 1]$. When the maximization of social surplus within this set is the efficiency criterion, the best conceivable performance would be delivered by a *first best rule* - i.e. a rule $q \in Q$ such that $x(q) = x^{fb}$.

Does a first best rule always exists? Under a symmetric distribution the first best is 1/2 and $x(q) = 1/2$ for all $q \in Q$, and therefore all majorities are (trivially) first best rules. For asymmetric populations, a first best rule generically does not exist for small populations: Since Q is finite x^{fb} will generically not be feasible. We next turn to examine the existence and properties the first best rule for large asymmetric populations.

A transparent analysis requires the specification of utilities. Let us begin with the case of tent-shaped utilities, $u(x, i) = 1 - |x - i|$, where it is immediate to see that a first best rule exists for all F.

Proposition 6 A SIMPLE MAJORITY IS THE FIRST BEST RULE FOR TENT-SHAPED UTILITIES. *Consider a large population with F asymmetric and $u(x, i) = 1 - |x - i|$. a) If $i^m \neq 1/2$, then $x(q) = i^{fb} = i^m$ if and only if $q = 1/2$. b) If $i^m = 1/2$, then $x(1/2) = x(1) = i^{fb} = i^m$.*

Proof. The first order condition that characterizes x^{fb} is equivalent to $F(x) = 1 - F(x)$, and therefore $x^{fb} = i^m$. Since $x(1/2) = i^m$ for all F, $q = 1/2$ is a first best rule. If $i^m = 1/2$, then $x(1) = 1/2$, and consequently the unanimity rule also delivers the first best. For generic distributions such that $i^m \neq 1/2$, no other rule delivers i^m. ∎

The optimality of the simple majority does not extend beyond the tent-shaped specification. If individuals suffer a strictly convex cost when the selected alternative is distant form the peak, then the rule that maximizes social surplus (i.e. the first best rule when it exists) is a strict super-majority.

Proposition 7 Optimal super-majorities. *Assume the distribution of peaks F is symmetric regular and $v'' < 0$. Then, the rule that maximizes social surplus is a strict super-majority.*

Proof. We prove the result for scenarios where $i^m < 1/2$. The symmetric argument applies when $i^m > 1/2$.

From density f construct its induced symmetric density function $\widehat{f}$. Observe that i^m is the first best allocation under $\widehat{f}$. Since utilities are strictly concave and f has more mass of agents with high peaks it must be that

$$S'(i^m) = \int_0^1 u_x(i^m, i) f(i)di > \int_0^1 u_x(i^m, i) \widehat{f}(i)di = 0,$$

that is, social surplus is increasing at i^m. On the other hand, $x(1)$ solves

$$F(x) \frac{u_x(x,0)}{u(x,0)} + [1 - F(x)] \frac{u_x(x,1)}{u(x,1)} = 0.$$

The concavity of utilities implies that the left hand side of the preceding equality is strictly decreasing in x. Since $i^m < 1/2$, it follows that

$$F(i^m) \frac{u_x(i^m,0)}{u(i^m,0)} + [1 - F(i^m)] \frac{u_x(i^m,1)}{u(i^m,1)} = \frac{1}{2}\left(\frac{u_x(i^m,0)}{u(i^m,0)} + \frac{u_x(i^m,1)}{u(i^m,1)}\right) > 0,$$

implying that $i^m < x(1)$. Hence, as $x(q)$ is continuous, social surplus is maximal at some $q \in (1/2, 1]$. ■

To conclude we examine the case of quadratic utilities $u(x,i) = 1 - (x-i)^2$. For this specification mean-median regularity assures that a first best rule exist, and that it is a supermajority weaker than unanimity.

Proposition 8 THE FIRST BEST RULE FOR QUADRATIC UTILITIES. *Consider a large population where F is mean-median regular and $u(x,i) = 1-(x-i)^2$. Then $x(q) = i^{fb} = i^e$ for some $1/2 < q < 1$.*

Proof. We first give a sufficient condition that assures existence of the first best rule for all utilities. The concavity of u implies that $F(x)\, u_x(x,l) + [1-F(x)]\, u_x(x,r)$ is decreasing in x. Denote by $\widetilde{x}$ the (unique) solution to $F(x)\, u_x(x,0) + [1-F(x)]\, u_x(x,1) = 0$. If $i^m \leq 1/2$ and $x^{fb} \in [i^m, \widetilde{x}]$, or $i^m \geq 1/2$ and $x^{fb} \in [\widetilde{x}, i^m]$, then there is a $q \in [1/2, 1]$ such that $x(q) = x^{fb}$. Consider w.l.o.g. the case $i^m \leq 1/2$. Since $x(1)$ solves $F(x)\, \frac{u_x(x,0)}{u(x,0)} + [1-F(x)]\, \frac{u_x(x,1)}{u(x,1)} = 0$, it must be that $x(1) < 1/2$ and therefore $u(x(1), 0) > u(x(1), 1)$. Thus, $x(1) > \widetilde{x} \geq x^{fb} \geq x(1/2) = i^m$ and, by continuity there is q such that $x(q) = x^{fb}$.

For $u(x,i) = 1-(x-i)^2$, the first order condition for social surplus maximization is $\int_0^1 (x-i)dF(i) = 0$, which implies that $x^{fb} = \int_0^1 i dF(i) = i^e$. By mean-median regularity either $i^m < i^e < 1/2$ and $F(i^e) + i^e < 1$, or the reverse inequalities hold. Assume, w.l.o.g. the first. Then, $F(i^e) u_x(i^e, 0) + [1-F(i^e)]\, u_x(i^e, 1) = 1 - i^e - F(i^e) > 0$ so that $i^e < \widetilde{x}$, and the sufficient condition for existence holds. ■

7 Final remarks

We have assumed a general distributions of peaks and a random protocol that selects all players with equal probability. Thus, the distribution of peaks determines both the distribution of proposals and the decisive players l and r. Our results generalize easily to a set up where the selection of the proposer does not follow the uniform distribution. One would simply need to redefine the expected payoffs given an approval set, $U[x,y]$, to account for the new proposal distribution. Clearly, proposal rights are important to determine the equilibrium.[9] However, under non-degenerate random protocols the bound-

[9] The importance of proposal rights over voting rights and impatience is stressed by Kalandrakis (2006).

ary players l and r remain determinant, even as players become perfectly patient: The asymptotic outcome must always lie within $[l, r]$. An obvious but important implication is that, since under a simple majority $l = r = i^m$, the median peak prevails regardless of the distribution of proposal rights.

An important feature of our model is that an immediate agreement prevails in equilibrium regardless of the majority required. In real negotiations, disagreements are a real possibility, and proposals failure to get approval seems more likely the greater the majority needed. This suggests that in bargaining processes with inefficient outcomes, the forces favoring supermajorities are more limited. We plan to explore this issue in future research.

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Appendix

On bargaining to distribute surplus. Consider the game to split one unit of surplus among a finite population where bargaining follows the usual protocol: A proposer is selected with probability $1/n$, and approval requires the acceptance of k responders, $k = (n-1)/2, ..., n-1$. In equilibrium the proposer must offer a distribution of the surplus among the individuals $x = (x_1, ...x_n)$, such that $\sum_{i \in I} x_i = 1, x_i \geq 0$. Assume that benefits upon approval of (x, t) are $\delta^t u(x_i)$,where $u(0) = 0, u' > 0, u'' < 0$.

Proposition 9 *Consider a finite population that bargains to split one unit of surplus. Then the following holds:*

1. *In equilibrium the proposer offers a distribution where the share of k opponents is x_r such that*

$$u(x_r) = \delta \left(\frac{1}{n} u(1 - kx_r) + \frac{k}{n} u(x_r) \right),$$

   the share to the remaining respondents is $x_j = 0$, and her own share is $x_i = 1 - kx_r$. Individuals that are offered x_r accept, and the proposal is approved.

2. *Prior to the selection of the proposer individual expected utility is strictly increasing in k, so that unanimity $q = 1$ is the unique Pareto optimal rule.*

Proof. The characterization of equilibrium is standard. The share x_r must leave the responders indifferent to their expected continuation payoff. That is $x_r = x$, where $u(x) = \delta \left[\frac{1}{n} u(1 - kx) + \frac{k}{n} u(x) \right]$, that is

$$\frac{u(1 - kx)}{u(x)} = \frac{n - \delta k}{\delta}.$$

Note that $u'' < 0$ implies that x_r increases in k. Let $A(k, x) \equiv \frac{u(1-kx)}{u(x)}$ and $B(k) \equiv \frac{n - \delta k}{\delta}$. For any k there is a unique solution, since $\frac{\partial A}{\partial x} = \frac{-ku'(1-kx)u(x) - u'(x)u(1-kx)}{u(x)^2} < 0$ and $\frac{\partial B}{\partial x} = 0$.

Moreover, $\frac{\partial B}{\partial k} = -1$ and $\frac{\partial A}{\partial k} = \frac{-xu'(1-kx)}{u(x)} = -\frac{u'(1-kx)}{\frac{u(x)}{x}} > -\frac{u'(1-kx)}{u'(x)} > -1$, implying that $\frac{\partial x_r}{\partial k} > 0$.

Now the ex-ante expected utility under rule that requires k votes is

$$Eu(k) = \frac{1}{n}u(1 - kx_r(k)) + \frac{k}{n}u(x_r(k)).$$

Substituting $u(1-(k-1)x_r) = u(x_r)\frac{n-\delta(k-1)}{\delta}$, yields

$$\begin{aligned} Eu(k) &= \frac{1}{n}u(x_r(k))\frac{n-\delta(k-1)}{\delta} + \frac{k}{n}u(x_r(k)) \\ &= u(x_r(k))\left(\frac{n-\delta(k-1)}{\delta n} + \frac{k}{n}\right) \\ &= u(x_r(k))\frac{n+\delta}{n\delta}, \end{aligned}$$

which is increasing in k provided that $x_r(k)$ increases in k. ■

Lemma 10 *Consider $\overline{s}(x,i)$ and $\underline{s}(x,i)$ defined as*

$$\overline{s}(x,i) = z \in (x,1] \text{ such that } u(z,i) = \delta U_i[x,z],$$

and

$$\underline{s}(x,i) = z \in [0,x) \text{ such that } u(z,i) = \delta U_i[z,x].$$

1. *If $\overline{s}(x,i)$ exists, then it is unique, and similarly for $\underline{s}(x,i)$.*
2. *There exists some threshold values $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in [0,1]$ such that $\overline{s}(x,i)$ exists if and only if $x \in [\underline{a}, \overline{a}]$ and $\underline{s}(x,i)$ exists if and only if $x \in [\underline{b}, \overline{b}]$.*

Proof. 1. Fix $x \in [0,1]$ and define $\kappa(z;x) = u(z,i) - \delta U_i[x,z]$, so that $\kappa'(z;x) = u_x(z,i)(1-\delta(1-F(z)))$, which is positive when $z < i$, and negative at $z > i$. Hence by single-peakedness $\kappa(z;x) = 0$ has at most two solutions: one with $z \leq i$ and another with $z > i$. Notice, however, that $\kappa(x;x) = u(x,i)(1-\delta) > 0$ so that at most one solution $z > x$ exists.

2. Next, we show that when $\overline{s}(x,i)$ and $\overline{s}(y,i)$ do exist, with $x \leq y$, then $\overline{s}(w,i)$ exists for any $w \in [x,y]$.

Assume otherwise, i.e., $\kappa(z;w) = u(z,i) - \delta U_i[w,z] > 0$ for all $z \geq i$. Then, in particular

$$\kappa(1;w) = u(1,i) - \delta U_i[w,1] > 0, \tag{3}$$

$$\kappa(1;x) = u(1,i) - \delta U_i[x,1] \leq 0, \tag{4}$$

$$\kappa(1;y) = u(1,i) - \delta U_i[y,1] \leq 0. \tag{5}$$

We distinguish two cases: either (a) $u(w,i) \geq u(y,i)$ or (b) $u(x,i) < u(w,i) < u(y,i)$

In case (a)

$$U_i[w,1] = F(w)\,u(w,i) + \int_w^1 u(z,i)\,dF(z) > F(w)\,u(y,i) + \int_w^y u(y,i)\,dF(z) + \int_y^1 u(z,i)\,dF(z) =$$

$$= F(w)\,u(y,i) + [F(y) - F(w)]\,u(y,i) + \int_y^1 u(z,i)\,dF(z) = F(y)\,u(y,i) + \int_y^1 u(z,i)\,dF(z)$$

Now, using (5) it is immediate that $\kappa(1;w) \leq 0$, contradicting (3).

In case (b) from (4) we have that

$$U_i[x,1] = F(x)\,u(x,i) + \int_x^1 u(z,i)\,dF(z) = F(x)\,u(x,i) + \int_x^w u(z,i)\,dF(z) + \int_w^1 u(z,i)\,dF(z) \geq \frac{u(1,i)}{\delta}.$$

Moreover, from (3),

$$U_i[x,1] = F(w)\,u(w,i) + \int_w^1 u(z,i)\,dF(z) < \frac{u(1,i)}{\delta}.$$

Thus, using (4)

$$F(x)\,u(x,i) + \int_x^w u(z,i)\,dF(z) + \int_w^1 u(z,i)\,dF(z) > F(w)\,u(w,i) + \int_w^1 u(z,i)\,dF(z).$$

I.e.,

$$F(x)\,u(x,i) + [F(w) - F(x)]\,u(w,i) > F(x)\,u(x,i) + \int_x^w u(z,i)\,dF(z) > F(w)\,u(w,i)$$

so that

$$F(x)\,[u(x,i) - u(w,i)] > 0$$

which is a contradiction. ■

Lemma 11 *Fix an equilibrium and let* $[\underline{x}, \overline{x}]$ *be the approval set. Then, the following hold:*

1. *If* $u(\underline{x}, i) \leq \delta U_i$ *then* $u(\underline{x}, j) < \delta U_j$ *for any* j *such that* $j > i$.
2. *If* $u(\underline{x}, i) \geq \delta U_i$ *then* $u(\underline{x}, j) > \delta U_j$ *for any* j *such that* $j < i$.
3. *If* $u(\overline{x}, i) \leq \delta U_i$ *then* $u(\overline{x}, j) < \delta U_j$ *for any* j *such that* $j < i$.
4. *If* $u(\overline{x}, i) \geq \delta U_i$ *then* $u(\overline{x}, j) > \delta U_j$ *for any* j *such that* $j > i$.

Proof. We prove statement 1. A similar argument applies to prove 2, 3 and 4.

Assume otherwise, i.e., $u(\underline{x}, j) \geq \delta U_j$ for some $j > i$ such that $u(\underline{x}, i) \leq \delta U_i$. Hence,

$$u(\underline{x}, i) \leq \delta(F(\underline{x})\,u(\underline{x}, i) + \int_{\underline{x}}^{\overline{x}} u(z,i)\,dF(z) + (1 - F(\overline{x}))\,u(\overline{x}, i)), \tag{6}$$

and

$$u(\underline{x}, j) \geq \delta(F(\underline{x})\,u(\underline{x}, j) + \int_{\underline{x}}^{\overline{x}} u(z,j)\,dF(z) + (1 - F(\overline{x}))\,u(\overline{x}, j)). \tag{7}$$

Let $d_{ij} = j - i$ denote the distance between the peaks of the players i and j. Because utilities are identical among players except by the location of their peak $u(x,j) =$

$u(x-d_{ij},i)$.[10] Moreover, by concavity it must be that $u(x,i)-u(x-d_{ij},i)$ is decreasing in x. Adding up equations (6) and (7) we have that

$$u(\underline{x},i)-u(\underline{x},j)\leq$$

$$\leq\delta(F(\underline{x})(u(\underline{x},i)-u(\underline{x},j))+\int_{\underline{x}}^{\overline{x}}(u(z,i)-u(z,j))\,dF(z)+(1-F(\overline{x}))(u(\overline{x},i)-u(\overline{x},j)))=$$

$$=\delta(F(\underline{x})(u(\underline{x},i)-u(\underline{x}-d_{ij},i))+\int_{\underline{x}}^{\overline{x}}(u(z,i)-u(z-d_{ij},i))\,dF(z)+$$

$$+(1-F(\overline{x}))(u(\overline{x},i)-u(\overline{x}-d_{ij},i)))$$

$$\leq\delta(u(\underline{x},i)-u(\underline{x}-d_{ij},i))=\delta[u(\underline{x},i)-u(\underline{x},j)],$$

(where the last inequality follows by concavity) which is a contradiction. ■

Lemma 12 *For each environment* (F,u,δ,q)*, condition (1) admits one and only one solution.*

Proof. Note that $\overline{\zeta}$ and $\underline{\zeta}$ are continuous and differentiable (with respect to x) almost everywhere. More precisely, $\overline{\zeta}$ is differentiable in $(0,1)-\{\underline{a}(i),\overline{a}(i)\}$ and $\underline{\zeta}$ is differentiable in $(0,1)-\{\underline{b}(i),\overline{b}(i)\}$. Direct computations establish that, $\overline{\zeta}_x(\underline{x},l)=\dfrac{\delta F(\underline{x})}{1-\delta+\delta F(\overline{x})}\dfrac{u_x(\underline{x},l)}{u_x(\overline{x},l)}$, for $\underline{x}\in(\underline{a}(l),\overline{a}(l))$.[11] And clearly $\overline{\zeta}_x(\underline{x},l)=0$, elsewhere. Now, for $\underline{x}\in(\underline{a}(l),\overline{a}(l))$, notice that $F(r)=q$ and $F(l)=1-q$ implies that $l\leq r$ and therefore $u(\overline{x},l)\leq u(\underline{x},l)$, since otherwise, by Lemma 11 we contradict Proposition 2. By symmetry and concavity of the utilities the preceding inequality implies that $|u_x(\overline{x},l)|\geq|u_x(\underline{x},r)|$. Hence, in any case $\left|\overline{\zeta}_x(\underline{x},l)\right|<1$. A similar argument proves that $\left|\underline{\zeta}_x(\overline{x},l)\right|<1$. Thus, (1) admits at most one solution.

[10] Note that $u(x-d_{ij},i)$ is well defined even if $x-d_{ij}<0$.

[11] Note that the existence of $\overline{s}(\underline{x},l)$ implies $u_x(\overline{x},l)<0$, and the existence of $\underline{s}(\overline{x},r)$ implies $u_x(\underline{x},r)>0$.

To see that a solution indeed exists, note that $\overline{\zeta}(.,l)$ and $\underline{\zeta}(.,r)$ are continuous, with $\overline{\zeta}(0,l) \leq 1$, $\overline{\zeta}(1,l) = 1$, $\underline{\zeta}(0,r) = 0$ and $\underline{\zeta}(1,r) \geq 0$. ■

Lemma 13 *As $\delta \to 1$ the approval set converges to a singleton.*

Proof. STEP 1: If there exists $\varepsilon > 0$ such that for all $\overline{\delta} \in (0,1)$ there exists $\delta > \overline{\delta}$ such that $\overline{x}(q) - \underline{x}(q) > \varepsilon$, then both $\overline{x}(q) < 1$ and $\underline{x}(q) > 0$.

Assume $\underline{x}(q) = 0$, and therefore $\overline{x}(q) > \varepsilon$,which implies that for all $\overline{\delta} \in (0,1)$ there exists $\delta > \overline{\delta}$ such that $u(\overline{x}, r) \geq u(\underline{x}, r) + \eta$, where $\eta = u(\varepsilon, r) - u(0, r) > 0$. In this case,

$$\begin{aligned} u(\underline{x}, r) &= \delta F(\underline{x}) u(\underline{x}, r) + \delta \int_{\underline{x}}^{\overline{x}} u(z, r)\, dF(z) + \delta (1 - F(\overline{x})) u(\overline{x}, r) > \\ &> \delta \left[F(\underline{x}) u(\underline{x}, r) + (F(\overline{x}) - F(\underline{x})) u(\underline{x}, r) + (1 - F(\overline{x})) u(\overline{x}, r) \right] \\ &= \delta F(\overline{x}) u(\underline{x}, r) + \delta (1 - F(\overline{x})) u(\overline{x}, r) \\ &\geq \delta F(\overline{x}) u(\underline{x}, r) + \delta (1 - F(\overline{x})) u(\underline{x}, r) + \delta (1 - F(\overline{x})) \eta \\ &= \delta u(\underline{x}, r) + \delta (1 - F(\overline{x})) \eta \end{aligned}$$

Thus,

$$u(\underline{x}, r)[1 - \delta] > \delta \eta [1 - F(\overline{x})] \tag{8}$$

Thus, there is $\overline{\delta} \in (0,1)$ such that if for all $\delta > \overline{\delta}$ we have that $\overline{x}(q) - \underline{x}(q) > \varepsilon$ we contradict inequality (8).[12]

STEP 2: If $u(\overline{x}, l) = \delta U_l$, $u(\underline{x}, r) = \delta U_r$ and $l < r$ then the approval set converges to a singleton as $\delta \to 1$.

Note that $u(\overline{x}, l) = \delta U_l$ implies that $u(\overline{x}, r) > \delta U_r$, since otherwise (by Lemma 16) $u(\overline{x}, l) < \delta U_l$. Also, $u(\underline{x}, r) = \delta U_r$ implies $u(\underline{x}, l) > \delta U_l$. Hence, $u(\underline{x}, l) > u(\overline{x}, l)$ and

[12] It is immediate to see that $\overline{x} = 1$ and $\underline{x} = 0$ cannot be simultaneously satisfied when δ is big enough. If $u(0, r) < u(1, r)$ just chose $\delta \geq u(0, r) / u(x_r^c, r) < 1$ to get a contradiction. Otherwise, $u(0, r) \geq u(1, r)$ implies that $u(0, l) \geq u(1, l)$ and we get also a contradiction by choosing $\delta \geq u(1, l) / u(x_l^c, l) < 1$. Hence, $1 - F(\overline{x}) > 0$.

$u(\underline{x}, r) < u(\overline{x}, r)$. Therefore, single-peakedness implies $u(\overline{x}, l) < u(z, l)$ and $u(\underline{x}, r) < u(z, r)$ for any $z \in (\underline{x}, \overline{x})$. We know that

$$u(\overline{x}, l) = \delta F(\underline{x})\, u(\underline{x}, l) + \delta \int_{\underline{x}}^{\overline{x}} u(z, l)\, dF(z) + \delta (1 - F(\overline{x}))\, u(\overline{x}, l) \geq$$

$$\geq \delta F(\underline{x})\, u(\underline{x}, l) + \delta (1 - F(\underline{x}))\, u(\overline{x}, l).$$

Thus,

$$(1-\delta)\, u(\overline{x}, l) \geq \delta F(\underline{x}) \left[u(\underline{x}, l) - u(\overline{x}, l)\right] \geq 0.$$

Therefore, as $\lim_{\delta \to 1} (1-\delta)\, u(\overline{x}, l) = 0$ it must be that either

1. $\lim_{\delta \to 1} F(\underline{x}) = 0$, or

2. $\lim_{\delta \to 1} u(\underline{x}, l) - u(\overline{x}, l) = 0$.

Similarly, using $u(\underline{x}, r) = \delta U_r$ we obtain

$$(1-\delta)\, u(\underline{x}, r) \geq \delta (1 - F(\overline{x})) \left[u(\overline{x}, r) - u(\underline{x}, r)\right] \geq 0.$$

Again, as $\lim_{\delta \to 1} (1-\delta)\, u(\underline{x}, r) = 0$ it must be that either

3. $\lim_{\delta \to 1} F(\overline{x}) = 1$, or

4. $\lim_{\delta \to 1} u(\underline{x}, r) - u(\overline{x}, r) = 0$.

In case that (2) and (4) occur, it is immediate that it must be that $\lim_{\delta \to 1} \underline{x} = \lim_{\delta \to 1} \overline{x}$.

If (1) occurs then $\lim_{\delta \to 1} \underline{x} = 0$ and therefore $\lim_{\delta \to 1} \int_0^{\overline{x}} u(z, l)\, dF(z) - F(\overline{x})\, u(\overline{x}, l) = 0$, implying that $\lim_{\delta \to 1} \overline{x} = 0$ since otherwise, as $l \in [0, \overline{x}]$ and $u(\overline{x}, l) \leq u(\underline{x}, l)$, single-peakedness would imply that $\int_0^{\overline{x}} u(z, l)\, dF(z) > F(\overline{x})\, u(\overline{x}, l)$.

Similarly (3), implies that $\lim_{\delta \to 1} \underline{x} = 1$.

Step 3. If $u(\overline{x}, l) = \delta U_l$, $u(\underline{x}, r) = \delta U_r$ and $l = r$, then the approval set converges to a singleton as $\delta \to 1$.

In this case, it must be that $u(\overline{x}, l) = u(\underline{x}, l)$. Thus,

$$u(\overline{x}, l) = \delta F(\underline{x}) u(\overline{x}, l) + \delta \int_{\underline{x}}^{\overline{x}} u(z, l) \, dF(z) + \delta (1 - F(\overline{x})) u(\overline{x}, l),$$

or equivalently,

$$u(\overline{x}, l)(1 - \delta) = \delta \int_{\underline{x}}^{\overline{x}} u(z, l) \, dF(z) - \delta [F(\overline{x}) - F(\underline{x})] u(\overline{x}, l).$$

Therefore, as $\lim_{\delta \to 1} u(\overline{x}, l)(1 - \delta) = 0$, we must have

$$\lim_{\delta \to 1} \delta \int_{\underline{x}}^{\overline{x}} [u(z, l) - u(\overline{x}, l)] \, dF(z) = 0$$

By single-peakedness, $u(z, l) > u(\overline{x}, l)$ for all $z \in (\underline{x}, \overline{x})$. Thus, we must have $\lim_{\delta \to 1} \overline{x} - \underline{x} = 0$. ■

Lemma 14 *Fix δ and q and let $[\underline{x}, \overline{x}]$ be the approval set. Let $K(x) \equiv F(x) \dfrac{u_x^+(x, l)}{u(x, l)} + [1 - F(x)] \dfrac{u_x^-(x, r)}{u(x, r)}$.*

1. *If there is an x^* such that $K(x^*) = 0$, then x^* is unique and $x^* \in [\underline{x}, \overline{x}]$.*
2. *Otherwise, there exist a unique $i^* \in I \cap [l(q), r(q)]$ such that $K(x) > 0$ for $x \in [l(q), i^*)$ and $K(x) < 0$ for $x \in [i^*, r(q)]$, and $i^* \in [\underline{x}, \overline{x}]$.*

Proof. First, note that the concavity of u implies that K is decreasing in x. Furthermore, $K(z) > 0$ for all $z < l$ and $K(z) < 0$ for all $z > r$. Therefore, $K(x) = 0$ admits at most one solution in $[l, r]$. Otherwise, K changes sign once, and this must occur at a point where F jumps, i.e. I is finite and the jump takes place at a unique $i \in I \cap [l, r]$.

We next claim that $K(\underline{x}) > 0$ and $K(\overline{x}) < 0$. Recall that x^e denotes the expected value the equilibrium outcome,

$$x^e = F(\underline{x}) \underline{x} + \int_{\underline{x}}^{\overline{x}} z dF(z) + (1 - F(\overline{x})) \overline{x},$$

and let x_r^c and x_l^c denote the certainty equivalents for the equilibrium for players r and l, respectively, that is

$$u\left(x_l^c, l\right) = F\left(\underline{x}\right) u\left(\underline{x}, l\right) + \int_{\underline{x}}^{\overline{x}} u\left(z, l\right) dF(z) + \left(1 - F\left(\overline{x}\right)\right) u\left(\overline{x}, l\right),$$
$$u\left(x_r^c, r\right) = F\left(\underline{x}\right) u\left(\underline{x}, r\right) + \int_{\underline{x}}^{\overline{x}} u\left(z, r\right) dF(z) + \left(1 - F\left(\overline{x}\right)\right) u\left(\overline{x}, r\right).$$

Note that the following inequalities are satisfied: $\underline{x} < x_r^c \leq x^e \leq x_l^c < \overline{x}$. By the equilibrium condition $u\left(\underline{x}, r\right) = \delta u\left(x_r^c, r\right)$, and by concavity

$$\frac{u\left(x_r^c, r\right) - u\left(\underline{x}, r\right)}{x_r^c - \underline{x}} = \frac{\left(1 - \delta\right) u\left(x_r^c, r\right)}{x_r^c - \underline{x}} \leq u_x^-\left(\underline{x}, r\right),$$

and also, by concavity,

$$\frac{\left(1 - \delta\right) u\left(x^e, r\right)}{x^e - \underline{x}} \leq \frac{\left(1 - \delta\right) u\left(x_r^c, r\right)}{x_r^c - \underline{x}}$$

Hence, as $u\left(\underline{x}, r\right) < u\left(x^e, r\right)$ we obtain

$$\frac{\left(1 - \delta\right) u\left(\underline{x}, r\right)}{x^e - \underline{x}} < u_x^-\left(\underline{x}, r\right). \tag{9}$$

On the other hand, $u\left(\overline{x}, l\right) = \delta u\left(x_l^c, l\right)$ and concavity imply

$$\frac{u\left(x_l^c, l\right) - u\left(\overline{x}, l\right)}{x_l^c - \overline{x}} = \frac{\left(1 - \delta\right) u\left(x_l^c, l\right)}{x_l^c - \overline{x}} \leq u_x^+\left(\underline{x}, l\right)$$

and, by concavity again (noting that $u\left(x_l^c, l\right) \leq u\left(x^e, l\right)$), 1)

$$\frac{\left(1 - \delta\right) u\left(x^e, l\right)}{x^e - \overline{x}} \leq \frac{\left(1 - \delta\right) u\left(x_l^c, l\right)}{x_l^c - \overline{x}}$$

and therefore, since $u\left(x^e, l\right) < u\left(\underline{x}, l\right)$ and $x^e - \overline{x} < 0$, we obtain

$$\frac{\left(1 - \delta\right) u\left(\underline{x}, l\right)}{x^e - \overline{x}} < u_x^+\left(\underline{x}, l\right). \tag{10}$$

Thus,

$$
\begin{aligned}
(1-\delta) K(\underline{x}) &= F(\underline{x}) \frac{u_x^+(\underline{x}, l)}{(1-\delta) u(\underline{x}, l)} + [1 - F(\underline{x})] \frac{u_x^-(\underline{x}, r)}{(1-\delta) u(\underline{x}, r)} = \\
&> F(\underline{x}) \frac{1}{x^e - \overline{x}} + [1 - F(\underline{x})] \frac{1}{x^e - \underline{x}} = \\
&= \frac{1}{(x^e - \overline{x})(x^e - \underline{x})} \left(F(\underline{x})(x^e - \underline{x}) + [1 - F(\underline{x})](x^e - \overline{x}) \right) \\
&= \frac{1}{(x^e - \underline{x})(x^e - \overline{x})} \left(F(\underline{x})(\overline{x} - \underline{x}) + (x^e - \overline{x}) \right)
\end{aligned}
$$

Since $x^e < F(\underline{x})\, \underline{x} + (1 - F(\underline{x}))\, \overline{x}$, we have

$$F(\underline{x})(\overline{x} - \underline{x}) + (x^e - \overline{x}) < 0.$$

Thus, as $(x^e - \underline{x})(x^e - \overline{x}) < 0$ we obtain

$$(1-\delta) K(\underline{x}) > 0.$$

Similarly, it can be shown that $K(\overline{x}) < 0$.

Therefore since K is decreasing, if $K(x) = 0$ has a unique solution x^*, then $\underline{x} < x^* < \overline{x}$. For $i^* \in I \cap [l, r]$ such that $K(i^* - \epsilon) > 0$ for all $\epsilon > 0$ and $K(i^*) < 0$, then $\underline{x} < i^* \leq \overline{x}$. ■

Lemma 15 *$\overline{\zeta}(\underline{x}, l)$ is increasing in l, and $\underline{\zeta}(\overline{x}, r)$ is increasing in r.*

Proof. Fix $\underline{x}$ and consider $\overline{\zeta}(\underline{x}, l)$ at two possible values of $l = a, b$ with $a < b$. Notice that $\overline{\zeta}(\underline{x}, a) < 1$ implies $u\left(\overline{\zeta}(\underline{x}, a), a\right) = \delta U_a$. In this case, if $\overline{\zeta}(\underline{x}, b) = 1$ the result follows. Otherwise, $\overline{\zeta}(\underline{x}, b)$ is defined by $u\left(\overline{\zeta}(\underline{x}, b), b\right) = \delta U_b$. Moreover, by Lemma 11(2), $u\left(\overline{\zeta}(\underline{x}, b), a\right) > \delta U_a$. Thus, $u\left(\overline{\zeta}(\underline{x}, b), a\right) > u\left(\overline{\zeta}(\underline{x}, a), a\right)$. Moreover, as $\overline{\zeta}(\underline{x}, b) > b > a$ we have that $\overline{\zeta}(\underline{x}, a) < \overline{\zeta}(\underline{x}, b)$. In case that $\overline{x} = \overline{\zeta}(\underline{x}, a) = 1$, it must be that $u(1, a) \geq \delta U_a$. Therefore, by Lemma 11 (4), $u(1, b) > \delta U_b$ so that $\overline{\zeta}(\underline{x}, b) = 1$. ■