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Abstract

The product test asks the product of a quantity index number and a price index number to equal the corresponding value change. The literature treats the product test as being so important that it is used to identify acceptable index number pairs, and to construct implicit index numbers when an otherwise desirable pair fails the test. We treat the product test as a hypothesis to be tested, and we provide an empirical application.

Keywords: product test, index numbers
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Testing the Product Test

1. introduction

Consider a value ratio V^1/V^o, the observed magnitude of which Fisher (1922;75) called “indubitable and undebatable.” Fisher continued “…[T]he problem then is to find a form of index number such that, applied alike to prices and quantities, it shall correctly ‘factor’ any such value ratio.” This is Fisher’s factor reversal test, a stringent test because it requires the product of a price index and a quantity index *of the same functional form* to equal the value ratio. Noting that so few functional forms satisfy the factor reversal test, Fisher proposed using the test not just as a screening device, but also as a way of deriving implicit index numbers: dividing an observed value ratio by a price (quantity) index generates an implicit quantity (price) index, and both pairs satisfy the factor reversal test by construction.

Frisch (1930) noted the restrictiveness of the factor reversal test, and introduced a weak factor reversal test, which he called the product test, and which requires that the product of a price and quantity index equal the value ratio, without requiring that two indexes have the same functional form. Even this weakened product test is too challenging for several popular index number pairs, including Edgeworth-Marshall and Törnqvist. Because a Törnqvist index has other desirable properties, resort to an implicit Törnqvist index is widespread.

As an empirical matter, however, failure of a particular index number pair to satisfy the product test may be acceptably small or unacceptably large. Particularly if an index number pair has other desirable properties, it may be appropriate to treat satisfaction of the product test as a hypothesis to be tested rather than a condition to be imposed. Our objective is to conduct a statistical test of the hypothesis that a desirable quantity index and an equally desirable price index, which are known to fail the product test, fail to a statistically acceptable degree. We conduct the test using a Malmquist quantity index and a Fisher price index. The desirability of a Malmquist quantity index is based on its satisfaction of a number of theoretical properties enumerated by Balk (1998) and its decomposability into the product of economic drivers of quantity change. The desirability of a Fisher price index is based on its being a superlative index, as demonstrated by Diewert (1992), and its decomposability into the product of individual price changes.

We test the product test using a large state-by-year panel of prices and quantities in US agriculture. We conduct the test within an analytical framework that expresses profitability change as the product of productivity change, the ratio of quantity indexes, and price recovery change, the ratio of price indexes. We are unable to reject the hypotheses that the product test is satisfied for output prices and quantities, and for input prices and quantities. We are, however, able to reject the hypothesis that the product test is satisfied for price ratios and quantity ratios. In all three cases the extent of failure is numerically small.

2. Profitability Change and the Product Test

We use profitability as an indicator of financial performance. Profitability, the ratio of revenue to cost, is less popular than profit, the difference between revenue and cost, but it has a long history as a financial performance indicator. Bliss (1923) recommended its use over long time periods because fluctuations in prices are likely to cancel out of numerator and denominator. Davis (1947) and Downie (1958) treated profitability as an indicator of the efficiency with which business converts expenditure to revenue. Georgescu-Roegen (1951) noted its independence of the scale of production, a desirable property not shared by profit.

Let $R = \Sigma p_m y_m$ denote revenue and $C = \Sigma w_n x_n$ denote cost, and define $\Pi = R/C$ as profitability. Change in profitability from base situation 0 to comparison situation 1 is

$$\Pi^1/\Pi^o = (R^1/R^o) \times (C^1/C^o). \tag{1}$$

Values R and C change because quantities change and prices change, and we want to convert (1) to an expression showing profitability change as the product of productivity change, the ratio of an output quantity index and an input quantity index, and price recovery change, the ratio of an output price index and an input price index.

On the output side we seek price and quantity indexes such that

$$R^1/R^o = P(p^1,p^o,\bullet) \times Y(y^1,y^o,\bullet), \tag{2}$$

and on the input side we seek price and quantity indexes such that

$$C^1/C^o = W(w^1,w^o,\bullet) \times X(x^1,x^o,\bullet), \tag{3}$$

where we are deliberately vague about the remaining arguments of the four indexes. If both equalities hold, then

$$\Pi^1/\Pi^o = [Y(y^1,y^o,\bullet)/X(x^1,x^o,\bullet)] \times [P(p^1,p^o,\bullet)/W(w^1,w^o,\bullet)], \tag{4}$$

which meets our objective of expressing profitability change as the product of productivity change and price recovery change.

Unfortunately equalities (2) and (3) do not necessarily hold. If the price indexes have Konüs form, then $P(p^1,p^o,\bullet) = P_K(p^1,p^o,x^1,x^o)$ and $W(w^1,w^o,\bullet) = W_K(w^1,w^o,y^1,y^o)$, and if the quantity indexes have Malmquist form, then $Y(y^1,y^o,\bullet) = Y_M(y^1,y^o,x^1,x^o)$ and $X(x^1,x^o,\bullet) = X_M(x^1,x^o,y^1,y^o)$, but neither equality holds. Our best theoretical indexes fail the product test. If price and quantity indexes have Törnqvist form, then $P(p^1,p^o,\bullet) = P_T(p^1,p^o,y^1,y^o)$, $W(w^1,w^o,\bullet) = W_T(w^1,w^o,x^1,x^o)$, $Y(y^1,y^o,\bullet) = Y_T(y^1,y^o,p^1,p^o)$ and $X(x^1,x^o,\bullet), = X_T(x^1,x^o,w^1,w^o)$, but neither equality holds. One of our best empirical indexes fails the product test. If price and quantity indexes have Fisher form, both equalities hold and consequently expression (4) provides an exact decomposition of profitability change.

For reasons mentioned above, we prefer to pair Fisher price indexes with Malmquist quantity indexes. However this pairing fails the product test, which we express as

$$R^1/R^o \neq P_F(p^1,p^o,y^1,y^o) \times Y_M(y^1,y^o,x^1,x^o), \quad (5)$$

$$C^1/C^o \neq W_F(w^1,w^o,x^1,x^o) \times X_M(x^1,x^o,y^1,y^o), \quad (6)$$

and consequently

$$\Pi^1/\Pi^o \neq [Y_M(y^1,y^o,x^1,x^o)/X_M(x^1,x^o,y^1,y^o)] \times [P_F(p^1,p^o,y^1,y^o)/W_F(w^1,w^o,x^1,x^o)], \quad (7)$$

which states that profitability change cannot be expressed as the (exact) product of a Malmquist productivity index and a Fisher price recovery index.

However we have theoretical and empirical reasons to expect that the failures in (5) – (7) may be acceptably small. Balk (1998) collects two sets of results. The first is based on Mahler's inequality, and yields $Y_M(y^1,y^o,x^1,x^o) \approx Y_F(y^1,y^o,p^1,p^o)$ and $X_M(x^1,x^o,y^1,y^o) \approx X_F(x^1,x^o,w^1,w^o)$. The two product tests are approximately satisfied, the approximation error depending on the extent of resource misallocation. The second relies on a flexible specification of the structure of technology, and on allocative efficiency, to obtain $Y_M(y^1,y^o,x^1,x^o) = Y_F(y^1,y^o,p^1,p^o)$ and $X_M(x^1,x^o,y^1,y^o) = X_F(x^1,x^o,w^1,w^o)$. The two product tests are exactly satisfied, but under functional form and allocative efficiency restrictions.

But matters are a bit more complicated, and our expectation must be tempered. The value ratios R^1/R^o, C^1/C^o, and consequently Π^1/Π^o, are "indubitable and undebatable." The Fisher price indexes are calculated from observed data. But the Malmquist quantity indexes must be estimated. Whatever we know about (5) – (7) is compounded by estimation error. Referring to the result based on Mahler's inequality, estimation error may moderate or compound optimization error. An empirical test of the product test is needed.

3. Testing the Product Test

We test the product test using data provided by the Economic Research Service of the US Department of Agriculture. The data are a panel of agriculture production covering 48 states during 1960-2004. The data include price indexes and quantity indexes for three outputs (livestock, crops and other output) and four inputs (capital, land, labor and materials). We calculate the value ratios R^1/R^o and C^1/C^o and the price indexes $P_F(p^1,p^o,y^1,y^o)$ and $W_F(w^1,w^o,x^1,x^o)$ directly from the data, and we use linear programming techniques to estimate the quantity indexes $Y_M(y^1,y^o,x^1,x^o)$ and $X_M(x^1,x^o,y^1,y^o)$.

We base our statistical tests on the ratios of the left side to the right side of (5) – (7). The output product test approximation error is defined from (5) as $\varepsilon_Y = (R^1/R^o) / [Y_M(y^1,y^o,x^1,x^o) \times P_F(p^1,p^o,y^1,y^o)]$. Under the null hypothesis of no output product test approximation error, ε_Y is a unit vector. We apply the same strategy to (6) and (7).

Insert Table 1 About Here

The evidence summarized in Table 1 is based on 2,112 observations (48 states and 44 annual ratios of comparison period to base period variables). The evidence is encouraging. It is not possible to reject, at a 95% confidence level, either the hypotheses that $Y_M(x^1,x^o,y^1,y^o) = Y_F$ or the hypothesis that $X_M(y^1,y^o,x^1,x^o) = X_F$. An implication is that estimated Malmquist quantity indexes provide excellent approximations to calculated Fisher quantity indexes that do satisfy the product test with their counterpart price indexes. This is a reversal of the usual line of reasoning, which states that Fisher quantity indexes provide approximations to their theoretical Malmquist counterparts. However our estimate of $Y_M(x^1,x^o,y^1,y^o)$ has slightly smaller mean than that of Y_F, and our estimate of $X_M(y^1,y^o,x^1,x^o)$ has slightly larger mean than that of X_F. This makes it possible to reject the hypothesis that $M(x^1,x^o,y^1,y^o) = Y_F/X_F$ at the same confidence level.

From a statistical perspective, our evidence is mixed: estimated Malmquist quantity indexes satisfy the product test with Fisher price indexes, although an estimated Malmquist productivity index does not satisfy the product test with a Fisher price recovery index. However the mean product test approximation error is extremely small, less than 0.4%, probably far smaller than what Fisher (1922) calls the formula error associated with the Fisher price recovery index and the estimation error associated with the Malmquist productivity index. In light of the importance we attach to our dual objective of decomposing a Malmquist productivity index by economic driver and decomposing a Fisher price recovery index by individual price change, we are willing to live with a 0.4% product test approximation error.

4. Conclusions

Most researchers are, by necessity, not purists. They have come to accept *formula error* associated with the use of empirical index numbers, although they prefer superlative index numbers because under certain conditions they provide closer approximations to the truth than other index numbers do. They also accept *estimation error* associated with econometric or mathematical programming estimates of the truth, although they attempt to minimize estimation error through the specification of functional form and estimation technique. We are willing to live with a third type of error, which we call *product test approximation error*, which we attempt to minimize by pairing our best theoretical quantity indexes with our best empirical price indexes.

We have provided one piece of empirical evidence bearing on the magnitude of the product test approximation error. We fail to reject the hypothesis of no product test approximation error on the revenue side and on the cost side, but we do reject the hypothesis at the profitability level. In all three cases the approximation error is numerically small.

One interpretation of our findings is that empirical estimates of theoretical indexes are, statistically and numerically, close to calculated values of their empirical counterparts. This interpretation provides a reverse spin on the superlative index number literature. An alternative interpretation is that if one has good reason for preferring a pair of indexes that fail the product test (and the Malmquist/Fisher pairing clearly qualifies), then one might well prefer living with the resulting product test approximation error to living with an implicit index that satisfies the product test by construction but is otherwise not very informative.

Fisher called his index number formula "ideal" because it satisfies so many tests. However it fails one important test, the circularity test. It is worth quoting Fisher on this embarrassment: "I aim to show that the circular test is theoretically a mistaken one, that a necessary irreducible minimum of divergence from such fulfillment is entirely right and proper, and, therefore, that a *perfect* fulfillment of this so-called circular test should really be taken as proof that the formula which fulfils it is erroneous." (p. 271, italics in the original)

Applying Fisher's unwillingness to require exact fulfilment of the circular test to the product test, we suggest that "…a necessary irreducible minimum of divergence from such fulfilment is entirely right and proper…."

References

Balk, B. M., 1998, Industrial Price, Quantity, and Productivity Indexes. (Kluwer Academic Publishers, Boston).

Bliss, J. H., 1923, Financial and Operating Ratios in Management. (The Ronald Press Co., New York).

Davis, H. S., 1947, The Industrial Study of Economic Progress. (University of Pennsylvania Press, Philadelphia).

Diewert, W. E., 1992, Fisher Ideal Output, Input, and Productivity Indexes Revisited, Journal of Productivity Analysis 3, 211-248.

Downie, J., 1958, The Competitive Process. (Gerald Duckworth & Co., London).

Fisher, I., 1922, The Making of Index Numbers. (Houghton Mifflin, Boston).

Frisch, R., 1930, Necessary and Sufficient Conditions Regarding the Form of an Index Number which Shall Meet Certain of Fisher's Tests, Journal of the American Statistical Association 25, 397-406.

Georgescu-Roegen, N., 1951, The Aggregate Linear Production Function and its Applications to von Neumann's Economic Model, in: T. C. Koopmans, ed., Activity Analysis of Production and Allocation. (John Wiley & Sons, New York) 98-115.

	$\varepsilon_Y = R^1/R^o$ / $Y_M(x^1,x^o,y^1,y^o)\times P_F$	$\varepsilon_X = C^1/C^o$ / $X_M(y^1,y^o,x^1,x^o)\times W_F$	$\varepsilon_Y/\varepsilon_X = \Pi^1/\Pi^o$ / $M(x^1,x^o,y^1,y^o)\times(P_F/W_F)$
Mean	1.0014	0.9987	1.0041
Standard Dev.	0.0501	0.0391	0.0616
Maximum	1.3175	1.2484	1.3461
Minimum	0.6072	0.8425	0.6160
Observations	2,112	2,112	2,112
95% Conf. Int.	[1.0035,0.9992]	[1.0004,0.9970]	[1.0058,1.0015]

Table 1. Statistical Tests of the Product Test Approximation Error