Simultaneous Nash Bargaining with Consistent Beliefs*

Roberto Burguet[†] and Ramon Caminal[‡]

This draft: November 2010

Abstract

We propose and analyze a new solution concept, the $R-$solution, for three-person, transferable utility, cooperative games. In the spirit of the Nash Bargaining Solution, our concept is founded on the predicted outcomes of simultaneous, two-party negotiations that would be the alternative to the grand coalition. These possibly probabilistic predictions are based on consistent beliefs. We analyze the properties of the $R-$solution and compare it with the Shapley value and other concepts. The $R-$solution exists and is unique. It belongs to the bargaining set and to the core whenever the latter is not empty. In fact, when the grand coalition can simply execute one of the three possible bilateral trades, the $R-$solution is the most egalitarian selection of the bargaining set. Finally, we discuss how the $R-$solution changes important conclusions of several well known Industrial Organization models.

KEYWORDS: cooperative games, bargaining, endogenous fallback options, consistent beliefs, $R-$solution.

JEL classification numbers: C71, C78, L14.

---

*We thank Albert Banal, Olivier Compte, Matthew Ellman, Jordi Massó, Clara Ponsatí, Debraj Ray and József Sákovics for useful comments. Also, we acknowledge the support of the Barcelona GSE, Generalitat de Catalunya, and Spanish Ministry of Science and Innovation (project ECO2008-01850).

[†]Institut d'Anàlisi Econòmica CSIC, and Barcelona GSE, e-mail: roberto.burguet@iae.csic.es

[‡]Institut d'Anàlisi Econòmica CSIC, and Barcelona GSE, e-mail: ramon.caminal@iae.csic.es

1 Introduction

When bilateral bargaining is one of the components of an economic model, most authors use the Nash Bargaining Solution (NBS) as a reduced form that maps the fundamentals of the model into negotiated outcomes. Since we often know very little about how agents actually bargain in the real world, a black-box approach seems justified. After all, the principles and intuitions implicit in the NBS are very convincing. However, such broad consensus does not exist when bargaining involves three players and different pairs of players can achieve by themselves different agreements.[1] This is the case when one (or more) player(s) may trade or reach an agreement with two alternative, potential partners. When analyzing such problems, some authors take a non-cooperative approach and assume a particular bargaining protocol. An alternative is to invoke solution concepts borrowed from cooperative game theory. The Shapley value is the most popular choice, as a simple value characterized by seemingly natural axioms. Yet, the Shapley value predicts outcomes that in some cases are controversial, to say the least.[2]

This paper presents a new solution concept for three-player cooperative games that can be readily applied to predicting the outcome of three-party negotiations. Instead of attempting to identify sensible axioms that single out one outcome or considering a particular protocol that would do the job, our approach is based on a few mainstream ideas in economics. The first is that the NBS is a satisfactory prediction for two-player bargaining or in general for what are called pure bargaining games, where the only coalition that adds some surplus is the grand coalition.[3] The second is that when players bargain they also form beliefs about what would happen if agreement is not reached in that particular negotiation. The third one is

[1] Examples of economic models that include three-player bargaining abound. In Section 4 we discuss in detail some particular examples.

[2] See, for instance, De Meza and Selvaggi (2007), page 89.

[3] See Krishna and Serrano (1996) for a non-cooperative motivation for this solution.

that these beliefs should satisfy some notion of consistency with payoffs. [4]

Consider one of the simplest of these three-person bargaining situations, that of a buyer that has to choose among two potential sellers. A prediction for any such model should include a (possibly probabilistic) prediction of which of the two trades will take place and how players would split the surplus in each of the two potential trades.[5] Also, if the latter prediction is to be made according to the NBS, then disagreement points for each of the two negotiations should be specified. For the buyer, the disagreement payoffs should be endogenous. Indeed, the fallback option in each negotiation is the possibility to trade with the alternative seller.

As we allow for more complex interactions, we will need to consider the case where all two-player negotiations result in some positive surplus. This is known as the three-player/three-cake problem (see Binmore, 1985). In this case, disagreement points and payoffs will need to be simultaneously and endogenously determined for all three players in all three alternative two-player negotiations. Moreover, now the (possibly probabilistic) prediction of what negotiation will end in an agreement will be necessary in order to consistently calculate (expected) fallback options.

Finally, what is predicted for the three-player/three-cake problem may leave gains that the three players may realize by coordinating. In other words, the total surplus that the grand coalition can realize may exceed the surplus expected from bilateral negotiations. That may be so because of synergies that can be realized only with the participation of all three players or just because, absent coordination, players anticipate that inefficient bi-

[4] In our previous research on labor contracting (Burguet et al., 2002) we also had to decide how to predict the outcome of negotiations among three players. In fact, in the Appendix of that paper we timidly started to outline some of the ideas that we fully develop here.

[5] The Shapley value predicts that the buyer will buy from the most efficient seller, yet the non trading seller will still receive a positive payment at the expense of the trading partners. Such a positive payoff is sometimes interpreted as the bribe that the non-trading seller receives in order to allow the implementation of the efficient trade. We will show that such a justification makes sense only in some games but not in this particular example.

lateral agreements may occur with positive probability. In this case, players may be able to avoid inefficient outcomes through three-party negotiations and then we expect them to share the extra surplus according to the (generalized) NBS.[6] In particular, the disagreement point for this three-player negotiation should be the players' expected payoffs in the alternative to the grand coalition agreement: the predicted outcome for bilateral negotiations.

As we have mentioned, our solution concept requires that agents form (and share) beliefs on the probabilities of success of each alternative negotiation. This is an important feature of our concept. In addition, we will impose a consistency requirement on this system of beliefs: parties should not expect a two-player negotiation to succeed when both parties to that negotiation prefer their alternative one. In Section 2 we present our solution concept, the $R-$solution, as a formalization of these ideas. We show that the $R-$solution exists and is unique. That is, it turns out that these simple ideas are sufficient to predict the division of surplus in these games. Moreover, computing the $R-$solution is a straightforward exercise. We provide these computations for all parameter values.

The idea that disagreement points in three-party negotiations should emanate from the alternative to these negotiations, that is, the predicted outcomes of simultaneous, bilateral negotiations, is probably non controversial. The same applies to assuming that disagreement points in simultaneous, bilateral negotiations should be endogenous. Moreover, the ideas are not novel. Bennett's (1997) approach to the analysis of such negotiations is the closest to ours in spirit (also, see Binmore, 1985, and references in Bennett, 1997). Indeed, Bennett also argues that disagreement payoffs should be obtained endogenously, but in her solution players do not form and share beliefs about the probability of success of each bilateral negotia-

[6]Three-party negotiations may not be feasible due to outside constraints. In Section 4, we consider one case when this is so.

tion.[7] Indeed, in Bennett's approach, when two parties negotiate both use as a fallback option their own agreement with the third player. That is equivalent to assuming that different players assign probability one to two different, mutually exclusive outcomes. On the contrary, a central piece of our concept is the endogenously determined, coherent system of beliefs that players use to compute their endogenous fallback options.[8]

We analyze the properties of the $R-$solution in Section 3. We show that the $R-$solution satisfies symmetry, efficiency, and the dummy player axioms. Thus, it has to violate the additivity axiom since the Shapley value is the only solution concept that satisfies all four. Indeed, the $R-$solution is not additive. We argue that, rather than a weakness, this non additivity is a desirable property of the concept for problems like the one discussed above. The seemingly innocuous additivity axiom implicitly imposes too much structure on what "protocols" are feasible for the players. For instance, in our one-buyer, two-sellers example, it implicitly imposes that the buyer cannot attempt bundling or make joint offers for two goods when dealing with the same two potential sellers of these two goods. The $R-$solution lets the primitives of the problem speak about such possibilities.

Contrary to the Shapley value, the $R-$solution is a selection of the core when the latter is not empty. When the core is empty, the Aumann-Maschler bargaining set (BS) is the most popular generalization. The BS contains the core and is never empty. We show that, again contrary to the Shapley value, the $R-$solution is a selection of the BS. In fact, for superadditive,

[7] In Bennett (1997), a *solution* should specify the division of surplus in each alternative bilateral negotiation. The disagreement point in each negotiation is the payoff that each player would obtain in her alternative negotiation. Thus, the disagreement point in some negotiations may be outside the *feasible* set of that negotiation, which Bennett interprets as failure of the negotiation. A predicted *outcome* specifies what negotiation will succeed and then sharing of the surplus according to the NBS (or any other concept) given the corresponding disagreement point.

[8] In Section 3 we also discuss alternative approaches to endogenizing fallback options, which are implicit in the notion of consistency proposed by Hart and Mas-Colell (1989) and Serrano and Shimomura (1998).

three-player TU-games, the BS (for the grand coalition) coincides with the core when the latter is not empty, and is a singleton when the core is empty. Thus, the $R-$solution coincides with the BS in the latter case. Moreover, if bilateral bargaining is all there is in the game, that is, if the grand coalition does not add any additional surplus, the $R-$solution is the most *egalitarian* selection in the BS. Thus, it is more egalitarian than other, different selections of the core or the BS, like the nucleolus.[9]

We postulate the $R-$solution as a satisfactory, unifying concept that can be used to analyze models that include three-party negotiations. In Section 4 we illustrate the use of our concept in some leading models in the Industrial Organization literature. Exclusive contracts (Segal and Whinston, 2000), endogenous mergers (Horn and Persson, 2001), and the property-rights theory of the firm (Hart and Moore, 1990) have been analyzed in models with a renegotiation stage, but using some other, diverse solution concepts. In Section 4 we also discuss the use and implications of the $R-$solution in these cases. Section 5 offers some closing discussions. Finally, most of the proofs are relegated to an Appendix.

2 The $R-$solution of a three-person game

Let $N = \{1, 2, 3\}$ be the set of players, and let 2^N represent the set of subsets of N. An element $Z \in 2^N$ represents a coalition. A TU game in characteristic form is the pair (N, v), where $v : 2^N \to R$ satisfies $v(\varnothing) = 0$. We assume v to be *superadditive.*

Assumption 1 (superadditivity): If $Z, Z' \subset 2^N$ and $Z \cap Z' = \varnothing$, then $v(Z) + v(Z') \leq v(Z \cup Z')$.

To save some space, we will use an abbreviated notation for the v function. Thus, we will let $v_{ij} = v(\{i, j\})$, $v_i = v(\{i\})$ and $V = v(\{1, 2, 3\})$.

[9] The nucleolus is also a selection of the BS. Thus, when the core is empty, the nucleolus and the $R-$Solution coincide. However, when the core is not empty and set-valued, the two concepts differ. More on this in Section 3.

Also, every time we write "for all i, j" or "for all i, j, k" we mean for all $i, j = 1, 2, 3, i \neq j$, and for all $i, j, k = 1, 2, 3,\ i \neq j \neq k, i \neq k$, respectively. That is, different sub/superindices in the same expression will always denote different players. Without loss of generality, we will assume that $v_{12} - v_1 - v_2 \geq v_{13} - v_1 - v_3 \geq v_{23} - v_2 - v_3$. In other words, coalition $\{1, 2\}$ is the (weakly) most "efficient" among the two-player coalitions and coalition $\{2, 3\}$ is the (weakly) least efficient.

The heart of our solution concept is a prediction of the outcomes of the three possible bilateral negotiations, including a prediction of which of these negotiations would succeed (with what probability), should three-player negotiations fail.[10] In many cases this is in fact all that will be needed for predicting the outcome of the whole game.

We begin by defining this prediction for the outcome of simultaneous, bilateral negotiations. For each player i in each bilateral negotiation ij, we denote i's predicted payoff by u_i^{ij}. Also, we represent by p_{ij} the predicted probability that players i and j are the ones whose negotiation succeeds and then "trade". Finally, since our concept is based on the two-player NBS, for each player i in each bilateral negotiation ij, we will define i's disagreement payoff or fallback option, which we will represent by t_i^{ij}. Before defining our solution, we explain the consistency requirements on these values that will define our solution concept for simultaneous, bilateral negotiations.

i) Given the fallback options, t_i^{ij}, players i and j share any extra surplus equally, provided this surplus is positive. That is, $u_i^{ij} = t_i^{ij} + \frac{1}{2}\left(v_{ij} - t_i^{ij} - t_j^{ij}\right) = \frac{1}{2}\left(v_{ij} + t_i^{ij} - t_j^{ij}\right)$, if $v_{ij} \geq t_i^{ij} + t_j^{ij}$. However, if their disagreement payoffs sum up to an amount in excess of the worth of the coalition, $v_{ij} < t_i^{ij} + t_j^{ij}$, then players will not be willing to reach an agreement. In this case, $u_i^{ij} = t_i^{ik}$. In the next paragraph we discuss the reasons and interpretation of this spec-

[10] As in the one-buyer/two-sellers example or in the three-player/three-cake game, we assume that only one of the two-player coalitions could form, if the grand coalition cannot form. See Sections 4 and 5 for more on this.

ification.

ii) The disagreement payoffs are computed according to the payoffs predicted in, and the probability distribution over alternative, two-party negotiations. In particular, assume that the negotiation between i and j flounders, and players contemplate their options in the larger picture of all two-player negotiations. As players calculate what they expect to get in this scenario, t_i^{ij}, they predict that, (a) with probability p_{ij} what they face is precisely this default, t_i^{ij}; (b) with probability p_{ik} coalition (i,k) will reach an agreement, and player i's payoff will be u_i^{ij}; and (c) with probability p_{jk} it will be coalition (j,k) who will agree, and hence i's payoff will be v_i. Thus, $t_i^{ij} = p_{ij}t_i^{ij} + p_{ik}u_i^{ik} + p_{jk}v_i$. If $p_{ij} < 1$ we can rewrite this expression as:

$$t_i^{ij} = \frac{p_{ik}u_i^{ik} + p_{jk}v_i}{1 - p_{ij}}.$$

Thus, player i's fallback option in her negotiation with j is the expected payoff in alternative negotiations, where the expectation is "conditional" on her negotiation with j having come to a halt.[11]

If the sum of the disagreement points in the negotiation between players i and k exceeds the worth of that coalition, $v_{ik} < t_i^{ik} + t_k^{ik}$, then $u_i^{ij} = t_i^{ik}$, and then the definition above implies that $t_i^{ij} = v_i$. In other words, if an agreement between i and k is not viable, then when players i and j negotiate they anticipate that if they do not reach an agreement then players j and k will do so and share v_{jk} with probability one, so that player i's payoff will be v_i.[12] Thus, player i's payoff when (hipothetically) dealing with k if her negotiation with j are suspended coincides with her payoff when dealing with j under the same assumption, i.e., t_i^{ij}.

iii) p_{ij} is (virtually) zero if $u_i^{ik} \geq u_i^{ij}$ and $u_j^{jk} \geq u_j^{ij}$, with one strict in-

[11] In contrast to our approach, Benett (1997) assume that players i and j believe that each one of them will be able to reach an agreement with player k with probability one, in case negotiations between i and j fail.

[12] Note that if coalition (i,k) is not viable then u_i^{ik} will not enter into the computations of expected payoffs, and will only matter in the determination of t_i^{ij}.

equality. That is, an agreement between players i and j cannot be reached (with non-negligible probability) if both players prefer their alternative agreement, one of them strictly.

Thus, we will build on the NBS by defining endogenous fallback options for each negotiation. Often, our solution will predict that some coalition would form with probability one, should the three-player coalition fail to form. However, probability one events leave too many degrees of freedom with respect to what are consistent outcomes in the rest of events. In order to avoid this indeterminacy, we will proceed in the standard way of first considering only probability distributions that assign to each two-player negotiation a probability of success bounded away from 1.

Definition 1 *For $\epsilon > 0$, an $\epsilon-$Prediction for simultaneous, bilateral negotiations for the three-player game (N, v), $\epsilon-$PSBN for short, is a triple $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}_{i,j=1,2,3}$ that satisfies:*

1)

$$u_i^{ij}(\epsilon) = \begin{cases} \frac{1}{2}\left(v_{ij} + t_i^{ij}(\epsilon) - t_j^{ij}(\epsilon)\right) & \text{if } v_{ij} \geq t_i^{ij}(\epsilon) + t_j^{ij}(\epsilon), \\ t_i^{ik}(\epsilon) & \text{otherwise;} \end{cases}$$

2) $t_i^{ij}(\epsilon) = p_{ij}(\epsilon)\, t_i^{ij}(\epsilon) + p_{ik}(\epsilon)\, u_i^{ik}(\epsilon) + p_{jk}(\epsilon)\, v_i$*, for all i, j, k;*

3) $p_{12}(\epsilon) + p_{13}(\epsilon) + p_{23}(\epsilon) = 1$; $p_{ij}(\epsilon) \leq 1 - \epsilon$ for all i, j; and for all i, j, k, $p_{ij}(\epsilon) < \epsilon$ if $u_i^{ij}(\epsilon) \leq u_i^{ik}(\epsilon)$ and $u_j^{ij}(\epsilon) \leq u_j^{jk}(\epsilon)$, with one strict inequality.

Our prediction for simultaneous, bilateral negotiations is the limiting value of predictions as the upper bound on p_{ij} tends to 1.

Definition 2 *A Prediction for simultaneous, bilateral negotiations for the three-player game (N, v), PSBN for short, is a triple $\left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}_{i,j=1,2,3}$ that satisfies* $\lim_{\epsilon \to 0} \left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}_{i,j=1,2,3} = \left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}_{i,j=1,2,3}$.

Note that, implicit in the definition of a PSBN is that an $\epsilon-$PSBN exists for any ϵ small, and also that $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}$ converges to the same values $\left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}$ independent of the path by which ϵ converges to 0, or the $\epsilon-$PSBN selected for each ϵ. The next proposition, the main result of this paper, states that these conditions are met. It also computes the PSBN.

In order to simplify the presentation of our results, we normalize $v_i = 0$ for all $i = 1, 2, 3$, although we will still offer the definitions for the general case. Also, at the end of this section we explain how all results and computations can be straightforwardly extended to the general case. Meanwhile, payoffs and disagreement points should be interpreted as net of one-player coalitions' payoffs.

Proposition 1 *The PSBN exists for the game (N, v). Moreover,*

(Region 1) if $v_{12} \geq v_{13} + v_{23}$ and $v_{13} \leq \frac{1}{2}v_{12}$, then $p_{12} = 1$, and $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$;

(Region 2) if $v_{12} \geq v_{13} + v_{23}$ and $v_{13} \geq \frac{1}{2}v_{12}$, then $u_1^{12} = u_1^{13} = v_{13}$, $u_2^{12} = v_{12} - v_{13}$,$u_3^{13} = 0$, $p_{23} = 0$ and if $v_{13} < v_{12}$ then $p_{12} = 1$; and

(Region 3) if $v_{12} \leq v_{13} + v_{23}$ then $u_i^{ij} = u_i^{ik} \equiv u_i = \frac{v_{ij}+v_{ik}-v_{jk}}{2}$, for all i, j, k, and $p_{ij} \equiv p_{ij} = \frac{u_i u_j}{u_1 u_2 + u_1 u_3 + u_2 u_3}$.

Proof. See Appendix. ■

In regions 1 and 2, the surplus that players 1 and 2 obtain if they agree is sufficiently high as compared to the alternative bilateral negotiations so that we predict that players 1 and 2 "trade" with probability one (except in the limit case of $v_{12} = v_{13}$ where these two trades are equivalent). The way they split the surplus depends on whether any player (player 1, given our notation) has a sufficiently important alternative. In particular, our solution concept conforms to the "outside option principle" (see Shaked and Sutton 1984, and Binmore, Shaked, and Sutton, 1989): the payoffs of players 1 and 2 coincide with the NBS of their bilateral negotiation in isolation unless one has an outside option that is binding, in which case this player obtains a payoff equal to that outside option. In a PSBN this outside option (for a bilateral negotiation) is "endogenously" determined (by all simultaneous, bilateral negotiations).[13] In Region 3 the PSBN predicts that any of the three bilateral negotiations may succeed. They all have positive probability of success since all three players are indifferent among their two partners.

There are many situations where bilateral trade is the only feasible outcome and side payments between the trading partners and the non trading player are not feasible. In Section 4 we will discuss an application to merger analysis that has this characteristic. In these cases, we claim that:

Remark 1 *If the grand coalition cannot form then the PSBN is the right solution concept.*

The predicted outcome of bilateral negotiations determines the fallback options in the three-player negotiation. Therefore, the last step in defining a solution concept for the game (N, v), the $R-$solution, is straightforward.

Definition 3 *A $R-$solution for the three-player game in characteristic form (N, v) is a triple (U_1, U_2, U_3) that satisfies: a) $U_i = \frac{1}{3}(V + 2T_i - T_j - T_k)$ for all i, j, k, b) $T_i = p_{ij}u_i^{ij} + p_{ik}u_i^{ik} + p_{jk}v_i$, where $\left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}_{i,j=1,2,3}$ is the PSBN for the game (N, v).*

The grand coalition shares the surplus $V - T_1 - T_2 - T_3$ according to the generalized NBS (part a). Player $i's$ fallback option, T_i, is her expected payoff in bilateral negotiations as computed from the PSBN (part b). Characterizing this solution, in particular its existence and uniqueness, requires characterizing $\left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}$. This was done in Proposition 1. Therefore T_i,

[13] This is another difference between the $R-$solution and the concept(s) defined in Bennett (1997). That is, by introducing coherent conjectures with respect to the probabilities of success, we endogenously obtain the "outside option principle" as a natural outcome of bagaining à la Nash.

$i = 1, 2, 3$ exist and are unique, and then the proof of the following theorem is straightforward.

Theorem 1 *The $R-$solution exists and is unique.*

Computing the $R-$solution is in fact extremely easy. What we offer below can be considered a user's manual. In regions 1 and 2 defined in Proposition 1, the PSBN predicts that players 1 and 2 would trade with probability 1. Thus, the expected payoffs for players 1 and 2 add up to v_{12} and player 3's payoff is zero in both regions. These payoffs are the disagreement payoffs in the three-player negotiation. Then in the $R-$solution each player obtains her disagreement payoff plus one third of any worth of the grand coalition in excess of v_{12}, if there is any.

In Region 3 all two-player coalitions have a positive probability to form, should the grand coalition fail to agree. The PSBN satisfies some interesting properties that make the computations simple. First, the payoff for a player in each of the two-player coalitions of which she is a part is the same: $u_i^{ij} = u_i^{ik} = u_i$. Also, for each game there exists a number Ψ such that

$$p_{ij} u_k = \Psi \text{ for all } i, j, k. \tag{1}$$

Note that p_{ij} is the probability that player k does not get u_k. Therefore, condition (1) indicates that the "loss" experienced by player i with respect to the benchmark where she is able to secure u_i with probability one, is the same for all $i = 1, 2, 3$. This property drastically simplifies the computation of final payoffs. More specifically, player i's expected payoff in the PSBN is:

$$T_i = (p_{ij} + p_{ik})\, u_i = u_i - \Psi,$$

where we have used (1) in the last equality. As a result the $R-$solution for player i is given by:

$$U_i = \frac{1}{3}(V + 2T_i - T_j - T_k) = \frac{1}{3}\left(V + 2u_i - u_j - u_k\right).$$

This makes it possible to compute the $R-$solution without computing the probabilities p_{ij}. Using Proposition 1 we obtain the final expression:

$$U_i = \frac{1}{3}\left(V + v_{ij} + v_{ik} - 2v_{jk}\right).$$

When $v_i \neq 0$, all these computations carry through with only substituting $v_{ij} - v_i - v_j$ for v_{ij}, for all i, j, and $V - v_1 - v_2 - v_3$ for V, and also adding v_i to all values u, t, U and T. The following Table 1 contains the expression for the $R-$solution in this general case.

Table 1: The $R-$solution

	Region 1	Region 2	Region 3
U_1	$\frac{2V+v_{12}+3v_1-3v_2-2v_3}{6}$	$v_{13} - \frac{4v_3}{3} + \frac{V-v_{12}}{3}$	$\frac{V+v_{12}+v_{13}-2v_{23}}{3}$
U_2	$\frac{2V+v_{12}-3v_1+3v_2-2v_3}{6}$	$\frac{2(v_{12}-v_{13})}{3} + \frac{2}{3}v_3 + \frac{V-v_{13}}{3}$	$\frac{V+v_{12}+v_{23}-2v_{13}}{3}$
U_3	$v_3 + \frac{V-v_{12}-v_3}{3}$	$\frac{2}{3}v_3 + \frac{V-v_{12}}{3}$	$\frac{V+v_{13}+v_{23}-2v_{12}}{3}$

Consider the case $V = v_{12} + v_3$. Note that the non-participating player (3) is able to appropriate a positive surplus ($U_3 > v_3$) only in Region 3. It is precisely in this region where there is a potential bargaining coordination problem and hence players 1 and 2 are willing to "bribe" player 3 out of the way.

3 Properties of the $R-$solution

In this section we study the properties of the $R-$solution by discussing its relation with key concepts in cooperative game theory: Shapley value, core, bargaining set, and nucleolus.

The Shapley value and the $R-$solution coincide only at two points of the parameter space: $v_{13} = v_{23} = 0$ and $v_{13} = v_{23} = v_{12}$.[14] For the rest of the parameter space, the comparison is straightforward and some regularities can be noticed. With respect to the Shapley value, according to the

[14]In the first point ($v_{13} = v_{23} = 0$) the $R-$solution coincides with the NBS of the game for players 1 and 2. In this sense, both the Shapley value and the $R-$solution are generalizations of the NBS to the case of three players.

$R-$solution: (i) Player 3's payoff is always lower, (ii) Player 2's payoff is lower if and only if v_{13} is sufficiently high, (iii) Player 1's payoff is lower if and only if both v_{13} and v_{23} are sufficiently small.[15]

In the next section we will use the predictions of these two solution concepts to discuss investment incentives. This requires studying the marginal effect of an increase in the worth of a coalition on the payoffs of the players that it contains. In contrast to the Shapley value (which is linear), the $R-$solution is piece-wise linear in the worth of different coalitions. In other words, when we cross borders between regions, then the marginal effect of the worth of a coalition on payoffs changes. For instance, if $v_{13} < \frac{v_{12}}{2}$ (Region 1) then $\frac{dU_2}{dv_{12}} = \frac{1}{2}$, but if $v_{13} > \frac{v_{12}}{2}$ (Region 2) then $\frac{dU_2}{dv_{12}} = 1$.

The Shapley value is the only value that satisfies the axioms of efficiency, symmetry, dummy player, and additivity (see for instance Winter, 2002). That means that the $R-$solution violates at least one of these axioms. The $R-$solution satisfies efficiency, that is, for any game $U_1+U_2+U_3 = V$. It also satisfies symmetry. That is, if U is the $R-$solution of (N, v) and U' is the $R-$solution of (N, v') where $v'(Z) = v(Z')$ and $Z' = \{i \in N \,| \mu(i) \in Z\}$, for some bijection $\mu : N \to N$ then $U_i = U'_{\mu(i)}$ for all $i \in N$. In other words, the name of the player has no effect on her value. Also, the $R-$solution satisfies the dummy axiom. That is, if $v(S \cup i) - v(S) = 0$ for every $S \subset N$, then $U_i = 0$. Therefore, the $R-$solution must violate the additivity axiom. Formally, if (N, v) and (N, v') are two games with solutions U and U' respectively, and we consider the game (N, v'') where $v''(Z) = v(Z) + v'(Z)$ for all $Z \subset N$, it may be that its $R-$solution U'' does not satisfy $U''_i = U_i + U'_i$.

We will argue that for the class of problems that we are envisioning this is a strength of the concept rather that a weakness.

[15]This discussion implies that the outcome predicted by the Shapley value does not always Lorentz dominate the outcome predicted by the $R-$solution. However, if we compute, for instance, the variance of the outcomes, the Shapley value is always less disperse than the $R-$Solution. Thus, the Shapley value is more egalitarian in this sense.

Example 1 *One buyer, B, can trade with two potential sellers, S and E. There are two goods and the buyer demands one unit of each. In the production of the first, S has a cost advantage, so that $v(B,S) = 1$ and $v(B,E) = \alpha \in \left(\frac{1}{2}, 1\right)$, whereas in the production of the second it is E who has the cost advantage, so that $v'(B,E) = 1$ and $v'(B,S) = \alpha$. According to the $R-$solution, E obtains 0 in the first game and $1-\alpha$, in the second. The game $v'' = v + v'$ satisfies $v''(B,E) = v''(B,S) = 1+\alpha$, and $v(B,S,E) = 2$.*

Additivity implies that player E, for instance, should still fetch $1-\alpha$ in game v''. In fact, in v'' the $R-$solution grants her one third of that amount. Note that additivity amounts to assuming that the negotiations over the two goods are conducted independently. Thus, by imposing additivity, as the Shapley value does, we would be implicitly allowing sellers to commit to negotiate over each of the two goods only through independent agents who would not listen to anything related to the other good. The $R-$solution does not presume any ability of any party to preclude the two negotiations to interact, and so does not assume such commitment power for any player.[16]

It is well known that the Shapley value is not necessarily in the core or the bargaining set (BS) for the grand coalition. This is another difference between the $R-$solution and the Shapley value or any probabilistic value.[17] The following simple lemma simplifies the discussion of these facts.

Lemma 1 *For three-player, superadditive TU-games, the bargaining set of*

[16] In the previous example, the reader may conclude that the fact that S may also supply the good for which it has a competitive disadvantage is a handicap. This is not so. Consider the game $\widetilde{v} = v''$ except that $\widetilde{v}(B,S) = 1$. In this case, S's payoff is still a third of $1-\alpha$ in the $R-$Solution. Thus, the $R-$solution, contrary to the Shapley value, implicitly postulates that S has no commitment device stronger than simply this sort of "burning the ships": destroying one's ability to deliver what is not going to be delivered. As this example shows, this is in particular weaker, not stronger, than schemes like delegation to independent agents.

[17] As shown by Weber (1988), a probabilistic value is efficient only if it is a *random-order value*, and in our superadditive setting efficiency is a condition for an allocation to be in the core. The set of all random-order values contains the core, but no single one is "always" contained in the core even if we restrict attention to three player games.

the grand coalition coincides with the core if the latter is not empty. If the core is empty, then the bargaining set of the grand coalition is a singleton.

The proof of this popular lemma is given in the Appendix. This lemma allows us to consider only the relationship between the $R-$solution and the BS.

Proposition 2 *The $R-$solution belongs to the bargaining set (for the grand coalition) and so to the core if the latter is not empty.*

Proof. First, we study the core. An element of the core is a positive vector (x_1, x_2, x_3) such that: (i) $x_1 + x_2 + x_3 = V$ and (ii) $x_i + x_j \geq v_{ij}$ for all i, j. Adding up these last three conditions, we obtain $x_1 + x_2 + x_3 \geq \frac{v_{12}+v_{13}+v_{23}}{2}$, which combined with condition (i) gives:

$$V \geq \frac{v_{12} + v_{13} + v_{23}}{2}. \tag{2}$$

When $v_{12} \geq v_{13} + v_{23}$, i.e., in Regions 1 and 2, this is satisfied trivially. It is then immediate to check that the $R-$solution satisfies (i) and (ii) in Regions 1 and 2. Thus, in Regions 1 and 2 the $R-$solution belongs to the core and then to the BS. In Region 3 the core may be empty, that is, (2) may not hold. Thus, we will show that the $R-$solution belongs to the BS. Remember that in Region 3 $U_i = \frac{V+v_{ij}+v_{ik}-2v_{jk}}{3}$. Since $U_i \geq v_i$ for all i, and since the grand coalition cannot be part of an objection, we need only consider objections that use two-player coalitions. Thus, consider an objection of i against j, for $i = 1, 2, 3$, and $j \neq i$. That is, consider a division of v_{ik}, $x = (x_i, x_k)$ where $k \neq i, j$: $x_i + x_k = v_{ik}$, such that $x_i > U_i$, and $x_k > U_k$. We show that there is a counter-objection of j, that is, a division $y = (y_j, y_k)$ of v_{jk} where $y_j + y_k = v_{jk}$, such that $y_j \geq U_j$ and $y_k \geq x_k$. Consider in particular $y_j = U_j$, so that $y_k = v_{jk} - U_j$. If $x_i > U_i$, then $x_k = v_{ik} - x_i < v_{ik} - U_i$. But then

$$y_k - x_k > v_{jk} - U_j - (v_{ik} - U_i) = 0.$$

Thus, if x is an objection then y is a counter-objection. QED. ■

In Region 3, when $V < \frac{v_{12}+v_{13}+v_{23}}{2}$, since the BS is a singleton and the $R-$solution belongs to the BS, we conclude that the $R-$solution *coincides* with the BS, and so with any selection or subset of the BS, in particular the nucleolus and the kernel (Nash set). We next discuss the $R-$solution with regard to these concepts and the "consistency" motivations behind them. For the rest of this section, let us restrict attention to the case $V = v_{12}$. i.e., suppose that the grand coalition does not add surplus. In this domain, the $R-$solution can be characterized from a perhaps surprising perspective. Indeed, let us label an allocation as the most egalitarian in a set if it Lorentz-dominates the rest of allocations in the set.

Proposition 3 *If $V = v_{12}$, the $R-$solution coincides with the selection of the most egalitarian allocation in the bargaining set. Thus, it also coincides with the selection of the most egalitarian allocation in the core, when the core is not empty.*

Proof. Note that $U_1 \geq U_2 \geq U_3$. Thus, a more egalitarian allocation would require to increase the payoff of player 3 or, at least, to increase the payoff of player 2 by reducing the payoff of player 1. We show first that in Region 1 and Region 2 any allocation x in the BS or, equivalently in these regions, in the core assigns a payoff $x_3 = 0$. Assume otherwise $x_3 > 0$. Then $x_1+x_2 = v_{12}-x_3 < v_{12}$, so that the allocation would not be in the core. This immediately proves that the $R-$solution is the most egalitarian allocation in the BS for Region 1. Now suppose that we are in Region 2 and that there is an allocation x that is more egalitarian than the $R-$solution. Since $x_3 = 0$, this implies that $x_2 > v_{12}- v_{13}$, so that $x_1 + x_3 = v_{12} - x_2 < v_{13}$ violating the conditions for x to be in the core. Thus, the $R-$solution is the most egalitarian allocation in the BS in Region 2. Finally, in Region 3 the core is empty, so that the BS is a singleton. Thus, the $R-$solution is the *only* allocation in the BS. QED ■

Thus, the $R-$solution is the most egalitarian among the *stable* (in the sense of Aumman-Maschler) allocations. That is, the most egalitarian among the allocations that cannot be blocked in the sense of the (grand coalition) BS.

An alternative selection in the BS is the nucleolus. For these games, the nucleolus is also a selection of the kernel, itself a subset of the BS. Thus, as we mentioned above, in Region 3 the four concepts, BS, nucleolus, kernel, and $R-$solution, coincide. In regions 1 and 2 and when $v_{12} = V$ (the core is not empty), the nucleolus is $(x_1, x_2, x_3) = \left(\frac{1}{2}(v_{12} + v_{13} - v_{23}), \frac{1}{2}(v_{12} + v_{23} - v_{13}), 0\right)$.[18]

Both the kernel and the Shapley value coincide with the NBS for two-player, TU games. Moreover, each of the two concepts has been shown to be the unique generalization of the NBS, in the sense that each satisfies a different concept of internal consistency (Serrano and Shimomura, 1998; Hart and Mas-Colell, 1989).[19] For the present discussion, the concept of internal consistency means that if $x = (x_1, x_2, x_3)$ is the corresponding solution it satisfies the following property: for any pair of players i, j, (x_i, x_j) is the NBS of a reduced game (N', v'), where $N' = \{i, j\}$ and $v'(N') = x_i + x_j$. The difference between the two consistency criteria lies in what $v'(\{i\})$ and $v'(\{j\})$ are. That is, the disagreement point in the reduced negotiation between i and j. Keeping the normalization $v_i = 0$, for the kernel (and nucleolus, since for these games both concepts coincide), $v'(\{i\}) = \max\{v_{ik} - x_k, 0\}$ (Serrano and Shimomura, 1998), whereas for the Shapley value $v'(\{i\}) = \frac{1}{2}v_{ik}$ (Hart and Mas-Colell, 1989). That is, in both cases if two players i, j bargain over how to share the total that the solution allocates to them, $x_i + x_j$, they still agree on the division (x_i, x_j), provided the disagreement point is as speci-

[18] See Leng and Parlar, 2010.

[19] Compte and Jehiel (2010) define another extension of the NBS, the Coalitional Nash Bargaining Solution, as the allocation that maximizes the product of payoffs in the core. Note that, with three players and $v_{12} = V$, all core allocations give a product of payoffs equal to 0. When the core contains an interior (which requires $V > v_{12}$), the Coalitional Nash Bargaining Solution and the most egalitatian selection of the core coincide. Yet the $R-$solution is not the most egalitarian selection in this case.

fied. This latter point is the crucial difference between the kernel and the Shapley value on one hand, and the $R-$solution on the other. Just like in the concepts proposed by Bennett (1997), neither of the disagreement payoffs for the negotiation between i and j defined above come from a feasible, alternative agreement. For instance, for $i, j = 1, 2$, the disagreement point that sustains the kernel is (v_{13}, v_{23}) and the one that sustains the Shapley value is $(\frac{v_{13}}{2}, \frac{v_{23}}{2})$. Although they imply a different division of the surplus with player 3, in both cases the disagreement payoffs result from player 1 *and also* player 2 "trading" with player 3. But those two trades are mutually exclusive, and in that sense players' expectations are not consistent. Instead, in the $R-$solution, if two players i, j bargain over how to share their total payoff, $U_i + U_j$, they still agree on the division (U_i, U_j) provided that the disagreement point is a lottery over the payoffs $(v_{ik} - U_k, 0)$ and $(0, v_{jk} - U_k)$, where the lottery is part of the solution. That is, the NBS and the $R-$solution are also consistent, but in a way that is itself based on consistently computed disagreement points.[20]

4 Applications

In this section we study in some detail how our solution concept changes the predictions of well-known Industrial Organization models in which bargaining among three players plays a crucial role. We start with a model (Segal and Whinston, 2000) that fits perfectly within the set of games considered in previous sections. Next, we discuss an example (Horn and Persson, 2001) where bilateral agreements generate externalities (the worth of an individual coalition depends on whether or not the other two players reach an agreement). We argue that the $R-$solution can also be applied to this type of games (partition function form) by simply taking into account the value of

[20] In Region 3, all concepts coincide. The reason is that in that case the "feasible" disagreement point and the "infeasible" one lie on the same 45 degree line in the payoff space for any pair i, j.

individual coalitions conditional on the agreement between the other two players. Moreover, in this example the grand coalition cannot form and hence in this case the natural solution concept is not the $R-$solution but the PSBN. Finally, we argue that the ideas contained in the $R-$solution can easily be extended to match the three-player example discussed by Hart and Moore (1990). The main issue in this example is that bilateral trades are not mutually exclusive.

4.1 Exclusive contracts

Segal and Whinston (2000), SW, study the impact of exclusive contracts. Their main insight is that an exclusive contract enhances the ability of the incumbent seller to capture rents in the ex-post bargaining game, but it is irrelevant in protecting his relation-specific investment, unless such investment generates an externality on the entrant. This is a somewhat counter intuitive result that contradicts the conventional wisdom (see, for instance, Klein, 1988, Marvel, 1982, or Masten and Sneyder, 1993).

Here we discuss a version of the model presented in their Section 2. There are three players B, S, and E. Player B (buyer) derives a potential utility of 1 from one unit of the good that can be provided by either S (the incumbent seller) or by E (the entrant). There are three periods: $0, 1$, and 2. In period 0, S and B may or may not sign an exclusive contract. In period 1, player S takes a costly investment decision, $x \in [0, 1]$, which affects the incumbent seller's costs. Also in period 1, once x is fixed, players learn the realization of a random variable $y \in [0, 1]$, which influences the entrant's cost and is distributed according to the cumulative function $H(y)$ and has expectation $\widehat{y}$. In period 2 production and trade take place, and players receive their payoffs. Players S and E can produce one unit of the good at a cost $c_s(x)$ and $c_e(y)$, respectively. For simplicity, we assume $c_s(x) = 1 - x$ and $c_e(y) = 1 - y$. If in period 0 players S and B had signed an exclusive contract, then in period 2 player B cannot purchase from E without S's permission. In both cases, with and without an exclusive contract, B, S, and E bargain in period 2 about who produces the good and how the surplus is distributed.

In their general model SW use a generalization of the Shapley value as the solution concept for the renegotiation in period 2. As an illustration of their ideas let us apply the Shapley value to the above simple version of their model.[21]

In the absence of any contract, the worth of various coalitions is as follows:

$$V = \max\{x, y\}, v_{SB} = x, v_{BE} = y. \tag{3}$$

The rest of coalitions have a worth of 0. Under the exclusive contract, the only difference is that $v_{BE} = 0$.

According to the Shapley value, without exclusivity S's payoff equals $U_S^{ne} = \frac{1}{3}\max\{x - y, 0\} + \frac{x}{6}$. Thus, S's marginal return on investment is $\frac{1}{3}H(x) + \frac{1}{6}$. Note that the marginal return on investment for the pair (B, S) is $\frac{2}{3}H(x) + \frac{1}{3}$. Hence, from the point of view of the pair (B, S) there is underinvestment (the classic hold up problem).

Surprisingly, under the Shapley value an exclusive contract does not help reducing the underinvestment problem. More specifically, under exclusivity player S's payoff is equal to $U_S^e = \frac{1}{3}\max\{x, y\} + \frac{x}{6}$ and the marginal return on investment is also $\frac{1}{3}H(x) + \frac{1}{6}$.[22]

Conclusion 1 *Under the Shapley value, an exclusive contract does not affect investment incentives.*

[21] In their Section 2, SW consider the case of a competitive entrant who is willing to supply the good at a price $p_e = 1 - y$, and given such an outside option players B and S engage in bargaining and the outcome is determined by the NBS. It turns out that exclusivity is also neutral with respect to investment incentives.

[22] The marginal social return on investment is $H(x)$. Hence, the equilibrium level of investment may be below or above the first best level.

It is important to emphasize that the neutrality result hinges on the specific way the Shapley value is computed. An exclusive contract only changes player S's payoff by changing his marginal contribution to the grand coalition. In the absence of exclusivity S's marginal contribution to the grand coalition is $\max\{x-y,0\}$ and under exclusivity it is $\max\{x,y\}$. The difference between these two values is y. Hence, under exclusivity $S's$ payoff increases by $\frac{1}{3}y$ (where $\frac{1}{3}$ is the weight of the grand coalition in payoffs), but S's marginal return on investment remains unchanged.

Let us now analyze the same problem when we use the $R-$solution to predict payoffs in period 2. In this case, player S's payoff is

$$U_s^{ne}=\begin{cases}\frac{x}{2}, \text{ if } y\leq\frac{x}{2}\\ x-y, \text{ if } x\geq y\geq\frac{x}{2}\\ 0, \text{ if } x<y\end{cases}$$

Thus, S's marginal return on investment is $H(x)-\frac{1}{2}H\left(\frac{x}{2}\right)$. Once again, there is underinvestment from the point of view of the pair (B,S): the marginal return on investment for the pair (B,S) is equal to $\min\{H(2x),1\}$.

Under exclusivity, player S's payoff is the same that we found when we used the Shapley value:

$$U_S^e=\begin{cases}\frac{x}{2}, \text{ if } y\leq x,\\ \frac{x}{6}+\frac{y}{3}, \text{ if } y\geq x.\end{cases}$$

Thus, S's marginal return on investment is $\frac{1}{3}H(x)+\frac{1}{6}$. Therefore, under exclusivity investment incentives may be enhanced or depressed with respect to the no contract case.

Note that under exclusivity the $R-$solution and the Shapley value coincide. Hence, we need to understand why these two solution concepts deliver different payoffs in the absence of a contract. In the latter case, if $y>x$ the Shapley value grants player S a payoff of $\frac{x}{6}$. If we think in terms of the sequential arrival interpretation of the Shapley value, such payoff results from the fact that S makes a positive contribution in case he arrives second after player B. However, according to the $R-$solution player S is redundant and should get a zero payoff (he is player 3 and the only thing that he might do is to influence the way y is split between B and E). Thus, under the $R-$solution incentives to invest will be enhanced if $y > x$ is a likely scenario; i.e., if investment costs are relatively high so that x is low. However, if $\frac{x}{2} < y < x$ then player S's marginal contribution to the coalition with B is x (weight $\frac{1}{6}$) and the marginal contribution to the grand coalition is $x - y$ (weight $\frac{1}{3}$). Hence, according to the Shapley value S is able to appropriate one half of his investment efforts. In contrast, the $R-$solution grants player S a payoff of $x - y$ (in this case player S is player 2), and hence he is able to appropriate the entire return on investment. Thus, under the $R-$solution incentives to invest are depressed if $\frac{x}{2} < y < x$ is a likely scenario; i.e., if investment costs are relatively low and x is high. In this case, the paradoxical result obtained by SW is magnified.[23]

Conclusion 2 *Under the $R-$solution, an exclusive contract enhances investment incentives if the cost of investment is relatively high, but the opposite holds if the cost is relatively low.*

In other words, under the $R-$solution exclusivity helps protecting relation-specific investments only when the seller's competitive position is sufficiently weak. Exclusivity is useful only when there is a lot to protect.[24],[25]

4.2 Endogenous mergers

Horn and Persson (2001), HP, present a model of endogenous merger formation. Here we focus on the example discussed in their Section 2.1, which

[23] If $y < \frac{x}{2}$ S's marginal return on investment is equal to $\frac{1}{2}$, under both the Shapley value and the $R-$solution.

[24] If investment costs are sufficiently high (x low), then the level of investment under exclusivity is inefficiently high. In other words, from a social point of view an exclusive contract may actually overprotect relation-specific investments.

[25] De Meza and Selvaggi (2007) also show that SW's conclusions are not robust to changes in the solution concept for the bargaining game. They set up a non-cooperative bargaining game that delivers different predictions than the $R-$solution and show that exclusivity always enhances investment incentives.

considers a market initially populated by three oligopolistic firms, 1, 2, and 3. They are allowed to merge, but not to form a monopoly. In other words, there are four possible market structures: no merger, 1 and 2 merge, 1 and 3 merge, and 2 and 3 merge. Although firms are symmetric before any merger, the synergies generated by alternative mergers are asymmetric. Firms' profits in the no merger case are normalized to 0. Profits of the firm resulting from the merger between firms i and j, and the non-merged firm k are denoted by π_{ij} and π_k respectively and are:

$$\pi_{12} = 70, \pi_3 = 50,$$

$$\pi_{13} = 100, \pi_2 = 0,$$

$$\pi_{23} = 90, \pi_1 = 5.$$

In previous sections we defined the $R-$solution for games in characteristic form, where the value of a coalition is independent of the agreements reached by players not included in the coalition. However, in HP the value of stand-alone coalitions do depend on whether or not the other two players have reached an agreement. Thus, this model can be described as a game in partition function form (Lucas and Thrall, 1963). In a three-player game, we need to specify what player i can obtain if no coalition is formed, $w^i_{\{\{i\},\{j\},\{k\}\}}$, and what player i obtains if the other two players do form a coalition, $w^i_{\{\{i\},\{j,k\}\}}$. Myerson (1977) extended the Shapley value for partition function form games. In his extension, player i's payoff depends on both, $w^i_{\{\{i\},\{j\},\{k\}\}}$ and $w^i_{\{\{i\},\{j,k\}\}}$. On the contrary, the definition of the $R-$solution already takes into account possible externalities. The stand-alone worth plays a role only in the definition of the values t^{ij}_i and t^{ik}_i. These values are obtained as a probability distribution over the events that can be expected as an alternative to i forming coalition with j or k, respectively. The only such event that has i standing alone is the formation of coalition $\{j,k\}$. Thus, only $w^i_{\{\{i\},\{j,k\}\}}$ matters, and then v_i should be interpreted as this value. Summarizing:

Remark 2 *The $R-$solution defined for games in characteristic form can also be applied to games in partition function form, simply by replacing the worth of individual coalitions, v_i, with the worth of individual coalitions conditional on the other two players forming a coalition, $w^i_{\{\{i\},\{j,k\}\}}$.*

The net surplus created by each merger is given by:

$$\pi_{12} - \pi_1 - \pi_2 = 65,$$

$$\pi_{13} - \pi_1 - \pi_3 = 45,$$

$$\pi_{23} - \pi_2 - \pi_3 = 35.$$

Thus, the most efficient merger (from the point of view of firms' profits) is the one between firms 1 and 2. HP use as a solution concept the set of market structures that are not dominated from the point of view of decisive players. In other words, in an *Equilibrium Ownership Structure* (EOS) the sum of profits achieved by all decisive players must be at least as high as in any other market structure. Since in this example all players are decisive when we compare alternative duopolies (all firms have a different position in each possible market structure resulting from a merger) then the only market structure which is undominated is the resulting from the merger between firms 1 and 2. In other words, HP predict that the most efficient market structure will occur with certainty.

Conclusion 3 *Under the notion of Equilibrium Ownership Structure the efficient merger occurs with probability one.*

HP do not allow any transfer between the merged and non-merged firms. Hence, since the grand coalition cannot be formed, we cannot directly apply the $R-$solution to this particular model. However, we can still predict the outcome of simultaneous bilateral negotiations (PSBN). Note that this example lies in Region 3 ($\pi_{12} - \pi_1 - \pi_2 < \pi_{13} - \pi_1 - \pi_3 + \pi_{23} - \pi_2 - \pi_3$). Thus, using the PSBN we predict that the efficient merger will take place with probability less than one. In fact, conditional on being part of the successful coalition, players obtain: $u_1 = 30$, $u_2 = 40$, $u_3 = 60$, and the probability that the merger between 1 and 2 is successful is given by:

$$p_{12} = \frac{(u_1 - \pi_1)(u_2 - \pi_2)}{\sum_{i<j,i,j=1}^{3} (u_i - \pi_i)(u_j - \pi_j)} = \frac{21}{34}.$$

According to our solution concept, the probability that an inefficient merger takes place is almost forty per cent. When all three mergers generate substantial surpluses and side payments to firms not participating in the merger are not feasible, then our theory predicts that there may exist a bargaining coordination problem: any deal that firms 1 and 2 may attempt to strike can be credibly challenged by firm 3. Thus, there are gains from forming the grand coalition and avoiding such coordination problems. However, this would require side payments, which may not be feasible in the context of mergers.

Conclusion 4 *Under the PSBN, any merger (including inefficient ones) may occur with positive probability.*

4.3 Allocation of property rights

Hart and Moore (1990), HM, study how the allocation of property rights over assets affects the ex-post relative bargaining position of different players, which in turn determines ex-ante incentives to undertake asset-specific investments. The example discussed in Section 4.1 is actually closely related to HM's ideas. As noted by SW, their own insights on the neutrality of exclusive contracts can be interpreted as an application of HM's theory. More specifically, assume that there exists one asset a and let $v(Z, a \mid x)$ and $v(Z, \varnothing \mid x)$ be the worth of coalition Z, conditional on investment x, when

Z can and when Z cannot use the asset, respectively. Then, the example discussed in Section 4.1 can be written as follows:

$$v(\{S,B\}, a \mid x) = v(\{S,B\}, \varnothing \mid x) = x,$$

$$v(\{E,B\}, a \mid x) = y,$$

$$v(\{S,B,E\}, a \mid x) = \max\{x,y\}, \tag{4}$$

and $v(\cdot,\cdot \mid x) = 0$ for all x in all other cases. That is, B's ownership of the asset is equivalent to the no contract situation in SW, and S's ownership corresponds to S and B having signed an exclusivity contract. In the language of HM, player B is an indispensable player: the marginal return on investment for members of coalitions that do not contain B is independent on whether or not the asset is used. Also, player S is the only one who takes an investment decision. According to Propositions 2 and 6 in HM, incentives to invest are identical when player B and when player S have the property rights over the asset. As shown in Section 4.1 this result holds when we use the Shapley value as the solution concept, but not when we use the $R-$solution.[26]

Let us take this discussion one step further. Suppose that we replace (4) by:

$$v(\{S,B,E\}, a \mid x) = x + y. \tag{5}$$

We can interpret this game as one where the cost of producing the good can be reduced with respect to the cost of either of the two sellers if the two of them cooperate. In this game, the same conclusions apply: HM's results are not robust to the application of the $R-$solution, and incentives to invest depend on whether the indispensable player, B, or the investor, S, own the asset. However, this game admits a different interpretation along the lines of HM's famous introductory example. Indeed, as in HM, let the three players

[26] De Meza and Loockwood (1998) and Chiu (1998) also show that HM's conclusions are not robust to changes in the solution concept for the bargaining game.

B, S, and E be a tycoon, a chef, and a skipper, respectively. Also, let the single asset be a yacht. The chef is able to offer a service in the yacht worth x to the tycoon. Similarly, the skipper is able to offer a service in the yacht worth y to the tycoon. Hence, there are two possible trades. If these two trades are mutually exclusive (the tycoon can enjoy a dinner in the yacht or enjoy sailing, but not both) then the value of the grand coalition is well represented by (4) and, under the $R-$solution, investment incentives change when we transfer property rights between the tycoon and the skipper. On the other hand, it could be that these two trades are not incompatible, and may take place simultaneously, as HM assume in their introduction. The difference would be irrelevant if we apply the Shapley value, since all relevant information is already contained in the characteristic function, which has not changed. More specifically, under (5) if the tycoon owns the yacht then the Shapley value grants the chef a payoff of $\frac{1}{2}x$, and if the chef owns the yacht he receives a payoff of $\frac{1}{2}x + \frac{1}{3}y$. Consequently:

Conclusion 5 *Under the Shapley value investment incentives do not change if we transfer property rights from an indispensable player to the player undertaking investment.*

Shouldn't payoffs depend on whether the two trades are compatible or not? The $R-$solution, as defined in Section 2, is a solution concept only for games where bilateral trades are mutually exclusive. In an article in progress, we extend the definition to more general cases, that include this example by HM with compatible bilateral trades. The main generalization is the definition of *feasible events*, a subset of the power set of $\{(1,2),(1,3),(2,3)\}$. An event describes the outcome of bilateral negotiations. For instance, the event $[(1,2),(1,3)]$ corresponds to 1 agreeing to trade with 2 *and* 1 also agreeing to trade with 3. If we represent with square brackets the events, in this paper the set of feasible events was $\{[(1,2)],[(1,3)],[(2,3)]\}$. In the example of HM, the set of events is $\{[(1,2),(1,3)],[(1,2)],[(1,3)],[(2,3)]\}$). That is, it contains the new event $[(1,2),(1,3)]$. In a case like that, the extended $\epsilon-$PSBN would still be a triple $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p(\epsilon)\right\}$ except that $p(\epsilon)$ is a probability distribution over the set of feasible events. That is, over four events. For our present purpose, let us keep the notation $p_{ij}(\epsilon)$ for the event $[(i,j)]$, and add $p(\epsilon)$ to denote the probability of event $[(1,2),(1,3)]$. Then, condition 1 in the definition of $\epsilon-$PSBN remains unchanged. For condition 2 we need to introduce the concept of *mutually excluding* pairs. Pairs (i,j) and (k,l) are mutually excluding if there is no feasible event that contains both. In the previous sections, all three pairs were mutually excluding with each other. In the present example, however, $(1,2)$ and $(2,3)$ are mutually excluding, as $(1,3)$ and $(2,3)$, but $(1,2)$ and $(1,3)$ are not. The relation is symmetric. Then condition 2 in the definition of an $\epsilon-$PSBN reads that $t_i^{ij}(\epsilon)$ equals the expected payoff for player i in pairs that are mutually excluding with (i,j), conditional on events that do not contain (i,j). In the present example, dropping the index ϵ for clarity,

$$t_i^{1i} = \frac{p_{23}u_i^{23}}{1-(p+p_{1i})},$$

for $i=2,3$. On the other hand, since $(1,3)$ and $(1,2)$ are not mutually excluding, and 1 is not in the pair $(2,3)$, we have $t_1^{1i}=0$. Also,

$$t_i^{23} = \frac{(p+p_{1i})u_i^{1i}}{1-p_{23}},$$

for $i=2,3$.

Finally, condition 3, still dropping the index ϵ, should now include $p_{12}+p_{13}+p_{23}+p=1$ and the probability of a *trade* between players i and j cannot be higher than $1-\epsilon$ for all i,j.[27] But now, the probability of an event should not be larger than ϵ if all players that are part of a pair included in

[27] In particular, $p+p_{12}, p+p_{13}$, and p_{23} all must be lower or equal than $1-\epsilon$.

that event obtain a higher payoff (one of them strictly so) in some other event. In the present example $u_1^{12} + u_1^{13} \geq u_1^{1i}$ for $i = 1, 2$. Thus, $p \leq \epsilon$ if $u_1^{12} + u_1^{13} = u_1^{1i}$ (which implies $u_1^{1j} = 0$ for $j \neq i$), and $u_i^{1i} < u_i^{23}$ for $i = 2, 3$. Also, $p_{23} \leq \epsilon$ if $u_2^{12} \geq u_2^{23}$ and $u_3^{13} \geq u_3^{23}$, with one strict inequality. Finally, $p_{1i} \leq \epsilon$ for $i = 1, 2$ if $u_1^{1j} > 0$ for $j \neq i$, or $u_1^{12} = u_1^{13} = 0$ and $u_i^{23} > u_i^{1i}$.

With this definition of an $\epsilon-$PSBN, we can define the PSBN and the $R-$solution for this more general case, just as in Section 2.

Remark 3 *If the two most efficient trades* $\{(1,2), (1,3)\}$ *can occur simultaneously and* $V = v_{12} + v_{13}$*, then the* $R-$*solution exists and is unique. In particular,* $p = 1$*,* $U_1 = \frac{v_{12}+v_{13}}{2}$*, and* $U_i = \frac{v_{1i}}{2}$*, for* $i = 2, 3$*.*

The proof can be found in the Appendix.

Now we are ready to apply the $R-$solution to the example in HM with non-exclusive bilateral trades. It turns out that, in this case, the $R-$solution and the Shapley value offer the same prediction:

Conclusion 6 *Under the* $R-$*solution and two compatible trades, investment incentives do not change when we transfer property rights from the indispensable player to the one undertaking investment.*

In contrast to the Shapley value, in order to apply the $R-$solution we need a more complete description of the game: besides the characteristic function we need to know which bilateral trades are compatible. In the previous sections we focused in the simplest case where all bilateral trades are mutually excluding. In this section we have shown that the ideas behind the $R-$solution can be extended to encompass more general situations. The payoffs predicted by the $R-$solution are sensitive to the set of alternatives to the grand coalition, i.e., sensitive to which bilateral trades are compatible and which ones are not. In particular, this set of alternatives to the grand coalition will affect whether HM's results hold or not under the $R-$solution.

5 Concluding remarks

In this paper we provide a new solution concept for three-player bargaining games, the $R-$solution, that can be interpreted as a generalization of the NBS with endogenous disagreement points. In particular, our solution identifies the outcomes of negotiations within any possible coalition, not only the grand coalition, and disagreement points in each negotiation are determined by consistent conjectures on the consequences of suspending that negotiation. These conjectures include a (probabilistic) prediction of what alternative negotiation will succeed in that case. We show that the $R-$solution always exists and is unique. Moreover, it belongs to the core if it is non-empty. In general, it belongs to the Aumann-Machler's bargaining set.

It turns out that the consistency requirement behind the $R-$solution causes this concept to conform with the "outside option principle" (see Binmore, Shaked, and Sutton, 1989) for bilateral negotiations. Indeed, when her endogenously determined fallback option is binding in a bilateral negotiation, a player's payoff in that negotiation is predicted to coincide with that fallback option. Otherwise, the option does not affect her payoff. Thus, although the noncooperative implementation of the $R-$solution is beyond the scope of this paper, protocols in the spirit of this principle should be appropriate instruments for this goal.[28]

Our solution concept is motivated by simultaneous bilateral negotiations that are mutually exclusive. That is, we allow for the grand coalition to add value, but absent an agreement in the grand coalition, the alternative is that one, and no more than one, two-player coalition forms and realizes its worth. It is not difficult to think of examples where this is not the case. We have discussed one such case, the famous skipper-chef-tycoon example in Hart and

[28] For instance, it is easy to check that in the three-player/three-cake game when $v_{23} = 0$, the "auctioning" protocol discussed in Binmore, Osborne and Rubinstein (1992) implements the $R-$solution as the limit when delay approaches zero.

Moore (1990) paper. In the spirit of our solution concept, the set of possible events when the grand coalition fails to form should influence the way parties share the surplus if the grand coalition does form. In a companion paper, we extend the $R-$solution for games with any such set of possible events. In general, the information contained in the characteristic function of a game is not sufficient to determine that set. Therefore concepts that are defined on only the information contained in the characteristic function, like the Shapley value, will be insensitive to variations in the set of possible events.

The study of games involving more than three players poses new questions that are not present in the current analysis. One set of such questions has to do with the hierarchy of coalitions and is related to the discussion in the previous paragraph. As we have just mentioned, in this paper we have assumed that if the grand coalition breaks down then only one trade between two players can be realized. In fact, there are three alternative two-player coalitions and each one of them is expected to strike a deal with certain probability. Therefore, computing the fallback option of each player in each coalition is relatively straightforward. However, in a four-player game, if the grand coalition fails then the relevant alternatives are not so easy to obtain even if we impose that only disjoint coalitions can form. The alternative to the grand coalition may be a one three-player coalition, excluding the fourth player but it may also be two disjoint two-player coalitions. Specifying the fallback option of a particular player in an arbitrary coalition can still be done along the lines discussed in Subsection 4.3, but it involves a higher degree of complexity. We leave the analysis of games with more than three players for future research.

6 References

Bennett, E. (1997), "Multilateral Bargaining Problems", *Games and Economic Behavior* 19, 151-179.

Binmore, K. (1985) "Bargaining and Coalitions." Chapter 13 in *Game-theoretic Models of Bargaining*, ed. Alvin Roth. Cambridge: Cambridge University Press, 269-304.

Binmore, K., A. Shaked, and J. Sutton (1989) "An Outside Option Experiment", *Quarterly Journal of Economics* 104 (4), 753-770.

Binmore, K., M. Osborne, and A. Rubinstein (1992) "Noncooperative Models of Bargaining", Chapter 7 in *Handbook of Game Theory*, Vol. 1, ed. R. Aumann and S. Hart, Elsevier Science Publishers, 179-225.

Burguet, R., R. Caminal, and C. Matutes (2002), "Golden cages for showy birds: optimal switching costs in labor contracts", *European Economic Review* 67 (7), 1153-1185.

Compte, O., and P. Jehiel (2010), "The Coalitional Nash Bargaining Solution", *Econometrica*, 78 (5), 1593-1623.

Chiu, S. (1998), "Noncooperative Bargaining, Hostages, and Optimal Asset Ownership", *American Economic Review* 88 (4), 882-901.

De Meza, D. and B. Lookwood (1998), "Does Ownership Always Motivate Managers? Outside Options and the Property Rights Theory of the Firm", *Quarterly Journal of Economics* 113 (2), 361-386.

De Meza, D. and M. Selvaggi (2007), "Exclusive Contracts Foster Relationship-Specific Investment", *The RAND Journal of Economics* 38 (1) (Spring), 85-97.

Grossman, S., and O. Hart (1986), “The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration”, *Journal of Political Economy* 94, 691-719.

Hart, O. and J. Moore (1990), "Property Rights and the Nature of the Firm", *Journal of Political Economy* 98 (6), 1119-1158.

Hart, S. and A. Mas-Colell (1989), "Potential, Value, and Consistency", *Econometrica* 57 (3), 589-614.

Horn, H. and L. Persson (2001), "Endogenous mergers in concentrated markets", *International Journal of Industrial Organization* 19, 1213-1244.

Klein, B. (1988), "Vertical Integration as Organizational Ownership: The Fisher Body-General Motors Relationship Revisited." *Journal of Law, Economics and Organization* 4, 199-213.

Krishna, V. and R. Serrano (1996), "Multilateral Bargaining", *Review of Economic Studies* 63, 61-80.

Leng, M. and M. Parlar (2010), "Analytic Solution for the Nucleolus of a Three-Player Cooperative Game", *Naval Research Logistics* 57, 667672.

Lucas, W., and R. Thrall (1963), "n-Person Games in Partition Function Form", *Naval Research Logistics Quarterly* 10, 281-298.

Marvel, H.P (1982), "Exclusive Dealing." *Journal of Law and Economics* 25, 1-25.

Masten, S.E. and E.A. Snyder (1993), "United States versus United Shoe Machinery Corporation: On the Merits." *Journal of Law and Economics* 36, 33-70.

Myerson, R. (1977), "Values of Games in Partition Function Form", *International Journal of Game Theory* 6 (1), 23-31.

Segal, I. and M. Whinston (2000), "Exclusive Contracts and Protection of Investments", *The RAND Journal of Economics* 31, (4) (Winter), 603-633.

Serrano, R. and K. Shimomura (1998), Beyond Nash Bargaining Theory: The Nash Set, *Journal of Economic Theory* 83, 286-307.

Shaked, A. and J. Sutton (1984), "Involuntary unemployment as a perfect equilibrium in a bargaining model", *Econometrica* 52, 1351-1364.

Spier, K. and M. Whinston (1995), On the Efficiency of Privately Stipulated Damages for Breach of Contract: Entry Barriers, Reliance, and Renegotiation, *The RAND Journal of Economics* 26, (2) (Summer), 180-202.

Winter, E (2002) "The Shapley value", chapter 53 in R.J. Aumann & S. Hart (ed.) *Handbook of Game Theory with Economic Applications*, 2025-

2054.

Weber, R. (1988) "Probabilistic Values for Games", chapter 7 in A. E. Roth (ed.) *The Shapley value: essays in honor of Lloyd S. Shapley*, 101-120.

7 Appendix

7.1 Proof of Proposition 1:

First we propose an $\epsilon-$PSBN for the game (N, v) for ϵ small enough. This will show existence. To save in notation, we will dispose of the (ϵ) index of the solution, and specify if we refer to the limit instead.

1) Let $\frac{1}{2}v_{12} \geq v_{13}$.

1.a) If $\frac{1}{2}v_{12} > v_{13}$ (so that $v_{12} \geq v_{13} + v_{23}$ is also satisfied), consider $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$, and $u_i^{ij} = 0$ for all other values of i, j. Also, let $p_{12} = 1-\epsilon$, $p_{13} = p_{23} = \frac{\epsilon}{2}$. Finally, let $t_1^{12} = t_2^{12} = t_3^{i3} = 0$ and $t_i^{i3} = \frac{1-\epsilon}{2-\epsilon}v_{12}$ for $i = 1, 2$. Note that $\lim_{\epsilon \to 0} \frac{1-\epsilon}{2-\epsilon}v_{12} = \frac{1}{2}v_{12} > v_{13} \geq v_{13}$. Thus, for ϵ sufficiently small, this satisfies the definition of an $\epsilon-$PSBN.

1.b) If $\frac{1}{2}v_{12} = v_{13} > v_{23}$, consider $u_1^{12} = u_1^{12} = \frac{1}{2}v_{12}$ $(= v_{13})$, and $u_2^{23} = u_3^{23} = 0$. Also, let $p_{12} = 1 - \epsilon$ and $p_{13} = 0$, $p_{23} = \epsilon$. Then, $t_1^{12} = t_3^{23} = t_2^{12} = t_3^{13} = 0$ and $t_2^{23} = \frac{1}{2}v_{12} > v_{23}$. To complete the definition of an $\epsilon-$PSBN we need only $t_1^{13} = (1-\epsilon)\frac{1}{2}v_{12}$, $u_1^{13} = \frac{1}{2}(v_{13} + t_1^{13} - t_3^{13}) = (1-\frac{\epsilon}{2})v_{13}$ and $u_3^{13} = \frac{\epsilon}{2}v_{13}$.

1.c) If $v_{13} = v_{23} = \frac{1}{2}v_{12}$, consider $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$ $(= v_{i3},\ i = 1, 2)$, $p_{12} = 1 - \epsilon$ and $p_{13} = p_{23} = \frac{\epsilon}{2}$. Then $t_i^{i3} = \frac{(1-\epsilon)v_{12}}{2-\epsilon} < v_{i3}$, $i = 1, 2$. Also, consider $u_3^{13} = u_3^{23} = A > 0$. Thus, t_3^{i3}, $i = 1, 2$, will have to satisfy:

$$t_3^{i3} = \frac{\epsilon A}{2-\epsilon}, \text{ and}$$

$$A = \frac{1}{2}\left(v_{i3} - \frac{(1-\epsilon)v_{12}}{2-\epsilon} + \frac{\epsilon A}{2-\epsilon}\right),$$

and solving for A taking into account that $\frac{1}{2}v_{12} = v_{i3}$, we obtain

$$A = \frac{\epsilon v_{i3}}{4-3\epsilon},$$

which is smaller than v_{i3} for small ϵ. Note that for ϵ small $t_3^{i3} + t_i^{i3} < v_{i3}$, $i = 1, 2$. Also, note that given these values for u_i^{i3}, we should define $t_1^{12} = t_2^{12} = \frac{\frac{\epsilon}{2}(v_{i3}-A)}{\epsilon} = \frac{(v_{12}-2A)}{4}$, and $t_1^{12} + t_2^{12} < v_{12}$. This satisfies the definition of an $\epsilon-$PSBN.

2) If $v_{12} \geq v_{13} + v_{23}$ but $v_{13} > \frac{1}{2}v_{12}$, then consider $u_1^{12} = u_1^{13} = u_1$, to be obtained later, with $0 < u_1 < v_{13}$, and $u_2^{23} = u_3^{23} = 0$. Thus, $u_2^{12} = v_{12} - u_1 > u_2^{23}$ and $u_3^{13} = v_{13} - u_1 > u_3^{23}$. Consequently, let $p_{23} = \epsilon$. Then

$p_{12} = 1 - \epsilon - p_{13}$. Finally, $u_2^{23} = u_3^{23} = 0$ implies that $t_2^{12} = t_3^{13} = 0$, and we can then check that $t_2^{12} + t_2^{12} < v_{12}$, whereas

$$\begin{aligned} t_1^{13} + t_3^{13} &= \frac{p_{12}u_2^{12}}{1-\epsilon} + \frac{p_{13}u_3^{13}}{1-\epsilon} \\ &= (v_{12} - u_1) - \frac{p_{13}(v_{12} - v_{13})}{1-\epsilon}. \end{aligned}$$

We will propose u_1 sufficiently close to v_{13} so that $t_1^{13} + t_3^{13} \leq v_{13}$. In that case, u should satisfy

$$u_1 = \frac{1}{2}(v_{13} + \frac{(1-\epsilon-p_{13})u_1}{1-p_{13}}) = \frac{1}{2}(v_{12} + \frac{p_{13}u_1}{\epsilon + p_{13}}).$$

This is a system of two equations with two unknowns. Note that if we have a (valid) solution to this system, then as ϵ approaches 0 the first equation approaches $u_1 = \frac{1}{2}(v_{13} + u_1)$ whose only solution is $v_{13} = u_1$. (For positive ϵ, indeed $u_1 < v_{13}$.) Thus, for ϵ small enough, $t_1^{13} + t_3^{13} < v_{12} - u_1 = v_{12} - 2v_{13} + v_{13} + (v_{13} - u_1)$ and the right hand side converges to $v_{12} - 2v_{13} + v_{13} < v_{13}$. Also, solving for u_1, we can write the system as

$$v_{13}\left(1 + \frac{\epsilon}{p_{13} + \epsilon}\right) = v_{12}\left(1 + \frac{\epsilon}{1 - p_{13}}\right).$$

This is a quadratic equation in p_{13} with one positive root that converges to 0 as ϵ converges to zero. Thus, we have an $\epsilon-$PSBN for ϵ small enough. And for ϵ small, p_{12} is close to 1.

3) If $v_{12} < v_{13} + v_{23}$, then propose $u_i^{ij} = u_i^{ik} = u_i > 0$, for all i, j, k. Then the definition of u_i^{ij} requires that $u_i + u_j = v_{ij}$ for all i, j. This is a system of three linear (independent) equations with solution $u_i = \frac{v_{ij} + v_{ik} - v_{jk}}{2}$. Also, $t_i^{ij} = \frac{p_{ik}u_i}{1-p_{ij}}$. Finally, p should satisfy

$$u_i = \frac{1}{2}(v_{ij} + \frac{p_{ik}u_i}{p_{ik} + p_{jk}} - \frac{p_{jk}u_j}{p_{ik} + p_{jk}})$$

for all i, j, k. Taking into account $u_i + u_j = v_{ij}$, these equations can be written as

$$\begin{aligned} -p_{13}u_2 + p_{23}u_1 &= 0, \\ -p_{12}u_3 + p_{13}u_2 &= 0, \\ -p_{12}u_3 + p_{23}u_1 &= 0. \end{aligned}$$

Note that the third equation is simply the sum of the previous two. That is, there are only two linearly independent equations. Thus, two of these equations plus $p_{13} + p_{23} + p_{23} = 1$ form a linear system with a unique solution. The solution is a probability distribution, since all three variables take positive values. Indeed, the first two equations can be written as $\frac{p_{13}}{u_1} =$

$\frac{p_{23}}{u_2}$ and $\frac{p_{12}}{u_2} = \frac{p_{13}}{u_3}$, so that all solution vectors to these two equations have either all positive components or all negative. And no solution with all negative components satisfies the equation $p_{13} + p_{23} + p_{23} = 1$. Finally, note that $t_j^{ij} + t_i^{ij} = \frac{p_{jk} u_j}{p_{jk} + p_{ik}} - \frac{p_{ik} u_i}{p_{jk} + p_{ik}}$, so that since both $u_j, u_i < v_{ij}$, indeed $t_j^{ij} + t_i^{ij} < v_{ij}$.

This concludes the proof of existence. Next, we can simply check that if we select the $\epsilon-$PSBN that we have just characterized for each possible values of v_{ij} for all ij, then the $\lim_{\epsilon \to 0} \{u(\epsilon), t(\epsilon), p(\epsilon)\}$ is as stated in the Proposition. Thus, we only need showing that there is no other triple $\{u, t, p\}$ that is the limit of a sequence of $\epsilon-$PSBN as ϵ approaches 0. First we prove a handy result.

Lemma 2 *In a $\epsilon-$PSBN, cycles cannot occur. That is, it cannot be that $u_i^{ij} \geq u_i^{ik}$; $u_j^{jk} \geq u_j^{ij}$; $u_k^{ik} \geq u_k^{jk}$ for some values of i, j, k. Moreover, $u_i^{ij} = u_i^{ik}$; $u_j^{jk} = u_j^{ij}$; $u_k^{ik} = u_k^{jk}$ can only occur if $v_{12} \leq v_{13} + v_{23}$.*

Proof of Lemma: First, assume that we have such cycle with at least one strict inequality, and such that $t_i^{ij} + t_j^{ij} \leq v_{ij}$ for all ij. In any such cycle, $u_i^{ij} = \frac{1}{2}\left(v_{ij} + t_i^{ij} - t_j^{ij}\right)$ for all i, j, k. Substituting for $v_{ij} = u_i^{ij} + u_j^{ij}$, and also substituting for

$$t_i^{ij} = \frac{p_{ik}}{1 - p_{ij}} u_i^{ik} \tag{6}$$

we can write this expression as

$$(u_i^{ij} - u_j^{ij})(1 - p_{ij}) = p_{ik} u_i^{ik} - p_{jk} u_j^{jk} \tag{7}$$

Adding these three equations, for all three pairs, this implies that

$$(u_i^{ij} - u_j^{ij}) + (u_k^{ik} - u_i^{ik}) + (u_j^{jk} - u_k^{jk}) = 0,$$

that is, $u_i^{ij} + u_k^{ik} + u_j^{jk} = u_j^{ij} + u_i^{ik} + u_k^{jk}$, which violates the inequalities defining the cycle if there is one that is strict.

Second, assume that $t_i^{ij} + t_j^{ij} > v_{ij}$ for some ij, but $t_i^{ik} + t_k^{ik} \leq v_{ik}$, and $t_j^{jk} + t_k^{jk} \leq v_{jk}$. Given the cycle, this implies that $u_i^{ij} = u_j^{ij} = u_i^{ik} = 0$, so that also $u_k^{ik} = v_{ik}$. Thus, equations (7) for the pair jk become

$$(u_j^{jk} - u_k^{jk})(1 - p_{jk}) = -p_{ik} v_{ik}.$$

Since $p_{jk} < 1$, that implies $u_k^{jk} \geq u_j^{jk}$. Note, however, that $t_j^{jk} = 0$, since $u_j^{ij} = 0$, so that $u_j^{jk} \geq u_k^{jk}$. These two inequalities then imply both $u_j^{jk} = u_k^{jk} = \frac{v_{jk}}{2}$, and $p_{ik} = 0$. Since the cycle inequalities include $u_k^{ik} \geq u_k^{jk}$, then we must have $v_{ik} \geq \frac{v_{jk}}{2}$. But substituting for $u_k^{jk} = \frac{v_{jk}}{2}$ and $p_{ik} = 0$ in

(6) corresponding to t_k^{ik}, we also have that $t_k^{ik} = p_{jk}\frac{v_{jk}}{2} < \frac{v_{jk}}{2} \leq v_{ik}$. This contradicts that $u_k^{ik} = v_{ik}$.

Third, assume that $t_i^{ij} + t_j^{ij} > v_{ij}$ and $t_i^{ik} + t_k^{ik} > v_{ik}$ for some ij and ik but $t_j^{jk} + t_k^{jk} \leq v_{jk}$. That implies that $u_i^{ij} = u_j^{ij} = u_i^{ik} = u_k^{ik} = 0$, which implies that $t_j^{jk} = t_k^{jk} = 0$, so that $u_k^{jk} = \frac{v_{jk}}{2} > u_k^{ik}$, which contradicts the inequalities in the cycle.

Thus, the only cycle that may exist is $u_i^{ij} = u_i^{ik}$; $u_j^{jk} = u_j^{ij}$; $u_k^{ik} = u_k^{jk}$, with $t_i^{ij} + t_j^{ij} \leq v_{ij}$ for all ij. But the system $u_i + u_j = v_{ij}$, for all ij has a valid solution only in Region 3, and coincides with the one found above. QED

Thus, an $\epsilon-$PSBN must satisfy:

$$u_i^{ij} \geq u_i^{ik}; u_j^{jk} \leq u_j^{ij}; u_k^{ik} \leq u_k^{jk}, \tag{8}$$

and except for the one we used in 3) above, at least two inequalities must be strict. Also, given part three of the definition of $\epsilon-$PSBN, $p_{ik} < \epsilon$ unless $u_i^{ij} = u_i^{ik}$ and $u_k^{ik} = u_k^{jk}$. Thus, in any but the $\epsilon-$PSBN constructed in 3) above, $p_{ik} < \epsilon$. Thus, in a sequence that converges as $\epsilon \to 0$, we must have $\lim_{\epsilon\to 0} p_{ik} = 0$.

Consider such a sequence of $\epsilon-$PSBN so that $\lim_{\epsilon\to 0} p_{ij} > 0$ and $\lim_{\epsilon\to 0} p_{jk} > 0$. From (8) and part three of the definition of $\epsilon-$PSBN, that implies that for ϵ small $u_j^{ij} = u_j^{jk}$. Thus, since at least two inequalities need to be strict, $u_i^{ij} > u_i^{ik}$ and $u_k^{ik} < u_k^{jk}$. These last inequalities imply that $u_k^{jk} + u_j^{jk} = v_{jk}$ and $u_i^{ij} + u_j^{ij} = v_{ij}$. Also, as ϵ approaches 0, $\frac{p_{jk}}{p_{jk}+p_{ik}}$ approaches 1, as does $\frac{p_{ij}}{p_{ij}+p_{ik}}$, so that applying part one of the definition of a $\epsilon-$PSBN,

$$u_j^{ij} = u_j^{jk} \to u_j = \frac{1}{2}(v_{ij} + u_j) = \frac{1}{2}(v_{jk} + u_j).$$

This cannot occur unless $v_{ij} = v_{jk}$. In the latter case, $u_j = v_{ij} = v_{jk}$, which implies that both u_i^{ij} and u_k^{jk} converge to 0, and so $t_i^{ik} + t_k^{ik}$ converges to 0, in which case u_i^{ik} converges to $\frac{v_{ik}}{2} > u_i^{ij}$ for ϵ small and when $v_{ik} > 0$. This is a contradiction unless $v_{ik} = 0$. But if $v_{ij} = v_{jk}$, $v_{ik} = 0$, the limit of such a sequence coincides with the $\epsilon-$PSBN constructed in 3) above.

Thus, we must have that both $\lim_{\epsilon\to 0} p_{ik} = 0$, and either $\lim_{\epsilon\to 0} p_{jk} = 0$ or $\lim_{\epsilon\to 0} p_{ij} = 0$. But if $\lim_{\epsilon\to 0} p_{ij} = 0$ then $\lim_{\epsilon\to 0} p_{jk} > 0$, and this contradicts part 3 of the definition of an $\epsilon-$PSBN since $u_i^{ij} \geq u_i^{ik}$ and $u_j^{ij} \geq u_j^{jk}$ with at least one inequality. Thus, assume that $\lim_{\epsilon\to 0} p_{ik} = \lim_{\epsilon\to 0} p_{jk} = 0$. We consider two possible cases:

1) Assume that $t_j^{jk} + t_k^{jk} > v_{jk}$ in all the terms of the sequence[29] as ϵ converges to 0, so that $u_j^{jk} = u_k^{jk} = 0 = t_j^{ij}$, for each ϵ small enough in the

[29] Note, in general, that except in trivial cases, either this is satisfied for ϵ close to 0 or else the sequence cannot converge.

sequence considered. Thus, from (8), we must also have that $u_k^{ik} = 0$. Since only one inequality in (8) may be non strict, and $u_k^{jk} = u_k^{ik}$ we must have $u_i^{ij} > u_i^{ik}$, and since $u_j^{jk} = 0$, we must also have that $u_j^{ij} > u_j^{jk}$. These two inequalities imply that $u_i^{ij} + u_j^{ij} = v_{ij}$. Since $t_j^{ij} = 0$, we must then have that $u_i^{ij} \geq \frac{v_{ij}}{2}$. Thus:

1.a) If $\frac{v_{ij}}{2} > v_{ik}$, since t_i^{ik} converges to $u_i^{ij} \geq \frac{v_{ij}}{2}$, then for ϵ small we must also have that $t_i^{ik} + t_k^{ik} > v_{ik}$, so that $u_i^{ik} = 0$, and then $t_j^{ij} = t_i^{ij} = 0$, and then $u_j^{ij} = u_i^{ij} = \frac{v_{ij}}{2}$. Note that $t_j^{jk} \leq u_j^{ij}$ and $t_k^{jk} = 0$. Thus, for $t_j^{jk} + t_k^{jk} > v_{jk}$, it must be that $\frac{v_{ij}}{2} > v_{jk}$. This requires that $ij = 1, 2$ and also that we are in Region 1. Thus, the limit of such sequence is the one stated in the Proposition.

1.b) If $\frac{v_{ij}}{2} \leq v_{ik}$, as before, if $t_i^{ik} + t_k^{ik} > v_{ik}$, then $u_j^{ij} = \frac{v_{ij}}{2}$, and since $t_k^{ik} = 0$, this would imply that $\frac{v_{ij}}{2} \geq t_i^{ik} > v_{ik}$ which is a contradiction. Thus, we must have $t_i^{ik} + t_k^{ik} \leq v_{ik}$. Thus, since $u_i^{ik} = 0$, we must have $u_k^{ik} = v_{ik}$. Since $u_k^{jk} = 0$, this contradicts the inequality $u_k^{ik} \leq u_k^{jk}$ in (8) unless $v_{ik} = 0$. In the latter case, since $\frac{v_{ij}}{2} \leq v_{ik}$, $v_{ij} = 0$ and we have a contradiction with $t_j^{jk} + t_k^{jk} = 0 > v_{jk}$.

2) Assume that $t_j^{jk} + t_k^{jk} \leq v_{jk}$ in all the terms of the sequence as ϵ converges to 0.

2.a) If $t_i^{ij} + t_j^{ij} \leq v_{ij}$, then

$$u_j^{jk} \geq t_j^{jk} = \frac{p_{ij}}{p_{ij} + p_{ik}} u_j^{ij},$$

where the right hand side converges to u_j^{ij}. From, (8), $u_j^{ij} \geq u_j^{jk}$. Thus, the limit of any such sequence should satisfy $\lim_{\epsilon \to 0} u_j^{ij} = \lim_{\epsilon \to 0} u_j^{jk} = \lim_{\epsilon \to 0} t_j^{jk}$. That implies that $\lim_{\epsilon \to 0} u_k^{jk} = 0$, and requires that $v_{ij} \geq v_{jk}$, and $\lim_{\epsilon \to 0} u_i^{ij} = v_{ij} - v_{jk}$. Since $u_k^{jk} \geq u_k^{ik}$, then we also have $\lim_{\epsilon \to 0} u_k^{ik} = 0$. But if $\lim_{\epsilon \to 0} u_k^{ik} = 0$, then $\lim_{\epsilon \to 0} t_k^{ik} = 0$, whereas $t_i^{ik} \leq u_i^{ik}$. Thus, if $v_{ik} > v_{ij} - v_{jk}$, then $\lim_{\epsilon \to 0} t_i^{ik} + t_k^{ik} < v_{ik}$ and then $\lim_{\epsilon \to 0} u_k^{ik} > \frac{v_{ik} - (v_{ij} - v_{jk})}{2} > 0$, which is a contradiction. Therefore, $v_{ik} \leq v_{ij} - v_{jk}$. Since $\lim_{\epsilon \to 0} t_i^{ik} = \lim_{\epsilon \to 0} u_i^{ij} = v_{ij} - v_{jk}$, then $\lim_{\epsilon \to 0} u_k^{ik} = 0 = \lim_{\epsilon \to 0} u_k^{jk}$. Thus, $\lim_{\epsilon \to 0} u_i^{ik} > 0$, only if $\lim_{\epsilon \to 0} u_i^{ik} = v_{ij} - v_{jk}$. In this case, we would have $u_i = \lim_{\epsilon \to 0} u_i^{ij} = \lim_{\epsilon \to 0} u_i^{ik}$, $u_k = \lim_{\epsilon \to 0} u_k^{ik} = \lim_{\epsilon \to 0} u_k^{jk}$, and $u_j = \lim_{\epsilon \to 0} u_j^{ij} = \lim_{\epsilon \to 0} u_j^{jk}$. This equation, together with $u_i + u_k = v_{ik}$, $u_i + u_j = v_{ij}$ $u_j + u_k = v_{jk}$ has a solution only in Region 3 ($v_{ik} = v_{ij} - v_{jk}$). Thus, if $v_{ik} < v_{ij} - v_{jk}$, $u_i^{ik} = 0$ for ϵ small, so that $t_i^{ik} = 0$, so that $u_j^{ij} \geq \frac{v_{ij}}{2}$, and then $v_{jk} \geq \frac{v_{ij}}{2}$. This is Region 2, and the limit coincides with the one stated in the Proposition.

2.b) If $t_i^{ij} + t_j^{ij} > v_{ij}$, then u_i^{ij} $(= u_j^{ij}) = 0$, so that $t_i^{ik} = 0$, and since from (8) $u_i^{ij} \geq u_i^{ik}$, then $u_i^{ik} = 0$. On the other hand, t_k^{ik} approaches 0 as $\epsilon \to 0$, and then $t_i^{ik} + t_k^{ik}$ approaches 0, which contradicts $u_i^{ik} = 0$ unless $v_{ik} = 0$. Moreover in this latter case $t_j^{jk} = t_k^{jk} = 0$, so that $u_j^{jk} = u_k^{jk} = \frac{v_{jk}}{2}$, so that $u_j^{jk} > u_j^{ij}$ which contradicts (8).

7.2 Proof of Lemma 1

Without loss of generality, assume that $v_i = 0$, for all $i = 1, 2, 3$. Assume the core is not empty, that is, condition (2) holds, and that x does not belong to the core. We will show that x does not belong to the BS of the grand coalition. We do not need to consider allocations where $x_i < 0$ for some i, or where $x_1 + x_2 + x_3 < V$, since they cannot be in the BS. Thus, assume that $x_i + x_j < v_{ij}$ for some i, j, so that $x_k > V - v_{ij}$, for $k \neq i, j$. Consider an objection y of i against k where $y_i + y_j = v_{ij}$, with $y_i > x_i$ and $y_j > x_j$. A counter-objection z of k against i would have to satisfy that $z_j \geq y_j$, and $z_j + z_k = v_{jk}$, so that $z_k \leq v_{jk} - y_j = v_{jk} - (v_{ij} - y_i)$. Also, $z_k \geq x_k > V - v_{ij}$. Therefore, if

$$v_{jk} - (v_{ij} - y_i) < V - v_{ij},$$

or

$$y_i < V - v_{jk},$$

then the objection y would have no counter-objection and x would not belong to the BS. If $x_i < V - v_{jk}$ we can always construct such y, and then a necessary condition for x to belong to the BS is that $x_i \geq V - v_{jk}$. Switching the subscripts i and j, we could consider an objection y' of j against k, and repeat the argument to show that a necessary condition for x to belong to the BS is that $x_j \geq V - v_{ik}$. Thus, a necessary condition is that

$$x_i + x_j \geq 2V - v_{jk} - v_{ik} \geq v_{ij},$$

where the last inequality follows from condition (2). This contradicts that $x_i + x_j < v_{ij}$ and proves that the BS coincides with the core when the latter is not empty. Now assume that condition (2) is not satisfied. In particular, this implies that we are in Region 3. We have shown above that the $R-$solution belongs to the BS. So we only need to show that any other allocation does not belong to the BS. Note that (2) implies that for any feasible allocation (including the efficient ones), if $x_i = U_i + \epsilon$ (in Region 3), then $x_j + x_k \leq v_{jk} - \epsilon$, for any $\epsilon > 0$. So, consider an efficient allocation such that this is the case for some ϵ, and an objection y of j against i, with $y_j = x_j + \frac{\epsilon}{2}$ and $y_k = v_{jk} - y_j = v_{jk} - x_j - \frac{\epsilon}{2}$. A counter-objection z of i against j should satisfy that $z_k \geq y_k$ but also $z_i \geq x_i$, so that $z_k \leq v_{ik} - x_i$. Thus, for i to indeed have a counter-objection against j it is required that

$$v_{ik} - x_i = v_{ik} - U_i - \epsilon \geq y_k = v_{jk} - x_j - \frac{\epsilon}{2},$$

that is, $x_j \geq v_{jk} - v_{ik} + U_i + \frac{\epsilon}{2} = U_j + \frac{\epsilon}{2}$, where the last equality follows from the definition of U_i. Thus, this is a necessary condition for x to be in the BS. Switching the subscripts j and k, we would also conclude that another necessary condition is that $x_k \geq U_k + \frac{\epsilon}{2}$. Thus, a necessary condition is that $x_i = V - x_j - x_k \leq V - U_j - U_k - \epsilon = U_i - \epsilon$. And this contradiction proves the result. QED

7.3 Proof of Remark 4

Existence: For ϵ small, let $u_1^{1i} = u_i^{1i} = \frac{v_{1i}}{2}$ for $i = 2, 3$, $p = 1 - 2\epsilon$, and $p_{1i} = \epsilon$ for $i = 1, 2$. Thus, $t_i^{1i} = 0$, and $u_i^{1i} = u_1^{1i} = \frac{v_{1i}}{2}$. Also, $t_i^{23} = (1 - \epsilon)\frac{v_{1i}}{2}$, so that $t_2^{23} + t_3^{23} = (1 - \epsilon)\frac{v_{12}+v_{13}}{2}$. If $\frac{v_{12}+v_{13}}{2} > v_{23}$, then for ϵ small $u_i^{23} = 0$. Hence, $t_2^{12} = t_3^{13} = 0$. If $\frac{v_{12}+v_{13}}{2} = v_{23}$, then it must be that $v_{12} = v_{13} = v_{23}$. Then $u_i^{23} = \frac{v_{23}}{2}$. Still, $t_2^{12} = t_3^{13} = 0$. In both cases, we have an $\epsilon-$PSBN with u independent of ϵ. In the limit, $p = 1$. Thus, applying the definition of the $R-$solution, $U_1 = \frac{v_{12}+v_{13}}{2}$, and $U_i = \frac{v_{1i}}{2}$.

Uniqueness: Consider a limit of $\epsilon-$PSBN where $u_1^{12} = u_2^{12} = 0$. This means that for ϵ small $t_1^{12} + t_2^{12} = t_2^{12} = \frac{p_{23}u_2^{23}}{1-(p+p_{12})} > v_{12}$. This is a contradiction, unless $p_{23} = p_{13} = 0$. since $u_2^{23} \leq v_{23} \leq v_{12}$ and $p_{23} \leq 1 - (p + p_{12})$. Also $p_{23} = p_{13} = 0$ implies $p + p_{13} = 1$, which violates the third condition in the definition of an $\epsilon-$PSBN, so indeed $u_1^{12} = u_2^{12} = 0$ cannot be the limit of a sequence of $\epsilon-$PSBN. For the same argument, we cannot have $u_1^{13} = u_3^{13} = 0$. Thus, since $t_1^{1i} = 0$, we have that $u_2^{12} \geq \frac{v_{12}}{2}$, and $u_3^{13} \geq \frac{v_{13}}{2}$. Also, $u_1^{1i} > 0$, since $p_{23} \leq 1 - \epsilon$, and so $t_i^{1i} < v_{23}$ for all $\epsilon > 0$. Thus, $p_{1i} \leq \epsilon$, for $i = 2, 3$. Thus, $t_2^{23} + t_3^{23} = \frac{(p+p_{12})u_2^{12}}{1-p_{23}} + \frac{(p+p_{13})u_3^{13}}{1-p_{23}}$ converges to $u_2^{12} + u_3^{13}$ as ϵ converges to 0, so that for ϵ small, $u_2^{12} + u_3^{13} > \frac{v_{12}}{2} + \frac{v_{13}}{2} \geq v_{23}$. Thus, for ϵ small, $u_i^{23} = 0$, and then $t_i^{1i} = 0$, for $i = 2, 3$, and $p_{23} \leq \epsilon$. Thus u_1^{1i}, u_i^{1i} should converge to $\frac{v_{1i}}{2}$, and p should converge to 1. This completes the proof. QED