Pseudomarkets with Priorities in Large Random Assignment Economies

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Abstract

I study large random assignment economies with a continuum of agents and a finite number of object types. I consider the existence of weak priorities discriminating among agents with respect to their rights concerning the final assignment. The respect for priorities ex ante (ex-ante stability) usually precludes ex-ante envy-freeness. Therefore I define a new concept of fairness, called no unjustified lower chances: priorities with respect to one object type cannot justify different achievable chances regarding another object type. This concept, which applies to the assignment mechanism rather than to the assignment itself, implies ex-ante envy-freeness among agents of the same priority type. I propose a variation of Hylland and Zeckhauser's (1979) pseudomarket that meets ex-ante stability, no unjustified lower chances and ex-ante efficiency among agents of the same priority type. Assuming enough richness in preferences and priorities, the converse is also true: any random assignment with these properties could be achieved through an equilibrium in a pseudomarket with priorities. If priorities are acyclical (the ordering

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of agents is the same for each object type), this pseudomarket achieves ex-ante efficient random assignments.

Keywords: Random Assignment; Fairness; Stability; School Choice.

1 Introduction

In 1979, Hylland and Zeckhauser proposed a pseudomarket mechanism that solved the ex-ante efficient random assignment problem. This result could be compatible with ex-ante envy-freeness should the agents face identical budgets. From Thomson and Zhou (1993) it was understood that any ex-ante efficient and envy-free random assignment could be obtained through a pseudomarket with identical budgets in large economies (continnum of agents). This paper constitutes an extension of pseudomarkets to environments where priorities, or agents' rights concerning the final assignment, exist that have to be respected.

Concerns about justice and efficiency arise in many assignment problems. Examples include the assignment of children to public schools, students to college residences, patients to public health care services, etc. In this paper, I consider the fact that the existence of priorities in many assignment problems precludes standard fairness desiderata such as the absence of ex-ante envy (i.e. no agent should prefer any other agent's assignment probabilities to her own) from being achievable. Therefore I construct a weaker notion of fairness that is compatible with the respect for priorities. It is based on the idea that priorities with respect to one object type cannot justify differences in chances with respect to another object type. I call it *no unjustified lower chances.* I propose a variation of the pseudomarket mechanism *à la* Hylland and Zeckhauser (1979) that respects priorities while meeting the new fairness condition. Under acyclical priority structures, that is, when agents are equally priority-ordered with respect to every object type, this pseudomarket achieves ex-ante efficient random assignments.

The presence of priorities is a recurrent feature in assignment problems. An agent has priority over another agent with respect to an object type when a unit of this object type must be given to the former rather than the latter if both agents claim for the same unit. Examples of priorities are those given by the presence of a sibling at and living at walking distance from the school in children-to-school assignment problems, seniority and existing tenants in residence assignment and medical emergency in health care services. Respect for priorities is understood as the stability of any ex-post (or final) assignment. The term stability, from the marriage market literature (Gale and

Shapley, 1962), indicates that an agent with priority over another agent for some object type must obtain a unit of that object type or a preferred one in the case that the latter agent obtains a unit of that object. Since priorities could be regarded as privileges with regard to the final assignment, ex-ante envy-freeness cannot be guaranteed.

The new concept of fairness I propose here, no unjustified lower chances, affects the assignment mechanism itself rather than the random assignment. It states that any agent should be able to obtain, *if she wishes*, at least the assignment probabilities of any other agent, ignoring those object types for which the latter has priority over the former. More formally, for each feasible random assignment, the assignment mechanism provides each agent with a *menu* of assignment probability vectors from which the agent freely chooses. According to the proposed concept, this menu should include any other agent's assigned probabilities (or higher), ignoring the object types for which the latter has priority over the former. This concept implies ex-ante envy-freeness among agents of the same *priority type* (i.e. those who have equal priority levels for each object type).

In this paper, I propose a *pseudomarket with priorities*, a variation of Hylland and Zeckhauser's suggested mechanism where the prices each agent pays depending on her priority type. For each object type, the priority type of an agent can be summarized into one of the following three statuses: *guaranteed, pivotal and banned*. Guaranteed agents pay zero price, pivotal agents pay the market price and banned agents pay an infinite price. A stable final assignment always arises from any equilibrium in a pseudomarket with priorities. Moreover, a pseudomarket with priorities guarantees no unjustified lower chances. It also obtains ex-ante efficient random assignments among agents of the same priority type.

In some scenarios, priority orderings coincide across object types. That is, priorities are acyclical. Seniority rights in residence assignment, or low-income priorities in school choice, are examples of such a priority structure. In such cases, a pseudomarket with priorities can be understood as a *sequential pseudomarket*: those agents with the highest priority level attend the pseudomarket, buy their assignment probabilities and leave; then those with the second-highest priority level attend the pseudomarket for the remaining object units, and so on. A sequential pseudomarket obtains ex-ante efficient random assignments. The reason is that the price discrimination is such that no agent of a higher priority type would have an incentive to trade with agents of lower types.

Related literature and comments.

The debate: Since the seminal paper by Abdulkadiroğlu and Sönmez (2003), there has been a lively debate on the relative value of the assignment mechanisms that are used in practice. Most of the interest has been centered on school choice mechanisms (i.e. the assignment of children to public schools), and more specifically on the comparison between strategy-proof mechanisms (with truth-telling as weakly dominant strategy) such as Deferred Acceptance and non-strategy-proof mechanisms such as the Boston Mechanism.[1] The strategic simplicity of Deferred Acceptance inspires both a justice argument (protection of naïve agents, see Abdulkadiroğlu, Pathak, Roth and Sömez, 2006, and Pathak and Sönmez, 2008) and also an efficiency argument (avoiding coordination failures, see Ergin and Sönmez, 2006) against non-strategy-proof mechanisms.[2] However, strategy-proofness may come at too high a cost in terms of ex-ante efficiency, as evidenced in Abdulkadiroğlu, Che and Yasuda (2008), Miralles (2008) and Abdulkadiroğlu, Che and Yasuda (2009). Some non-strategy-proof mechanisms such as the Boston Mechanism and related mechanisms achieve better qualitative ex-ante efficiency results because they impose some market (or trade-off) incentives. Inspired by this idea, I consider here how we can combine market incentives with respect for priorities.

Justice and efficiency: The main recent reference is the survey by Thomson (2007). For large economies with a continuum of agents, Varian (1976) and Zhou (1992) provide general results relating fairness and efficiency to Walrasian markets with equal endowments. Varian assumes strictly concave preferences and Zhou analyzes strictly positive consumption sets. Both authors conclude that the set of fair allocations coincides with the set of allocations arising from Walrasian market equilibria with equal endowments.

The no-envy requirement is one of several concepts of justice that one could use. *Egalitarian equivalence* (Pazner and Schmeidler, 1978), that is, everyone's indifference to some (not necessarily feasible) equal split allocation, could alternatively have been chosen among other concepts. Thomson and Zhou (1993) give a wide result on that subject: all egalitarian (with respect to equal split), consistent (i.e. holding for any subset of agents) and efficient allocations are obtained by Walrasian markets with equal endowments, even if preferences are satiated. I easily extend that result to the kind of assignment problems I analyze. Assignment probabilities are the goods in this economy. The fact that probabilities must add up to one could be modeled as a case of satiation. In this

[1] Both mechanisms are ranking mechanisms in that parents (the agents) are first requested to submit a ranking over the schools (the object types). The assignment algorithm uses the provided information in several rounds. In the first round, students are considered for the schools parents ranked first. In schools with excess demand some students are rejected (following priority criteria and tie-breaking lotteries) and go to the next round, where they are considered for the schools that were ranked in second position. Accepted students are definitely accepted in the Boston Mechanism, whereas in Deferred Acceptance they are only reconsidered for that school in the next round. The algorithms likewise follow a finite number of rounds until all students are finally accepted at some school.

[2] For these reasons, the Boston Mechanism was replaced by Deferred Acceptance in Boston (!).

economy, ex-ante envy-freeness is equivalent to consistency and egalitarian equivalence with respect to equal split. Thus, among agents of the same priority type, ex-ante envy-freeness and efficiency are obtained through a Walrasian market with equal endowment and priority-dependent prices. The pseudomarket with priorities I propose meets these properties.

Ex ante and ex post: The importance of ex-ante efficiency (in which random assignments are compared to each other) compared to ex-post efficiency (where final assignments are compared) is stressed when priorities are rather coarse, that is, when massive sets of agents belong to the same priority type. An example is the case of elementary school choice in Boston, where there are only four priority categories (combining sibling and walking zone priorities) for more than one thousand new entrants a year. Priority ties are typically solved by some sort of lottery and thus we talk about *random assignments* when this uncertainty has not yet been resolved. In this context, ex-ante efficiency is a refinement of ex-post efficiency. Since any feasible random assignment could be understood as a lottery over feasible sure assignments, it is easy to see that no ex-post efficiency implies no ex-ante efficiency. The converse is not true. An example is the mechanism known as *random serial dictatorship*, in which agents are strictly ranked according to an even lottery, and then the top-ranked agent picks a unit from her most-preferred object type, the second-ranked agent picks among the available units, and so on. This mechanism always obtains ex-post efficient assignments. However, it can be shown that it is ex-ante inefficient (see for instance Bogomolnaia and Moulin, 2001).

Ex-ante envy-freeness is however a weaker concept than ex-post envy-freeness. In effect, a random assignment could be ex-ante, yet not ex-post, envy-free, whereas the absence of ex-post envy implies its absence ex ante. However, the concept of ex-post envy-freeness is so tight that it could either be unattainable or yield non-sensible assignments. For instance, if an object type is popular (i.e. the number of agents who prefer it over every other object type exceeds the number of units of this type), then ex-post envy-freeness implies that no unit of that object type could be assigned to any agent who prefers it. In a similar way, the concept of no unjustified lower chances makes sense only when applied to assignment probabilities. In conclusion, the ex-ante approach to efficiency and no envy seems recommendable, and this motivates my focus on random assignments.

Priorities and fairness: In several cases, priority structures are designed to provide agents with a chance to prove the intensity of their preferences. In school choice, having a sibling at the school and living nearby is positively correlated with the parents' preference for the school, and this justifies giving priority on the basis of these variables for the sake of a more efficient allocation.

Similarly, previous tenants may have a preference for staying where they are, concerning residence allocation. Nevertheless, in many other cases priorities arise for other reasons. As an example, the San Francisco school authority gives priority to those applicants who "depart more" from the average pool with respect to some socioeconomic indicators, in order to lessen the concentration of minority students in a few schools. My results in this paper indicate that when priorities are single-ordered (e.g. seniority rights), it is possible to obtain a random assignment satisfying ex-post stability, no unjustified lower chances and ex-ante efficiency. In the literature on ex-post assignments (e.g. Ergin, 2002), single-ordered priority rules or similar requirements (e.g. "one step away from single ordering") are also needed to satisfy justice and efficiency properties.

More centrally related to the present paper is the recent work by Kesten and Ünver (2010). Acknowledging the need to respect priorities, they conceive a weak notion of fairness which they call *no ex-ante discrimination*. There is ex-ante discrimination of an agent with respect to another agent and a certain object type if: 1) both agents are at the same priority level concerning that object type, 2) the former agent obtains lower chances than the latter to be assigned an unit of that object type, and 3) the former agent has positive probability to be assigned an unit of an object type that is less preferred to the previous object type. It is a sound concept of fairness in that agents with same priority level for some object type should have the same chances with respect to that object type unless there is "enough" compensation. Moreover, it implies ex-ante envy-freeness among agents of the same priority type. Based on that concept, the authors propose a modification of the Deferred Acceptance algorithm that meets ex-ante stability and no ex-ante discrimination while Pareto-dominating all other assignments meeting the same conditions.

Nevertheless, the concept of no ex-ante discrimination is not exempt of discussion. A first point is, what they understand as "enough compensation" (not being possibly assigned a worst object type) is not the only way to conceive it. For instance, "enough compensation" could just mean that the probability of being assigned to the considered object type *or* a preferred one should not be less that the probability that the other agent has to be assigned to that object type.

A second point is that this concept of justice could come at a high price in terms of (ex-ante) efficiency. Consider the extreme case where there is only one priority level for all schools, that is, the no-priority case. In such an environment, and with a continuum of agents, any ex-ante envy-free and efficient random assignment is obtained through a pseudomarket with equal budgets (once again citing Thomson and Zhou, 1993). However, no ex-ante discrimination is tighter than no ex-ante envy, thus in many cases the pseudomarket assignments are precluded and ex-ante efficiency is not achievable. A simple example contains three object types a, b, c with equilibrium prices $3/2, 1/2, 0$ respectively. Agents have unit budgets. Depending on preferences, some agents would buy $2/3$ probability at a and $1/3$ at c, and some others would buy $1/2$ probability at a and $1/2$ at b. The latter agents prefer a to b, otherwise buying sure assignment at b would have been a better option. Consequently, the latter agents would be ex-ante discriminated with respect to the former agents and object type a. The alternative notion of fairness proposed here, no unjustified lower chances, is intead compatible with these pseudomarkets.

Priorities and property rights: A last observation concerning the presence of priorities is the fact that they could be considered as externalities. It could then be conceivable, *à la* Coase, to convert the priorities into property rights and to let then the agents trade them (Abdulkadiroğlu and Sönmez, 2003). However, apart from legal issues (this procedure does not guarantee ex-post stability, and unstable assignments have been legaly disputed), other theoretical concerns arise here. First of all is the question on how priorities are converted into property rights, that is, into probability endowments that agents trade. Even thoguh a satisfactory answer is possible, a second concern arises when assignment probabilities are traded instead of bought using "fake" monetary income. Hylland and Zeckhauser (1979) have argued that a pseudomarket equilibrium may not exist if agents trade probability endowments. Key in their argument is the fact that each agent ends up with a probability bundle that must add up to one. As an example, consider an environment with two object types a and b where a set of agents who prefer a to b are given an endowment of a sure assignment (probability 1) to b. As long as the price of b lies below the price of a, these agents cannot trade probabilities of b for probabilities of a (demanded probabilities would not add up to one). As soon as the prices equal each other, however, these agents trade all their endowment for a sure assignment at a. This generates an hemi-discontinuity in aggregrate demand, thus Kakutani's fixed-point theorem may not apply. The example could be easily extended to scenarios with more than two object types.

The paper is structured as follows. The second section presents the assignment problem with and without priorities, and the concepts of justice and efficiency are brought into play. The third section introduces and analyzes an extended version of Hylland and Zeckhauser's (1979) pseudomarkets. The fourth section presents the results and discussion. The last section concludes.

2 The (random) assignment problem

There is a finite set S of $J > 2$ object types $S = \{1, ..., J\}$. For simplicity I assume that there is no outside option.[3] Each object type j has capacity mass $\eta_j > 0$. The total sum of capacities across object types is at least 1. Let $\vec{\eta} = (\eta_1, ..., \eta_J)$. There is a mass 1 of agents $x \in X \equiv [0, 1]$ where X is the set of agents endowed with the Lebesgue (uniform) measure λ. There is a measurable von Neumann-Morgenstein (vNM) valuation function $v : X \to V \subset \mathbb{R}^J_+$, where $v(x) = (v_1(x), ..., v_J(x))$ denotes agent x's valuations for object types 1 to J. Each agent is indifferent between any two units of the same object type. The function is assumed to have a range that intersects with any positive ray from the origin (i.e. all relative preferences belong to the image). It is also assumed that $\lambda(\{x \in X : v(x) \in V'\}) = 0$ whenever $\dim(V') < \dim(V)$.[4]

A *random assignment* is a measurable function $q : X \to \Delta(S)$, where $q(x) = (q_1(x), ..., q_J(x))$ denotes agent x's assignment chances for object types 1 to J. A random assignment q is feasible at $(v, \vec{\eta})$ if $\int_X q(x)d\lambda \leq \vec{\eta}$. An (*ex-post or final*) *assignment* is a measurable function $a : X \to S$. Let A be the family of all feasible final assignments (i.e. the mass of agents who are assigned to each object type j does not exceed η_j). By the Birkhoff - von Neumann theorem, any feasible random assignment can be implemented as a (not necessarily unique) lottery $l \in \Delta(A)$ over feasible assignments. In some results, we select l such that it puts positive weight only on those final assignments a whose assignment frequencies coincide with the random assignment probabilities: $\forall X' \subset X$, $\lambda(X') > 0$, $\int_{X'} a(x)d\lambda = \int_{X'} q(x)d\lambda$. This is regarded as the *frequency-probability condition.* Let F denote the set of all feasible random assignments (which obviously depends on $\vec{\eta}$).

Each agent x's expected payoff from the random assignment q is equal to $q(x) \cdot v(x)$. A feasible random assignment is *ex-post efficient* at $(v, \vec{\eta})$ if it can be implemented as a lottery over Pareto-optimal assignments. That is, for any possible lottery outcome, the resulting assignment is Pareto-optimal. A feasible random assignment is *ex-ante efficient* at $(v, \vec{\eta})$ if there is no other feasible random assignment at $(v, \vec{\eta})$ that provides each agent with a weakly higher expected payoff and a positive-measure set of agents obtains a strictly higher payoff. Ex-ante efficiency implies ex-post efficiency, but the converse may not be true. The concept could be extended to groups: for instance, q is *ex-ante efficient within* $X' \subset X$ if there is no feasible reassignment affecting only agents in X' such that there is a Pareto-improvement.

A random assignment q is *ex-ante envy-free* if for any $x, y \in X$, $q(x) \cdot v(x) \geq q(y) \cdot v(x)$. An

[3] The results presented here could easily be extended to include that option.

[4] I am abusing notation: the dimension of a set is in reality taken with respect to the closure of its interior.

assignment is envy-free if for any $x, y \in X$, $v_{a(x)}(x) \geq v_{a(y)}(x)$. A random assignment is ex-post envy free if for any lottery outcome, any final assignment is envy-free. Ex-post envy-freeness implies ex-ante envy-freeness. However, it is easy to see that ex-post absence of envy is too a harsh condition. For any set of agents preferring the same object, either none or all of them have to be assigned to that object type. A within-groups version of the envy-freeness definition could also apply here.

A *priority structure* is a function $P : X \to \Pi \equiv \prod_{j \in S} \{0, ..., G_j\}$, where $P(x) = (P_1(x), ..., P_J(x))$ denotes agent x's *priority type* with respect to object types 1 to J, and $G_j \in \mathbb{N}$ is arbitrarily large. We say that agent x has priority over agent y with respect to object type j if $P_j(x) > P_j(y)$. The triple $(v, \vec{\eta}, P)$ defines the *economy*. Let $\pi \in \Pi$ denote a generic priority type, $\pi = (\pi_1, ..., \pi_J)$, and let X_π be the set of agents of priority type π. In some result we will use the following assumption:

Definition 1 *The priority structure P satisfies the* priority-bridge condition *if for any pair $\pi, \pi' \in \Pi$, $\lambda(X_\pi), \lambda(X_{\pi'}) > 0$, and for any triple $i, j, k \in S$ such that $\pi_i > \pi'_i$, $\pi_j = \pi'_j$ and $\pi_k < \pi'_k$, there exists $\pi'' \in \Pi$, $\lambda(X_{\pi''}) > 0$, such that $\pi_i = \pi''_i$, $\pi_j = \pi'_j = \pi''_j$ and $\pi'_k = \pi''_k$.*

So if two priority types are tied with respect to some object type, and one priority type has higher priority with respect to a second object type while the other priority type has higher priority with respect to a third object type, then there is a "bridge" priority type which has the highest priority of the two in both three object types. The priority structure is required to be rich in that sense. A particular case of a priority structure satisfying the priority-bridge condition would be the join-semilattice structure: for any pair $\pi, \pi' \in \Pi$, $\lambda(X_\pi), \lambda(X_{\pi'}) > 0$, we have $\lambda(X_{\pi \vee \pi'}) > 0$.

Furthermore, we will impose an assumption on the preference profile in some of the results. In words, the assumption states that all preferences are possible regardless the priority type.

Definition 2 *The economy $(v, \vec{\eta}, P)$ satisfies the* preference-richness condition *if for any $\pi \in \Pi$ such that $\lambda(X_\pi) > 0$, we have $v(X_\pi) = V$.*

We say that an assignment a is *stable* (or it respects priorities) given P if $\forall x, y \in X, P_j(x) > P_j(y)$ and $a(y) = j \Longrightarrow v_{a(x)}(x) \geq v_j(x)$. A random assignment is *ex-post stable* if it can be defined as a lottery $l \in \Delta(A)$ whose support is constituted by stable assignments. Following Kesten and Ünver (2010), a random assignment q is defined as *ex-ante stable* given P if and only if $\forall x, y \in X, \{P_j(x) > P_j(y)$ and $q_j(y) > 0\} \Longrightarrow \{q_i(x) = 0 \ \forall i \in S : v_i(x) < v_j(x)\}$. That is, if agent x has priority over agent y with respect to object type j and y obtains chances of being assigned to j, then x cannot be possibly assigned to an object type that is less preferred than j. Ex-ante stability implies ex-post stability. For the converse, the frequency-probability condition is needed.

Lemma 1 *Under the frequency-probability condition, a random assignment q is ex-post stable given P if and only if it is ex-ante stable given P.*

I omit the easy proof given the frequency-probability condition. In words, when an agent with lower priority status for some object type obtains a positive probability of being assigned to that object type, no agent with a higher priority level can be assigned to a strictly less preferred object type (her assignment probabilities must weakly first-order stochastically dominate a sure assignment to that object type). It can be seen that the absence of envy and ex-post stability are generally incompatible features. From now on I will restrict attention to ex-ante stability, taking into account that the frequency-probability condition extends to ex-post stability. It can be seen that the absence of envy and ex-post (or ex-ante) stability are generally incompatible features. Hence, I propose a new and weaker notion of envy-freeness that could be compatible with the respect for priorities.

First, let us define a (*reduced-form*) *mechanism* as a correspondence $Q : X \times F \rightrightarrows \Delta(S)$. Obviously, this is not a standard (nor even precise) definition of a mechanism. It could be understood as a reduced form of a game, where q is given by the strategy profile and Q is given by the agent's achievable assignment probability vectors given the other agents' strategies and her own strategy space. $Q(x, q)$ is understood as the *menu* offered to agent x given the feasible random assignment q. In the pseudomarket analyzed in this paper, an agent's menu consists of her budget set. An optimal choice set $C(Q(x, q))$ is the set of elements in $Q(x, q)$ maximizing x's expected utility. q^* is an *equilibrium random assignment* if $q^*(x) \in C(Q(x, q^*))$ for almost every $x \in X$. Let Q^* denote the set of equilibrium random assignments under Q. Although I skip notation, it is understood that this set depends on the mechanism and on the analyzed economy.

Second, $\forall x, y \in X$, and defining $S_{yx} \equiv \{j \in S : P_j(y) > P_j(x)\}$, the set of object types for which y has priority over x, consider the transformation $\rho(x, y, q^*)$ defined as $\rho_j(x, y, q) = q_j(y)$ if $j \notin S_{yx}$, $\rho_j(x, y, q) = 0$ if $j \in S_{yx}$. That is, this transformation is the vector of y's assignment probabilities, ignoring those object types for which y has priority over x.

Definition 3 *A random assignment reduced-form mechanism Q satisfies* no unjustified lower chances *given P if for almost every $x, y \in X$ and $\forall q^* \in Q^*$, $\exists \tilde{q}(x) \in Q(x, q^*)$ such that $\rho(x, y, q^*) \leq \tilde{q}(x)$.*

In other words, it is required, following the definition, that priorities with respect to some object types shall not justify better chances concerning any of the other object types. Each agent must have (if she wishes) the chance to obtain at least as much as any other agent, ignoring the object types for which the latter has priority over the former. If P is constant across a group of agents, this concept implies ex-ante envy-freeness inside this group (given that agents make optimal choices). In the next section, I extend the pseudomarket mechanism proposed by Hylland and Zeckhauser (1979) to a version that satisfies ex-ante stability, no unjustified lower chances and ex-ante efficiency within each priority type group.

3 Pseudomarkets

Hylland and Zeckhauser (1979) proposed this solution for the assignment of workers to positions more than thirty years ago. They pointed out that even in environments without monetary incentives one could create a fake numeraire good and use it as a means to partially elicit information on agents' preferences over the goods. What they propose is actually very simple: they endow each agent with a budget of fake money and allow her to buy allocation probabilities for each object type. The condition must be met that the purchased probabilities add up to 1. Each object type has its price for each probability unit. A Walrasian market could be simulated that reaches a price equilibrium such that all agents maximize their utilities given the budget constraint, and markets clear (the aggregate demand of each object type does not exceed its supply). The authors find that a price equilibrium always exists,[5] and that each equilibrium random assignment is ex-ante efficient. If all agents start from the same budget, any equilibrium outcome is also naturally envy-free.

This paper builds upon Hylland and Zeckhauser's ideas. It generalizes the concept of a pseudomarket to include the respect for priorities as a objective to accomplish. Unsurprisingly, the respect for priorities entails priority-dependent prices, giving rise to what I call a *pseudomarket with priorities*. Let a *price matrix* p be defined as an element of $\bar{\mathbb{R}}_+^{J\cdot G}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, such that for each $j \in S$ there is $g_j \in \{0, ..., G\} : p_{jg} = 0 \ \forall g > g_j$, $p_{jg} = \infty \ \forall g < g_j$. Each $g > g_j$ will be regarded as a *guaranteed priority type* for object type j, g_j will be called *pivotal priority type*, and each $g < g_j$ will be named a *banned priority type* for object type j. Each agent x is endowed with the same budget, say 1. She faces her own price vector $p(x) = (p_{jP_j(x)})_{j \in S}$ and buys the assignment probability vector $q^*(x,p) \in \Delta(S)$ that maximizes $q(x,p) \cdot v(x)$ subject to $q(x,p) \cdot p(x) \leq 1$. $q^*(x,p)$ is singleton for almost every $x \in X$. A *pseudomarket equilibrium (with priorities)* is a price matrix p^* such that $\int_X q^*(x,p^*) d\lambda \leq \vec{\eta}$.

For any price matrix, each agent's optimal consumption decision can be partially characterized.

[5] This cannot be guaranteed if agents are first given actual assignment probability endowments and they are then allowed to trade them. See Hylland and Zeckhauser (1979).

Almost every agent x, given $p(x)$, buys positive probabilities for at most two object types. Ideally, given the linear utility form that serves to evaluate lotteries, any agent would choose to spend her whole budget (or else buy sure assignment) on only one object type. The additional constraint that the sum of purchased probabilities must be 1 forces the agent to acquire probabilities from a back-up option. This back-up option must have a price lower than 1. A back-up option always exists in equilibrium. Indeed, given that the overall supply of units is never less than its overall demand, without loss of generality we can assume that there exists a "worst object type" w whose price could be normalized to zero for every agent (all agents have guaranteed assignment at w). If the agent does not spend her whole budget, it must be the case that she has bought a sure assignment to her preferred object type.

Given the linear expected utility form, it is the case that almost every agent optimally chooses among kink points in her budget set, thus the number of relevant choices is finite. At a kink point in which agent x buys from object types i and j (the latter being the back-up option), it must be the case that $p_i(x) > 1 > p_j(x)$ and $v_i(x) > v_j(x)$. Let us denote such a kink point lottery as ij. Denoting $O_x \equiv \{i \in S : p_i(x) > 1\}$ (*overdemanded*, or unaffordable, object types) and $U_x \equiv S \backslash O_x$ (*underdemanded*, or affordable, object types), there are at most $(\#O_x + 1) \cdot \#U_x$ relevant choices for agent x. Moreover, it is easy to check that if agent x (weakly) prefers ij to hj and ij to ik, she must (weakly) prefer ij to hk. That simplifies the agent's optimization algorithm. In sum, at least in environments with infinitely many agents, there exist reasons to believe that a pseudomarket mechanism is not strategically cumbersome. This is important due to the current concern about so-called naïve agents versus sophisticated ones (Pathak and Sönmez, 2008). One could also use a direct mechanism in which agents are required to submit information about their preferences and then the planner simulates a pseudomarket and assigns probabilities.

Lemma 2 *For any economy $(v, \vec{\eta}, P)$, a pseudomarket equilibrium (with priorities) p^* exists.*

Proof. Given a vector of prices, the number of relevant (i.e. possibly best) consumption choices is finite for each agent. Thus the pseudomarket could be modeled as a game with a continuum of agents and a finite strategy space for each agent, embedding all singletons and pairs of object types. Payoffs are continuous on the vector of measures of priority-type/strategy tuples (i.e. for each priority type, the measure of agents playing a specific strategy). Thus conditions apply to use Theorem 2 in Mas-Colell (1984) and state the existence of an equilibrium in pure strategies. ■

Implementing the probabilities the pseudomarket assigns is certainly easy. Since almost every agent buys from at most two object types, the assignment algorithm proceeds in two rounds. First notice that the agents could be classified into a finite number of sets according to their assigned probabilities. Consider a set with mass μ_{ij}^{π} of agents of priority type π who are assigned to object type i with probability q_{ij}^{π}, and to object type j with probability $1 - q_{ij}^{\pi}$. Take a mass $q_{ij}^{\pi}\mu_{ij}^{\pi}$ of units of object type i, and randomly assign them to the agents from that group. Those agents who were not assigned to i go to a second round, where they are assigned to object type j. Do the same for each group. Those agents who were indifferent among kink points and bought a convex combination of them are randomly classified into one of the groups the kink point choices belong to, with probabilities mimicking the weights of the convex combination.[6]

A final note on pseudomarkets will clarify why it is not always correct to first give agents property rights over the assignment probabilities and then let them trade these rights. A simple example is where there are only two object types. Since any agent's purchased bundle of probabilities must add up to one, those agents preferring the most expensive object type would not be able to buy additional probability units of this object type. Hylland and Zeckhauser (1979) have an example in which no market equilibrium exists following this procedure, while an equilibrium always exists when agents buy assignment probabilities out of a fictitious budget. Other discontinuity problems arise if agents are given property rights over sure assignments. They will not trade against more preferred but more expensive object types out of budgetary impossibility, and suddenly they will trade all of their endowment as soon as the preferred goods become at least as cheap as the endowed object types. In sum, by organizing a market with budgets instead of real endowments, one avoids discontinuity problems and guarantees the existence of price equilibria.

4 Results

I list my results in this section. First I consider scenarios without restrictions on the priority structure, where I establish the properties of the pseudomarket mechanism with priorities. Secondly, I study scenarios with acyclical priorities. In these, if an agent has priority over some other agent with respect to one object type, she does too for any other object type. With acyclical priorities, the pseudomarket with priorities obtains overall ex-ante efficiency properties.

[6] The implementation process is a bit more complex with a finite number of agents. Hylland and Zeckhauser (1979) provided a solution in such a case.

4.1 General case

In this subsection, I state and discuss the main result of this paper. The proposed pseudomarket with priorities guarantees ex-post stability and satisfies some efficiency and fairness properties discussed in the preceding sections.

Theorem 1 *For any economy $(v, \vec{\eta}, P)$, the pseudomarket with priorities satisfies no unjustified lower chances and obtains (in equilibrium) random assignments that are both ex-ante stable and ex-ante efficient among agents of the same priority type.*

Proof. Ex-ante efficiency within priority type: for each priority type π, we can analyze a separate economy in which a vector $\vec{\eta}_\pi = \int_{X_\pi} q(x) d\lambda$ characterizes the supply side. For each such economy, the price vector is identical across agents, therefore Hylland and Zeckhauser (1979) applies here to show that the pseudomarket generates ex-ante efficient random assignments in equilibrium.

Ex-ante stability: consider a school $j \in S$ and two agents $x, y \in X$ such that $P_j(x) > P_j(y)$. If $p_j(y) = \infty$, agent y cannot buy probability units from that object type, and we are done. If $p_j(y) < \infty$, it must be the case that $p_j(x) = 0$. Then optimally x buys an assignment probability vector that first order stochastically dominates a sure assignment to j. Therefore it is impossible that x ends up being assigned to a school that is less preferred than j.

No unjustified lower chances: consider two agents $x, y \in X$ and $\rho(x, y)$ as defined before. Almost every y buys positive assignment probabilities from at most two object types. First, consider the case in which y buys sure assignment for a unique object type j (therefore $p_j(y) \leq 1$). If $j \in S_{yx}$, then $\rho(x, y) = \vec{0}$, and we are done. If $j \notin S_{yx}$, then $p_j(x) \leq p_j(y)$, hence a sure assignment to j is also available to x. Second, consider the case in which y buys probabilities from two object types i ($p_i(y) > 1$) and j ($p_j(y) < 1$). If both i and j belong to S_{yx}, once again $\rho(x, y) = \vec{0}$, and we are done. If neither belong, then $p_i(x) \leq p_i(y)$ and $p_j(x) \leq p_j(y)$, thus $\rho(x, y)$ is affordable for x. If only $j \in S_{yx}$, then $p_i(x) \leq p_i(y)$. Consider the "worst object type" w whose price is zero for every agent. In that case, x can afford to buy at least $\rho(x, y)$ by spending her budget on i and completing the bundle with w. Finally, consider the case where only i belongs to S_{yx}. Then $p_j(x) \leq p_j(y) \leq 1$, and $\rho(x, y)$ is available to x (who can indeed buy a sure assignment to j). ■

A first question with respect to this result is whether the converse is true. That is, whether no unjustified lower chances, ex-post stability and ex-ante efficiency within priority types can be characterized by the equilibrium outcomes of the pseudomarket mechanism here proposed. It turns out that the converse is not necessarily true. For any two agents $x, y \in X$, define $E_{xy} = S \backslash (S_{xy} \cup S_{yx})$, the set of object types for which both agents have the same priority level. Ideally, we would like no unjustified lower chances to imply equal prices for E_{xy} for both agents. However, if $p_j(y) \leq 1$ for any $j \in E_{xy}$, this fairness property only implies $p_j(x) \leq 1$ (affordability of sure assignments) for any $j \in E_{xy}$.

Nevertheless, the converse becomes true if we add the priority-bridge and preference-richness conditions.

Theorem 2 *Given $\vec{\eta}$, fix P satisfying the priority-bridge condition and consider the set $\Omega \equiv \{v : (v, \vec{\eta}, P)$ satisfies the preference richness condition$\}$. Then the following statement is true for any $v \in \Omega$: any (equilibrium) random assignment generated by a mechanism satisfying no unjustified lower chances that is both ex-ante stable and ex-ante efficient among agents of the same priority type can be generated by an equilibrium in a pseudomarket with priorities. If P is fixed so that it does not satisfy the priority-bridge condition, there exists $v \in \Omega$ such that the previous statement is not true.*

Proof. A mechanism satisfying no unjustified lower chances obtains (in equilibrium) random assignments that are ex-ante envy-free among agents of the same priority type. Jointly with ex-ante efficiency within the set of agents of the same priority type, this allows me to invoke Thomson and Zhou (1993): in equilibrium, the random assignment is generated by optimal consumption decisions given a matrix of priority-type dependent price vectors $(p(\pi))_{\pi \in \Pi}$ and an equal income 1 for every agent. This is so because, once again, for each priority type π, we can analyze a separate economy with a set of agents X_π in which a vector $\vec{\eta}_\pi = \int_{X_\pi} q(x) d\lambda$ characterizes the supply side.

I next show that ex-ante stability implies that for any school $j \in S$ and any two priority types $\pi, \pi' \in \Pi$ such that $\pi_j > \pi'_j$, we must obtain (WLOG) the following double implication: $p_j(\pi') < \infty \iff p_j(\pi) = 0$. Consider $p_j(\pi') < \infty, p_j(\pi) > 0$. Given the preference-richness assumption, there is $y \in X_{\pi'}$ that optimally buys positive probabilities for object type j. Now suppose $p_j(\pi) > 1$. Consider the "worst" object type w that has zero price regardless the priority type. By the preference-richness condition, some agent $x \in X_\pi$ will optimally buy positive probabilities of both j and w while preferring j to w. This violates ex-ante stability when comparing x to y. Suppose instead that $p_j(\pi) \leq 1$. If there is $i \in S$ such that $1 < p_i(\pi) < \infty$, by the aforementioned assumption there will be $x \in X_\pi$ who will optimally buy positive probabilities of both i and w while preferring j to w. This once again violates ex-ante stability when comparing x to y. If there is no such an i, prices that are not higher than 1 could be normalized to 0 (including $p_j(\pi)$), since the budget set would not be affected.

In the next two paragraphs, I show that for any school $j \in S$ and any two priority types $\pi, \pi' \in X$ such that $\pi_j = \pi'_j$, we must have (WLOG) $p_j(\pi) = p_j(\pi')$. Let us suppose $p_j(\pi) > p_j(\pi')$. If $p_j(\pi) > 1$, some agent $x \in X_\pi$ that prefers j to any other object type will be able to buy less assignment probability at j than some agent $y \in X_{\pi'}$ who optimally chooses to spend her budget on j and completes her bundle with w (the "worst" object type, with zero price). This violates no unjustified lower chances. If $p_j(\pi) \leq 1$, I consider several cases. First, if either x or y (or both) is facing prices that are either not higher than 1 or ∞, then these not-above-one prices could be normalized making $p_j(\pi) = p_j(\pi')$ with no alteration of the budget set (WLOG). Second, if there is another object type $h : \pi_h = \pi'_h$, $p_h(\pi) = p_h(\pi') > 1$, (we must have $p_h(\pi) = p_h(\pi')$ as seen before in the case $p_j(\pi) > 1$) then $p_j(\pi) > p_j(\pi')$ implies, under the preference-richness assumption, that some agent $y \in X_{\pi'}$ optimally buys a bundle with $\frac{1-p_j(\pi')}{p_h(\pi)-p_j(\pi')} > \frac{1-p_j(\pi)}{p_h(\pi)-p_j(\pi)}$ probability units of being assigned at h and the remaining probability at j. This bundle does not belong to the budget set of agents in X_π, hence violating no unjustified lower chances.

Thus I consider a last set of scenarios where $\pi_j = \pi'_j$ and $\exists i, k \in S \backslash E_{\pi'\pi} : \infty > p_i(\pi) > 1$, $\infty > p_k(\pi') > 1$, where $E_{\pi'\pi} \equiv \{h \in S : \pi'_h = \pi_h\}$. Here I use the priority-bridge condition imbedded in the richness condition. There is a priority type π'' such that $i, j \in E_{\pi''\pi} \equiv \{h \in S : \pi''_h = \pi_h\}$ and $j, k \in E_{\pi'\pi''} \equiv \{h \in S : \pi'_h = \pi''_h\}$ with a positive mass of agents of that type. By the previous paragraph we must have both $p_j(\pi'') = p_j(\pi')$ and $p_j(\pi'') = p_j(\pi')$, contradicting $p_j(\pi) > p_j(\pi')$. The necessity argument is also seen here: if there were no such a π'', there would be a preference profile v such that a vector of equilibrium prices exists satisfing $p_j(\pi) > p_j(\pi')$ (while all the stated properties are met). ■

A second question here is whether the frequency-probability condition is relevant. This condition provides some structure to ex-post assignments. It has to be said, though, that this condition reduces the set of ex-ante assignments that are ex-post stable *under a propery designed lottery* over final assignments. Consider the following example with three object types a, b, c where a (with capacity 1/2) is preferred to b (with capacity 1/4) and b to c (with capacity 1/4) by every agent. There is a measure 1/2 of agents of priority type π and another measure 1/2 of agents of priority type π'. The only difference between π and π' is that those agents of type π have priority over those of type π' with respect to object type b. Consider a random assignment in which all agents obtain the same assignment probabilities 1/2,1/4,1/4 for a, b and c respectively. According to the frequency-probability condition, this random assignment is not ex-post stable. However, consider an ex-post implementation where with probability 1/2, all agents of type π are assigned to a and the other agents are randomly split between b and c, and with probability 1/2 all agents of type π' are assigned to a whereas those of type π are randomly split between b and c. All the ex-post assignments here are stable. The example could be slightly modified to acommodate the richness assumption. This suggests that a relaxation of the frequency-probability condition could enrich the analysis of ex-post stable random assignments.

4.2 Acyclical priorities

Consider a specific priority structure P such that for any $x, y \in X, i, j \in S$, $P_i(x) > P_i(y) \Leftrightarrow P_j(x) > P_j(y)$. This would be regarded as *acyclical priorities*, since there is a unique ordering among agents that applies to all object types. One example in real scenarios could be the assignment of college students to residences, where seniority may give priority. In these cases, I could then generally talk about top-ranked agents, second-ranked agents and so on, with no mention of object types. Let us define a *sequential pseudomarket* in the following way: first, top-ranked agents attend a pseudomarket with equal budgets, buy their assignment probabilities at equilibrium prices and leave; then, second-ranked agents attend a pseudomarket with equal budgets for the remaining object units; and so on. It is easy to see that the sequential pseudomarket is equivalent to the more general pseudomarket with priorities when these are acyclical.

Theorem 3 *Fix an acyclical priority structure P. Under the frequency-probability condition, a sequential pseudomarket guarantees stability and obtains ex-ante efficient random assignments.*

Proof. Let $\pi \gg \pi'$, and consider any $x \in X_\pi$ and any $y \in X_{\pi'}$. Consider any object type j from which y buys assignment probabilities in the market equilibrium. Then it must be the case that $p_j(x) = 0$, since that object type was necessarily oversupplied for $\pi-$ranked agents. This guarantees ex-post stability, since x will optimally buy assignment probabilities from object types that are at least as preferred as j. Since ex-ante efficiency is guaranteed among agents of the same priority type, potentially mutually beneficial trade could only arise between priority groups. Once again, however, no set of agents $x \in X_\pi$ could have an interest in trading with any set of agents $y \in X_{\pi'}$. All object types from which any such y buys are (weakly) less preferred for x than the object types from which x buys, since x is facing a zero price for the former goods. Note that any trade has to keep every agent with assignment probabilities adding up to one. ■

Notice here that we have not mentioned the fairness condition proposed in this paper, no unjustified lower chances. It turns out that this property is not informative here, since it holds only trivially. In fact, for $\pi > \pi'$, this property just implies that agents in π' obtain a non-negative assignment probability vector in the menu offered, which is the budget set.

5 Conclusion

In this paper, I have analyzed large random assignment economies with a continuum of agents and a finite number of object types. I have introduced the realistic assumption that some priority criteria might alter the symmetry of agents in their rights regarding the final assignment. Without these priorities, ex-ante efficiency and envy-freeness is characterized by the equilibrium outcomes from a pseudomarket with equal budgets, *à la* Hylland and Zeckhauser (1979), as pointed out by Thomson and Zhou (1993). In a pseudomarket, each agent is endowed with a fictitious budget that allows her to buy assignment probabilities, where each object type has its own price. In this paper, I have proposed an extension of this pseudomarket mechanism that takes the existence of priorities into account. For each object type, there is a priority level that is pivotal, that is, agents at that level pay the market price. Agents at a higher priority level (guaranteed) pay zero price, and those at lower levels (banned) pay infinite price.

Once priorities are considered, ex-ante envy-freeness is typically not achievable. I have alternatively searched for a notion of "maximal fairness" subject to respect for priorities. I have based this notion on the fact that priorities regarding one object type cannot justify any privileges concerning another object type, and I have named the observation of this desideratum *no unjustified lower chances*. This property implies ex-ante envy freeness among agents of the same priority type. It states that, for any feasible random assignment, each agent must be offered a *menu* of assignment probability vectors such that each other agent's assignment probabilities (or higher) are included in that menu, ignoring the objects for which the latter agent has priority over the former. It will be noticed that this fairness condition not only affects random assignment but also the process (the mechanism) generating it. The menu is a reduced form of the assignment probabilities that the agent can obtain given her strategy space and the other agents' strategy profile.

In general scenarios where I do not impose any structure on priorities, I show that a pseudomarket with priorities respects priorities (it is ex-post stable), guarantees no unjustified lower chances and obtains ex-ante efficient allocations among agents of the same priority type. Assuming enough richness in preferences and priorities, the converse is also true: any random assignment with these properties could be achieved through an equilibrium in a pseudomarket with priorities. If priorities are constrained to be acyclical, that is, the ranking of agents' priority levels does not vary across object types, then the pseudomarket with priorities can be implemented via a sequential pseudomarket. In such a mechanism, agents with the highest priority level attend the pseudomarket, buy their assignment probabilities and leave. Agents at the second-highest priority level attend the pseudomarket for the remaining slots and leave. And so on. This sequential pseudomarket obtains in equilibrium overall ex-ante efficient random assignments, since agents from a higher priority type would never have an incentive to trade their assigned probabilities with agents in lower priority levels.

There are several other interesting features in assignment problems that have been skipped here in order to obtain more concise results. Among them, the finiteness of the number of agents, and the presence of peer-group effects on agents' preferences. The first element is often found in the theoretical literature. Following Hylland and Zeckhauser's (1979) seminal work, the results found in the present paper do not substantially differ from what we could expect in environments with sufficiently many agents per object type. However, interesting departures from competitive market behavior deserve more careful analysis, when the number of agents per object type is low enough. For instance, in the children-to-school assignment problem, is the reasonable 25-30 new students per school and year ratio sufficiently high? The answer will come from empirical evidence. The second element I mention is specially interesting in that it has not received proper attention from the theoretical literature despite the empirical evidence. It could be argued that a model that incorporates peer-group effects adds much more complexity to already cumbersome problems. However, further inclusion of this feature is relevant since it may affect what we know from the mechanism I suggest in this paper and other mechanisms that have been suggested in the literature.

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