Stable Partitions in Many Division Problems:

The Proportional and the Sequential Dictator Solutions

Gustavo Bergantiños[1], Jordi Massó[2], Inés Moreno de Barreda[3] and Alejandro Neme[4]

October 2013

Abstract: We study how to partition a set of agents in a stable way when each coalition in the partition has to share a unit of a perfectly divisible good, and each agent has symmetric single-peaked preferences on the unit interval of his potential shares. A rule on the set of preference profiles consists of a partition function and a solution. Given a preference profile, a partition is selected and as many units of the good as the number of coalitions in the partition are allocated, where each unit is shared among all agents belonging to the same coalition according to the solution. A rule is stable at a preference profile if no agent strictly prefers to leave his coalition to join another coalition and all members of the receiving coalition want to admit him. We show that the proportional solution and all sequential dictator solutions admit stable partition functions. We also show that stability is a strong requirement that becomes easily incompatible with other desirable properties like efficiency, strategy-proofness, anonymity, and non-envyness.

Journal of Economic Literature Classification Number: D71.
Keywords: Division Problem, Symmetric Single-peaked Preferences, Stable Partition.

[1]Research Group in Economic Analysis. Facultad de Económicas, Universidad de Vigo. 36310, Vigo (Pontevedra), Spain. E-mail: gbergant@uvigo.es

[2]Universitat Autònoma de Barcelona and Barcelona GSE. Departament d'Economia i d'Història Econòmica, 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

[3]Nuffield College, Oxford University, New Road Oxford OX1 1NF, United Kingdom. E-mail: ines.morenodebarreda@economics.ox.ac.uk.

[4]Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina. E-mail: aneme@unsl.edu.ar

1 Introduction

Consider the division problem faced by a set of agents who have to share a unit of an homogeneous and perfectly divisible good. For instance, a group of agents participate in an activity that requires a fixed amount of labor measured in units of time. Given a wage, classical monotonic and quasi-concave preferences on the set of bundles of money and leisure generate single-peaked preferences on the set of potential shares, where the best share is the amount of working time associated to the optimal bundle and in both sides of the best share the preference is strictly monotonic, decreasing at its right and increasing at its left. Similarly, a group of agents join a partnership to invest in a project (an indivisible bond with a face value, for example) that requires a fixed amount of money, neither more nor less. Their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. As in the previous examples, there are other social choice settings for which the division problem appears as its reduced problem (see for example, Barberà and Jackson (1995)).

A solution is a family of mappings that select for each instance of division problem (a set of agents and their single-peaked preferences) a vector of shares, one for each agent. But for most single-peaked preference profiles, the sum of the best shares will be either larger or smaller than the total amount to be allocated. A positive or negative rationing problem emerges depending on whether the sum of the best shares exceeds or falls short of the fixed amount. Sprumont (1991) started a large literature characterizing solutions in terms of alternative sets of properties. These solutions differ on the underlying principles guiding how the rationing problem has to be solved.[1]

In this paper we study the division problem when the good to be allocated also comes with fixed amounts but now agents may share several units, whose number is endogenous because it may depend on agents' preferences. Consider for example a group of entrepreneurs examining several business opportunities. Each entrepreneur is willing to devote himself to at most one of those business opportunities and as before, their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. We let agents partition themselves into coalitions in such a way that agents in each coalition will have to share one and only one unit of the good. An allocation is a pair consisting of a partition of the set of agents and a vector of allotments specifying for each coalition in the partition a vector of shares, one for each agent in the coalition, whose components add up to one unit. A rule is a mapping that selects for each profile of single-peaked preferences an allocation; *i.e.*, a partition and a vector of allotments. Thus, a rule can be decomposed into two procedures. For each profile of single-peaked preferences, the first procedure is a function that selects a partition of the set of agents while the second procedure is a solution to be applied to the subprofile of single-peaked preferences of the agents in each coalition of the partition. We restrict ourselves to second procedures that select the allotment by means of a unique solution applied to each rationing problem faced by each coalition in the partition. This restriction implies that the same principles are used across coalitions and it can be interpreted as a consistency requirement. Thus, a rule can be identified with a partition function (mapping single-peaked preference profiles into partitions of the set of agents) and a solution (to be applied to each coalition of the selected partition).

[1] For axiomatic characterizations of solutions see for example Barberà, Jackson, and Neme (1997), Ching (1992, 1994), Dagan (1996), Ehlers (2002a, 2002b), Herrero and Villar (2000), Schummer and Thomson (1997), Sönmez (1994), and Thomson (1994, 1995, 1997, 2003).

Our main concern in this paper is the stability of rules.[2] Specifically, fix a solution. We want to know whether there exists a partition function that, together with the fixed solution, constitute an stable rule. Our notion of stability is based on the principle that the allocations proposed by the rule have to be voluntarily accepted by the agents in the following sense. Consider a rule and a profile of single-peaked preferences. Apply the rule to the profile, thereby obtaining a partition and a vector of allotments. Take an agent in a coalition and another coalition (which may be empty), and suppose that (1) the agent wants to leave his original coalition to join the other one because the share assigned to him by the solution applied to the subprofile of preferences of the agents in the new coalition is strictly preferred to his former share, and (2) all agents of the receiving coalition want to admit the agent because the shares assigned to them by the solution applied to the subprofile of preferences of the agents in the new coalition are weakly preferred to their respective former shares. In this case, the original chosen allocation would be instable at the profile to which the rule has been applied. A rule is stable if it chooses stable allocations at each profile of single-peaked preferences. Three remarks are in order. First, to generate an instability we are requiring that the moving agent has to obtain an strictly preferred share while the agents of the receiving coalition have to obtain a weakly preferred share. This captures the idea that to move from one coalition to another (the origin of the instability) requires a bit more than just to admit a new member in the coalition. Second, instabilities are generated only by one agent moving to a new coalition. In this case, the needed coordination among agents to fulfill the instability is minimal compared with the coordination needed if non-singleton subcoalitions would be allowed to change coalitions. Third, the receiving coalition may be empty, in which case the instability would be produced only by the agent that by leaving his current coalition could be strictly better off; *i.e.*, the agent would strictly prefer the full share of one unit of the good to the share he had been assigned in his original coalition.

In a similar setting Gensemer et al (1996, 1998) study another concept of stability that they call "migration equilibrium". Agents with single-peaked preferences are partitioned into several local economies, each of which has an endowment that is allocated among its participants following a given solution. A migration equilibrium requires that no agent will be better off by leaving his economy to join another. They show that when the solution applied to each local economy is well behaved[3] there might not exist a migration equilibria. Note that the receiving economy cannot ban the arrival of a new agent and hence the migration equilibrium is a stronger stability condition than the one studied in this paper. Which stability condition to apply depends on the applications. In an environment with a small number of agents with decision power such as in joint ventures, our concept is more appealing whereas for movements across countries or big societies the migration equilibrium is the one to be considered.

[2] Kar and Kibris (2008) consider the efficiency of such rules in a setting where the number of units to share is fixed rather than endogenous. They show that for the domain of single-peaked preferences and for well behaved solutions (efficient, non dictatorial, strategy proof, resource monotonic and consistent), it is not possible to find a partition such that the final allocation is efficient.

[3] In particular, in Gensemer et al (1996) they show that a migration equilibrium might fail to exists if the solution applied to the local economies satisfies two of the following three properties: Pareto efficiency, strategy proofness and no-envy. In Gensemer et al (1998) they show that a migration equilibrium might fail to exists whenever the solution applied to the local economies is either the Proportional, the Sequential Dictator, the Uniform or the Egalitarian rules.

We found that in general, finding partition functions for well-known and simple solutions, to constitute together stable rules, is not an easy task. Indeed, it may become extremely complex in the general setting of the division problem. Thus, we have simplified the problem by assuming that agents' single-peaked preferences are in addition symmetric.[4] A single-peaked preference is symmetric if the following additional condition holds: a share is strictly preferred to another one if and only if the former is strictly closer to the best share. Observe that in many applications the linear order structure on the set of potential shares, relative to which single-peakedness is defined, conveys to agents' preferences more than just an ordinal content. Often, an agent's preference on the set of shares is responsive also to the notion of distance, embedding to the preference its corresponding property of symmetry (see Massó and Moreno de Barreda (2011) for the use of symmetric single-peaked preferences in the context of selecting a public good, as in Moulin (1980)). The use of symmetric single-peaked preferences has the additional advantage that, without loss of generality, the domain of the rule is the set of vectors of best shares, instead of the set of profiles of full preferences.

Our main results establish that, provided that agents' preferences are symmetric single-peaked, the proportional solution (Proposition 1) and all sequential dictator solutions (Proposition 2) have the property that for each one of them there exists a partition function that, together with the corresponding solution, constitute an stable rule.[5] The proportional solution of the division problem assigns to each agent, given a vector of agents' best shares, a share that is equal to his best share divided by the sum of all the best shares. Remember that the solution is applied to each coalition in the partition selected by the partition function at the vector of agents' best shares. Given an ordering on the set of agents, the sequential dictator solution associated with this ordering, and applied to a vector of agents' best shares, let each agent, except the last one, choose sequentially (following the ordering) his share of what is left of the good (if anything) by his predecessors. The last agent in the ordering gets the remainder. Observe that (i) each ordering on the set of agents define a different solution of the division problem, and (ii) the order is fixed and used in each of the coalitions selected by the partition function at the same vector of agents' best shares. The proofs of the two results are constructive and proceed by induction on the number of agents. In addition, we exhibit examples showing that for both rules stability is an strong requirement incompatible with many other desirable properties like efficiency, strategy-proofness, anonymity, and non-envyness.

We also show that there are simple solutions for which there do not exist partition functions that together constitute stable rules. For this purpose we exhibit as example a weighted proportional solution; namely, we show that there is a vector of weights, one for each agent, with the property that the corresponding weighted proportional solution has no partition function that together constitute an stable rule.

Amorós (2002), Adachi (2010), and Morimoto, Serizawa, and Ching (2013) study also multi-dimensional extensions of Sprumont (1991)'s division problem. They extend the uniform solution of a division problem to many division problems (Amorós (2002) does it for problems with only

[4] Kar and Kibris (2008) show that in the domain of symmetric singled-peaked preferences, and when the number of units of the good to be shared is fixed, whenever a (local) solution is efficient, there exists a partition such that the final allocation is efficient.

[5] Note that for both the proportional solution and the sequential dictator solution a migration equilibrium [Gensemer et al (1996,1998)] might fail to exists even when we restrict the preferences to symmetric single-peaked domain.

two agents). Their approach is different to ours because they consider problems where the goods to be allocated may be different and each agent has preferences on vectors of his potential shares (one for each different good). Their main contribution is to extend and axiomatically characterize the uniform solution to the multiple goods setting.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we state and prove our main results. In Section 4 we present some final comments.

2 Preliminaries

Agents are indexed by the elements of a finite set $N = \{1, ..., n\}$, where $n \geq 1$. They have to partition themselves in such a way that the agents of each coalition of the partition have to share one unit of a perfectly divisible good (as in Sprumont (1991)). Let Π denote the set of partitions of N. Given $\pi = \{S_1, ..., S_K\} \in \Pi$ and $i \in N$, let $S_\pi(i)$ denote the set $S_k \in \pi$ such that $i \in S_k$. An *allocation* is a pair (π, x) where $\pi = \{S_1, ..., S_K\} \in \Pi$ is a partition and $x = (x_1, ..., x_n) \in [0, 1]^n$ is a vector of allotments such that, for each $k = 1, ..., K$,

$$\sum_{i \in S_k} x_i = 1.$$

Let A be the set of allocations. We assume that each agent $i \in N$ has a complete, reflexive and transitive *preference relation* R_i on the set of his potential shares $[0, 1]$. Let P_i and I_i be its associated strict and indifference relations, respectively. Namely, for each pair $x_i, y_i \in [0, 1]$, $x_i P_i y_i$ if and only if $y_i R_i x_i$ does not hold, and $x_i I_i y_i$ if and only if $x_i R_i y_i$ and $y_i R_i x_i$ hold. In addition, we assume that R_i is *symmetric single-peaked*; that is, there exists the best share $b_i \in [0, 1]$ (*i.e.*, $b_i P_i x_i$ for all $x_i \in [0, 1] \backslash \{b_i\}$) and $x_i P_i y_i$ if and only if $|x_i - b_i| < |y_i - b_i|$. Therefore, we can identify a profile of preference relations $R = (R_1, ..., R_n)$ by the vector of their corresponding best shares $b = (b_1, ..., b_n)$, which we call a *profile*. The set of all profiles is $[0, 1]^n$. Given a non-empty subset $S \subseteq N$, we denote its cardinality by its lower case representation, *i.e.*, $s = \#S$. Given a profile $b \in [0, 1]^n$ and a non-empty subset $S \subseteq N$ we denote by $b_S = (b_i)_{i \in S} \in [0, 1]^s$ the subprofile of best shares of agents in S.

A *rule* is a function $\Phi : [0, 1]^n \to A$ selecting, for each profile $b \in [0, 1]^n$, an allocation $\Phi(b) = (\pi, x) \in A$. Given a non-empty subset of agents $S \subseteq N$, a *solution for* S is a function $f^S : [0, 1]^s \to [0, 1]^s$ that selects for each subprofile an allotment for S; namely, for each $b_S \in [0, 1]^s$, $f^S(b_S) \in [0, 1]^s$ has the property that

$$\sum_{i \in S} f_i^S(b_S) = 1.$$

A *solution* f is a family $\{f^S\}_{S \subseteq N}$, where each f^S is a solution for S. Given a solution f, a partition π, and a profile b, denote by $f(\pi, b) = (f_i(\pi, b))_{i \in N}$ the following vector of allotments: for each $i \in N$,

$$f_i(\pi, b) = f_i^{S_\pi(i)}(b_{S_\pi(i)}).$$

A *partition function* is a mapping $\mu : [0, 1]^n \to \Pi$ that selects, for each profile $b \in [0, 1]^n$, a partition $\mu(b) \in \Pi$.

A rule $\Phi : [0,1]^n \to A$ can be decomposed as $\Phi = (\mu, f)$, where $\mu : [0,1]^n \to \Pi$ is a partition function and f is a solution. Then, for all $b \in [0,1]^n$, $\Phi(b) = (\mu(b), f(\mu(b), b))$. We can also apply a rule to a subprofile in the obvious way.

In the next section we will focus on the stability of some rules. We now define this property. Consider a rule. Given a profile, apply the rule to the profile, thereby obtaining a partition and a vector of allotments. Now, imagine that an agent moves to another coalition in the partition and, after applying the solution to this new subset, he obtains an strictly better share and all former members of this receiving coalition (which may be empty) are at least as well as they were before. Then, the rule would not be stable at the profile. To define formally a stable rule we have to describe, given a partition, an agent and a coalition receiving this agent, the new partition after the agent moves from the former coalition to the new one. Given $\pi = \{S_1, ..., S_K\}$, $i \in N$, and $k \in \{1, ..., K\}$ define

$$\pi_{-i,k} = \begin{cases} [\pi \backslash (\{S_\pi(i)\} \cup \{S_k\})] \cup \{\{S_\pi(i)\} \backslash \{i\}\} \cup \{S_k \cup \{i\}\} & \text{if } S_k \neq S_\pi(i) \\ [\pi \backslash \{S_\pi(i)\}] \cup \{\{S_\pi(i)\} \backslash \{i\}\} \cup \{i\} & \text{if } S_k = S_\pi(i). \end{cases}$$

Observe that if $i \notin S_k$ then $\pi_{-i,k}$ is the new partition where all coalitions remain the same except that $S_\pi(i)$ looses i and S_k gains i. But if $i \in S_k$ then $\pi_{-i,k}$ is the new partition where all coalitions remain the same except that $S_\pi(i)$ looses i and $\{i\}$ is itself one of the elements of the partition.

Definition 1 Let $\Phi = (\mu, f)$ be a rule and let b be a profile. Take $\mu(b) = \{S_1, ..., S_K\} \equiv \pi$. We say that Φ is *stable at* $b \in [0,1]^n$ if there do not exist $i \in N$ and $k \in \{1, ..., K\}$ such that:

(1) $f_i(\pi_{-i,k}, b) P_i f_i(\pi, b)$, and

(2) if $S_k \neq S_\pi(i)$ then, for all $j \in S_k$, $f_j(\pi_{-i,k}, b) R_j f_j(\pi, b)$.[6]

A rule $\Phi = (\mu, f)$ is *stable* if it is stable at all $b \in [0,1]^n$.

Up to now all the definitions allowed for each subset $S \subseteq N$ to have its own distinct solution f^S. As we have already said in the Introduction, we restrict our analysis to the case in which a unique solution is applied to every coalition in the partition. This requirement implies that the same principles are used across coalitions and can be interpreted as a consistency requirement.

3 Stable Rules

In the following subsections we study the stability of rules associated to two well-known solutions: the proportional and the sequential dictator solutions.

3.1 Proportional Solution

The *proportional solution* $p = \{p^S\}_{S \subseteq N}$ is defined as follows: for each non-empty subset of agents $S \subseteq N$, each $b_S \in [0,1]^s$ and $i \in S$,

$$p_i^S(b_S) = \begin{cases} \frac{b_i}{\sum_{j \in S} b_j} & \text{if } \sum_{j \in S} b_j \neq 0 \\ \frac{1}{\# S} & \text{otherwise.} \end{cases}$$

[6] If there exist i and k such that (1) and (2) hold we say i wants to leave $S_\pi(i)$ to join S_k and all agents in S_k want to admit i.

Proposition 1 *There exists a partition function μ^p such that the rule $P = (\mu^p, p)$ is stable.*

Proof By induction on n.

► Assume $n = 1$. The stability of $P = (\mu^p, p)$ is obvious.

Induction Hypothesis: For all S with $1 \leq \#S < n$ there exists a partition function μ^S such that $(\mu^S, p = \{p^T\}_{T \subseteq S})$ is stable.

► Consider N and an arbitrary $b \in [0,1]^n$. Define $\mu^p(b)$ as follows.

First, assume $b_i = 0$ for all $i \in N$. Then, set $\mu^p(b) = \{\{N\}\}$. Obviously, (μ^p, p) is stable at b.

Assume now that $b_i > 0$ for some $i \in N$. Take an arbitrary $S_1 \subseteq N$ with the property that

$$S_1 \in AM \equiv \arg \min_{\substack{S \subseteq N \\ \exists j \in S \text{ s.t. } b_j > 0}} \frac{\left|\sum_{j \in S} b_j - 1\right|}{\sum_{j \in S} b_j}$$

and $\#S_1 \geq \#T$ for all $T \in AM$. Observe that if $b_i = 0$, $i \in S_1$. If $S_1 = N$, set $\mu^p(b) = \{\{N\}\}$. By the definition of S_1, $P(b)$ is stable at b. Assume now that $N \backslash S_1 \neq \varnothing$ and consider the subprofile $b_{N \backslash S_1}$. Define $\mu^p(b) = \left\{\mu^{N \backslash S_1}(b_{N \backslash S_1}), S_1\right\}$. We next show that $(\mu^p(b), p(b))$ is an stable allocation at b.

First, by definition of S_1, for all $i \in S_1$ and for any $T \in \mu^{N \backslash S_1}(b_{N \backslash S_1})$,

$$\frac{\left|\sum_{j \in S_1} b_j - 1\right|}{\sum_{j \in S_1} b_j} \leq \frac{\left|\sum_{j \in T} b_j + b_i - 1\right|}{\sum_{j \in T} b_j + b_i}.$$

Then,

$$\left|\frac{b_i}{\sum_{j \in S_1} b_j} - b_i\right| = b_i \left|\frac{\sum_{j \in S_1} b_j - 1}{\sum_{j \in S_1} b_j}\right| \leq b_i \left|\frac{\sum_{j \in T} b_j + b_i - 1}{\sum_{j \in T} b_j + b_i}\right| = \left|\frac{b_i}{\sum_{j \in T} b_j + b_i} - b_i\right|.$$

Thus, $p_i^{S_1}(b_{S_1}) R_i p_i^{T \cup \{i\}}(b_{T \cup \{i\}})$. Hence, i does not want to leave S_1 to join T.

Second, take $k \in N \backslash S_1$. By definition of S_1,

$$\frac{\left|\sum_{j \in S_1} b_j + b_k - 1\right|}{\sum_{j \in S_1} b_j + b_k} > \frac{\left|\sum_{j \in S_1} b_j - 1\right|}{\sum_{j \in S_1} b_j}.$$

Notice that the inequality is strict since S_1 has the largest size among all sets in AM. Then, consider any $i \in S_1$ such that $b_i > 0$. By the definition of S_1, there exists at least one agent with this property. Then,

$$\left|\frac{b_i}{\sum_{j \in S_1} b_j + b_k} - b_i\right| > \left|\frac{b_i}{\sum_{j \in S_1} b_j} - b_i\right|.$$

Thus, $p_i^{S_1}(b_{S_1}) P_i p_i^{S_1 \cup \{k\}}(b_{S_1 \cup \{k\}})$. Hence, $i \in S_1$ does not want to admit $k \in N \backslash S_1$ in S_1.

Third, by the induction hypothesis, $(\mu^{N \backslash S_1}(b_{N \backslash S_1}), p^{N \backslash S_1}(b_{N \backslash S_1}))$ is an stable allocation at $b_{N \backslash S_1}$; namely, for all $S \in \pi^{N \backslash S_1}(b_{N \backslash S_1})$ and $k \in N \backslash S_1$ such that $k \notin S$, either k does not want to join S or there is some agent in S that does not want to admit k.

Finally, we check that no agent in S_1 wants to leave and form a singleton coalition; namely, for all $i \in S_1$, $p_i^{S_1}(b_{S_1}) R_i p_i^{\{i\}}(b_i)$. If $b_i = 0$, the weak preference follows immediately. Assume $b_i > 0$.

Then, the weak preference also holds because, by the definition of S_1,

$$\left|\frac{b_i}{\sum_{j\in S_1} b_j} - b_i\right| = b_i \frac{\left|1-\sum_{j\in S_1} b_j\right|}{\sum_{j\in S_1} b_j} \le b_i \frac{|1-b_i|}{b_i} = |1-b_i|.$$

Hence, $P = (\mu^p, p)$ is an stable rule. ■

We finish this subsection by showing that not all weighted proportional rules are stable. To see that, let $w = (w_1, ..., w_n) \in (0,1)^n$ be a vector of weights such that $\sum_{i\in N} w_i = 1$. The *weighted proportional solution* $wp = \{wp^S\}_{S\subseteq N}$ is defined as follows: for each non-empty subset of agents $S \subseteq N$, each $b_S \in [0,1]^s$ and $i \in S$,

$$wp_i^S(b_S) = \begin{cases} \frac{w_i b_i}{\sum_{j\in S} w_j b_j} & \text{if } \sum_{j\in S} b_j \neq 0 \\ \frac{w_i}{\sum_{j\in S} w_j} & \text{otherwise.} \end{cases}$$

Given a partition function μ, define the *weighted proportional rule* as $W^\mu = (\mu, wp)$.

The following example shows that there are vectors of weights w such that there is no partition function μ for which the weighted proportional rule $W^\mu = (\mu, wp)$ is stable.

Example 1 Let $N = \{1,2,3\}$ and consider the vector of weights $w = (0.4, 0.2, 0.4)$. Take the profile $b = (0.8, 0.5, 0.4)$. Then, the allocations corresponding to the five possible partitions functions are: $W^{\mu_1}(b) = (\{1,2,3\}, (0.552, 0.172, 0.276))$, $W^{\mu_2}(b) = (\{\{1\},\{2,3\}\}, (1, 0.385, 0.615))$, $W^{\mu_3}(b) = (\{\{1,3\},\{2\}\}, (0.667, 1, 0.333))$, $W^{\mu_4}(b) = (\{\{1,2\},\{3\}\}, (0.762, 0.238, 1))$ and $W^{\mu_5}(b) = (\{\{1\},\{2\},\{3\}\}, (1,1,1))$. But the strict preference $wp_1(\mu_2(b), b) P_1 wp_1(\mu_1(b), b)$ implies that if all agents were in the same coalition, agent 1 would prefer to leave and set a new coalition by himself. Thus, W^{μ_1} is not an stable rule. The two strict preferences $wp_3(\mu_3(b), b) P_3 wp_3(\mu_2(b), b)$ and $wp_1(\mu_3(b), b) P_1 wp(\mu_2(b), b)$ imply that if the partition was $\{\{1\},\{2,3\}\}$ then agent 3 would rather join agent 1 and agent 1 would be happy to admit him. Thus, W^{μ_2} is not an stable rule. Similarly, the three pairs (i) $wp_1(\mu_4(b), b) P_1 wp_1(\mu_3(b), b)$ and $wp_2(\mu_4(b), b) P_2 wp_2(\mu_3(b), b)$, (ii) $wp_2(\mu_2(b), b) P_2 wp_2(\mu_4(b), b)$ and $wp_3(\mu_2(b), b) P_3$ $wp_3(\mu_4(b), b)$, and (iii) $wp_1(\mu_3(b), b) P_1 wp_1(\mu_5(b), b)$ and $wp_3(\mu_3(b), b) P_3 wp_3(\mu_5(b), b)$ imply that W^{μ_3}, W^{μ_4}, and W^{μ_5} are not stable rules, respectively. □

3.2 Sequential Dictator Solutions

Let $\sigma : N \to N$ be a one-to-one mapping defining an ordering on the set of agents N; namely, for $i, j \in N$, $\sigma(i) < \sigma(j)$ means that i goes before j in the ordering σ. Fix σ and $S \neq \varnothing$. Define the *sequential dictator solution associated to* σ *for* S, denoted by $\sigma d^S : [0,1]^s \to [0,1]^s$, as follows: for each $b_S \in [0,1]^s$ and $i \in S$,

$$\sigma d_i^S(b_S) = \begin{cases} \min\{b_i, \max\{1 - \sum_{\{j\in S|\sigma(j)<\sigma(i)\}} b_j, 0\}\} & \text{if } i \text{ is s.t. } \exists j \in S,\ \sigma(i) < \sigma(j) \\ \max\{1 - \sum_{\{j\in S|\sigma(j)<\sigma(i)\}} b_j, 0\} & \text{otherwise.} \end{cases}$$

The *sequential dictator solution associated to* σ is the family $\sigma d = \{\sigma d^S\}_{S\subseteq N}$, where for each non-empty subset $S \subseteq N$, σd^S is a sequential dictator solution associated to σ for S.

Proposition 2 *Let* σ *be an ordering on* N. *Then, there exists a partition function* $\mu^{\sigma d}$ *such that the rule* $\sigma D = (\mu^{\sigma d}, \sigma d)$ *is stable.*

Proof Without loss of generality we assume that σ is such that $\sigma(i) = i$ for all $i \in N$. In the proof we will omit the reference to the ordering σ. The proof is by induction on n.

▶ Assume $n = 1$. The stability of $\sigma D = (\mu^{\sigma d}, \sigma d)$ is obvious.

Induction Hypothesis: For all $S \subset N$ with $1 \leq \#S < n$, there exists a partition function $\mu^{d,S}$ such that $(\mu^{d,S}, d = \{d^T\}_{T \subseteq S})$ is stable.

▶ Given N and $b \in [0,1]^n$, select

$$\bar{S} \in \overline{AM} = \arg \min_{\substack{S \subset N \\ n \in S}} \left| 1 - \sum_{i \in S} b_i \right|$$

with the property that $\#\bar{S} \geq \#S$ for all $S \in \overline{AM}$. Hence, if $b_j = 0$ then,

$$j \in \bar{S}. \tag{1}$$

Let $v = \left|1 - \sum_{i \in \bar{S}} b_i\right| = \min_{S \subset N \backslash \{n\}} \left|1 - \sum_{i \in S} b_i - b_n\right|$. In particular

$$v \leq |1 - b_n|. \tag{2}$$

We will consider the following cases:

Case 1: $v \geq b_n$.
Notice that $\left|1 - \sum_{i \in T} b_i - b_n\right| \geq v$ for all $T \subseteq N \backslash \{n\}$. By the induction hypothesis the allocation $(\mu^{d,N\backslash\{n\}}(b_{N\backslash\{n\}}), d^{N\backslash\{n\}}(b_{N\backslash\{n\}}))$ is stable. Let $\hat{S}$ be such that $\hat{S} \in \mu^{d,N\backslash\{n\}}(b_{N\backslash\{n\}})$ and, for all $S \in \mu^{d,N\backslash\{n\}}(b_{N\backslash\{n\}})$, $\sum_{i \in S} b_i \leq \sum_{i \in \hat{S}} b_i$. Define $\mu^{d,N}(b) = \left\{\left\{\mu^{d,N\backslash\{n\}}(b_{N\backslash\{n\}}) \backslash \hat{S}\right\}, \hat{S} \cup \{n\}\right\}$. We want to show that $(\mu^{d,N}(b), d^N(b))$ is stable at b.

Assume first that $\hat{S} \cup \{n\} = N$. Observe that

$$d_n^N(b) = \begin{cases} 0 & \text{if } \sum_{i \in N \backslash \{n\}} b_i \geq 1 \\ 1 - \sum_{i \in N \backslash \{n\}} b_i & \text{otherwise.} \end{cases}$$

Since $v \geq b_n$, for all $j < n$, $d_j^{N\backslash\{n\}}(b_{N\backslash\{n\}}) = d_j^N(b)$. By the induction hypothesis, $\left|d_j^{N\backslash\{n\}}(b_{N\backslash\{n\}}) - b_j\right| \leq |1 - b_j|$. Thus, $\left|d_j^N(b) - b_j\right| \leq |1 - b_j|$; namely, j does not want to leave the set $\hat{S} \cup \{n\} = N$.

We will show that n does not want to leave N either. Assume the contrary,

$$|1 - b_n| < \left|d_n^N(b) - b_n\right|. \tag{3}$$

If $d_n^N(b) = 0$ then, $|1 - b_n| < |0 - b_n|$; *i.e.*, $b_n > \frac{1}{2}$. By the hypothesis of Case 1 and (2), $\frac{1}{2} < b_n \leq v \leq 1 - b_n$, a contradiction. If $d_n^N(b) = 1 - \sum_{i \in N \backslash \{n\}} b_i > 0$ then, by (3), $b_n \leq v \leq |1 - b_n| < \left|1 - \sum_{i \in N \backslash \{n\}} b_i - b_n\right|$. If $1 - \sum_{i \in N\backslash\{n\}} b_i - b_n > 0$ then, $1 - b_n < 1 - \sum_{i \in N \backslash \{n\}} b_i - b_n$, a contradiction. If $1 - \sum_{i \in N \backslash \{n\}} b_i - b_n < 0$ then, by the alternative definition of v as the $\min_{S \subset N \backslash \{n\}} \left|1 - \sum_{i \in S} b_i - b_n\right|$ and the fact that $v \geq b_n$, $b_n \leq \sum_{i \in N \backslash \{n\}} b_i + b_n - 1$. Hence, $1 - \sum_{i \in N \backslash \{n\}} b_i \leq 0$, a contradiction. Thus, $(\{\{N\}\}, d^N(b))$ is an stable allocation at b.

Assume now that $N \backslash (\hat{S} \cup \{n\}) \neq \varnothing$. We distinguish between the following two subcases.

Subcase 1.1: $\sum_{i \in \hat{S}} b_i \geq 1$.

Then, $d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) = 0$ and for all $i \in \hat{S}$,

$$d_i^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) = d_i^{\hat{S}}(b_{\hat{S}}). \tag{4}$$

First, by (4) and the induction hypothesis, no agent in $\hat{S}$ wants to leave $\hat{S}$. Moreover, by the hypothesis of Case 1 and (2), $b_n \leq \frac{1}{2}$ and hence, n does not want to leave $\hat{S} \cup \{n\}$ and to form a singleton coalition.

Second, take any $j \in N \backslash (\hat{S} \cup \{n\})$. Observe that $j < n$. Then, since $\sum_{i \in \hat{S}} b_i \geq 1$,

$$d_j^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}}) = d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}). \tag{5}$$

By the induction hypothesis, the allocation $(\mu^{d,N\backslash\{n\}}(b_{N\backslash\{n\}}), d^{N\backslash\{n\}}(b_{N\backslash\{n\}}))$ is stable. Define $\pi^{N\backslash\{n\}} = \mu^{d,N\backslash\{n\}}(b_{N\backslash\{n\}})$. Hence, and since $\hat{S} \in \pi^{N\backslash\{n\}}$ and $j \notin \hat{S}$, either there exists $i \in \hat{S}$ such that

$$d_i^{\hat{S}}(b_{\hat{S}}) P_i d_i^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}})$$

or else

$$d_j^{S_{\pi^{N\backslash\{n\}}}(j)}(b_{S_{\pi^{N\backslash\{n\}}}(j)}) R_j d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}).$$

Thus, by (4) and (5), either

$$d_i^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) P_i d_i^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}})$$

or else

$$d_j^{S_{\pi^{N\backslash\{n\}}}(j)}(b_{S_{\pi^{N\backslash\{n\}}}(j)}) R_j d_j^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}}).$$

Namely, either j is not admitted in $\hat{S} \cup \{n\}$ or else j does not want to leave $S_{\pi^{N\backslash\{n\}}}(j)$ to join $\hat{S} \cup \{n\}$.

Third, take any $T \in \mu^{d,N}(b) \backslash (\hat{S} \cup \{n\})$ and consider the coalition $T \cup \{n\}$. If $\sum_{i \in T} b_i \geq 1$ then $d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) = d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) = 0$ and n does not want to leave $\hat{S}\cup\{n\}$ to join T. If $\sum_{i\in T} b_i < 1$ then, $d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) = 1 - \sum_{i\in T} b_i$. Since $\left| d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) - b_n \right| = \left| 1 - \sum_{i\in T} b_i - b_n \right| \geq v \geq b_n = \left| d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) - b_n \right|$, $d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) R_n d_n^{T\cup\{n\}}(b_{T\cup\{n\}})$. Thus, n does not want to leave $\hat{S} \cup \{n\}$ to join T.

SUBCASE 1.2: $\sum_{i\in\hat{S}} b_i < 1$.
Notice that, by the definition of $\hat{S}$, for all $S \in \pi^{N\backslash\{n\}}$,

$$\sum_{i \in S} b_i < 1. \tag{6}$$

First, take $j \in S' \in \mu^{d,N}(b) \backslash (\hat{S} \cup \{n\})$. Using the fact that, by the induction hypothesis, j and $\hat{S}$ did not generate an instability in the allocation $(\pi^{N\backslash\{n\}}, d^{N\backslash\{n\}}(b_{N\backslash\{n\}}))$ we will show that j and $\hat{S} \cup \{n\}$ do not generate an instability in the allocation $(\mu^{d,N}(b), d^N(b))$. Assume

$$d_j^{S'}(b_{S'}) \neq b_j; \tag{7}$$

otherwise, j does not want to leave S'. By (6), $\sum_{i\in S'} b_i < 1$. Hence, $j = \max_{i\in S'} i$ and $d_j^{S'}(b_{S'}) = 1 - \sum_{i \in S' \backslash \{j\}} b_i > b_j$.

Assume $\sum_{i\in\hat{S}} b_i + b_j \geq 1$. Then, since $\sum_{i\in\hat{S}} b_i < 1$ and $i < n$ for all $i \in \hat{S}$,

$$d_i^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) = d_i^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}}). \tag{8}$$

By the stability of the allocation $(\pi^{N\setminus\{n\}}, d^{N\setminus\{n\}}(b_{N\setminus\{n\}}))$ either

$$\left|d_j^{S'}(b_{S'}) - b_j\right| \leq \left|d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) - b_j\right| \tag{9}$$

or else there must exist $i' \in \hat{S}$ such that

$$\left|d_{i'}^{\hat{S}}(b_{\hat{S}}) - b_{i'}\right| < \left|d_{i'}^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) - b_{i'}\right|. \tag{10}$$

If (9) holds then, by (8),

$$\left|d_j^{S'}(b_{S'}) - b_j\right| \leq \left|d_j^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}}) - b_j\right|, \tag{11}$$

namely, j does not want to leave S' to join $\hat{S}\cup\{n\}$. Assume (9) does not hold; *i.e.* j wants lo leave S' to join $\hat{S}$. Then, (10) holds. Assume that j wants to leave S' to join $\hat{S}\cup\{n\}$; that is,

$$\left|d_j^{S'}(b_{S'}) - b_j\right| > \left|d_j^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}}) - b_j\right|.$$

Then, by (10), (8) and $d_{i'}^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) = b_{i'}$,

$$\begin{aligned}\left|d_{i'}^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) - b_{i'}\right| &= 0 \leq \left|d_{i'}^{\hat{S}}(b_{\hat{S}}) - b_{i'}\right| \\ &< \left|d_{i'}^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) - b_{i'}\right| \\ &= \left|d_{i'}^{\hat{S}\cup\{j\}\cup\{n\}}(b_{\hat{S}\cup\{j\}\cup\{n\}}) - b_{i'}\right|.\end{aligned}$$

Thus, i' does not want to admit j in the coalition $\hat{S}\cup\{n\}$.

Assume $\sum_{i\in\hat{S}} b_i + b_j < 1$. Then,

$$d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) = \begin{cases} b_j & \text{if } j \text{ is not the last in } \hat{S}\cup\{j\} \\ 1 - \sum_{i\in\hat{S}} b_i & \text{if } j \text{ is the last in } \hat{S}\cup\{j\}. \end{cases}$$

Assume first that j is not the last in $\hat{S}\cup\{j\}$. Then, $d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) = b_j$. By (7),

$$d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) P_j d_j^{S'}(b_{S'}). \tag{12}$$

Let $j^* = \max_{i\in\hat{S}} i > j$. For all $i \in \hat{S}\setminus\{j^*\}$,

$$d_i^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) = d_i^{\hat{S}}(b_{\hat{S}}). \tag{13}$$

Moreover, since $d_{j^*}^{\hat{S}}(b_{\hat{S}}) = 1 - \sum_{i\in\hat{S}} b_i + b_{j^*}$ and $d_{j^*}^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) = 1 - \sum_{i\in\hat{S}} b_i + b_{j^*} - b_j$,

$$\left|d_{j^*}^{\hat{S}}(b_{\hat{S}}) - b_{j^*}\right| = \left|1 - \sum_{i\in\hat{S}} b_i\right| \geq \left|1 - \sum_{i\in\hat{S}} b_i - b_j\right| = \left|d_{j^*}^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) - b_{j^*}\right|.$$

Hence,

$$d_{j^*}^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}})R_{j^*}d_{j^*}^{\hat{S}}(b_{\hat{S}}). \tag{14}$$

Conditions (12), (13), and (14) imply that j wants to leave S' to join $\hat{S}$ and all agents in $\hat{S}$ want to admit j, contradicting that $(\pi^{N\setminus\{n\}}, d^{N\setminus\{n\}}(b_{N\setminus\{n\}}))$ is an stable allocation.

Assume now that j is the last in $\hat{S}\cup\{j\}$. Hence, for all $i \in \hat{S}$, $d_i^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}})R_i d_i^{\hat{S}}(b_{\hat{S}})$. Moreover, since $\sum_{i\in S'} b_i \leq \sum_{i\in\hat{S}} b_i$ and $\sum_{i\in\hat{S}} b_i + b_j < 1$, if either $b_j \neq 0$ or $\sum_{i\in S'} b_i < \sum_{i\in\hat{S}} b_i$ then,

$$1 - \sum_{i\in S'} b_i + b_j > 1 - \sum_{i\in\hat{S}} b_i > b_j.$$

Since $d_j^{S'}(b_{S'}) = 1 - \sum_{i\in S'} b_i + b_j$ and $d_j^{\hat{S}\cup\{j\}}(b_{\hat{S}\cup\{j\}}) = 1 - \sum_{i\in\hat{S}} b_i$, j wants to leave S' to join $\hat{S}$ and all agents in $\hat{S}$ want to admit j, contradicting that $(\pi^{N\setminus\{n\}}, d^{N\setminus\{n\}}(b_{N\setminus\{n\}}))$ is an stable allocation. Assume now that $b_j = 0$ and $\sum_{i\in S'} b_i = \sum_{i\in\hat{S}} b_i$. We can prove that j^* wants to leave $\hat{S}$ to join S' and no agent of S' rejects j^*. If $b_{j^*} = 0$ then, we get a contradiction of the stability of $(\pi^{N\setminus\{n\}}, d^{N\setminus\{n\}}(b_{N\setminus\{n\}}))$ proceeding as above. If $b_{j^*} > 0$ then, changing the roles of $\hat{S}$ and S' and j^* and j we will get a contradiction of the stability of $(\pi^{N\setminus\{n\}}, d^{N\setminus\{n\}}(b_{N\setminus\{n\}}))$. Observe that this can be done since $b_j = 0$ and $\sum_{i\in S'} b_i = \sum_{i\in\hat{S}} b_i$.

Second, take any $T \in \mu^{d,N}(b)\setminus(\hat{S}\cup\{n\})$. We want to check that n does not want to leave $\hat{S}\cup\{n\}$ to join T. Since, by definition of $\hat{S}$, $\sum_{i\in T} b_i \leq \sum_{i\in\hat{S}} b_i$,

$$d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) = 1 - \sum_{i\in T} b_i \geq 1 - \sum_{i\in\hat{S}} b_i = d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) > b_n,$$

where the strict inequality follows because otherwise, if $d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}}) \leq b_n$ then, $1 - \sum_{i\in\hat{S}} b_i - b_n \leq 0$, implying that

$$b_n \leq v = \left|1 - \sum_{i\in\bar{S}} b_i\right| \leq \left|1 - \sum_{i\in\hat{S}} b_i - b_n\right| = \sum_{i\in\hat{S}} b_i + b_n - 1,$$

a contradiction with the hypothesis of Subcase 1.2 stating that $\sum_{i\in\hat{S}} b_i < 1$. Thus, by single-peakedness, $d_n^{\hat{S}\cup\{n\}}(b_{\hat{S}\cup\{n\}})R_n d_n^{T\cup\{n\}}(b_{T\cup\{n\}})$. Hence, n does not want to leave $\hat{S}\cup\{n\}$ to join coalition T.

Third, we show that n does not want to leave $\hat{S}\cup\{n\}$ to form a singleton coalition. Assume otherwise; *i.e.*,

$$|1 - b_n| < \left|1 - \sum_{i\in\hat{S}} b_i - b_n\right| \tag{15}$$

holds. If $1 - \sum_{i\in\hat{S}} b_i - b_n > 0$ then, (15) implies that $\sum_{i\in\hat{S}} b_i < 0$, a contradiction. If $1 - \sum_{i\in\hat{S}} b_i - b_n \leq 0$ then, and since $b_n \leq v \leq |1 - b_n|$, (15) implies that $b_n \leq \sum_{i\in\hat{S}} b_i + b_n - 1$, a contradiction with $\sum_{i\in\hat{S}} b_i < 1$.

Case 2: $v < b_n$.

Recall that $v = \left|1 - \sum_{i\in\bar{S}} b_i\right|$ where $\bar{S} \in \overline{AM} = \arg\min_{\substack{S\subset N\\ n\in S}} \left|1 - \sum_{i\in S} b_i\right|$ with the property that $\#\bar{S} \geq \#S$ for all $S \in \overline{AM}$. Observe that

$$\sum_{i\in\bar{S}\setminus\{n\}} b_i < 1; \tag{16}$$

otherwise, if $\sum_{i\in\bar{S}\backslash\{n\}} b_i \geq 1$ then, $v = \sum_{i\in\bar{S}\backslash\{n\}} b_i + b_n - 1$. Hence, $v - b_n = \sum_{i\in\bar{S}\backslash\{n\}} b_i - 1 \geq 0$, which contradicts the assumption that $v < b_n$.

First, assume that $\bar{S} = N$. Define $\mu^d(b) = \{\{N\}\}$. To obtain a contradiction, suppose that $(\{\{N\}\}, d^N(b))$ is not an stable allocation. By (16), for all $i \neq n$, $d_i^N(b) = b_i$. Hence, it has to be agent n who wants to leave N to form a singleton coalition; that is, $d_n^{\{n\}}(b_n) P_n d_n^N(b)$, or equivalently, $|1 - b_n| < \left|1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i - b_n\right| = v$, a contradiction with (2).

Thus assume that $\bar{S} \subsetneq N$. By the induction hypothesis, $(\mu^{d,N\backslash\bar{S}}(b_{N\backslash\bar{S}}), d^{N\backslash\bar{S}}(b_{N\backslash\bar{S}}))$ is an stable allocation. Define $\mu^{d,N}(b) = \mu^{d,N\backslash\bar{S}}(b_{N\backslash\bar{S}}) \cup \bar{S} \equiv \pi^N$. To prove that the allocation $(\pi^N, d^N(b))$ is stable, we first check that n does not want to leave $\bar{S}$ to form a singleton coalition. Assume otherwise; then, by (16), $|1 - b_n| < \left|1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i - b_n\right| = v$, a contradiction with (2).

We now distinguish between the following two subcases.

Subcase 2.1: $\sum_{i\in\bar{S}} b_i > 1$.

By (16), $d_n^{\bar{S}}(b_{\bar{S}}) = b_n - v > 0$.

First, take $j \in N\backslash\bar{S}$ and consider the coalition $\bar{S} \cup \{j\}$. Then, $d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}}) = \max\{1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i - b_j, 0\} < b_n - v = d_n^{\bar{S}}(b_{\bar{S}}) < b_n$, where the first strict inequality holds because by (1), $j \notin \bar{S}$ implies $b_j > 0$ and $v = \sum_{i\in\bar{S}} b_i - 1$ implies that $1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i = b_n - v$. Thus, by symmetric single-peakedness, $d_n^{\bar{S}}(b_{\bar{S}}) P_n d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}})$. Hence, n does not want to admit j in $\bar{S}$.

Second, by (16), $d_i^{\bar{S}}(b_{\bar{S}}) = b_i$, for all $i \in \bar{S}\backslash\{n\}$. Hence, no agent in $\bar{S}\backslash\{n\}$ wants to leave $\bar{S}$ to join any other coalition.

Third, let $T \in \mu^{d,N\backslash\bar{S}}(b_{N\backslash\bar{S}})$ and consider the coalition $T \cup \{n\}$. Then, $d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) = \max\{1 - \sum_{i\in T} b_i, 0\}$. If $1 - \sum_{i\in T} b_i \geq 0$ then, by the definition of $\bar{S}$, $\left|d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) - b_n\right| \geq \left|1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i - b_n\right| = v = \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$, and if $1 - \sum_{i\in T} b_i < 0$, then $|0 - b_n| = b_n > v = \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$. In the two cases we have that $d_n^{\bar{S}}(b_{\bar{S}}) R_n d_n^{T\cup\{n\}}(b_{T\cup\{n\}})$. Hence, n does not want to leave $\bar{S}$ to join T.

Subcase 2.2: $\sum_{i\in\bar{S}} b_i < 1$.

By definition of $\bar{S}$, $d_n^{\bar{S}}(b_{\bar{S}}) = b_n + v = 1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i > b_n$.

First, take $j \in N\backslash\bar{S}$ and consider the coalition $\bar{S} \cup \{j\}$. Then, $d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}}) = \max\{1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i - b_j, 0\}$. If $d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}}) = 0$ then, $\left|d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}}) - b_n\right| = b_n > v = \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$. If $d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}}) = 1 - \sum_{i\in\bar{S}\backslash\{n\}} b_i - b_j > 0$ then, $\left|d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\cup\{j\}}) - b_n\right| = \left|1 - \sum_{i\in\bar{S}\cup\{j\}} b_i\right| > v = \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$, where the strict inequality holds because, by (1), $j \notin \bar{S}$ implies $b_j > 0$. Thus, in both cases $d_n^{\bar{S}}(b_{\bar{S}}) P_n d_n^{\bar{S}\cup\{j\}}(b_{\bar{S}\{j\}})$. Hence, n does not want to admit j in $\bar{S}$.

Second, and since $d_i^{\bar{S}}(b_{\bar{S}}) = b_i$ for all $i \in \bar{S}\backslash\{n\}$, no agent in $\bar{S}\backslash\{n\}$ wants to leave $\bar{S}$ to join any other coalition.

Third, let $T \in \mu^{d,N\backslash\bar{S}}(b_{N\backslash\bar{S}})$ and consider the coalition $T \cup \{n\}$. Then, $d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) = \max\{1 - \sum_{i\in T} b_i, 0\}$. If $1 - \sum_{i\in T} b_i > 0$, and since $\left|1 - \sum_{i\in T} b_i - b_n\right| \geq v$ by definition of $\bar{S}$, we have that $\left|1 - \sum_{i\in T} b_i - b_n\right| \geq v = \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$. If $1 - \sum_{i\in T} b_i \leq 0$ then, $|0 - b_n| = b_n > v = \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$. Thus, in both situations $\left|d_n^{T\cup\{n\}}(b_{T\cup\{n\}}) - b_n\right| \geq \left|d_n^{\bar{S}}(b_{\bar{S}}) - b_n\right|$, which implies that $d_n^{\bar{S}}(b_{\bar{S}}) R_n d_n^{T\cup\{n\}}(b_{T\cup\{n\}})$. Hence, n does not want to leave $\bar{S}$ to join T. ■

4 Final Comments

We finish the paper with three comments.

First, we show in Remarks 1 and 2 below that stability is incompatible with other desirable properties such as strategy-proofness, efficiency, anonymity and envy-freeness, bit for the proportional and the sequential dictator solutions. But before, we state these properties formally.

Strategy-proofness says that no agent obtains a better share by misreporting his best share.

Definition 2 A rule $\Phi = (\mu, f)$ is *manipulable* at $b \in [0,1]^n$ if there exist $i \in N$ and $b_i' \in [0,1]$ such that

$$f_i(\mu(b_i', b_{-i}), (b_i', b_{-i})) P_i f_i(\mu(b_i, b_{-i}), (b_i, b_{-i})).$$

A rule $\Phi = (\mu, f)$ is *strategy-proof* if it is not manipulable at any $b \in [0,1]^n$.

Efficiency says that the rule always selects efficient allocations.

Definition 3 An allocation $(\pi, x) \in A$ is *efficient* at profile $b \in [0,1]^n$ if it does not have a Pareto improvement; that is, there does not exist another allocation $(\gamma, y) \in A$ such that $y_i R_i x_i$ for all $i \in N$ and $y_j P_j x_j$ for at least one $j \in N$.

A rule $\Phi = (\mu, f)$ is *efficient* if for all $b \in [0,1]^n$, $\Phi(b)$ is an efficient allocation at b.

Anonymity says that the name of the agents should not matter. Let $\sigma : N \to N$ be a one-to-one mapping and let $b \in [0,1]^n$ be a profile. Define the profile $b^\sigma = (b_{\sigma(1)}, ..., b_{\sigma(n)}) \in [0,1]^n$.

Definition 4 A rule $\Phi = (\mu, f)$ is *anonymous* if for all one-to-one mapping $\sigma : N \to N$ and all $b \in [0,1]^n$, $f_i(\mu(b), b) = f_{\sigma(i)}(\mu(b^\sigma), b^\sigma)$ for all $i \in N$.

Envy-freeness says that no agent strictly prefers the share of another agent.

Definition 5 A rule $\Phi = (\mu, f)$ is *envy-free* if for all $b \in [0,1]^n$ and all $i, j \in N$, $f_i(\mu(b), b) R_i f_j(\mu(b), b)$.

Remark 1 There is no partition function μ for which the rule (μ, p) is stable and satisfies one of the following properties: strategy-proofness, efficiency, anonymity and envy-freeness.

To see that stability and strategy-proofness are incompatible, let $N = \{1, 2\}$ and consider the profile $b = (0.5, 0.4)$. The only stable allocation is $(\mu^p(b), p(\mu^p(b), b)) = (\{\{1,2\}\}, (0.56, 0.44))$. Let $b_2' = 0,33$. Then, $\mu^p(b_1, b_2') = \{\{1,2\}\}$ and $p(\mu^p(b_1, b_2'), (b_1, b_2')) = (0.6, 0.4)$. Thus, $0.4 = p(\mu^p(b_1, b_2'), (b_1, b_2')) P_2 p(\mu^p(b), b) = 0.44$ and hence (μ^p, p) is not strategy-proof.

To see that stability and efficiency are incompatible, let $N = \{1, 2\}$ and consider the profile $b = (0.4, 0.9)$. Now, the only stable allocation is $(\mu^p(b), p(\mu^p(b), b)) = (\{\{1\}, \{2\}\}, (1, 1))$. However, the allocation $(\{\{1, 2\}\}, (0.1, 0.9))$ is a Pareto improvement.

To see that stability and anonymity or envy-freeness are incompatible, let $N = \{1, 2, 3\}$ and consider the profile $b = (0.3, 0.3, 0.8)$. It is easy to see that the only stable allocations are $(\{\{1\}, \{2, 3\}\}, (1, \frac{0.3}{1.1}, \frac{0.8}{1.1})$ and $(\{\{2\}, \{1, 3\}\}, (\frac{0.3}{1.1}, 1, \frac{0.8}{1.1})$. Agent 1 and agent 2 have the same best-shares but in both allocations they receive a different share and either agent 1 envies agent 2 (first allocation) or the opposite (second allocation). □

Remark 2 Fix an ordering σ on N. There is no partition function μ for which the rule $(\mu, \sigma d)$ is stable and satisfies one of the following properties: strategy-proofness, efficiency, anonymity and envy-freeness.

To see that stability and strategy-proofness are incompatible, we take $N = \{1,2,3\}$, without loss of generality $\sigma(i) = i$ for each $i = 1,2,3$, and $b = (0.6, 0.5, 0.6)$. The only stable allocation at b is $(\mu^{\sigma d}(b), \sigma d(b)) = (\{\{1\},\{2,3\}\}, (1, 0.5, 0.5))$. Take $b_1' = 0.4$. Since the only stable allocation at $b' = (b_1', b_2, b_3)$ is $(\mu^{\sigma d}(b'), \sigma d(b')) = (\{\{1,3\},\{2\}\}, (0.4, 1, 0.6))$ and $0.4 P_1 1$, we conclude that for every μ for which $(\mu, \sigma d)$ is stable, $(\mu, \sigma d)$ is not strategy-proof.

Moreover, stability and efficiency are incompatible. To see this, we take $N = \{1,2\}$, let $\sigma(1) = 1$, $\sigma(2) = 2$, and $b = (0.5, 0.8)$. The only stable allocation at b is $(\mu^{\sigma d}(b), \sigma d(b)) = (\{\{1\},\{2\}\}, (1,1))$, but the allocation $(\{\{1,2\}\}, (0.3, 0.7))$ is a Pareto improvement.

To see that stability and anonymity or envy-freeness are incompatible, let $N = \{1,2\}$, $\sigma(1) = 1$, $\sigma(2) = 2$ and consider the profile $b = (0.6, 0.6)$. Then, the stable allocation $(\{1,2\}, (0.6, 0.4))$ is stable but the two agents have the same best-shares, receive a different share and agent 2 envies agent 1. Finally, the allocation $(\{\{1\},\{2\}\}, (1,1))$ is not stable since 1 wants to join $\{2\}$ and 2 wants to admit 1. □

Second, and based on simulations, we conjecture that there exist stable rules associated to the uniform and the equal gain-losses solutions.

Third, it is possible to show that the following result holds. Let f be an efficient solution; then, there exists a partition function μ^{ef} such that (μ^{ef}, f) is efficient.[7] However, (μ^{ef}, f) may not be stable.

Acknowledgements

The work of G. Bergantiños is partially supported by research grant ECO2011-23460 from the Spanish Ministry of Science and Innovation and FEDER. J. Massó acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2011-0075) and through grant ECO2008-0475-FEDER (Grupo Consolidado-C), and from the Generalitat de Catalunya, through the prize "ICREA Academia" for excellence in research and grant SGR2009-419. The work of A. Neme is partially supported by the Universidad Nacional de San Luis, through grant 319502, and by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), through grant PIP 112-200801-00655. Part of this research was done while A. Neme visited the UAB thanks to the Generalitat de Catalunya grant SGR2009-419.

References

[1] Adachi, T. (2010). "The Uniform Rule with Several Commodities: A Generalization of Sprumont's Characterization," *Journal of Mathematical Economics* 46, 952-964.

[2] Amorós, P. (2002). "Single-peaked Preferences with Several Commodities," *Social Choice and Welfare* 19, 57-67.

[3] Barberà, S., Jackson, M. (1995)."Strategy-proof Exchange," *Econometrica* 63, 51-87.

[7] Kar and Kibris (2008) have a similar result in a setting where the number of goods to be shared is fixed. In our case however, the number of goods, and hence the number of coalitions in the partition, is endogenous and depend on the preferences of the agents.

[4] Barberà, S., Jackson, M., Neme, A. (1997). "Strategy-proof Allotment Rules," *Games and Economic Behavior* 18, 1-21.

[5] Ching, S. (1992). "A Simple Characterization of the Uniform Rule," *Economics Letters* 40, 57-60.

[6] Ching, S. (1994). "An Alternative Characterization of the Uniform Rule," *Social Choice and Welfare* 11, 131-136.

[7] Dagan, N. (1996)."A Note on Thomson's Characterizations of the Uniform Rule," *Journal of Economic Theory* 69, 255-261.

[8] Ehlers, L. (2002a)."On Fixed-Path Rationing Methods," *Journal of Economic Theory* 106, 172-177.

[9] Ehlers,. L. (2002b). "Resource-monotonic Allocation when Preferences Are Single-peaked," *Economic Theory* 20, 113-131.

[10] Gensemer, S., Hong, L., Kelly, J. (1996). "Division Rules and Migration Equilibrium," *Journal of Economic Theory* 69, 104-116.

[11] Gensemer, S., Hong, L., Kelly, J. (1998). "Migration Disequilibrium and Specific Division Rules," *Social Choice and Welfare* 15, 201-209.

[12] Herrero, C., Villar, A. (2000). "An Alternative Characterization of the Equal-distance Rule for Allocation Problems with Single-peaked Preferences,"*Economics Letters* 66, 311-317.

[13] Kar, A., Kibris, O. (2008)."Allocating Multiple Estates among Agents with Single-peaked Preferences," *Social Choice and Welfare* 31, 641-666.

[14] Massó, J., Moreno de Barreda, I. (2011). "On Strategy-proofness and Symmetric Single-peakedness," *Games and Economic Behavior* 72, 467-484.

[15] Morimoto, S., Serizawa, S., Ching, S. (2013). "A Characterization of the Uniform Rule with Several Commodities and Agents," *Social Choice and Welfare* 40, 871-911.

[16] Moulin, H. (1980). "On Strategy-proofness and Single Peakedness," *Public Choice* 35, 437-455.

[17] Schummer, J., Thomson, W. (1997). "Two Derivations of the Uniform Rule and an Application to Bankruptcy," *Economics Letters* 55, 333-337 .

[18] Sönmez, T. (1994). "Consistency, Monotonicity, and the Uniform Rule," *Economics Letters* 46, 229-235.

[19] Sprumont, Y. (1991). "The Division Problem with Single-peaked Preferences: A Characterization of the Uniform Allocation Rule," *Econometrica* 59, 509-519.

[20] Thomson, W. (1994). "Consistent Solutions to the Problem of Fair Division when Preferences are Single-peaked," *Journal of Economic Theory* 63, 219-245.

[21] Thomson, W. (1995). "Population Monotonic Solutions to the Problem of Fair Division when Preferences are Single-peaked," *Economic Theory* 5, 229-246.

[22] Thomson, W. (1997). “The Replacement Principle in Economies with Single-peaked Preferences,” *Journal of Economic Theory* 76, 145-168.

[23] Thomson, W. (2003). “Axiomatic and Game-theoretic Analysis of Bankruptcy and Taxation Problems: A Survey,” *Mathematical Social Sciences* 45, 249-297.