On the Structure of Cooperative and Competitive Solutions for a Generalized Assignment Game*

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Abstract: We study cooperative and competitive solutions for a many-to-many generalization of Shapley and Shubik (1972)'s assignment game. We consider the Core, three other notions of group stability and two alternative definitions of competitive equilibrium. We show that (i) each group stable set is closely related with the Core of certain games defined using a proper notion of blocking and (ii) each group stable set contains the set of payoff vectors associated to the two definitions of competitive equilibrium. We also show that all six solutions maintain a strictly nested structure. Moreover, each solution can be identified with a set of matrices of (discriminated) prices which indicate how gains from trade are distributed among buyers and sellers. In all cases such matrices arise as solutions of a system of linear inequalities. Hence, all six solutions have the same properties from a structural and computational point of view.

Keywords: Assignment Game; Competitive Equilibrium; Core; Group Stability.
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1 Introduction

Gale and Shapley (1962) introduce ordinal two-sided matching models to study assignment problems between two disjoint sets of agents like men and women, buyers and sellers, firms and workers, students and colleges, and so on. In the marriage model, where matchings are one-to-one, each agent has to be matched to at most an agent on the opposite set. In the college admissions model, where matchings are many-to-one, each student has to be matched to at most a college, while each college can be assigned to a set of students as long as its cardinality does not exceed the number of the college's available seats. It is assumed that each agent has strict ordinal preferences over the set of agents that he does not belong to plus the prospect of remaining unmatched or, in the case of a college, over the family of all subsets of students, identifying the empty set with the prospect of not being matched to any student. These models are ordinal and money do not play any role; in particular, money can not be used to compensate an agent in the case he has to be matched to an agent (or set of agents) at the bottom of the agent's preference list. Ordinal models have been enormously useful and extensively used in Economics to study situations where the assignment problem has only one issue: who is matched to whom.[1] In these models, and given a preference profile (a preference for each agent), a matching is stable if it is individually rational (no agent is assigned to a partner that is worse than to remain unmatched) and pair-wise stable (there is no pair of agents that are not matched to each other but they would prefer to be so rather than to be matched to the partner proposed by the matching, or to one of them if the agent is a college). Gale and Shapley (1962) show that, for every preference profile, the set of stable matchings is non-empty, it coincides with the Core of the associated cooperative game with non-transferable utility (and hence, coalitions with two or more agents from the same set of agents do not have additional blocking power), and there exist two stable matchings, μ_1 and μ_2, with the properties that all agents in one set agree that the partner they receive at μ_1 (at μ_2) is the best (worst) among all partners that they receive at any stable matching and, simultaneously, all agents in the other set agree that the partner they receive at μ_2 (at μ_1) is the best (worst) among all partners that they receive at any stable matching.[2] Finally, they define the deferred acceptance algorithm to compute, depending on the set of agents that make offers, each of the two extreme stable matchings μ_1 and μ_2.

However, there are many assignment problems (solved by markets) where money plays a significant role; for instance, through salaries or prices. Hence, in those cases agents' preferences may be cardinal. But then, to describe a solution of the problem

[1] Roth and Sotomayor (1990) contains a masterful presentation of the most relevant matching models and some of their applications.

[2] Knuth (1976) shows that the set of stable matchings is a (dual) complete lattice with the unanimous partial ordering of the agents in one set.

(in particular, to unsure its stability) it is not sufficient to specify the matching between the two sides of the market because it is also required to describe how each pair of assigned agents share the gains of being matched to each other. Shapley and Shubik (1972) propose the assignment game as an appropriated tool to study one-to-one matching problems with money (*i.e.*, with transferable utility). The prototypical and most simple example of an assignment game is a market with sellers and buyers in which each seller owns one indivisible unit of a good and each buyer wants to buy at most one unit of one good. This setting differs from the marriage model of Gale and Shapley (1962) by the fact that there exists money used as a means of exchange. In addition money is also used to determine buyers' valuations (or maximal willingness to pay) of each unit of the available goods and sellers' reservation prices (or minimal amounts at which they are willing to sell the unit of the good they own). Shapley and Shubik (1972) show that the assignment game has the following properties. (i) There exists at least one competitive equilibrium price vector, with a price for each of the goods, and an assignment between buyers and sellers such that, at those prices, each buyer is assigned to the seller that owns the good (namely, the buyer buys the unit of the good that the seller has, and pays its price) that gives him the maximal net valuation (the difference between his valuation and the price of the good). (ii) The set of competitive equilibrium prices is a complete lattice (with the natural order of vectors in an n-dimensional Euclidian space, where n is the number of goods). (iii) The lattice structure on the set of competitive equilibrium prices is embedded into the set of induced payoffs (or utilities or net gains). (iv) The set of competitive equilibrium payoffs coincides with the Core of the cooperative game with transferable utility induced by the assignment game. And finally, (v) the Core coincides with the set of individually rational and pair-wise stable payoff vectors. In this model, a solution is not only an assignment (who buys to whom, or equivalently, who sells to whom) but it is also a description of how each assigned pair of agents splits the gains generated by their trade (the difference between the valuation that the buyer assigns to the good and the seller's reservation price).[3]

Sotomayor (1992, 1999, 2002, 2007, 2009 and 2011), Camiña (2006), Milgrom (2009), Fagebaume, Gale and Sotomayor (2010), Jaume, Massó and Neme (2012) and Massó and Neme (2013) are some of the papers that extend the one-to-one Shapley and Shubik (1972)'s assignment game by allowing that buyers can buy different goods and/or that sellers can own and sell units of different goods to different buyers. Most of those papers show that some of the properties of the one-to-one model also hold for the generalized versions. In particular, Jaume, Massó and Neme (2012) and

[3]Observe that competitive equilibrium assignments are optimal in the sense that they maximize the sum of all net gains. Thus, and since they are solutions of a linear problem, they are generically unique. The complete lattice structure of the set of stable matchings in the ordinal model is translated to the complete lattice structure of the set of competitive payoff vectors in the cardinal model.

Massó and Neme (2013) consider a cardinal two-sided many-to-many generalized assignment game (a *market*) in which, given a set of goods, each seller may own (and hence, may sell) several units of different goods and each buyer may buy several units of several goods, and buyers and sellers trade indivisible units of these goods by money. Namely, buyers pay money to sellers who deliver in exchange units of the goods. Jaume, Massó and Neme (2012) show that the set of (relevant) competitive equilibrium price vectors is a complete lattice (with the natural order of vectors). Hence, there exist a sellers-optimal competitive equilibrium price vector (which is the best from the point of view of the sellers and it is the worse from the point of view of the buyers) and a buyers-optimal competitive equilibrium price vector (with the symmetric property). In addition, most of the previously cited papers propose and study cooperative solution concepts that are natural in the many-to-one or many-to-many contexts. The Core is the most studied solution concept. Given a payoff vector and an associated assignment (the payoffs are obtained after distributing among players the net gains generated from each trade specified by the assignment) a coalition Core-blocks the payoff vector if all its agents, by breaking all their trades with all agents outside the coalition, may improve upon their payoffs by reorganizing new trades, performed only among themselves. The Core is the set of payoff vectors that are not Core-blocked by any coalition. Another group stability notion naturally arises after giving to coalitions a stronger power to block by admitting that each member of the blocking coalition keeps, partially or totally, some of his trades with agents outside the coalition.[4] Massó and Neme (2013) study the convergence of the set of group stable payoffs and the Core through a replicated sequence of markets to the set of competitive equilibrium payoff vectors. They show that when the market is replicated twice the set of group stable payoffs already coincides with the set of competitive equilibrium payoffs and that when the number of replications tends to infinity the Core converges to the set of competitive equilibrium payoffs.

However, in this setting there are other alternative notions of group stability. They differ on the type of transactions that agents in a blocking coalition are allowed to perform with agents outside. That is, the notions depend on how sale contracts have been specified and hence, on how they can be broken. The Core concept assumes that agents in a blocking coalition can only trade among themselves, without being able to keep any trade with agents outside the blocking coalition; thus, when a coalition of agents Core-blocks a proposed payoff vector they have to break all contracts with agents outside the coalition. In the group stability notion defined in Massó and Neme (2013) it is assumed that sale contracts are unit-by-unit. A trade of a unit of a good between a buyer and a seller is performed independently of the other traded units of the same good as well as of the traded units of the other goods. An agent of a blocking coalition can reduce (but not increase) the trade, with members outside the

[4]Sotomayor (1999) introduces group stability as an alternative and more natural concept than the Core.

coalition, of a given good in the number of units that he wishes, but without being forced for this reason to reduce neither the number of traded units of the same good nor the number of units of the other goods. In this paper we consider the other two alternative notions of group stability. They are more appropriated for those cases where sale contracts are written good-by-good or globally. In the good-by-good case, the sale contract between a buyer and a seller includes all traded units of only one good, and it is independent of their trade on the other goods. Thus, when an agent belongs to a blocking coalition and the other does not, either they keep the trade of all units of the good specified in the sale contract or they completely eliminate the trade of this good. In the global case, the sale contract between a buyer and a seller includes all trades on all goods and thus, when an agent belongs to a blocking coalition and the other does not, either they keep all trades or they have to be eliminated all together.

Jaume, Massó and Neme (2012), when defining competitive equilibrium for this generalized assignment game, consider that given a price vector (a price for each of the goods) agents demand and supply those units of the goods that maximize their total payoff without taking into account the aggregate feasibility constraints. The supply or demand of each agent only depends on the price vector and his *individual* feasibility constraints. The fact that, at a given price vector, all supply and demand plans are mutually compatible is an equilibrium question, rather than a restriction on the individual maximization problems. On the other hand, the competitive equilibrium notion studied by Sotomayor (2007, 2009 and 2011) in related models assume that individual demands and supplies have to be feasible for the market. Agents, to be able to formulate their demands or supplies at a given price vector, have to know all aggregate feasibility constraints in the market. Namely, when obtaining their optimal demands and supplies it is assumed that agents can not demand or supply more than the available amounts present in the market.

The most important results of this paper are the following.

First, we show that each one of the sets of payoffs corresponding to the three group stability notions can be directly identified with the union of Cores of particular cooperative games with transferable utility, where the blocking power of coalitions is inherited from the corresponding nature of the sale contracts between buyers and sellers (unit-by-unit, good-by-good, or global). Therefore, the stability notion associated to the Core is closely related to the group stability notions, provided that the games for which we obtain the Core are properly defined.

Second, and using this identification, we show that the following properties connected with the structure of the set of payoffs corresponding to the alternative group stability notions hold. (a) The three notions of group stability are supported by a Cartesian product structure between a given set of *matrices of prices* (that can be interpreted as a set of discriminated buyer-seller prices) and the set of optimal assignments. (b) All payoff vectors in any of the sets corresponding to the three group stability notions are fully identified by a set of matrices of prices (defined accordingly to the nature of sale contracts). Each price in the matrix indicates how the gains from trade of each unit of each good are distributed between the buyer and the seller. (c) All payoff vectors in any of the sets corresponding to the three group stability notions are completely identified with the solutions of a system of bounded linear inequalities.

Third, using a similar technique to that already used for the group stability notions we show that each of the two competitive equilibrium notions can be directly identified with the union of Cores of certain cooperative games with transferable utility. This result allows us to obtain for the two competitive equilibrium concepts the same conclusions that we have already obtained for the three group stability notions. Hence, cooperative as well as competitive solutions have all the same properties from a structural and computational point of view. Furthermore, all studied solutions maintain a strictly nested relationship.

In short, the paper contributes to the study of markets with indivisible goods. In particular, it shows that the two competitive equilibrium notions are immune with respect to the secession of subgroups of agents. It also identifies some structural properties that hold for competitive equilibrium solutions as well as for different notions of group stability.

The paper is organized as follows. In the next section we present the model introduced in Jaume, Massó and Neme (2012). In Section 3 we define three notions of group stability and study the equivalence of each of these notions with the Cores of their corresponding cooperative games with transferable utility. We show that the three group stability sets of payoffs have a Cartesian product structure and that they can be identified as the solutions of a system of linear inequalities. In Section 4 we perform a similar analysis for the two notions of competitive equilibria. In Section 5 we compare the three notions of group stability with the two notions of competitive equilibria. Section 6 contains an Appendix with the proofs of three results omitted in the main text.

2 Preliminaries

A generalized assignment game (a *market*) consists of three finite and disjoint sets: the set $\mathcal{B}$ of B *buyers*, the set $\mathcal{G}$ of G *goods*, and the set $\mathcal{S}$ of S *sellers*. We denote a generic buyer by i, a generic good by j, and a generic seller by k. Buyers have a constant marginal valuation of each good. Let $v_{ij} \geq 0$ be the monetary valuation that buyer i assigns to each unit of good j; namely, v_{ij} is the maximum price that buyer i is willing to pay for each unit of good j. Denote by $V = (v_{ij})_{(i,j)\in\mathcal{B}\times\mathcal{G}}$ the *matrix of valuations*. We assume that buyer $i \in \mathcal{B}$ can buy at most $d_i \in \mathbb{Z}_+\backslash\{0\}$ units in total, where $\mathbb{Z}_+$ is the set of non-negative integers. The strictly positive integer d_i should be interpreted as a capacity constraint due to limits on i's ability for storage, transport, etc. Denote by $d = (d_i)_{i\in\mathcal{B}}$ the *vector of maximal demands.* Each seller $k \in \mathcal{S}$ has $q_{jk} \in \mathbb{Z}_+$ indivisible units of each good $j \in \mathcal{G}$. Denote by $Q = (q_{jk})_{(j,k)\in\mathcal{G}\times\mathcal{S}}$ the *matrix of capacities.* We assume that there is a strictly amount of each good; namely,

$$\text{for each } j \in \mathcal{G} \text{ there exists } k \in \mathcal{S} \text{ such that } q_{jk} > 0. \tag{1}$$

Let $r_{jk} \geq 0$ be the monetary valuation that seller k assigns to each unit of good j; that is, r_{jk} is the reservation (or minimum) price that seller k is willing to accept for each unit of good j. Denote by $R = (r_{jk})_{(j,k)\in\mathcal{G}\times\mathcal{S}}$ the *matrix of reservation prices.*

A *market* M is a 7-tuple $(\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ satisfying condition (1). Shapley and Shubik (1972)'s (one-to-one) assignment game is a special case of a market where each buyer can buy at most one unit, there is only one unit of each good, and each seller only owns one unit of one of the goods; *i.e.*, $d_i = 1$ for all $i \in \mathcal{B}$, $G = S$, and for all $(j,k) \in \mathcal{G}\times\mathcal{S}$, $q_{jk} = 1$ if $j = k$ and $q_{jk} = 0$ if $j \neq k$.

Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market. An *assignment* for market M is a three-dimensional integer matrix (*i.e.*, a 3^{rd}-order tensor) $A = (A_{ijk})_{(i,j,k)\in\mathcal{B}\times\mathcal{G}\times\mathcal{S}} \in \mathbb{Z}_+^{B\times G\times S}$ describing a collection of deliveries of units of the goods from sellers to buyers. Each A_{ijk} should be interpreted as "buyer i receives A_{ijk} units of good j from seller k." We often omit the sets to which the subscripts belong to and write, for instance, $\sum_{ijk} A_{ijk}$ and $\sum_i A_{ijk}$ instead of $\sum_{(i,j,k)\in\mathcal{B}\times\mathcal{G}\times\mathcal{S}} A_{ijk}$ and $\sum_{i\in\mathcal{B}} A_{ijk}$, respectively.

The assignment A is *feasible* for market M if each buyer i buys at most d_i units and each seller k sells at most q_{jk} units of each good j. We are only interested in feasible assignments; namely in the set

$$\{A \in \mathbb{Z}_+^{B\times G\times S} \mid \textstyle\sum_{jk} A_{ijk} \leq d_i \text{ for all } i \in \mathcal{B} \text{ and } \sum_i A_{ijk} \leq q_{jk} \text{ for all } (j,k) \in \mathcal{G}\times\mathcal{S}\}.$$

For further reference, we denote this set of feasible assignments for market M by $\mathcal{F}^0(M)$ (or simply by $\mathcal{F}^0$).

The *total gain from trade of market M at assignment A* is

$$T^M(A) = \textstyle\sum_{ijk}(v_{ij} - r_{jk}) \cdot A_{ijk}.$$

Definition 1 A feasible assignment A is *optimal* for market M if, for any feasible assignment A', $T^M(A) \geq T^M(A')$.

Example 1 below contains an instance of a market with a unique optimal assignment.

Example 1 Let $(\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market where $\mathcal{B} = \{b_1, b_2\}$, $\mathcal{G} = \{g_1, g_2, g_3\}$, $\mathcal{S} = \{s_1\}$, $V = \begin{pmatrix} 6 & 4 & 4 \\ 7 & 3 & 5 \end{pmatrix}$, $d = (10, 10)$, $Q = (10, 5, 1)$ and $R = (5, 2, 1)$. For any $A' \in \mathcal{F}^0$,

$$\begin{aligned} T^M(A') &= (6-5)\cdot A'_{111} + (4-2)\cdot A'_{121} + (4-1)\cdot A'_{131} + (7-5)\cdot A'_{211} \\ &\quad +(3-2)\cdot A'_{221} + (5-1)\cdot A'_{231} \\ &= A'_{111} + 2\cdot A'_{121} + 3\cdot A'_{131} + 2\cdot A'_{211} + A'_{221} + 4\cdot A'_{231}. \end{aligned}$$

It is easy to check that $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$ is the unique optimal assignment for M and $T^M(A) = 1 + 2 \cdot 5 + 2 \cdot 9 + 4 = 33$. $\square$

Let $\mathcal{F}(M)$ (or simply $\mathcal{F}$) be the set of all optimal assignments for market M. The set $\mathcal{F}$ is always non-empty.[5] Denote by T^M the total gain from trade of market M at any optimal assignment.

Fix a market $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$. Denote by $G^{>}$ the set of goods that are exchanged at some optimal assignment. Namely,

$$G^{>} = \{j \in \mathcal{G} \mid \text{there exists } A \in \mathcal{F} \text{ such that } A_{ijk} > 0 \text{ for some } (i,k) \in \mathcal{B} \times \mathcal{S}\}.$$

Moreover, for each buyer $i \in \mathcal{B}$ and each seller $k \in \mathcal{S}$, define

$$G^{>}_{ik} = \{j \in \mathcal{G} \mid \text{there exists } A \in \mathcal{F} \text{ such that } A_{ijk} > 0\}$$

as the set of goods that i buys to k at some optimal assignment.

3 Cooperative Solutions: Core and Group Stability

Massó and Neme (2013) define, for any market M, two cooperative solutions: the Core and a group stable set (they call it set-wise stable). As described in the Introduction the two concepts are based on the idea that a coalition will object to a proposed payoff vector if all agents in the coalition can improve upon their payoffs, but differ in that, when objecting, the Core requires that all members of the blocking coalition break their exchanges with agents outside the coalition while group stability (which we shall call it here type 1−group stability) allows that the exchanges of an agent in the blocking coalition with agents outside the coalition are maintained or reduced (since sale contracts are unit-by-unit). Here we propose two alternative notions of group stability. Type 2−group stability makes sense when sale contracts are performed good-by-good and therefore an agent in the blocking coalition can maintain with an agent outside the coalition the exchange of all units of the good or else delete them all. Type 3−group stability makes sense when between a buyer and a seller there exists only a sale contract and therefore an agent in the blocking coalition can maintain with an agent outside the coalition all exchanges or delete them all.

Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market and $C \subset \mathcal{B} \cup \mathcal{S}$ be a coalition. Denote the sets of buyers and sellers in C by $\mathcal{B}^C = C \cap \mathcal{B}$ and $\mathcal{S}^C = C \cap \mathcal{S}$, respectively.

[5] See Milgrom (2009) for a proof of this statement, based on a fix point argument, in a more general model. Jaume, Massó and Neme (2012) contains a proof of the statement, using only linear programming arguments, in the same model as the one studied here.

Definition 2 Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market and $C \subset \mathcal{B} \cup \mathcal{S}$ be a coalition. A feasible assignment $\widehat{A} \in \mathcal{F}^0$ is $1-$*group compatible with* C if there exists an optimal assignment $A \in \mathcal{F}$ such that
(i) for all $i \in \mathcal{B}^C$, $\widehat{A}_{ijk} > 0$ implies that either $k \in \mathcal{S}^C$ or else $\widehat{A}_{ijk} \leq A_{ijk}$, and
(ii) for all $k \in \mathcal{S}^C$, $\widehat{A}_{ijk} > 0$ implies that either $i \in \mathcal{B}^C$ or else $\widehat{A}_{ijk} \leq A_{ijk}$.[6]

We want to emphasize that the above definition considers as compatible any reallocation of goods between the agents within the coalition and only decreases (with respect of some optimal assignment) the trade, of any good, between an agent in the coalition with another agent outside. The next two definitions of group compatibility limits the reallocations of goods between members of the blocking coalition and outsiders depending on whether sale contracts are good-by-good or global.

Definition 3 Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market and $C \subset \mathcal{B} \cup \mathcal{S}$ be a coalition. A feasible assignment $\widehat{A} \in \mathcal{F}^0$ is $2-$*group compatible with* C if there exists an optimal assignment$A \in \mathcal{F}$ such that
(i) for all $i \in \mathcal{B}^C$, $\widehat{A}_{ijk} > 0$ implies that either $k \in \mathcal{S}^C$ or else $\widehat{A}_{ijk} = A_{ijk}$, and
(ii) for all $k \in \mathcal{S}^C$, $\widehat{A}_{ijk} > 0$ implies that either $i \in \mathcal{B}^C$ or else $\widehat{A}_{ijk} = A_{ijk}$.

Definition 4 Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market and $C \subset \mathcal{B} \cup \mathcal{S}$ be a coalition. A feasible assignment $\widehat{A} \in \mathcal{F}^0$ is $3-$*group compatible with* C if there exists an optimal assignment $A \in \mathcal{F}$ such that
(i) for all $i \in \mathcal{B}^C$, $\widehat{A}_{ijk} > 0$ implies that either $k \in \mathcal{S}^C$ or else $\widehat{A}_{ij'k} = A_{ij'k}$ for all $j' \in \mathcal{G}$, and
(ii) for all $k \in \mathcal{S}^C$, $\widehat{A}_{ijk} > 0$ implies that either $i \in \mathcal{B}^C$ or else $\widehat{A}_{ij'k} = A_{ij'k}$ for all $j' \in \mathcal{G}$.

Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market, $C \subset \mathcal{B} \cup \mathcal{S}$ a coalition and $t \in \{1, 2, 3\}$. Denote by $\mathcal{F}^t(C)$ the set of all feasible assignments that are $t-$group compatible with C.

Example 1 (continued) To see the differences among the three types of group compatibility, consider the coalition $C = \{b_1, s_1\}$ in market M of Example 1. Then,

$$\begin{aligned}
\mathcal{F}^1(C) &= \{\widehat{A} \in \mathcal{F}^0 \mid 0 \leq \widehat{A}_{211} \leq 9,\ \widehat{A}_{221} = 0 \text{ and } 0 \leq \widehat{A}_{231} \leq 1\}. \\
\mathcal{F}^2(C) &= \{\widehat{A} \in \mathcal{F}^0 \mid \widehat{A}_{211} \in \{0, 9\},\ \widehat{A}_{221} = 0 \text{ and } \widehat{A}_{231} \in \{0, 1\}\}. \\
\mathcal{F}^3(C) &= \{\widehat{A} \in \mathcal{F}^0 \mid (\widehat{A}_{211}, \widehat{A}_{221}, \widehat{A}_{231}) = (9, 0, 1) \text{ or } (\widehat{A}_{211}, \widehat{A}_{221}, \widehat{A}_{231}) = (0, 0, 0)\}.
\end{aligned}$$

Thus, $\mathcal{F}^3(C) \subset \mathcal{F}^2(C) \subset \mathcal{F}^1(C)$ and

[6] Massó and Neme (2013) add a third condition requiring that for all $i \notin \mathcal{B}^C$ and $k \notin \mathcal{S}^C$, $\widehat{A}_{ijk} = 0$ for all $j \in \mathcal{G}$. Since the exchanges between two agents outside the blocking coalition are irrelevant for describing the payoffs that agents in the blocking coalition can obtain, here we will dispense with this condition, since often will be useful that the assignment $\widehat{A}$ be an optimal one.

$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 1 \end{pmatrix} \in \mathcal{F}^1(C) \backslash \mathcal{F}^2(C)$, $\begin{pmatrix} 1 & 5 & 1 \\ 9 & 0 & 0 \end{pmatrix} \in \mathcal{F}^2(C) \backslash \mathcal{F}^3(C)$, and $\begin{pmatrix} 4 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{F}^3(C)$. □

Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market. A 3^{rd}-order tensor $\Gamma = (\Gamma_{ijk})_{(i,j,k) \in \mathcal{B}\times\mathcal{G}\times\mathcal{S}} \in \mathbb{R}_+^{B\times G\times S}$ is a *distribution matrix* for market M if for all $(i,j,k) \in \mathcal{B}\times\mathcal{G}\times\mathcal{S}$ such that $v_{ij} \geq r_{jk}$ and $j \in G_{ik}^{>}$, $v_{ij} \geq \Gamma_{ijk} \geq r_{jk}$ holds. Let Γ be a distribution matrix for market M and assume that $v_{ij} \geq r_{jk}$ for some $(i,j,k) \in \mathcal{B}\times\mathcal{G}\times\mathcal{S}$ and $j \in G_{ik}^{>}$. Then, Γ_{ijk} describes a possible way of how buyer i and seller k can split the gain $v_{ij} - r_{jk} \geq 0$ they could obtain by exchanging one unit of good j: buyer i receives $v_{ij} - \Gamma_{ijk}$ and seller k receives $\Gamma_{ijk} - r_{jk}$. If $j \notin G_{ik}^{>}$ the value Γ_{ijk} will be irrelevant since i and k will not exchange any unit of good j in any optimal assignment. Observe that distribution matrices are not necessarily anonymous because a buyer may obtain different gains per unit of good j if he buys the same good from different sellers, and viceversa. Denote by $\mathcal{D}(M)$ (or simply by $\mathcal{D}$) the set of all distribution matrices for market M.

Definition 5 A vector $(u_i, w_k)_{(i,k)\in\mathcal{B}\times\mathcal{S}} \in \mathbb{R}^{B\times S}$ is a *feasible payoff* for market M if

$$\sum_{i\in\mathcal{B}} u_i + \sum_{k\in\mathcal{S}} w_k = T^M.$$

Denote by $\mathcal{X}(M)$ (or simply by $\mathcal{X}$) the set of all feasible payoffs for market M.

Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market and $C \subset \mathcal{B} \cup \mathcal{S}$ a coalition. For every $\Gamma \in \mathcal{D}$ and $\widehat{A} \in \mathcal{F}^0$, define the *gain for* C *at* $\widehat{A}$ *according to* Γ by the expression[7]

$$\begin{aligned}\phi^M(C,\widehat{A},\Gamma) \equiv & \sum_{(i,j,k)\in\mathcal{B}^C\times\mathcal{G}\times\mathcal{S}^C} (v_{ij} - r_{jk}) \cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in\mathcal{B}^C\times\mathcal{G}\times(\mathcal{S}^C)^c} (v_{ij} - \Gamma_{ijk}) \cdot \widehat{A}_{ijk} \\ & + \sum_{(i,j,k)\in(\mathcal{B}^C)^c\times\mathcal{G}\times\mathcal{S}^C} (\Gamma_{ijk} - r_{jk}) \cdot \widehat{A}_{ijk}.\end{aligned} \quad (2)$$

Observe that $\phi^M(C,\widehat{A},\Gamma)$ is independent of $t \in \{1,2,3\}$.

We are now ready to define the blocking notions according to the assignments that the coalition can use.

Definition 6 Let M be a market and $t \in \{1,2,3\}$. A payoff $(u,w) \in \mathcal{X}(M)$ *is not* $t-$*group blocked* if there exists a distribution matrix $\Gamma = (\Gamma_{ijk})_{(i,j,k)\in\mathcal{B}\times\mathcal{G}\times\mathcal{S}} \in \mathcal{D}(M)$ such that for all coalition $C \subset \mathcal{B}\cup\mathcal{S}$ and $\widehat{A} \in \mathcal{F}^t(C)$,

$$\sum_{i\in\mathcal{B}^C} u_i + \sum_{k\in\mathcal{S}^C} w_k \geq \phi^M(C,\widehat{A},\Gamma).$$

It is useful to point out that the definition depends on $t \in \{1,2,3\}$ since the gain for C depends on the set $\mathcal{F}^t(C)$ of feasible assignments (that is, $t-$group compatible) with C. Finally, we define the three notions of group stability.

[7] Given a set Y we denote its complement by Y^c. The reader should not be confused when Y is $\mathcal{B}^C$ or $\mathcal{S}^C$, whose complements are denoted by $(\mathcal{B}^C)^c$ and $(\mathcal{S}^C)^c$, respectively.

Definition 7 Let M be a market and $t \in \{1,2,3\}$. A payoff $(u,w) \in \mathcal{X}(M)$ is *$t-$group stable for M* if it is not $t-$group blocked.[8]

Denote by $\mathcal{GS}^t(M)$ (or simply $\mathcal{GS}^t$) the set of payoffs that are $t-$group stable for M. Since $\mathcal{F}^3(C) \subset \mathcal{F}^2(C) \subset \mathcal{F}^1(C)$ for all $C \subset \mathcal{B} \cup \mathcal{S}$, it follows that

$$\mathcal{GS}^1 \subset \mathcal{GS}^2 \subset \mathcal{GS}^3.$$

Moreover, there are markets for which these inclusions are strict and hence,[9]

$$\mathcal{GS}^1 \subsetneq \mathcal{GS}^2 \subsetneq \mathcal{GS}^3. \tag{3}$$

By the above remark and the fact that $\mathcal{GS}^1 \neq \emptyset$ (see Massó and Neme (2013)) all $t-$group stable sets are non-empty. For further reference, we present this result as Proposition 1 below.

Proposition 1 *For any market M and $t \in \{1,2,3\}$, $\mathcal{GS}^t(M) \neq \emptyset$.*

Massó and Neme (2013) define the Core of market M as the Core of the cooperative game with transferable utility induced by M. They show first that the $1-$group stable set is a strict subset of the Core and strictly contains the set of competitive equilibrium payoffs. Second, the $1-$group stable set converges in the second replica to the set of competitive equilibrium payoffs while the Core does not converge to it in a finite number of replica. Hence, one may infer from the two results that the two cooperative notions are essentially different. We will see here that the difference does not refer so much to the solution concept but rather on how the game for which the Core is obtained is defined. Massó and Neme (2013) define the cooperative game by assuming that the assignment $\widehat{A}$ is feasible for a coalition $C \subset \mathcal{B} \cup \mathcal{S}$ if and only if members of C only exchange goods among themselves.

Definition 8 Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market and $C \subset \mathcal{B} \cup \mathcal{S}$ be a coalition. A feasible assignment $\widehat{A} \in \mathcal{F}^0$ is *Core-compatible with C* if
(i) for all $i \in \mathcal{B}^C$, $\widehat{A}_{ijk} > 0$ implies $k \in \mathcal{S}^C$, and
(ii) for all $k \in \mathcal{S}^C$, $\widehat{A}_{ijk} > 0$ implies $i \in \mathcal{B}^C$.

Given $C \subset \mathcal{B} \cup \mathcal{S}$, the set of all Core-compatible assignments with C will be denoted by $\mathcal{F}^{Co}(C)$. Using this notion, we define the cooperative game with transferable utility $(\mathcal{B} \cup \mathcal{S}, v)$ where, for every $C \subset \mathcal{B} \cup \mathcal{S}$,[10]

$$v(C) = \max_{\widehat{A} \in \mathcal{F}^{Co}(C)} \phi^M(C, \widehat{A}, \Gamma). \tag{4}$$

[8] The notion of $1-$group stability corresponds to set-wise stability defined in Massó and Neme (2013).

[9] In the Appendix in Section 6 we show that this property holds for the market M of Example 1.

[10] Observe that if $\widehat{A} \in \mathcal{F}^{Co}(C)$, then $\phi^M(C, \widehat{A}, \Gamma)$ is independent of Γ since $\phi^M(C, \widehat{A}, \Gamma) = \sum_{(i,j,k) \in \mathcal{B}^C \times \mathcal{G} \times \mathcal{S}^C} (v_{ij} - r_{jk}) \cdot \widehat{A}_{ijk}$. For those cases we could simply write $\phi^M(C, \widehat{A})$.

Then, the *Core of market* M, denoted by $\mathcal{C}(M)$, is the Core of the game $(\mathcal{B}\cup\mathcal{S},v)$; namely,

$$\mathcal{C}(M)=\{(u,w)\in\mathcal{X}(M)\mid v(C)\leq\sum_{i\in\mathcal{B}^C}u_i+\sum_{k\in\mathcal{S}^C}w_k \text{ for all } C\subset\mathcal{B}\cup\mathcal{S}\}.$$

Now, if we accept the notions of group stability as reasonable solutions, we can define new cooperative games with transferable utility where compatible assignments with a coalition C admit that its members may have certain exchanges with agents outside C. For this purpose it is necessary to consider a distribution matrix $\Gamma\in\mathcal{D}$ indicating how the gains from trade are distributed with members outside coalition C. We now present these notions formally.

Definition 9 Let $M=(\mathcal{B},\mathcal{G},\mathcal{S},V,d,R,Q)$ be a market, $\Gamma\in\mathcal{D}$ and $t\in\{1,2,3\}$. The *cooperative game with transferable utility associated to* t *and* Γ, denoted by $(\mathcal{B}\cup\mathcal{S},v^{t\Gamma})$, is defined as follows: for every $C\subset\mathcal{B}\cup\mathcal{S}$,

$$v^{t\Gamma}(C)=\max_{\widehat{A}\in\mathcal{F}^t(C)}\phi^M(C,\widehat{A},\Gamma).$$

If $\Gamma\in\mathcal{D}$ is given and we allow C to choose among the set of assignments in $\mathcal{F}^t(C)$, the game $(\mathcal{B}\cup\mathcal{S},v^{t\Gamma})$ can be interpreted in a similar way as we interpreted the game defined in (4), where each coalition maximizes the total payoff since $\phi^M(C,\widehat{A},\Gamma)$ is the total gain received by members of C under $\widehat{A}$. We will denote by $\mathcal{C}^{t\Gamma}(M)$ (or simply by $\mathcal{C}^{t\Gamma}$) the Core of the game $(\mathcal{B}\cup\mathcal{S},v^{t\Gamma})$.

Remark 1 Note that for all $\Gamma\in\mathcal{D}$ and $t\in\{1,2,3\}$,

$$T^M=v(\mathcal{B}\cup\mathcal{S})=v^{1\Gamma}(\mathcal{B}\cup\mathcal{S})=v^{2\Gamma}(\mathcal{B}\cup\mathcal{S})=v^{3\Gamma}(\mathcal{B}\cup\mathcal{S}).$$

Hence, (u,w) is a feasible payoff (*i.e.*, $(u,w)\in\mathcal{X}$) if and only if $\sum_{i\in\mathcal{B}}u_i+\sum_{k\in\mathcal{S}}w_k=v^{t\Gamma}(\mathcal{B}\cup\mathcal{S})$.

Using the games $(\mathcal{B}\cup\mathcal{S},v^{t\Gamma})$ associated to M we can now see that the notions of Core and group stability are extremely related. Indeed, the following result holds.

Theorem 1 *Let M be a market. Then, for all $t\in\{1,2,3\}$,*

$$\mathcal{GS}^t(M)=\bigcup_{\Gamma\in\mathcal{D}(M)}\mathcal{C}^{t\Gamma}(M).$$

Proof Fix M and t. We first show that for all $\Gamma\in\mathcal{D}$, $\mathcal{C}^{t\Gamma}\subset\mathcal{GS}^t$. Let $(u,w)\in\mathcal{C}^{t\Gamma}$. By Remark 1, (u,w) is a feasible payoff. Moreover, for all $C\subset\mathcal{B}\cup\mathcal{S}$, $\sum_{i\in\mathcal{B}^C}u_i+\sum_{k\in\mathcal{S}^C}w_k\geq v^{t\Gamma}(C)$. Hence, for all C and all $\widehat{A}\in\mathcal{F}^t(C)$, $\sum_{i\in\mathcal{B}^C}u_i+\sum_{k\in\mathcal{S}^C}w_k\geq\phi^M(C,\widehat{A},\Gamma)$. Thus, $(u,w)\in\mathcal{GS}^t$. Namely, $\bigcup_{\Gamma\in\mathcal{D}(M)}\mathcal{C}^{t\Gamma}\subset\mathcal{GS}^t$.

Take now a payoff $(u,w)\in\mathcal{GS}^t$. Since (u,w) is a feasible payoff, by Remark 1, $\sum_{i\in\mathcal{B}}u_i+\sum_{k\in\mathcal{S}}w_k=v^{t\Gamma}(\mathcal{B}\cup\mathcal{S})$ for all $\Gamma\in\mathcal{D}$. Moreover, and since (u,w) is not $\mathcal{GS}^t-$blocked, there exists $\Gamma\in\mathcal{D}$ such that for all $C\subset\mathcal{B}\cup\mathcal{S}$ and all $\widehat{A}\in\mathcal{F}^t(C)$,

$$\sum_{i\in\mathcal{B}^C}u_i+\sum_{k\in\mathcal{S}^C}w_k\geq\phi^M(C,\widehat{A},\Gamma).$$

Hence, there exists $\Gamma \in \mathcal{D}$ such that $\sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k \geq v^{t\Gamma}(C)$ for all $C \subset \mathcal{B} \cup \mathcal{S}$; namely, $(u, w) \in \mathcal{C}^{t\Gamma}$. Thus, $(u, w) \in \bigcup_{\Gamma \in \mathcal{D}(M)} \mathcal{C}^{t\Gamma}$. ■

In the Appendix in Section 6 we show, using the market of Example 1, that the sets $\mathcal{C}^{t\Gamma}$ may be empty for some Γ.

3.1 Cartesian Product Structure and Computation of the Group Stable Solutions

In this section we present, using Theorem 1, results on the structure of the $t-$group stable set of payoffs for $t = 1, 2, 3$ and how to compute them.

Fix $\Gamma \in \mathcal{D}$ and $A \in \mathcal{F}^0$. Define the *utility of buyer* $i \in \mathcal{B}$ at the pair (Γ, A) as the total net gain obtained by i from his exchanges specified by A and the distribution of gains given by Γ. Denote such utility by $u_i(\Gamma, A)$; namely,

$$u_i(\Gamma, A) = \sum_{jk} (v_{ij} - \Gamma_{ijk}) \cdot A_{ijk}. \tag{5}$$

Similarly, define the *utility of seller* $k \in \mathcal{S}$ at the pair (Γ, A) as the total net gain obtained by k from his exchanges specified by A and the distribution of gains given by Γ. Denote such utility by $w_k(\Gamma, A)$; namely,

$$w_k(\Gamma, A) = \sum_{ij} (\Gamma_{ijk} - r_{jk}) \cdot A_{ijk}. \tag{6}$$

Given (Γ, A), we will denote by $u(\Gamma, A) = (u(\Gamma, A))_{i \in \mathcal{B}}$ and $w(\Gamma, A) = (w_k(\Gamma, A))_{k \in \mathcal{S}}$ the vectors of utilities of buyers and sellers at (Γ, A), respectively.

Proposition 2 *Let M be a market, Γ a distribution matrix and $t \in \{1, 2, 3\}$. Then,*

$$\mathcal{C}^{t\Gamma} \neq \emptyset \text{ if and only if } \mathcal{C}^{t\Gamma} = \{(u(\Gamma, A), w(\Gamma, A)) \mid A \in \mathcal{F}\}.$$

Proof It is immediate to check that $\mathcal{C}^{t\Gamma} = \{(u(\Gamma, A), w(\Gamma, A)) \mid A \in \mathcal{F}\}$ implies $\mathcal{C}^{t\Gamma} \neq \emptyset$. To show that the other implication holds, assume $\mathcal{C}^{t\Gamma} \neq \emptyset$. We first check that $(u(\Gamma, A), w(\Gamma, A)) \in \mathcal{C}^{t\Gamma}$ for all $A \in \mathcal{F}$. Let $A \in \mathcal{F}$ be arbitrary and let $(u, w) \in \mathcal{C}^{t\Gamma}$. Consider any coalition $C = \{i\}$ with $i \in \mathcal{B}$. Then, $A \in \mathcal{F}^t(\{i\})$. Hence, since $(u, w) \in \mathcal{C}^{t\Gamma}$ and the definition of $v^{t\Gamma}$,

$$u_i \geq \phi^M(C, A, \Gamma) = \sum_{(j,k) \in \mathcal{G} \times \mathcal{S}} (v_{ij} - \Gamma_{ijk}) \cdot A_{ijk}. \tag{7}$$

Similarly, and considering any coalition $C = \{k\}$ with $k \in \mathcal{S}$,

$$w_k \geq \phi^M(C, A, \Gamma) = \sum_{(i,j) \in \mathcal{B} \times \mathcal{G}} (\Gamma_{ijk} - r_{jk}) \cdot A_{ijk}. \tag{8}$$

Moreover, by Remark 1, $\sum_{i\in\mathcal{B}} u_i + \sum_{k\in\mathcal{S}} w_k = v^{t\Gamma}(\mathcal{B}\cup\mathcal{S}) = T^M$. But $T^M = \sum_{i\in\mathcal{B}} \sum_{(j,k)\in\mathcal{G}\times\mathcal{S}} (v_{ij} - \Gamma_{ijk})\cdot A_{ijk} + \sum_{k\in\mathcal{S}} \sum_{(i,j)\in\mathcal{B}\times\mathcal{G}} (\Gamma_{ijk} - r_{jk})\cdot A_{ijk}$. Hence, (7) and (8) imply

$$\begin{aligned} u_i &= \sum_{(j,k)\in\mathcal{G}\times\mathcal{S}} (v_{ij} - \Gamma_{ijk})\cdot A_{ijk} \text{ for all } i\in\mathcal{B} \text{ and} \\ w_k &= \sum_{(i,j)\in\mathcal{B}\times\mathcal{G}} (\Gamma_{ijk} - r_{jk})\cdot A_{ijk} \text{ for all } k\in\mathcal{S}. \end{aligned}$$

Thus, $(u,w) = (u(\Gamma,A), w(\Gamma,A))$. Therefore, $(u(\Gamma,A), w(\Gamma,A)) \in \mathcal{C}^{t\Gamma}$. Now it remains to be proven that if $(u,w)\in\mathcal{C}^{t\Gamma}$, then there exists $A\in\mathcal{F}$ such that $(u,w) = (u(\Gamma,A), w(\Gamma,A))$, but observing that $\mathcal{F} = \mathcal{F}^t(\mathcal{B}\cup\mathcal{S})$, it is proven similarly as we did previously. ■

Denote by $\mathcal{D}^t(M) = \{\Gamma : \mathcal{C}^{t\Gamma}(M) \neq \emptyset\}$ (or simply by $\mathcal{D}^t$) the set of distribution matrices whose associated game $v^{t\Gamma}$ has a non-empty Core. By Theorem 1 and Proposition 2, the set $\mathcal{GS}^t$ has the following Cartesian product structure.

Corollary 1 *Let M be a market and $t\in\{1,2,3\}$. Then,*

$$\mathcal{GS}^t = \{(u(\Gamma,A), w(\Gamma,A)) \mid (\Gamma,A)\in\mathcal{D}^t\times\mathcal{F}\}.$$

We will refer to the set $\mathcal{D}^t$ as the set of $t-$*distributions by groups*. The above Corollary establishes that $\mathcal{GS}^t$ has a similar structure to the set of competitive equilibrium payoffs.[11]

Lemma 1 *Let $t\in\{1,2,3\}$ and $\Gamma\in\mathcal{D}^t$ be such that $\mathcal{C}^{t\Gamma}\neq\emptyset$. Then, $(u(\Gamma,A), w(\Gamma,A)) = (u(\Gamma,A'), w(\Gamma,A'))$ for all $A, A'\in\mathcal{F}$.*

Proof Observe that the proof of Proposition 2 does not depend on the particular optimal assignment $A\in\mathcal{F}$. Hence, fixed Γ, if $\mathcal{C}^{t\Gamma}\neq\emptyset$ then the vector of utilities $(u(\Gamma,A), w(\Gamma,A))$ at the pair (Γ,A) is independent of the chosen optimal assignment $A\in\mathcal{F}$. ■

By Lemma 1, for $\Gamma\in\mathcal{D}^t$ and $A\in\mathcal{F}$ we can write $(u(\Gamma), w(\Gamma))$ instead of $(u(\Gamma,A), w(\Gamma,A))$. Hence, the following result follows immediately from Theorem 1 and Lemma 1.

Corollary 2 *Let M be a market and $t\in\{1,2,3\}$. Then,*

$$\mathcal{GS}^t = \{(u(\Gamma), w(\Gamma)) \mid \Gamma\in\mathcal{D}^t\}.$$

The above corollary establishes that each payoff vector in $\mathcal{GS}^t$ comes from a distribution matrix $\Gamma\in\mathcal{D}^t$. Again, Jaume, Massó and Neme (2012) show that a similar

[11] Jaume, Massó and Neme (2012) show that the set of competitve equilibrium payoffs is the Cartesian product of the set of competitive equilibrium prices and the set of optimal assignments $\mathcal{F}$.

result holds for the set of competitive equilibrium payoffs when the gains from trade are determined by an equilibrium price vector (a price for each good).

Proposition 3 below gives necessary and sufficient conditions under which a distribution matrix Γ is a $t-$*distribution by groups*. But to state it, we present, given an optimal assignment $A \in \mathcal{F}$, the following system of inequalities on Γ:

$$\phi^M(C, \hat{A}, \Gamma) \leq \phi^M(C, A, \Gamma) \text{ for all } C \subset \mathcal{B} \cup \mathcal{S} \text{ and all } \hat{A} \in \mathcal{F}^t(C). \tag{9}$$

Proposition 3 *Let M be a market and $t \in \{1, 2, 3\}$. Then, the following statements are equivalent:*

(i) Γ *is a $t-$distribution by groups.*

(ii)

$$v^{t\Gamma}(\mathcal{B} \cup \mathcal{S}) = \sum_{i \in \mathcal{B}} v^{t\Gamma}(\{i\}) + \sum_{k \in \mathcal{S}} v^{t\Gamma}(\{k\}) \textit{ and} \tag{10}$$

$$v^{t\Gamma}(C) \leq \sum_{i \in \mathcal{B}^C} v^{t\Gamma}(\{i\}) + \sum_{k \in \mathcal{S}^C} v^{t\Gamma}(\{k\}) \textit{ for all } C \subset \mathcal{B} \cup \mathcal{S}. \tag{11}$$

(iii) *There exists $A \in \mathcal{F}$ such that $v^{t\Gamma}(C) = \phi^M(C, A, \Gamma)$ for all $C \subset \mathcal{B} \cup \mathcal{C}$.*

(iv) *For all $A \in \mathcal{F}$, $v^{t\Gamma}(C) = \phi^M(C, A, \Gamma)$ for all $C \subset \mathcal{B} \cup \mathcal{C}$.*

(v) Γ *solves the system in* (9).

Proof The equivalence between (iii) and (v) is immediate. That (ii) implies (i) is immediate since, by (10) and (11), $(v^{t\Gamma}(\{i\}), v^{t\Gamma}(\{k\})_{(i,k) \in \mathcal{B} \cup \mathcal{S}} \in \mathcal{C}^{t\Gamma}$. By the definition of $v^{t\Gamma}$, we have that (iii) implies (ii). That (iv) implies (iii) is also immediate. It remains to be proven that (i) implies (iv).

Assume $\mathcal{C}^{t\Gamma} \neq \emptyset$ and let $A \in \mathcal{F}$. By Proposition 2, $(u(\Gamma, A), w(\Gamma, A)) \in \mathcal{C}^{t\Gamma}$. Hence,

$$\begin{aligned} u_i(\Gamma, A) &\geq v^{t\Gamma}(\{i\}) \text{ for all } i \in \mathcal{B} \text{ and} \\ w_k(\Gamma, A) &\geq v^{t\Gamma}(\{k\}) \text{ for all } k \in \mathcal{S}. \end{aligned}$$

Thus, by the definition of $v^{t\Gamma}$,

$$\begin{aligned} u_i(\Gamma, A) &= v^{t\Gamma}(\{i\}) \text{ for all } i \in \mathcal{B} \text{ and} \\ w_k(\Gamma, A) &= v^{t\Gamma}(\{k\}) \text{ for all } k \in \mathcal{S}. \end{aligned}$$

Hence,

$$\begin{aligned} v^{t\Gamma}(\{i\}) &= \phi^M(\{i\}, A, \Gamma) \text{ for all } i \in \mathcal{B} \text{ and} \\ v^{t\Gamma}(\{k\}) &= \phi^M(\{k\}, A, \Gamma) \text{ for all } k \in \mathcal{S}. \end{aligned}$$

Now, since $(u(\Gamma, A), w(\Gamma, A)) \in \mathcal{C}^{t\Gamma}$ holds, by the definition of $v^{t\Gamma}(C)$ it follows that

$$v^{t\Gamma}(\mathcal{B} \cup \mathcal{S}) = \sum_{i\in\mathcal{B}} u_i(\Gamma, A) + \sum_{k\in\mathcal{S}} w_k(\Gamma, A) \text{ and, for all } C \subset \mathcal{B} \cup \mathcal{S},$$
$$\phi^M(C, A, \Gamma) \leq v^{t\Gamma}(C) \leq \sum_{i\in\mathcal{B}^C} u_i(\Gamma, A) + \sum_{k\in\mathcal{S}^C} w_k(\Gamma, A) = \phi^M(C, A, \Gamma).$$

Thus, $v^{t\Gamma}(C) = \phi^M(C, A, \Gamma)$ for all $C \subset \mathcal{B} \cup \mathcal{S}$. ■

4 Competitive Solutions

4.1 Two Competitive Equilibrium Notions

In this section we first present two already known competitive solutions for generalized assignment games. Using a similar approach to the one already used with $t-$group stability we will see how competitive equilibria are related with the notions of Core, provided that the cooperative games with transferable utility are defined properly. This will allow us to draw conclusions with regard to the structure of competitive solutions and how to compute them.

The first competitive solution was presented by Jaume, Massó and Neme (2012). We will see how we can obtain some of the their results using the approach used in the previous section. This solution assumes that buyers and sellers exchange goods through competitive markets. Namely, there is a unique market for each of the goods (with its corresponding price). Hence, a price vector is an $n-$dimensional vector of non-negative real numbers. Buyers and sellers are price-takers in the following sense. Given a price vector $p = (p_j)_{j\in G} \in \mathbb{R}^n_+$ each seller offers units of the goods he owns (up to his capacity) to maximize his net gains and each buyer demands units of the goods (up to his maximal capacity) to maximize his total net valuation. The unique information that each agent has about the markets, besides the price vector, is his per unit valuations of the goods and his capacity of maximal demand (if the agent is a buyer) and his reservation prices and number units owned of each of the goods. Agents do not know the aggregate capacities.

In the second notion we will assume that the aggregate capacities of the market are known by the agents. For instance, because the market is small and the transactions take place all at the same time in a small place. Hence, given a price vector p, agents will maximize their utility taking into account the market aggregate capacities. Namely, a buyer i will never demand of good j a quantity larger than $\sum_k q_{jk}$, eventhough this amount is smaller than d_i and the net valuation $(v_{ij} - p_j)$ of good j is strictly larger than the net valuations of all the other goods. This notion can be seen as an extension of the competitive equilibrium notions introduced and studied in Sotomayor (2007), in an assignment model with indivisible goods and by Sotomayor (2009 and 2011), in a model with infinitely divisible goods, but in both cases and in contrast with our model, it is assumed that sellers only own units of the same good. In these three papers, given a price vector p, agents' demands and supplies are obtained by solving their maximizing problems over the set of *feasible* assignments; that is, it is assumed that agents know the aggregate capacities.

It is also possible to consider the case where only buyers know the aggregate capacities and only they adjust their demands to such constraints, and viceversa. Our proofs could be adapted easily to these two settings to obtain similar conclusions for them.

To present the first approach, we transcribe some definitions in Jaume, Massó and Neme (2012).

Supply of seller k: For each price vector $p = (p_j)_{j \in \mathcal{G}} \in \mathbb{R}^G_+$, seller k offers of each good j any feasible amount that maximizes his gain; namely,

$$\mathcal{S}_{jk}(p_j) = \begin{cases} \{q_{jk}\} & \text{if } p_j > r_{jk} \\ \{0, 1, ..., q_{jk}\} & \text{if } p_j = r_{jk} \\ \{0\} & \text{if } p_j < r_{jk}. \end{cases} \tag{12}$$

To define the demand of buyer $i \in \mathcal{B}$, we will use the following notation. Let $p \in \mathbb{R}^G_+$ and let

$$\nabla_i^>(p) = \{j \in \mathcal{G} \mid v_{ij} - p_j = \max_{j' \in \mathcal{G}}\{v_{ij'} - p_{j'}\} > 0\} \tag{13}$$

be the set of goods that give to buyer i the maximal (and strictly positive) net valuation at p. Obviously, for some p, the set $\nabla_i^>(p)$ may be empty. Let

$$\nabla_i^{\geq}(p) = \{j \in \mathcal{G} \mid v_{ij} - p_j = \max_{j' \in \mathcal{G}}\{v_{ij'} - p_{j'}\} \geq 0\} \tag{14}$$

be the set of goods that give to buyer i the maximal (and strictly positive) net valuation at p. Obviously, for some p, the set $\nabla_i^{\geq}(p)$ may be empty. It is obvious that for all $p \in \mathbb{R}^G_+$ and all $i \in \mathcal{B}$,

$$\nabla_i^>(p) \subseteq \nabla_i^{\geq}(p). \tag{15}$$

Demand of buyer i: For each price vector $p = (p_j)_{j \in \mathcal{G}} \in \mathbb{R}^G_+$, buyer i demands any feasible amount of goods that maximize his net valuation at p; namely,

$$\begin{aligned} D_i(p) = \{\alpha = (\alpha_{jk})_{(j,k) \in \mathcal{G} \times \mathcal{S}} \in \mathbb{Z}^{G \times S} \mid\ & \text{(D.a) } \alpha_{jk} \geq 0 \text{ for all } (j,k) \in \mathcal{G} \times \mathcal{S}, \\ & \text{(D.b) } \textstyle\sum_{jk} \alpha_{jk} \leq d_i, \\ & \text{(D.c) } \nabla_i^>(p) \neq \emptyset \Longrightarrow \textstyle\sum_{jk} \alpha_{jk} = d_i \text{ and} \\ & \text{(D.d) } \textstyle\sum_k \alpha_{jk} > 0 \Longrightarrow j \in \nabla_i^{\geq}(p)\}. \end{aligned}$$

Given $A \in \mathcal{F}^0$ and $i \in \mathcal{B}$, denote by $A(i) = (A(i)_{jk})_{(j,k) \in \mathcal{G} \times \mathcal{S}}$ the element in $\mathbb{Z}^{G \times S}_+$ such that, for all $(j,k) \in \mathcal{G} \times \mathcal{S}$, $A(i)_{jk} = A_{ijk}$.

Definition 10 A -1$-$*competitive equilibrium*[12] of market M is a pair $(p, A) \in \mathbb{R}^G_+ \times \mathcal{F}^0$ such that

[12] Jaume, Massó and Neme (2012) refer to this notion as competitive equilibrium; here we will refer to it as -1$-$competitive equilibrium to have available in this way a notation that will help us to compare it with other solutions.

(E.D) for all $i \in \mathcal{B}$, $A(i) \in D_i(p)$, and

(E.S) for all $j \in \mathcal{G}$ and all $k \in \mathcal{S}$, $\sum_i A_{ijk} \in \mathcal{S}_{jk}(p_j)$.

Next, we present the second competitive solution related to situations where agents, given a price vector, adjust their demands and supplies to the aggregate restrictions of the market. Given a price vector $p = (p_j)_{j\in\mathcal{G}} \in \mathbb{R}^G_+$ sellers will offer units of the goods (below their capacities) to maximize the net gains at p, but knowing that buyers will be able buy at most $D = \sum_{i\in\mathcal{B}} d_i$ units in total, and buyers will demand units of the goods (below their capacities) to maximize the net valuations at p, but knowing that they will be able to buy at most $Q_j = \sum_{k\in\mathcal{S}} q_{jk}$ units of each good j. To define the supply of seller $k \in \mathcal{S}$, we will need the following notation. Let $p \in \mathbb{R}^G_+$ be a price vector and let

$$\begin{array}{c}
\nabla_k^{1>}(p) = \{j \in \mathcal{G} \mid p_j - r_{jk} = \max_{j'\in\mathcal{G}}\{p_{j'} - r_{j'k}\} > 0\} \\
\nabla_k^{2>}(p) = \{j \in \mathcal{G}\backslash\nabla_k^{1>}(p) \mid p_j - r_{jk} = \max_{j'\in\mathcal{G}\backslash\nabla_k^{1>}(p)}\{p_{j'} - r_{j'k}\} > 0\} \\
\vdots \\
\nabla_k^{z>}(p) = \{j \in \mathcal{G}\backslash \cup_{m=1}^{z-1} \nabla_k^{m>}(p) \mid p_j - r_{jk} = \max_{j'\in\mathcal{G}\backslash\cup_{m=1}^{z-1}\nabla_i^{m>}(p)}\{p_{j'} - r_{j'k}\} > 0\} \\
\vdots \\
\nabla_k^{J>}(p) = \{j \in \mathcal{G}\backslash \cup_{m=1}^{J-1} \nabla_k^{m>}(p) \mid p_j - r_{jk} = \max_{j'\in\mathcal{G}\backslash\cup_{m=1}^{J-1}\nabla_k^{m>}(p)}\{p_{j'} - r_{j'k}\} > 0\}
\end{array}$$

be the sets of goods that give to seller k an strictly positive net gain at p, ordered in such a way that goods in $\nabla_k^{z>}(p)$ give a larger net gain than goods in $\nabla_k^{z'>}(p)$ if and only if $z < z'$. Obviously, for some p, the set $\nabla_k^{z>}(p)$ may be empty from a given z on.

Since seller k knows the market constraints, k knows that the maximal possible demand is $D = \sum_{i\in\mathcal{B}} d_i$. Hence, k will adjust his supply to this demand. Now define

$$\begin{array}{c}
s_{1k}(p) = \min\{\sum_{j\in\nabla_k^{1>}(p)} q_{jk}, D\} \\
s_{2k}(p) = \min\{\sum_{j\in\nabla_k^{2>}(p)} q_{jk}, D - s_{1k}(p)\} \\
\vdots \\
s_{zk}(p) = \min\{\sum_{j\in\nabla_k^{z>}(p)} q_{jk}, D - \sum_{m=1}^{z-1} s_{mk}(p)\} \\
\vdots \\
s_{Jk}(p) = \min\{\sum_{j\in\nabla_k^{J>}(p)} q_{jk}, D - \sum_{m=1}^{J-1} s_{mk}(p)\}.
\end{array}$$

We may have $s_{zk}(p) = 0$ from some z on.

Now, let

$$\nabla_k^{\geq}(p) = \{j \in \mathcal{G} \mid p_j - r_{jk} \geq 0\} \tag{16}$$

be the set of goods that give to seller k a non-negative net gain at p. Obviously, for some p, the set $\nabla_k^{\geq}(p)$ may be empty. It is obvious that for all $p \in \mathbb{R}^G_+$ and all $k \in \mathcal{S}$,

$$\nabla_k^{z>}(p) \subseteq \nabla_k^{\geq}(p) \text{ for all } z = 1, ..., J. \tag{17}$$

Supply-0 of seller k: For each price vector $p = (p_j)_{j\in\mathcal{G}} \in \mathbb{R}_+^G$, seller k supplies any feasible amount for the market of the goods that maximize his net gain at p; namely

$$S_k^0(p) = \{\beta = (\beta_j)_{j\in\mathcal{G}} \in \mathbb{Z}^G \mid \begin{array}{l} \text{(S.a0) } \beta_j \geq 0 \text{ for all } j \in \mathcal{G}, \\ \text{(S.b0) } \beta_j \leq q_{jk} \text{ for all } j \in \mathcal{G}, \\ \text{(S.c0) } \nabla_k^{z>}(p) \neq \emptyset \Longrightarrow \sum_{j\in\nabla_k^{z>}(p)} \beta_j = s_{zk}(p) \\ \qquad\qquad \text{for } z = 1, ..., J \text{ and} \\ \text{(S.d0) } \beta_j > 0 \Longrightarrow j \in \nabla_k^{\geq}(p)\}. \end{array}$$

Therefore, $S_k^0(p)$ describes the set of sales that maximize the net gain of seller k at p (taking into account the market constraints).[13] Observe that the set of sales described by each element in $S_k^0(p)$ gives, to seller k, the same net gain; namely, k is indifferent among all sales in $S_k^0(p)$.

To define the demand of buyer $i \in \mathcal{B}$, we will need the following notation. Let $p \in \mathbb{R}_+^G$ be a price vector and let

$$\begin{array}{c} \nabla_i^{1>}(p) = \{j \in \mathcal{G} \mid v_{ij} - p_j = \max_{j'\in\mathcal{G}}\{v_{ij'} - p_{j'}\} > 0\} \\ \nabla_i^{2>}(p) = \{j \in \mathcal{G}\backslash\nabla_i^{1>}(p) \mid v_{ij} - p_j = \max_{j'\in\mathcal{G}\backslash\nabla_i^{1>}(p)}\{v_{ij'} - p_{j'}\} > 0\} \\ \vdots \\ \nabla_i^{z>}(p) = \{j \in \mathcal{G}\backslash \cup_{m=1}^{z-1} \nabla_i^{m>}(p) \mid v_{ij} - p_j = \max_{j'\in\mathcal{G}\backslash\cup_{m=1}^{z-1}\nabla_i^{m>}(p)}\{v_{ij'} - p_{j'}\} > 0\} \\ \vdots \\ \nabla_i^{J>}(p) = \{j \in \mathcal{G}\backslash \cup_{m=1}^{J-1} \nabla_i^{m>}(p) \mid v_{ij} - p_j = \max_{j'\in\mathcal{G}\backslash\cup_{m=1}^{J-1}\nabla_i^{m>}(p)}\{v_{ij'} - p_{j'}\} > 0\} \end{array}$$

be the sets of goods that give to buyer i an strictly positive net valuation at p, ordered in such a way that goods in $\nabla_i^{z>}$ give a larger net valuation than goods in $\nabla_i^{z'>}$ if and only if $z < z'$. Obviously, for some p, the set $\nabla_i^{z>}(p)$ may be empty from some z on.

Now we define

$$\begin{array}{c} d_{1i}(p) = \min\{d_i, \sum_{j\in\nabla_i^{1>}(p)} Q_j\} \\ d_{2i}(p) = \min\{d_i - d_{1i}(p), \sum_{j\in\nabla_i^{2>}(p)} Q_j\} \\ \vdots \\ d_{zi}(p) = \min\{d_i - \sum_{m=1}^{z-1} d_{mi}(p), \sum_{j\in\nabla_i^{z>}(p)} Q_j\} \\ \vdots \\ d_{Ji}(p) = \min\{d_i - \sum_{m=1}^{J-1} d_{mi}(p), \sum_{j\in\nabla_i^{J>}(p)} Q_j\}. \end{array}$$

Obviously, for some p, we may have $d_{zi}(p) = 0$ from some z on. Also, for all $p \in \mathbb{R}_+^G$ and all $i \in \mathcal{B}$,

$$\nabla_i^{z>}(p) \subseteq \nabla_i^{\geq}(p) \text{ for all } z = 1, ..., J. \tag{18}$$

Demand-0 of buyer i: For each price vector $p = (p_j)_{j\in\mathcal{G}} \in \mathbb{R}_+^G$, buyer i demands any feasible amount for the market that maximizes his net valuation at p; namely,

[13] When $s_{zk}(p) = \sum_{j\in\nabla_k^{z>}(p)} q_{jk}$ for all $z = 1, ..., J$, the supply-0 of seller k coincides with that presented in Jaume, Massó and Neme (2012).

$$D_i^0(p) = \{\alpha = (\alpha_{jk})_{(j,k)\in\mathcal{G}\times\mathcal{S}} \in \mathbb{Z}^{G\times S} \mid \begin{array}{l} \text{(D.a0) } \alpha_{jk} \geq 0 \text{ for all } (j,k)\in\mathcal{G}\times\mathcal{S}, \\ \text{(D.b0) } \sum_{jk}\alpha_{jk} \leq d_i, \\ \text{(D.c0) } \nabla_i^{z>}(p) \neq \emptyset \Longrightarrow \sum_{j\in\nabla_i^{z>}(p)}\sum_k \alpha_{jk} = d_{zi}(p) \\ \qquad\qquad\qquad\qquad\qquad \text{for } z = 1,...,J \text{ and} \\ \text{(D.d0) } \sum_k \alpha_{jk} > 0 \Longrightarrow j \in \nabla_i^{\geq}(p)\}. \end{array}$$

Thus, $D_i^0(p)$ describes the set of all purchases that maximize the net valuation of buyer i at p, taking into account the aggregate constraints of the market.[14] Observe that the set of purchases described by each element in $D_i^0(p)$ give to i the same net valuation; namely, i is indifferent among all purchases in $D_i^0(p)$.

Definition 11 A $0-$*competitive equilibrium* of market M is a pair $(p, A) \in \mathbb{R}_+^G \times \mathcal{F}^0$ such that

(E.D0) for all $i \in \mathcal{B}$, $A(i) \in D_i^0(p)$, and

(E.S0) for all $k \in \mathcal{S}$, $(\sum_i A_{ijk})_{j\in\mathcal{G}} \in S_k^0(p)$.

In the remaining of this section, t will be an index in $\{-1, 0\}$. We say that the vector $p \in \mathbb{R}_+^G$ is a $t-$*competitive equilibrium price* (or simply a $t-$equilibrium price) of market M if there exists $A \in \mathcal{F}^0$ such that (p, A) is a $t-$competitive equilibrium of M (or simply a $t-$equilibrium). Denote by $\mathcal{P}^t$ to the set of all $t-$equilibrium prices of market M.

Fix a price vector $p \in \mathbb{R}_+^G$ and a feasible assignment $A \in \mathcal{F}^0$. According to (5) and (6), the utility of buyer $i \in \mathcal{B}$ at (p, A) is

$$u_i(p, A) = \sum_{jk}(v_{ij} - p_j) \cdot A_{ijk}$$

and the utility of seller $k \in \mathcal{S}$ at (p, A) is

$$w_k(p, A) = \sum_{ij}(p_j - r_{jk}) \cdot A_{ijk}.$$

Definition 12 Let M be a market and $t \in \{-1, 0\}$. The set of $t-$*competitive equilibrium payoffs* is given by

$$\mathcal{CE}^t = \{(u, w) \in \mathbb{R}^B \times \mathbb{R}^S \mid (u, w) = (u(p, A), w(p, A)) \text{ for some } t-\text{equilibrium } (p, A)\}.$$

We now define a cooperative game with transferable utility that will allow us to draw conclusions about $\mathcal{P}^t$ and $\mathcal{CE}^t$, for $t = -1, 0$, similarly as we did for $\mathcal{D}^t$ and $\mathcal{GS}^t$, for $t = 1, 2, 3$.

Definition 13 Let M be a market. A pair $(A^B, A^S) \in \mathbb{Z}_+^{B\times G\times S} \times \mathbb{Z}_+^{B\times G\times S}$ is -1$-$*compatible* in M if

(i) for each $i \in \mathcal{B}$, $\sum_{jk} A_{ijk}^B \leq d_i$, and

(ii) for each $k \in \mathcal{S}$ and $j \in \mathcal{G}$, $\sum_i A_{ijk}^S \leq q_{jk}$.

[14] When $d_{1i}(p) = d_i$ the demand-0 coincides with the definition in Jaume, Massó and Neme (2012).

The set of pairs -1−compatible in M will be denoted by $\mathcal{F}^{-1}$. Moreover, and with an abuse of notation, we will denote by $\mathcal{F}^0 = \{(A, A) \mid (A, A) \in \mathcal{F}^{-1}\}$ the set of 0−*compatible assignments* in M.[15]

Definition 14 Let M be a market, $t \in \{-1, 0\}$, $p \in \mathbb{R}^G_+$ a price vector, $C \subset \mathcal{B} \cup \mathcal{S}$ a coalition and $(A^B, A^S) \in \mathcal{F}^t$. Define the *net gain for C at (A^B, A^S) according to p* by

$$\varphi^M(C, (A^B, A^S), p) = \sum_{i \in \mathcal{B}^C} \sum_{jk} (v_{ij} - p_j) \cdot A^B_{ijk} + \sum_{k \in \mathcal{S}^C} \sum_{ij} (p_j - r_{jk}) \cdot A^S_{ijk}.$$

Note that if $(A, A) \in \mathcal{F}^0$ then $\varphi^M(C, (A, A), p) = \phi^M(C, A, p)$, where ϕ^M is given by (2) after setting, for all $j \in \mathcal{G}$, $\Gamma_{ijk} = p_j$ for all $(i, k) \in \mathcal{B} \times \mathcal{S}$. For each price vector p, we can define the following associated games to market M.

Definition 15 Let M be a market, $t = \{-1, 0\}$ and p a price vector. The *cooperative game $(\mathcal{B} \cup \mathcal{S}, v^{tp})$ with transferable utility associated to t and p* is defined as follows:

$$v^{tp}(C) = \begin{cases} \max_{(A^B, A^S) \in \mathcal{F}^t} \varphi^M(C, (A^B, A^S), p) & \text{if } C \subsetneq \mathcal{B} \cup \mathcal{S} \\ T^M & \text{if } C = \mathcal{B} \cup \mathcal{S}. \end{cases}$$

We denote by $\mathcal{C}^{tp}(M)$ (or simply by $\mathcal{C}^{tp}$) the Core of the game $(\mathcal{B} \cup \mathcal{S}, v^{tp})$. We now see that these Cores are intimately related with the corresponding notions of competitive equilibria.

Theorem 2 *Let M be a market and $t = \{-1, 0\}$. Then,*

$$p \in \mathcal{P}^t \text{ if and only if } \mathcal{C}^{tp} \neq \emptyset.$$

To prove Theorem 2 we need the following two results.

Lemma 2 *Let M be a market and $t = \{-1, 0\}$. Then, (p, A) is a $t-$equilibrium if and only if, for all $(A^B, A^S) \in \mathcal{F}^t$,*

$$\sum_{jk} (v_{ij} - p_j) \cdot A_{ijk} \geq \sum_{jk} (v_{ij} - p_j) \cdot A^B_{ijk} \text{ for all } i \in \mathcal{B} \tag{19}$$

and

$$\sum_{ij} (p_j - r_{jk}) \cdot A_{ijk} \geq \sum_{ij} (p_j - r_{jk}) \cdot A^S_{ijk} \text{ for all } k \in \mathcal{S}. \tag{20}$$

Proof See the Appendix in Section 6. ■

Parallel to Proposition 2 , we now have Proposition 4.

Proposition 4 *Let M be a market, $t = \{-1, 0\}$ and $p \in \mathbb{R}^G_+$ a price vector. Then,*

$$\mathcal{C}^{tp} \neq \emptyset \text{ if and only if } \mathcal{C}^{tp} = \{(u(p, A), w(p, A)) \mid A \in \mathcal{F}\}.$$

[15] Although, by the notation used in the previous section, we have that $\mathcal{F}^0 = \{A \mid (A, A) \in \mathcal{F}^{-1}\}$ the abuse of notation when writing $\mathcal{F}^0 = \{(A, A) \mid (A, A) \in \mathcal{F}^{-1}\}$ does not produce any trouble and helps to present the results.

Proof It is similar to the proof of Proposition 2 and therefore it is omitted. ■

Proof of Theorem 2 Assume $p \in \mathcal{P}^t$ and let A be such that (p, A) is a $t-$equilibrium. Then, by the definition of v^{tp} and Lemma 2, $(u(p, A), w(p, A)) \in \mathcal{C}^{tp}$. To see that the other implication holds, let p be such that $\mathcal{C}^{tp} \neq \emptyset$ and let $A \in \mathcal{F}$. By Proposition 4, $(u(p, A), w(p, A)) \in \mathcal{C}^{tp}$. Hence, for all $(A^B, A^S) \in \mathcal{F}^t$,

$$\sum_{jk}(v_{ij} - p_j) \cdot A_{ijk} \geq \sum_{jk}(v_{ij} - p_j) \cdot A^B_{ijk} \text{ for all } i \in \mathcal{B} \text{ and}$$
$$\sum_{ij}(p_j - r_{jk}) \cdot A_{ijk} \geq \sum_{ij}(p_j - r_{jk}) \cdot A^S_{ijk} \text{ for all } k \in \mathcal{S}.$$

Thus, by Lemma 2, (p, A) is a $t-$equilibrium and hence, $p \in \mathcal{P}^t$. ■

It is easy to check that, for all $p \in \mathbb{R}^G_+$,

$$v^{-1p}(C) \geq v^{0p}(C) \text{ for all } C \subsetneq \mathcal{B} \cup \mathcal{S} \text{ and} \quad (21)$$
$$v^{-1p}(\mathcal{B} \cup \mathcal{S}) = v^{0p}(\mathcal{B} \cup \mathcal{S}) \quad (22)$$

hold. Hence, $\mathcal{C}^{-1p} \subset \mathcal{C}^{0p}$ for all $p \in \mathbb{R}^G_+$. Thus, by Theorem 2, the following result holds.

Corollary 3 *Let M be a market. Then, $\emptyset \neq \mathcal{P}^{-1} \subsetneq \mathcal{P}^0$.*

Proof Jaume, Massó and Neme (2012) show that $\emptyset \neq \mathcal{P}^{-1}$. The inclusion follows from Theorem 2, (21) and (22). The strict inclusion follows from Example 2 below.■

Example 2 Let $M = (\mathcal{B}, \mathcal{G}, \mathcal{S}, V, d, R, Q)$ be a market where $B = \{1, 2\}$, $G = \{1, 2\}$, $S = \{1\}$, $V = \begin{pmatrix} 6 & 4 \\ 7 & 0 \end{pmatrix}$, $d = (7, 5)$ $Q = (8, 4)$ and $R = (5, 2)$. The unique optimal assignment is $A = \begin{pmatrix} 3 & 4 \\ 5 & 0 \end{pmatrix}$. Consider the price vector $p = (5, 2)$. Then, $v^{0p}(\{b_1, b_2, s_1\}) = T(A) = 1 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 21$, $v^{0p}(\{b_1, s_1\}) = 1 \cdot 3 + 2 \cdot 4 = 11$, $v^{0p}(\{b_2, s_1\}) = 2 \cdot 5 = 10$, $v^{0p}(\{s_1\}) = 0$, $v^{0p}(\{b_1\}) = 1 \cdot 3 + 2 \cdot 4 = 11$, $v^{0p}(\{b_2\}) = 2 \cdot 5 = 10$. Thus, $(u(p, A), w(p, A)) = (11, 10, 0) \in C^{0p}$ and hence, $(5, 2) \in \mathcal{P}^0$. But $(5, 2) \notin \mathcal{P}^{-1}$, since at $p = (5, 2)$ buyer b_1 would demand 7 units of good 2. □

The next proposition follows immediately from Lemma 2 and the fact that if $\widehat{A} \in \mathcal{F}^0$, then $(\widehat{A}, \widehat{A}) \in \mathcal{F}^t$ for all $t \in \{-1, 0\}$.

Proposition 5 *Let M be a market and $t \in \{-1, 0\}$. Then, (p, A) is a $t-$equilibrium if and only if $p \in \mathcal{P}^t$ and $A \in \mathcal{F}$.*[16]

A result, similar to Theorem 1 for group stable sets, hold for the sets of competitive equilibrium payoffs.

[16] Jaume, Masso y Neme (2012) prove the result in another way when $t = -1$.

Theorem 3 *Let M be a market. Then, for $t \in \{-1, 0\}$,*[17]

$$,\mathcal{CE}^t = \bigcup_{p \in \mathbb{R}^G_+} \mathcal{C}^{tp}.$$

Proof That $\mathcal{CE}^t \subset \bigcup_{p \in \mathbb{R}^G_+} \mathcal{C}^{tp}$ holds follows from Theorem 2 and Propositions 4 and 5. To see that the other inclusion holds, let $(u, w) \in \bigcup_{p \in \mathbb{R}^G_+} \mathcal{C}^{tp}$. By Proposition 4, there exists $(p, A) \in \mathbb{R}^G_+ \times \mathcal{F}$ such that $(u, w) = (u(p, A), w(p, A)) \in \mathcal{C}^{tp}$. Hence, by Lemma 2 and Theorem 2, $(u, w) \in \mathcal{CE}^t$. ■

Corollary 4 *Let M be a market. Then, $\emptyset \neq \mathcal{CE}^{-1} \subsetneq \mathcal{CE}^0$.*

Proof Jaume, Massó and Neme (2012) show that $\emptyset \neq \mathcal{CE}^{-1}$. The inclusion follows from Theorem 3, (21) and (22). Example 2 below shows that the inclusion may be strict. ■

Example 2 (continued) We already saw that $p = (5, 2) \in \mathcal{P}^0 \backslash \mathcal{P}^{-1}$. Hence, $(11, 10, 0) \in C^{0p}$ and $(11, 10, 0) \in \mathcal{CE}^0$. Moreover, we have that $(u(p^*, A), w(p^*, A)) = (11, 10, 0)$ if and only if $p^* = (5, 2)$. But since $(5, 2) \notin \mathcal{P}^{-1}$, $(11, 10, 0) \notin \mathcal{CE}^{-1}$. Namely, $\mathcal{CE}^{-1} \subsetneq \mathcal{CE}^0$.

4.2 Cartesian Product Structure and Computation of Competitive Equilibria

We have already seen that for $t \in \{-1, 0\}$ the set $\mathcal{CE}^t$ is a Cartesian product in the following sense:

$$\mathcal{CE}^t = \{(u, w) \in \mathbb{R}^{B \times S} \mid \text{for some } (p, A) \in \mathcal{P}^t \times \mathcal{F},\ (u, w) = (u(p, A), w(p, A))\}.$$

Now, parallel to Lemma 1, the following result holds.

Lemma 3 *Let M be a market, $t \in \{-1, 0\}$ and $p \in \mathbb{R}^G_+$ a price vector. If $\mathcal{C}^{tp} \neq \emptyset$, then $(u(p, A), w(p, A)) = (u(p, A'), w(p, A'))$ for all pairs $A, A' \in \mathcal{F}$.*

Proof The proof proceeds similarly to the proof of Proposition 2, using the fact that if $A \in \mathcal{F}$, then $(A, A) \in \mathcal{F}^t$ for $t \in \{-1, 0\}$. ■

Thus, if $\mathcal{C}^{tp} \neq \emptyset$ and $A \in \mathcal{F}$ we will write $(u(p, A), w(p, A))$ simply by $(u(p), w(p))$, without any reference to A. We present this fact in the following Corollary.

[17] For the case $t = -1$, if we extend Definition 14 to all $\Gamma \in \mathcal{D}$, we can show that

$$\mathcal{CE}^{-1} = \bigcup_{\Gamma \in \mathcal{D}} \mathcal{C}^{-1\Gamma}$$

holds. Indeed, if $\mathcal{C}^{-1\Gamma} \neq \emptyset$ then Γ is essentially a price vector; namely, for every pair $(i, j, k), (i', j, k') \in \mathcal{B} \times \mathcal{G} \times \mathcal{S}$ such that $j \in \mathcal{G}^{>}_{ik} \cap \mathcal{G}^{>}_{i'k'}$, $\Gamma_{ijk} = \Gamma_{i'jk'}$.

Corollary 4 *Let M be a market and $t \in \{-1,0\}$. Then, $\mathcal{CE}^t = \{(u(p), w(p)) \mid p \in \mathcal{P}^t\}$.*

Parallel to Proposition 2, we present several necessary and sufficient conditions for $\mathcal{C}^{tp} \neq \emptyset$ (one of them can be used to check whether or not p belongs to $\mathcal{P}^t$). Observe that the condition

$$v^{tp}(C) \leq \sum_{i \in \mathcal{B}^C} v^{tp}(\{i\}) + \sum_{k \in \mathcal{S}^C} v^{tp}(\{k\}) \text{ for all } C \subset \mathcal{B} \cup \mathcal{S}$$

is trivially satisfied for $t \in \{-1,0\}$.

Fix $A \in \mathcal{F}$ and consider the system on p of lineal inequalities given by

$$\varphi^M(C,(A^B,A^S),p) \leq \varphi^M(C,(A,A),p) \quad \begin{array}{l}\text{for all } C \subset \mathcal{B} \cup \mathcal{C} \text{ with } \#C = 1 \\ \text{and for all } (A^B, A^A) \in \mathcal{F}^t.\end{array} \tag{23}$$

Proposition 6 *Let M be a market, $t \in \{-1,0\}$ and $p \in \mathbb{R}^G_+$ a price vector. Then, the following statements are equivalent.*

(i) *p is a $t-$equilibrium price.*

(ii) $\mathcal{C}^{tp} \neq \emptyset$.

(iii)

$$v^{tp}(\mathcal{B} \cup \mathcal{S}) = \sum_{i \in \mathcal{B}} v^{tp}(\{i\}) + \sum_{k \in \mathcal{S}} v^{tp}(\{k\}). \tag{24}$$

(iv) *There exists $A \in \mathcal{F}$ such that $v^{tp}(C) = \varphi^M(C,(A,A),p)$, for all $C \subset \mathcal{B} \cup \mathcal{S}$ with $\#C = 1$.*

(v) *For all $A \in \mathcal{F}$, $v^{tp}(C) = \varphi^M(\mathbf{C},(A,A),p)$ for all $C \subset \mathcal{B} \cup \mathcal{S}$ with $\#C = 1$.*

(vi) *p solves system* (23).

Proof The equivalence between (i) and (ii) follows from Theorem 2. The equivalence between (iv) and (vi) is immediate. That (iii) implies (ii) follows from the fact that $(v^{tp}(\{i\}), v^{tp}(\{k\})_{(i,k) \in \mathcal{B} \cup \mathcal{S}} \in \mathcal{C}^{t\Gamma}$. That (iv) implies (iii) follows easily from the definition v^{tp}. That (v) implies (iv) is also immediate. Hence, it only remains to be proved that (ii) implies (v).

Assume $\mathcal{C}^{tp} \neq \emptyset$. By Proposition 2, if $A \in \mathcal{F}$ then $(u(p,A), w(p,A)) \in \mathcal{C}^{tp}$. Hence,

$$\begin{aligned} u_i(p) &\geq v^{tp}(\{i\}) \text{ for all } i \in \mathcal{B} \text{ and} \\ w_k(p) &\geq v^{tp}(\{k\}) \text{ for all } k \in \mathcal{S}. \end{aligned}$$

By the definition of v^{tp},

$$\begin{aligned} \varphi^M(\{i\},(A,A),p) &= u_i(p,A) = v^{tp}(\{i\}) \text{ for all } i \in \mathcal{B} \text{ and} \\ \varphi^M(\{k\},(A,A),p) &= w_k(p,A) = v^{tp}(\{k\}) \text{ for all } k \in \mathcal{S}. \end{aligned}$$

■

The above proposition gives criteria and procedures to compute price vectors in $\mathcal{P}^t$ and therefore payoff vectors in $\mathcal{CE}^t$.

5 Comparison and Relationships among Solutions

Our notation will facilitate us to compare the solutions and to show how the group stability notions, the notions of competitive equilibria and the Core of a market are related. We first observe that for all $C \subset \mathcal{B} \cup \mathcal{S}$,

$$\mathcal{F}^{\mathcal{C}}(C) \times \mathcal{F}^{\mathcal{C}}(C) \subset \mathcal{F}^{3}(C) \times \mathcal{F}^{3}(C) \subset \mathcal{F}^{2}(C) \times \mathcal{F}^{2}(C) \subset \mathcal{F}^{1}(C) \times \mathcal{F}^{1}(C) \subset \mathcal{F}^{0} \subset \mathcal{F}^{-1}.$$

Moreover, if $(A, A) \in \mathcal{F}^t(C) \times \mathcal{F}^t(C)$ then $\varphi^M(C, (A, A), p) = \phi^M(C, A, p)$. Hence, for all p and all $C \subsetneq \mathcal{B} \cup \mathcal{S}$,

$$v(C) \leq v^{3p}(C) \leq v^{2p}(C) \leq v^{1p}(C) \leq v^{0p}(C) \leq v^{-1p}(C)$$

and

$$v(\mathcal{B} \cup \mathcal{S}) = v^{3p}(\mathcal{B} \cup \mathcal{S}) = v^{2p}(\mathcal{B} \cup \mathcal{S}) = v^{1p}(\mathcal{B} \cup \mathcal{S}) = v^{0p}(\mathcal{B} \cup \mathcal{S}) = v^{-1p}(\mathcal{B} \cup \mathcal{S}).$$

Thus, for all p,

$$\mathcal{C} \supset \mathcal{C}^{3p} \supset \mathcal{C}^{2p} \supset \mathcal{C}^{1p} \supset \mathcal{C}^{0p} \supset \mathcal{C}^{-1p}, \tag{25}$$

and therefore,

$$\text{if } \mathcal{C}^{t'p} \neq \emptyset, \text{ then } \mathcal{C}^{tp} = \mathcal{C}^{t'p} \text{ for } t \geq t'. \tag{26}$$

It is easy to describe markets for which there exists p such that $C^{1p} \neq \emptyset$ and $C^{0p} = \emptyset$.

Now, we state a result showing that the set of payoffs associated to all six solutions are non-empty and have a strictly nested structure.

Theorem 4 *Let M be a market. Then,*

$$\emptyset \neq \mathcal{CE}^{-1} \subsetneq \mathcal{CE}^{0} \subsetneq \mathcal{GS}^{1} \subsetneq \mathcal{GS}^{2} \subsetneq \mathcal{GS}^{3} \subsetneq \mathcal{C}.$$

Proof By Corollary 4, (3), Theorems 1 and 3, and (25) it only remains to be proven that the inclusion of $\mathcal{CE}^0$ in $\mathcal{GS}^1$ is strict. But Example 2 below will show that. ■

Example 2 (continued) Consider $p = (5, 4)$. Then, $v^{1p}(\{b_1, s_1\}) = 11$, $v^{1p}(\{b_2, s_1\}) = 18$, $v^{1p}(\{s_1\}) = 8$, $v^{1p}(\{b_1\}) = 3$, $v^{1p}(\{b_2\}) = 10$. Hence, $(u(p, A), w(p, A)) = (3, 10, 8) \in C^{1p}$. Thus, $(3, 10, 8) \in \mathcal{GS}^1$. But $p \notin \mathcal{P}^0$, since b_1 would demand 8 units of good 1. Moreover, $(u(p^*, A), w(p^*, A)) = (3, 10, 8)$ if and only if $p^* = (5, 4)$. That is, $(3, 10, 8) \notin \mathcal{CE}^0$. □

Massó and Neme (2013) show that $\mathcal{CE}^{-1} \subsetneq \mathcal{GS}^1$ using an alternative proof. Moreover, from the inclusion relationships established in Theorem 4, and by Theorems 1 and 3, we observe that all solutions have a similar structure because to compute the payoff vectors in the solutions it is sufficient to identify the appropriated Γ (or p). Namely,

$$\mathcal{GS}^t = \{(u(\Gamma), w(\Gamma)) \mid \Gamma \in \mathcal{D}^t\} \text{ for } t = 1, 2, 3$$

and

$$\mathcal{CE}^t = \{(u(p), w(p)) \mid p \in \mathcal{P}^t\} \text{ for } t = -1, 0.$$

By Propositions 3 and 6, the elements in $\mathcal{D}^t$ and $\mathcal{P}^t$ are solutions of a system of non-strict lineal inequalities (the functions ϕ^M and φ^M are lineal and continuous in Γ and p, respectively). Hence, a procedure to compute payoff vectors in $\mathcal{GS}^t$ and $\mathcal{CE}^t$ is by solving the respective systems. In addition, the sets of solutions of such systems are convex and closed. Thus, $\mathcal{D}^t$ and $\mathcal{P}^t$ are convex and closed sets. But since the functions $(u(\Gamma), w(\Gamma))$ are lineal and continuos in Γ, it follows that $\mathcal{GS}^t$ and $\mathcal{CE}^t$ are convex and closed sets. Moreover, $\mathcal{GS}^t$ and $\mathcal{CE}^t$ are compact sets since $\mathcal{GS}^t \subset \mathcal{C}(M)$ and $\mathcal{CE}^t \subset \mathcal{C}(M)$. Thus, the inclusions given in Theorem 4 constitute a chain of nested convex sets.

6 Appendix

6.1 $\mathcal{GS}^1 \subsetneqq \mathcal{GS}^2 \subsetneqq \mathcal{GS}^3$ in Example 1

We want to show that $\mathcal{GS}^1 \subsetneqq \mathcal{GS}^2 \subsetneqq \mathcal{GS}^3$ holds for the market M of Example 1.

a) First, we will see that $(u, w) = (11, 16, 6) \in \mathcal{GS}^3 \backslash \mathcal{GS}^2$. Let $\Gamma = \begin{pmatrix} 5 & 2 & 4 \\ \frac{17}{3} & 3 & 1 \end{pmatrix}$ and $C \subset \mathcal{B} \cup \mathcal{S}$. We distinguish among five different cases.

(I) If $C = \{s_1\}$ and $\widehat{A} \in \mathcal{F}^t(C)$,

$$\begin{aligned} \phi^M(C, \widehat{A}, \Gamma) &= (\Gamma_{111} - r_{11}) \cdot \widehat{A}_{111} + (\Gamma_{121} - r_{21}) \cdot \widehat{A}_{121} + (\Gamma_{131} - r_{31}) \cdot \widehat{A}_{131} + \\ &\qquad (\Gamma_{211} - r_{11}) \cdot \widehat{A}_{211} + (\Gamma_{221} - r_{21}) \cdot \widehat{A}_{221} + (\Gamma_{231} - r_{31}) \cdot \widehat{A}_{231} \\ &\leq 0 \cdot A_{111} + 0 \cdot A_{121} + 3 \cdot A_{131} + \tfrac{2}{3} \cdot A_{211} + 1 \cdot A_{221} + 0 \cdot A_{231} \\ &= 0 + 0 + \tfrac{2}{3} \cdot 9 \\ &= w_1. \end{aligned}$$

(II) If $C = \{b_i\}$ and $\widehat{A} \in \mathcal{F}^t(C)$,

$$\begin{aligned} \phi^M(C, \widehat{A}, \Gamma) &= (v_{i1} - \Gamma_{i11}) \cdot \widehat{A}_{i11} + (v_{i2} - \Gamma_{i21}) \cdot \widehat{A}_{i21} + (v_{i3} - \Gamma_{i31}) \cdot \widehat{A}_{i31} \\ &\leq (v_{i1} - \Gamma_{i11}) \cdot A_{i11} + (v_{i2} - \Gamma_{i21}) \cdot A_{i21} + (v_{i3} - \Gamma_{i31}) \cdot A_{i31} \\ &= u_i. \end{aligned}$$

(III) If $C = \{b_1, s_1\}$ and $\widehat{A} \in \mathcal{F}^0$,

$$\begin{aligned} \phi^M(C, \widehat{A}, \Gamma) &= (v_{11} - r_{11}) \cdot \widehat{A}_{111} + (v_{12} - r_{21}) \cdot \widehat{A}_{121} + (v_{13} - r_{31}) \cdot \widehat{A}_{131} + \\ &\quad (\Gamma_{211} - r_{11}) \cdot \widehat{A}_{211} + (\Gamma_{221} - r_{21}) \cdot \widehat{A}_{221} + (\Gamma_{231} - r_{31}) \cdot \widehat{A}_{231} \\ &= 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} + \tfrac{2}{3} \cdot \widehat{A}_{211} + 1 \cdot \widehat{A}_{221} + 0 \cdot \widehat{A}_{231}. \end{aligned} \tag{27}$$

If $\widehat{A} \in \mathcal{F}^3(C)$, we have two possibilities:

(i) $\widehat{A}_{211} = \widehat{A}_{221} = \widehat{A}_{231} = 0$, in which case,

$$\begin{aligned} \phi^M(C, \widehat{A}, \Gamma) &= 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} \\ &\leq 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 1 \\ &= 17 \\ &\leq u_1 + w_1. \end{aligned}$$

(ii) $\widehat{A}_{211} = 9$, $\widehat{A}_{221} = 0$, $\widehat{A}_{231} = 1$, in which case,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &= 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} + \tfrac{2}{3} \cdot 9 + 1 \cdot 0 + 0 \cdot 1 \\ &\leq 1 \cdot 1 + 2 \cdot 5 + 3 \cdot 0 + \tfrac{2}{3} \cdot 9 + 1 \cdot 0 + 0 \cdot 1 \\ &= 17 \\ &= u_1 + w_1.\end{aligned}$$

(IV) If $C = \{b_2, s_1\}$ and $\widehat{A} \in \mathcal{F}^0$,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &= (v_{21} - r_{11}) \cdot \widehat{A}_{211} + (v_{22} - r_{21}) \cdot \widehat{A}_{221} + (v_{23} - r_{31}) \cdot \widehat{A}_{231} + \\ &(\Gamma_{111} - r_{11}) \cdot \widehat{A}_{111} + (\Gamma_{121} - r_{21}) \cdot \widehat{A}_{121} + (\Gamma_{131} - r_{31}) \cdot \widehat{A}_{131} \\ &= 2 \cdot \widehat{A}_{211} + 1 \cdot \widehat{A}_{221} + 4 \cdot \widehat{A}_{231} + 0 \cdot \widehat{A}_{111} + 0 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131}.\end{aligned}$$

If $\widehat{A} \in \mathcal{F}^3(C)$, we have two possibilities:
(i) $\widehat{A}_{111} = \widehat{A}_{121} = \widehat{A}_{131} = 0$, in which case,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &= 2 \cdot \widehat{A}_{211} + 1 \cdot \widehat{A}_{221} + 4 \cdot \widehat{A}_{231} \\ &\leq 2 \cdot 9 + 1 \cdot 0 + 4 \cdot 1 \\ &= 22 \\ &\leq u_2 + w_1.\end{aligned}$$

(ii) $\widehat{A}_{111} = 1$, $\widehat{A}_{121} = 5$, $\widehat{A}_{131} = 0$, in which case,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &= 2 \cdot \widehat{A}_{211} + 1 \cdot \widehat{A}_{221} + 4 \cdot \widehat{A}_{231} + 0 \cdot 1 + 0 \cdot 5 + 3 \cdot 0 \\ &\leq 2 \cdot 9 + 1 \cdot 0 + 4 \cdot 1 + 0 \cdot 1 + 0 \cdot 5 \\ &= 22 \\ &= u_2 + w_1.\end{aligned}$$

(V) If $C = \{b_1, b_1, s_1\}$ and $\widehat{A} \in \mathcal{F}^3(C)$ then, $\phi^M(C, \widehat{A}, \Gamma) = T(\widehat{A})$. Hence, $\phi^M(C, \widehat{A}, \Gamma) \leq T(A) = u_1 + u_2 + w_1$.

Thus, we can conclude that for all $C \subset \mathcal{B} \cup \mathcal{S}$ and all $\widehat{A} \in \mathcal{F}^3(C)$, $\sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k \geq \phi^M(C, \widehat{A}, \Gamma)$ holds. Hence, $(11, 16, 6) \in \mathcal{GS}^3$.

We now check that $(11, 16, 6) \notin \mathcal{GS}^2$. Assume there exists $\Gamma' \in \mathcal{D}$ such that for all $C \subset \mathcal{B} \cup \mathcal{S}$ and all $\widehat{A} \in \mathcal{F}^2(C)$,

$$\sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k \geq \phi^M(C, \widehat{A}, \Gamma') \tag{28}$$

holds. Consider $\{b_1, s_1\} \subset \mathcal{B} \cup \mathcal{S}$ and $\widehat{A} = \begin{pmatrix} 1 & 5 & 1 \\ 9 & 0 & 0 \end{pmatrix}$. Observe that $\widehat{A} \in \mathcal{F}^2(\{b_1, s_1\})$. By (28),

$$\phi^M(\{b_1, s_1\}, \widehat{A}, \Gamma') = 1 \cdot 1 + 2 \cdot 5 + 3 \cdot 1 + (\Gamma'_{211} - 5) \cdot 9 \leq 11 + 6. \tag{29}$$

Now, consider $\{b_2\} \subset \mathcal{B} \cup \mathcal{S}$ and $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$. Observe that $A \in \mathcal{F}^2(\{b_2\})$. By (28),

$$\phi^M(\{b_2\}, A, \Gamma') = (7 - \Gamma'_{211}) \cdot 9 + (5 - \Gamma'_{231}) \cdot 1 \leq 16, \tag{30}$$

and hence, by (29) and (30),

$$1 + 10 + 3 + (\Gamma'_{211} - 5) \cdot 9 + (7 - \Gamma'_{211}) \cdot 9 + (5 - \Gamma'_{231}) \leq 33,$$

which means that $\Gamma'_{231} \geq 4$. Consider now the assignment $\widehat{A} = \begin{pmatrix} 5 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and observe that $\widehat{A} \in \mathcal{F}^2(\{b_1, s_1\})$. By (28),

$$\phi^M(\{b_1, s_1\}, \widehat{A}, \Gamma') = 5 + 10 + (\Gamma'_{231} - 1) \leq 6 + 11, \tag{31}$$

and hence, by (31) and (30),

$$5 + 10 + (\Gamma'_{231} - 1) + (7 - \Gamma'_{211}) \cdot 9 + (5 - \Gamma'_{231}) \leq 33,$$

which means that $\Gamma'_{231} \geq \frac{49}{9}$. Finally, consider $\{s_1\} \subset \mathcal{B} \cup \mathcal{S}$ and $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$. Observe that $A \in \mathcal{F}^2(\{s_1\})$ and

$$\begin{aligned} \phi^M(\{s_1\}, A, \Gamma') &= (\Gamma'_{111} - 5) \cdot 1 + (\Gamma'_{121} - 5) \cdot 5 + (\Gamma'_{211} - 5) \cdot 9 + (\Gamma'_{231} - 1) \cdot 1 \\ &\geq (\Gamma'_{211} - 5) \cdot 9 + (\Gamma'_{231} - 1) \\ &\geq (\tfrac{49}{9} - 5) \cdot 9 + 4 - 1 \\ &= 7. \end{aligned}$$

Hence, $\phi^M(\{s_1\}, A, \Gamma') \geq 7 > 6 = w_1$, which contradicts (28). Thus, $(11, 16, 6) \in \mathcal{GS}^3 \backslash \mathcal{GS}^2$ holds.

b) Second, we will see that $(u, w) = (11, 13, 9) \in \mathcal{GS}^2 \backslash \mathcal{GS}^1$. Let $\Gamma = \begin{pmatrix} 6 & 4 & 4 \\ \frac{17}{3} & 3 & 4 \end{pmatrix}$ and $C \subset \mathcal{B} \cup\, \mathcal{S}$. We distinguish between two cases:

(I) If $C \subset \mathcal{B} \cup \mathcal{S}$, $C \neq \{b_1, s_1\}$ and $\widehat{A} \in \mathcal{F}^t(C)$, we can show using a similar argument to the one used in case **a**) that $\phi^M(C, \widehat{A}, \Gamma) \leq \sum\limits_{i \in \mathcal{B}^C} u_i + \sum\limits_{k \in \mathcal{S}^C} w_k$ holds as well.

(II) If $C = \{b_1, s_1\}$ and $\widehat{A} \in \mathcal{F}^0$,

$$\begin{aligned} \phi^M(C, \widehat{A}, \Gamma) &= (v_{11} - r_{11}) \cdot \widehat{A}_{111} + (v_{12} - r_{21}) \cdot \widehat{A}_{121} + (v_{13} - r_{31}) \cdot \widehat{A}_{131} + \\ &\quad (\Gamma_{211} - r_{11}) \cdot \widehat{A}_{211} + (\Gamma_{221} - r_{21}) \cdot \widehat{A}_{221} + (\Gamma_{231} - r_{31}) \cdot \widehat{A}_{231} \\ &= 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} + \tfrac{2}{3} \cdot \widehat{A}_{211} + 1 \cdot \widehat{A}_{221} + 3 \cdot \widehat{A}_{231}. \end{aligned} \tag{32}$$

If $\widehat{A} \in \mathcal{F}^2(C)$, we have three possibilities:

(i) If $\widehat{A}_{231} = 1$ and $\widehat{A}_{211} = 9$,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &\leq 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} + \tfrac{2}{3} \cdot 9 + 1 \cdot 0 + 3 \cdot 1 \\ &\leq 1 + 10 + 6 + 3 \\ &= 20 \\ &= u_1 + w_1.\end{aligned}$$

(ii) If $\widehat{A}_{231} = 1$ and $\widehat{A}_{211} = 0$,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &\leq 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} + \tfrac{2}{3} \cdot 0 + 1 \cdot 0 + 3 \cdot 1 \\ &\leq 1 \cdot 5 + 10 + 3 \\ &= 18 \\ &\leq u_1 + w_1.\end{aligned}$$

(iii) If $\widehat{A}_{231} = 0$,

$$\begin{aligned}\phi^M(C, \widehat{A}, \Gamma) &= 1 \cdot \widehat{A}_{111} + 2 \cdot \widehat{A}_{121} + 3 \cdot \widehat{A}_{131} + \tfrac{2}{3} \cdot \widehat{A}_{211} + 1 \cdot 0 + 3 \cdot 0 \\ &= 1 \cdot \widehat{A}_{111} + 2 \cdot 5 + 3 \cdot 0 + \tfrac{2}{3} \cdot \widehat{A}_{211} + 1 \cdot 0 \\ &\leq u_1 + w_1,\end{aligned}$$

where the last inequality follows from what we have established in cases (i) and (ii) above.

Thus, we can conclude that for all $C \subset \mathcal{B} \cup \mathcal{S}$ and all $\widehat{A} \in \mathcal{F}^2(C)$, $\sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k \geq \phi^M(C, \widehat{A}, \Gamma)$ holds. Hence, $(11, 13, 9) \in \mathcal{GS}^2$.

We now check that $(11, 13, 9) \notin \mathcal{GS}^1$. Assume there exists $\Gamma' \in \mathcal{D}$ such that for all $C \subset \mathcal{B} \cup \mathcal{S}$ and all $\widehat{A} \in \mathcal{F}^1(C)$,

$$\sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k \geq \phi^M(C, \widehat{A}, \Gamma') \tag{33}$$

holds. Consider $\{b_1, s_1\} \subset \mathcal{B} \cup \mathcal{S}$ and $\widehat{A} = \begin{pmatrix} 5 & 5 & 0 \\ 4 & 0 & 1 \end{pmatrix}$. Observe that $\widehat{A} \in \mathcal{F}^1(\{b_1, s_1\})$. By (33),

$$\phi^M(\{b_1, s_1\}, \widehat{A}, \Gamma') = 1 \cdot 5 + 2 \cdot 5 + 3 \cdot 0 + (\Gamma'_{211} - 5) \cdot 4 + (\Gamma'_{231} - 1) \cdot 1 \leq 11 + 9. \tag{34}$$

Consider now $\{b_2\} \subset \mathcal{B} \cup \mathcal{S}$ and $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$. Observe that $A \in \mathcal{F}^2(\{b_2\})$. By (33),

$$\phi^M(\{b_2\}, A, \Gamma') = (7 - \Gamma'_{211}) \cdot 9 + (5 - \Gamma'_{231}) \cdot 1 \leq 13, \tag{35}$$

and hence, by (34) and (35),

$$5 + 10 + (\Gamma'_{211} - 5) \cdot 4 + (\Gamma'_{231} - 1) \cdot 1 + (7 - \Gamma'_{211}) \cdot 9 + (5 - \Gamma'_{231}) \leq 33,$$

which means that $\Gamma'_{211} \geq 6$. Consider now the assignment $\widehat{A} = \begin{pmatrix} 1 & 5 & 1 \\ 9 & 0 & 0 \end{pmatrix}$. Observe that $\widehat{A} \in \mathcal{F}^1(\{b_1, s_1\})$ and

$$\phi^M(\{b_1, s_1\}, \widehat{A}, \Gamma') = 1 + 10 + 3 + (\Gamma'_{211} - 5) \cdot 9 \leq 11 + 9. \qquad (36)$$

Hence, by (36) and (35),

$$1 + 10 + 3 + (\Gamma'_{211} - 5) \cdot 9 + (7 - \Gamma'_{211}) \cdot 9 + (5 - \Gamma'_{231}) \leq 33,$$

which means that $\Gamma'_{231} \geq 4$. Finally, consider $\{s_1\} \subset \mathcal{B} \cup \mathcal{S}$ and $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$. Observe that $A \in \mathcal{F}^2(\{s_1\})$ and

$$\begin{aligned} \phi^M(\{s_1\}, A, \Gamma') &= (\Gamma'_{111} - 5) \cdot 1 + (\Gamma'_{121} - 5) \cdot 5 + (\Gamma'_{211} - 5) \cdot 9 + (\Gamma'_{231} - 1) \cdot 1 \\ &\geq (\Gamma'_{211} - 5) \cdot 9 + (\Gamma'_{231} - 1) \\ &\geq (6 - 5) \cdot 9 + 4 - 1 \\ &= 12. \end{aligned}$$

Hence, $\phi^M(\{s_1\}, A, \Gamma') \geq 12 > 9 = w_1$, which contradicts (33). Thus, $(11, 13, 9) \in \mathcal{GS}^2 \backslash \mathcal{GS}^1$.

c) To finish, we will exhibit a vector in $\mathcal{GS}^1$. Let $(u, w) = (0, 0, 33)$, $\Gamma = \begin{pmatrix} 6 & 4 & 4 \\ 7 & 3 & 5 \end{pmatrix}$ and $C \subset \mathcal{B} \cup \mathcal{S}$. We distinguish between two cases.
(I) If $C \subset \mathcal{B}$ then, $\phi^M(C, \widehat{A}, \Gamma) = 0$ holds for all $\widehat{A} \in \mathcal{F}^1(C)$. Hence, $\phi^M(C, \widehat{A}, \Gamma) \leq \sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k$.
(II) If $s_1 \in C$ and $\widehat{A} \in \mathcal{F}^1(C)$ then, $\phi^M(C, \widehat{A}, \Gamma) \leq T^M(\widehat{A}) \leq 33$ (since $\widehat{A} \in \mathcal{F}^0$ holds). Hence,

$$\phi^M(C, \widehat{A}, \Gamma) \leq 33 = w_1 = \sum_{i \in \mathcal{B}^C} u_i + \sum_{k \in \mathcal{S}^C} w_k,$$

which means that $(u, w) = (0, 0, 33) \in \mathcal{GS}^1$.

6.2 $\mathcal{C}^{t\Gamma} = \emptyset$ in Example 1

Remember that the unique optimal assignment in the market of Example 1 is $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix}$ with $T^M(A) = 33$. Let $\Gamma = \begin{pmatrix} 6 & 4 & 4 \\ \frac{17}{3} & 3 & 4 \end{pmatrix}$. By Remark 1, $v^{1\Gamma}(\{b_1, b_2, s_1\}) = 33$. Observe that $\widehat{A} = \begin{pmatrix} 5 & 5 & 0 \\ 4 & 0 & 1 \end{pmatrix} \in \mathcal{F}^1(\{b_1, s_1\})$, thus

$$v^{t\Gamma}(\{b_1, s_1\}) \geq \phi^M(\{b_1, s_1\}, \widehat{A}, \Gamma) = 5 + 10 + \left(\tfrac{17}{3} - 5\right) \cdot 4 + (4 - 1) \cdot 1 = \tfrac{62}{3}.$$

Now, consider $\{b_2\}$. We have $A = \begin{pmatrix} 1 & 5 & 0 \\ 9 & 0 & 1 \end{pmatrix} \in \mathcal{F}^1(\{b_2\})$, thus

$$v^{t\Gamma}(\{b_2\}) \geq \phi^M(\{b_2\}, A, \Gamma) = (7 - \tfrac{17}{3}) \cdot 9 + (5 - 4) \cdot 1 = 13.$$

Therefore, $v^{1\Gamma}(\{b_1, s_1\}) + v^{1\Gamma}(\{b_2\}) \geq \frac{62}{3} + 13 = \frac{101}{3} > 33 = v^{1\Gamma}(\{b_1, b_2, s_1\})$, where we deduce that the game $(\mathcal{B} \cup \mathcal{S}, v^{1\Gamma})$ has empty Core.

6.3 Proof of Lemma 2

We first prove the statement in Lemma 2 for $t = -1$. For this purpose we will use the following notation. Fix $p \in \mathbb{R}_+^G$. Define for every $i \in \mathcal{B}$

$$\gamma_i(p) = \begin{cases} v_{ij} - p_j & \text{if there exists } j \in \nabla_i^{>}(p) \\ 0 & \text{otherwise,} \end{cases} \tag{37}$$

and for every $(j,k) \in \mathcal{G} \times \mathcal{S}$

$$\pi_{jk}(p) = \begin{cases} p_j - r_{jk} & \text{if } p_j - r_{jk} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

The number $\gamma_i(p)$ is the net valuation obtained by buyer i from each unit of the goods that he wants to buy at p and the number $\pi_{jk}(p)$ is the net gain obtained by seller k from each unit of good j that he want to sell at p.

Let $(A^B, A^S) \in \mathcal{F}^{-1}$. Since (p, A) is a -1−equilibrium, for each $i \in \mathcal{B}$,

$$\sum_{jk} (v_{ij} - p_j) \cdot A_{ijk} = \gamma_i(p) \cdot d_i.$$

But $d_i \geq \sum_{jk} A_{ijk}^B$ and $(v_{ij} - p_j) \leq \gamma_i(p)$ for all j. Hence, for each $i \in \mathcal{B}$,

$$\sum_{jk} (v_{ij} - p_j) \cdot A_{ijk} \geq \sum_{jk} (v_{ij} - p_j) \cdot A_{ijk}^B.$$

Thus, (19) holds. The proof that (20) holds as well proceeds similarly and therefore it is omitted.

To prove the other implication, consider a pair (p, A) satisfying (19) and (20) for all $(A^B, A^S) \in \mathcal{F}^{-1}$. We will show that (p, A) is a -1-competitive equilibrium.

First, we will check that (E.D) holds. Since A is feasible, (D.a) and (D.b) hold.

To check that (D.c) holds assume that for $i \in \mathcal{B}$, $\nabla_i^{>}(p) \neq \emptyset$. We want to show that $\sum_{j \in \nabla_i^{>}(p)} \sum_k A_{jk} = d_i$. Assume there exists i' such that $\nabla_{i'}^{>}(p) \neq \emptyset$ but $\sum_{j \in \nabla_{i'}^{>}(p)} \sum_k A_{i'jk} < d_{i'}$. Let $j' \in \nabla_{i'}^{>}(p)$ and let A^B be such that

$$\sum_k A_{i'jk}^B = \begin{cases} d_{i'} & \text{if } j = j' \\ 0 & \text{if } j \neq j'. \end{cases}$$

It is clear that $(A^B, A^S) \in \mathcal{F}^{-1}$ for some A^S. Now we have that $\sum_{jk} (v_{i'j} - p_j) \cdot A_{i'jk}^B = \gamma_{i'}(p) \cdot d_{i'}$. We distinguish between two cases.

<u>Case 1</u>: $\sum_{jk} A_{i'jk} < d_{i'}$. Then,

$$\sum_{jk} (v_{i'j} - p_j) \cdot A_{i'jk}^B = \gamma_{i'}(p) \cdot d_{i'} > \gamma_{i'}(p) \cdot \sum_{jk} A_{i'jk} \geq \sum_{jk} (v_{i'j} - p_j) \cdot A_{i'jk},$$

which contradicts (19).

Case 2: $\sum_{jk} A_{i'jk} = d_{i'}$. Then,

$$\begin{aligned}\sum_{jk}(v_{i'j}-p_j)\cdot A^B_{i'jk} &= \gamma_{i'}(p)\cdot d_{i'} \geq \gamma_{i'}(p)\cdot \textstyle\sum_j\sum_k A_{i'jk} \\ &= \gamma_{i'}(p)\cdot(\textstyle\sum_{j\in\nabla^{\geq}_{i'}(p)}\sum_k A_{i'jk}) + \gamma_{i'}(p)\cdot(\sum_{j\notin\nabla^{\geq}_{i'}(p)}\sum_k A_{i'jk}) \\ &> \textstyle\sum_{j\in\nabla^{\geq}_{i'}(p)}\sum_k(v_{i'j}-p_j)\cdot A_{i'jk} + \sum_{j\notin\nabla^{\geq}_{i'}(p)}(v_{i'j}-p_j)\cdot\sum_k A_{i'jk} \\ &= \textstyle\sum_{jk}(v_{i'j}-p_j)\cdot A_{i'jk},\end{aligned}$$

which contradicts (19).

To check that (D.d) holds, assume that for $i \in \mathcal{B}$, $\sum_k A_{ijk} > 0$. We want to show that $j \in \nabla^{\geq}_i(p)$. Assume there exist $i' \in \mathcal{B}$, $j' \in \mathcal{G}$ and $k' \in \mathcal{S}$ such that $A_{i'j'k'} > 0$, but $j' \notin \nabla^{\geq}_{i'}(p)$. Define

$$A^B_{ijk} = \begin{cases} A_{ijk} & \text{if } (i,j,k) \neq (i',j',k') \\ 0 & \text{if } (i,j,k) = (i',j',k'). \end{cases}$$

We have that $(A^B, A^S) \in \mathcal{F}^{-1}$ for some A^S and in addition,

$$\sum_{jk}(v_{i'j}-p_j)\cdot A^B_{i'jk} = \sum_{\substack{jk:\\(j,k)\neq(j',k')}}(v_{i'j}-p_j)\cdot A_{i'jk} > \sum_{jk}(v_{i'j}-p_j)\cdot A_{i'jk}.$$

Hence, (p, A) does not satisfy (19). Thus, (E.D) holds.

Proceeding similarly, we can check that (E.S) holds, since for (p, A) to satisfy (20), it is necessary that each seller $k \in \mathcal{S}$ sells all the units he owns of each good that produce a strict positive net gain and no unit of the goods producing negative net gains.

We now proceed to prove Lemma 2 for the case $t = 0$. For this purpose we will use the following notation. Fix $p \in \mathbb{R}^G_+$ and $j \in \nabla^{z>}_i(p)$ for $z = 1, ..., J$, define $\gamma_{zi}(p) = (v_{ij} - p_j)$. Moreover, if $\nabla^{z>}_i(p) = \emptyset$ define $\gamma_{zi}(p) = 0$. Let (p, A) be a $0-$competitive equilibrium and assume there exist $i \in \mathcal{B}$ and $A^* \in \mathcal{F}^0$ such that

$$\sum_{jk}(v_{ij}-p_j)\cdot A^*_{ijk} > \sum_{jk}(v_{ij}-p_j)\cdot A_{ijk}.$$

If $(v_{ij} - p_j) < 0$, then $A_{ijk} = 0$ for all k since (p, A) is a $0-$competitive equilibrium. Hence,

$$\begin{aligned}\sum_{z=1,...,J}\gamma_{zi}(p)\sum_{\substack{j\in\nabla^{z>}_i(p)\\k\in\mathcal{S}}}A^*_{ijk} + \sum_{\substack{j\notin\cup\nabla^{t>}_i(p)\\k\in\mathcal{S}}}(v_{ij}-p_j)\cdot A^*_{ijk} &= \sum_{jk}(v_{ij}-p_j)\cdot A^*_{ijk} \\ &> \sum_{jk}(v_{ij}-p_j)\cdot A_{ijk} \\ &= \sum_{z=1,...,J}\gamma_{zi}(p)\sum_{\substack{j\in\nabla^{z>}_i(p)\\k\in\mathcal{S}}}A_{ijk}.\end{aligned}$$

Then, since $\sum_{\substack{j\notin\cup\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}(v_{ij}-p_j)\cdot A^*_{ijk}\leq 0$ holds,

$$\sum_{z=1,...,J}\gamma_{it}(p)\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A^*_{ijk}>\sum_{z=1,...,J}\gamma_{zi}(p)\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A_{ijk}. \tag{39}$$

Assume $\sum_{j\in\nabla_i^{1>}(p)\ k}A^*_{ijk}>\sum_{j\in\nabla_i^{1>}(p)\ k}A_{ijk}$. Since A and A^* are feasible, $\nabla_i^{1>}(p)\neq\emptyset$ and $\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A_{ijk}<\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A^*_{ijk}\leq d_{1i}$. Then, $A(i)\notin D_i(p)$. Hence,

$$\sum_{\substack{j\in\nabla_i^{1>}(p)\\ k\in\mathcal{S}}}A^*_{ijk}\leq\sum_{\substack{j\in\nabla_i^{1>}(p)\\ k\in\mathcal{S}}}A_{ijk}. \tag{40}$$

Let z^* be the minimum $z=1,...,J$ such that $\sum_{z=1,...,z^*}\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A^*_{ijk}>\sum_{z=1,...,z^*}\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A_{ijk}$ (z^* exists by (39) and (40)). Clearly, $\nabla_i^{z^*>}(p)\neq\emptyset$. Thus,

$$\sum_{z=1,...,z^*}\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A_{ijk}=\sum_{z=1,...,z^*}d_{it}.$$

We distinguish between two cases.

<u>Case 1</u>: $\sum_{z=1,...,z^*}d_{zi}=d_i$. Then, $\sum_{z=1,...,z^*}\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A^*_{ijk}>d_i$, which contradicts that A^* is feasible.

<u>Case 2</u>: $\sum_{z=1,...,z^*}d_{zi}<d_i$. Then, $d_{zi}=\min\{d_i-\sum_{m=1}^{z-1}d_{mi},\sum_{j\in\nabla_i^{z>}(p)}Q_j\}=\sum_{j\in\nabla_i^{z>}(p)}Q_j$ for all $z=1,...,z^*$. Hence, $\sum_{z=1,...,z^*}\sum_{\substack{j\in\nabla_i^{t>}(p)\\ k\in\mathcal{S}}}A_{ijk}=\sum_{z=1,...,z^*}\sum_{j\in\nabla_i^{z>}(p)}Q_j$. Thus, $\sum_{z=1,...,z^*}\sum_{\substack{j\in\nabla_i^{z>}(p)\\ k\in\mathcal{S}}}A^*_{ijk}>\sum_{z=1,...,z^*}\sum_{j\in\nabla_i^{z>}(p)}Q_j$, which again contradicts that A^* is feasible.

The fact that $\sum_{ij}(p_j-r_{jk})\cdot A_{ijk}\geq\sum_{ij}(p_j-r_{jk})\cdot A^*_{ijk}$ holds for all $k\in\mathcal{S}$, can be deduced similarly.

To verify that the other implication holds as well, assume that the pair (p,A) satisfies (19) and (20) for all feasible A^*. We want to show that (p,A) is a $0-$competitive equilibrium. First, we check that (E.D0) holds. Since A is feasible, (D.a0) and (D.b0) hold.

To check that (D.c) holds, assume $\nabla_i^{z>}(p)\neq\emptyset$. We want to show that $\sum_{j\in\nabla_i^{z>}(p)}\sum_k A_{jk}=d_{zi}$. Assume there exist i' and z^* such that $\nabla_{i'}^{z^*>}(p)\neq\emptyset$ but

$$\sum_{j\in\nabla_{i'}^{z^*>}(p)}\sum_k A_{i'jk}<d_{i'z^*}. \tag{41}$$

Without loss of generality, we may assume that $\sum_{j\in\nabla_{i'}^{z>}(p)}\sum_k A_{jk}=d_{zi'}$ holds for all $z<z^*$.We have $d_{z^*i'}\leq\sum_{j\in\nabla_{i'}^{z^*>}(p)}Q_j$. By (41), there exist $k^*\in\mathcal{S}$ and $j^*\in\nabla_{i'}^{z^*>}$ such that $A_{i'j^*k^*}<q_{j^*k^*}$. We distinguish between two cases.

Case 1: $\sum_{jk} A_{i'jk} < d_{i'}$. Define A^* as follows:

$$A^*_{ijk} = \begin{cases} A_{ijk} & \text{if } i = i',\ j \in \nabla_i^{z>}(p) \text{ for some } z < z^* \text{ or } z^* < z \text{ for all } k \\ A_{ijk} + 1 & \text{if } i = i', j = j^* \text{ and } k = k^* \\ A_{ijk} & \text{if } i = i', j \in \nabla_i^{z^*>}(p),\ j \neq j^* \text{ and } k \neq k^* \\ 0 & \text{otherwise.} \end{cases}$$

We have that A^* is feasible. Moreover,

$$\begin{aligned} \sum_{jk}(v_{ij} - p_j) \cdot A^*_{ijk} &= \sum_{z=1,\dots,J} \gamma_{zi}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ k \in \mathcal{S}}} A^*_{ijk} > \sum_{z=,1\dots,J} \gamma_{zi}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ k \in \mathcal{S}}} A_{ijk} \\ &\geq \sum_{jk}(v_{ij} - p_j) \cdot A_{ijk}, \end{aligned}$$

which contradicts (19).

Case 2: $\sum_{jk} A_{i'jk} = d_{i'}$. Then, by (41), there exist $\tilde{z} > z$, $\tilde{j} \in \mathcal{G}$ and $\tilde{k} \in \mathcal{S}$ such that $\tilde{j} \in \nabla_{i'}^{\tilde{z}>}(p)$ and $A_{i'\tilde{j}\tilde{k}} > 0$. Now define A^* as follows:

$$A^*_{ijk} = \begin{cases} A_{ijk} & \text{if } i = i',\ j \in \nabla_i^{z>}(p) \text{ for some } z < z^* \text{ and for all } k \\ A_{ijk} + 1 & \text{if } i = i', j = j^* \text{ and } k = k^* \\ A_{ijk} - 1 & \text{if } i = i', j = \tilde{j} \text{ and } k = \tilde{k} \\ A_{ijk} & \text{if } i = i', j \in \nabla_i^{t^*>}(p) \text{ and } (j,k) \neq (j^*, k^*) \\ A_{ijk} & i = i',\ j \in \nabla_i^{z>}(p) \text{ for some } z > z^* \text{ and } (j,k) \neq (\tilde{j}, \tilde{k}). \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate to check that A^* is feasible. Moreover,

$$\begin{aligned} \sum_{jk}(v_{ij} - p_j) \cdot A^*_{ijk} &= \sum_{z=1,\dots,J} \gamma_{zi}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ k \in \mathcal{S}}} A^*_{ijk} > \sum_{z=1,\dots,J} \gamma_{zi}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ k \in \mathcal{S}}} A_{ijk} \\ &\geq \sum_{jk}(v_{ij} - p_j) \cdot A_{ijk}. \end{aligned}$$

which contradicts (19).

To check that (D.d0) holds, assume $\sum_k A_{ijk} > 0$. We want to show that $j \in \nabla_i^{\geq}(p)$ for all $i \in \mathcal{B}$. Assume there exist $i' \in \mathcal{B}$ and $j' \in \mathcal{G}$ such that $\sum_k A_{i'j'k} > 0$ but $j' \notin \nabla_{i'}^{\geq}(p)$. Define

$$A^*_{ijk} = \begin{cases} 0 & \text{if } i = i' \text{ and } j \notin \nabla_{i'}^{\geq}(p) \text{ for all } k \in \mathcal{S} \\ A_{ijk} & \text{if } i = i' \text{ and } j \in \nabla_i^{z>}(p) \text{ for some } z \text{ for all } k \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate to check that A^* is feasible. Moreover,

$$\begin{aligned} \sum_{jk}(v_{ij} - p_j) \cdot A^*_{ijk} &= \sum_{z=1,\dots,J} \gamma_{zi}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ k \in \mathcal{B}}} A^*_{ijk} = \sum_{z=1,\dots,J} \gamma_{zi}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ k \in \mathcal{B}}} A_{ijk} \\ &> \sum_{jk}(v_{ij} - p_j) \cdot A_{ijk}, \end{aligned}$$

which contradicts (19). Namely, (E.D0) holds.

Now we check that (E.S0) holds. That is, for each seller $k \in \mathcal{S}$, $(\sum_i A_{ijk})_j \in S_k^0(p)$. Since A is feasible, (S.a0) and (S.b0) holds.

To check that (S.c0) holds, assume $\nabla_k^{z>}(p) \neq \emptyset$ for some $z = 1, ..., J$. We want to show that $\sum_{j \in \nabla_i^{z>}(p)} \beta_j = s_{zk}(p)$. Assume there exist k' and z^* such that $\nabla_{k'}^{z^*>}(p) \neq \emptyset$ but for $z = 1, ..., J$,

$$\sum_{j \in \nabla_{i'}^{z^*>}(p)} \sum_i A_{ijk'} < s_{z^*k'}(p). \tag{42}$$

Without loss of generality we may assume that $\sum_{j \in \nabla_{k'}^{z>}(p)} \sum_i A_{ijk'} = s_{zk'}(p)$ for all $z < z^*$. We have $s_{z^*k'}(p) \leq \min\{\sum_{j \in \nabla_{k'}^{tz^*>}(p)} q_{jk'}, D - \sum_{m=1}^{z-1} s_{mk'}(p)\}$. Then, by (42),

$$\sum_{j \in \nabla_{i'}^{z^*>}(p)} \sum_i A_{ijk'} < D - \sum_{m=1}^{z-1} s_{mk'}(p) = D - \sum_{m=1}^{z^*-1} \sum_{j \in \nabla_{k'}^{m>}(p)} \sum_i A_{ijk'}.$$

Hence, $\sum_{n=1}^{z^*} \sum_{j \in \nabla_{k'}^{n>}(p)} \sum_i A_{ijk'} < D$. Thus, $\sum_{i \in \mathcal{B}} \sum_{n=1}^{z^*} \sum_{j \in \nabla_{k'}^{n>}(p)} A_{ijk'} < \sum_{i \in \mathcal{B}} d_i$. Then, there exists $i^* \in \mathcal{B}$ such that $\sum_{n=1}^{z^*} \sum_{j \in \nabla_{k'}^{n>}(p)} A_{i^*jk'} < d_{i^*}$. Moreover, by (42), we know $\sum_{j \in \nabla_{k'}^{z^*>}(p)} \sum_i A_{ijk'} < \sum_{j \in \nabla_{k'}^{z^*>}(p)} q_{jk'}$. Then, there exists $j^* \in \nabla_{k'}^{z^*>}$ such that $\sum_i A_{ij^*k'} < q_{j^*k'}$. Define A^* as follows:

$$A^*_{ijk} = \begin{cases} A_{ijk} & \text{if } k = k' \text{ and } j \in \nabla_k^{z>}(p) \text{ for some } z < z^* \text{ or } z^* < z \text{ for all } i \\ A_{ijk} + 1 & \text{if } i = i^*, j = j^* \text{ and } k = k' \\ A_{ijk} & \text{if } i = i', j \in \nabla_i^{z^*>}(p),\ j \neq j^* \text{ and } k \neq k^* \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate to check that A^* is feasible. Moreover,

$$\begin{aligned} \sum_{ij}(p_j - r_{jk}) \cdot A^*_{ijk} &= \sum_{z=1,...,J} \pi_{zk}(p) \sum_{\substack{j \in \nabla_k^{z>}(p) \\ i \in \mathcal{B}}} A^*_{ijk} \\ &> \sum_{z=1,...,J} \pi_{zk}(p) \sum_{\substack{j \in \nabla_i^{z>}(p) \\ i \in \mathcal{B}}} A_{ijk} \geq \sum_{ij}(p_j - r_{jk}) \cdot A_{ijk}, \end{aligned}$$

which contradicts (20).

The proof that (S.d0) holds as well is similar, and therefore omitted. ■

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