Comparing Generalized Median Voter Schemes According to their Manipulability*

R. Pablo Arribillaga[†] Jordi Massó[‡]

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Abstract: We propose a simple criterion to compare generalized median voter schemes according to their manipulability. We identify three necessary and sufficient conditions for the comparability of two generalized median voter schemes in terms of their vulnerability to manipulation. The three conditions are stated using the two associated families of monotonic fixed ballots and depend very much on the power each agent has to unilaterally change the outcomes of the two generalized median voter schemes. We perform a specific analysis of all median voter schemes, the anonymous subfamily of generalized median voter schemes.

Keywords: Generalized Median Voting Schemes; Strategy-proofness; Anonymity.
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[†]Instituto de Matemática Aplicada San Luis (UNSL-CONICET). Ejército de los Andes 950, 5700 San Luis, Argentina. E-mail: rarribi@unsl.edu.ar

[‡]Universitat Autònoma de Barcelona and Barcelona GSE. Departament d'Economía i Història Econòmica. Edifici B, UAB. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

1 Introduction

Consider a set of agents that has to collectively choose an alternative. Each agent has a preference relation on the set of alternatives. We would like the chosen alternative to depend on the preference profile (a list of preference relations, one for each agent). But preference relations are private information and, to be used to choose the alternative, they have to be revealed by the agents. A social choice function collects individual preference relations and selects an alternative for each declared preference profile. Hence, a social choice function induces a game form that generates, at every preference profile, a strategic problem to each agent. An agent manipulates a social choice function if there exist a preference profile and a different preference relation for the agent such that, if submitted, the social choice function selects a strictly better alternative according to the preference relation of the agent of the original preference profile. A social choice function is strategy-proof if no agent can manipulate it. That is, the game form induced by a strategy-proof social choice function has the property that, at every preference profile, to declare the true preference relation is a weakly dominant strategy for all agents. Hence, each agent has an optimal strategy (to truth-tell) independently of the agent's beliefs about the other agents' declared preference relations. This absence of any informational hypothesis about the others' preference relations is one of the main reasons of why strategy-proofness is an extremely desirable property of social choice functions.

However, the Gibbard-Satterthwaite Theorem establishes that nontrivial strategy-proof social choice functions do not exist on universal domains. Strategy-proofness is a strong requirement since a social choice function is not longer strategy-proof as soon as there exist a preference profile and an agent that can manipulate the social choice function by submitting another preference relation that if submitted, the social choice function selects another alternative that is strictly preferred by the agent. Nevertheless, there are many social choice problems where the structure of the set of alternatives restricts the set of conceivable preference relations, and hence the set of strategies available to agents. For instance, when the set of alternatives has a natural order, in which all agents agree upon. The localization of a public facility, the temperature of a room, the platform of political parties in the left-right spectrum, or the income tax rate are all examples of such structure that imposes natural restrictions on agents' preference relations. Black (1948) was the first to argue that in those cases agents' preference relations have to be single-peaked (relative to the unanimous order on the set of alternatives). A preference relation is single-peaked if there exists a top alternative that is strictly preferred to all other alternatives and at each of the two sides of the top alternative the preference relation is monotonic, increasing in the left and decreasing in the right.

A social choice function operating only on a restricted domain of preference profiles may become strategy-proof. The elimination of preference profiles restricts the normal form game induced by the social choice function, and strategies (*i.e.*, preference relations) that were not dominant may become dominant. Consider any social choice problem where the set of alternatives can be identified with the interval $[a, b]$ of real numbers and where single-peaked preference relations are defined on $[a, b]$. For this set up Moulin (1980) characterizes all strategy-proof and tops-only social choice functions on the domain of single-peaked preference relations as the class of all generalized median voter schemes.[1] In addition, Moulin (1980) also characterizes the subclass of median voter schemes as the set of all strategy-proof, tops-only and anonymous social choice functions on the domain of single-peaked preference relations; and this is indeed a large class of social choice functions. A median voter scheme can be identified with a vector $x = (x_1, ..., x_{n+1})$ of $n+1$ numbers in $[a, b]$, where n is the cardinality of the set of agents N and $x_1 \leq ... \leq x_{n+1}$. Then, for each preference profile, the median voter scheme identified with x selects the alternative that is the median among the n top alternatives of the agents and the $n + 1$ fixed numbers $x_1, ..., x_{n+1}$. Since $2n + 1$ is an odd number, this median always exists and belongs to $[a, b]$. Observe that median voter schemes are tops-only and anonymous by definition. They are strategy-proof on the domain of single-peaked preference relations because, given a preference profile, each agent can only change the chosen alternative by moving his declared top away from his true top; thus, no agent can manipulate a median voter scheme at any preference profile. A median voter scheme distributes the power to influence the outcome among agents according to its associated vector x in an anonymous way. Generalized median voter schemes constitute non-anonymous extensions of median voter schemes. A generalized median voter scheme can be identified with a set of fixed ballots $\{p_S\}_{S \subseteq N}$ on $[a, b]$, one for each subset of agents S. Then, for each preference profile, the generalized median voter scheme identified with $\{p_S\}_{S \subseteq N}$ selects the alternative α that is the smallest one with the following two properties: (i) there is a subset of agents S whose top alternatives are smaller or equal to α and (ii) the fixed ballot p_S associated to S is also smaller or equal to α.

Generalized median voter schemes are strategy-proof on the domain of single-peaked preference profiles, but manipulable on the universal domain. There are several papers that have identified, in our or similar settings, maximal domains under which social choice functions are strategy-proof but, as soon as the domain is enlarged with a preference outside the domain, the social choice function becomes manipulable. Barberà, Massó and Neme (1998), Barberà, Sonnenschein and Zhou (1992),

[1] A social choice function is tops-only if it only depens on the profile of top alternatives.

Berga and Serizawa (2000), Bochet and Storcken (2009), Ching and Serizawa (1998), Hatsumi, Berga, and Serizawa (2014), Kalai and Müller (1977), and Serizawa (1995) are some examples of these papers. Our contribution on this paper builds upon this literature and has the objective of giving a criterion to compare generalized median voter schemes according to their manipulability. We want to emphasize the fact that the manipulability of a social choice function does not indicate the degree of its lack of strategy-proofness. There may be only one instance at which the social choice function is manipulable or there may be many such instances. The mechanism design literature that has focused on strategy-proofness has not distinguished between these two situations; it has declared both social choice functions as being not strategy-proof, dot![2]

Our criterion to compare two social choice functions takes the point of view of individual agents. We say that an agent is able to manipulate a social choice function at a preference relation (the true one) if there exist a list of preference relations, one for each one of the other agents, and another preference relation for the agent (the strategic one) such that if submitted, the agent obtains a strictly better alternative according to the true preference relation. Consider two generalized median voter schemes, f and g, that can operate on the universal domain of preference profiles. Assume that for each agent the set of preference relations under which the agent is able to manipulate f is contained in the set of preference relations under which the agent is able to manipulate g. Then, from the point of view of all agents, g is more manipulable than f. Hence, we think that f is unambiguously a better generalized median voter scheme than g according to the strategic incentives induced to the agents. Often, it may be reasonable to think that agents' preferences are single-peaked, but if the designer foresees that agents may have also non single-peaked preferences, then f may be a better choice than g if strategic incentives are relevant and important for the designer.

Before presenting our general result in Theorem 2, we focus on median voter schemes, the subclass of anonymous generalized median voter schemes. In Theorem 1 we provide two necessary and sufficient conditions for the comparability of two median voter schemes in terms of their manipulability. Let f and g be two (non-constant) median voter schemes and let $x = (x_1, ..., x_{n+1})$ and $y = (y_1, ..., y_{n+1})$ be their associated vectors of fixed ballots, x to f and y to g, where $x_1 \leq ... \leq x_{n+1}$ and $y_1 \leq ... \leq y_{n+1}$. Then, g is at least as manipulable as f if and only if $[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$ and $[x_2, x_n] \subset [y_2, y_n]$. Using this characterization we are able to establish

[2] Kelly (1977) is an exemption although, to compare social chocie functions according to their manipulability, it uses a counting criteria. Pathak and Sönmez (2013) is a recent exemption and we will refer to it later on.

simple comparability tests for the subclass of unanimous and efficient median voter schemes. Using the partial order "to be equally manipulable as" obtained in Theorem 1 we show that the set of equivalence classes of median voter schemes has a complete lattice structure with the partial order "to be as manipulable as"; the supremum is the equivalence class containing all median voter schemes with $x_1 = x_2 = a$ and $x_n = x_{n+1} = b$,[3] and the infimum is the equivalence class with all constant median voter schemes; *i.e.*, for all $k = 1, ..., n+1$, $x_k = \alpha$ for some $\alpha \in [a, b]$.

In Theorem 2 we provide three necessary and sufficient conditions for the comparability of two generalized median voter schemes in terms of their manipulability. The three conditions are stated using the two associated families of monotonic fixed ballots and depend very much on the power each agent has to unilaterally change the outcome of the two generalized median voter schemes (*i.e.*, the intervals of alternatives where agents are non-dummies). Obviously, Theorem 2 is more general than Theorem 1. However, our analysis can be sharper and deeper on the subclass of anonymous generalized median voter schemes. In addition, Theorem 1 can be seen as a first step to better understand the general characterization of Theorem 2.

Before finishing this Introduction we want to relate our comparability notion with another one recently used in centralized matching markets. Pathak and Sönmez (2013) apply a different notion to compare the manipulability of some specific matching mechanisms in school choice problems. Their notion is based on the inclusion of preference profiles at which there exists a manipulation, while our notion is based on the inclusion of preference relations at which an agent is able to manipulate. In applications, preference profiles are not common knowledge while, in contrast, each agent knows his preference relation (and he may only know that). To use a more manipulable generalized median voter scheme means that each agent has to worry about his potential capacity to manipulate in a larger set. Again, the use of the inclusion of preference relations as a basic criterion to compare generalized median voter schemes according to their manipulability do not require any informational hypothesis. Thus, we find it more appealing. Moreover, we show that if two generalized median voter schemes are comparable according to Pathak and Sönmez's notion, then they are also comparable according to our notion. Furthermore, Example 1 shows that our notion is indeed much weaker than Pathak and Sönmez's notion.

The paper is organized as follows. Section 2 contains preliminary notation and definitions. Section 3 describes the family of anonymous generalized median voter schemes and compares them according to their manipulability. Section 4 extends the analysis to all generalized median voter schemes. Section 5 contains a final remark comparing Pathak and Sönmez's criterion with ours. Sections 6 and 7 contain two

[3] When n is odd, this class contains the *true* median voter scheme.

appendices that collect all omitted proofs.

2 Preliminaries

Agents are the elements of a finite set $N = \{1, ..., n\}$. The set of *alternatives* is the interval of real numbers $[a, b] \subseteq \mathbb{R}$. We assume that $n \geq 2$ and $a < b$. Generic agents will be denoted by i and j and generic alternatives by α and β. Subsets of agents will be represented by S and T.

The (weak) *preference* of each agent $i \in N$ on the set of alternatives $[a, b]$ is a complete, reflexive, and transitive binary relation (a complete preorder) R_i on $[a, b]$. As usual, let P_i and I_i denote the strict and indifference preference relations induced by R_i, respectively; namely, for all $\alpha, \beta \in [a, b]$, $\alpha P_i \beta$ if and only if $\neg \beta R_i \alpha$, and $\alpha I_i \beta$ if and only if $\alpha R_i \beta$ and $\beta R_i \alpha$. The *top of* R_i is the unique alternative $\tau(R_i) \in [a, b]$ that is strictly preferred to any other alternative; *i.e.*, $\tau(R_i) P_i \alpha$ for all $\alpha \in [a, b] \backslash \{\tau(R_i)\}$. Let $\mathcal{U}$ be the set of preferences with a unique top on $[a, b]$. A *preference profile* $R = (R_1, ..., R_n) \in \mathcal{U}^n$ is a n-tuple of preferences. To emphasize the role of agent i or subset of agents S, a preference profile R will be represented by (R_i, R_{-i}) or (R_S, R_{-S}), respectively.

A subset $\widehat{\mathcal{U}}^n \subseteq \mathcal{U}^n$ of preference profiles (or the set $\widehat{\mathcal{U}}$ itself) will be called a *domain.* A *social choice function* is a function $f : \widehat{\mathcal{U}}^n \to [a, b]$ selecting an alternative for each preference profile in the domain $\widehat{\mathcal{U}}^n$. The range of a social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is denoted by r_f. That is,

$$r_f = \{\alpha \in [a, b] \mid \text{there exists } R = (R_1, ..., R_n) \in \widehat{\mathcal{U}}^n \text{ s.t. } f(R_1, ..., R_n) = \alpha\}.$$

Social choice functions require each agent to report a preference on a domain $\widehat{\mathcal{U}}$. A social choice function is strategy-proof on $\widehat{\mathcal{U}}$ if it is always in the best interest of agents to reveal their preferences truthfully. Formally, a social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *strategy-proof* if for all $R \in \widehat{\mathcal{U}}^n$, all $i \in N$, and all $R'_i \in \widehat{\mathcal{U}}$,

$$f(R_i, R_{-i}) R_i f(R'_i, R_{-i}). \tag{1}$$

In the sequel we will say that a social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *not manipulable by* $i \in N$ *at* $R_i \in \mathcal{U}$ if (1) holds for all $(R'_i, R_{-i}) \in \widehat{\mathcal{U}}^n$. To compare social choice functions according to their manipulability, our reference set of preferences will be the full set $\mathcal{U}$.

The set of *manipulable preferences of agent* $i \in N$ *for* $f : \mathcal{U}^n \to [a, b]$ is given by

$$\mathcal{M}_i^f = \{R_i \in \mathcal{U} \mid f(R'_i, R_{-i}) P_i f(R_i, R_{-i}) \text{ for some } (R'_i, R_{-i}) \in \mathcal{U}^n\}.$$

Obviously, a social choice function $f : \mathcal{U}^n \to [a, b]$ is strategy-proof if and only if $\mathcal{M}_i^f = \emptyset$ for all $i \in N$. We say that $f : \mathcal{U}^n \to [a, b]$ is *more manipulable than* $g : \mathcal{U}^n \to [a, b]$ *for* $i \in N$ if $\mathcal{M}_i^g \subsetneq \mathcal{M}_i^f$.

Now, we introduce our criterion to compare social choice functions according to their manipulability.

Definition 1 *A social function* $f : \mathcal{U}^n \to [a, b]$ *is* at least as manipulable as *social function* $g : \mathcal{U}^n \to [a, b]$ *if* $\mathcal{M}_i^g \subseteq \mathcal{M}_i^f$ *for all* $i \in N$.

Definition 2 *A social function* $f : \mathcal{U}^n \to [a, b]$ *is* equally manipulable as *social function* $g : \mathcal{U}^n \to [a, b]$ *if* f *is at least as manipulable as social function* g *and vice versa; i.e.,* $\mathcal{M}_i^g = \mathcal{M}_i^f$ *for all* $i \in N$.

Definition 3 *A social function* $f : \mathcal{U}^n \to [a, b]$ *is* more manipulable than *a social function* $g : \mathcal{U}^n \to [a, b]$ *if* f *is at least as but not equally manipulable as social function* g*; i.e.,* $\mathcal{M}_i^g \subseteq \mathcal{M}_i^f$ *for all* $i \in N$ *and there exists* $j \in N$ *such that* $\mathcal{M}_j^g \subsetneq \mathcal{M}_j^f$.

Given two social choice functions $f : \mathcal{U}^n \to [a, b]$ and $g : \mathcal{U}^n \to [a, b]$ we write (i) $f \succsim g$ to denote that f is at least as manipulable as g, (ii) $f \approx g$ to denote that f is equally manipulable as g, and (iii) $f \succ g$ to denote that f is more manipulable than g. Obviously, there are many pairs of social choice functions that can not be compared according to their manipulability.

Strategy-proofness is not the unique property we will look at. A social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *anonymous* if it is invariant with respect to the agents' names; namely, for all one-to-one mappings $\sigma : N \to N$ and all $R \in \widehat{\mathcal{U}}^n$, $f(R_1, ..., R_n) = f(R_{\sigma(1)}, ..., R_{\sigma(n)})$. A social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *dictatorial* if there exists $i \in N$ such that for all $R \in \widehat{\mathcal{U}}^n$, $f(R) R_i \alpha$ for all $\alpha \in r_f$. A social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *efficient* if for all $R \in \widehat{\mathcal{U}}^n$, there is no $\alpha \in [a, b]$ such that, for all $i \in N$, $\alpha R_i f(R)$ and $\alpha P_j f(R)$ for some $j \in N$. A social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *unanimous* if for all $R \in \widehat{\mathcal{U}}^n$ such that $\tau(R_i) = \alpha$ for all $i \in N$, $f(R) = \alpha$. A social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *onto* if for all $\alpha \in [a, b]$ there is $R \in \widehat{\mathcal{U}}^n$ such that $f(R) = \alpha$ (*i.e.*, $r_f = [a, b]$). A social choice function $f : \widehat{\mathcal{U}}^n \to [a, b]$ is *tops-only* if for all $R, R' \in \widehat{\mathcal{U}}^n$ such that $\tau(R_i) = \tau(R'_i)$ for all $i \in N$, $f(R) = f(R')$.

In our setting the Gibbard-Satterthwaite Theorem states that a social choice function $f : \mathcal{U}^n \to [a, b]$, with $\#r_f \neq 2$, is strategy-proof if and only if it is dictatorial (see Barberà and Peleg (1990)). An implicit assumption is that the social choice function operates on all preference profiles on $\mathcal{U}^n$, because all of them are reasonable. However, for many applications, a linear order structure on the set of alternatives naturally induces a domain restriction in which for each preference R_i in the domain not only there exists a unique top but also that at each of the sides of the top of R_i the preference is monotonic. A well-known domain restriction is the set of single-peaked preferences on an interval of real numbers.

Definition 4 *A preference* $R_i \in \mathcal{U}$ *is* single-peaked *on* $A \subseteq [a,b]$ *if for all* $\alpha, \beta \in A$ *such that* $\beta \leq \alpha < \tau(R_i)$ *or* $\tau(R_i) < \alpha \leq \beta$*,* $\tau(R_i) P_i \alpha R_i \beta$*.*

We will denote the domain of all single-peaked preferences on $[a,b]$ by $\mathcal{SP} \subset \mathcal{U}$. Moulin (1980) characterizes the family of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences. This family contains many non-dictatorial social choice functions. All of them are extensions of the median voter. Following Moulin (1980), and before presenting the general result, we first compare in Section 3, the anonymous subclass according to their manipulability on the full domain of preferences $\mathcal{U}$. In Section 4 we will give a general result to compare according to their manipulability *all* strategy-proof and tops-only social choice functions on $\mathcal{SP}^n$ when they operate on the domain $\mathcal{U}^n$.

3 Anonymity: Comparing Median Voter Schemes

3.1 Median Voter Schemes

Assume first that n is odd and let $f : \mathcal{U}^n \to [a,b]$ be the social choice function that selects, for each preference profile $R = (R_1, ..., R_n) \in \mathcal{U}^n$, the median among the top alternatives of the n agents; namely, $f(R) = med\{\tau(R_1), ..., \tau(R_n)\}$.[4] This social choice function is anonymous, efficient, tops-only, and strategy-proof on $\mathcal{SP}$. Add now, to the n agents' top alternatives, $n+1$ fixed ballots: $\frac{n+1}{2}$ ballots at alternative a and $\frac{n+1}{2}$ ballots at alternative b. Then, the median among the n top alternatives, and the median among the n top alternatives and the $n+1$ fixed ballots coincide since the $\frac{n+1}{2}$ ballots at a and the $\frac{n+1}{2}$ ballots at b cancel each other; namely, for all $R = (R_1, ..., R_n) \in \mathcal{U}^n$,

$$f(R) = med\{\tau(R_1), ..., \tau(R_n), \underbrace{a, ..., a}_{\frac{n+1}{2}-\text{times}}, \underbrace{b, ..., b}_{\frac{n+1}{2}\text{-times}}\} = med\{\tau(R_1), ..., \tau(R_n)\}.$$

To proceed, and instead of adding $n+1$ fixed ballots at the extremes of the interval, we can add, regardless of whether n is odd or even, $n+1$ fixed ballots at any of the alternatives in $[a,b]$. Then, a social choice function $f : \mathcal{U}^n \to [a,b]$ is a *median*

[4] Given a set of real numbers $\{x_1, ..., x_K\}$, define its *median* as $med\{x_1, ..., x_K\} = y$, where y is such that $\#\{1 \leq k \leq K \mid x_k \leq y\} \geq \frac{K}{2}$ and $\#\{1 \leq k \leq K \mid x_k \geq y\} \geq \frac{K}{2}$. If K is odd the median is unique and belongs to the set $\{x_1, ..., x_K\}$.

voter scheme if there exist $n+1$ fixed ballots $(x_1, ..., x_{n+1}) \in [a,b]^{n+1}$ such that for all $R \in \mathcal{U}^n$,

$$f(R) = med\{\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}\}. \tag{2}$$

Hence, each median voter scheme can be identified with its vector $x = (x_1, ..., x_{n+1}) \in [a,b]^{n+1}$ of fixed ballots. Moulin (1980) shows that the class of all tops-only, anonymous and strategy-proof social choice functions on the domain of single-peaked preferences coincides with all median voter schemes.

Proposition 1 *(Moulin, 1980) A social choice function $f : \mathcal{SP}^n \to [a,b]$ is strategy-proof, tops-only and anonymous if and only if f is a median voter scheme; namely, there exist $n+1$ fixed ballots $(x_1, ..., x_{n+1}) \in [a,b]^{n+1}$ such that for all $R \in \mathcal{SP}^n$,*

$$f(R) = med\{\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}\}.$$

Median voter schemes are tops-only and anonymous by definition. To see that they are strategy-proof, let $f : \mathcal{SP}^n \to [a,b]$ be any median voter scheme and fix $R \in \mathcal{SP}^n$ and $i \in N$. If $f(R) = \tau(R_i)$, i can not manipulate f. Assume $\tau(R_i) < f(R)$ (the other case is symmetric). Agent i can only modify the chosen alternative by declaring a preference $R_i' \in \mathcal{SP}$ with the property that $f(R) < \tau(R_i')$. But then, either $f(R) = f(R_i', R_{-i})$ or $f(R) < f(R_i', R_{-i})$. Hence, $\tau(R_i) < f(R) \leq f(R_i', R_{-i})$. Since R_i is single-peaked, $f(R)R_i f(R_i', R_{-i})$. Thus, i can not manipulate f. It is less immediate to see that the set of all median voter schemes (one for each vector of $n+1$ fixed ballots) coincides with the class of all tops-only, anonymous and strategy-proof social choice functions on the domain of single-peaked preferences. The key point in the proof is to identify, given a tops-only, anonymous and strategy-proof social choice function $f : \mathcal{SP}^n \to [a,b]$, the vector $x = (x_1, ..., x_{n+1}) \in [a,b]^{n+1}$ of fixed ballots. To identify each x_k with $1 \leq k \leq n+1$, consider any preference profile $R \in \mathcal{SP}^n$ with the property that $\#\{i \in N \mid \tau(R_i) = a\} = n-k+1$ and $\#\{i \in N \mid \tau(R_i) = b\} = k-1$ and define $x_k = f(R)$. The proof concludes by checking that indeed f satisfies (2) with this vector $x = (x_1, ..., x_{n+1}) \in [a,b]^{n+1}$ of identified fixed ballots.

To see that in the statement of Proposition 1 tops-onlyness does not follow from strategy-proofness and anonymity consider the social choice function $f : \mathcal{SP}^n \to [a,b]$ where for all $R \in \mathcal{SP}^n$,

$$f(R) = \begin{cases} a & \text{if } \#\{i \in N \mid aR_ib\} \geq \#\{i \in N \mid bP_ia\} \\ b & \text{otherwise.} \end{cases}$$

Notice that f is strategy-proof and anonymous but it is not tops-only. It also violates efficiency, unanimity, and ontoness.

We finish this subsection with a useful remark stating that median voter schemes are monotonic.

Remark 1 *Let $f : \mathcal{U}^n \to [a, b]$ be a median voter scheme and let $R, R' \in \mathcal{U}^n$ be such that $\tau(R_i) \leq \tau(R'_i)$ for all $i \in N$. Then, $f(R) \leq f(R')$.*

3.2 Main result with anonymity

Median voter schemes are strategy-proof on the domain $\mathcal{SP}^n$ of single-peaked preferences. However, when they operate on the larger domain $\mathcal{U}^n$ they may become manipulable. Then, all median voter schemes are equivalent from the classical manipulability point of view. In this subsection we give a simple test to compare two median voter schemes according to their manipulability. Given a vector $x = (x_1, ..., x_{n+1}) \in [a, b]^{n+1}$ we will denote by f^x its associated median voter scheme on $\mathcal{U}^n$; namely, for all $R \in \mathcal{U}^n$,

$$f^x(R) = med\{\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}\}.$$

Given $x = (x_1, ..., x_{n+1}) \in [a, b]^{n+1}$, we will assume that $x_1 \leq ... \leq x_{n+1}$. This can be done without loss of generality because the social choice function associated to any reordering of the components of x coincides with f^x. Obviously, the rang of f^x is $[x_1, x_{n+1}]$, *i.e.*, $r_{f^x} = [x_1, x_{n+1}]$. Any constant social choice function, $f(R) = \alpha$ for all $R \in \mathcal{U}^n$, can be described as a median voter scheme by setting, for all $1 \leq k \leq n+1$, $x_k = \alpha$. We denote it by f^α. Trivially, any constant social choice function f^α is strategy proof on $\mathcal{U}^n$. Then, for any $\alpha \in [a, b]$ and any social choice function $g : \mathcal{U}^n \to [a, b]$ we have that g is at least as manipulable as f^α (*i.e.*, $g \succsim f^\alpha$). Furthermore, all non-constant median voter schemes are manipulable on $\mathcal{U}^n$. Hence, any non-constant median voter scheme f^x is more manipulable than f^α (*i.e.*, $f^x \succ f^\alpha$). Theorem 1 below gives an easy and operative way of comparing non-constant median voter schemes according to their manipulability.

Theorem 1 *Let $x = (x_1, ..., x_{n+1}) \in [a, b]^{n+1}$ and $y = (y_1, ..., y_{n+1}) \in [a, b]^{n+1}$ be two vectors of fixed ballots such that f^x and f^y are not constant; i.e., $x_1 < x_{n+1}$ and $y_1 < y_{n+1}$. Then, f^y is at least as manipulable as f^x if and only if $[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$ and $[x_2, x_n] \subset [y_2, y_n]$.*

3.3 Proof of Theorem 1

In the proof of Theorem 1 the following option set will play a fundamental role.

Definition 5 *Let $f : \mathcal{U}^n \to [a, b]$ be a social choice function and let $R_i \in \mathcal{U}$. The* set of options left open by *$R_i \in \mathcal{U}$ is defined as follows:*

$$o^f(R_i) = \{\alpha \in [a, b] \mid f(R_i, R_{-i}) = \alpha \text{ for some } R_{-i} \in \mathcal{U}^{n-1}\}.$$

If f^x is a median voter scheme, we denote $o^{f^x}(R_i)$ by $o^x(R_i)$.

Before proving Theorem 1 we state three useful lemmata, whose proofs are in Appendix 1.

Lemma 1 *Let* $f^x : \mathcal{U}^n \to [a,b]$ *be a median voter scheme associated with* $x = (x_1, ..., x_{n+1}) \in [a,b]^{n+1}$. *Then,* f^x *is not manipulable by* $i \in N$ *at* $R_i \in \mathcal{U}$ *if and only if* R_i *is single-peaked on* $o^x(R_i) \cup \{\tau(R_i), \alpha\}$ *for all* $\alpha \in r_{f^x}$.

Lemma 2 *Let* $f^x : \mathcal{U}^n \to [a,b]$ *be a median voter scheme associated with* $x = (x_1, ..., x_{n+1}) \in [a,b]^{n+1}$. *Then,*

$$o^x(R_i) = \begin{cases} [x_1, x_n] & \text{if } a \le \tau(R_i) < x_1 \\ [\tau(R_i), x_n] & \text{if } x_1 \le \tau(R_i) < x_2 \\ [x_2, x_n] & \text{if } x_2 \le \tau(R_i) \le x_n \\ [x_2, \tau(R_i)] & \text{if } x_n < \tau(R_i) \le x_{n+1} \\ [x_2, x_{n+1}] & \text{if } x_{n+1} < \tau(R_i) \le b. \end{cases}$$

Lemma 3 *Let* $f^x : \mathcal{U}^n \to [a,b]$ *and* $f^y : \mathcal{U}^n \to [a,b]$ *be two median voter schemes associated with* $x = (x_1, ..., x_{n+1}) \in [a,b]^{n+1}$ *and* $y = (y_1, ..., y_{n+1}) \in [a,b]^{n+1}$ *such that* $[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$ *and* $[x_2, x_n] \subset [y_2, y_n]$. *Then,* $o^x(R_i) \subset o^y(R_i)$ *for all* $R_i \in \mathcal{U}$.

Lemma 1 plays a key role in the proof of Theorem 1. To understand it notice that it roughly says that whether or not agent i can manipulate f^x at R_i depends on the fact that R_i should *only* be like single-peaked on the set of alternatives that may be selected by f^x for some subprofile R_{-i}, given R_i. The comparison, in terms of R_i, of pairs of alternatives that will never be selected once R_i is submitted, is irrelevant in terms of agent i's power to manipulate f^x. To illustrate that, consider the case where $n = 3$, $x_1 = a$, $x_2 = \frac{a+b}{3}$, $x_3 = \frac{2(a+b)}{3}$ and $x_4 = b$. Then, $r_{f^x} = [a,b]$. Let $R_i \in \mathcal{U}$ be any preference with $\tau(R_i) \in \left(\frac{a+b}{3}, \frac{2(a+b)}{3}\right)$. By Lemma 2, $o^x(R_i) = \left[\frac{a+b}{3}, \frac{2(a+b)}{3}\right]$. Lemma 1 says that R_i should be single-peaked on this interval and that the preference away from $\tau(R_i)$ towards the direction of $\frac{a+b}{3}$ has to be monotonically decreasing until alternative $\frac{a+b}{3}$ and that all alternatives further away have to be worse than $\frac{a+b}{3}$ but they can be freely ordered among themselves; and symmetrically from $\tau(R_i)$ towards the direction of $\frac{2(a+b)}{3}$. Figure 1 illustrates a preference that is single-peaked on $o^x(R_i) \cup \{\tau(R_i), \alpha\}$ for all $\alpha \in r_{f^x}$. It also shows that this set may be significantly larger than the set of single-peaked preferences.

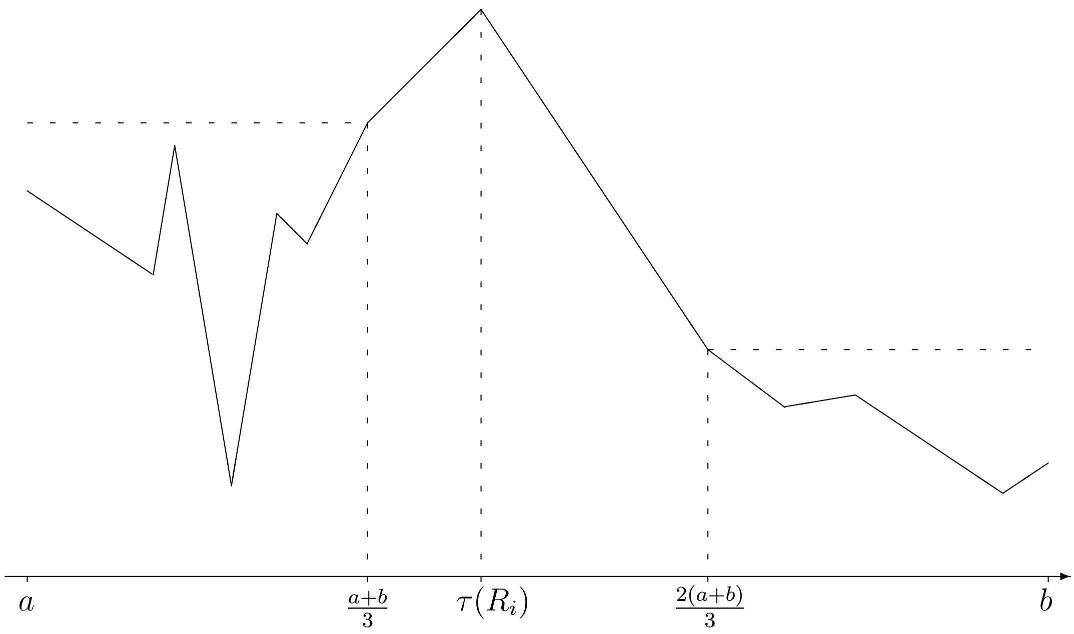

Figure 1

Proof of Theorem 1 First, we will prove that if $[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$ and $[x_2, x_n] \subset [y_2, y_n]$, then f^y is at least as manipulable as f^x. Suppose that $R_i \in \mathcal{M}_i^{f^x}$. By Lemma 1, there exists $\alpha^* \in r_{f^x}$ such that R_i is not single-peaked on $o^x(R_i) \cup \{\tau(R_i), \alpha^*\}$. By Lemma 3, $o^x(R_i) \subset o^y(R_i)$. Since $r_{f^x} = [x_1, x_{n+1}] \subset [y_1, y_{n+1}] = r_{f^y}$, we have that $\alpha^* \in r_{f^y}$. Hence, R_i is not single-peaked on $o^y(R_i) \cup \{\tau(R_i), \alpha^*\}$, where $\alpha^* \in R_{f^y}$. Thus, by Lemma 1, $R_i \in \mathcal{M}_i^{f^y}$. Therefore, f^y is at least as manipulable as f^x.

To prove the other implication assume that f^y is at least as manipulable as f^x. Hence,

$$\mathcal{M}_i^{f^x} \subset \mathcal{M}_i^{f^y} \text{ for all } i \in N. \tag{3}$$

To obtain a contradiction assume that $[x_1, x_{n+1}] \nsubseteq [y_1, y_{n+1}]$ or $[x_2, x_n] \nsubseteq [y_2, y_n]$. We will divide the proof between two cases.

Case 1: $[x_1, x_{n+1}] \nsubseteq [y_1, y_{n+1}]$. In particular, suppose that $x_1 < y_1$; the proof for the case $y_{n+1} < x_{n+1}$ proceeds similarly and therefore it is omitted. We will divide the proof between two cases again, depending on whether $x_1 < x_2$ or $x_1 = x_2$.

Case 1.1: $x_1 < x_2$. Let $\alpha, \beta, \gamma \in [a, b]$ be such that $x_1 < \alpha < \beta < \gamma < \min\{x_2, y_1\}$ and let $R_i \in \mathcal{U}$ be such that:
i) $\tau(R_i) = \alpha$,
ii) $\gamma P_i \beta$ and
iii) if $\rho, \delta \in [a, b]$ and $y_1 < \rho < \delta$, then $\rho R_i \delta$.
Since $x_1 < \tau(R_i) < x_2$ and $x_1 < \tau(R_i) < y_1$, by Lemma 2,

$$o^x(R_i) = [\tau(R_i), x_n] \text{ and } o^y(R_i) = [y_1, y_n].$$

Hence, and since $\tau(R_i), \beta, \gamma \in o^x(R_i)$ and *ii)* holds, R_i is not single-peaked on $o^x(P_i)$ and, for all $\alpha' \in r_{f^y}$, R_i is single-peaked on $o^y(R_i) \cup \{\tau(R_i)\} \cup \{\alpha'\}$ because $r_{f^y} =$

$[y_1, y_{n+1}]$. Thus, by Lemma 1, $R_i \in \mathcal{M}_i^{f^x} \backslash \mathcal{M}_i^{f^y}$ which contradicts (3).

Case 1.2: $x_1 = x_2$. Since f^x is not constant and $x_1 < y_1$, $x_1 < \min\{y_1, x_{n+1}\}$. Let $\alpha, \beta, \gamma \in [a, b]$ be such that $x_1 < \alpha < \beta < \gamma < \min\{y_1, x_{n+1}\}$ and let $R_i \in \mathcal{U}$ be such that:
i) $\tau(R_i) = \gamma$,
ii) $\alpha P_i \beta$ and
iii) if $\rho, \delta \in [a, b]$ and $y_1 < \rho < \delta$, then $\rho P_i \delta$.
Since $x_1 < \tau(R_i) < y_1$ and $x_1 = x_2 < \tau(R_i)$, by Lemma 2,

$$o^x(R_i) = \begin{cases} [x_2, x_n] & \text{if } x_2 \leq \tau(R_i) \leq x_n \\ [x_2, \tau(R_i)] & \text{if } x_n < \tau(R_i) \leq x_{n+1} \end{cases} \quad \text{and } o^y(R_i) = [y_1, y_n].$$

Hence, and since $\alpha, \beta, \tau(R_i) \in o^x(R_i)$ and *ii)* holds, R_i is not single-peaked on $o^x(R_i)$ and, for all $\alpha' \in r_{f^y}$, R_i is single-peaked on $o^y(R_i) \cup \{\tau(R_i)\} \cup \{\alpha'\}$ because $r_{f^y} = [y_1, y_{n+1}]$. Thus, by Lemma 1, $R_i \in \mathcal{M}_i^{f^x} \backslash \mathcal{M}_i^{f^y}$ which contradicts (3).

Case 2: $[x_2, x_n] \nsubseteq [y_2, y_n]$ and $[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$. In particular, suppose that $x_2 < y_2$; the proof for the case $y_n < x_n$ proceeds similarly and therefore it is omitted. Let $\alpha, \beta \in [a, b]$ be such that $x_2 < \alpha < \beta < \frac{x_2+y_2}{2} < y_2$ and let $R_i \in \mathcal{U}$ be such that:
i) $\tau(R_i) = \frac{x_2+y_2}{2}$,
ii) $\alpha P_i \beta$ and
iii) if $\gamma, \delta \in [a, b]$ and $\tau(R_i) < \gamma < \delta$, then $\gamma P_i \delta$.
Since $y_1 \leq x_1 \leq x_2 < \tau(R_i) < y_2$, by Lemma 2,

$$o^x(R_i) = \begin{cases} [x_2, x_n] & \text{if } x_2 \leq \tau(R_i) \leq x_n \\ [x_2, \tau(R_i)] & \text{if } x_n < \tau(R_i) \leq x_{n+1} \\ [x_2, x_{n+1}] & \text{if } \tau(R_i) > x_{n+1} \end{cases} \quad \text{and } o^y(R_i) = [\tau(R_i), y_n].$$

Hence, and since $\alpha, \beta, \tau(R_i) \in o^x(R_i)$ and *ii)* holds, R_i is not single-peaked on $o^x(R_i)$ and, for all $\alpha' \in r_{f^y}$, R_i is single-peaked on $o^y(R_i) \cup \{\tau(R_i), \alpha'\}$. Thus, by Lemma 1, $R_i \in \mathcal{M}_i^{f^x} \backslash \mathcal{M}_i^{f^y}$ which contradicts (3). ■

For further reference, let MVS denote the set of all median voting schemes from $\mathcal{U}^n$ to $[a, b]$. An immediate consequence of Theorem 1 is that if median voter scheme f is at least as manipulable as median voter scheme g, then the range of g is contained in the range of f. The improvement in terms of the strategy-proofness of median voter schemes necessarily requires the corresponding reduction of their ranges since smaller ranges reduce agents' power to manipulate. The corollary below, that follows from Theorem 1 and the fact that for all $f^x \in MVS$, $r_{f^x} = [x_1, x_{n+1}]$, states this observation formally.

Corollary 1 *Let $f, g \in MVS$. If $f \succsim g$, then $r_g \subset r_f$.*

Consider a problem where the range of the social choice has to be fixed *a priori* to be a subinterval $[c,d] \subset [a,b]$. Let $MVS_{[c,d]}$ be the set of all median voter schemes with range $[c,d]$ (i.e., $f^x \in MVS_{[c,d]}$ if and only if $x_1 = c$ and $x_{n+1} = d$). Theorem 1 gives criteria to compare the elements in $MVS_{[c,d]}$.

Corollary 2 *Let $f^y, f^x \in MVS_{[c,d]}$.*
a) Then, $f^y \succsim f^x$ if and only if $[x_2, x_n] \subset [y_2, y_n]$.
b) If $y_2 = y_n$, then there does not exist $g \in MVS_{[c,d]}$ such that $f^y \succ g$.

Statement *b*) identifies the median voter schemes in $MVS_{[c,d]}$ that do not admit a less manipulable median voter scheme in $MVS_{[c,d]}$.

3.4 Unanimity

According to Proposition 1 in Moulin (1980), a median voter scheme $f^x : \mathcal{SP}^n \to [a,b]$ is efficient (on the single-peaked domain) if and only if $x_1 = a$ and $x_{n+1} = b$; namely, f^x can be described as the median of the n top alternatives submitted by the agents and *only* $n-1$ fixed ballots since $x_1 = a$ and $x_{n+1} = b$ cancel each other in (2). But this subclass of median voter schemes is appealing because it coincides with the class of all unanimous median voter schemes ($MVS_{[a,b]}$ using the notation introduced in the previous subsection).[5] Corollary bellow shows that Theorem 1 has clear implications on how unanimous and non-unanimous median voter schemes can be ordered according to their manipulability. In particular, given a unanimous median voter scheme there is always a non-unanimous median voter scheme that is less manipulable. Moreover, if a unanimous median voter scheme and a non-unanimous median voter scheme are comparable according to their manipulability, then the former is more manipulable than the later.

Corollary 3 *Let $f^y \in MVS$ be unanimous.*
a) Then, for all $f^x \in MVS$, $f^y \succsim f^x$ if and only if $[x_2, x_n] \subset [y_2, y_n]$.
b) There exists a non-constant and non-unanimous $f^x \in MVS$ such that $f^y \succ f^x$.
c) Let $f^x \in MVS$ be non-unanimous and assume f^x and f^y are comparable according to their manipulability. Then, $f^y \succ f^x$.

Proof Let $f^y \in MVS$ be unanimous. Hence, $y_1 = a$ and $y_{n+1} = b$.

a) The statement follows immediately from Theorem 1.

[5] Observe that when unanimous median voter schemes operate on the full domain $\mathcal{U}^n$ they are not anymore efficient. In the next subsection we will provide some simple criteria to compare efficient median voter schemes on the full domain $\mathcal{U}^n$ according to their manipulability.

b) We distinguish between two cases.

Case 1: Assume $y_2 < y_n$ and let $\alpha, \beta, \gamma \in [a, b]$ be such that $y_2 < \alpha < \beta < \gamma < y_n$. Consider $x = (\alpha, \beta, ..., \beta, \gamma) \in [a, b]^{n+1}$. Then, $[x_2, x_n] = \{\beta\} \subset [y_2, y_n]$. By Theorem 1, f^y is at least as manipulable as f^x and since $[y_2, y_n] \nsubseteq [x_2, x_n]$, f^x is not at least as manipulable as f^y. Hence, f^y is more manipulable than f^x and f^x is neither constant nor unanimous since $a < x_1 < x_{n+1} < b$.

Case 2: Assume $y_2 = y_n$. Furthermore, suppose that $a < y_2$; the proof when $y_n < b$ proceeds symmetrically and therefore it is omitted. Let $\alpha \in (a, y_2)$ and consider $x = (\alpha, y_2, ..., y_2, b) \in [a, b]^{n+1}$. Then, $[x_2, x_n] = \{y_2\}$. By Theorem 1, f^y is at least as manipulable as f^x and, since $[y_1, y_{n+1}] = [a, b] \nsubseteq [x_1, x_{n+1}]$, f^x is not at least as manipulable as f^y. Hence, f^y is more manipulable than f^x. Furthermore, and since $a < x_1 = ... = x_n < x_{n+1} = b$, f^x is neither constant nor unanimous.

c) Assume $f^x \in MVS$ is not unanimous. Then, $[x_1, x_{n+1}] \subsetneq [y_1, y_{n+1}] = [a, b]$. By Theorem 1, f^x is not at least as manipulable as f^y. Furthermore, as f^x and f^y are comparable, $f^y \succ f^x$ must hold. ■

We conclude this subsection with a corollary that identifies the unanimous median voter schemes that do not admit a less manipulable unanimous median voter scheme. The statement also follows immediately from Theorem 1.

Corollary 4 *Let f^y be a unanimous median voter scheme such that $y_2 = y_n$. Then, there does not exist an unanimous median voting scheme g such that $f^y \succ g$.*

3.5 Efficiency

A median voter scheme $f^x : \mathcal{U}^n \to [a, b]$ (operating on the full domain of preferences) is efficient if and only if $x_1 = a$, $x_{n+1} = b$ and $x_k \in \{a, b\}$ for all $2 \leq k \leq n$.[6] This is because on the larger domain, if a median voter scheme f^x has an interior fixed ballot $x_k \in (a, b)$ it is always possible to find a preference profile R with $f^x(R) = x_k$ such that there exists an alternative y that is unanimously strictly preferred by all agents; namely, $yP_i f^x(R)$ for all $i \in N$. Moreover, all efficient median voter schemes are unanimous.

We now present simple criteria that are useful to compare efficient median voter schemes with other unanimous median voter schemes according to their manipulability. But before, we need a bit of additional notation.

[6]Hence, an efficient median voter scheme $f^x : \mathcal{U}^n \to [a, b]$ has the property that for all $(R_1, ..., R_n) \in \mathcal{U}^n$,

$$f^x(R_1, ..., R_n) \in \{\tau(R_1), ..., \tau(R_n)\}.$$

Miyagawa (1998) and Heo (2013) have studied this property under the name of *peak-selection.*

Let k be an integer such that $1 \leq k \leq n$ and $(\alpha_1, ..., \alpha_n) \in [a,b]^n$. Denote by $\pi^k(\alpha_1, ..., \alpha_n)$ the k-th ranked number; namely, $\#\{\alpha_i \in \{\alpha_1, ..., \alpha_n\} \mid \alpha_i \leq \pi^k(\alpha_1, ..., \alpha_n)\} \leq n - k + 1$ and $\#\{\alpha_i \in \{\alpha_1, ..., \alpha_n\} \mid \alpha_i \geq \pi^k(\alpha_1, ..., \alpha_n)\} \leq k$. In particular, for $k = 1$ and $k = n$,

$$\begin{aligned} \pi^1(\alpha_1, ..., \alpha_n) &= \max\{\alpha_1, ..., \alpha_n\} \\ \pi^n(\alpha_1, ..., \alpha_n) &= \min\{\alpha_1, ..., \alpha_n\}. \end{aligned}$$

Let $f^x : \mathcal{U}^n \to [a,b]$ be an efficient median voter scheme. Then, $x = (\underbrace{a, ..., a}_{k}, \underbrace{b, ..., b}_{n+1-k})$ for some $1 \leq k \leq n$ and, for all $R \in \mathcal{U}^n$,

$$f^x(R_1, ..., R_n) = \pi^k(\tau(R_1), ..., \tau(R_n)).$$

We denote the efficient median voter scheme f^x with k fixed ballots at a by f^k.

Corollary 5 *Let $f^k : \mathcal{U}^n \to [a,b]$ be an efficient median voter scheme such that $k \notin \{1, n\}$. Then, the following hold.*
a) For any $f^x \in MVS$, $f^k \succsim f^x$.
b) If $1 < k' < n$, then $f^k \approx f^{k'}$.
c) $f^k \succ f^1$ and $f^k \succ f^n$.
d) If f^x is non-unanimous, then $f^k \succ f^x$.
e) There exists a non-efficient and unanimous $f^x \in MVS$ such that $f^k \succ f^x$.

Corollary 5 says the following. Statement a) states that any efficient median voter scheme $f \notin \{f^1, f^n\}$ belongs to the set of the most manipulable median voter schemes. Statement c) states that the two efficient median voter schemes f^1 and f^n are less manipulable than any other efficient median voter scheme $f \notin \{f^1, f^n\}$. Statement d) states that any non-unanimous median voter scheme is less manipulable that any efficient median voter scheme $f \notin \{f^1, f^n\}$. Statement e) states that given an efficient median voter scheme $f \notin \{f^1, f^n\}$ there is always a (non-efficient) unanimous median voter scheme that is less manipulable. Moreover, Corollary 5 has the following two implications when n is odd. First, for any $f^x \in MVS$, $f^{\frac{n+1}{2}} \succsim f^x$, and second, for all non-unanimous $f^x \in MVS$, $f^{\frac{n+1}{2}} \succ f^x$.

Proof Let y be the vector of fixed ballots associated to f^k. Since $k \notin \{1, n\}$,

$$y_1 = y_2 = a \text{ and } y_n = y_{n+1} = b. \tag{4}$$

a) It follows from (4) and Theorem 1.

b) It follows from a).

c) Let z be the vector of fixed ballots associated to f^1; namely, $z_1 = a$ and $z_2 = ... = z_{n+1} = b$. Hence, by (4) and Theorem 1, f^k is more manipulable than f^1. Using a similar argument, it also follows that $f^k \succ f^n$.

d) Let f^x be a non-unanimous median voter scheme. Then, either $a < x_1$ or $x_{n+1} < b$. Hence, by (4) and Theorem 1, f^k is more manipulable than f^x

e) Consider any $\alpha \in (a, b)$ and define $x = (a, \underbrace{\alpha, ..., \alpha}_{k-1\text{-times}}, b, ..., b)$. Then, f^x is unanimous but it is not efficient. By (4) and Theorem 1, $f^k \succ f^x$. ■

Corollary 6 *Let $f \in MVS$ be efficient and such that either $f = f^1$ or $f = f^n$.*
a) Then, there exists a non-efficient and non-constant $f^x \in MVS$ such that $f \succ f^x$.
b) If f^x and f are comparable and f^x is non-efficient, then $f \succ f^x$.

Corollary 6 says the following. Statement a) states that there exists a non-efficient and non-constant median voter scheme that is less manipulable than f^1 (or f^n). Statement b) says that if the efficient median voter scheme f^1 (or f^n) and a non-efficient median voter scheme f are comparable according to their manipulability, then the former is more manipulable than the later. Corollaries 5 and 6 make clear the well-known trade-off between strategy-proofness and efficiency.

Proof Consider $f^1 \in MVS$ and let $y = (a, b, ..., b)$ be its associated vector of fixed ballots. The case $f^n \in MVS$ proceeds symmetrically.

a) Define $x = (a, \alpha, b, ..., b)$, where $\alpha \in (a, b)$. Then, by Theorem 1, $f^1 \succ f^x$ and it is clear that f^x is non-efficient.

b) Since $[y_2, y_n] = \{b\}$, and f^x and f^1 are comparable, Theorem 1 implies that $f^1 \succ f^x$. ■

3.6 Complete lattice structure

Using Theorem 1 we can partition the set of median voter schemes MVS into equivalence classes in such a way that each equivalence class contains median voter schemes that are all equally manipulable. Denote the (cocient) set of those equivalence classes by $MVS/\approx$. Furthermore, we can extend $\succsim$ on MVS to the set of equivalence classes $MVS/\approx$ in a natural way. Denote this extension by $[\succsim]$. In this subsection we will show that the pair $(MVS/\approx, [\succsim])$ is a complete lattice; namely, any nonempty subset of equivalence classes in $MVS/\approx$ has a supremum and an infimum according to $[\succsim]$. Formally, given $f^x \in MVS$, denote by $[f^x]$ the equivalence class of f^x with respect to $\approx$; *i.e.*,

$$[f^x] = \{g \in MVS \mid g \approx f^x\}.$$

Let $[c]$ be the class of all constant median voter schemes.[7] Assume that $[f^x] \neq [c]$. By Theorem 1, $[f^x]$ can be identified with the four-tuple (x_1, x_2, x_n, x_{n+1}).

Denote by $MVS/\approx$ the set of all equivalence classes induced by $\approx$ on MVS and consider the binary relation $[\succsim]$ on $MVS/\approx$ defined as follows. For any pair $[f^x], [f^y] \in MVS/\approx$, set

$$[f^x][\succsim][f^y] \text{ if and only if } f^x \succsim f^y.$$

Since $\succsim$ is a preorder on MVS, it follows that $[\succsim]$ is a partial order on $MVS/\approx$. Furthermore, by Theorem 1, if $[f^x] \neq [c]$ and $[f^y] \neq [c]$, then

$$[f^x][\succsim][f^y] \text{ if and only if } x_1 \leq y_1, x_2 \leq y_2, x_n \geq y_n \text{ and } x_{n+1} \geq y_{n+1}.$$

We can now state and prove the result of this subsection.

Proposition 2 *The pair $(MVS/\approx, [\succsim])$ is a complete lattice.*

Proof Let $\emptyset \neq Z \subseteq MVS/\approx$. Define

$$(x_1^{SZ}, x_2^{SZ}, x_n^{SZ}, x_{n+1}^{SZ}) = (\inf_{x_1:[f^x]\in Z} x_1, \inf_{x_2:[f^x]\in Z} x_2, \sup_{x_n:[f^x]\in Z} x_n, \sup_{x_{n+1}:[f^x]\in Z} x_{n+1})$$

and

$$(x_1^{IZ}, x_2^{IZ}, x_n^{IZ}, x_{n+1}^{IZ}) = \begin{cases} (\sup_{x_1:[f^x]\in Z} x_1, \sup_{x_2:[f^x]\in Z} x_2, \inf_{x_n:[f^x]\in Z} x_n, \inf_{x_{n+1}:[f^x]\in Z} x_{n+1}) & \text{if } [c] \notin Z \\ \\ [c] & \text{if } [c] \in Z. \end{cases}$$

Observe that if $[f^x] \in Z$, then $x_k \in [a, b]$ for all $k = 1, 2, n, n+1$. Hence, $(x_1^{SZ}, x_2^{SZ}, x_n^{SZ}, x_{n+1}^{SZ})$ and $(x_1^{IZ}, x_2^{IZ}, x_n^{IZ}, x_{n+1}^{IZ})$ are well defined and $x_k^{SZ}, x_k^{IZ} \in [a, b]$ for all $k = 1, 2, n, n+1$. Consider the equivalence classes $[f^{SZ}]$ and $[f^{IZ}]$ associated to $(x_1^{SZ}, x_2^{SZ}, x_n^{SZ}, x_{n+1}^{SZ})$ and $(x_1^{IZ}, x_2^{IZ}, x_n^{IZ}, x_{n+1}^{IZ})$, respectively. That is, $f^y \in [f^{SZ}]$ if and only if $y_k = x_k^{SZ}$ for $k = 1, 2, n, n+1$ and $f^y \in [f^{IZ}]$ if and only if $y_k = x_k^{IZ}$ for $k = 1, 2, n, n+1$. Since $x_k^{SZ}, x_k^{IZ} \in [a, b]$ for all $k = 1, 2, n, n+1$, we have that

$$[f^{SZ}], [f^{IZ}] \in MVS/\approx . \tag{5}$$

Moreover, if $Z = MVS/\approx$ then $[f^{SZ}] = (a, a, b, b)$ and $[f^{IZ}] = [c]$.

Now we show that $(MVS/\approx, [\succsim])$ is a complete lattice. Let $\emptyset \neq Z \subseteq MVS/\approx$. By (5), $[f^{SZ}], [f^{IZ}] \in MVS/\approx$. By Theorem 1 and the definition of $[f^{SZ}]$ and $[f^{IZ}]$, $lub\ Z = [f^{SZ}]$ and $llb\ Z = [f^{IZ}]$ are, respectively, the least upper bound and the

[7] Remember that all constant median voter schemes (excluded in the statement of Theorem 1) are equally manipulable since all of them are strategy-proof on $\mathcal{U}^n$.

largest lower bound with respect to $[\precsim]$. Hence, $\sup_{[\precsim]} Z = [f^{SZ}]$ and $\inf_{[\precsim]} Z = [f^{IZ}]$. Thus, $(MVS/\approx, [\precsim])$ is a complete lattice. ■

Two immediate consequences follow from the proof of Proposition 2. First, and since $[c]$ is the smallest equivalence class in $MVS/\approx$ according to $[\precsim]$, all constant median voter schemes are less manipulable than any other non-constant median voter scheme (*i.e.*, $[c] = \inf_{[\precsim]} MSV/\approx$). Second, and since the equivalence class containing all median voter schemes identified with the four-tuple (a, a, b, b) is the largest equivalence in $MVS/\approx$ according to $[\precsim]$ (*i.e.*, this equivalence class is the $\sup_{[\precsim]} MSV/\approx$), any median voter scheme f^x such that $x_1 = x_2 = a$ and $x_n = x_{n+1} = b$ is more manipulable than any other MVS outside this class. Observe that this class includes all efficient median voting schemes except f^1 and f^n.

Finally, if $n \leq 3$ and $f^x \in MVS$ is non-constant, then $[f^x] = \{f^x\}$. Thus, the pair $(MVS, \precsim)$ is like a complete lattice (it is not because the equivalence class of constant median voter schemes is not degenerated).

Figure 2 summarizes the complete lattice structure of the pair $(MVS/\approx, [\precsim])$ for any $n \geq 2$, whose properties have been collected along Corollaries 3, 4, 5, and 6.

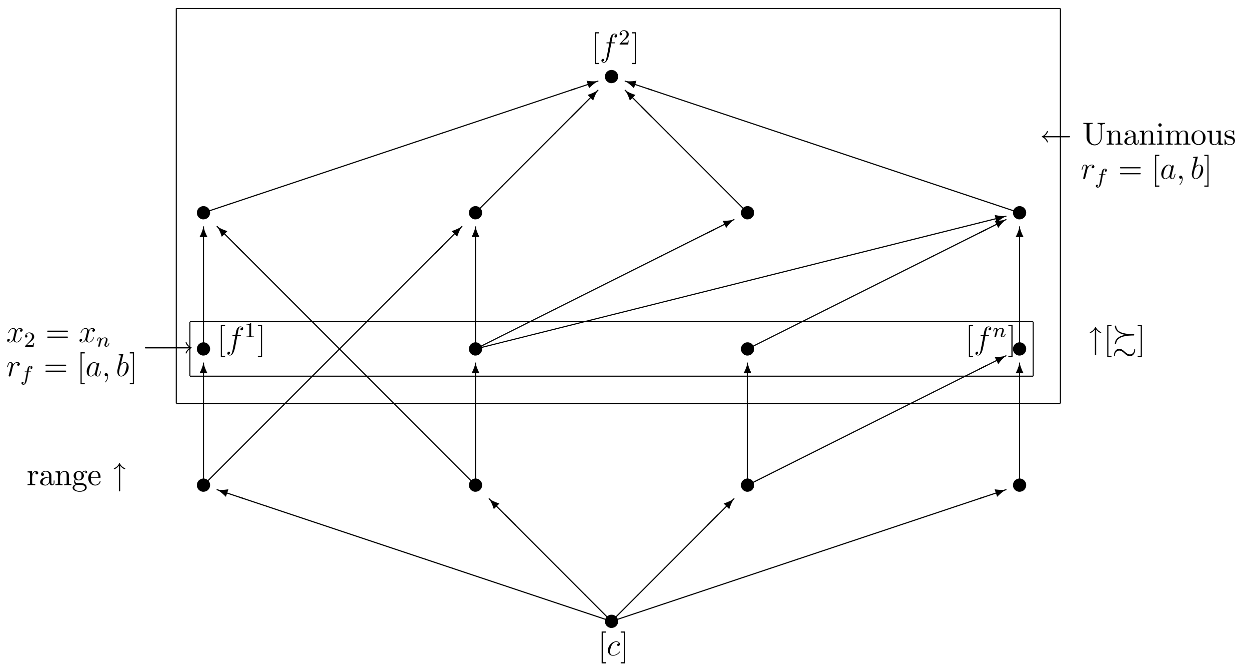

Figure 2

4 Comparing All Generalized Median Voter Schemes

4.1 Generalized Median Voter Schemes

Median voter schemes are anonymous. All agents have the same power to influence the outcome of a given median voter scheme f^x, although this power depends on the distribution of its associated fixed ballots $x = (x_1, ..., x_{n+1})$. *Generalized median voter schemes* admit the possibility that different agents may have different power to influence its outcome. This power will be described by a monotonic family of fixed ballots, one for each coalition (subset) of agents. To develop a useful intuition to understand the class of all generalized median voter schemes, consider first the case $n = 2$. Given a monotonic family of fixed ballots $\{p_{\{1,2\}}, p_{\{1\}}, p_{\{2\}}, p_{\{\emptyset\}}\}$, one for each coalition of agents, such that $a \leq p_{\{1,2\}} \leq p_{\{1\}} \leq p_{\{2\}} \leq p_{\{\emptyset\}} \leq b$, we define the social choice function $f : \mathcal{U}^2 \to [a, b]$ as follows: for each $R \in \mathcal{U}^2$,

$$f(R) = \begin{cases} p_{\{1,2\}} & \text{if } \tau(R_1), \tau(R_2) \leq p_{\{1,2\}} \\ \tau(R_2) & \text{if } \tau(R_1) \leq p_{\{1,2\}} \leq \tau(R_2) \leq p_{\{1\}} \\ p_{\{1\}} & \text{if } \tau(R_1) \leq p_{\{1,2\}} \leq p_{\{1\}} \leq \tau(R_2) \\ med\{\tau(R_1), \tau(R_2), p_{\{1\}}\} & \text{if } p_{\{1,2\}} \leq \tau(R_1) \leq p_{\{1\}} \\ \tau(R_1) & \text{if } p_{\{1\}} \leq \tau(R_1) \leq p_{\{2\}} \\ med\{\tau(R_1), \tau(R_2), p_{\{2\}}\} & \text{if } p_{\{2\}} \leq \tau(R_1) \leq p_{\{\emptyset\}} \\ p_{\{2\}} & \text{if } p_{\{\emptyset\}} \leq \tau(R_1) \text{ and } \tau(R_2) \leq p_{\{2\}} \\ \tau(R_2) & \text{if } p_{\{2\}} \leq \tau(R_2) \leq p_{\{\emptyset\}} \leq \tau(R_1) \\ p_{\{\emptyset\}} & \text{if } p_{\{\emptyset\}} \leq \tau(R_1), \tau(R_2). \end{cases}$$

Observe that $r_f = [p_{\{1,2\}}, p_{\{\emptyset\}}]$. We can interpret this function as a way of assigning to agents 1 and 2 the power to select the alternative in the subset $r_f = [p_{\{1,2\}}, p_{\{\emptyset\}}]$. For instance, agent 1 can make sure that the outcome is at most $p_{\{1\}}$ by voting below $p_{\{1\}}$ and at most $\tau(R_1)$ by voting above $p_{\{1\}}$ and agent 1 is a dictator on $[p_{\{1\}}, p_{\{2\}}]$ (*i.e.*, $f(R) = \tau(R_1)$ whenever $\tau(R_1) \in [p_{\{1\}}, p_{\{2\}}]$). It is easy to check that f can be rewritten as

$$f(R) = \min_{S \subseteq \{1,2\}} \max_{i \in S} \{\tau(R_i), p_S\}.$$

To present the characterization of all strategy-proof and tops-only social choice functions on the domain of single-peaked preferences for all $n \geq 2$, we say that a collection $\{p_S\}_{S \in 2^N}$ is a *monotonic family of fixed ballots* if (i) $p_S \in [a, b]$ for all $S \in 2^N$ and (ii) $T \subset Q$ implies $p_Q \leq p_T$. The characterization is the following.

Proposition 3 *(Moulin, 1980) A social choice function $f : \mathcal{SP}^n \to [a, b]$ is strategy-proof and tops-only if and only if there exists a monotonic family of fixed ballots*

$\{p_S\}_{S\in 2^N}$ *such that for all* $R \in \mathcal{SP}^n$,

$$f(R) = \min_{S\in 2^N} \max_{i\in S}\{\tau(R_i), p_S\}.$$

The social choice functions identified in Proposition 3 are called *generalized median voter schemes*. A simple way of interpreting them is as follows. Each generalized median voting scheme (and its associated monotonic family of fixed ballots) can be understood as a particular way of distributing the power among coalitions to influence the social choice. To see that, take an arbitrary coalition S and its fixed ballot p_S. Then, coalition S can make sure that, by all of its members reporting a top alternative below p_S, the social choice will be at most p_S, independently of the reported top alternatives of the members of the complementary coalition.[8] An alternative way of describing this distribution of power among coalitions is as follows. Fix a monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$ (*i.e.*, a generalized median voter scheme) and take a vector of tops $(\tau(R_1), ..., \tau(R_n))$. Start at the left extreme of the interval a and push the outcome to the right until it reaches an alternative α for which the following two things happen simultaneously: (i) there exists a coalition of agents S such that all its members have reported a top alternative below or equal to α (*i.e.*, $\tau(R_i) \leq \alpha$ for all $i \in S$) and (ii) the fixed ballot p_S associated to S is located also below α (*i.e.*, $p_S \leq \alpha$). Median voter schemes are the anonymous subclass of generalized median voter schemes. Hence, the fixed ballots of any two coalitions with the same cardinality of any anonymous generalized median voter scheme are equal. From a monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$ associated to an anonymous generalized median voter scheme $f : \mathcal{U}^n \to [a, b]$ we can identify the $n+1$ ballots $x_1 \leq ... \leq x_{n+1}$ needed to describe f as a median voter scheme as follows: for each $1 \leq k \leq n+1$, $x_k = p_S$ for all $S \in 2^N$ such that $\#S = n - k + 1$. Moreover, the onto social choice function $f : \mathcal{U}^n \to [a, b]$ where agent $j \in N$ is the dictator (*i.e.*, for all $R \in \mathcal{U}^n$, $f(R) = \tau(R_j)$) can be described as a generalized median voter scheme by setting $p_T = a$ for all $T \subset N$ such that $j \in T$ and $p_S = b$ for all $S \subset N$ such that $j \notin S$. Then, for any $R \in \mathcal{U}^n$, (i) $\max\{\tau(R_j), p_{\{j\}}\} = \tau(R_j)$; $\tau(R_j) \leq \max_{i\in T}\{\tau(R_i), p_T\}$ for any $T \subset N$ such that $j \in T$; and (iii) $\max_{i\in S}\{\tau(R_i), p_S\} = b$ for any $S \subset N$ such that $j \notin S$. Thus, $\min_{S'\in 2^N} \max_{i'\in S'}\{\tau(R_{i'}), p_{S'}\} = \tau(R_j)$.

Given a monotonic family of fixed ballots $p = \{p_S\}_{S\subset N}$, let f^p denote the generalized median voter scheme associated to p.

[8] See Barberà, Massó, and Neme (1997) for a similar interpretation for the case of a finite number of ordered alternatives.

4.2 Main result

Our main result will provide a systematic way of comparing non-constant and non-dictatorial generalized median voter schemes according to their manipulability. It turns out that to perform this comparison it is crucial to identify, for each agent $i \in N$, the subintervals where i is a non-dummy agent; *i.e.*, the subset of alternatives that are eventually chosen at some profile but agent i is able to change the chosen alternative by reporting a different preference relation. We define formally below the general notion of a non-dummy agent at an alternative in a social choice function.

Definition 6 *Let $f : \mathcal{U}^n \to [a,b]$ be a social choice function. Agent i is* non-dummy at $\alpha \in [a,b]$ in f *if there exists $R \in \mathcal{U}^n$ and $R_i' \in \mathcal{U}$ such that*

$$f(R_i, R_{-i}) = \alpha \text{ and}$$
$$f(R_i', R_{-i}) \neq \alpha.$$

The lemma below characterizes non-dummyness at an alternative in a generalized median voter scheme $f^p : \mathcal{U}^n \to [a,b]$ in terms of the monotonic family of fixed ballots p. This characterization will be useful in the sequel.

Lemma 4 *Let $f^p : \mathcal{U}^n \to [a,b]$ be a generalized median voter scheme. Then, i is non-dummy at α in f^p if and only if there exists $S \subset N$ such that $i \in S$, $p_S < p_{S\backslash\{i\}}$ and $p_S \leq \alpha \leq p_{S\backslash\{i\}}$.*

Proof See Appendix 2 at the end of the paper.

The set of all $\alpha \in [a,b]$ such that i is non-dummy at α in $f^p : \mathcal{U}^n \to [a,b]$ is denoted by ND_p^i. By Lemma 4,

$$ND_p^i = \bigcup_{\{S \subset N | i \in S \text{ and } p_S < p_{S\backslash\{i\}}\}} [p_S, p_{S\backslash\{i\}}]. \qquad (6)$$

We are now ready to state the main result of the paper.

Theorem 2 *Let $p = \{p_S\}_{S\subset N}$ and $\bar{p} = \{\bar{p}_S\}_{S\subset N}$ be two monotonic families of fixed ballots and assume that the two associated generalized median voter schemes $f^p : \mathcal{U}^n \to [a,b]$ and $f^{\bar{p}} : \mathcal{U}^n \to [a,b]$ are neither constant nor dictatorial. Then,*

$$[p_N, p_{\{i\}}] \cap ND_p^i \subset [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i, \qquad (7)$$

$$[p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i, \qquad (8)$$

and

$$[p_{\{i\}}, p_{N\backslash\{i\}}] \subset ND_{\bar{p}}^i \qquad (9)$$

hold for all $i \in N$ if and only if $f^{\bar{p}}$ is at least as manipulable as f^p.

Before presenting three lemmata used in the proof of Theorem 2 few remarks are in order.

First, conditions (7), (8), and (9) say that the relevant information to compare two generalized median voter schemes according to their manipulability for agent $i \in N$ lies in the values of the fixed ballots associated to coalitions N, $N \backslash \{i\}$, $\{i\}$ and $\{\emptyset\}$ and in i's non-dummy sets.

Second, observe that condition (9) is only relevant when $p_{\{i\}} < p_{N\backslash\{i\}}$ because if $p_{N\backslash\{i\}} < p_{\{i\}}$, then $[p_{\{i\}}, p_{N\backslash\{i\}}] = \emptyset$ and if $p_{N\backslash\{i\}} = p_{\{i\}}$, then (9) follows from (7) and (8) since f^p is not constant and $p_{N\backslash\{i\}} = p_{\{i\}} \in ND_p^i$.

Third, if the non constant generalized median voter schemes associated to the monotonic families of fixed ballots $p = \{p_S\}_{S\in 2^N}$ and $\bar{p} = \{\bar{p}_S\}_{S\subset N}$ are anonymous, then $ND_p^i = [p_N, p_{\{\emptyset\}}]$, $ND_{\bar{p}}^i = [\bar{p}_N, \bar{p}_{\{\emptyset\}}]$ (i is non-dummy in the full ranges of f^p and $f^{\bar{p}}$), $p_{N\backslash\{i\}} \leq p_{\{i\}}$ and $\bar{p}_{N\backslash\{i\}} \leq \bar{p}_{\{i\}}$ for all $i \in N$. Therefore, conditions (7), (8), and (9) are equivalent to

$$[p_N, p_{\{i\}}] \subset [\bar{p}_N, \bar{p}_{\{i\}}]$$

and

$$[p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}]$$

or

$$[p_N, p_{\{\emptyset\}}] \subset [\bar{p}_N, \bar{p}_{\{\emptyset\}}]$$

and

$$[p_{N\backslash\{i\}}, p_{\{i\}}] \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{i\}}].$$

Now, if x and y are the $n+1$ vectors associated to f^p and $f^{\bar{p}}$, respectively, then $x_1 = p_N$, $x_2 = p_{N\backslash\{i\}}$, $x_n = p_{\{i\}}$, $x_{n+1} = p_{\{\emptyset\}}$, $y_1 = \bar{p}_N$, $y_2 = \bar{p}_{N\backslash\{i\}}$, $y_n = \bar{p}_{\{i\}}$ and $y_{n+1} = \bar{p}_{\{\emptyset\}}$. Thus, conditions (7), (8), and (9) are equivalent to

$$[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$$

and

$$[x_2, x_n] \subset [y_2, y_n],$$

which is what Theorem 1 says. Hence, Theorem 1 can be seen as a corollary of Theorem 2.

We will say that an interval $I_i = [c, d]$ with $c < d$ is *a non-dummy interval for i in f^p* if $I_i \subset ND_p^i$. Whenever we refer to an interval as a non-dummy interval we exclude the possibility that the interval contains only one alternative. If $i \in S$ with $p_S < p_{S\backslash\{i\}}$, then $[p_S, p_{S\backslash\{i\}}]$ is a non-dummy interval for i in f^p and we denote it by I_i^S. We will write $\bar{I}_i^S$ when the median voter scheme used as reference is $f^{\bar{p}}$ instead of f^p.

We state now the three lemmata, whose proofs are in Appendix 2, that will be used in the proof of Theorem 2. To simplify notation, given $p = \{p_S\}_{S \subset N}$ and $R_i \in \mathcal{U}$, we denote $o^{f^p}(R_i)$ by $o^p(R_i)$.

Lemma 5 *Let $f^p : \mathcal{U}^n \to [a, b]$ be a non-constant generalized median voter scheme. Then, f^p is not manipulable by i at R_i if and only if, for all I_i^S, R_i is single-peaked on $(o^p(R_i) \cap I_i^S) \cup \{\tau(R_i), \alpha^*\}$ for all $\alpha^* \in I_i^S$.*

Lemma 6 *Let $p = \{p_S\}_{S \subset N}$ be a monotonic family of fixed ballots and $R_i \in \mathcal{U}$. If $p_{\{i\}} < p_{N \setminus \{i\}}$, then*

$$o^p(R_i) = \begin{cases} [p_N, p_{\{i\}}] & \text{if } a \le \tau(R_i) \le p_N \\ [\tau(R_i), p_{\{i\}}] & \text{if } p_N < \tau(R_i) \le p_{\{i\}} \\ \{\tau(R_i)\} & \text{if } p_{\{i\}} < \tau(R_i) \le p_{N\setminus\{i\}} \\ [p_{N\setminus\{i\}}, \tau(R_i)] & \text{if } p_{N\setminus\{i\}} < \tau(R_i) \le p_{\{\emptyset\}} \\ [p_{N\setminus\{i\}}, p_{\{\emptyset\}}] & \text{if } p_{\{\emptyset\}} < \tau(R_i). \end{cases}$$

If $p_{N\setminus\{i\}} \le p_{\{i\}}$, then

$$o^p(R_i) = \begin{cases} [p_N, p_{\{i\}}] & \text{if } a \le \tau(R_i) \le p_N \\ [\tau(R_i), p_{\{i\}}] & \text{if } p_N < \tau(R_i) \le p_{N\setminus\{i\}} \\ [p_{N\setminus\{i\}}, p_{\{i\}}] & \text{if } p_{N\setminus\{i\}} < \tau(R_i) \le p_{\{i\}} \\ [p_{N\setminus\{i\}}, \tau(R_i)] & \text{if } p_{\{i\}} < \tau(R_i) \le p_{\{\emptyset\}} \\ [p_{N\setminus\{i\}}, p_{\{\emptyset\}}] & \text{if } p_{\{\emptyset\}} < \tau(R_i). \end{cases}$$

Lemma 7 *Let $p = \{p_S\}_{S \subset N}$ and $\bar{p} = \{\bar{p}_S\}_{S \subset N}$ be two monotonic families of fixed ballots such that f^p and $f^{\bar{p}}$ are not constant. Assume (7), (8), and (9) in Theorem 2 hold. Then, for any non-dummy interval I_i^S and for all $\alpha^* \in I_i^S$ there exists a non dummy interval $\hat{I}_i$ for i in $f^{\bar{p}}$ such that $\alpha^* \in \hat{I}_i$ and $(o^p(R_i) \cap I_i^S) \subset (o^{\bar{p}}(R_i) \cap \hat{I}_i)$ for all $R_i \in \mathcal{U}$.*[9]

Definition 7 *Let $f : \mathcal{U}^n \to [a, b]$ be a social choice function. Agent i is a* dictator at $\alpha \in [a, b]$ in f *if for all $R_i \in \mathcal{U}$ such that $\tau(R_i) = \alpha$,*

$$f(R_i, R_{-i}) = \alpha \text{ for all } R_{-i} \in \mathcal{U}^{n-1}.$$

Let $f^p : \mathcal{U}^n \to [a, b]$ be a generalized median voter scheme and $i \in N$ be an agent. Denote the set of all $\alpha \in [a, b]$ such that i is a dictator at α in f^p, by DT_p^i. By Lemma 6, $DT_p^i = [p_{\{i\}}, p_{N\setminus\{i\}}]$. Observe that if $p_{N\setminus\{i\}} < p_{\{i\}}$, then i is not a dictator at any $\alpha \in [a, b]$ in f^p. Furthermore, if $p_{\{i\}} < p_{N\setminus\{i\}}$, then, by monotonicity, $p_{N\setminus\{j\}} \le p_{\{i\}} < p_{N\setminus\{i\}} \le p_{\{j\}}$ for all $j \neq i$. Therefore, if $p_{\{i\}} < p_{N\setminus\{i\}}$, then j is not a dictator at any $\alpha \in [a, b]$ in f^p for all $j \neq i$.

[9] Note that $\hat{I}_i$ does not have to be necessarily written as $\bar{I}_i^{S'}$ for some $S' \ni i$.

Definition 8 *Let* $p = \{p_S\}_{S \subset N}$ *and* $\bar{p} = \{\bar{p}_S\}_{S \subset N}$ *be two monotonic families of fixed ballots. The generalized median voter scheme* $f^p : \mathcal{U}^n \to [a,b]$ *is at least more (or more) dictatorial for* i *than the generalized median voter scheme* $f^{\bar{p}} : \mathcal{U}^n \to [a,b]$ *if* $\emptyset \neq DT^i_{\bar{p}} \subset DT^i_p$ *(or* $\emptyset \neq DT^i_{\bar{p}} \subsetneq DT^i_p$*).*

Proposition below formalizes the trade-off between dictatorialness and manipulability.

Proposition 4 *Let* $p = \{p_S\}_{S \subset N}$ *and* $\bar{p} = \{\bar{p}_S\}_{S \subset N}$ *be two monotonic families of fixed ballots. Assume that* $f^p : \mathcal{U}^n \to [a,b]$ *and* $f^{\bar{p}} : \mathcal{U}^n \to [a,b]$ *are non-constant, non-dictatorial and comparable according to their manipulability. If* f^p *is more dictatorial for* i *than* $f^{\bar{p}}$*, then* $f^{\bar{p}}$ *is more manipulable than* f^p*.*

Proof Since f^p is more dictatorial than $f^{\bar{p}}$ for i, $\emptyset \neq DT^i_{\bar{p}} \subsetneq DT^i_p$. Then, $[\bar{p}_{\{i\}}, \bar{p}_{N\setminus\{i\}}] \subsetneq [p_{\{i\}}, p_{N\setminus\{i\}}]$ and $\bar{p}_{\{i\}} \leq \bar{p}_{N\setminus\{i\}}$. Therefore, $p_{\{i\}} < \bar{p}_{\{i\}}$ and $\bar{p}_{N\setminus\{i\}} \leq p_{N\setminus\{i\}}$ or $p_{\{i\}} \leq \bar{p}_{\{i\}}$ and $\bar{p}_{N\setminus\{i\}} < p_{N\setminus\{i\}}$. Assume that $p_{\{i\}} < \bar{p}_{\{i\}}$ and $\bar{p}_{N\setminus\{i\}} \leq p_{N\setminus\{i\}}$ hold; the proof for the other case proceeds similarly and therefore it is omitted. Since $DT^i_p \neq \emptyset$ and $p = \{p_S\}_{S \subset N}$ is monotonic, $ND^i_p = [p_N, p_{\{\emptyset\}}]$ holds by (6). Thus,

$$[p_N, p_{\{i\}}] \cap ND^i_p = [p_N, p_{\{i\}}].$$

Similarly, and since $DT^i_{\bar{p}} \neq \emptyset$,

$$[\bar{p}_N, \bar{p}_{\{i\}}] \cap ND^i_{\bar{p}} = [\bar{p}_N, \bar{p}_{\{i\}}].$$

Since f^p and $f^{\bar{p}}$ are comparable according to their manipulability and $p_{\{i\}} < \bar{p}_{\{i\}}$,

$$[p_N, p_{\{i\}}] \cap ND^i_p = [p_N, p_{\{i\}}] \subsetneq [\bar{p}_N, \bar{p}_{\{i\}}] = [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND^i_{\bar{p}}.$$

Thus, by Theorem 2, $f^{\bar{p}}$ is more manipulable than f^p. ■

5 Final remark

Before moving to the omitted proofs we finish with a final remark relating our comparability notion with the one proposed by Pathak and Sönmez. Pathak and Sönmez (2013) propose an inclusion criterion to compare two social choice functions according to their manipulability. In general, the social choice function ψ is at least as manipulable as the social choice function φ according to Pathak and Sönmez if φ is manipulable at profile R, then ψ is also manipulable at profile R. Proposition 5 shows that in our setting Pathak and Sönmez criterion is weaker than ours.

Proposition 5 *Let f and g be two generalized median voter schemes and assume that g is at least as manipulable as f according to Pathak and Sönmez's notion. Then, g is at least as manipulable as f.*

Proof Assume that g is at least as manipulable as f according to Pathak and Sönmez's notion. Fix $i \in N$ and let $R_i \in \mathcal{M}_i^f$. There exists $(R_i', R_{-i}) \in \mathcal{U}^n$ such that

$$f(R_i', R_{-i}) P_i f(R_i, R_{-i}). \tag{10}$$

Since f is tops-only, we may assume that $R_{-i} \in \mathcal{SP}^{n-1}$. By (10), f is manipulable at profile R. Hence, by assumption, g is manipulable at profile R. Since $R_{-i} \in \mathcal{SP}^{n-1}$, by Lemma 5, agent i manipulates g at profile R. Hence, $R_i \in \mathcal{M}_i^g$. Thus, g is at least as manipulable as f. ■

Example 1 below shows that the reverse implication does not hold; that is, Pathak and Sönmez's notion is strictly weaker than ours and leaves many pairs of generalized median voter schemes as being non-comparable while they are according to our notion.

Example 1 Let $n = 3$ and f^x and f^y be two median voter schemes associated to $x = (0, \frac{1}{2}, \frac{1}{2}, 1)$ and $y = (0, 0, 1, 1)$, respectively. By Theorem 1, and since $[x_1, x_{n+1}] \subset [y_1, y_{n+1}]$ and $[x_2, x_n] \subsetneq [y_2, y_n]$, f^y is more manipulable that f^x. In one hand, consider any profile $R = (R_1, R_2, R_3) \in \mathcal{U}^3$ and any preference $R_3' \in \mathcal{U}$ such that (i) $\tau(R_i) = 1$ for $i = 1, 2$, (ii) $\tau(R_3) = \frac{1}{4}$ and $\frac{3}{4} R_3 \frac{1}{2}$, and (iii) $\tau(R_3') = \frac{3}{4}$. Therefore, $f^x(R_1, R_2, R_3') = \frac{3}{4} P_3 \frac{1}{2} = f^x(R)$ and hence, f^x is manipulable at profile R. Moreover, $f^y(R) = 1$ and f^y is not manipulable at profile R. Hence, f^y is not more manipulable than f^x according to Pathak and Sönmez's notion. On the other hand, consider any profile $\widehat{R} = (\widehat{R}_1, \widehat{R}_2, \widehat{R}_3) \in \mathcal{U}^3$ and any preference $\widehat{R}_3' \in \mathcal{U}$ such that (i) $\tau(\widehat{R}_1) = \frac{1}{2}$, (ii) $\tau(\widehat{R}_2) = \frac{1}{4}$, (iii) $\tau(\widehat{R}_3) = \frac{3}{4}$ and $\frac{1}{4} \widehat{R}_3 \frac{1}{2}$, and (iv) $\tau(\widehat{R}_3') = \frac{1}{4}$. Therefore, $f^x(\widehat{R}) = \frac{1}{2}$ and f^x is not manipulable at profile $\widehat{R}$. Moreover, $f^y(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3') = \frac{1}{4} \widehat{R}_3 \frac{1}{2} = f^y(\widehat{R})$ and hence, f^y is manipulable at profile $\widehat{R}$. Hence, f^x is not more manipulable than f^y according to Pathak and Sönmez's notion. Thus, f^x and f^y are not comparable according to Pathak and Sönmez's notion of manipulability, although they are according to our notion. □

Example 1 illustrates the fact that our comparability notion is based on the inclusion of the maximal domains of preferences under which each of the two generalized median voter schemes are strategy-proof. In this case, the maximal domain of preferences under which f^y is strategy-proof is the set of single-peaked preferences on $[0, 1]$ while f^x admits a much larger maximal domain, the union of the following three sets:

$\{R_i \in \mathcal{U} \mid 0 \le \tau(R_i) < \frac{1}{2}, \tau(R_i) < \alpha < \beta \le \frac{1}{2} \Rightarrow \alpha R_i \beta$, and $\frac{1}{2} < \alpha \Rightarrow \frac{1}{2} R_i \alpha\}$,

$\{R_i \in \mathcal{U} \mid \frac{1}{2} < \tau(R_i) \le 1, \frac{1}{2} \le \beta < \alpha < \tau(R_i) \Rightarrow \alpha R_i \beta$, and $\alpha < \frac{1}{2} \Rightarrow \frac{1}{2} R_i \alpha\}$, and

$\{R_i \in \mathcal{U} \mid \tau(R_i) = \frac{1}{2}\}$.

6 Appendix 1

Proof of Lemma 1

$\Rightarrow$) Suppose there exists $\alpha^* \in r_{f^x}$ such that R_i is not single-peaked on $o^x(R_i) \cup \{\tau(R_i), \alpha^*\}$. We will prove that there exist $R'_i \in \mathcal{U}$ and $R_{-i} \in \mathcal{U}^{n-1}$ such that $f^x(R'_i, R_{-i})P_i f^x(R_i, R_{-i})$. We will divide the proof into three different cases.

Case 1: Suppose $\alpha^* \in o^x(R_i)$ and there exists $\beta \in o^x(R_i)$ such that $\alpha^* < \beta < \tau(R_i)$ and $\alpha^* P_i \beta$; the other case where $\tau(R_i) < \alpha^* < \beta$ and $\beta P_i \alpha^*$ is similar and therefore it is omitted. Let $\hat{R} \in \mathcal{U}^n$ be such that $\tau(\hat{R}_j) = \alpha^*$ for all $j \in N$. Since $\alpha^* \in o^x(R_i)$, and f^x is a median voter scheme, $f^x(R_i, \hat{R}_{-i}) = \alpha^*$. Similarly, let $\bar{R} \in \mathcal{U}^n$ be such that $\tau(\bar{R}_j) = \beta$ for all $j \in N$. Since $\beta \in o^x(R_i)$, $f^x(R_i, \bar{R}_{-i}) = \beta$. Since $f^x(R_i, \hat{R}_{-i}) = \alpha^* P_i \beta = f^x(R_i, \bar{R}_{-i})$, by the definition of f^x, there must exist $S \subset N \backslash \{i\}$ and $j' \notin S$ such that

$$f^x(R_i, \hat{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}})P_i f^x(R_i, \bar{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}). \tag{11}$$

Now, let $R'_i \in \mathcal{U}$ be such that $\tau(R'_i) = f^x(R_i, \hat{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j\}})$. Since $\tau(\hat{R}_j) = \alpha^* < \beta = \tau(\bar{R}_{j'})$ for all $j \in N$,

$$\begin{aligned}
\tau(\hat{R}_{j'}) = \alpha^* = f^x(R_i, \hat{R}_{-i}) \quad &\leq f^x(R_i, \hat{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}) \\
&\leq f^x(R_i, \bar{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}) \\
&\leq f^x(R_i, \bar{R}_{-i}) \\
&= \beta \\
&= \tau(\bar{R}_{j'}) \\
&< \tau(R_i).
\end{aligned} \tag{12}$$

Then, by (12) and the definition of f^x,

$$f^x(R'_i, \bar{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}) = f^x(R_i, \hat{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}).$$

Hence, by (11),

$$f^x(R'_i, \bar{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}})P_i f^x(R_i, \bar{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}).$$

Thus, f^x is manipulable by i at R_i with any R'_i with the property that $\tau(R'_i) = f^x(R_i, \hat{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j\}})$.

Case 2: Suppose $\alpha^* \notin o^x(R_i)$ and there exists $\beta \in o^x(R_i)$ such that $\alpha^* < \beta < \tau(R_i)$ and $\alpha^* P_i \beta$; the other case where $\tau(R_i) < \beta < \alpha^*$ and $\alpha^* P_i \beta$ proceeds similarly and it is therefore omitted. Let $\bar{R} \in \mathcal{U}^n$ be such that $\tau(\bar{R}_j) = \beta$ for all $j \in N$. Since $\beta \in o^x(R_i)$,

$$f^x(R_i, \bar{R}_{-i}) = \beta. \tag{13}$$

Let $\hat{R} \in \mathcal{U}^n$ be such that $\tau(\hat{R}_j) = \beta$ for all $j \in N$. If there exist $S \subset N\backslash\{i\}$ and $j' \notin S$ such that

$$f^x(R_i, \hat{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}})P_i f^x(R_i, \bar{R}_{j'}, \hat{R}_S, \bar{R}_{-S\cup\{i,j'\}}) \qquad (14)$$

holds, the proof proceeds as in Case 1. Hence, assume that there do not exist $S \subset N\backslash\{i\}$ and $j' \notin S$ satisfying (14). Let $N\backslash\{i\} = \{j_1, ..., j_{n-1}\}$. Then,

$$\begin{array}{lll}
\beta & = f^x(R_i, \bar{R}_{-i}) & \text{by (13)} \\
& R_i f^x(R_i, \hat{R}_{j_1}, \bar{R}_{-\{i,j_1\}}) & \text{consider } S_1 = \emptyset, \\
& & \quad j' = j_1 \notin S_1, \text{ and } \neg(14) \\
& R_i f^x(R_i, \hat{R}_{j_2}, \hat{R}_{j_1}, \bar{R}_{-\{j_1\}\cup\{i,j_2\}}) & \text{consider } S_2 = \{j_1\}, \\
& & \quad j' = j_2 \notin S_2, \text{ and } \neg(14) \\
& R_i f^x(R_i, \hat{R}_{j_3}, \hat{R}_{\{j_1,j_2\}}, \bar{R}_{-\{j_1,j_2\}\cup\{i,j_3\}}) & \text{consider } S_3 = \{j_1, j_2\}, \\
& & \quad j' = j_3 \notin S_3, \text{ and } \neg(14) \\
& \vdots & \vdots \\
& R_i f^x(R_i, \hat{R}_{j_{n-2}}, \hat{R}_{\{j_1,j_2,...,j_{n-3}\}}, \bar{R}_{-\{j_1,j_2,...,j_{n-3}\}\cup\{i,j_{n-2}\}}) & \text{consider } S_{n-1} = \{j_1, j_2, ..., j_{n-3}\}, \\
& & \quad j' = j_{n-2} \notin S_{n-1}, \text{ and } \neg(14) \\
& R_i f^x(R_i, \hat{R}_{j_{n-1}}, \hat{R}_{\{j_1,j_2,...,j_{n-2}\}}, \bar{R}_{-\{j_1,j_2,...,j_{n-2}\}\cup\{i,j_{n-1}\}}) & \text{consider } S_n = \{j_1, j_2, ..., j_{n-2}\}, \\
& & \quad j' = j_{n-1} \notin S_n, \text{ and } \neg(14) \\
& = f^x(R_i, \hat{R}_{-i}) & \{j_1, j_2, ..., j_{n-2}\} \cup \{i, j_{n-1}\} = N.
\end{array}$$

Hence, as $\alpha^* P_i \beta$,

$$\alpha^* P_i f^x(R_i, \hat{R}_{-i}). \qquad (15)$$

Since $\alpha^* \in r_{f^x}$, $f^x(\hat{R}_i, \hat{R}_{-i}) = \alpha^*$. Thus, by (15), $f^x(\hat{R}_i, \hat{R}_{-i})P_i f^x(R_i, \hat{R}_{-i})$, which means that f^x is manipulable by i at R_i with any $\hat{R}_i$ such that $\tau(\hat{R}_i) = \alpha^*$.

Case 3: Suppose $\alpha^* \notin o^x(R_i)$ and there exists $\beta \in o^x(R_i)$ such that $\beta < \alpha^* < \tau(R_i)$ and $\beta P_i \alpha^*$; the other case where $\tau(R_i) < \alpha^* < \beta$ and $\beta P_i \alpha^*$ proceeds similarly and it is therefore omitted. We will prove that this case is not possible. Consider the profile $\hat{R}$ such that $\tau(\hat{R}_j) = \alpha^*$ for all $j \in N$. Since $\alpha^* \notin o^x(R_i)$, $\beta \in o^x(R_i)$ and $o^x(R_i)$ is an interval (see Lemma 2), $f(R_i, \hat{R}_{-i}) < \alpha^*$. Furthermore, and since $\alpha^* \leq \tau(R_i)$, $f^x(\hat{R}_i, \hat{R}_{-i}) \leq f^x(R_i, \hat{R}_{-i}) < \alpha^*$. Hence, $f^x(\hat{R}) < \alpha$. Thus, $\alpha^* \notin r_{f^x}$ which contradicts the initial hypothesis.

$\Leftarrow$) Suppose f^x is manipulable by i at R_i; that is, there exist $R'_i \in \mathcal{U}$ and $R_{-i} \in \mathcal{U}^{n-1}$ such that

$$f^x(R'_i, R_{-i})P_i f^x(R_i, R_{-i}). \qquad (16)$$

Consider the case $\tau(R'_i) < \tau(R_i)$; the other case is similar and therefore it is omitted. We distinguish among three different cases.

Case 1: $\tau(R_i) < f^x(R_i, R_{-i})$. Since f^x is a median voter scheme and $\tau(R'_i) < \tau(R_i)$, $f^x(R'_i, R_{-i}) = f^x(R_i, R_{-i})$. But this contradicts (16).

Case 2: $\tau(R_i) = f^x(R_i, R_{-i})$. Then, $f^x(R_i, R_{-i}) P_i f^x(R'_i, R_{-i})$ which also contradicts (16).

Case 3: $f^x(R_i, R_{-i}) < \tau(R_i)$. Since $\tau(P'_i) < \tau(R_i)$ and (16), $f^x(R'_i, R_{-i}) < f^x(R_i, R_{-i})$. Hence, $f^x(R'_i, R_{-i}) < f^x(R_i, R_{-i}) < \tau(R_i)$ and $\tau(R_i) P_i f^x(R'_i, R_{-i}) P_i f^x(R_i, R_{-i})$. Thus, and since $f^x(R_i, R_{-i}), \tau(R_i) \in o^x(R_i) \cup \{\tau(R_i)\}$ and $f^x(R'_i, R_{-i}) \in r_{f^x}$, R_i is not single-peaked on $o^x(R_i) \cup \{\tau(R_i), f^x(R'_i, R_{-i})\}$. ■

Proof of Lemma 2 We divide the proof into three cases.

Case 1: Suppose $\tau(R_i) < x_1$. The case $x_{n+1} < \tau(R_i)$ is symmetric and its proof proceeds similarly; therefore, it is omitted. We prove that $o^x(R_i) = [x_1, x_n]$. Let $\alpha \in o^x(R_i)$ be arbitrary. Then, there exists $R_{-i} \in \mathcal{U}^{n-1}$ such that

$$med\{\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}\} = \alpha.$$

Redefine $y = (y_1, ..., y_{2n+1}) \equiv (\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}) \in [a, b]^{2n+1}$. If $y_{s^*} < x_1$, and since $\tau(R_i) < x_1 \leq .... \leq x_{n+1}$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq y_{s^*}\} \geq n+2$. Hence, $\alpha \neq y_{s^*}$. If $x_n < y_{s^*}$, and since $\tau(R_i) < x_1 \leq x_2 \leq .... \leq x_n$, $\#\{s \in \{1, ...2n+1\} \mid y_s \leq y_{s^*}\} \geq n+2$. Hence, $\alpha \neq y_{s^*}$. Thus, $\alpha \in [x_1, x_n]$. Now, let $\alpha \in [x_1, x_n]$, $\hat{R}_i = R_i$ and for all $j \in N\backslash\{i\}$ let $\hat{R}_j \in \mathcal{U}$ be such that $\tau(\hat{R}_j) = \alpha$. Redefine $y = (y_1, ..., y_{2n+1}) \equiv (\tau(\hat{R}_1), ..., \tau(\hat{R}_n), x_1, ..., x_{n+1}) \in [a, b]^{2n+1}$. Since $\alpha \leq x_n \leq x_{n+1}$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq \alpha\} \geq n+1$. Furthermore, and since $\tau(R_i) < x_1 \leq \alpha$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \leq \alpha\} \geq n+1$. Hence, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq \alpha\} = \#\{s \in \{1, ..., 2n+1\} \mid y_s \leq \alpha\} = n+1$. Thus, $med\{\tau(\hat{R}_1), ..., \tau(\hat{R}_n), x_1, ..., x_{n+1}) = \alpha$. Since $\hat{R}_i = R_i$, $\alpha \in o^x(R_i)$. Therefore, $o^x(R_i) = [x_1, x_n]$.

Case 2: Suppose $x_1 \leq \tau(R_i) < x_2$. The case $x_n < \tau(R_i) \leq x_{n+1}$ is symmetric and its proof proceeds similarly; therefore, it is omitted. We prove that $o^x(R_i) = [\tau(R_i), x_{n+1}]$. Let $\alpha \in o^x(R_i)$ be arbitrary. Then, there exists $R_{-i} \in \mathcal{U}^{n-1}$ such that

$$med\{\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}\} = \alpha.$$

Redefine $y = (y_1, ..., y_{2n+1}) = (\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}) \in [a, b]^{2n+1}$. If $y_{s^*} < \tau(R_i)$, and since $\tau(R_i) < x_2 \leq .... \leq x_{n+1}$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq y_{s^*}\} \geq n+2$. Hence, $\alpha \neq y_{s^*}$. If $x_n < y_{s^*}$, and since $\tau(R_i) < x_2 \leq .... \leq x_{n+1}$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \leq y_{s^*}\} \geq n+2$. Hence, $\alpha \neq y_{s^*}$. Thus, $\alpha \in [\tau(R_i), x_n]$. Now, let $\alpha \in [\tau(R_i), x_n]$, $\hat{R}_i = R_i$ and for all $j \in N\backslash\{i\}$ let $\hat{R}_j \in \mathcal{U}$ be such that $\tau(\hat{R}_j) = \alpha$. Redefine $y = (y_1, ..., y_{2n+1}) \equiv (\tau(\hat{R}_1), ..., \tau(\hat{R}_n), x_1, ..., x_{n+1}) \in [a, b]^{2n+1}$. Since $\alpha \leq x_n \leq x_{n+1}$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq \alpha\} \geq n+1$. Furthermore, and since $x_1 \leq \tau(R_i) \leq \alpha$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \leq \alpha\} \geq n+1$. Hence, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq \alpha\} = \#\{s \in \{1, ..., 2n+1\} \mid y_s \leq \alpha\} = n+1$. Thus, $med\{\tau(\hat{R}_1), ..., \tau(\hat{R}_n), x_1, ..., x_{n+1}) = \alpha$. Since $\hat{R}_i = R_i$, $\alpha \in o^x(R_i)$. Therefore, $o^x(R_i) = [\tau(R_i), x_n]$.

Case 3: Suppose $x_2 \leq \tau(R_i) \leq x_n$. We prove that $o^x(R_i) = [x_2, x_n]$. Let $\alpha \in o^x(R_i)$ be arbitrary. Then, there exists $R_{-i} \in \mathcal{U}^{n-1}$ such that

$$med\{\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}\} = \alpha.$$

Redefine $y = (y_1, ..., y_{2n+1}) = (\tau(R_1), ..., \tau(R_n), x_1, ..., x_{n+1}) \in [a, b]^{2n+1}$. If $y_{s^*} < x_2$, and since $x_2 \leq .... \leq x_{n+1}$ and $x_2 \leq \tau(R_i)$, we have that $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq y_{s^*}\} \geq n+2$. Hence, $\alpha \neq y_{s^*}$. If $x_n < y_{s^*}$, and since $x_1 \leq .... \leq x_n$ and $\tau(R_i) \leq x_n$, we have that $\#\{s \in \{1, ..., 2n+1\} \mid y_s \leq y_{s^*}\} \geq n+2$. Hence, $\alpha \neq y_{s^*}$. Thus, $\alpha \in [x_2, x_n]$. Now, let $\alpha \in [x_2, x_n]$, $\hat{R}_i = R_i$ and for all $j \in N \backslash \{i\}$ let $\hat{R}_j \in \mathcal{U}$ be such that $\tau(\hat{R}_j) = \alpha$. Redefine $y = (y_1, ..., y_{2n+1}) \equiv (\tau(\hat{R}_1), ..., \tau(\hat{R}_n), x_1, ..., x_{n+1}) \in [a, b]^{2n+1}$. Since $\alpha \leq x_n \leq x_{n+1}$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq \alpha\} \geq n+1$. Furthermore, and since $x_1 \leq x_2 \leq \alpha$, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \leq \alpha\} \geq n+1$. Hence, $\#\{s \in \{1, ..., 2n+1\} \mid y_s \geq \alpha\} = \#\{s \in \{1, ..., 2n+1\} \mid y_s \leq \alpha\} = n+1$. Thus, $med\{\tau(\hat{R}_1), ..., \tau(\hat{R}_n), x_1, ..., x_{n+1}) = \alpha$. Since $\hat{R}_i = R_i$, $\alpha \in o^x(R_i)$. Therefore, $o^x(R_i) = [x_2, x_n]$. ■

Proof of Lemma 3 We divide the proof into five cases.

Case 1: Suppose $\tau(R_i) < x_1$. Then, by Lemma 2, $o^x(R_i) = [x_1, x_n]$. Since $\tau(R_i) < x_1 \leq x_n \leq y_n$,

$$o^y(R_i) = \begin{cases} [y_1, y_n] & \text{if } \tau(R_i) < y_1 \\ [\tau(R_i), y_n] & \text{if } y_1 \leq \tau(R_i) < y_2 \\ [y_2, y_n] & \text{if } y_2 \leq \tau(R_i) \leq y_n. \end{cases}$$

Hence, $o^x(R_i) \subset o^y(R_i)$.

Case 2: Suppose $x_1 \leq \tau(R_i) < x_2$. Then, by Lemma 2, $o^x(R_i) = [\tau(R_i), x_n]$. Since $y_1 \leq x_1 \leq \tau(R_i) < x_2 \leq x_n \leq y_n$,

$$o^y(R_i) = \begin{cases} [\tau(R_i), y_n] & \text{if } y_1 \leq \tau(R_i) < y_2 \\ [y_2, y_n] & \text{if } y_2 \leq \tau(R_i) \leq y_n. \end{cases}$$

Hence, $o^x(R_i) \subset o^y(R_i)$.

Case 3: Suppose $x_2 \leq \tau(R_i) \leq x_n$. Then, $y_2 \leq \tau(R_i) \leq y_n$. By Lemma 2, $o^x(R_i) = [x_2, x_n]$ and $o^y(R_i) = [y_2, y_n]$. Hence, $o^x(R_i) \subset o^y(R_i)$.

Case 4: Suppose $x_n < \tau(R_i) \leq x_{n+1}$. Then, by Lemma 2, $o^x(R_i) = [x_2, \tau(R_i)]$. Since $y_2 \leq x_2 \leq x_n < \tau(R_i) \leq x_{n+1} \leq y_{n+1}$,

$$o^y(R_i) = \begin{cases} [y_2, y_n] & \text{if } y_2 \leq \tau(R_i) \leq y_n \\ [y_2, \tau(R_i)] & \text{if } y_n < \tau(R_i) \leq y_{n+1}. \end{cases}$$

Hence, $o^x(R_i) \subset o^y(R_i)$.

Case 5: Suppose $x_{n+1} < \tau(R_i)$. Then, by Lemma 2, $o^x(R_i) = [x_2, x_{n+1}]$. Since $y_2 \leq x_2 \leq x_{n+1} < \tau(R_i)$,

$$o^y(R_i) = \begin{cases} [y_2, y_n] & \text{if } y_2 \leq \tau(R_i) \leq y_n \\ [y_2, \tau(R_i)] & \text{if } y_n < \tau(R_i) \leq y_{n+1} \\ [y_2, y_{n+1}] & \text{if } y_{n+1} < \tau(R_i). \end{cases}$$

Hence, $o^x(R_i) \subset o^y(R_i)$. ■

7 Appendix 2

We start with two preliminary notions and several remarks.

First, a generalized median voter scheme $f^p : \mathcal{U}^n \to [a,b]$ can alternatively be represented by a monotonic family of right fixed ballots $p^r = \{p^r_S\}_{S \in 2^N}$, where (i) for all $S \in 2^N$, $p^r_S \in [a,b]$, (ii) $S \subset T$ implies $p^r_S \leq p^r_T$, (iii) for all $S \in 2^N$, $p^r_S = p_{N \backslash S}$, and (iv) for all $R \in \mathcal{U}^n$, $f^p(R) = \max_{S \in 2^N} \min_{j \in S}\{\tau(R_j), p^r_S\} \equiv f^{p^r}(R)$.

Second, a non-dummy interval I_i is a *maximal non-dummy interval for* i if there is no non-dummy interval I'_i such that $I_i \subsetneq I'_i$. Since the number of coalitions that contain a player is finite, any maximal non-dummy interval I_i can be written as the union of a family of intervals; namely, $I_i = \cup_{k=1}^K I_i^{S_k}$, where $i \in S_k$ for all $k = 1, ..., K$.

Before moving to the proof of the four lemmata used to prove Theorem 2, we state without proof the following facts.

Remark 2 *Let $f^p : \mathcal{U}^n \to [a,b]$ be a generalized median voter scheme and let $R_i \in \mathcal{U}$. Then, R_i is single-peaked on $(o^p(R_i) \cap I_i) \cup \{\tau(R_i), \alpha^*\}$ for all $\alpha^* \in I_i$, for all maximal non-dummy interval I_i if and only if R_i is single-peaked on $(o^p(R_i) \cap I_i^S) \cup \{\tau(R_i), \alpha^*\}$ for all $\alpha^* \in I_i^S$, for all non-dummy interval I_i^S.*

Remark 3 *If $p_{\{i\}} < p_{\{\emptyset\}}$, then $[p_{\{i\}}, p_{\{\emptyset\}}]$ is a non-dummy interval for i in $f^p : \mathcal{U}^n \to [a,b]$. If $p_N < p_{N\backslash\{i\}}$, then $[p_N, p_{N\backslash\{i\}}]$ is a non-dummy interval for i in f^p.*

Remark 4 *If $\alpha \in [p_N, p_{N\backslash\{i\}}]$, $\beta \in [p_{\{i\}}, p_{\{\emptyset\}}]$ and I_i is a maximal non-dummy interval for i in $f^p : \mathcal{U}^n \to [a,b]$ such that $\alpha, \beta \in I_i$, then $I_i = [p_N, p_{\{\emptyset\}}]$.*

Remark 5 *If $p_{\{i\}} < p_{N\backslash\{i\}}$, then $[p_N, p_{\{\emptyset\}}]$ is a (maximal) non-dummy interval for i in $f^p : \mathcal{U}^n \to [a,b]$.*

Remark 6 *If $p_N = p_{\{i\}} < p_{N\backslash\{i\}} = p_{\{\emptyset\}}$, then i is a dictator in $f^p : \mathcal{U}^n \to [a,b]$.*

Proof of Lemma 4 Let $f^p : \mathcal{U}^n \to [a,b]$ be a generalized median voter scheme. We will denote f^p simply by f.

$\Rightarrow$) Assume i is non-dummy at α in f. Then, there exist $R \in \mathcal{U}^n$ and $R'_i \in \mathcal{U}$ such that $f(R_i, R_{-i}) = \alpha$ and $f(R'_i, R_{-i}) \neq \alpha$. We distinguish between two cases.

Case 1: Assume $f(R_i, R_{-i}) = \alpha < f(R'_i, R_{-i})$. Since f is a generalized median voter scheme, $\tau(R_i) \leq \alpha < \tau(R'_i)$. Let $S = \{j \in N \mid \tau(R_j) \leq \alpha\}$. Observe that $i \in S$. First, we prove that $p_S \leq \alpha$. Suppose otherwise, $\alpha < p_S$; then, $\max_{j \in S}\{\tau(R_j), p_S\} = p_S > \alpha$. By the definition of S and f, $f(R_i, R_{-i}) > \alpha$, a contradiction with $f(R_i, R_{-i}) = \alpha$. Now, we prove that $\alpha < p_{S\backslash\{i\}}$. Suppose otherwise, $p_{S\backslash\{i\}} \leq \alpha$. For all $j \in S\backslash\{i\}$, $\tau(R'_j) = \tau(R_j) \leq \alpha$. Hence, $\max_{j \in S\backslash\{i\}}\{\tau(R'_j), p_{S\backslash\{i\}}\} \leq \alpha$. Thus, $f(R'_i, R_{-i}) \leq \alpha$, a contradiction with $f(R'_i, R_{-i}) > \alpha$. Therefore, $p_S \leq \alpha \leq p_{S\backslash\{i\}}$. Since $f(R_i, R_{-i}) < f(R'_i, R_{-i})$, $p_S < p_{S\backslash\{i\}}$.

Case 2: Assume $f(R'_i, R_{-i}) < \alpha = f(R_i, R_{-i})$. The proof proceeds symmetrically to Case 1 using the right phantom representation of f.

$\Leftarrow$) Assume there exists $S \subset N$ such that $i \in S$, $p_S < p_{S\backslash\{i\}}$ and $p_S \leq \alpha \leq p_{S\backslash\{i\}}$. We distinguish between two cases.

Case 1: Assume $p_S \leq \alpha < p_{S\backslash\{i\}}$. Let $R \in \mathcal{U}^n$ be such that $\tau(R_j) = \alpha$ for all $j \in S$ and $\tau(R_j) = b$ for all $j \notin S$. Then, $f(R) = \alpha$. Let $R'_i \in \mathcal{U}$ be such that $\alpha < \tau(R'_i) < p_{S\backslash\{i\}}$. Hence, $f(R'_i, R_{-i}) = \tau(R'_i) \neq \alpha$. Thus, i is non-dummy at α in f.

Case 2: Assume $p_S < \alpha \leq p_{S\backslash\{i\}}$. Let $R \in \mathcal{U}^n$ be such that $\tau(R_j) = p_S$ for all $j \in S\backslash\{i\}$, $\tau(R_i) = \alpha$ and $\tau(R_j) = b$ for all $j \notin S$. Then, $f(R) = \alpha$. Let $R'_i \in \mathcal{U}$ be such that $p_S < \tau(R'_i) < \alpha$. Hence, $f(R'_i, R_{-i}) = \tau(R'_i) \neq \alpha$. Thus, i is non-dummy at α in f. ■

Proof of Lemma 5 We will denote f^p and $o^p(R_i)$ simply by f and $o(R_i)$, respectively.

$\Rightarrow$) Assume f is not manipulable by i at R_i and let $I_i^S = [p_S, p_{S\backslash\{i\}}]$ be a non dummy interval for i in f. Fix $\alpha^* \in I_i^S$ and let $\beta \in (o(R_i) \cap I_i^S) \cup \{\tau(R_i)\}$. We distinguish among four cases.

Case 1: Assume $\alpha^* \in (o(R_i) \cap I_i^S) \cup \{\tau(R_i)\}$ and $\alpha^* < \beta \leq \tau(R_i)$ (if $\beta < \alpha^*$ the proof is similar changing the role of α^* and β). We will show that $\beta R_i \alpha^*$. If $\beta = \tau(R_i)$ the statement holds immediately. Assume $\beta < \tau(R_i)$. Then, $\alpha^*, \beta \in I_i^S$. Hence, and since $\alpha^* < \beta$, $p_S \leq \alpha^* < p_{S\backslash\{i\}}$. Consider any $R_{-i} \in \mathcal{U}^{n-1}$ with the property that for every $j \in N\backslash\{i\}$,

$$\tau(R_j) = \begin{cases} \alpha^* & \text{if } j \in S\backslash\{i\} \\ \beta & \text{if } j \in N\backslash S. \end{cases} \tag{17}$$

Let $\bar{R} \in \mathcal{U}^n$ be such that $\tau(\bar{R}_j) = \beta$ for all $j \in N\backslash\{i\}$ and $\tau(\bar{R}_i) = \tau(R_i)$. Since $\beta \in o(R_i) = o(\bar{R}_i)$, $f(\bar{R}) = \beta$. As $\tau(R_j) \leq \tau(\bar{R}_j)$ for $j \in N$, by Remark 1, $f(R) \leq f(\bar{R}) = \beta$. Moreover, $\alpha^* \leq f(R)$. Hence,

$$\alpha^* \leq f(R) \leq \beta.$$

If $S' \subset S\backslash\{i\}$, then $\alpha^* < p_{S\backslash\{i\}} \leq p_{S'}$ because p is monotonic. Hence, $\max_{j \in S'}\{\tau(R_j), p_{S'}\} > \alpha^*$. If $S' \nsubseteq S\backslash\{i\}$, then $\max_{j\in S'}\{\tau(R_j), p_{S'}\} \geq \beta > \alpha^*$. Thus, $\alpha^* < f(R) \leq \beta$. We proceed by distinguishing between two subcases.

Subcase 1.1: Assume $f(R) = \beta$. Consider any $\hat{R}_i \in \mathcal{U}$ such that $\tau(\hat{R}_i) = \alpha^*$. Since $\alpha^* < f(R)$, $\alpha^* \leq f(\hat{R}_i, R_{-i})$. Furthermore, since $p_S \leq \alpha^* = \tau(\hat{R}_i)$ and $\tau(R_j) = \alpha^*$ for all $j \in S\backslash\{i\}$, $f(\hat{R}_i, R_{-i}) \leq \alpha^*$. Hence, $f(\hat{R}_i, R_{-i}) = \alpha^*$. Since f is not manipulable by i at R_i, $\beta R_i \alpha^*$ holds.

Subcase 1.2: Assume $f(R) < \beta$. Then, $f(R) \notin \{\alpha^*, \beta, \tau(R_i)\} = \{\tau(R_j) \mid j \in N\}$. Thus, $f(R) \in \{p_S \mid S \subset N\}$. Set $R^1 \equiv R$ and $\alpha_1^* \equiv f(R^1)$. Observe that $\alpha^* < \alpha_1^* < \beta$ and since f is not manipulable by i at R_i, $\alpha_1^* = f(R^1)R_i\alpha^*$ (because $f(\hat{R}_i, R_{-i}^1) = \alpha^*$ if $\tau(\hat{R}_i) = \alpha^*$). Since $\{p_S \mid S \subset N\}$ is finite, we apply successively the previous argument starting with $\alpha_1^* < \beta$ and obtaining $R^1, R^2, ..., R^K$ where (i) $K \leq 2^n$, (ii) $R_i^k = R_i$ for all $k = 1, ..., K$, (iii) $\alpha^* < f(R^k) < f(R^{k+1}) < \beta$ for all $k = 1, ..., K-1$, (iv) $f(R^1)R_i\alpha^*$ and $f(R^k)R_if(R^{k-1})$ for all $k = 1, ..., K$, (v) $f(R^k) \in \{p_S \mid S \subset N\}$ and (vi) $f(R^K) = \beta$. Then, by transitivity of R_i, $\beta R_i \alpha^*$.

Case 2: Assume $\alpha^* \in (o(R_i) \cap I_i^S) \cup \{\tau(R_i)\}$ and $\tau(R_i) \leq \beta < \alpha^*$. The proof proceeds as in *Case 1* using the right phantom representation of f.

Case 3: Assume $\alpha^* \notin o(R_i)$ and $\alpha^* < \beta \leq \tau(R_i)$ (if $\tau(R_i) \leq \beta < \alpha^*$ the proof is similar using the right phantom representation of f). We will show that $\beta R_i \alpha^*$. If $\beta = \tau(R_i)$ the statement holds immediately. Assume $\beta < \tau(R_i)$ and consider any profile $\bar{R} \in \mathcal{U}^n$ where, for every $j \in N$, $\tau(\bar{R}_j) = \alpha^*$. Since $\alpha^* \in I_i^S \subset r_f$, $f(\bar{R}) = \alpha^*$. We will show that $\alpha^* \leq f(R_i, \bar{R}_{-i}) \leq \beta$. Let $\hat{R} = (R_i, \bar{R}_{-i})$. Since $\alpha^* \leq \tau(\hat{R}_j)$ for all $j \in N$, $\alpha^* \leq f(\hat{R})$. Consider any subprofile $\tilde{R}_{-i} \in \mathcal{U}^{n-1}$ where, for every $j \in N\backslash\{i\}$, $\tau(\tilde{R}_j) = \beta$. Since $\beta \in o(R_i)$, $f(R_i, \tilde{R}_{-i}) = \beta$. As $\tau(\bar{R}_j) = \alpha^* < \beta = \tau(\tilde{R}_j)$ for all $j \in N\backslash\{i\}$, by Remark 1, $f(R_i, \bar{R}_{-i}) \leq \beta$. Since f is not manipulable by i at R_i and $f(R_i, \bar{R}_{-i}) \neq \alpha^*$ (because $\alpha^* \notin o(R_i)$) we have that $f(R_i, \bar{R}_{-i})R_if(\bar{R}) = \alpha^*$. Define $\alpha' = f(R_i, \bar{R}_{-i})$. Notice that $\alpha' \leq \beta \leq \tau(R_i)$ and $\alpha' \in o(R_i) \cap I_i^S$. Therefore, by *Case 1*, $\beta R_i \alpha'$. By transitivity of R_i, $\beta R_i \alpha^*$.

Case 4: Assume $\alpha^* \notin o(R_i)$ and $\beta < \alpha^* \leq \tau(R_i)$. (if $\tau(R_i) \leq \alpha^* < \beta$ the proof is similar changing the role of α^* by β). We will show that this case is not possible. Consider any profile $R' \in \mathcal{U}^n$ such that $\tau(R'_j) = \alpha^*$ for all $j \in N$. Since $\alpha^* \notin o(R_i)$, $\beta \in o(R_i)$ and $o(R_i)$ is an interval, $f(R_i, R'_{-i}) < \alpha^*$. Furthermore, as $\alpha^* \leq \tau(R_i)$ and

Remark 1 holds, $f(R_i', R_{-i}') \leq f(R_i, R_{-i}') < \alpha^*$. Hence, $f(R') < \alpha^*$. Thus, $\alpha^* \notin r_f$ which contradicts the fact that $\alpha^* \in I_i^S$.

$\Leftarrow$) Assume f is manipulable by i at R_i. Then there exist $R_i' \in \mathcal{U}$ and $R_{-i} \in \mathcal{U}^{n-1}$ such that

$$f(R_i', R_{-i}) P_i f(R_i, R_{-i}). \tag{18}$$

We assume that $\tau(R_i') < \tau(R_i)$ (if $\tau(R_i) < \tau(R_i')$ the proof is similar using the right phantom representation of f). Set $R' = (R_i', R_{-i})$. We distinguish among three cases.

Case 1: Assume $\tau(R_i) < f(R)$. Since f is a generalized median voter scheme and $\tau(R_i') < \tau(R_i)$, $f(R') = f(R)$, which contradicts (18).

Case 2: Assume $\tau(R_i) = f(R)$. Then $f(R) R_i f(R')$, which also contradicts (18).

Case 3: Assume $f(R) < \tau(R_i)$. Since $\tau(R_i') < \tau(R_i)$, by Remark 1, $f(R') \leq f(R)$ and (18), $f(R') < f(R)$ holds. Hence, $f(R') < f(R) < \tau(R_i)$ and $\tau(R_i) P_i f(R') P_i f(R)$. Thus, as $f(R), \tau(R_i) \in o(R_i) \cup \{\tau(R_i)\}$, R_i is not single-peaked on $o(R_i) \cup \{\tau(R_i), f(R')\}$. We will show that there exists $S \subset N$ such that $i \in S$ and $f(R'), f(R) \in I_i^S = [p_S, p_{S\backslash\{i\}}]$. Set $\alpha^* \equiv f(R') < f(R) \equiv \beta$. Since $f(R') < f(R)$ and f is a generalized median voter scheme, $\tau(R_i') \leq f(R') = \alpha^*$ Define $\bar{S} = \{j \in N \mid \tau(R_j) \leq \alpha^*\}$. Then, $i \notin \bar{S}$ and because $\beta = f(R)$,

$$p_{\bar{S}} \geq \beta. \tag{19}$$

Set, $S \equiv \bar{S} \cup \{i\}$. Hence, $S = \{j \in N \mid \tau(R_j') \leq \alpha^*\}$. Suppose $p_S > \alpha^*$. Then, for all $S' \subset S$ $\max_{j \in S'}\{\tau(R_j'), p_{S'}\} \geq p_{S'} \geq p_S > \alpha^*$ and for all $S^* \nsubseteq S$, $\max_{j \in S^*}\{\tau(R_j'), p_{S^*}\} > \alpha^*$ because if $j \notin S$, then $\tau(R_j') > \alpha^*$. Thus, $\alpha^* < f(R')$, which is a contradiction. Hence $p_S \leq \alpha^*$. Therefore, $i \in S$ and

$$p_S \leq \alpha^* < \beta \leq p_{S\backslash\{i\}},$$

since $S\backslash\{i\} = \bar{S}$ and (19) hold. Thus, there exist a non dummy interval $[p_S, p_{S\backslash\{i\}}]$ and $\alpha^* = f(R') \in [p_S, p_{S\backslash\{i\}}]$ such that R_i is not single-peaked on $(o(R_i) \cap [p_S, p_{S\backslash\{i\}}]) \cup \{\tau(R_i), \alpha^*\}$. ■

Proof of Lemma 6 The proof is omitted since it consists of verifying that the option set can be written as stated. ■

Proof of Lemma 7 Let $i \in S \subset N$, I_i^S a non-dummy interval for i in f^p and $\alpha^* \in I_i^S$ be arbitrary. The proof proceeds by looking at different cases that can be grouped into two main cases depending on whether $p_{N\backslash\{i\}} \leq p_{\{i\}}$ (Case 1) or $p_{\{i\}} < p_{N\backslash\{i\}}$ (Case 2).

Case 1: Assume $p_{N\backslash\{i\}} \leq p_{\{i\}}$. Since $[p_N, p_{\{i\}}] \cup [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] = r_f$ and $I_i^S \subseteq ND_p^i \subset r_f$, either $\alpha^* \in [p_N, p_{\{i\}}] \cap ND_p^i$ or $\alpha^* \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i$. Hence, by (7) and (8), either $\alpha^* \in [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$ or $\alpha^* \in [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$. Thus, there exists a non-dummy interval $\bar{I}_i$ for i in $f^{\bar{p}}$ such that $\alpha^* \in \bar{I}_i$. Let $\hat{I}_i$ be a maximal non-dummy interval for i in $f^{\bar{p}}$ such that $\bar{I}_i \subset \hat{I}_i$. We have that $\alpha^* \in \hat{I}_i$. We will show that $(o^p(R_i) \cap I_i^S) \subset (o^{\bar{p}}(R_i) \cap \hat{I}_i)$ for all $R_i \in \mathcal{U}$, showing that for all $\beta \in o^p(R_i) \cap I_i^S$ two things happen simultaneously: $\beta \in \hat{I}_i$ (Claim a) and $\beta \in o^{\bar{p}}(R_i)$ (Claim A), for all $R_i \in \mathcal{U}$.

Claim a: $\beta \in \hat{I}_i$.

Proof of Claim a: We distinguish among five cases.

Case a.1: $\alpha^* \in [p_N, p_{\{i\}}] \backslash [p_{N\backslash\{i\}}, p_\emptyset]$ and $p_N \leq \beta \leq p_{\{i\}}$. Assume $\beta \leq \alpha^*$ (the proof of the other case proceeds similarly). As $\beta, \alpha^* \in [p_N, p_{\{i\}}] \cap I_i^S$ and I_i^S is a interval, $[\beta, \alpha^*] \subset [p_N, p_{\{i\}}] \cap I_i^S$. Hence, by (7), $[\beta, \alpha^*] \subset [p_N, p_{\{i\}}] \cap ND_p^i \subset [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$. Then, $[\beta, \alpha^*] \subset ND_{\bar{p}}^i$. As $\hat{I}_i$ is a maximal non-dummy interval and $\alpha^* \in \hat{I}_i$, $[\beta, \alpha^*] \subset \hat{I}_i$. Therefore, $\beta \in \hat{I}_i$.

Case a.2: $\alpha^* \in [p_N, p_{\{i\}}] \backslash [p_{N\backslash\{i\}}, p_{\{\emptyset\}}]$ and $p_{\{i\}} < \beta \leq p_{\{\emptyset\}}$. As $p_N \leq \alpha^* < p_{N\backslash\{i\}}$, $p_{\{i\}} < \beta \leq p_{\{\emptyset\}}$ and I_i^S is a non-dummy interval such that $\alpha^*, \beta \in I_i^S$, we have that by Remark 4, $ND_p^i = [p_N, p_{\{\emptyset\}}]$. Then, by (7) and (8), $[p_N, p_{\{i\}}] \subset [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$ and $[p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$. Hence, $[p_N, p_{\{i\}}] \cup [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \subset ND_{\bar{p}}^i$. Thus, $[\alpha^*, \beta] \subset ND_{\bar{p}}^i$. As $\hat{I}_i$ is a maximal non-dummy interval and $\alpha^* \in \hat{I}_i$, $[\alpha^*, \beta] \subset \hat{I}_i$. Therefore, $\beta \in \hat{I}_i$.

Case a.3: $\alpha^* \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \backslash [p_N, p_{\{i\}}]$ and $p_{N\backslash\{i\}} \leq \beta \leq p_{\{\emptyset\}}$. Assume $\alpha^* < \beta$ (the proof of the other case proceeds similarly). Since $\beta, \alpha^* \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap I_i^S$ and I_i^S is an interval, by (8), $[\alpha^*, \beta] \subset [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap I_i^S \subset [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$. Hence, $[\alpha^*, \beta] \subset ND_{\bar{p}}^i$. As $\hat{I}_i$ is a maximal non-dummy interval and $\alpha^* \in \hat{I}_i$, $[\alpha^*, \beta] \subset \hat{I}_i$. Therefore, $\beta \in \hat{I}_i$.

Case a.4: $\alpha^* \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \backslash [p_N, p_{\{i\}}]$ and $p_N \leq \beta < p_{N\backslash\{i\}}$. Since $p_{\{i\}} < \alpha^* \leq p_{\{\emptyset\}}$, $p_N \leq \beta < p_{N\backslash\{i\}}$ and I_i^S is a non-dummy interval such that $\alpha^*, \beta \in I_i^S$, by Remark 4, $ND_p^i = [p_N, p_{\{\emptyset\}}]$. Hence, by (7) and (8) $[p_N, p_{\{i\}}] \subset [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$ and $[p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$. Hence, $[p_N, p_{\{i\}}] \cup [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \subset ND_{\bar{p}}^i$ and $[\beta, \alpha^*] \subset ND_{\bar{p}}^i$. As $\hat{I}_i$ is a maximal non-dummy interval and $\alpha^* \in \hat{I}_i$, $[\beta, \alpha^*] \subset \hat{I}_i$. Therefore, $\beta \in \hat{I}_i$.

Case a.5: $\alpha^* \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap [p_N, p_{\{i\}}]$. Hence, $\alpha^* \in [p_{N\backslash\{i\}}, p_{\{i\}}] \cap ND_p^i$. Thus, by (7) and (8), $\alpha^* \in [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_p^i$. Assume $\alpha^* < \beta$ (the proof of the other case proceeds similarly). Since $p_{N\backslash\{i\}} < \alpha^* < \beta \leq p_{\{\emptyset\}}$, and I_i^S is a an interval, $[\alpha^*, \beta] \subset [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap I_i^S$. Hence, by (8), $[\alpha^*, \beta] \subset [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$. Thus $[\alpha^*, \beta] \subset ND_{\bar{p}}^i$. As $\hat{I}_i$ is a maximal non-dummy interval and $\alpha^* \in \hat{I}_i$, $[\alpha^*, \beta] \subset \hat{I}_i$. Therefore, $\beta \in \hat{I}_i$.

Claim A: $\beta \in o^{\bar{p}}(R_i)$.

Proof of Claim A: We proceed by first distinguishing between Case A.1 and Case A.2, and in turn for each one of them, the proof is divided in 5 subcases.

Case A.1: $\bar{p}_{N\setminus\{i\}} \leq \bar{p}_{\{i\}}$. By Lemma 6,

$$o^p(R_i) = \begin{cases} [p_N, p_{\{i\}}] & \text{if } a \leq \tau(R_i) \leq p_N \\ [\tau(R_i), p_{\{i\}}] & \text{if } p_N < \tau(R_i) \leq p_{N\setminus\{i\}} \\ [p_{N\setminus\{i\}}, p_{\{i\}}] & \text{if } p_{N\setminus\{i\}} < \tau(R_i) \leq p_{\{i\}} \\ [p_{N\setminus\{i\}}, \tau(R_i)] & \text{if } p_{\{i\}} < \tau(R_i) \leq p_{\{\emptyset\}} \\ [p_{N\setminus\{i\}}, p_{\{\emptyset\}}] & \text{if } p_{\{\emptyset\}} < \tau(R_i). \end{cases}$$

and

$$o^{\bar{p}}(R_i) = \begin{cases} [\bar{p}_N, \bar{p}_{\{i\}}] & \text{if } a \leq \tau(R_i) \leq \bar{p}_N \\ [\tau(R_i), \bar{p}_{\{i\}}] & \text{if } \bar{p}_N < \tau(R_i) \leq \bar{p}_{N\setminus\{i\}} \\ [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\{i\}}] & \text{if } \bar{p}_{N\setminus\{i\}} < \tau(R_i) \leq \bar{p}_{\{i\}} \\ [\bar{p}_{N\setminus\{i\}}, \tau(R_i)] & \text{if } \bar{p}_{\{i\}} < \tau(R_i) \leq \bar{p}_{\{\emptyset\}} \\ [\bar{p}_{N\setminus\{i\}}, p_{\{\emptyset\}}] & \text{if } \bar{p}_{\{\emptyset\}} < \tau(R_i). \end{cases} \tag{20}$$

Case A.1.1: $a \leq \tau(R_i) \leq p_N$. Then, $\beta \in [p_N, p_{\{i\}}]$. Since $\beta \in I_i^S$, $\beta \in [p_N, p_{\{i\}}] \cap ND_p^i$. By (7), $\beta \in [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\beta \in [\tau(R_i), \bar{p}_{\{i\}}]$ and $\bar{p}_N \leq \beta$. Therefore, by the first three rows in (20), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case A.1.2: $p_N < \tau(R_i) \leq p_{N\setminus\{i\}}$. Then, $\beta \in [\tau(R_i), p_{\{i\}}]$. Since $\beta \in I_i^S$, $\beta \in [p_N, p_{\{i\}}] \cap ND_p^i$. By (7), $\beta \in [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\beta \in [\tau(R_i), \bar{p}_{\{i\}}]$ and $\bar{p}_N \leq \beta$. Therefore, by the first three rows in (20), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case A.1.3: $p_{N\setminus\{i\}} < \tau(R_i) \leq p_{\{i\}}$. Then, $\beta \in [p_{N\setminus\{i\}}, p_{\{i\}}]$. Since $\beta \in I_i^S$, $\beta \in [p_{N\setminus\{i\}}, p_{\{i\}}] \cap ND_p^i$. By (7) and (8), $\beta \in [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\{i\}}]$. By (20), $\beta \in o^{\bar{p}}(R_i)$.

Case A.1.4: $p_{\{i\}} < \tau(R_i) \leq p_{\{\emptyset\}}$. Then, $\beta \in [p_{N\setminus\{i\}}, \tau(R_i)]$. Since $\beta \in I_i^S$, $\beta \in [p_{N\setminus\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i$. By (8), $\beta \in [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\emptyset}]$. Then, $\beta \in [\bar{p}_{N\setminus\{i\}}, \tau(R_i)]$ and $\beta \leq \bar{p}_{\{\emptyset\}}$. Therefore, by the last three rows in (20), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case A.1.5: $p_{\{\emptyset\}} < \tau(R_i)$. Then, $\beta \in [p_{N\setminus\{i\}}, p_{\{\emptyset\}}]$. Since $\beta \in I_i^S$, $\beta \in [p_{N\setminus\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i$. By (8), $\beta \in [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\emptyset}]$. Then, $\beta \in [\bar{p}_{N\setminus\{i\}}, \tau(R_i)]$ and $\beta \leq \bar{p}_{\{\emptyset\}}$. Therefore, by the last three rows in (20), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case A.2: $\bar{p}_{\{i\}} < \bar{p}_{N\setminus\{i\}}$. By Lemma 6,

$$o^{\bar{p}}(R_i) = \begin{cases} [\bar{p}_N, \bar{p}_{\{i\}}] & \text{if } a \leq \tau(R_i) \leq \bar{p}_N \\ [\tau(R_i), \bar{p}_{\{i\}}] & \text{if } \bar{p}_N < \tau(R_i) \leq \bar{p}_{\{i\}} \\ \{\tau(R_i)\} & \text{if } \bar{p}_{\{i\}} < \tau(R_i) \leq \bar{p}_{N\setminus\{i\}} \\ [\bar{p}_{N\setminus\{i\}}, \tau(R_i)] & \text{if } \bar{p}_{N\setminus\{i\}} < \tau(R_i) \leq \bar{p}_{\{\emptyset\}} \\ [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\{\emptyset\}}] & \text{if } \bar{p}_{\{\emptyset\}} < \tau(R_i). \end{cases} \tag{21}$$

Case A.2.1: $a \leq \tau(R_i) \leq p_N$. The proof proceeds as in Case A.1.1.

Case A.2.2: $p_N < \tau(R_i) \leq p_{N\setminus\{i\}}$. The proof proceeds as in Case A.1.2.

Case A.2.3: $p_{N\setminus\{i\}} < \tau(R_i) \leq p_{\{i\}}$. Then, $\beta \in [p_{N\setminus\{i\}}, p_{\{i\}}]$. By (7), (8) and $\beta \in ND^i_p$, $\beta \in [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\{i\}}] \cap ND^i_{\bar{p}}$, contradicting that $\bar{p}_{\{i\}} < \bar{p}_{N\setminus\{i\}}$. Then, in this case, $I^S_i \cap o^p(R_i) = \emptyset$ and the proof is trivial.

Case A.2.4: $p_{\{i\}} < \tau(R_i) \leq p_{\{\emptyset\}}$. The proof proceeds as in Case A.1.4.

Case A.2.5: $p_{\{\emptyset\}} < \tau(R_i)$. The proof proceeds as in Case A.1.5.

Case 2: Assume $p_{\{i\}} < p_{N\setminus\{i\}}$. Then, by Remark 5, $ND^i_p = [p_N, p_{\{\emptyset\}}]$ and itself is a maximal non-dummy interval for i in f^p. As $[p_N, p_{\{\emptyset\}}] = [p_N, p_{\{i\}}] \cup [p_{\{i\}}, p_{N\setminus\{i\}}] \cup [p_{N\setminus\{i\}}, p_{\{\emptyset\}}]$, by (7), (8) and (9), we have that there exists a non-dummy interval for i in $f^{\bar{p}}$ such that $[p_N, p_{\{\emptyset\}}] \subset \hat{I}_i$. Let $\alpha^* \in [p_N, p_{\{\emptyset\}}]$ be arbitrary. Then, $\alpha^* \in \hat{I}_i$. We will show that

$$(o^p(R_i) \cap [p_N, p_{\{\emptyset\}}]) \subset (o^{\bar{p}}(R_i) \cap \hat{I}_i) \text{ for all } R_i \in \mathcal{U}. \tag{22}$$

Then, and since $I^S_i \subset [p_N, p_{\{\emptyset\}}]$ for any $S \subset N$, the statement of Lemma 7 will follow immediately since $(o^p(R_i) \cap I^S_i) \subset (o^p(R_i) \cap [p_N, p_{\{\emptyset\}}]) \subset (o^{\bar{p}}(R_i) \cap \hat{I}_i)$. To prove that (22) holds observe first that $o^p(R_i) \cap [p_N, p_{\{\emptyset\}}] \subset \hat{I}_i$. It remains to be proven that if $\beta \in o^p(R_i) \cap [p_N, p_{\{\emptyset\}}]$, then $\beta \in o^{\bar{p}}(R_i)$. We proceed by distinguishing between two cases.

Case 2.1: $\bar{p}_{N\setminus\{i\}} \leq \bar{p}_{\{i\}}$. By Lemma 6,

$$o^p(R_i) = \begin{cases} [p_N, p_{\{i\}}] & \text{if } a \leq \tau(R_i) \leq p_N \\ [\tau(R_i), p_{\{i\}}] & \text{if } p_N < \tau(R_i) \leq p_{\{i\}} \\ \{\tau(R_i)\} & \text{if } p_{\{i\}} < \tau(R_i) \leq p_{N\setminus\{i\}} \\ [p_{N\setminus\{i\}}, \tau(R_i)] & \text{if } p_{N\setminus\{i\}} < \tau(R_i) \leq p_{\{\emptyset\}} \\ [p_{N\setminus\{i\}}, p_{\{\emptyset\}}] & \text{if } p_{\{\emptyset\}} < \tau(R_i) \end{cases} \tag{23}$$

and

$$o^{\bar{p}}(R_i) = \begin{cases} [\bar{p}_N, \bar{p}_{\{i\}}] & \text{if } a \leq \tau(R_i) \leq \bar{p}_N \\ [\tau(R_i), \bar{p}_{\{i\}}] & \text{if } \bar{p}_N < \tau(R_i) \leq \bar{p}_{N\setminus\{i\}} \\ [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\{i\}}] & \text{if } \bar{p}_{N\setminus\{i\}} < \tau(R_i) \leq \bar{p}_{\{i\}} \\ [\bar{p}_{N\setminus\{i\}}, \tau(R_i)] & \text{if } \bar{p}_{\{i\}} < \tau(R_i) \leq \bar{p}_{\{\emptyset\}} \\ [\bar{p}_{N\setminus\{i\}}, \bar{p}_{\{\emptyset\}}] & \text{if } \bar{p}_{\{\emptyset\}} < \tau(R_i). \end{cases} \tag{24}$$

We distinguish among five subcases.

Case 2.1.1: $a \leq \tau(R_i) \leq p_N$. Then $\beta \in [p_N, p_{\{i\}}]$. Since $\beta \in I^S_i$, $\beta \in [p_N, p_{\{i\}}] \cap ND^i_p$. By (7), $\beta \in [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\beta \in [\tau(R_i), \bar{p}_{\{i\}}]$ and $\bar{p}_N \leq \beta$. Therefore, by the first three rows in (24), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case 2.1.2: $p_N < \tau(R_i) \leq p_{\{i\}}$. Then, $\beta \in [\tau(R_i), p_{\{i\}}]$. Since $\beta \in I^S_i$, $\beta \in [p_N, p_{\{i\}}] \cap ND^i_p$. By (7), $\beta \in [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\beta \in [\tau(R_i), \bar{p}_{\{i\}}]$ and $\bar{p}_N \leq \beta$. Therefore, by the first three rows in (24), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case 2.1.3: $p_{\{i\}} < \tau(R_i) \leq p_{N\backslash\{i\}}$. Then, $\beta = \tau(R_i) \in [p_N, p_{\{\emptyset\}}] \subset \hat{I}_i \subset [\bar{p}_N, \bar{p}_{\{\emptyset\}}]$. Since $\beta = \tau(R_i) \in [\bar{p}_N, \bar{p}_\emptyset]$, $\beta \in o^{\bar{p}}(R_i)$ because $f^{\bar{p}}$ in unanimous on $r_{f^{\bar{p}}} = [\bar{p}_N, \bar{p}_\emptyset]$.

Case 2.1.4: $p_{N\backslash\{i\}} < \tau(R_i) \leq p_{\{\emptyset\}}$. Then, $\beta \in [p_{N\backslash\{i\}}, \tau(R_i)]$. Since $\beta \in I_i^S$, $\beta \in [p_{N\backslash\{i\}}, p_\emptyset] \cap ND_p^i$. By (8), $\beta \in [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}]$. Then, $\beta \in [\bar{p}_{N\backslash\{i\}}, \tau(R_i)]$ and $\beta \leq \bar{p}_{\{\emptyset\}}$. Therefore, by the last three rows in (24), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case 2.1.5: $p_{\{\emptyset\}} < \tau(R_i)$. Then, $\beta \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}]$. Since $\beta \in I_i^S$, $\beta \in [p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i$. By (8), $\beta \in [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}]$. Then, $\beta \in [\bar{p}_{N\backslash\{i\}}, \tau(R_i)]$ and $\beta \leq \bar{p}_{\{\emptyset\}}$. Therefore, by the last three rows in (24), $\beta \in o^{\bar{p}}(R_i)$ holds.

Case 2.2: $\bar{p}_{\{i\}} < \bar{p}_{N\backslash\{i\}}$. By Lemma 6,

$$o^{\bar{p}}(R_i) = \begin{cases} [\bar{p}_N, \bar{p}_{\{i\}}] & \text{if } a \leq \tau(R_i) \leq \bar{p}_N \\ [\tau(R_i), \bar{p}_{\{i\}}] & \text{if } \bar{p}_N < \tau(R_i) \leq \bar{p}_{\{i\}} \\ \{\tau(R_i)\} & \text{if } \bar{p}_{\{i\}} < \tau(R_i) \leq \bar{p}_{N\backslash\{i\}} \\ [\bar{p}_{N\backslash\{i\}}, \tau(R_i)] & \text{if } \bar{p}_{N\backslash\{i\}} < \tau(R_i) \leq \bar{p}_{\{\emptyset\}} \\ [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] & \text{if } \bar{p}_{\{\emptyset\}} < \tau(R_i). \end{cases} \tag{25}$$

The proof follows similar arguments to the ones already used in Case 2.1. ■

Proof of Theorem 2

$\Rightarrow$) Suppose $R_i \in \mathcal{M}_i^{f^p}$. By Lemma 5, there exist a non-dummy interval $I_i^S = [p_S, p_{S\backslash\{i\}}]$ for i in f^p and $\alpha^* \in I_i^S$ such that R_i is not single-peaked on $(o^p(R_i) \cap I_i^S) \cup \{\tau(R_i), \alpha^*\}$. Hence, by Lemma 7, there exists a maximal non-dummy interval $\hat{I}_i$ for i in $f^{\bar{p}}$ such that $\alpha^* \in \hat{I}_i$ and $(o^p(R_i) \cap I_i^S) \cup \{\tau(R_i), \alpha^*\} \subset (o^{\bar{p}}(R_i) \cap \hat{I}_i) \cup \{\tau(R_i), \alpha^*\}$. Thus, R_i is not single-peaked on $(o^{\bar{p}}(R_i) \cap \hat{I}_i) \cup \{\tau(R_i), \alpha^*\}$. Then by Lemma 5 and Remark 2, $R_i \in \mathcal{M}_i^{f^{\bar{p}}}$.

$\Leftarrow$) Assume $f^{\bar{p}}$ is at least as manipulable as f^p. Then,

$$\mathcal{M}_i^{f^p} \subset \mathcal{M}_i^{f^{\bar{p}}} \text{ for all } i \in N. \tag{26}$$

To obtain a contradiction assume $[p_N, p_{\{i\}}] \cap ND_p^i \not\subseteq [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$ or $[p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i \not\subseteq [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$ or $[p_{\{i\}}, p_{N\backslash\{i\}}] \not\subseteq ND_{\bar{p}}^i$. We proceed by distinguishing among the three cases.

Case 1: $[p_N, p_{\{i\}}] \cap ND_p^i \not\subseteq [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$. Then, there exists a maximal non-dummy interval I for i in f^p such that $[p_N, p_{\{i\}}] \cap I \not\subseteq [\bar{p}_N, \bar{p}_{\{i\}}] \cap ND_{\bar{p}}^i$. Let $\sigma_1 \leq \sigma_2$ be such that $[p_N, p_{\{i\}}] \cap I = [\sigma_1, \sigma_2]$. Let $\{\bar{I}_t^i\}_{t=1,...,T}$ be the collection of all maximal non-dummy intervals for i in $f^{\bar{p}}$; in particular, by the definition of $ND_{\bar{p}}^i$ and the fact that they are maximal intervals, $ND_{\bar{p}}^i = \bigcup_{t=1,...,T} \bar{I}_t^i$ and for all $t, t' = 1, ..., T$ such that $t \neq t'$, $\bar{I}_t^i \cap \bar{I}_{t'}^i = \emptyset$. Then, for any maximal non-dummy interval $\bar{I}_t^i$ for i in $f_{\bar{p}}$ we have that

$$[\sigma_1, \sigma_2] \not\subseteq [\bar{p}_N, \bar{p}_{\{i\}}] \cap \bar{I}_t^i. \tag{27}$$

We distinguish between two subcases.

Case 1.a: $ND^i_{\bar{p}} = \emptyset$. Two further subcases are possible.

Case 1.a.1: $\sigma_1 < \sigma_2$. Let $\alpha, \beta, \gamma \in [a,b]$ and $R_i \in \mathcal{U}$ be such that $\sigma_1 < \alpha < \beta < \gamma < \sigma_2$, $\tau(R_i) = \alpha$, and $\gamma P_i \beta$.[10] Hence, $\tau(R_i) \in [\sigma_1, \sigma_2] \subset [p_N, p_{\{i\}}]$. By Lemma 6,

$$o^p(R_i) = \begin{cases} [p_{N\backslash\{i\}}, p_{\{i\}}] \text{ if } p_{N\backslash\{i\}} < \tau(R_i) \leq p_{\{i\}} \\ [\tau(R_i), p_{\{i\}}] \text{ otherwise.} \end{cases}$$

Then, and because $\beta, \gamma \in [\sigma_1, \sigma_2] \subset I$ and $\beta, \gamma \in [\tau(R_i), \sigma_2] \subset [\tau(R_i), p_{\{i\}}] \subset o^p(R_i)$, R_i is not single-peaked on $(o^p(R_i) \cap I) \cup \{\tau(R_i)\}$ since $\gamma P_i \beta$. But for all $t = 1, ..., T$ and all $\alpha' \in \bar{I}^i_t$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}^i_t) \cup \{\tau(R_i), \alpha'\}$ trivially since $o^{\bar{p}}(R_i) \cap \bar{I}^i_t = \emptyset$. Thus, by Lemma 5, $R_i \in \mathcal{M}^{f^p}_i \backslash \mathcal{M}^{f^{\bar{p}}}_i$ which contradicts (26).

Case 1.a.2: $\sigma_1 = \sigma_2$. Since $I \subset [p_N, p_{\{\emptyset\}}]$, $[p_N, p_{\{i\}}] \cap I = \{\sigma_1\}$ and I is a (non degenerated) interval (since I is a non-dummy interval), $p_{\{i\}} = \sigma_1 = \sigma_2$. Therefore, $I = [p_{\{i\}}, p_{\{\emptyset\}}]$ because $I \subset [p_N, p_{\{\emptyset\}}]$, I is a maximal non-dummy interval and by Remark 3, $[p_{\{i\}}, p_{\{\emptyset\}}]$ is a non-dummy interval of i in f^p. Hence, as I is a non degenerated interval,

$$p_{\{i\}} = \sigma_1 < p_{\{\emptyset\}}.$$

Two subcases are possible.

Case 1.a.2.a: $p_{N\backslash\{i\}} < p_{\{\emptyset\}}$. Let $\alpha, \beta, \gamma \in [a,b]$ and $R_i \in \mathcal{U}$ be such that $\max\{p_{N\backslash\{i\}}, p_{\{i\}}\} < \alpha < \beta < \gamma < p_{\{\emptyset\}}$, $\tau(R_i) = \gamma$, and $\alpha P_i \beta$.[11] Hence, $\tau(R_i) \in [\max\{p_{N\backslash\{i\}}, p_{\{i\}}\}, p_{\{\emptyset\}}]$. By Lemma 6,

$$o^p(R_i) = [p_{N\backslash\{i\}}, \tau(R_i)].$$

Then, and because $\alpha, \beta, \tau(R_i) \in o^p(R_i) \cap I \cup \{\tau(R_i)\}$, R_i is not single-peaked on $o^p(R_i) \cap I \cup \{\tau(R_i)\}$ since $\alpha P_i \beta$. But for $t = 1, ..., T$ and all $\alpha \in \bar{I}^i_t$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}^i_t) \cup \{\tau(R_i), \alpha'\}$ trivially since $o^{\bar{p}}(R_i) \cap \bar{I}^i_t = \emptyset$. Thus, by Lemma 5, $R_i \in \mathcal{M}^{f^p}_i \backslash \mathcal{M}^{f^{\bar{p}}}_i$ which contradicts (26).

Case 1.a.2.b: $p_{\{i\}} < p_{N\backslash\{i\}} = p_{\{\emptyset\}}$. Then, by Remark 5, $[p_N, p_{\{\emptyset\}}]$ is a non-dummy interval of i in f^p. As $I = [p_{\{i\}}, p_{\{\emptyset\}}]$ is a maximal non-dummy interval of i in f^p, we must have $I = [p_N, p_{\{\emptyset\}}]$. Therefore, $p_N = p_{\{i\}}$. Hence, $p_N = p_{\{i\}}$ and $p_{N\backslash\{i\}} = p_{\{\emptyset\}}$. By Remark 6, i is a dictator in f^p, which is a contradiction.

Case 1.b: $ND^i_{\bar{p}} \neq \emptyset$. Then, $[\bar{p}_N, \bar{p}_{\{i\}}] \cap \bar{I}^i_t \neq \emptyset$ for all $t = 1, ..., T$. To see that, observe that it holds immediately if $\bar{p}_{\{i\}} = \bar{p}_{\{\emptyset\}}$. Assume $\bar{p}_{\{i\}} < \bar{p}_{\{\emptyset\}}$. Then, there exists $\bar{I}^i_{t'} \supseteq \bar{I}^{\{i\}}_i = [\bar{p}_{\{i\}}, \bar{p}_{\{\emptyset\}}]$ because, by Remark 3, $[\bar{p}_{\{i\}}, \bar{p}_{\{\emptyset\}}]$ is a non-dummy interval for i in $f^{\bar{p}}$. Then, $[\bar{p}_N, \bar{p}_{\{i\}}] \cap \bar{I}^i_{t'} \neq \emptyset$. Furthermore, for all $t \neq t'$, $[\bar{p}_N, \bar{p}_{\{i\}}] \cap \bar{I}^i_t \neq \emptyset$, since

[10] R_i is defined in any arbitrary way in $[a,b] \backslash \{\gamma, \beta\}$.

[11] R_i is defined in any arbitrary way in $[a,b] \backslash \{\alpha, \beta\}$.

$\bar{I}_t^i \cap \bar{I}_i^{t'} = \emptyset$. For each $t = 1, ..., T$, let $\eta_1^t \leq \eta_2^t$ be such that $[\bar{p}_N, \bar{p}_{\{i\}}] \cap \bar{I}_t^i = [\eta_1^t, \eta_2^t]$. Then, by (27), $[\sigma_1, \sigma_2] \nsubseteq [\eta_1^t, \eta_2^t]$ for all $t = 1, ..., T$. Hence,

$$\sigma_1 < \eta_1^t \text{ or } \sigma_2 > \eta_2^t \text{ for all } t = 1, ..., T. \tag{28}$$

Assume, without loss of generality, that $\eta_1^1 < \eta_1^2 < ... < \eta_1^T$ (and $\eta_2^1 < \eta_2^2 < ... < \eta_2^T$). We distinguish among four different cases.

Case 1.b.1: There exists $t' \in \{1, ..., T\}$ such that $\sigma_1 < \eta_1^{t'} \leq \sigma_2 \leq \eta_2^{t'}$. This t' is unique, because the family $\{\bar{I}_t^i\}_{t=1,...,T}$ is pair-wise disjoint. Let

$$\eta_2 = \begin{cases} \max\{\alpha \in \bar{I}_{t'-1}^i\} & \text{if } t' \neq 1 \\ a & \text{if } t' = 1 \end{cases}$$

and

$$\eta_1 = \begin{cases} \min\{\alpha \in \bar{I}_{t'+1}^i\} & \text{if } t' \neq T \\ b & \text{if } t' = T. \end{cases}$$

Thus, $\eta_2 < \eta_1^{t'}$ (if $\eta_2 \neq a$, then proof is trivial and if $\eta_2 = a$, then $a \leq \sigma_1 < \eta_1^{t'}$) and $\eta_1 \geq \eta_2^{t'}$. Let $R_i \in \mathcal{U}$ and $\alpha, \beta, \gamma \in [a, b]$ be such that (i) $\max\{\sigma_1, \eta_2\} < \alpha < \beta < \gamma < \eta_1^{t'}$, (ii) $\tau(R_i) = \alpha$, (iii) $\gamma P_i \beta$, (iv) if $\rho, \delta \in [a, b]$ and $\eta_1^{t'} < \rho < \delta$, then $\eta_1^{t'} R_i \rho R_i \delta$, and (v) if $\rho, \delta \in [a, b]$ and $\delta < \rho < \max\{\sigma_1, \eta_2\}$, then $\max\{\sigma_1, \eta_2\} R_i \delta R_i \rho$.[12] Hence, $\tau(R_i) \in (\sigma_1, \sigma_2) \subset [p_N, p_{\{i\}}]$ and $\tau(R_i) < \eta_1^{t'} < \bar{p}_{\{i\}}$, where the last inequality follows from the fact that $[\bar{p}_N, \bar{p}_{\{i\}}] \cap \bar{I}_{t'}^i = [\eta_1^{t'}, \eta_2^{t'}]$. By Lemma 6, and since if $p_{\{i\}} < p_{N \setminus \{i\}}$ then $p_N \leq \sigma_1 \leq \tau(R_i) \leq \sigma_2 \leq p_{\{i\}} < p_{N \setminus \{i\}}$, and if $\bar{p}_{\{i\}} < \bar{p}_{N \setminus \{i\}}$ then $\leq \tau(R_i) \leq \eta_1^{t'} \leq \bar{p}_{\{i\}} < \bar{p}_{N \setminus \{i\}}$,

$$o^p(R_i) = \begin{cases} [p_{N \setminus \{i\}}, p_{\{i\}}] & \text{if } p_{N \setminus \{i\}} < \tau(R_i) \leq p_{\{i\}} \\ [\tau(R_i), p_{\{i\}}] & \text{otherwise} \end{cases} \quad \text{and}$$

$$o^{\bar{p}}(R_i) = \begin{cases} [\bar{p}_N, \bar{p}_{\{i\}}] & \text{if } \tau(R_i) < \bar{p}_N \\ [\bar{p}_{N \setminus \{i\}}, \bar{p}_{\{i\}}] & \text{if } \bar{p}_{N \setminus \{i\}} < \tau(R_i) \leq \bar{p}_{\{i\}} \\ [\tau(R_i), \bar{p}_{\{i\}}] & \text{otherwise.} \end{cases} \tag{29}$$

Then, R_i is not single-peaked on $(o^p(R_i) \cap I) \cup \{\tau(R_i)\}$ because $\beta, \gamma \in [\sigma_1, \sigma_2] \subset I$ and $\beta, \gamma \in [\tau(R_i), \sigma_2] \subset [\tau(R_i), p_{\{i\}}] \subset o^p(R_i)$. We will now show that, for all $t = 1, ..., T$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}_t^i) \cup \{\tau(R_i), \alpha'\}$ for all $\alpha' \in \bar{I}_t^i$. We distinguish between two subcases.

Case 1.b.1.a: $t \neq t'$. By the definition of R_i and the fact that either $\bar{I}_t^i \subset [\bar{p}_N, \eta_2] \subset [\bar{p}_N, \max\{\sigma_1, \eta_2\}]$ or $\bar{I}_t^i \subset [\eta_1, \bar{p}_{\{\emptyset\}}] \subset [\eta_1, \bar{p}_{\{\emptyset\}}] \subset [\eta_1^{t'}, \bar{p}_{\{\emptyset\}}]$, R_i is single-peaked on $\bar{I}_t^i \cup \{\tau(R_i)\}$. Thus, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}_t^i) \cup \{\tau(R_i), \alpha'\}$ for all $\alpha' \in \bar{I}_t^i$.

[12] R_i is defined in any arbitrary way in $[\max\{\sigma_1, \eta_2\}, \eta_1^{t'}] \setminus \{\gamma, \beta\}$.

Case 1.b.1.b: $t = t'$. By (29), $o^{\bar{p}}(R_i) \subset [\bar{p}_N, \bar{p}_{\{i\}}]$. Hence, $o^{\bar{p}}(R_i) \cap \bar{I}^i_{t'} \subset [\eta_1^{t'}, \eta_2^{t'}]$. Thus, by its definition, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}^i_{t'}) \cup \{\tau(R_i)\}$. Let $\alpha' \in \bar{I}^i_{t'}$. Two further subcases are distinguished.

Case 1.b.1.b.1: $\alpha' \in [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\alpha' \in [\eta_1^{t'}, \eta_2^{t'}]$ because $\alpha' \in \bar{I}^i_{t'}$. Hence, by the definition of R_i and the fact that $o^{\bar{p}}(R_i) \cap \bar{I}^i_{t'} \subset [\eta_1^{t'}, \eta_2^{t'}]$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}^i_{t'}) \cup \{\tau(R_i), \alpha'\}$.

Case 1.b.1.b.2: $\alpha' \notin [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\alpha' > \bar{p}_{\{i\}} \geq \eta_2^{t'} \geq \eta_1^{t'}$. Hence, by the definition of R_i and the fact that $o^{\bar{p}}(R_i) \cap \bar{I}^i_{t'} \subset [\eta_1^{t'}, \eta_2^{t'}]$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}^i_{t'}) \cup \{\tau(R_i), \alpha'\}$.

Then, by Lemma 5, $R_i \in \mathcal{M}_i^{f^p} \backslash \mathcal{M}_i^{f^{\bar{p}}}$ which contradicts (26).

Case 1.b.2: There exists $t' \in \{1, ..., T\}$ such that $\eta_1^{t'} \leq \sigma_1 \leq \eta_2^{t'} < \sigma_2$. This t' is unique, because the family $\{\bar{I}^i_t\}_{t=1,...,T}$ is pair-wise disjoint. The proof of this case is similar to *Case 1.b.1*, because the problem is symmetric, and therefore it is omitted.

Case 1.b.3: $[\sigma_1, \sigma_2] \cap [\eta_1^t \eta_2^t] = \emptyset$ for all $t \in \{1, ..., T\}$. The proof of this case is similar to *Case 1.a* and therefore it is omitted.

Case 1.b.4: Assume that neither *Case 1.b.1* nor *Case 1.b.2* nor *Case 1.b.3* hold. By (28), for all $t \in \{1, ..., T\}$,

$$\eta_1^t > \sigma_1 \text{ and } \eta_2^t < \sigma_2.$$

Let $\eta_1 = \eta_1^1$ and $\eta_2 = \eta_2^T$. Then,

$$\sigma_1 < \eta_1 \leq \eta_2 < \sigma_2.$$

Let $R_i \in \mathcal{U}$ and $\alpha, \beta, \gamma \in [a, b]$ be such that (i) $\sigma_1 < \alpha < \beta < \gamma < \eta_1$ (ii) $\tau(R_i) = \alpha$, (iii) $\gamma P_i \beta$, and (iv) if $\rho, \delta \in [a, b]$ and $\eta_1 < \rho < \delta$, then $\eta_1 R_i \rho R_i \delta$.[13] Hence, $\tau(R_i) \in [\sigma_1, \sigma_2] \subset [p_N, p_{\{i\}}]$ and $\tau(R_i) < \eta_1 \leq \bar{p}_{\{i\}}$. By Lemma 6, and similarly as in *Case 1.b.1*,

$$o^p(R_i) = \begin{cases} [p_{N\backslash\{i\}}, p_{\{i\}}] & \text{if } p_{N\backslash\{i\}} < \tau(R_i) \leq p_{\{i\}} \\ [\tau(R_i), p_{\{i\}}] & \text{otherwise} \end{cases} \quad \text{and}$$
$$o^{\bar{p}}(R_i) = \begin{cases} [\bar{p}_N, \bar{p}_{\{i\}}] & \text{if } \tau(R_i) < \bar{p}_N \\ [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{i\}}] & \text{if } \bar{p}_{N\backslash\{i\}} < \tau(R_i) \leq \bar{p}_{\{i\}} \\ [\tau(R_i), \bar{p}_{\{i\}}] & \text{otherwise.} \end{cases} \tag{30}$$

Then, R_i is not single-peaked on $(o^p(R_i) \cap I) \cup \{\tau(R_i)\}$ because $\beta, \gamma \in o^p(R_i) \cap I$. We will now show that, for all $t = 1, ..., T$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}^i_t) \cup \{\tau(R_i), \alpha'\}$

[13] R_i is defined in any arbitrary way in $[a, \eta_1] \backslash \{\gamma, \beta\}$.

for all $\alpha' \in \bar{I}_t^i$. Fix $t = 1, ..., T$. Since $o^{\bar{p}}(R_i) \subset [\bar{p}_N, \bar{p}_{\{i\}}]$, $o^{\bar{p}}(R_i) \cap \bar{I}_t^i \subset [\eta_1^t, \eta_2^t] \subset [\eta_1, \eta_2]$. Then, by its definition, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}_t^i) \cup \{\tau(R_i)\}$. We will now show that R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}_t^i) \cup \{\tau(R_i), \alpha'\}$ for all $\alpha' \in \bar{I}_t^i$. We distinguish between two subcases.

Case 1.b.4.a: $\alpha' \in [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\alpha' \in [\eta_1^t, \eta_2^t]$ because $\alpha' \in \bar{I}_t^i$. Hence, $\alpha' \in [\eta_1, \eta_2]$. Therefore, by definition of R_i and the fact that $o^{\bar{p}}(R_i) \cap \bar{I}_t^i \subset [\eta_1, \eta_2]$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}_t^i) \cup \{\tau(R_i), \alpha'\}$.

Case 1.b.4.b: $\alpha' \notin [\bar{p}_N, \bar{p}_{\{i\}}]$. Then, $\alpha' > \bar{p}_{\{i\}} \geq \eta_2 \geq \eta_1$ because $\alpha' \in \bar{I}_t^i \subset r_{f^{\bar{p}}})$. Hence, by definition of R_i and the fact that $o^{\bar{p}}(R_i) \cap \bar{I}_t^i \subset [\eta_1, \eta_2]$, R_i is single-peaked on $(o^{\bar{p}}(R_i) \cap \bar{I}_t^i) \cup \{\tau(R_i), \alpha'\}$.

Therefore, by Lemma 5, $R_i \in \mathcal{M}_i^{f^p} \backslash \mathcal{M}_i^{f^{\bar{p}}}$ which contradicts (26).

Case 2: $[p_{N\backslash\{i\}}, p_{\{\emptyset\}}] \cap ND_p^i \not\subseteq [\bar{p}_{N\backslash\{i\}}, \bar{p}_{\{\emptyset\}}] \cap ND_{\bar{p}}^i$. Since the problem is symmetric, the proof is similar to the one used in *Case 1*.

Case 3: $[p_{\{i\}}, p_{N\backslash\{i\}}] \not\subseteq ND_{\bar{p}}^i$. Then $p_{\{i\}} \leq p_{N\backslash\{i\}}$. We proceed by distinguishing among four subcases.

Case 3.a: $p_{\{i\}} = p_{N\backslash\{i\}}$. Then, we can apply either *Case 1* or *Case 2*.

Hence, assume $p_{\{i\}} < p_{N\backslash\{i\}}$ and let $\gamma \in [p_{\{i\}}, p_{N\backslash\{i\}}] \backslash ND_{\bar{p}}^i$.

Case 3.b: Either $p_{\{i\}} = \gamma$ or $p_{N\backslash\{i\}} = \gamma$ hold. Then, we can apply either *Case 1* or *Case 2*.

Case 3.c: $p_{\{i\}} < \gamma < p_{N\backslash\{i\}}$ and $p_N < p_{\{i\}}$. Let $R_i \in \mathcal{U}$ and $\alpha, \beta \in [a, b]$ be such that (i) $p_N < \alpha < \beta < p_{\{i\}}$, (ii) $\tau(R_i) = \alpha$, (iii) $\gamma P_i \beta$ and (iv) if $\rho, \delta \in [a, b] \backslash \{\gamma\}$ and $\alpha < \rho < \delta$ or $\delta < \rho < \alpha$, then $\rho R_i \delta$. By Lemma 6,

$$o^p(R_i) = [\tau(R_i), p_{\{i\}}]. \tag{31}$$

Since $p_{\{i\}} < p_{N\backslash\{i\}}$, $ND_i^p = [p_N, p_{\{\emptyset\}}]$ holds. As R_i is not single-peaked on $(o^p(R_i) \cap [p_N, p_{\{\emptyset\}}]) \cup \{\tau(R_i), \gamma\}$ and $\gamma \in [p_N, p_{\{\emptyset\}}] = ND_i^p$, by Lemma 5, $R_i \in \mathcal{M}_i^{f^p}$. Furthermore, as R_i is single-peaked on $[a, b] \backslash \{\gamma\}$ and $\gamma \notin ND_{\bar{p}}^i$, by Lemma 5, $R_i \notin \mathcal{M}_i^{f^{\bar{p}}}$. Thus, $R_i \in \mathcal{M}_i^{f^p} \backslash \mathcal{M}_i^{f^{\bar{p}}}$ which contradicts (26).

Case 3d: $p_{\{i\}} < \gamma < p_{N\backslash\{i\}}$ and $p_N = p_{\{i\}}$. Then, $p_{N\backslash\{i\}} < p_{\{\emptyset\}}$ (otherwise i is a dictator). Let $R_i \in \mathcal{U}$ and $\alpha, \beta \in [a, b]$ be such that (i) $p_{N\backslash\{i\}} < \beta < \alpha < p_{\{\emptyset\}}$, (ii) $\tau(R_i) = \alpha$, (iii) $\gamma P_i \beta$ and (iv) if $\rho, \delta \in [a, b] \backslash \{\gamma\}$ and $\alpha < \rho < \delta$ or $\delta < \rho < \alpha$, then $\rho R_i \delta$. By Lemma 6,

$$o^p(R_i) = [p_{N\backslash\{i\}}, \tau(R_i)]. \tag{32}$$

Since $p_{\{i\}} < p_{N\backslash\{i\}}$, $ND_i^p = [p_N, p_{\{\emptyset\}}]$ holds. As R_i is not single-peaked on $(o^p(R_i) \cap [p_N, p_{\{\emptyset\}}]) \cup \{\tau(R_i), \gamma\}$ and $\gamma \in [p_N, p_{\{\emptyset\}}] = ND_i^p$, by Lemma 5, $R_i \in \mathcal{M}_i^{f^p}$. Furthermore, as R_i is single-peaked on $[a,b]\backslash\{\gamma\}$ and $\gamma \notin ND^i_{\bar{p}}$, by Lemma 5, $R_i \notin \mathcal{M}_i^{f^{\bar{p}}}$. Thus, $R_i \in \mathcal{M}_i^{f^p} \backslash \mathcal{M}_i^{f^{\bar{p}}}$ which contradicts (26). ■

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