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Abstract

In this paper we develop exact relationships between empirical Fisher indexes and their theoretical Malmquist and Konüs counterparts. We begin by using implicit Malmquist price and price recovery indexes to establish exact relationships between Malmquist quantity and productivity indexes and Fisher quantity and productivity indexes. We then show that Malmquist quantity and productivity indexes and Fisher price and price recovery indexes "almost" satisfy the product test with the relevant value change, and we derive a quantity mix function that ensures satisfaction of the product test. We next use implicit Konüs quantity and productivity indexes to establish exact relationships between Konüs price and price recovery indexes and Fisher price and price recovery indexes. We then show that Konüs price and price recovery indexes and Fisher quantity and productivity indexes "almost" satisfy the product test with the relevant value change, we derive a price mix function that ensures satisfaction of the product test, and we show that this price mix function differs fundamentally from the quantity mix function relating Malmquist and Fisher indexes.

Keywords: implicit Malmquist indexes, implicit Konüs indexes, Fisher indexes, quantity mix and price mix functions

JEL Classification codes: C43, D24, D61

Exact Relationships between Fisher Indexes and Theoretical Indexes

1. Introduction

Theoretical Malmquist quantity and productivity indexes differ from empirical Fisher quantity and productivity indexes. This matters because Malmquist indexes can be estimated using empirical data, and empirical Malmquist studies are proliferating. Our first objective is to relate theoretical Malmquist quantity and productivity indexes to empirical Fisher quantity and productivity indexes, and to provide economically meaningful expressions for the relationships. These expressions also enable Malmquist quantity and productivity indexes and Fisher price and price recovery indexes to satisfy the product test with the relevant value change. The key ingredients in this analysis are *implicit* Malmquist price and price recovery indexes.

Similarly, theoretical Konüs price and price recovery indexes differ from empirical Fisher price and price recovery indexes. This also matters, because Konüs indexes also can be estimated using empirical data, although to date this has not become a popular exercise. Nonetheless our second objective is to relate theoretical Konüs price and price recovery indexes to empirical Fisher price and price recovery indexes, and to provide (fundamentally different) economically meaningful expressions for the relationships. These expressions also enable Konüs price and price recovery indexes and Fisher quantity and productivity indexes to satisfy the product test with the relevant value change. The key ingredients in this analysis are *implicit* Konüs quantity and productivity indexes.

The literature relating theoretical and empirical index numbers has taken two approaches. One approach seeks restrictions on the structure of production technology, in conjunction with a form of optimizing behavior, that *equate* an empirical index with a corresponding theoretical index. Diewert (1992) follows this approach to provide "a strong economic justification" for the use of Fisher quantity and productivity indexes. A second approach imposes relatively weak regularity conditions on the structure of production technology, sufficient for duality to hold, augmented with Mahler inequalities, to establish *approximate* relationships between empirical and theoretical indexes. Balk (1998) makes extensive use of this approach.

Our paper fits into neither category. Our analysis begins with implicit theoretical price and quantity indexes. We use these implicit indexes to derive functions that link Fisher indexes with Malmquist and Konüs indexes, and that guarantee satisfaction of the analogous product tests. We provide economic intuition behind the content of these functions, which characterize variation in the mix of choice variables, either quantities or prices.[1]

Our paper unfolds as follows. In Section 2 we provide some background to motivate our analysis relating empirical and theoretical index numbers. In Section 3 we use implicit Malmquist price and price recovery indexes to relate Fisher quantity and productivity indexes to Malmquist quantity and productivity indexes. We also show that Malmquist quantity and productivity indexes and Fisher price and price recovery indexes "almost" satisfy the product test with the relevant value change, and we derive and provide economic interpretations of quantity mix functions that guarantee satisfaction of the product test. In Section 4 we use implicit Konüs quantity and productivity indexes to relate Fisher price and price recovery indexes to Konüs price and price recovery indexes. We also show that Konüs price and price recovery indexes and Fisher quantity and productivity indexes "almost" satisfy the product test with the relevant value change, we derive price mix functions that guarantee satisfaction of the product test, and we show that these functions differ fundamentally from the analogous functions relating Malmquist and Fisher indexes. Section 5 concludes.

2. Background

Let $y^t \in R_+^M$ and $x^t \in R_+^N$ be output and input quantity vectors with corresponding price vectors $p^t \in R_{++}^M$ and $w^t \in R_{++}^N$, and let revenue $R^t = p^{tT}y^t$, cost $C^t = w^{tT}x^t$, and profitability (or cost recovery) $\Pi^t = R^t/C^t$, all for two time periods, a base period t=0 and a comparison period t=1. Let the technology T^t = {(y,x): x can produce y in period t}, the convex output set $P^t(x) = \{y: (y,x) \in T^t\}$ with frontier $IP^t(x) = \{y: y \in P^t(x), \lambda y \notin P^t(x), \lambda > 1\}$, and the convex input set $L^t(y) = \{x: (x,y) \in T^t\}$ with frontier $IL^t(y) = \{x: x \in L^t(y), \lambda x \notin L^t(y), \lambda < 1\}$. Finally let the revenue frontier $r^t(x,p) = \max_y\{p^Ty: y \in P^t(x)\} \geq R^t$ and the cost frontier $c^t(y,w) = \min_x\{w^Tx: x \in L^t(y)\} \leq C^t$.

We know from Balk (1998) that our best empirical and theoretical quantity and productivity indexes are related by

$$Y_F \cong Y_M(x^1,x^0,y^1,y^0)$$

$$X_F \cong X_M(y^1,y^0,x^1,x^0)$$

$$\frac{Y_F}{X_F} \cong \frac{Y_M(x^1,x^0,y^1,y^0)}{X_M(y^1,y^0,x^1,x^0)}, \tag{1}$$

where Y_F, X_F and Y_F/X_F are Fisher output quantity, input quantity and productivity indexes, and $Y_M(x^1,x^0,y^1,y^0)$, $X_M(y^1,y^0,x^1,x^0)$ and $Y_M(x^1,x^0,y^1,y^0)/X_M(y^1,y^0,x^1,x^0)$ are Malmquist output quantity, input quantity and productivity indexes in geometric mean form.[2]

It follows that[3]

$$P_F \times Y_M \cong P_F \times Y_F = \frac{R^1}{R^0}$$

$$W_F \times X_M \cong W_F \times X_F = \frac{C^1}{C^0}$$

$$\frac{P_F}{W_F} \times \frac{Y_M}{X_M} \cong \frac{P_F}{W_F} \times \frac{Y_F}{X_F} = \frac{\Pi^1}{\Pi^0}. \tag{2}$$

Results (1) and (2) are based on Mahler inequalities, which use distance functions to bound the allocative efficiencies of quantity vectors [$r^t(x,p) \geq p^Ty/D_o^t(x,y)$

$\forall$p,y,x and $c^t(y,w) \le w^T x/D_i^t(y,x)$ $\forall$w,x,y], with an assumption of within-period allocative efficiency [$p^{tT}y^t/D_o^t(y^t,x^t) = r^t(x^t,p^t)$ and $w^{tT}x^t/D_i^t(x^t,y^t) = c^t(y^t,w^t)$, t=0,1], where $D_o^t(x,y) = \min\{\phi>0: y/\phi \in P^t(x)\} \le 1$ $\forall y \in P^t(x)$ are output distance functions and $D_i^t(y,x) = \max\{\theta>0: x/\theta \in L^t(y)\} \ge 1$ $\forall x \in L^t(y)$ are input distance functions.

We also know that our best empirical and theoretical price and price recovery indexes are related by

$$P_F \cong P_K(x^1,x^0,p^1,p^0)$$

$$W_F \cong W_K(y^1,y^0,w^1,w^0)$$

$$\frac{P_F}{W_F} \cong \frac{P_K(x^1,x^0,p^1,p^0)}{W_K(y^1,y^0,w^1,w^0)}, \tag{3}$$

where P_F, W_F and P_F/W_F are Fisher output price, input price and price recovery indexes, and $P_K(x^1,x^0,p^1,p^0)$, $W_K(y^1,y^0,w^1,w^0)$ and $P_K(x^1,x^0,p^1,p^0)/W_K(y^1,y^0,w^1,w^0)$ are Konüs output price, input price and price recovery indexes.[4]

It follows that

$$Y_F \times P_K \cong Y_F \times P_F = \frac{R^1}{R^0}$$

$$X_F \times W_K \cong X_F \times W_F = \frac{C^1}{C^0}$$

$$\frac{Y_F}{X_F} \times \frac{P_K}{W_K} \cong \frac{Y_F}{X_F} \times \frac{P_F}{W_F} = \frac{\square^1}{\square^0}. \tag{4}$$

Results (3) and (4) are not based on Mahler inequalities. These results are based on inequalities having similar form [$r^t(x,p) \ge p^T y$ $\forall$p,y,x and $c^t(y,w) \le w^T x$ $\forall$w,x,y], but they use revenue and cost frontiers to bound the overall efficiencies, and the efficiencies being bounded are those of price vectors rather than quantity vectors.

In Sections 3 and 4 we derive exact relationships between empirical and theoretical index numbers, and we provide economic interpretations of the mix functions that convert the approximations to equalities. We also show that the economic content of the mix functions that convert the approximations in (1) and (2) to equalities coincide, and they differ fundamentally from the economic content of the mix functions that convert the approximations in (3) and (4) to equalities, which also coincide. The quantity mix functions in Section 3, but not the price mix functions in Section 4, are ratio analogues to the product mix and resource mix effects in Grifell-Tatjé and Lovell (1999;1182,1184).

The starting points in our analyses are implicit Malmquist output and input price indexes in Section 3, and implicit Konüs output and input quantity indexes in Section 4. Neither set of implicit indexes satisfies the fundamental homogeneity property in prices or quantities, respectively (Diewert (1981;174,176)). However we do not treat these implicit indexes as price or quantity indexes; we use them for other purposes, to convert the economic approximations in (1) and (3) to exact relationships, which in turn eliminates the product test gaps in (2) and (4), and to provide economic interpretations of the gaps they eliminate.

3. Implicit Malmquist Price and Price Recovery Indexes

In this section we exploit implicit Malmquist output price, input price and price recovery indexes. These indexes enable us to derive exact relationships between Fisher and Malmquist output quantity, input quantity and productivity indexes, and exact decompositions of revenue change, cost change and profitability change.

3.1 The Output Side

A base period implicit Malmquist output price index is

$$PI_M^0(x^0,p^1,p^0,y^1,y^0) \equiv \frac{R^1/R^0}{Y_M^0(x^0,y^1,y^0)}$$

$$= \frac{p^{1T}y^1/D_o^0(x^0,y^1)}{p^{0T}y^0/D_o^0(x^0,y^0)}, \tag{5}$$

in which $Y_M^0(x^0,y^1,y^0) = D_o^0(x^0,y^1)/D_o^0(x^0,y^0)$ is a base period Malmquist output quantity index. Multiplying and dividing by $p^{0T}y^1/D_o^0(x^0,y^1)$ yields

$$PI_M^0(x^0,p^1,p^0,y^1,y^0) = P_P \times \frac{p^{0T}[y^1/D_o^0(x^0,y^1)]}{p^{0T}[y^0/D_o^0(x^0,y^0)]}$$

$$= P_P \times \frac{Y_L}{Y_M^0(x^0,y^1,y^0)}$$

$$= P_P \times YM_M^0(x^0,p^0,y^1,y^0), \tag{6}$$

in which $P_P = p^{1T}y^1/p^{0T}y^1$ is a Paasche output price index, $Y_L = p^{0T}y^1/p^{0T}y^0$ is a Laspeyres output quantity index, and $YM_M^0(x^0,p^0,y^1,y^0) = [p^{0T}y^1/D_o^0(x^0,y^1)]/[p^{0T}y^0/D_o^0(x^0,y^0)]$ is a base period Malmquist output quantity mix function, so named because it is based on output distance functions that scale output vectors y^1 and y^0 to the base period frontier $IP^0(x^0)$, thereby eliminating any magnitude difference between them, leaving only difference in their mix. This function is the ratio of the revenue generated by $y^1/D_o^0(x^0,y^1)$ to that generated by $y^0/D_o^0(x^0,y^0)$ when both are valued at base period output prices. The second equality in (6) provides an exact decomposition of a base period implicit Malmquist output price index. The third equality demonstrates that the base period Malmquist output quantity mix function is the ratio of a Laspeyres output quantity index to a base period Malmquist output quantity index. In the presence of base period prices we expect normalized base period quantities $y^0/D_o^0(x^0,y^0)$ to generate at least as much revenue as normalized comparison period quantities $y^1/D_o^0(x^0,y^1)$, and so we expect $YM_M^0(x^0,p^0,y^1,y^0) \leq 1$, and thus $Y_L \leq Y_M^0(x^0,y^1,y^0)$.

Revenue change is

$$\frac{R^1}{R^0} \equiv Y_M^0(x^0,y^1,y^0) \times PI_M^0(x^0,p^1,p^0,y^1,y^0)$$

$$= [Y_M^0(x^0,y^1,y^0) \times P_P] \times YM_M^0(x^0,p^0,y^1,y^0), \quad (7)$$

which uses (6) to provide an exact decomposition of revenue change, showing that the product of a base period Malmquist output quantity index, a Paasche output price index, and a base period Malmquist output quantity mix function satisfies the product test with revenue change.

The base period output quantity mix function has a value of unity if M=1, or if M>1 and $y^1 = \lambda y^0$, $\lambda>0$, which effectively converts the problem to a single output problem. If $YM_M^0(x^0,p^0,y^1,y^0) = 1$, $PI_M^0(x^0,p^1,p^0,y^1,y^0) = P_P$ and $Y_L = Y_M^0(x^0,y^1,y^0)$ in (6) and $R^1/R^0 = Y_M^0(x^0,y^1,y^0) \times P_P$ in (7), so that, under either of the stipulated conditions, a base period implicit Malmquist output price index is equal to a Paasche output price index, a base period Malmquist output quantity index is equal to a Laspeyres output quantity index, and the product of a base period Malmquist output quantity index and a Paasche output price index satisfies the product test with revenue change.

If neither of these conditions holds, $YM_M^0(x^0,p^0,y^1,y^0) < 1$. Base period output allocative efficiency (but not necessarily technical efficiency) of y^0 relative to p^0 [i.e., $p^{0T}y^0/D_o^0(x^0,y^0) = r^0(x^0,p^0)$ in (6)] is sufficient for $YM_M^0(x^0,p^0,y^1,y^0) < 1$, and thus for $PI_M^0(x^0,p^1,p^0,y^1,y^0) < P_P$, $Y_L < Y_M^0(x^0,y^1,y^0)$, and $R^1/R^0 \leq Y_M^0(x^0,y^1,y^0) \times P_P$. A less restrictive sufficient condition for all three inequalities requires only that y^0 be more allocatively efficient than y^1 relative to (x^0,p^0) on base period technology [i.e., $p^{0T}y^1/D_o^0(x^0,y^1) < p^{0T}y^0/D_o^0(x^0,y^0) \leq r^0(x^0,p^0)$ in (6)]. This assumption is weaker than one of base period output allocative efficiency (e.g., Balk (1998)) or of base period revenue maximization (e.g., Diewert (1981)).

A comparison period implicit Malmquist output price index is

$$PI_M^1(x^1,p^1,p^0,y^1,y^0) \equiv \frac{R^1/R^0}{Y_M^1(x^1,y^1,y^0)}$$

$$= \frac{p^{1T}y^1/D_O^1(x^1,y^1)}{p^{0T}y^0/D_O^1(x^1,y^0)}, \quad (8)$$

in which $Y_M^1(x^1,y^1,y^0) = D_o^1(x^1,y^1)/D_o^1(x^1,y^0)$ is a comparison period Malmquist output quantity index. Multiplying and dividing by $p^{1T}y^0/D_o^1(x^1,y^0)$ yields

$$PI_M^1(x^1,p^1,p^0,y^1,y^0) = P_L \times \frac{p^{1T}[y^1/D_o^1(x^1,y^1)]}{p^{1T}[y^0/D_o^1(x^1,y^0)]}$$

$$= P_L \times \frac{Y_P}{Y_M^1(x^1,y^1,y^0)}$$

$$= P_L \times YM_M^1(x^1,p^1,y^1,y^0), \qquad (9)$$

in which $P_L = p^{1T}y^0/p^{0T}y^0$ is a Laspeyres output price index, $Y_P = p^{1T}y^1/p^{1T}y^0$ is a Paasche output quantity index, and $YM_M^1(x^1,p^1,y^1,y^0) = [p^{1T}y^1/D_o^1(x^1,y^1)]/[p^{1T}y^0/D_o^1(x^1,y^0)]$ is a comparison period Malmquist output quantity mix function that is the ratio of the revenue generated by $y^1/D_o^1(x^1,y^1)$ to that generated by $y^0/D_o^1(x^1,y^0)$ when both are valued at comparison period output prices. The second equality in (9) provides an exact decomposition of a comparison period implicit Malmquist output price index. The third equality shows that the comparison period Malmquist output quantity mix function is the ratio of a Paasche output quantity index to a comparison period Malmquist output quantity index. In the presence of comparison period prices we expect normalized comparison period quantities $y^1/D_o^1(x^1,y^1)$ to generate at least as much revenue as normalized base period quantities $y^0/D_o^1(x^1,y^0)$, and so we expect $YM_M^1(x^1,p^1,y^1,y^0) \geq 1$, and thus $Y_P \geq Y_M^1(x^1,y^1,y^0)$.

Revenue change is

$$\frac{R^1}{R^0} \equiv Y_M^1(x^1,y^1,y^0) \times PI_M^1(x^1,p^1,p^0,y^1,y^0)$$

$$= [Y_M^1(x^1,y^1,y^0) \times P_L] \times YM_M^1(x^1,p^1,y^1,y^0), \qquad (10)$$

which provides a second exact decomposition of revenue change, in which the product of a comparison period Malmquist output quantity index, a Laspeyres output price index, and a comparison period Malmquist output quantity mix function also satisfies the product test with revenue change.

The comparison period output quantity mix function has a value of unity if M=1, or if M>1 and $y^1 = \lambda y^0$, $\lambda>0$. Under either of these conditions a comparison period implicit Malmquist output price index is equal to a Laspeyres output price index, a comparison period Malmquist output quantity index is equal to a Paasche output quantity index; and the product of a comparison period Malmquist output quantity index and a Laspeyres output price index satisfies the product test with revenue change.

If neither of these conditions holds, comparison period output allocative efficiency of y^1 relative to p^1 [i.e., $p^{1T}y^1/D_o^1(x^1,y^1) = r^1(x^1,p^1)$ in (9)] is sufficient for $YM_M^1(x^1,p^1,y^1,y^0) > 1$, and thus for $PI_M^1(x^1,p^1,p^0,y^1,y^0) > P_L$, $Y_P > Y_M^1(x^1,y^1,y^0)$, and $R^1/R^0 > Y_M^1(x^1,y^1,y^0) \times P_L$. A less restrictive sufficient condition for all three inequalities requires only that y^1 be more allocatively efficient than y^0 relative to (x^1,p^1) on comparison period technology [i.e., $p^{1T}y^0/D_o^1(x^1,y^0) < p^{1T}y^1/D_o^1(x^1,y^1) \leq r^1(x^1,p^1)$ in (9)].

Figure 1 illustrates the base period and comparison period output quantity mix functions for M=2. Convexity of the output sets guarantees that $YM_M^0[x^0,p^0,y^1/D_o^0(x^0,y^1),y^0/D_o^0(x^0,y^0)] \leq 1$ and that $YM_M^1[x^1,p^1,y^1/D_o^1(x^1,y^1),y^0/D_o^1(x^1,y^0)] \geq 1$. Within period allocative efficiency is sufficient but not necessary; all that is required is that $y^0/D_o^0(x^0,y^0)$ be more allocatively efficient than $y^1/D_o^0(x^0,y^1)$ relative to p^0, and that $y^1/D_o^1(x^1,y^1)$ be more allocatively efficient than $y^0/D_o^1(x^1,y^0)$ relative to p^1.

Insert Figure 1 about here

An implicit Malmquist output price index is the geometric mean of (6) and (9), and so

$$PI_M(x^1,x^0,p^1,p^0,y^1,y^0) = P_F \times YM_M(x^1,x^0,p^1,p^0,y^1,y^0)$$

$$= P_F \times \frac{Y_F}{Y_M(x^1,x^0,y^1,y^0)}, \qquad (11)$$

in which $P_F = [P_P \times P_L]^{1/2}$ is a Fisher output price index, $Y_F = [Y_L \times Y_P]^{1/2}$ is a Fisher output quantity index, $Y_M(x^1,x^0,y^1,y^0) = [Y_M^0(x^0,y^1,y^0) \times Y_M^1(x^1,y^1,y^0)]^{1/2}$ is a Malmquist output quantity index, and the Malmquist output quantity mix function $YM_M(x^1,x^0,p^1,p^0,y^1,y^0) = [YM_M^0(x^0,p^0,y^1,y^0) \times YM_M^1(x^1,p^1,y^1,y^0)]^{1/2}$.

It follows from (11) that

$$Y_F = Y_M(x^1,x^0,y^1,y^0) \times YM_M(x^1,x^0,p^1,p^0,y^1,y^0), \qquad (12)$$

which provides an exact relationship between the empirical Fisher output quantity index and the theoretical Malmquist output quantity index.

Revenue change is the geometric mean of (7) and (10), and so

$$\frac{R^1}{R^0} = [Y_M(x^1,x^0,y^1,y^0) \times P_F] \times YM_M(x^1,x^0,p^1,p^0,y^1,y^0), \qquad (13)$$

which provides an exact decomposition of revenue change.

The output quantity mix function has a value of unity if M=1, or if $y^1 = \lambda y^0$, $\lambda>0$. Under either of these conditions $PI_M(x^1,x^0,p^1,p^0,y^1,y^0) = P_F$ in (11), $Y_F = Y_M(x^1,x^0,y^1,y^0)$

in (12), and $R^1/R^0 = Y_M(x^1,x^0,y^1,y^0) \times P_F$ in (13). If neither of these conditions holds, $YM_M(x^1,x^0,p^1,p^0,y^1,y^0) \cong 1$ provides an economically meaningful characterization of the differences $PI_M(x^1,x^0,p^1,p^0,y^1,y^0) \cong P_F$ in (11), $Y_F \cong Y_M(x^1,x^0,y^1,y^0)$ in (12), and $R^1/R^0 \cong Y_M(x^1,x^0,y^1,y^0) \times P_F$ in (13).

3.2 The Input Side

We exploit the implicit Malmquist input price index in a similar manner, using the same strategies and the same quantity mix logic. The base period implicit Malmquist input price index is $WI_M^0(y^0,w^1,w^0,x^1,x^0) \equiv (C^1/C^0)/X_M^0(y^0,x^1,x^0)$ and the comparison period implicit Malmquist input price index is $WI_M^1(y^1,w^1,w^0,x^1,x^0) \equiv (C^1/C^0)/X_M^1(y^1,x^1,x^0)$. We omit all intermediate steps and arrive at the geometric mean of the two, the implicit Malmquist input price index

$$WI_M(y^1,y^0,w^1,w^0,x^1,x^0) = W_F \times \left[\frac{w^{0T}[x^1/D_i^0(y^0,x^1)]}{w^{0T}[x^0/D_i^0(y^0,x^0)]} \times \frac{w^{1T}[x^1/D_i^1(y^1,x^1)]}{w^{1T}[x^0/D_i^1(y^1,x^0)]}\right]^{1/2}$$

$$= W_F \times \frac{X_F}{X_M(y^1,y^0,x^1,x^0)}$$

$$= W_F \times XM_M(y^1,y^0,w^1,w^0,x^1,x^0), \qquad (14)$$

in which the Fisher input price index $W_F = [W_P \times W_L]^{1/2}$, the Fisher input quantity index $X_F = [X_L \times X_P]^{1/2}$, and the Malmquist input quantity index $X_M(y^1,y^0,x^1,x^0) = [X_M^0(y^0,x^1,x^0) \times X_M^1(y^1,x^1,x^0)]^{1/2}$. The Malmquist input quantity mix function $XM_M(y^1,y^0,w^1,w^0,x^1,x^0)$ is the geometric mean of a base period Malmquist input quantity mix function that is the ratio of the cost incurred at $x^1/D_i^0(y^0,x^1)$ to that at $x^0/D_i^0(y^0,x^0)$ when both are valued at base period input prices, and a comparison period Malmquist input quantity mix function that is the ratio of the cost incurred at $x^1/D_i^1(y^1,x^1)$ to that at $x^0/D_i^1(y^1,x^0)$ when both are valued at comparison period input prices. The second equality in (14) provides an exact decomposition of the implicit Malmquist input price index. The third equality shows that the Malmquist input quantity mix function is the ratio of a Fisher input quantity index to a Malmquist input quantity index, from which it follows that

$$X_F = X_M(y^1,y^0,x^1,x^0) \times XM_M(y^1,y^0,w^1,w^0,x^1,x^0), \qquad (15)$$

which provides an exact relationship between an empirical Fisher input quantity index and a theoretical Malmquist input quantity index.

Since cost change can be expressed as $C^1/C^0 = X_F \times W_F$, it follows from (15) that

$$\frac{C^1}{C^0} = [X_M(y^1,y^0,x^1,x^0) \times W_F] \times XM_M(y^1,y^0,w^1,w^0,x^1,x^0), \qquad (16)$$

which provides an exact decomposition of cost change.

The input quantity mix function has a value of unity if N=1, or if $x^1 = \mu x^0$, μ>0, which effectively converts the problem to a single input problem. Under either of these conditions $WI_M(y^1,y^0,w^1,w^0,x^1,x^0) = W_F$ in (14), $X_F = X_M(y^1,y^0,x^1,x^0)$ in (15), and $C^1/C^0 = X_M(y^1,y^0,x^1,x^0) \times W_F$ in (16). If neither of these conditions holds, we exploit the expectation that $XM_M(y^1,y^0,w^1,w^0,x^1,x^0) \cong 1$, even in the absence of within-period input allocative efficiency, which generates $WI_M(y^1,y^0,w^1,w^0,x^1,x^0) \cong W_F$ in (14), $X_F \cong X_M(y^1,y^0,x^1,x^0)$ in (15), and $C^1/C^0 \cong X_M(y^1,y^0,x^1,x^0) \times W_F$ in (16).

Figure 2 illustrates the base period and comparison period input quantity mix functions with N=2. Convexity of the input sets guarantees that $XM_M^0[y^0,w^1,w^0,x^1/D_i^0(y^0,x^1),x^0/D_i^0(y^0,x^0)] \geq 1$ and that $XM_M^1[y^1,w^1,w^0,x^1/D_i^1(y^1,x^1),x^0/D_i^1(y^1,x^0)] \leq 1$. As with an output orientation, within period allocative efficiency is sufficient, but not necessary, for $XM_M(y^1,y^0,w^1,w^0,x^1,x^0) \cong 1$.

Insert Figure 2 about here

3.3 Combining the Output Side and the Input Side

We ignore base period and comparison period indexes and proceed directly to an implicit Malmquist price recovery index. The ratio of (11) and (14) is

$$\frac{PI_M(x^1,x^0,p^1,p^0,y^1,y^0)}{WI_M(y^1,y^0,w^1,w^0,x^1,x^0)} = \frac{P_F}{W_F} \times M_M(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0), \qquad (17)$$

in which $M_M(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0) = YM_M(x^1,x^0,p^1,p^0,y^1,y^0)/XM_M(y^1,y^0,w^1,w^0,x^1,x^0)$ is a Malmquist quantity mix function that provides an economic characterization of the gap, if any, between P_F/W_F and $PI_M(x^1,x^0,p^1,p^0,y^1,y^0)/WI_M(y^1,y^0,w^1,w^0,x^1,x^0)$. From (11) – (13) we expect $YM_M(x^1,x^0,p^1,p^0,y^1,y^0) \cong 1$, and from (14) – (16) we expect $XM_M(y^1,y^0,w^1,w^0,x^1,x^0) \cong 1$. Consequently we expect $M_M(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0) \cong 1$, in which case a Fisher price recovery index is approximately equal to an implicit Malmquist price recovery index. A unitary ratio would require equality between the output quantity mix function and the input quantity mix function, a sufficient but not necessary condition for which is $y^1 = \lambda y^0$, λ>0 and $x^1 = \mu x^0$, μ>0, which converts a multiple output, multiple input problem to a single output, single input problem that does not require index numbers of either sort.

An expression for productivity change is given by the ratio of (12) and (15), and is

$$\frac{Y_F}{X_F} = \frac{Y_M(x^1,x^0,y^1,y^0)}{X_M(y^1,y^0,x^1,x^0)} \times M_M(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0), \qquad (18)$$

which provides an exact relationship between a Fisher productivity index and a Malmquist productivity index, with the Malmquist quantity mix function providing an economic interpretation of the (presumably small) gap between the two.

An expression for profitability change is given by the ratio of (13) and (16), and is

$$\frac{\Pi^1}{\Pi^0} = \left[\frac{Y_M(x^1,x^0,y^1,y^0)}{X_M(y^1,y^0,x^1,x^0)} \times \frac{P_F}{W_F}\right] \times M_M(y^1, y^0, x^1, x^0, p^1, p^0, w^1, w^0), \qquad (19)$$

and if the Malmquist quantity mix function is approximately unity a Malmquist productivity index and a Fisher price recovery index approximately satisfy the product test with profitability change.[5]

In this section we have used implicit Malmquist price and price recovery indexes to relate Malmquist quantity and productivity indexes to Fisher quantity and productivity indexes. The important findings are contained in (11), (14) and (17); (12), (15) and (18); and (13), (16) and (19). The first set of results relates implicit theoretical price and price recovery indexes to their explicit empirical counterparts, and establishes the foundations for the second and third sets of results. (12), (15) and (18) clarify the sense in which Fisher quantity and productivity indexes and Malmquist quantity and productivity indexes are approximately equal. (13), (16) and (19) clarify the sense in which Malmquist quantity and productivity indexes approximately satisfy the relevant product test with Fisher price and price recovery indexes. Each of these sets of results depends fundamentally on Malmquist output and input quantity mix functions, which have clear economic interpretations. It is worth emphasizing that the quantity mix functions compare the allocative efficiencies of pairs of quantity vectors, which are the choice variables in the exercises.

(13), (16) and (19) warrant special emphasis from an empirical perspective, because of their decomposability properties. $Y_M(x^1,x^0,y^1,y^0)$, $X_M(y^1,y^0,x^1,x^0)$ and $Y_M(x^1,x^0,y^1,y^0)/X_M(y^1,y^0,x^1,x^0)$ decompose into the product of economic drivers of productivity change: technical change, technical efficiency change, mix efficiency change and scale efficiency change (O'Donnell (2012)). In contrast, P_F, W_F and P_F/W_F decompose into contributions of individual output and input price changes (Balk (2004)). These two features enable a decomposition of value (revenue, cost and profitability) change into the economic drivers of quantity change and the individual price drivers of price change.

4. Implicit Konüs Quantity and Productivity Indexes

In this section we exploit implicit Konüs output quantity, input quantity and productivity indexes. These indexes lead us to exact relationships between Fisher and Malmquist output price, input price and price recovery indexes, and to exact decompositions of revenue change, cost change and profitability change. Both sets of results differ from analogous results in Section 3.

4.1 The Output Side

We begin with a base period implicit Konüs output quantity index

$$YI_K^0(x^0,p^1,p^0,y^1,y^0) \equiv \frac{R^1/R^0}{P_K^0(x^0,p^1,p^0)}$$

$$= \frac{p^{1T}y^1/r^0(x^0,p^1)}{p^{0T}y^0/r^0(x^0,p^0)}, \tag{20}$$

in which $P_K^0(x^0,p^1,p^0) = r^0(x^0,p^1)/r^0(x^0,p^0)$ is a base period Konüs output price index. Multiplying and dividing by $p^{1T}y^0/r^0(x^0,p^1)$ yields

$$YI_K^0(x^0,p^1,p^0,y^1,y^0) = Y_P \times \frac{p^{1T}y^0/r^0(x^0,p^1)}{p^{0T}y^0/r^0(x^0,p^0)}$$

$$= Y_P \times \frac{P_L}{P_K^0(x^0,p^1,p^0)}$$

$$= Y_P \times PM_K^0(x^0,y^0,p^1,p^0), \tag{21}$$

in which $Y_P = p^{1T}y^1/p^{1T}y^0$ is a Paasche output quantity index, $P_L = p^{1T}y^0/p^{0T}y^0$ is a Laspeyres output price index, and $PM_K^0(x^0,y^0,p^1,p^0) = [y^{0T}p^1/r^0(x^0,p^1)]/[y^{0T}p^0/r^0(x^0,p^0)]$ is a base period Konüs output price mix function, so named because it is a function of revenue functions that coincide apart from their output price vectors. This function is the ratio of the revenue generated by y^0 at normalized comparison period output prices $p^1/r^0(x^0,p^1)$ to that generated by y^0 at normalized base period output prices $p^0/r^0(x^0,p^0)$. The two normalized price vectors differ only in their output price mix. An alternative interpretation of the base period Konüs output price mix function is that it is the ratio of two revenue efficiencies, both with base period technology and quantity vectors but with different output price vectors.

The second equality in (21) provides an exact decomposition of a base period implicit Konüs output quantity index. The third equality demonstrates that the base period Konüs output mix function is the ratio of a Laspeyres output price index and a base period Konüs output price index. This mix function is bounded above by unity if y^0 is revenue efficient relative to (x^0,p^0) on base period technology [i.e., $p^{0T}y^0 = r^0(x^0,p^0)$ in (21)], or if y^0 is more revenue efficient relative to (x^0,p^0) than to (x^0,p^1) on base period technology [i.e., $r^0(x^0,p^0) \geq p^{0T}y^0/r^0(x^0,p^0) \geq p^{1T}y^0/r^0(x^0,p^1)$ in (21)]. In either case $P_L \leq P_K^0(x^0,p^1,p^0)$ and $YI_K^0(x^0,p^1,p^0,y^1,y^0) \leq Y_P$. $YI_K^0(x^0,p^1,p^0,y^1,y^0) = Y_P$ if either M=1 or $p^1 = \lambda p^0$, $\lambda>0$, which essentially converts the problem to a single input problem. These bounds do not require base period revenue maximizing behavior, or even base period allocative efficiency.

Revenue change is

$$\frac{R^1}{R^0} = P_K^0(x^0,p^1,p^0) \times YI_K^0(x^0,p^1,p^0,y^1,y^0)$$

$$= [P_K^0(x^0,p^1,p^0) \times Y_P] \times PM_K^0(x^0,y^0,p^1,p^0), \qquad (22)$$

which states that the product of a base period Konüs output price index, a Paasche output quantity index and a base period Konüs output price mix function satisfies the product test with R^1/R^0. As above we expect $R^1/R^0 \le P_K^0(x^0,p^1,p^0) \times Y_P$. However if either M=1 or $p^1 = \lambda p^0$, $\lambda>0$, (21) and (22) collapse to $YI_K^0(x^0,p^1,p^0,y^1,y^0) = Y_P$ and $R^1/R^0 = P_K^0(x^0,p^1,p^0) \times Y_P$, in which case a base period implicit Konüs output quantity index is equal to a Paasche output quantity index, and consequently a Konüs output price index and a Paasche output quantity index satisfy the product test with R^1/R^0.

We now sketch the results of a comparison period implicit Konüs output quantity index. Following the same procedures as above, after multiplying and dividing by $p^{0T}y^1/r^1(x^1,p^0)$ we have

$$YI_K^1(x^1,p^1,p^0,y^1,y^0) = \frac{R^1/R^0}{P_K^1(x^1,p^1,p^0)}$$

$$= Y_L \times \frac{y^{1T}p^1/r^1(x^1,p^1)}{y^{1T}p^0/r^1(x^1,p^0)}$$

$$= Y_L \times \frac{P_P}{P_K^1(x^1,p^1,p^0)}$$

$$= Y_L \times PM_K^1(x^1,y^1,p^1,p^0), \qquad (23)$$

in which $Y_L = p^{0T}y^1/p^{0T}y^0$ is a Laspeyres output quantity index, $P_P = y^{1T}p^1/y^{1T}p^0$ is a Paasche output price index, and $P_K^1(x^1,p^1,p^0) = r^1(x^1,p^1)/r^1(x^1,p^0)$ is a comparison period Konüs output price index. The comparison period Konüs output price mix function $PM_K^1(x^1,y^1,p^1,p^0)$ is the ratio of the revenue efficiency of two output price vectors, given comparison period technology and quantity vectors. If y^1 is more revenue efficient relative to (x^1,p^1) than to (x^1,p^0) on comparison period technology, then $PM_K^1(x^1,y^1,p^1,p^0) \ge 1$, $YI_K^1(x^1,y^1,y^0) \ge Y_L$ and $P_P \ge P_K^1(x^1,p^1,p^0)$.

Revenue change is

$$\frac{R^1}{R^0} = P_K^1(x^1,p^1,p^0) \times YI_K^1(x^1,p^1,p^0,y^1,y^0)$$

$$= [P^1_K(x^1,p^1,p^0) \times Y_L] \times PM^1_K(x^1,y^1,p^1,p^0), \tag{24}$$

which states that the product of a comparison period Konüs output price index, a Laspeyres output quantity index and a comparison period Konüs output price mix function satisfies the product test with R^1/R^0. Under the conditions above, $R^1/R^0 \geq P^1_K(x^1,p^1,p^0) \times Y_L$. If either M=1 or $p^1 = \lambda p^0$, $\lambda>0$, $R^1/R^0 = P^1_K(x^1,p^1,p^0) \times Y_L$.

Figure 3 illustrates the base period and comparison period output price mix functions for M=2. It is not necessary that y^0 be revenue efficient relative to (x^0,p^0) on base period technology; all that is required is that y^0 be more revenue efficient relative to (x^0,p^0) than to (x^0,p^1) on base period technology. A similar remark applies to y^1.

Insert Figure 3 about here

The geometric mean of (21) and (23) is an implicit Konüs output quantity index

$$YI_K(x^1,x^0,p^1,p^0,y^1,y^0) = Y_F \times [PM^0_K(x^0,y^0,p^1,p^0) \, \square \, PM^1_K(x^1,y^1,p^1,p^0)]^{1/2}$$

$$= Y_F \times \frac{P_F}{P_K(x^1,x^0,p^1,p^0)}$$

$$= Y_F \times PM_K(x^1,x^0,y^1,y^0,p^1,p^0), \tag{25}$$

which states that an implicit Konüs output quantity index is the product of a Fisher output quantity index and a Konüs output price mix function. Because one component of the output price mix function is bounded above by unity and the other is bounded below by unity we expect $YI_K(x^1,x^0,p^1,p^0,y^1,y^0) \approx Y_F$.

It follows from the second and third equalities in (25) that

$$P_F = P_K(x^1,x^0,p^1,p^0) \times PM_K(x^1,x^0,y^1,y^0,p^1,p^0), \tag{26}$$

which enables us to calculate the gap between the theoretical and empirical output price indexes.

The geometric mean of (22) and (24) yields the expression for revenue change

$$\frac{R^1}{R^0} = [P_K(x^1,x^0,p^1,p^0) \times Y_F] \times PM_K(x^1,x^0,y^1,y^0,p^1,p^0), \tag{27}$$

which states that a Konüs output price index and a Fisher output quantity index approximately satisfy the product test with R^1/R^0, the approximation becoming an equality if either M=1 or $p^1 = \lambda p^0$, $\lambda>0$.

4.2 The Input Side

We now consider the implicit Konüs input quantity index. The base period implicit Konüs input quantity index is $XI_K^0(y^0,w^1,w^0,x^1,x^0) = (C^1/C^0)/W_K^0(y^0,w^1,w^0)$ and the comparison period implicit Konüs input quantity index is $XI_K^1(y^1,w^1,w^0,x^1,x^0) = (C^1/C^0)/W_K^1(y^1,w^1,w^0)$. The geometric mean of the two, the implicit Konüs input quantity index, is

$$XI_K(y^1,y^0,w^1,w^0,x^1,x^0) = X_F \times [WM_K^0(y^0,x^0,w^1,w^0) \times WM_K^1(y^1,x^1,w^1,w^0)]^{1/2}$$

$$= X_F \times \frac{W_F}{W_K(y^1,y^0,w^1,w^0)}$$

$$= X_F \times WM_K(y^1,y^0,x^1,x^0,w^1,w^0), \qquad (28)$$

in which the Konüs input price mix function $WM_K(y^1,y^0,x^1,x^0,w^1,w^0)$ measures the gap between $XI_K(y^1,y^0,w^1,w^0,x^1,x^0)$ and X_F, and is defined analogously to the output price mix function in (25).

From the second and third equalities in (28)

$$W_F = W_K(y^1,y^0,w^1,w^0) \times WM_K(y^1,y^0,x^1,x^0,w^1,w^0), \qquad (29)$$

which provides an exact relationship between empirical Fisher and theoretical Konüs input price indexes.

Cost change is

$$\frac{C^1}{C^0} = [W_K(y^1,y^0,w^1,w^0) \times X_F] \times WM_K(y^1,y^0,x^1,x^0,w^1,w^0). \qquad (30)$$

The base period and comparison period input price mix functions are illustrated in Figure 4, in which cost efficiency of x^0 and x^1 is not required; all that is required is that x^0 be more cost efficient relative to (y^0,w^0) than to (y^0,w^1) on base period technology, and that x^1 be more cost efficient relative to (y^1,w^1) than to (y^1,w^0) on comparison period technology.

Insert Figure 4 about here

4.3 Combining the Output Side and the Input Side

We now construct an implicit Konüs productivity index. We ignore base period and comparison indexes and proceed directly to an implicit Konüs productivity index. The ratio of (25) and (28) is

$$\frac{YI_K(x^1,x^0,p^1,p^0,y^1,y^0)}{XI_K(y^1,y^0,w^1,w^0,x^1,x^0)} = \frac{Y_F}{X_F} \times M_K(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0), \tag{31}$$

in which the Konüs price mix function $M_K(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0) = PM_K(x^1,x^0,y^1,y^0,p^1,p^0)/WM_K(y^1,y^0,x^1,x^0,w^1,w^0)$ measures the gap between $YI_K(x^1,x^0,p^1,p^0,y^1,y^0)/XI_K(y^1,y^0,w^1,w^0,x^1,x^0)$ and Y_F/X_F. Because we expect $PM_K(x^1,x^0,y^1,y^0,p^1,p^0) \approx 1$ and we expect $WM_K(y^1,y^0,x^1,x^0,w^1,w^0) \approx 1$ we also expect $M_K(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0) \approx 1$, in which case the implicit Konüs productivity index is approximately equal to a Fisher productivity index. Equality would require the output and input price mix functions to be equal, a sufficient but not necessary condition for which is $p^1 = \lambda p^0$, $\lambda>0$ and $w^1 = \mu w^0$, $\mu>0$, which converts the problem to a single output, single input problem for which index numbers are unnecessary.

The ratio of (26) and (29)

$$\frac{P_F}{W_F} = \frac{P_K(x^1,x^0,p^1,p^0)}{W_K(y^1,y^0,w^1,w^0)} \times M_K(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0), \tag{32}$$

provides an exact relationship between an empirical Fisher price recovery index and a theoretical Konüs price recovery index.

The ratio of (27) and (30) provides an implicit Konüs measure of profitability change

$$\frac{\Pi^1}{\Pi^0} = \frac{R^1/R^0}{C^1/C^0}$$

$$= \left[\frac{P_K(x^1,x^0,p^1,p^0)}{W_K(y^1,y^0,w^1,w^0)} \times \frac{Y_F}{X_F}\right] \times M_K(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0), \tag{33}$$

and if $M_K(y^1,y^0,x^1,x^0,p^1,p^0,w^1,w^0) \approx 1$ a Konüs price recovery index and a Fisher productivity index approximately satisfy the product test with profitability change.[6]

In this section we have used implicit Konüs quantity and productivity indexes to relate Konüs price and price recovery indexes to Fisher price and price recovery indexes. The important findings are contained in (25), (28) and (31); (26), (29) and (32); and (27), (30) and (33). The first three relate implicit theoretical quantity and productivity indexes to their explicit empirical counterparts, and establish the foundations for the second and third sets of results. (26), (29) and (32) clarify the sense in which Konüs price and price recovery indexes approximate Fisher price and price recovery indexes. (27), (30) and (33) clarify the sense in which Konüs price and price recovery indexes approximately satisfy the relevant product test with Fisher quantity and productivity indexes. In both the second and third sets of results clarity is provided by the relevant Konüs price mix function.

In Section 3 the product test expressions (13), (16) and (19) have useful empirical applications, since Malmquist quantity and productivity indexes decompose by economic driver and Fisher price and price recovery indexes decompose by individual prices. Here the product test expressions (27), (30) and (33) are of potential, but as yet unrealized, empirical value. The Fisher quantity and productivity indexes have been decomposed by economic driver of productivity change, although agreement on a preferred decomposition remains elusive.[7] The Konüs price and price recovery indexes have yet to be decomposed by economic drivers of price change (rather than, as commonly practiced, by individual prices), although research on this issue is underway.

We emphasize that the Konüs price mix functions differ significantly from the Malmquist quantity mix functions in Section 3, although they serve the same purposes, to convert approximations to exact relationships and to close product test gaps. The Malmquist quantity mix functions are ratios of values generated by two normalized quantity vectors weighted by a common price vector. The Konüs price mix functions are ratios of values generated by a single quantity vector weighted by two normalized price vectors.

5. Summary and Conclusions

We have exploited implicit Malmquist price and price recovery indexes to derive exact relationships between Malmquist and Fisher quantity and productivity indexes, and to derive economically meaningful functions describing the ability of Malmquist quantity and productivity indexes to satisfy product tests with Fisher price and price recovery indexes. The key to these exact relationships is the concept of Malmquist output and input quantity mix functions, in which quantities are allowed to vary between base and comparison periods but prices are fixed at either base period values or comparison period values. The smaller the variation in the quantity mix between base and comparison periods, the smaller the gap between Fisher and Malmquist quantity and productivity indexes.

We also have exploited implicit Konüs quantity and productivity indexes to derive exact relationships between Konüs and Fisher price and price recovery indexes, and to derive fundamentally different, but nonetheless economically meaningful functions describing the ability of Konüs price and price recovery indexes to satisfy product tests with Fisher quantity and productivity indexes. The key to these exact relationships is the concept of Konüs output and input price mix functions, in which prices are allowed to vary between base and comparison periods but quantities are fixed at either base period values or comparison period values. The smaller the variation in the price mix between base and comparison periods, the smaller the gap between Fisher and Konüs price and price recovery indexes.

The exact relationships have clear economic interpretations, as allocative efficiency effects, although these effects differ between Sections 3 and 4. These allocative efficiency effects are easy to calculate, using data required to calculate

Fisher indexes and estimate Malmquist and Konüs indexes, as Brea et al. (2011) have demonstrated for Fisher/Malmquist pairings.

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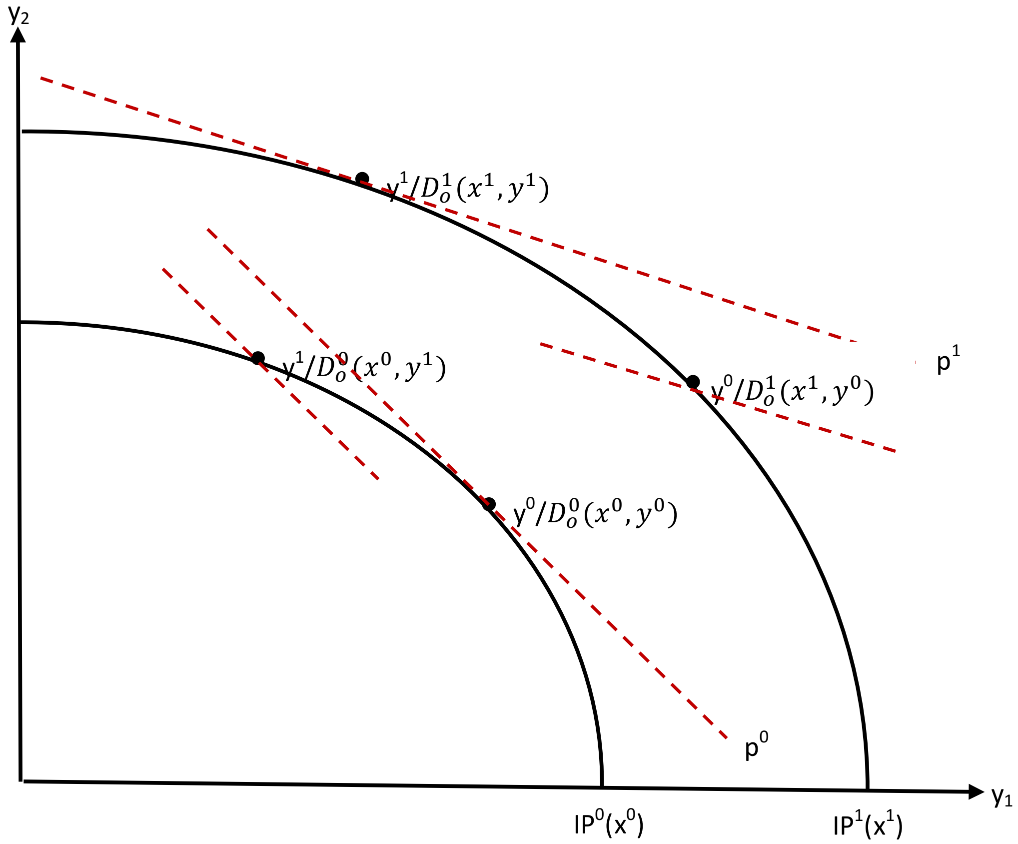

Figure 1 Output Quantity Mix Functions

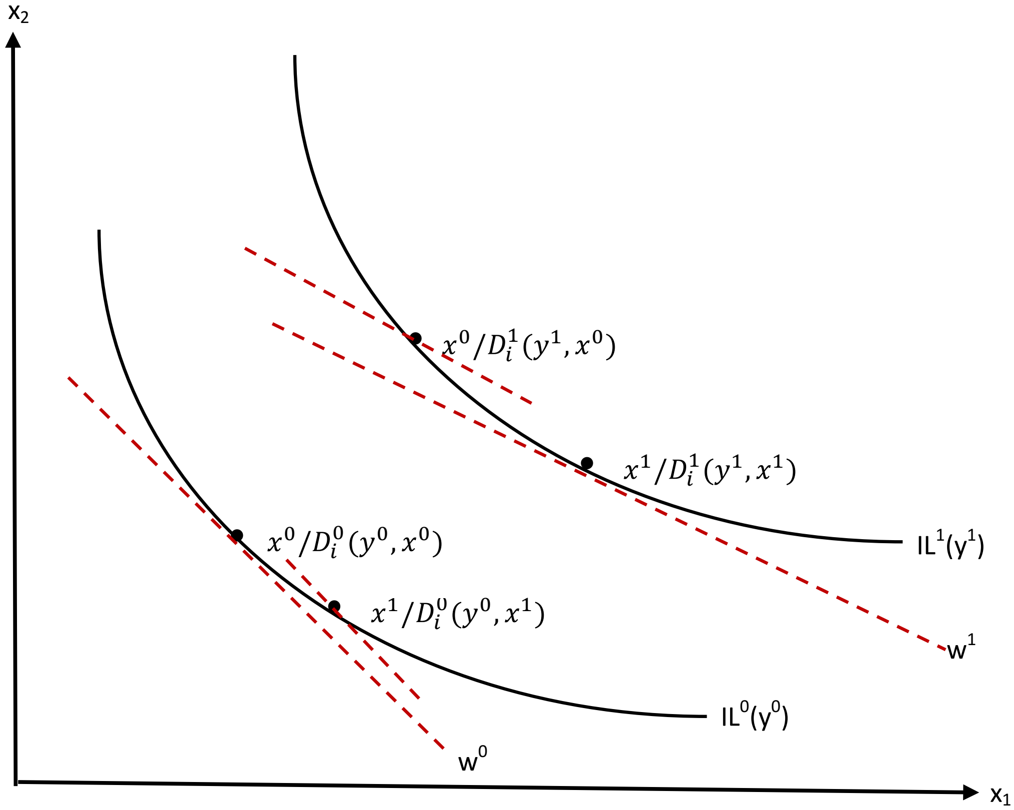

Figure 2 Input Quantity Mix Functions

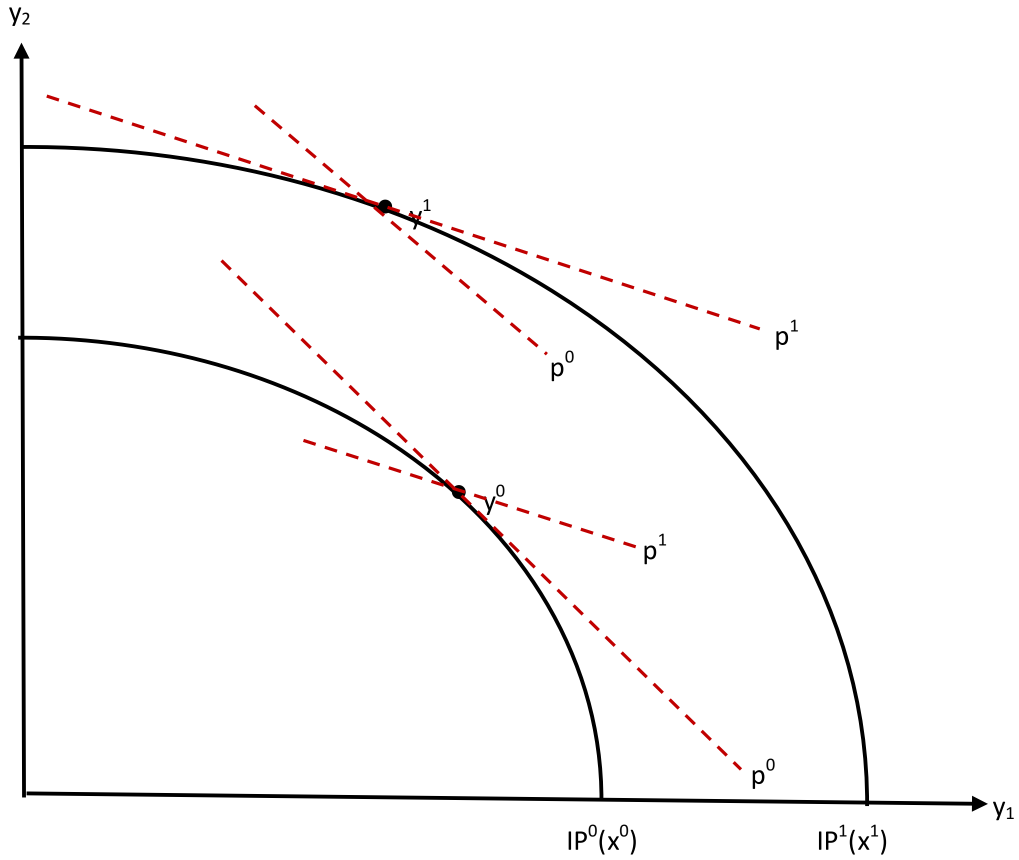

Figure 3 Output Price Mix Functions

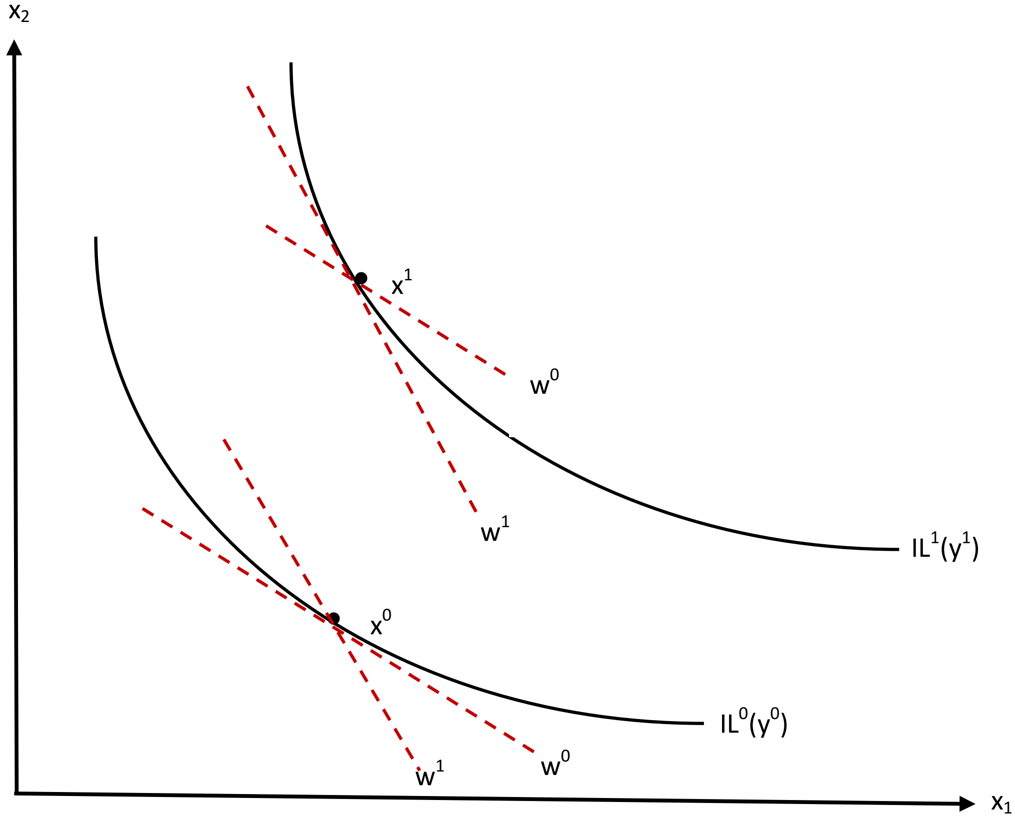

Figure 4 Input Price Mix Functions

Endnotes

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[1] Our analysis extends results in Grifell-Tatjé and Lovell (2015; Chapter 3).

[2] The Fisher quantity indexes are defined as $Y_F = \left[\left(\frac{p^{0T}y^1}{p^{0T}y^0}\right) \times \left(\frac{p^{1T}y^1}{p^{1T}y^0}\right)\right]^{1/2}$ and $X_F = \left[\left(\frac{w^{0T}x^1}{w^{0T}x^0}\right) \times \left(\frac{w^{1T}x^1}{w^{1T}x^0}\right)\right]^{1/2}$, and the Malmquist quantity indexes are defined as $Y_M = \left[\left(\frac{D_o^0(x^0,y^1)}{D_o^0(x^0,y^0)}\right) \times \left(\frac{D_o^1(x^1,y^1)}{D_o^1(x^1,y^0)}\right)\right]^{1/2}$ and $X_M = \left[\left(\frac{D_i^0(y^0,x^1)}{D_i^0(y^0,x^0)}\right) \times \left(\frac{D_i^1(y^1,x^1)}{D_i^1(y^1,x^0)}\right)\right]^{1/2}$.

[3] The Fisher price indexes are defined as $P_F = \left[\left(\frac{y^{0T}p^1}{y^{0T}p^0}\right) \times \left(\frac{y^{1T}p^1}{y^{1T}p^0}\right)\right]^{1/2}$ and $W_F = \left[\left(\frac{x^{0T}w^1}{x^{0T}w^0}\right) \times \left(\frac{x^{1T}w^1}{x^{1T}w^0}\right)\right]^{1/2}$.

[4] The Konüs price indexes are defined as $P_K = \left[\frac{r^0(x^0,p^1)}{r^0(x^0,p^0)} \times \frac{r^1(x^1,p^1)}{r^1(x^1,p^0)}\right]^{1/2}$ and $W_K = \left[\frac{c^0(y^0,w^1)}{c^0(y^0,w^0)} \times \frac{c^1(y^1,w^1)}{c^1(y^1,w^0)}\right]^{1/2}$.

[5] All approximation results in this section also can occur if the technologies allow infinite output substitution possibilities along $IP^0(x^0)$ and $IP^1(x^1)$ between output rays defined by y^1 and y^0 in Figure 1, and infinite input substitution possibilities along $IL^0(y^0)$ and $IL^1(y^1)$ between input rays defined by x^1 and x^0 in Figure 2.

[6] All approximation results in this section can also occur if y^0 and y^1 in Figure 3 and x^0 and x^1 in Figure 4 are vertices of piecewise linear technologies that allow $p^{1T}y^0 = p^{0T}y^0$, $p^{0T}y^1 = p^{1T}y^1$ and $w^{1T}x^0 = w^{0T}x^0$, $w^{0T}x^1 = w^{1T}x^1$, as might occur with DEA.

[7] Compare, for example, the decompositions proposed by Ray and Mukherjee (1996) and by Kuosmanen and Sipiläinen (2009).