Betting under Subjective Uncertainty

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Abstract

This paper shows that uncertainty-averse agents may trade extrinsic variables on efficient markets. The finding is robust to identical beliefs and strictly convex preferences. Conditional on a realization of fundamentals, the distribution of an otherwise irrelevant variable may depend on the underlying probability regime. This dependence cannot be exploited through trade on fundamentals. I provide necessary and sufficient conditions for the irrelevance of non-fundamental variables under maxmin, smooth, and variational preferences. These conditions are found to be stringent, except for multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), for which it suffices that the reference priors agree conditional on fundamentals.

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1 Introduction

Most models of behavior under *uncertainty* (ambiguity) allow for a conservative or worst-case view on probabilities.[1] The resulting beliefs are *subjective*–two agents may disagree on how to rank priors from best to worst–and *endogenous*–an agent's choices may determine state-payoffs and, in turn, her ranking of priors. For instance, an investor may fear different probability models, depending on whether she takes a long or short position on the S&P 500.

Tracking these beliefs across interacting agents is a challenge towards a comprehensive theory of trade under uncertainty. On one hand, subjective beliefs may lead to disagreement and speculative trade. At the same time, as trade allows to eliminate idiosyncratic risks, the personal views on probabilities may align, thereby reinforcing mutual insurance.

Previous literature has shown that the latter phenomenon may indeed prevail. If aggregate resources are riskless, a consensus belief (and full insurance) may arise endogenously, even if agents disagree about priors (Billot, Chateauneuf, Gilboa, and Tallon, 2000; Rigotti, Shannon, and Strzalecki, 2008). For instance, under strictly convex "maxmin" preferences (see Gilboa and Schmeidler, 1989), efficient trade provides full insurance unless the individual sets of priors have an empty intersection.[2]

However, recent papers show that stringent conditions on priors are needed to guarantee a consenus under aggregate uncertainty (Strzalecki and Werner, 2011; De Castro and Chateauneuf, 2011).[3] As a result, equilibrium consumption need no longer comove across agents. Moreover, efficient trade need not eliminate diversifiable risks. That is, consumption may vary on events which do not feature aggregate risk.[4] The nature of trade under aggregate uncertainty is not fully understood. Despite their failure to pool all idiosyncratic risks, agents may still engage in eliminating risk through mutual insurance. At the same time, the presence of diversifiable risk may be a symptom of speculation, thereby indicating a more profound departure from expected utility.

[1] Some aspect of the underlying probabilities is not known with certainty by the decision maker.

[2] This phenomenon is related to the characteristic reluctance to trade at riskless positions, associated with portfolio inertia, home-bias, or bid-ask spreads. Examples include Dow and Werlang (1992); Epstein and Wang (1994); Epstein and Miao (2003); Illeditsch (2011).

[3] Intuitively, as they are exposed to aggregate risk, traders rank priors subjectively, thereby limiting the scope for a consensus "worst".

[4] For instance, a non-empty intersection of the sets of priors is no longer sufficient for risk pooling under "maxmin" preferences.

This paper studies whether uncertainty-averse traders speculate on extrinsic variables.[5] I pursue two concrete goal in a standard setting of ex-ante trade in finite Arrow-Debreu exchange economies with strictly convex preferences. The first is to answer whether trade in non-fundamental variables may be an outcome on efficient markets under common priors. I will show that the answer is positive, precisely because equilibrium beliefs do not tend to converge under aggregate uncertainty. The second goal is to identify restrictions which guarantee purely fundamental trade for multiple priors "maxmin" expected utility (MEU), for the family of variational preferences (Maccheroni, Marinacci, and Rustichini, 2006), and for smooth ambiguity preferences (Klibanoff, Marinacci, and Mukerji, 2005).

To construct the fundamental events, I introduce a partition whose cells collect all states which specify the same distribution of individual endowments. That is, any agent's endowment is measurable with respect to the partition, while any variation within a cell is extrinsic. First, I consider a *coarse equilibrium*, in which agents trade fundamentals alone; that is, they trade freely across the cells, but not within. The property of interest is whether coarse equilibria are robust (immune) to trade on extrinsic variables.

The relevance of beliefs can be illustrated in an expected utility (EU) economy with risk-averse agents. At a coarse equilibrium, the marginal-utility-corrected priors need to agree on fundamentals. For the equilibrium to be immune to extrinsic trades, priors need to also agree about extrinsic variables, conditional on fundamentals, i.e., within the cells of the partition. This property lies at the heart of various sunspot immunity theorems (Cass and Shell, 1983; Milgrom and Stokey, 1982).[6]

To generalize these results for a wide range of (not necessarily differentiable) preferences, I follow Rigotti, Shannon, and Strzalecki (2008) and Strzalecki and Werner (2011) in appealing to basic convexity properties of preferred sets.[7] At any payoff, an agent's unwillingness to take either side of a small bet reveals a supporting *subjective belief*. Before a normalization and the usual marginal utility correction, these subjective beliefs correspond to an agent's unique prior (EU), a set of minimizing priors (MEU, Variational preferences), or a weighted average prior (Smooth preferences), respectively. By an argument which parallels the second welfare theorem, a coarse equilibrium must be supported by

[5] In exchange economies, I call a variable extrinsic if it does not affect the distribution of endowments.

[6] Since marginal utilities are constant within each cell at a coarse equilibrium, the equilibrium is robust if and only if agents hold common conditional priors.

[7] Preferences need to satisfy only basic properties next to strict convexity. Except for incomplete preferences (Bewley, 1986), all major specifications in the literature are compatible.

an agreed-upon subjective belief about fundamentals.[8] Similarly, to make the coarse equilibrium immune to extrinsic trade, the supporting subjective beliefs also need to agree on the distribution of extrinsic variables conditional on fundamentals.

To illustrate, it is useful to differentiate two kinds of extrinsic variables, depending on their conditional distribution being known or uncertain. Consider an informal example.

> A grain trader is uncertain about the probability regime governing the fundamentals (output) at harvest: either she faces a drought regime or a rain regime. On top of output, two extrinsic variables are observed: sunspot activity and a statistic about realized precipitation. Conditional on any output realization, the probability of observing sunspot activity is identical in both regimes. In contrast, low precipitation is weak evidence in favor of the drought regime: (upon some output levels) the conditional probability of observing increased precipitation is higher in the rain regime.

An extrinsic variable whose conditional distribution does not depend on the unknown model–such as the sunspot activity–ensures that the subjective beliefs which support a coarse equilibrium also agree within each cell of the partition. In the absence of conditional uncertainty, I will show that trade is purely fundamental under common conditional priors, just as under expected utility.

In contrast, the presence of a conditionally uncertain variable–such as precipitation–makes the relevant conditional prior an endogenous object. Under MEU, for instance, the subjective beliefs which support a coarse equilibrium inherit the conditional probabilities from a minimizing prior. The examples in this paper show that even if the sets of priors are identical across all agents, no common minimizing prior may exist, and immunity may fail.[9] In the context of the previous example, two MEU traders who consider an identical set of priors may disagree about the worst regime at a coarse equilibrium. As a result, there exist odds, such that taking opposite sides of a bet on precipitation has positive expectation under both agents' respective worst-case regime.[10]

[8] While in a different setting, analogous results obtain in Rigotti, Shannon, and Strzalecki (2008) and Strzalecki and Werner (2011).

[9] Since MEU preference belong to the variational preferences, identical cost functions are not sufficient for immunity. Similarly, identical first-order and second-order beliefs are not sufficient for immunity with smooth preferences.

[10] Put differently, a conditionally uncertain variable carries weak evidence ("statistical fingerprints") of the

To address the second goal of the paper, I build on results in Strzalecki and Werner (2011), who characterize restrictions on beliefs which guarantee the stronger property of constant consumption on events with constant aggregate resources. Under MEU, extrinsic betting can be ruled out if the sets of priors are rich enough to satisfy a rectangularity condition (see Epstein and Schneider, 2003; Kajii and Ui, 2009). For smooth preference, the differentiability implies locally unique subjective beliefs. Any consensus is therefore vulnerable to small changes in the specification of the economy. As a result, any coarse equilibria is fragile unless all priors in the support of second-order beliefs agree on the conditional distribution of extrinsic variables.

I provide a novel characterization for the a subclass of variational preferences with strictly convex divergence penalties which may be of interest on its own. This family encompasses multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), which are popular in robust control theory. I find that trade is purely fundamental if and only if the reference models agree on conditional probabilities. This is due to the strict convexity of divergence penalty (e.g., relative entropy penalties), which ensures that any belief which supports a coarse equilibrium, inherits the conditional probabilities from the reference model.

While detailed predictions about equilibrium portfolios are beyond the scope of this paper, the present results links uncertainty with trade on variables which are statistically linked with the unknown regime. In an environment of stochastic variance, uncertainty-averse agents may trade securities whose payoffs are reminiscent of *variance swaps*, where the long side of the swap is exposed to realized variance of an underlying.[11] Similarly, Drechsler (2012) shows that uncertainty-averse investors may help explain high variance premia on index options as uncertainty-averse traders demand them to hedge against stochastic volatility. This paper provides a general equilibrium foundation for non-zero trade in seemingly redundant or speculative securities under uncertainty.

The paper is organized as follows. Section 2 provides the general framework and examples with extrinsic bets under common priors. Section 3 derives the necessary and suffi-

underlying regime. Betting against precipitation allows to improve the expected utility under the drought regime at the expense of rain regime.

[11] More recent financial innovations, such as *VIX futures*, are typically advertised as providing exposure to statistical properties of an index, stock, or a currency, with minimal or no exposure to the underlying. VIX futures entitle the holder to a Dollar notional times the opening quote of the VIX index–which measures the options-implied expected volatility of the *S&P 500*–at a predetermined date.

cient conditions for immunity; first in terms of subjective beliefs, then for various popular representations. Before concluding, Section 4 characterizes global immunity properties for specific families of preferences.

2 Set-up

2.1 Preferences and Equilibrium

Let ω be a generic element of the finite state space $\{1,2,...,\Omega\}$ and denote the endowment of an agent $i \in I := \{1,\ldots,N\}$ in terms of the single consumption good by $e_i : \{1,2,...,\Omega\} \to \mathbb{R}_+$. It is convenient to define two partitions of $\{1,2,...,\Omega\}$. First, denote by $\mathscr{S}$ the partition into fundamental events. Any two outcomes ω, ω' belong to the same cell $s \in \mathscr{S}$ if and only if $e_i(\omega) = e_i(\omega')$ for all $i \in I$.

Some cells s may not be singleton sets. In this case, there are non-fundamental variables –such as horse-races, sunspot activity, data, statistics– which are not constant on s. Call $\mathscr{A}$ the partition of the state space into elementary non-fundamental or *statistical events*: any two states ω, ω' belong to the same cell $\alpha \in \mathscr{A}$ if and only if they induce identical non-fundamental outcomes. In the terminology of Milgrom and Stokey (1982), the cells in $\mathscr{A}$ are *payoff-irrelevant* events. For simplicity, I assume that the state space $\{1,2,...,\Omega\}$ is coarse enough for any two states to differ in either fundamentals, statistics, or both; such that any state ω may be represented by a characteristic vector (s,α).

Consider the set of strictly positive payoffs (strictly positive acts) $\mathscr{X} = \left\{x \in \mathbb{R}^{|\Omega|}_{++}\right\}$. Next to transitivity, completeness, and continuity, I assume that the preference relation $\succeq$ on $\mathscr{X}$ satisfies *monotonicity*, i.e., $x \geq x' \Rightarrow x \succeq x'$ and *strict convexity*, i.e., if $x \succeq x'$ then $tx + (1-t)x' \succ x'$ for all $t \in (0,1)$, whenever $x \neq x'$. By the above, $\succeq$ has a representation $V : \mathscr{X} \to \mathbb{R}$ which is increasing, continuous, and strictly quasi-concave. To summarize, I make the following assumptions throughout the paper.

Assumption. *Preferences $\succeq$ are complete, transitive, continuous, monotone, and strictly convex.*

Investors are price takers on competitive securities markets. The usual definitions for an interior equilibrium apply. I say that $\boldsymbol{x}$ is interior if it contains strictly positive payoffs for all agents $x_i \in \mathscr{X}$. Abusing notation, I denote the strictly positive elements of the simplex by $\Delta \equiv \{P : \{1,2,...,\Omega\} \to (0,1], \sum_\omega P(\omega) = 1\}$.

Definition. *Given* $\boldsymbol{e}$*, an interior allocation* $\boldsymbol{x} = (x_1, ..., x_I)$ *and a normalized price vector* $\Pi \in \Delta$ *constitute an* equilibrium *if the following conditions hold for all* i*:*

(i) $\Pi \cdot (e_i - x_i) = 0$,

(ii) $\sum_{i \in I}(e_i - x_i) = \mathbf{0}$.

(iii) $x \succ_i x_i \Longrightarrow \Pi \cdot (x - x_i) > 0$ *for all* $x \in \mathscr{X}$.

An allocation and prices $\{\boldsymbol{x}, \Pi\}$ are called *feasible*, if $\boldsymbol{x}$ is affordable (i) and Π is market-clearing (ii).[12] Condition (iii) means that demand is optimal, given prices.

2.2 An MEU Example

In this section, I preview the case of multiple-priors expected utility (Gilboa and Schmeidler, 1989), to be discussed in Section 3.2. Any two alternatives are ranked by the lowest expected utility induced by a member of a closed and convex set of priors $\mathscr{P} \subseteq \Delta$

$$V(x) = \min_{P \in \mathscr{P}} E_P u(x). \tag{1}$$

The set $\mathscr{P}$ disciplines the choice of worst case beliefs. In the limit, if $\mathscr{P} = \{P\}$ is a singleton set, preferences satisfy expected utility. In line with our assumptions $u : \mathbb{R} \to \mathbb{R}$ is increasing and strictly quasi-concave. The following examples illustrate how the present paper relates to existing literature.

Example 1. *Two MEU agents* $i = 1, 2$ *share a common outcome utility* $u(x) = \log x$*. Let there be three endowment states* $\mathscr{S} = \{\underline{b}, \overline{b}, g\}$ *with* $e_1 = (1, 0, 0)$ *and* $e_2 = (0, 1, 4)$*. Agent* 1 *is an EU maximizer* $\mathscr{P}_1 = \{P_1\}$*, with* $P_1 = (1/5, 2/5, 2/5)$*. The set of priors of agent* 2 *consists of the convex combinations between* $P_2 = (1/3, 1/3, 1/3)$ *and* $Q_2 = (1/6, 5/12, 5/12)$*,* $\mathscr{P}_2 = \{\lambda P_2 + (1-\lambda) Q_2 | \lambda \in [0, 1]\}$*.*

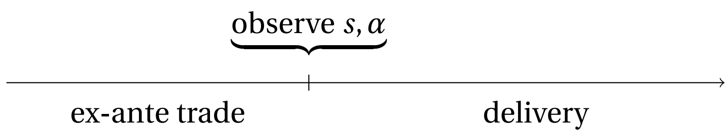
The Appendix shows that equilibrium allocation $\boldsymbol{x}$ is not constant[13] on the event $\{\underline{b}, \overline{b}\}$, even though the aggregate resources are constant and equal to 1. Equilibrium consumption is therefore not measurable with respect to aggregate resources, despite the presence of a common prior $\{P_1\} = \mathscr{P}_1 \cap \mathscr{P}_2$. In equilibrium, the unique minimizing prior of agent 2

[12] The equality in the budget constraint (i) is without loss of generality due to monotonicity.

[13] The agents take opposing sides on a "bet" on $\underline{b}$ vs $\overline{b}$ with $\boldsymbol{x} = (x_1, x_2) = \big((1/5, 1/3, 4/3), (4/5, 2/3, 8/3)\big)$.

is P_2. Since P_1 and P_2 disagree on $\{\underline{b}, \overline{b}\}$, this is an instance of the results in Strzalecki and Werner (2011).

This paper asks whether the above equilibrium is immune to the addition of new assets, whose payoffs do not depend on fundamentals $\mathscr{S}$ alone. The following Example 2 features refined assets. From here onwards, I use $P(\alpha|s) \equiv \frac{P(s,\alpha)}{P(s)}$.



Example 2. *The single difference compared to Example 1 is a contractible extrinsic variable with realizations $\alpha \in \{\beta, \gamma\}$. Conditional on states $s \in \{\underline{b}, \overline{b}\}$, all models agree on $P(\alpha|s)$ for any α. Conditional on state g, models disagree: $P_1(\beta|g) = 1/5$, $P_2(\beta|g) = 1/3$, and $Q_2(\beta|g) =$ 1/6.*

For immunity to hold, no mutually acceptable trades can exist at $\boldsymbol{x}$, if new assets are introduced. The unique minimizing priors at $\boldsymbol{x}$ remain P_1 and P_2, respectively. Consider a bet on β vs γ, conditional on g. Since x_i are constant on g, there exists a mutually acceptable bet if and only if P_1 and P_2 disagree conditional on g. Since $P_1(\beta|g) < P_2(\beta|g)$, immunity fails.

Intuitively, at $\boldsymbol{x}$, agent 2 values protection against the worst model P_2 highly. With only standard assets at hand, she achieves protection by transferring consumption towards $\overline{b}$, whose realization is likely under P_2. With the new assets, models can be targeted in an alternative way. Observing β and g together is evidence in favor P_2. Therefore, agent 2 is willing to exchange (g, γ) for (g, β) at rates which are acceptable for agent 1.

Even if agents agree on their relevant sets of priors, there may be gains from trade if either u_i or e_i are not identical. This is shown subsequently in Example 3, Section 3.2, for MEU preferences. Example 3 in the Appendix, features the identical equilibrium $\boldsymbol{x}$ from above, but derived with smooth ambiguity preferences Klibanoff et al. (2005) (Section 3.4). Agents have identical beliefs on all levels, but still, immunity fails.

3 Local Immunity

3.1 Subjective Beliefs

This section provides local conditions which are necessary and sufficient for immunity to refined trades. Beforehand, I introduce a generalized notion of risk-neutral probabilities due to Rigotti, Shannon, and Strzalecki (2008) in order to encompass kinked models, such as the MEU class.

Definition. *$\pi^x \in \Delta$ is a* subjective belief *at x if $E_{\pi^x}[x'] \geq E_{\pi^x}[x]$ for all $x' \succeq x$.*

A subjective belief at x supports the upper contour set relative to payoff $x \in \mathscr{X}$. Since, for all $x' \in \mathscr{X}$,

$$V(x) + \pi^x \cdot (x' - x) \geq V(x'), \tag{2}$$

any subjective belief coincides with a normalized *supergradient* (Rockafellar, 1970). The set $\boldsymbol{\pi}^x$ of all subjective beliefs is therefore equivalent to the normalized *superdifferential* $\partial V(x)$. By the assumptions on preferences, there exists at least one non-zero π^x for all finite payoffs in $\mathscr{X}$, which is unique if and only if V is differentiable. In the latter case, the subjective belief coincides with the normalized gradient $DV(x)$ (Rockafellar, 1970). In the polar case of differentiable expected utility, subjective beliefs are unique and simplify to the vector of normalized marginal-utility-weighted probabilities, with $u'(x(\omega))P(\omega)$ being the ωth component, before normalization.

Since subjective beliefs support the upper contour sets, the presence of a common subjective belief guarantees the absence of gains from trade. Versions of the following result, and its counterparts for Pareto optimal allocations, appear in Billot, Chateauneuf, Gilboa, and Tallon (2000), Rigotti and Shannon (2005), Rigotti, Shannon, and Strzalecki (2008), and Strzalecki and Werner (2011).

Lemma 1. *An interior and feasible $\{\boldsymbol{x}, \Pi\}$ is an equilibrium if and only if $\Pi \in \cap_{i=1}^{I} \boldsymbol{\pi}_i^{x_i}$.*

Proof. Since $\boldsymbol{x}$ is interior and feasible, conditions (i) and (ii) are satisfied. First, I show that $\Pi \in \cap_{i=1}^{I} \boldsymbol{\pi}_i^{x_i}$ implies (iii). If $\Pi \in \boldsymbol{\pi}_i^{x_i}$ for all i, then $V_i(x') > V_i(x_i) \Rightarrow E_\Pi[x'] > E_\Pi[x_i] \Rightarrow \Pi \cdot (x' - x_i) > 0$ for all i, which implies (iii). To prove that (iii) implies $\Pi \in \cap_{i=1}^{I} \boldsymbol{\pi}_i^{x_i}$, suppose instead that $\Pi \notin \boldsymbol{\pi}_i^{x_i}$ for some i. Since Π is not a subjective belief at x_i, there exists a $x' \succ_i x_i$ such that $E_\Pi[x'] \leq E_\Pi[x_i]$. Therefore $\Pi \cdot (x' - x_i) \leq 0$, a contradiction to (iii). □

If a payoff is constant on an event, then π^x may inherit conditional probabilities from $\mathscr{P}$. In particular, if all agents have MEU preferences, then $\boldsymbol{\pi}^x = \mathscr{P}$ at any risk-less x. Lemma 1 therefore confirms full insurance in the absence of aggregate risk under a common prior $\cap_{i=1}^{I}\mathscr{P}_i \neq \emptyset$. At risky payoffs, the subjective beliefs are "marginal-utility-corrected". However, within cells of $\mathscr{S}$ the correction has no effect on subjective conditional beliefs if payoffs are constant. In particular, I say that function $f : \{1,2,...,\Omega\} \to \mathbb{R}_+$ is measurable relative to partition $\mathscr{S}$ if $f(\omega) = f(\omega')$ holds whenever ω and ω' belong to the same cell $s \in \mathscr{S}$. Endowments e_i, are by definition measurable. Further, let $\overline{\mathscr{X}} \subset \mathscr{X}$ be the subset of strictly positive payoffs which are measurable.

Suppose a set of models $\mathscr{M}^x \subseteq \Delta$ and a differentiable u, strictly increasing and strictly quasi-concave, such that the subjective beliefs at x satisfy

$$\boldsymbol{\pi}^x = \Big\{\frac{\boldsymbol{q}}{\sum_\omega q(\omega)} : q(\omega) = u'(x(\omega))M(\omega)\ ;\ M \in \mathscr{M}^x\Big\}. \tag{3}$$

If condition (3) holds, then, for any $x \in \overline{\mathscr{X}}$, the marginal outcome utility function $u' \circ x$ is also measurable.[14] Any subjective belief satisfies therefore $\pi^x(\alpha|s) = M(\alpha|s)$ for some $M \in \mathscr{M}^x$.

Finally, a weaker equilibrium notion is introduced for the case when the only trades which are allowed are measurable.

Definition. $\mathscr{E} = \{\boldsymbol{x}, \Pi\}$ *is a* coarse equilibrium *if it is feasible ((i),(ii)) and the following two conditions hold:*

(iii^a) $x \succ_i x_i \Longrightarrow \Pi \cdot (x - x_i) > 0$ *for all* $x \in \overline{\mathscr{X}}$,

(iv) $x_i \in \overline{\mathscr{X}}$.

Condition (iii^a) is a weaker than optimality condition (iii) from the unrestricted equilibrium definition. It is immediate that every $\{\boldsymbol{x}, \Pi\}$ which satisfies the original equilibrium conditions (i), (ii), and (iii), can be sustained as a coarse equilibrium if it is measurable (condition (iv)). Example 2 illustrates that the distinction between a coarse equilibrium and the equilibrium is indeed meaningful.[15] The following holds thanks to the convexity assumption.

[14] I depart from Rigotti et al. (2008), whose normalization $\pi^x = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|}$ resizes vectors $\boldsymbol{q}$ to belong to the unit-sphere rather than the space of probability measures.

[15] For coarse equilibria, if $\Pi \in \Delta$ is an equilibrium price, then any $\Pi' \in \Delta$ which satisfies $\Pi(s) = \Pi'(s)$ for all s, is also an equilibrium price.

Lemma 2. *Let* $\{\boldsymbol{x},\Pi\}$ *be interior, measurable, and feasible.* $\mathscr{E}=\{\boldsymbol{x},\Pi\}$ *is a coarse equilibrium if and only if there exist* $\pi_i\in\boldsymbol{\pi}_i^{x_i}$ *such that* $\Pi(s)=\pi_i(s)$ *for all* i *and all* $s\in\mathscr{S}$*.*

Proof. Define subjective beliefs for measurable payoffs as $\overline{\boldsymbol{\pi}}^x\subseteq\Delta$, with $\pi\in\overline{\boldsymbol{\pi}}^x$, such that $E_\pi[x']>E_\pi[x]$ for all $x'\in\overline{\mathscr{X}}$ with $x'\succ x$. First, I show that $\overline{\boldsymbol{\pi}}^x=\boldsymbol{\pi}^x$ for any $x\in\overline{\mathscr{X}}$. By construction, $\overline{\boldsymbol{\pi}}^x\supseteq\boldsymbol{\pi}^x$, since $\overline{\mathscr{X}}\subset\mathscr{X}$. To show $\overline{\boldsymbol{\pi}}^x\subseteq\boldsymbol{\pi}^x$, suppose $\pi\in\overline{\boldsymbol{\pi}}^x$, but $\pi\notin\boldsymbol{\pi}^x$: then there exists a $x''\in\mathscr{X}\setminus\overline{\mathscr{X}}$, such that $x''\succ x$, but $E_\pi[x'']\leq E_\pi[x]$. Define $\overline{x}''$ through $\overline{x}''(s)\equiv E_\pi[x''|s]$. By strict convexity, $\overline{x}''\succ x''\succ x$, but $E_\pi[x'']=E_\pi[x]$. A contradiction to $\pi\in\overline{\boldsymbol{\pi}}^x$. The rest of the proof is analogous to the proof of Lemma 1, when $\boldsymbol{\pi}_i^{x_i}$ is used as the set of subjective beliefs at measurable payoffs. □

By Lemma 2, the presence of at least one consensus subjective belief $\pi=\Pi$ is necessary. The rest of subjective beliefs may be idle in equilibrium. If (3) holds, one can identify the underlying priors which actively support an equilibrium $\mathscr{E}=\{\boldsymbol{x},\Pi\}$ in the following way.[16]

Definition. $M_i^{\mathscr{E}}\in\mathscr{M}_i^{x_i}$ *is an* active prior *at* $\mathscr{E}=\{\boldsymbol{x},\Pi\}$ *if* $\Pi(s)=\frac{M_i^{\mathscr{E}}(s)u_i'(x_i(s))}{\sum_{s\in\mathscr{S}}M_i^{\mathscr{E}}(s)u_i'(x_i(s))}$ *for all* s*.*

Before stating a necessary and sufficient condition for immunity if property (3) holds, it is useful to characterize when statistics contain useful information about the true model.[17]

Definition. *Two models* $P_1\in\Delta$ *and* $P_2\in\Delta$ *are said to be* indistinguishable *if* $P_1(\alpha|s)=P_2(\alpha|s)$ *for all* $s\in\mathscr{S}$ *and any* $\alpha\in\mathscr{A}$*.*

Proposition 1. *Let the marginal utility correction* (3) *hold, and let* $\mathscr{E}$ *be a coarse equilibrium. The following conditions are equivalent:*

1. $\mathscr{E}$ *is immune.*

2. *There exists a set of indistinguishable active priors* $\{M_i^{\mathscr{E}}\}_{i\in I}$*.*

Proof. First, I show 2. ⇒ 1. Condition 2. means $M_1^{\mathscr{E}},\ldots,M_I^{\mathscr{E}},\Pi$ are indistinguishable and, therefore, $\pi_1^x,\ldots,\pi_I^x,\Pi$ are indistinguishable. Since $\mathscr{E}$ is coarse, the above implies $\Pi\in\cap_{i=1}^I\boldsymbol{\pi}_i^{x_i}\neq\emptyset$, which is a sufficient for condition 1. to hold. Now we show 1. ⇒ 2. by contraposition. If 2. is violated, then there exist at least one i, such that any $\pi\in\boldsymbol{\pi}_i^{x_i}$

[16] No matter if the alternatives belong to $\overline{\mathscr{X}}$ or to $\mathscr{X}$, the same set of beliefs in Δ supports a measurable x in the sense of (2). Therefore, it is without loss of generality to use subjective beliefs $\boldsymbol{\pi}_i^{x_i}$ for local properties at measurable payoffs for both coarse and unrestricted equilibria.

[17] Milgrom and Stokey (1982) and Strzalecki and Werner (2011) call this property *concordance*.

which satisfies $\pi(s) = \Pi(s)$ for all s, violates $\pi(\alpha|s) = \Pi(\alpha|s)$ for some α, s. This renders $\Pi \notin \cap_{i=1}^{I} \boldsymbol{\pi}_i^{x_i} \neq \emptyset$ impossible and condition 1. fails. $\square$

Proposition 1 says that, if given the chance, investors trade away from any coarse equilibrium allocation $\boldsymbol{x}$ unless there exist active priors which generate consensus conditional distributions $M_1(\cdot|s) = \ldots = M_I(\cdot|s)$ for all $s \in \mathscr{S}$. I proceed to apply these results to several classes of uncertainty preferences.

3.2 Multiple-Priors Expected Utility

In the following sections, I use of the results in Rigotti et al. (2008) on how representations shape subjective beliefs. To single out the kinks which are due to uncertainty aversion, I assume from here onwards that the outcome utility $u : \mathbb{R}_+ \to \mathbb{R}$ be differentiable.

Once corrected for marginal utility, the worst case priors are the sole determinants of subjective beliefs and condition (3) is satisfied with

$$\mathscr{M}^x = \arg\min_{P \in \mathscr{P}} E_P u(x). \tag{4}$$

This allows to use Lemma 1 to state the following result.

Proposition 2. *Let $\mathscr{E} = \{\boldsymbol{x}, \Pi\}$ be an interior coarse equilibrium and let preferences belong to the MEU class. The following conditions are equivalent:*

1. *$\mathscr{E}$ is immune.*

2. *There exists a set $\{P_i\}_{i \in I}$ such that*

   (i) for all $i \in I$: $P_i \in \mathscr{P}_i$ and $\Pi(s) = \frac{P_i(s) u_i'(x_i(s))}{E_{P_i} u_i'(x_i)}$ for all $s \in \mathscr{S}$;

   (ii) any two members of $\{P_i\}_{i \in I}$ are indistinguishable.

Proof. By Lemma 1, the proof is completed if P_i can be shown to be an active prior. By definition of an active prior at $\mathscr{E}$, $\Pi(s) = \frac{M_i^{\mathscr{E}}(s) u_i'(x_i(s))}{\sum_{s \in \mathscr{S}} M_i^{\mathscr{E}}(s) u_i'(x_i(s))}$ for all s. Hence, P_i agrees with any active prior on marginals. Since $\boldsymbol{x}$ is coarse, x_i is measurable, and, hence, if $M_i^{\mathscr{E}}$ is a minimizing prior at x_i, then so is P_i. Therefore P_i is an active prior. $\square$

A necessary and sufficient condition for $\mathscr{E}$ to be immune is that, among the minimizing priors $P_i \in \mathscr{M}_i^{x_i}$ which generates the consensus of subjective beliefs $\pi_i^{x_i}(s) = \pi_j^{x_j}(s)$ through

(3), there exists a consensus Q such that $P_i(\alpha|s) = Q(\alpha|s)$ for all i, s, α. As a prerequisite for immunity, there must therefore exist indistinguishable priors $\{P_i\}_{i\in I}$ in the respective set of priors with $P_i \in \mathscr{P}_i$. If the latter is violated, then none of the coarse equilibria is immune, irrespective of the endowments specification.

This is relevant, particularly, in the presence of an expected utility maximizer. In such a case, any agent must hold an active prior which is indistinguishable from the EU agent's prior belief. Example 1 can be used to illustrate. The unique minimizing priors at $\boldsymbol{x}$ are P_1 and P_2, respectively. These models are distinguishable, conditional on cell g. It is therefore individually optimal for $i = 1,2$ to take opposite sides of a bet on state (g, β).

For MEU preferences, and therefore the family of variational preferences at large, consider the following example, illustrated in Figure 1.

Example 3. *Two agents are identical up to the concavity of their outcome utility, with* $u_2(x) = 0.5\sqrt{x}$ *and* $u_2(x) = \log x$*. Their endowments are identical,* $e_i = (1,2,4)$*, in fundamental cells* $s \in \{a, b, c\}$*. Their sets of priors are identical. The prior beliefs about fundamentals are depicted in Figure 1. They correspond to a circle in the Machina triangle, with* $\sqrt{(P(a)-1/3)^2 + (P(c)-1/3)^2} \leq 1/6$.

Suppose for a moment that the sets of minimizing priors $\mathscr{M}_i$ had a nonempty intersection. In this case, the equilibrium unravels according to standard expected utility risk sharing with identical priors. At such an allocation, the same element of $\mathscr{P}$ minimizes both agents' expected utility if and only if the slope of their indifference curves in the Machina triangle are identical, with the slope $\frac{u_i(x_i(c))-u_i(x_i(b))}{u_i(x_i(b))-u_i(x_i(a))}$. Equilibrium consumption is a convex function of total resources for the agent whose risk-tolerance increases more with wealth (i.e., agent 1) (Leland, 1980). Therefore, by market-clearing, $\frac{x_1(b)}{x_1(a)} < 2 < \frac{x_2(b)}{x_2(a)}$ and $\frac{x_1(c)}{x_1(b)} > 2 > \frac{x_2(c)}{x_2(b)}$. For agent 1, simple algebra yields a slope exceeding $\sqrt{2}$, while it must be lower than unity for agent 2. It is easy to see that immunity fails in economies where no two measures in $\mathscr{P}$ are indistinguishable.

3.3 Variational Preferences

The representation of variational preferences (Maccheroni, Marinacci, and Rustichini, 2006) satisfies

$$V(x) = \min_{P\in\Delta} \left(E_P u(x) + c(P)\right) \tag{5}$$

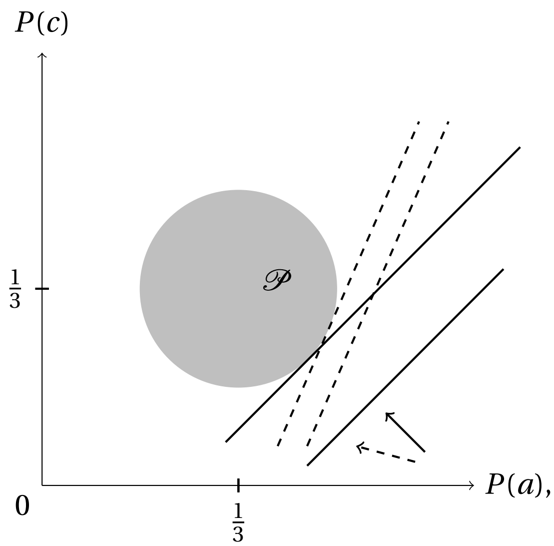


Figure 1: Iso-utility lines at an EU efficient allocation in Example 3 (solid for agent 2).

for some nonnegative, convex, lowersemicontinuous function c such that $c(P) = 0$ for some $P \in \Delta$. As before, the decision maker ranks payoffs using a worst case prior. However, the choice is disciplined by a cost function function c. An MEU preference can therefore be nested by cost function: $c(P) = 0$ if $P \in \mathscr{P}$, $c(P) = \infty$ otherwise. Accordingly, there is no hope for coarse equilibria to be immune in general, since we did not find immunity for MEU agents.

With differentiable functions u, the marginal utility correction for subjective beliefs (3) is satisfied, and the set of supporting priors minimizes the penalized expected utility

$$\mathscr{M}^x = \arg\min \left(E_P u(x) + c(P) \right). \tag{6}$$

This allows to state the following result.

Proposition 3. *Let $\mathscr{E} = \{\boldsymbol{x}, \Pi\}$ be a coarse equilibrium and let all preferences belong to the variational class. The following conditions are equivalent:*

1. *$\mathscr{E}$ is immune.*

2. *There exists a set $\{P_i\}_{i \in I}$ such that*

   (i) for all i: $P_i \in \arg\min_P c_i(P)$, subject to $\{P \in \Delta : \Pi(s) = \frac{P(s) u_i'(x_i(s))}{E_P u_i'(x_i)}$ for all $s \in \mathscr{S}\}$;

   (ii) any two members of $\{P_i\}_{i \in I}$ are indistinguishable.

Proof. Since $x \in \overline{\mathscr{X}}$, we have $E_P u(x) = E_{P_i} u(x)$ for all P_i, P which satisfy $\{P \in \Delta$ and $\Pi(s) = \frac{P(s)u_i'(x_i(s))}{E_P u_i'(x_i)}$ for all $s \in \mathscr{S}\}$. Therefore, P_i is an active prior if and only if $c_i(P_i) \leq c_i(P)$ for any model P which satisfies the previous constraint. The rest of the proof follows from Lemma 1. □

At any measurable x, the expected utility of any prior depends on marginals alone. Therefore, the burden of creating the necessary consensus lies entirely on the cost functions c_i. Specifically, these functions must all agree that they assign lowest costs to the same conditionals. Again, if there exists an expected utility agent, then all cost function must be minimized at measures which share the EU agent's conditional beliefs. Section 4 provides a more detailed discussion for popular cost-specifications.

3.4 Smooth Preferences

The axiomatization of the smooth model (Klibanoff, Marinacci, and Mukerji, 2005) employs the expected utility proposition twice. In terms of its representation, this means that the expected utility indices of candidate models $P \in \mathscr{P}$ be aggregated using weights $\mu \in \Delta(|\mathscr{P}|)$ and a strictly increasing function ϕ, to obtain

$$V(x) = E_\mu \phi(E_P u(x)). \tag{7}$$

Strict convexity can be obtained through strict convexity of the outcome utility function u. Again, agents may be heterogeneous with respect to functions u_i and ϕ_i, where concave functions imply risk aversion and ambiguity aversion, respectively.[18]

If u is differentiable, then the marginal utility correction (3) is satisfied. If also ϕ is differentiable, then there exists a unique normalized gradient. As is common in the literature, I assume that both u and ϕ are differentiable, such that (3) is satisfied with a single model $\mathscr{M}^x = \{\widehat{P}^x\}$, with

$$\widehat{P}^x = \frac{E_\mu \left[\phi'(E_P u(x))\, P\right]}{E_\mu \phi'(E_P u(x))}. \tag{8}$$

One interpretation of (8) is that of a pessimistic aggregation with weights $\mu(P)\phi'(E_P u(x))$, where low expected utility models are overweighted due to the concavity of ϕ.

[18]Two expected utility assumptions underlie representation (7). Risky payoffs are ranked by $E_P u(x)$. Payoffs which depend on θ (*second order acts* $h : \mathscr{P} \to \mathbb{R}$) are ranked by $E_\mu \widehat{u}(h(\theta))$. The standard SEU case can be nested by $\widehat{u} = u$. Otherwise, function ϕ in (7) satisfies $\widehat{u} = \phi \circ u$.

Based on Example 2, the Appendix provides an example with logarithmic u and exponential ϕ (Example 3). The parameters are calibrated to generate weighted average beliefs $\widehat{P}_i^{x_i}$ which correspond to the minimizing priors in Example 2 and the coarse equilibrium is again not robust since the measures $\pi_i^{x_i}$ are not aligned.[19] The following provides the necessary and sufficient conditions for smooth preferences.

Proposition 4. *Let $\mathscr{E} = \{\boldsymbol{x}, \Pi\}$ be a coarse equilibrium and let all preferences belong to the* Smooth *class. The following conditions are equivalent:*

1. *$\mathscr{E}$ is immune.*

2. *The weighted average beliefs $\{\widehat{P}_i^x\}_{i\in I}$ are indistinguishable.*

For linear ϕ, or for payoffs x whose expected utility does not depend on the model, the weighted average belief $\widehat{P}^x$ simplifies to $E_\mu[P]$. In general, however, the measure $\widehat{P}^x$ is endogenous and it need not belong to $\mathscr{P}$, unless $\mathscr{P}$ is convex.

If statistics do not contain extra information, i.e., if any two models $P, P' \in \cap_{i=1}^I \mathscr{P}_i$ are indistinguishable, then Proposition 1 predicts immunity. The opposite case obtains if there exist two individuals and a state s, such that no $P \in \mathscr{P}_i$ agrees with any $P' \in \mathscr{P}_j$, conditional on s. In the latter case immunity fails everywhere.

4 Global Immunity

4.1 Consistent Beliefs

This section proposes general immunity propositions for trade which hold for any specification of endowments. The strategy parallels the one proposed in Strzalecki and Werner (2011). I seek conditions such that some subjective conditional beliefs appear at all measurable payoffs. The following adapts the notion of *consistency* from Strzalecki and Werner (2011) for partition $\mathscr{S}$.

[19] Since smooth preferences with exponential ϕ belong to the variational class, Example 3 may also be reinterpreted to illustrate Proposition 4. Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) show that the costs $c(Q)$ of considering Q instead of the compounded $E_\mu[P]$ is $c(Q) = \min_{\mu'\in\Delta(|\mathscr{P}|):E_{\mu'}[P]=Q} 0.5\mathrm{R}(\mu'\|\mu)$, where R is the entropy of μ' relative to μ and where the intensity 0.5 is determined by $-\frac{\phi_2'}{\phi_2''} = 0.5$.

Definition. π_i *is an* consistent belief *for* i *if, at any* $x \in \overline{\mathscr{X}}$*, some* $\pi \in \boldsymbol{\pi}_i^x$ *is indistinguishable from* π_i*. Her beliefs are* strongly consistent *if, at any* $x \in \overline{\mathscr{X}}$*, all* $\pi \in \boldsymbol{\pi}_i^x$ *are consistent beliefs.*

Two measures are indistinguishable if they agree on conditionals. Consistency requires that there some $\pi_i(\alpha|s)$ appears at any $x \in \overline{\mathscr{X}}$. Strong consistency means that all subjective beliefs at any $x \in \overline{\mathscr{X}}$ are consistent.

The most important difference to Strzalecki and Werner (2011) is that their analysis is focused on the partition imposed by *aggregate* endowment, which satisfies $\sum_i e_i(\omega) = \sum_i e_i(\omega')$ for any two states ω, ω' which belong to the same cell in a partition I call $\mathscr{W}$. Their notions are therefore implied by the corresponding definition above. From here on, measurability and consistency is understood to apply to partition $\mathscr{S}$. The stronger properties are marked as $\mathscr{W}$-measurability and $\mathscr{W}$-consistency, respectively.

At this stage, I can link Lemma 1 with the notions of consistency and strong consistency. I say that an equilibrium $\{\boldsymbol{x}, \Pi\}$ is measurable, if all x_i are measurable. In the remainder I focus on interior equilibria.[20]

Lemma 3. *If there exists a common consistent belief, then all equilibria are measurable. If all beliefs are strongly consistent and a measurable equilibrium exists, then there exists a common consistent belief.*

Proof. Suppose a violation of condition 1. Then there exists an equilibrium $\{\boldsymbol{x}, \Pi\}$, which is not measurable, despite a common consistent belief. Call such a belief π_c. Define a feasible alternative allocation $\boldsymbol{a}$ through $a_i(\omega) = a_i(s) = E_{\pi_c}[x_i|s]$ for all $\omega \in s$ and all cells s. First I show that $\boldsymbol{a}$ is strictly preferred to $\boldsymbol{x}$ by all i whose payoffs x_i were not measurable; and weakly preferred by everyone else. Since all a_i are measurable, a common consistent belief means that at least one subjective belief is indistinguishable from π_c for all i. For all i whose payoff was not measurable, since $a_i(s) = E_{\pi_c}[x_i|s]$ and $a_i \neq x_i$, we obtain $a_i \succ_i x_i$ by strict convexity. The remaining i are indifferent as $a_i = x_i$. Finally, I show that $\{\boldsymbol{x}, \Pi\}$ violates equilibrium condition (iii) since $\Pi \cdot (a_i - x_i) \leq 0$ for some i, while $a_i \succ x_i$. By construction, $\sum_i a_i(\omega) = \sum_i x_i(\omega)$ for all i. If $\Pi \cdot (a_j - x_j) \geq 0$ for all $j \neq i$ and $\Pi \cdot (a_j - x_j) > 0$ for some j, then $\Pi \cdot (a_i - x_i) < 0$. Since there exist at least two agents whose allocation was not measurable, we have $a_i \succ x_i$ for both, a contradiction to $\{\boldsymbol{x}, \Pi\}$ being an equilibrium. To prove 2., take any measurable interior equilibrium $\{\boldsymbol{x}, \Pi\}$. By Lemma 1, there exists a

[20] This is without loss of generality if Inada conditions $\lim_{x \to 0} u'(x) = +\infty$ and $\lim_{x \to +\infty} u'(x) = 0$ are imposed on the outcome utility functions u.

common subjective belief, since $\Pi \in \cap_{i=1}^{I} \boldsymbol{\pi}_i^{x_i} \neq \emptyset$. By strong consistency, this shared belief is consistent for all i. □

The presence of a shared consistent belief means that any equilibrium, irrespective of the distribution of endowments, must be measurable. The converse requires that all individuals have strongly consistent beliefs. This set of conditions has its counterparts in Theorem 2 of Strzalecki and Werner (2011). Moreover, since $\mathscr{S}-$measurability is stronger than $\mathscr{W}-$measurability, the following link can be established directly.

Corollary 1. *If a common consistent belief exists, then all equilibria are $\mathscr{W}$-measurable.*

Finally, the information in Lemma 3 can be arranged to provide a necessary and sufficient condition for all equilibria to be measurable under strong consistency.

Lemma 4. *Suppose strongly consistent beliefs for all i. The following statements are equivalent:*

1. *There exists a measurable equilibrium.*
2. *All equilibria are measurable.*
3. *There exists a common consistent belief.*

For large classes of decision criteria the consistency properties are easily verified. An expected utility maximizer with differentiable u has unique subjective beliefs $\pi^x(\alpha|s) = P(\alpha|s)$. The unique subjective prior in $\mathscr{M}^x = \{P\}$ establishes strong consistency. Lemma 4 allows to restate the classical benchmarks from Milgrom and Stokey (1982) and Cass and Shell (1983). In the terminology of the former, statistics are *payoff-irrelevant*, in the latter statistics are a source of *extrinsic uncertainty*.

Corollary 2. *Under differentiable subjective expected utility, the following conditions are equivalent: (1) There exists a measurable equilibrium; (2) All equilibria are measurable; (3) All priors $P_1, \ldots, P_I$ are indistinguishable.*

4.2 Multiple-Priors Expected Utility

The previous sections established that consistency is not a general feature under uncertainty aversion, since $\mathscr{M}^x$, and, thus, $\boldsymbol{\pi}^x$, vary with x.[21]

At any measurable payoff $x \in \overline{\mathscr{X}}$, whether P belongs to $\mathscr{M}^x$ depends on marginals alone.[22]

Remark 1. *For MEU preferences, and any two models $P, P' \in \mathscr{P}$ which satisfy $P(s) = P'(s)$ for all $s \in \mathscr{S}$: If $x \in \overline{\mathscr{X}}$, then $P \in \mathscr{M}^x \Leftrightarrow P' \in \mathscr{M}^x$.*

Consistency requires a belief Q, whose conditionals are featured in $\mathscr{M}^x$ for all members of $\overline{\mathscr{X}}$. This can be obtained by rich enough priors in the sense that, for every $P \in \mathscr{P}$, there exist a version $P^Q \in \mathscr{P}$ which takes marginals from P and conditionals from Q with

$$P^Q(E) = \sum_{s \in \mathscr{S}} Q(E|s)P(s)$$

for any event $E \subset \{1, ..., \Omega\}$ such that $P^Q(\alpha|s) = Q(\alpha|s)$, $P^Q(s) = P(s)$. If P and Q are indistinguishable, then $P^Q = P$.[23] This property is implied by the notion of Q-stability, due to (Werner, 2011). It requires that there exist rich enough priors to satisfy the previous condition for *any* partition of $\{1, \ldots, \Omega\}$. Below, we consider the weaker form of Q-stability, which holds for $\mathscr{S}$ alone.

Definition. *The set $\mathscr{P}$ is Q-stable on $\mathscr{S}$, if there exist a $Q \in \Delta$, such that $P^Q \in \mathscr{P}$ for any $P \in \mathscr{P}$. The set $\mathscr{P}$ is rectangular, if it is Q-stable on $\mathscr{S}$ relative to any $Q \in \mathscr{P}$.*

An instance of Q-stability obtains if the set of priors is constructed from an f-divergence criterion, with $\mathscr{P} = \left\{P \in \Delta \middle| D_f(P \| Q) \leq \varepsilon\right\}$, and

$$D_f(P \| Q) = E_P f\left(\frac{P(\omega)}{Q(\omega)}\right), \tag{9}$$

[21]Take agent 2 from Example 1. At a measurable payoff with consumption (1,2,3) on $s \in (\underline{b}, \overline{b}, g)$, the unique subjective belief π_2^x inherits conditional beliefs from P. At (3,2,1), the unique subjective belief inherits conditionals from Q. Since P and Q are not indistinguishable, no consistent belief exists.

[22]The existence of a common subjective belief is therefore vulnerable to small changes in conditional beliefs for some i, leaving beliefs about $\mathscr{S}$ constant. All $\pi_i^{x_i}(s)$ remain unaffected while $\pi_i^{x_i}(\alpha|s)$ change, thus, any coarse equilibrium may cease to be an equilibrium after small perturbations of conditionals.

[23]This property is immediately satisfied if there is no conditional uncertainty, i.e., if an agents holds a unique Q.

for f increasing, convex, with $f(1) = 0$. The criterion assigns a measure of difference, relative to some reference model Q. Notable examples include the Kulback-Leibler relative entropy ($f(x) = x\ln(x)$) and total variation ($f(x) = |x-1|$). Q-stability also holds for the Choquet expected utility family Schmeidler (1989) with convex capacities, known from decision theory.

The stronger property of rectangular beliefs requires $\mathscr{P}$ to be formed through the permutations of a set of marginals with a set of conditionals. This property has been used in Epstein and Schneider (2003) in the context of belief updating and dynamic consistency. If there is no conditional uncertainty, i.e., if all models are indistinguishable, then rectangularity holds, since $P^Q = P$ for all $P, Q \in \mathscr{P}$.

In the following, I adapt the results in Proposition 4 and Theorem 5 from Strzalecki and Werner (2011) to obtain necessary and sufficient conditions for (strong) consistency.

Lemma 5. *Suppose an MEU agent with differentiable u.*

1. *There exists a consistent belief if and only if $\mathscr{P}$ is Q-stable on $\mathscr{S}$.*
2. *Beliefs are strongly consistent if and only if $\mathscr{P}$ is rectangular.*
3. *If beliefs are strongly consistent, then any $P \in \mathscr{P}$ is a consistent belief.*

Proof. First, I show property 1. Sufficiency is immediate from the utility correction (3) and Remark 1. For necessity, suppose that Q is a consistent belief, but that Q-stability fails. Then there exists a $P \in \mathscr{P}$, such that $P^Q \notin \mathscr{P}$. By strict convexity, there exists a payoff $x \in \overline{\mathscr{X}}$, such that, for all $R \in \mathscr{M}^x$, $P(s) = R(s)$ for all $s \in \mathscr{S}$. By consistency, there exists an $R \in \mathscr{M}^x$ with $R^Q \in \mathscr{M}^x$. Yet, the two measures agree on marginals, hence, $R^Q = P^Q$, and thus $P^Q \in \mathscr{M}^x \subseteq \mathscr{P}$, a contradiction. Property 2. follows from the definition of strong consistency and property 1. By Remark 1, strong consistency means that, at every $x \in \overline{\mathscr{X}}$, and for every $P \in \mathscr{P}$, there exists a subjective belief with $\pi^x(\alpha|s) = P(\alpha|s)$. Thus, every P is a consistent belief, which proves property 3. □

If $\mathscr{P}$ is Q-stable on $\mathscr{S}$, then $P^Q \in \mathscr{M}^x$ for some $P \in \mathscr{P}$, at any measurable x. If $\mathscr{P}$ is rectangular, then all conditionals which are featured in $\mathscr{P}$ are also featured in $\mathscr{M}^x$, at any measurable x. The first properties of Lemmas 3 and 5 combined, yield the following result.

Proposition 5. *Let preferences belong to the MEU class. If all $\mathscr{P}_i$ are Q-stable on $\mathscr{S}$ relative to a common belief Q, then all equilibria are measurable with respect to the aggregate endowments partition $\mathscr{W}$.*

Proof. If the sets of priors $\mathscr{P}_i$ are stable on $\mathscr{S}$ relative to a common Q, then all agents share a common consistent belief. By Lemma 3, every equilibrium must be measurable. If Q is a common consistent belief, then it is also a common $\mathscr{W}$-consistent belief. The analogous argument delivers the result. □

Since partition $\mathscr{S}$ is finer than the aggregate endowments partition $\mathscr{W}$, the Q-stability on $\mathscr{S}$ implies both consistency and $\mathscr{W}$-consistency of the subjective beliefs. In the presence of a common theory Q, on how fundamentals relate to statistics, all coarse equilibria are robust, and statistics do not offer any gains from trade. Similarly, Lemmas 4 and 5 can be combined to obtain the following.

Proposition 6. *Suppose MEU preferences and let all sets of priors $\mathscr{P}_i$ be rectangular. The following statements are equivalent.*

1. *There exists an interior measurable equilibrium.*
2. *All interior equilibria are measurable.*
3. *There exists a set $\{P_i\}_{i \in I}$, such that*
   - *(i) $P_i \in \mathscr{P}_i$ for all i;*
   - *(ii) any two members are indistinguishable.*

4.3 Variational Preferences

Before stating consistency properties, I generalize Remark 1 from the MEU model.

Remark 2. *Suppose variational preferences. For any P and Q which agree on marginals, if $c(P) \geq c(Q)$, then $P \in \mathscr{M}^x$ implies $Q \in \mathscr{M}^x$.*

Intuitively, for consistency to obtain, the disciplining function c should punish deviations from the conditional beliefs of Q. The first part of the following result is due to Theorem 5 in Strzalecki and Werner (2011). The second part strengthens the above if, controlling for marginals, the cost function obtain its unique minimum at conditional belief Q.

Lemma 6. *Let preferences belong to the variational class.*

1. *If there exists a* $Q \in \Delta$*, such that* $c(P) \geq c(P^Q)$ *for all* $P \in \Delta$*, then beliefs are consistent.*
2. *If there exist a unique* $Q \in \Delta$*, such that* $c(P) > c(P^Q)$ *for all* $P \neq P^Q$*, then beliefs are strongly consistent.*

Proof. 1. By $c(P) \geq c(P^Q)$, we obtain $E_P u(x) + c(P) \geq E_{P^Q} u(x) + c(P^Q)$ for any $x \in \overline{\mathscr{X}}$. Therefore $P \in \mathscr{M}^x \Rightarrow P^Q \in \mathscr{M}^x$. 2. suppose $P \in \mathscr{M}^x$ and $P^Q \neq P$. Since $c(P) > c(P^Q)$ implies $E_P u(x) + c(P) > E_{P^Q} u(x) + c(P^Q)$ for any $x \in \overline{\mathscr{X}}$, we have a contradiction. Therefore $P \in \mathscr{M}^x \Rightarrow P^Q = P$. □

In line with Strzalecki and Werner (2011) (Proposition 5), I use that, if c belongs to the class of f-divergence functions (9), with $c(P) = \delta D_f(P \| Q)$, then 1. is satisfied. This allows to state the following result.

Proposition 7. *Suppose divergence preferences with differentiable utility. If the reference models* $\{Q_i\}_{i \in I}$ *are indistinguishable, then all interior equilibria are* $\mathscr{W}$*-measurable.*

Proof. Since $c(P) = \theta D_f(P \| Q) = \theta E_Q f\left(\frac{P(\omega)}{Q(\omega)}\right)$ and $\theta > 0$, we only need to show that the convexity of f implies $E_Q f\left(\frac{P(\omega)}{Q(\omega)}\right) \geq E_Q f\left(\frac{P^Q(\omega)}{Q(\omega)}\right)$. Measurability follows from Lemma 3. By Jensen's inequality

$$E_Q f\left(\frac{P(\omega)}{Q(\omega)}\right) = \sum_{s=1}^{S} Q(s) \sum_{\alpha=1}^{A} Q(\alpha|s) f\left(\frac{P(\alpha|s)}{Q(\alpha|s)} \frac{P(s)}{Q(s)}\right) \geq \sum_{s=1}^{S} Q(s) f\left(\frac{P(s)}{Q(s)}\right) = E_Q f\left(\frac{P^Q(\omega)}{Q(\omega)}\right).$$

The result obtains from by Corollary 1. □

Divergence preferences are among the most widely-used variational preferences in Macroeconomics and Finance.[24] The above result states that, if the reference models Q conditionally agree on statistics, immunity obtains. More generally, any equilibrium is measurable with respect to aggregate endowments $\mathscr{W}$.

Many specifications of these preferences are based on strictly convex divergence functions f. By Jensen's inequality and Lemma 6, the resulting strong consistency can be used to state the following.

[24] Maccheroni, Marinacci, Rustichini, and Taboga (2009) show that, under the appropriate domain restrictions, and combined with a linear outcome utility function u, the Gini concentration delivers standard mean variance preferences $V(x) = E_Q[z] - \frac{\theta}{2}\mathrm{Var}_Q(x)$ due to Markowitz (1952)

Proposition 8. *Suppose divergence preferences with a strictly convex f. The following three conditions are equivalent.*

1. *There exists a measurable equilibrium.*
2. *All equilibria are measurable.*
3. *The reference models $\{Q_i\}_{i\in I}$ are indistinguishable.*

Proof. Following analogous steps as in the proof of Proposition 7, one obtains the strict inequality $E_Q f\left(\frac{P(\omega)}{Q(\omega)}\right) > E_Q f\left(\frac{P^Q(\omega)}{Q(\omega)}\right)$ for all $P \neq Q$ by Jensen's inequality and strict convexity of f. □

The most important example with strictly convex functions are multiplier preferences, axiomatized in Strzalecki (2011). These represent the robust control criterion of relative entropy $f(x) = x \log x$ (Hansen and Sargent, 2001; Hansen et al., 1999; Hansen and Sargent, 2007). An important exception obtains for total variation $f(x) = |x-1|$, as employed in Routledge and Zin (2009), which is not strictly convex.

4.4 Smooth Preferences

Even if all agents i agree on the sets of relevant models, $\mathscr{P}_i = \mathscr{P}$, in the support of μ_i, and even if the weights μ_i are identical, the weighted average beliefs $\widehat{P}_i^x$ from(8) are equilibrium objects whose properties depend on $\boldsymbol{e}$ (see Example 3). Immunity results which hold for any endowment specification are therefore unavailable, except if all beliefs in the support of the respective μ_i are indistinguishable. In the latter case, beliefs are strongly consistent since the weighted average $\widehat{P}_i^x$ shares the conditionals. The expected utility immunity proposition can therefore be extended

Proposition 9. *Suppose that all preferences satisfy smooth ambiguity aversion and that statistics have a known conditional distribution with all P in the support of μ_i indistinguishable. The following are equivalent*

1. *There exists a measurable equilibrium.*
2. *All equilibria are measurable.*

3. *All models with positive support* $\{P \in \Delta : \mu_i(P) > 0$ *for some* $i \in I\}$ *are indistinguishable.*

This has to be contrasted with the special case in the absence of aggregate risk, as discussed in Rigotti et al. (2008). There, full insurance, and therefore immunity, holds if and only if the average beliefs $\overline{P}_i = E_{\mu_i}[P]$ are identical.

5 Conclusion

Market participants are confronted with a variety of data and statistics. Within the common priors expected utility model, the relevance of any of these can be determined through a simple test: is this variable essential to determine the state of fundamentals, i.e., endowments, technology, tastes?

This paper finds that agents trade in variables for which the answer to the previous question is negative. The new reason for trade is an endogenous formation of priors under uncertainty aversion which is unrelated to frictions, divergent opinions, or non-convexities. For a wide range of settings, I find the sufficient conditions for immunity to extrinsic trades to be stringent.

The basic techniques are not specific to the atemporal setting. It is natural to conjecture that the same forces may lead to a desire to hold assets whose payoff is path dependent, such as lookback options (Conze and Vishwanathan, 1991).

A Examples

Example 1. *Suppose two agents $i = \{1,2\}$ and three endowment states $\mathscr{S} = \{a,b,g\}$. Let agent 1 be endowed with $e_1 = (e_1(a), e_1(b), e_1(g)) = (1,0,0)$ and $e_2 = (0,1,4)$. Both agents have a common outcome utility function $u(x) = \log x$. Agent 1 is ambiguity-neutral with $P_1 = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$. Agent 2 is an MEU agent with $\mathscr{P}_2 = \{\lambda P_2 + (1-\lambda)Q_2 : \lambda \in [0,1]\}$, where $P_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $Q_2 = (\frac{2}{12}, \frac{5}{12}, \frac{5}{12})$.*

First I show that $x_2(g) > x_2(b)$ for $b \in \{\underline{b}, \overline{b}\}$. This in turn determines the minimizing prior $M_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = P_2$. Suppose instead an equilibrium with $x_2(g) \leq x_2(b)$ for some b. Equilibrium price ratios equal marginal rates of substitution $\frac{\Pi(g)}{\Pi(b)} = \frac{M_i(g)}{M_i(b)} \frac{x_i(b)}{x_i(g)}$ for any minimizing belief M_i. By construction, $\frac{P(g)}{P(b)} \frac{x_2(b)}{x_2(g)} \geq 1$ for all $P \in \mathscr{P}_2$, while $\frac{P_1(b)}{P_1(g)} \frac{1-x_2(b)}{4-x_2(g)} \leq \frac{1}{4}$, a contradiction. The marginal effect of λ on expected utility is $\frac{1}{12}(u(x_2(\underline{b}) + u(x_2(\overline{b})) - 2(u(x_2(g)) < 0$. Hence the unique minimizing prior P_2. To solve for the equilibrium, I proceed as in a case with subjective expected utility beliefs P_1, P_2. With log-utility the market values of the respective asset holdings are proportional to beliefs. For agent 1 the total market value is $\Pi(\underline{b})$. Hence the equilibrium unravels via $x_1(\underline{b}) = \frac{1}{5}$. Successively using budget constraints and first-order conditions, I obtain $\{x_1, x_2, \Pi\} = \left\{(\frac{1}{5}, \frac{1}{3}, \frac{4}{3}), (\frac{4}{5}, \frac{2}{3}, \frac{8}{3}), (\frac{20}{49}, \frac{24}{49}, \frac{5}{49})\right\}$. $\square$

Example 3. *Reconsider the endowments of Example 1 and outcome utility functions u_i. Let agent 2 have smooth preferences with $\phi(u) = \frac{\exp(-4.3u)}{-4.3}$. Agents share beliefs $\mathscr{P}_1 = \mathscr{P}_2 = \{P', Q'\}$ with $P' = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $Q' = (0, \frac{1}{2}, \frac{1}{2})$. They agree on the second-order weights $\mu(P') = \frac{2}{5}$.*

It suffices to show that $\widehat{P}_2^x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = P_2$ at $x = x_2 = (\frac{4}{5}, \frac{2}{3}, \frac{8}{3})$. Let $\Delta P = P' - Q'$. By construction

$$\frac{\phi_2'(E_{P'} u(x_2))}{\phi_2'(E_{Q'} u(x_2))} = \left(x_2(a)^{\Delta P(a)} x_2(b)^{\Delta P(b)} x_2(g)^{\Delta P(g)}\right)^{-4.3} = \left(\sqrt{3/5}\right)^{-4.3} = 3.$$

The resulting corrected second-order weights satisfy $\widehat{\mu}(P') = \frac{3\mu(P')}{3\mu(P')+\mu(Q')} = \frac{2}{3}$. The weighted average beliefs therefore become $\widehat{P}_2^{x_2} = \frac{2}{3}P' + \frac{1}{3}Q' = P_2$. $\square$

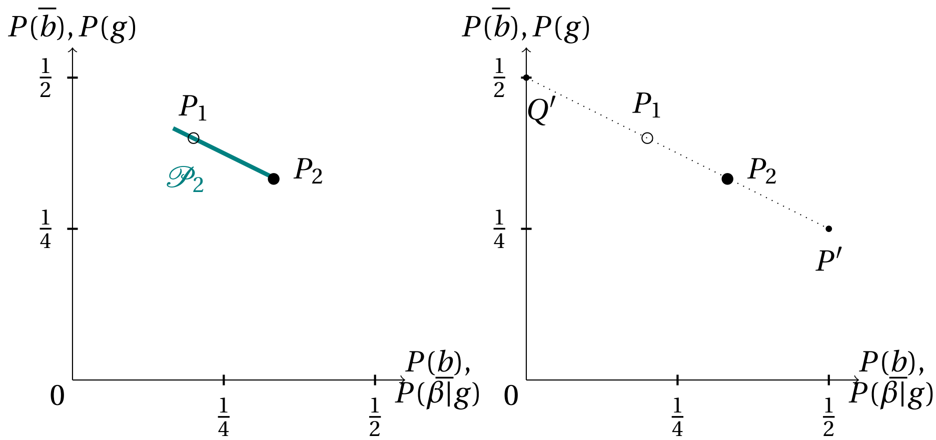


Figure 2: The priors which are effectively used in equilibrium are identical in Examples 1 (left) and 3.

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