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Economic growth and productive structure in an input–output model: An alternative coefficient sensitivity analysis.

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Abstract

Taking into account the relationship between the eigenvalue of the matrix of technical coefficients and the rate of growth in a simple Leontief model, a measure of the sensitivity of the coefficients of such matrix is proposed. This allows to determine the importance of the impact of changes in the different coefficients on the rate of production growth of the system. Moreover, an extension of this approach to the analysis of the growth of other variables linked to production is proposed.

Key words: Growth, input–output model, coefficients sensitivity, elasticities.

1. Introduction

The relationship between the maximum eigenvalue of the matrix of technical coefficients, in the Leontief model, and the degree of efficiency of the productive system it represents is well-known. The lower the dominant eigenvalue of such matrix the more productive is the system. That is, the greater the ability to generate a surplus allowing growth of the system. Furthermore, as we will see, there is a clear relationship between this eigenvalue and the rate of growth of the economic system.

It is not surprising that since the work of Sherman and Morrison (1950), who analyzed the relationship between the coefficients of the inverse of a matrix and the changes in the original matrix without recalculating the inverse, various applications were developed in the input–output framework. Applications designed to gauge the effect of technical coefficients change on the elements of the Leontief inverse. Since the pioneering work of Evans (1954) there have been several applications aimed at determining the "important coefficients" of a technological matrix[1].

We propose an alternative (or complementary) methodological approach to analyze the sensitivity of technical coefficients in the Leontief model, as a basis for determining the important coefficients. The method is based on the relationship between the technical coefficients as elements of a matrix and the maximum eigenvalue of such matrix, considering the importance of such value for system growth. In this regard, we are interested in the role played by the different coefficients in production structure from the perspective of production expansion.

Moreover, there is a close relationship between production levels and the size of other variables. From this point of view, it is convenient to analyze the importance of the different transactions within the production system in relation to the likely expansion or contraction of the volume of transactions.

In the following discussion we will show first, Section 2, the relationship between the matrix of technical coefficients and the rate of growth of the system in a simple Leontief model. Section 3 shows the process to compute coefficients sensitivity matrix and its interpretation. Section 4 proposes an extension of the approach to other variables.

2. Balanced growth in a simple Leontief model

Our starting point is the well-known open Leontief model:

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{y} \qquad (1)$$

where $\mathbf{x}$ is a column vector whose elements show the production of the n sectors or productive branches constituting the economy. The column vector $\mathbf{y}$ denotes the

[1] An interesting review of this issue can be found in Tarancón Morán et al. (2008).

productive surplus of the n sectors, which constitute the final demand of the economy. **A** is an n x n matrix whose characteristic element a_{ij} denotes the production units of sector i used in the production of one unit of sector j. It is then a matrix showing the technology of the productive system analyzed. Therefore, **Ax** is the vector of the production used as intermediate inputs. The solution of the system (1) if the technical coefficients of matrix **A** remain unchanged is then:

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \qquad (2)$$

where $(\mathbf{I} - \mathbf{A})^{-1}$ is the Leontief inverse.

From expression (1) it is possible to design a simple model of balanced growth and establish the conditions under which such growth is possible[2].

In expression (1) total output is consumed in the same period as intermediate inputs and final demands. However, as noted by Nikaido (1978; 134), "generally, consumption comes after production". A lag that causes production to take place in t and consumption in $t + 1$. Equation (1) can be rewritten as:

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t+1} + \mathbf{y}_{t+1} \qquad (3)$$

If the economy expands at a uniform rate α, then:

$$\begin{aligned}(1+\alpha)x_{i,t} &= x_{i,(t+1)} \\ (1+\alpha)y_{i,t} &= y_{i,(t+1)}\end{aligned} \qquad (4)$$

and we can write:

$$\mathbf{x}_t = (1+\alpha)(\mathbf{A}\mathbf{x}_t + \mathbf{y}_t) \qquad (5)$$

Defining $\rho = 1/(1+\alpha)$, we obtain:

$$(\rho\mathbf{I} - \mathbf{A})\mathbf{x}_t = \mathbf{y}_t \qquad (6)$$

[2] For a simple growth model in the input–output framework see e.g., Nikaido (1978), Takayama (1985); and in a Leontief-Sraffa model, see Pasinetti (1975).

That is a relationship between production and demand in the same period. The only condition for a non-negative solution for $\mathbf{x_t}$ when $\mathbf{y}_t > 0$ is:

$$\rho = \frac{1}{1+\alpha} > \lambda_{máx}(A) \qquad (7)$$

Where λ_{max} is the máximum eigenvalue of matrix **A**.

If $\mathbf{y} = 0$, we can write:

$$\mathbf{Ax} = \lambda_{max}\mathbf{x} \qquad (8)$$

Then, it can be deduced that the maximum rate of growth of the system, α_{max}, is:

$$\alpha_{máx} = \frac{1}{\lambda_{máx}} - 1 \qquad (9)$$

There is an inverse relationship between the maximum rate of growth of the system and the spectral radius of the matrix of technical coefficients. The eigenvalue of matrix **A** depends on its elements, the technical coefficients. Any small change in one of these coefficients will cause a general change in the eigenvalues of the matrix and, in short, in its maximum eigenvalue, affecting thus to the rate of growth of the system. The impact of the change in a coefficient on the maximum eigenvalue of **A** and, thus, in the rate of maximum growth of the system, will depend on the position and the role that such coefficient plays in the productive structure of such system[3]. As Dietzenbacher (1992) notes, there is a correspondence, taking into account Perron-Frobenious theorem, between the dominant eigenvalue of the matrix and forward and backward linkages. As the author highlights, an increase (decrease) in any element a_{ij} of **A** induces an increase (decrease) in the dominant eigenvalue λ. This has an evident relationship with the variations experienced by the abovementioned linkages.

Given that α is the rate of uniform growth of the system, our problem is: which is the importance of each one of the coefficients for growth? We want to show the sensitivity of λ in response to a small change in any coefficient of matrix **A**. That is, we can

[3] The relevant role that plays the disturbance of the input–output matrix in the Perron-Frobenius root and the Perron vector is widely developed in Dietzenbacher (1988).

undertake a sensitivity analysis showing the importance of each coefficient as regards the growth of the system. In the next section we propose an analytical technique to address this sensitivity analysis.

3. Growth and change in productive structure

We will first focus on the increase in production. We will show later that the application can be extended to any variable related with production structure.

Let **q** and **z**, respectively, the left and right eigenvectors associated to λ_{max}. Then, if $\eta = \lambda_{max}$ we can write:

$$\mathbf{Az} = \eta \mathbf{z} \qquad (10)$$

as regards the left eigenvector:

$$\mathbf{q}'\mathbf{A} = \eta \mathbf{q}' \qquad (11)$$

The total differential of (10) leads to:

$$\dot{\mathbf{A}}\mathbf{z} + \mathbf{A}\dot{\mathbf{z}} = \dot{\eta}\mathbf{z} + \eta\dot{\mathbf{z}} \qquad (12)$$

Multiplying by the left eigenvector **q**¸we obtain:

$$\mathbf{q}'\dot{\mathbf{A}}\mathbf{z} + \mathbf{q}'\mathbf{A}\dot{\mathbf{z}} = \mathbf{q}'\dot{\eta}\mathbf{z} + \mathbf{q}'\eta\dot{\mathbf{z}} \qquad (13)$$

Notice that the second terms in both sides of the equation are the same and thus:

$$\mathbf{q}'\dot{\mathbf{A}}\mathbf{z} = \mathbf{q}'\dot{\eta}\mathbf{z} \qquad (14)$$

$$\dot{\eta} = \frac{\mathbf{q}'\dot{\mathbf{A}}\mathbf{z}}{\mathbf{q}'\mathbf{z}} \qquad (15)$$

Equation (15) provides the change experienced by the dominant eigenvalue and the rate of growth (following equation (9)) when there is a change in technical coefficients. Therefore, we have a measure of the importance of each coefficient according to the magnitude of its impact on system growth.

If only one technical coefficient changes, the sensitivity of a coefficient s_{ij} is the partial derivative of λ_{max} with respect to a_{ij}. That is:

$$s_{ij} = \frac{\partial \eta}{\partial a_{ij}} = \frac{q_i z_j}{\mathbf{q}'\mathbf{z}} \qquad (16)$$

We can then obtain the sensitivity matrix:

$$\mathbf{S} = \frac{1}{\mathbf{q}'\mathbf{z}}\mathbf{q}\mathbf{z}' \qquad (17)$$

Matrix **S** has often been used in Biology for the analysis of populations[4].

However, the sensitivity matrix presents the problem of non-standardization of its elements and so the problem of comparability. Indeed, from a comparative perspective, the results in absolute terms may lead to a wrong assessment of the importance of the various coefficients s_{ij}. The problem is easy to solve, it requires to turn the sensitivity matrix into an elasticity matrix. The elasticity would measure the proportional change experienced by λ_{max} in response to a proportional change of any coefficient. That is:

$$\varepsilon_{ij} = \frac{a_{ij}}{\eta}\frac{\partial \eta}{\partial a_{ij}} \qquad (18)$$

The matrix of elasticities can be computed as:

$$\mathbf{E} = \frac{1}{\eta}\mathbf{S} \circ \mathbf{A} \qquad (19)$$

where $\circ$ expresses the Hadamard product.

A relevant characteristic of the matrix of elasticities is that the sum of elasticities is equal to one:

$$\sum_{ij} \varepsilon_{ij} = 1 \qquad (20)$$

[4] See e.g., Caswell (2001) and Caswell and Trevisan (1994).

The proposed an analysis can be extended to analyze the relationship between the productive structure and other variables closely related to this structure. This is the case of employment, pollution, income, etc. Next section develops this extension.

4. Extensions of the analysis

Let f_i be the quantity of employment, energy, value-added, pollution, etc., required or generated per unit or production of sector *i*. Let **f** be the column vector (*n x 1*), assuming that the economy is composed of *n* productive sectors, whose characteristic element is f_i. We can then run the following similarity transformation:

$$\mathbf{F} = \hat{\mathbf{f}}\mathbf{A}\hat{\mathbf{f}}^{-1} \qquad (21)$$

^ shows the expression of a vector as a diagonal matrix.

The characteristic element f_{ij} of matrix **F** denotes now, for example, the quantity of energy consumed by sector *i* per unit of energy consumed by sector *j* in the process of obtaining one unit of product. The same can be computed for employment or any other variable linked to the production of *j*.

A and **F** are similar matrixes and, therefore, their eigenvalues are equal. That is $\lambda_{\max}(\mathbf{A}) = \lambda_{\max}(\mathbf{F}) = \eta$.

Let $q^{f\,\prime}$ and z^f be the eigenvectors associated to the maximum eigenvalue of matrix **F**. Then:

$$\mathbf{S}^f = \frac{\mathbf{1}}{\mathbf{q}^{f\,\prime}\mathbf{z}^f}\mathbf{q}^f\mathbf{z}^{f\,\prime} \qquad (22)$$

This new sensitivity matrix indicates the degree of importance of each coefficient of the productive structure with respect to the variable *f*.

We can then obtain the new matrix of elasticities:

$$\mathbf{E}^f = \frac{1}{\eta}\mathbf{S}^f \circ \mathbf{F} = \frac{1}{\eta}\mathbf{S}^f \circ \hat{\mathbf{f}}\mathbf{A}\hat{\mathbf{f}}^{-1} \qquad (23)$$

5. Conclusions

We have obtained an assessment of the technical coefficients of an input-output matrix in relation to the production of goods and services in a particular economic system. We have proposed a measure that indicates the importance of technological coefficients independently of the structure of final demand at a given time. Assessment that links the maximum eigenvalue of the matrix of technical coefficients, assuming a stable growth path of the economy, with the rate of growth of the economic system.

Using the characteristics of the similar matrixes, we have extended the analysis to other variables such as employment, electricity consumption, pollution, income, etc. The proposal could serve as basis for more sophisticated developments.

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