An Armington-Leontief Model

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Abstract: We develop a novel linear equilibrium model with an Armington flavor. We show (1) that the standard multiplier matrix arises as a special case of this new model and (2) that this model allows the computation of multiplier effects with no external output bias, which is particularly relevant for applied analysis. We also provide (3) a mathematical proof of the solvability of the model and the non-negativity of the newly derived multiplier matrix that results from the model's equilibrium solution.

Keywords: Armington principle; extended linear equilibrium model; technological productivity.

JEL codes: C62, C67, D57

Highlights:

- We incorporate the empirically relevant distinction between domestic and imported production,
- We propose a more precise calculation of multiplier effects for empirical applications,
- We provide a mathematical proof of the model's economic consistency.

1. Introduction

Linear general equilibrium models provide a simple and transparent platform for economic analysis and policy evaluation. The best-known linear model is the classic Leontief model (Leontief, 1966; Miller and Blair, 2009). This model has the nice property that yields a reduced form that allows for the calculation of output multipliers in response to demand-driven changes, such as those initiated by discretionary government expenditure policies. We sometimes fail to distinguish that the total supply of output is the aggregation of domestic and imported outputs. From an evaluation perspective, however, the relevant triggered effect that one wishes to measure is on the domestic component of output, not on total output. Indeed, domestic output summarizes the internal economic response to any changes originating in final demand, once the economy has absorbed all the general equilibrium interactions.

The Armington (1969) principle captures that total supply in an economy is, in fact, an aggregate of domestic and imported foreign outputs. In its most general formulation, domestic and foreign goods are substitutes and the final

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composition of total output responds to both market signals (relative prices) and technology (substitution elasticities). We therefore incorporate the Armington principle into the linear Leontief model in a manner that is conformal with the classical structure of the linear model: we assume perfect complementarity between domestic and foreign outputs (no substitution is permitted). In the next Section, we formulate some preliminaries needed later on. In Section 3 we undertake the economics of the model whereas in Section 4 we show the mathematics of the main theorem justifying the existence and non-negativity of the equilibrium in this variant of the linear model.

2. Preliminaries

Definition 1: A *linear* economy is a pair (**A**, **f**) with **A** being a $(n \times n)$ non-negative square matrix and **f** a $(n \times 1)$ non-negative column vector[1]. Matrix **A** $=(a_{ij})$ represents the available technology with a_{ij} indicating the minimal amount of good i (as input) needed to generate a unit of good j (as output). Vector **f**, in turn, represents final demand for goods.

Definition 2: The economy (**A**, **f**) is in *balance* if for the given final demand vector **f** there is a non-negative vector **x** such that $\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f}$. Vector **x** represents total output in the economy and $\mathbf{A} \cdot \mathbf{x}$ indicates the part of total output that is needed to produce it (intermediate demand). In a balanced state, total supply **x** is equal to total demand, i.e. the sum of intermediate $\mathbf{A} \cdot \mathbf{x}$ and final **f** demands.

Definition 3: The technology **A** is *productive* if for any non-negative vector of final demand **f** there is a non-negative output vector **x** such that the economy is in balance.

Property 1: These three statements are equivalent (Nikaido, 1972, thm. 17.1):

(i) The technology **A** is productive;

(ii) The maximal eigenvalue of **A** satisfies $\lambda(\mathbf{A}) < 1$;

(iii) The inverse matrix $(\mathbf{I} - \mathbf{A})^{-1}$ exists and is non-negative.

Definition 4: An *input-output table* is a collection of economic data that satisfies $\mathbf{x} = \mathbf{Z} \cdot \vec{1} + \mathbf{f} = \vec{1} \cdot \mathbf{Z}^{\mathrm{T}} + \mathbf{v}$, where **x, f** and **v** are observed vectors of total output, final demand and value-added. In empirical applications, aggregation is such that all these vectors are typically positive. The non-negative matrix **Z**$=(z_{ij})$ shows all intermediate transactions taking place between sectors i and j.

[1] Notational conventions: For two vectors **x** and **y**, $\mathbf{x} < \mathbf{y}$ means $x_i < y_i$ for all i; $\mathbf{x} \leq \mathbf{y}$ means $x_i \leq y_i$ for all i with $x_k \neq y_k$ for some k; finally, $\mathbf{x} \leqq \mathbf{y}$ means $x_i \leq y_i$ for all i. The same type of considerations applies to matrices. The vector $\vec{1}$ represents the column unit vector. Any vector **x** can be rewritten in the format of a diagonal matrix $\hat{\mathbf{X}}$. Given a square matrix **S**, its inverse is given by $\mathbf{S}^{-1}$ (if it exists). If **x**>0, then $\hat{\mathbf{X}}^{-1}$ exists. **I** denotes the identity matrix and $\mathbf{S}^{\mathrm{T}}$ the transpose of **S**.

Property 2: Let us consider an empirical input-output table that presents positive levels of total output and final demand; if we define the technology by $\mathbf{A} = \mathbf{Z} \cdot \hat{\mathbf{X}}^{-1}$ and assume constant returns to scale we obtain a balanced empirical economy.

Proof: We first observe that $\vec{1} = \hat{\mathbf{X}}^{-1} \cdot \mathbf{x}$ (since all observed outputs are assumed to be positive). Hence $\mathbf{x} = \mathbf{Z} \cdot \vec{1} + \mathbf{f} = \mathbf{Z} \cdot \hat{\mathbf{X}}^{-1} \cdot \mathbf{x} + \mathbf{f} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f}$.

Notice that in this empirical economy all technical coefficients $a_{ij}=z_{ij}/x_j$ in matrix $\mathbf{A}$ are well defined since for all j $x_j>0$.

Property 3: If $\mathbf{A}$ is the technology matrix of a balanced empirical economy, then $\mathbf{A}$ is productive.

Proof: Post-multiply $\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f}$ by $\hat{\mathbf{X}}^{-1}$ and obtain $\vec{1} = \mathbf{A} \cdot \vec{1} + \mathbf{f} \cdot \hat{\mathbf{X}}^{-1} > \mathbf{A} \cdot \vec{1}$ since both $\mathbf{f}$ and $\mathbf{x}$ are assumed positive. From the inequality $\vec{1} > \mathbf{A} \cdot \vec{1}$ we verify that the Brauer-Solow sufficient condition (Solow, 1952) holds, which implies $\lambda(\mathbf{A}) < 1$.

3. The Armington-Leontief model

In this Section we assume we can perform all the required matrix operations and algebra. The standard linear model outlined above corresponds to a fully closed (no trade) economy. We now introduce the empirically relevant distinction that there are two sources of output, domestic output $\mathbf{x}^d$ and imports $\mathbf{x}^m$. Total output satisfies:

$$\mathbf{x} = \mathbf{x}^d + \mathbf{x}^m = \mathbf{Z} \cdot \vec{1} + \mathbf{f} \tag{1}$$

The technology matrix $\mathbf{A}$ must now capture the domestic production function and for this we need to define $\mathbf{A}$ in relation to domestic output $\mathbf{x}^d$, not total output $\mathbf{x}$. We now have:

$$\mathbf{A} = (a_{ij}) = z_{ij}/ x_j^d$$

Alternatively:

$$\mathbf{Z} \cdot \vec{1} = \mathbf{A} \cdot \mathbf{x}^d \tag{2}$$

From expressions (1) and (2) we find:

$$\mathbf{f} = \mathbf{x} - \mathbf{Z} \cdot \vec{1} = (\mathbf{x}^d + \mathbf{x}^m) - \mathbf{A} \cdot \mathbf{x}^d = (\mathbf{I} - \mathbf{A}) \cdot \mathbf{x}^d + \mathbf{x}^m \tag{3}$$

If the inverse of ($\mathbf{I}$ – $\mathbf{A}$) exists we obtain:

$$(\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{f} = \mathbf{x}^d + (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{x}^m \tag{4}$$

We introduce the Leontief inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ and we use it solve for domestic output:

$$\mathbf{x}^d = \mathbf{L} \cdot \mathbf{f} - \mathbf{L} \cdot \mathbf{x}^m \tag{5}$$

We now invoke the Armington (1969) assumption in a fixed coefficients setting. Domestic and imported output will now be linear functions of total output:

$$x_j^d = \alpha_j^d \cdot x_j$$
$$x_j^m = \alpha_j^m \cdot x_j$$

with the proportionality factors being the shares of domestic and imported output over total output. If we write the shares in two diagonal matrices $\hat{\alpha}^d, \hat{\alpha}^m$ we obtain:

$$\begin{aligned} \mathbf{x}^d &= \hat{\alpha}^d \cdot \mathbf{x} \\ \mathbf{x}^m &= \hat{\alpha}^m \cdot \mathbf{x} \end{aligned} \tag{6}$$

Let us assume for the time being that all shares are positive, i.e. $\hat{\alpha}^d, \hat{\alpha}^m > 0$. We now substitute the first equation in (6) into the second one:

$$\mathbf{x}^m = \hat{\alpha}^m \cdot \mathbf{x} = \hat{\alpha}^m \cdot (\hat{\alpha}^d)^{-1} \cdot \mathbf{x}^d = \hat{\beta} \cdot \mathbf{x}^d \tag{7}$$

Here in expression (7) $\hat{\boldsymbol{\beta}}$ is a positive diagonal matrix with entries $\beta_{jj} = \alpha_j^m / \alpha_j^d$. We now use (7) to transform expression (4):

$$\mathbf{x}^d = \mathbf{L} \cdot \mathbf{f} - \mathbf{L} \cdot \mathbf{x}^m = \mathbf{L} \cdot \mathbf{f} - \mathbf{L} \cdot \hat{\beta} \cdot \mathbf{x}^d \tag{8}$$

Solving for domestic output we find:

$$\mathbf{x}^d = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} \cdot \mathbf{f} \tag{9}$$

We will refer to the matrix given by:

$$\mathbf{M} = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} \tag{10}$$

as the Armington-Leontief multiplier matrix. It links the vector of final demand **f** with the vector of domestic output $\mathbf{x}^d$.

We now explore the relationship between **M** and the standard multiplier matrix as captured by the Leontief inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. We consider two polar cases to provide limit bounds for **M**; first we go to one extreme and make all $\beta_{jj} \to 0$, and then we consider the other extreme case with all $\beta_{jj} \to \infty$.

Property 4: Limit bounds for **M**:

(i) if $\beta_{jj} \to 0$ for all j then $\mathbf{M} \to \mathbf{L}$,

(ii) if $\beta_{jj} \to \infty$ for all j then $\mathbf{M} \to \mathbf{0}$.

Proof: The first statement turns out to be trivial and follows directly from expression (10). To check statement (*ii*) we will assume for the sake of the current argument that the inverse matrix $(\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1}$ exists. Since the identity **I** is (trivially) invertible and $\mathbf{L} \cdot \hat{\beta}$ is invertible (provided **A** is productive, Property 1, and $\hat{\beta} > 0$), we can use a version of the matrix inversion lemma of Henderson and Searle (1981) that states that the inverse of a sum of invertible matrices can be written as:

$$(\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} = (\mathbf{L} \cdot \hat{\beta})^{-1} \cdot (\mathbf{I}^{-1} + (\mathbf{L} \cdot \hat{\beta})^{-1})^{-1} \cdot \mathbf{I}^{-1}$$

We reorder and simplify a little bit:

$$\mathbf{M} = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} = \hat{\beta}^{-1} \cdot \mathbf{L} \cdot (\mathbf{I} + \hat{\beta}^{-1} \cdot \mathbf{L})^{-1} \cdot \mathbf{L}$$

Notice that $\beta_{jj} \to \infty$ implies $\hat{\beta}^{-1} \to \mathbf{0}$ and then $\mathbf{M} \to \mathbf{0}$.

The economic interpretation is straightforward. In a fully closed (no trade) economy, i.e. $\hat{\beta} \to \mathbf{0}$, the Armington-Leontief multiplier matrix $\mathbf{M}$ coincides with the standard Leontief inverse $\mathbf{L}$. Should all the domestic production be progressively eliminated and imports be increasingly dominant, $\hat{\beta}^{-1} \to \mathbf{0}$, then there would be no domestic multiplier effect whatsoever as a result of changes in final demand. All impulses from final demand would leak outside the economy.

4. The main analytical result

<u>Property 5</u>: If the non-negative matrix $\mathbf{A}$ is productive and the shares satisfy $\hat{\beta} > \mathbf{0}$, then the multiplier matrix $\mathbf{M}$ exists and is non-negative.

<u>Proof</u>: Recall first that if $\mathbf{A}$ is productive the inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ exists and is non-negative. Hence trivially $\mathbf{L}^{-1}$ also exists and is equal to $(\mathbf{I} - \mathbf{A})$. Additionally, it is always the case that:

$$\mathbf{I} - \mathbf{A} + \hat{\beta} = \mathbf{L}^{-1} + \hat{\beta} = \mathbf{L}^{-1}(\mathbf{I} + \mathbf{L} \cdot \hat{\beta})$$

If $\mathbf{I} - \mathbf{A} + \hat{\beta}$ should happen to be invertible, then we would have:

$$(\mathbf{I} - \mathbf{A} + \hat{\beta})^{-1} = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} = \mathbf{M}$$

and the multiplier matrix $\mathbf{M}$ would be recovered. We therefore need to see that matrix $\mathbf{I} - \mathbf{A} + \hat{\beta}$ is indeed invertible. Notice that $\hat{\beta} > \mathbf{0}$ implies the diagonal matrix $\hat{\rho}$ defined by $\rho_{ii} = 1 + \beta_{ii}$ satisfies $\hat{\rho} > \mathbf{I}$. From here we can write:

$$\mathbf{I} - \mathbf{A} + \hat{\beta} = (\mathbf{I} + \hat{\beta}) - \mathbf{A} = \hat{\rho} - \mathbf{A} = \hat{\rho} \cdot (\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})$$

Since the diagonal matrix $\hat{\rho}$ is clearly invertible, non-negative, and $\mathbf{0} < \hat{\rho}^{-1} < \mathbf{I}$ all that remains to check is that matrix $(\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})$ is invertible too. For this we invoke the property that eigenvalues for non-negative matrices are a non-decreasing function of the matrix coefficients (Nikaido, 1972, thm. 17.1). In this case from $\mathbf{0} < \hat{\rho}^{-1} < \mathbf{I}$ we verify that $\mathbf{0} < \hat{\rho}^{-1} \cdot \mathbf{A} < \mathbf{A}$. From this result and the fact that $\mathbf{A}$ is productive it follows that:

$$\lambda(\hat{\rho}^{-1} \cdot \mathbf{A}) \leq \lambda(\mathbf{A}) < 1$$

Property 1.*iii* now implies that the inverse of matrix $(\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})$ exists and is non-negative. Therefore the multiplier matrix $\mathbf{M}$ exists, is non-negative and equal to:

$$\mathbf{M} = (\mathbf{I} - \mathbf{A} + \hat{\beta})^{-1} = (\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})^{-1} \cdot \hat{\rho}^{-1}$$

Remark 1: The result that **M** exists and is non-negative ensures that the inverse of $(\mathbf{I}+\mathbf{L}\cdot\hat{\beta})$ also exists.

Remark 2: Notice that without loss of generality we can relax the restriction that $\hat{\beta}>\mathbf{0}$ to $\hat{\beta}\geq\mathbf{0}$. The only change would be that now $\hat{\rho}^{-1}\cdot\mathbf{A}\leq\mathbf{A}$ but the maximal eigenvalue of matrix $\hat{\rho}^{-1}\cdot\mathbf{A}$ would still be less than 1. Hence, productivity of matrix $\hat{\rho}^{-1}\cdot\mathbf{A}$ is guaranteed. The relaxation is relevant for empirical analysis since in these cases sectoral aggregation is selected such that for all i $x_i^d>0$ whereas $x_i^m\geq 0$.

Remark 3: If matrices $\hat{\rho}^{-1}\cdot\mathbf{A}$ and $\mathbf{A}$ are both productive they can be expanded in convergent matrix series. Since $\hat{\rho}^{-1}\cdot\mathbf{A}\leq\mathbf{A}$ it follows that:

$$\sum_{k=0}^{\infty}(\hat{\rho}^{-1}\cdot\mathbf{A})^k=(\mathbf{I}-\hat{\rho}\cdot\mathbf{A})^{-1}\leq\sum_{k=0}^{\infty}\mathbf{A}^k=\mathbf{L}$$

Now $(\mathbf{I}-\hat{\rho}\cdot\mathbf{A})^{-1}\leq\mathbf{L}$ and $\mathbf{0}<\hat{\rho}^{-1}<\mathbf{I}$ imply that $\mathbf{M}\leq\mathbf{L}$. Thus matrix **L** is effectively an upper bound for matrix **M**. If in empirical analysis we use **L** when **M** is in fact called for, an evaluation error will ensue for we would be upward biasing the multiplier estimates and any results that derive from them. The size of the error will depend on the degree of openness of the economy. The more open to trade the economy, the larger the evaluation bias.

Remark 4: The standard Leontief system $(\mathbf{I}-\mathbf{A})\cdot\mathbf{x}=\mathbf{f}$ is a particular case of the more general equation $(\rho\cdot\mathbf{I}-\mathbf{A})\cdot\mathbf{x}=\mathbf{f}$ when the real number ρ satisfies $\rho=1$. This more general system is said to be *solvable* if for any non-negative vector **f** there is a non-negative vector **x** such that $(\rho\cdot\mathbf{I}-\mathbf{A})\cdot\mathbf{x}=\mathbf{f}$ holds. A well-known theorem (Nikaido, 1968, thm. 15.3) establishes that solvability is equivalent to the matrix $(\rho\cdot\mathbf{I}-\mathbf{A})$ satisfying the Hawkins-Simon (1949) conditions. This property is readily extended to our Armington-Leontief model. Indeed, equation (9) can be rewritten as $\mathbf{x}^d=(\hat{\rho}-\mathbf{A})^{-1}\cdot\mathbf{f}$ and invertibility of $(\hat{\rho}-\mathbf{A})$ produces the linear system $(\hat{\rho}-\mathbf{A})\cdot\mathbf{x}^d=\mathbf{f}$. This system is solvable if and only if matrix $(\hat{\rho}-\mathbf{A})$ satisfies the Hawkins-Simon condition.

Remark 5: Thm. 6.4 in Nikaido (1968) can also be painstakingly extended to the more general case we study here. Let us define the matrix sequence

$$\mathbf{T}_k=\hat{\rho}^{-1}\cdot\sum_{s=0}^{k}(\hat{\rho}^{-1}\cdot\mathbf{A})^k$$

If matrix $(\hat{\rho}-\mathbf{A})$ has a non-negative inverse, then the diagonal matrix satisfies $\hat{\rho}>\mathbf{0}$ and the sequence $\{\mathbf{T}_k\}$ converges to $(\hat{\rho}-\mathbf{A})^{-1}$. And reciprocally, if $\hat{\rho}>\mathbf{0}$ and the sequence $\{\mathbf{T}_k\}$ is convergent, then $(\hat{\rho}-\mathbf{A})$ has a non-negative inverse and the limit of the sequence is the inverse $(\hat{\rho}-\mathbf{A})^{-1}$.

Remark 6: For the general linear equilibrium case of the Armington-Leontief model, with equation $(\hat{\rho} - \mathbf{A}) \cdot \mathbf{x}^d = \mathbf{f}$, no condition relating the maximal eigenvalue of $\mathbf{A}$ in relation to the eigenvalues of matrix $\hat{\rho}$ seem to arise (or we have not been able to find). In the standard linear system case, it is known that solvability of the system is equivalent to the maximal eigenvalue of matrix $\mathbf{A}$ satisfying the condition $\lambda(\mathbf{A}) < \rho$ (Nikaido, 1972, thm. 17.1). This provision is clearly and trivially the same as $\lambda(\mathbf{A}) < \lambda(\rho \cdot \mathbf{I})$. However, the conjecture that $\lambda(\mathbf{A}) < \lambda(\hat{\rho})$ would also suffice for solvability in the new setup does not hold. See the counterexample in Section 5.

5. Examples[2]

Example 1: Consider a simple input-output (IOT) table with two sectors ("iron" and "wheat") and total output comprising domestic and imported outputs:

IOT	Iron	Wheat	Demand	Output
Iron	40	10	50	100
Wheat	30	50	20	100
Labor	20	10		
Imports	10	30		
Output	100	100		

For this linear economy we have total output, imports, domestic output, and final demand and intermediate flows equal to:

$$\mathbf{x} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} \quad \mathbf{x}^m = \begin{pmatrix} 10 \\ 30 \end{pmatrix} \quad \mathbf{x}^d = \mathbf{x} - \mathbf{x}^m = \begin{pmatrix} 90 \\ 70 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 50 \\ 20 \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} 40 & 10 \\ 30 & 50 \end{pmatrix}$$

The technology matrices $\mathbf{A}$, $\hat{\beta}$ and $\hat{\rho}$ are given by:

$$\mathbf{A} = \mathbf{Z} \cdot (\hat{\mathbf{X}}^d)^{-1} = \begin{pmatrix} 4/9 & 1/7 \\ 1/3 & 5/7 \end{pmatrix} \quad \hat{\beta} = \begin{pmatrix} 1/9 & 0 \\ 0 & 3/7 \end{pmatrix} \quad \hat{\rho} = \begin{pmatrix} 10/9 & 0 \\ 0 & 10/7 \end{pmatrix}$$

Matrix $\mathbf{A}$ verifies the eigenvalue productivity condition ($\lambda(\mathbf{A}) = 0.836 < 1$). From here we can calculate the Leontief multiplier matrix $\mathbf{L}$ and the new multiplier matrix $\mathbf{M}$:

[2] We have constructed these examples using Smath Studio—a wonderfully simple but amazingly powerful and free piece of software.

$$\mathbf{L} = \begin{pmatrix} 2.571 & 1.286 \\ 3 & 5 \end{pmatrix} \geq \mathbf{M} = \begin{pmatrix} 1.667 & 0.333 \\ 0.778 & 1.556 \end{pmatrix}$$

<u>Example 2:</u> We verify that matrix **M** exists and is non-negative should the above economy stop trading in either wheat or iron. In the first case, β_{11}=0 and β_{22}=3/7 whereas in the second β_{11}=1/9 and β_{22}=0. We obtain respectively:

$$\mathbf{M} = \begin{pmatrix} 2.046 & 0.409 \\ 0.955 & 1.591 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 2 & 1 \\ 2.333 & 4.667 \end{pmatrix}$$

In both calculations we still have **L** ≥ **M** but notice that as the economy restricts trade flows the internal multiplier effect magnifies and gets closer to **L**.

<u>Example 3:</u> Take matrices **A** and $\hat{\rho}$ defined by:

$$\mathbf{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.6 \end{pmatrix} \qquad \hat{\rho} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.9 \end{pmatrix}$$

We can check that $\lambda(\mathbf{A}) = 0.8$, $\lambda(\hat{\rho}) = 0.9$ and so $\lambda(\mathbf{A}) < \lambda(\hat{\rho})$. When we calculate **M**, however, we find a non-positive matrix:

$$\mathbf{M} = \begin{pmatrix} -5 & -3.333 \\ -5 & 0 \end{pmatrix}$$

Hence, the system $(\hat{\rho} - \mathbf{A}) \cdot \mathbf{x}^d = \mathbf{f}$ would not be solvable. In light of Remark 4 this system would not satisfy the Hawkins-Simon condition. This is indeed the case as we can easily check. The eigenvalue condition is not sufficient for solvability. For our empirically-based matrices, this possible negativity problem of the generalized multiplier matrix does not arise as Property 5 demonstrates.

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References

Armington, P. (1969), "A theory of production distinguished by place of production", *IMF Staff Papers*, 16(1): 158-178.

Hawkins, D. and H.A. Simon (1949), "Some conditions on macroeconomic stability", *Econometrica*, 17(3/4): 245-248.

Henderson, H.V. and S.R. Searle (1981), "On deriving the inverse of a sum of matrices", *SIAM Review*, 23(1): 53-60.

Leontief, W. (1966), *Input output economics*, New York, Oxford University Press.

Miller, R. and P. Blair (2009), *Input output analysis: foundations and extensions*, New York, Cambridge University Press (2nd edition).

Nikaido, H. (1968), *Convex structures and economic theory*, New York, Academic Press.

Nikaido, H. (1972), *Introduction to sets and mappings in modern economics*, Amsterdam, North-Holland.

Smath Studio (2017), https://en.smath.info.

Solow, R. (1952), "On the structure of linear models", *Econometrica*, 20(1): 29-46.