On Obvious Strategy-proofness and Single-peakedness*

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Abstract: We characterize the set of all obviously strategy-proof and onto social choice functions on the domain of single-peaked preferences. Since obvious strategy-proofness implies strategy-proofness, and the set of strategy-proof and onto social choice functions on this domain coincides with the class of generalized median voter schemes, we focus on this class. We identify a condition on generalized median voter schemes for which the following characterization holds. A generalized median voter scheme is obviously strategy-proof if and only if it satisfies the increasing intersection property. Our proof is constructive; for each generalized median voter scheme that satisfies the increasing intersection property we define an extensive game form that implements it in obviously dominant strategies.

Keywords: Obvious Strategy-proofness, Generalized Median Voters, Single-peakedness.
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1 Introduction

The purpose of this paper is to identify the class of all obviously strategy-proof and onto social choice functions on the domain of single-peaked preferences. Specifically, we consider social choice problems where a set of agents has to choose an alternative from a finite and linearly ordered set of alternatives. For instance, when alternatives are possible levels or locations of a public good, political parties' platforms, temperatures in a room, etc. In these cases, and in a broad variety of economic and political settings, it is natural and meaningful to assume that agents have strict single-peaked preferences over alternatives. A preference is *single-peaked* if there is a best alternative, or top, and alternatives that are further away from this top are progressively less preferred. A central result in the mechanism design literature studying strategy-proof social choice functions on restricted domains of preferences is that a social choice function is strategy-proof and onto on the domain of single-peaked preferences if and only if it is a *generalized median voter scheme.*[1]

But in general, the mechanism design literature has mainly neglected the question of how easy is for the agents to realize that truth-telling is indeed weakly dominant (*i.e.*, how much contingent reasoning is required to do so). Li (2017) proposes the notion of obvious strategy-proofness as a criterion to deal with this question. Obvious strategy-proofness has already been used to identify, among the class of strategy-proof mechanisms in different settings, those mechanisms that are "easy to play" because truth-telling is an undoubtedly optimal decision. Here, we answer the following question: what is the property that a generalized median voter scheme has to satisfy to be obviously strategy-proof.

A social choice function is *obviously strategy-proof* if there exists an extensive game form, whose set of players is the set of agents and its outcomes are alternatives (*i.e.*, there exists a sequential mechanism), with two properties. First, for each preference profile one can identify a profile of truth-telling (behavioral) strategies with the property that if agents play the extensive game form according to it, the outcome of the game is the alternative selected by the social choice function at the preference profile (*i.e.*, the extensive game form induces the social choice function). Second, agents use the two most extreme behavioral assumptions when comparing the truth-telling strategy with any other strategy; agents are absolutely pessimistic when assessing the consequence of truth-telling and absolutely optimistic when assessing the consequence of any other behavior, and they weakly prefer the former to the latter. Whenever an agent has to play along the sequential mechanism, truth-telling appears then as being obviously optimal.

Obvious strategy-proofness is stronger than strategy-proofness. Hence, to describe

[1] See for instance Moulin (1980) or Barberà, Gül and Stacchetti (1993). Generalized median voter schemes are extensions of the median voter rule and since they respect unanimity they are onto.

the class of all obviously strategy-proof and onto social choice functions on the domain of single-peaked preferences we must restrict our search into the class of generalized median voter schemes. A generalized median voter scheme can be described as a sequence of electoral confrontations between pairs of correlative alternatives. Each electoral confrontation is settled by a *committee*, a monotone family of winning coalitions, associated to one of the two alternatives (call it x). Then, given a profile of single-peaked preferences, x is selected if and only if the set of agents that prefer x to the other alternative belongs to the committee. For instance, if the number of agents is odd, majority voting between two alternatives is the committee that associates to one of the two alternatives all coalitions with more than half of the agents. More specifically, a generalized median voter scheme can be represented by a coalition system that associates to each alternative a committee and operates as follows. Fix a profile of single-peaked preferences over the set of alternatives.[2] At any generic alternative x, and starting at the smallest one, agents face two possibilities. Either to select the current alternative x as the one finally chosen or else to select, tentatively, $x + 1$. If the set of agents that prefer x to $x + 1$, according to the preference profile, is a winning coalition at x (that is, it is a member of the committee at x), then x is selected, and finally chosen; otherwise, $x + 1$ becomes the new current alternative that is confronted with $x + 2$ by applying the committee at $x + 1$.

Our contribution is two-fold. First, we give the explicit description of each obviously strategy-proof and onto social choice function on the domain of single-peaked preferences. We do it by showing that a generalized median voter scheme is obviously strategy-proof if and only if its associated coalition system satisfies the *increasing intersection property.* The property has two parts, both applied to each alternative and related with the cardinalities of the intersections of (minimal) winning coalitions. Second, we propose an *algorithm* that, when applied to each coalition system with the increasing intersection property, defines an extensive game form that implements in obviously dominant strategies the corresponding social choice function. The algorithm is based on the description of generalized median voter schemes as a sequence of electoral confrontations between pairs of correlative alternatives and it uses the increasing intersection property of their associated coalition systems.

Literature review

There is a large literature, prior to Li (2017), dealing with the difficulties that agents might have when trying to identify that truth-telling is dominant in strategy-proof mechanisms. See for instance Attiyeh, Franciosi and Isaac (2000), Cason, Saijo, Sjöström and Yamato (2006), Friedman and Schenker (1998), Kawagoe and Mori (2001) and Yamamura and Kawasaki (2013). Even earlier, Kagel, Harstad and Levin (1987) interpret their

[2] Without loss of generality, we may assume that the set of alternatives is a finite set of correlative integers.

experimental results as suggesting that the breakdown of the equivalence between the English ascending clock and the second-price sealed-bid auctions "on a behavioral level can be attributed to differential information flows inherited in the structure of the two auctions." Glazer and Rubinstein (1996) already argues that complexity considerations may suggest the convenience of using extensive game forms to facilitate the identification of the set of strategies that survive iterative elimination of dominated strategies.

Li (2017)'s notion of obvious strategy-proofness is based on an extreme and strong behavioral criterion. Thus, it is not surprising that the literature has already identified settings for which either none of the strategy-proof social choice functions are obviously strategy-proof or only a very special and small subset of them satisfy the stronger requirement. For instance, in the complete impossibility case, Li (2017) already shows that the top-trading cycles algorithm in the house allocation problem of Shapley and Scarf (1974) is not obviously strategy-proof. Ashlagi and Gonczarowski (2018) shows that the deferred acceptance algorithm in the marriage model is not obviously strategy-proof for the agents belonging to the offering side.

In the partial (or total) possibility case, Li (2017) characterizes the monotone price mechanisms (generalizations of ascending auctions) as those that are obviously strategy-proof on the domain of quasi-linear preferences. Li (2017) also shows that, for online advertising auctions, the Vickrey-Clarke-Groves mechanism is obviously strategy-proof. Ashlagi and Gonczarowski (2018) shows however that the deferred acceptance algorithm becomes obviously strategy-proof, for the agents belonging to the offering side, on the restricted domain of acyclic preferences introduced by Ergin (2002).[3] Arribillaga, Massó and Neme (2019) surprisingly finds that, for the discrete division problem with single-peaked preferences, each sequential allotment rule (*i.e.*, each strategy-proof, efficient and replacement monotonic social choice function) is indeed obviously strategy-proof. This is shown by means of an algorithm that, for each sequential allotment rule, delivers the extensive game form that implements the rule in obviously dominant strategies.

But the closest paper to ours is Bade and Gonczarowski (2017). They establish a general revelation principle like result for obvious strategy-proofness: a social choice function is implementable in obviously dominant strategies if and only if some obviously incentive compatible gradual mechanism implements it. For the problem of assigning a set of objects to a set of agents, Bade and Gonczarowski (2017) shows that an efficient social choice function is obviously strategy-proof if and only if it can be implemented by an extensive game form with sequential barters with lurkers; this class consists of generalizations of serial dictatorships. They also show that Li (2017)'s positive result on monotone price mechanisms for binary allocation problems does not hold for more general problems with

[3]For other partially positive or revelation principle like results see also Bade and Gonczarowski (2017), Pycia and Troyan (2018) and Troyan (2019).

two or more goods. For the case of voting over two alternatives, Bade and Gonczarowski (2017) shows that if a social choice function is obviously strategy-proof and onto then it can be implemented by a proto-dictatorship. Finally, for the problem of an infinite and linearly ordered set of alternatives with single-peaked preferences, Bade and Gonczarowski (2017) shows that if a social choice function is obviously strategy-proof and onto then it can be implemented by an extensive game form consisting of dictatorships with safeguards against extremisms (and arbitration via proto-dictatorships, if the set of alternatives is discrete).

Bade and Gonczarowski (2017) and our paper have important overlaps regarding the two-alternative case and the model with single-peaked preferences.[4] The main differences between the two papers are the following. First, in the single-peaked case, our assumption that the set of alternatives is finite is important and becomes crucial for the construction of the algorithm. On the contrary, Bade and Gonczarowski (2017) assumes that the set of alternatives is infinite. Our finite assumption allows us to obtain the result for the two-alternative case as a particular instance of our general result (see Corollary 1 and subsequent comments in Section 6) without having to look at it as a separate model, as in Bade and Gonczarowski (2017). Second, our approach, proposed extensive game forms and proofs of the results differ from theirs because we formally describe and characterize obviously strategy-proof and onto social choice functions as generalized median voter schemes. In contrast, Bade and Gonczarowski (2017) describe their class of dictatorships with safeguards against extremisms directly and verbally. Third, in contrast with Bade and Gonczarowski (2017), and the existing positive results described above (and except the result in Arribillaga, Massó and Neme (2019)), our characterization is not a revelation principle like result identifying a class of extensive game forms where, without loss of generality (but not necessarily), the designer has to look for in order to implement in obviously dominant strategies a particular and given social choice function. But these revelation principle like results do not identify the specific mechanism, among all in the class, that has to be used in order to implement that given social choice function; and this is important because different mechanisms in the class may implement different social choice functions. Instead, our proof is constructive. We propose and algorithm that, for each obviously strategy-proof and onto social choice function, generates (and shows how to construct) an extensive game form that implements the social choice function in obviously dominant strategies. For the important class of social choice functions defined on the domain of single-peaked preferences, our characterization identifies the increasing intersection property as being necessary and sufficient for obvious strategy-proofness. Given a generalized median voter scheme, one can easily check whether or not it is obviously

[4] We have obtained our results in an independent way, before knowing the existence of the first version of Bade and Gonczarowski (2017), as well as those of Pycia and Troyan (2018) and Mackenzie (2018).

strategy-proof by using our property, since it is short and reasonably transparent.

To state and prove our results we will use two previous general results that simplify the search for a specific extensive game form that can be used to implement in obviously dominant strategies a given social choice function. First, we will use a revelation principle like result saying that in our setting, and without loss of generality, we can assume that the extensive game form that induces the social choice function has perfect information (see Ashlagi and Gonczarowski (2018) and Mackenzie (2018)). Second, and following Mackenzie (2018), the new notion of obvious strategy-proofness can be fully captured by the classical notion of strategy-proofness applied to extensive form games with perfect information. In addition, we use in one of our proofs the proto-dictatorship revelation principle like result, established by Bade and Gonczarowski (2017) for the two-alternative case.

The paper is organized as follows. Section 2 contains the basic notation and definitions. In Section 3 we present the notion of obvious strategy-proofness applied to our context. In Section 4 we define the increasing intersection property and state in Theorem 1 the characterization result. In Section 5 we construct the algorithm that, taking as input a generalized median voter scheme satisfying the increasing intersection property, gives as output the extensive game form that implements it in obviously dominant strategies (this result is stated in Theorem 2). In Section 6 we apply our general results to the two-alternative case and/or to anonymous social choice functions. In Section 7 we conclude. An Appendix collects the proofs of the two results, omitted in the main text.

2 Preliminaries

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and linearly ordered set $X = \{x_1, \ldots, x_M\}$, with $M \geq 2$. Without loss of generality, we will often assume that X is the set of correlative integers $\{1, \ldots, M\}$. Each agent $i \in N$ has a strict *preference* P_i (a linear order) over X. We denote by R_i the weak preference over X associated to P_i; *i.e.,* for all $x, y \in X$, xR_iy if and only if either $x = y$ or xP_iy. There is a rich literature studying this class of problems when agents' preferences are single-peaked. Agent i's preference P_i over X is *single-peaked* if (i) there exists $t(P_i) \in X$, called the top of P_i, such that $t(P_i)P_ix$ for all $x \in X \backslash \{t(P_i)\}$ and (ii) for all $x, y \in X$, $x < y \leq t(P_i)$ or $t(P_i) \leq y < x$ implies yP_ix. Given $i \in N$ and $x \in X$ we write P_i^x to denote a generic single-peaked preference such that $t(P_i^x) = x$. Let $\mathcal{P}$ be the set of single-peaked preferences over X. When $|X| = 2$, the linear order structure of X plays no role and the set of single-peaked preferences is simply the universal domain of strict preferences over X. A (preference) *profile* is a n-tuple $P = (P_1, \ldots, P_n)$, an ordered list of n preferences, one for each agent. Let $\mathcal{P}^N$ be the set of single-peaked preference profiles. Given $P = (P_1, \ldots, P_n) \in \mathcal{P}^N$, we denote the vector of tops at P by $t(P) = (t(P_1), \ldots, t(P_n))$. Given a profile P and an agent i, P_{-i} denotes the subprofile in $\mathcal{P}_{-i} = \mathcal{P}^{N\backslash\{i\}}$ obtained by removing P_i from P.

A *social choice function* (SCF) $f : \mathcal{P}^N \to X$ selects, for each preference profile $P \in \mathcal{P}^N$, an alternative $f(P) \in X$.

A SCF $f : \mathcal{P}^N \to X$ is *strategy-proof* (SP) if for all $P \in \mathcal{P}^N$, all $i \in N$ and all $P'_i \in \mathcal{P}$,

$$f(P_i, P_{-i}) R_i f(P'_i, P_{-i}).$$

The literature refers to a strategy-proof SCF as being implementable in dominant strategies (or SP-implementable) in the following sense. Let $f : \mathcal{P}^N \to X$ be a SCF. Construct its associated normal game form, where N is the set of players, $\mathcal{P}$ is the set of strategy profiles and f is the outcome function, mapping strategy profiles into the set of alternatives. Then, f is SP-implementable if the normal game form has the property that, for all $P \in \mathcal{P}^N$ and all $i \in N$, P_i is a weakly dominant strategy for i in the game in normal form, where each $i \in N$ uses P_i to evaluate the outcomes of strategy profiles. The normal game form is known as the direct revelation mechanism that SP-implements f.

We define several properties that a SCF $f : \mathcal{P}^N \to X$ may satisfy and that we will use in the sequel. We say that f is (i) *onto* if for each $x \in X$ there exists $P \in \mathcal{P}^N$ such that $f(P) = x$,[5] and (ii) *anonymous* if for all $P \in \mathcal{P}^N$ and all one-to-one mapping $\pi : N \to N$, $f(P) = f(P^\pi)$ where, for all $i \in N$, $P_i^\pi = P_{\pi(i)}$.

The description of the family of all strategy-proof and onto SCFs $f : \mathcal{P}^N \to X$ is based on the notion of a committee. Let 2^N denote the family of all subsets of N (we call them coalitions). A non-empty family $\mathcal{C} \subset 2^N \backslash \{\emptyset\}$ of non-empty coalitions is a *committee* if it is (coalition) monotonic in the sense that for each pair $S, T \subseteq N$ such that $S \in \mathcal{C}$ and $S \subsetneq T$, we have $T \in \mathcal{C}$. Coalitions in $\mathcal{C}$ are called *winning*. Given $\mathcal{C}$, denote by $\mathcal{C}^m$ the family of *minimal winning coalitions* of $\mathcal{C}$; namely,

$$\mathcal{C}^m = \{S \in \mathcal{C} \mid \text{there is no } S' \in \mathcal{C} \text{ such that } S' \subsetneq S\}.$$

Observe that specifying $\mathcal{C}^m$ is enough to completely determine $\mathcal{C}$.

We define now a class of SCFs, known as generalized median voter schemes, by means of a coalition system. A family of committees $\{\mathcal{C}_x\}_{x \in X}$, one for each alternative in X, is a *coalition system* if (i) it is (outcome) monotonic in the sense that, for each pair $x, x' \in X$ such that $x < x'$, $S \in \mathcal{C}_x$ implies $S \in \mathcal{C}_{x'}$, and (ii) $\mathcal{C}_M = 2^N \backslash \{\emptyset\}$.

Definition 1 A SCF $f : \mathcal{P}^N \to X$ is a *generalized median voter scheme* if there exists a coalition system $\{\mathcal{C}_x\}_{x \in X}$ such that, for all $P \in \mathcal{P}^N$,

$$f(P) = x \quad \text{if and only if} \quad \begin{array}{l} \text{(i) } \{i \in N \mid t(P_i) \leq x\} \in \mathcal{C}_x \text{ and} \\ \text{(ii) for all } x' < x,\ \{i \in N \mid t(P_i) \leq x'\} \notin \mathcal{C}_{x'}. \end{array}$$

[5] A SCF $f : \mathcal{P}^N \to X$ is *unanimous* if, for all $P \in \mathcal{P}^N$ such that $t(P_i) = x$ for all $i \in N$, $f(P) = x$. Although ontoness is weaker than unanimity, it is easy to see that among the class of all strategy-proof SCFs, the classes of unanimous and onto SCFs coincide.

Namely, the alternative x selected by the generalized median voter scheme f at P is the smallest one for which the top alternatives of all agents of a winning coalition at x are smaller than or equal to x.[6]

Alternatively, and more metaphorically, a generalized median voter scheme described by a coalition system might be understood as a force that, starting at the lowest alternative, pushes up towards the highest possible alternative. However, the coalition system distributes among agents the power to stop this force in such a way that all members of a winning coalition at x can make sure that, by declaring that their top alternative is smaller than or equal to x, the pushing force of f will not overcome x.

It is well-known that a SCF $f : \mathcal{P}^N \to X$ is strategy-proof and onto if and only if f is a generalized median voter scheme.[7] By definition, all generalized median voter schemes are unanimous, and so they are onto.

Example 1 contains a generalized median voter scheme that illustrates Definition 1 and that we will use in the sequel.

Example 1 Assume $X = \{x_1, x_2, x_3\}$ and $n = 5$. Consider the coalition system $\mathcal{C} = \{\mathcal{C}_{x_1}, \mathcal{C}_{x_2}, \mathcal{C}_{x_3}\}$ where

$$\begin{aligned} \mathcal{C}^m_{x_1} &= \{\{1\}, \{2, 3, 4\}, \{2, 3, 5\}\} \\ \mathcal{C}^m_{x_2} &= \{\{1\}, \{2\}, \{3\}, \{4, 5\}\} \\ \mathcal{C}^m_{x_3} &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}, \end{aligned}$$

and let $f : \mathcal{P}^N \to X$ be the generalized median voter scheme defined by $\mathcal{C} = \{\mathcal{C}_{x_1}, \mathcal{C}_{x_2}, \mathcal{C}_{x_3}\}$. Consider any profile $P \in \mathcal{P}^N$ whose vector of tops is $t(P) = (x_3, x_1, x_2, x_1, x_3)$. Then, since $\{i \in N \mid t(P_i) \le x_1\} = \{2, 4\} \notin \mathcal{C}_{x_1}$ and $\{i \in N \mid t(P_i) \le x_2\} = \{2, 3, 4\} \in \mathcal{C}_{x_2}$, $f(P) = x_2$. Consider now any profile $P' \in \mathcal{P}^N$ whose vector of tops is $t(P') = (x_3, x_1, x_1, x_1, x_3)$. Then, since $\{i \in N \mid t(P'_i) \le x_1\} = \{2, 3, 4\} \in \mathcal{C}_{x_1}$, $f(P') = x_1$. $\square$

3 Obvious strategy-proofness

We briefly describe the notion of obvious strategy-proofness, adapted to our setting. Li (2017) proposes this notion with the aim of reducing the contingent reasoning required by agents to identify that truth-telling is a weakly dominant strategy. A SCF $f : \mathcal{P}^N \to X$

[6] The term generalized median voter scheme is used in the literature to refer to a minimax rule (introduced in Moulin (1980) for the case $X = \mathbb{R} \cup \{-\infty, +\infty\}$) when applied to a finite and multidimensional set of alternatives; see for instance Barberà, Gül and Stacchetti (1993) or Barberà, Massó and Neme (1997). Since we represent strategy-proof SCFs on the domain of single-peaked preferences by means of coalition systems (instead of using the equivalent representation by collections of fixed ballots, as first used by Moulin (1980)), we adopt this terminology here.

[7] See Barberà, Gül and Stacchetti (1993).

is obviously strategy-proof if there exists an extensive game form Γ, with N as the set of players and X as the set of outcomes, with two properties. First, for each profile $P = (P_1, \ldots, P_n) \in \mathcal{P}^N$ one can identify a behavioral strategy profile, to be interpreted as being truth-telling, such that if agents played Γ according to such strategy the outcome would be $f(P)$, the alternative selected by the SCF f at P; that is, Γ induces f. Second, whenever agent i with preference P_i has to play at a history in Γ, i evaluates the consequence of choosing the action prescribed by i's truth-telling strategy according to the *worse* possible outcome, among all outcomes that may occur as an effect of later actions made by the other agents along the rest of Γ. In contrast, i evaluates the consequence of choosing an action different from the one prescribed by i's truth-telling strategy according to the *best* possible outcome, among all outcomes that may occur again as an effect of later actions chosen by the other agents along the rest of Γ. Then, i's truth-telling strategy is obviously dominant in Γ if, at all histories where i has to play, its pessimistic outcome is at least as preferred as the optimistic outcome used to evaluate any other strategy. If Γ induces f and for each agent truth-telling is obviously dominant, then f is obviously strategy-proof. Obvious strategy-proofness is stronger than strategy-proofness (see Corollary 1 in Li (2017)).

Two important simplifications related to obvious strategy-proofness have been identified in the literature that follows from Li (2017), and that we can use in our context. First, without loss of generality we can assume that the extensive game form that induces the rule has perfect information (see Ashlagi and Gonczarowski (2018) and Mackenzie (2018)). Second, the new notion of obvious strategy-proofness can be fully captured by the classical notion of strategy-proofness applied to extensive form games with perfect information. This last observation essentially follows from the fact that, the best possible outcome obtained when agent i chooses an action different from the one prescribed by i's truth-telling strategy and the worst possible outcome obtained when agent i chooses the action prescribed by i's truth-telling strategy, are both obtained with only one strategy profile of the other agents. This holds because the perfect information implies that all information sets are singleton sets (and each one belongs to either the subgame that follows the truth-telling choice or else to the subgame that follows the alternative choice).[8] Then, for easy presentation and following this literature, we will say that a SCF is obviously strategy-proof if it is implemented by an extensive game form with perfect information for which truth-telling is a weakly dominant strategy (see Definition 2 below). We present the general notion of an extensive game form that will be used here to state and prove our results.

[8]Mackenzie (2018) formally proves this statement for a special class of extensive game forms with perfect information, called round table mechanisms, but its proof can be adapted to any extensive game form with perfect information.

An *extensive game form* with perfect information associated to (N, X) consists of the following elements.

1. A finite and partially ordered set of histories $(H, \prec)$, where:

   (a) $\emptyset \in H$ is the empty history for which $\emptyset \prec h$ for all $h \in H\backslash\{\emptyset\}$.

   (b) For each $h \in H\backslash\{\emptyset\}$, there is a unique h', the immediate predecessor of h, such that $h' \prec h$ and there is no $\bar{h}$ such that $h' \prec \bar{h} \prec h$ (that is, $(H, \prec)$ can be seen as a rooted tree).

   (c) H can be partitioned into two sets, the set of terminal histories $H_T = \{h \in H \mid \text{there is no } \bar{h} \in H \text{ such that } h \prec \bar{h}\}$ and the set of non-terminal histories $H_{NT} = \{h \in H \mid \text{there is } \bar{h} \in H \text{ such that } h \prec \bar{h}\}$.

2. A mapping $\mathcal{N} : H_{NT} \rightarrow N$ that assigns to each non-terminal history $h \in H_{NT}$ the agent $\mathcal{N}(h)$ that has to play at history h. For each $i \in N$, define $H_i = \{h \in H_{NT} \mid \mathcal{N}(h) = i\}$.

3. A set of actions A and a correspondence $\mathcal{A} : H_{NT} \twoheadrightarrow A\backslash\{\emptyset\}$ where, for each $h \in H_{NT}$, $\mathcal{A}(h)$ is the non-empty set of actions available to player $\mathcal{N}(h)$ at h.

4. An outcome function $o : H_T \rightarrow X$ that assigns an alternative $o(h) \in X$ to each terminal history $h \in H_T$.

An extensive game form with perfect information associated to (N, X) is a six-tuple $\Gamma = (N, X, (H, \prec), \mathcal{N}, \mathcal{A}, o)$ with the above properties.[9] The set of agents N and the set of alternatives X will be fixed throughout the paper. Let $\mathcal{G}$ be the class of all extensive game forms satisfying conditions 1 to 4 above.[10]

Fix an extensive game form $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral and pure) *strategy* of i in Γ is a function $\sigma_i : H_i \rightarrow A$ such that, for each $h \in H_i$, $\sigma_i(h) \in \mathcal{A}(h)$; namely, σ_i selects at each history h where i has to play one of i's available actions at h. Let Σ_i be the set of i's strategies in Γ. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_1 \times \cdots \times \Sigma_n \equiv \Sigma$ is an ordered list of strategies, one for each agent. Given $i \in N$, $\sigma \in \Sigma$ and $\sigma'_i \in \Sigma_i$ we

[9] Note that the set of actions A is embedded in the definition of $\mathcal{A}$. Moreover, Γ is not yet a game in extensive form because agents' preferences over alternatives are still unspecified. But given a game Γ and a preference profile P, the pair (Γ, P) defines a game in extensive form where each agent i uses P_i to evaluate alternatives, associated to terminal histories, induced by strategy profiles.

[10] According to Mackenzie (2018) a game $\Gamma \in \mathcal{G}$ is a *round table mechanism* if the set of actions A is the family of all non-empty subsets of preference relations $2^{\mathcal{P}}\backslash\{\emptyset\}$ and (i) the set of actions at any history are disjoint subsets of preferences, (ii) when a player has to play for the first time the set of actions is a partition of $\mathcal{P}$, and (iii) later, the set of actions at history h is the intersection of the actions taken by agent $\mathcal{N}(h)$ at all predecessors that lead to h.

often write (σ'_i, σ_{-i}) to denote the strategy profile where σ_i is replaced in σ by σ'_i. Let $h^\Gamma(\sigma)$ be the terminal history that results in Γ when agents play Γ according to $\sigma \in \Sigma$.

Fix an extensive game form $\Gamma \in \mathcal{G}$ and a preference $P_i \in \mathcal{P}$. A strategy σ_i is *weakly dominant* in Γ at P_i if, for all σ_{-i} and all σ'_i,

$$o(h^\Gamma(\sigma_i, \sigma_{-i}))R_i o(h^\Gamma(\sigma'_i, \sigma_{-i})).$$

We are now ready to define obvious strategy-proofness in our context.

Definition 2 A SCF $f : \mathcal{P}^N \to X$ is *obviously strategy-proof* if there is an extensive game form $\Gamma \in \mathcal{G}$ associated to (N, X) such that, for each $P \in \mathcal{P}^N$, there exists a strategy profile $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ with the properties that
(i) $f(P) = o(h^\Gamma(\sigma^P))$ and
(ii) for all $i \in N$ and all $P_i \in \mathcal{P}$, $\sigma_i^{P_i}$ is weakly dominant in Γ at P_i.

When (i) holds we say that Γ *induces* f. When (i) and (ii) hold we say that Γ OSP-*implements* f.

4 The increasing intersection property and the characterization result

We present the key definition of the paper and our characterization of the class of all obviously strategy-proof and onto SCFs on the domain of single-peaked preferences. To state the property and the result, we need the following notation.

For each $x \in X$, let k^x denote the cardinality of the coalitions in $\mathcal{C}_x^m$ with maximal cardinality; namely,

$$k^x = \max\{|S| \in \{1, \ldots, n\} \mid S \in \mathcal{C}_x^m\}.$$

For any $k \geq 1$, denote by I_x^k the intersection of the coalitions in $\mathcal{C}_x^m$ with cardinality greater than or equal to k; namely,

$$I_x^k = \bigcap_{S \in \mathcal{C}_x^m : |S| \geq k} S.$$

Of course, $I_x^k = \emptyset$ for all $k > k^x$. By convention, we set $I_x^0 = \emptyset$. In Example 1, $k^{x_1} = 3$, $k^{x_2} = 2$, $k^{x_3} = 1$, and $I_{x_1}^1 = \emptyset$, $I_{x_1}^2 = I_{x_1}^3 = \{2, 3\}$, $I_{x_2}^1 = \emptyset$, $I_2^2 = \{4, 5\}$ and $I_{x_3}^1 = \emptyset$.

Definition 3 A coalition system $\{\mathcal{C}_x\}_{x \in X}$ satisfies the *Increasing Intersection* (InIn) property if, for each $x \in \{1, \ldots, M-1\}$,
(a) $\left|I_x^k\right| \geq k - 1$ for all $k \leq k^x$, and
(b) if $k^x > 1$, there exists $i \in I_x^2$ such that $I_{x+1}^1 \cup \{i\} \in \mathcal{C}_{x+1}^m$.

To describe the definition, and its role in our results, fix an alternative x and let k^x be the largest cardinality of minimal winning coalitions in the committee at x. Part (a) requires that, for each integer $k \leq k^x$, the cardinality of the intersection of all coalitions with more than k agents that belong to the committee at x is larger than or equal to $k-1$; namely, all minimal winning coalitions at x of a given cardinality can diverge at most by one agent. This property will allow us to distinguish, at each alternative x, those agents that are able to impose x in its pairwise electoral confrontation with a contiguous alternative, from those that are able to veto x (and so, transforming the contiguous alternative with the one used as reference in the new electoral confrontation). Part (b) requires that if the committee at x has a winning coalition with at least two agents, then the committee at $x+1$ contains a minimal winning coalition formed by an agent that belongs to all minimal winning coalitions with more than two agents at x (such agent does exist by part (a)) and all agents that belong to all minimal winning coalitions at $x+1$. This property ensures that the agent that has the power to veto the current alternative will not regret of doing so because the agent will have the power to make the new current alternative the finally selected one, if the agent wishes to do so.

Theorem 1 *A social choice function $f : \mathcal{P}^N \to X$ is obviously strategy-proof and onto if and only if f is a generalized median voter scheme whose associated coalition system $\mathcal{C} = \{\mathcal{C}_x\}_{x \in X}$ satisfies the increasing intersection property.*

Proof See the Appendix.

The proof of the sufficiency part of Theorem 1 will be constructive. For each generalized median voter scheme f whose associated coalition system $\mathcal{C}$ satisfies the (InIn) property we will construct an extensive game form $\Gamma^{\mathcal{C}}$ that OSP-implements f. In Section 5 below we will define an algorithm that takes $\mathcal{C}$ as input and delivers as output the extensive game form $\Gamma^{\mathcal{C}}$. However, before moving to Section 5, we illustrate the (InIn) property, introduce additional notation, and present a preliminary result and another example.

Given a coalition system $\{\mathcal{C}_x\}_{x \in X}$ we say that condition (a) of the (InIn) property holds at x if (a) holds for $x \in X$. Similarly for (b). We will say that the (InIn) property holds at x if conditions (a) and (b) hold at x. We say that a generalized median voter scheme satisfies the (InIn) property if its associated coalition system satisfies it.

The agent identified in condition (b) of the (InIn) property is not necessarily unique, and we denote one of such agents by i^x; for instance, in Example 1, i^{x_1} could be agent 2 or 3 and i^{x_2} could be agent 4 or 5.[11]

Example 1 (continued) The two tables below might help the reader to check that

[11] Whenever we want to identify a single agent satisfying a property that several agents may satisfy, we could select the smallest agent (according to the order $1 < \cdots < n$) among the set of agents that satisfy the property, and this will be without loss of generality.

the coalition system $\mathcal{C} = \{\mathcal{C}_{x_1}, \mathcal{C}_{x_2}, \mathcal{C}_{x_3}\}$ of Example 1 satisfies the (InIn) property.

$x_1, k^{x_1} = 3$	$k = 1$	$k = 2$	$k = 3$
Intersections	$I^1_{x_1} = \emptyset$	$I^2_{x_1} = \{2, 3\}$	$I^3_{x_1} = \{2, 3\}$
(a) of (InIn)	$\left\|I^1_{x_1}\right\| = 0 \geq 0$	$\left\|I^2_{x_1}\right\| = 2 > 1$	$\left\|I^3_{x_1}\right\| = 2 \geq 2$
(b) of (InIn)	$i^{x_1} \in \{2, 3\}$ and $I^1_{x_2} \cup \{i^{x_1}\} = \{\emptyset\} \cup \{i^{x_1}\} \in \mathcal{C}^m_{x_2}$		

$x_2, k^{x_2} = 2$	$k = 1$	$k = 2$
Intersections	$I^1_{x_2} = \emptyset$	$I^2_{x_2} = \{4, 5\}$
(a) of (InIn)	$\left\|I^1_{x_2}\right\| = 0 \geq 0$	$\left\|I^2_{x_2}\right\| = 2 > 1$
(b) of (InIn)	$i^{x_2} \in \{4, 5\}$ and $I^1_{x_3} \cup \{i^{x_2}\} = \{\emptyset\} \cup \{i^{x_2}\} \in \mathcal{C}^m_{x_3}$	

□

Remark 1 Let $\{\mathcal{C}_x\}_{x \in X}$ be a coalition system. Then, the following properties hold.

(1.1) If $k^x = 1$, the (InIn) property holds at x. To see that, observe that $k^x = 1$ implies $\mathcal{C}^m_x \subseteq \{\{i\} \mid i \in N\}$. Hence, $0 \leq |I^1_x| \leq 1$ and condition (a) of the (InIn) property holds at x. Moreover, since $k^x = 1$, condition (b) of the (InIn) property at x does not apply.

(1.2) If $|\mathcal{C}^m_x| = 1$, condition (a) of the (InIn) property holds at x. To see that, let S be the unique coalition in $\mathcal{C}^m_x$. Hence, $k^x = |S|$ and, for all $k \leq k^x$, $I^k_x = S$. Then, for all $k \leq k^x$, $\left|I^k_x\right| = |S| = k^x > k - 1$.

(1.3) If $X = \{x_1, x_2\}$, condition (b) of the (InIn) property holds at x_1. This is because $I^1_{x_2} = \emptyset$ and $\{i\} \in \mathcal{C}^m_{x_2}$ for all $i \in N$.

To highlight the additional requirements of obvious strategy-proofness with respect to strategy-proofness, we exhibit a simple example with a SCF that is SP but not OSP-implementable.

Example 2 Assume $X = \{x, x + 1\}$ and $n = 5$. Consider the SCF $f : \mathcal{P}^N \to X$ defined by the coalition system $\mathcal{C} = \{\mathcal{C}_x, \mathcal{C}_{x+1}\}$, where $\mathcal{C}^m_x = \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}$ and $\mathcal{C}_{x+1} = 2^N \backslash \{\emptyset\}$. We already know that f is SP-implementable because it is a generalized median voter scheme but f is not OSP-implementable because it does not satisfy the (InIn) property because $k^x = 2$ and $|I^2_x| = 0 < 1 = k^x - 1$. In the direct revelation mechanism that SP-implements f, truth-telling is a weakly dominant strategy: to give support to the top is always optimal independently of whether or not the top is selected. In contrast, consider any extensive game form Γ that could OSP-implement f. The notion requires that (i) Γ induces f and (ii) truth-telling is weakly (*i.e.*, obviously) dominant in Γ. In the example, (i) requires that the agent that has to move first in Γ has to have available two actions, both inducing x and $x + 1$ as possible outcomes, since for all $i \in N$, it holds simultaneously that $\{i\} \notin \mathcal{C}_x$ (i can not impose x) and there exists $S \in \mathcal{C}^m_x$ such that $i \notin S$ (i can not impose $x + 1$). But then, for the agent that has to move first the outcome associated to the optimistic view of not truth-telling is strictly preferred to the outcome associated to the pessimistic view of truth-telling, and so truth-telling is not an obvious optimal decision for this agent. SCFs that are OSP-implementable have to exclude this possibility. The (InIn) property is the condition that does that, and so it discriminates the SCFs that are OSP-implementable from those that are not. □

5 The extensive game form

To prove the necessity part of Theorem 1, we will define an algorithm that takes each coalition system $\mathcal{C}$ satisfying the (InIn) property and delivers an extensive game form $\Gamma^{\mathcal{C}}$ that OSP-implements the generalized median voter scheme associated to $\mathcal{C}$. The algorithm will be based on a collection of elections confronting x and $x+1$ (for $x < M$) by means of an extensive form game, defined also by an algorithm and denoted by Γ^x. The specific sequence along which these elections take place will be determined later, in Subsection 5.2.

To proceed, and given a committee $\mathcal{C}_x$, we need the following notation. For each $k \leq k^x$, let F_x^k be the subset of agents not in I_x^k with the property that each of them completes, together with those in I_x^k, a minimal winning coalition at x; namely,

$$F_x^k = \{i \in N \backslash I_x^k \mid I_x^k \cup \{i\} \in \mathcal{C}_x^m\}. \tag{1}$$

By convention, we set $F_x^0 = \emptyset$. It can be shown that if condition (a) of the (InIn) property holds at x, each minimal winning coalition at x can be written as the union of I_x^k and $\{i\}$ for some $k \leq k^x$ and $i \in F_x^k$, or just as $I_x^{k^x}$ (see (f) and (g) in Remark 2 at the beginning of the Appendix). Moreover, for all $1 < k \leq k^x$,

$$\text{if } F_x^k \backslash F_x^{k-1} = \emptyset \text{ then either } F_x^k = \emptyset \text{ or } F_x^k = F_x^{k-1}. \tag{2}$$

To see that, assume $F_x^k \neq \emptyset$. Since $F_x^k \backslash F_x^{k-1} = \emptyset$, $i \in F_x^k$ implies that $i \in F_x^{k-1}$. Therefore, by definition of F_x^k, $I_x^k \cup \{i\} \in \mathcal{C}_x^m$ and $i \notin I_x^k$ imply $I_x^{k-1} \cup \{i\} \in \mathcal{C}_x^m$ and $i \notin I_x^{k-1}$. Since $I_x^{k-1} \subset I_x^k$, $I_x^{k-1} = I_x^k$; otherwise, there would exists $i \in I_x^k \backslash I_x^{k-1}$ such that $I_x^{k-1} \cup \{i\} \subsetneq I_x^k \cup \{i\}$, contradicting that $I_x^k \cup \{i\} \in \mathcal{C}_x^m$. Therefore, by (1), $F_x^k = F_x^{k-1}$.

Example 3 contains a committee $\mathcal{C}_x$ that illustrates the above definition, and that we will use in the sequel.

Example 3 Let $n = 10$ and $\mathcal{C}_x^m = \{\{1\}, \{2,3\}, \{2,4\}, \{2,5,6\}, \{2,5,7,8,9\}, \{2,5,7,8,10\}\}$. Note that $k^x = 5$. Then,

$$\begin{array}{ll} I_x^1 = \emptyset & F_x^1 = \{1\} \\ I_x^2 = \{2\} & F_x^2 = \{3,4\} \\ I_x^3 = \{2,5\} & F_x^3 = \{6\} \\ I_x^4 = \{2,5,7,8\} & F_x^4 = \{9,10\} \\ I_x^5 = \{2,5,7,8\} & F_x^5 = \{9,10\}. \end{array}$$

Observe that condition (a) of the (InIn) property holds at x since $k^x = 5 > 1$ and $|I_x^1| = 0 \geq 0$, $|I_x^2| = 1 \geq 1$, $|I_x^3| = 2 \geq 2$, $|I_x^4| = 4 \geq 3$ and $|I_x^5| = 4 \geq 4$ hold. Moreover, any $S \in \mathcal{C}_x^m$ can be written as $S = I_x^k \cup \{i\}$ for some $i \in F_x^k$ and $k \leq k^x$. □

5.1 The algorithm confronting x and $x+1$ (for $x < M$)

Here, we focus only on the election confronting x and $x+1$, for $x < M$, by means of $\mathcal{C}_x$.

Fix $\mathcal{C}_x$. The algorithm consists of two types of **Stages**, **A** and **B**, that are played alternately, and each with (potentially) several steps. Agents play sequentially at most once, and when they do, their choice set is $\{x, x+1\}$. Agents playing in steps of **Stage A** (agents belonging to $I_x^1, \ldots, I_x^{k^x}$) can either impose $x+1$ (by choosing $x+1$) or let the extensive game form proceed (by choosing x). Agents playing in steps of **Stage B** (agents belonging to the sets $F_x^1, \ldots, F_x^{k_x}$) can either impose x (by choosing x) or let the extensive game form proceed (by choosing $x+1$). The agent playing in the last step can impose x (by choosing x) or $x+1$ (by choosing $x+1$).

The algorithm defining the extensive game form Γ^x

Input: A committee $\mathcal{C}_x$ satisfying condition (a) of the (InIn) property at x.

Initialization: Identify the integer k^x and, for each $1 \leq k \leq k^x$, the subsets of agents I_x^k and F_x^k. Set $k = 1$ and go to **Stage A.1**.

Stage A.k ($1 \leq k \leq k^x$)**.**

If $I_x^k \backslash I_x^{k-1} \neq \emptyset$, agents in $I_x^k \backslash I_x^{k-1}$ play sequentially in any order choosing an action in the set $\{x, x+1\}$.

If one agent chooses $x+1$, Γ^x ends with outcome $x+1$.

If all agents choose x, go to **Stage B.k**.

If $I_x^k \backslash I_x^{k-1} = \emptyset$, go to **Stage B.k**.

Stage B.k ($1 \leq k \leq k^x$)**.**

(i) Assume $1 \leq k < k^x$.

If $F_x^k \backslash F_x^{k-1} \neq \emptyset$, agents in $F_x^k \backslash F_x^{k-1}$ play sequentially in any order choosing an action in the set $\{x, x+1\}$.

If one agent chooses x, Γ^x ends with outcome x.

If all agents choose $x+1$, go to **Stage A.k+1**.

If $F_x^k \backslash F_x^{k-1} = \emptyset$, go to **Stage A.k+1**.

(ii) Assume $k = k^x$.

If $F_x^k \backslash F_x^{k-1} \neq \emptyset$, agents in $F_x^k \backslash F_x^{k-1}$ play sequentially in any order choosing an action in the set $\{x, x+1\}$.

If one agent chooses x, Γ^x ends with outcome x.

If all agents choose $x+1$, Γ^x ends with outcome $x+1$.

If $F_x^k \backslash F_x^{k-1} = \emptyset$.

If $F_x^k = \emptyset$, Γ^x ends with outcome x.

If $F_x^k = F_x^{k-1} \neq \emptyset$, Γ^x ends with outcome $x+1$.[12]

Output: Γ^x.

The extensive game form Γ^x is a proto-dictatorship, as defined by Bade and Gonczarowski (2017). Each agent plays at most once by choosing either x or $x+1$ and, except for the last player, one and only one of the two choices induces a terminal history while for the last player both choices induce a terminal history.

Example 3 (continued) Figure 1 represents the extensive game form Γ^x for the committee $\mathcal{C}_x$ of Example 3, where agents play from left to right, with the order $1, \ldots, 10$, and the set of actions is $\{x, x+1\}$ for all agents.

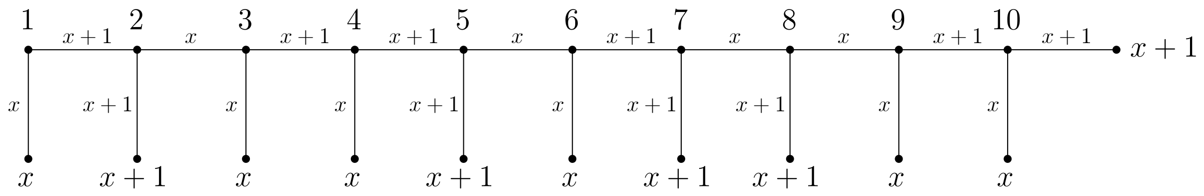


Figure 1

Below, we apply the algorithm to $\mathcal{C}_x$ to obtain Γ^x depicted in Figure 1.

At **Stage A.1**, since $I_x^1 = \emptyset$, go to **Stage B.1**. At **Stage B.1**, since $F_x^1 = \{1\}$, only 1 plays. If 1 chooses x, Γ^x ends with x, and if 1 chooses $x+1$, go to **Stage A.2**. At **Stage A.2**, since $I_x^2 \backslash I_x^1 = \{2\}$, only 2 plays. If 2 chooses $x+1$, Γ^x ends with $x+1$, and if 2 chooses x, go to **Stage B.2**. At **Stage B.2**, since $F_x^2 \backslash F_x^1 = \{3, 4\}$, 3 and 4 play (in Figure 1, 3 plays before 4). If 3 chooses x, Γ^x ends with x, and if 3 chooses $x+1$, 4 plays. If 4 chooses x, Γ^x ends with x, and if 4 chooses $x+1$, go to **Stage A.3**. At **Stage A.3**, since $I_x^3 \backslash I_x^2 = \{5\}$, only 5 plays. If 5 chooses $x+1$, Γ^x ends with $x+1$, and if 5 chooses x, go to **Stage B.3**. At **Stage B.3**, since $F_x^3 \backslash F_x^2 = \{6\}$, only 6 plays. If 6 chooses x, Γ^x ends with x, and if 6 chooses $x+1$, go to **Stage A.4**. At **Stage A.4**, since $I_x^4 \backslash I_x^3 = \{7, 8\}$, 7 and 8 play (in Figure 1, 7 plays before 8). If 7 chooses $x+1$, Γ^x ends with $x+1$, and if 7 chooses x, 8 plays. If 8 chooses $x+1$, Γ^x ends with $x+1$, and if 8 chooses x, go to **Stage B.4**. At **Stage B.4**, since $F_x^4 \backslash F_x^3 = \{9, 10\}$, 9 and 10 play (in Figure 1, 9 plays before 10). If 9 chooses x, Γ^x ends with x, and if 9 chooses $x+1$, agent 10 plays. If 10 chooses x, Γ^x ends with x, and if 10 chooses $x+1$, go to **Stage A.5**. At **Stage A.5,** since

[12] By (2), these two cases are the only possible ones.

$I_x^5 \backslash I_x^4 = \emptyset$, go to **Stage B.5**. At **Stage B.5**, since $k = k^x = 5$ and $F_x^5 = F_x^4 = \{9, 10\}$, Γ^x ends with outcome $x + 1$ and the algorithm stops after **Stage B.5** with output Γ^x. □

5.2 The extensive game form $\Gamma^{\mathcal{C}}$

This subsection contains the description of the algorithm defining the full extensive game form that OSP-implements a given generalized median voter scheme $f : \mathcal{P}^N \to X$ satisfying the (InIn) property. This description will require to identify, given the coalition system $\{\mathcal{C}_x\}_{x \in X}$ associated to f, (i) the smallest alternative $x^* \in X$ with the property that its committee $\mathcal{C}_{x^*}$ has a singleton set and (ii) one of the agents that alone is a minimal winning coalition at x^*, denoted by i^*. Namely,

$$x^* = \arg\min\{x \in X \mid \{i\} \in \mathcal{C}_x^m \text{ for some } i \in N\}.$$

The alternative x^* is well defined since $\mathcal{C}_M^m = \{\{1\}, \ldots, \{n\}\}$. Define[13]

$$i^* = \begin{cases} \arg\min\{i \in N \mid \{i\} \in \mathcal{C}_{x^*}^m\} & \text{if } x^* = 1 \\ i^{x^*-1} & \text{otherwise.} \end{cases}$$

The algorithm defining the extensive game form $\Gamma^{\mathcal{C}}$

Input: A coalition system $\{\mathcal{C}_x\}_{x \in X}$ satisfying the (InIn) property.

Initialization: Identify the alternative x^*, the agent i^* and, for each $x < M$, the integer k^x, the agent i^x (if $k^x > 1$) and, for each $1 \leq k \leq k^x$, the subsets of agents I_x^k and F_x^k. Go to **Stage I**.

Stage I. The first agent to play is $\mathcal{N}(\emptyset) = i^*$ choosing an action in the set $\mathcal{A}(\emptyset)$, where

$$\mathcal{A}(\emptyset) = \begin{cases} \{x^*, x^* + 1\} & \text{if } x^* = 1 \\ \{x^* - 1, x^*, x^* + 1\} & \text{if } 1 < x^* < M \\ \{x^* - 1, x^*\} & \text{if } x^* = M. \end{cases}$$

If i^* chooses x^*, $\Gamma^{\mathcal{C}}$ ends with outcome x^*.
If i^* chooses $x^* + 1$, go to **Stage Up.1**.
If i^* chooses $x^* - 1$, go to **Stage Down.1**.

Stage Up.k $(k \geq 1)$**.** Set $x = x^* + (k - 1)$.

(i) Assume $x + 1 < M$.

If $k^x > 1$, agents play Γ^x as previously defined except that in **Stage A.2**, agent

[13] If $x^* > 1$, $k^{x^*-1} > 1$ holds since no singleton coalition belongs to $\mathcal{C}_{x^*-1}^m$. Observe also that, since condition (b) of the (InIn) property holds at $x^* - 1$, $\{i^{x^*-1}\} \in \mathcal{C}_{x^*}^m$ because, by the definitions of x^* and i^{x^*-1}, either $I_{x^*}^1 = \emptyset$ or $I_{x^*}^1 = \{i^{x^*-1}\}$ and so $\mathcal{C}_{x^*}^m = \{\{i^{x^*-1}\}\}$.

$i^x \in I_x^2 \backslash I_x^1$ plays first.[14]

If i^x chooses $x+1$, go to **Stage Up.k+1**.

If i^x chooses x, the other agents in $I_x^2 \setminus I_x^1$ play sequentially, in any order. The outcome of Γ^x is the outcome of $\Gamma^{\mathcal{C}}$.

If $k^x = 1$, agents play Γ^x.

If the outcome of Γ^x is x, then $\Gamma^{\mathcal{C}}$ ends with outcome x.

If the outcome of Γ^x is $x+1$, go to **Stage Up.k+1**.

(ii) Assume $x+1 = M$. Agents play Γ^x and the outcome of Γ^x is the outcome of $\Gamma^{\mathcal{C}}$.

Stage Down.k ($k \geq 1$)**.** Set $x = x^* - k$.

(i) Assume $x > 1$.

If $|\mathcal{C}_x^m| > 1$, agents play Γ^x as previously defined except that in **Stage B.1** agent $i^{x-1} \in F_x^1$ plays first.[15]

If i^{x-1} chooses x, go to **Stage Down.k+1**.

If i^{x-1} chooses $x+1$, the other agents in F_x^1 play sequentially, in any order. The outcome of Γ^x is the outcome of $\Gamma^{\mathcal{C}}$.

If $|\mathcal{C}_x^m| = 1$, agents play Γ^x.

If the outcome of Γ^x is $x+1$, then $\Gamma^{\mathcal{C}}$ ends with outcome $x+1$.

If the outcome of Γ^x is x, go to **Stage Down.k+1**.

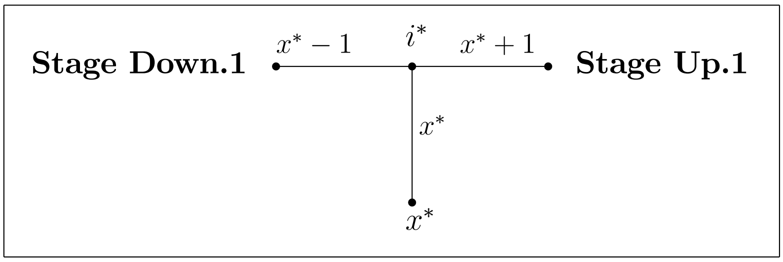
(ii) Assume $x = 1$. Agents play Γ^x and the outcome of Γ^x is the outcome of $\Gamma^{\mathcal{C}}$.

Output: $\Gamma^{\mathcal{C}}$.

The following figures represent the building blocks that make up the algorithm.

Stage I.

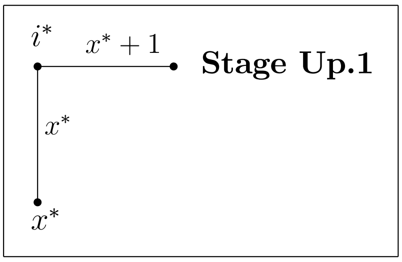
If $1 < x^* < M$

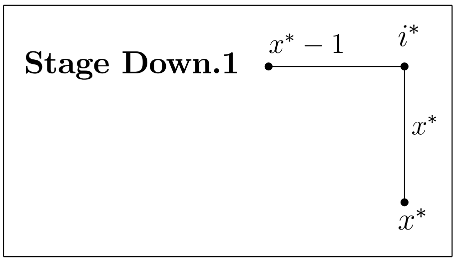


[14] Observe that since $k^x > 1$, $x^* \leq x < M$, and condition (b) of the (InIn) property holds at x, we have that $I_x^1 = \emptyset$ and $i^x \in I_x^2 \backslash I_x^1$.

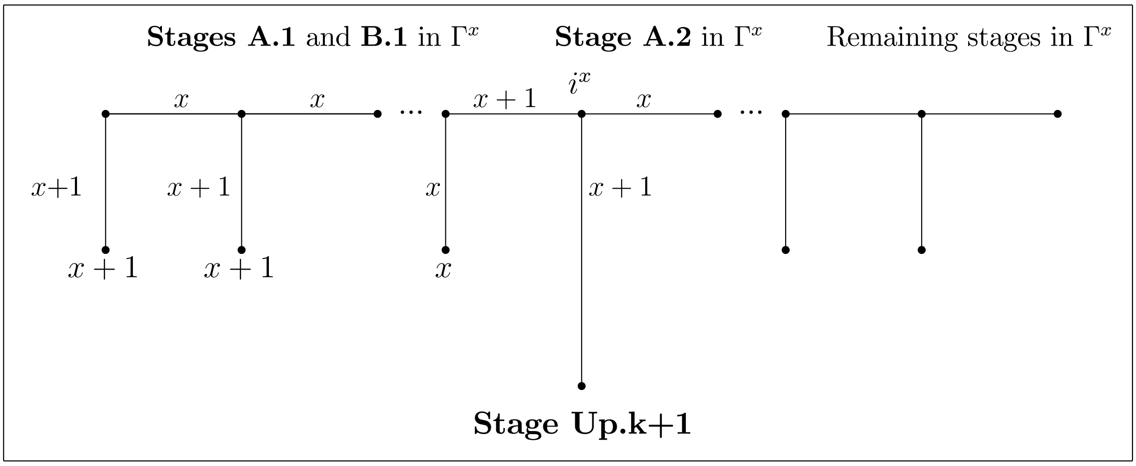
[15] Observe that, since $1 < x < x^*$ and condition (b) of the (InIn) property holds at $x-1$, we have that $\{i^{x-1}\} \notin \mathcal{C}_x^m$ and $I_x^1 \cup \{i^{x-1}\} \in \mathcal{C}_x^m$. Since $|\mathcal{C}_x^m| > 1$, $I_x^1 \notin \mathcal{C}_x^m$ holds, and so $i^{x-1} \notin I_x^1$ and $i^{x-1} \in F_x^1$.

If $x^* = 1$ If $x^* = M$

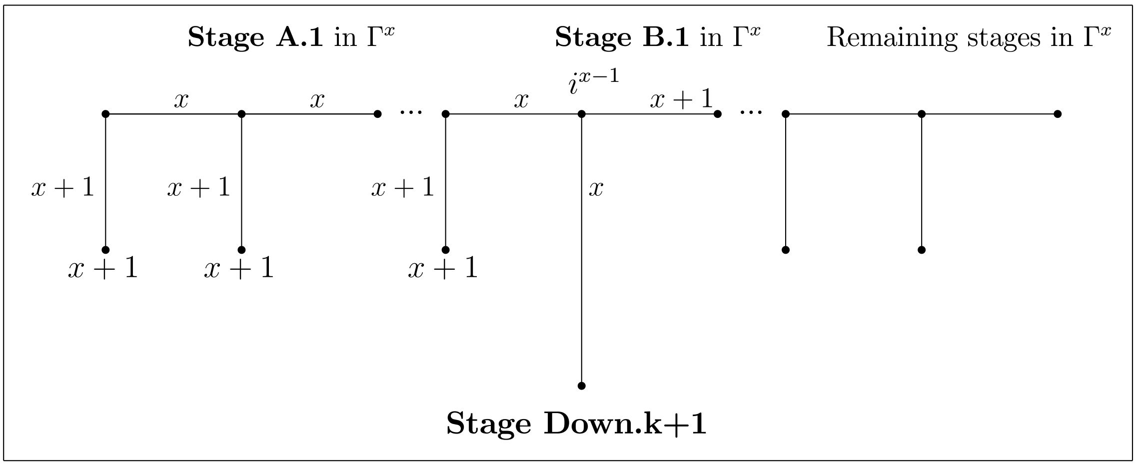




Stage Up.k. For $x = x^* + (k - 1)$



Stage Down.k. For $x = x^* - k$



The algorithm can be seen as a sequence of electoral confrontations between x and $x+1$, each by means of Γ^x. However, obvious strategy-proofness requires that the transition from Γ^x to Γ^{x+1}, if $x^* \leq x$, or from Γ^x to Γ^{x-1}, if $x < x^*$, can not just depend on the outcome of Γ^x. When Γ^x is played in an up stage (*i.e.*, $x^* \leq x$), and the outcome of Γ^x is $x + 1$ after i^x chooses x, the overall game $\Gamma^{\mathcal{C}}$ does not move to Γ^{x+1} but instead it finishes with final outcome $x+1$. Similarly, when Γ^x is played in a down stage (*i.e.*, $x < x^*$), and the outcome of Γ^x is x after i^{x-1} chooses $x+1$, the overall game $\Gamma^{\mathcal{C}}$ does not move to Γ^{x-1} but instead it finishes with final outcome x. Observe that by definitions of i^x and i^{x-1} the outcome of Γ^x is respectively $x+1$ if $x^* \leq x$ and i^x chooses $x+1$ or x if $x < x^*$ and i^{x-1} chooses x; and then the corresponding Γ^{x+1} or Γ^{x-1} will be played after Γ^x. To preserve obvious strategy-proofness, agent i^x or agent i^{x-1} has to be the first to choose respectively in the corresponding stages **A.2** or **B.1** of Γ^x.

We now illustrate the algorithm by applying it to the coalition system $\mathcal{C} = \{\mathcal{C}_{x_1}, \mathcal{C}_{x_2}, \mathcal{C}_{x_3}\}$ of Example 1. We have already checked that $\mathcal{C}$ satisfies the (InIn) property.

Example 1 (continued) Remember that $X = \{x_1, x_2, x_3\}$, $n = 5$,

$$\begin{aligned} \mathcal{C}^m_{x_1} &= \{\{1\}, \{2,3,4\}, \{2,3,5\}\} \\ \mathcal{C}^m_{x_2} &= \{\{1\}, \{2\}, \{3\}, \{4,5\}\} \\ \mathcal{C}^m_{x_3} &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \end{aligned}$$

and, without loss of generality, assume $x_2 = x_1 + 1$ and $x_3 = x_2 + 1$.

The application of the algorithm to obtain $\Gamma^{\mathcal{C}}$ (see Figure 2 below)

Input: The coalition system $\mathcal{C} = \{\mathcal{C}_{x_1}, \mathcal{C}_{x_2}, \mathcal{C}_{x_3}\}$ that satisfies the (InIn) property.

Initialization: Identify the alternative $x^* = x_1$, the agent $i^* = 1$, and the cardinalities, subsets of agents and agents shown in the table below.

x_1		x_2		x_3	
$k^{x_1} = 3$		$k^{x_2} = 2$		$k^{x_3} = 1$	
$I^1_{x_1} = \emptyset$	$F^1_{x_1} = \{1\}$	$I^1_{x_2} = \emptyset$	$F^1_{x_2} = \{1,2,3\}$	$I^1_{x_3} = \emptyset$	$F^1_{x_3} = \{1,2,3,4,5\}$.
$I^2_{x_1} = \{2,3\}$	$F^2_{x_1} = \{4,5\}$	$I^2_{x_2} = \{4,5\}$	$F^2_{x_2} = \emptyset$		
$I^3_{x_1} = \{2,3\}$	$F^3_{x_1} = \{4,5\}$				
$i^{x_1} = 2$		$i^{x_2} = 4$			

Go to **Stage I**.

Stage I. Agent 1 is the first to play choosing an action in the set $\{x_1, x_2\}$.

If 1 chooses x_1, $\Gamma^{\mathcal{C}}$ ends with outcome x_1.

If 1 chooses x_2, go to **Stage Up.1.**

Stage Up.1. Set $x = x_1$.

Since $x_2 < x_3$ and $k^{x_1} = 3 > 1$, agents play Γ^{x_1} with the modification that $i^{x_1} = 2$ plays first in **Stage A.2**.

$\boxed{\Gamma^{x_1}}$

Stage A.1. Since $I^1_{x_1} \backslash I^0_{x_1} = \emptyset$, go to **Stage B.1**.

Stage B.1. Since $k^{x_1} = 3 > 1$ and $F^1_{x_1} \backslash F^0_{x_1} = \{1\}$, agent 1 plays choosing an action in the set $\{x_1, x_2\}$.

If 1 chooses x_1, $\Gamma^{\mathcal{C}}$ ends with outcome x_1.

If 1 chooses x_2, go to **Stage A.2**.

Stage A.2. Since $I^2_{x_1} \backslash I^1_{x_1} = \{2, 3\}$ and $i^{x_1} = 2$, agents 2 and 3 play in this order by choosing an action in the set $\{x_1, x_2\}$.

If 2 chooses x_2, go to **Stage Up.2**.

If 2 chooses x_1, 3 plays.

If 3 chooses x_2, $\Gamma^{\mathcal{C}}$ ends with outcome x_2.

If 3 chooses x_1, go to **Stage B.2**.

Stage B.2. Since $k^{x_1} = 3 > 2$ and $F^2_{x_1} \backslash F^1_{x_1} = \{4, 5\}$, agents 4 and 5 play in any other by choosing an action in the set $\{x_1, x_2\}$. Set the order $4, 5$.

If 4 chooses x_1, $\Gamma^{\mathcal{C}}$ ends with outcome x_1.

If 4 chooses x_2, 5 plays.

If 5 chooses x_1, $\Gamma^{\mathcal{C}}$ ends with outcome x_1.

If 5 chooses x_2, go to **Stage A.3**.

Stage A.3. Since $I^3_{x_1} \backslash I^2_{x_1} = \emptyset$, go to **Stage B.3**.

Stage B.3. Since $k^{x_1} = 3$ and $F^3_{x_1} = F^1_{x_1} = \{4, 5\}$, $\Gamma^{\mathcal{C}}$ ends with x_2.

Stage Up.2. Set $x = x_2$.

Since $x_2 + 1 = x_3$, agents play Γ^{x_2} and the outcome of Γ^{x_2} is the outcome of $\Gamma^{\mathcal{C}}$.

$\boxed{\Gamma^{x_2}}$

Stage A.1. Since $I^1_{x_2} \backslash I^0_{x_2} = \emptyset$, go to **Stage B.1**.

Stage B.1. Since $k^{x_2} = 2 > 1$ and $F^1_{x_2} \backslash F^0_{x_2} = \{1, 2, 3\}$, agents 1, 2 and 3 play in any order by choosing an action in the set $\{x_2, x_3\}$. Set the order $1, 2, 3$.

If 1 chooses x_2, $\Gamma^{\mathcal{C}}$ ends with outcome x_2.

If 1 chooses x_3, 2 plays.

If 2 chooses x_2, $\Gamma^{\mathcal{C}}$ ends with outcome x_2.

If 2 chooses x_3, 3 plays.

If 3 chooses x_2, $\Gamma^{\mathcal{C}}$ ends with outcome x_2.

If 3 chooses x_3, go to **Stage A.2**.

Stage A.2. Since $I^2_{x_2} \backslash I^1_{x_2} = \{4, 5\}$, agents 4 and 5 play in any order by choosing and action in the set $\{x_2, x_3\}$. Set the order $4, 5$.

If 4 chooses x_3, $\Gamma^{\mathcal{C}}$ ends with outcome x_3.

If 4 chooses x_2, 5 plays.

If 5 chooses x_3, $\Gamma^{\mathcal{C}}$ ends with outcome x_3.

If 5 chooses x_2, go to **Stage B.2**.

Stage B.2. Since $k^{x_2} = 2$ and $F^2_{x_2} = \emptyset$, $\Gamma^{\mathcal{C}}$ ends with outcome x_2.

Output: $\Gamma^{\mathcal{C}}$.

Figure 2 depicts the extensive game form $\Gamma^{\mathcal{C}}$, output of the algorithm, that OSP-implements the generalized median voter scheme associated to $\mathcal{C}$.

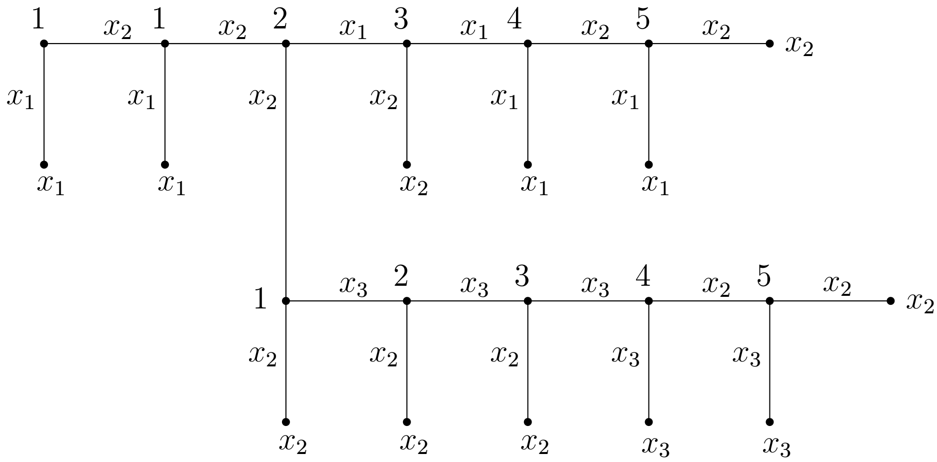


Figure 2

Two comments about Figure 2 are pertinent. First, at the beginning of the game, player 1 plays twice in a row with the same set of actions. The game without the first node is strategically equivalent to $\Gamma^{\mathcal{C}}$. We have maintained this potential redundancy in order to be consistent with the definition of the general algorithm which distinguishes between agent i^*, who moves first in **Stage I**, and the first agent to move in Γ^{x^*}, who moves just after i^* has chosen $x^* + 1$; in the example, these two agents coincide (both are player 1) but in general they may be different. Second, the example may help to clarify the role of the (InIn) property to guarantee that truth-telling is obviously dominant as well as why **Stage A.2** (in **Stage Up.k)** of Γ^x has to be modified, and the special role given to player i^x (player 2 in the example).[16] In Figure 2, and to see why truth-telling is an obviously dominant strategy in Γ^x for any $i \in \{1, 3, 4\}$, consider i's choice at any history where i plays (the case $i = 5$ is trivial). If i's top coincides with the alternative that i can induce as final outcome, then truth-telling is obviously dominant since the worse outcome is the top. If i's top does not coincide with the alternative that i can induce as final outcome, then truth-telling is obviously dominant since the worse outcome it induces coincides with the outcome of not truth-telling. Consider now agent 2 (in the role of player i^{x_1}) who plays first in **Stage A.2** (in **Stage Up.1)** in the modified Γ^{x_1}.

[16]The truth-telling strategies here consist of choosing always the preferred alternative on the set of available actions, either $\{x_1, x_2\}$ or $\{x_2, x_3\}$.

Observe that, despite the fact that none of 2's actions induces a terminal node, truth-telling is obviously dominant. If 2 chooses x_1, x_3 is not a possible outcome because Γ^{x_2} is not played after 2 chooses x_1. Moreover, when 2 chooses x_2, and Γ^{x_2} is played, x_1 is not a possible outcome but, at the same time, 2 has the power to avoid x_3. Otherwise, x_3 could be the worse outcome if 2, with the single-peaked preference $x_2P_2x_1P_2x_3$ chooses x_2 (*i.e.*, truth-tells) while x_2 could be the best outcome after choosing x_1. Condition (b) of the (InIn) property guarantees that 2 is a minimal winning coalition at x_2 and so, 2 can impose x_2 (*i.e.*, avoid x_3) after choosing x_2. When agent 2's preference is $x_1P_2x_2P_2x_3$, $x_2P_2x_3P_2x_1$ or $x_3P_2x_2P_2x_1$, x_2 is the worse outcome of truth-telling and the best of not doing so. Thus, truth-telling is obviously dominant for 2. □

We are now ready to state Theorem 2, the second main result of the paper. Theorem 2 implies the sufficiency part of Theorem 1 but, in addition, it gives for each obviously strategy-proof SCF an extensive game form that OSP-implements it.

Theorem 2 *Let $f : \mathcal{P}^N \to X$ be a generalized median voter scheme whose associated coalition system $\mathcal{C} = \{\mathcal{C}_x\}_{x\in X}$ satisfies the increasing intersection property. Then, $\Gamma^{\mathcal{C}}$ implements f in obviously dominant strategies.*

Proof See the Appendix.

6 Particular results: the two-alternative case and/or anonymity

We apply our results to special cases of our setting, those in which X only contains two alternatives and/or the SCFs are anonymous.

Assume $|X| = 2$ and, without loss of generality, let $X = \{x, x+1\}$. Then, the set $\mathcal{P}$ of single-peaked preferences over X is the universal domain of (strict) preferences over $\{x, x+1\}$. Let $f : \mathcal{P}^N \to \{x, x+1\}$ be a strategy-proof and onto SCF (*i.e.*, it is not constant) and let $\{\mathcal{C}_x, \mathcal{C}_{x+1}\}$ be its associated coalition system. By (1.3) in Remark 1, $\{\mathcal{C}_x, \mathcal{C}_{x+1}\}$ trivially satisfies condition (b) of the (InIn) property. Hence, we obtain as a corollary of our results the characterization of all obviously strategy-proof and onto SCFs for the two-alternative case.

Corollary 1 *Assume $X = \{x, x+1\}$. Then, a social choice function $f : \mathcal{P}^N \to X$ is obviously strategy-proof and onto if and only if the committee $\mathcal{C}_x$ associated to f satisfies condition* (a) *of the* (InIn) *property at x. Moreover, the extensive game form Γ^x, outcome of the algorithm applied to $\mathcal{C}_x$, implements f in obviously dominant strategies.*

Corollary 1 helps to further clarify the boundary between Bade and Gonczarowski (2017) and our work. We can present in an unified way the two-alternative result and the single-peaked result into a sole result about single-peaked preferences. Bade and Gonczarowski (2017) cannot do this, as their single-peaked result is for infinite sets of alternatives. For this reason, they have to treat the two cases separately (their Theorem 4.1 refers to the two-alternative case). In addition to the fact that the approaches of the two papers are different,[17] this is an additional evidence that the results of the two independent papers are distinct and complement each other well.

A committee $\mathcal{C}_x$ is *anonymous* if $\mathcal{C}_x = \{S \in 2^N \mid |S| \geq q\}$ for some $q \in \{1, \ldots, n\}$. The associated anonymous SCF and $\mathcal{C}_x$ itself are named *voting by quota* q (see Barberà, Sonnenschein and Zhou (1991)). The two special and extreme cases $q = n$ and $q = 1$ correspond to the two unanimity cases. Unanimity for x when $q = n$ (*i.e.*, to be elected, x needs n votes) and unanimity for $x + 1$ when $q = 1$ (*i.e.*, to be elected, $x + 1$ needs n votes). Among all voting by quota, these two extreme cases are the unique ones for which condition (a) of the (InIn) property holds at x. Indeed, if $q = 1$, then $k^x = 1$ and $|I_x^1| = 0$. If $q = n$, then $k^x = n$ and, for all $1 \leq k \leq n$, $\left|I_x^k\right| = n > k - 1$. In contrast, if $n > 2$ and $1 < q < n$, then $k^x = q$ and, for all $1 < k \leq q$, $\left|I_x^k\right| = 0 < k - 1$; hence, condition (a) of the (InIn) property does not hold at x. We state as corollary of our results the following characterization of all obviously strategy-proof, anonymous, and onto SCFs for the two-alternative case.

Corollary 2 *Assume* $X = \{x, x + 1\}$*. Then, a social choice function* $f : \mathcal{P}^N \to X$ *is obviously strategy-proof, anonymous and onto if and only if* f *is either voting by quota* 1 *or voting by quota* n*.*

The reason of why voting by quota 1 is obviously strategy-proof is as follows. Let Γ^x be the extensive game form that OSP-implements voting by quota 1. When agent i has to move, i has two choices: voting for x (*i.e.*, vetoing $x + 1$), and so ending the game with x, or voting for $x + 1$, and so passing to the next agent in the sequence (if any) the power to impose x. If i prefers x, truth-telling (voting for x) gives to i the top alternative, at least as preferred as the outcome of not truth-telling. If i prefers $x + 1$, not truth-telling (voting for x) gives to i the worse alternative, indifferent or less preferred to the outcome of truth-telling (voting for $x + 1$). Hence, truth-telling is obviously dominant. Symmetrically for voting by quota n. The reason of why any voting by quota $1 < q < n$ is not obviously strategy-proof is as follows. Let Γ be an extensive game form that induces voting by quota q. Look at the first agent (called i) who has available a set of two actions.[18] None of them can be decisive (both have to leave as possible outcomes x and $x + 1$), as otherwise Γ would not induce voting by quota q. Hence, the other agents can always impose both outcomes

[17] Bade and Gonczarowski (2017) gives revelation principle like results while our approach, based on the algorithm, identifies for each obviously strategy-proof and onto SCF an extensive game form that implements it in obviously dominant strategies.

[18] By Bade and Gonczarowski (2017), this simplification can be done without loss of generality.

on i, irrespective of i' choice. Thus, the worse possible outcome of i's truth-telling strategy is strictly worse than the best possible outcome of any alternative strategy. Hence, voting by quota $1 < q < n$ is not obviously strategy-proof.

As a consequence of our results, we finally obtain Corollary 3 characterizing the class of all obviously strategy-proof, anonymous and onto SCFs on the domain of single-peaked preferences over an arbitrary finite set of alternatives $X = \{1, \ldots, M\}$, with $M \geq 2$. The result follows as a consequence of two observations. By Corollary 2, condition (a) of the (InIn) property requires that, for all $x \in \{1, \ldots, M-1\}$, $\mathcal{C}_x$ is either voting by quota 1 or voting by quota n (observe that $\mathcal{C}_M^m = \{\{1\}, \ldots, \{n\}\}$ is voting by quota 1). Moreover, outcome monotonicity of the coalition system requires that it should exists $x^* \in X$ such that, for all $x < x^*$ (if any), $\mathcal{C}_x$ is voting by quota n and, for all $x \geq x^*$, $\mathcal{C}_x$ is voting by quota 1. Namely,

Corollary 3 *A social choice function $f : \mathcal{P}^N \to X$ is obviously strategy-proof, anonymous and onto if and only if f is a generalized median voter scheme whose associated coalition system $\{\mathcal{C}_x\}_{x \in X}$ has the property that there exists $x^* \in X$ such that* (i) *for all $1 \leq x < x^*$ (if any), $\mathcal{C}_x^m = \{N\}$ and* (ii) *for all $x^* \leq x \leq M$, $\mathcal{C}_x^m = \{\{1\}, \ldots, \{n\}\}$.*

Note that if $M = 2$, then $x^* = 1$ corresponds to the case of voting by quota 1 and $x^* = 2$ corresponds to the case of voting by quota n. Figure 3 below represents one of those anonymous generalized median voter schemes for the case where $M = 5$, $n = 3$ and $x^* = x_3$. For each $x \in X$, $\mathcal{C}_x^m$ is depicted on the top of x.

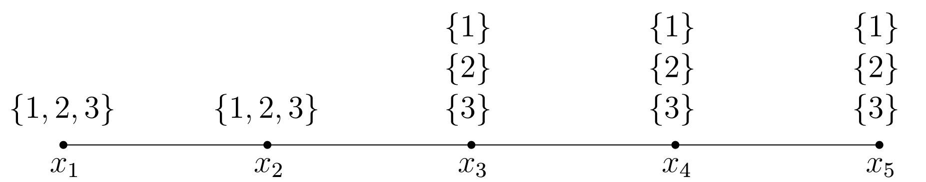


Figure 3

Observe that in general, the two cases $x^* = 1$ and $x^* = M$ correspond to the SCFs that select the minimum and maximum top alternative, respectively. Corollary 3 says that there are still other obviously strategy-proof, anonymous and onto SCFs different of these two extremes. For instance, in the example depicted in Figure 3, at any $P = (P_1, P_2, P_3)$ with the property that $t(P) = (x_1, x_4, x_5)$, $f(P) = x_3$ is neither the maximum nor the minimum top alternative but f is somehow simple, and far of being a dictatorship. In fact, f can be described as follows: $f(P)$ is the maximum top, as long as all tops are below x_3, $f(P)$ is the minimum top, as long as all tops are above x_3, and $f(P) = x_3$, as long as there are tops below and above x_3.

7 Conclusion

For the class of social choice problems where a set of agents have to select an alternative from a finite and linearly ordered set of alternatives over which agents have single-peaked preferences, we have characterized the set of all obviously strategy-proof and onto social choice functions. Our contribution is to identify the (InIn) property as being necessary and sufficient for OSP-implementation. Moreover, we use the property to define an algorithm that for each obviously strategy-proof social choice function delivers an extensive game form that OSP-implements it. This is in contrast with a major part of the literature on obvious strategy-proofness containing revelation principle like results.

The (InIn) property is restrictive and substantially reduces the class of strategy-proof social choice functions in this setting. Often, apparently a simple mechanism (*e.g.*, in the two-alternative case, voting by quota q when $1 < q < n$) that seems to suggest that truth-telling is clearly dominant, nonetheless the mechanism is not obviously strategy-proof. Our paper confirms the conviction that obvious strategy-proofness is a very restrictive notion. However, our companion paper (Arribillaga, Massó and Neme (2019)) indicates that in another setting this is not necessarily the case; *e.g.*, when alternatives have private components, OSP may not have any additional bite at all. This means that for each specific setting a particular analysis has to be carried out. Our two papers are two examples of those, each with two extreme and different conclusions: restrictive in the public-good case and not at all in the private-good case.

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Appendix

We start with the proof of Theorem 2 since it implies the sufficiency part of Theorem 1. Observe that Theorem 2 does not only guarantee that a generalized median voter scheme that satisfies the (InIn) property is obviously strategy-proof but it also says that the extensive game form defined in Subsection 5.2 OSP-implements it.

We state a remark that will be used in the sequel.

Remark 2 Let $\mathcal{C} = \{\mathcal{C}_x\}_{x \in X}$ be a coalition system that satisfies the (InIn) property and let $x \in X$. Then, the following statements hold.

(a) Assume $x^* \leq x$. Then, $\{i\} \in \mathcal{C}_x^m$ if and only if $i \in I_x^1 \cup F_x^1$.

(b) If $S \in \mathcal{C}_x^m$ is such that $|S| \geq 2$, then $i^x \in S$.

(c) If $1 < k^x$ and $x^* \leq x < x'$, then $\{i^x\} \in \mathcal{C}_{x'}^m$.

(d) If $x' < x \leq x^*$, then $i^{x-1} \in I_{x'}^1$.

(e) If $x < x^*$, then $I_{x+1}^1 \cup \{i^x\} \subseteq S$ for every $S \in \mathcal{C}_x^m$.

(f) If $F_x^{k^x} \neq \emptyset$, then

$$\mathcal{C}_x^m = \{S \subset N \mid S = I_x^k \cup \{i\} \text{ for some } 1 \leq k \leq k^x \text{ and } i \in F_x^k\}. \tag{3}$$

(g) If $F_x^{k^x} = \emptyset$, then

$$\mathcal{C}_x^m = \{S \subset N \mid S = I_x^k \cup \{i\} \text{ for some } 1 \leq k \leq k^x - 1 \text{ and } i \in F_x^k\} \cup \{I_x^{k^x}\}. \tag{4}$$

(h) $|\mathcal{C}_x^m| = 1$ if and only if $I_x^1 \in \mathcal{C}_x^m$.

We now argue why the statements of Remark 2 hold.

To see that (a) holds, notice that $x^* \leq x$ together with outcome monotonicity of $\mathcal{C}$ imply that $\mathcal{C}_x^m$ contains at least one singleton coalition. Hence, either I_x^1 is a singleton set and $\mathcal{C}_x^m = \{I_x^1\}$, in which case $F_x^1 = \emptyset$, or else $I_x^1 = \emptyset$. Then, by definition of F_x^1, the statement follows.

Statement (b) holds because, by hypothesis, $k^x > 1$ and, by the definitions of i^x and I_x^2, $i^x \in I_x^2$ and $I_x^2 \subseteq S$.

Statement (c) holds because, by hypothesis, $I_x^1 = \emptyset$ and so, by outcome monotonicity of the coalition system, $I_{x+1}^1 = \emptyset$. By condition (b) of the (InIn) property, $\{i^x\} \in \mathcal{C}_{x+1}^m$ and, by outcome monotonicity of the coalition system, $\{i^x\} \in \mathcal{C}_{x'}^m$ if $x < x'$.

Statement (d) holds because, by definition of x^*, $k^{x-1} > 1$. By the definition of i^{x-1} and condition (b) of the (InIn) property, $i^{x-1} \in I_{x-1}^2$. By definition of x^* and the hypothesis $x' < x \leq x^*$, $I_{x-1}^1 = I_{x-1}^2$ and $I_{x'}^1 = I_{x'}^2$ hold. By outcome monotonicity of the coalition system, $I_{x-1}^1 \subset I_{x'}^1$. Hence, $i^{x-1} \in I_{x'}^1$.

To see that (e) holds, assume $x < x^*$. By definition of x^*, $k^x > 1$. Let $S \in \mathcal{C}_x^m$ be arbitrary. Notice first that $I_x^2 \subseteq S$ and $i^x \in I_x^2$, and so $i^x \in S$. Second, by outcome monotonicity of the coalition system, $S \in \mathcal{C}_{x+1}$. Hence, $I_{x+1}^1 \subseteq S$. Thus, $I_{x+1}^1 \cup \{i^x\} \subseteq S$.

Statement (f) holds because the fact that $\mathcal{C}_x^m$ includes the set in the right side of (3) follows from the definition of F_x^k. To see that the other inclusion in (3) holds as well, let $S \in \mathcal{C}_x^m$ and set $k = |S|$. Observe that $F_x^{k^x} \neq \emptyset$ implies that there exists $S' \in \mathcal{C}_x^m$ such that $|S'| \geq k$, and so $I_x^k \subsetneq S$ and, by condition (a) of the (InIn) property, $\left|I_x^k\right| = k - 1$. By the definition of F_x^k, there exists $i \in F_x^k$ such that $S = I_x^k \cup \{i\}$.

To see that (g) holds, observe first that $F_x^{k^x} = \emptyset$ implies $I_x^{k^x} \in \mathcal{C}_x^m$. Now, the fact that the union of the two sets in the right side of (4) is included in $\mathcal{C}_x^m$ follows from the definition of F_x^k. To see that the other inclusion in (4) holds as well, let $S \in \mathcal{C}_x^m$ and set $k = |S|$. If $k < k^x$, the inclusion follows by the same argument used to show that (f) holds. If $k = k^x$, $F_x^{k^x} = \emptyset$ implies $S = I_x^{k^x}$ and so S belongs to the union of the two sets.

Statement (h) follows immediately from the definition of I_x^1.

Proof of Theorem 2 Let $f : \mathcal{P}^N \to X$ be a generalized median voter scheme whose associated coalition system $\mathcal{C} = \{\mathcal{C}_x\}_{x \in X}$ satisfies the (InIn) property. Let $\Gamma^{\mathcal{C}}$ be the extensive game form obtained by the algorithm defined in Subsection 5.2. For each $P \in \mathcal{P}^N$, define the profile of *truth-telling strategies* $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n})$ in $\Gamma^{\mathcal{C}}$ as follows. For each $i \in N$, let h be a history with the property that $\mathcal{N}(h) = i$. Suppose h is a history in **Stage I** (namely, $h = \emptyset$ and $i = i^*$). Then,

$$\sigma_i^{P_i}(h) = \begin{cases} x^* - 1 & \text{if } x^* > 1 \text{ and } t(P_i) \leq x^* - 1 \\ x^* & \text{if } t(P_i) = x^* \\ x^* + 1 & \text{if } x^* < M \text{ and } t(P_i) \geq x^* + 1. \end{cases}$$

Suppose h is a history in **Stage Up.k** or in **Stage Down.k**, for some $k \geq 1$. Then,

$$\sigma_i^{P_i}(h) = \begin{cases} x & \text{if } t(P_i) \leq x \\ x + 1 & \text{if } t(P_i) \geq x + 1, \end{cases}$$

where $x = x^* + k - 1$ if h belongs to **Stage Up.k** and $x = x^* - k$ if h belongs to **Stage Down.k**. Namely, $\sigma_i^{P_i}$ chooses always the best available action according to P_i.

We prove Theorem 2 by showing that, for each profile $P \in \mathcal{P}^N$, the following two statements hold.

(I.a) $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.

(I.b) σ^P is weakly dominant in $\Gamma^{\mathcal{C}}$.

Proof of (I.a) Let $P = (P_1, \ldots, P_n) \in \mathcal{P}^N$ be an arbitrary profile and let $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$ be the outcome of the extensive game form $\Gamma^{\mathcal{C}}$ when agents play it according to σ^P. We will distinguish among three cases depending on whether $h^{\Gamma^{\mathcal{C}}}(\sigma^P)$ is a terminal history in **Stage I**, **Stage Up.k** (for some $k \geq 1$) or **Stage Down.k** (for some $k \geq 1$).

Case I: Assume $h^{\Gamma^{\mathcal{C}}}(\sigma^P)$ is a terminal history in **Stage I**. Then, $x^* = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$. Let $i = \mathcal{N}(\emptyset)$ be the agent that has chosen the terminal action x^* in **Stage I** (namely, $i = i^*$). By the definition of σ^P, $t(P_i) = x^*$. Since $\{i\} \in \mathcal{C}^m_{x^*}$, $f(P) \leq x^*$. If $x^* = 1$, $f(P) = x^*$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$. Assume now $x^* > 1$. By the definition of x^*, $\mathcal{C}^m_{x^*-1}$ does not contain any singleton coalition. By definition, $i = i^{x^*-1} \in I^2_{x^*-1}$. Therefore, $i \in S$ for all $S \in \mathcal{C}_{x^*-1}$. Since $t(P_i) = x^*$, $f(P) \geq x^*$. Hence, $f(P) = x^*$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.

Case II: Assume $h^{\Gamma^{\mathcal{C}}}(\sigma^P)$ is a terminal history in **Stage Up.k** for some $k \geq 1$. Let $x = x^* + (k-1)$. By definition of $\Gamma^{\mathcal{C}}$, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) \in \{x, x+1\}$.

We first show that

$$f(P) \in \{x, x+1\}.$$

We start by showing that $f(P) \geq x$, which is immediate if $x = 1$. Consider the case $x > 1$. When considering the reasons why $\Gamma^{\mathcal{C}}$ has reached **Stage Up.k** we distinguish between the cases $k = 1$ and $k > 1$.

Assume $k = 1$, *i.e.*, $x = x^*$. By construction of $\Gamma^{\mathcal{C}}$, agent i^{x^*-1} has chosen $x^* + 1$ in **Stage I**. Since i^{x^*-1} is playing according to the truth-telling strategy, $t(P_{i^{x^*-1}}) \geq x^* + 1$. Since $i^{x^*-1} \in I^2_{x^*-1}$ and, by definition of x^*, $\mathcal{C}^m_{x^*-1}$ has no minimal winning coalition of cardinality equal to one, $f(P) \geq x^* = x$.

Assume $k > 1$, *i.e.*, $x > x^*$. We distinguish between the two cases in **Stage Up.k-1** that lead $\Gamma^{\mathcal{C}}$ to reach **Stage Up.k**. Suppose $k^{x-1} > 1$ and so $I^1_{x-1} = \emptyset$. Since $\Gamma^{\mathcal{C}}$ has reached **Stage Up.k**, each agent $i \in F^1_{x-1} \cup \{i^{x-1}\}$ has chosen x when playing (the modified) Γ^{x-1} in **Stage Up.k-1**. By the definition of σ^P, $t(P_i) \geq x$ for all $i \in F^1_{x-1} \cup \{i^{x-1}\}$. Then, by (a) and (b) in Remark 2, $f(P) \geq x$. Suppose $k^{x-1} = 1$. Since $\Gamma^{\mathcal{C}}$ has reached **Stage Up.k**, each agent $i \in I^1_{x-1} \cup F^1_{x-1}$ has chosen x when playing (the modified) Γ^{x-1} in **Stage Up.k-1**, because the outcome of Γ^{x-1} must be x. Therefore, by the definition of σ^P, $t(P_i) \geq x$ for all $i \in I^1_{x-1} \cup F^1_{x-1}$. Since $k^{x-1} = 1$ holds, by (f) and (g) in Remark 2, $S \in \mathcal{C}^m_{x-1}$ if and only if $S = \{i\}$ for some $i \in I^1_{x-1} \cup F^1_{x-1}$. Then, $f(P) \geq x$. Hence, and independently of whether $k^{x-1} > 1$ or $k^{x-1} = 1$,

$$f(P) \geq x. \tag{5}$$

We now proceed by showing that $f(P) \leq x + 1$, which is immediate if $x + 1 = M$. Consider the case $x + 1 < M$. We distinguish between the two circumstances under which $\Gamma^{\mathcal{C}}$ has ended in **Stage Up.k**. Suppose $k^x > 1$. Since $x^* \leq x$, by outcome monotonicity of $\mathcal{C}$, $\mathcal{C}^m_x$ contains at least a minimal winning coalition of cardinality equal to one and so, by assumption, $I^1_x = \emptyset$. Therefore, there exists $\bar{\imath} \in F^1_x \cup \{i^x\}$ that has chosen x in Γ^x, *i.e.*, $t(P_{\bar{\imath}}) \leq x$. Therefore, by (a) and (c) in Remark 2, either $\{\bar{\imath}\} \in \mathcal{C}^m_x$ (and $f(P) = x$) or $\{\bar{\imath}\} \in \mathcal{C}_{x+1}$ (*i.e.*, $\bar{\imath} = i^x$ and $f(P) \leq x + 1$). Therefore, $f(P) \leq x + 1$. Suppose $k^x = 1$. Since $\Gamma^{\mathcal{C}}$ has ended in **Stage Up.k**, at least one $i \in I^1_x \cup F^1_x$ has chosen x in Γ^x. Therefore, by the definition of σ^P, $t(P_i) \leq x$ for at least one $i \in I^1_x \cup F^1_x$. By (f) and (g) in Remark

2, $S \in \mathcal{C}_x^m$ if and only if $S = \{i\}$ for some $i \in I_x^1 \cup F_x^1$. Then, $f(P) \leq x$. Hence, and independently of whether $k^x > 1$ or $k^x = 1$,

$$f(P) \leq x + 1. \tag{6}$$

Thus, by (5) and (6),

$$f(P) \in \{x, x + 1\}. \tag{7}$$

Consider now (the modified) Γ^x played in **Stage Up.k**. By hypothesis, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$ is the outcome of Γ^x when agents play it according to σ^P. We show that $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$ by distinguishing between two cases.

First assume that the outcome of (the modified) Γ^x takes place in **Stage A.k**, with $1 \leq k \leq k^x$. This implies that $I_x^k \backslash I_x^{k-1} \neq \emptyset$, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) = x + 1$ and the following two conditions hold.

(1.A) There exists $i \in I_x^k \backslash I_x^{k-1}$ that has chosen $x + 1$, *i.e.*, $x + 1 \leq t(P_i)$.

(2.A) For all $k' < k$, each $i \in F_x^{k'}$ has chosen $x + 1$, *i.e.*, $x + 1 \leq t(P_i)$.

Let $S \in \mathcal{C}_x^m$ be such that $|S| \geq k$. Then $I_x^k \subseteq S$, and by (1.A) above, there exists $i \in S$ such that $x + 1 \leq t(P_i)$. Thus, there is no $S \in \mathcal{C}_x^m$ such that $|S| \geq k$ and $t(P_i) \leq x$ for all $i \in S$.

Let $S \in \mathcal{C}_x^m$ be such that $|S| = \overline{k} < k$. Then, as $\overline{k} < k \leq k^x$, by (f) and (g) in Remark 2, $S = I_x^{\overline{k}} \cup \{i\}$ for some i such that $i \in F_x^{\overline{k}}$. By (2.A) above, there exists $i \in S$ such that $x + 1 \leq t(P_i)$. Thus, there is no $S \in \mathcal{C}_x^m$ such that $|S| < k$ and $t(P_i) \leq x$ for all $i \in S$.

Therefore, $f(P) \geq x + 1$. By (7), $f(P) = x + 1$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.

Assume now that the outcome of (the modified) Γ^x takes place in **Stage B.k**, with $1 \leq k \leq k^x$. We proceed by distinguishing among several cases and subcases.

Case 1: $k < k^x$. Then, $F_x^k \backslash F_x^{k-1} \neq \emptyset$, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) = x$ and the following two conditions hold.

(1.B.1) There exists $\bar{\imath} \in F_x^k \backslash F_x^{k-1}$ that has chosen x, *i.e.*, $t(P_{\bar{\imath}}) \leq x$.

(2.B.1) Each $i \in I_x^k$ has chosen x, *i.e.*, $t(P_i) \leq x$.

By definition of F_x^k, it holds that for agent $\bar{\imath}$ identified in (1.B.1), $I_x^k \cup \{\bar{\imath}\} \in \mathcal{C}_x^m$. By (1.B.1) and (2.B.1), $t(P_i) \leq x$ for all $i \in I_x^k \cup \{\bar{\imath}\}$, implying that $f(P) \leq x$. Hence, by (7), $f(P) = x$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.

Case 2: $k = k^x$. The following two conditions hold.

(1.B.2) Each $i \in I_x^{k^x}$ has chosen x, *i.e.*, $t(P_i) \leq x$.

(2.B.2) For all $k' < k^x$, each $i \in F_x^{k'}$ has chosen $x + 1$, *i.e.*, $x + 1 \leq t(P_i)$.

Subcase 2.1: $F_x^k \backslash F_x^{k-1} \neq \emptyset$. We distinguish between two cases.

Subcase 2.1.1: There exists $\bar{\imath} \in F_x^k \setminus F_x^{k-1}$ that has chosen x, *i.e.*, $t(P_{\bar{\imath}}) \leq x$. Then, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) = x$. By definition of F_x^k, $I_x^k \cup \{\bar{\imath}\} \in \mathcal{C}_x^m$. By (1.B.2), $t(P_i) \leq x$ for all $i \in I_x^k \cup \{\bar{\imath}\}$, implying that $f(P) \leq x$. Hence, by (7), $f(P) = x$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.

Subcase 2.1.2: Each $i \in F_x^k \backslash F_x^{k-1}$ has chosen $x+1$, *i.e.*, $x+1 \leq t(P_i)$ for all $i \in F_x^k \backslash F_x^{k-1}$. Then, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) = x+1$. By condition (2.B.2), for all $k' \leq k^x$, each $i \in F_x^{k'}$ has chosen $x+1$. By (f) in Remark 2, there is no $S \in \mathcal{C}_x^m$ such that $t(P_i) \leq x$ for all $i \in S$. Thus, $x+1 \leq f(P)$. Hence, by (7), $f(P) = x+1$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.
Case 2.2: $F_x^k \backslash F_x^{k-1} = \emptyset$. By (2), we distinguish only between two cases.
Subcase 2.2.1: $F_x^{k^x} = \emptyset$. Then, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) = x$. By (g) in Remark 2, $I_x^{k^x} \in \mathcal{C}_x^m$, which implies, by (1.B.2) above, that $f(P) \leq x$. Hence, by (7), $f(P) = x$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.
Subcase 2.2.2: $F_x^{k^x} = F_x^{k^x-1} \neq \emptyset$, Then, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) = x+1$. Condition (2.B.2) implies that, for all $k' \leq k^x$, each $i \in F_x^{k'}$ has chosen $x+1$. By (f) in Remark 2, there is no $S \in \mathcal{C}_x^m$ such that $t(P_i) \leq x$ for all $i \in S$. Thus, $x+1 \leq f(P)$. Hence, by (7), $f(P) = x+1$ and so $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$.
Case III: Assume $h^{\Gamma^{\mathcal{C}}}(\sigma^P)$ is a terminal history in **Stage Down.k** for some $k \geq 1$. Let $x = x^* - k$. By definition of $\Gamma^{\mathcal{C}}$, $o(h^{\Gamma^{\mathcal{C}}}(\sigma^P)) \in \{x, x+1\}$.

We first show that

$$f(P) \in \{x, x+1\}.$$

We start be showing that $f(P) \leq x+1$, which is immediate if $x+1 = M$. Consider the case $x+1 < M$. When considering the reasons why $\Gamma^{\mathcal{C}}$ has reached **Stage Down.k** we distinguish between the cases $k=1$ and $k>1$.

Assume $k=1$, *i.e.*, $x = x^*-1$. By construction of $\Gamma^{\mathcal{C}}$, agent i^{x^*-1} has chosen x^*-1 in **Stage I**. Since i^{x^*-1} is playing according to the truth-telling strategy, $t(P_{i^{x^*-1}}) \leq x^*-1$. Since $\{i^{x^*-1}\} \in \mathcal{C}_{x^*}^m$ (see footnote 13), $f(P) \leq x^* = x+1$.

Assume $k > 1$, *i.e.*, $x < x^*-1$. We distinguish between the two cases in **Stage Down.k-1** that lead $\Gamma^{\mathcal{C}}$ to reach **Stage Down.k**. Suppose $\left|\mathcal{C}_{x+1}^m\right| > 1$. Since $\Gamma^{\mathcal{C}}$ has moved to **Stage Down.k**, each agent $i \in I_{x+1}^1 \cup \{i^x\}$ has chosen $x+1$ when playing (the modified) Γ^{x+1} in **Stage Down.k-1**. By definition of σ^P, $t(P_i) \leq x+1$ for all $i \in I_{x+1}^1 \cup \{i^x\}$. Since $x+1 < x^*$, $k^{x+1} > 1$. By condition (b) of the (InIn) property, $I_{x+1}^1 \cup \{i^x\} \in \mathcal{C}_{x+1}^m$ holds, and then $f(P) \leq x+1$. Suppose $\left|\mathcal{C}_{x+1}^m\right| = 1$. Then, in **Stage Down.k-1** the outcome of Γ^{x+1} is $x+1$, which means that each $i \in I_{x+1}^1$ has chosen $x+1$ in Γ^{x+1}. By definition of σ^P, $t(P_i) \leq x+1$ for all $i \in I_{x+1}^1$. Since $\left|\mathcal{C}_{x+1}^m\right| = 1$, by (h) in Remark 2, $I_{x+1}^1 \in \mathcal{C}_{x+1}^m$, and then $f(P) \leq x+1$. Hence, and independently of whether $\left|\mathcal{C}_{x+1}^m\right| > 1$ or $\left|\mathcal{C}_{x+1}^m\right| = 1$,

$$f(P) \leq x+1. \tag{8}$$

We now proceed by showing that $f(P) \geq x$, which is immediate if $x = 1$. Consider the case $1 < x$. We distinguish between the two circumstances under which $\Gamma^{\mathcal{C}}$ has ended at **Stage Down.k**. Suppose $|\mathcal{C}_x^m| > 1$. Then, there exists $\bar{\imath} \in I_x^1 \cup \{i^{x-1}\}$ that has chosen $x+1$ when playing (the modified) Γ^x in **Stage Down.k**. By definition of σ^P, $t(P_{\bar{\imath}}) \geq x+1$. By (e) in Remark 2, there is no $S \in \mathcal{C}_{x-1}^m$ such that $t(P_i) \leq x-1$ for all $i \in S$, and then

$f(P) \geq x$. Suppose $|\mathcal{C}_x^m| = 1$. Then, in **Stage Down.k** the outcome of Γ^x is $x+1$. Then, since $|\mathcal{C}_x^m| = 1$, by (h) in Remark 2, there exists $\bar{\imath} \in I_x^1$ that has chosen $x+1$ when playing Γ^x in **Stage Down.k**. By definition of σ^P, $t(P_{\bar{\imath}}) \geq x+1$. By (h) in Remark 2, $\mathcal{C}_x^m = \{I_x^1\}$ holds, and so there is no $S \in \mathcal{C}_x^m$ such that $t(P_i) \leq x$ for all $i \in S$, and then $f(P) \geq x$. Hence, and independently of whether $|\mathcal{C}_x^m| > 1$ or $|\mathcal{C}_x^m| = 1$,

$$f(P) \geq x. \tag{9}$$

Thus, by (8) and (9),

$$f(P) \in \{x, x+1\}. \tag{10}$$

Now, the proof that $f(P) = o(h^{\Gamma^{\mathcal{C}}}(\sigma^P))$ follows as in Case II.

Proof of (I.b) We show that, for each $i \in N$ and $P_i \in \mathcal{P}$, the strategy $\sigma_i^{P_i}$ is weakly dominant in $\Gamma^{\mathcal{C}}$. Fix $i \in N$ and $P_i \in \mathcal{P}$, and let $\sigma_i' \neq \sigma_i^{P_i}$ be arbitrary. We consider three cases depending on the stage at which σ_i' chooses for the first time an action different from the one that $\sigma_i^{P_i}$ would choose.

Case 1: Assume σ_i' chooses a different action than $\sigma_i^{P_i}$ in **Stage I**, *i.e.*, $\mathcal{N}(\emptyset) = i = i^*$ and $\sigma_i'(\emptyset) \neq \sigma_i^{P_i}(\emptyset)$. We distinguish among three cases.

Subcase 1.1: $t(P_i) = x^*$. Then, $\sigma_i^{P_i}$ chooses x^*, the outcome of $\Gamma^{\mathcal{C}}$ is x^* and so $\sigma_i^{P_i}$ is trivially weakly dominant.

Subcase 1.2: $t(P_i) \geq x^*+1$. Then, $\sigma_i^{P_i}$ chooses x^*+1. By (c) in Remark 2, $\{i\} \in \mathcal{C}_{x'}^m$ for all $x' \geq x^*$. Hence, the outcome of $\Gamma^{\mathcal{C}}$ is greater than or equal to x^* and smaller than or equal to $t(P_i)$. Furthermore, $\sigma_i'(\emptyset) \in \{x^*-1, x^*\}$ in **Stage I** and so the outcome of $\Gamma^{\mathcal{C}}$ when i plays according to σ_i' is smaller than or equal to x^*. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ is weakly dominant.

Subcase 1.3: $t(P_i) \leq x^*-1$. Then, $\sigma_i^{P_i}$ chooses x^*-1. By (d) in Remark 2, $i \in I_{x'}^1$ for all $x' < x^*$. Hence, the outcome of $\Gamma^{\mathcal{C}}$ is smaller than or equal to x^*-1 and larger than or equal to $t(P_i)$. Furthermore, $\sigma_i'(\emptyset) \in \{x^*, x^*+1\}$ in **Stage I** and so the outcome of $\Gamma^{\mathcal{C}}$ when i plays according to σ_i' is larger than or equal to x^*. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ is weakly dominant.

Case 2: Assume σ_i' chooses a different action than $\sigma_i^{P_i}$ in **Stage Up.k** for some $k \geq 1$. Let $x = x^* + (k-1)$. We distinguish between two cases.

Subcase 2.1: $i \in I_x^{k^x}$. Observe two things. First, $I_x^k \subseteq I_x^{k^x}$ for all $1 \leq k \leq k^x$ and i plays in some **Stage A.k′** for some $k' \geq 1$ in (the modified) Γ^x. We distinguish between two cases.

Subcase 2.1.1: $t(P_i) \leq x$. Then, $\sigma_i^{P_i}$ chooses x in (the modified) Γ^x. We distinguish between the cases $k^x = 1$ and $k^x > 1$.

Assume $k^x = 1$. Then, since $i \in I_x^{k^x}$, $\{\{i\}\} = \mathcal{C}_x^m$ and the outcome of $\Gamma^{\mathcal{C}}$ is x. Since σ_i' chooses $x+1$, the outcome of $\Gamma^{\mathcal{C}}$ is now larger than or equal to $x+1$. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ weakly dominates σ_i'.

Assume $k^x > 1$. Since i plays in **Stage A.k$'$** for some $k' \geq 1$, agent i, and every agent $j \in I_x^{k^x}$ that has played before i, have chosen x. Hence, when i plays according to $\sigma_i^{P_i}$, $\Gamma^{\mathcal{C}}$ ends with the outcome of Γ^x, which is either x or $x+1$, regardless of whether or not $i = i^x$. In contrast, when i plays according to σ_i', i chooses $x+1$ and then the outcome of $\Gamma^{\mathcal{C}}$ is greater than or equal to $x+1$. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ weakly dominates σ_i'.
Subcase 2.1.2: $t(P_i) \geq x+1$. Then, $\sigma_i^{P_i}$ chooses $x+1$ in (the modified) Γ^x, and the outcome of $\Gamma^{\mathcal{C}}$ is greater than or equal to $x+1$. We distinguish between the cases $i = i^x$ and $i \neq i^x$. Suppose $i = i^x$. Then, by (c) in Remark 2, $\{i\} \in \mathcal{C}_{x'}$ for all $x' > x$, the outcome of $\Gamma^{\mathcal{C}}$ is smaller than or equal to $t(P_i)$. Suppose $i \neq i^x$. Since agent i plays $x+1$ in **Stage A.k** and agent i^x has chosen x and $\{i^x\} \in \mathcal{C}_{x+1}$, the outcome of $\Gamma^{\mathcal{C}}$ is $x+1$. Now, and independently of whether $i = i^x$ or $i \neq i^x$, σ_i' chooses x and the outcome of $\Gamma^{\mathcal{C}}$ is x or $x+1$. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ weakly dominates σ_i'.
Subcase 2.2: $i \in F_x^{k'} \setminus F_x^{k'-1}$ for $k' \leq k$. Observe that i plays in some **Stage B.k$'$** for some $k' \geq 1$ in (the modified) Γ^x. We distinguish between two cases.
Subcase 2.2.1: $t(P_i) \leq x$. Then, $\sigma_i^{P_i}$ chooses x in (the modified) Γ^x and the outcome of $\Gamma^{\mathcal{C}}$ is x. Furthermore, σ_i' chooses $x+1$ and the outcome of $\Gamma^{\mathcal{C}}$ is greater than or equal to x. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ weakly dominates σ_i'.
Subcase 2.2.2: $t(P_i) \geq x+1$. We distinguish between the cases $k' = 1$ and $k' > 1$. Suppose $k' = 1$. Then, $i \in F_x^1$ implies $\{i\} \in C_{x'}^m$ for $x' \geq x$. Since $\sigma_i^{P_i}$ chooses $x+1$ in (the modified) Γ^x, the outcome of $\Gamma^{\mathcal{C}}$ is larger than or equal to x and smaller than or equal to $t(P_i)$. Suppose $k' > 1$. Then, i^x has chosen x in Γ^x and so the outcome of $\Gamma^{\mathcal{C}}$ is either x or $x+1$. Hence, and independently of whether $k' = 1$ or $k' > 1$, σ_i' chooses x and then the outcome of $\Gamma^{\mathcal{C}}$ is x. Hence, since P_i is single-peaked, $\sigma_i^{P_i}$ is weakly dominant.

Case 3: Assume σ_i' chooses a different action than $\sigma_i^{P_i}$ in **Stage Down.k** for some $k \geq 1$. This case is similar to Case 2, replacing the role of (c) by (d) in Remark 2, and therefore its proof is omitted. ■

Proof of Theorem 1 The sufficiency part follows from Theorem 2.

To prove necessity, assume $f : \mathcal{P}^N \to X$ is obviously strategy-proof and onto. By Corollary 1 in Li (2017), f is strategy-proof. By Barberà, Gül and Stacchetti (1993), f is a generalized median voter scheme. Let $\{\mathcal{C}_x\}_{x \in X}$ be the coalition system associated to f. We have to show that $\{\mathcal{C}_x\}_{x \in X}$ satisfies the (InIn) property. To do so we will use the fact that, similarly to what happens with SP-implementability, OSP-implementability is a hereditary property in the following sense. If f is OSP-implementable in a domain, then the restriction of f on any of its subdomains is also OSP-implementable.[19]

The subdomains that we will consider are those obtained by considering subsets of single-peaked preferences over two or at most three consecutive alternatives in X, with

[19]The proof of Proposition 5 in Li (2017) contains this observation.

tops on one of those alternatives. We now show that condition (a) of the (InIn) property holds for every $x < M$. Fix $x \in \{1, \ldots, M-1\}$, denote by $\mathcal{P}_x$ the set of the two preferences over $\{x, x+1\}$ and consider the generalized median voter scheme $\overline{f} : \mathcal{P}_x^N \to \{x, x+1\}$ defined by the coalition system $\overline{\mathcal{C}} = \{\overline{\mathcal{C}}_x, \overline{\mathcal{C}}_{x+1}\}$, where $\overline{\mathcal{C}}_x = \mathcal{C}_x$ and $\overline{\mathcal{C}}_{x+1} = \{\{i\} \mid i \in N\}$. Since f is obviously strategy-proof so is $\overline{f}$.[20] As we have already mentioned in the Introduction, Bade and Gonczarowski (2017) show that to OSP-implement $\overline{f}$ one can restrict attention only to proto-dictatorship mechanisms (see also the proof of Proposition 1 in Arribillaga, Massó and Neme (2016)). That is, we can assume that the extensive game form that OSP-implements $\overline{f}$ has the following properties. Agents play sequentially, at most once, and have to choose either x or $x+1$. Moreover, they are grouped into alternate subsets in which each agent has either the choice between implementing x (by choosing it) or letting the game continue (by choosing $x+1$) or the choice between implementing $x+1$ (by choosing it) or letting the game continue (by choosing x), except the last player in the sequence who has the choice between implementing x (by choosing x) or implementing $x+1$ (by choosing $x+1$).

Let X_1 be the first group of agents in the sequence that can implement x or let the game continue. Let Y_1 be the second group of agents in the sequence that play after the agents in X_1, and can implement $x+1$ or let the game continue. In general, for $t \in \{2, \ldots, \overline{t}\}$, let X_t the group of agents in the sequence that play after the agents in Y_{t-1}, and can implement x or let the game continue. Let Y_t the group of agents in the sequence that play after the agents in X_t, and can implement $x+1$ or let the game continue. Finally, let $\widehat{j}$ be the last agent in the sequence that plays after the agents in $Y_{\overline{t}}$, and can implement either x or $x+1$. Hence, the order of play of the subsets of agents is given by $X_1, Y_1, X_2, \ldots, X_t, Y_t, X_{t+1}, \ldots, X_{\overline{t}}, Y_{\overline{t}}, \widehat{j}$, and agents in each subset can play in any order. Observe that X_1 and/or $Y_{\overline{t}}$ could be empty.[21]

Since the proto-dictatorship OSP-implements $\overline{f}$ and $\overline{\mathcal{C}}_x = \mathcal{C}_x$, it can be checked that $\mathcal{C}_x^m$ can be written as the following collection of subsets

$$\begin{aligned} \mathcal{C}_x^m &= \{\{i\} \mid i \in X_1\} \\ &\bigcup \{S \mid S = \bigcup_{t=1}^{\widehat{t}} Y_t \cup \{i\} \text{ s.t. } i \in X_{\widehat{t}+1} \text{ for some } 1 \le \widehat{t} \le \overline{t} - 1\} \\ &\bigcup \{\bigcup_{t=1}^{\overline{t}} Y_t \cup \{\widehat{j}\}\}. \end{aligned}$$

If $k = 1 \le k^x$, $|I_x^1| \ge 0$ holds trivially. Let $1 < k \le k^x$ and $\widehat{S} \in \mathcal{C}_x^m$ be such that $\left|\widehat{S}\right| \ge k$ and $\left|\widehat{S}\right| \le |S|$ for all $S \in \mathcal{C}_x^m$ such that $|S| \ge k$. That is, $\widehat{S}$ is one of the subsets

[20] Since generalized median voter schemes are tops only and onto, $\overline{f}$ is the restriction of f into the subdomain $\mathcal{P}_x^N$.

[21] Figure 1 in Example 2 represents the proto-dictatorship mechanism where $\overline{t} = 4$ and $X_1 = \{1\}$, $Y_1 = \{2\}$, $X_2 = \{3, 4\}$, $Y_2 = \{5\}$, $X_3 = \{6\}$, $Y_3 = \{7, 8\}$, $X_4 = \{9\}$, $Y_4 = \emptyset$ and $\widehat{j} = 10$.

with the smallest cardinality among all subsets in $\mathcal{C}_x^m$ with cardinality larger than or equal to k. Clearly, $\widehat{S} \notin \{\{i\} \mid i \in X_1\}$ and let $\widehat{t} \in \{1, \ldots, \overline{t}\}$ be such that $\widehat{S} = \bigcup_{t=1}^{\widehat{t}} Y_t \cup \{\widehat{i}\}$. Observe that if $S \in \mathcal{C}_x^m$ and $|S| \geq k$, $S = \bigcup_{t=1}^{t'} Y_t \cup \{i\}$ with $t' \geq \widehat{t}$ and $i \in X_{t'+1} \cup \{\widehat{j}\}$. Then, $I_x^k = \bigcup_{t=1}^{\widehat{t}} Y_t$ and so $\left|I_x^k\right| = \left|\bigcup_{t=1}^{\widehat{t}} Y_t\right| \geq \left|\widehat{S}\right| - 1 \geq k - 1$.

Now we show that condition (b) of the (InIn) property holds at x. Assume $k^x > 1$. Then, there exists $S \in \mathcal{C}_x^m$ such that $|S| \geq 2$ and, by condition (a) of the (InIn) property holds at x, $I_x^2 \neq \emptyset$. We distinguish between two cases.

Case 1: $x+1 = M$. Then $\mathcal{C}_{x+1}^m = \{\{i\} \mid i \in N\}$ and $I_{x+1}^1 = \emptyset$. Therefore, $I_{x+1}^1 \cup \{i\} \in \mathcal{C}_{x+1}^m$ for each $i \in I_x^2$, which means that condition (b) of the (InIn) property holds at x.

Case 2: $x+1 < M$. We distinguish between two cases.

Subcase 2.1: $\left|\mathcal{C}_{x+1}^m\right| = 1$. Let $\{S'\} = \mathcal{C}_{x+1}^m$, and so $I_{x+1}^1 = S'$. By outcome monotonicity of the coalition system, $S \in \mathcal{C}_{x+1}$ for all $S \in \mathcal{C}_x^m$ with $|S| \geq 2$. Hence,

$$S' = \bigcap_{\substack{S \in \mathcal{C}_{x+1} \\ |S| \geq 2}} S \subset \bigcap_{\substack{S \in \mathcal{C}_x^m \\ |S| \geq 2}} S = I_x^2.$$

Then, there exists $i \in S' \subset I_x^2$ such that $S' \cup \{i\} = S' \in \mathcal{C}_{x+1}^m$. Thus, condition (b) of the (InIn) property holds at x.

Subcase 2.2: $\left|\mathcal{C}_{x+1}^m\right| > 1$. We distinguish between two cases.

Subcase 2.2.1: There exists j' such that $\{j'\} \in \mathcal{C}_{x+1}^m$. Define $\widetilde{\mathcal{P}}_1 \times \cdots \times \widetilde{\mathcal{P}}_n \equiv \widetilde{\mathcal{P}} \subseteq \mathcal{P}^N$ as follows.

i) If $\{i\} \in \mathcal{C}_x^m$, then $\widetilde{\mathcal{P}}_i = \{P_i \in \mathcal{P} \mid t(P_i) \in \{x+1, x+2\}\}$.

ii) If $\{i\} \notin \mathcal{C}_x^m$, then $\widetilde{\mathcal{P}}_i = \{P_i \in \mathcal{P} \mid t(P_i) \in \{x, x+1, x+2\}\}$.

Let $\widetilde{f}$ be the restriction of f to the set of profiles in $\widetilde{\mathcal{P}}$. Since f is OSP-implementable, so is $\widetilde{f}$. Let $\widetilde{\Gamma}$ be an extensive game form that OSP-implements $\widetilde{f}$. From now on, we will use a tilde to refer to the components of $\widetilde{\Gamma}$, and set $\widetilde{N} = N$. Hence, for every $P \in \widetilde{\mathcal{P}}$, there exists σ^P such that $\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma^P)) = \widetilde{f}(P)$. For $P_i \in \widetilde{\mathcal{P}}_i$, denote $\sigma_i^{P_i}$ by σ_i^z where $t(P_i) = z$.

Let j be the first agent that has to play in $\widetilde{\Gamma}$ (*i.e.*, $\widetilde{\mathcal{N}}(\emptyset) = j$). By Mackenzie (2018), we can assume without loss of generality that j has at least two actions available at $\emptyset$ (*i.e.*, $\left|\widetilde{\mathcal{A}}(\emptyset)\right| \geq 2$); that is,

$$\sigma_j^z(\emptyset) \neq \sigma_j^{z'}(\emptyset) \tag{11}$$

for $z, z' \in \{x, x+1, x+2\}$. We claim that $j \in I_x^2$.

Claim 1 $\quad j \in I_x^2$.

Proof of Claim 1 $\quad$ Suppose otherwise. We distinguish between two cases, depending on whether or not $\{j\}$ is a minimal winning coalition at x.

(1.i) $\{j\} \in \mathcal{C}_x^m$. Since $k^x > 1$, there exists $S \in \mathcal{C}_x^m$ such that $|S| \geq 2$ and $j \notin S$. By the definition of $\widetilde{\mathcal{P}}_j$, $\{j\} \in \mathcal{C}_x^m$ and (11), $\sigma_j^{x+1}(\emptyset) \neq \sigma_j^{x+2}(\emptyset)$.

For each $i \in S$ and history h in $\widetilde{\Gamma}$ such that $\widetilde{\mathcal{N}}(h) = i$, define

$$\widetilde{\sigma}_i(h) = \begin{cases} \sigma_i^{x+1}(h) & \text{if } \sigma_j^{x+1}(\emptyset) \preceq h \\ \sigma_i^x(h) & \text{if } \sigma_j^{x+2}(\emptyset) \preceq h. \end{cases}$$

Since $\widetilde{\Gamma}$ induces $\widetilde{f}$,

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \widetilde{\sigma}_S, \sigma_j^{x+2})) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \sigma_S^x, \sigma_j^{x+2})) = \widetilde{f}(P_{-S-\{j\}}^{x+2}, P_S^x, P_j^{x+2}) = x$$

and

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \widetilde{\sigma}_S, \sigma_j^{x+1})) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \sigma_S^{x+1}, \sigma_j^{x+1})) = \widetilde{f}(P_{-S-\{j\}}^{x+2}, P_S^{x+1}, P_j^{x+1}) = x+1.$$

By single-peakedness, $(x+1)\, P_j^{x+2} x$ holds, which implies that σ_j^{x+2} is not weakly dominant. Hence, $j \in I_x^2$ if $\{j\} \in \mathcal{C}_x^m$.

(1.ii) $\{j\} \notin \mathcal{C}_x^m$. Then, by our contradiction hypothesis stating that $j \notin I_x^2$, there exists $S \in \mathcal{C}_x^m$ such that $|S| \geq 2$ and $j \notin S$. By the definition of $\widetilde{\mathcal{P}}_j$, $\{j\} \notin \mathcal{C}_x^m$ and (11), there exists $y \in \{x, x+1\}$ such that $\sigma_j^y(\emptyset) \neq \sigma_j^{x+2}(\emptyset)$.

For each $i \in S$ and history h in $\widetilde{\Gamma}$ such that $\widetilde{\mathcal{N}}(h) = i$, define

$$\widetilde{\sigma}_i(h) = \begin{cases} \sigma_i^{x+1}(h) & \text{if } \sigma_j^{y}(\emptyset) \preceq h \\ \sigma_i^x(h) & \text{if } \sigma_j^{x+2}(\emptyset) \preceq h. \end{cases}$$

Since $\widetilde{\Gamma}$ induces $\widetilde{f}$,

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \widetilde{\sigma}_S, \sigma_j^{x+2})) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \sigma_S^x, \sigma_j^{x+2})) = \widetilde{f}(P_{-S-\{j\}}^{x+2}, P_S^x, P_j^{x+2}) = x$$

and

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \widetilde{\sigma}_S, \sigma_j^{y})) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+2}, \sigma_S^{x+1}, \sigma_j^{y})) = \widetilde{f}(P_{-S-\{j\}}^{x+2}, P_S^{x+1}, P_j^{y}) = x+1.$$

By single-peakedness, $(x+1)\, P_j^{x+2} x$ holds, which implies that σ_j^{x+2} is not weakly dominant. Hence, $j \in I_x^2$ if $\{j\} \notin \mathcal{C}_x^m$, and this concludes the proof of Claim 1. □

To proceed with the proof for this Subcase 2.2.1, assume that condition (b) of the (InIn) property does not hold at x. By Claim 1, $j \in I_x^2$, and so $I_{x+1}^1 \cup \{j\} \notin \mathcal{C}_{x+1}^m$. Since $\left|\mathcal{C}_{x+1}^m\right| > 1$ and there exists j' such that $\{j'\} \in \mathcal{C}_{x+1}^m$, $I_{x+1}^1 = \emptyset$, and so $\{j\} \notin \mathcal{C}_{x+1}^m$. By outcome monotonicity of the coalition system, $\{j\} \notin \mathcal{C}_x^m$. By the definition of $\widetilde{\mathcal{P}}_j$, $\{j\} \notin \mathcal{C}_x^m$ and (11), there exists $y \in \{x+1, x+2\}$ such that $\sigma_j^x(\emptyset) \neq \sigma_j^y(\emptyset)$.

For each $i \neq j$ and history h in $\widetilde{\Gamma}$ such that $\widetilde{\mathcal{N}}(h) = i$, define

$$\widetilde{\sigma}_i(h) = \begin{cases} \sigma_i^{x+2}(h) & \text{if } \sigma_j^{x}(\emptyset) \preceq h \\ \sigma_i^{x+1}(h) & \text{if } \sigma_j^{y}(\emptyset) \preceq h. \end{cases}$$

Since $\widetilde{\Gamma}$ induces $\widetilde{f}$,

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\widetilde{\sigma}_{-j}, \sigma_j^x)) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-j}^{x+2}, \sigma_j^x)) = \widetilde{f}(P_{-j}^{x+2}, P_j^x) = x+2$$

and

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\widetilde{\sigma}_{-j}, \sigma_j^y)) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-j}^{x+1}, \sigma_j^y)) = \widetilde{f}(P_{-j}^{x+1}, P_j^y) = x+1.$$

By single-peakedness, $(x+1)\, P_j^x (x+2)$ holds, which implies that σ_j^x is not weakly dominant. This contradiction implies that condition (b) of the (InIn) property holds at x for the Subcase 2.2.1.

Subcase 2.2.2: There is no j' such that $\{j'\} \in \mathcal{C}_{x+1}^m$. By outcome monotonicity of the coalition system, there is no j' such that $\{j'\} \in \mathcal{C}_x^m$. Hence, $I_{x+1}^1 = I_{x+1}^2$. Since condition (a) of the (InIn) property holds at $x+1$, $I_{x+1}^1 = I_{x+1}^2 \neq \emptyset$. Define $\widetilde{\mathcal{P}}_1 \times \cdots \times \widetilde{\mathcal{P}}_n \equiv \widetilde{\mathcal{P}} \subseteq \mathcal{P}^N$ as follows.

i) If $i \in I_{x+1}^1$, then $\widetilde{\mathcal{P}}_i = \{P_i \in \mathcal{P} \mid t(P_i) \in \{x, x+1\}\}$.

ii) If $i \notin I_{x+1}^1$, then $\widetilde{\mathcal{P}}_i = \{P_i \in \mathcal{P} \mid t(P_i) \in \{x, x+1, x+2\}\}$.

Let $\widetilde{f}$ be the restriction of f to the set of profiles in $\widetilde{\mathcal{P}}$. Since f is OSP-implementable, so is $\widetilde{f}$. Let $\widetilde{\Gamma}$ be an extensive game form that OSP-implements $\widetilde{f}$. Hence, for every $P \in \widetilde{\mathcal{P}}$, there exists σ^P such that $\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma^P)) = \widetilde{f}(P)$. For $P_i \in \widetilde{\mathcal{P}}_i$, denote $\sigma_i^{P_i}$ by σ_i^z where $t(P_i) = z$.

Let j be the first agent that has to play in $\widetilde{\Gamma}$ (*i.e.*, $\widetilde{\mathcal{N}}(\emptyset) = j$). By Mackenzie (2018), we can assume without loss of generality that j has at least two actions available at $\emptyset$ (*i.e.*, $\left|\widetilde{\mathcal{A}}(\emptyset)\right| \geq 2$); that is,

$$\sigma_j^z(\emptyset) \neq \sigma_j^{z'}(\emptyset) \tag{12}$$

for $z, z' \in \{x, x+1, x+2\}$. We claim that $j \in I_x^2$.

Claim 2 $\quad j \in I_x^2$.

Proof of Claim 2 $\quad$ Assume otherwise. Then, there exists $S \in \mathcal{C}_x^m$ such that $|S| \geq 2$ and $j \notin S$. By outcome monotonicity of the coalition system, $S \in \mathcal{C}_{x+1}$, and so $j \notin I_{x+1}^1$. By (12), there exists $y \in \{x, x+1\}$ such that $\sigma_j^y(\emptyset) \neq \sigma_j^{x+2}(\emptyset)$.

For each $i \in S$ and history h in $\widetilde{\Gamma}$ such that $\widetilde{\mathcal{N}}(h) = i$, define

$$\widetilde{\sigma}_i(h) = \begin{cases} \sigma_i^{x+1}(h) & \text{if } \sigma_j^y(\emptyset) \preceq h \\ \sigma_i^x(h) & \text{if } \sigma_j^{x+2}(\emptyset) \preceq h. \end{cases}$$

Since $\widetilde{\Gamma}$ induces $\widetilde{f}$,

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+1}, \widetilde{\sigma}_S, \sigma_j^{x+2})) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+1}, \sigma_S^x, \sigma_j^{x+2})) = \widetilde{f}(P_{-S-\{j\}}^{x+1}, P_S^x, P_j^{x+2}) = x$$

and

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+1}, \widetilde{\sigma}_S, \sigma_j^y)) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-S-\{j\}}^{x+1}, \sigma_S^{x+1}, \sigma_j^y)) = \widetilde{f}(P_{-S-\{j\}}^{x+1}, P_S^{x+1}, P_j^y) = x+1.$$

By single-peakedness, $(x+1)\, P_j^{x+2} x$ holds, which implies that σ_j^{x+2} is not weakly dominant. A contradiction. $\square$

To proceed with the proof for this Subcase 2.2.2, assume that condition (b) of the (InIn) property does not hold at x. Since by Claim 1, $j \in I_x^2$,

$$I_{x+1}^1 \cup \{j\} \notin \mathcal{C}_{x+1}^m. \tag{13}$$

We distinguish between two cases, depending on whether or not j belongs to I_{x+1}^1.

(2.i) $j \in I_{x+1}^1$. By (13), $I_{x+1}^1 \notin \mathcal{C}_{x+1}^m$. By (12), $\sigma_j^x(\emptyset) \neq \sigma_j^{x+1}(\emptyset)$.

For each $i \notin I_{x+1}^1$ and history h in $\widetilde{\Gamma}$ such that $\mathcal{N}(h) = i$, define

$$\widetilde{\sigma}_i(h) = \begin{cases} \sigma_i^{x+2}(h) & \text{if } \sigma_j^x(\emptyset) \preceq h \\ \sigma_i^{x+1}(h) & \text{if } \sigma_j^{x+1}(\emptyset) \preceq h. \end{cases}$$

Since $\widetilde{\Gamma}$ induces $\widetilde{f}$ and $I_{x+1}^1 \notin \mathcal{C}_{x+1}^m$,

$$\widetilde{o}(h^{\widetilde{\Gamma}}((\widetilde{\sigma}_{-I_{x+1}^1-\{j\}}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^x)) = \widetilde{o}(h^{\widetilde{\Gamma}}((\sigma_{-I_{x+1}^1-\{j\}}^{x+2}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^x)) = \widetilde{f}(P_{-I_{x+1}^1-\{j\}}^{x+2}, P_{I_{x+1}^1}^{x+1}, P_j^x) = x+2$$

and

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\widetilde{\sigma}_{-I_{x+1}^1-\{j\}}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^{x+1})) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-I_{x+1}^1-\{j\}}^{x+1}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^{x+1})) = \widetilde{f}(P_1^{x+1}, \ldots, P_n^{x+1}) = x+1.$$

By single-peakedness, $(x+1)\, P_j^x\, (x+2)$ holds, which implies that σ_j^x is not weakly dominant. A contradiction.

(2.ii) $j \notin I_{x+1}^1$. By (12), there exists $y \in \{x+1, x+2\}$ such that $\sigma_j^x(\emptyset) \neq \sigma_j^y(\emptyset)$.

For every $i \notin I_{x+1}^1 \cup \{j\}$ and history h such that $\widetilde{\mathcal{N}}(h) = i$, define

$$\widetilde{\sigma}_i(h) = \begin{cases} \sigma_i^{x+2}(h) & \text{if } \sigma_j^x(\emptyset) \preceq h \\ \sigma_i^{x+1}(h) & \text{if } \sigma_j^y(\emptyset) \preceq h. \end{cases}$$

Since $\widetilde{\Gamma}$ induces $\widetilde{f}$ and (13) holds,

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\widetilde{\sigma}_{-I_{x+1}^1-\{j\}}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^x)) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-I_{x+1}^1-\{j\}}^{x+2}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^x)) = \widetilde{f}(P_{-I_{x+1}^1-\{j\}}^{x+2}, P_{I_{x+1}^1}^{x+1}, P_j^x) = x+2$$

and

$$\widetilde{o}(h^{\widetilde{\Gamma}}(\widetilde{\sigma}_{-I_{x+1}^1-\{j\}}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^y)) = \widetilde{o}(h^{\widetilde{\Gamma}}(\sigma_{-I_{x+1}^1-\{j\}}^{x+1}, \sigma_{I_{x+1}^1}^{x+1}, \sigma_j^y)) = \widetilde{f}(P_{-I_{x+1}^1-\{j\}}^{x+1}, P_{I_{x+1}^1}^{x+1}, P_j^y) = x+1.$$

By single-peakedness, $(x+1)\, P_j^x\, (x+2)$ holds, which implies that σ_j^x is not weakly dominant.

Thus, condition (b) of the (InIn) property holds at x. $\blacksquare$