Stone-Geary meets CES:

An extended linear expenditure system

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Abstract: We reformulate the Stone-Geary utility function to incorporate non-unitary elasticities of substitution. We show that this extended linear expenditure system eliminates some of the restrictions implied by the Stone-Geary function. In particular, and most significantly, the proportions of expenditure over supernumerary income become price responsive under the extended demand system and their price derivatives behave coherently with the degree of substitution.

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1. Introduction

The Stone-Geary utility function (Geary, 1950; Stone, 1954) is an extension of the Cobb-Douglas function that incorporates minimum (or subsistence) levels of consumption. The Stone-Geary function is a shifted Cobb-Douglas with the shifting defined by the minimum consumptions. The nicest property of the Stone-Geary function is that its demand function yields a linear expenditure system (LES; Deaton and Muellbauer, 1980, chapter 3). For any of the goods, total expenditure includes the expenditure for its minimum consumption plus a fixed proportion of the supernumerary income. The Stone-Geary utility function is commonly used in the field of numerical general equilibrium to overcome the limitations of the easy-to-apply but more restrictive Cobb-Douglas functions, in particular its homotheticity. Even so, the Stone-Geary function still retains limitations that are a bit at odds with the empirical evidence. One is the unitary elasticity of substitution implicit in the base Cobb-Douglas function; another is that the proportion of supernumerary income in the demand function is fixed and not sensitive to price changes. We show in this note that these limitations are removed if we use a constant elasticity of substitution (CES; Arrow et al, 1961) shifted function.

In Section 2, we recall the essential properties of the LES system. In Section 3 we present the properties of the CES shifted expenditure system and compare it with the properties of standard LES. Section 4 presents an example of a calibrated demand function under the CES shifted property. Section 5 concludes.

2. The Cobb-Douglas shifted Linear Expenditure System

For the n-good case the Stone-Geary utility function is defined by:

$$u(x_1, x_2, \dots, x_n) = \prod_{j=1}^{n} (x_j - z_j)^{\alpha_j} \tag{1}$$

where $z_j \geq 0$ are minimum levels of consumption and α_j nonnegative weights that add up to 1. The utility maximization problem under the budget constraint imposed by income level m and prices p_j and given minimum consumptions z_j is:

$$\begin{aligned} & Max\ u(x_1, x_2, \dots, x_n) = \prod_{j=1}^{n} (x_j - z_j)^{\alpha_j} \\ & st. \sum_{j=1}^{n} p_j \cdot x_j = m \end{aligned} \tag{2}$$

The solution of this problem is straightforward, and yields demand functions that satisfy linearity in expenditure:

$$p_j \cdot x_j = p_j \cdot z_j + \alpha_j \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i) \tag{3}$$

The expenditure on good j includes a fixed part –given by value at the current price of the minimal consumption of j– and a variable part that is a fixed proportion α_j of the supernumerary (i.e., leftover) income.

3. Extension to CES utility functions

The CES utility function takes the form:

$$u(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} a_j \cdot x_j^{\theta} \right)^{1/\theta} \tag{4}$$

In this expression $\theta = (\sigma - 1) / \sigma$ and $\sigma = 1 / (1 - \theta)$ where $0 \le \sigma < \infty$ is the elasticity of substitution, which takes non-negative values only, and thus $-\infty < \theta \le 1$. We now shift the CES function using the minimum consumption levels $z_j \ge 0$:

$$u(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} a_j \cdot (x_j - z_j)^{\theta} \right)^{1/\theta} \tag{5}$$

The shifted CES function inherits most of the properties of the standard CES function (monotonicity, convexity, differentiability but not homotheticity). The solution of the utility maximization problem:

$$\begin{aligned} & u(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} a_j \cdot x_j^{\theta} \right)^{1/\theta} \\ & st. \sum_{j=1}^{n} p_j \cdot x_j = m \ \text{ and } \ x_j \ge z_j \end{aligned} \tag{6}$$

involves combining the standard procedure[1] under a simplifying change of variable. We obtain the demand functions for the CES shifted utility:

$$x_j = z_j + \frac{a_j^{\sigma} \cdot p_j^{-\sigma} \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i)}{\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma}} \tag{7}$$

The expenditure system becomes now:

$$p_j \cdot x_j = p_j \cdot z_j + s_j(p) \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i) \tag{8}$$

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[1] See Varian (1992), Chapter 7, and Jehle and Reny (2011), Chapter 1.

with:

$$s_j(p) = \frac{a_j^{\sigma} \cdot p_j^{1-\sigma}}{\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma}} \tag{9}$$

The following properties hold:

Property 1. The terms $s_j(p)$ are non-negative proportions. Their sum over j is clearly 1. But unlike the proportions in the standard LES system (4), these proportions $s_j(p)$ are price and substitution elasticity dependent.

Property 2. When σ=1, which represents the LES Cobb-Douglas case, the proportions become constant. Indeed:

$$s_j(p) = \frac{a_j^{\sigma} \cdot p_j^{1-\sigma}}{\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma}} = \frac{a_j}{\sum_{i=1}^{n} a_i} = \alpha_j$$

Property 3. The own derivatives have opposite signs depending on the substitution elasticity:

$$\frac{\partial s_j(p)}{\partial p_j} > 0 \text{ for } 0 < \sigma < 1 \text{ and } \frac{\partial s_j(p)}{\partial p_j} < 0 \text{ for } \sigma > 1$$

Take the derivative from expression (9) to obtain:

$$\frac{\partial s_j(p)}{\partial p_j} = \frac{(1-\sigma) \cdot a_j^{\sigma} \cdot \sum_{i \neq j} a_i^{\sigma} \cdot p_i^{1-\sigma}}{\left(\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma} \right)^2}$$

The sign of the derivative depends only on whether σ<1 or σ>1. For complementary goods (0<σ<1) any increase in the price of good j will require a larger fraction of the leftover income to be devoted to the good getting more expensive. The reason is that for complements consumptions tend to move in the same direction and the good getting relatively more expensive will be the most affected in terms of expenditure. The consumer needs to devote a greater proportion of the leftover income to purchase the good in question. We can see this more clearly in the limit case of perfect complements. In this extreme case, consumption proportions are constant and, even if the allotted income falls, the share of leftover income needed for the good whose price increase becomes larger. The opposite occurs when goods are substitutes (σ>1).

Property 4. The cross derivatives have opposite signs depending on the substitution elasticity:

$$\frac{\partial s_j(p)}{\partial p_i} < 0 \text{ for } 0 < \sigma < 1 \text{ and } \frac{\partial s_j(p)}{\partial p_i} > 0 \text{ for } \sigma > 1$$

The same intuition as in Property 3 helps explaining why is so. If good i becomes more expensive, and goods are complements, demand for j will fall too but the share $s_j(p)$ becomes smaller for good j since it is getting relatively cheaper than good i.

Property 5. Similar to the standard LES, the CES utility function with minimal consumptions is no longer homothetic. We calculate the marginal rate of substitution $MRS_{i,j}$ for the utility function in (5) and along a ray $x_j = \beta \cdot x_i$ with $\beta>0$ we obtain:

$$MRS_{i,j} = \frac{a_i \cdot (x_i - z_i)^{\theta-1}}{a_j \cdot (\beta \cdot x_i - z_j)^{\theta-1}}$$

whose value depends on the value of x_i and thus the marginal rate of substitution is not constant. One of the criticisms of the use (or abuse) of homothetic utilities in applied work is that the real world does not seem to be homothetic. The CES utility function displaced with minimal consumption levels does not suffer from this problem.

4. A calibrated demand function.

When econometric estimates are not available, a circumstance we often face, an alternative to implement demand systems in applied general equilibrium models (Shoven and Whalley, 1984) is to use the simpler method of calibration. This method uses all available information to derive demand functions that are consistent with the data and the restrictions on parameters implied by utility maximization (Dawkins et al, 2001). The goal of calibration is to have an empirical specification of expression (5) that has the property that when used to maximize utility, the solution endogenously yields the observed empirical data registered in some database.

A look at (5) shows that we need numerical values for the n coefficients a_j, the elasticity of substitution σ (which gives us θ) and the n minimum consumption levels z_j. In total, $2n+1$ parameters. If the value of σ is known, or assumed, this leaves $2n$ parameters to be determined, or "calibrated". If we use I-O data, however, only n consumption observations are available.

We need some calibration tricks. The first one is to assume that all prices reflected in the database are unitary $p_j=1$. This entails a redefinition of the units in such a way that one physical unit has the worth of one currency unit. With this redefinition, all observed data in the database are now both value as well as physical units. This allows us to simplify expression (9) to:

$$s_j(1) = \frac{a_j^\sigma}{\sum_{i=1}^{n} a_i^\sigma} \qquad j = 1,2,..,n \tag{10}$$

The second trick involves, when they are available, the use of income elasticities of demand η_j. From (7) we can easily check that

$$s_j(1) = \alpha_j(1) \cdot \eta_j \tag{11}$$

where $\alpha_j(1)$ is the share of expenditure on good j over total income at the initial unitary prices. Substituting (11) into (10) we obtain a system of linear equations that can be solved conditional on the value of σ. The system, however, has a redundant equation since if a_j is a solution so is $\mu \cdot a_j$ for any scalar μ. We can therefore add an extra equation to make the sum of all the a_j coefficients to be 1.

The final trick needs to deal with the determination of the subsistence levels z_j. This step requires the use of the Frisch parameter ϕ (Frisch, 1959) which measures the flexibility in the distribution of income between the fixed and the variable parts of consumption. At the initial unitary prices this gives:

$$-\phi = \frac{m}{m - \sum_{i=1}^{n} 1 \cdot z_j} \tag{12}$$

Substituting into expression (8) we can solve for the subsistence levels:

$$z_j = x_j - s_j(1) \cdot \frac{m}{-\phi} \tag{13}$$

As an illustration of the calibration procedure, we borrow income elasticities calculated for the two-digit 12 ECOICOP sectors for Spain for the Spanish economy from Garcia-Villar (2018). The fact that these estimated income elasticities are not unitary challenges the typical use of homothetic utility functions in numerical general equilibrium since, most commonly, the selection of preferences give rise to unitary income elasticity[2]. This justifies departing from homothetic functions and endorses the use of LES or the here proposed CES extended Stone-Geary utility functions. We use the reported value of the Frisch parameter from Deaton and Muellbauer (1980) along with two sensible small deviations around it. The central elasticity of substitution value corresponds to the widely used Cobb-Douglas case but, again, we introduce deviations from this unitary elasticity value to appraise the sensitivity of the results. Finally, we use expenditure data for the same classification of goods taken from the ECOICOP data published by the National

[2] The econometrics literature provides ample evidence for non-unitary income elasticities. See Lecocq & Robin (2006), Christensen (2014) and García-Enriquez & Echevarría (2016).

Institute of Statistics for 2017[3]. Table 1 shows the calibration of the utility function coefficients whereas Table 2 shows the calibrated minimum levels of consumption.

Table 1: Utility coefficients for alternative substitution elasticity values and $\phi = -2$

Goods	**Coefficients**	$\sigma = 0.75$	$\sigma = 1$	$\sigma = 1.25$
1.-Food and non-alcoholic beverages	a_1	0.087	0.093	0.095
2.-Alcoholic beverages and tobacco	a_2	0.006	0.013	0.019
3.-Clothing	a_3	0.058	0.069	0.075
4.-Housing	a_4	0.165	0.151	0.140
5.-Household articles	a_5	0.051	0.062	0.069
6.-Health	a_6	0.032	0.044	0.052
7.-Transportation	a_7	0.259	0.211	0.183
8.-Communication services	a_8	0.009	0.016	0.024
9.-Recreational services	a_9	0.070	0.079	0.084
10.-Education	a_{10}	0.013	0.022	0.030
11.-Hotels and restaurants	a_{11}	0.175	0.157	0.145
12.-Other services	a_{12}	0.074	0.083	0.086

Source: Our computations

Table 2: Minimum consumptions for alternative values of the Frisch parameter ϕ

Goods	**Expenditure data 2017**	**Minima**	ϕ = -1.75	ϕ = -2	ϕ = -2.25
1.-Food and non-alcoholic beverages	76.042	z_1	47.268	50.864	53.662
2.-Alcoholic beverages and tobacco	9.927	z_2	6.054	6.538	6.914
3.-Clothing	28.043	z_3	6.781	9.439	11.506
4.-Housing	162.431	z_4	115.939	121.750	126.270
5.-Household articles	24.762	z_5	5.542	7.944	9.813
6.-Health	18.149	z_6	4.502	6.208	7.535
7.-Transportation	67.890	z_7	2.819	10.953	17.279
8.-Communication services	17.209	z_8	12.188	12.816	13.304
9.-Recreational services	30.770	z_9	6.289	9.349	11.729
10.-Education	7.668	z_{10}	0.855	1.707	2.369
11.-Hotels and restaurants	55.588	z_{11}	7.008	13.081	17.804
12.-Other services	41.864	z_{12}	16.332	19.524	22.006

Source: National Institute of Statistics, García-Villar (2018) for the income elasticities, and our computations. Expenditure data in millions of current Euros.

[3] https://www.ine.es/jaxiT3/Tabla.htm?t=24765&L=0

5. Concluding remarks

The Stone-Geary linear expenditure system correctly captures some rigidity properties of consumption demand, namely, the likely existence of minimum levels of consumption for some goods. Under standard LES, excess consumption over these minimal levels is apportioned using fixed share coefficients. When we contemplate wider substitution possibilities, as is the case with CES displaced utility functions, the shares of excess consumption become price and elasticity of substitution sensitive, capturing a more realistically empirical property. The CES displaced utility function gives rise to price responsive shares, hence improving the reaction capacities of consumers when prices change. This may enrich the modeling of numerical general equilibrium providing more reliable, and more real-world grounded, welfare assessment of policies. Additionally, the LES or the proposed CES extended function, being non-homothetic, both reflect the empirical nature of income elasticities, since plenty of econometrics evidence suggest their values are not unitary.

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