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Identifying strong voter support: Condorcet and Smith revisited*

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Abstract. The conditions of strong Condorcet winner consistency and strong Condorcet loser consistency are, in essence, universally accepted as attractive criteria to evaluate the performance of social choice functions. However, there are many situations in which these conditions are silent because such winners and losers may not exist. Hence, weakening these desiderata in order to extend the domain of profiles where they apply is an appealing task. Yet, the often-proposed and accepted weak counterparts of these properties suffer from the shortcoming that a weak Condorcet winner can be a weak Condorcet loser at the same time. We propose new notions of Condorcet-type winners and losers that are between these two extremes: they share the intuitive appeal of strong Condorcet winner consistency and strong Condorcet loser consistency and avoid the contradictory recommendations that would derive from the double identification of candidates as being weak Condorcet winners and losers at the same time. We provide a thorough examination of the extent to which some important reference properties are satisfied by social choice functions that are consistent with our new proposals. In addition, we revisit the concept of Smith sets (Smith, 1973) and examine a possible modification. As is the case for our intermediate Condorcet winners and losers, these notions are intended to generalize Condorcet's ideas. Using our reference properties again, we discuss the social choice functions that are consistent with the selection of candidates from these sets. By contrasting the consequences of using the suggestions inspired by Smith with those that are implied by the intermediate Condorcet consistency conditions that we propose, we hope to shed new light on the possibilities of extending Condorcet's principles to a larger set of circumstances. *Journal of Economic Literature* Classification Nos.: D71, D72, D63.

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1 Introduction

The classical contribution of Condorcet (1785) continues to play a major role in the analysis of collective decision procedures. Along with Condorcet's seminal work, numerous pioneering advancements in the theory of voting appeared around the time of the French Revolution; these include the essays of Borda (1781), Morales (1797), and Daunou (1803), to name some of the most prominent examples. It is not very surprising that, reflecting the turbulent events that transpired during that time period, the thoughts on collective choice expressed by these scholars were mostly phrased in terms of the election of candidates for public office by the citizenry. We follow this convention here, keeping in mind that our observations as well as those of the above-mentioned classics are applicable to a broader variety of collective decision problems.

Generally speaking, an election proceeds by selecting one or several of the feasible candidates on the basis of the voters' relative assessments of these candidates. This process can be formalized by means of a voting rule, also referred to as a social choice function. For a specific election, the set of feasible candidates is a subset of a universal set of candidates that could, in principle, be available. Each voter is assumed to compare the candidates using a goodness ordering: for any two candidates in the universal set, a voter can declare one of them to be better than the other, or state that (s)he considers the two equally good. A social choice function specifies, for every profile of the voters' goodness orderings and for every feasible set of candidates in its domain, the non-empty subset of elected candidates if the voters' assessments are represented by the goodness orderings that make up the profile.

Much of Condorcet's work is concerned with the formulation of conditions that, loosely speaking, guarantee the election of candidates who are strongly supported by the electorate and prevent the election of candidates who are firmly rejected by the voters. Perhaps the most prominent notions of such candidates are those of a strong Condorcet winner and a strong Condorcet loser. A strong Condorcet winner is a candidate who wins against every other feasible candidate in a pairwise comparison and, analogously, a strong Condorcet loser loses against every other feasible candidate in a pairwise contest. To clarify our use of the terms in question, we say that a candidate wins (loses) against another candidate if the number of voters who consider the former better than the latter is higher (lower) than the number of voters who consider the latter better than the former; if these two numbers are the same, the two candidates tie with each other in a pairwise comparison. If a strong Condorcet winner (loser) exists, this candidate is unique; this follows immediately from the observation that at most one candidate can accumulate more (fewer) votes than every feasible opponent in a pairwise contest. The underlying motivation is that a strong Condorcet winner has very solid support from the voters and should be uniquely selected and, likewise, a strong Condorcet loser is firmly rejected by the voters and should never be chosen. The two conditions just described are what we refer to as strong Condorcet winner consistency and strong Condorcet loser consistency. Unfortunately, while highly appealing, these requirements are quite demanding and, in many situations, strong Condorcet winners or strong Condorcet losers do not exist.

An alternative version of Condorcet winners (losers) defines these candidates by means of the more modest requirement that a candidate must win (lose) against or tie with each of the other candidates. This definition of weak Condorcet winners and losers is unsatisfactory, however, because it is possible for a candidate to simultaneously be a weak Condorcet winner and a weak Condorcet loser. It is even possible that every candidate is a weak Condorcet winner and a weak Condorcet loser at the same time. We emphasize that this phenomenon is not restricted to degenerate cases in which every voter assesses the feasible candidates by means of the universal equal-goodness relation, and we illustrate this observation by presenting some concrete examples later on.

We introduce two new notions of Condorcet winners that are less demanding than the strong one, while harder to meet than the potentially inconsistent weak version we just described. Candidates satisfying the first notion are called intermediate Condorcet winners, whereas those in the second category are referred to as maximal intermediate Condorcet winners. By definition, maximal intermediate Condorcet winners are contained in the set of intermediate Condorcet winners and, in view of this subset relationship, we concentrate our discussion on the notion of intermediate Condorcet winners and its dual counterpart, that of intermediate Condorcet losers, in the remainder of this introduction. A detailed motivation and analysis of maximal intermediateness is postponed until Section 4.

Our first proposal defines intermediate Condorcet winners (losers) as those candidates who receive at least (at most) as many votes as their opponent in every pairwise comparison with at least one instance of a strict inequality. That is, unlike a weak Condorcet winner (loser), an intermediate Condorcet winner (loser) has to win (lose) against at least one other candidate. As a consequence, intermediate Condorcet winners and losers do not lead to the type of contradictory recommendations that weak Condorcet winners and losers are afflicted with. It is straightforward to verify that an equivalent definition of an intermediate Condorcet winner (loser) is that this candidate is a weak Condorcet winner (loser) but not a weak Condorcet loser (winner), an equivalence that further underlines the intuitive appeal of the definition.

It may be tempting to see our proposal as no more than the alternative use of weak or strict inequalities in otherwise similar definitions. But such a seemingly minor modification can have substantial ramifications. As a prominent example, consider Suzumura's (1976) property of consistency, which strengthens that of acyclicity. The latter condition merely prevents cycles in which all links represent betterness, whereas Suzumura's also rules out all at-least-as-good-as cycles that involve at least one instance of betterness. This change has profound consequences. As shown by Suzumura (1976), consistency is necessary and sufficient for the existence of an ordering extension (Szpilrajn, 1930). Moreover, unlike acyclicity, the property of consistency has a well-defined closure operation; see Bossert, Sprumont, and Suzumura (2005). Another setting in which the choice of a weak versus a strict inequality can make a major difference appears in Donaldson and Weymark (1986). They show that, in the context of poverty measurement, it frequently matters whether income recipients at the poverty line are considered poor or non-poor. As they illustrate, some properties that are accommodated by one version lead to impossibilities if the alternative definition is employed. A similar observation applies to the von Neumann and Morgenstern abstract stable set that turns out to be remarkably sensitive to the choice of dominance property employed; see Greenberg (1992). The basic question in the context of stable sets is whether everyone in a coalition has to benefit from a deviation in order for this coalition to object to a proposed outcome, or only one member of the coalition needs to be made better off as long as no other member ends up worse off. Clearly, there is a strong parallel between this question and the issue addressed in our contribution.

Inspired by the observation that strong Condorcet winners or strong Condorcet losers do not exist under many circumstances, there have been earlier attempts to modify the corresponding consistency conditions. As is the case for our proposal, these approaches represent an effort to extend the possibilities of identifying candidates that enjoy overwhelming support from the voters while preserving the spirit of Condorcet's consistency conditions; see Fishburn (1973, 1977) for detailed discussions. To the best of our knowledge, there are two major proposals of this nature, developed by Smith (1973) and by Demange (1983). Smith (1973) proposes a consistency condition that extends the original property of strong Condorcet winner consistency to sets of candidates. Instead of a single candidate who wins against every other candidate, Smith's (1973) condition allows for a (not necessarily single-valued) subset of candidates each of which wins against every candidate in the complement of this subset. Demange's (1983) definition of a strong (weak) quasi-Condorcet winner is, in general, more inclusive than that of a strong (weak) Condorcet winner. If the voters' goodness relations are assumed to be antisymmetric, strong (weak) quasi-Condorcet winners and strong (weak) Condorcet winners are equivalent; without the antisymmetry assumption, the quasi-Condorcet versions are more accommodating than their respective original counterparts. Because Smith's proposal is closely aligned with our purposes, we provide a thorough comparative examination of it. Demange's notion of quasi-Condorcet winners appears to be of less direct relevance. In our context, there is the additional concern that the application of her proposal may recommend the choice of additional candidates even if a strong Condorcet winner exists—and these additional candidates may include a strong Condorcet loser. As a consequence, we do not discuss quasi-Condorcet winners and losers in detail. We stress, however, that this takes nothing away from their important role in other settings, such as the spatial models analyzed by Demange (1983).

An important aspect of intermediate Condorcet winners and losers is that they generalize the notion of strong Condorcet winners and losers in the sense that the former may exist when the latter do not. One of our results addresses the existence issue directly by proving that the set of intermediate Condorcet winners (losers) can only be empty if the profile under consideration is either such that all pairwise comparisons result in a tie, or the profile is cyclical in the sense that there is a chain of candidates such that the first element in the chain wins against the second element, the second wins against the third and so on, and the last element wins against the first, thus completing the cycle.

For the most part, we work with a general domain of possible profiles of orderings. However, there are numerous results in the theory of social choice that can be obtained on specific restricted domains, perhaps most notably those that apply to single-peaked or single-plateaued profiles; see Black (1948) and Sen (1970, Chapter 10*), for example. These types of profiles are employed in various settings. They are particularly relevant in political environments that allow the candidates to be ranked according to a one-dimensional criterion, such as one candidate being located more to the left in the political spectrum than another. In addition, single-peaked profiles appear in spatial choice models. We emphasize that our observations are robust in the sense that they also apply under domain restrictions of this nature. This is evidenced by our choice of examples; whenever possible, we employ profiles that are single-peaked or single-plateaued. The basic reason why single-peaked profiles cannot be found for the remaining examples is that they involve empty sets of intermediate Condorcet winners. As pointed out in the previous paragraph, this is only possible if either the profile under consideration is such that each candidate is in a tie with every other candidate, or the profile is cyclical in a feasible set. Because such cycles cannot occur if a profile is single-peaked, it follows immediately that single-peakedness cannot be satisfied in these instances.

We begin in Section 2 with a formal definition of social choice functions, which are mappings that assign a non-empty set of selected candidates to each profile of the voters' goodness orderings and each feasible set of candidates. In addition, we introduce three reference properties that we think are well aligned with the spirit of Condorcet's contribution. To conclude the section, we discuss the notion of strong voter support and review Condorcet's methods of defining this type of support. Our first proposal of intermediate Condorcet winners constitutes the main subject matter of Section 3, along with the consistency conditions they induce. As announced, Section 4 suggests a criterion that focuses on a subset of the set of all intermediate Condorcet winners, the ones we call maximal intermediate Condorcet winners. The above-mentioned reference properties of a social choice function play a vital role in the analysis of our intermediate notions of consistency. In Section 5, we review a related earlier proposal to generalize the classical Condorcet consistency conditions, introduced by Smith (1973). We establish some links between our suggestion and that of Smith. In addition, we employ the three reference properties of a social choice function again to assess the merits of Smith's suggestion. Section 6 examines an intermediate variant of Smith's original definition and its ramifications. Section 7 contains an explicit method of constructing a large class of social choice functions that satisfy intermediate Condorcet winner consistency and intermediate Condorcet loser consistency. Section 8 concludes.

2 Social choice functions and strong voter support

There is a finite set of candidates X with cardinality $|X| \geq 2$, and we use $\mathcal{X}$ to denote the set of all non-empty subsets of X. The finite set of voters is $N = \{1, \ldots, n\}$ with $n \geq 2$. The set of all orderings on X (that is, the set of all reflexive, complete, and transitive binary relations on X) is $\mathcal{R}$. We assume that each voter $i \in N$ has an ordering $R_i \in \mathcal{R}$, interpreted as a relation that expresses the relative goodness of the candidates. The asymmetric and symmetric parts of R_i are denoted by P_i and I_i. Because the voters' goodness relations are orderings, there is no risk of ambiguity in employing simplified notation; for instance, we use xP_iyI_iz to indicate that, according to voter $i \in N$, candidate x is better than candidates y and z, and y and z are equally good.

A profile is an n-tuple $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{R}^n$. For a profile $\mathbf{R} \in \mathcal{R}^n$ and two distinct candidates x and y in X, we use

$$p(\mathbf{R}; x, y) = |\{i \in N : xP_iy\}|$$

to indicate the number of voters who consider x better than y in a profile $\mathbf{R}$.

Let $\mathcal{D} \subseteq \mathcal{R}^n \times \mathcal{X}$ be a non-empty domain of pairs of profiles and feasible sets. A social choice function is a mapping $F \colon \mathcal{D} \to \mathcal{X}$ such that $F(\mathbf{R}, S) \subseteq S$ for all $(\mathbf{R}, S) \in \mathcal{D}$.

The function F assigns a set of selected candidates to each pair of a profile of the voters' goodness orderings and a non-empty feasible set of candidates within the domain of F.

We employ three reference properties of a social choice function to assess the relative performance of the alternative notions of strong voter support considered in this paper. In our opinion, these properties are well-suited for this purpose because they capture the spirit of the consistency conditions motivated by Condorcet's proposals.

The first of these is the fairly uncontroversial requirement that dominated candidates are not to be selected by a social choice function. That is, if there are two candidates z and w such that all voters consider z better than w, then w cannot be chosen.

Exclusion of dominated candidates. For all $(\mathbf{R}, S) \in \mathcal{D}$ and for all $w \in S$, if there exists $z \in S$ such that zP_iw for all $i \in N$, then $w \notin F(\mathbf{R}, S)$.

A commonly-employed requirement is that the presence of candidates who are considered uniquely worst by all voters has no influence on the set of chosen candidates. We use this condition as our second reference property.

Independence of unanimously worst candidates. For all $(\mathbf{R}, S) \in \mathcal{D}$ and for all $w \in X \setminus S$, if $(\mathbf{R}, S \cup \{w\}) \in \mathcal{D}$ and zP_iw for all $i \in N$ and for all $z \in S$, then

$$F(\mathbf{R}, S \cup \{w\}) = F(\mathbf{R}, S).$$

Our third reference property is a suitable version of pairwise justifiability, an axiom introduced and discussed by Barberà, Berga, Moreno, and Nicolò (2022). Consider two pairs $(\mathbf{R}, S), (\mathbf{R}', S) \in \mathcal{D}$ and suppose that $x \in F(\mathbf{R}, S)$. Suppose now that a change from $\mathbf{R}$ to $\mathbf{R}'$ has the effect of removing x from the choice set for the feasible set S—that is, $x \notin F(\mathbf{R}', S)$. Pairwise justifiability requires that the demotion of x must be caused by this candidate having lost ground in the move from $\mathbf{R}$ to $\mathbf{R}'$. More precisely, there must be a voter $j \in N$ and a candidate z in $S \setminus \{x\}$ such that x is at least as good as z according to R_j but not according to R'_j, or x is better than z according to R_j but not according to R'_j.

Pairwise justifiability. For all $(\mathbf{R}, S), (\mathbf{R}', S) \in \mathcal{D}$ and for all $x \in F(\mathbf{R}, S)$, if $x \notin F(\mathbf{R}', S)$, then there exist $j \in N$ and $z \in S \setminus \{x\}$ such that

$$[xR_jz \text{ and } \neg(xR'_jz)] \text{ or } [xP_jz \text{ and } \neg(xP'_jz)].$$

The conjunction of independence of unanimously worst candidates and pairwise justifiability implies exclusion of dominated candidates. This implication is true on all domains that are relevant for our purposes, and we prove it after the requisite definitions have been introduced; see Theorem 9 in Section 6.

There are two additional requirements that are linked to the reference properties of exclusion of dominated candidates and independence of unanimously worst candidates. Independence of dominated candidates (Ching, 1996) rules out not only the choice of dominated candidates but also their influence on the selected set of candidates, thereby strengthening both exclusion of dominated candidates and independence of unanimously worst candidates. Because none of the definitions of strong voter support that we examine here satisfies this requirement, we do not employ it. Analogously, we do not use the condition of exclusion of unanimously worst candidates, which is weaker than exclusion of dominated candidates and independence of unanimously worst candidates. All of the notions of strong voter support that we analyze comply with it and, therefore, the requirement is not of much use in discriminating between them.

As alluded to in the introduction, Condorcet was concerned with the formulation of conditions that guarantee the election of candidates who enjoy strong support on the part of the voters. The underlying motivation of this requirement can be captured by identifying, for each pair $(\mathbf{R}, S)$ in the domain of a social choice function, the set of candidates that are deemed to have sufficient support for them to be chosen, to the exclusion of all others. The consistency requirement that applies to these privileged candidates demands that, whenever this set is non-empty, the candidates chosen by the social choice function for $\mathbf{R}$ and S must be contained in this set of those with strong voter support. A dual definition identifies the set of candidates that are firmly rejected by the voters for a pair $(\mathbf{R}, S)$. The corresponding consistency property requires that these strongly rejected candidates not be selected by a social choice function.

The criteria proposed by Condorcet are based on the method of majority decision: if a candidate beats every other candidate in a pairwise majority contest, this candidate—and only this candidate—is to be elected. If such a candidate exists, it is referred to as a strong Condorcet winner. Because at most one candidate can beat every other candidate, the set of strong Condorcet winners is either empty or a singleton set. The counterpart of this definition declares a candidate to be a strong Condorcet loser if this candidate loses against every other candidate in a pairwise majority comparison and, because such a candidate is firmly rejected by the electorate, it should never be chosen if it exists. As is the case for strong Condorcet winners, the set of strong Condorcet losers is either empty or a singleton set. To introduce these properties formally, let $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$. A candidate $x \in S$ is a strong Condorcet winner for $\mathbf{R}$ in S if

$$p(\mathbf{R}; x, z) > p(\mathbf{R}; z, x) \text{ for all } z \in S \setminus \{x\}, \tag{1}$$

and $x \in S$ is a strong Condorcet loser for $\mathbf{R}$ in S if the inequality in (1) is reversed. For any profile $\mathbf{R} \in \mathcal{R}^n$ and for any feasible set $S \in \mathcal{X}$, there is at most one strong Condorcet winner for $\mathbf{R}$ in S and at most one strong Condorcet loser for $\mathbf{R}$ in S. We denote the set of strong Condorcet winners for $\mathbf{R}$ in S by $SCW(\mathbf{R}, S)$, and $SCL(\mathbf{R}, S)$ is the set of strong Condorcet losers for $\mathbf{R}$ in S.

The notion of a strong Condorcet winner gives rise to the condition of strong Condorcet winner consistency, which states that if there is a strong Condorcet winner, then this candidate—and only this candidate—should be chosen.

Strong Condorcet winner consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$, if $SCW(\mathbf{R}, S) \neq \emptyset$, then

$$F(\mathbf{R}, S) \subseteq SCW(\mathbf{R}, S).$$

Because there can be at most one strong Condorcet winner for a profile $\mathbf{R}$ in a feasible set S, the set inclusion in this definition must be satisfied with an equality.

Analogously, strong Condorcet loser consistency postulates that a strong Condorcet loser should never be chosen.

Strong Condorcet loser consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$,

$$F(\mathbf{R}, S) \cap SCL(\mathbf{R}, S) = \emptyset.$$

There are numerous profiles and feasible sets for which strong Condorcet winners or strong Condorcet losers do not exist. A weaker notion of Condorcet winners and losers is obtained if the strict inequalities in the above definitions are replaced with weak inequalities. A candidate $x \in S$ is a weak Condorcet winner for $\mathbf{R}$ in S if

$$p(\mathbf{R}; x, z) \geq p(\mathbf{R}; z, x) \text{ for all } z \in S \setminus \{x\}, \tag{2}$$

and $x \in S$ is a weak Condorcet loser for $\mathbf{R}$ in S if the inequality in (2) is reversed. The set of weak Condorcet winners for $\mathbf{R}$ in S is denoted by $WCW(\mathbf{R}, S)$, and the set of weak Condorcet losers for $\mathbf{R}$ in S is $WCL(\mathbf{R}, S)$. These definitions lead to the following consistency conditions.

Weak Condorcet winner consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$, if $WCW(\mathbf{R}, S) \neq \emptyset$, then

$$F(\mathbf{R}, S) \subseteq WCW(\mathbf{R}, S).$$

Because there may be multiple weak Condorcet winners for a profile $\mathbf{R}$ in a feasible set S, the set inclusion in this definition need not be satisfied with an equality; it is possible that only some but not all weak Condorcet winners are selected. The condition does require, though, that no candidates other than weak Condorcet winners are chosen if such winners exist.

Weak Condorcet loser consistency precludes the choice of weak Condorcet losers.

Weak Condorcet loser consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$,

$$F(\mathbf{R}, S) \cap WCL(\mathbf{R}, S) = \emptyset.$$

The consistency conditions that are based on weak Condorcet winners and weak Condorcet losers suffer from an unfortunate shortcoming. As the example below illustrates, it is possible for a candidate to be a weak Condorcet winner and a weak Condorcet loser at the same time—and, therefore, it is impossible to satisfy both of the two consistency properties.

Example 1 *The set of feasible candidates is $S = X = \{x, y\}$ and the set of voters is $N = \{1, 2\}$. Define the profile $\mathbf{R}$ by xP_1y and yP_2x. Each candidate wins against or ties with the other candidate with a score of one to one, and loses against or ties with the other candidate with the same score. Thus, both candidates are weak Condorcet winners and weak Condorcet losers at the same time.*

The profile of this example is composed of antisymmetric relations, which implies that it is not degenerate in the sense of displaying universal equal goodness on the part of every voter.

The above example can be generalized to arbitrary numbers of candidates and voters. If the number of voters n is even, assign an arbitrary antisymmetric ordering to $n/2$ voters and its inverse to the remaining $n/2$ voters. It follows that each candidate wins against or ties with every other candidate, and also loses against or ties with every other candidate with a score of $n/2$ to $n/2$. If n is odd, assign an arbitrary antisymmetric ordering to $(n-1)/2$ voters, its inverse to $(n-1)/2$ voters, and the universal equal-goodness relation to the remaining voter. Now each candidate wins against or ties with every other candidate, and also loses against or ties with every other candidate with a score of $(n-1)/2$ to $(n-1)/2$. These generalized variants involve only single-peaked profiles if the number of voters is even, and single-plateaued profiles for an odd number of voters. Thus, if the voters' goodness relations are restricted to be antisymmetric, an example can be constructed for all even numbers of voters but not if the number of voters is odd because, in this case, the inequalities in the definition of weak Condorcet winners and weak Condorcet losers cannot but be strict.

We chose the profile of the above example for its simplicity and easy generalizability. It is possible to find more intricate examples in which not every candidate emerges as a weak Condorcet winner and a weak Condorcet loser at the same time. For instance, the following example displays a profile of six antisymmetric orderings such that only one out of four candidates is a weak Condorcet winner and a weak Condorcet loser at the same time; it is analogous to the example that appears on page 267 in Barberà, Bossert, and Suzumura (2021).

Example 2 *Suppose that the set of feasible candidates is $S = X = \{x, y, z, w\}$ and the set of voters is $N = \{1, 2, 3, 4, 5, 6\}$. Define the profile $\mathbf{R}$ by*

$$
\begin{gathered}
xP_1yP_1zP_1w, \\
xP_2yP_2zP_2w, \\
xP_3zP_3wP_3y, \\
zP_4wP_4yP_4x, \\
wP_5yP_5zP_5x, \\
wP_6yP_6zP_6x.
\end{gathered}
$$

Candidate x is the unique weak Condorcet winner and the unique weak Condorcet loser for $\mathbf{R}$ in S because x ties with every other candidate with a score of three to three. It is straightforward to verify that none of the other candidates can be a weak Condorcet winner or a weak Condorcet loser.

It is immediate that contradictory recommendations of the above-described nature cannot occur if strong Condorcet winners and strong Condorcet losers are considered; by definition, a strong Condorcet winner cannot be a strong Condorcet loser, and *vice versa.*

Fishburn (1977, Section 4) examines alternative Condorcet consistency conditions that he refers to as the inclusive Condorcet principle, the exclusive Condorcet principle, and the strict Condorcet principle; see also Fishburn (1973, Chapter 12) for a thorough treatment. The strict Condorcet principle requires that the set of weak Condorcet winners be chosen whenever this set is non-empty. This condition is equivalent to the conjunction of the inclusive Condorcet principle and the exclusive Condorcet principle, each of which takes care of one of the two weak set inclusions required in the strict Condorcet principle. Because all of these three conditions involve the choice of weak Condorcet winners, the above observation that it is possible for a candidate to be a weak Condorcet winner and a weak Condorcet loser at the same time calls all of them into question.

3 Intermediate Condorcet winners

This section is devoted to our first proposal of generalizing the strong Condorcet consistency conditions, thereby making Condorcet's ideas applicable in a larger set of circumstances. We define what we refer to as intermediate Condorcet winners by requiring weak inequalities with at least one instance of strict inequality in the requisite definition. Thus, letting $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$, a candidate $x \in S$ is an intermediate Condorcet winner for $\mathbf{R}$ in S if

$$p(\mathbf{R}; x, z) \geq p(\mathbf{R}; z, x) \text{ for all } z \in S \setminus \{x\} \text{ with at least one strict inequality,} \tag{3}$$

and $x \in S$ is an intermediate Condorcet loser for $\mathbf{R}$ in S if the inequality in (3) is reversed. We denote the set of intermediate Condorcet winners for $\mathbf{R}$ in S by $ICW(\mathbf{R}, S)$, and $ICL(\mathbf{R}, S)$ is the set of intermediate Condorcet losers for $\mathbf{R}$ in S. As is the case for strong and weak Condorcet winners (losers), it is possible that the set of intermediate Condorcet winners (losers) is empty. An intermediate Condorcet winner (loser) can alternatively be described as a candidate who is a weak Condorcet winner (loser) but not a weak Condorcet loser (winner); the requisite equivalence follows immediately from the above definitions.

By definition, every strong Condorcet winner (loser) is an intermediate Condorcet winner (loser) but, as illustrated in the following example, the converse set inclusion is not valid in general.

Example 3 *Suppose that the set of feasible candidates is $S = X = \{x, y, z, w\}$ and the set of voters is $N = \{1, 2\}$. Define the profile $\mathbf{R}$ by $xP_1zP_1yP_1w$ and $yP_2xP_2zP_2w$. Candidate x ties with candidate y with a score of one to one, and x wins against z and against w with a score of two to zero each. Candidate y ties with candidates x and z with a score of one to one each, and y wins against w with a score of two to zero. Neither z nor w can be an intermediate Condorcet winner for $\mathbf{R}$ in S because each of these candidates loses against at least one other candidate in S. Thus, the set of intermediate Condorcet winners for $\mathbf{R}$ in S consists of candidates x and y. There is no strong Condorcet winner because no candidate wins against every other candidate. The profile of this example is single-peaked, as illustrated in Figure 1.*

Likewise, an intermediate Condorcet winner (loser) is a weak Condorcet winner (loser) by definition but the reverse set inclusion is not valid in general, as established by Example 2. In the example, candidate x is a weak Condorcet winner and a weak Condorcet loser but the set of intermediate Condorcet winners and the set of intermediate Condorcet losers are empty.

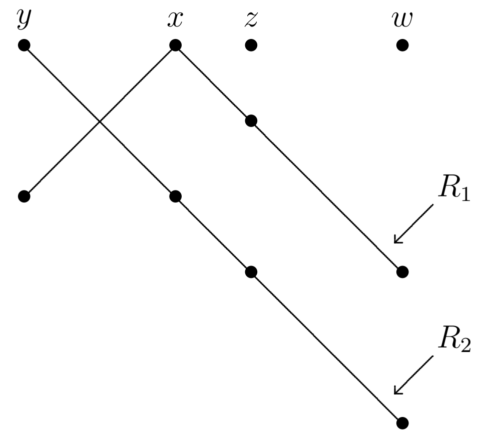

Figure 1: The profile $\mathbf{R}$ of Example 3.

As is the case for strong Condorcet winners and losers, intermediate Condorcet winners and losers do not suffer from the shortcoming associated with weak Condorcet winners and losers—it is not possible for a candidate to be an intermediate Condorcet winner and an intermediate Condorcet loser at the same time. Again, this observation follows immediately from the definition of intermediate Condorcet winners and intermediate Condorcet losers: an intermediate Condorcet winner (loser) is a candidate who is a weak Condorcet winner but not a weak Condorcet loser (winner) and, therefore, the requisite sets cannot but be disjoint.

In this context, we stress that a marked difference between weak Condorcet winners and intermediate Condorcet winners can be present even under comparatively rich and well-established restrictions on the domain of possible profiles. An interesting example consists of a domain employed by Barberà and Ehlers (2011) in the context of characterizing circumstances under which the majority rule always generates quasi-transitive aggregate goodness relations.

The idea underlying the domain assumption of Barberà and Ehlers (2011) is that there are objective instances of equal goodness that may, for instance, have their origins in a voter's inability to meaningfully distinguish some of the candidates. This is modeled by using, for each voter $i \in N$, a partition $\mathcal{C}_i$ of the universal set of candidates X. The interpretation is that, for each element of the partition $C \in \mathcal{C}_i$, voter i cannot distinguish between the candidates in C. More precisely, if $\{x, y\} \subseteq C$ for some $x, y \in X$ and some $C \in \mathcal{C}_i$, it must be the case that xI_iy; that is, candidates x and y must be (objectively) equally good according to voter i.

The requisite domain-restriction assumptions can now be expressed as properties of the partitions $\mathcal{C}_i$ of the voters. A special case consists of requiring admissible partition profiles to satisfy the following property of $(n-1)$ dichotomous goodness; see Barberà and Ehlers (2011, p. 563).

$(n-1)$ **dichotomous goodness.** For all pairwise distinct $x, y, z \in X$, there exists $M \subseteq N$ such that $|M| \geq n-1$ and, for all $i \in M$, there exists $C \in \mathcal{C}_i$ such that $|C \cap \{x, y, z\}| \geq 2$.

In words, the condition requires that, for each triple of candidates, at least $n-1$ voters cannot distinguish at least two candidates. Following Barberà and Ehlers (2011), we assume that there are only instances of objective equal goodness, that is, any two candidates $x, y \in X$ such that x and y do not belong to the same constituent set of the partition $\mathcal{C}_i$ are such that we have xP_iy or yP_ix.

The following example illustrates that the divergence of weak and intermediate Condorcet winners is not restricted to degenerate cases but can be accommodated in settings with well-established and plausible domain restrictions.

Example 4 *Suppose that the set of feasible candidates is $S = X = \{x, y, z\}$ and the set of voters is $N = \{1, 2\}$. Furthermore, suppose that the partitions $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by*

$$\mathcal{C}_1 = \{\{x\}, \{y\}, \{z\}\},$$
$$\mathcal{C}_2 = \{\{x, y\}, \{z\}\}.$$

This partition satisfies 1 dichotomous goodness. There is only one triple, and one out of two voters (voter 2) cannot distinguish at least two candidates in the triple.

Now define a profile $\mathbf{R}$ by

$$xP_1yP_1z,$$
$$zP_2yI_2x.$$

This profile respects the domain restriction induced by the partitions $\mathcal{C}_1$ and $\mathcal{C}_2$. Candidate x is the unique intermediate Condorcet winner because it beats y with a score of one to zero and ties with z with a score of one to one. In addition to x, z is a weak Condorcet winner because it ties with x and with y with a score of one to one.

For simplicity, Example 4 involves a mere two voters. We note that additional voters whose relations are given by universal equal goodness could be added without changing the desired result; this is worth mentioning because Barberà and Ehlers (2011) assume that the are at least three voters.

The restricted domain examined by Barberà and Ehlers (2011) is but one example; it is not difficult to find others that share the feature of being rich enough for a significant difference between weak and intermediate Condorcet winners (and losers) to emerge.

The condition of intermediate Condorcet winner consistency requires that only members of the set $ICW(\mathbf{R}, S)$ can be chosen, provided that this set is non-empty.

Intermediate Condorcet winner consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$, if $ICW(\mathbf{R}, S) \neq \emptyset$, then

$$F(\mathbf{R}, S) \subseteq ICW(\mathbf{R}, S).$$

In analogy to strong Condorcet loser consistency, intermediate Condorcet loser consistency precludes intermediate Condorcet losers from being selected.

Intermediate Condorcet loser consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$,

$$F(\mathbf{R}, S) \cap ICL(\mathbf{R}, S) = \emptyset.$$

We emphasize that the conjunction of intermediate Condorcet winner consistency and intermediate Condorcet loser consistency does not preclude the choice of weak Condorcet winners if there are no intermediate Condorcet winners. Of course, these weak Condorcet winners may also be weak Condorcet losers, which is why we evidently cannot endorse the conjunction of weak Condorcet winner consistency and weak Condorcet loser consistency. We certainly do not exclude the possibility of choosing weak Condorcet winners in the absence of intermediate Condorcet winners but consider it inappropriate to assign a form of privileged status to weak Condorcet winners from the outset.

The following theorem establishes that the set of intermediate Condorcet winners (losers) for a profile $\mathbf{R}$ in a feasible set S can only be empty if either each candidate is tied with every other candidate, or the profile $\mathbf{R}$ is cyclical in the feasible set S. A profile $\mathbf{R} \in \mathcal{R}^n$ is cyclical in a feasible set $S \in \mathcal{X}$ if there exist a number $K \in \{3, \ldots, |S|\}$ and K pairwise distinct candidates $x^1, \ldots, x^K \in S$ such that

$$p(\mathbf{R}; x^k, x^{k+1}) > p(\mathbf{R}; x^{k+1}, x^k) \text{ for all } k \in \{1, \ldots, K-1\}$$

and

$$p(\mathbf{R}; x^K, x^1) > p(\mathbf{R}; x^1, x^K).$$

Theorem 1 *Let* $\mathbf{R} \in \mathcal{R}^n$ *and* $S \in \mathcal{X}$. *If* $ICW(\mathbf{R}, S) = \emptyset$ *or* $ICL(\mathbf{R}, S) = \emptyset$, *then* $p(\mathbf{R}; x, y) = p(\mathbf{R}; y, x)$ *for all distinct* $x, y \in X$ *or the profile* $\mathbf{R}$ *is cyclical in* S.

Proof. Suppose that a profile $\mathbf{R}$ and a feasible set S are such that $ICW(\mathbf{R}, S) = \emptyset$ or $ICL(\mathbf{R}, S) = \emptyset$. Without loss of generality, suppose that $ICW(\mathbf{R}, S) = \emptyset$; the proof for the case $ICL(\mathbf{R}, S) = \emptyset$ is obtained by reversing all inequalities. It is immediate that each candidate being in a tie with every other candidate is one possible way of obtaining an empty set of intermediate Condorcet winners. Now suppose that this case does not apply so that there exist two distinct candidates x^1 and x^2 in S such that

$$p(\mathbf{R}; x^2, x^1) > p(\mathbf{R}; x^1, x^2). \tag{4}$$

If $p(\mathbf{R}; x^2, z) \geq p(\mathbf{R}; z, x^2)$ for all $z \in S \setminus \{x^2\}$, candidate x^2 is an intermediate Condorcet winner for $\mathbf{R}$ in S because of (4). This is not possible because $ICW(\mathbf{R}, S)$ is assumed to be empty. Therefore, there exists $x^3 \in S$ such that $p(\mathbf{R}; x^3, x^2) > p(\mathbf{R}; x^2, x^3)$. Because S is finite, this process can be repeated until we reach a number $K \geq 3$ and a candidate $x^K \in S$ such that $p(\mathbf{R}; x^K, x^{K-1}) > p(\mathbf{R}; x^{K-1}, x^K)$ and x^K also appears earlier in the iterative process as one of the candidates that loses against another candidate. This means that the profile $\mathbf{R}$ is cyclical in S, as was to be established. ■

The result of Theorem 1 explains why the profile of Example 2 is not single-peaked. Recall that the example is intended to be such that there is one candidate (candidate x in the example) who is a weak Condorcet winner and a weak Condorcet loser at the same time, and there are no other weak Condorcet winners or losers. This means that not all pairwise comparisons can result in a tie; if that were the case, all candidates would be weak Condorcet winners and weak Condorcet losers, and this is to be avoided for the purposes of the example. Clearly, an intermediate Condorcet winner (loser) is a weak Condorcet winner (loser) by definition and, therefore, there cannot be any intermediate Condorcet winners or intermediate Condorcet losers. Thus, Theorem 1 implies that the profile must be cyclical in a feasible set; in the example, the cycle involves the candidates y, z, and w. As is well-known, such cycles cannot occur if the profile is single-peaked.

We now use the three reference properties of a social choice function introduced in Section 2 to provide an assessment of intermediate Condorcet winner consistency. A consistency property regarding sets of candidates with strong voter support is silent when it comes to pairs of profiles and feasible sets for which such candidates do not exist. As a consequence, it is convenient to think of these sets as the outcomes assigned by a social choice function the domain of which is restricted to pairs $(\mathbf{R}, S)$ such that the requisite set of privileged candidates is non-empty. In the case of intermediate Condorcet winners, define the domain $\mathcal{D}^{IC}$ as the set of all pairs $(\mathbf{R}, S) \in \mathcal{R}^n \times \mathcal{X}$ for which the set $ICW(\mathbf{R}, S)$ is non-empty. The social choice function $F^{ICW} \colon \mathcal{D}^{IC} \to \mathcal{X}$ can now be defined by letting, for all $(\mathbf{R}, S) \in \mathcal{D}^{IC}$, $F^{ICW}(\mathbf{R}, S) = ICW(\mathbf{R}, S)$.

The following theorem shows how intermediate Condorcet winners fare when assessed by means of the three reference properties defined in Section 2.

Theorem 2 *The social choice function F^{ICW}*

(a) satisfies exclusion of dominated candidates;

(b) violates independence of unanimously worst candidates;

(c) satisfies pairwise justifiability.

Proof. (a) To prove that F^{ICW} satisfies exclusion of dominated candidates, let $(\mathbf{R}, S) \in \mathcal{D}^{IC}$, and suppose that $z, w \in S$ are such that zP_iw for all $i \in N$. This implies $p(\mathbf{R}; z, w) = n$ and $p(\mathbf{R}; w, z) = 0$ and, therefore, $p(\mathbf{R}; w, z) < p(\mathbf{R}; z, w)$. This inequality implies that w cannot be an intermediate Condorcet winner for $\mathbf{R}$ in S, that is, $w \notin F^{ICW}(\mathbf{R}, S)$.

(b) Example 3 can be used to demonstrate that F^{ICW} does not satisfy independence of unanimously worst candidates. Using the profile $\mathbf{R}$ of the example, it follows that $F^{ICW}(\mathbf{R}, \{x, y, z\}) = \{x\}$ and $F^{ICW}(\mathbf{R}, \{x, y, z, w\}) = \{x, y\}$. Thus, candidate y enters the set of chosen candidates as a consequence of adding candidate w, who is uniquely worst according to all voters, to the feasible set.

(c) Finally, we show that F^{ICW} satisfies pairwise justifiability. Let $(\mathbf{R}, S), (\mathbf{R}', S) \in \mathcal{D}^{IC}$, and suppose that $x \in F^{ICW}(\mathbf{R}, S)$ and $x \notin F^{ICW}(\mathbf{R}', S)$. If

$$[xR_jz \text{ and } \neg(xR'_jz)] \ \text{ or } \ [xP_jz \text{ and } \neg(xP'_jz)] \tag{5}$$

is violated for all $j \in N$ and for all $z \in S \setminus \{x\}$, it follows that x does not lose against any of the other candidates in S and wins against at least one of them for the profile $\mathbf{R}'$; this follows immediately because x is an intermediate Condorcet winner for $\mathbf{R}$ in S. Thus, x is an intermediate Condorcet winner for $\mathbf{R}'$ in S and, therefore, $x \in F^{ICW}(\mathbf{R}', S)$. This is a contradiction and, therefore, (5) must be true. ■

4 Maximal intermediate Condorcet winners

By definition, an intermediate Condorcet winner wins against or ties with every other candidate in a pairwise contest. This suggests a possible modification by focusing on those intermediate Condorcet winners that register the highest number of wins. Example 3 illustrates this distinction: candidate x wins against two other candidates in S, whereas y only wins against one other candidate and, therefore, x is maximal within the set of intermediate Condorcet winners when assessed by means of the number-of-wins criterion.

To provide a formal definition, let $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$. A candidate $x \in ICW(\mathbf{R}, S)$ is a maximal intermediate Condorcet winner for $\mathbf{R}$ in S if x scores the maximal number of wins among the candidates in $ICW(\mathbf{R}, S)$; that is, if

$$\begin{aligned} |\{z \in S \setminus \{x\} : p(\mathbf{R}; x, z) > p(\mathbf{R}; z, x)\}| \quad &\geq \quad |\{z \in S \setminus \{y\} : p(\mathbf{R}; y, z) > p(\mathbf{R}; z, y)\}| \\ &\quad\ \text{for all } y \in ICW(\mathbf{R}, S). \end{aligned}$$

We denote the set of maximal intermediate Condorcet winners for $\mathbf{R}$ in S by $MCW(\mathbf{R}, S)$. By definition, $MCW(\mathbf{R}, S)$ is non-empty if and only if $ICW(\mathbf{R}, S)$ is non-empty. Returning to Example 3, it follows that x is a maximal intermediate Condorcet winner for $\mathbf{R}$ in S but y is not.

The axiom of maximal intermediate Condorcet winner consistency is obtained by replacing the set of intermediate Condorcet winners with its maximal variant.

Maximal intermediate Condorcet winner consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$, if $MCW(\mathbf{R}, S) \neq \emptyset$, then

$$F(\mathbf{R}, S) \subseteq MCW(\mathbf{R}, S).$$

In analogy to the social choice function that selects the set of intermediate Condorcet winners, a corresponding function F^{MCW} for maximal intermediate Condorcet winners can be defined. The function has the same domain $\mathcal{D}^{IC}$ as F^{ICW} because maximal intermediate Condorcet winners exist if and only if intermediate Condorcet winners exist. Formally, this social choice function is defined by $F^{MCW}(\mathbf{R}, S) = MCW(\mathbf{R}, S)$ for all $(\mathbf{R}, S) \in \mathcal{D}^{IC}$. The following theorem establishes which of the reference properties of Section 2 are satisfied by this social choice function.

Theorem 3 *The social choice function F^{MCW}*

(a) satisfies exclusion of dominated candidates;

(b) satisfies independence of unanimously worst candidates;

(c) violates pairwise justifiability.

Proof. (a) To prove that F^{MCW} satisfies exclusion of dominated candidates, let $(\mathbf{R}, S) \in \mathcal{D}^{IC}$, and suppose that $z, w \in S$ are such that zP_iw for all $i \in N$. This implies $p(\mathbf{R}; z, w) = n$ and $p(\mathbf{R}; w, z) = 0$ and, therefore, $p(\mathbf{R}; w, z) < p(\mathbf{R}; z, w)$. This inequality implies that w cannot be an intermediate Condorcet winner for $\mathbf{R}$ in S. Therefore, w cannot be a maximal intermediate Condorcet winner for $\mathbf{R}$ in S so that $w \notin F^{MCW}(\mathbf{R}, S)$.

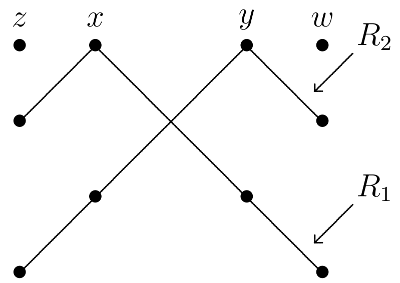

Figure 2: The profile $\mathbf{R}$ of Example 5.

(b) To prove that F^{MCW} satisfies independence of unanimously worst candidates, let $(\mathbf{R}, S), (\mathbf{R}, S \cup \{w\}) \in \mathcal{D}^{IC}$, where $w \in X \setminus S$ is such that zP_iw for all $i \in N$ and for all $z \in S$.

To show that $F^{MCW}(\mathbf{R}, S) \subseteq F^{MCW}(\mathbf{R}, S \cup \{w\})$, observe first that the addition of candidate w who is worst according to all voters only has the effect of increasing the number of candidates who lose against a member of $F^{MCW}(\mathbf{R}, S)$ by one. Thus, every maximal intermediate Condorcet winner for $\mathbf{R}$ in S is also a maximal intermediate Condorcet winner for $\mathbf{R}$ in $S \cup \{w\}$ and, therefore, any candidate who is a member of $F^{MCW}(\mathbf{R}, S)$ must also be a member of $F^{MCW}(\mathbf{R}, S \cup \{w\})$, which establishes the set inclusion.

To prove the reverse set inclusion, suppose that $x \in F^{MCW}(\mathbf{R}, S \cup \{w\})$. Because every candidate in S wins against w with a score of n to zero, removing w from $S \cup \{w\}$ reduces, for each candidate in $F^{MCW}(\mathbf{R}, S)$, the number of wins of this candidate by one. Therefore, because candidate x wins against the maximal number of other candidates within $ICW(\mathbf{R}, S \cup \{w\})$, x also wins against the maximal number of candidates within $ICW(\mathbf{R}, S)$; note that the latter set is non-empty because $MCW(\mathbf{R}, S)$ is non-empty. Therefore, x is a candidate that records the maximal number of wins within the non-empty set $ICW(\mathbf{R}, S)$ which, by definition, implies that $x \in MCW(\mathbf{R}, S) = F^{MCW}(\mathbf{R}, S)$.

(c) To show that F^{MCW} violates pairwise justifiability, consider the following example.

Example 5 *Let $S = X = \{x, y, z, w\}$, $N = \{1, 2\}$, and define the profile $\mathbf{R}$ by $xP_1zP_1yP_1w$ and $yP_2wP_2xP_2z$. Both x and y are maximal intermediate Condorcet winners and, therefore, $F^{MCW}(\mathbf{R}, S) = \{x, y\}$. This profile is single-peaked, as illustrated in Figure 2.*

Now consider the profile $\mathbf{R}'$ given by $xP'_1yP'_1zP'_1w$ and $R'_2 = R_2$. The only change when moving from $\mathbf{R}$ to $\mathbf{R}'$ is that the relative position of y and z is reversed; there is no change that involves candidate x—and, therefore, no deterioration of x relative to any other candidate. Both x and y are intermediate Condorcet winners for $\mathbf{R}'$ in S but x no longer is maximal—it registers only one win (against w), whereas y now wins against two other candidates (z and w). Therefore, $x \notin F^{MCW}(\mathbf{R}', S) = \{y\}$ so that x is removed from the set of selected candidates even though its relative position to the other candidates did not deteriorate. This is a violation of pairwise justifiability. As is the case for $\mathbf{R}$, the profile $\mathbf{R}'$ is single-peaked; see Figure 3. ■

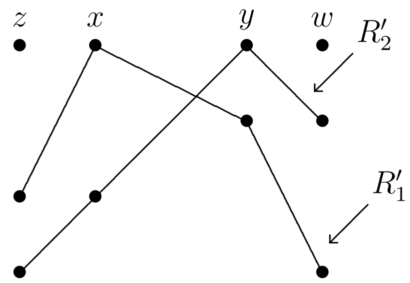

Figure 3: The profile $\mathbf{R}'$ of Example 5.

Comparing Theorems 2 and 3, there is no immediately obvious way of favoring one of the two notions of strong voter support over the other. Both F^{ICW} and F^{MCW} satisfy exclusion of dominated candidates. Choosing the entire set of intermediate Condorcet winners conflicts with independence of unanimously worst candidates and complies with pairwise justifiability. Conversely, selecting only the set of maximal intermediate Condorcet winners satisfies independence of unanimously worst candidates and violates pairwise justifiability. The failure of the maximal variant to comply with pairwise justifiability is rooted in the dependence of the criterion on other candidates. Whereas the privileged status of a candidate as an intermediate Condorcet winner can be identified exclusively in terms of this candidate's performance against others, this information no longer suffices to determine whether an intermediate Condorcet winner possesses the maximality property. In addition to calculating the number of pairwise wins achieved by the candidate in question, the pairwise contests that involve all other intermediate Condorcet winners must be consulted as well. Especially in the context of defining a notion of privileged status, the observation that a candidate x can achieve such a position on its own without having to consult comparisons that do not involve x, this can be seen as an argument in favor of the entire set of intermediate Condorcet winners. On the other hand, independence of unanimously worst candidates has considerable appeal and is often regarded as being highly desirable because it permits the elimination of unambiguously undesirable options without influencing the resulting choice.

We do not propose a dual maximal intermediate Condorcet loser principle because there is an important asymmetry between intermediate Condorcet winners and intermediate Condorcet losers. While it may be desirable to select some but not all intermediate Condorcet winners, all intermediate Condorcet losers can be considered unappealing and, therefore, we think that it is appropriate to remove all of them from consideration.

5 Smith sets

An interesting alternative generalization of strong Condorcet winner consistency appears in Smith (1973, p. 1038); see also Fishburn (1977, p. 478). Let $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$. A non-empty strict subset T of S is a Smith set for $\mathbf{R}$ and S if

$$p(\mathbf{R}; x, z) > p(\mathbf{R}; z, x) \text{ for all } x \in T \text{ and for all } z \in S \setminus T.$$

There are profiles and feasible sets for which a Smith set does not exist. Moreover, even if such a set exists, it need not be unique. For instance, if $S = X = \{x, y, z\}$, N is an arbitrary set of voters, and the profile $\mathbf{R}$ is given by xP_iyP_iz for all $i \in N$, it follows that both $T = \{x\}$ and $T' = \{x, y\}$ are Smith sets: for each of these sets, its members win against all candidates in the respective complement. However, it must be the case that multiple Smith sets for a profile $\mathbf{R}$ and a feasible set S are nested and, therefore, there is a unique minimal Smith set. This is established in the following theorem.

Theorem 4 *Let $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$. If T and T' are Smith sets for $\mathbf{R}$ and S, then*

$$T \subseteq T' \quad \text{or} \quad T' \subseteq T.$$

Proof. By way of contradiction, suppose that there exist a profile $\mathbf{R} \in \mathcal{R}^n$, a feasible set $S \in \mathcal{X}$, and two Smith sets T and T' for $\mathbf{R}$ and S such that

$$T \setminus T' \neq \emptyset \quad \text{and} \quad T' \setminus T \neq \emptyset.$$

Letting $x \in T \setminus T'$ and $y \in T' \setminus T$, it follows that $p(\mathbf{R}; x, y) > p(\mathbf{R}; y, x)$ because T is a Smith set for $\mathbf{R}$ and S, and that $p(\mathbf{R}; y, x) > p(\mathbf{R}; x, y)$ because T' is a Smith set for $\mathbf{R}$ and S. This is a contradiction and, therefore, one of the two sets T and T' must be a subset of the other. ■

Because the number of candidates is finite, Theorem 4 implies that if there exists a Smith set for $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$, there is a unique minimal Smith set for $\mathbf{R}$ and S, and we write it as $MSW(\mathbf{R}, S)$. With this notational convention, we formulate the corresponding consistency condition as follows.

Strong Smith consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$, if there exists a minimal Smith set $MSW(\mathbf{R}, S)$ for $\mathbf{R}$ and S, then

$$F(\mathbf{R}, S) \subseteq MSW(\mathbf{R}, S).$$

Strong Smith consistency is equivalent to the requirement that the choice set $F(\mathbf{R}, S)$ cannot contain any elements that are in the complement of the minimal Smith set if such a set exists; the latter is Smith's (1973) original formulation of the consistency condition. This also means that, in the case of minimal Smith sets, there is no need for two separate consistency properties. Because the complement of the minimal Smith set contains those candidates that are viewed as being inferior according to this criterion, their choice is already precluded by strong Smith consistency. Clearly, strong Condorcet winner consistency is obtained as the special case in which $MSW(\mathbf{R}, S)$ is a singleton set.

We note that the non-emptiness of the set of (maximal) intermediate Condorcet winners and the existence of a minimal Smith set are independent, as demonstrated by means of the following two examples.

Example 6 *Consider the set of candidates* $S = X = \{x, y, z, w\}$*, the set of voters* $N = \{1, 2, 3\}$*, and the profile* $\mathbf{R}$ *given by*

$$xP_1yP_1zP_1w,$$
$$zP_2xP_2yP_2w,$$
$$yP_3zP_3xP_3w.$$

The set $ICW(\mathbf{R}, S)$ *is empty because every candidate loses against at least one other candidate in a pairwise contest but there exists a minimal Smith for* $\mathbf{R}$ *and* S*. The set* $\{x, y, z\}$ *is a Smith set because every candidate in this set beats every candidate in the complement* $S \setminus \{x, y, z\} = \{w\}$*. Moreover,* $\{x, y, x\}$ *is the unique Smith set for* $\mathbf{R}$ *and* S *and, therefore, the minimal Smith set for* $\mathbf{R}$ *and* S *is given by* $MSW(\mathbf{R}, S) = \{x, y, z\} \neq \emptyset$*.*

The profile of Example 6 is not single-peaked. The set of intermediate Condorcet winners is empty and there exists a minimal Smith set. The latter observation implies that there exist two distinct feasible candidates that are not in a tie with each other according to the profile to be constructed and, by Theorem 1, it follows that the profile must be cyclical in the feasible set. As a consequence, the profile cannot be single-peaked.

Example 7 *Let* $S = X = \{x, y, z, w\}$*,* $N = \{1, 2\}$*, and define the profile* $\mathbf{R}$ *by* $xP_1zP_1yP_1w$ *and* $yP_2wP_2xP_2z$*; this is the profile that appears in Example 5. There is no Smith set for* $\mathbf{R}$ *and* S *because, for every non-empty strict subset* T *of* S*, at least one of the members of its complement wins against or ties with each of the members of* T*. But we have* $ICW(\mathbf{R}, S) = \{x, y\} \neq \emptyset$*. As mentioned earlier, the profile of the example is single-peaked; see again Figure 2.*

Although the non-emptiness of the set of intermediate Condorcet winners does not imply the existence of a Smith set, there is a subset relationship between the requisite sets, provided that a (minimal) Smith set exists. In particular, the set inclusion

$$ICW(\mathbf{R}, S) \subseteq MSW(\mathbf{R}, S)$$

is valid whenever $MSW(\mathbf{R}, S)$ exists for $\mathbf{R}$ and S. To prove this statement, suppose that the profile $\mathbf{R}$ and the feasible set S are such that there exists a minimal Smith set $MSW(\mathbf{R}, S)$ for $\mathbf{R}$ and S. The set inclusion is trivially satisfied if $ICW(\mathbf{R}, S) = \emptyset$. Now suppose that $ICW(\mathbf{R}, S) \neq \emptyset$ and let $x \in ICW(\mathbf{R}, S)$. If $x \notin MSW(\mathbf{R}, S)$, it follows that there exists $z \in MSW(\mathbf{R}, S) \subseteq S$ such that $p(\mathbf{R}; z, x) > p(\mathbf{R}; x, z)$, a contradiction to the assumption that x is an intermediate Condorcet winner for $\mathbf{R}$ in S. Thus, $x \in MSW(\mathbf{R}, S)$.

In addition, Theorem 1 can be strengthened as follows.[1]

Theorem 5 *Let* $\mathbf{R} \in \mathcal{R}^n$ *and* $S \in \mathcal{X}$*. If* $ICW(\mathbf{R}, S) = \emptyset$ *or* $ICL(\mathbf{R}, S) = \emptyset$ *and* $MSW(\mathbf{R}, S)$ *exists, then* $p(\mathbf{R}; x, y) = p(\mathbf{R}; y, x)$ *for all distinct* $x, y \in X$ *or the profile* $\mathbf{R}$ *is cyclical in* $MSW(\mathbf{R}, S)$ *or in* $S \setminus MSW(\mathbf{R}, S)$*.*

[1]The above set inclusion and the result of Theorem 5 were provided by the reviewer mentioned in the acknowledgment footnote.

Proof. Suppose that a profile $\mathbf{R}$ and a feasible set S are such that $ICW(\mathbf{R}, S) = \emptyset$ or $ICL(\mathbf{R}, S) = \emptyset$. Moreover, suppose that a minimal Smith set $MSW(\mathbf{R}, S)$ exists for $\mathbf{R}$ and S. By Theorem 1, $p(\mathbf{R}; x, y) = p(\mathbf{R}; y, x)$ for all distinct $x, y \in X$ or the profile $\mathbf{R}$ is cyclical in S. By way of contradiction, suppose that $\mathbf{R}$ is cyclical in S but there is neither a cycle in $MSW(\mathbf{R}, S)$ nor in $S \setminus MSW(\mathbf{R}, S)$. This implies that there is a cycle that contains two elements $x \in S \setminus MSW(\mathbf{R}, S)$ and $y \in MSW(\mathbf{R}, S)$ such that $p(\mathbf{R}; x, y) > p(\mathbf{R}; y, x)$. This contradicts the definition of a Smith set. ■

To assess minimal Smith sets with respect to their compliance with the three reference properties of Section 2, we again define the requisite social choice function. Denoting the domain on which a minimal Smith set exists by $\mathcal{D}^{MS}$, the function $F^{MSW} \colon \mathcal{D}^{MS} \to \mathcal{X}$ is defined by $F^{MSW}(\mathbf{R}, S) = MSW(\mathbf{R}, S)$ for all $(\mathbf{R}, S) \in \mathcal{D}^{MS}$. We obtain the following result.

Theorem 6 *The social choice function F^{MSW}*

(a) violates exclusion of dominated candidates;

(b) satisfies independence of unanimously worst candidates;

(c) violates pairwise justifiability.

Proof. (a) To prove that F^{MSW} violates exclusion of dominated candidates, consider Example 3. The universal set of the example is $X = \{x, y, z, w\}$, the set of voters is $N = \{1, 2\}$, and the profile $\mathbf{R}$ is given by $xP_1zP_1yP_1w$ and $yP_2xP_2zP_2w$. It follows that $F^{MSW}(\mathbf{R}, \{x, y, z, w\}) = \{x, y, z\}$ because each of the candidates x, y, and z wins against w and no strict subset of $\{x, y, z\}$ shares this property. But candidate z is dominated by candidate x and, therefore, exclusion of dominated candidates is violated.

(b) We show that F^{MSW} satisfies independence of unanimously worst candidates. Let $(\mathbf{R}, S), (\mathbf{R}, S \cup \{w\}) \in \mathcal{D}^{MS}$, where $w \in X \setminus S$ is such that zP_iw for all $i \in N$ and for all $z \in S$.

To show that $F^{MSW}(\mathbf{R}, S) \subseteq F^{MSW}(\mathbf{R}, S \cup \{w\})$, observe that the minimal Smith set is unchanged if a candidate w who is uniquely worst according to all voters is added to S; this follows immediately because each candidate in $MSW(\mathbf{R}, S)$ wins against each candidate in $S \setminus MSW(\mathbf{R}, S)$ and, by definition, also wins against w. Thus, every member of the minimal Smith set for $\mathbf{R}$ and S is also a member of the minimal Smith set for $\mathbf{R}$ and $S \cup \{w\}$, establishing the desired set inclusion.

To prove the reverse set inclusion, suppose that $x \in F^{MSW}(\mathbf{R}, S \cup \{w\})$. It follows that $p(x, z) > p(z, x)$ for all $x \in F^{MSW}(\mathbf{R}, S)$ and for all $z \in S \cup \{w\} \setminus F^{MSW}(\mathbf{R}, S \cup \{w\})$. Because w is the worst candidate according to all voters, it follows that removing w from $S \cup \{w\}$ preserves these inequalities for all $z \in S$. Therefore, x continues to be in the minimal Smith set once w is removed from $S \cup \{w\}$. It follows that $x \in F^{MSW}(\mathbf{R}, S)$ and the proof is complete.

(c) We use Example 3 again to show that F^{MSW} violates pairwise justifiability. As in part (a) of the proof, we have $F^{MSW}(\mathbf{R}, \{x, y, z, w\}) = \{x, y, z\}$. Now define the profile $\mathbf{R}'$ by letting $R'_1 = R'_2 = R_1$. As compared to $\mathbf{R}$, the position of x relative to y has improved in voter 2's relation, and the position of z relative to y has improved in voter 2's relation. It follows that $F^{MSW}(\mathbf{R}', S) = \{x\}$ because x wins against all other candidates. This means that z has been eliminated from the chosen set even though its position did not deteriorate with respect to any of the other candidates. That the profile $\mathbf{R}'$ is single-peaked is immediate because the two voters' goodness relations are identical. ■

In view of Theorems 3 and 6, a comparative assessment based on our three reference properties demonstrates that Smith sets are dominated by maximal intermediate Condorcet winners. Both satisfy independence of unanimously worst candidates but only the social choice function defined in terms of maximal intermediate Condorcet winners satisfies exclusion of dominated candidates. A comparison of intermediate Condorcet winners and Smith sets is less decisive because, in this case, there is no set inclusion regarding the reference properties that are satisfied by the requisite social choice functions. As demonstrated in Theorems 2 and 6, maximal Smith sets satisfy independence of unanimously worst candidates, whereas the remaining two desiderata are satisfied by the function defined in terms of the entire set of intermediate Condorcet winners.

6 Intermediate Smith sets

In analogy to intermediate Condorcet winners, an intermediate variant of a Smith set can be defined.[2] Let $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$. A non-empty strict subset T of S is an intermediate Smith set for $\mathbf{R}$ and S if

$p(\mathbf{R}; x, z) \geq p(\mathbf{R}; z, x)$ for all $x \in T$ and for all $z \in S \setminus T$ with at least one strict inequality.

As is the case for Smith sets, there are profiles and feasible sets for which intermediate Smith sets do not exist. By definition, if a Smith set exists for a profile $\mathbf{R}$ and a feasible set S, it follows that this set is also an intermediate Smith set for $\mathbf{R}$ and S.

Another analogy with Smith sets is that intermediate Smith sets need not be unique. However, there is an additional complexity in the intermediate case because multiple intermediate Smith sets need not be nested—and, therefore, a unique minimal intermediate Smith set is not well-defined. For example, suppose that $S = X = \{x, y, z\}$, N is a set of voters, and the profile $\mathbf{R}$ is defined by $xI_i yP_i z$ for all $i \in N$. It follows that there are three intermediate Smith sets, given by $\{x\}$, $\{y\}$, and $\{x, y\}$. Clearly, there is no unique minimal intermediate Smith set for $\mathbf{R}$ and S.

Because there is no unique minimal intermediate Smith set, the corresponding consistency condition cannot be obtained by means of a straightforward reformulation of strong Smith consistency. An alternative possibility is to consider the union of all cardinally minimal intermediate Smith sets. Suppose that $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$, and let T be an intermediate Smith set for $\mathbf{R}$ and S. The set T is a cardinally minimal intermediate Smith set for $\mathbf{R}$ and S if $|T| \leq |T'|$ for all intermediate Smith sets T' for $\mathbf{R}$ and S. The collection of all cardinally minimal intermediate Smith sets for $\mathbf{R}$ and S is denoted by $mISW(\mathbf{R}, S)$,

[2]Most of the observations reported in this section were suggested by the reviewer mentioned in the acknowledgment footnote.

and the union $ISW(\mathbf{R}, S)$ of these sets is given by

$$ISW(\mathbf{R}, S) = \bigcup_{T \in mISW(\mathbf{R}, S)} T.$$

Clearly, this union exists whenever an intermediate Smith set exists. The intermediate notion of Smith consistency is defined as follows.

Intermediate Smith consistency. For all $(\mathbf{R}, S) \in \mathcal{D}$, if there exists an intermediate Smith set for $\mathbf{R}$ and S, then

$$F(\mathbf{R}, S) \subseteq ISW(\mathbf{R}, S).$$

There is an interesting link between intermediate Condorcet winners and the union of all cardinally minimal intermediate Smith sets, established in the following theorem.

Theorem 7 *(a) For all* $\mathbf{R} \in \mathcal{R}^n$ *and for all* $S \in \mathcal{X}$*,*

$$ICW(\mathbf{R}, S) \subseteq ISW(\mathbf{R}, S). \tag{6}$$

(b) For all $(\mathbf{R}, S) \in \mathcal{D}^{IC}$*,*

$$ICW(\mathbf{R}, S) = ISW(\mathbf{R}, S). \tag{7}$$

Proof. (a) First, observe that the set inclusion in (6) is trivially satisfied if there are no intermediate Condorcet winners. Now suppose that $\mathbf{R} \in \mathcal{R}^n$, $S \in \mathcal{X}$, and $x \in ICW(\mathbf{R}, S)$. By definition, this means that

$$p(\mathbf{R}; x, z) \geq p(\mathbf{R}; z, x) \text{ for all } z \in S \setminus \{x\} \text{ with at least one strict inequality.}$$

This immediately implies that $\{x\}$ is an intermediate Smith set. Because this set contains only one element, it must be a cardinally minimal intermediate Smith set and, therefore, x is an element of the union of all cardinally minimal intermediate Smith sets.

(b) Suppose that $(\mathbf{R}, S) \in \mathcal{D}^{IC}$, that is, the profile $\mathbf{R}$ and the feasible set S are such that the set of intermediate Condorcet winners $ICW(\mathbf{R}, S)$ for $\mathbf{R}$ in S is non-empty. To prove (7), recall first that every singleton that consists of an intermediate Condorcet winner must be a cardinally minimal intermediate Smith set; see the argument employed in the proof of part (a). Thus, every cardinally minimal intermediate Smith set must be a singleton. By definition, the member of a singleton that is a cardinally minimal intermediate Smith set must be an intermediate Condorcet winner. Thus, the union of all cardinally minimal intermediate Smith sets is equal to the set of intermediate Condorcet winners, and (7) is established. ■

A set inclusion analogous to (6) is not valid if the set of intermediate Condorcet winners is replaced with the minimal Smith set; see the following example.

Example 8 *Suppose that the set of feasible candidates is $S = X = \{x, y, z, w, v\}$ and the set of voters is $N = \{1, 2\}$. Define the profile $\mathbf{R}$ by $xP_1zP_1yP_1wP_1v$ and $yP_2wP_2xP_2zP_2v$. This is analogous to the profile employed in Examples 5 and 7; note that candidate v is added at the bottom of both goodness relations. The minimal Smith set is given by $MSW(\mathbf{R}, S) = \{x, y, z, w\}$. The collection of intermediate Smith sets is given by*

$$\{\{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, w\}, \{x, y, z\}, \{x, y, w\}, \{x, y, z, w\}\}$$

and, therefore, the union of all cardinally minimal intermediate Smith sets is $\{x, y\}$. Thus, z and w are in the minimal Smith set for $\mathbf{R}$ and S but not in the union of all cardinally minimal intermediate Smith sets. The profile of this example is single-peaked; this follows immediately by appending candidate v at the right of the illustration in Figure 2.

We use $\mathcal{D}^{IS}$ to denote the domain on which intermediate Smith sets exist. By definition, a Smith set is also an intermediate Smith set and, therefore, this domain is a superset of the domain $\mathcal{D}^{MS}$. The social choice function $F^{ISW} \colon \mathcal{D}^{IS} \to \mathcal{X}$ is defined by $F^{ISW}(\mathbf{R}, S) = ISW(\mathbf{R}, S)$ for all $(\mathbf{R}, S) \in \mathcal{D}^{IS}$. As shown in the next theorem, this function violates all three of our reference properties.

Theorem 8 *The social choice function F^{ISW}*

(a) violates exclusion of dominated candidates;

(b) violates independence of unanimously worst candidates;

(c) violates pairwise justifiability.

Proof. (a) The following example proves that the union of all cardinally minimal intermediate Smith sets violates exclusion of dominated candidates.

Example 9 *Suppose that the feasible set of candidates is $S = X = \{x, y, z, w, v\}$ and the set of voters is $N = \{1, 2, 3\}$. Define the profile $\mathbf{R}$ by*

$$zP_1xP_1yP_1wP_1v,$$
$$wP_2zP_2xP_2yP_2v,$$
$$yP_3wP_3zP_3xP_3v.$$

It follows that the unique intermediate Smith set (and, in this case, the minimal Smith set) for $\mathbf{R}$ and S is given by $\{x, y, z, w\}$. Therefore, this is also the union of all cardinally minimal intermediate Smith sets. Because x is dominated by z, this is a violation of exclusion of dominated candidates.

(b) Next, we show that F^{ISW} violates independence of unanimously worst candidates.

Example 10 *Suppose that the universal set of candidates is* $X = \{x, y, z, w, v\}$ *and the set of voters is* $N = \{1, 2, 3, 4, 5, 6\}$*. Define the profile* $\mathbf{R}$ *by*

$$\begin{aligned}
&xP_1yP_1zP_1wP_1v,\\
&xP_2yP_2zP_2wP_2v,\\
&xP_3zP_3wP_3yP_3v,\\
&zP_4wP_4yP_4xP_4v,\\
&wP_5yP_5zP_5xP_5v,\\
&wP_6yP_6zP_6xP_6v;
\end{aligned}$$

this is based on Example 2 with candidate v *added as the worst candidate according to all voters. Consider first the feasible set* $S = \{x, y, z, w\}$*. There is a unique intermediate Smith set for* $\mathbf{R}$ *and* S*, given by* $\{y, z, w\}$*. It follows that* $F^{ISW}(\mathbf{R}, S) = \{y, z, w\}$*.*

Now consider the set $S \cup \{v\} = X$*. The added candidate* v *is considered worse than all others by all voters. Candidate* x *is the unique intermediate Condorcet winner for* $\mathbf{R}$ *in* $S \cup \{v\}$ *and, therefore,* $\{x\}$ *is the unique cardinally minimal intermediate Smith set. Therefore,* $F^{ISW}(\mathbf{R}, S \cup \{v\}) = \{x\}$*, a violation of independence of unanimously worst candidates.*

(c) Finally, we present an example that establishes a violation of pairwise justifiability.

Example 11 *Suppose that the feasible set of candidates is* $S = X = \{x, y, z, w\}$ *and the set of voters is* $N = \{1, 2, 3\}$*. Define the profile* $\mathbf{R}$ *by*

$$\begin{aligned}
&xP_1yP_1zP_1w,\\
&yP_2zP_2xP_2w,\\
&zP_3xP_3yP_3w.
\end{aligned}$$

It follows that the unique intermediate Smith set (and, in this case, the minimal Smith set) for $\mathbf{R}$ *and* S *is given by* $\{x, y, z\}$*. Therefore, this is also the unique cardinally minimal intermediate Smith set and, therefore, it follows that* $F^{ISW}(\mathbf{R}, S) = \{x, y, z\}$*.*

Now consider the profile $\mathbf{R}'$ *defined by* $R'_1 = R_1$*,* $R'_2 = R_2$*, and*

$$xP_3zP_3yP_3w.$$

It follows that the unique intermediate Smith set (and, in this case, the minimal Smith set) is given by $\{x\}$*. Therefore, this is also the unique cardinally minimal intermediate Smith set and, as a consequence,* $F^{ISW}(\mathbf{R}', S) = \{x\}$*. In the change from* $\mathbf{R}$ *to* $\mathbf{R}'$*, candidate* y *has been removed from the choice set, even though its relative position to each of the remaining candidates is not affected by the change. This is a violation of pairwise justifiability.* ■

The profiles $\mathbf{R}$ that appear in the proofs of parts (a) and (c) of this theorem are not single-peaked. The reason why single-peaked profiles cannot be employed in these cases is the equality (7) established in part (b) of Theorem 7. If intermediate Condorcet winners exist, the union of all cardinally minimal intermediate Smith sets must be equal to the set of intermediate Condorcet winners. As demonstrated in Theorem 2, the social choice function defined in terms of intermediate Condorcet winners satisfies the two reference properties of parts (a) and (c). This means that profiles that establish violations of these reference properties cannot permit the existence of intermediate Condorcet winners and, by Theorem 1, they cannot be single-peaked.

Observe that the examples used in the proofs of parts (a) and (c) of Theorem 8 also apply to minimal Smith sets. The reason why we do not use them in Theorem 6 is that an equality analogous to (7) does not apply in this case and, therefore, single-peaked profiles are available when constructing the requisite examples.

The proof of part (b) of Theorem 8 involves a relatively elaborate example. Our reason for this choice is that it allows us to make another point. In addition to proving the requisite claim, the example serves to illustrate that the union of all cardinally minimal intermediate Smith sets possesses another attribute that does not work in favor of this criterion. As pointed out in the context of Example 2, x is the unique weak Condorcet winner and the unique Condorcet loser for $\mathbf{R}$ in S. Moreover, there is no intermediate Condorcet winner for $\mathbf{R}$ in S. According to F^{ISW}, the set of candidates $\{y, z, w\}$ must be selected, thereby ruling out the choice of the weak Condorcet winner x. As alluded to earlier, intermediate Condorcet winner consistency is silent in the absence of intermediate Condorcet winners. In particular, the property does not preclude the choice of weak Condorcet winners (even if they are weak Condorcet losers at the same time), provided that the set of intermediate Condorcet winners is empty.

As shown in Theorem 2, F^{ICW} also violates the reference property of independence of unanimously worst candidates. Because a candidate who ties with all other candidates becomes an intermediate Condorcet winner once a unanimously worst candidate is added to a feasible set, this violation is an immediate consequence of the definition of the set of intermediate Condorcet winners. We note that the possible dependence on unanimously worst candidates can take more dramatic forms in the case of the union of all cardinally minimal intermediate Smith sets. As illustrated by the above example, the selected sets of candidates before and after the addition of a unanimously worst candidate may be disjoint, even though both of them are non-empty. This quite extreme form of a violation of the independence condition cannot occur with intermediate Condorcet winners. If an intermediate Condorcet winner exists before the addition of a unanimously worst candidate, this candidate must remain an intermediate Condorcet winner after the worst candidate is added.

As we see it, Theorem 8 provides a forceful argument against the use of intermediate Smith sets. Although there may be other features that are more appealing, violating all three of our reference properties seems to be a disadvantage that is hard to overcome. If, in addition, the above observation regarding the possible choice of weak Condorcet winners is taken into consideration, the picture becomes even more discouraging.

To conclude this section, we show that the conjunction of independence of unanimously worst candidates and pairwise justifiability implies exclusion of dominated candidates on any of the domains $\mathcal{D}^{IC}$, $\mathcal{D}^{MS}$, and $\mathcal{D}^{IS}$. This result has no effect on the theorems that identify the reference properties satisfied by our four social choice functions; note that none of the four satisfies both independence of unanimously worst candidates and pairwise justifiability. That no additional implications are valid follows from Theorems 2 and 3.

The social choice function F^{ICW} satisfies exclusion of dominated candidates and pairwise justifiability but not independence of unanimously worst candidates, and F^{MCW} satisfies independence of unanimously worst candidates and exclusion of dominated candidates but not pairwise justifiability.

The result of the following theorem is valid on a variety of domains and, in particular, on the three domains that correspond to our four generalizations of strong Condorcet winners. We explicitly mention these domains in the theorem statement but note that the profiles and feasible sets employed in the proof are available under numerous alternative domain specifications.

Theorem 9 *Suppose that $\mathcal{D} \in \{\mathcal{D}^{IC}, \mathcal{D}^{MS}, \mathcal{D}^{IS}\}$. If a social choice function $F: \mathcal{D} \to \mathcal{X}$ satisfies independence of unanimously worst candidates and pairwise justifiability, then F satisfies exclusion of dominated candidates.*

Proof. Suppose that F satisfies independence of unanimously worst candidates and pairwise justifiability. By way of contradiction, suppose that F violates exclusion of dominated candidates. Then there exist $(\mathbf{R}, S) \in \mathcal{D}$ and $z, w \in S$ such that zP_iw for all $i \in N$ and $w \in F(\mathbf{R}, S)$.

If w is unanimously worst for $\mathbf{R}$ in S, we obtain a contradiction to independence of unanimously worst candidates because, in this case, this reference property requires that $F(\mathbf{R}, S \setminus \{w\}) = F(\mathbf{R}, S)$ which is not possible if $w \in F(\mathbf{R}, S)$. Therefore, there exist $y^1 \in S \setminus \{w\}$ and $j \in N$ such that wR_jy^1. Clearly, $y^1 \neq z$. Observe that the pair $(\mathbf{R}, S \setminus \{w\})$ is also in the domain $\mathcal{D}^{IC}$ (or $\mathcal{D}^{MS}$ or $\mathcal{D}^{IS}$, respectively) because maximal Condorcet winners (or Smith sets or intermediate Smith sets, respectively) continue to exist after the removal of a unanimously worst candidate.

Now consider a profile $\mathbf{R}^1$ that is obtained from $\mathbf{R}$ by moving y^1 to the bottom of each voter's ordering. The position of w relative to each of the other candidates has not deteriorated when changing from $\mathbf{R}$ to $\mathbf{R}^1$ and, therefore, $w \in F(\mathbf{R}^1, S)$; otherwise, we would obtain a contradiction to pairwise justifiability. By independence of unanimously worst candidates, we obtain $F(\mathbf{R}^1, S \setminus \{y^1\}) = F(\mathbf{R}^1, S)$.

If w is unanimously worst for $\mathbf{R}^1$ in $S \setminus \{y^1\}$, we again obtain a contradiction to independence of unanimously worst candidates because, in this case, this reference property requires that $F(\mathbf{R}^1, S \setminus \{y^1, w\}) = F(\mathbf{R}^1, S \setminus \{y^1\})$ which is not possible if $w \in F(\mathbf{R}^1, S \setminus \{y^1\})$. Therefore, there exist $y^2 \in S \setminus \{y^1, w\}$ and $j \in N$ such that $wR^1_jy^2$.

Because S is finite, the above argument can be repeated as often as required until we reach a step $K \geq 2$ and a pair $(\mathbf{R}^K, S) \in \mathcal{D}$ such that $w \in F(\mathbf{R}^K, S \setminus \{y^1, \ldots, y^K\})$ and w is the unanimously worst candidate for $\mathbf{R}^K$ in $S \setminus \{y^1, \ldots, y^K\}$; this occurs by the latest when $S \setminus \{y^1, \ldots, y^K\} = \{z, w\}$. By independence of unanimously worst candidates, it follows that $F(\mathbf{R}^K, S \setminus \{y^1, \ldots, y^K, w\}) = F(\mathbf{R}^K, S \setminus \{y^1, \ldots, y^K\})$, which is impossible because $w \in F(\mathbf{R}^K, S \setminus \{y^1, \ldots, y^K\})$. This is a contradiction. To complete the proof, we observe that the pairs of profiles and feasible sets used in all of the above steps clearly can be chosen to be within any of the domains specified in the theorem statement. ■

7 A class of social choice functions

A natural class of social choice functions that satisfy intermediate Condorcet winner consistency and intermediate Condorcet loser consistency is obtained by assigning priorities in a lexicographic manner. This method parallels the definition of some social choice functions that can be found in the earlier literature. Daunou (1803), a strong supporter of Condorcet's views, suggests such a social choice function. According to the interpretation of Barberà, Bossert, and Suzumura (2021), Daunou's method proceeds as follows. Strong Condorcet winner consistency is used as the primary criterion—that is, if a strong Condorcet winner exists, this candidate—and only this candidate—is to be chosen. If a strong Condorcet winner does not exist, a second stage is reached in which the best candidates according to the plurality rule are selected after the iterative elimination of the strong Condorcet losers. Black (1958, p. 66) suggests to employ the Borda (1781) rule in place of the plurality rule if a strong Condorcet winner does not exist. See Morales (1797) for an elaborate endorsement of Borda's method. A characterization of Daunou's proposal and a general discussion of the lexicographic assignment of priorities can be found in Barberà, Bossert, and Suzumura (2021).

If the set of intermediate Condorcet losers is removed from a set of candidates $S \in \mathcal{X}$, new intermediate Condorcet losers may appear in the reduced set. Therefore, we propose to eliminate intermediate Condorcet losers in a cumulative fashion; see Barberà, Bossert, and Suzumura (2021) for an analogous observation that applies to strong Condorcet losers. We employ an iterative procedure and determine, after each step, whether there are candidates that have become intermediate Condorcet losers as a consequence of removing others in earlier steps. Because the set of candidates is finite, this procedure can be continued until no further intermediate Condorcet losers remain. Furthermore, because intermediate Condorcet losers must lose against some other candidate(s), the set that remains is non-empty. That the iterative procedure may indeed be necessary is illustrated by the following example, taken from Barberà, Bossert, and Suzumura (2021, p. 268).

Example 12 *Suppose that the feasible set of candidates is given by $S = X = \{x, y, z, w, v\}$ and the set of voters is $N = \{1, 2, 3\}$. Define the profile $\mathbf{R}$ by*

$$xP_1yP_1zP_1wP_1v,$$
$$yP_2zP_2xP_2wP_2v,$$
$$wP_3zP_3xP_3yP_3v.$$

It follows that $ICW(\mathbf{R}, S) = \emptyset$, $ICL(\mathbf{R}, S) = \{v\}$, and $ICL(\mathbf{R}, S \setminus \{v\}) = \{w\}$. Thus, it is possible that a new intermediate Condorcet loser (here, candidate w) emerges once an intermediate Condorcet loser (here, candidate v) is eliminated from the original feasible set. The set of intermediate Condorcet winners is empty, which ensures that the example is relevant in the sense that the second stage of the lexicographic procedure alluded to earlier is indeed reached.

The profile of the above example is not single-peaked. Because we want the set of intermediate Condorcet losers to be non-empty, the profile to be constructed cannot be such that all pairs of distinct candidates are in a tie with each other. Therefore, because the set of intermediate Condorcet winners is supposed to be empty, Theorem 1 implies that the requisite profile must be cyclical in the feasible set so that the profile cannot be single-peaked.

We define $CICL(\mathbf{R}, S)$, the cumulative set of intermediate Condorcet losers for a profile $\mathbf{R} \in \mathcal{R}^n$ in the feasible set $S \in \mathcal{X}$, iteratively as follows (again, see Barberà, Bossert, and Suzumura, 2021, for an analogous procedure in the context of strong Condorcet losers). If $ICL(\mathbf{R}, S) = \emptyset$, let $CICL(\mathbf{R}, S) = \emptyset$. If $ICL(\mathbf{R}, S) \neq \emptyset$, let $S^1 = S \setminus ICL(\mathbf{R}, S)$. If $ICL(\mathbf{R}, S^1) \neq \emptyset$, let $S^2 = S^1 \setminus ICL(\mathbf{R}, S^1)$ and so on until we reach a step K such that no intermediate Condorcet losers remain—that is, $ICL(\mathbf{R}, S^K) = \emptyset$. Because S is finite, such a step K must exist. The cumulative set of intermediate Condorcet losers for $\mathbf{R}$ in S is given by

$$CICL(\mathbf{R}, S) = ICL(\mathbf{R}, S) \cup ICL(\mathbf{R}, S^1) \cup \ldots \cup ICL(\mathbf{R}, S^{K-1})$$

and, because the last iteratively eliminated intermediate Condorcet losers must lose against some candidate(s) who are not intermediate Condorcet losers, the set $S \setminus CICL(\mathbf{R}, S)$ of remaining candidates is non-empty.

As an illustration, consider Example 12. Candidate v is the unique intermediate Condorcet loser for $\mathbf{R}$ in $S = \{x, y, z, w, v\}$ and, therefore, $ICL(\mathbf{R}, S) = \{v\}$. After eliminating v, we obtain the reduced set $S^1 = S \setminus ICL(\mathbf{R}, S) = \{x, y, z, w\}$. It follows that $ICL(\mathbf{R}, S^1) = ICL(\mathbf{R}, \{x, y, z, w\}) = \{w\}$ and, after removing w, the remaining set is $S^2 = S^1 \setminus ICL(\mathbf{R}, S^1) = \{x, y, z\}$. There are no intermediate Condorcet losers for $\mathbf{R}$ in $\{x, y, z\}$ so that $ICL(\mathbf{R}, S^2) = \emptyset$ and, therefore, the cumulative set of intermediate Condorcet losers for $\mathbf{R}$ in S is $CICL(\mathbf{R}, S) = \{w, v\}$. The set of remaining candidates is given by $S \setminus CICL(\mathbf{R}, S) = \{x, y, z\}$.

In order to obtain a sensible lexicographic assignment of priorities that selects the set of intermediate Condorcet winners in the first instance, it is important to ensure that the successive elimination of intermediate Condorcet losers does not generate intermediate Condorcet winners that were not intermediate Condorcet winners in a larger set obtained at an earlier stage in the iteration. This is indeed the case, as established in the following theorem.

Theorem 10 *For all* $\mathbf{R} \in \mathcal{R}^n$ *and for all* $S \in \mathcal{X}$,

$$ICW(\mathbf{R}, S \setminus ICL(\mathbf{R}, S)) \subseteq ICW(\mathbf{R}, S). \tag{8}$$

Proof. Let $\mathbf{R} \in \mathcal{R}^n$ and $S \in \mathcal{X}$, and suppose that $x \in ICW(\mathbf{R}, S \setminus ICL(\mathbf{R}, S))$. By definition,

$p(\mathbf{R}; x, z) \geq p(\mathbf{R}; z, x)$ for all $z \in S \setminus (\{x\} \cup ICL(\mathbf{R}, S))$ with at least one strict inequality.

If $x \notin ICW(\mathbf{R}, S)$, there exists $y \in ICL(\mathbf{R}, S)$ such that $p(\mathbf{R}; x, y) < p(\mathbf{R}; y, x)$. This inequality is a contradiction because y is an intermediate Condorcet loser for $\mathbf{R}$ in S. Thus, $x \notin ICW(\mathbf{R}, S)$, which proves the set inclusion in (8). ■

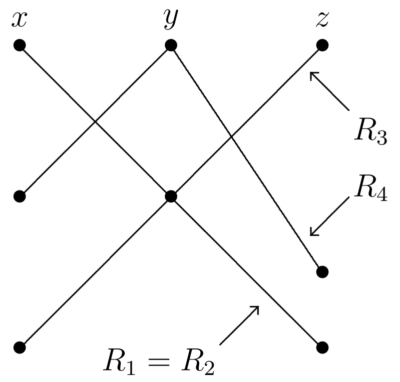

Figure 4: The profile $\mathbf{R}$ of Example 13.

The reverse set inclusion of (8) is not valid. As shown in the following example, it is possible that an intermediate Condorcet winner for a profile $\mathbf{R}$ in a feasible set S is not an intermediate Condorcet winner for $\mathbf{R}$ in $S \setminus ICL(\mathbf{R}, S)$.

Example 13 *Suppose that the set of feasible candidates is $S = X = \{x, y, z\}$ and the set of voters is $N = \{1, 2, 3, 4\}$. Define the profile $\mathbf{R}$ by*

$$\begin{gathered} xP_1yP_1z, \\ xP_2yP_2z, \\ zP_3yP_3x, \\ yP_4xP_4z. \end{gathered}$$

Candidate x ties with candidate y with a score of two to two, and x wins against z with a score of three to one. Likewise, candidate y ties with candidate x with a score of two to two, and y wins against z with a score of three to one. It follows that $ICW(\mathbf{R}, S) = \{x, y\}$ and $ICL(\mathbf{R}, S) = \{z\}$. After eliminating the intermediate Condorcet loser z from S, we obtain $S \setminus ICL(\mathbf{R}, S) = \{x, y\}$ and

$$ICW(\mathbf{R}, S \setminus ICL(\mathbf{R}, S)) = ICW(\mathbf{R}, \{x, y\}) = \emptyset$$

so that x and y are intermediate Condorcet winners for $\mathbf{R}$ in $S = \{x, y, z\}$ but not in $S \setminus ICL(\mathbf{R}, S) = \{x, y\}$. The profile defined in this example is single-peaked, as illustrated in Figure 4.

In analogy to the social choice functions advocated by Daunou (1803) and by Black (1958), we propose the following general class of intermediate lexicographic social choice functions. In the first stage, the set of all intermediate Condorcet winners is chosen if this set is non-empty. If there are no intermediate Condorcet winners, any non-empty subset from the set of those who are not cumulative intermediate Condorcet losers is selected. To define this class formally, let G be an arbitrary social choice function. The corresponding intermediate lexicographic social choice function F^G is defined as follows. For all $\mathbf{R} \in \mathcal{R}^n$ and for all $S \in \mathcal{X}$,

(i) if $ICW(\mathbf{R}, S) \neq \emptyset$, then $F^G(\mathbf{R}, S) = ICW(\mathbf{R}, S)$;

(ii) if $ICW(\mathbf{R}, S) = \emptyset$, then $F^G(\mathbf{R}, S) = G(\mathbf{R}, S \setminus CICL(\mathbf{R}, S))$.

All intermediate lexicographic social choice functions satisfy intermediate Condorcet winner consistency and intermediate Condorcet loser consistency.

As illustrated by Example 12, removing cumulative intermediate Condorcet losers before the application of the tie-breaking criterion is essential for some social choice functions G. Observe that candidate w in the example is a plurality winner and, therefore, it would be selected if G is the method of plurality decision, in spite of it being an intermediate Condorcet loser (in fact, even a strong Condorcet loser) once candidate v is eliminated.

In contrast, if G is the Borda rule, our modification of Black's (1958) rule can be defined without this cumulative elimination process because the Borda method (and, therefore, the intermediate variant of Black's rule) satisfies intermediate Condorcet loser consistency. To see that an intermediate Condorcet loser cannot be a Borda winner, suppose that candidate x is an intermediate Condorcet loser for a profile $\mathbf{R} \in \mathcal{R}^n$ in a feasible set $S \in \mathcal{X}$. By definition, this means that

$$p(\mathbf{R}; x, z) \leq p(\mathbf{R}; z, x) \quad \text{for all } z \in S \setminus \{x\} \quad \text{with at least one strict inequality.}$$

Therefore,

$$p(\mathbf{R}; x, z) - p(\mathbf{R}; z, x) \leq 0 \quad \text{for all } z \in S \setminus \{x\} \quad \text{with at least one strict inequality.}$$

Adding over all candidates $z \in X \setminus \{x\}$, we obtain

$$\sum_{z \in X \setminus \{x\}} [p(\mathbf{R}; x, z) - p(\mathbf{R}; z, x)] < 0 \tag{9}$$

because at least one of the inequalities is strict. The sum in (9) is the Borda score of x. The average Borda score is zero so a candidate with a negative Borda score cannot be a Borda winner. The argument just employed is taken from Moulin's (1988, p. 249) proof that a strong Condorcet loser cannot be a Borda winner.

If the set of intermediate Condorcet winners is replaced with the set of maximal intermediate Condorcet winners in the definition of our class of social choice functions, the members of the resulting class satisfy maximal intermediate Condorcet winner consistency and intermediate Condorcet loser consistency. Thus, our lexicographic method is applicable no matter which of the two variants is favored.

8 Concluding remarks

Table 1 summarizes our comparative assessment of the four definitions of strong voter support considered in this paper. Of the social choice functions corresponding to these definitions, only that induced by the entire set of intermediate Condorcet winners satisfies the reference property of pairwise justifiability. There is an intuitive explanation of this observation. Unlike the remaining three contenders, membership in this set can be determined exclusively on the basis of pairwise comparisons of the candidate in question with other candidates. In all other cases, additional comparisons have to be invoked. To determine whether a candidate is a maximal intermediate Condorcet winner, this candidate's performance compared to others is not sufficient—the pairwise contests that involve all other intermediate Condorcet winners must be consulted as well. Likewise, a candidate that belongs to a(n intermediate) Smith set cannot achieve privileged status on its own; it is only through membership in this set that it can be determined whether it has strong voter support according to the requisite criterion.

As far as a verdict regarding the relative desirability of the four definitions is concerned, the union of all cardinally minimal intermediate Smith sets performs rather poorly when assessed by means of our three reference properties—it fails to satisfy even a single one of them. Minimal Smith sets do better because they at least comply with independence of unanimously worst candidates (but not with the remaining two desiderata). However, they are dominated by the set of maximal intermediate Condorcet winners because the corresponding social choice function satisfies exclusion of dominated candidates in addition to independence of unanimously worst candidates. Therefore, only the entire set of intermediate Condorcet winners and the set of maximal intermediate Condorcet winners remain as the potentially most appealing options. An ultimate choice between the two of them comes down to the relative importance that is assigned to the reference properties of independence of unanimously worst candidates and pairwise justifiability. If the former is considered more significant than the latter, the set of maximal intermediate Condorcet winners comes out on top; if the reverse judgment prevails, the entire set of Condorcet winners emerges as the favorite method of generalizing the set of strong Condorcet winners. Alternatively, instead of selecting a single notion of strong voter support, it is perfectly plausible to make the final selection on a case-by-case basis, depending on the specific application that one may have in mind.

Table 1: Notions of strong voter support and their properties.

	ICW	MCW	MSW	ISW
Exclusion of dominated candidates	Yes	Yes	No	No
Independence of unanimously worst candidates	No	Yes	Yes	No
Pairwise justifiability	Yes	No	No	No

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