Simple Market Structures are Incomplete*

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Abstract

We study how risk is shared using simple assets, each of which pays on a specific subset of payoff-relevant variables. We show that a market composed of simple assets is incomplete – no matter how many or which assets exist– because it can never address the joint realizations of some distinct risks. We also show that this inability to refine trades generates spill overs between agents with different types of financial constraints. Agents of type i are exposed to uninsurable income risk unless there exists another type j who can condition on all variables affecting i's income.

Keywords: Risk sharing, Incomplete markets, Market structure

JEL Codes: D11, D52, D53, G52

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1 Introduction

Households and firms face multiple sources of risk. While each shock matters on its own, simultaneous bad outcomes can have a disproportionate effect. Nonetheless, distinct risks are often managed by separate instruments. Inflation-indexed bonds, mutual funds, or credit-swaps, for instance, all respond to a specific set of random variables while not tracking others. Even the most complex over-the-counter financial vehicles do not address the interaction of all risks, from global exchange rate risk down to a particular worker's health status. Insurance contracts are similarly segmented, in that multiple kinds of losses require separate contracts. For instance, the copay in an individual's health insurance does not depend on whether she recently also had water damage in her home.

There are good reasons why financial markets favor standardized assets which target a limited set of risk factors. An instrument whose payoff depends on the joint realization of many variables may be costly to operate. For instance, imagine a firm that relies on ten essential input suppliers, each of which may deliver or fail to deliver. If each of these risks are addressed with separate contracts, the insurance payments depend on 20 different clauses. On the other hand, a single insurance contract that conditions on all joint shocks would specify more than one thousand (i.e. 2^{10}) contingent payments.[1] The cost of contracting and executing assets therefore increases with the complexity of the arrangement.

In this paper we study financial market structures which are *simple* in the sense that each instruments conditions on a specific subset of risks. Our notion of simplicity is separate from the more common notion of *coarse* assets, which pay along events that lump multiple states together. Although all simple assets are necessarily coarse, the converse is not true since describing an event may require information on all sources of risk. It is well known that a small collection of coarse assets can be rich enough to complete the market. Very little is known, however, about how simple market structures affect insurance outcomes.

We consider an economy where each state of the world is uniquely determined by the realization of a collection of random variables that we call fundamental risks. Agents have state-dependent income and trade contingent-claims in a competitive market. We assume that if the financial market contains a *simple* asset that conditions on a specific subset of risks, then it contains the *class* of all assets that condition on this subset. Hence, a financial market structure is defined by a specific collection of *asset classes*, each of which responds to a different

[1] This example is adapted from Gollier and Schlesinger (1995) and inspired by the supply-chain model of Elliott, Golub, and Leduc (2022).

subset of fundamental risks. For any given market structure, we characterize the joint insurance possibilities obtained through the combination of its existing instruments.

Our results are the first to show that simple financial structures systematically underprovide insurance against joint shocks. Adding another simple asset to a market structure expands trades without *refining* them. As a result, market structures are only as precise as its most precise element, and markets are complete if and only if they contain non-simple assets (which condition on all variables). We illustrate significant welfare effects to risk-sharing with simple assets in several examples.

In the second part of the paper we introduce heterogeneity in financial access. Agents are assigned a personal subset of risks that they can trade. Any assets which condition on risks outside their set cannot be traded. For markets to clear, insurance demands must not only satisfy one's own restriction: they must also satisfy others'. Therefore, certain trades may be ruled out from any equilibrium, even though they are allowed by an agent's own trading restriction and would lead to a Pareto improvement. This may happen because individuals' constraints *spill over* to one another. When an agent is only restricted by her own constraint, we say that she is *resilient*.

Echoing our results on market structure, we find that resilience is inherently difficult to obtain. An agent can address a joint risk if and only if there exists another agent that can also trade it. Therefore, unless another agent can individually provide insurance against a joint shock, there is no hope that this insurance can be provided collectively. This can lead to uninsurable income risk if an agent is exposed to multiple risks. We derive an important income decomposition result. For each agent we can extract the component of her income that she is resilient to and the component that she is not. The latter identifies the income fluctuations that she cannot diversify, given the structure of everyone else's trading restrictions. In the context of a simple example we compute important welfare differences from acquiring resilience to multiple risks.

This paper is the first to document that all simple market structures are incomplete. The results reveal a strong form of linear dependence across asset payoffs which render larger markets incapable of transferring resources to worst case scenarios – where multiple bad outcomes coincide. Our results are general: they hold across many different preference classes (even outside of expected utility) and many income processes. They highlight a hereto unknown friction that is inextricably linked to insuring simultaneous risks.

1.1 Discussion of the Main Modelling Assumptions.

We briefly discuss some of our key modeling choices. First, our market structure is composed of *asset classes*, each of them rich enough to insure against any joint realization of a specific set of fundamental risks. Formally, each asset class spans all payoffs measurable with respect to the *coarse partition* induced by ignoring some risks. We assume this in order to isolate how simple assets restrict insurance in and of themselves, without imposing additional portfolio constraints such as short-selling or limited liability. Our model therefore deviates minimally from the complete markets model: we only simplify assets, but otherwise allow the market to be rich. This gives markets the best chance to generate flexible payoffs. Additional portfolio restrictions would just make insurance with simple assets even more restrictive. This upper bound is not only of theoretical interest. Starting with Ross (1976), the literature has shown that the put and call options on an underlying asset span a partitional payoff space. Thus, our market structure can be interpreted as one in which there exist simple underlying assets and their derivatives.

In the second half of the paper we assume that agents are unable to condition on certain fundamental risks. There are many reasons why different agents may have different constraints. An agent may lack access to timely information about a sector-specific risk in a foreign country.[2] Alternatively, an agent may be subject to cognitive constraints, such as limited awareness, or they may process information in lower-dimensional categories.[3] Even among equally sophisticated investors, the set of available financial instruments may vary substantially due to legal requirements, like a narrow mandate to manage a company's risk.[4] Any one or a combination of these circumstances reduces the set of tradable risks.

Finally, although we assume that each market participant may have a different trading restriction, our model can also represent a large population organized along different types. Each type represents a different trading restriction. No matter how many agents there are of each type, as long as agents with similar constraints have similar income shocks, our results on

[2] This could also be reflecting soft constraints like a financial "home bias" where households focus endogenously on sectors or geographical areas they are familiar with (Massa and Simonov, 2006; Van Nieuwerburgh and Veldkamp, 2009).

[3] A large literature has documented limited awareness in financial markets. Guiso and Jappelli (2005), for instance, find that financial literacy rates vary widely across Italian households, while Van Rooij, Lusardi, and Alessie (2011) find that financial literacy affects financial decision-making in the Netherlands. Auster and Pavoni (2020) model the incentives of financial intermediaries in this context. Gul, Pesendorfer, and Strzalecki (2017) model households' behavioral constraints when paying attention to very rare events is costly.

[4] Indeed, multinational companies manage third-party risk under strict corporate guidelines regularly approved by the sitting board of directors (Aebi, Sabato, and Schmid, 2012). In another example, tax-deductible retirement plans typically rule out investing in certain classes of risky assets (Atkins, 2011).

resilience follow. Intuitively, if incomes respond to the same set of underlying risks, then any gains from trading these risks within types vanish. Resilience can therefore be thought of as the ability to obtain insurance against type-specific income risk.

1.2 Relation to the Literature

There is a long literature interested in what it takes for market structures to promote insurance. On the one hand, fund separation theorems show that optimal exposure to risk can be implemented with a small number of assets (Cass, Stiglitz et al., 1970; Rubinstein, 1974; Dybvig and Liu, 2018). However, next to requiring suitable risk preferences, these results only relate to the sharing of aggregate risk. If income risks are complex, additional specialized instruments will be required to reach such an equilibrium. On the other hand, the spanning literature shows that a set of options on a single fundamental asset can replicate a complete market (Ross, 1976; Duffie and Rahi, 1995). Our paper can be seen as addressing this question whenever underlying assets are themselves simple in the sense described above. Our negative result shows that there is indeed an inherent lack of richness in any simple market structure.

There is also an active literature on market outcomes when each individual takes coarse actions coming from bounded rationality or lack of awareness. Agents may be constrained to choose from less than the full set (or history) of states. Frequently, these constraints are combined with an aspect of rationality where agents trade off the gains from separating two states against a mental cost. This leads to an optimal coarsening along aggregate phenomena, such as the price or the level of total resources (Gul et al., 2017; Auster, Kettering, and Kochov, 2021; Al-Najjar and Pai, 2014; Gabaix, 2014). Grouping states by an aggregate statistic satisfy simplicity – as we define it – since it often requires specifying many payoff-relevant variables. In our examples, we find that similar extreme price phenomena can be generated through exogenous simplicity constraints.

This paper is not the first to acknowledge that financial access may not be universal. Merton (1987) showed that if agents must share risk by interacting across different submarkets, asset prices and risk exposure get distorted. Moreover, if these markets clear separately, there is a trade off between flexibility and market power (Malamud and Rostek, 2017; Rostek and Yoon, 2021). In contrast, we seek to characterize the space of trading possibilities in which such markets must clear, without committing to specific protocols, distributions, or preferences. Similar to this paper, Guerdjikova and Quiggin (2019) consider differential measurability constraints to explore survival of incorrect beliefs in an otherwise classical setup. The lack of trading flexibility may prevent agents from engaging in wealth-depleting bets. To our knowledge, ours is the first paper to formalize a link between the trading limitations of others and the component of income risk that cannot be sold to the market, no matter the preferences or beliefs.

2 The Model

2.1 Basic Environment

Consider ex-ante trade in an economy with finite state space Ω, where each agent $i \in N = \{1, 2, ..., N\}$ has a state-contingent endowment in terms of the single consumption good, $y_i : \Omega \to \mathbb{R}$. All agents have identical expected utility preferences over final payoffs, $c_i : \Omega \to \mathbb{R}$. We assume agents hold common beliefs $p : \Omega \to [0, 1]$ and that the utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and strictly concave.[5]

Fundamental risks are described by a set Q of payoff-relevant random variables. For each *fundamental risk* $q \in Q$, denote its possible realizations by $Z_q \subseteq \mathbb{R}$ and let $q(\omega)$ represent the realization of $q : \Omega \to Z_q$ in state ω. A typical q can represent payoff-relevant fluctuations at any level of economic activity.[6] We assume that each ω can be associated with a characteristic Q-dimensional vector $\omega = (q_1(\omega), \dots, q_Q(\omega))$.[7] For simplicity, we assume that p assigns a non-zero probability to all joint q realization: $\Omega = Z_1 \times \dots \times Z_Q$. In section 5 we show that our results extend to general state spaces which are not the product space of all q risks.

Many standard settings fit this general specification. In a Lucas tree economy, each individual i is endowed with an individual stream of random output y_i. In this case, the set of payoff-relevant variables Q simply corresponds to the collection of the N individual incomes $q_i = y_i$. A state of the world would be characterized by the distribution of realized output $\omega = (y_1(\omega), ..., y_n(\omega))$. On the other hand, consider a standard shareholder economy in which there are L firms with output $F_l(Q_l)$ determined by a set Q_l of factors from some set $Q = \cup_{l=1}^{L} Q_l$. Household income $y_i(\omega) = \sum_{l=1}^{L} \theta_{il} F_l(Q_l)$ is constructed from a portfolio of shares $\theta_i = (\theta_{i1}, ..., \theta_{iL})$ for each household $i \in N$.

In this paper, we consider *simple* assets whose payoffs depend on the realization of some (but

[5] Our spanning results do not require assumptions on homogeneity or even expected utility.

[6] It could represent, for instance, an idiosyncratic risk factor such as local rainfall or a health shock affecting specific workers/farmers. It could represent input shortages or a productivity shock to a specific firm, affecting an entire sector of the economy. It could also capture an exchange rate risk or a shock to the global supply chain, affecting global markets.

[7] More concretely, for any two distinct states $\omega, \omega' \in \Omega$, there exists a $q \in Q$ such that $q(\omega) \neq q(\omega')$.

not all) risks $q \in Q$. Consider an asset which only conditions on a subset $M \subseteq Q$. Define by $\mathcal{L}_M$ the partition of Ω induced by grouping those states which are *indistinguishable* according to M. The different elements of $\mathcal{L}_M$ correspond to specific realizations of the risks $q \in M$. Formally, $L_M(\omega) \subseteq \Omega$ denotes the cell of $\mathcal{L}_M$ containing ω:

$$L_M(\omega) \equiv \{\omega' \in \Omega : q(\omega') = q(\omega) \text{ for all } q \in M\}. \tag{1}$$

We call the event $L_M(\omega) \in \mathcal{L}_M$ an *M-local state.*[8] A simple asset which conditions on $M \subseteq Q$ is therefore a function that is measurable with respect to $\mathcal{L}_M$.

The set of all M-measurable assets form a subspace of $\mathbb{R}^{|\Omega|}$. It is convenient to refer to this set in terms of a basis that generalizes Arrow securities. For any event $E \subseteq \Omega$, denote by a_E the *unit claim* on E, which pays 1 unit in event E and 0 otherwise:

$$a_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We can also describe this asset by a column vector $\mathbf{a}_E \in \mathbb{R}^{|\Omega|}$ whose ω-th position coordinate is $a_E(\omega)$. Therefore, asset $\mathbf{a}_{L_M(\omega)}$ represents the unit claim on the M-local state $L_M(\omega) \in \mathcal{L}_M$. It pays 1 unit in all states where every variable $q \in M$ assigns $q(\omega)$. We call $\mathbf{a}_{L_M(\omega)}$ an *M-local asset.*[9] The payoff matrix which collects all M-local assets defines an *M-asset class*,

$$\mathbf{A}_M = [\mathbf{a}_L]_{L \in \mathcal{L}_M}. \tag{3}$$

Notice that all M-measurable assets are contained in $\langle \mathbf{A}_M \rangle$, where we denote by $\langle \mathbf{A} \rangle = \text{span}(\mathbf{A})$ the column span of a matrix $\mathbf{A}$.

We assume that markets consist of a collection of M-asset classes, as defined in (3). As discussed in Section 1.1, to depart minimally from the classical assumptions, we allow markets to generate any measurable payoff in a given class. This does not require that each local asset be traded as its own separate instrument. In particular, Ross (1976), shows that any M-asset class in (3) can be spanned by the call and put options on a single underlying asset. For instance, if q_i represents the payoff of a firm, then the firm's equity and its derivatives form a linear subspace spanned by $\mathbf{A}_{\{q_i\}}$.

[8]By definition, $L_M(\omega) = L_M(\omega')$ for any $\omega' \in L_M(\omega)$.

[9]Notice that if $M = Q$, then there are $|\Omega|$ *M-local assets* corresponding to the standard Arrow securities. Alternatively, if $M = \emptyset$ then there is just one *M-local asset*, which corresponds to a riskless bond, $\mathbf{a}_{L_\emptyset}$, paying equally in all states.

$\mathcal{L}_{M_a}$	$(b,b,b),(b,b,g)$	$(b,g,b),(b,g,g)$	$(g,b,b),(b,b,g)$	$(g,g,b),(g,g,g)$
$\mathcal{L}_{M_b}$	$(b,b,b),(g,b,b)$	$(b,b,g),(g,b,g)$	$(b,g,b),(g,g,b)$	$(b,g,g),(g,g,g)$
$\mathcal{L}_{M_c}$	$(b,b,b),(b,g,b)$	$(b,b,g),(b,g,g)$	$(g,b,b),(g,g,b)$	$(g,b,g),(g,g,g)$

Table 1: Each asset class in Example 1 ignores one source of risk.

We consider a general setup where the market structure $\mathbf{J}_K$ is constructed from an arbitrary set K of different asset classes, in which each class k has a characteristic set $M_k \subseteq Q$,

$$\mathbf{J}_K = [\,\mathbf{A}_{M_k}\,]_{k\in K}. \tag{4}$$

Our model is entirely flexible on which M-asset classes make up market structure $\mathbf{J}_K$. On one extreme, we can accommodate a static version of the Bewley incomplete markets model, in which trades have to be constant across all states of the world, (Bewley, 1977; Aiyagari, 1994). Such a bond economy corresponds to a single asset class where M_k is the empty set: $\mathbf{J}_K = \mathbf{A}_\emptyset$. At the other extreme, a market that conditions on all risks is one where there exists a $k \in K$ such that $M_k = Q$. This market structure is not simple, since it allows for trades that depend on all risks. Our setup allows us to consider the interesting cases in between. A market structure could require that each risk be insured separately. Thus, K may contain a separate class for every $q \in Q$ that assigns the singleton $M_k = \{q\}$. Similarly, K could consist of all classes induced by the pairs of risk, where $M_k \subset Q$ is contained if $|M_k| = 2$. But K can contain any variety of classes k, where M_k need not have a fixed cardinality. We say that an market structure is *simple* if no class can complete the market on its own.

Definition 1 (**Simple Market Structures**). *A market structure, $\mathbf{J}_K$, defined in* (4) *is a simple market structure if $\nexists k \in K$ such that $M_k = Q$.*

Before characterizing the space of all trades compatible with simple market structures we consider an example with two agents which shows that simple assets severely limit the scope for insurance.

2.2 A Motivating Example

Consider a simplified instance of the setup described above with only three fundamental risks, $Q = \{q_r, q_s, q_t\}$. Variable q_r could represent a rainfall index, q_s shipping disruptions, and q_t timber prices. For simplicity, let each of them have two possible outcomes $Z_q = \{g, b\}$ and

	(b,b,b)	(b,b,g)	(b,g,b)	(b,g,g)	(g,b,b)	(g,b,g)	(g,g,b)	(g,g,g)
$\mathbf{y}_1$	4	2	2	4	2	4	4	2
$\mathbf{y}_2$	2	4	4	2	4	2	2	4

Table 2: Endowments $y_i(\omega)$ for a Simple Example

assume that the common belief p assigns equal probability to each of the eight states of the world: $p(\omega) = 1/8$.[10]

Assume that almost all joint risks can be traded, except for $M = Q$. That is, assume that market structure $\mathbf{J}_K$ contains instruments which condition on joint realization of rainfall and shipping $M_a = \{q_r, q_s\}$, of rainfall and timber $M_b = \{q_r, q_t\}$, and of shipping and timber $M_c = \{q_s, q_t\}$, respectively. The corresponding partitions of Ω are described in Table 1.

Let there be two agents, $i = 1, 2$ with endowments $\mathbf{y}_i$ and $\mathbf{y}_2$, described in Table 2. Since aggregate resources are state-independent (i.e. $y_1(\omega) + y_2(\omega) = 6$ for all ω), all Pareto optimal allocations imply full insurance. Consider a classical competitive equilibrium on ex-ante trades, supported by a state price vector $\mathbf{\Pi} \in \mathbb{R}^{|\Omega|}$, with the only difference being that all net-trades must belong to $\langle \mathbf{J}_K \rangle$.[11]

Even though both agents could enjoy full insurance by pooling their risks, the competitive equilibrium under the simple asset structure features no trade. This allocation is supported by a vector of constant state prices $\Pi(\omega) = 1$ for all ω. Comparing within and across asset classes, all cells in Table 1 are symmetric, featuring exactly one state in which income equals 4 and another one where it equals 2. Hence, the ratio of marginal expected utilities must be equated across any two cells, and no agent trades away from her endowment at constant prices. As a result, market structure $\mathbf{J}_K$ cannot sustain Pareto optimal allocations.

The insurance market does not break down because assets are coarse, but because they are *simple*. Coarse instruments can indeed be combined to compensate for missing Arrow securities. Consider a slight modification in the market structure. Replace asset class $k = c$ by an equally coarse (but not equally simple) partition $\mathcal{L}_{\text{mix}}$ shown in (5) below. The only difference with

[10] None of our results depend on equal state-probabilities.

[11] Formally, given initial endowment $\{\mathbf{y}_i\}_{i \in N}$, an allocation $\{\mathbf{c}_i\}_{i \in N}$ and a state-price $\Pi : \Omega \to \mathbb{R}$ form a *competitive equilibrium* if the following four conditions hold: (i) Budget constraints $\mathbf{\Pi}(\mathbf{y}_i - \mathbf{c}_i) \geq 0$ are satisfied for all $i \in N$; (ii) Markets are cleared $\sum_{i \in N} [y_i(\omega) - c_i(\omega)] = 0$ for all $\omega \in \Omega$; (iii) For every agent $i \in N$, net-trade $\mathbf{c}_i - \mathbf{y}_i \in \langle \mathbf{J}_K \rangle$ is optimal, $\mathbf{x} \succ_i \mathbf{c}_i \Longrightarrow \Pi(\mathbf{x} - \mathbf{y}_i) > 0$, among all *measurable* net-trades $\mathbf{x} - \mathbf{y}_i \in \langle \mathbf{J}_K \rangle$.

$\mathcal{L}_{M_c}$ is that states (b, g, b) and (b, g, g) switch cells.

$$\begin{array}{|c||c|c|c|c|} \hline \mathcal{L}_{\text{mix}} & (b,b,b),(\mathbf{b},\mathbf{g},\mathbf{g}) & (b,b,g),(\mathbf{b},\mathbf{g},\mathbf{b}) & (g,b,b),(g,g,b) & (g,b,g),(g,g,g) \\ \hline \end{array} \tag{5}$$

The asset classes associated with partitions $\mathcal{L}_{\text{mix}}$ and $\mathcal{L}_{M_c}$ both force payments to be constant across pairs of states. However, determining the payments of the former requires knowing the joint realization of all three risks q_r, q_s, and q_t, whereas the payments of the latter only depend on the outcome of q_s and q_t. Therefore, an M_c-local asset is simpler.

Now consider a financial market structure $\mathbf{J}_K = [\mathbf{A}_{M_a}, \mathbf{A}_{M_b}, \mathbf{A}_{\text{mix}}]$ which replaces asset class $\mathbf{A}_{M_c}$ with $\mathbf{A}_{\text{mix}}$. It can be shown that this market structure – with the same number of coarse instruments as the previous one – now corresponds to a *complete market*: $\langle \mathbf{J}_{\text{mix}} \rangle = \mathbb{R}^{|\Omega|}$. The competitive equilibrium therefore features full insurance, where $c_1(\omega) = 3 = c_2(\omega)$ in all $\omega = \Omega$ is supported by constant prices $\Pi(\omega) = 1$. Relative to the previous case, the current market structure leads to higher overall welfare. For instance, with logarithmic utility, the no-trade equilibrium with the original financial structure yields a certainty-equivalent consumption of 2.83 compared to 3 when $\mathcal{L}_{mix}$ is included. Even with a low coefficient of relative risk aversion, this small modification generates a sizable and economically significant increase in certainty-equivalent consumption.

3 Simple Insurance

We now describe the joint insurance possibilities obtained from combining simple asset classes. We show that incorporating additional asset classes to a market $\mathbf{J}_K$ enriches the payoff space in a particular – and ultimately very restrictive – direction. For clarity, we focus on the simplest possible case in which each variables $q \in Q$ represents a binary risk $Z_q = \{b, g\}$. However, all our results are proved for the general case with Z_q finite.

Consider a trade $\mathbf{x} \in \langle \mathbf{A}_{M_k} \rangle$ which uses assets of class k. By definition, any two states ω and ω' which belong to the same M_k-local state (i.e., $\omega' \in L_{M_k}(\omega)$) are indistinguishable in the sense that $x(\omega) = x(\omega')$. We can extend this notion to events. If two events E, E' of equal size belong to the same M_k-local state $L \in \mathcal{L}_{M_k}$, then any measurable trade $\mathbf{x}$ must be *balanced* between them: $\sum_{\omega \in E} x(\omega) = \sum_{\omega \in E'} x(\omega)$. When this happens, no combination of claims in $\mathbf{A}_{M_k}$ can transfer resources from E to E'. More generally, trades must be balanced as long as the two events *cover* every local-state equally. In this case, we say the events are *linked*.

Definition 2 (**Linked events**). *Two disjoint events E, E' are* linked *by partition $\mathcal{L}_{M_k}$ if, for every local state $L \in \mathcal{L}_{M_k}$, their intersection is of equal size: $|L \cap E| = |L \cap E'|$. In this case we write $E \sim_k E'$.*

If two events are linked by k, no asset trades of class k can shift resources across these events. This is the operational notion of insurance limits that we need in order to describe how adding an asset class to market $\mathbf{J}_K$ expands trades. To achieve this, we first show that any possible trade $\mathbf{x} \in \mathbb{R}^{|\Omega|}$ can be uniquely decomposed into simple unit transfers across linked events, which we call *joint bets*. We then show that no trade compatible with market $\mathbf{J}_K$ can involve a joint bet across events that are linked by all its asset classes $k \in K$. This allows us to completely characterize the set of $\mathbf{J}_K$-compatible trades (Proposition 1) and show that expanding the set of asset classes in $\mathbf{J}_K$ does not generate more precise trades (Corollary 1). With all this we prove that no simple market structure is complete (Proposition 2).

For any $M \subseteq Q$, define a pair of events which split Ω evenly into two sets E_M and E'_M, depending on whether an even or an odd number of risks in M assigns outcome b:

$$E_M \equiv \{\omega \in \Omega | \text{ an even number of } q \in M \text{ assign } q(\omega) = b\}, \tag{6}$$

where E'_M is defined analogously for odd.[12] In the example of Section 2.2, $E_{\{q_r\}}$ would be the event in which rainfall takes value g, and event $E'_{\{q_r\}}$ stands for weather b. Similarly, event $E_{\{q_r, q_s\}}$ is one in which rainfall and shipping both have the same realization, i.e., both take b or both g, while they have opposing realizations in $E'_{\{q_r, q_s\}}$.

Consider a *joint bet*, $\mathbf{w}_M$, which uniformly transfers one unit of consumption from the states in E'_M to those in E_M:

$$w_M(\omega) = \begin{cases} +1 & \text{if } \omega \in E_M \\ -1 & \text{if } \omega \in E'_M. \end{cases} \tag{7}$$

The full set of joint bets across M-linked events for every subset $M \subseteq Q$ produces the following matrix

$$\mathbf{W} = [\mathbf{w}_M]_{M \subseteq Q}\,. \tag{8}$$

$\mathbf{W}$ is an $|\Omega| \times |\Omega|$ matrix whose first column is the constant vector $\mathbf{w}_\emptyset = (1, 1, ..., 1)^T$, followed by joint bets across the M-specific linked events for nonempty subsets M. The following Lemma

[12] In the non-binary setting (i.e. $|Z_q| > 2$) we proceed similarly, but we construct more than two linked events per set M. Concretely, take any two realizations for $q \in M$ and define one as b the other as g. A pair of linked events can be constructed using (6) among those states whose q-th coordinates take values b or g. All details can be found in Appendix A.1.

shows that any two distinct joint bets are orthogonal $\mathbf{w}_M \perp \mathbf{w}_R$ and that any arbitrary payoff in $\mathbb{R}^{|\Omega|}$ can be expressed as a linear combination of joint bets.

Lemma 1. *The full set of joint bets across all linked events,* $\mathbf{W}$*, forms an orthogonal basis of* $\mathbb{R}^{|\Omega|}$.

Proof. By construction, $\mathbf{W}$ has $|\Omega| = 2^Q$ columns. It remains to show that for all $M, R \subseteq Q$ with $M \neq R$, the inner product $\mathbf{w}_M \cdot \mathbf{w}_R$ is Zero. By their definition in equation (7), the inner product between two joint bets subtracts the number of states in which both vectors pay equally from the states in which their payoffs differ in sign. Denote by $T = (M \cup R) \setminus (M \cap R)$ the set of variables q belonging to only one of the two sets. The inner product can be rewritten as $\mathbf{w}_M \cdot \mathbf{w}_R = |E_T| - |E'_T|$. Since E_T and E'_T are of equal cardinality, $\mathbf{w}_M \cdot \mathbf{w}_R = 0$. ■

Thanks to Lemma 1, we are now able to decompose any net-trade $\mathbf{x} \in \mathbb{R}^{|\Omega|}$ into a sum of contribution terms increasing in complexity, similar to the Sobol indeces used in functional ANOVA decompositions (Sobol, 1993),

$$\mathbf{x} = \sum_{M \subseteq Q} \alpha_M(x) \mathbf{w}_M. \tag{9}$$

We can obtain the weights in (9) by a simple application of orthogonal projection, where $\alpha_M(x)$ measures the response of trade $\mathbf{x}$ to the joint realization of $q \in M$,

$$\alpha_M(x) = \frac{\mathbf{x} \cdot \mathbf{w}_M}{\mathbf{w}_M \cdot \mathbf{w}_M} = \frac{1}{|\Omega|} \left(\sum_{\omega \in E_M} x(\omega) - \sum_{\omega \in E'_M} x(\omega) \right). \tag{10}$$

For instance, if $\alpha_{\{q,r\}}(x) = 0$, then $\mathbf{x}$'s reaction to q does not depend on the realization of r, and vice versa. More generally, $\alpha_M(x) = 0$ implies that $\mathbf{x}$ cannot make its response to $q \in M$ depend on the joint realization of *all* other variables in M. Intuitively, equations (9) and (10) show whether a trade pays on the interaction of multiple risks. Similarly, if an individual's income risk contains many higher-order terms, she needs flexible trades in order to reduce her exposure.

Consider a trade $\mathbf{x} \in \langle \mathbf{A}_{M_k} \rangle$ in class k. Are there joint risks $M \subseteq Q$ for which the interaction component in (9) is always zero? Notice that equation (10) assigns weight $\alpha_M(x) = 0$ to any M for which E_M and E'_M are linked by class k (since the two sums in (10) would be equal). We now determine for which sets M this is the case.

Lemma 2. *Events E_M and E'_M defined in (6) are linked by class k if and only if M is not contained in M_k*

$$E_M \sim_k E'_M \iff M \nsubseteq M_k.$$

Proof. Consider any variable q in M which does not belong to M_k. For every state $\omega \in E_M$, there exists another state $\omega' \in E'_M$ which only differs by the realization of variable q. These two states are indistinguishable according to $\mathcal{L}_{M_k}$. Hence E_M and E'_M must cover all of its cells equally. Conversely, if E_M and E'_M are linked by k, then for other M_k-local states to share elements with E_M and E'_M, there must exist a variable $q \in M$ which does not belong to M_k. If E_M and E'_M are linked by asset class k then M_k-measurable trades cannot condition on the joint realization of $q \in M$. Therefore $\sum_{\omega \in E_M} x(\omega) = \sum_{\omega \in E'_M} x(\omega)$ ■

Following Lemma 2, all M_k-measurable trades $\mathbf{x} \in \langle \mathbf{A}_{M_k} \rangle$ can be uniquely decomposed by (9) using only those interaction components associated with subsets $M \subseteq M_k$. In order to extend this argument to an entire collection K of asset classes, let $\mathcal{M}_K$ collect all subsets M contained in M_k for at least one $k \in K$:

$$\mathcal{M}_K = \{M \subseteq Q \,:\, \exists k \in K \text{ such that } M \subseteq M_k\}. \tag{11}$$

For instance, the set $\mathcal{M}_{\{k\}}$ collects all subsets of M_k, $\mathcal{M}_{\{k,k'\}}$ forms the union of $\mathcal{M}_k$ and $\mathcal{M}'_k$, and so forth.[13] In Section 2.2's example, for instance, the set $\mathcal{M}_K$ consists of all subsets of $Q = \{q_r, q_s, q_t\}$ except the grand set Q itself, since none of the three asset classes contained Q.

Take from $\mathbf{W}$ those joint bets associated to members of $\mathcal{M}_K$:

$$\mathbf{W}_K = [\mathbf{w}_M]_{M \in \mathcal{M}_K}. \tag{12}$$

By our previous argument, the span of this set must contain $\langle \mathbf{J}_K \rangle$. Indeed, we can now show that it is a minimal spanning set. As a result, we obtain a simple description of all trades compatible with market structure $\mathbf{J}_K$.

Proposition 1. *The joint bets in $\mathbf{W}_K$ form an orthogonal basis of the payoff space, $\langle \mathbf{J}_K \rangle = \langle \mathbf{W}_K \rangle$.*

Proof. We first show the result for $\mathbf{W}_K$ with a single class $K = \{k\}$. Note that $\mathbf{A}_{M_k}$ and $\mathbf{W}_{\{k\}}$ both have $2^{|M_k|}$ columns. Since all the columns in $\mathbf{W}_K$ are orthogonal, and therefore linearly

[13] This is not equivalent to the collection of subsets of $M_k \cup M'_k$ (except if one set weakly contains the other).

independent, we only need to show that every column in $\mathbf{W}_K$ belongs to $\langle \mathbf{A}_{M_k} \rangle$. However, by definition the set $\mathcal{M}_{\{k\}}$ consists of all subsets of M_k, hence, for all $M \in \mathcal{M}_{\{k\}}$, $\mathbf{w}_M$ lies in $\langle \mathbf{A}_{M_k} \rangle$. Finally, to show that the result holds for any K, notice that $\langle \mathbf{W}_K \rangle \subseteq \langle \mathbf{J}_K \rangle$ follows immediately from $\mathbf{w}_M \in \langle \mathbf{A}_M \rangle \subseteq \langle \mathbf{J}_K \rangle$ for all $M \subseteq \mathcal{M}_K$. Conversely, the spanning set of $\mathbf{J}_K$ consist of all local assets for any $k \in K$. But by the above result for $K = \{k\}$, each of these local assets belong to $\langle \mathbf{W}_{\{k\}} \rangle$. Since $\langle \mathbf{W}_K \rangle \supseteq \langle \mathbf{W}_{\{k\}} \rangle$ for all $k \in K$, we have $\langle \mathbf{W}_K \rangle \supseteq \langle \mathbf{J}_K \rangle$ and the result obtains. ∎

Proposition 1 implies redundancies between asset classes. Consider classes k and k', each conditioning on risks the other does not – and therefore generating net trades the other cannot. Combining the two classes $\mathbf{J}_{\{k,k'\}}$ allows for trades $\mathbf{x} \in \langle \mathbf{J}_{\{k,k'\}} \rangle$ which were neither available in $\langle \mathbf{J}_{\{k\}} \rangle$ nor in $\langle \mathbf{J}_{\{k'\}} \rangle$.[14] However, Proposition 1 shows that this expansion does not increase the set of joint bets. That is, any $\mathbf{w}_M$ available in the combined market must have been available in either $\langle \mathbf{J}_{\{k\}} \rangle$, $\langle \mathbf{J}_{\{k'\}} \rangle$, or both.[15] This impossibility of generating new joint bets is at the core of our subsequent results on the stark limits to improving the quality of insurance.

Our example in Section 2.2 showed that the novel trades available when merging a new class $\mathcal{L}_{mix}$ with an existing market structure allowed for finer contingent claims. Formally, we say that a unit claim $\mathbf{a}_E$ on event E is finer than $\mathbf{a}_{\widetilde{E}}$ if $E \subset \widetilde{E}$. In contrast, our next result shows that this is impossible when classes are simple.

Corollary 1. *Adding a class k' to a set of asset classes K does not generate finer local assets. Formally, every local asset $\mathbf{a}_{L_M(\omega)}$ in the expanded set $\langle \mathbf{J}_{K \cup \{k'\}} \rangle$ must have been available in $\langle \mathbf{J}_K \rangle$ or $\langle \mathbf{J}_{\{k'\}} \rangle$.*

Proof. By (9), any local asset $\mathbf{a}_{L_M(\omega)}$ puts non-zero weight $\alpha_R(a_{L_M(\omega)}) = (\mathbf{a}_{L_M(\omega)} \cdot \mathbf{w}_R)/(\mathbf{w}_R \cdot \mathbf{w}_R)$ on $\mathbf{w}_R$, if and only if $R \subseteq M$. Therefore, $\mathbf{a}_{L_M(\omega)}$ belongs to $\langle \mathbf{W}_{K \cup \{k'\}} \rangle = \langle \mathbf{J}_{K \cup \{k'\}} \rangle$ if and only if $\mathbf{w}_R$ belongs to it for all $R \subseteq M$. By Proposition 1, the expanded set of asset classes $K' = K \cup \{k'\}$ contains $\mathbf{w}_M$ if $M \subseteq \mathcal{M}_{K'}$. For this to be the case, it must be that either $M \subseteq \mathcal{M}_K$ or $M \subseteq \mathcal{M}_{\{k'\}}$. ∎

[14] For instance, the sum of two joint bets $\mathbf{w}_{M_k} + \mathbf{w}_{M_{k'}}$. By Lemma 6, payoff $\mathbf{w}_{M_k}$ can be generated by class k and payoff $\mathbf{w}_{M_{k'}}$ by class k'. Therefore, their sum must be available in the combined market structure $\mathbf{J}_{\{k,k'\}}$. But by (9), and the fact that both classes condition on variables the other does not, this combination $\mathbf{w}_{M_k} + \mathbf{w}_{M_{k'}}$ was not available in either class k or class k'. This follows from the fact that $\mathcal{M}_{\{k,k'\}} \supseteq \mathcal{M}_k$ and $\mathcal{M}_{\{k,k'\}} \supseteq \mathcal{M}_{k'}$.

[15] For any $M \subseteq Q$ not contained in M_k or in $M_{k'}$, the two events E_M and E'_M must still be linked. As a result, now new $\mathbf{w}_M$ can be generated.

	(b,b,b)	(b,b,g)	(b,g,b)	(b,g,g)	(g,b,b)	(g,b,g)	(g,g,b)	(g,g,g)
$\mathbf{y}_1$	3	5	5	5	5	5	5	15
$\mathbf{y}_2$	5	5	5	10	5	10	10	10

Table 3: Modified endowments $y_i(\omega)$ for Example 1: No trade

Corollary 1 provides a stark negative result. No matter how many coarse instruments from different assets classes in K are combined, the asset structure is never rich enough to compensate for the absence of local assets which do not belong to any class $k \in K$. By Corollary 1, whenever there exists an $M \notin \mathcal{M}_K$, then neither the M-local assets nor the trades $\mathbf{w}_M$ belong to $\langle \mathbf{J}_K \rangle$. In other words, all $\mathbf{J}_K$ markets are incomplete unless one asset class $k \in K$ directly conditions on all risks (i.e., $M_k = Q$). Extending this argument, we show that simple market structures can never condition on all risks simultaneously.

Proposition 2. *No simple market structure is complete.*

Proof. By Proposition 1 and Corollary 1, if $M \notin \mathcal{M}_K$, then no M-local assets belongs to $\langle \mathbf{J}_K \rangle$. Therefore, $\langle \mathbf{J}_K \rangle \subset \mathbb{R}^{|\Omega|}$ unless there exists a k with $M_k = Q$. ∎

Consider again Example 1 from Section 2.2. For agents 1 and 2 to pool and thereby eliminate their idiosyncratic income risk, agent 1 would need to give up 1 unit of consumption in all states where $y_1(\omega) = 4$ is high and receive 1 in those where $y_1(\omega) = 2$. Her endowment was chosen so that this trade is precisely $\mathbf{w}_Q$ (i.e. the *joint bet* across events that vary all risks). Since no simple asset class conditions on Q, the event E_Q remains linked with E'_Q in market structure $\mathbf{J}_K$. Thus, since agents 1 and 2 do not have access to contracts which condition on how the three sources of risk interact, they cannot realize their gains from trade.

The lack of flexibility in transferring risks can also lead to a sub-optimal sharing of aggregate risk. To see this, consider a modified version of the example where we vary agents' incomes, as shown in Table 3. Agent 1 enjoys constant income except at the two extreme states where either all shocks coincide to be b or g. Agent 2, on the other hand, has high income whenever two or more realizations are g. At the initial endowment, the marginal rates of substitution are not equated across states. Take states (b,b,b) and (b,b,g), for example. Agent 1 has to bear the aggregate shock on her own with no help from agent 2. On the other hand, between (g,b,b) and (g,b,g), all risk is borne by agent 2. However, there is no trade in equilibrium, since the marginal rates of substitution are equated across any two cells between the partitions that determine the asset structure.[16] By our decomposition result (10), the resulting uninsurable components in the income shocks amount to $\boldsymbol{\varepsilon}_1 = 1.5\mathbf{w}_Q$ and $\boldsymbol{\varepsilon}_2 = -1.2\mathbf{w}_Q$ for agents 1 and 2, respectively.[17]

4 Heterogeneous Constraints

So far, we have focused on the limits imposed by a simple market structure, $\mathbf{J}_K$. We now consider how heterogeneous trading restrictions jointly determine insurance outcomes if each of them is simple. There are many reasons why agents may be unable to trade against the joint realization of all risks, Q. As discussed in Section 1.1, lack of access to information, bounded rationality, legal constraints, or even physical transaction costs can all reduce the set of tradable risks.

Assume that each agent i is assigned a set $Q_i \subseteq Q$. This set represents i's trading possibilities. Formally, we assume that agent i's net-trades $x_i \equiv c_i - y_i$ must be *measurable* with respect to $\mathcal{L}_{Q_i}$.[18] To simplify things, we assume that each Q_i must include at least those risks which affect agent i's income. In other words, we assume that y_i is measurable with respect to $\mathcal{L}_{Q_i}$ and hence that income risk is not uninsurable by assumption. Our decomposition results in Section 4.3 extend readily to an arbitrary income $y_i(\omega)$.

Agents must transfer income across states via net-trades $\mathbf{x}_i$ that are not only compatible with the market $\mathbf{J}_K$ (as before) but also compatible with their trading restrictions, Q_i. Since we want to focus our attention on how individual trading constraints interact, we assume for now that the market structure $\mathbf{J}_K$ is complete. We consider the interaction of coarse market structures and heterogeneous financial constraints at the end of section 4.2.

Denote by X_i the subspace of trades satisfying i's access constraints. Formally, let

$$X_i = \langle \mathbf{A}_{Q_i} \rangle. \tag{13}$$

We call X_i the set of i's *compatible* trades. If agent i trades away from her initial endowment,

[16]On any cell of the partitions in Table 1, the ratio of expected marginal utilities between agent 1 and 2 equals 4/3 using endowments in Table 3. For instance, $Eu'(y_1(\omega)|\{(b,b,b),(b,b,g)\}) = 1/2(1/3+1/5) = 4/15$ while $Eu'(y_2(\omega)|\{(b,b,b),(b,b,g)\}) = 1/2(1/5+1/5) = 3/15$.

[17]The projection weight $\alpha_Q(y_1) = 1.5 = (1/8)(30-18)$ is positive since rainfall $q_r(\omega) = g$ increases 1's income more if the two remaining shocks are aligned between them (both assigning g or both b).

[18]If $Q_i = \emptyset$, the set $\mathcal{L}_\emptyset = \{\{\Omega\}\}$ corresponds to the trivial partition. In this case, agent i is unable to distinguish any two states. At the other extreme, $Q_i = Q$ implies that $\mathcal{L}_{Q_i} = \{\{\omega\}_{\omega\in\Omega}\}$ partitions Ω into singleton cells, representing an agent without trading restrictions.

	(b,b,b)	(b,b,g)	(b,g,b)	(b,g,g)	(g,b,b)	(g,b,g)	(g,g,b)	(g,g,g)
f_{Q_1}	1	1	1	1	1	1	3	3
f_{Q_2}	1	1	1	3	1	1	1	3
f_{Q_3}	1	1	1	1	1	3	1	3

Table 4: Example 3, Technology

for markets to clear, the remaining agents must collectively meet her demand. Therefore, since $\mathbf{x}_i = -\sum_{j\neq i} \mathbf{x}_j$ some of i's compatible trades may fail to satisfy the constraints of the remaining agents.[19] The space of trades satisfying the constraints of agents $I \subseteq N$ can be easily represented by a market structure, as we did for $\mathbf{J}_K$ in section 3:

$$\mathbf{J}_I = [\mathbf{A}_{Q_i}]_{i\in I}.$$

Therefore, a net trade $\mathbf{x}_i \in \langle \mathbf{A}_{Q_i} \rangle$ is feasible if it can be constructed by the remaining agents $\mathbf{x}_i \in \langle \mathbf{J}_{N\setminus\{i\}} \rangle$.[20] Accordingly, we can define i's feasible trades as follows:

$$X_i^* = \langle \mathbf{A}_{Q_i} \rangle \cap \langle \mathbf{J}_{N\setminus\{i\}} \rangle. \tag{14}$$

This simple setup presents several important questions. Under which circumstances are all of i's compatible trades also feasible (i.e. $X_i^\star = X_i$) and when are they not (i.e. $X_i^\star \subset X_i$) ? How does this depend on the structure of others' trading restrictions (i.e. on the Q_j's for $j \neq i$)? What are the implications for agent i's insurance possibilities and individual welfare? Is i able to trade away all of her income risk in any feasible allocation? We provide answers to all these questions by fully characterizing $X_i^\star$ as a function of $\mathbf{J}_K$ and $\{Q_i\}_{i\in N}$. First, we consider a simple example with 4 agents which shows how constraints interact and limit the scope for mutual insurance.

4.1 An Example with Spillovers

Consider an extended version of Example 1 in Section 2.2. Assume three agents $i = 1, 2, 3$, each with a different trading constraint: $Q_1 = \{q_r, q_s\}$, $Q_2 = \{q_r, q_t\}$, $Q_3 = \{q_s, q_t\}$. These constraints correspond to the three asset classes in the market structure from Example 1 shown

[19] We assume that resource constraints, $\sum_i x_i(\omega) = 0$, $\forall\omega \in \Omega$, hold with equality. Given the monotonicity of preferences, this is without loss of generality in any competitive equilibrium, but also holds in various other assignment mechanisms such as Nash bargaining.

[20] We use the linear subspace property $-\mathbf{x} \in \langle \mathbf{J}_{N\setminus\{i\}} \rangle \Leftrightarrow \mathbf{x} \in \langle \mathbf{J}_{N\setminus\{i\}} \rangle$.

	(b,b,b)	(b,b,g)	(b,g,b)	(b,g,g)	(g,b,b)	(g,b,g)	(g,g,b)	(g,g,g)
$\mathbf{c_1}$	**0.71**	**0.71**	**0.84**	**0.84**	**0.84**	**0.84**	**1.93**	**1.93**
y_1	0.75	0.75	0.75	0.75	0.75	0.75	2.25	2.25
y_4	0.75	0.75	0.75	1.25	0.75	1.25	1.25	2.25

Table 5: Example 4, Preview of the shareholder equilibrium.

in Table 1.

Imagine every individual engaging in a productive activity related to the sources of risk relevant to her. Specifically, let the couple of risks in Q_i fix i's income via perfect substitutes $y_i(\omega) = \min_{q \in Q_i} q(\omega)$. Table 4 shows this. Setting $q(b) = 1$ and $q(g) = 3$ for all $q \in Q$, each technology corresponds to a Leontief production function $f_{Q_i}(\omega) = \min\limits_{q \in Q_i} q(\omega)$, a special case of a constant elasticity of substitution technology.[21]

All agents could gain from pooling the risk involved with having one rather than two good realizations. For example, agent 1 could shift resources from cell $\{(g,g,b),(g,g,g)\}$ to $\{(g,b,b),(g,b,g)\}$ without violating her own trading constraints Q_1. Instead, the unique equilibrium in this example prescribes that all individuals remain in autarky.[22] This happens because every trade that leads to a mutual improvement conflicts with the constraint of at least one other agent. Any equilibrium trade $\mathbf{x}_1$, for example, must be reciprocated by a combination of trades available to agents 2 and 3. But while $\mathbf{A}_{Q_2}$ contains trades which target q_s and $\mathbf{A}_{Q_3}$ trades which target q_r, none of these trades can be combined to target a particular joint realization of q_r and q_s. That is, although the trade described above is compatible with i's *individual constraints*, it is incompatible with *feasibility*. As a result, $X_1^\star \subset X_1$ and trading restrictions *spill over* because the market must discipline agent 1 to demanding insurance which respects the constraints of her counterparties.

Consider a version of the previous economy in which there is a fourth individual $i = 4$. Assume that he is not constrained, $Q_4 = \{q_r, q_s, q_t\}$, and entitled to 25% of each of the three productive activities. That is, let $y_i(\omega) = \frac{3}{4} f_{Q_i}(\omega)$ for $i = 1,2,3$ and $y_4(\theta) = \sum_{i=1}^{3} \frac{1}{4} f_{Q_i}(\omega)$. Table 5 describes income for agents 1 and 4. If markets were efficient, the heterogeneity between the four different individuals would be inconsequential. That is, since all states are equally likely

[21] Note that we obtain the function by letting $\rho \to -\infty$ with $f_{Q_1}(\omega) = \left(\frac{1}{2} q_r(\omega)^\rho + \frac{1}{2} q_s(\omega)^\rho\right)^{1/\rho}$. The no trade property shown below holds for any substitution parameter ρ.

[22] Take for instance logarithmic utility. Set state prices equal to $\Pi(\omega) = 1$ except when ω assigns g exactly once, in which case set $\Pi(\omega) = 5$. It is easy to check that across any two cells of i's partition, the ratios of marginal utility equal the ratio of event prices. For instance $\frac{u'(y_1(L_{\{q_r,q_s\}}(b,b,b))}{u'(y_1(L_{\{q_r,q_s\}}(g,g,g))} = \frac{1}{3} = \frac{(\Pi(b,b,b)+\Pi(b,b,g))}{(\Pi(g,g,b)+\Pi(g,g,g))} = \frac{1+1}{1+5}$.

and preferences are homogeneous, all agents would equally share the joint output in equilibrium $c_i(\omega) = \sum_{i=1}^{3} \frac{3}{4} f_{Q_i}(\omega)$ and agent 4 would not facilitate additional trades.

However, agent 4 extends the set of available trades if the other agents are constrained, as reported in the equilibrium consumption c_i under logarithmic utility in Table 5. Thanks to 4's presence, agents $i = 1, 2, 3$ no longer remain in autarky, but manage their exposure to the joint risks in Q_i. Even with a modest degree of relative risk aversion, the new trades are sizeable and are used to address joint risks. We discuss spillovers in detail in Section 4.2

To summarize, Example 3 shows that even as agent 1 is only restricted against trading q_t, others' constraints further limit *how* she can respond to q_r and q_s. In the following section, we will apply the results from Section 3 to characterize the effective trading spaces X_i^* and when they are (not) constrained by spillovers, $X_i^\star = X_i$.

4.2 Spillovers and Resilience

Formally, $X_i^* \subseteq X_i$ forms another linear subspace of $\mathbb{R}^{|\Omega|}$. Notice that X_i^* allows i to trade a joint risk M if it contains the corresponding contingent claim space $\langle \mathbf{A}_M \rangle \subseteq X_i^*$. In this case, we say that i can *address* this joint risk. Moreover, if $X_i^* = \langle \mathbf{A}_{M_i} \rangle = X_i$, there are no spillovers. In this case, we say that i is *resilient*.

Definition 3 (**Resilience**). *Agent i can* address *a joint risk $M \subseteq Q_i$ if the remaining group $N \setminus \{i\}$ can jointly trade M, i.e., $\langle \mathbf{A}_M \rangle \subseteq X_i^*$. If i can address Q_i, we say that i is* resilient.

A *resilient* agent does not face additional restrictions beyond her own trading constraint Q_i. As long as i's income $\mathbf{y}_i$ satisfies i's measurability constraint, resilience implies that she can sell all her income risk to the market. Conversely, agents that are not resilient $X_i^\star \subset X_i$ may end up holding non-diversifiable risk in any equilibrium. For some economies, it is immediately clear whether i is resilient. For instance, if there is an individual j who can trade every risk that i can (i.e., $Q_i \subseteq Q_j$), then all trades available to i must be available to j: $\langle \mathbf{A}_{Q_i} \rangle \subseteq \langle \mathbf{A}_{Q_j} \rangle \subseteq \langle \mathbf{J}_{N \setminus \{i\}} \rangle$. In this case we say that j covers i.

Definition 4 (**Cover**). *Agent i is covered if there exist an agent $j \neq i$ such that $Q_i \subseteq Q_j$.*

In order to characterize X_i^* for any arbitrary set of trading constraints, $\{Q_i\}_{i \in N}$, we proceed as in section 3. We show that any trade $\mathbf{x} \in X_i^\star$ can be described as the combination of joint bets $\mathbf{w}_M \in \mathbf{W}$ across events that are not linked by i but also not linked by the remaining agents. The collection of all such joint bets contains those elements of $\mathbf{W}_{\{i\}}$ which also appear in $\mathbf{W}_{\{j\}}$ for at least one $j \neq i$. Formally, we obtain the matrix $\mathbf{W}^*_{\{i\}} = [\mathbf{w}_M]_{M \in \mathcal{M}^*_i}$ where

$$\mathcal{M}^*_{\{i\}} \equiv \mathcal{M}_{\{i\}} \cap \mathcal{M}_{N \setminus \{i\}} \tag{15}$$

is defined similarly to $\mathcal{M}_N$ in (11). Following Section 3 we show that $\langle \mathbf{W}^*_{\{i\}} \rangle = X^*_i$. This leads to the following result.

Proposition 3. *Agent i can address a joint risk $M \subseteq Q_i$ if and only if there exist an agent $j \neq i$ who can also address M. Therefore, agent i is resilient if and only if i is covered.*

Proof. We need to show $X^*_i = \langle \mathbf{W}^*_{\{i\}} \rangle \supseteq \langle \mathbf{A}_{\{M\}} \rangle$ if and only if there exists a $j \neq i$ with $M \subseteq Q_i \cap Q_j$. First, note that the presence of j with $Q_i \cap Q_j \supseteq M$ implies $M \subseteq \mathcal{M}_{\{i\}} \cap \mathcal{M}_{\{j\}} \subseteq \mathcal{M}^*_i$, where the last inclusion uses the definition (15). Hence, $\langle \mathbf{A}_M \rangle \subseteq \mathbf{W}^*_{\{i\}}$. Finally, note that for $M \in \mathcal{M}^*_{\{i\}}$, there must exist a $j \in N$ such that $M \in \mathcal{M}_{\{j\}}$. ■

In contrast to X_i, the set of feasible trades, X^*_i, are not represented as contingent claims on a single partition of Ω. Indeed, if two sets M and M' belong to $\mathcal{M}^*_i$ the agent may be able to generate payoffs which are not measurable with respect to either of the corresponding partitions. However, she will typically be unable to generate payoffs which condition on the joint realization of $M \cup M'$. More formally, this means that combining the M-local and M'-local assets generates a space which is greater than the union of the respective measurable trades but smaller than conditioning on the coarsest common refinement: $\langle \mathbf{A}_M \rangle \cup \langle \mathbf{A}_{M'} \rangle \subseteq \langle [\mathbf{A}_M, \mathbf{A}_{M'}] \rangle \subseteq \langle \mathbf{A}_{M \cup M'} \rangle$.

Finally, we can also consider how individuals' trading constraints interact with market structure $\mathbf{J}_K$. Our results from the previous sections can be readily applied to characterize the effective trading possibilities $X^{**}_i = X^*_i \cap \langle \mathbf{J}_K \rangle$. The sets of tradable joint risks $\mathcal{M}^{**}_{\{i\}}$ for i simply correspond to those which are simultaneously compatible with the market constraints $\mathcal{M}_K$ from (12) and the spillovers $\mathcal{M}^*_{\{i\}}$ from (15), so that only the risks $\mathcal{M}^{**}_{\{i\}} = \mathcal{M}^*_{\{i\}} \cap \mathcal{M}_K$ can be traded by i.

4.3 Risk Decomposition

Recall that we have assumed $\mathbf{y}_i$ satisfies i's trading constraint. This guarantees that the set of compatible payoffs C_i – which contains the elements $\mathbf{c}_i = \mathbf{y}_i - \mathbf{x}_i$ – is fully described by the set of compatible net trades X_i (i.e., $X_i = C_i$). However, the set of feasible trades does not include all compatible trades (i.e., $X^\star_i \subset X_i$) unless i is totally resilient. Therefore, agent

i's income may vary across events that i cannot feasible trade against. The set of i's feasible payoff profiles C_i^* satisfies

$$C_i^* = \mathbf{y}_i + X_i^*. \tag{16}$$

and it is possible that $X_i^* \neq C_i^*$. We therefore investigate whether i is exposed to *uninsurable* endowment risk whenever others' constraints do not allow i to diversify this risk.[23]

Any payoff $\mathbf{x} \in \mathbb{R}^{|\Omega|}$ can be easily decomposed into a component $\mathbf{x}^*$ which belongs to X_i^* and an orthogonal residual ε which cannot be traded in a market-clearing allocation,

$$\mathbf{x}_i^* = \sum_{M \in \mathcal{M}_i^*} \alpha_M(x)\mathbf{w}_M,$$

and

$$\varepsilon = \sum_{M \notin \mathcal{M}_i^*} \alpha_M(x)\mathbf{w}_M,$$

where the weights from (10) are governed by the response to a joint shock in M.

Consider again the examples from Section 4.1. In Example 3, agent 1's suffers from spill overs which are binding in equilibrium. While aggregate resources increase gradually as the number of good realizations g surpass one, her own income remains constant before abruptly jumping when q_r and q_s are both equal to g. Her own trading restrictions do allow her to move consumption from the four extreme states where q_r and q_s are aligned, $E_{Q_1} = \{(b,b,b),(b,b,g),(g,g,b),(g,g,g)\}$, to those where they are opposed, E'_{Q_1}. Indeed, for her, E'_{Q_1} is worse since it fixes her income to the lowest possible level. However, since the remaining agents $j = 2,3$ link the two events $E_{Q_1} \sim_j E'_{Q_1}$, this component of her income risk, $\varepsilon = 0.5\mathbf{w}_{Q_1}$, cannot be passed on to others.

In contrast, in Example 4, agent 4 does not link events which are relevant to individual. Indeed, none of the agents (including agent 4) has an uninsurable income component. In equilibrium, agents $i = 1,2,3$ trade across events E_{Q_i} and E'_{Q_i} so that their constrained consumption now increases with every good realization in Q_i. This is reported for agent 1 in Table 6, but holds symmetrically for agents 2 and 3. In order to allow for this, agent 4 lowers consumption in states where there is a single good realization. In fact, aggregate resources remain low but two of his trading partners enjoy an increase in consumption. As usual when a small share of the population has to be incentivized to bear the majority of the social risk, prices need to

[23]Formally, if $\mathbf{y}_i$ does not belong to X_i^*, then C_i^* becomes an affine space which shifts the subspace X_i^* to pass through $\mathbf{y}_i$.

	(b,b,b)	(b,b,g)	(b,g,b)	(b,g,g)	(g,b,b)	(g,b,g)	(g,g,b)	(g,g,g)
y_1	0.75	0.75	0.75	0.75	0.75	0.75	2.25	2.25
$\mathbf{c_1}$	**0.71**	**0.71**	**0.84**	**0.84**	**0.84**	**0.84**	**1.93**	**1.93**
y_4	0.75	0.75	0.75	1.25	0.75	1.25	1.25	2.25
$\mathbf{c_4}$	**0.86**	**0.61**	**0.61**	**1.38**	**0.61**	**1.38**	**1.38**	**3.20**
$\mathbf{\Pi}$	**1**	**1.43**	**1.43**	**0.62**	**1.43**	**0.62**	**0.62**	**0.27**

Table 6: Example 4, Equilibrium allocations and prices in the shareholder economy.

become extreme, similar to what Gul et al. (2017) find related to endogenous categorization. Table 6 shows that prices also become non-monotone in aggregate risk. Even though aggregate resources are constant between (b, b, b) and (b, b, g), prices must increase for agent 4 to clear the market.

5 Extending to a General State Space

Our results in Section 3 and 4 show that synthesizing precise instruments from a variety of simple elements is difficult. None of these results rely on the binary setting. In Appendix A.1, we shows that all of our results and extend to the case where each variable $q \in Q$ has an arbitrary finite range Z_q with $\Omega = Z_1 \times ... \times Z_Q$. The difference is now that a partition $\mathcal{L}_M$ links not only a single characteristic pair of events but several of them since risks can vary between any two elements in their range Z_q. We provide both an orthogonal basis $\mathbf{W}$ to mirror the proofs of Section 3 and an alternative basis $\mathbf{B}$ of simple unit claims which may be of independent interest and is more intuitive. Moreover, Proposition 4 shows that Ω need not correspond to a product space. If agents only attach positive probability to a particular set of joint realizations, our results extend as long as a minimum amount of conditional risk remains.

But the nature of our results can be made transparent on a more abstract level with arbitrary state spaces and partitions. For each class, different sets of states could be indistinguishable for various reasons, related to information costs, lack of expertise, skills, or the legal means to trade across them. Let $\mathcal{L}^k$ be an arbitrary partition with cells $L^k(\omega)$ and let $\mathbf{a}_{L^k(\omega)}$ be a unit claim in the partitional spanning set $\mathbf{A}_{L^k}$ analogous to (3).

Clearly, if ever a collection K induces a common pair of linked events, the market remains incomplete. To see that this is not a pathological feature of our Q-induced product space, consider any two mutually exclusive events E and E' with $|E| = |E'|$ in an abstract state space Ω. Assume that there exists a class k which allows for a bijection between the two sets $\mu^k : E \to E'$, where $\mu^k(\omega) = \omega'$ means that the two states belong to the same cell $\{\omega\} \sim_k \{\omega'\}$. In this case, the events E and E' must be linked and any trade must be balanced between them. Note that, for any such pair of events, the number of possible bijections μ, each associated with a partition, is $|E|!$.

To illustrate, consider the binary example with $Q = \{r, s, t\}$. With the full set of classes conditioning on pairs of risk, $M_a = \{q_r, q_s\}$, $M_b = \{q_s, q_t\}$, $M_c = \{q_r, q_t\}$, the two linked events were $E_Q = \{\omega_1, \omega_4, \omega_6, \omega_7\}$ and $E'_Q = \{\omega_2, \omega_3, \omega_5, \omega_8\}$. Even if we only limit attention to partitions which lump eight states into four pairs, 4! among them link E_Q and E'_Q. Put differently, even after adding twenty-one new abstract asset classes to the three above, the market may still not be complete.

Conversely, our results can be used to identify sparse asset structures, consisting of coarse instruments only, which promote resilience. If a pair of events is linked by an existing financial structure, a novel coarse class can break this link if one of the cells in its partition intersects unequally with E and E'. For instance, in the example in Table 1, adding a fourth class through a coarse partition $\mathcal{L} = \{E_Q, E'_Q\}$ would break the link and complete the market. Similarly, in Example 1, we used that replacing class $k = c$ with $\mathcal{L}_{mix} = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}, \{\omega_5, \omega_7\}, \{\omega_6, \omega_8\}\}$ completes the market. To see why it does, note that between classes $k = a, b$, there are two pairs of commonly linked events. The first pair is $E_{M_c} \sim_k E'_{M_c}$, where $E_{M_c} = \{\omega_1, \omega_3, \omega_6, \omega_8\}$ and $E'_{M_c} = \{\omega_2, \omega_4, \omega_5, \omega_7\}$. The second pair is $E_Q \sim_k E'_Q$. To see why the former link is broken, notice that cell $\{\omega_6, \omega_8\} \subset E_{M_c}$ does not intersect with E'_{M_c}. Similarly, $\{\omega_1, \omega_4\} \subset E_Q$ does not intersect with E'_Q.[24]

To summarize, our results show that in the context of coarse financial structures, be they induced by fundamental risks or not, resilience is limited through events being linked. Addressing these limitations may only require targeted and coarse interventions.

6 Conclusion

We have analyzed the scope for sharing risk through a combination of simple arrangements, each measurable with respect to a partition. We showed that if the partitions are induced by subsets of payoff-relevant variables, there is a systematic underprovision of insurance against joint risks. We showed that the same mechanism can explain spill overs between individuals if

[24]Alternatively, we could have argued with $\{\omega_5, \omega_7\} \subset E'_{M_c}$ and $\{\omega_2, \omega_3\} \subset E'_Q$, respectively.

they face differential trading constraints. The transfer of income risk is limited by the trading possibilities of the most flexible individual.

Our results suggest that expanding the variety of financial products need not allow for better insurance against extreme events. Instead, managing an agent's exposure to how multiple risks interact requires a small number of specific contracts or interventions.

The present framework could be extended to allow agents some degree of control over their own constraints Q_i, akin to the rational inattention literature. Similarly, agents may enable or prevent access of others at a cost. Our examples show that heterogeneity in constraints can have important distributional effects. In our example 4, the certainty-equivalent consumptions of agents $i = 1, 2, 3$ decrease by 4.6% compared to the first best while the unconstrained agent 4's equivalent increases by 6.3% due to the constraints of others.

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A Generalized State Space

A.1 Arbitrary Random Variables

Consider the case where each variable can have a finite range Z_q which may be q-specific while maintaining the assumption that each state in the product space occurs with non-zero probability $\Omega = Z_1 \times ... \times Z_Q$. All of the concepts we presented can be suitably adapted for this specification. First, our results on the linear dependence between different asset classes in Section 3 were using an argument of linear dependence on linked events. More concretely, combining a set of classes where none of them conditions on M produces payoffs which are constant across events linked by set M. If each q takes an arbitrary number of values, the same result holds, only that a conditioning subset M typically induces not only one pair, but a variety of pairs of linked events. Take any two realizations for $q \in M$, defining one as q_b the other as q_g, then a pair of linked events can be constructed by adapting (6). Put differently, take any two states ω_b and ω_g with the property that all variables $q \in M$ differ across them, $q(\omega_b) \neq q(\omega_g)$. Then $E_M(\omega_b, \omega_g)$, which contains all M-local states in which all $q \in M$ take values q_b or q_g and the number of b realizations is even. Similarly, $E'_M(\omega_b, \omega_g)$ is defined according to the odd-numbered criterion.

Below, we show that generalized versions of the orthogonal bases $\mathbf{W}$ and $\mathbf{W}_K$ exist. However, for ease of exposition, we prove the general results using a non-orthogonal basis of local assets, $\mathbf{B}$ and $\mathbf{B}_K$, respectively. They share the features underlying our proofs above. Namely that $\langle \mathbf{B}_K \rangle = \langle \mathbf{J}_K \rangle$, where $\mathbf{B}_K$ selects exactly those columns of $\mathbf{B}$ induced by $\mathcal{M}_K$. Moreover, we show that in order to replicate a local asset $\mathbf{a}_{L_M(\omega)}$ using the spanning set $\mathbf{B}$, positive weight is put on the basis vectors defined through set M.

Constructing suitable basis vectors requires two auxiliary objects. First, consider any references state ω_0. For the sake of exposition, we select a prominent state $\omega_0 \in \arg\min_{\omega \in \Omega} \sum_i y_i(\omega)$ which minimizes aggregate income and we refer to ω_0 as *the* worst state and the corresponding $q(\omega_0) = q_b$ as *the* worst realization of variable q. Second, for every possible subset $M \subseteq Q$, we collect the set of states which assign this worst realization q_b to all risks q outside of M but to none inside of set M:

$$O_M \equiv \{\omega \in \Omega \mid \forall q \in M : q(\omega) \neq q_b \text{ and } \forall r \notin M : r(\omega) = r_b\}. \tag{17}$$

Clearly, any state can be uniquely described by the two unique pieces of information: the set

M of variables which do not obtain their worst realization, and their M-local state $L \in \mathcal{L}_M$.

For every group M, the number of M-local states, $|\mathcal{L}_M|$ – each of which defines a particular realization of risks in group M – corresponds to the number of elements in the sets O_K (as defined in (17)) across all its subsets $K \subseteq M$. To construct our basis $\mathbf{B}$, we collect, for each possible subset $M \subseteq Q$, exactly the subset of M-local assets $\mathbf{a}_{L_M(\omega)}$ induced by O_M:

$$\mathbf{B} = \left[\left[\mathbf{a}_{L_M(\omega)} \right]_{\omega \in O_M} \right]_{M \subseteq Q}. \tag{18}$$

To prove that $\mathbf{B}$ is indeed a basis for $\mathbb{R}^{|\Omega|}$, we need to show that it consist of $|\Omega|$ linearly independent assets. Lemma 5 shows that $\mathbf{B}$ has exactly $|\Omega|$ columns. Intuitively, each ω points to exactly one column $\mathbf{a}_{L_M(\omega)}$ where M is defined by the variables whose value is different from $\underline{z}_q$. The following lemma proves that the set of columns is indeed linearly independent.

Lemma 3. *The column vectors in* $\mathbf{B}$ *form a basis of* $\mathbb{R}^{|\Omega|}$*.*

To see why none of the assets in $\mathbf{B}$ are redundant, recall that each of them corresponds to a contingent claim on an event in which all the variables in a set M take a better-than-worst realization. Moreover, notice that only observing a set $K \not\supseteq M$ of variables does not reveal the joint realization of M. Therefore, no linear combination of any such $K \not\supseteq M$-local instruments can ever replicate a finer M-local asset. At the same time, none of the finer $K \supset M$-local assets which belong to $\mathbf{B}$ pay in states where any $q \in K$ takes its worst value. However, in order to replicate an M-local asset requires a payment in states that do assign the worst value to the variables $K \setminus M$. As a result, no linear combination of these K-local assets with $K \supset M$ can ever generate a payoff that is measurable with respect to $\mathcal{L}_M$. Indeed, we can state the following result.

Lemma 4. *Every* M*-local asset corresponds to a linear combination of columns in* $\mathbf{B}$ *associated with sets* $R \subseteq M$ *with non-zero weight on at least one* M*-local asset in* $\mathbf{B}$*.*

We are now presented with the convenient property of the $\mathbf{W}$ basis in the main section that every partition can be associated with a subset of the spanning vectors. Proceeding analogously, construct for each K a submatrix of $\mathbf{B}$ which only keeps those M-local assets associated with $\mathcal{M}_K$ from (11)

$$\mathbf{B}_K = \left[\mathbf{a}_{L_M(\overline{\omega})} \right]_{M \in \mathcal{M}_K}. \tag{19}$$

We are now ready to state the desired result.

Lemma 5. *The local assets in* $\mathbf{B}_K$ *form a basis of the payoff space,* $\langle \mathbf{J}_K \rangle = \langle \mathbf{B}_K \rangle$.

Thanks to Lemmas 3, 4, and 5, all our results from Section 3 extend: the absence of M-local assets can never be compensated. That is, the collection of simple asset classes is complete if and only if one of them is complete individually. Analogously, all results on resilience and spill overs apply.

Orthogonal Basis

Consider first the binary case and (8), where each q takes a value in $\{b, g\}$. Define for each $q \in Q$ and every $\omega \in \Omega$, the following function:

$$h(\omega) = \begin{cases} -1 & \text{if } q(\omega') = b \\ +1 & \text{otherwise.} \end{cases}$$

Given h, we can rewrite the joint bet $\mathbf{w}_M$ as $w_M(\omega) = \prod_{q \in M}$ for all nonempty sets M. In the case of the three variables $|Q| = 3$, the orthogonal matrix is as follows:

$$\mathbf{W} = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

In the general setting when each q takes an arbitrary number of realizations $|Z_q| = Z_q$, an orthogonal basis can be constructed on the same principles. It should correspond to a simple trade which transfers resources to a given joint realization based on the interaction of variables. For each $q \in Q$ denote the realization in the reference state ω_0 by b_q. Then, fixing a state $\omega \in \Omega$ and any variable q, the following function $h_{L_{\{q\}}(\omega)}$ assigns a number based on each realization of q:

$$h_{L_{\{q\}(\omega)}}(\omega') = \begin{cases} 1 - Z_q & \text{if } q(\omega') = b_q \\ 1 + \sqrt{Z_q}(Z_q - 2) & \text{if } q(\omega') = q(\omega) \\ 1 - \sqrt{Z_q} & \text{otherwise.} \end{cases}$$

If $Z_q > 2$, the function $h_{L_{\{q\}(\omega)}}$ can be interpreted as a trade which, for a fixed payment of $(1 - \sqrt{Z_q})$, shifts resources from the local states where q assigns the same realization as in ω to the worst local state where q agrees with $\omega_0 = (b_q)_{q \in Q}$. Indeed, the trade can be written as

$$\boldsymbol{h}_{L_{\{q\}(\omega)}} = ((Z_q - 1)\sqrt{Z_q})\mathbf{a}_{L_{\{q\}(\omega)}} - (Z - \sqrt{Z_q})\mathbf{a}_{L_{\{q\}(b_q)}} + (1 - \sqrt{Z_q})$$

In the binary case, the function simplifies to what we had above when interpreting $b_q = b$. The orthogonal matrix $\mathbf{W}$ can then be obtained by adapting (8) to

$$w_{L_M(\omega)}(\omega') = \begin{cases} \prod_{q \in M} h_{L_{\{q\}(\omega)}}(\omega') & \text{if } M \neq \emptyset \\ 1 & \text{if } M = \emptyset. \end{cases}$$

and

$$\mathbf{W} = \left[\left[\mathbf{w}_{L_M(\omega)} \right]_{\omega \in O_M} \right]_{M \subseteq Q}. \tag{21}$$

Lemma 6. *The elements in* $\mathbf{W}$ *defined in* (21) *form an orthogonal basis of* $\mathbb{R}^{|\Omega|}$.

A.2 Perfect Correlations

Our assumption of a rich state space $\Omega = Z_1 \times \cdots Z_Q$ amounts to the joint assumption of two properties. The first requires the set of fundamental risks to be exhaustive, so that the finest partition $\mathcal{L}_Q$ is the trivial partition with singleton cells. In other words, observing the joint realizations of the risks in Q is sufficient to identify the true state. The second property requires that all joint realizations occur with positive probability. This implies that any refinement must be strict, with $M \supset M' \Rightarrow \mathcal{L}_M < \mathcal{L}_{M'}$. In other words, observing the joint realizations of the risks in Q is necessary to identify the true state.

While the first condition merely requires an accurate specification of Q relative to Ω, the second is indeed restrictive. Without it, even observing a strict subset of variables $M \subset Q$ could reveal some states of the world. Such a situation arises when there is a perfect conditional correlation between some of the variables in Q.

However, our qualitative results on completeness and resilience hold if this revelation is imperfect. Specifically, assume alternatively that the state space corresponds to a subset of the Cartesian product $\Omega \subset Z_1 \times \cdots Z_{\overline{q}}$. If there exist two realizations $\widehat{Z}_q = \{b_q, g_q\}$ for each variable $q \in Q$ such that all their joint realizations occur with positive probability, $\Omega \supseteq \widehat{Z}_1 \times \cdots \widehat{Z}_Q$, our results on completeness follow.

Proposition 4. *If there exists a pair of realizations $\widehat{Z}_q = \{b_q, g_q\}$ for each $q \in Q$ such that $\Omega \supseteq \widehat{Z}_1 \times \cdots \widehat{Z}_Q$, then:*

1. *The market structure $\mathbf{J}_K$ is complete if and only if there exists a class k which conditions on all risks, such that $M_k = Q$.*
2. *Agent i is resilient if and only if there exists an agent $j \neq i$ who can trade all risks that she can, $Q_j \subseteq Q_i$.*

Proof. Consider a subset $\widehat{\Omega} \subseteq \Omega$ with $\widehat{\Omega} = \widehat{Z}_1 \times \cdots \widehat{Z}_{\overline{q}}$. Applying Proposition 2 to $\widehat{\Omega}$ says that there exists an instrument which separates $\omega \in \widehat{\Omega}$ from a distinct $\omega' \in \widehat{\Omega}$ if and only if there exists a class k with $M_k = Q$. Expanding the state space to Ω expands the number of columns in $\mathbf{A}_{M_k}$ for each k, but any new columns would assign 0 to the states in $\widehat{\Omega}$. As a result, any two states $\omega \in \widehat{\Omega} \subseteq \Omega$ and $\omega' \in \widehat{\Omega} \subseteq \Omega$ can still only be separated if and only if there exists an k with $M_k = Q$, which proves the first statement. Proceeding analogously for isolating a local state $L_{Q_i}(\omega)$ with $\omega \in \widehat{\Omega}$ proves the second statement. ■

A.3 Proofs for Section A.1

Lemma 5. *The column vectors in $\mathbf{B}$ form a basis of $\mathbb{R}^{|\Omega|}$.*

Proof. We begin by showing that the cardinality of sets O_M are compatible with the number of local assets.

Lemma. *For any set of risks $M \subseteq Q$, the number of M-local states satisfies $|\mathcal{L}_M| = \sum_{K \subseteq M} |O_K|$. In particular, each state $\omega \in \Omega$ belongs to O_M for exactly one set $M \subseteq Q$, such that $|\Omega| = \sum_{M \subseteq N} |O_M|$.*

Proof. Consider a set of risks $M \subseteq Q$. We need to show that every state in $\cup_{K \subseteq M} O_K$ belongs to exactly one cell $L_M \in \mathcal{L}_M$, and that each cell $L_M \in \mathcal{L}_M$ contains exactly one element of

$\cup_{K\subseteq M} O_K$. The former is immediate since $\mathcal{L}_M$ is a partition of the state space $\Omega \supseteq \cup_{K\subseteq M} O_K$. To show the latter, notice that every cell $L_M \in \mathcal{L}_M$ contains any state which assign the L_M-characteristic outcome for all $q \in M$. Among them, there exists exactly one state $\omega \in L_M$ which assigns the worst outcome to the members of the out-group, $z_r(\omega) = z_r(\omega_0)$ for all $r \in Q \setminus M$. This state ω must belong to exactly one set O_K with $K \subseteq M$, where K is the subset of risks in M whose outcome is different from the worst state. And, by definition of (17), all other element of L_M can only belong to O_K if $K \nsubseteq M$. ■

Thanks to the Lemma above, what remains to be shown is that all $|\Omega|$ columns of $\mathbf{B}$ are linearly independent. That is, for any $|\Omega| \times 1$ vector α, and for $\mathbf{x} = \mathbf{B}\alpha$, we have $\mathbf{x} = \mathbf{0} \Rightarrow \alpha = \mathbf{0}$. Consider any arbitrary state ω and the ω-th row of the basis $\mathbf{B}$. By the second part of Lemma above, ω belongs to a unique set O_M for $M \subseteq Q$. For convenience, from now on, we keep track of the relevant set M by referring to our state by ω_M. Since $\mathbf{B}$ consists of local assets, for a column to pay in ω_M, it must be a K-local assets $\mathbf{a}_{L_K(\omega_M)}$ which agrees with ω_M for the coordinates in K. However, since $\omega_M \in O_M$, it assigns the worst realization for all $Q \setminus M$ coordinates. Hence, whenever $K \nsubseteq M$, the candidate column $\mathbf{a}_{L_K(\omega_M)}$ cannot belong to $\mathbf{B}$ because it assigns the worst realization for all coordinates in $K \setminus M$. In contrast, for any remaining set $K \subseteq M$, the set O_K contains a state $\omega_K \in O_K \cap L_K(\omega_M)$ which agrees with ω_M for all K (while all coordinates outside K take the worst value, by definition of O_K). Therefore, the ω_M-th row of $\mathbf{B}$ has 0 in all columns except for assigning a 1 to every K-local asset that agrees with ω_M on K and where K must be a weak subset of M. As a result, the ω_M-th coordinate of the multiplication $\mathbf{x} = \mathbf{B}\alpha$ equals

$$x_{\omega_M} = \alpha_{L_M(\omega_M)} + \sum_{K\subset M} \alpha_{L_K(\omega_K)} \mathbf{1}_{\omega_K \in \{L_K(\omega_M)\cap O_K\}},$$

where $\alpha_{L_M(\omega)}$ is the weight that α assigns to the column $\mathbf{a}_{L_M(\omega)}$ in $\mathbf{B}$. We now show that $\mathbf{x} = \mathbf{0} \Rightarrow \alpha = \mathbf{0}$ using the above definition and proceeding by induction, starting from $x_{\omega_\emptyset}$ – i.e., starting from the set $M = \emptyset$ with no strict subset. In fact, given our definition for O_M defined in (17), we have that $\omega_\emptyset = \omega_0$ corresponds to the reference state. Notice that for all ω_M, the reference state, ω_0, satisfies $\{\omega_0\} = L_\emptyset(\omega_M) \cap O_\emptyset$. Moreover, since only the column $\mathbf{a}_{L_\emptyset(\omega_0)}$ pays on the ω_0-coordinate, we get $x_{\omega_0} = \alpha_{L_\emptyset(\omega_0)} = 0$. Similarly, for the next bigger subset $K = \{q\}_{q\in M}$, only the column $\mathbf{a}_{L_{\{q\}}(\omega_{\{q\}}}$ and $\mathbf{a}_{L_\emptyset(\omega_0)}$ can pay in the state $\omega_{\{q\}} \in O_{\{q\}}$. And since $\alpha_{L_\emptyset(\omega_0)} = 0$, it follows that also $\alpha_{L_{\{q\}}(\omega_{\{i\}}} = 0$. Finally, since M is the maximal element in this partially ordered set of variables $K \subseteq M$, proceeding analogously yields that $x_{\omega_M} = \alpha_{L_M(\omega_M)} + \sum_{K\subset M} \left(\mathbf{1}_{\{\omega_K \in \{L_K(\omega_M)\cap O_K\}\}}\right) \alpha_{L_K(\omega_K)} = \alpha_{L_M(\omega_M)} = 0$, where the second equality follows from successive application of the previous argument which fixes the coefficients on the local assets for all mentioned sets to 0. Therefore, for all local assets $\alpha_{L_M(\omega_M)}$ in $\mathbf{B}$, whenever $\mathbf{x} = 0$, the coefficient $\alpha_{L_M(\omega_M)}$ must be 0. ■

Lemma 6. *Every M-local asset corresponds to a linear combination of columns in $\mathbf{B}$ associated with sets $R \subseteq M$ with non-zero weight on at least one M-local asset in $\mathbf{B}$.*

Proof. First, we show that, for any $M \subseteq M_k$, an asset $\mathbf{a}_{L_M(\omega)}$ belongs to $\mathbf{B}_{\{k\}}$ if and only if $L_M(\omega)$ has a nonempty intersection with O_M. If $L_M(\omega) \cap O_M \neq \emptyset$, by its definition, $\mathbf{B}_{\{k\}}$ contains column $\mathbf{a}_{L_M(\omega)}$ whenever $M \subseteq M_k$. Conversely, if $\mathbf{a}_{L_M(\omega)}$ belongs to $\mathbf{B}_{\{k\}}$, by definition, a state $\widetilde{\omega} \in L_M(\omega)$ must exist such that $q(\widetilde{\omega}) \neq q(\omega_0)$ for all $q \in M$ and $r(\widetilde{\omega}) = r(\omega_0)$ for all $r \in Q \setminus M$. Therefore, $\widetilde{\omega} \in L_M(\omega) \cap O_M \neq \emptyset$. Which proves our first claim.

We now prove the result for an arbitrary state $\widetilde{\omega} \in \Omega$. Generically, it has $r \geq 0$ variables in M_k which generate the worst outcome. Let us carry this information through superscript $\widetilde{\omega}^r$. Consider first the case with $r = 0$. The corresponding local state $L_{M_k}(\widetilde{\omega}^0)$ must contain an element $\omega' \in L_{M_k}(\widetilde{\omega}^0)$ which fixes the outcome of all variables outside of M_k to their worst level: $t(\omega') = t(\omega_0)$ for all $t \notin M_k$ and $t(\omega') = t(\widetilde{\omega}^0)$. By definition of O_{M_k} in (17), $\omega' \in O_{M_k}$. Therefore, by Lemma 5, the local asset for $L_{M_k}(\widetilde{\omega}^0) = L_{M_k}(\omega')$ must be represented as a column in both $\mathbf{A}_{M_k}$ and $\mathbf{B}_{\{k\}}$.

Next, consider the case $r = 1$, a state $\widetilde{\omega}^1$, and let $t \in M_k$ be the variable taking the worst value. Since O_{M_k} requires that no risk in M_k generate its worst outcome, none of the states in $L_{M_k}(\widetilde{\omega}^1)$ belongs to O_{M_k}, so the column $\mathbf{a}_{L_{M_k}(\widetilde{\omega}^1)}$ does not appear directly in $\mathbf{B}_{\{k\}}$. Still, this column can be constructed from a linear combination of columns of $\mathbf{B}_{\{k\}}$. To see this, note that the coarser event $L_{M_k \setminus \{t\}}(\widetilde{\omega}^1)$ corresponds to the union of all $L_{M_k}(\omega)$ across all ω that fix the outcome of $M_k \setminus \{t\}$ to agree with $\widetilde{\omega}^1$. We now show that combining this coarser asset with the M_k-local assets represented in $\mathbf{B}_{\{k\}}$ generates $L_{M_k}(\widetilde{\omega}^1)$. Consider any state that agrees with $\widetilde{\omega}^1$ for $M_k \setminus \{t\}$ while assigning $q \neq q(\omega_0)$ to $q = t$. Since all outcomes in M_k are different from ω_0 the set O_{M_k} must contain one such state, the one which assigns the worst outcome to all $q \notin M_k$. Indeed, the set of states $O_{M_k} \cap L_{M_k \setminus \{t\}}(\widetilde{\omega}^1)$ selects all states ω with the above property, varying only the realization of t. So we obtain the following expression:

$$L_{M_k}(\widetilde{\omega}^1) = L_{M_k \setminus \{t\}}(\widetilde{\omega}^1) \setminus \cup_{\omega \in \left(O_{M_k} \cap L_{M_k \setminus \{t\}}(\widetilde{\omega}^1)\right)} L_{M_k}(\omega). \tag{22}$$

Translating the above equality into local assets, we obtain that the column $\mathbf{a}_{L_{M_k}(\widetilde{\omega}^1)}$ of $\mathbf{A}_{M_k}$

is a linear combination of columns in $\mathbf{B}_{\{k\}}$ as follows:

$$\mathbf{a}_{L_{M_k}(\widetilde{\omega}^1)} = \mathbf{a}_{L_{M_{\setminus\{t\}}}(\widetilde{\omega})} - \sum_{\omega \in O_{M_k} \cap L_{M_k \setminus \{t\}}(\widetilde{\omega})} \mathbf{a}_{L_{M_k}(\omega)}.$$

To show the result for any $r \geq 2$, notice that any $\widetilde{\omega} \in \Omega$ can be expressed as an element of a sequence of states $\widetilde{\omega}^0, \widetilde{\omega}^1, \widetilde{\omega}^2, ..., \widetilde{\omega}^{|M_k|}$, indexed by the number of risks in M_k which generate their worst outcome, where $\widetilde{\omega}^{r+1}$ relates to its predecessor $\widetilde{\omega}^r$ by one additional risk taking its worst outcome. We now proceed by induction on this sequence to show that the local asset for any element must be a linear combination of columns in $\mathbf{B}_{\{k\}}$.

Let $R \subseteq M_k$ be a set of $r = |R|$ variables with worst outcomes. The appropriate $L_{M_k}(\omega)$ events to subtract from $L_{M_k \setminus R}(\widetilde{\omega}^k)$ must include any instance where a strict subset $M \subset R$ assign the worst outcome. Proceeding as before for the case $r = 1$, we obtain the following generalization of (22):

$$L_{M_k}(\widetilde{\omega}^r) = L_{M_k \setminus R}(\widetilde{\omega}^r) \setminus \cup_{M \subset R} \left(\cup_{\omega \in O_{M_k \setminus R} \cap L_{M_k \setminus R}(\widetilde{\omega}^r)} L_{M_k}(\omega) \right).$$

The first term $L_{M_k \setminus R}$ has a nonempty intersection with $O_{M_k \setminus R}$ and therefore corresponds to a local state with a column in $\mathbf{B}_{\{k\}}$. For the remaining events $L_{M_k}(\omega)$ in the parentheses, assume by induction that, for any state ω in which a subset of variables $M \subset R$ assign the worst outcome, then the corresponding local asset $\mathbf{a}_{L_{M_k}(\omega)}$ can be expressed from columns in $\mathbf{B}_{\{k\}}$. Then, by the above equation, $L_{M_k}(\widetilde{\omega}^r)$ can also be expressed from local states that have a corresponding column in $\mathbf{B}_{\{k\}}$. In other words, provided that the result holds when any strict subset M of risks in R assign the worst outcome, the result must also hold whenever all risks in $R \subset M_k$ assign the worst outcome. This proves the inductive step, and together with the result above for $r = 0$ and $r = 1$, we obtain the result for any $r \geq 0$. Converting the local states to their local assets, we can therefore express $\mathbf{a}_{L_{M_k}(\omega)}$ of $\mathbf{A}_{M_k}$ as follows:

$$\mathbf{a}_{L_{M_k}(\widetilde{\omega})} = \mathbf{a}_{L_{M_k \setminus R}(\widetilde{\omega})} - \sum_{M \subset R} \left(\sum_{\omega \in O_{M_k \setminus M} \cap L_{M_k \setminus R}(\widetilde{\omega})} \mathbf{a}_{L_{M_k}(\omega)} \right).$$

The same induction argument applied to the local assets $\mathbf{a}_{L_{M_k}(\omega)}$ implies that, for any $\omega \in O_{M_k \setminus M} \cap L_{M_k \setminus R}$, the local assets must be a linear combination of the columns in $\mathbf{B}_{\{k\}}$ associated to sets $M_k \setminus T$ with $T \subset M$. ■

Lemma 7. *The local assets in* $\mathbf{B}_K$ *form a basis of the payoff space* $\langle \mathbf{J}_K \rangle = \langle \mathbf{B}_K \rangle$.

Proof. We prove our result for the general case, using definition of $\mathbf{B}$ in (18) and O_M in (17). As an intermediate step, we prove the desired result for singleton sets $K = \{k\}$, i.e., that $B_{\{k\}}$ is a basis for $\langle \mathbf{A}_{M_k} \rangle$. Lemma 4 showed that each $\langle B_{\{k\}} \rangle \subseteq \langle \mathbf{A}_{M_k} \rangle$. To prove the that $\langle \mathbf{B}_{\{k\}} \rangle = \langle \mathbf{A}_{M_k} \rangle$, we need to also show that each column in $\mathbf{B}_{\{k\}}$ can be expressed as a linear combination of the columns in $\mathbf{A}_{M_k}$. Notice that, by definition of $\mathcal{M}_{\{k\}}$, for each column $\mathbf{a}_{L_M}(\widetilde{\omega})$ in $\mathbf{B}_{\{k\}}$, since k can generate any arbitrary payoff as long as it only conditions on the outcomes of risks in M_k, the column $\mathbf{a}_{L_M}(\widetilde{\omega})$ must belong to $\langle \mathbf{A}_{M_k} \rangle$.

Finally, we extend the result to K of arbitrary size. The matrix $\mathbf{B}_K$ contains every column of $\mathbf{B}_{\{k\}}$ for any $k \in K$. Moreover, each column in $\mathbf{J}_K$ must belong to $\mathbf{A}_{M_k}$ for some $k \in K$. By the above result for singleton sets, each column in any $\mathbf{A}_{M_k}$ must belong to $\langle \mathbf{B}_{\{k\}}) \rangle$. Therefore, any column in $\mathbf{J}_I$ must belong to $\langle \mathbf{B}_K) \rangle$. Finally, each column in $\mathbf{B}_K$ must belong to $\mathbf{B}_{\{k\}}$ for some $k \in K$. Each column in $\mathbf{B}_{\{k\}}$ must belong to $\langle \mathbf{A}_{M_k} \rangle$ by the result for singleton sets. Therefore, any column in $\mathbf{B}_{\{k\}}$ must belong to $\langle \mathbf{J}_K \rangle$. Consequently, every column in $\mathbf{B}_K$ must belong to $\langle \mathbf{J}_K \rangle$. ■

Lemma 8. *The elements in* $\mathbf{W}$ *defined in (21) form an orthogonal basis of* $\mathbb{R}^{|\Omega|}$.

Proof. By definition of the set O_M in (17), the number of elements in $\mathbf{W}$ equals $|\Omega|$. We need to show that these columns are orthogonal.

Consider first any singleton sets $M = \{q\}$ and a local state $L_{\{q\}}(s)$ where $s \in O_{\{q\}}$. The inner product with the constant column fro $M = \emptyset$ corresponds to the sum $\mathbf{1} \cdot \mathbf{w}_{L_{\{q\}}(s)} = \sum_{\omega \in \Omega} h_{L_{\{q\}}(s)}(\omega)$. By design,

$$\mathbf{1} \cdot \mathbf{w}_{L_{\{q\}}(s)} = \frac{\Omega}{Z_q}[(1 - Z_q) + (1 - \sqrt{Z_q}(Z_q - 2)) + (Z_q - 2)(1 + \sqrt{Z_q})] = 0.$$

Similarly, take any two columns for singleton sets $\{q\}$ and $\{r\}$. Let q and r be different first. Conditional on realization of r, the possible realizations of q appear in the same proportion as in Ω. As a result, $\sum_{\omega \in L_{\{q\}}(\omega')} h_{L_{\{r\}}(t)}(\omega) = 0$. Orthogonality follows from the fact that $\mathbf{w}_{L_{\{q\}}(s)} \cdot \mathbf{w}_{L_{\{r\}}(t)} = \sum_{z \in Z_q} \left(h_{L_{\{q\}}(s)}(\omega_z) \sum_{\omega \in L_{\{q\}}(\omega_z)} h_{L_{\{r\}}(t)}(\omega) \right) = 0$, where ω_z is a state inducing realization $q(\omega_z) = z$ for q. If $q = r$, then the states s and t must induce different realizations of q. The inner product yields

$$\mathbf{w}_{L_{\{q\}}(s)} \cdot \mathbf{w}_{L_{\{q\}}(t)} = \frac{\Omega}{Z_q}[(1 - Z_q)^2 + 2(1 - \sqrt{Z_q}(Z_q - 2))(1 + \sqrt{Z_q}) + (Z_q - 3)(1 + \sqrt{Z_q})^2] = 0,$$

where the first partial sum corresponds to the case when $q(\omega) = b_g$, which happens in Ω/Z_q states, each contributing a term $(1 - \sqrt{Z_q})^2$. The remaining two parts of the sum correspond to the two local states when q takes either $q(s)$ or $q(t)$ and, finally, the $Z_q - 3$ remaining cases where q takes any other value.

Orthogonality of the finer columns follows a similar argument. Take any two vectors $\mathbf{w}_{L_M(s)}$ and $\mathbf{w}_{L_R(t)}$. By definition, there must exist at least one variable q for which either q is not contained in both M and R, or $q(s) \neq q(t)$. Consider first the case where q only belongs to one set M or R. Fixing the realizations in $M \cup R \setminus \{q\}$, the inner product can be decomposed into partial sums, one for each $M \cup R \setminus \{q\}$-local state. Conditional on each of them, $h_{L_{\{q\}}}(s)$ takes the values in the same proportion as on Ω, while all other terms $h_{L_{\{r\}}}(\cdot)$ for any $r \in M \cup R \setminus \{q\}$ remain constant and can be factored out in the inner product. Therefore, the partial sum in the inner product associated with every $M \cup R \setminus \{q\}$-local state must equal Zero. Hence, the inner product must be Zero. Finally, if q belongs to both M and R, it must be true that $q(s) \neq q(t)$. Proceeding analogously, all factors remain constant except for the product $h_{L_{\{q\}}}(s)(\omega)h_{L_{\{q\}}}(t)(\omega)$. However, fixing any of these local states, the joint distribution of the two factors must be identically distributed as in Ω. Therefore, by the previous result for singletons, this partial sum must again equal Zero. Hence the result. ■