Calibrated Cobb-Douglas functions: Summary and examples

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March 2025

Abstract: Cobb-Douglas functions are highly useful and conveniently simple. They are widely used in microeconomics to illustrate a multitude of economic phenomena and properties. A key characteristic of these functions is that their isoquants are smoothly convex, allowing for continuous substitution along them as the prices of goods or factors change. They are also popular in research, particularly in computable general equilibrium analysis. However, practical references on the generation and use of calibrated Cobb-Douglas functions are scarce. In these notes, we aim to fill that gap.

Keywords: Cobb-Douglas, cost functions, calibrated cost functions.

Partial support from Research Project PID2023-147183NB-I00, funded by the Ministry of Science of Spain, and from SGR-2021-00194, funded by the Department of Universities and Research of Catalonia is gratefully acknowledged.

1. Introduction

The Cobb-Douglas function (Cobb and Douglas, 1928) is a particular case of the constant elasticity of substitution (CES) functions (Arrow et al, 1961) when the said elasticity is unitary. It is strictly convex, differentiable, and homothetic. See any micro textbook for a proof of these properties.

The Cobb-Douglas (CD) function is used both in production and demand theory. For simplicity, we will assume two inputs in production theory and two consumptions in demand theory. In the first case, we have the production function:

$$y = \mu \cdot l^{\alpha} \cdot k^{\beta} \tag{1}$$

with y being output and l and k being labor and capital services, respectively. The parameter μ is an efficiency parameter whereas α, and β indicate input contribution coefficients to production. All these parameters are positive. Under constant returns to scale (CRS), $\alpha+\beta=1$, or $\beta = 1 - \alpha$. All these parameters have a physical interpretation.

In demand theory, the CD utility function u takes the form:

$$u = c_1^{\alpha_1} \cdot c_2^{\alpha_2} \tag{2}$$

Notice that there is no equivalent to the efficiency parameter in (1), the reason being that utility is ordinal: we can multiply by any positive number without affecting the underlying preference relation. Similarly, the sum of the coefficients can be 1, or not. Provided the sum $\alpha_1+\alpha_2$ is re-scaled up or down, nothing changes in the preference relation. Again, this is because utility is ordinal.

2. The CD cost function in production theory

CRS is a popular assumption in input-output (IO) and computable general equilibrium (CGE) analyses for modeling production, in part because estimates of sectoral returns to scale are not particularly abundant. For simplicity, we will refer to IO and CGE models as “multisectoral” models, as they are indeed models with many sectors. We will maintain the CRS assumption here (but see Guerra and Sancho, 2012, and Liboreiro, 2019, for exceptions) and, for the most part, omit the indexing of sectors to keep notation manageable and improve transparency. When we refer to “output” and “inputs,” keep in mind that they always apply to all sectors, even if the notation does not explicitly indicate this.

That being said, let us return to the Cobb-Douglas production function from expression (1) and examine production under CRS and competitive behavior. Output y is priced at p, while factors l and k are compensated at rates w and r, respectively. From microeconomic theory, we know that under these assumptions, the optimal production plan results in zero economic profits. In other words, total revenue precisely covers total costs, ensuring that price equals average (and marginal) cost at any output level.

Given this framework, the firm's optimization problem reduces to minimizing the cost of production for any given output level:

$$\begin{aligned} &\text{Min} \quad w \cdot l + r \cdot k \\ &\text{ST} \quad y = \mu \cdot l^{\alpha} \cdot k^{(1-\alpha)} \text{ with } y, w \text{ and } r \text{ given.} \end{aligned} \tag{3}$$

We will not work out the solution details and refer the reader to any microeconomics text (see Varian, 1992, chapter 4). The solution of this problem turns out to be:

$$\begin{aligned} l &= \mu^{-1} \cdot \left(\frac{\alpha}{1-\alpha}\right)^{(1-\alpha)} \cdot \left(\frac{r}{w}\right)^{1-\alpha} \cdot y \\ k &= \mu^{-1} \cdot \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \cdot \left(\frac{w}{r}\right)^{\alpha} \cdot y \end{aligned} \tag{4}$$

Substituting (4) into the minimizing goal we (can) obtain the cost function for any level of output:

$$c(w, r, y) = \mu^{-1} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \cdot w^{\alpha} \cdot r^{(1-\alpha)} \cdot y \tag{5}$$

The initial multiplication of coefficients in (5) is kind of ugly so we can hide it using the substitution:

$$\rho = \mu^{-1} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \tag{6}$$

There is a very interesting property of the cost function known as “Shephard’s lemma” that states that the derivatives of the cost function in relation to the prices of the inputs give back the optimal solutions for l and k. Taking the derivative of $c(w,r,y)$ with respect to w and after some simplifications we obtain:

$$l = \frac{\partial c(w, r, y)}{\partial w} = \alpha \cdot \rho \cdot w^{\alpha-1} \cdot r^{1-\alpha} \cdot y = \frac{\alpha}{w} \cdot (\rho \cdot w^{\alpha} \cdot r^{1-\alpha} \cdot y) = \frac{\alpha}{w} \cdot c(y) \tag{7}$$

From here we find that the coefficient α is nothing but the share of the labor cost over total cost:

$$\alpha = \frac{w \cdot l}{c(w, r, y)} \tag{8}$$

For the capital factor, we would similarly obtain:

$$1 - \alpha = \frac{r \cdot k}{c(w, r, y)} \tag{9}$$

Summing up, the coefficients α and (1-α) in the CD production function indicate relevant economic information, namely, the proportion of costs borne by the two factors of production.

3. The calibrated CD functions on the production side

Multisectoral models are constructed by integrating economic theory with available statistical data, primarily derived from input-output tables (IOT) and social accounting matrices (SAM). The latter extend the former by detailing the distribution of factor incomes—such as labor and capital—into household incomes, acknowledging that households ultimately own most factors of production. However, from the perspective of production and cost modeling, the information contained in both databases is essentially equivalent.

Thus, our focus will be on explaining the process of transforming available production and cost data into functional expressions that can be used as equations in an economic model. This process, known as *calibration*, involves using the data—subject to the restrictions imposed by the theory—to calculate the parameters that define the production technology.

A notable challenge in this process is that IOT and SAM data are expressed in monetary units (e.g., euros, dollars), whereas the Cobb-Douglas (CD) production function in expression (1) is defined in terms of physical quantities. The table below illustrates the typical structure of the available data:

Table 1: Production and factors

$p \cdot y$	100€
$w \cdot l$	60€
$r \cdot k$	40€

Output y with value $p{\cdot}y$ = 100€ has been produced using labor l with labor payments $w{\cdot}l$ = 60€ and capital k with capital payments $r{\cdot}k$ = 40€. We know the currency values involved but we do not know the prices or the quantities. The question is how to derive the CD production function that fits these data.

The road to the solution is surprisingly straightforward. Suppose we happened to know that $p{\cdot}y$ = 100€ comes from p = 5€/kg and y = 20kg. It is obvious that if 1 kg costs 5 €, "one fifth" of a kg will cost 1€. And in 20kg there are 100 "one fifths" of a kg. In other words, we can redefine units so that p' = 1€/"fifth of kg" and y' = 100 "fifths of kg" and $p{\cdot}y$ = 5€/kg · 20kg = 1€/"fifth of kg" · 100"fifths of kg" = $p'{\cdot}y'$.

We have found a redefinition of the initial units so that in the new redefined units the number of physical units coincides with the given currency value. The value of the produced output is 100€ and the number of produced units is 100"fifths of kg". This redefinition of units is called the *standard normalization*. The same applies to the value paid for labor and capital services. All units are redefined in such a way that all redefined units have a value of 1€.

Thanks to this conjuring trick, we can read the values in the table as if they were units of output, labor and capital in the redefined units: $y' = 100$, $l' = 60$ and $k' = 40$, with each of these redefined units having a price of 1€. Without loss of generality, we can forget about the 'primes' in the redefined magnitudes and simply use: $y = 100$, $l = 60$ and $k = 40$ knowing that we also have $p = w = r = 1$. The amazing thing is that we do not need to know the name of the new units. They always exist, and this is all that matters.

3.1 The calibrated CD production function

We can now use expressions (8) and (9) to derive the coefficients in the production function. Labor cost for producing output $y = 100$ is $w{\cdot}l = 60$€ while total cost at prices w and r is $c(y,w,r) = w{\cdot}l + r{\cdot}k = 100$€. Therefore the share of labor cost on total cost is:

$$\alpha = \frac{w \cdot l}{c(w,r,y)} = \frac{60}{100} = 0.6$$

Similarly for capital:

$$1 - \alpha = \frac{r \cdot l}{c(w,r,y)} = \frac{40}{100} = 0.4$$

We have so far advanced to having:

$$y = \mu \cdot l^{0.6} \cdot k^{0.4}$$

The calibration of μ is now simple:

$$\mu = \frac{y}{l^{0.6} \cdot k^{0.4}} = \frac{100}{60^{0.6} \cdot 40^{0.4}} = 1.9601$$

This is the fully calibrated CD production function:

$$y = 1.9601 \cdot l^{0.6} \cdot k^{0.4}$$

Plug $l = 60$ and $k = 40$ into it, and it yields $y = 100$. The calibration is correct.

3.2 *The calibrated CD cost function*

We go back to micro theory and recall the cost function in (5) plus the substitution in (6):

$$c(w,r,y) = \mu^{-1} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \cdot w^{\alpha} \cdot r^{(1-\alpha)} \cdot y = \rho \cdot w^{\alpha} \cdot r^{(1-\alpha)} \cdot y$$

We now claim that for the calibrated CD cost function we will always have $\rho = 1$.

The reason has to do, once again, with the implicit selection of units. On the one hand, all prices are 1 and, on the other hand, total cost equals total income since there are no profits. From:

$$c(w,r,y) = \rho \cdot w^{\alpha} \cdot r^{(1-\alpha)} \cdot y = p \cdot y$$

and $p = w = r = 1$ it necessarily follows that $\rho = 1$. We can also numerically verify that this is so using the parameter values for α and μ we obtained during the calibration. The calibrated cost function is:

$$c(w,r,y) = w^{0.6} \cdot r^{0.4} \cdot y$$

Using Shephard's Lemma we can derive the calibrated conditional demand functions for labor and capital:

$$l = \frac{\partial c(w,r,y)}{\partial w} = 0.6\left(\frac{r}{w}\right)^{0.4} \cdot y$$

$$k = \frac{\partial c(w,r,y)}{\partial r} = 0.4\left(\frac{w}{r}\right)^{0.6} \cdot y$$

Conditional demands for labor and capital increase when output increases, and all the derivatives have the correct sign: $\partial l / \partial w < 0$, $\partial k / \partial w > 0$, $\partial l / \partial r > 0$, $\partial k / \partial r < 0$. When the wage rate increases, demand for labor falls and demand for capital increases along the isoquant of level y. Labor becomes relatively more expensive than capital, which induces the substitution of labor for capital. Similar considerations, although in the opposite direction, apply to increases in r. When capital becomes relatively more expensive, labor will substitute for capital.

3.3 Here comes taxation!

Unfortunately, things in the real world are a bit more complicated. The exchange of goods is often affected by indirect taxes. In the case of labor, it is common for payments generated through its use in production to be subject to taxation. The total wages paid, which include both the amount received by the worker and the tax collected by the government, constitute the total cost borne by the producer. The key question is whether, and if so, how the presence of indirect taxes affects the calibration process. Consider Table 1, which has been modified to reflect an ad-valorem tax on labor payments at rate $t = 0.20$.

Table 2: Production and factors with a tax on labor

$p \cdot y$	100€
$w \cdot l$	50€
$t \cdot w \cdot l$	10€
$r \cdot k$	40€

Total labor cost borne by the producer is still 60€ but only 50€ go to the labor factor. We can start by distinguishing the net wage rate received by the factor, i.e. w, and the gross wage rate paid by the producer, i.e. w_g. They are related by $w_g = (1+t) \cdot w$. We use the standard normalization so that all prices (net of taxes) are 1 and currency values indicate physical output and inputs in redefined units: $y = 100$, $l = 50$, $k = 40$.

Using (8) and (9) and remembering that now the wage rate relevant for the producer is the gross wage rate, we find the shares:

$$\alpha = \frac{w_g \cdot l}{c(w_g, r, y)} = \frac{50 + 10}{100} = 0.6 \qquad 1 - \alpha = \frac{r \cdot k}{c(w_g, r, y)} = \frac{40}{100} = 0.4$$

From expression (1) we obtain:

$$\mu = \frac{y}{l^{\alpha} \cdot k^{1-\alpha}} = \frac{100}{50^{0.6} \cdot 40^{0.4}} = 2.1867$$

Back to the cost function, we recall expression (6) and plug in the values just obtained:

$$\rho = \mu^{-1} \cdot \alpha^{-\alpha} \cdot (1 - \alpha)^{-(1-\alpha)} = \frac{0.6^{-0.6} \cdot 0.4^{-0.4}}{2.1867} = 0.8964$$

We have the production and cost functions fully calibrated:

$$y = \mu \cdot l^{\alpha} \cdot k^{\beta} = 2.1867 \cdot l^{0.6} \cdot k^{0.4}$$

$$c(w, r, y) = \rho \cdot w_g^{\alpha} \cdot r^{(1-\alpha)} \cdot y = \rho \cdot (1 + t)^{\alpha} \cdot w^{\alpha} \cdot r^{1-\alpha} \cdot y =$$
$$= 0.8964 \cdot (1 + t)^{0.6} \cdot w^{0.6} \cdot r^{0.4} \cdot y$$

The reader can verify that substituting the parameters, unitary prices, factors' use, and indirect tax rate in these expressions results in the correct values for both output (in the production function) and total cost (in the cost function).

4. The CD functions in demand theory

The solution of the CD utility maximization problem is straightforward. Given an income level m and consumption prices p_1 and p_2 we formulate:

$$\begin{array}{ll} \text{Max} & u = c_1^{\alpha_1} \cdot c_2^{\alpha_2} \\ \text{ST} & m = p_1 \cdot c_1 + p_2 \cdot c_2 \text{ with } m, p_1 \text{ and } p_2 \text{ given.} \end{array}$$

The solution to this problem gives us (Marshallian) consumption levels:

$$c_1 = \frac{\alpha_1 \cdot m}{p_1}$$
$$c_2 = \frac{\alpha_2 \cdot m}{p_2}$$

Observe that the parameters α_j are the proportion of total income devoted to the purchase of each of the consumption goods. Therefore, they are always nonnegative and add up to 1.

We can also formulate the expenditure minimization problem:

$$\begin{array}{ll} \text{Min} & e(p,u) = p_1 \cdot c_1 + p_2 \cdot c_2 \\ \text{ST} & u = c_1^{\alpha_1} \cdot c_2^{\alpha_2} \quad \text{with } u,\ p_1 \text{ and } p_2 \text{ given.} \end{array}$$

This problem is formally equivalent to the cost minimization problem (3) whose solution we already know. All we have to do is change the notation. The expenditure function will be:

$$e(p,u) = \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot u$$

Using Shephard's Lemma once again, we derive conditional demand functions—also known as Hicksian or compensated consumption functions in demand theory:

$$\frac{\partial e(p,u)}{\partial p_1} = c_1 = \rho \cdot \alpha_1 \cdot \left(\frac{p_2}{p_1}\right)^{\alpha_2} \cdot u$$
$$\frac{\partial e(p,u)}{\partial p_2} = c_2 = \rho \cdot \alpha_2 \cdot \left(\frac{p_1}{p_2}\right)^{\alpha_1} \cdot u$$

Observe that both the Marshallian and Hicksian demand functions behave properly in regard to price changes. Demand for good 1 falls when good 1 becomes relatively more expensive than good 2, i.e. p_1 goes up and/or p_2 goes down.

5. The calibrated CD functions on the demand side

Suppose now that we are now given consumption expenditure and income data as reported in Table 3.

Table 3: Household expenditure

m	100€
$p_1 \cdot c_1$	80€
$p_2 \cdot c_2$	20€

Total expenditure in consumption is 100€ and is equal to income m = 100€. If the utility function is assumed to be CD, then all we need to do is calculate the expenditure shares:

$$\alpha_1 = \frac{p_1 \cdot c_1}{m} = \frac{80}{100} = 0.8$$
$$\alpha_2 = \frac{p_2 \cdot c_2}{m} = \frac{20}{100} = 0.2$$

The calibrated CD utility function will be:

$$u = c_1^{\alpha_1} \cdot c_2^{\alpha_2} = c_1^{0.8} \cdot c_2^{0.2}$$

Or any monotone positive transformation of u such as:

$$v = 123 \cdot c_1^{0.8} \cdot c_2^{0.2}$$

$$v = 0.27 \cdot c_1^{1.6} \cdot c_2^{0.4}$$

The structure of the expenditure function is also immediate:

$$e(p,u) = \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot u = 1.6494 \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot u$$

Since all income is spent, i.e. $m = e(p,u) =$ 100€, this gives us the utility level when prices are unitary as $u = m/1.6194 = 60.6287$. The same utility value is obtained if we plug in the consumption vector $(c_1,\ c_2) = (20,\ 20)$ into the calibrated CD utility function.

Let is now face a situation with ad-valorem taxes on consumption. Good 1 is taxed at rate $t_1 = 0.20$ and good 2 at rate $t_2 = 0.25$. Table 4 describes the situation. Total expenditure and income is 100€ with a total tax collection of 18€. Prices p_j are net prices (i.e. before taxes).

Table 4: Consumption with sales taxes.

m	100€
$p_1 \cdot c_1$	50€
$t_1 \cdot p_1 \cdot c_1$	10€
$p_2 \cdot c_2$	32€
$t_1 \cdot p_1 \cdot c_1$	8€

From the solution of the utility maximization problem, we can find the shares:

$$\alpha_1 = \frac{p_1 \cdot (1 + t_1) \cdot c_1}{m} = \frac{60}{100} = 0.6$$
$$\alpha_2 = \frac{p_2 \cdot (1 + t_2) \cdot c_2}{m} = \frac{40}{100} = 0.6$$

This gives us the utility function and initial utility level:

$$u = c_1^{\alpha_1} \cdot c_2^{\alpha_2} = 50^{0.6} \cdot 32^{0.4} = 41.8256$$

The expenditure function from the expenditure minimization problem is:

$$e(p, u) = \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot \left[p_1 \cdot (1 + t_1)\right]^{\alpha_1} \cdot \left[p_2 \cdot (1 + t_2)\right]^{\alpha_2} \cdot u$$

Since expenditure equals income and initial net prices are 1, we can obtain the utility level from:

$$u = \frac{m}{\alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot (1 + t_1)^{\alpha_1} \cdot (1 + t_2)^{\alpha_2}} =$$
$$\frac{100}{0.6^{-0.6} \cdot 0.4^{-0.4} \cdot 1.2^{0.6} \cdot 1.25^{0.4}} = 41.8256$$

Everything works out.

The calibration of the utility and expenditure functions is correct. The calibration on the demand side is somewhat simpler than on the production side for the simple reason that utility is ordinal. As a result, the parameter μ, which is related to the units in which inputs are measured, does not appear in the utility function. Alternatively, we can express this by stating that, due to ordinality, we can arbitrarily set the parameter value to $\mu = 1$. However, when this parameter has a structural meaning, as in the case of the production function, its value must be determined and preserved during calibration, which complicates the determination of the remaining parameters.

6. Final remarks

The calibration of parameters to data gives us a simple procedure for determining utility and production functions of the Cobb-Douglas class. These calibrated functions have the property that they reproduce the available data under the standard normalization. They also allow for the calculation of consumption demand and inputs demand when, for some reason, prices for consumption goods or for primary inputs change. This is typically the case in multisectoral models when equilibrium prices adjust to external changes that affect structural or fiscal parameters.

The calibration procedure is often described for CES function. See Sancho (2009) for regular CES functions and Rutherford (2008) for shared CES functions. However, the calibration details for CES functions are significantly more complex. Ultimately, we must weigh this added technical complexity against the fact that, in many cases, reliable estimates of substitution elasticities are either unavailable or conflicting. In such situations, we resort to unitary elasticities, meaning Cobb-Douglas functions, making the additional calibration effort unnecessary. If we are going to use Cobb-Douglas functions regardless, then the simpler procedure outlined here is sufficient.

References

Arrow, K.J., Chenery, H.B., Minhas, B.S. and Solow, R.M. (1961). "Capital-Labor Substitution and Economic Efficiency". *Review of Economics and Statistics*, 43, 225-250.

Cobb, C.W. and Douglas, P. H. (1928). "A Theory of Production". *American Economic Review*, 18(1), Supplement, 139–165.

Guerra, A.I. and Sancho, F. (2014). "An operational, nonlinear input-output system" *Economic Modelling*, 41, 99-108.

Liboreiro, P. (2023). "Estimating disguised unemployment in major middle-income countries by means of non-linear input–output analysis, 2000–2014". *Economic Systems Research*, 35(4), 634-657.

Sancho, F. (2009). "Calibration of CES functions for real-world multisectoral modeling". *Economic Systems Research*, 21(1), 45-58.

Rutherford, T. (2008). "Calibrated CES utility functions. A worked example", Mimeo, Department of Management, Technology and Economics, ETH Zürich

Varian, H. (1992). Microeconomic Analysis, third edition. Norton, New York.