A Characterization of Black's Voting Rule

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Abstract. In his 1958 classic, The Theory of Committees and Elections, Duncan Black proposed the following lexicographic rule: for any set of feasible alternatives, and any profile of voters' goodness relations, choose the strong Condorcet winner if it exists, and select the set of Borda winners otherwise. We provide what we think is the first axiomatic characterization of this rule. We do so through the intermediary study of the generalized social welfare functions that underlie the rule's choices, and the use of axioms that emphasize what is common and what is different in the spirit of the amply debated proposals made by these two 18^{th}-century authors. *Journal of Economic Literature* Classification Nos.: D71, D72, D63.

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1 Introduction

The conflicting views of Condorcet and Borda regarding what candidates deserve to be chosen in an election were very alive at the time when these two authors and intellectual rivals were active in the French Academy, during the final two decades of the 18^{th}th century. Their proposals are still very much in apparent conflict today, almost 250 years later. Condorcet's (1785) basic and very attractive principle postulates that, if a candidate beats all others in a pairwise majority contest, this candidate—and only this candidate—should be chosen. The main drawback of this idea is that such a candidate does not exist in some situations. Borda's (1781) contention is that, even when a candidate with this characteristic is available, it need not be appropriate to declare its choice to be the unique election outcome. This opposing view is based on the hypothesis that what matters most is whether an alternative receives the maximal number of total supporting votes when competing with every other candidates in a pairwise comparison.

The merits of each proposal have been debated at length, and the resulting collective choice rules that they inspire have been characterized by using different axiomatic systems; see, for instance, Young (1974), Hansson and Sahlquist (1976), Nitzan and Rubinstein (1981), Mihara (2017), and Barberà and Bossert (2023). The advantage of Borda's proposal is that it always identifies one or several candidates with maximal support, while a strong Condorcet winner may not always exist. When there is a strong Condorcet winner, however, many classical and contemporary authors would be happy to let Condorcet's proposal prevail. But this still leaves the question of how to choose in its absence. Black, one of the most conspicuous social choice theorists of the twentieth century, suggests that the most reasonable solution to this question might well be to use these two apparently irreconciliable principles sequentially. If there is a strong Condorcet winner, choose it. If not, select the Borda winners. This is the choice-theoretic formulation of Black's rule, as proposed in his seminal monograph (Black, 1958). In this paper, we characterize a variant of this rule in terms of aggregate rankings.

Black's strong stand in favor of the Condorcet principle, when it applies, and his specific proposal of using Borda's rule in a subsidiary way, places this author among those adopting what Fishburn (1977) refers to as a lexicographic view. It is, in this respect, analogous to the position that was adopted by Daunou (1803) in the early 19^{th} century. Daunou suggests that, in the absence of a strong Condorcet winner, a plurality vote among the candidates who do not lose against all others in a pairwise comparison should be used to determine the choice. Daunou's voting rule is characterized in Barberá, Bossert, and Suzumura (2021) and, although the approach that we follow here is quite different, it is based on the same principle of applying a lexicographic procedure. Given that the use of Condorcet's principle lexicographically prevails whenever it can be applied, the characterization of rules of this nature requires the careful selection of axioms that apply on the part of the domain where Condorcet's criterion keeps silent.

In the case of Black's rule, the challenge posed by a characterization is attractive, because it is well understood that the principles underlying Condorcet and Borda's proposals are often at odds. This is reflected by the fact that they frequently recommend dramatically different choices; see, for instance, Moulin (1988, Part IV) for a brilliant summary and comparison. And yet, our approach will start from the observation that, under specific circumstances, the two are closely related, in a manner that is easy to understand and natural to justify.

We depart from the traditional description of these two principles in terms of collective choices and, instead, focus on the pairwise rankings of the candidates that each of them induces. This is accomplished by means of an aggregation procedure that we refer to as a generalized social welfare function, an extension of Arrow's (1951; 1963) classical tool to environments in which the set of feasible candidates may vary. The application of this method allows us to observe that, in some situations, the two principles are not in a state of conflict. Indeed, whenever two candidates x and y are contiguous in the rankings of all voters, Condorcet and Borda agree on their relative ranking, independent of the positions of all other candidates. This initial observation leads us to consider other situations where the positions of two candidates x and y are, in a specific sense, close to being contiguous. We call these situations quasi-majoritarian, and they apply when, in the rankings of all voters that rank one of the two candidates above the other (candidate x above y, say), the two candidates are contiguous, while some interstitial alternatives may be present between y and x in the rankings of the voters who rank y ahead of x. In such cases, the votes for and against each of these two candidates is still the basis for the Condorcet comparison, but their Borda ranking combines, at the same level of importance, the sizes of the majority support and that of the set of interstitial alternatives between y and x.

These elementary observations regarding the similarities and the differences between Borda's and Condorcet's rankings in well-defined situations allow us to characterize Borda's generalized social welfare function on the universal set of situations by using an axiom that connects the pairwise rankings in these special cases to those in general situations. An additional difficulty arises when characterizing Black's rule, caused by the residual character attached to the situations in which the Borda rule applies. In order to guarantee that all transformations of situations needed for our proofs remain in the subdomain of those for which no strong Condorcet winner exists, we augment the set of candidates in order to be able to construct auxiliary situations under which no such winner can ever arise. The details of this construction are introduced later, when we proceed to our characterization of Black's rule. Suffice it to say at this point that the augmentation just alluded to hinges crucially on our modeling choice of an aggregation procedure as a generalized social welfare function. In this context, we emphasize that once an aggregate weak order of the candidates is established for a feasible set, it is straightforward to define a corresponding choice function.

A very large body of literature discusses the use of different voting principles, be they Condorcet's, Borda's, or others following a variety of strategies. If some authors definitely prioritize Condorcet's criterion above all, other writers definitely favor the use of Borda's. Morales's (1797) contribution constitutes a classical defense of this method, strongly supported by Saari (1994) through more rigorous and contemporary arguments. However, most of the literature adopts a less radical position. Even Black (1976) makes a case in favor of the Borda rule by examining some of its important characteristics. And many authors suggest the use of rules that combine, to a certain extent, the attractive features inspiring each of the two historical principles, often at the cost of failing to fully satisfy either one of them. Laslier's (1997) monograph provides a catalog and comparative assessment of a large variety of works in that line that start from data defining tournaments. Fishburn (1977) contains different methods to complement the use of Condorcet's criterion when it keeps silent. Specific discussions centering on the connections between the spirit of Borda and that of Condorcet span many decades. See, for instance, Young and Levenglick (1978) or Herrero and Villar (2021). There is also a branch of the literature that starts from an identification of those situations for which the application of some principles appears to be more desirable than the use of others, and proposes the use of each method depending on circumstances. Ultimately, this can lead to a principle that minimizes, in some well-defined sense, the risks of malfunctioning implied by its choice. The latter method is applied, for example, in contributions such as Baharad and Nitzan (2007, 2011) and Nitzan and Nitzan (2024). This variety of approaches attest to the persistent challenges to be addressed when discussing the use of fundamental principles of collective choice.

Our motivation to characterize Black's rule is, to a large extent, based on historical considerations, since we are not aware of any existing axiomatization of this aggregation procedure. But it is also of methodological significance, because we employ several conceptually novel proof techniques that may turn out to be successful tools in a variety of contexts that include, but need not be restricted to, lexicographic considerations. The necessity of developing proof methods that are applicable in this context is a consequence of the difficulties inherent in adapting earlier axiomatizations of the Borda rule to the problem of characterizing Black's proposal. Other innovative techniques such as the use of amplifications, pioneered by Hansson and Sahlquist (1976), fail in our setting, because they cannot but take us out of the subdomain in which no strong Condorcet winner exists. For the same reason, Barberà and Bossert's (2023) approach that is based on states of opinion rather than individual goodness relations turns out to be unsuitable when it comes to an axiomatization of Black's rule.

The complexities involved in axiomatizing Black's rule are succinctly summarized by Young and Levenglick (1978, p. 285), who write that

> "The challenge of combining the regularity of Borda's approach with Condorcet's principle into a unified method is a long-standing problem in the theory of elections. A variety of proposals have been made over the years; many of which are surveyed and compared in a recent paper by Fishburn [(1977)]. One of the earliest is due to Black [(1958)], who proposed that the Condorcet alternative be chosen when one exists, and otherwise that the Borda method be reverted to. This somewhat ad hoc proposal avoids the fundamental issue of choosing *properties* that are natural in the context of election, and then asking what (if any) methods have these properties."

The use of our relatively unconventional method of characterization can also be interpreted as a response to Young and Levenglick's remarks in that we adjust to the idiosyncrasy of Black's rule. Starting out with specific situations that allow for intuitively appealing comparisons of two candidates based on both Condorcet's and Borda's principles, we then extend these comparisons to more general circumstances.

2 Generalized social welfare functions

There is a countably infinite universal set of candidates X. The set of all finite subsets of X with at least two elements is denoted by $\mathcal{X}$. For all $S \in \mathcal{X}$, the set of all weak orders (that is, the set of all transitive and complete binary relations) $R \subseteq S \times S$ on S is denoted by $\mathcal{R}^S$. Furthermore, we define $\mathcal{R} = \cup_{S \in \mathcal{X}} \mathcal{R}^S$.

The (fixed) finite set of voters is $N = \{1, \ldots, n\}$ with $n \geq 2$. Each voter $i \in N$ uses a strict total order (that is, an asymmetric, transitive, and connected binary relation) $P_i \subseteq X \times X$ to compare the candidates. Because we restrict attention to strict total orders, there is no danger of ambiguity in using simplified notation to indicate a voter i's relation P_i. For example, if the strict total order P_i of a voter $i \in N$ is such that xP_iy, yP_iz, and xP_iz, this relationship can equivalently be expressed as $P_i : xyz$. A profile is an n-tuple $P = (P_1, \ldots, P_n)$, and the set of all profiles composed of n strict total orders is denoted by $\mathcal{P}$.

We follow Young (1974) in assuming that the voters' relation are strict total orders. The central assumption is that the candidates are strictly ranked, which could equivalently be achieved by replacing asymmetry with the conjunction of reflexivity and antisymmetry. Reflexivity is not required in any of our results, and the assumption of asymmetry simplifies our exposition because this property makes it unnecessary to explicitly require that two candidates are distinct. That aggregate relations are assumed to be weak orders is standard; as usual, forcing any two candidates to be strictly ranked is too restrictive because it would make it difficult—if not impossible—in some situations to ensure the equal treatment of voters and the equal treatment of candidates.

Ordered pairs (S, P) formed by a given set of candidates and a given profile are called situations; see Hansson and Sahlquist (1976, p. 193) and Fishburn (1977, p. 470) for the use of this term.

The notion of a strong Condorcet winner is a cornerstone of the majoritarian principles advocated by Condorcet (1785). Let $(S, P) \in \mathcal{X} \times \mathcal{P}$. A candidate $x \in S$ is a strong Condorcet winner for (S, P) if

$$|\{i \in N \mid xP_iz\}| - |\{i \in N \mid zP_ix\}| > 0 \text{ for all } z \in S \setminus \{x\}.$$

For any situation $(S, P) \in \mathcal{X} \times \mathcal{P}$, there is at most one strong Condorcet winner for (S, P). We denote the set of strong Condorcet winners for (S, P) by $SCW(S, P)$.

We will have occasion to focus on specific subsets of the set $\mathcal{X} \times \mathcal{P}$ of all possible situations, to be interpreted as families of situations about which we develop a special interest, either because they are the feasible ones for society, or, as is the case here, because they are deemed deserving of special consideration when defining collective choice criteria.

A generalized social welfare function assigns, to each situation in its domain, a weak order defined on the feasible set under consideration. Thus, a generalized social welfare function is a mapping $F\colon \mathcal{X} \times \mathcal{P} \to \mathcal{R}$ such that, for all $(S, P) \in \mathcal{X} \times \mathcal{P}$, $F(S, P) \in \mathcal{R}^S$. Unlike a traditional social welfare function in the sense of Arrow (1951; 1963), our generalized variant is not restricted to a fixed feasible set.

In addition to the full domain that allows for all possible situations, we are particularly interested in two subdomains, which form a partition of the full domain. The first if these is the Condorcet domain $\mathcal{C}$, composed of all situations (S, P) such that there is a (unique) strong Condorcet winner for (S, P), the second is its complement $\mathcal{NC}$ in $\mathcal{X} \times \mathcal{P}$—that is, the family of situations (S, P) such that there is no strong Condorcet winner for (S, P). Formally, these subdomains are defined by

$$\mathcal{C} = \{(S, P) \in \mathcal{X} \times \mathcal{P} \mid SCW(S, P) \neq \emptyset\}$$

and

$$\mathcal{NC} = \{(S, P) \in \mathcal{X} \times \mathcal{P} \mid SCW(S, P) = \emptyset\}.$$

A prominent example of a generalized social welfare function is based on the Borda (1781) rule. For all $(S, P) \in \mathcal{X} \times \mathcal{P}$ and for all $x \in S$, the Borda score $b(x; S, P)$ of x for (S, P) is defined by

$$b(x; S, P) = \sum_{i \in N} |\{z \in S \setminus \{x\} \mid x P_i z\}|.$$

Thus, the Borda score of a candidate x for the situation (S, P) is given by the total number of other candidates in S that are beaten by x in a pairwise majority contest according to the situation (S, P). The Borda generalized social welfare function F^B is obtained by defining, for all $(S, P) \in \mathcal{X} \times \mathcal{P}$ and for all $x, y \in S$,

$$x \, F^B(S, P) \, y \;\Leftrightarrow\; b(x; S, P) \geq b(y; S, P).$$

Because we assume the voters' relations to be strict total orders, this definition of F^B is equivalent to the universally applicable version that subtracts the total number of losses from the total number of wins to calculate the Borda score of each candidate. Note that the Borda generalized social welfare function is defined on the full domain, but it can just as well be defined for any of its subdomains—particularly the family $\mathcal{NC}$ that only includes situations without a strong Condorcet winner.

Black's (1958) lexicographic social choice function chooses the strong Condorcet winner whenever such a candidate exists and, as a secondary criterion, the set of Borda winners is selected if there is no strong Condorcet winner. We adapt this rule to our setting in terms of aggregate rankings. For all situations such that there is a strong Condorcet winner, the resulting aggregate weak order places this strong Condorcet winner uniquely at the top. If there is no strong Condorcet winner, the corresponding aggregate weak order is obtained by comparing the Borda scores of the feasible candidates. Thus, a generalized social welfare function F is a Black generalized social welfare function if

(i) for all $(S, P) \in \mathcal{X} \times \mathcal{P}$ such that there exists $x^c \in S$ with $SCW(S, P) = \{x^c\}$, for all $y \in S \setminus \{x^c\}$,

$$x^c \, F(S, P) \, y \;\text{ and }\; \neg(y \, F(S, P) \, x^c);$$

(ii) for all $(S, P) \in \mathcal{NC}$, $F(S, P) = F^B(S, P)$.

In our formulation of the first part of Black's rule, we do not specify the relative rankings of the remaining feasible candidates. This means that we end up with an entire class of Black generalized social welfare functions. All of them uniquely induce Black's social choice function in the sense that the strong Condorcet winner is the unique greatest element for any of the Black generalized social welfare functions. For this reason, we are confident that our definition is firmly within the spirit of Black's proposal. One possibility consists of declaring, in the presence of a strong Condorcet winner, all remaining candidates to be pairwise equally good. Another special case ranks all candidates that are not strong Condorcet winners according to their Borda scores. This second option constitutes an interesting alternative, because it aligns well with an interesting alternative expression of Black's rule. It is perfectly legitimate to describe the Black rule by stating that it selects the candidates with the highest Borda score, except in those circumstances where there is a strong Condorcet winner—in which case the latter is to be chosen uniquely. We reiterate that the Black social choice function—the choice function that selects the unique strong Condorcet winner as the only chosen candidate whenever such a candidate exists—is induced by any Black generalized social welfare function as defined above.

3 Borda

It is well-known that there is a pronounced conflict between the recommendations of the Borda rule and the majoritarian ideas on which the property of strong Condorcet winner consistency is based. There are, however, situations for which the two coincide. To describe them, we first define the notion of contiguous candidates in a voter's relation for a given feasible set.

Definition 1. *Let $S \in \mathcal{X}$ be a feasible set of candidates, and let $P_i \subseteq X \times X$ be the strict total order of a voter $i \in N$. Two candidates $x, y \in S$ are contiguous in P_i for S if*

$$\{z \in S \mid xP_iz \text{ and } zP_iy\} = \{z \in S \mid yP_iz \text{ and } zP_ix\} = \emptyset.$$

That is, two candidates are contiguous in a voter's strict total order for a feasible set if there are no (interstitial) candidates in the feasible set that are ranked between them.

It is immediate that the rankings recommended by Borda and by Condorcet are the same for any two feasible candidates x and y if the situation under consideration is such that x and y are contiguous in all of the voters' strict total orders for the corresponding feasible set. If there are no interstitial candidates for any of the voters, the Borda count cannot but generate the same ranking as a majority comparison between x and y. These situations can be described as majoritarian with respect to candidates x and y because the majority rule is the natural criterion to rank them, both from the perspective of Borda and that of Condorcet. The central axiom that we employ in our characterizations extends this intuitively appealing idea to situations that we refer to as irreducibly quasi-majoritarian for a pair of candidates—situations that are, in a well-defined sense, as close as we can get (if not equal) to a majoritarian situation.

Definition 2. *Let $(S, P) \in \mathcal{X} \times \mathcal{P}$ and $x, y \in S$. The situation $(S, P) \in \mathcal{P}$ is irreducibly xy-quasi-majoritarian if, for all sets of voters $L \subseteq N$ such that x and y are not contiguous in P_i for S for all $i \in L$,*

$$[xP_iy \text{ for all } i \in L] \text{ or } [yP_ix \text{ for all } i \in L].$$

The following definitions apply to any situation in the universal domain. We first use them to complete the statement of our first axiom, when restricted to irreducibly xy-quasi-majoritarian situations, and later on show their general link with the Borda scores.

Let $(S, P) \in \mathcal{X} \times \mathcal{P}$, $x, y \in S$, and $i \in N$. The majority advantage of x over y for (S, P) is defined as

$$M(x, y; S, P) = |\{i \in N \mid xP_iy\}|.$$

The interstitial advantage of x over y for (S, P) according to voter i is given by

$$G_i(x, y; S, P) = |\{z \in S \mid xP_iz \text{ and } zP_iy\}|,$$

and the aggregate interstitial advantage of x over y for (S, P) is

$$G(x, y; S, P) = \sum_{i \in N} G_i(x, y; S, P).$$

As alluded to earlier, our first axiom prescribes the relative ranking of two candidates x and y if the situation under consideration is irreducibly xy-quasi-majoritarian.

Quasi-majoritarianism. For all $(S, P) \in \mathcal{X} \times \mathcal{P}$ and for all $x, y \in S$, if (S, P) is irreducibly xy-quasi-majoritarian, then

$$x\, F(S, P)\, y \;\Leftrightarrow\; M(x, y; S, P) + G(x, y; S, P) \geq M(y, x; S, P) + G(y, x; S, P).$$

The intuition underlying the axiom is transparent and natural: if a situation is, in a precisely defined sense, as close as we can get to a majoritarian situation with respect to two candidates x and y, the recommended ranking of these two candidates is to be based on two crucial attributes—the majority advantages to take account of the majoritarian component, and the interstitial advantages to account for the principles represented by the Borda rule. This is in line with numerous axiomatizations that proceed by first identifying circumstances in which it can plausibly be argued that a specific aggregate ranking of two candidates should result, and then using additional axioms to extend the requisite ranking rule to more general situations.

Indeed, as demonstrated in the following lemma, there is a close connection between the Borda scores and the functions M and G.

Lemma 1. *For all $(S, P) \in \mathcal{X} \times \mathcal{P}$ and for all $x, y \in S$,*

$$b(x; S, P) \geq b(y; S, P) \;\Leftrightarrow\; M(x, y; S, P) + G(x, y; S, P) \geq M(y, x; S, P) + G(y, x; S, P). \quad (1)$$

Proof. By definition of the Borda scores,

$$b(x; S, P) \geq b(y; S, P) \;\Leftrightarrow\; \sum_{i \in N} |\{z \in S \setminus \{x\} \mid xP_iz\}| \geq \sum_{i \in N} |\{z \in S \setminus \{y\} \mid yP_iz\}|.$$

The inequality on the right side of this equivalence can be rewritten as

$$\begin{aligned} & |\{i \in N \mid xP_iy\}| + \sum_{i \in N} |\{z \in X \setminus \{x,y\} \mid xP_iz \text{ and } zP_iy\}| \\ & \qquad + \sum_{i \in N} |\{z \in X \setminus \{x,y\} \mid xP_iz \text{ and } yP_iz\}| \\ \geq \; & |\{i \in N \mid yP_ix\}| + \sum_{i \in N} |\{z \in X \setminus \{x,y\} \mid yP_iz \text{ and } zP_ix\}| \\ & \qquad + \sum_{i \in N} |\{z \in X \setminus \{x,y\} \mid yP_iz \text{ and } xP_iz\}|. \end{aligned}$$

Simplifying, this is equivalent to

$$\begin{aligned} & |\{i \in N \mid xP_iy\}| + \sum_{i \in N} |\{z \in X \setminus \{x,y\} \mid xP_iz \text{ and } zP_iy\}| \\ \geq \; & |\{i \in N \mid yP_ix\}| + \sum_{i \in N} |\{z \in X \setminus \{x,y\} \mid yP_iz \text{ and } zP_ix\}|. \end{aligned}$$

Substituting the definitions of M and G and combining the above equivalences, we obtain (1). ■

Note that, in this section, we only employ axioms the scope of which is the full domain $\mathcal{X} \times \mathcal{P}$. We now move on to our second axiom. It is based on a well-established compensation principle that is known to be a central characteristic of the Borda rule; see, for example, Young (1974).

Definition 3. *Let $(S,P) \in \mathcal{X} \times \mathcal{P}$ and $x, y \in S$. A profile $P^c \in \mathcal{P}$ is an xy-compensating profile for (S,P) if there exist $i, j \in N$ and $z, w \in S$ such that*

(i) x and z are contiguous in P_i for S and x and w are contiguous in P_j for S

and

(ii) xP_iz and zP_iy and yP_jw and wP_jx

and

(iii) zP_i^cx and xP_j^cw

and

(iv) $sP_i^ct \Leftrightarrow sP_it$ for all $s, t \in S$ such that $\{s,t\} \neq \{x,z\}$

and

(v) $sP_j^ct \Leftrightarrow sP_jt$ for all $s, t \in S$ such that $\{s,t\} \neq \{x,w\}$

and

(vi) $P_k^c = P_k$ for all $k \in N \setminus \{i,j\}$.

The axiom of pairwise compensation requires that the relative ranking of any two candidates x and y according to $F(S,P)$ be unchanged if P is replaced with an xy-compensating profile for (S,P).

Pairwise compensation. For all $(S,P) \in \mathcal{X} \times \mathcal{P}$, for all $x,y \in S$, and for all xy-compensating profiles P^c for (S,P),

$$x\,F(S,P^c)\,y \;\Leftrightarrow\; x\,F(S,P)\,y.$$

The following auxiliary result shows that comparisons made by means of the sum of the values of M and G are preserved when we move to an xy-compensating profile.

Lemma 2. *For all $(S,P) \in \mathcal{X} \times \mathcal{P}$, for all $x,y \in S$, and for all xy-compensating profiles P^c for (S,P),*

$$\begin{aligned} & M(x,y;S,P^c) + G(x,y;S,P^c) \geq M(y,x;S,P^c) + G(y,x;S,P^c) \\ \Leftrightarrow\; & M(x,y;S,P) + G(x,y;S,P) \geq M(y,x;S,P) + G(y,x;S,P). \end{aligned}$$

Proof. Let $(S,P) \in \mathcal{X} \times \mathcal{P}$ and $x,y \in S$, and suppose that P^c is an xy-compensating profile for (S,P). By definition, the majority advantage of x over y for (S,P^c) is identical to the majority advantage of x over y for (S,P), and the same is true for the requisite majority advantages of y over x. Thus,

$$M(x,y;S,P^c) = M(x,y;S,P) \;\text{ and }\; M(y,x;S,P^c) = M(y,x;S,P).$$

The aggregate interstitial advantage of candidate x over y decreases by one when moving from (S,P) to (S,P^c) because of the change in P_i, and the aggregate interstitial advantage of y over x in S decreases by one when moving from (S,P) to (S,P^c) because of the change in P_j. Therefore,

$$G(x,y;S,P^c) = G(x,y;S,P) - 1 \;\text{ and }\; G(y,x;S,P^c) = G(y,x;S,P) - 1.$$

Substituting and simplifying, it follows that the inequality

$$M(x,y;S,P^c) + G(x,y;S,P^c) \geq M(y,x;S,P^c) + G(y,x;S,P^c)$$

is equivalent to

$$M(x,y;S,P) + G(x,y;S,P) \geq M(y,x;S,P) + G(y,x;S,P). \;\blacksquare$$

The final lemma to be used in the proof of our characterization of the Borda generalized social welfare function establishes that we can find an xy-compensating profile for a situation (S,P), as long as (S,P) is not irreducibly xy-quasi-majoritarian.

Lemma 3. *If $(S,P) \in \mathcal{X} \times \mathcal{P}$ and $x,y \in S$ are such that (S,P) is not irreducibly xy-quasi-majoritarian, then there exists an xy-compensating profile P^c for (S,P).*

Proof. Suppose that $(S, P) \in \mathcal{X} \times \mathcal{P}$ and $x, y \in S$ are such that (S, P) is not irreducibly xy-quasi-majoritarian. This implies that there exist voters $i, j \in N$ such that x and y are not contiguous in P_i and not contiguous in P_j for S, and xP_iy and yP_jx. Because x and y are not contiguous in P_i and not contiguous in P_j for S, there exist candidates z and w in S such that

$$x \text{ and } z \text{ are contiguous in } P_i \text{ for } S \text{ and } x \text{ and } w \text{ are contiguous in } P_j \text{ for } S$$

and

$$xP_iz \text{ and } zP_iy \text{ and } yP_jw \text{ and } wP_jx.$$

It follows immediately that there exists a profile $P^c \in \mathcal{P}$ that differs from P only in that the positions of x and z are reversed in P_i, and the positions of w and x are reversed in P_j. By definition, P^c is an xy-compensating profile for (S, P). ■

As alluded to earlier, the conjunction of pairwise compensation and quasi-majoritarianism characterizes the Borda generalized social welfare function.

Theorem 1. *A generalized social welfare function F satisfies quasi-majoritarianism and pairwise compensation if and only if $F = F^B$.*

Proof. That the Borda generalized social welfare function F^B satisfies quasi-majoritarianism and pairwise compensation is straightforward to verify.

Now suppose that F is a generalized social welfare function that satisfies the two axioms. Let $(S, P) \in \mathcal{X} \times \mathcal{P}$ and $x, y \in S$.

(i) If (S, P) is irreducibly xy-quasi-majoritarian, quasi-majoritarianism implies

$$x\, F(S, P)\, y \;\Leftrightarrow\; M(x, y; S, P) + G(x, y; S, P) \geq M(y, x; S, P) + G(y, x; S, P).$$

By Lemma 1,

$$x\, F(S, P)\, y \;\Leftrightarrow\; b(x; S, P) \geq b(y; S, P)$$

and, by definition,

$$x\, F(S, P)\, y \;\Leftrightarrow\; x\, F^B(S, P)\, y. \tag{2}$$

(ii) If (S, P) is not irreducibly xy-quasi-majoritarian, Lemma 3 implies that there exists an xy-compensating profile P^{c1} for (S, P). This argument can be applied iteratively a finite number $K \geq 1$ of times (recall that the feasible set S is finite) until we reach a profile $P^c := P^{cK}$ such that (S, P^c) is irreducibly xy-quasi-majoritarian. By quasi-majoritarianism,

$$x\, F(S, P^c)\, y \;\Leftrightarrow\; M(x, y; S, P^c) + G(x, y; S, P^c) \geq M(y, x; S, P^c) + G(y, x; S, P^c). \tag{3}$$

The relative ranking of x and y according to $F(S, P^c)$ in (3) is unambiguous. This is the case because the ranking does not depend on the choice from among the possible xy-compensating profiles employed in each step of the iterative construction of the irreducibly xy-quasi-majoritarian situation (S, P^c). To see that this is indeed the case, suppose that (S, P) is not irreducibly xy-quasi-majoritarian. The path from (S, P) to an irreducibly xy-quasi-majoritarian situation (S, P^c) is not necessarily unique (and neither is the profile

P^c), but the comparison of the resulting sums of the values of M and G is—and, therefore, so is the relative ranking of x and y expressed in (3). By definition, a move from a given profile to an associated xy-compensating profile changes neither the majority advantage of x over y nor the majority advantage of y over x. Moreover, such a move reduces both the aggregate interstitial advantage of x over y and the aggregate interstitial advantage of y over x by a value of one; see also the proof of Lemma 2. Therefore, the comparison of the sums of the requisite values of M and G is unchanged. This means that the ranking of x and y that is obtained by means of the sum of the values of M and G must remain the same in every step of the iteration that takes us from (S, P) to (S, P^c). The choice of a specific xy-compensating profile in each step has no impact on the comparisons arrived at in (3); all that matters is that the iteration terminates once at least one of the two aggregate interstitial advantages is equal to zero. An illustration is provided after this proof by means of an example.

Applying Lemma 2 and pairwise compensation in each step of the above iteration, it follows that

$$x\, F(S,P)\, y \;\Leftrightarrow\; M(x,y;S,P) + G(x,y;S,P) \geq M(y,x;S,P) + G(y,x;S,P).$$

Using Lemma 1 and the definition of the Borda generalized social welfare function, we obtain (2). ■

To illustrate the proof technique applied in Theorem 1, consider the following example.

Example 1. *Suppose that there are five voters $N = \{1, \ldots, 5\}$. Consider the feasible set $S = \{x, y, z, w, v\}$ and the profile P such that the restriction of P to S is given by*

$$\begin{aligned} P_1 &: wyzxv, \\ P_2 &: xwzvy, \\ P_3 &: yvxwz, \\ P_4 &: xzywv, \\ P_5 &: wyxzv. \end{aligned}$$

The Borda scores of the candidates for (S, P) are $b(x; S, P) = 13$, $b(y; S, P) = 12$, $b(z; S, P) = 8$, $b(w; S, P) = 13$, and $b(v; S, P) = 4$.

Consider the two candidates x and y. It is immediate that

$$M(x,y;S,P) = 2 \;\; \textit{and} \;\; M(y,x;S,P) = 3$$

and

$$G(x,y;S,P) = 4 \;\; \textit{and} \;\; G(y,x;S,P) = 2.$$

The situation (S, P) is not irreducibly xy-quasi-majoritarian because, for example,

$$yP_1z \;\textit{and}\; zP_1x \;\textit{and}\; xP_2w \;\textit{and}\; wP_2y.$$

An xy-compensating profile P^{c1} for (S, P) is obtained by replacing zP_1x with $xP_1^{c1}z$ and xP_2w with $wP_2^{c1}x$; this is well-defined because x and z are contiguous in P_1 for S, and x

and w are contiguous in P_2 for S. Thus, we obtain

$$\begin{aligned} P_1^{c1} &: wyxzv, \\ P_2^{c1} &: wxzvy, \\ P_3^{c1} &: yvxwz, \\ P_4^{c1} &: xzywv, \\ P_5^{c1} &: wyxzv. \end{aligned}$$

It follows that

$$M(x, y; S, P^{c1}) = 2 \;\; \text{and} \;\; M(y, x; S, P^{c1}) = 3$$

and

$$G(x, y; S, P^{c1}) = 3 \;\; \text{and} \;\; G(y, x; S, P^{c1}) = 1.$$

Thus, both values of M are unchanged, and both G values are reduced by one as a consequence of the move from P to P^{c1}. By pairwise compensation,

$$x\, F(S, P)\, y \;\Leftrightarrow\; x\, F(S, P^{c1})\, y.$$

The situation (S, P^{c1}) is not irreducibly xy-quasi-majoritarian because, for example,

$$y P_3^{c1} v \;\text{and}\; v P_3^{c1} x \;\text{and}\; x P_4^{c1} z \;\text{and}\; z P_4^{c1} y.$$

An xy-compensating profile P^{c2} for (S, P^{c1}) is obtained by replacing $vP_3^{c1}x$ with $xP_3^{c2}v$ and $xP_4^{c1}z$ with $zP_4^{c2}x$, leading to

$$\begin{aligned} P_1^{c2} &: wyxzv, \\ P_2^{c2} &: wxzvy, \\ P_3^{c2} &: yxvwz, \\ P_4^{c2} &: zxywv, \\ P_5^{c2} &: wyxzv. \end{aligned}$$

We obtain

$$M(x, y; S, P^{c2}) = 2 \;\; \text{and} \;\; M(y, x; S, P^{c2}) = 3$$

and

$$G(x, y; S, P^{c2}) = 2 \;\; \text{and} \;\; G(y, x; S, P^{c2}) = 0.$$

By repeated application of pairwise compensation,

$$x\, F(S, P)\, y \;\Leftrightarrow\; x\, F(S, P^{c1})\, y \;\Leftrightarrow\; x\, F(S, P^{c2})\, y.$$

The situation (S, P^{c2}) is irreducibly xy-quasi-majoritarian and, letting $P^c = P^{c2}$, quasi-majoritarianism implies

$$\begin{aligned} x\, F(S, P^c)\, y \;&\Leftrightarrow\; M(x, y; S, P^c) + G(x, y; S, P^c) \geq M(y, x; S, P^c) + G(y, x; S^a, P^c) \\ &\Leftrightarrow\; 2 + 2 \geq 3 + 0. \end{aligned}$$

Because the inequality on the right side of this equivalence is satisfied, it follows that $x\,F(S,P^c)\,y$*. Combining the results of Lemmas 4 and 2, we obtain* $x\,F(S,P)\,y$*. Using Lemma 1, it follows that* $x\,F^B(S,P)\,y$*.*

It is straightforward to verify that the choice of the xy*-compensating profiles in each step does not affect the conclusion—the iteration always terminates with an irreducibly* xy*-quasi-majoritarian situation such that the aggregate interstitial advantage of* y *over* x *for this situation is equal to zero.* ■

The result of Theorem 1 can equivalently be established in an environment with a fixed feasible set. This observation follows because the axioms and the arguments used in the proof do not involve any changes in the feasible set.

4 Black

To characterize the class of Black generalized social welfare function, we first amend the two axioms that we employ in our axiomatization of the Borda generalized social welfare function. This is necessary because their scope has to be restricted to situations for which there is no strong Condorcet winner.

Quasi-majoritarianism on $\mathcal{NC}$. For all $S \in \mathcal{X}$, for all $x, y \in S$, and for all irreducibly xy-quasi-majoritarian profiles P for S such that $(S,P) \in \mathcal{NC}$,

$$x\,F(S,P)\,y \;\Leftrightarrow\; M(x,y;S,P) + G(x,y;S,P) \geq M(y,x;S,P) + G(y,x;S,P).$$

Pairwise compensation on $\mathcal{NC}$. For all $(S,P) \in \mathcal{NC}$, for all $x, y \in S$, and for all xy-compensating profiles P^c for (S,P) such that $(S,P^c) \in \mathcal{NC}$,

$$x\,F(S,P^c)\,y \;\Leftrightarrow\; x\,F(S,P)\,y.$$

We introduce two further axioms that appear in our main result. The first of these ensures that we stay within the domain $\mathcal{NC}$ when moving across different profiles in our proof. Note that, in Example 1, the situations resulting from the repeated application of pairwise compensating changes turn from an initial one in which no strong Condorcet winner exists, to new ones where a strong Condorcet winner is present. This is inconsequential when it comes to characterizing the Borda generalized social welfare function on the full domain, but must be prevented in our characterization of Black's generalized social welfare functions, where we need to restrict the use of the Borda rule and its predicated properties to the subdomain $\mathcal{NC}$. To avoid such an occurrence and still be able to apply similar reasoning, we propose to employ the following natural procedure.

Given a situation $(S,P) \in \mathcal{NC}$, augment the feasible set S by a non-empty set S' of candidates drawn from the set $X \setminus S$ to arrive at the set $S^a = S \cup S'$. Now define the profile P^a such that $(S^a,P^a) \in \mathcal{NC}$, all candidates in S' are ranked above all candidates in S according to P^a, and all voters' rankings of the candidates within the original set S are unchanged. The following definition introduces top augmentations formally.

Definition 4. *Let $(S,P) \in \mathcal{NC}$. A situation $(S^a,P^a) \in \mathcal{X} \times \mathcal{P}$ is a top augmentation of (S,P) if there exists a non-empty set $S' \subseteq X \setminus S$ such that*

(i) $S^a = S \cup S'$;

(ii) $(S^a,P^a) \in \mathcal{NC}$;

(iii) $zP_i^a x$ *for all* $x \in S$, *for all* $z \in S'$, *and for all* $i \in N$;

(iv) $[xP_i^a y \;\Leftrightarrow\; xP_i y]$ *for all* $x,y \in S$ *and for all* $i \in N$.

Clearly, the restriction that $(S^a,P^a) \in \mathcal{NC}$ is crucial in order to achieve the desired objective—a top augmentation must ensure that we remain within the subdomain $\mathcal{NC}$ when modifying the rankings within S. It is immediate that such top augmentations are easily found. For example, choosing the set $S' \subseteq X \setminus S$ and the profile P^a such that S' forms a top Latin square of size n for P^a will do. (A top Latin square of size n for P^a is a set S' of n candidates such that each candidate appears once in each position from 1 to n according to the profile P^a.)

The following axiom states that moving from a situation $(S,P) \in \mathcal{NC}$ to a top augmentation (S^a,P^a) of (S,P) does not alter the relative rankings of the candidates in S according the generalized social welfare function F.

Independence of top augmentations on $\mathcal{NC}$. For all $(S,P) \in \mathcal{NC}$, for all top augmentations (S^a,P^a) of (S,P), and for all $x,y \in S$,

$$x\, F(S^a,P^a)\, y \;\Leftrightarrow\; x\, F(S,P)\, y.$$

Our final axiom is the well-established property of strong Condorcet winner consistency. Phrased in terms of a generalized social welfare function, it requires that if there is a strong Condorcet winner for a situation, then the resulting aggregate weak order must be such that this strong Condorcet winner is uniquely at the top.

Strong Condorcet winner consistency. For all $(S,P) \in \mathcal{X} \times \mathcal{P}$, if there exists $x^c \in S$ such that $SCW(S,P) = \{x^c\}$, then, for all $y \in S \setminus \{x^c\}$,

$$x^c\, F(S,P)\, y \;\text{ and }\; \neg(y\, F(S,P)\, x^c).$$

The class of Black generalized social welfare functions can be characterized by adding independence of top augmentations on $\mathcal{NC}$ and strong Condorcet winner consistency to the modified axioms of the previous theorem. Before stating and proving this result, we establish one further preliminary observation.

Lemma 4. *For all $(S,P) \in \mathcal{X} \times \mathcal{P}$, for all $x,y \in S$, and for all top augmentations (S^a,P^a) of (S,P),*

$$\begin{aligned} & M(x,y;S^a,P^a) + G(x,y;S^a,P^a) \geq M(y,x;S^a,P^a) + G(y,x;S^a,P^a) \\ \Leftrightarrow\;\; & M(x,y;S,P) + G(x,y;S,P) \geq M(y,x;S,P) + G(y,x;S,P). \end{aligned}$$

Proof. Suppose that $(S, P) \in \mathcal{X} \times \mathcal{P}$, $x, y \in S$, and (S^a, P^a) is a top augmentation of (S, P). Because the values of $M(x, y; S, P)$, $G(x, y; S, P)$, $M(y, x; S, P)$, and $G(y, x; S, P)$ are unchanged if (S, P) is replaced by any top augmentation (S^a, P^a) of (S, P), it follows that

$$M(x, y; S^a, P^a) = M(x, y; S, P) \text{ and } G(x, y; S^a, P^a) = G(x, y; S, P)$$

and

$$M(y, x; S^a, P^a) = M(y, x; S, P) \text{ and } G(y, x; S^a, P^a) = G(y, x; S, P).$$

The conjunction of these equalities immediately implies the equivalence claimed in the lemma. ■

Theorem 2. *A generalized social welfare function F satisfies quasi-majoritarianism on $\mathcal{NC}$, pairwise compensation on $\mathcal{NC}$, independence of top augmentations on $\mathcal{NC}$, and strong Condorcet winner consistency if and only if F is a Black generalized social welfare function.*

Proof. That the Black generalized social welfare functions satisfy pairwise compensation on $\mathcal{NC}$, quasi-majoritarianism on $\mathcal{NC}$, independence of top augmentations on $\mathcal{NC}$, and strong Condorcet winner consistency is straightforward to verify.

Conversely, suppose that F is a generalized social welfare function that satisfies the four axioms.

(i) If $(S, P) \notin \mathcal{NC}$, there exists $x^c \in S$ with $SCW(S, P) = \{x^c\}$. Strong Condorcet winner consistency implies that, for all $y \in S \setminus \{x^c\}$,

$$x^c \, F(S, P) \, y \text{ and } \neg(y \, F(S, P) \, x^c),$$

as prescribed by the definition of the Black generalized social welfare functions.

(ii) If $(S, P) \in \mathcal{NC}$, let $x, y \in S$. Suppose that (S^a, P^a) is a top augmentation of (S, P), and let S' be the set identified in the definition of a top augmentation. By definition, none of the candidates in S' can be a strong Condorcet winner for (S^a, P^a). Moreover, because $zP_i^a x$ for all $x \in S$, for all $z \in S'$, and for all $i \in N$, none of the candidates in S can be a strong Condorcet winner for (S^a, P^a). Therefore, $SCW(S^a, P^a) = \emptyset$ and, as a consequence, (S^a, P^a) is an element of the subdomain $\mathcal{NC}$. By independence of top augmentations on $\mathcal{NC}$, it follows that

$$x \, F(S, P) \, y \;\Leftrightarrow\; x \, F(S^a, P^a) \, y. \tag{4}$$

If P^a is an irreducibly xy-quasi-majoritarian profile for S^a, let $P^{ac} = P^a$.

If P^a is not irreducibly xy-quasi-majoritarian for S^a, we iteratively define as many xy-compensating profiles as required to arrive at an irreducibly xy-quasi-majoritarian profile P^{ac} for S^a; see the proof of Theorem 1. Observe that, in each step of this iteration, the requisite set of strong Condorcet winners is empty. By definition, no member of S' can be a strong Condorcet winner because each one of them is beaten by another member of S' in a pairwise majority contest; this is immediate because none of them can be beaten by a member of S by definition. The changes in moving from one profile to the next in the iterative process can only affect candidates in S. To see that this is indeed the case, observe that part (ii) of the definition of an xy-compensating profile requires that z is ranked between x and y in P_i, and w is ranked between y and x in P_j. This implies that neither z nor w can be a member of S', because neither x nor y belongs to S'. Therefore, none of the candidates in S can become a strong Condorcet winner because all of them continue to be beaten in a pairwise majority contest by all candidates in S'. This iteration terminates because the set S^a is finite. Repeated application of pairwise compensation on $\mathcal{NC}$ implies that

$$x\,F(S^a, P^a)\,y \;\Leftrightarrow\; x\,F(S^a, P^{ac})\,y. \tag{5}$$

By quasi-majoritarianism on $\mathcal{NC}$, we obtain

$$x\,F(S^a, P^{ac})\,y \;\Leftrightarrow\; M(x, y; S^a, P^{ac}) + G(x, y; S^a, P^{ac}) \geq M(y, x; S^a, P^{ac}) + G(y, x; S^a, P^{ac}). \tag{6}$$

Combining (4), (5), and (6), we obtain

$$x\,F(S, P)\,y \;\Leftrightarrow\; M(x, y; S^{ac}, P^{ac}) + G(x, y; S^{ac}, P^{ac}) \geq M(y, x; S^{ac}, P^{ac}) + G(y, x; S^{ac}, P^{ac}).$$

Applying Lemma 2 and Lemma 4, it follows that

$$x\,F(S, P)\,y \;\Leftrightarrow\; M(x, y; S, P) + G(x, y; S, P) \geq M(y, x; S, P) + G(y, x; S, P).$$

Using Lemma 1, we obtain

$$x\,F(S, P)\,y \;\Leftrightarrow\; b(x; S, P) \geq b(y; S, P) \;\Leftrightarrow\; x\,F^B(S, P)\,y,$$

which is the ranking according to the Black generalized social welfare functions. ■

References

Arrow, K.J. (1951, second ed. 1963). Social Choice and Individual Values. Wiley, New York.

Baharad, E. and S. Nitzan (2007), The cost of implementing the majority principle: the golden voting rule. Economic Theory, 31, 69–84.

Baharad, E. and S. Nitzan (2011). Condorcet vs. Borda in light of a dual majoritarian approach. Theory and Decision, 71, 151–162.

Barberà, S. and W. Bossert (2023). Opinion aggregation: Borda and Condorcet revisited. Journal of Economic Theory, 210, Article 105654.

Barberà, S., W. Bossert, and K. Suzumura (2021). Daunou's voting rule and the lexicographic assignment of priorities. Social Choice and Welfare, 56, 259–289.

Black, D. (1958). The Theory of Committees and Elections. Cambridge University Press, Cambridge, UK.

Black, D. (1976). Partial justification of the Borda count. Public Choice, 28, 1–15.

Borda, J.-C. de (1781). Mémoire sur les élections au scrutin. Mémoires de l'Académie Royale des Sciences année 1781, 657–665. Translated and reprinted in I. McLean and A.B. Urken, eds. (1995). Classics of Social Choice. University of Michigan Press, Ann Arbor, Chapter 5.

Condorcet, M.J.A.N. de (1785). Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. Imprimerie Royale, Paris. Translated and reprinted in part in I. McLean and A.B. Urken, eds. (1995). Classics of Social Choice. University of Michigan Press, Ann Arbor, Chapter 6.

Daunou, P.C.F. (1803). Mémoire sur les élections au scrutin. Baudouin, imprimeur de l'Institut National, Paris. Translated and reprinted in I. McLean and A.B. Urken, eds. (1995). Classics of Social Choice. University of Michigan Press, Ann Arbor, Chapter 11.

Fishburn, P.C. (1977), Condorcet social choice functions, SIAM Journal on Applied Mathematics, 33, 469–489.

Hansson, B. and H. Sahlquist (1976). A proof technique for social choice with variable electorate. Journal of Economic Theory, 13, 193–200.

Herrero, C. and A. Villar (2021). Group decisions from individual rankings: the Borda-Condorcet rule. European Journal of Operational Research, 291, 757–765.

Laslier, J.-F. (1997). Tournament solutions and majority voting. Springer, Berlin.

Mihara, H.R. (2017). Characterizing the Borda ranking rule for a fixed population. MPRA Paper No. 78093, available at `https://mpra.ub.uni-muenchen.de/78093/`.

Morales, J.I. (1797). Memoria matemática sobre el cálculo de la opinion en las elecciones. Imprenta Real, Madrid. Translated and reprinted in I. McLean and A.B. Urken, eds. (1995). Classics of Social Choice. University of Michigan Press, Ann Arbor, Chapter 10.

Moulin, H. (1988). Axioms of cooperative decision making. Cambridge University Press, New York.

Nitzan, A.D. and S. Nitzan (2024). Balancing democracy: majoritarianism vs. expression of preference intensity. Public Choice, 200, 149–171.

Nitzan, S. and A. Rubinstein (1981). A further characterization of Borda ranking method. Public Choice, 36, 153–158.

Saari, D. (1994). Geometry of Voting. Springer, Berlin.

Young, H.P. (1974). An axiomatization of Borda's rule. Journal of Economic Theory, 9, 43–52.

Young, H.P and A. Levenglick (1978). A consistent extension of Condorcet's election principle. SIAM Journal on Applied Mathematics, 35, 285–300.